BOOTSTRAPPING GAGA THEOREMS

ANIRUDDH VENKATESAN

Abstract. Let X be a noetherian scheme with affine diagonal, and let $\lambda : \mathcal{X} \to X$ be a morphism of locally ringed spaces from a quasi-compact quasi-separated locally strongly noetherian analytic adic space. We prove that an equivalence on coherent sheaves, together with comparison of their cohomology, extends to every proper X-scheme. As an application, we obtain GAGA for proper schemes over the Fargues–Fontaine curve in equal and mixed characteristic.

Contents

1. Introduction 1
2. The comparison on the base 4
3. Relative full faithfullness 11
4. Localization at the boundary 18
5. Pseudocoherent GAGA 30
6. Coherent sheaves and classical GAGA 35
7. The Fargues–Fontaine curve 46
Appendix A. Recollections on condensed mathematics 50
Appendix B. Descent 60
References 71

hexagon:2611.00363v1

1. Introduction

We prove that, in the adic setting below, a GAGA comparison on a base extends to every proper scheme over it. Given an equivalence on coherent sheaves and a comparison of their cohomology, . For the schematic and adic Fargues–Fontaine curves, this gives GAGA in equal and mixed characteristic.

1.1. **The relative problem.** Let X be a noetherian scheme with affine diagonal, let $\mathcal{X}$ be a quasi-compact quasi-separated locally strongly noetherian analytic adic space, and let $\lambda : \mathcal{X} \to X$ be a morphism of locally ringed spaces. Assume that pullback gives an exact equivalence

$$\lambda^* : \mathrm{Coh}(X) \xrightarrow{\sim} \mathrm{Coh}(\mathcal{X})$$

and that the canonical maps

$$H^i(X, \mathcal{F}) \longrightarrow H^i(\mathcal{X}, \lambda^* \mathcal{F})$$

are isomorphisms for every coherent $\mathcal{F}$ and every $i \geq 0$. We keep these hypotheses throughout Sections 2–6.

For a proper X-scheme Y, write $c_Y : Y_{\mathcal{X}}^{\mathrm{ad}} \to Y$ for its relative Huber analytification. Over an affinoid $V = \mathrm{Spa}(R, R^+) \subset \mathcal{X}$ mapping into an affine open of X, this is the analytification of $Y \times_X \operatorname{Spec} R$ over V. The local spaces agree under rational base change and hence glue [Hub94, Proposition 3.8].

Theorem 6.14. *For every proper $f : Y \to X$, analytification induces equivalences*

$$\begin{aligned}\mathrm{Coh}(Y) &\xrightarrow{\sim} \mathrm{Coh}(Y_{\mathcal{X}}^{\mathrm{ad}}),\\ \mathrm{D}^b_{\mathrm{coh}}(Y) &\xrightarrow{\sim} \mathrm{D}^b_{\mathrm{coh}}(Y_{\mathcal{X}}^{\mathrm{ad}}),\\ \mathrm{Perf}(Y) &\xrightarrow{\sim} \mathrm{Perf}(Y_{\mathcal{X}}^{\mathrm{ad}}),\\ \mathrm{Vect}(Y) &\xrightarrow{\sim} \mathrm{Vect}(Y_{\mathcal{X}}^{\mathrm{ad}}).\end{aligned}$$

For $\mathcal{F} \in \mathrm{Coh}(Y)$ and $i \geq 0$, the canonical maps are isomorphisms:

$$\begin{aligned}(R^i f_* \mathcal{F})^{\mathrm{ad}} &\xrightarrow{\sim} R^i f_*^{\mathrm{ad}}(\mathcal{F}^{\mathrm{ad}}),\\ H^i(Y, \mathcal{F}) &\xrightarrow{\sim} H^i(Y_{\mathcal{X}}^{\mathrm{ad}}, \mathcal{F}^{\mathrm{ad}}).\end{aligned}$$

1.2. **The solid comparison.** Write $\mathrm{D}_{\mathrm{qc}}(T)$ for the category of quasicoherent complexes and $\mathrm{D}_{\mathrm{pc}}(T)$ for its pseudocoherent subcategory; on a noetherian scheme, $\mathrm{D}_{\mathrm{pc}}(T) = \mathrm{D}^-_{\mathrm{coh}}(T)$. On the analytic side, let $\mathcal{C}_{\square}(\mathcal{Z})$ denote solid quasicoherent complexes, and let $\mathsf{D}^{\square}_{\mathrm{pc}}(\mathcal{Z})$ be the subcategory defined by nuclearity and pseudocompactness on bounded charts. On $\mathrm{Spa}(R, R^+)$ this subcategory is $\mathrm{Cond}_R(\mathrm{D}_{\mathrm{pc}}(R))$ [And21, Theorem 5.50]. The definitions and comparison functors are recalled in Appendix A.

A separated finitely presented T/X has an algebraic realization $T_{\mathcal{X}}^{\mathrm{alg}}$ and an analytic locus $T_{\mathcal{X}}^{\mathrm{an}}$. They are analytic stacks and coincide for proper T/X [LH26, A.9–A.13]. The comparison on their solid quasicoherent categories gives the stronger statement below.

Theorem 5.5. *For proper $f : Y \to X$, the comparison*

$$\alpha_Y : \mathrm{D}_{\mathrm{qc}}(Y) \longrightarrow \mathcal{C}_{\square}(Y_{\mathcal{X}}^{\mathrm{an}})$$

is fully faithful, colimit-preserving, and strong symmetric monoidal. It takes perfect complexes to compact objects and induces a symmetric monoidal equivalence

$$\mathrm{D}_{\mathrm{pc}}(Y) \xrightarrow{\sim} \mathsf{D}^{\square}_{\mathrm{pc}}(Y_{\mathcal{X}}^{\mathrm{an}}).$$

For every $M \in \mathrm{D}_{\mathrm{qc}}(Y)$, the canonical comparisons are equivalences:

$$\begin{aligned}\alpha_X(\mathbf{R}f_* M) &\xrightarrow{\sim} \mathbf{R}f_*^{\mathrm{an}} \alpha_Y(M),\\ \mathbf{R}\Gamma(Y, M) &\xrightarrow{\sim} \mathbf{R}\Gamma_{\square}(Y_{\mathcal{X}}^{\mathrm{an}}, \alpha_Y M).\end{aligned}$$

1.3. **The boundary calculation.** The main calculation is a relative localization formula. Let $f : Y \to X$ be proper, let $j : U \hookrightarrow Y$ be quasi-compact open, and put $q = f \circ j$. Blowing up an ideal cutting out $Y \setminus U$ gives $p : \widetilde{Y} \to Y$, an isomorphism over U, with effective Cartier boundary D. Put $\widetilde{f} = f \circ p$. For $C \in \mathcal{C}_{\square}(Y_{\mathcal{X}}^{\mathrm{an}})$, write $\widetilde{C} = (p^{\mathrm{an}})^* C$ and set

$$E_n(C) = \mathbf{R}\widetilde{f}_*^{\mathrm{an}}\big(\widetilde{C} \otimes \mathcal{O}_{\widetilde{Y}}(nD)\big).$$

The canonical section of $\mathcal{O}(D)$ gives the transition maps.

Theorem 5.2. *With this notation, there is a canonical equivalence, natural in C,*

$$\mathbf{R}q_*^{\mathrm{alg}} j^{\mathrm{alg},*} C \simeq \operatorname*{colim}_{n \geq 0} \mathbf{R}\widetilde{f}_*^{\mathrm{an}}\big(\widetilde{C} \otimes \mathcal{O}_{\widetilde{Y}}(nD)\big).$$

If C is pseudocoherent, the sequence $E_\bullet(C)$ lifts functorially to a sequence $M_\bullet(C)$ in $\mathrm{D}_{\mathrm{pc}}(X)$. Its colimit satisfies

$$\alpha_X\Big(\operatorname*{colim}_n M_n(C)\Big) \simeq \mathbf{R}q_*^{\mathrm{alg}} j^{\mathrm{alg},*} C$$

and, with this comparison, is canonically independent of the blowup.

Locally, the formula is the identity $A[t^{-1}] = \operatorname{colim}(A \xrightarrow{t} A \xrightarrow{t} \cdots)$. For pseudocoherent C, each proper direct image $E_n(C)$ is pseudocoherent and can therefore be algebraized on X. We take the colimit of the resulting sequence in $\mathrm{D}_{\mathrm{qc}}(X)$. When q is affine, full faithfulness lifts the $\alpha_X(q_*\mathcal{O}_U)$-action to this colimit and so produces a complex on U. Its pseudocoherence follows by faithfully flat descent.

1.4. **Outline of the proof.** The first step is to extend the comparison on the base to complexes. Exact coherent pullback makes the chart maps flat, so scalar extension followed by condensification defines α_X. Coherent-cohomology comparison gives full faithfulness on perfect complexes. Their analytic images are compact by finite open-cover descent, which allows us to pass to filtered colimits and obtain full faithfulness on $\mathrm{D}_{\mathrm{qc}}(X)$. We then algebraize bounded coherent complexes by their truncation triangles. For a pseudocoherent complex, the bounded truncations form a tower whose cohomology is eventually constant in each degree; taking its inverse limit on ordinary affine module categories gives the algebraization.

Over the general base, the module description gives full faithfulness on affine schemes, and Zariski descent extends it to relative schemes. For a proper scheme, the boundary formula supplies algebraic preimages on affine opens. A finite faithfully flat family detects pseudocoherence, while full faithfulness recovers the gluing data. Applying the comparison to the algebraic counit defines the proper-direct-image map. We prove that it is an equivalence first on pseudocoherent complexes and then on all quasicoherent complexes by passage to colimits. Finally, sheafification of the ordinary affinoid representatives identifies the coherent and bounded categories with their Huber counterparts.

For the application, let E be a nonarchimedean local field with residue field $\mathbf{F}_q$, and let F be a complete perfectoid field of characteristic p containing $\mathbf{F}_q$. The schematic and adic curves $X^{\mathrm{sch}}_{F,E}$ and $X^{\mathrm{ad}}_{F,E}$ satisfy the base hypotheses [KL19, Theorems 4.7.3–4.7.5]. Section 7 recalls their constructions and verifies these hypotheses, giving Theorem 7.4.

1.5. **Comparison with the work Clausen–Scholze.** Our proof follows the boundary-localization argument of [CS26, Theorem 13.10], in the Tate-base form of [LH26, A.13 and A.16–A.18]. There the complex-field case begins with the comparison over $\operatorname{Spec}\mathbf{C}$. Here the base may be nonaffine, and the direct images of the boundary twists must be algebraized as complexes of sheaves on X. The comparison on the base supplies these complexes, their transition maps, and the algebra action on their colimit. We use the affinoid results of [And21]; other GAGA theorems and criteria appear in [Ser56, FK18, Hal22, Wan26].

Section 2 proves the comparison on the base. Sections 3–4 construct the relative realizations and establish proper direct image and localization, and Section 5 proves the bootstrapping theorem. Section 6 recovers ordinary GAGA and compares proper direct images between proper X-schemes in Corollary 6.16. The Fargues–Fontaine application occupies Section 7. The appendices recall the analytic categories and give the descent arguments.

Notation and conventions. Derived categories are stable ∞-categories, with cohomological indexing $H^i(K[n]) = H^{i+n}(K)$. A subscript conn means cohomologically nonpositive. We write Map for mapping spectra in stable categories and mapping spaces otherwise. Tensor products are derived, except on coherent hearts; in complete-module categories they

include analytic completion. A finite module means a finitely generated module. Over a coherent ring, coherent cohomology means finitely presented cohomology modules.

For a topological ring R, R_{disc} and $\underline{R}$ denote its discrete and topological condensed rings. We write $\mathsf{D}_{\mathrm{cond}}(B)$ for the derived category of condensed B_{disc}-modules, $\mathsf{D}(\mathcal{A})$ for complete modules over an analytic ring, and $\mathsf{D}_{\square}(R, R^+) = \mathsf{D}((R, R^+)^{\square})$. Condensification is $\mathrm{Cond}_R(M) = \underline{R} \otimes^{\mathbf{L}}_{R_{\mathrm{disc}}} M_{\mathrm{disc}}$. An affinoid representative of C on $V = \mathrm{Spa}(R, R^+)$ is a pseudocoherent R-complex M with $\mathrm{Cond}_R(M) \simeq C|_V$.

The symbols alg, an, and ad distinguish the two analytic-stack realizations from ordinary Huber analytification. We use α_T for the comparison on algebraic realizations and α_T^{an} for its restriction to the analytic locus; for proper T they agree. Algebraization includes a specified equivalence with the analytic image. For an adic space Z, $\mathcal{C}_{\square}(Z)$ abbreviates $\mathcal{C}_{\square}(Z^{\square})$. The notation $\mathrm{Cart}_V \mathcal{C}_V$ means families with compatible pullback equivalences. In Section 7, standard affines refer to the Proj presentation of the curve. We write $K(nD) = K \otimes \mathcal{O}(nD)$.

1.6. **Acknowledgements.** This project was conceived in the summer of 2025 while the author was a Summer Undergraduate Research Fellow at UC Berkeley. We thank the organizers and other participants of the program for helping to foster a productive research environment. We thank Zeyu Liu for mentoring us on the initial portions of this work. We thank Andrew Cheng, Sug Woo Shin, and Avi Zeff for discussions regarding this work.

1.7. **AI Usage Statement.** AI agents (specifically GPT 6 Pro) were used in the preparation of this manuscript for brainstorming, proofreading, and consistency-checking. In particular, the AI suggested that a separatedness hypothesis in an earlier version of the main result could be weakened to merely requiring an affine diagonal, and suggested references useful for the writing of Appendix B. The author takes full responsibility for the completeness and correctness of all of its content, mathematical and otherwise.

2. The comparison on the base

Throughout this section, $\lambda : \mathcal{X} \to X$ is as in Section 1.1. We construct α_X on quasicoherent complexes and prove full faithfulness. The coherent equivalence will then allow us to recover pseudocoherent complexes from their bounded truncations.

We use affine opens of X and their intersections to describe scalar extension. These intersections are affine [Stacks, Tag 0BDX], and every morphism from an affine scheme to X is affine [Stacks, Lemma 29.11.12]. The same applies to a separated X-scheme T, since its diagonal factors as

$$T \longrightarrow T \times_X T \longrightarrow T \times T.$$

Here the first map is a closed immersion and the second is a base change of Δ_X. Thus T has affine diagonal, and its affine opens also have affine intersections.

2.1. **Affinoid charts and scalar extension.** Let $\mathcal{B} = \mathrm{RatAff}_X(\mathcal{X})$ be the basis of strongly noetherian affinoid opens $V \subset \lambda^{-1}(W)$, where $W \subset X$ is affine. Its restriction maps are rational inclusions. We use finite covers from this basis, refining intersections by common rational subdomains.

Lemma 2.1. *Let $W = \operatorname{Spec} A \subset X$ be affine and let $V = \mathrm{Spa}(R, R^+) \subset \lambda^{-1}(W)$ be a strongly noetherian affinoid open. Then $A \to R$ is flat, and $\lambda : \mathcal{X} \to X$ is flat on local rings.*

Proof. By the ideal criterion for flatness, it suffices to show that

$$I \otimes_A R \longrightarrow R$$

is injective for every finitely generated ideal $I \subset A$.

The open immersion $W \hookrightarrow X$ is quasi-compact, so the coherent ideal $\widetilde{I} \subset \mathcal{O}_W$ extends to a coherent ideal $\mathcal{J} \subset \mathcal{O}_X$ with $\mathcal{J}|_W = \widetilde{I}$ [Stacks, Section 28.23, especially Lemma 28.23.2]. Exactness of λ^* preserves the injection $\mathcal{J} \hookrightarrow \mathcal{O}_X$. On V we therefore have

$$(\lambda^*\mathcal{J})|_V \longrightarrow \mathcal{O}_V.$$

Pull back a finite presentation of I to V. The same matrix over R has cokernel $I \otimes_A R$, so the affinoid coherent-sheaf theorem [Zav25, Theorem 4.2] identifies the induced map on sections with $I \otimes_A R \to R$. Taking sections preserves the injection above, and the ideal criterion gives flatness.

For $x \in V$ with $\mathfrak{p} = \lambda(x)$, the map $R \to \mathcal{O}_{V,x}$ is flat by [Zav25, Lemma 6.7]. Its composite with $A \to R$ remains flat. Localizing at $\mathfrak{p}$ gives the asserted flatness on local rings. □

Let $M \in \mathrm{D}_{\mathrm{qc}}(X)$. For an affine open $W = \operatorname{Spec} A$, put

$$M_W = \mathbf{R}\Gamma(W, M|_W) \in \mathrm{D}(A).$$

On $V = \operatorname{Spa}(R, R^+) \subset \lambda^{-1}(W)$, define

$$\alpha_{W,V}(M) = \mathrm{Cond}_R(R \otimes^{\mathbf{L}}_A M_W).$$

Thus the local comparison extends scalars from A to R and then condensifies. On a complex of finite free modules it retains the differential matrices, now with entries in $\underline{R}$.

Suppose V maps into two affine opens $W_i = \operatorname{Spec} A_i$, and write their affine intersection as $W_{12} = \operatorname{Spec} A_{12}$. Quasicoherent restriction gives

$$A_{12} \otimes^{\mathbf{L}}_{A_i} M_{W_i} \simeq M_{W_{12}}.$$

Since each $A_i \to R$ factors through A_{12}, tensoring with R identifies both local definitions with

$$\mathrm{Cond}_R(R \otimes^{\mathbf{L}}_{A_{12}} M_{W_{12}}).$$

Next let $V' = \operatorname{Spa}(R', R'^+) \subset V$ be rational. Condensification commutes with scalar extension, so

$$\begin{aligned} \alpha_{W,V}(M)|_{V'} &\simeq \mathrm{Cond}_{R'}\big(R' \otimes^{\mathbf{L}}_R (R \otimes^{\mathbf{L}}_A M_W)\big) \\ &\simeq \mathrm{Cond}_{R'}(R' \otimes^{\mathbf{L}}_A M_W) = \alpha_{W,V'}(M). \end{aligned}$$

For successive rational inclusions, the comparisons are the associativity equivalences of the same tensor product. The comparisons obtained by changing the schematic affine have the same compatibility. Thus the local objects form a cartesian section, which descends to $\alpha_X(M)$.

The construction is functorial in M. Its tensor-product and unit comparisons commute with restriction, so descent makes α_X strong symmetric monoidal. On each chart it is exact and preserves filtered colimits [And21, Theorem 5.9 and Lemma 5.10]. It therefore preserves arbitrary coproducts, expressed as filtered colimits of finite partial sums, and hence all small colimits. Since colimits of cartesian sections are computed on charts, α_X has the same properties.

2.2. **Finite descent and full faithfulness.** For perfect complexes, duality expresses the mapping spectrum as global sections of a tensor product. We first prove that these sections commute with filtered colimits; this will allow us to pass from perfect to arbitrary quasicoherent complexes.

Lemma 2.2. *For every quasi-compact quasi-separated analytic adic space $\mathcal{Z}$, the tensor unit of $\mathcal{C}_\square(\mathcal{Z})$ is compact. Thus $\mathbf{R}\Gamma_\square(\mathcal{Z}, -)$ preserves filtered colimits.*

Proof. Choose a finite affinoid cover (V_i) of $\mathcal{Z}$, and put $V_I = \bigcap_{i\in I} V_i$ for nonempty I. Each V_I is quasi-compact and admits a finite cover by rational subdomains of an affinoid containing it. The intersections in this cover are again rational affinoids.

On an analytic affine, the unit is the measure module for the one-point set and is compact [And21, Theorem 3.28 and Proposition 2.11]. Proposition B.6 therefore shows that $\mathbf{R}\Gamma_\square(V_I, -)$ preserves filtered colimits.

Let P be the poset of nonempty subsets of the covering indices. Finite descent gives

$$\mathbf{R}\Gamma_\square(\mathcal{Z}, K) \simeq \lim_{I\in P} \mathbf{R}\Gamma_\square(V_I, K|_{V_I})$$

by Lemma B.5. The nerve of P has finitely many nondegenerate simplices, so the limit is finite. Restrictions preserve colimits, and filtered colimits of spectra commute with finite homotopy limits. It follows that $\mathbf{R}\Gamma_\square(\mathcal{Z}, -)$ preserves filtered colimits, or equivalently, that the unit is compact. □

Duality reduces the comparison of maps between perfect complexes to the following comparison of global sections.

Lemma 2.3. *For every bounded coherent complex K on X, the natural maps*

$$\mathbf{R}\Gamma_\square(\mathcal{X}, \alpha_X K) \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{adic}}(\mathcal{X}, \lambda^* K) \xleftarrow{\sim} \mathbf{R}\Gamma(X, K)$$

are equivalences.

Proof. We extend the coherent-cohomology comparison to K by applying it to successive truncation triangles. Choose $a \le b$ such that $\mathcal{H}^i(K) = 0$ outside $[a, b]$, and put $K_r = \tau_{\le r} K$. The successive triangles are

$$K_{r-1} \longrightarrow K_r \longrightarrow \mathcal{H}^r(K)[-r] \longrightarrow K_{r-1}[1].$$

By Lemma 2.1, pullback is flat and preserves these triangles. Applying the two section functors gives a comparison of triangles whose third term is the coherent-sheaf comparison of Section 1.1. Beginning with $K_{a-1} = 0$, induction on r gives the comparison for $K_b = K$.

To compare solid and ordinary analytic sections, take $V = \mathrm{Spa}(R, R^+) \subset \lambda^{-1}(W)$ with $W = \operatorname{Spec} A$, and set

$$K_R = R \otimes^{\mathbf{L}}_A \mathbf{R}\Gamma(W, K|_W).$$

Flatness gives finite R-cohomology in $[a, b]$. The restriction of $\alpha_X K$ is $\mathrm{Cond}_R(K_R)$, so full faithfulness of condensification gives

$$\begin{aligned} \mathbf{R}\Gamma_\square(V, \alpha_X K) &\simeq \mathrm{Map}_{\mathsf{D}_\square(R,R^+)}(\mathrm{Cond}_R R, \mathrm{Cond}_R K_R) \\ &\simeq K_R \end{aligned}$$

[And21, Theorem 5.9]. Ordinary analytic sections give the same complex: for a finite module this is coherent affinoid acyclicity, and the bounded case follows from the truncation triangles above.

Both local identifications commute with rational restriction, since both are induced by scalar extension of K_R. Choose a finite affinoid cover subordinate to affine opens of X, and cover each intersection by finitely many rational subdomains of one of its members. Taking limits of the local comparisons, first on these intersections and then on $\mathcal{X}$, gives the assertion. Solid sections are computed by Lemma B.5, and ordinary sections by the finite alternating Čech complex [Stacks, Tag 08BZ]. □

Combining the section comparison with compactness gives full faithfulness on quasicoherent complexes.

Theorem 2.4. *The functor $\alpha_X : \mathrm{D}_{\mathrm{qc}}(X) \longrightarrow \mathcal{C}_{\square}(\mathcal{X})$ is fully faithful. It sends perfect complexes to compact objects, and its essential image is closed under arbitrary colimits. The same conclusions hold if the coherent-cohomology hypothesis is replaced by the canonical comparison*

$$\mathbf{R}\Gamma(X, P) \xrightarrow{\sim} \mathbf{R}\Gamma(\mathcal{X}, \lambda^* P)$$

for every $P \in \mathrm{Perf}(X)$.

Proof. Let P, Q be perfect complexes. Strong monoidality identifies $\alpha_X(P^\vee)$ with the dual of $\alpha_X P$, so

$$\begin{aligned}\mathrm{Map}_{\mathcal{C}_{\square}(\mathcal{X})}(\alpha_X P, \alpha_X Q) &\simeq \mathbf{R}\Gamma_{\square}\big(\mathcal{X}, \alpha_X(P^\vee \otimes Q)\big)\\ &\simeq \mathbf{R}\Gamma(X, P^\vee \otimes Q)\\ &\simeq \mathrm{Map}_{\mathrm{D}_{\mathrm{qc}}(X)}(P, Q).\end{aligned}$$

The middle equivalence is Lemma 2.3 applied to $P^\vee \otimes Q$. Hence α_X is fully faithful on perfect complexes.

For a filtered diagram (N_j) in $\mathcal{C}_{\square}(\mathcal{X})$, duality gives

$$\mathrm{Map}(\alpha_X P, N_j) \simeq \mathbf{R}\Gamma_{\square}(\mathcal{X}, \alpha_X(P^\vee) \otimes N_j).$$

Tensor product preserves colimits, and global sections preserve filtered colimits by Lemma 2.2. Thus

$$\operatorname*{colim}_j \mathrm{Map}(\alpha_X P, N_j) \simeq \mathrm{Map}(\alpha_X P, \operatorname*{colim}_j N_j).$$

This proves compactness of $\alpha_X P$.

By [Stacks, Theorem 36.15.3, Tag 09IS, and Proposition 36.17.1, Tag 09M0], $\mathrm{D}_{\mathrm{qc}}(X) \simeq \mathrm{Ind}(\mathrm{Perf}(X))$. Write $M \simeq \operatorname{colim}_i P_i$ and $N \simeq \operatorname{colim}_j Q_j$ as filtered colimits of perfect complexes. Then

$$\begin{aligned}\mathrm{Map}(\alpha_X M, \alpha_X N) &\simeq \lim_i \operatorname*{colim}_j \mathrm{Map}(\alpha_X P_i, \alpha_X Q_j)\\ &\simeq \lim_i \operatorname*{colim}_j \mathrm{Map}(P_i, Q_j)\\ &\simeq \mathrm{Map}(M, N).\end{aligned}$$

The first equivalence uses colimit preservation and compactness of the perfect images; the second is the comparison just proved. The last uses compactness of perfect complexes on X. This proves full faithfulness on $\mathrm{D}_{\mathrm{qc}}(X)$.

By full faithfulness and Lemma B.2, every diagram in the essential image lifts, with its maps and homotopies, to a diagram (M_i) on X. Colimit preservation gives

$$\operatorname*{colim}_i \alpha_X(M_i) \simeq \alpha_X(\operatorname*{colim}_i M_i).$$

Thus the essential image is closed under colimits.

Now suppose that cohomology comparison is given for perfect complexes. Apply it to $P^\vee \otimes Q$, and use the affinoid section calculation in Lemma 2.3 to compare solid and ordinary analytic sections. This gives the same mapping-spectrum calculation for P, Q, and compactness again extends full faithfulness to $\mathrm{D}_{\mathrm{qc}}(X)$. Mapping from the structure sheaf then gives, for every coherent $\mathcal{F}$,

$$\mathbf{R}\Gamma(X, \mathcal{F}) \simeq \mathrm{Map}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}, \alpha_X \mathcal{F}) \simeq \mathbf{R}\Gamma(\mathcal{X}, \lambda^* \mathcal{F}).$$

The second equivalence is the local section comparison. Both maps are induced by pullback, so this identifies the canonical coherent-cohomology map. □

2.3. **Pseudocoherent complexes.** We turn to essential surjectivity. The bounded coherent truncations of an analytic pseudocoherent complex can be algebraized one at a time. To recover the whole complex, we first express analytic descent in terms of ordinary complexes on affinoids.

Put $\mathcal{B} = \mathrm{RatAff}_X(\mathcal{X})$ and $R_V = \Gamma(V, \mathcal{O}_V)$. Rational restriction gives the diagram

$$\begin{aligned} \mathcal{B}^{\mathrm{op}} &\longrightarrow \mathrm{Cat}_\infty, \\ V &\longmapsto \mathrm{D}_{\mathrm{pc}}(R_V), \end{aligned}$$

with transition functor $M \mapsto R_{V'} \otimes^{\mathbf{L}}_{R_V} M$ for $V' \subset V$. Write $\mathrm{Cart}_{V \in \mathcal{B}} \mathrm{D}_{\mathrm{pc}}(R_V)$ for its limit.

A cartesian section consists of complexes $M_V \in \mathrm{D}_{\mathrm{pc}}(R_V)$ and equivalences

$$\theta_{V',V} : R_{V'} \otimes^{\mathbf{L}}_{R_V} M_V \xrightarrow{\sim} M_{V'}.$$

For $V'' \subset V' \subset V$, these satisfy $\theta_{V'',V} \simeq \theta_{V'',V'} \circ (1 \otimes \theta_{V',V})$ under tensor associativity, together with the unit and higher compatibility homotopies. A morphism is a family $M_V \to N_V$ respecting these data. Condensification turns the scalar-extension equivalences into analytic gluing data.

On $V = \mathrm{Spa}(R, R^+)$, we identify $\mathrm{D}_{\mathrm{pc}}(R)$ with the nuclear and pseudocompact subcategory of $\mathsf{D}_\square(R, R^+)$ through condensification.

Proposition 2.5. *Restriction induces an equivalence*

$$\mathsf{D}^\square_{\mathrm{pc}}(\mathcal{X}) \xrightarrow{\sim} \mathrm{Cart}_{V \in \mathrm{RatAff}_X(\mathcal{X})} \mathrm{D}_{\mathrm{pc}}(R_V),$$

independent of the rational basis and bounded-chart presentations.

Proof. On each $V = \mathrm{Spa}(R, R^+)$, the affinoid comparison gives

$$\mathrm{Cond}_R : \mathrm{D}_{\mathrm{pc}}(R) \xrightarrow{\sim} \mathsf{D}^\square_{\mathrm{pc}}(V) \subset \mathsf{D}_\square(R, R^+)$$

[And21, Theorem 5.50] [LH26, A.15]. These equivalences also identify the transition functors. If $u : V' \hookrightarrow V$ is rational, then

$$u^* \mathrm{Cond}_R(M) \simeq \mathrm{Cond}_{R'}(R' \otimes^{\mathbf{L}}_R M)$$

by [And21, Lemma 5.10]. The scalar extension of a bounded-above finite-projective resolution is again such a resolution, so the object on the right is pseudocoherent. Tensor associativity makes these identifications compatible with successive restrictions.

Given a family (M_V), condensify its terms. The scalar-extension equivalences become pullback equivalences, so Proposition B.1 glues them to $C \in \mathcal{C}_\square(\mathcal{X})$ with $C|_V \simeq \mathrm{Cond}_R(M_V)$. These restrictions show that C belongs to $\mathsf{D}^\square_{\mathrm{pc}}(\mathcal{X})$.

Conversely, choose ordinary representatives for the restrictions of $C \in \mathsf{D}^{\square}_{\mathrm{pc}}(\mathcal{X})$. Full faithfulness of condensification lifts the restriction equivalences and their homotopies to a cartesian family. This inverts the preceding construction on objects; the limit of the local mapping-space equivalences gives the assertion on morphisms.

Common rational refinements and [And21, Theorem 5.3] identify the descent descriptions for different bases. On each affinoid, the subcategory is $\mathrm{Cond}_R(\mathrm{D}_{\mathrm{pc}}(R))$, independently of the bounded-chart presentation. The global subcategory is therefore independent of both choices. □

We will use the following inverse-limit calculation. Recall that a ring is coherent if every finitely generated ideal is finitely presented. For such a ring A, the notation $\mathrm{D}^b_{\mathrm{coh}}(A)$ refers to complexes with finitely presented cohomology in a finite interval.

Lemma 2.6. *Let A be a coherent ring and $b \in \mathbf{Z}$. Let $(K_n)_{n\geq 0}$ be an inverse system with $K_n \in \mathrm{D}^b_{\mathrm{coh}}(A)^{[b-n,b]}$ whose transitions induce equivalences $\tau_{\geq b-n}K_{n+1} \simeq K_n$. Then $K = \mathbf{R}\lim_n K_n$ is pseudocoherent and $\tau_{\geq b-n}K \simeq K_n$. For every ring map $A \to A'$, the canonical map*

$$A' \otimes^{\mathbf{L}}_A K \longrightarrow \mathbf{R}\lim_n (A' \otimes^{\mathbf{L}}_A K_n)$$

is an equivalence.

Proof. Exactness of products of ordinary A-modules gives the derived-limit exact sequence

$$0 \longrightarrow \lim\nolimits^1_n H^{j-1}(K_n) \longrightarrow H^j(K) \longrightarrow \lim_n H^j(K_n) \longrightarrow 0$$

[Stacks, Tag 08U5]. For fixed j, the transition maps on $H^j(K_n)$ are isomorphisms once $b-n \leq j$. Both systems in this sequence are eventually constant, so the $\lim^1$ term vanishes and $H^j(K) \simeq H^j(K_n)$ for sufficiently large n. These modules are finitely presented and vanish for $j > b$. Since A is coherent, this makes K pseudocoherent [Stacks, Tag 0EWZ]. For fixed n, the projection to K_n induces isomorphisms in degrees at least $b-n$, proving the truncation assertion.

Put $F_n = \mathrm{fib}(K \to K_n)$. The truncation assertion gives

$$F_n \in \mathrm{D}^{\leq b-n-1}(A).$$

Derived scalar extension along any ring map $A \to A'$ preserves upper cohomological bounds. Indeed, choose a bounded-above free resolution with the same upper bound and tensor it with A' [Stacks, Section 15.60]. It follows that

$$A' \otimes^{\mathbf{L}}_A F_n \in \mathrm{D}^{\leq b-n-1}(A').$$

The long exact sequence of the tensored fiber triangle now shows that

$$H^j(A' \otimes^{\mathbf{L}}_A K) \longrightarrow H^j(A' \otimes^{\mathbf{L}}_A K_n)$$

is an isomorphism once $b-n \leq j$. These maps respect the transitions, so the base-changed cohomology systems are eventually constant. The derived-limit exact sequence identifies their limit with $H^j(A' \otimes^{\mathbf{L}}_A K)$, proving that the canonical comparison is an equivalence. Naturality in the tower and compatibility with successive scalar extensions follow from the corresponding properties of the tower maps and tensor associativity. □

Proposition 2.7. *The ordinary affinoid t-structures descend to* $\mathsf{D}^{\square}_{\mathrm{pc}}(\mathcal{X})$*, and comparison induces a t-exact equivalence*

$$\alpha_X : \mathrm{D}_{\mathrm{pc}}(X) \xrightarrow{\sim} \mathsf{D}^{\square}_{\mathrm{pc}}(\mathcal{X}).$$

Proof. On each $V = \mathrm{Spa}(R, R^+)$, choose ordinary representatives $C_V \in \mathrm{D}_{\mathrm{pc}}(R)$. Flat rational localization preserves their degree conditions and truncation triangles, which glue by Proposition 2.5. Orthogonality follows by taking limits of the contractible local mapping spaces from nonpositive objects to objects in degrees at least one. These constructions define the t-structure, whose heart is $\mathrm{Coh}(\mathcal{X})$ by the coherent affinoid theorem and sheaf descent.

On charts, α_X is flat scalar extension followed by condensification, so it preserves pseudocoherence and truncations. It is fully faithful by Theorem 2.4 and induces the given equivalence on coherent hearts. We first prove essential surjectivity for bounded complexes.

Let C have cohomology in $[a, b]$. We argue by induction through the truncation triangles

$$\tau_{\leq r-1}C \longrightarrow \tau_{\leq r}C \longrightarrow \mathcal{H}^r(C)[-r] \longrightarrow \tau_{\leq r-1}C[1],$$

starting with $\tau_{\leq a-1}C = 0$. If the first term has been algebraized, then so has the third, by coherent GAGA. Full faithfulness lifts the connecting map between their preimages. Its fiber algebraizes $\tau_{\leq r}C$, and the long exact cohomology sequence makes this preimage bounded coherent. This proves the bounded equivalence. Since α_X commutes with cohomology and the coherent equivalence detects vanishing, the bounded equivalence also preserves and reflects cohomological bounds.

For $C \in \mathsf{D}^{\square}_{\mathrm{pc}}(\mathcal{X})$, choose a common upper bound b on a finite affinoid cover and put

$$C_n = \tau_{\geq b-n}C.$$

The bounded equivalence lets us algebraize these objects together with the maps $C_{n+1} \to C_n$. By Lemma B.2, we obtain a tower (M_n) of bounded coherent complexes on X with $\alpha_X(M_\bullet) \simeq C_\bullet$. Its transitions satisfy

$$\tau_{\geq b-n}M_{n+1} \xrightarrow{\sim} M_n,$$

and M_n has cohomology in $[b - n, b]$.

For each affine $W = \operatorname{Spec} A \subset X$, set

$$M_W = \mathbf{R}\lim_n \mathbf{R}\Gamma(W, M_n|_W).$$

This is pseudocoherent by Lemma 2.6. For an affine inclusion $W' \subset W$, that lemma identifies the restriction of M_W with $M_{W'}$. The identifications commute with successive restrictions and their associativity homotopies, so Zariski descent gives $M \in \mathrm{D}_{\mathrm{pc}}(X)$.

To identify its image, take $V = \mathrm{Spa}(R, R^+) \subset \lambda^{-1}(W)$ and an ordinary representative $P_V \in \mathrm{D}_{\mathrm{pc}}(R)$ of $C|_V$. The bounded comparisons give an equivalence of towers

$$R \otimes^{\mathbf{L}}_A \mathbf{R}\Gamma(W, M_n|_W) \simeq \tau_{\geq b-n}P_V.$$

Indeed, their condensifications are the restrictions of $\alpha_X(M_n) \simeq C_n$, and full faithfulness lifts this equivalence of diagrams. Lemma 2.6 then gives

$$\begin{aligned} R \otimes^{\mathbf{L}}_A M_W &\simeq \mathbf{R}\lim_n \big(R \otimes^{\mathbf{L}}_A \mathbf{R}\Gamma(W, M_n|_W)\big) \\ &\simeq \mathbf{R}\lim_n \tau_{\geq b-n} P_V \\ &\simeq P_V. \end{aligned}$$

The last equivalence is the reconstruction of P_V from its truncations. Condensifying gives $\alpha_X(M)|_V \simeq C|_V$.

The identifications commute with rational restriction by Lemma 2.6. They glue to $\alpha_X(M) \simeq C$, proving essential surjectivity. Together with the truncation compatibility already established, this gives the asserted t-exact equivalence. □

2.4. **A faithfully flat family.** The coordinate spectra of a finite affinoid cover form a faithfully flat family over X. This will allow us to test pseudocoherence after passing to the affinoid bases.

Lemma 2.8. *There is a finite affinoid cover $V_i = \mathrm{Spa}(R_i, R_i^+)$ of $\mathcal{X}$, with $V_i \subset \lambda^{-1}(W_i)$ for affine opens $W_i \subset X$, such that*

$$\coprod_i \operatorname{Spec} R_i \longrightarrow X$$

is fpqc.

Proof. Choose a finite affine cover of X, refine its inverse image by strongly noetherian affinoids, and take a finite subcover of $\mathcal{X}$. Discard the empty members and write $W_i = \operatorname{Spec} A_i$.

By Lemma 2.1, each $A_i \to R_i$ is flat. The composite $\operatorname{Spec} R_i \to X$ is affine and flat because $W_i \hookrightarrow X$ is affine, and the finite disjoint union is quasi-compact. It remains to prove surjectivity.

Let $x \in X$ be closed. Its coherent skyscraper $\kappa(x)$ is nonzero, so coherent GAGA makes its pullback nonzero. This pullback is nonzero on some V_i. Hence $x \in W_i$, and on that chart the sheaf is represented by the nonzero ring

$$R_i \otimes_{A_i} \kappa(x).$$

A prime of this ring gives a prime of R_i over the maximal ideal defining x in A_i.

For any $y \in X$, choose a closed point $x \in \overline{\{y\}}$ and a lift of x to some $\operatorname{Spec} R_i$. The open W_i contains y, and the corresponding primes satisfy $\mathfrak{p}_y \subseteq \mathfrak{p}_x$. Flat going-down gives a prime of R_i below the chosen lift of x and lying over y. Thus the morphism is surjective and hence fpqc. □

3. Relative full faithfullness

Let T/X be separated and finitely presented. We construct its realizations over $\mathcal{X}$ by gluing those of $T_R = T \times_X \operatorname{Spec} R$ over affinoid charts. For an affine morphism $q : U \to X$, a quasicoherent complex on U is a complex on X equipped with a $q_*\mathcal{O}_U$-action. Applying α_X to this description gives the comparison on U; we then glue over a finite affine cover of T.

3.1. **Relative algebraic and analytic realizations.** We use the rational-affinoid basis $\mathcal{B} = \mathrm{RatAff}_X(\mathcal{X})$ introduced in Section 2.1.

Choose $V = \mathrm{Spa}(R, R^+) \subset \mathcal{X}$ mapping into $W = \operatorname{Spec} A \subset X$, and set $T_R = T \times_X \operatorname{Spec} R$. By [LH26, Theorem A.13], there are an algebraic realization $(T_R)^{\mathrm{alg}}$ over $(\operatorname{Spec} R)^{\mathrm{alg}}$ and an analytic locus $(T_R)^{\mathrm{an}}$ over $V = (\operatorname{Spec} R)^{\mathrm{an}}$. Define

$$(T^{\mathrm{alg}}_{\mathcal{X}})|_V = (T_R)^{\mathrm{alg}} \times_{(\operatorname{Spec} R)^{\mathrm{alg}}} V,$$
$$(T^{\mathrm{an}}_{\mathcal{X}})|_V = (T_R)^{\mathrm{an}}.$$

Lemma 3.1. *Let $u : V' = \mathrm{Spa}(R', R'^+) \to V = \mathrm{Spa}(R, R^+)$ be a rational localization in $\mathrm{RatAff}_X(\mathcal{X})$. For every separated finitely presented X-scheme T,*

$$u^*\big((T_R)^{\mathrm{alg}} \times_{(\operatorname{Spec} R)^{\mathrm{alg}}} V\big) \simeq (T_{R'})^{\mathrm{alg}} \times_{(\operatorname{Spec} R')^{\mathrm{alg}}} V',$$
$$u^*(T_R)^{\mathrm{an}} \simeq (T_{R'})^{\mathrm{an}}.$$

These equivalences are natural in T, compatible with Zariski restriction, and coherent under composition of rational localizations.

Proof. If $\operatorname{Spec} R \to X$ factors through two affine neighborhoods, it factors through their intersection. Both descriptions of T_R are therefore the fiber product $T \times_X \operatorname{Spec} R$.

If $T_R = \operatorname{Spec} B$, then $T_{R'} = \operatorname{Spec}(B \otimes_R R')$. The first comparison is obtained by applying AnSpec to Lemma B.9(i), and the second is part (iii) of that lemma. The same result gives functoriality and compatibility with successive rational localizations.

Choose a finite affine cover (T_a) of T_R. Its multiple intersections are affine by separatedness. The realizations have the Zariski presentations [LH26, Theorem A.13]

$$(T_R)^{\mathrm{alg}} \simeq \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \coprod_{(a_0,\dots,a_n)} (T_{a_0\dots a_n})^{\mathrm{alg}},$$
$$(T_R)^{\mathrm{an}} \simeq \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \coprod_{(a_0,\dots,a_n)} (T_{a_0\dots a_n})^{\mathrm{an}}.$$

Base change preserves these colimits. Replacing each affine term by its base change therefore gives precisely the two presentations for $T_{R'}$.

The affine comparisons commute with the face and degeneracy maps, morphisms of schemes, and successive base changes. These compatibilities pass to the colimit, and a common affine refinement identifies the constructions from different covers. □

The rational base-change identifications give the required descent data over $\mathcal{X}$.

Proposition 3.2. *For a separated finitely presented X-scheme T, the local realizations glue functorially to analytic stacks $T^{\mathrm{alg}}_{\mathcal{X}}$ and $T^{\mathrm{an}}_{\mathcal{X}}$, with a natural map*

$$T^{\mathrm{an}}_{\mathcal{X}} \longrightarrow T^{\mathrm{alg}}_{\mathcal{X}}$$

that is an equivalence for proper T/X. The construction commutes with rational base change and Zariski gluing. For an open immersion $T \hookrightarrow X$, we have $T^{\mathrm{an}}_{\mathcal{X}} \simeq \lambda^{-1}(T)$.

Proof. Choose a rational-affinoid cover $V_i = \mathrm{Spa}(R_i, R_i^+)$ of $\mathcal{X}$. On a common rational subdomain of $V_i \cap V_j$, Lemma 3.1 identifies the two local realizations. On triple and higher intersections, all comparisons factor through the construction on a common rational chart, with the composition homotopies supplied by the same lemma. These are descent data for $T^{\mathrm{alg}}_{\mathcal{X}}$ and $T^{\mathrm{an}}_{\mathcal{X}}$.

The descent data are functorial in T and commute with rational restriction. Zariski compatibility follows by applying the presentation of [LH26, Theorem A.13] on each base chart before gluing over $\mathcal{X}$.

For an open immersion $T \hookrightarrow X$, the local realization is $V_i \cap \lambda^{-1}(T)$, and these opens glue to $\lambda^{-1}(T)$.

It remains to compare the two realizations. For an affine scheme, the bounded charts have natural maps to the algebraic realization [LH26, A.11–A.13]. Lemma B.9 makes these maps compatible with restriction. We glue first over a Zariski cover of T_R and then over the base cover, obtaining

$$T_{\mathcal{X}}^{\mathrm{an}} \longrightarrow T_{\mathcal{X}}^{\mathrm{alg}}.$$

If T/X is proper, then each T_R/R is proper. The proper analytic-locus theorem makes the comparison an equivalence on every base chart. These local inverses agree on overlaps, and hence give a global inverse. □

The pseudocoherent subcategories also glue over the affinoid charts.

Put $T_R = T \times_X \operatorname{Spec} R$ and $\mathcal{P}_T(V) = \mathsf{D}_{\mathrm{pc}}^{\square}((T_R)^{\mathrm{an}})$. Nuclearity and pseudocompactness of a complex on $(T_R)^{\mathrm{an}}$ are tested on the bounded charts of [LH26, A.15]. For a rational inclusion $u : V' \hookrightarrow V$, Lemma 3.1 gives the pullback u_T^*, computed on analytic affines by derived scalar extension followed by analytic completion.

As before, $\mathrm{Cart}_{V\in\mathcal{B}}\,\mathcal{P}_T(V)$ denotes the limit. Its objects are families C_V on $(T_R)^{\mathrm{an}}$ with equivalences $u_T^* C_V \simeq C_{V'}$ and the unit, composition, and higher homotopies. Its morphisms are maps of families respecting these identifications.

Proposition 3.3. *For a separated finitely presented X-scheme T, restriction gives*

$$\mathsf{D}_{\mathrm{pc}}^{\square}(T_{\mathcal{X}}^{\mathrm{an}}) \xrightarrow{\sim} \mathrm{Cart}_{V\in \mathrm{RatAff}_X(\mathcal{X})}\, \mathsf{D}_{\mathrm{pc}}^{\square}((T_R)^{\mathrm{an}}),$$

with rational pullbacks as transition functors, independently of the basis and bounded-chart presentations. For proper T, the same description holds on $T_{\mathcal{X}}^{\mathrm{alg}}$.

Proof. We show that descent for $\mathcal{C}_{\square}$ restricts to $\mathsf{D}_{\mathrm{pc}}^{\square}$. Consider scalar extension on an analytic affine,

$$F : \mathsf{D}(\mathcal{A}) \longrightarrow \mathsf{D}(\mathcal{A}')$$

with restriction-of-scalars right adjoint G. The right adjoint preserves direct sums and lower cohomological bounds [Sch26a, Proposition 12.4, Remark 12.5, and Proposition 12.6]. For pseudocompact M and a family (N_a) with a common lower bound, adjunction gives

$$\begin{aligned}
\mathrm{Map}(FM, \bigoplus_a N_a) &\simeq \mathrm{Map}(M, G(\bigoplus_a N_a)) \\
&\simeq \mathrm{Map}(M, \bigoplus_a GN_a) \\
&\simeq \bigoplus_a \mathrm{Map}(M, GN_a) \\
&\simeq \bigoplus_a \mathrm{Map}(FM, N_a).
\end{aligned}$$

This is the mapping criterion for pseudocompactness of FM [LH26, A.5].

By [LH26, A.4], nuclear objects are colimits of sequential colimits of compact objects with trace-class transitions. Since G preserves filtered colimits, F preserves compact objects.

The natural map $F(P^\vee) \to F(P)^\vee$ carries a class defining a trace-class map to one defining its scalar extension. Thus F preserves these presentations and, being colimit-preserving, preserves nuclear objects.

We apply these observations to $V' \subset V$. Choose a finite affine cover of T_R. On each intersection, Lemma B.9 gives maps between the universal bounded charts, and the preceding adjunction calculation proves that their pullbacks preserve pseudocompactness.

For nuclearity, the defining charts use the analytic structure $\mathcal{Q}_R$ induced from $(R, R^+)^\square$ on $R\langle T_1, \dots, T_m\rangle$ [LH26, A.10 and A.15]. Consider the ring map

$$R\langle T_1, \dots, T_m\rangle \longrightarrow R'\langle T_1, \dots, T_m\rangle.$$

It respects these structures: a complete $\mathcal{Q}_{R'}$-module is complete over $(R', R'^+)^\square$, hence over $(R, R^+)^\square$, which is the completeness condition for $\mathcal{Q}_R$ [Sch26a, Proposition 12.8] [LH26, A.3]. Base change gives maps of the chart pushouts for each tuple $(\varpi^n b_i)$, and the preceding argument for F proves preservation of nuclearity.

Nuclearity and pseudocompactness descend along bounded charts and Zariski covers [LH26, A.15 and Theorem A.13]. Pullback therefore restricts to $\mathcal{P}_T(V) \to \mathcal{P}_T(V')$, with compatibility under composition given by associativity of the pushouts.

Restriction sends a global pseudocoherent object to a cartesian family. Conversely, regard a family (C_V) as descent data in the solid categories. The equivalence

$$\mathcal{C}_\square(T_{\mathcal{X}}^{\mathrm{an}}) \simeq \lim_{V \in \mathcal{B}^{\mathrm{op}}} \mathcal{C}_\square((T_R)^{\mathrm{an}}),$$

glues it to $C \in \mathcal{C}_\square(T_{\mathcal{X}}^{\mathrm{an}})$. Every defining-chart restriction is a restriction of some C_V, and hence is nuclear and pseudocompact. Thus $C \in \mathsf{D}_{\mathrm{pc}}^\square$. This construction inverts restriction on objects, and the same limit formula identifies the mapping spaces.

Common rational and affine refinements identify different choices of base and Zariski covers. Independence of bounded charts follows from [LH26, A.15] together with the restriction compatibility above. If T is proper, Proposition 3.2 identifies the two realizations chart by chart, so the description applies equally to $T_{\mathcal{X}}^{\mathrm{alg}}$. □

We will also use descent in the scheme variable.

Lemma 3.4. *Let $(U_i)_{i\in I}$ be a finite Zariski cover of a separated finitely presented X-scheme T, and put $U_{i_0\dots i_n} = U_{i_0} \cap \dots \cap U_{i_n}$. Restriction gives*

$$\mathrm{D}_{\mathrm{qc}}(T) \simeq \lim_{[n]\in\Delta} \prod_{(i_0,\dots,i_n)\in I^{n+1}} \mathrm{D}_{\mathrm{qc}}(U_{i_0\dots i_n}),$$

$$\mathcal{C}_\square(T_{\mathcal{X}}^{\mathrm{alg}}) \simeq \lim_{[n]\in\Delta} \prod_{(i_0,\dots,i_n)\in I^{n+1}} \mathcal{C}_\square(U_{i_0\dots i_n}^{\mathrm{alg}}\mathcal{X}).$$

The comparison functors form a map of these diagrams. Degreewise full faithfulness, equivalences, and colimit preservation descend, as do tensor structures, module objects, and pullback transformations.

Proof. The first formula is ordinary Zariski descent: a complex on each U_i, equivalences on overlaps, and compatible homotopies on higher intersections. Allowing ordered tuples with repetitions gives the full Čech diagram in the statement.

The same intersections present the algebraic realization as a geometric realization [LH26, Theorem A.13], compatibly with base charts by Lemma 3.1. Proposition B.1 gives the second formula.

Refine the cover by affine opens. On the resulting affine intersections, comparison commutes with restriction by associativity of scalar extension. Lemma B.2 passes full faithfulness and equivalences to the limits. Colimits, tensor products, and module actions are computed componentwise, so they descend with the functors and their pullback comparisons. Passing to common affine refinements gives these assertions for the original cover. □

3.2. **Ordinary relative Huber analytification.** The ordinary adic realization is obtained from Huber's relative analytification on the same base charts.

Proposition 3.5. *Let T be an X-scheme locally of finite type. The Huber analytifications $(T_R)^{\mathrm{ad}/V}$, for $V = \mathrm{Spa}(R, R^+)$ and $T_R = T \times_X \operatorname{Spec} R$, glue canonically to a locally strongly noetherian adic space $T^{\mathrm{ad}}_{\mathcal{X}}/\mathcal{X}$ and a morphism of locally ringed spaces*

$$c_T : T^{\mathrm{ad}}_{\mathcal{X}} \longrightarrow T.$$

The construction commutes with rational base change and Zariski gluing, and preserves properness over the base.

Proof. For $V = \mathrm{Spa}(R, R^+) \in \mathrm{RatAff}_X(\mathcal{X})$, the scheme T_R is locally of finite type over R, so its relative Huber analytification $(T_R)^{\mathrm{ad}/V}$ exists [Hub94, Proposition 3.8]. For rational $V' = \mathrm{Spa}(R', R'^+) \subset V$, the base change

$$(T_R)^{\mathrm{ad}/V} \times_V V'$$

has the mapping property of $(T_{R'})^{\mathrm{ad}/V'}$: a compatible map from an analytic test space over V' to T_R factors through $T_{R'}$. We obtain a canonical isomorphism

$$(T_R)^{\mathrm{ad}/V} \times_V V' \xrightarrow{\sim} (T_{R'})^{\mathrm{ad}/V'}.$$

For $V'' \subset V' \subset V$, both composites represent the identity of the same functor over V'', so they agree.

Choose a rational-affinoid cover (V_i). The isomorphisms just constructed agree on common rational refinements of $V_i \cap V_j$. They therefore glue over each overlap, and their uniqueness gives the cocycle condition on triple intersections.

Gluing the spaces, structure sheaves, and valuations gives $T^{\mathrm{ad}}_{\mathcal{X}}$, with c_T induced by the local maps to T. Each piece is locally of finite type over V_i, so the resulting space is locally strongly noetherian. Common refinements compare different base covers, and the same mapping property gives compatibility with Zariski gluing in T.

Finally, if T/X is proper, then each $(T_{R_i})^{\mathrm{ad}/V_i} \to V_i$ is proper [Zav25, Remark 6.3]. As properness is local on the target, the glued morphism $T^{\mathrm{ad}}_{\mathcal{X}} \to \mathcal{X}$ is proper. □

3.3. **Module categories and relative full faithfulness.** Let $q : U \to X$ be affine and finitely presented, and put $\mathcal{A} = q_*\mathcal{O}_U \in \mathrm{CAlg}(\mathrm{D}_{\mathrm{qc}}(X))$. The usual equivalence $\mathrm{D}_{\mathrm{qc}}(U) \simeq \mathrm{Mod}_{\mathcal{A}}(\mathrm{D}_{\mathrm{qc}}(X))$ has an analytic counterpart, which describes the relative comparison as a functor on modules.

Lemma 3.6. *There is a canonical equivalence, compatible with rational restriction,*

$$\mathcal{C}_{\square}(U^{\mathrm{alg}}_{\mathcal{X}}) \simeq \mathrm{Mod}_{\alpha_X\mathcal{A}}(\mathcal{C}_{\square}(\mathcal{X})).$$

It identifies $\mathbf{R}q^{\mathrm{alg}}_*$ with the forgetful functor and α_U with the module functor induced by α_X.

Proof. Take a base chart $V = \mathrm{Spa}(R, R^+) \subset \lambda^{-1}(W)$ with $W = \operatorname{Spec} A$, and write $U_W = \operatorname{Spec} B_0$ and $U_R = \operatorname{Spec} B$. Flatness identifies $B = B_0 \otimes_A R$ with the derived tensor product. Put $\mathcal{R} = (R, R^+)^\square$ and

$$\mathcal{R}_0 = (R_{\mathrm{disc}}, R^+_{\mathrm{disc}})^\square,$$
$$\mathcal{B}_0 = (B_{\mathrm{disc}}, B^+_{\mathrm{disc}})^\square,$$

where B^+ is the integral closure of the image of R^+ in B.

The algebraic realization on this chart is defined by

$$B_R^{\mathrm{alg}} = \mathcal{R} \otimes^{\mathrm{AnRing}}_{\mathcal{R}_0} \mathcal{B}_0$$

[LH26, A.9]. Passing to module categories gives

$$\begin{aligned} \mathsf{D}(B_R^{\mathrm{alg}}) &\simeq \mathsf{D}(\mathcal{R}) \otimes_{\mathsf{D}(\mathcal{R}_0)} \mathsf{D}(\mathcal{B}_0) \\ &\simeq \mathsf{D}(\mathcal{R}) \otimes_{\mathsf{D}(\mathcal{R}_0)} \mathrm{Mod}_B(\mathsf{D}(\mathcal{R}_0)) \\ &\simeq \mathrm{Mod}_{\mathrm{Cond}_R(B)}(\mathsf{D}_\square(R, R^+)). \end{aligned}$$

The first line is the analytic-ring pushout formula [LH26, Lemma A.1]. The second uses Lemma B.7 to regard complete B-modules as complete base modules equipped with a B-action. The final line extends this module category from $\mathsf{D}(\mathcal{R}_0)$ to $\mathsf{D}(\mathcal{R})$ [Lur17, Section 4.8.5]; the algebra acting after extension is $\mathrm{Cond}_R(B)$.

Under $\mathrm{D}_{\mathrm{qc}}(W) \simeq \mathrm{D}(A)$, the restriction $\mathcal{A}|_W$ corresponds to B_0. Hence

$$(\alpha_X \mathcal{A})|_V = \mathrm{Cond}_R(R \otimes^{\mathbf{L}}_A B_0) \simeq \mathrm{Cond}_R(B).$$

This identifies the acting algebra with $(\alpha_X \mathcal{A})|_V$.

Under rational base change, Lemma B.8 identifies the same calculation with the one for $B \otimes_R R'$. Reassociating tensor products makes the identifications compatible with successive restrictions. The module categories, with their actions, therefore glue by Proposition B.1.

In this description, structural pullback is the free-module functor $N \mapsto \mathrm{Cond}_R(B) \otimes N$. Its right adjoint forgets the $\mathrm{Cond}_R(B)$-action. These adjoint pairs commute with restriction, so the forgetful functor is $\mathbf{R}q^{\mathrm{alg}}_*$ after gluing. Applying α_X to an algebraic module and its action gives the local comparison and hence, by descent, the global one. □

We apply the module description on affine opens to extend the comparison to T.

Proposition 3.7. *For every separated finitely presented X-scheme T, scalar extension gives a fully faithful, colimit-preserving, strong symmetric monoidal functor*

$$\alpha_T : \mathrm{D}_{\mathrm{qc}}(T) \longrightarrow \mathcal{C}_\square(T^{\mathrm{alg}}_{\mathcal{X}}),$$

compatible with pullback. For proper T/X, its target identifies with $\mathcal{C}_\square(T^{\mathrm{an}}_{\mathcal{X}})$.

Proof. Suppose first that $q : T \to X$ is affine, and put $\mathcal{A} = q_*\mathcal{O}_T$. By Lemma 3.6, the two categories are $\mathrm{Mod}_{\mathcal{A}}(\mathrm{D}_{\mathrm{qc}}(X))$ and $\mathrm{Mod}_{\alpha_X \mathcal{A}}(\mathcal{C}_\square(\mathcal{X}))$. Theorem 2.4 and Lemma B.3 make the induced functor fully faithful. Colimits are computed on underlying objects, so it preserves them. It preserves relative tensor products as well, because α_X is strong symmetric monoidal and preserves colimits.

For general T, choose a finite affine cover (U_i). The separated morphism $T \to X$ and the affine diagonal of X give an affine diagonal on T. Hence every intersection U_I is affine, and its map to X is affine by [Stacks, Lemma 29.11.12]. The preceding argument therefore applies on every intersection.

Tensor associativity makes the local functors commute with restriction. They therefore define a map of the descent diagrams in Lemma 3.4. Taking limits of their mapping-space equivalences proves full faithfulness. Since colimits are computed on the intersections and preserved by each local functor, they are preserved globally as well.

The tensor comparisons agree under restriction and supply the monoidal structure. For a morphism $T' \to T$, choose source affines mapping into target affines. Pullback and comparison are scalar extensions on these charts. Their associativity identifications agree on intersections, so they give the required pullback compatibility, including compatibility with composition. □

On the analytic locus, the comparison preserves pseudocoherence, as one sees on finite-free resolutions.

Proposition 3.8. *Let T be a separated finitely presented X-scheme, with analytic-locus map $\jmath_T : T_{\mathcal{X}}^{\mathrm{an}} \to T_{\mathcal{X}}^{\mathrm{alg}}$. Then*

$$\alpha_T^{\mathrm{an}} := \jmath_T^* \alpha_T : \mathrm{D}_{\mathrm{pc}}(T) \longrightarrow \mathsf{D}_{\mathrm{pc}}^{\square}(T_{\mathcal{X}}^{\mathrm{an}})$$

is well-defined. For proper T, it is the restriction of the fully faithful α_T and is denoted by α_T.

Proof. Take $M \in \mathrm{D}_{\mathrm{pc}}(T)$, a rational-affinoid base chart, and an affine $\operatorname{Spec} B \subset T_R$. Pulling back a bounded-above finite-projective resolution shows that the pullback of M is pseudocoherent. After a further restriction we may represent it by

$$\cdots \longrightarrow P^{b-2} \longrightarrow P^{b-1} \longrightarrow P^{b} \longrightarrow 0.$$

Each P^i is finite free over B.

On a bounded chart, scalar extension and analytic completion replace the finite free terms by finite sums of the tensor unit, with the same differential matrices. By [LH26, A.15 and the paragraph preceding Theorem A.18], using [And21, Theorem 5.50 and Proposition 5.33], the resulting complexes are nuclear and pseudocompact on the respective defining charts.

All these local complexes are restrictions of $\jmath_T^* \alpha_T(M)$, with the scalar-extension identifications used in its construction. Descent along the bounded and Zariski covers therefore gives

$$\jmath_T^* \alpha_T(M) \in \mathsf{D}_{\mathrm{pc}}^{\square}(T_{\mathcal{X}}^{\mathrm{an}}).$$

When T is proper, $\jmath_T$ is an equivalence by Proposition 3.2, so this is the restriction of α_T itself. □

3.4. **Flatness.** Flatness of ordinary analytification follows from the flatness of the base charts and the local Huber comparison.

Proposition 3.9. *For every X-scheme T locally of finite type, the morphism $c_T : T_{\mathcal{X}}^{\mathrm{ad}} \longrightarrow T$ is flat.*

Proof. Fix $z \in T_{\mathcal{X}}^{\mathrm{ad}}$ and write $t = c_T(z)$. Choose an affine $W = \operatorname{Spec} A \subset X$ containing the image of t, a rational affinoid $V = \operatorname{Spa}(R, R^+) \subset \lambda^{-1}(W)$ containing the image of z, and an affine neighborhood $U \subset T \times_X W$ of t. On these neighborhoods the map factors as

$$U_R^{\mathrm{ad}/V} \longrightarrow U_R \longrightarrow U,$$

where $U_R = U \times_A \operatorname{Spec} R$.

The second map is a base change of the flat map from Lemma 2.1. The first is flat on local rings by [Zav25, Lemma 6.7]. Their composite is flat at z, which proves the assertion. □

4. Localization at the boundary

We now construct the direct images that will be algebraized on X. The main ingredients are proper direct image and localization at a boundary. We establish these first, then identify the analytic-stack realization with Huber analytification. This last comparison supplies the ordinary affinoid representatives used in faithfully flat descent.

4.1. **Proper direct images of solid quasicoherent complexes.**

4.1.1. *The affinoid case.* The affinoid proper-direct-image theorem follows from the adjunction argument of [LH26, A.16–A.17].

Let (R, R^+) be strongly noetherian and complete Tate. For S locally of finite type over R, let S^{an} be its analytic-stack analytification [LH26, Theorem A.13], with structural map $\pi : S^{\mathrm{an}} \to \mathrm{AnSpec}(R, R^+)^{\square}$. The notation $\mathsf{D}_{\mathrm{pc}}^{\square}(S^{\mathrm{an}})$ uses the nuclearity and pseudocompactness conditions of [LH26, A.15]. On the base we use $\mathrm{Cond}_R : \mathrm{D}_{\mathrm{pc}}(R) \hookrightarrow \mathsf{D}_{\square}(R, R^+)$ [And21, Theorem 5.9].

Theorem 4.1. *Let (R, R^+) be a strongly noetherian complete Tate Huber pair and let S/R be proper and finitely presented. For every $K \in \mathsf{D}_{\mathrm{pc}}^{\square}(S^{\mathrm{an}})$, there is $M \in \mathrm{D}_{\mathrm{pc}}(R)$ with*

$$\mathbf{R}\pi_* K \simeq \mathrm{Cond}_R(M),$$

unique up to contractible choice together with this equivalence.

Proof. We apply the exceptional-pullback calculation of [LH26, Proposition A.16] on polynomial and quotient charts. We first give the finite resolutions entering that calculation.

Write $B = P/I$ with $P = R[T_1, \dots, T_m]$. Since P is noetherian, a finite free surjection onto B has finitely generated kernel. Repeating this for the successive kernels gives a resolution

$$\cdots \longrightarrow P^{r_2} \longrightarrow P^{r_1} \longrightarrow P^{r_0} \longrightarrow B \longrightarrow 0$$

with every r_i finite. In particular, $B \in \mathrm{D}_{\mathrm{pc}}(P)$ [Stacks, Tag 064T].

Put $D = R\langle T_1, \dots, T_m\rangle$. For a finitely generated ideal $J \subset P$, pull $\widetilde{J} \hookrightarrow \mathcal{O}_{\mathbf{A}_R^m}$ back to relative analytification and restrict to the closed unit polydisc. Flatness on local rings preserves the injection [Zav25, Lemma 6.7]. A finite presentation of J and the affinoid coherent-sheaf theorem [Zav25, Theorem 4.2] identify the map on sections with

$$J \otimes_P D \longhookrightarrow D.$$

The ideal criterion gives flatness of $P \to D$ [Stacks, Tag 00HD]. Consequently,

$$D \otimes_P^{\mathbf{L}} B \simeq (D/ID)[0],$$

and the tensored resolution makes D/ID pseudocoherent. Equipped with its complete quotient topology, this module has nuclear and pseudocompact condensification [And21, Theorem 5.9, Lemma 5.23, and Theorem 5.50]. The substitution $U_i = \varpi^n T_i$ gives the same calculation on any bounded polydisc. It also remains valid after rational localization, which preserves strong noetherianity.

On a quotient chart, exceptional pullback is internal derived Hom from $(D/ID)[0]$ [LH26, Proposition A.16]. When the targets have a common lower bound, only finitely many terms

of the finite-free resolution contribute in each total degree. Pseudocompactness therefore gives the direct-sum comparison, while concentration of the quotient in degree zero gives the lower bound. On a polynomial chart, the cited calculation gives ordinary pullback shifted by $[m]$.

Applying these calculations on the bounded charts and a finite affine cover of S shows that, whenever (M_a) has a common lower bound, so does $(\pi^! M_a)$ on the defining charts. Moreover,

$$\bigoplus_a \pi^! M_a \xrightarrow{\sim} \pi^! \left(\bigoplus_a M_a \right).$$

For proper S, the identification $S^{\mathrm{an}} \simeq S^{\mathrm{alg}}$ gives a finite cover by the induced bounded charts of [LH26, A.10 and A.15]. The chart algebras are nuclear over the base, and restriction of scalars preserves nuclear objects by [LH26, Lemma A.3 and Proposition A.10]. The finite-cover argument in the proof of [LH26, Corollary A.17], using [CS26, Theorem 8.6], gives nuclearity of $\mathbf{R}\pi_* K$. The same proof identifies $\mathbf{R}\pi_*$ with $\mathbf{R}\pi_!$, so that $\mathbf{R}\pi_* \dashv \pi^!$ on the full solid categories. For a family (M_a) with a common lower bound, we may therefore compute

$$\begin{aligned} \mathrm{Map}(\mathbf{R}\pi_* K, \bigoplus_a M_a) &\simeq \mathrm{Map}(K, \pi^! \bigoplus_a M_a) \\ &\simeq \mathrm{Map}(K, \bigoplus_a \pi^! M_a) \\ &\simeq \bigoplus_a \mathrm{Map}(K, \pi^! M_a) \\ &\simeq \bigoplus_a \mathrm{Map}(\mathbf{R}\pi_* K, M_a). \end{aligned}$$

The outside equivalences are adjunction. The second uses the direct-sum comparison for $\pi^!$, and the third uses pseudocompactness of K and the common lower bound on $(\pi^! M_a)$ [CS26, Definition 9.9]. Hence $\mathbf{R}\pi_* K$ is pseudocompact as well as nuclear.

The affinoid comparison identifies these objects with $\mathrm{Cond}_R(\mathrm{D}_{\mathrm{pc}}(R))$ [LH26, A.15] [And21, Theorem 5.50]. Thus M exists. Full faithfulness of condensification makes M, equipped with its comparison equivalence, unique up to contractible choice. □

The same adjoint pair gives the colimit comparison.

Corollary 4.2. *Under the hypotheses of Theorem 4.1,* $\mathbf{R}\pi_*$ *preserves arbitrary colimits on* $\mathcal{C}_\square(S^{\mathrm{an}})$.

Proof. For proper π, the identification $\mathbf{R}\pi_! \simeq \mathbf{R}\pi_*$ holds on all solid quasicoherent complexes [LH26, proof of Corollary A.17]. Thus $\mathbf{R}\pi_*$ is also a left adjoint, with right adjoint $\pi^!$.

Explicitly, for a diagram (K_i) and a base object N, the adjunction gives

$$\begin{aligned} \mathrm{Map}(\mathbf{R}\pi_* \operatorname*{colim}_i K_i, N) &\simeq \mathrm{Map}(\operatorname*{colim}_i K_i, \pi^! N) \\ &\simeq \lim_i \mathrm{Map}(K_i, \pi^! N) \\ &\simeq \mathrm{Map}(\operatorname*{colim}_i \mathbf{R}\pi_* K_i, N). \end{aligned}$$

The composite is induced by the canonical colimit map and is natural in N. Yoneda therefore proves the assertion. □

4.1.2. *Globalization.* We pass to $\mathcal{X}$ by restricting direct images to its affinoid opens. The restriction formula follows from extension by zero.

Lemma 4.3. *Let*

$$\begin{array}{ccc} \mathcal{Z}' & \xrightarrow{i'} & \mathcal{Z} \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ \mathcal{V} & \xrightarrow{i} & \mathcal{S} \end{array}$$

be a cartesian square of analytic stacks. Suppose $i_! \dashv i^*$ and $i'_! \dashv i'^*$ are extension-by-zero adjoint pairs and the canonical map $i'_!(g')^* \to g^* i_!$ is an equivalence. Then the canonical map

$$i^* \mathbf{R} g_* \longrightarrow \mathbf{R} g'_* i'^*$$

is an equivalence on $\mathcal{C}_\square(\mathcal{Z})$.

Proof. For $A \in \mathcal{C}_\square(\mathcal{Z})$ and $B \in \mathcal{C}_\square(\mathcal{V})$, adjunction gives

$$\begin{aligned} \mathrm{Map}_{\mathcal{V}}(B, i^* \mathbf{R} g_* A) &\simeq \mathrm{Map}_{\mathcal{S}}(i_! B, \mathbf{R} g_* A) \\ &\simeq \mathrm{Map}_{\mathcal{Z}}(g^* i_! B, A) \\ &\simeq \mathrm{Map}_{\mathcal{Z}}(i'_!(g')^* B, A) \\ &\simeq \mathrm{Map}_{\mathcal{Z}'}((g')^* B, i'^* A) \\ &\simeq \mathrm{Map}_{\mathcal{V}}(B, \mathbf{R} g'_* i'^* A). \end{aligned}$$

The middle equivalence is the comparison for extension by zero. The units and counits identify the composite with the canonical base-change map. Since the calculation is natural in B, Yoneda proves the lemma. □

For quasi-compact opens in the class I of [LH26, A.2], extension by zero and its pullback comparison are supplied by [Sch26b, Proposition 5.16 and Theorem 5.19]. This includes rational-affinoid inclusions [RC24, Proposition 2.3.2]. Lemma 4.3 therefore identifies the restriction of $\mathbf{R} g^{\mathrm{an}}_*$ to each base affinoid with the direct image there. For proper g, we can apply the affinoid theorem on these charts.

Proposition 4.4. *For proper finitely presented $g : Z \to X$, the functor*

$$\mathbf{R} g^{\mathrm{an}}_* : \mathcal{C}_\square(Z^{\mathrm{an}}_{\mathcal{X}}) \longrightarrow \mathcal{C}_\square(\mathcal{X})$$

preserves arbitrary colimits and carries $\mathsf{D}^\square_{\mathrm{pc}}(Z^{\mathrm{an}}_{\mathcal{X}})$ into $\mathsf{D}^\square_{\mathrm{pc}}(\mathcal{X})$.

Proof. Both assertions can be checked on rational base charts. Take $V = \mathrm{Spa}(R, R^+)$ and set $Z_R = Z \times_X \operatorname{Spec} R$. Its inverse image is Z^{an}_R, and Lemma 4.3 identifies

$$(\mathbf{R} g^{\mathrm{an}}_* K)|_V \simeq \mathbf{R} (g_R)^{\mathrm{an}}_* (K|_{Z^{\mathrm{an}}_R}).$$

For pseudocoherent K, Theorem 4.1 applies to its restriction on the proper scheme Z_R over strongly noetherian R. The direct image belongs to $\mathrm{Cond}_R(\mathrm{D}_{\mathrm{pc}}(R))$. The affinoid criterion therefore gives $\mathbf{R} g^{\mathrm{an}}_* K \in \mathsf{D}^\square_{\mathrm{pc}}(\mathcal{X})$.

For a diagram (K_a), consider the canonical map

$$\delta : \operatorname*{colim}_a \mathbf{R}g_*^{\mathrm{an}} K_a \longrightarrow \mathbf{R}g_*^{\mathrm{an}}\Big(\operatorname*{colim}_a K_a\Big).$$

Restriction to V preserves colimits, so open base change identifies $\delta|_V$ with

$$\operatorname*{colim}_a \mathbf{R}(g_R)_*^{\mathrm{an}}(K_a|_{Z_R^{\mathrm{an}}}) \longrightarrow \mathbf{R}(g_R)_*^{\mathrm{an}}\Big(\operatorname*{colim}_a K_a|_{Z_R^{\mathrm{an}}}\Big).$$

Corollary 4.2 makes this an equivalence. The cofiber of δ thus vanishes on an affinoid cover and hence globally. □

4.2. **Algebraic localization.** On $\operatorname{Spec} A$, restriction to $D(t)$ is scalar extension to $A[t^{-1}]$, and its right adjoint is restriction of scalars. For a section of a line bundle, these local calculations glue to a sequence of twists.

We use the same notation for line bundles and sections and their images under the comparison functors.

Lemma 4.5. *Let (R, R^+) be a complete Tate Huber pair and let S/R be separated and finitely presented. For a section $s : \mathcal{O}_S \to \mathcal{L}$ of a line bundle, let $j : S_s \hookrightarrow S$ be its nonvanishing open. Then $j^{\mathrm{alg},*}$ has a right adjoint and, for every $K \in \mathcal{C}_\square(S^{\mathrm{alg}})$,*

$$\mathbf{R}j_*^{\mathrm{alg}} j^{\mathrm{alg},*} K \simeq \operatorname*{colim}_{n\geq 0}\big(K \otimes \mathcal{L}^{\otimes n}\big),$$

naturally in K, with transition maps induced by s.

Proof. Choose a finite affine cover trivializing $\mathcal{L}$. On an intersection $\operatorname{Spec} A$, write the section as t; its nonvanishing locus is $\operatorname{Spec} A[t^{-1}]$. Lemma B.10 identifies restriction with scalar extension along $A \to A[t^{-1}]$, whose right adjoint is restriction of scalars.

For a further restriction $A \to A'$, flatness of localization gives

$$A'[t^{-1}] \simeq A' \otimes_A^{\mathbf{L}} A[t^{-1}],$$

so, for an $A[t^{-1}]$-module M, tensor associativity gives

$$A' \otimes_A^{\mathbf{L}} \operatorname{Res}_{A[t^{-1}]/A} M \simeq \operatorname{Res}_{A'[t^{-1}]/A'}\big(A'[t^{-1}] \otimes_{A[t^{-1}]}^{\mathbf{L}} M\big).$$

This is the comparison defined by the scalar-extension adjoint pairs, and tensor associativity gives compatibility with successive restrictions. Lemma B.4, applied to the Zariski presentation of [LH26, Theorem A.13], glues the right adjoints to $\mathbf{R}j_*^{\mathrm{alg}}$.

To compute the composite, send the nth term of

$$A \xrightarrow{t} A \xrightarrow{t} A \xrightarrow{t} \cdots$$

to $A[t^{-1}]$ by $a \mapsto a/t^n$. Every fraction appears, and it is zero exactly when a later multiplication by a power of t kills the numerator. Hence the colimit is $A[t^{-1}]$. Filtered colimits of modules are exact, so this also computes the derived colimit.

Tensoring with K gives

$$\mathbf{R}j_*^{\mathrm{alg}} j^{\mathrm{alg},*} K \simeq K \otimes_A^{\mathbf{L}} A[t^{-1}] \simeq \operatorname{colim}(K \xrightarrow{t} K \xrightarrow{t} \cdots).$$

In the chosen trivialization, the transition on $K \otimes \mathcal{L}^n$ is multiplication by t. Changing the basis multiplies t by a unit and the nth basis by the corresponding power, identifying the sequences and their localization maps on overlaps. The formulas therefore glue. □

Using the relative module description, we obtain the same localization formula over X.

Lemma 4.6. *Let Z be a separated finitely presented X-scheme, let $s : \mathcal{O}_Z \to \mathcal{L}$ be a line-bundle section, and put $j : U = Z_s \hookrightarrow Z$. Then $j^{\mathrm{alg},*}$ has a right adjoint and, naturally for $K \in \mathcal{C}_\square(Z^{\mathrm{alg}}_{\mathcal{X}})$,*

$$\mathbf{R}j^{\mathrm{alg}}_* j^{\mathrm{alg},*} K \simeq \operatorname*{colim}_{n\geq 0}\big(K \otimes \mathcal{L}^{\otimes n}\big),$$

with transitions induced by s. For the canonical section of an effective Cartier divisor D, the terms are $K \otimes \mathcal{O}_Z(nD)$.

Proof. Choose a finite affine cover of Z subordinate to affines of X and trivializing $\mathcal{L}$. On an intersection $\operatorname{Spec} A$, choose a basis e and write $s = te$. Then $U \cap \operatorname{Spec} A = \operatorname{Spec} A[t^{-1}]$. By Lemma 3.6, restriction is scalar extension to $A[t^{-1}]$ and its right adjoint is restriction of scalars.

On a further intersection with ring A', flatness of localization gives

$$A'[t^{-1}] \simeq A' \otimes^{\mathbf{L}}_A A[t^{-1}].$$

For an $A[t^{-1}]$-module M, we therefore compute

$$A' \otimes^{\mathbf{L}}_A \operatorname{Res}_{A[t^{-1}]/A} M \simeq \operatorname{Res}_{A'[t^{-1}]/A'}\big(A'[t^{-1}] \otimes^{\mathbf{L}}_{A[t^{-1}]} M\big).$$

The displayed map is the comparison defined by the scalar-extension adjunctions. Tensor associativity makes the comparisons agree under further restriction, so Lemmas 3.4 and B.4 glue the right adjoints to $\mathbf{R}j^{\mathrm{alg}}_*$.

The maps from the nth term to $A[t^{-1}]$ given by $a \mapsto a/t^n$ identify

$$A[t^{-1}] \simeq \operatorname{colim}(A \xrightarrow{t} A \xrightarrow{t} \cdots).$$

Every fraction occurs. Two fractions a/t^n and b/t^m agree exactly when $t^{k-n}a = t^{k-m}b$ for some $k \geq \max(m, n)$, which is precisely equality at a later stage of the sequence. Exactness of filtered colimits makes the same sequence a derived colimit, and tensoring gives

$$\begin{aligned}\mathbf{R}j^{\mathrm{alg}}_* j^{\mathrm{alg},*} K &\simeq K \otimes^{\mathbf{L}}_A A[t^{-1}] \\ &\simeq \operatorname{colim}(K \xrightarrow{t} K \xrightarrow{t} \cdots).\end{aligned}$$

Using the bases $e^{\otimes n}$, multiplication by s on the twists is multiplication by t. Thus the last line is the sequence of twists in the statement.

To check independence of the basis, write $e' = ue$. The coordinate of s becomes $u^{-1}t$, while the nth basis changes by u^n. These changes identify the sequences and their localization maps, so the local equivalences glue.

For an effective Cartier divisor with $\mathcal{O}_Z(-D) = (t)$, the basis t^{-1} of $\mathcal{O}_Z(D)$ writes the canonical section as $1 = t \cdot t^{-1}$. Its coordinate is therefore t, so the sequence consists of the twists $K \otimes \mathcal{O}_Z(nD)$ with the stated transition maps [LH26, proof of Theorem A.18]. □

To apply the localization formula to a quasi-compact open, we first realize it as the complement of a Cartier divisor on a blowup. We begin with the noetherian case.

Lemma 4.7. *Let $U \subset Y$ be a quasi-compact open in a noetherian scheme. There is a coherent ideal $\mathcal{J}$ cutting out $Y \setminus U$ whose blowup $p : \widetilde{Y} \to Y$ is projective, finitely presented, and an isomorphism over U, with*

$$\mathcal{J}\mathcal{O}_{\widetilde{Y}} = \mathcal{O}_{\widetilde{Y}}(-D)$$

for an effective Cartier divisor D with complement U. If Y/X is proper and finitely presented, so is $\widetilde{Y}/X$.

Proof. Choose a coherent ideal $\mathcal{J}$ whose vanishing locus is $Y \setminus U$, and form

$$\widetilde{Y} = \operatorname{Proj}_Y \left(\bigoplus_{n \geq 0} \mathcal{J}^n \right).$$

Because the Rees algebra is generated in degree one by the finite ideal $\mathcal{J}$, this morphism is projective; noetherianity gives finite presentation. Over U, the ideal is the unit ideal, so the blowup is $\operatorname{Proj}_U \mathcal{O}_U[T] = U$.

On $\operatorname{Spec} A$, write $\mathcal{J} = (a_1, \ldots, a_r)$. The blowup charts have rings $A[\mathcal{J}/a_i] \subset A[a_i^{-1}]$, generated by the fractions a_j/a_i. Since $a_j = a_i(a_j/a_i)$, the extended ideal is (a_i). Multiplication by a_i is injective in this subring, so these ideals define effective Cartier divisors and glue to $\mathcal{J}\mathcal{O}_{\widetilde{Y}}$ [Stacks, Section 31.33, especially Lemma 31.33.4].

The resulting divisor has support equal to the inverse image of $Y \setminus U$, so its complement is the copy of U. The final assertions about properness and finite presentation over X follow by composition. □

For a quasi-compact quasi-separated scheme, descend the closed complement to a noetherian model and pull back the resulting modification. The line bundle and its section pull back with it.

Lemma 4.8. *Let S be a quasi-compact quasi-separated scheme and let $U \subset S$ be quasi-compact open. There are a projective finitely presented $p : \widetilde{S} \to S$, an isomorphism over U, and a line-bundle section $s : \mathcal{O}_{\widetilde{S}} \to \mathcal{L}$ with nonvanishing locus $p^{-1}(U) \simeq U$.*

Proof. Choose a finite-type quasicoherent ideal $\mathcal{J}$ defining $S \setminus U$ [Stacks, Tag 01PH]. By the approximation in [Stacks, Tag 0EVA], write S as an inverse limit of noetherian schemes with affine transition maps and descend $V(\mathcal{J})$ to one stage S_0. Let $\mathcal{J}_0$ be the ideal of that closed subscheme. Then $U_0 = S_0 \setminus V(\mathcal{J}_0)$ pulls back to U.

The blowup $\widetilde{S}_0 \to S_0$ of $\mathcal{J}_0$ is projective and finitely presented, is an isomorphism over U_0, and has effective Cartier boundary D_0 [Stacks, Section 31.33]. The canonical section of $\mathcal{O}_{\widetilde{S}_0}(D_0)$ has nonvanishing locus U_0.

Pull back the morphism, line bundle, and section to S. Projectivity, finite presentation, and the isomorphism over U are preserved. Locally the nonvanishing locus is a principal open $D(t)$ and therefore commutes with base change. Thus the pulled-back section has nonvanishing locus $p^{-1}(U) \simeq U$. □

4.3. **Comparison with Huber analytification.** We next compare the bounded charts with those of Huber analytification. Identifying their rings and analytic structures will give an affinoid description of the pseudocoherent subcategory.

The assignment $\operatorname{Spa}(A, A^+) \mapsto \operatorname{AnSpec}(A, A^+)^{\square}$ sends rational covers to steady-localization covers and glues to a functor $(-)^{\square}$ from analytic adic spaces to analytic stacks [LH26, A.19].

Lemma 4.9. *For an open cover $(Z_i)_{i\in I}$ of a quasiseparated analytic adic space Z, put $Z_{i_0\dots i_n} = Z_{i_0} \cap \cdots \cap Z_{i_n}$. The canonical map*

$$\left| [n] \longmapsto \coprod_{(i_0,\dots,i_n)\in I^{n+1}} Z^{\square}_{i_0\dots i_n} \right| \xrightarrow{\sim} Z^{\square}$$

is an equivalence. In particular, an increasing affinoid cover (Z_n) with rational inclusions satisfies $\operatorname{colim}_n Z_n^{\square} \simeq Z^{\square}$. *Every quasi-compact affinoid $W \subset Z$ then lies in some Z_N and has a finite cover by common rational subdomains of W and Z_N.*

Proof. The functor $(-)^{\square}$ already commutes with rational gluing [LH26, A.19]. To handle an arbitrary open cover, choose affinoid charts in each Z_i and regard them also as charts of Z. Quasiseparatedness makes their intersections quasi-compact, so we may refine those intersections by finitely many rational affinoids. Repeat this on the multiple intersections of the Čech diagram.

The two covers give a bisimplicial diagram. Realizing in the rational direction first gives the stacks $Z^{\square}_{i_0\dots i_n}$ and then the source of the displayed map. Reversing the order gives the affinoid presentation of $Z^{\square}$. Since the two colimits commute, the map is an equivalence. Common refinements identify the construction for different choices of charts.

For an increasing cover, the rational inclusions give monomorphisms $Z_n^{\square} \to Z_{n+1}^{\square}$ [RC24, Proposition 2.3.2]. Their colimit is their union in the sheaf topos, and the first assertion identifies that union with $Z^{\square}$.

Finally, $(W \cap Z_n)$ is an increasing open cover of the quasi-compact space W. A finite subcover is contained in one member, so $W \subset Z_N$ for some N. Rational subdomains of Z_N contained in W cover W; after pullback to W they are still rational. A finite selection gives the common cover in the statement. □

Write $\alpha_T^{\mathrm{an}} = \jmath_T^* \alpha_T^{\mathrm{alg}}$, where $\jmath_T : T^{\mathrm{an}} \to T^{\mathrm{alg}}$ is the analytic-locus map. We compare the bounded charts and record the resulting formula for α_T^{an} on a Huber affinoid.

Proposition 4.10. *Let $V = \operatorname{Spa}(R, R^+)$ be a strongly noetherian complete Tate affinoid, and let T/R be separated and finitely presented. There is a canonical equivalence over V,*

$$\vartheta_{T,V} : T^{\mathrm{an}} \xrightarrow{\sim} (T^{\mathrm{ad}/V})^{\square},$$

natural in T and coherently compatible with Zariski restriction and rational base change.

Under $\vartheta_{T,V}$, an object belongs to $\mathsf{D}_{\mathrm{pc}}^{\square}$ exactly when its restriction to every strongly noetherian affinoid $W = \operatorname{Spa}(C, C^+)$ belongs to $\operatorname{Cond}_C(\mathrm{D}_{\mathrm{pc}}(C))$. *If $W \subset c_{T/V}^{-1}(U)$ for an affine $U = \operatorname{Spec} A \subset T$, then for $M \in \mathrm{D}_{\mathrm{qc}}(T)$,*

$$\alpha_T^{\mathrm{an}}(M)|_{W^{\square}} \simeq \operatorname{Cond}_C\big(C \otimes_A^{\mathbf{L}} \mathbf{R}\Gamma(U, M|_U)\big).$$

Proof. We follow the bounded-chart comparison of [LH26, proof of Theorem A.19]. For $T = \operatorname{Spec} B$, choose a presentation

$$B = R[T_1, \dots, T_m]/I.$$

Let b_i be the image of T_i, and choose a topologically nilpotent unit $\varpi \in R^+$. The inequalities $|\varpi^n b_i| \le 1$ define Huber affinoids W_n. Relative affine space is covered by the corresponding polydiscs, and the equations of I cut out $T^{\mathrm{ad}/V}$ in them [Hub94, Proposition 3.8] [Zav25,

Examples 6.4–6.5]. Hence

$$T^{\mathrm{ad}/V} = \bigcup_{n\geq 0} W_n.$$

Inside W_{n+1}, the subspace W_n is defined by $|\varpi^{n+1}b_i| \leq |\varpi|$. Since ϖ is a unit, the inclusions form an increasing system of rational opens.

The coordinate rings are

$$C_{0,n} = R\langle \varpi^n T_1, \ldots, \varpi^n T_m\rangle,$$
$$C_n = C_{0,n}/IC_{0,n}.$$

Set $C_{0,n}^+ = R^+\langle \varpi^n T_1, \ldots, \varpi^n T_m\rangle$, and let C_n^+ be the integral closure of its image in C_n. To write the equations in bounded coordinates, put $U_i = \varpi^n T_i$. They then generate

$$I_n = (f(\varpi^{-n}U_1, \ldots, \varpi^{-n}U_m) : f \in I).$$

Thus $C_n = R\langle \mathbf{U}\rangle / I_n R\langle \mathbf{U}\rangle$. The restriction from level $n+1$ to level n is the substitution $U_i \mapsto \varpi U_i$.

Put $P = R[T_1, \ldots, T_m]$. For a finite ideal $J \subset P$, pull $\widetilde{J} \hookrightarrow \mathcal{O}_{\mathbf{A}^m_R}$ back to the polydisc. Flatness of analytification preserves the injection [Zav25, Lemma 6.7]. A finite presentation and the affinoid coherent-sheaf theorem [Zav25, Theorem 4.2] identify its map on sections with

$$J \otimes_P C_{0,n} \longhookrightarrow C_{0,n}.$$

The ideal criterion proves flatness [Stacks, Tag 00HD]. Hence

$$C_{0,n} \otimes^{\mathbf{L}}_P B \simeq C_n[0]$$

in the ordinary derived category.

The ideal $IC_{0,n}$ is finite since $C_{0,n}$ is noetherian. The finite-module completeness and open mapping theorems show that it is closed and that C_n is complete for the quotient topology [KL19, Theorem 1.2.9 and Corollaries 1.2.11–1.2.12]. Hence $W_n = \mathrm{Spa}(C_n, C_n^+)$.

Let B_n^{bd} be the first bounded analytic ring, denoted by an underline in [LH26, A.15]. For the polynomial algebra it is $(C_{0,n}, C_{0,n}^+)^{\square}$ [LH26, Lemma A.8]. On passing to the quotient, the induced analytic structure has underlying condensed ring, before completion,

$$E_n = C_{0,n} \otimes^{\mathbf{L}}_P B,$$

where $C_{0,n}$ has its topological condensed structure and P, B are discrete [LH26, proof of Proposition A.16] [Sch26a, Propositions 12.8 and 12.12].

Choose a bounded-above finite-free P-resolution $F^\bullet \to B$. Discrete condensification is exact [And21, Lemma 5.7], so this remains a free condensed resolution. It is K-flat, meaning that tensoring with an acyclic complex remains acyclic, and hence computes E_n. Tensoring with $\underline{C_{0,n}}$ gives the condensification of the ordinary complex $C_{0,n} \otimes_P F^\bullet$. The latter is quasi-isomorphic to $C_n[0]$ by flatness. Applying condensification to this quasi-isomorphism and identifying the finite module C_n with its complete quotient topology gives

$$E_n \xrightarrow{\sim} C_n[0]$$

[And21, Theorem 5.9 and Lemma 5.23]. This is the canonical quotient map, so it is also an equivalence of the underlying animated condensed rings.

The analytic structure on C_n is induced from $(C_{0,n}, C_{0,n}^+)^\square$, so completeness is tested after restriction to $C_{0,n}$, or equivalently on the discrete plus ring by [And21, Proposition 3.29]. Factor its map as

$$C_{0,n}^+ \twoheadrightarrow D_n^+ := \mathrm{im}(C_{0,n}^+ \to C_n) \longrightarrow C_n^+.$$

Both maps are integral; the first is a quotient map. The restriction criterion and invariance under integral extensions [And21, Propositions 3.21–3.22] therefore identify all three completeness conditions with completeness for $(C_n, C_n^+)^\square$.

The unit C_n is complete [And21, Lemma 3.24], so analytic completion leaves the underlying ring unchanged. The two analytic structures have the same ring and complete connective modules; uniqueness [Sch26a, Proposition 12.20] gives

$$B_n^{\mathrm{bd}} \simeq (C_n, C_n^+)^\square,$$
$$\mathrm{AnSpec}\, B_n^{\mathrm{bd}} \simeq W_n^\square.$$

This also identifies the transition maps: on both sides they are the identity on B together with $U_i \mapsto \varpi U_i$ between consecutive levels.

Put $\underline{\mathcal{U}}_n = \mathrm{AnSpec}\, B_n^{\mathrm{bd}}$. The definition of the analytic locus also uses the second chart system $\mathcal{U}_n$ from [LH26, A.15]. The two systems are related by maps

$$\underline{\mathcal{U}}_n \longrightarrow \mathcal{U}_n \longrightarrow \underline{\mathcal{U}}_{n+1}$$

[LH26, A.11]. The composites are the respective transition maps. On colimits they shift the index by one, hence act as the identity. Lemma 4.9 gives

$$T^{\mathrm{an}} \simeq \operatorname*{colim}_n \underline{\mathcal{U}}_n \simeq \operatorname*{colim}_n W_n^\square \simeq (T^{\mathrm{ad}/V})^\square.$$

This defines $\vartheta_{T,V}$ for affine T. To compare the pseudocoherent subcategories, we also need the behavior of nuclearity under the maps between the two chart systems. Scalar extension along these maps preserves compact objects, since it sends measure generators to the corresponding target generators. It is monoidal and colimit-preserving. A class $\mathbf{1} \to P^\vee \otimes Q$ defining a trace-class map is carried, via the natural map to the dual of the image, to a class defining its scalar extension. Hence scalar extension preserves trace-class transitions, basic nuclear objects, and their colimits [And21, Propositions 2.9 and 2.11, Definition 5.32, and Proposition 5.33] [CS26, Definition 8.5 and Theorem 8.6].

Let $W = \mathrm{Spa}(C, C^+) \subset c_{T/V}^{-1}(U)$ with $U = \mathrm{Spec}\, A$, and put $K = \mathbf{R}\Gamma(U, M|_U)$. By Lemma 4.9, W lies in some W_N and has a finite cover $W_j = \mathrm{Spa}(C_j, C_j^+)$ rational in both. On W_N, comparison is scalar extension followed by condensification. Restricting to W_j and to the algebraic open U gives

$$\alpha_T^{\mathrm{an}}(M)|_{W_j^\square} \simeq \mathrm{Cond}_{C_j}(C_j \otimes_A^{\mathbf{L}} K)$$

[And21, Lemma 5.10]. On the other hand, restricting $\mathrm{Cond}_C(C \otimes_A^{\mathbf{L}} K)$ to W_j gives the same object, because

$$C_j \otimes_C^{\mathbf{L}} (C \otimes_A^{\mathbf{L}} K) \simeq C_j \otimes_A^{\mathbf{L}} K.$$

These identifications agree on rational intersections by tensor associativity. Descent therefore gives the claimed formula on W.

Now suppose $K \in \mathsf{D}_{\mathrm{pc}}^\square(T^{\mathrm{an}})$. By definition, its restriction is pseudocompact on $\underline{\mathcal{U}}_N$ and nuclear on $\mathcal{U}_N$. Pulling back along $\underline{\mathcal{U}}_N \to \mathcal{U}_N$ makes it nuclear on $W_N^\square = \underline{\mathcal{U}}_N$ as well. For a

Huber affinoid $W \subset W_N$, restrict to the common rational cover above. There the objects are both nuclear and pseudocompact. Nuclearity descends by [And21, Theorem 5.42], and pseudocompactness by [LH26, Proposition A.5] and [RC24, Proposition 2.3.2]. Thus $K|_{W^\square} \in \mathrm{Cond}_C(\mathrm{D}_{\mathrm{pc}}(C))$ by [LH26, A.15].

Conversely, suppose ordinary pseudocoherent representatives exist on all Huber affinoids. In particular, on $W_n^\square = \underline{\mathcal{U}}_n$ the object is nuclear and pseudocompact. The first chart system then gives pseudocompactness on T^{an} [LH26, A.15 and Proposition A.5]. Pullback along $\mathcal{U}_n \to \underline{\mathcal{U}}_{n+1}$ gives nuclearity on the second system and hence on T^{an} [LH26, A.15 and Proposition A.4]. This proves that the two descriptions of the pseudocoherent subcategory agree; see also [And21, Theorems 5.42, 5.44, and 5.50].

Changing the presentation or ϖ gives the common bounded refinements of Lemma B.9. On these refinements, the comparisons are defined by the same ring maps and scalar extensions. They are therefore independent of the choices and commute with morphisms and rational base change; tensor associativity gives the compatibilities for successive restrictions.

Finally, let T be separated and choose a finite affine cover with affine intersections. Huber analytification glues the local adic spaces, and Lemma 4.9 glues their associated stacks. The same intersections present T^{an} [LH26, Theorem A.13], so the affine equivalences glue to $\vartheta_{T,V}$. The formula for complexes and the nuclearity and pseudocompactness conditions can all be checked on this cover. Their comparison, together with the functoriality and base-change identifications, therefore descends to T. □

On these affinoid representatives, pullback and tensor product are computed by ordinary derived tensor products.

Lemma 4.11. *Let (R, R^+) be a strongly noetherian complete Tate Huber pair and let $h : S' \to S$ be a morphism of separated finitely presented R-schemes. For $K \in \mathsf{D}_{\mathrm{pc}}^\square(S^{\mathrm{an}})$ and $L \in \mathsf{D}_{\mathrm{pc}}^\square((S')^{\mathrm{an}})$,*

$$h^{\mathrm{an},*}K \otimes L \in \mathsf{D}_{\mathrm{pc}}^\square((S')^{\mathrm{an}}).$$

These pullbacks are compatible with composition.

Proof. Put $V = \mathrm{Spa}(R, R^+)$. Proposition 4.10 lets us work on Huber affinoids. Given a strongly noetherian affinoid $W' \subset (S')^{\mathrm{ad}/V}$, choose a finite rational cover $W'_\nu = \mathrm{Spa}(C'_\nu, C'^+_\nu)$ whose members map into affinoids $W_\nu = \mathrm{Spa}(C_\nu, C_\nu^+)$ of $S^{\mathrm{ad}/V}$. On these charts, choose ordinary pseudocoherent representatives P_ν, Q_ν such that

$$K|_{W_\nu^\square} \simeq \mathrm{Cond}_{C_\nu}(P_\nu),$$
$$L|_{(W'_\nu)^\square} \simeq \mathrm{Cond}_{C'_\nu}(Q_\nu).$$

Compatibility of condensification with pullback and tensor product gives

$$(h^{\mathrm{an},*}K \otimes L)|_{(W'_\nu)^\square} \simeq \mathrm{Cond}_{C'_\nu}\big((C'_\nu \otimes^{\mathbf{L}}_{C_\nu} P_\nu) \otimes^{\mathbf{L}}_{C'_\nu} Q_\nu\big).$$

Choose bounded-above finite-free resolutions $P_\nu^\bullet, Q_\nu^\bullet$ with upper bounds a, b. Scalar extension of the first remains finite free in each degree, and their tensor product has degree-d term

$$\bigoplus_{d-b \le i \le a} (C'_\nu \otimes_{C_\nu} P_\nu^i) \otimes_{C'_\nu} Q_\nu^{d-i}.$$

Only finitely many summands occur, all finite free, and the term is zero for $d > a + b$. This complex computes the derived tensor product and is a bounded-above finite-free resolution. Thus the tensor product is pseudocoherent [Stacks, Tags 0650 and 0DJE].

Rational descent gives the assertion on W' and hence on $(S')^{\mathrm{an}}$. Setting L equal to the unit gives preservation under pullback; setting h equal to the identity gives closure under tensor product. Compatibility with composition is the associativity of scalar extension [And21, Lemma 5.10]. □

Lemma 4.12. *Let $h : Z' \to Z$ be a morphism of separated finitely presented X-schemes. For $K \in \mathsf{D}^{\square}_{\mathrm{pc}}(Z^{\mathrm{an}}_{\mathcal{X}})$ and $L \in \mathsf{D}^{\square}_{\mathrm{pc}}((Z')^{\mathrm{an}}_{\mathcal{X}})$,*

$$h^{\mathrm{an},*}K \otimes L \in \mathsf{D}^{\square}_{\mathrm{pc}}((Z')^{\mathrm{an}}_{\mathcal{X}}).$$

Proof. This is local on the analytic base. Choose a rational-affinoid $V = \mathrm{Spa}(R, R^+) \subset \mathcal{X}$ and let $h_R : Z'_R \to Z_R$ be the base change of h. On this chart, the complex in the statement becomes

$$h_R^{\mathrm{an},*}(K|_{Z_R^{\mathrm{an}}}) \otimes L|_{(Z'_R)^{\mathrm{an}}}.$$

Both restrictions are pseudocoherent by Proposition 3.3. The schemes Z_R, Z'_R are separated and finitely presented over the strongly noetherian ring R, so Lemma 4.11 applies to this tensor product.

These are restrictions of the given global complex, with its overlap maps. Proposition 3.3 therefore proves the global assertion. As before, taking the unit or the identity gives pullback preservation and tensor closure. Proposition 3.8 also allows L to be the image of an algebraic pseudocoherent complex. □

For proper T/R, a finite Huber affinoid cover gives an fpqc family of ordinary coordinate spectra.

Lemma 4.13. *Let $V = \mathrm{Spa}(R, R^+)$ be a strongly noetherian complete Tate affinoid and let T/R be proper and finitely presented. There is a finite strongly noetherian affinoid cover $T^{\mathrm{ad}/V} = \bigcup_i W_i$, with $W_i = \mathrm{Spa}(B_i, B_i^+) \subset c_{T/V}^{-1}(U_i)$ for affine opens $U_i \subset T$, such that*

$$\coprod_i \mathrm{Spec}\, B_i \longrightarrow T$$

is fpqc.

Proof. The proper adic space $T^{\mathrm{ad}/V}$ is quasi-compact [Zav25, Remark 6.3]. Choose a finite strongly noetherian affinoid cover $W_i = \mathrm{Spa}(B_i, B_i^+)$ subordinate to the inverse images of affine opens $U_i = \mathrm{Spec}\, A_i$ of T.

For a finite ideal $J \subset A_i$, flatness of analytification gives

$$c_{T/V}^* \widetilde{J}|_{W_i} \hookrightarrow \mathcal{O}_{W_i},$$

by [Zav25, Lemma 6.7]. Choose a finite presentation $A_i^r \to A_i^s \to J \to 0$. After pullback it is the same matrix over B_i, so the coherent-sheaf theorem [Zav25, Theorem 4.2] identifies the source with the sheaf associated to $J \otimes_{A_i} B_i$. Taking sections gives an injection $J \otimes_{A_i} B_i \to B_i$. The ideal criterion now proves flatness [Stacks, Tag 00HD].

For surjectivity, start with a closed point $x \in T$. The skyscraper $\kappa(x)$ is nonzero. By Proposition B.11 and the proper identification $T^{\mathrm{an}} \simeq T^{\mathrm{alg}}$, its image $\alpha_T^{\mathrm{an}}\kappa(x)$ is also nonzero.

It must therefore be nonzero on some $W_i^{\square}$. Proposition 4.10 identifies that restriction with

$$\mathrm{Cond}_{B_i}\big(B_i \otimes^{\mathbf{L}}_{A_i} \Gamma(U_i, \kappa(x)|_{U_i})\big).$$

Nonvanishing first forces $x \in U_i$. Flatness then identifies the ordinary complex with the nonzero ring $B_i \otimes_{A_i} \kappa(x)$ in degree zero. A prime of this ring gives a point of $\operatorname{Spec} B_i$ over x.

Every point $y \in T$ has a closed specialization x, since T is noetherian. Choose a lift of x to $\operatorname{Spec} B_i$. Then $y \in U_i$ and $\mathfrak{p}_y \subset \mathfrak{p}_x$. Flat going-down lifts $\mathfrak{p}_y$ to a prime below the chosen prime over x. Thus the family covers every point of T.

Each $\operatorname{Spec} B_i \to T$ is affine, as it factors through an affine open of the separated scheme T. Their finite disjoint union is therefore quasi-compact, flat, and surjective, hence fpqc. □

Combining localization with proper direct image gives the strongly noetherian case of [LH26, Theorem A.18]. We construct the preimage on affine opens and test its pseudocoherence on the preceding fpqc family.

Proposition 4.14. *Let (R, R^+) be a strongly noetherian complete Tate Huber pair and let S/R be proper and finitely presented. Then*

$$\alpha_S^{\mathrm{an}} : \mathrm{D}_{\mathrm{pc}}(S) \xrightarrow{\sim} \mathsf{D}^{\square}_{\mathrm{pc}}(S^{\mathrm{an}}).$$

Proof. Analytification preserves pseudocoherence by the bounded-chart calculation [LH26, A.15 and the paragraph preceding Theorem A.18]. Properness identifies the two realizations [LH26, Theorem A.13], and Proposition B.11 gives full faithfulness. Let $C \in \mathsf{D}^{\square}_{\mathrm{pc}}(S^{\mathrm{an}})$. We construct its algebraic preimage on affine opens.

Choose $j : U = \operatorname{Spec} A \hookrightarrow S$, with structural map $q : U \to \operatorname{Spec} R$. By Lemma 4.8, there is a proper finitely presented $p : \widetilde{S} \to S$, unchanged over U, and a section s of a line bundle $\mathcal{L}$ whose nonvanishing locus is U. Write $\widetilde{j} : U \hookrightarrow \widetilde{S}$, $\widetilde{\pi} : \widetilde{S} \to \operatorname{Spec} R$, and $\widetilde{C} = p^{\mathrm{an},*}C$. The scheme $\widetilde{S}$ is proper, so its algebraic and analytic realizations agree.

Lemma 4.11 makes each twist $\widetilde{C} \otimes \mathcal{L}^n$ pseudocoherent. Localization expresses the restriction to U in terms of these twists:

$$\mathbf{R}\widetilde{j}^{\mathrm{alg}}_* \widetilde{j}^{\mathrm{alg},*} \widetilde{C} \simeq \operatorname*{colim}_n (\widetilde{C} \otimes \mathcal{L}^n).$$

Apply $\mathbf{R}\widetilde{\pi}_*$. By Theorem 4.1, each

$$E_n = \mathbf{R}\widetilde{\pi}_*(\widetilde{C} \otimes \mathcal{L}^n)$$

is the condensification of a pseudocoherent R-complex, and Corollary 4.2 permits passage of the colimit through this direct image.

Full faithfulness of Cond_R and Lemma B.2 lift the sequence $E_\bullet$, with its transition maps, to $M_\bullet$ in $\mathrm{D}_{\mathrm{pc}}(R)$. Put $M = \operatorname{colim}_n M_n$ in $\mathrm{D}(R)$. Colimit preservation gives

$$\mathbf{R}\widetilde{\pi}_* \mathbf{R}\widetilde{j}^{\mathrm{alg}}_* \widetilde{j}^{\mathrm{alg},*} \widetilde{C} \simeq \operatorname*{colim}_n E_n \simeq \mathrm{Cond}_R(M).$$

Because p is the identity over U, the restricted complex is $j^{\mathrm{alg},*}C$. Also $\widetilde{\pi} \circ \widetilde{j} = q$, so the composite direct image is right adjoint to $q^{\mathrm{alg},*}$ and is therefore $\mathbf{R}q^{\mathrm{alg}}_*$. Thus $\mathrm{Cond}_R(M)$ is the underlying complete R-module complex of $j^{\mathrm{alg},*}C$.

Transfer the $\mathrm{Cond}_R(A)$-action of Lemma B.10 to $\mathrm{Cond}_R(M)$. Strong monoidality identifies it with a map $\mathrm{Cond}_R(A \otimes^{\mathbf{L}}_R M) \to \mathrm{Cond}_R(M)$. Full faithfulness lifts this map, and Lemma B.3 lifts its unit, associativity, and higher relations. The resulting A-complex N_U realizes $j^{\mathrm{alg},*}C$.

Carry out the construction on a finite affine cover (U_a). The restrictions of C identify the analytic images on overlaps. Full faithfulness lifts those identifications and the homotopies on higher intersections. Zariski descent therefore gives $N \in \mathrm{D}_{\mathrm{qc}}(S)$ with $\alpha_S^{\mathrm{an}}(N) \simeq C$.

To prove that N is pseudocoherent, use the family $s_i : \operatorname{Spec} B_i \to S$ from Lemma 4.13, with corresponding affinoids W_i, and put $N_i = \mathbf{L}s_i^* N$. The affinoid comparison gives

$$\mathrm{Cond}_{B_i}(N_i) \simeq C|_{W_i^{\square}} \simeq \mathrm{Cond}_{B_i}(P_i)$$

for some $P_i \in \mathrm{D}_{\mathrm{pc}}(B_i)$. Full faithfulness gives $N_i \simeq P_i$, so N is pseudocoherent after the fpqc pullback. Faithfully flat descent [Stacks, Tags 066D and 068R] gives $N \in \mathrm{D}_{\mathrm{pc}}(S)$ and proves essential surjectivity. □

5. Pseudocoherent GAGA

Let $Y \to X$ be proper. By Section 3, the comparison is already fully faithful. We prove essential surjectivity by constructing preimages on affine opens of Y and gluing them. The boundary formula recovers their direct images on X, and lifting the algebra actions recovers the complexes on the opens.

5.1. **Faithfully flat detection of pseudocoherence.** Before constructing the preimages, we show that they will be pseudocoherent. After pullback to the fpqc family on X, affinoid GAGA gives a pseudocoherent preimage. Full faithfulness identifies it with the pullback of the given complex, so pseudocoherence descends.

Proposition 5.1. *Let $f : Y \to X$ be proper and let $j : U \hookrightarrow Y$ be quasi-compact open. For $C \in \mathsf{D}_{\mathrm{pc}}^{\square}(Y_{\mathcal{X}}^{\mathrm{an}})$ and $N \in \mathrm{D}_{\mathrm{qc}}(U)$, an equivalence*

$$\alpha_U(N) \simeq j^{\mathrm{alg},*} C$$

implies $N \in \mathrm{D}_{\mathrm{pc}}(U)$, using the proper identification $Y_{\mathcal{X}}^{\mathrm{an}} \simeq Y_{\mathcal{X}}^{\mathrm{alg}}$.

Proof. We may check pseudocoherence locally on U. Restrict the given equivalence to a finite affine cover, using pullback compatibility of α. It is therefore enough to treat $U = \operatorname{Spec} B$. Since X has affine diagonal, the structural map $q : U \to X$ is affine. Choose the finite fpqc family from Lemma 2.8 and form

$$Y_i = Y \times_X \operatorname{Spec} R_i,$$
$$U_i = U \times_X \operatorname{Spec} R_i.$$

The map q is affine, so $U_i = \operatorname{Spec} B_i$. Their disjoint union is $\operatorname{Spec} B'$, where $B' = \prod_i B_i$. It is the base change of an fpqc family, hence $B \to B'$ is faithfully flat.

Put $N_i = N \otimes_B^{\mathbf{L}} B_i$ and $C_i = C|_{Y_i^{\mathrm{an}}}$. Since Y_i is proper over the strongly noetherian ring R_i, Proposition 4.14 gives $M_i \in \mathrm{D}_{\mathrm{pc}}(Y_i)$ with $\alpha_{Y_i}^{\mathrm{an}}(M_i) \simeq C_i$.

Rational base change identifies the image of N_i with $C_i|_{U_i^{\mathrm{alg}}}$, which is also the image of $M_i|_{U_i}$. Full faithfulness from Proposition B.11 gives

$$N_i \simeq M_i|_{U_i}.$$

Each N_i is therefore pseudocoherent.

Combining the members of the finite family gives

$$N \otimes_B^{\mathbf{L}} B' \simeq \prod_i N_i.$$

Under $\mathrm{D}(B') \simeq \prod_i \mathrm{D}(B_i)$, this is a finite family of pseudocoherent complexes. Their upper bounds have a common maximum, so the product is pseudocoherent over B'. Faithfully flat descent [Stacks, Tag 068R] now gives $N \in \mathrm{D}_{\mathrm{pc}}(B)$. □

5.2. **Boundary pushforward and affine algebraization.** We express direct image from an open as a colimit of proper direct images on a blowup, following [CS26, Theorem 13.10] and [LH26, Theorem A.18]. GAGA on X then algebraizes the terms together with their transition maps.

Let $f : Y \to X$ be proper, let $j : U \hookrightarrow Y$ be quasi-compact open, and put $q = f \circ j$. Lemma 4.7 gives a blowup $p : \widetilde{Y} \to Y$, unchanged over U, with effective Cartier boundary D. Write $\widetilde{f} = f \circ p$. Both Y and $\widetilde{Y}$ are proper over X, so their algebraic and analytic realizations agree.

For $C \in \mathcal{C}_{\square}(Y_{\mathcal{X}}^{\mathrm{an}})$, put $\widetilde{C} = (p^{\mathrm{an}})^* C$ and

$$E_n(C) = \mathbf{R}\widetilde{f}_*^{\mathrm{an}}\big(\widetilde{C} \otimes \mathcal{O}_{\widetilde{Y}}(nD)\big)$$

for $n \geq 0$. The canonical section of $\mathcal{O}(D)$ induces the maps $E_n(C) \to E_{n+1}(C)$.

Theorem 5.2. *With this notation, for every $C \in \mathcal{C}_{\square}(Y_{\mathcal{X}}^{\mathrm{an}})$ there is a canonical equivalence, natural in C,*

$$\mathbf{R}q_*^{\mathrm{alg}} j^{\mathrm{alg},*} C \simeq \operatorname*{colim}_{n \geq 0} \mathbf{R}\widetilde{f}_*^{\mathrm{an}}\big(\widetilde{C} \otimes \mathcal{O}_{\widetilde{Y}}(nD)\big).$$

If $C \in \mathsf{D}_{\mathrm{pc}}^{\square}(Y_{\mathcal{X}}^{\mathrm{an}})$, then $E_n(C) \in \mathsf{D}_{\mathrm{pc}}^{\square}(\mathcal{X})$ and the sequence lifts functorially to $M_\bullet(C) \in \mathrm{Fun}(\mathbf{N}, \mathrm{D}_{\mathrm{pc}}(X))$, with $\alpha_X M_\bullet(C) \simeq E_\bullet(C)$. The lift with its comparison is unique up to contractible choice. Its colimit $M_U(C) \in \mathrm{D}_{\mathrm{qc}}(X)$ satisfies

$$\alpha_X(M_U(C)) \simeq \mathbf{R}q_*^{\mathrm{alg}} j^{\mathrm{alg},*} C$$

and is canonically independent of the blowup, with this comparison.

Proof. Let $\widetilde{j} : U \hookrightarrow \widetilde{Y}$ be the open immersion, and take $C \in \mathcal{C}_{\square}(Y_{\mathcal{X}}^{\mathrm{an}})$. We use the identifications of the proper algebraic and analytic realizations from Proposition 3.2.

The canonical section of $\mathcal{O}_{\widetilde{Y}}(D)$ has nonvanishing locus U. Lemma 4.6 therefore gives

$$\mathbf{R}\widetilde{j}_*^{\mathrm{alg}} \widetilde{j}^{\mathrm{alg},*} \widetilde{C} \simeq \operatorname*{colim}_{n} \big(\widetilde{C} \otimes \mathcal{O}_{\widetilde{Y}}(nD)\big).$$

Since p is an isomorphism over U, we have $\widetilde{j}^{\mathrm{alg},*} \widetilde{C} \simeq j^{\mathrm{alg},*} C$. The identity $\widetilde{f} \circ \widetilde{j} = q$ identifies the composite pullback with $q^{\mathrm{alg},*}$. For $A \in \mathcal{C}_{\square}(U_{\mathcal{X}}^{\mathrm{alg}})$ and $B \in \mathcal{C}_{\square}(\mathcal{X})$, adjunction gives

$$\begin{aligned} \mathrm{Map}_{\mathcal{X}}(B, \mathbf{R}\widetilde{f}_*^{\mathrm{an}} \mathbf{R}\widetilde{j}_*^{\mathrm{alg}} A) &\simeq \mathrm{Map}_{U_{\mathcal{X}}^{\mathrm{alg}}}(\widetilde{j}^{\mathrm{alg},*} (\widetilde{f}^{\mathrm{an}})^* B, A) \\ &\simeq \mathrm{Map}_{U_{\mathcal{X}}^{\mathrm{alg}}}(q^{\mathrm{alg},*} B, A) \\ &\simeq \mathrm{Map}_{\mathcal{X}}(B, \mathbf{R}q_*^{\mathrm{alg}} A). \end{aligned}$$

By Yoneda, this identifies $\mathbf{R}\widetilde{f}_*^{\mathrm{an}} \mathbf{R}\widetilde{j}_*^{\mathrm{alg}}$ with $\mathbf{R}q_*^{\mathrm{alg}}$.

Applying $\mathbf{R}\widetilde{f}_*^{\mathrm{an}}$ to the localization formula and using its colimit preservation from Proposition 4.4, we obtain

$$\begin{aligned}\mathbf{R}q_*^{\mathrm{alg}}j^{\mathrm{alg},*}C &\simeq \mathbf{R}\widetilde{f}_*^{\mathrm{an}}\mathbf{R}\widetilde{j}_*^{\mathrm{alg}}\widetilde{j}^{\mathrm{alg},*}\widetilde{C}\\ &\simeq \mathbf{R}\widetilde{f}_*^{\mathrm{an}}\left(\operatorname*{colim}_n \widetilde{C}\otimes\mathcal{O}_{\widetilde{Y}}(nD)\right)\\ &\simeq \operatorname*{colim}_n E_n(C).\end{aligned}$$

Naturality follows from that of the localization map and the adjunctions.

Now suppose C is pseudocoherent. Lemma 4.12 and Proposition 3.8 show that

$$\widetilde{C}\otimes\mathcal{O}_{\widetilde{Y}}(nD)\in\mathsf{D}_{\mathrm{pc}}^{\square}(\widetilde{Y}_{\mathcal{X}}^{\mathrm{an}}).$$

Proposition 4.4 gives $E_n(C)\in\mathsf{D}_{\mathrm{pc}}^{\square}(\mathcal{X})$ for every n.

Applying an inverse of the equivalence in Proposition 2.7 to $E_\bullet(C)$ gives a sequence $M_\bullet(C)$ with $\alpha_X M_\bullet(C)\simeq E_\bullet(C)$. Lemma B.2 ensures that this equivalence includes the transition maps and their composition homotopies. The sequence is functorial in C and, together with its comparison, unique up to contractible choice.

Set $M_U(C)=\operatorname{colim}_n M_n(C)$ in $\mathrm{D}_{\mathrm{qc}}(X)$. Colimit preservation gives

$$\alpha_X(M_U(C))\simeq\operatorname*{colim}_n E_n(C)\simeq\mathbf{R}q_*^{\mathrm{alg}}j^{\mathrm{alg},*}C.$$

A second blowup gives a preimage of the same right-hand side. Full faithfulness lifts the specified equivalence between their images and all composition homotopies. The colimit with its comparison is therefore independent of the blowup. □

When $q:U\to X$ is affine, $j^{\mathrm{alg},*}C$ is determined by its direct image together with the $\alpha_X(q_*\mathcal{O}_U)$-action. The preceding theorem algebraizes the underlying complex; full faithfulness now lets us lift the action.

Proposition 5.3. *Let $f:Y\to X$ be proper and let $j:U\hookrightarrow Y$ be any affine open. For every $C\in\mathsf{D}_{\mathrm{pc}}^{\square}(Y_{\mathcal{X}}^{\mathrm{an}})$, there is $N_U\in\mathrm{D}_{\mathrm{pc}}(U)$ with*

$$\alpha_U(N_U)\simeq j^{\mathrm{alg},*}C$$

in $\mathcal{C}_{\square}(U_{\mathcal{X}}^{\mathrm{alg}})$.

Proof. Since U is affine and X has affine diagonal, $q=f\circ j:U\to X$ is affine. By Theorem 5.2, there is $M\in\mathrm{D}_{\mathrm{qc}}(X)$ with

$$\mathbf{R}q_*^{\mathrm{alg}}j^{\mathrm{alg},*}C\simeq\alpha_X(M).$$

Put $\mathcal{A}=q_*\mathcal{O}_U$. By Lemma 3.6, $j^{\mathrm{alg},*}C$ is an $\alpha_X(\mathcal{A})$-module whose underlying object is the displayed direct image. Transferring its action along this equivalence gives

$$\alpha_X(\mathcal{A})\otimes\alpha_X(M)\longrightarrow\alpha_X(M).$$

Strong monoidality identifies the source with $\alpha_X(\mathcal{A}\otimes M)$, so full faithfulness lifts the action to $\mathcal{A}\otimes M\to M$. Lemma B.3 lifts the unit, associativity, and higher homotopies. This defines $N_U\in\mathrm{Mod}_{\mathcal{A}}(\mathrm{D}_{\mathrm{qc}}(X))\simeq\mathrm{D}_{\mathrm{qc}}(U)$ with image $j^{\mathrm{alg},*}C$.

Proposition 5.1 then gives $N_U\in\mathrm{D}_{\mathrm{pc}}(U)$. □

5.3. **The stable equivalence and proper pushforward.** We assemble the affine algebraizations into a global equivalence and then compare proper direct images. The algebraic direct images lie in the required category by the following finiteness lemma.

Lemma 5.4. *Let $h : S \to T$ be a proper morphism between noetherian schemes. Then $\mathbf{R}h_* : \mathrm{D}^-_{\mathrm{coh}}(S) \longrightarrow \mathrm{D}^-_{\mathrm{coh}}(T)$.*

Proof. Choose a finite affine cover (T_a) of T and cover each $S_a = h^{-1}(T_a)$ by r_a affine opens. Since S_a is separated, their intersections are affine. The alternating Čech complex therefore computes quasicoherent cohomology on S_a in degrees $0, \dots, r_a - 1$.

This bound also holds after replacing T_a by an affine open inside it: the covering opens and their intersections pull back to affine schemes, since they are fiber products of affines over T_a. Thus the same calculation bounds the sheaves $R^i h_*$. Set $d = \max_a(r_a - 1)$.

For a complex in degrees at most q, sections on an affine intersection are given by its ordinary module complex. The Čech degree adds at most d, so

$$\mathbf{R}h_*\big(\mathrm{D}_{\mathrm{qc}}(S)^{\leq q}\big) \subset \mathrm{D}_{\mathrm{qc}}(T)^{\leq q+d}.$$

Let $M \in \mathrm{D}^-_{\mathrm{coh}}(S)$ with $M \in \mathrm{D}^{\leq N}$. Its direct image lies in degrees at most $N + d$. For coherence in degree n, use the triangle

$$\tau_{\leq n-d-1}M \longrightarrow M \longrightarrow \tau_{\geq n-d}M.$$

The direct image of the first term lies in degrees at most $n - 1$. The long exact cohomology sequence therefore identifies

$$H^n(\mathbf{R}h_*M) \simeq H^n\big(\mathbf{R}h_*\tau_{\geq n-d}M\big).$$

The complex $\tau_{\geq n-d}M$ has coherent cohomology in the finite interval $[n-d, N]$. Its Postnikov triangles express it as successive extensions of these shifted coherent sheaves. Proper direct image has coherent cohomology on each such sheaf, and coherence is preserved by exact triangles on the noetherian target [Stacks, Tags 0BCF and 02O5]. Hence the right side is coherent. Since n was arbitrary, $\mathbf{R}h_*M$ is bounded above with coherent cohomology. □

We can now prove the bootstrapping theorem.

Theorem 5.5. *Under the hypotheses of Section 1.1, let $f : Y \to X$ be proper. The functor*

$$\alpha_Y : \mathrm{D}_{\mathrm{qc}}(Y) \longrightarrow \mathcal{C}_{\square}(Y^{\mathrm{an}}_{\mathcal{X}})$$

is fully faithful, colimit-preserving, and strong symmetric monoidal. It takes perfect complexes to compact objects and induces a symmetric monoidal equivalence

$$\mathrm{D}_{\mathrm{pc}}(Y) \xrightarrow{\sim} \mathsf{D}^{\square}_{\mathrm{pc}}(Y^{\mathrm{an}}_{\mathcal{X}}).$$

For every $M \in \mathrm{D}_{\mathrm{qc}}(Y)$, the canonical comparisons are equivalences:

$$\alpha_X(\mathbf{R}f_*M) \xrightarrow{\sim} \mathbf{R}f^{\mathrm{an}}_*\alpha_Y(M),$$
$$\mathbf{R}\Gamma(Y, M) \xrightarrow{\sim} \mathbf{R}\Gamma_{\square}(Y^{\mathrm{an}}_{\mathcal{X}}, \alpha_Y M).$$

Proof. Use properness to identify $Y^{\mathrm{an}}_{\mathcal{X}}$ with $Y^{\mathrm{alg}}_{\mathcal{X}}$. Proposition 3.7 gives full faithfulness, colimit preservation, and the monoidal structure on α_Y, while Proposition 3.8 shows that it preserves pseudocoherence. We begin with essential surjectivity.

Let $C \in \mathsf{D}^{\square}_{\mathrm{pc}}(Y^{\mathrm{an}}_{\mathcal{X}})$ and choose a finite affine cover (U_a) of Y. On each member, Proposition 5.3 supplies a preimage $N_a \in \mathrm{D}_{\mathrm{pc}}(U_a)$ and an equivalence

$$e_a : \alpha_{U_a}(N_a) \xrightarrow{\sim} C|_{U_a^{\mathrm{alg}}}.$$

On $U_{ab} = U_a \cap U_b$, the composite of e_a with the inverse of e_b gives

$$\alpha_{U_{ab}}(N_a|_{U_{ab}}) \xrightarrow{\sim} \alpha_{U_{ab}}(N_b|_{U_{ab}}).$$

Full faithfulness lifts this to an equivalence θ_{ab} of algebraic complexes. On a triple intersection, the comparisons with C give a homotopy between the images of $\theta_{bc}\theta_{ab}$ and θ_{ac}. The same mapping-space equivalence lifts that homotopy. Applying Lemma B.2 to the full Čech nerve recovers all higher compatibilities as well.

These pseudocoherent descent data glue to $N \in \mathrm{D}_{\mathrm{pc}}(Y)$. The comparisons e_a descend at the same time, giving $\alpha_Y(N) \simeq C$ and proving essential surjectivity.

Locally, the tensor product of two bounded-above finite-free resolutions has finitely many finite free summands in each degree, so $\mathrm{D}_{\mathrm{pc}}(Y)$ is closed under tensor products [Stacks, Tag 0DJE]. The analytic subcategory is tensor-closed by Lemma 4.12. The strong symmetric monoidal structure on α_Y therefore restricts to the equivalence.

For $M \in \mathrm{D}_{\mathrm{qc}}(Y)$, apply α_Y to the counit $\mathbf{L}f^*\mathbf{R}f_*M \to M$. Compatibility with pullback gives

$$(f^{\mathrm{an}})^*\alpha_X(\mathbf{R}f_*M) \simeq \alpha_Y(\mathbf{L}f^*\mathbf{R}f_*M) \longrightarrow \alpha_Y(M).$$

By the adjoint pair $(f^{\mathrm{an}})^* \dashv \mathbf{R}f^{\mathrm{an}}_*$, this defines

$$\gamma_M : \alpha_X(\mathbf{R}f_*M) \longrightarrow E := \mathbf{R}f^{\mathrm{an}}_*\alpha_Y(M).$$

Suppose first that M is pseudocoherent. Lemma 5.4 gives $\mathbf{R}f_*M \in \mathrm{D}_{\mathrm{pc}}(X)$, while Propositions 3.8 and 4.4 give $E \in \mathsf{D}^{\square}_{\mathrm{pc}}(\mathcal{X})$. Choose $N \in \mathrm{D}_{\mathrm{pc}}(X)$ with $E \simeq \alpha_X(N)$ by base GAGA. Full faithfulness lifts γ_M to $u_M : \mathbf{R}f_*M \to N$.

For $P \in \mathrm{Perf}(X)$, the two adjunctions give

$$\begin{aligned}
\mathrm{Map}_X(P, \mathbf{R}f_*M) &\simeq \mathrm{Map}_Y(f^*P, M),\\
\mathrm{Map}_X(P, N) &\simeq \mathrm{Map}_{\mathcal{X}}(\alpha_X P, E)\\
&\simeq \mathrm{Map}_{Y^{\mathrm{an}}_{\mathcal{X}}}((f^{\mathrm{an}})^*\alpha_X P, \alpha_Y M)\\
&\simeq \mathrm{Map}_{Y^{\mathrm{an}}_{\mathcal{X}}}(\alpha_Y(f^*P), \alpha_Y M)\\
&\simeq \mathrm{Map}_Y(f^*P, M).
\end{aligned}$$

Under these identifications, $\mathrm{Map}_X(P, u_M)$ is the identity. Indeed, a map $P \to \mathbf{R}f_*M$ corresponds to $f^*P \to M$. Applying α_Y and analytic adjunction gives its composite with γ_M, by the definition of γ_M from the counit. The same calculation applies to the full mapping spectra.

It follows that the cofiber Q of u_M satisfies $\mathrm{Map}_X(P, Q) = 0$ for every perfect P. Perfect complexes generate $\mathrm{D}_{\mathrm{qc}}(X)$, so $Q = 0$. Both u_M and γ_M are therefore equivalences.

For arbitrary M, compare the two functors

$$\begin{aligned}
M &\longmapsto \alpha_X(\mathbf{R}f_*M),\\
M &\longmapsto \mathbf{R}f^{\mathrm{an}}_*\alpha_Y(M).
\end{aligned}$$

Both functors preserve small colimits. On the analytic side, apply Proposition 4.4 and colimit preservation of α_Y. On the algebraic side, $\mathbf{R}f_*$ preserves direct sums for quasi-compact

quasi-separated f [Stacks, Tag 08DZ]. As it is exact, it preserves all small colimits, and so does its composite with α_X.

Write $M \simeq \operatorname{colim}_i P_i$ as a filtered colimit of perfect complexes [Stacks, Tags 09IS and 09M0]. Naturality of the counit identifies γ_M with the composite

$$\alpha_X(\mathbf{R}f_*M) \simeq \operatorname*{colim}_i \alpha_X(\mathbf{R}f_*P_i) \xrightarrow{\operatorname{colim}_i \gamma_{P_i}} \operatorname*{colim}_i \mathbf{R}f_*^{\mathrm{an}}\alpha_Y(P_i) \simeq \mathbf{R}f_*^{\mathrm{an}}\alpha_Y(M).$$

Every γ_{P_i} is an equivalence by the pseudocoherent case. Taking their colimit proves the comparison for M.

For $P \in \mathrm{Perf}(Y)$ and a filtered diagram (N_i) in $\mathcal{C}_\square(Y^{\mathrm{an}}_{\mathcal{X}})$, duality gives

$$\operatorname{Map}(\alpha_Y P, N_i) \simeq \mathbf{R}\Gamma_\square\big(Y^{\mathrm{an}}_{\mathcal{X}}, \alpha_Y(P^\vee) \otimes N_i\big).$$

Compute these global sections by first applying $\mathbf{R}f_*^{\mathrm{an}}$ and then $\mathbf{R}\Gamma_\square(\mathcal{X}, -)$. Tensor product and proper direct image preserve colimits, and the last functor preserves filtered colimits by Lemma 2.2. Consequently,

$$\operatorname*{colim}_i \operatorname{Map}(\alpha_Y P, N_i) \simeq \operatorname{Map}(\alpha_Y P, \operatorname*{colim}_i N_i),$$

proving compactness of $\alpha_Y P$.

Finally, the cohomology comparison follows directly from full faithfulness, since α_Y sends the structure sheaf to the tensor unit:

$$\mathbf{R}\Gamma(Y, M) = \operatorname{Map}_Y(\mathcal{O}_Y, M) \simeq \operatorname{Map}_{Y^{\mathrm{an}}_{\mathcal{X}}}(\mathcal{O}, \alpha_Y M) = \mathbf{R}\Gamma(Y^{\mathrm{an}}_{\mathcal{X}}, \alpha_Y M).$$

This holds for every $M \in \mathrm{D}_{\mathrm{qc}}(Y)$ and completes the proof. □

6. Coherent sheaves and classical GAGA

We pass from the solid comparison to ordinary Huber GAGA. On an affinoid, sheafification and condensification apply to the same finite modules and bounded complexes. Gluing these identifications gives the coherent and bounded-derived comparisons. We identify the resulting functors with ordinary pullback, then restrict to vector bundles and perfect complexes using duality and Tor-amplitude. On coherent hearts, tensor product means the ordinary tensor product, equivalently H^0 of derived tensor product.

6.1. **The transported t-structure.** Let T/X be proper and finitely presented, and put $\mathcal{P}_T = \mathsf{D}^{\square}_{\mathrm{pc}}(T^{\mathrm{an}}_{\mathcal{X}})$. Transport the t-structure along the equivalence $\alpha_T : \mathrm{D}_{\mathrm{pc}}(T) \simeq \mathcal{P}_T$ of Theorem 5.5. Thus the degree conditions on $\alpha_T(M)$ are those on M. Write $\mathcal{P}_T^{\heartsuit}$ for the heart and $\mathcal{P}_T^b$ for the bounded subcategory.

Restriction to a base affinoid preserves these degree conditions.

Lemma 6.1. *For $V = \operatorname{Spa}(R, R^+) \in \mathrm{RatAff}_X(\mathcal{X})$ and $T_R = T \times_X \operatorname{Spec} R$, restriction*

$$\mathcal{P}_T \longrightarrow \mathsf{D}^{\square}_{\mathrm{pc}}(T^{\mathrm{an}}_R)$$

is t-exact for the structures transported from $\mathrm{D}_{\mathrm{pc}}(T)$ and $\mathrm{D}_{\mathrm{pc}}(T_R)$. These structures are also preserved by rational localization of V.

Proof. Take $M \in \mathrm{D}_{\mathrm{pc}}(T)$ and write $M_R = \mathbf{L}(T_R \to T)^* M$. The base-change construction gives

$$\rho_V \alpha_T(M) \simeq \alpha^{\mathrm{an}}_{T_R}(M_R).$$

On an affine $\operatorname{Spec} B \subset T$ over $\operatorname{Spec} A \subset X$, this pullback replaces B by $B \otimes_A R$. Lemma 2.1 makes $A \to R$ flat, and hence $B \to B \otimes_A R$ flat as well. Thus

$$\mathcal{H}^j(M_R) \simeq (T_R \to T)^* \mathcal{H}^j(M).$$

Thus pullback preserves both degree conditions and commutes with truncation, proving t-exactness for the transported structures. For rational $V' \subset V$, the same argument applies to the flat map $R \to R'$ and the pullback $T_{R'} \to T_R$. □

6.2. **Bounded and coherent comparison.** We first glue complexes with a fixed lower cohomological bound, then impose coherence and an upper bound.

Lemma 6.2. *Let $(Z_i)_{i \in I}$ cover a ringed space $(Z, \mathcal{O}_Z)$, and put $Z_{i_0 \dots i_n} = Z_{i_0} \cap \cdots \cap Z_{i_n}$. For every integer a, restriction gives*

$$\mathrm{D}^{\geq a}(\mathcal{O}_Z) \xrightarrow{\sim} \lim_{[n] \in \Delta} \prod_{(i_0, \dots, i_n) \in I^{n+1}} \mathrm{D}^{\geq a}(\mathcal{O}_{Z_{i_0 \dots i_n}}),$$

including descent of the inherited mapping spectra.

Proof. Let $H\mathcal{O}_Z$ denote the Eilenberg–MacLane sheaf of rings: its degree-zero homotopy sheaf is $\mathcal{O}_Z$, and its other homotopy sheaves vanish. We first work in

$$\mathrm{Mod}_{H\mathcal{O}_Z}(\mathrm{Shv}(Z, \mathrm{Sp}))$$

where open descent follows by gluing sheaves of spectra together with their module actions and homotopies. Equivalently, formation of module categories commutes with the limits of symmetric monoidal categories in the descent diagram [Lur17, Section 4.8.5].

Cohomology sheaves commute with restriction, so vanishing below a is local. Descent therefore restricts to the given lower bound.

For the topos of Z, [Lur11, Proposition 2.1.8] identifies bounded-below complexes of $\mathcal{O}_Z$-modules with the corresponding sheaves of $H\mathcal{O}_Z$-module spectra. This identification commutes with open restriction. Applying it on each intersection, with the bound $\geq a$ in our cohomological convention, gives the asserted equivalence.

The limit also computes mapping spaces. For any fixed shift, choose a lower bound containing the two complexes and that shift, and apply the same assertion. This identifies all homotopy groups of the inherited mapping spectra, proving the final claim. □

We apply this lower-bounded descent to complexes with coherent cohomology in a fixed interval. For a finite cover, a common interval can be chosen from the local bounds.

Lemma 6.3. *Let $(Z, \mathcal{O}_Z)$ be a ringed space with coherent structure sheaf. For a finite open cover $(Z_i)_{i \in I}$, put $Z_{i_0 \dots i_n} = Z_{i_0} \cap \cdots \cap Z_{i_n}$. Then*

$$\mathrm{D}^b_{\mathrm{coh}}(Z) \xrightarrow{\sim} \lim_{[n] \in \Delta} \prod_{(i_0, \dots, i_n)} \mathrm{D}^b_{\mathrm{coh}}(Z_{i_0 \dots i_n}).$$

For a fixed cohomological interval $[a, b]$, the same assertion holds for an arbitrary open cover.

Proof. First fix an interval $[a, b]$; the cover may then be arbitrary. Lemma 6.2 glues the datum to $K \in \mathrm{D}^{\geq a}(\mathcal{O}_Z)$. For $j > b$, the sheaf $\mathcal{H}^j(K)$ is zero on every Z_i and hence zero. In degrees $a \leq j \leq b$, its restrictions are coherent, so it is coherent. Thus the descended object belongs to $\mathrm{D}^b_{\mathrm{coh}}(Z)^{[a,b]}$. The lower-bounded descent lemma also identifies the mapping spaces.

Now let the cover be finite, without fixing an interval in advance. Choose $[a_i, b_i]$ for each local complex K_i and set $a = \min_i a_i$, $b = \max_i b_i$. The higher Čech components are restrictions of the K_i, so the entire datum has bounds $[a, b]$. The fixed-interval assertion therefore gives a bounded coherent global complex.

Conversely, restriction preserves coherence and bounds. For two global bounded coherent complexes, choose a common lower bound and apply Lemma 6.2 to their mapping spaces. This proves full faithfulness. □

The corresponding description on an open basis is as follows.

Lemma 6.4. *Let S be quasi-compact and quasiseparated, and let $\mathcal{B}$ be a quasi-compact open basis whose finite intersections have finite $\mathcal{B}$-covers. If a presheaf $\mathcal{F}$ of ∞-categories on the quasi-compact opens satisfies finite Čech descent, including mapping spaces, then*

$$\mathcal{F}(S) \xrightarrow{\sim} \mathrm{Cart}_{V\in\mathcal{B}}\,\mathcal{F}(V).$$

Proof. Let $(K_V)_{V\in\mathcal{B}}$ be a cartesian family, and choose a finite basis cover $S = V_1 \cup \cdots \cup V_r$.

Cover $V_i \cap V_j$ by finitely many basis opens W. Cartesianness gives

$$K_{V_i}|_W \xrightarrow{\sim} K_W \xleftarrow{\sim} K_{V_j}|_W.$$

Hence the two restrictions agree on each W. Their comparisons agree on smaller basis opens, again by cartesianness, so mapping-space descent glues an equivalence over $V_i \cap V_j$. Repeating the argument on triple and higher intersections glues the composition homotopies and their compatibilities.

Finite descent gives $K \in \mathcal{F}(S)$. For any $V \in \mathcal{B}$, choose a finite basis refinement of the cover $(V \cap V_i)$. On this refinement, $K|_V$ and K_V have the same descent data, so they are equivalent.

The same mapping-space descent glues compatible maps and homotopies, proving full faithfulness. Passing to a common refinement compares different choices of cover, so the equivalence is the restriction functor in the statement. □

For bounded coherent complexes, the cartesian identifications carry the bounds on a finite basis cover to every member of the basis.

Lemma 6.5. *Let $(Z, \mathcal{O}_Z)$ have coherent structure sheaf, let S be a quasi-compact quasi-separated space, and let $p : Z \to S$ be continuous. Let $\mathcal{B}$ be a basis of quasi-compact opens whose finite intersections have finite $\mathcal{B}$-covers. Put $Z_V = p^{-1}(V)$. Restriction induces an equivalence*

$$\mathrm{D}^b_{\mathrm{coh}}(Z) \xrightarrow{\sim} \mathrm{Cart}_{V\in\mathcal{B}}\big(\mathrm{D}^b_{\mathrm{coh}}(Z_V)\big).$$

Proof. Let $(K_V)_{V\in\mathcal{B}}$ be a cartesian family in $\mathrm{D}^b_{\mathrm{coh}}(Z_V)$. Choose a finite basis cover $S = \bigcup_i V_i$ and an interval $[a, b]$ containing the bounds of all K_{V_i}.

For any $V \in \mathcal{B}$, quasi-compactness and quasiseparatedness give a finite refinement

$$V = \bigcup_{i,\nu} W_{i\nu}$$

with $W_{i\nu} \in \mathcal{B}$ and $W_{i\nu} \subset V \cap V_i$. On each $W_{i\nu}$, cartesianness identifies the restrictions of K_V and K_{V_i}. The latter has bounds $[a, b]$. Since cohomology vanishing is local, so does K_V.

For quasi-compact $U \subset S$, put

$$\mathcal{F}_{[a,b]}(U) = \mathrm{D}^b_{\mathrm{coh}}(Z_U)^{[a,b]}.$$

Lemma 6.3 gives finite descent for these categories and their mapping spaces. The basis satisfies the finite-refinement hypothesis of Lemma 6.4, so that lemma glues the family to a bounded coherent complex on Z.

Conversely, a global bounded coherent complex restricts to such a family. Choosing a common interval for two global objects and applying the same mapping-space descent proves full faithfulness. □

We can now glue the affinoid identifications between sheafification and condensification.

Let Z be a quasi-compact quasiseparated locally strongly noetherian analytic adic space. An object of $\mathcal{C}_\square(Z^\square)$ belongs to $\mathsf{D}^\square_{\mathrm{pc}}(Z^\square)$ when, on every strongly noetherian affinoid $W = \mathrm{Spa}(A, A^+)$, it lies in $\mathrm{Cond}_A(\mathrm{D}_{\mathrm{pc}}(A))$. We use the subscript ch for the t-structure defined by the degrees of these ordinary representatives.

Proposition 6.6. *The ordinary affinoid t-structures descend to $\mathsf{D}^\square_{\mathrm{pc}}(Z^\square)$. Condensification gives canonical exact equivalences*

$$\mathrm{Coh}(Z) \xrightarrow{\sim} \mathsf{D}^\square_{\mathrm{pc}}(Z^\square)^\heartsuit_{\mathrm{ch}},$$
$$\mathrm{D}^b_{\mathrm{coh}}(Z) \xrightarrow{\sim} \mathsf{D}^\square_{\mathrm{pc}}(Z^\square)^b_{\mathrm{ch}},$$

compatible with affinoid restriction. The first is symmetric monoidal.

Proof. On a strongly noetherian affinoid $W = \mathrm{Spa}(A, A^+)$, the coherent-sheaf theorem gives an exact equivalence

$$\mathrm{Mod}_{\mathrm{fg}}(A) \xrightarrow{\sim} \mathrm{Coh}(W)$$

and vanishing of higher coherent cohomology [Zav25, Theorem 4.2].

Write $\mathrm{D}^b_{\mathrm{fg}}(A)$ for complexes with finite cohomology in a finite interval. Since A is noetherian, these are the objects coming from the bounded derived category of finite A-modules. Derived sheafification and sections form the adjoint pair

$$(-)^\sim : \mathrm{D}^b_{\mathrm{fg}}(A) \rightleftarrows \mathrm{D}^b_{\mathrm{coh}}(W) : \mathbf{R}\Gamma(W, -).$$

For $K \in \mathrm{D}^b_{\mathrm{coh}}(W)$, consider the spectral sequence

$$H^p(W, \mathcal{H}^q(K)) \Longrightarrow H^{p+q}\mathbf{R}\Gamma(W, K).$$

Coherent acyclicity makes the spectral sequence degenerate, with

$$H^q\mathbf{R}\Gamma(W, K) \simeq \Gamma(W, \mathcal{H}^q(K)),$$

a finite A-module. The counit $(\mathbf{R}\Gamma(W, K))^\sim \to K$ induces in degree q the coherent-sheaf isomorphism

$$\widetilde{\Gamma(W, \mathcal{H}^q(K))} \xrightarrow{\sim} \mathcal{H}^q(K).$$

The counit is therefore a quasi-isomorphism. For $M \in \mathrm{D}^b_{\mathrm{fg}}(A)$, the same calculation shows that the unit $M \to \mathbf{R}\Gamma(W, \widetilde{M})$ is an equivalence. We have proved

$$\mathrm{D}^b_{\mathrm{coh}}(W) \simeq \mathrm{D}^b_{\mathrm{fg}}(A).$$

Condensification identifies $\mathrm{D}_{\mathrm{pc}}(A)$ with the nuclear and pseudocompact subcategory [And21, Theorem 5.9, Proposition 5.22, and Theorem 5.50] [LH26, A.15]. Its heart and bounded part are represented by finite modules and $\mathrm{D}^b_{\mathrm{fg}}(A)$, respectively. Sheafification of these same representatives gives the ordinary categories.

For rational $A \to A'$, both constructions restrict by $M \mapsto A' \otimes^{\mathbf{L}}_{A} M$ [And21, Lemma 5.10]. Flatness makes this commute with truncation. The truncation triangles therefore glue, and the limit of the local mapping spaces gives orthogonality, defining the asserted t-structure.

Choose a finite affinoid cover of Z. Coherent sheaves descend on it, and bounded coherent complexes descend by Lemma 6.5. For solid complexes, apply Proposition B.1 and [And21, Theorems 5.3 and 5.44]. Taking limits of the compatible affinoid equivalences gives both global comparisons. On hearts, the tensor comparisons are those of ordinary finite modules and hence give a symmetric monoidal equivalence. The construction commutes with affinoid restriction. □

Over a Tate affinoid, the fpqc family identifies the degree conditions on an algebraic preimage with those on its ordinary analytic representatives.

Proposition 6.7. *Let $V = \mathrm{Spa}(R, R^+)$ be a strongly noetherian complete Tate affinoid and let T/R be proper and finitely presented. Under Proposition 4.10, the transported and chartwise t-structures on $\mathsf{D}^{\square}_{\mathrm{pc}}(T^{\mathrm{an}})$ agree. There are canonical exact equivalences*

$$\beta^{\heartsuit}_{T,V} : \mathsf{D}^{\square}_{\mathrm{pc}}(T^{\mathrm{an}})^{\heartsuit} \xrightarrow{\sim} \mathrm{Coh}(T^{\mathrm{ad}/V}),$$
$$\beta^{b}_{T,V} : \mathsf{D}^{\square}_{\mathrm{pc}}(T^{\mathrm{an}})^{b} \xrightarrow{\sim} \mathrm{D}^{b}_{\mathrm{coh}}(T^{\mathrm{ad}/V}),$$

with $\beta^{\heartsuit}_{T,V}$ symmetric monoidal. Their composites with α^{an}_{T} are $c^{*}_{T/V}$ and $\mathbf{L}c^{*}_{T/V}$, respectively; hence these ordinary pullbacks are equivalences as well.

Proof. Choose the fpqc family of Lemma 4.13, with affinoids $W_i = \mathrm{Spa}(B_i, B_i^+)$ lying over $U_i = \mathrm{Spec}\, A_i$. For $C \in \mathsf{D}^{\square}_{\mathrm{pc}}(T^{\mathrm{an}})$, take $M \in \mathrm{D}_{\mathrm{pc}}(T)$ with $\alpha^{\mathrm{an}}_{T}(M) \simeq C$ by Proposition 4.14. Proposition 4.10 gives

$$C|_{W_i^{\square}} \simeq \mathrm{Cond}_{B_i}\big(B_i \otimes^{\mathbf{L}}_{A_i} \mathbf{R}\Gamma(U_i, M)\big).$$

The ring maps $A_i \to B_i$ are flat, so

$$H^j\big(B_i \otimes^{\mathbf{L}}_{A_i} \mathbf{R}\Gamma(U_i, M)\big) \simeq B_i \otimes_{A_i} \Gamma(U_i, \mathcal{H}^j(M)).$$

In particular, vanishing of $\mathcal{H}^j(M)$ implies vanishing of the degree-j cohomology of every representative.

Conversely, if these cohomology modules vanish for every i, then $\mathcal{H}^j(M)$ vanishes on the fpqc family and hence on T. Applying this in positive and negative degrees identifies the two t-structures. Since the cover is finite, the same argument compares bounded objects with a common interval.

Proposition 6.6, transported through $\vartheta_{T,V}$, gives $\beta^{\heartsuit}_{T,V}$ and $\beta^{b}_{T,V}$. Their composites with α^{an}_{T} sheafify the displayed scalar extension on each W_i. These are the formulas for $c^{*}_{T/V}$ on coherent sheaves and $\mathbf{L}c^{*}_{T/V}$ on bounded complexes, and they agree on overlaps. Equality of the t-structures gives exactness; ordinary tensor products of finite modules give the tensor comparison on hearts. Composing with GAGA proves the final assertions. □

These equivalences commute with rational restriction.

Lemma 6.8. *The functors $\beta^{\heartsuit}_{T,V}$ and $\beta^{b}_{T,V}$ commute naturally with restriction along rational localizations $V' \to V$, coherently under composition.*

Proof. For an object in the domain of $\beta^{\heartsuit}_{T,V}$ or $\beta^{b}_{T,V}$, write its restriction to $W = \mathrm{Spa}(C, C^+)$ as $\mathrm{Cond}_C(P)$. Here P is a finite module in degree zero for the heart, and $P \in \mathrm{D}^{b}_{\mathrm{fg}}(C)$ for the bounded category. In either case, β sheafifies P.

After restricting to V', choose $W' = \mathrm{Spa}(C', C'^{+})$ mapping into W. Pullback compatibility in Proposition 4.10 gives the representative

$$\mathrm{Cond}_{C'}(C' \otimes^{\mathbf{L}}_{C} P).$$

Both applying β to this representative and restricting the sheafification of P to W' give the sheafification of $C' \otimes^{\mathbf{L}}_{C} P$. This identification is natural in P.

For a further restriction with ring C'', the comparison is

$$C'' \otimes^{\mathbf{L}}_{C'} (C' \otimes^{\mathbf{L}}_{C} P) \simeq C'' \otimes^{\mathbf{L}}_{C} P.$$

Associativity identifies the comparisons for successive restrictions, together with their homotopies. The formulas also agree on common affinoid refinements, so they descend to the relative spaces. Proposition 6.7 identifies them with the restriction comparisons for c^* and $\mathbf{L}c^*$. □

The rational-restriction identifications let us glue the local equivalences over $\mathcal{X}$. We identify their source categories using the fpqc family.

Proposition 6.9. *For proper T/X, the local comparisons glue to canonical exact equivalences*

$$\beta^{\heartsuit}_{T} : \mathcal{P}^{\heartsuit}_{T} \xrightarrow{\sim} \mathrm{Coh}(T^{\mathrm{ad}}_{\mathcal{X}}),$$
$$\beta^{b}_{T} : \mathcal{P}^{b}_{T} \xrightarrow{\sim} \mathrm{D}^{b}_{\mathrm{coh}}(T^{\mathrm{ad}}_{\mathcal{X}}).$$

The first is symmetric monoidal. Their composites with α_T are c^*_T and $\mathbf{L}c^*_T$, respectively.

Proof. The functors $\beta_{T,V}$ form a compatible diagram by Lemma 6.8. We identify its limit on hearts and bounded subcategories.

Write $C \simeq \alpha_T(M)$ with $M \in \mathrm{D}_{\mathrm{pc}}(T)$. By definition, C lies in $\mathcal{P}^{\heartsuit}_{T}$ precisely when $\mathcal{H}^{j}(M) = 0$ for $j \neq 0$. Consider the fpqc family

$$T \times_X \coprod_i \mathrm{Spec}\, R_i \longrightarrow T$$

Flat pullback commutes with cohomology, and faithful flatness detects its vanishing. Thus M is concentrated in degree zero if and only if all its pullbacks are. Equivalently, the analytic restrictions of C lie in the local transported hearts. Proposition 3.3 therefore identifies $\mathcal{P}^{\heartsuit}_{T}$ with their limit.

On the ordinary side, coherent sheaves descend on the corresponding cover of $T^{\mathrm{ad}}_{\mathcal{X}}$. Taking the limit of the functors $\beta^{\heartsuit}_{T,V}$ gives the heart equivalence.

For bounded objects, choose one interval $[a, b]$ containing the bounds on the finite base cover. Every rational chart has a finite refinement by intersections with this cover. Cartesianness gives the same bounds on that refinement, and therefore on the chart itself, as in Lemma 6.5.

Pseudocoherent descent gives a global object $C \simeq \alpha_T(M)$. The cohomology of M outside $[a, b]$ vanishes on the fpqc family, hence on T. Thus $C \in \mathcal{P}^{b}_{T}$. Conversely, a bounded algebraic preimage has bounded restrictions. This identifies the bounded source with the limit of the local bounded sources. For the ordinary target,

$$\mathrm{D}^{b}_{\mathrm{coh}}(T^{\mathrm{ad}}_{\mathcal{X}}) \simeq \mathrm{Cart}_{V \in \mathrm{RatAff}_X(\mathcal{X})}\, \mathrm{D}^{b}_{\mathrm{coh}}(T^{\mathrm{ad}/V}_{R})$$

by Lemma 6.5. The limit of the $\beta^{b}_{T,V}$ consequently gives β^{b}_{T}.

Locally, the composites with α_T are scalar extension followed by sheafification, exactly as in c_T^* and $\mathbf{L}c_T^*$. Lemma 6.8 shows that these identifications agree under restriction, so they identify the global functors. Exactness follows from the local degree conditions. The tensor comparisons for finite modules agree on overlaps as well, giving the symmetric monoidal equivalence on hearts. □

The restrictions to vector bundles and perfect complexes follow from duality and finite Tor-amplitude.

Corollary 6.10. *For proper T/X, analytification gives exact equivalences*

$$c_T^* : \mathrm{Coh}(T) \xrightarrow{\sim} \mathrm{Coh}(T_{\mathcal{X}}^{\mathrm{ad}}),$$
$$\mathbf{L}c_T^* : \mathrm{D}^b_{\mathrm{coh}}(T) \xrightarrow{\sim} \mathrm{D}^b_{\mathrm{coh}}(T_{\mathcal{X}}^{\mathrm{ad}}).$$

The first is symmetric monoidal, and these restrict to equivalences on vector bundles and perfect complexes.

Proof. Theorem 5.5 and Proposition 6.9 give the coherent and bounded-derived equivalences. The proposition identifies the composites with c_T^* and $\mathbf{L}c_T^*$, respectively.

A coherent sheaf is a vector bundle exactly when it is dualizable for ordinary tensor product. Indeed, locally a duality datum writes the identity of a finite module as $m \mapsto \sum_i \phi_i(m)m_i$, exhibiting it as a summand of a finite free module. A finite projective module, conversely, has its usual dual and evaluation. The tensor equivalence preserves and reflects these data, and hence identifies vector bundles.

Derived pullback preserves perfect complexes because it preserves bounded complexes of finite projectives. For the converse, first work over a base affinoid and let $M \in \mathrm{D}^b_{\mathrm{coh}}(T_R)$ have perfect analytification. The analytic space is proper and therefore quasi-compact. Choose a finite affinoid cover on which this perfect complex has finite Tor-amplitude, and choose $[a, b]$ containing all the resulting intervals.

For any $N \in \mathrm{Coh}(T_R)$, flatness gives

$$c_{T_R/V}^* H^j(M \otimes^{\mathbf{L}} N) \simeq H^j(\mathbf{L}c_{T_R/V}^* M \otimes^{\mathbf{L}} c_{T_R/V}^* N).$$

The right side vanishes for $j \notin [a, b]$, with bounds independent of N. The algebraic cohomology sheaf on the left is coherent: locally, tensor a bounded-above finite-free resolution of M with the finite module representing N, and use noetherianity for its kernels and quotients. Coherent GAGA detects zero objects, so $H^j(M \otimes^{\mathbf{L}} N) = 0$ outside the same interval.

Let $U = \operatorname{Spec} A \subset T_R$ and let Q be a finite A-module. Extend its associated sheaf to $N \in \mathrm{Coh}(T_R)$ using [Stacks, Tag 0G41]. Restricting the preceding vanishing to U gives

$$H^j\big(\mathbf{R}\Gamma(U, M|_U) \otimes_A^{\mathbf{L}} Q\big) = 0$$

for $j \notin [a, b]$. Every A-module is a filtered union of finite submodules, and derived tensor product preserves this colimit. Exactness of filtered colimits extends the vanishing to every A-module. Thus $M|_U$ has Tor-amplitude in $[a, b]$; being pseudocoherent, it is perfect by [Stacks, Tag 0658].

Finally, let $M \in \mathrm{D}^b_{\mathrm{coh}}(T)$ have perfect analytification. Pull it back along the finite fpqc family on X. The affinoid argument makes each pullback perfect on T_{R_i}, and perfectness descends faithfully flatly. Hence M itself is perfect, proving the remaining equivalence. □

6.3. **Ordinary proper pushforward.** We compare ordinary proper direct images on the rational opens of the base, using open base change.

Lemma 6.11. *Let*

$$\begin{array}{ccc} Z' & \xrightarrow{j'} & Z \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ V' & \xrightarrow{j} & V \end{array}$$

be a cartesian square of ringed spaces in which j is an open immersion. Then the canonical morphism $j^*\mathbf{R}h_* \longrightarrow \mathbf{R}h'_*j'^*$ is an equivalence on derived categories of modules.

Proof. For an open immersion j, extension by zero is an exact left adjoint $j_! \dashv j^*$. On stalks it is the given sheaf inside the open and zero outside. We first check

$$\mathbf{L}h^*j_! \simeq j'_!\mathbf{L}(h')^*.$$

Choose a K-flat resolution on V' and extend it by zero. The extension is still K-flat, since its tensor product with an acyclic complex is acyclic on every stalk. At a point of Z', the two displayed functors give the same scalar extension; outside Z', they both give zero. This proves the equivalence.

For $A \in \mathrm{D}(\mathcal{O}_Z)$ and $B \in \mathrm{D}(\mathcal{O}_{V'})$, adjunction gives

$$\begin{aligned} \mathrm{Map}_{V'}(B, j^*\mathbf{R}h_*A) &\simeq \mathrm{Map}_V(j_!B, \mathbf{R}h_*A) \\ &\simeq \mathrm{Map}_Z(\mathbf{L}h^*j_!B, A) \\ &\simeq \mathrm{Map}_Z(j'_!\mathbf{L}(h')^*B, A) \\ &\simeq \mathrm{Map}_{Z'}(\mathbf{L}(h')^*B, j'^*A) \\ &\simeq \mathrm{Map}_{V'}(B, \mathbf{R}h'_*j'^*A). \end{aligned}$$

The middle equivalence is the one just proved; the others use extension by zero and pullback–direct-image adjunction.

Tracing a map through the units and counits gives the canonical base-change morphism. It is an equivalence after mapping from every B, so Yoneda proves the claim. □

Over a Tate affinoid, full faithfulness compares global sections by mapping from the structure sheaf. Applying this comparison on rational opens identifies the proper direct images.

Proposition 6.12. *Let $V = \mathrm{Spa}(R, R^+)$ be a strongly noetherian complete Tate affinoid and let $h : Z \to \operatorname{Spec} R$ be proper and finitely presented. For $M \in \mathrm{D}^b_{\mathrm{coh}}(Z)$, the canonical map*

$$\mathbf{L}c_V^*\mathbf{R}h_*M \xrightarrow{\sim} \mathbf{R}h_*^{\mathrm{ad}}\mathbf{L}c_{Z/V}^*M$$

is an equivalence in $\mathrm{D}^b_{\mathrm{coh}}(V)$, natural in M and coherently compatible with rational localization.

Proof. Apply analytification to the algebraic counit. Compatibility with pullback gives

$$\mathbf{L}(h^{\mathrm{ad}})^*\mathbf{L}c_V^*\mathbf{R}h_*M \simeq \mathbf{L}c_{Z/V}^*\mathbf{L}h^*\mathbf{R}h_*M \longrightarrow \mathbf{L}c_{Z/V}^*M.$$

Its adjoint under $\mathbf{L}(h^{\mathrm{ad}})^* \dashv \mathbf{R}h_*^{\mathrm{ad}}$ is the map in the statement.

Take a rational open $V' = \mathrm{Spa}(R', R'^{+}) \subset V$, and put $Z_{R'} = Z \times_R \operatorname{Spec} R'$ and $M_{R'} = M \otimes^{\mathbf{L}}_R R'$. Flat base change and coherent affinoid acyclicity identify sections of the source on V' with $\mathbf{R}\Gamma(Z_{R'}, M_{R'})$. By Lemma 6.11, sections of the target are computed on the inverse-image analytic open. The resulting map is

$$\mathbf{R}\Gamma(Z_{R'}, M_{R'}) \longrightarrow \mathbf{R}\Gamma(Z_{R'}^{\mathrm{ad}/V'}, \mathbf{L}c^* M_{R'}).$$

By Proposition 6.7, analytification over R' is fully faithful on bounded coherent complexes and preserves the structure sheaf. We therefore have an equivalence

$$\mathrm{Map}_{Z_{R'}}(\mathcal{O}, M_{R'}) \xrightarrow{\mathbf{L}c^*} \mathrm{Map}_{Z_{R'}^{\mathrm{ad}/V'}}(\mathcal{O}, \mathbf{L}c^* M_{R'}).$$

Its source and target are the section complexes above, and it sends $s : \mathcal{O} \to M_{R'}$ to $\mathbf{L}c^*(s)$. After flat base change, the canonical comparison is defined by applying $\mathbf{L}c^*$ to

$$\mathbf{L}h_{R'}^* \mathbf{R}(h_{R'})_* M_{R'} \longrightarrow M_{R'},$$

and taking the adjoint. It therefore induces the same map on sections and is an equivalence on every rational V'.

Let Q be the cofiber. Choose a with $M \in \mathrm{D}^{\geq a}(Z)$. Flat analytification preserves this bound, and derived direct image is left t-exact, so both terms of the comparison lie in degrees at least a. Hence $Q \in \mathrm{D}^{\geq a-1}(V)$. Lemma 6.2 identifies it with a sheaf of module spectra whose values on the rational basis are

$$\mathbf{R}\Gamma(V', Q|_{V'}) = 0.$$

These values determine the sheaf, so $Q = 0$. Algebraic proper direct image and flat pullback make the source bounded coherent; the equivalent target is therefore bounded coherent as well.

Every restriction comparison comes from the same counit. Tensor associativity and naturality of the units and counits identify the composites, together with their higher homotopies. This proves compatibility with rational localization. □

For the base itself, Proposition 3.5 identifies $X_{\mathcal{X}}^{\mathrm{ad}} = \mathcal{X}$ and $c_X = \lambda$. Applying the affinoid comparison on its charts gives the global direct-image comparison.

Proposition 6.13. *For proper finitely presented $f : Y \to X$ and $M \in \mathrm{D}^b_{\mathrm{coh}}(Y)$, the canonical map*

$$\mathbf{L}c_X^* \mathbf{R}f_* M \xrightarrow{\sim} \mathbf{R}f_*^{\mathrm{ad}} \mathbf{L}c_Y^* M$$

is an equivalence in $\mathrm{D}^b_{\mathrm{coh}}(\mathcal{X})$.

Proof. As in the affinoid case, apply $\mathbf{L}c_Y^*$ to the algebraic counit. Pullback compatibility gives

$$\mathbf{L}(f^{\mathrm{ad}})^* \mathbf{L}c_X^* \mathbf{R}f_* M \simeq \mathbf{L}c_Y^* \mathbf{L}f^* \mathbf{R}f_* M \longrightarrow \mathbf{L}c_Y^* M.$$

Take its adjoint to define the global comparison, and restrict to a rational-affinoid chart.

Choose $V = \mathrm{Spa}(R, R^+) \subset \lambda^{-1}(W)$ with $W = \operatorname{Spec} A$, and put $Y_R = Y \times_X \operatorname{Spec} R$ and $M_R = \mathbf{L}(Y_R \to Y)^* M$. Lemma 2.1 makes $A \to R$ flat. Flat base change, Proposition 6.12,

and open base change then give

$$\begin{aligned}(\mathbf{L}c_X^*\mathbf{R}f_*M)|_V &\simeq \mathbf{L}c_V^*\mathbf{R}(f_R)_*M_R\\ &\simeq \mathbf{R}(f_R^{\mathrm{ad}})_*\mathbf{L}c_{Y_R/V}^*M_R\\ &\simeq (\mathbf{R}f_*^{\mathrm{ad}}\mathbf{L}c_Y^*M)|_V.\end{aligned}$$

The first map is obtained by flat base change of the counit, the second by the affinoid construction from that counit, and the last by open restriction. Their composite is therefore the restriction of our global map.

On a finite cover (V_i), the target is bounded coherent by the affinoid comparison. Taking the minimum of the lower bounds and maximum of the upper bounds gives a common interval. Coherence and cohomology vanishing are local, so the target lies in $\mathrm{D}^b_{\mathrm{coh}}(\mathcal{X})$.

The cofiber vanishes on every V_i, hence globally. Thus the comparison is an equivalence in $\mathrm{D}^b_{\mathrm{coh}}(\mathcal{X})$. □

6.4. **Classical GAGA.** The category equivalences and direct-image comparison give classical GAGA. Global cohomology follows by applying the base comparison to the proper direct image.

Theorem 6.14. *Under the hypotheses of Section 1.1, let $f : Y \to X$ be proper. Ordinary analytification $c_Y : Y_{\mathcal{X}}^{\mathrm{ad}} \to Y$ induces canonical equivalences*

$$\begin{aligned}\mathrm{Coh}(Y) &\xrightarrow{\sim} \mathrm{Coh}(Y_{\mathcal{X}}^{\mathrm{ad}}),\\ \mathrm{D}^b_{\mathrm{coh}}(Y) &\xrightarrow{\sim} \mathrm{D}^b_{\mathrm{coh}}(Y_{\mathcal{X}}^{\mathrm{ad}}),\\ \mathrm{Perf}(Y) &\xrightarrow{\sim} \mathrm{Perf}(Y_{\mathcal{X}}^{\mathrm{ad}}),\\ \mathrm{Vect}(Y) &\xrightarrow{\sim} \mathrm{Vect}(Y_{\mathcal{X}}^{\mathrm{ad}}).\end{aligned}$$

The coherent equivalence is exact and symmetric monoidal; the bounded-derived equivalence is exact. Under Theorem 5.5, they identify the transported heart and bounded part of $\mathsf{D}^{\square}_{\mathrm{pc}}(Y_{\mathcal{X}}^{\mathrm{an}})$ with the ordinary coherent and bounded coherent categories, through c_Y^* and $\mathbf{L}c_Y^*$.

For $\mathcal{F} \in \mathrm{Coh}(Y)$ and $i \geq 0$, the canonical maps are isomorphisms:

$$\begin{aligned}(R^i f_*\mathcal{F})^{\mathrm{ad}} &\xrightarrow{\sim} R^i f_*^{\mathrm{ad}}(\mathcal{F}^{\mathrm{ad}}),\\ H^i(Y,\mathcal{F}) &\xrightarrow{\sim} H^i(Y_{\mathcal{X}}^{\mathrm{ad}},\mathcal{F}^{\mathrm{ad}}).\end{aligned}$$

Proof. By Proposition 6.9, the transported heart and bounded part are the ordinary coherent and bounded coherent categories, and the functors are c_Y^* and $\mathbf{L}c_Y^*$. Corollary 6.10 gives the restrictions to vector bundles and perfect complexes, so all four equivalences follow.

For $\mathcal{F} \in \mathrm{Coh}(Y)$, apply Proposition 6.13 to $\mathcal{F}[0]$. Flatness identifies degree-i cohomology of the source with $c_X^*(R^i f_*\mathcal{F})$, while degree-$i$ cohomology of the target is $R^i f_*^{\mathrm{ad}}(\mathcal{F}^{\mathrm{ad}})$. The resulting isomorphism is the asserted canonical direct-image map.

The complex $\mathbf{R}f_*\mathcal{F}$ is bounded coherent: proper direct image gives coherence, Lemma 5.4 gives an upper bound, and negative cohomology vanishes. Applying the base comparison

yields

$$\begin{aligned}\mathbf{R}\Gamma(Y,\mathcal{F}) &\simeq \mathbf{R}\Gamma(X,\mathbf{R}f_*\mathcal{F})\\ &\simeq \mathbf{R}\Gamma_{\mathrm{ad}}(\mathcal{X},(\mathbf{R}f_*\mathcal{F})^{\mathrm{ad}})\\ &\simeq \mathbf{R}\Gamma_{\mathrm{ad}}(\mathcal{X},\mathbf{R}f_*^{\mathrm{ad}}\mathcal{F}^{\mathrm{ad}})\\ &\simeq \mathbf{R}\Gamma_{\mathrm{ad}}(Y_{\mathcal{X}}^{\mathrm{ad}},\mathcal{F}^{\mathrm{ad}}).\end{aligned}$$

The first and last maps compose direct images. The second is Lemma 2.3, and the third is the proper-direct-image comparison. Each is canonical, so taking degree-i cohomology gives the required isomorphism. □

6.5. **Changing the proper base.** A proper X-scheme, together with its analytification, is another base to which the theorem applies. To compare direct images after this change of base, we first identify the successive analytifications.

Lemma 6.15. *Let $Z \to X$ be proper, put $\mathcal{Z} = Z_{\mathcal{X}}^{\mathrm{ad}}$, and let $Y \to Z$ be separated and finitely presented. There is a canonical isomorphism*

$$Y_{\mathcal{Z}}^{\mathrm{ad}} \xrightarrow{\sim} Y_{\mathcal{X}}^{\mathrm{ad}},$$

compatible with the maps to Y and $\mathcal{Z}$. It induces an equivalence $Y_{\mathcal{Z}}^{\mathrm{an}} \simeq Y_{\mathcal{X}}^{\mathrm{an}}$ identifying the comparison functors α_Y^{an}.

Proof. A map from an analytic test space H to $Y_{\mathcal{X}}^{\mathrm{ad}}$ is a compatible pair $H \to \mathcal{X}$, $H \to Y$, where the latter is a morphism of locally ringed spaces. Composing with $Y \to Z$ gives the pair defining a unique map $H \to \mathcal{Z}$, compatible with $H \to Y$ over Z. Conversely, such a pair over $\mathcal{Z}$ gives one over $\mathcal{X}$. These constructions are inverse and natural in H.

Huber's mapping property [Hub94, Proposition 3.8], applied on affinoids and glued as in Proposition 3.5, therefore gives the displayed isomorphism. For a further change of base, both composites are specified by this same mapping property.

Since Y is separated and finitely presented over X, Proposition 4.10 applies over both locally strongly noetherian analytic bases. It identifies the analytic loci $Y_{\mathcal{Z}}^{\mathrm{an}}$ and $Y_{\mathcal{X}}^{\mathrm{an}}$ with the stack associated to this adic space. Their comparison functors agree as well: if $W = \mathrm{Spa}(D, D^+)$ maps into $U = \operatorname{Spec} A \subset Y$, both send $M \in \mathrm{D}_{\mathrm{qc}}(Y)$ to

$$\mathrm{Cond}_D\big(D \otimes_A^{\mathbf{L}} \mathbf{R}\Gamma(U, M|_U)\big)$$

on W. The formulas agree under restriction by tensor associativity and hence identify the global functors. □

Corollary 6.16. *Let Y, Z be proper X-schemes and let $g : Y \to Z$ be an X-morphism. For every $M \in \mathrm{D}_{\mathrm{qc}}(Y)$, the canonical map*

$$\alpha_Z(\mathbf{R}g_*M) \xrightarrow{\sim} \mathbf{R}g_*^{\mathrm{an}}\alpha_Y(M)$$

is an equivalence. For $M \in \mathrm{D}_{\mathrm{coh}}^b(Y)$, the canonical map

$$\mathbf{L}c_Z^*\mathbf{R}g_*M \xrightarrow{\sim} \mathbf{R}g_*^{\mathrm{ad}}\mathbf{L}c_Y^*M$$

is an equivalence in $\mathrm{D}_{\mathrm{coh}}^b(Z_{\mathcal{X}}^{\mathrm{ad}})$. Both comparisons commute with composition of such morphisms.

Proof. First observe that g is proper. Its graph in $Y \times_X Z$ is closed because $Z \to X$ is separated, and the projection to Z is a base change of the proper map $Y \to X$.

The pair $(Z, \mathcal{Z})$, with $\mathcal{Z} = Z_{\mathcal{X}}^{\mathrm{ad}}$, satisfies Section 1.1. Indeed, Z is noetherian and has affine diagonal by the factorization in Section 2. Its analytification is locally strongly noetherian and proper over the quasi-compact quasi-separated space $\mathcal{X}$, hence is quasi-compact and quasi-separated. Theorem 6.14 supplies exact coherent GAGA and the canonical coherent-cohomology comparison.

Apply Theorem 5.5 and Proposition 6.13 to g over $(Z, \mathcal{Z})$. Lemma 6.15 identifies their spaces and functors with the ones in the statement. Both constructions use the algebraic counit, so the resulting maps are the stated canonical maps. For successive morphisms, the composite counit is the counit of the composite adjoint pair. Hence the direct and iterated comparisons agree, proving compatibility with composition. □

7. The Fargues–Fontaine curve

For the Fargues–Fontaine curve, coherent GAGA and cohomology comparison provide the input to the bootstrapping theorem. We recall the schematic and adic constructions and their comparison map [FF18, FS24], then give the affinoid cover needed to apply it.

7.1. **The adic and schematic curves.** Let E be a nonarchimedean local field with residue field $\mathbf{F}_q$, and let F be a complete perfectoid field of characteristic p containing $\mathbf{F}_q$. Write $\mathcal{O}_E$ and $\mathcal{O}_F$ for their valuation rings, and choose a uniformizer ϖ_E of E and a nonzero topologically nilpotent element $\varpi_F \in \mathcal{O}_F$. In both constructions F has characteristic p; it is the characteristic of E that distinguishes the equal- and mixed-characteristic cases.

The starting ring is

$$A_{\mathrm{inf}} = \begin{cases} W(\mathcal{O}_F)\widehat{\otimes}_{W(\mathbf{F}_q)}\mathcal{O}_E, & \operatorname{char} E = 0, \\ \mathcal{O}_F[[\varpi_E]], & E = \mathbf{F}_q((\varpi_E)). \end{cases}$$

Here W denotes p-typical Witt vectors, and the tensor product is completed ϖ_E-adically. In either case, $A_{\mathrm{inf}}/\varpi_E \simeq \mathcal{O}_F$. For $x \in \mathcal{O}_F$, write $[x]$ for its Teichmüller lift; in equal characteristic, this is just the constant series x. Give A_{inf} the $(\varpi_E, [\varpi_F])$-adic topology. The q-power Frobenius on F then induces an automorphism φ satisfying

$$\varphi([x]) = [x^q],$$
$$\varphi(\varpi_E) = \varpi_E.$$

Thus in equal characteristic, Frobenius acts on the coefficients and fixes the variable. In mixed characteristic, the same formulas describe the ramified Witt-vector construction [FS24, Section II.1.1].

To obtain the adic curve, first consider the open period-ring space

$$\mathcal{Y} = \operatorname{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}}) \setminus V(\varpi_E[\varpi_F]).$$

Both ϖ_E and $[\varpi_F]$ have nonzero value on $\mathcal{Y}$. Inverting ϖ_E makes $\mathcal{Y}$ an analytic adic space over E. In equal characteristic, it is the punctured open unit disc over F with coordinate ϖ_E [FS24, Section II.1.1 and the proof of Proposition II.2.2].

For a rank-one point x, define its radius by

$$\operatorname{rad}(x) = \frac{\log |[\varpi_F](x)|}{\log |\varpi_E(x)|}.$$

For any other point, use its rank-one generalization in the formula. The identities for Frobenius give $\mathrm{rad}(\varphi x) = q\,\mathrm{rad}(x)$. Thus an annulus of sufficiently small multiplicative width does not meet any of its nontrivial Frobenius translates. The action is free and totally discontinuous, so we can form the quotient

$$\mathcal{X} = X_{F,E}^{\mathrm{ad}} = \mathcal{Y}/\varphi^{\mathbf{Z}},$$

with quotient map $\pi : \mathcal{Y} \to \mathcal{X}$ [FS24, Definition II.1.15 and Proposition II.1.16]. Annuli disjoint from their nontrivial translates give affinoid charts on this adic curve.

For the schematic construction, put $B = \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ and form the graded E-algebra of Frobenius eigenvectors

$$P = \bigoplus_{n \geq 0} P_n,$$
$$P_n = B^{\varphi = \varpi_E^n}.$$

Here $P_n = \{b \in B : \varphi(b) = \varpi_E^n b\}$. If $a \in P_m$ and $b \in P_n$, then $\varphi(ab) = \varpi_E^{m+n} ab$, so multiplication respects the grading. Define

$$X = X_{F,E}^{\mathrm{sch}} = \mathrm{Proj}(P).$$

The graded algebra P is also a section ring on the adic curve. Descending the trivial line on $\mathcal{Y}$ with semilinear Frobenius $\varpi_E^{-1}\varphi$ gives $\mathcal{O}_{\mathcal{X}}(1)$. Sections of its nth tensor power are functions b satisfying $\varphi(b) = \varpi_E^n b$, so $P_n = H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(n))$ [FS24, Section II.2 and Proposition II.2.7].

To describe the comparison map, let $s \in P_d$ be homogeneous of positive degree. The corresponding standard affine is

$$D_+(s) = \mathrm{Spec}\big(P[s^{-1}]_0\big),$$

where the subscript means degree zero. An element of this ring is a fraction a/s^k with $a \in P_{kd}$. Wherever s is nonzero, we compute

$$\varphi(a/s^k) = \frac{\varpi_E^{kd} a}{(\varpi_E^d s)^k} = a/s^k.$$

Thus the fraction descends to an analytic function on the nonvanishing open $\mathcal{X}_s$, giving $\mathcal{X}_s \to D_+(s)$. These opens cover $\mathcal{X}$ because $\mathcal{O}_{\mathcal{X}}(d)$ is globally generated for sufficiently large d [FS24, Theorem II.2.6]. On $\mathcal{X}_s \cap \mathcal{X}_t$, both maps are given by localization through $D_+(st)$. They glue to

$$\lambda : \mathcal{X} \longrightarrow X$$

as a morphism of locally ringed spaces [FS24, Proposition II.2.7 and its proof]. This is the map used in Section 2.1.

For nonzero homogeneous s of positive degree, the ring $P[s^{-1}]_0$ is a Dedekind domain, and the annular rings are strongly noetherian [KL19, Theorem 4.7.3]. In particular, X is a noetherian regular separated curve. The known GAGA theorem for λ gives

$$\lambda^* : \mathrm{Coh}(X) \xrightarrow{\sim} \mathrm{Coh}(\mathcal{X}),$$
$$H^i(X, \mathcal{F}) \xrightarrow{\sim} H^i(\mathcal{X}, \lambda^*\mathcal{F})$$

for coherent $\mathcal{F}$ and $i \geq 0$ [KL19, Theorems 4.7.4–4.7.5]. The following affinoid cover verifies quasi-compactness and quasiseparatedness of $\mathcal{X}$.

7.2. **A two-affinoid cover of the curve.** We use the cover of [Ked17, Remark 3.1.9]. It is obtained from two annuli disjoint from their nontrivial Frobenius translates; their intersections with one another's translates determine the overlap.

For an interval $I = [a, b] \subset (0, \infty)$ with rational endpoints, the corresponding rational affinoid in $\mathcal{Y}$ is

$$\mathcal{Y}_I = \left\{x \in \mathcal{Y} : |\varpi_E(x)|^b \leq |[\varpi_F](x)| \leq |\varpi_E(x)|^a\right\} \subset \mathrm{rad}^{-1}(I),$$

with $\mathrm{rad} \circ \varphi = q\ \mathrm{rad}$ [FS24, Definition II.1.15 and Proposition II.1.16]. Substituting $\varphi(\varpi_E) = \varpi_E$ and $\varphi([\varpi_F]) = [\varpi_F]^q$ in the inequalities gives $\varphi(\mathcal{Y}_I) = \mathcal{Y}_{qI}$.

Lemma 7.1. *For rational intervals $I = [a, b]$ and $J = [c, d]$ in $(0, \infty)$,*

$$\mathcal{Y}_I \cap \mathcal{Y}_J = \begin{cases} \mathcal{Y}_{[\max(a,c),\, \min(b,d)]}, & \max(a, c) \leq \min(b, d), \\ \varnothing, & \max(a, c) > \min(b, d). \end{cases}$$

For every $n \in \mathbf{Z}$, we also have $\varphi^n(\mathcal{Y}_I) = \mathcal{Y}_{q^n I}$.

Proof. At a point x, write $u = |\varpi_E(x)|$ and $v = |[\varpi_F](x)|$. To belong to both annuli, we need $u^b \leq v \leq u^a$ and $u^d \leq v \leq u^c$. Since $0 < u < 1$, these conditions combine into

$$u^{\min(b,d)} \leq v \leq u^{\max(a,c)}.$$

If $\max(a, c) > \min(b, d)$, the inequalities cannot hold. Otherwise they define exactly the claimed rational subspace, with its analytic structure.

Frobenius carries the interval $[a, b]$ to $[qa, qb]$ [FS24, Definition II.1.15 and Proposition II.1.16]. Applying it repeatedly, or applying its inverse, gives the formula for every $n \in \mathbf{Z}$. □

An open disjoint from its nontrivial translates descends to an open of the quotient.

Lemma 7.2. *Let $U \subset \mathcal{Y}$ be an open analytic subspace satisfying $U \cap \varphi^n(U) = \varnothing$ for every $n \in \mathbf{Z} \setminus \{0\}$. Then the restriction $\pi|_U : U \to \mathcal{X}$ is an open immersion.*

Proof. The quotient is formed from the free, totally discontinuous $\varphi^{\mathbf{Z}}$-action of [FS24, Definition II.1.15 and Proposition II.1.16]. Consider the saturation

$$U^{\mathrm{sat}} = \bigcup_{n \in \mathbf{Z}} \varphi^n(U).$$

Its translates are pairwise disjoint: applying φ^{-m} turns an intersection of two distinct translates into $U \cap \varphi^{n-m}(U)$, which is empty by hypothesis.

We have $\pi^{-1}(\pi(U)) = U^{\mathrm{sat}}$, so $\pi(U)$ is open. Every orbit in this saturation meets U exactly once. The same argument for an open subset of U shows that $|U| \to \pi(U)$ is an open bijection, hence a homeomorphism.

Powers of φ identify each component of U^{sat} with U, preserving the structure sheaf and valuations. The quotient therefore has the analytic structure of U, proving the open-immersion assertion. □

Choose one annulus crossing the ends of a fundamental interval and another contained in its interior.

Proposition 7.3. *Choose rational $1 < a' < a < b < b' < q$, and put $I_0 = [b/q, a]$ and $I_1 = [a', b']$. The restrictions $\pi|_{\mathcal{Y}_{I_0}}$ and $\pi|_{\mathcal{Y}_{I_1}}$ are open immersions whose images $\mathcal{U}, \mathcal{V}$ cover $\mathcal{X}$, with*

$$\mathcal{U} \cap \mathcal{V} \simeq \mathcal{Y}_{[a',a]} \amalg \mathcal{Y}_{[b,b']}.$$

Both components are rational in each covering affinoid; the second maps into $\mathcal{Y}_{I_0}$ by φ^{-1}.

Proof. An interval $[c, d]$ with $d < qc$ is disjoint from its positive q-translates, and scaling gives the same conclusion for negative translates. For our two intervals,

$$\frac{a}{b/q} = q\frac{a}{b} < q,$$
$$\frac{b'}{a'} < q.$$

Both multiplicative widths are therefore less than q. Lemmas 7.1 and 7.2 identify the annuli with open affinoids of $\mathcal{X}$.

Every Frobenius orbit meets $\mathcal{Y}_{[1,q]}$, so it is enough to cover this fundamental annulus. We claim

$$\mathcal{Y}_{[1,q]} = \mathcal{Y}_{[1,a]} \cup \mathcal{Y}_{[a',b']} \cup \mathcal{Y}_{[b,q]}.$$

Let $x \in \mathcal{Y}_{[1,q]}$, and put $u = |\varpi_E(x)|$ and $v = |[\varpi_F](x)|$. If x lies in neither the first nor the third subannulus, then $u^b < v < u^a$. Using $a' < a < b < b'$, we get

$$u^{b'} < u^b < v < u^a < u^{a'}.$$

Hence x lies in the middle subannulus. The reverse inclusion is immediate from the interval bounds, proving the claimed cover.

The first piece lies in $\mathcal{Y}_{I_0}$, while $\mathcal{Y}_{[b,q]} \subset \varphi(\mathcal{Y}_{I_0})$ because $qI_0 = [b, qa]$. The middle piece is $\mathcal{Y}_{I_1}$. Since an annulus and its translate have the same image in the quotient, this proves $\mathcal{U} \cup \mathcal{V} = \mathcal{X}$.

To compute the overlap, use representatives in $\mathcal{Y}_{I_1}$. Such a representative lies over $\mathcal{U}$ exactly when it belongs to some $\mathcal{Y}_{q^n I_0}$. If $n \leq -1$, the upper endpoint is at most $a/q < a'$; if $n \geq 2$, the lower endpoint is at least $qb > b'$. Only $n = 0, 1$ remain. Lemma 7.1 gives

$$\mathcal{Y}_{I_1} \cap \mathcal{Y}_{I_0} = \mathcal{Y}_{[a',a]},$$
$$\mathcal{Y}_{I_1} \cap \varphi(\mathcal{Y}_{I_0}) = \mathcal{Y}_{[b,b']}.$$

These pieces are disjoint because $a < b$, and $\mathcal{Y}_{I_1}$ embeds in the quotient. They are therefore exactly the two overlap components. Each inclusion is rational; to map into $\mathcal{Y}_{I_0}$, use the identity on the first component and φ^{-1} on the second. □

7.3. **A subordinate affinoid cover.** To obtain the charts used for scalar extension in Section 2.1, refine the two annuli by strongly noetherian rational affinoids $V \subset \lambda^{-1}(W)$ with W standard affine in X. These form a basis stable under rational localization, and their intersections admit finite covers by basis members.

Choose a finite standard affine cover (W_j) of X, and refine $\mathcal{U}$ and $\mathcal{V}$ by finite rational-affinoid covers subordinate to $(\lambda^{-1}(W_j))$. Together they give

$$\mathcal{X} = \bigcup_{i=1}^{N} V_i$$

with $V_i \subset \lambda^{-1}(W_{j(i)})$. We call this the subordinate cover. Intersections within either annulus are rational affinoid. A mixed intersection is rational affinoid on each of the two components of $\mathcal{U} \cap \mathcal{V}$. Thus every intersection is empty or a disjoint union of at most two strongly noetherian affinoids, each lying over a standard affine of X.

Both covers satisfy Proposition B.6. The subordinate one also gives exactly the charts used to construct scalar extension in Section 2.1.

7.4. **Application of the bootstrapping theorem.** The annular cover exhibits $\mathcal{X}$ as a quasi-compact quasi-separated locally strongly noetherian analytic adic space. Its schematic counterpart X is noetherian and separated, hence has affine diagonal. Together with the morphism λ and the coherent and cohomological comparisons of [KL19, Theorems 4.7.4–4.7.5], this gives the pair of Section 1.1. The bootstrapping theorem therefore applies.

Theorem 7.4. *Let E be a nonarchimedean local field with residue field $\mathbf{F}_q$, and let F be a complete perfectoid field of characteristic p containing $\mathbf{F}_q$. For $X = X^{\mathrm{sch}}_{F,E}$, $\mathcal{X} = X^{\mathrm{ad}}_{F,E}$, and every proper X-scheme Y, comparison induces a symmetric monoidal equivalence*

$$\mathrm{D}_{\mathrm{pc}}(Y) \xrightarrow{\sim} \mathsf{D}^{\square}_{\mathrm{pc}}(Y^{\mathrm{an}}_{\mathcal{X}}).$$

It extends to a fully faithful colimit-preserving comparison on $\mathrm{D}_{\mathrm{qc}}(Y)$, takes perfect complexes to compact objects, and commutes with proper direct images. Ordinary Huber analytification induces equivalences

$$\begin{aligned}
\mathrm{Coh}(Y) &\xrightarrow{\sim} \mathrm{Coh}(Y^{\mathrm{ad}}_{\mathcal{X}}),\\
\mathrm{D}^b_{\mathrm{coh}}(Y) &\xrightarrow{\sim} \mathrm{D}^b_{\mathrm{coh}}(Y^{\mathrm{ad}}_{\mathcal{X}}),\\
\mathrm{Perf}(Y) &\xrightarrow{\sim} \mathrm{Perf}(Y^{\mathrm{ad}}_{\mathcal{X}}),\\
\mathrm{Vect}(Y) &\xrightarrow{\sim} \mathrm{Vect}(Y^{\mathrm{ad}}_{\mathcal{X}}),
\end{aligned}$$

compatible with coherent cohomology and proper direct images.

Proof. Apply Theorems 5.5 and 6.14 to the pair just checked. □

This gives the relative comparison asked for in [Ked23, Ked24].

Appendix A. Recollections on condensed mathematics

We recall the categories and functors used in the proof. Starting with ordinary modules, we introduce condensed modules, solidity, and analytic completeness. We then explain how nuclearity and pseudocompactness characterize the objects coming from ordinary pseudocoherent complexes.

For the ordinary commutative algebra used here, see [AM69, Chapters 2–3, 5, and 7]; for complexes, derived functors, and derived categories, see [Wei94, Chapters 1–3 and 10]. We use the stable ∞-categorical conventions of [Lur09, Lur17].

For the condensed and analytic constructions we follow [Sch26a, And21, CS26] and [LH26, A.1–A.3], retaining cohomological indexing. The descent and base-change arguments are in Appendix B.

A.1. **Ordinary and condensed modules.** For an ordinary ring B, write Mod_B for its abelian category of modules and $\mathrm{D}(B)$ for the category of complexes with quasi-isomorphisms inverted. When B is topological, $\mathrm{D}(B)$ still refers to its underlying ring; its objects have no specified topology.

Condensed objects record continuous families indexed by profinite sets. A *condensed set* is a sheaf on profinite sets for finite jointly surjective families of continuous maps. Condensed abelian groups, rings, and modules are such sheaves with the corresponding algebraic structures. We fix a sufficiently large size bound on the profinite test objects, as in [Sch26a, Lecture I].

For a Hausdorff space T, put

$$\underline{T}(S) = \mathrm{Cont}(S, T).$$

At the one-point set $*$ we recover the underlying set of T. Taking $S = \mathbf{N} \cup \{\infty\}$, the one-point compactification, records a sequence together with its limit. The functor $T \mapsto \underline{T}$ is fully faithful on compactly generated Hausdorff spaces [Sch26a, Proposition 1.2].

For a topological ring R, distinguish the condensed rings

$$R_{\mathrm{disc}}(S) = \{\text{locally constant maps } S \to R\},$$
$$\underline{R}(S) = \mathrm{Cont}(S, R).$$

The identity on R induces $R_{\mathrm{disc}} \to \underline{R}$, a bijection at $*$ that need not be a bijection on other test sets. For example, the map $n \mapsto p^n$, $\infty \mapsto 0$ on $\mathbf{N} \cup \{\infty\}$ is continuous but not locally constant near ∞. It is therefore a section of $\underline{\mathbf{Q}_p}$ but not of $(\mathbf{Q}_p)_{\mathrm{disc}}$.

For a condensed ring A, let $\mathrm{Mod}_A^{\mathrm{cond}}$ denote condensed abelian groups with an A-action; maps are sheaf maps respecting that action. A topological module gives an example, but a general condensed module is specified as a sheaf, not as an ordinary module with a chosen topology. Its derived category is

$$\mathrm{D}(\mathrm{Mod}_A^{\mathrm{cond}}).$$

For ordinary B, we abbreviate this as

$$\mathrm{D}_{\mathrm{cond}}(B) = \mathrm{D}(\mathrm{Mod}_{B_{\mathrm{disc}}}^{\mathrm{cond}}).$$

For modules over $\underline{R}$, we write $\mathrm{D}(\mathrm{Mod}_{\underline{R}}^{\mathrm{cond}})$, specifying the topological rather than the discrete condensed ring.

A profinite set is *extremally disconnected* if the closure of every open subset is open. Every profinite set is a continuous quotient of an extremally disconnected one. The free condensed A-module $A[S]$ is characterized by

$$\mathrm{Hom}_{\mathrm{Mod}_A^{\mathrm{cond}}}(A[S], N) = N(S).$$

When S is extremally disconnected, evaluation is exact. The modules $A[S]$ form compact projective generators, so on the derived category we have

$$\mathbf{R}\,\mathrm{Hom}_A(A[S], K) \simeq K(S),$$

where $K(S)$ is obtained by evaluating the complex term by term. These compact projectives play the role of free modules in constructing resolutions [And21, Section 2].

Giving ordinary modules the discrete topology defines an exact fully faithful functor, which extends to

$$\begin{aligned} \mathrm{D}(B) &\longrightarrow \mathrm{D}_{\mathrm{cond}}(B), \\ M &\longmapsto M_{\mathrm{disc}}. \end{aligned}$$

We call this *discrete condensification*. It is computed term by term and preserves filtered colimits [And21, Lemma 5.7]. Its image does not contain every condensed module, since a general profinite family need not be locally constant.

A.2. **Derived operations and solid modules.** Descent for complexes includes equivalences on double intersections, homotopies between their composites on triple intersections, and higher compatibilities. Working in stable ∞-categories retains these data in the corresponding limit of categories.

For K, L in a stable category, the mapping spectrum satisfies

$$\pi_{-i} \operatorname{Map}(K, L) = \operatorname{Hom}(K, L[i]).$$

In our module categories, this is the spectrum associated with the derived Hom complex. Internal derived Hom $\mathbf{R}\underline{\mathrm{Hom}}(K, L)$ is instead an object of the module category. For condensed modules, its derived evaluation at $*$ gives the derived Hom complex of abelian groups.

An object P is *compact* when $\operatorname{Map}(P, -)$ preserves filtered colimits. It is *dualizable* when evaluation and coevaluation satisfy the duality identities. Whenever internal Hom exists, put $P^\vee = \mathbf{R}\underline{\mathrm{Hom}}(P, \mathbf{1})$. For dualizable P, evaluation gives

$$P^\vee \otimes N \xrightarrow{\sim} \mathbf{R}\underline{\mathrm{Hom}}(P, N).$$

In $\mathrm{D}(B)$, compact objects and dualizable objects are both precisely the perfect complexes. For the analytic comparisons, we use the displayed duality formula to compute maps by global sections, and compactness to commute those maps with filtered colimits [Lur17, Sections 1.1 and 4.6].

Solidity is defined using certain measure modules. Write a profinite set as $S = \lim_i S_i$ with each S_i finite, and form

$$\mathbf{Z}^\square[S] = \lim_i \mathbf{Z}[S_i].$$

The transition maps send each basis point to its image in a coarser quotient. Thus evaluation at $*$ gives a compatible system of integer coefficients: the coefficient at a point is the sum of the coefficients at its preimages in a finer quotient. A point of S determines such a system, its Dirac distribution, giving the map $\mathbf{Z}[S] \to \mathbf{Z}^\square[S]$.

A condensed abelian group M is *solid* if, for every profinite S, restriction induces a bijection

$$\operatorname{Hom}(\mathbf{Z}^\square[S], M) \xrightarrow{\sim} \operatorname{Hom}(\mathbf{Z}[S], M).$$

The right side is $M(S)$, so the condition says that a section indexed by S extends uniquely to the measure module. When S is finite the two modules already agree; the condition concerns infinite profinite families.

The solid abelian groups form a full abelian subcategory $\mathrm{Solid} \subset \mathrm{Cond}(\mathrm{Ab})$. Its inclusion has a left adjoint, solidification. Its derived category identifies with the full subcategory of

derived solid objects in D(Cond(Ab)), and its tensor product is

$$M \otimes^{\mathbf{L}}_{\mathbf{Z}^\square} N = \left(M \otimes^{\mathbf{L}}_{\mathbf{Z}} N\right)^\square,$$

where the inner tensor product is condensed and the outer operation is derived solidification [Sch26a, Lecture II].

Discrete abelian groups are solid. Modules over a solid ring are formed inside Solid with its tensor product. For Huber pairs, the analytic structure further specifies the complete modules.

A.3. **Analytic rings and complete modules.** An analytic ring specifies measure modules appropriate to a given ring, replacing the modules $\mathbf{Z}^\square[S]$ above. Write $\mathcal{A} = (A, \mathcal{M})$, with A an animated condensed commutative ring. For each extremally disconnected profinite set S, we have a connective condensed A-module $\mathcal{M}[S]$ and a natural Dirac map

$$A[S] \longrightarrow \mathcal{M}[S],$$

and $\mathcal{M}$ takes finite disjoint unions to finite direct sums.

An *animated ring* is obtained from simplicial commutative rings by inverting maps inducing isomorphisms on homotopy groups; the same construction applies in condensed objects. Positive homological degrees are negative cohomological degrees in our convention and record, for example, higher Tor groups of derived tensor products. Animated rings enter in the derived algebra of the analytic realizations [Sch26a, Lectures XI–XII].

An A-module complex N is *analytically complete* if the Dirac maps induce equivalences

$$\mathbf{R}\underline{\mathrm{Hom}}_A(\mathcal{M}[S], N) \xrightarrow{\sim} \mathbf{R}\underline{\mathrm{Hom}}_A(A[S], N)$$

for every extremally disconnected S. These are internal derived Hom objects in condensed abelian groups [Sch26a, Definitions 12.1 and 12.3, Remark 12.2]. Evaluating at $*$ gives $\mathbf{R}\,\mathrm{Hom}_A(\mathcal{M}[S], N) \simeq N(S)$. For the nonanimated Huber analytic rings, these evaluated comparisons also characterize completeness [And21, Theorem 2.7].

The analyticity axiom requires the internal comparison for connective objects generated from the $\mathcal{M}[S]$ by sifted colimits. A colimit is sifted when its indexing category makes colimits of spaces commute with finite products. Such colimits are generated by filtered colimits and geometric realizations [Sch26a, Lecture XI]. In the nonanimated setting, one may instead test the axiom on bounded-above complexes of direct sums of measure modules [And21, Definitions 2.5 and 2.14].

For commutative animated analytic rings, we also impose the Frobenius condition of [Sch26a, Definition 12.10]. For every prime ℓ, Frobenius from A to its derived reduction modulo ℓ must be a map of analytic rings, with measure modules reduced modulo ℓ. This gives compatibility of analytic completion with the animated commutative algebra structure.

Write the full stable category of complete modules as

$$\mathsf{D}(\mathcal{A}) = \mathsf{D}(A, \mathcal{M}).$$

For nonanimated A, this is a full subcategory of $\mathrm{D}(\mathrm{Mod}^{\mathrm{cond}}_A)$; for animated A, we use modules over the derived condensed ring. Completeness is a property of a module, not additional structure or a choice of topology.

The inclusion of complete modules has a left adjoint $L_{\mathcal{A}}$. Its defining property is

$$\mathrm{Map}_{\mathsf{D}(\mathcal{A})}(L_{\mathcal{A}} M, N) \simeq \mathrm{Map}(M, N)$$

for complete N, with the right side computed in condensed A-modules. This is *analytic completion*; in particular, $L_{\mathcal{A}}(A[S]) = \mathcal{M}[S]$. The measure modules are compact projective in the connective category and become compact generators after stabilization. We call the analytic ring complete if $A \to \mathcal{M}[*]$ is an equivalence, in which case A is the tensor unit. All analytic rings used here are complete [Sch26a, Proposition 12.4, Remark 12.5, and Definition 12.9].

Analytic completion is determined by the measure modules. For example, $\underline{\mathbf{Q}_p}$ is complete for $(\mathbf{Q}_p, \mathbf{Z}_p)^{\square}$, whereas its p-adic completion is zero: p is invertible, so every quotient by p^n vanishes.

To tensor complete modules, first form the condensed derived tensor product and then complete:

$$M \otimes^{\mathbf{L}}_{\mathcal{A}} N = L_{\mathcal{A}}\big(M \otimes^{\mathbf{L}}_{A} N\big).$$

Analytic completion preserves exact triangles, and $\mathsf{D}(\mathcal{A})$ admits all small limits and colimits [Sch26a, Proposition 12.4 and Remark 12.5].

For a map $\mathcal{A} = (A, \mathcal{M}) \to \mathcal{B} = (B, \mathcal{N})$, we have

$$B \otimes^{\mathbf{L}}_{\mathcal{A}} - : \mathsf{D}(\mathcal{A}) \rightleftarrows \mathsf{D}(\mathcal{B}) : \mathrm{Res}\,.$$

The left adjoint extends scalars along $A \to B$ and then applies $L_{\mathcal{B}}$. The right adjoint restricts the B-action to A; the definition of a map of analytic rings ensures that a complete module remains complete under this restriction. Extension of scalars is exact, strong symmetric monoidal, and colimit-preserving. On analytic affines, these two functors are pullback and direct image.

Suppose a connective animated A-algebra B is already $\mathcal{A}$-complete. Its induced analytic structure satisfies

$$\mathsf{D}(\mathcal{B}) \simeq \mathrm{Mod}_B\big(\mathsf{D}(\mathcal{A})\big).$$

A complete $\mathcal{B}$-module is therefore an $\mathcal{A}$-complete module equipped with a B-action. The underlying ring and its complete connective modules determine the complete analytic structure [Sch26a, Propositions 12.8 and 12.20]. Appendix B uses this description to identify analytic-ring pushouts.

A.4. **Huber pairs and condensification.** A Huber ring R has an open ring of definition with a finitely generated ideal of definition. It is *Tate* if it contains a topologically nilpotent unit. A Huber pair (R, R^+) specifies, in addition, an open integrally closed subring R^+ of power-bounded elements. An element is power-bounded if its powers form a bounded set, and topologically nilpotent if they tend to zero. The Tate algebra $R\langle T_1, \dots, T_n\rangle$ consists of restricted power series, whose coefficients tend to zero. The ring R is *strongly noetherian* if all these Tate algebras are noetherian. Our base charts are complete Tate pairs of this kind.

A complete Huber pair determines an analytic ring $(R, R^+)^{\square}$ whose underlying condensed ring is $\underline{R}$ [And21, Theorem 3.28]. Its complete modules form the category

$$\mathsf{D}_{\square}(R, R^+) = \mathsf{D}((R, R^+)^{\square}) \subset \mathrm{D}(\mathrm{Mod}^{\mathrm{cond}}_{\underline{R}}).$$

An object is therefore a condensed $\underline{R}$-complex complete for the measure modules of $(R, R^+)^{\square}$. The map from $\mathbf{Z}^{\square}$ makes its underlying complex solid. The notation $\square$ here refers to the Huber-pair analytic structure and its complete-module category.

To compare this with ordinary R-modules, take $M \in \mathrm{D}(R)$, give it the discrete condensed structure, and extend scalars from R_{disc} to $\underline{R}$. The resulting functor is

$$\mathrm{Cond}_R(M) = \underline{R} \otimes^{\mathbf{L}}_{R_{\mathrm{disc}}} M_{\mathrm{disc}}.$$

This condensed tensor product is already complete for $(R, R^+)^\square$ [And21, Definition 5.8 and Theorem 5.9]. Thus the two condensification constructions fit into the diagram

$$\begin{array}{ccc} \mathrm{D}(R) & \xrightarrow{\mathrm{Cond}_R} & \mathsf{D}_\square(R, R^+) \\ {\scriptstyle (-)_{\mathrm{disc}}}\downarrow & & \downarrow{\scriptstyle \text{inclusion}} \\ \mathrm{D}_{\mathrm{cond}}(R) & \xrightarrow{\underline{R}\otimes^{\mathbf{L}}_{R_{\mathrm{disc}}} -} & \mathrm{D}(\mathrm{Mod}^{\mathrm{cond}}_{\underline{R}}). \end{array}$$

The left arrow assigns the discrete topology; the top arrow also extends scalars to $\underline{R}$. The right arrow includes the full subcategory of complete modules without changing their underlying condensed modules.

For a finite free module, the formula reads $\mathrm{Cond}_R(R^n) = \underline{R}^n$, and a map is still given by the same matrix. An idempotent matrix then gives the corresponding construction for a finite projective summand. More generally, a bounded-above finite-free resolution, when available, computes derived scalar extension term by term.

Condensification is fully faithful, preserves exact triangles and colimits, and is strong symmetric monoidal [And21, Theorem 5.9 and Lemma 5.10]. Its essential image consists of the *relatively discrete* objects. Thus $\underline{R} = \mathrm{Cond}_R(R)$ is relatively discrete: the adjective records its origin in $\mathrm{D}(R)$ rather than a choice of discrete topology.

For strongly noetherian R, a finite module M has a complete quotient topology independent of the choice of finite free presentation. Choose a presentation

$$R^a \longrightarrow R^b \longrightarrow M \longrightarrow 0.$$

The same matrix over $\underline{R}$ has cokernel $\underline{M}$, by completeness of this quotient topology [And21, Lemma 5.23]. Consequently, a finite-free resolution computes a pseudocoherent object on either side using the same differentials and the same degree conditions on its ordinary representative.

For the infinite module $R[T] = \bigoplus_{n\geq 0} RT^n$, colimit preservation gives

$$\mathrm{Cond}_R(R[T]) \simeq \bigoplus_{n\geq 0} \underline{R}\, T^n.$$

The right side is the condensed polynomial module. By contrast, imposing a bound on T in the analytic-ring construction gives the closed unit disc, whose ring $\underline{R\langle T\rangle}$ contains restricted power series.

For rational localization $(R, R^+) \to (R', R'^+)$,

$$\mathrm{Cond}_R(M)|_{\mathrm{Spa}(R',R'^+)} \simeq \mathrm{Cond}_{R'}(R' \otimes^{\mathbf{L}}_R M).$$

This is the scalar-extension compatibility used throughout the proof. For strongly noetherian rings, $R \to R'$ is also flat, so restriction commutes with cohomology and truncation [And21, Lemma 5.10].

We also use discrete integral pairs (B, B^+). Their complete modules lie in $\mathrm{D}_{\mathrm{cond}}(B)$, since the underlying ring is discrete. Lemmas B.7 and B.8 describe these categories and their

Tor-independent base changes. The algebraic realization below combines such a discrete pair with a topological one.

A.5. **Analytic stacks and descent.** The analytic affine associated with $\mathcal{A}$ is denoted by $\operatorname{AnSpec}\mathcal{A}$. The construction is contravariant, with quasicoherent category $\mathsf{D}(\mathcal{A})$. A map of analytic rings induces pullback by extension of scalars and direct image by restriction of scalars.

A map $\mathcal{A} \to \mathcal{B}$ is a *localization* if restriction $\mathsf{D}(\mathcal{B}) \to \mathsf{D}(\mathcal{A})$ is fully faithful. It is *steady* if extension and restriction of scalars satisfy the following base-change condition. Let $f : \mathcal{U} \to \mathcal{Z}$ be the associated map of analytic affines. For every map $g : \mathcal{V} \to \mathcal{Z}$ from an analytic affine, the canonical map

$$f^*\mathbf{R}g_* \xrightarrow{\sim} \mathbf{R}g'_*f'^*$$

must be an equivalence of functors from $\mathcal{C}_\square(\mathcal{V})$ to $\mathcal{C}_\square(\mathcal{U})$, in the cartesian square

$$\begin{array}{ccc} \mathcal{U}' & \xrightarrow{g'} & \mathcal{U} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{V} & \xrightarrow{g} & \mathcal{Z}, \end{array}$$

This is the characterization of steadiness in [Sch26a, Proposition 12.14]. A *steady localization* satisfies both conditions. Steady maps are stable under composition and base change [Sch26a, Definitions 12.13 and 12.16 and Proposition 12.15].

An *analytic stack* is a sheaf of spaces for the topology generated by finite jointly conservative families of steady localizations. Joint conservativity means that a map is an equivalence if its pullback to every member is an equivalence; for these exact functors, it is enough to test zero objects. Rational covers of our Huber affinoids give examples of these covers [LH26, A.2].

We define quasicoherent complexes on a stack by specifying them on all its analytic affines:

$$\mathcal{C}_\square(\mathcal{Z}) = \lim_{(\operatorname{AnSpec}\mathcal{A}\to\mathcal{Z})^{\mathrm{op}}} \mathsf{D}(\mathcal{A}).$$

An object assigns a complete module to each analytic affine over $\mathcal{Z}$, with pullback equivalences and unit, composition, and higher homotopies; morphisms are compatible families of module maps. For the Huber-pair stacks considered here, these are the *solid quasicoherent complexes* [Sch26b, Proposition 5.16]. They are defined using complete modules on analytic affines, rather than arbitrary ordinary sheaves of modules on a topological space.

For $V = \operatorname{Spa}(R, R^+)$, put $V^\square = \operatorname{AnSpec}(R, R^+)^\square$. Then

$$\mathcal{C}_\square(V^\square) = \mathsf{D}_\square(R, R^+).$$

Gluing this assignment on an analytic adic space Z gives the analytic stack $Z^\square$. In the main text we abbreviate $\mathcal{C}_\square(Z^\square)$ to $\mathcal{C}_\square(Z)$. We will identify the subcategory corresponding to ordinary coherent sheaves in Section A.7.

Let $\mathcal{U}_0 \to \mathcal{Z}$ be an effective epimorphism, with Čech nerve $\mathcal{U}_\bullet$. Descent gives

$$\mathcal{C}_\square(\mathcal{Z}) \simeq \lim_{[n]\in\Delta} \mathcal{C}_\square(\mathcal{U}_n).$$

Here $\mathcal{U}_n$ is the $(n+1)$-fold fiber product over $\mathcal{Z}$, and effective epimorphism means $|\mathcal{U}_\bullet| \simeq \mathcal{Z}$. The coproduct of an open cover is an example. For two opens, descent gives a complex on each open and an equivalence on their intersection; for a larger cover, it also records the compatibility homotopies on higher intersections.

On an affinoid basis, we can express the same gluing data as *cartesian sections*. An object of $\mathrm{Cart}_V \mathrm{D}_{\mathrm{pc}}(R_V)$ consists of ordinary complexes $M_V \in \mathrm{D}_{\mathrm{pc}}(R_V)$ and equivalences

$$R_{V'} \otimes^{\mathbf{L}}_{R_V} M_V \xrightarrow{\sim} M_{V'}$$

for $V' \subset V$, compatible with successive restrictions. The pullback identifications and their compatibility homotopies are included in the cartesian section.

For a finite open cover, Lemma B.5 replaces the full Čech nerve by the finite poset of nonempty intersections. Filtered colimits commute with the resulting finite limit of mapping spectra, giving Proposition B.6.

Derived global sections are

$$\mathbf{R}\Gamma_\square(\mathcal{Z}, K) = \mathrm{Map}_{\mathcal{C}_\square(\mathcal{Z})}(\mathcal{O}_\mathcal{Z}, K).$$

On an affine, this is derived evaluation at $*$. On a stack, descent computes it as a limit of local section complexes. Pullback f^* is given by derived scalar extension on charts, and $\mathbf{R}f_*$ is its right adjoint. For the morphisms of Section 4, there is also an exceptional direct image $\mathbf{R}f_!$ with right adjoint $f^!$, called exceptional pullback. Properness identifies $\mathbf{R}f_!$ with $\mathbf{R}f_*$, making proper direct image a left adjoint as well [LH26, A.2 and A.16–A.17].

A.6. **The three realizations and the comparison functors.** Let (R, R^+) be a complete Tate pair, let B be a finitely presented R-algebra, and let B^+ be the integral closure of the image of R^+ in B. We describe the algebraic realization, analytic locus, and Huber analytification of $\operatorname{Spec} B$.

For the algebraic realization, form the analytic ring

$$B^{\mathrm{alg}}_R = (R, R^+)^\square \otimes^{\mathrm{AnRing}}_{(R_{\mathrm{disc}}, R^+_{\mathrm{disc}})^\square} (B_{\mathrm{disc}}, B^+_{\mathrm{disc}})^\square.$$

The subscript pair and the second factor use discrete rings; only the first factor carries the topology of R. Lemma B.10 identifies its modules as

$$\mathcal{C}_\square((\operatorname{Spec} B)^{\mathrm{alg}}) \simeq \mathrm{Mod}_{\mathrm{Cond}_R(B)}\big(\mathsf{D}_\square(R, R^+)\big)$$

Thus we have a complete condensed $\underline{R}$-complex together with a $\mathrm{Cond}_R(B)$-action. When $B = R[T]$, giving the action means giving an endomorphism in $\mathsf{D}_\square(R, R^+)$. For a quotient of a polynomial algebra, one also specifies the relations and their derived compatibilities.

Direct image forgets the $\mathrm{Cond}_R(B)$-action and retains the underlying object of $\mathsf{D}_\square(R, R^+)$. To algebraize the module, we first recover an ordinary R-complex and then lift the B-action by full faithfulness.

To form the analytic locus, choose generators $b_1, \dots, b_m$ and a topologically nilpotent unit ϖ. The bounded charts $|\varpi^n b_i| \le 1$ increase with n, and their union is $(\operatorname{Spec} B)^{\mathrm{an}}$. For example, if $B = R[T]$ with R strongly noetherian, the nth chart is the disc of radius $|\varpi|^{-n}$ with ring $R\langle \varpi^n T\rangle$. For a quotient, impose the defining equations on these polydiscs. The construction is independent of the chosen generators and extends by Zariski gluing to separated finitely presented schemes [LH26, A.9–A.13].

The Huber analytification $T^{\mathrm{ad}/V}$ over $V = \mathrm{Spa}(R, R^+)$ is instead an adic space. We associate to it the analytic stack $(T^{\mathrm{ad}/V})^{\square}$. Proposition 4.10 identifies

$$T^{\mathrm{an}} \simeq (T^{\mathrm{ad}/V})^{\square}$$

for strongly noetherian R, by identifying bounded chart rings and their complete modules. For proper T/R, the valuative criterion also gives $T^{\mathrm{an}} \simeq T^{\mathrm{alg}}$. The first equivalence compares solid complexes with ordinary adic sheaves; the second permits algebraic localization of an analytic complex.

Over $(X, \mathcal{X})$, perform these constructions on each $V = \mathrm{Spa}(R, R^+) \subset \lambda^{-1}(W)$ with $W = \operatorname{Spec} A$, and then glue. This gives $T^{\mathrm{alg}}_{\mathcal{X}}$, $T^{\mathrm{an}}_{\mathcal{X}}$, and $T^{\mathrm{ad}}_{\mathcal{X}}$. The comparison on the base is given by

$$\alpha_X(M)|_V = \mathrm{Cond}_R\big(R \otimes^{\mathbf{L}}_A \mathbf{R}\Gamma(W, M|_W)\big).$$

This is scalar extension of sections on W, followed by condensification. Its compatibility on overlaps is proved in Section 2.1.

For affine $q : U \to X$, put $\mathcal{A} = q_*\mathcal{O}_U$. The two module descriptions are

$$\mathrm{D}_{\mathrm{qc}}(U) \simeq \mathrm{Mod}_{\mathcal{A}}(\mathrm{D}_{\mathrm{qc}}(X)),$$
$$\mathcal{C}_{\square}(U^{\mathrm{alg}}_{\mathcal{X}}) \simeq \mathrm{Mod}_{\alpha_X\mathcal{A}}(\mathcal{C}_{\square}(\mathcal{X})).$$

Apply α_X to the algebra, the complex, and the action to define α_U. Zariski descent then defines α_T for general T. An *algebraization* is a preimage equipped with its comparison equivalence. We call the forward functor the comparison functor, or analytification when its target is the analytic locus.

A.7. **Finiteness conditions and the GAGA subcategories.** Over an ordinary ring, a perfect complex has a bounded resolution by finite projective modules. A pseudocoherent complex has a resolution of the same kind that is only required to be bounded above. On schemes these are local conditions, and for noetherian T we have

$$\mathrm{D}_{\mathrm{pc}}(T) = \mathrm{D}^-_{\mathrm{coh}}(T).$$

A complex $P \in \mathrm{D}(B)$ has Tor-amplitude in $[a, b]$ when $H^j(P \otimes^{\mathbf{L}}_B N) = 0$ for every B-module N and every $j \notin [a, b]$. On schemes, test this locally. Pseudocoherence together with finite Tor-amplitude gives perfectness. Derived pullback and tensor product preserve pseudocoherence [Stacks, Tags 064Q, 0650, 0658, and 0DJE].

For example, take $B = k[\epsilon]/(\epsilon^2)$. The module k has the resolution

$$\cdots \xrightarrow{\epsilon} B \xrightarrow{\epsilon} B \xrightarrow{\epsilon} B \longrightarrow k \longrightarrow 0.$$

The image and kernel of multiplication by ϵ are both (ϵ), so the displayed complex is a resolution by finite free modules. It makes k pseudocoherent. After tensoring with k, however, every differential is zero, giving $\mathrm{Tor}^B_i(k, k) = k$ for all $i \geq 0$. Hence k is not perfect.

There is an analogous finiteness condition in a derived category generated by compact projective modules: an object is *pseudocompact* if it has a bounded-above resolution by compact projectives. We use this condition for nonanimated Huber analytic rings. It can be tested by maps into families with a common lower cohomological bound: for every such

family (N_a),

$$\bigoplus_a \mathrm{Map}(P, N_a) \longrightarrow \mathrm{Map}\left(P, \bigoplus_a N_a\right)$$

is an equivalence [CS26, Definition 9.9]. In each total degree, only finitely many terms of a bounded-above projective resolution contribute against targets with a common lower bound. Compactness of those terms gives the direct-sum comparison. A compact P satisfies the analogous comparison for arbitrary filtered colimits, without a common degree bound.

For a Huber analytic ring $\mathcal{A} = (A, \mathcal{M})$, a complete module N is *nuclear* when

$$\big(\mathcal{M}[S]^\vee \otimes^{\mathbf{L}}_{\mathcal{A}} N\big)(*) \longrightarrow N(S)$$

is an equivalence for every extremally disconnected S [And21, Definition 5.30]. Here $\mathcal{M}[S]^\vee$ is the internal dual and $N(S)$ means derived evaluation. For finite S, the formula is the tensor–Hom comparison for a finite sum of units. Nuclearity requires the same comparison after evaluation for infinite profinite sets. It is a condition on these families, not a bound on the dimension of $N(*)$.

For compact P, Q, a map $P \to Q$ is *trace class* when a map $\mathbf{1} \to P^\vee \otimes Q$ induces it by evaluation. It therefore factors as

$$P \longrightarrow P \otimes P^\vee \otimes Q \longrightarrow Q.$$

A *basic nuclear object* is a sequential colimit of compact objects whose transition maps are trace class. Nuclear objects are closed under colimits and admit presentations by basic nuclear objects [And21, Definitions 5.31–5.32 and Proposition 5.33]. In Section 3, we use these presentations to prove that scalar extension on bounded charts preserves nuclearity.

The link with ordinary pseudocoherence is the following comparison:

$$\mathrm{Cond}_R(\mathrm{D}_{\mathrm{pc}}(R)) = \{N \in \mathsf{D}_\square(R, R^+) : N \text{ is nuclear and pseudocompact}\}.$$

This is the relative-discreteness theorem [And21, Proposition 5.22 and Theorem 5.50] [LH26, A.15]. The compact-projective resolution condition is called pseudocoherence in [And21, Section 5.2] for nonanimated Huber rings. We call it pseudocompactness here to distinguish it from ordinary pseudocoherence in $\mathrm{D}_{\mathrm{pc}}(R)$.

On a strongly noetherian affinoid, the ordinary representatives give

$$\begin{aligned}
\mathrm{Cond}_R(\mathrm{D}_{\mathrm{pc}}(R)) &= \mathsf{D}^\square_{\mathrm{pc}}(V),\\
\mathrm{Cond}_R(\mathrm{D}^b_{\mathrm{fg}}(R)) &= \mathsf{D}^\square_{\mathrm{pc}}(V)^b,\\
\mathrm{Cond}_R(\mathrm{Mod}_{\mathrm{fg}}(R)) &= \mathsf{D}^\square_{\mathrm{pc}}(V)^\heartsuit.
\end{aligned}$$

The subscript fg means finitely generated, and $\mathrm{D}^b_{\mathrm{fg}}(R)$ requires finite cohomology in a finite interval. The last two lines use the degrees of the ordinary representatives. Finite projective modules give vector bundles, while perfect complexes give dualizable complete modules [And21, Proposition 5.12 and Corollary 5.51.1].

For relative realizations, the two bounded-chart systems of [LH26, A.15] test nuclearity and pseudocompactness and thus define $\mathsf{D}^\square_{\mathrm{pc}}$. Proposition 4.10 gives an equivalent description: on each Huber affinoid $W = \mathrm{Spa}(C, C^+)$, the object is $\mathrm{Cond}_C(P_W)$ for some $P_W \in \mathrm{D}_{\mathrm{pc}}(C)$. Rational restriction is ordinary derived scalar extension of these representatives. Since it is flat, it commutes with cohomology and truncation, allowing the local t-structures to glue.

Sheafification and condensification of a finite C-module give its coherent sheaf and complete condensed module, respectively. The bounded comparisons use the same representatives with a common interval on a finite cover. Section 6 identifies these comparisons with c_T^* and $\mathbf{L}c_T^*$.

Appendix B. Descent

We collect the descent arguments used in the main text. We first pass from analytic affines to stacks, then treat the lifting of diagrams and module actions and the gluing of right adjoints. Finite open descent gives the compactness statements. The appendix ends with the integral and bounded-chart calculations underlying the affine module description.

For ordinary limits, adjunctions, and Kan extensions, see [Mac98, Chapters III–V and X]. The arguments below take place in ∞-categories, for which we use [Lur09, Lur17]. We retain the notation of Appendix A and the analytic-ring constructions of [Sch26a, Section 12] and [LH26, A.1–A.3].

B.1. **Descent and fully faithful functors.** Put $\mathcal{C} = \mathrm{AnRing}^{\mathrm{op}}$, with the topology $\mathcal{D}$ generated by finite jointly conservative families of steady localizations. Complete modules on analytic affines extend to compatible families on $\mathrm{Shv}_{\mathcal{D}}(\mathcal{C})$. The resulting functor takes colimits of stacks to limits of module categories, as we now prove. In particular, this applies to the simplicial presentations of algebraic realizations in [LH26, Theorem A.13].

Proposition B.1. *Let $a : \mathcal{U}_0 \to \mathcal{Z}$ be an effective epimorphism of analytic stacks with Čech nerve $\mathcal{U}_\bullet$. Pullback gives an equivalence in symmetric monoidal presentable stable ∞-categories*

$$\mathcal{C}_\square(\mathcal{Z}) \xrightarrow{\sim} \lim_{[n]\in\Delta} \mathcal{C}_\square(\mathcal{U}_n).$$

More generally, for every small diagram $(\mathcal{Z}_i)_{i\in I}$,

$$\mathcal{C}_\square\left(\operatorname*{colim}_{i\in I}^{\mathrm{Shv}_{\mathcal{D}}} \mathcal{Z}_i\right) \xrightarrow{\sim} \lim_{i\in I^{\mathrm{op}}} \mathcal{C}_\square(\mathcal{Z}_i).$$

For $A \in \mathrm{CAlg}(\mathcal{C}_\square(\mathcal{Z}))$, we also have

$$\mathrm{Mod}_A(\mathcal{C}_\square(\mathcal{Z})) \xrightarrow{\sim} \lim_{[n]\in\Delta} \mathrm{Mod}_{A|_{\mathcal{U}_n}}(\mathcal{C}_\square(\mathcal{U}_n)).$$

Colimits in these limits are computed componentwise. In particular, $\mathcal{C}_\square(|\mathcal{V}_\bullet|) \simeq \lim_{[n]\in\Delta} \mathcal{C}_\square(\mathcal{V}_n)$ for every simplicial analytic stack.

Proof. Let D_0 denote the functor on analytic affines

$$\begin{aligned} \mathcal{C}^{\mathrm{op}} &\longrightarrow \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}}), \\ \mathrm{AnSpec}(A, M) &\longmapsto \mathrm{D}(A, M). \end{aligned}$$

It satisfies descent for the generating covers [LH26, A.2] [Sch26a, Proposition 12.18].

For a presheaf F, define

$$\overline{D}(F) = \lim_{(U\to F)\in(\mathcal{C}_{/F})^{\mathrm{op}}} D_0(U).$$

An object of this right Kan extension is a compatible family of complete modules on analytic affines over F. Since every presheaf is a colimit of representables, the universal property of the presheaf category shows that $\overline{D}$ carries colimits to limits [Sch26b, Proposition 5.16].

Let L denote sheafification. We claim that $\overline{D}(F) \simeq \overline{D}(LF)$. Affine descent makes $\overline{D}$ invert every covering-sieve inclusion. These inclusions generate the sheafification equivalences under two-out-of-three, retracts, pushouts, and transfinite compositions [Lur09, Section 6.2.2]. The maps inverted by $\overline{D}$ are closed under these operations: pushouts and transfinite compositions become pullbacks and inverse limits of equivalences, while two-out-of-three and retracts follow from the same properties for equivalences. Thus $\overline{D}$ factors through sheafification, giving the functor $\mathcal{C}_\square$ on stacks.

Since a colimit of sheaves is the sheafification of the presheaf colimit,

$$\begin{aligned}\mathcal{C}_\square\left(\operatorname*{\overset{\text{Shv}}{colim}}_i \mathcal{Z}_i\right) &\simeq \overline{D}\left(L \operatorname*{\overset{\text{PSh}}{colim}}_i \mathcal{Z}_i\right)\\ &\simeq \overline{D}\left(\operatorname*{\overset{\text{PSh}}{colim}}_i \mathcal{Z}_i\right)\\ &\simeq \lim_{i\in I^{\mathrm{op}}} \overline{D}(\mathcal{Z}_i)\\ &\simeq \lim_{i\in I^{\mathrm{op}}} \mathcal{C}_\square(\mathcal{Z}_i).\end{aligned}$$

This proves the colimit formula, including the assertion for geometric realizations.

For a generating cover $(U_i \to U)$, let $S \subset y(U)$ be the sieve of maps factoring through some U_i. The coproduct map factors as

$$\coprod_i y(U_i) \longrightarrow S \longrightarrow y(U)$$

with the first arrow an effective epimorphism and the second a monomorphism [Lur09, Lemma 6.2.3.18]. Sheafification preserves finite limits and colimits, hence both the Čech nerve and its geometric realization. It therefore preserves effective epimorphisms. Since S is covering, $LS \simeq Ly(U)$, and

$$\coprod_i Ly(U_i) \longrightarrow Ly(U)$$

is an effective epimorphism [Lur09, Lemma 6.2.3.19]. For an open cover, its image is the union of the opens, namely the whole target. The argument remains valid after base change.

An effective epimorphism satisfies $|\mathcal{U}_\bullet| \simeq \mathcal{Z}$ [Lur09, Corollary 6.2.3.5]. Applying the colimit formula gives descent, with the overlap equivalences and all their higher homotopies.

Finally, pullback preserves colimits, so taking a colimit componentwise retains the pullback identifications and computes the colimit in the limit category. Tensor products are componentwise too. An algebra or module structure is therefore a compatible collection of local actions and their homotopies; formation of these structures commutes with limits [Lur17, Sections 3.2.2 and 4.2.3]. Applying this to A gives the module-category equivalence. □

We next record how full faithfulness behaves under descent, both for diagrams and for algebra actions.

Lemma B.2. *Let $F : \mathcal{C} \to \mathcal{D}$ be a natural transformation of diagrams $I \to \mathrm{Cat}_\infty$, with I small. Componentwise full faithfulness or equivalence passes to $\lim_I F$. In the fully faithful case, its essential image is detected componentwise.*

Natural transformations pass to limits. If the component categories admit small colimits and the transition functors preserve them, colimits in the limits are computed componentwise; if the F_i also preserve them, so does $\lim_I F$.

For fully faithful $H : \mathcal{A} \to \mathcal{B}$ and small J, the functor $\mathrm{Fun}(J, \mathcal{A}) \to \mathrm{Fun}(J, \mathcal{B})$ *is fully faithful, with essential image the diagrams valued in the essential image of H.*

Proof. An object of the source limit consists of $x_i \in \mathcal{C}_i$, transition equivalences, and their homotopies. Applying F_i gives the corresponding family in the target. For two such objects, maps are computed by

$$\mathrm{Map}_{\lim_I \mathcal{C}_i}(x, y) \simeq \lim_{i \in I} \mathrm{Map}_{\mathcal{C}_i}(x_i, y_i).$$

The transition maps on the right use the given identifications for x and y. Full faithfulness of each F_i identifies this diagram with the one in $\mathcal{D}$, so taking limits proves full faithfulness. If the F_i are equivalences, the resulting map of limits is an equivalence as well.

To identify the essential image, let $\mathcal{E}_i \subset \mathcal{D}_i$ be the full essential image of F_i. Naturality preserves these subcategories and gives a pointwise equivalence $\mathcal{C} \simeq \mathcal{E}$. A compatible family in $\mathcal{D}$ whose terms lie in the $\mathcal{E}_i$ is already a family in $\mathcal{E}$: fullness includes its maps and homotopies. It therefore lifts through

$$\lim_I \mathcal{C}_i \xrightarrow{\sim} \lim_I \mathcal{E}_i.$$

This proves the essential-image assertion.

Natural transformations act on the component families. For colimits, the transition functors preserve the componentwise colimits and hence their gluing identifications. Commuting the limits of local mapping spaces verifies the universal property. The same calculation shows that $\lim_I F$ preserves colimits whenever all the F_i do.

For the last assertion, identify $\mathcal{A}$ with its full essential image $\mathcal{E} \subset \mathcal{B}$. A diagram is valued in $\mathcal{E}$ exactly when all its objects are: its arrows and homotopies then lie there by fullness. The same is true of natural transformations. Hence

$$\mathrm{Fun}(J, \mathcal{A}) \simeq \mathrm{Fun}(J, \mathcal{E}) \subset \mathrm{Fun}(J, \mathcal{B})$$

is the asserted full subcategory. □

A fully faithful strong monoidal functor lifts a module structure whenever it lifts the underlying object.

Write LMod_A for left modules over an associative algebra A. When A is commutative and the category is symmetric monoidal, this is our usual Mod_A.

Lemma B.3. *Let $F : \mathcal{C} \to \mathcal{D}$ be fully faithful and strong monoidal between monoidal ∞-categories, and let A be an associative algebra in $\mathcal{C}$. Then*

$$\mathrm{LMod}_A(\mathcal{C}) \longrightarrow \mathrm{LMod}_{F(A)}(\mathcal{D})$$

is fully faithful, with essential image detected on underlying objects.

Proof. Let N be an $F(A)$-module with underlying object identified with $F(M)$. Strong monoidality and full faithfulness identify the mapping spaces for its action:

$$\mathrm{Map}_{\mathcal{C}}(A \otimes M, M) \simeq \mathrm{Map}_{\mathcal{D}}(F(A) \otimes F(M), F(M)).$$

The action therefore has a preimage $\mu : A \otimes M \to M$.

Associativity compares

$$A \otimes A \otimes M \xrightarrow{m_A \otimes 1} A \otimes M \xrightarrow{\mu} M,$$
$$A \otimes A \otimes M \xrightarrow{1 \otimes \mu} A \otimes M \xrightarrow{\mu} M.$$

The action on N provides a homotopy between their images. Full faithfulness on $\mathrm{Map}(A \otimes A \otimes M, M)$ lifts that homotopy. For the unit, the image of $M \to A \otimes M \to M$ is homotopic to the identity, and the same argument lifts this homotopy as well.

For the higher relations, use the equivalences

$$\mathrm{Map}_{\mathcal{C}}(A^{\otimes r} \otimes M, M) \simeq \mathrm{Map}_{\mathcal{D}}(F(A)^{\otimes r} \otimes F(M), F(M)).$$

Compatibility with composition and insertion of the unit identifies all the operations and homotopies of the module structure. These are encoded by the two-colored module operad $\mathrm{LM}^{\otimes}$, whose colors specify an algebra and its module; here the algebra is fixed to be A [Lur17, Sections 2.2.1 and 4.2.1]. The lifted data therefore make M an A-module with image N.

The same argument applies to module maps. Full faithfulness lifts the underlying map $F(M) \to F(M')$ and its linearity homotopy, and the mapping-space equivalences for all tensor powers lift the remaining relations and homotopies between maps. Thus the functor on modules is fully faithful. Conversely, every module in its image has underlying object in the image of F. □

The right adjoints in the localization argument are restriction-of-scalars functors on affines. They glue when the comparisons defined by their units and counits are invertible.

Lemma B.4. *Let $F : \mathcal{C} \to \mathcal{D}$ be a natural transformation of diagrams $I \to \mathrm{Cat}_\infty$, with I small, and suppose $F_i \dashv G_i$. For each $\theta : i \to j$, let c_θ, d_θ be the transition functors. If each comparison map obtained by adjunction*

$$c_\theta G_i \longrightarrow G_j d_\theta$$

is invertible, these maps assemble the G_i into a natural transformation and give a right adjoint to $\lim_I F$, computed componentwise.

Proof. For $\theta : i \to j$, the given natural transformation supplies $d_\theta F_i \simeq F_j c_\theta$. Let η_j be the unit of $F_j \dashv G_j$ and ε_i the counit of $F_i \dashv G_i$. The comparison of right adjoints is

$$\begin{aligned} \beta_\theta : c_\theta G_i &\xrightarrow{\eta_j} G_j F_j c_\theta G_i \\ &\simeq G_j d_\theta F_i G_i \xrightarrow{G_j d_\theta \varepsilon_i} G_j d_\theta. \end{aligned}$$

This is an equivalence by hypothesis.

For a compatible family $y = (y_i)$, set $(Gy)_i = G_i y_i$ and equip it with transition maps

$$c_\theta(G_i y_i) \xrightarrow{\beta_\theta} G_j(d_\theta y_i) \xrightarrow{\sim} G_j y_j.$$

For two successive arrows, the intermediate unit and counit cancel by the triangle identity, leaving the comparison for the composite. Naturality gives the higher homotopies. Thus the displayed maps make Gy a compatible family. Applying the G_i to the component maps defines G on morphisms.

By construction, the local adjunctions respect these transition maps. Therefore

$$\begin{aligned} \mathrm{Map}_{\lim_I \mathcal{D}_i}(Fx, y) &\simeq \lim_{i \in I} \mathrm{Map}_{\mathcal{D}_i}(F_i x_i, y_i) \\ &\simeq \lim_{i \in I} \mathrm{Map}_{\mathcal{C}_i}(x_i, G_i y_i) \\ &\simeq \mathrm{Map}_{\lim_I \mathcal{C}_i}(x, Gy). \end{aligned}$$

These equivalences are natural in x and y, proving that the componentwise functor G is right adjoint to the functor on limits. □

B.2. **Finite open-cover descent.** A finite open cover admits a finite descent diagram, indexed by its nonempty intersections. We use this description to test compactness locally.

Let $\mathcal{Z} = \bigcup_{i=1}^{r} \mathcal{U}_i$. Put $\mathcal{U}_I = \bigcap_{i\in I} \mathcal{U}_i$ for nonempty $I \subset \{1, \dots, r\}$, and let P_r be the poset of these subsets ordered by inclusion. Write $K_I = K|_{\mathcal{U}_I}$. An inclusion $I \subseteq J$ induces restriction from $\mathcal{U}_I$ to $\mathcal{U}_J$.

Lemma B.5. *Restriction gives natural equivalences*

$$\mathcal{C}_\square(\mathcal{Z}) \xrightarrow{\sim} \lim_{I\in P_r} \mathcal{C}_\square(\mathcal{U}_I),$$
$$\mathrm{Map}_{\mathcal{Z}}(K, L) \xrightarrow{\sim} \lim_{I\in P_r} \mathrm{Map}_{\mathcal{U}_I}(K_I, L_I)$$

for $K, L \in \mathcal{C}_\square(\mathcal{Z})$.

Proof. Put $Q = P_r^{\mathrm{op}}$. We first prove the union formula

$$\operatorname*{colim}_{I\in Q} \mathcal{U}_I \xrightarrow{\sim} \mathcal{Z}.$$

The map $\coprod_i \mathcal{U}_i \to \mathcal{Z}$ is an effective epimorphism by Proposition B.1, so we may check the formula after pulling back to each $\mathcal{U}_k$. Pullback commutes with colimits and replaces $\mathcal{U}_I$ by $\mathcal{U}_I \times_{\mathcal{Z}} \mathcal{U}_k = \mathcal{U}_{I\cup\{k\}}$.

Let $Q_k \subset Q$ consist of the subsets containing k. The functor $r_k(I) = I \cup \{k\}$ is cofinal. To check this, fix $J \in Q_k$. A map $J \to r_k(I)$ means $I \subseteq J$, so its indexing category is the reverse-inclusion poset of nonempty subsets of J. It has initial object J, and [Lur09, Theorem 4.1.3.1] applies. Since $\{k\}$ is terminal in Q_k, we obtain

$$\operatorname*{colim}_{I\in Q} \mathcal{U}_{I\cup\{k\}} \simeq \operatorname*{colim}_{J\in Q_k} \mathcal{U}_J \simeq \mathcal{U}_k.$$

This proves the union formula on the cover and hence globally. Proposition B.1 turns it into the limit formula for module categories. Mapping spaces in this limit are limits of the local mapping spaces; applying the same calculation to shifts gives the statement for mapping spectra.

For two opens, the limit is a homotopy pullback over the intersection, hence in spectra the fiber of the difference of restrictions. Taking $K = \mathcal{O}_{\mathcal{Z}}$ gives Mayer–Vietoris for global sections. □

The nerve of P_r has only finitely many nondegenerate simplices: they are the strictly increasing chains of nonempty subsets. Consequently, the limits above are finite homotopy limits and commute with filtered colimits of spectra.

Proposition B.6. *Let* $\mathcal{Z} = \bigcup_{i=1}^{r} \mathcal{U}_i$ *be a finite open cover of an analytic stack, and put* $\mathcal{U}_I = \bigcap_{i\in I} \mathcal{U}_i$ *for nonempty* I*. Then:*

(i) $\mathbf{R}\Gamma(\mathcal{Z}, -)$ *preserves filtered colimits if the unit of each* $\mathcal{C}_\square(\mathcal{U}_I)$ *is compact;*

(ii) $K \in \mathcal{C}_\square(\mathcal{Z})$ *is compact if every* $K|_{\mathcal{U}_I}$ *is compact.*

The condition in (i) *holds when the intersections are finite disjoint unions of affines* $\mathrm{AnSpec}(R, R^+)^\square$ *for complete Huber pairs.*

Proof. Suppose each K_I is compact, and let (L_a) be a filtered diagram. Restriction preserves colimits, so the finite descent formula gives

$$\begin{aligned}\operatorname{Map}_{\mathcal{Z}}(K, \operatorname*{colim}_a L_a) &\simeq \lim_{I\in P_r} \operatorname{Map}_{\mathcal{U}_I}(K_I, \operatorname*{colim}_a L_{a,I}) \\ &\simeq \lim_{I\in P_r} \operatorname*{colim}_a \operatorname{Map}_{\mathcal{U}_I}(K_I, L_{a,I}) \\ &\simeq \operatorname*{colim}_a \lim_{I\in P_r} \operatorname{Map}_{\mathcal{U}_I}(K_I, L_{a,I}) \\ &\simeq \operatorname*{colim}_a \operatorname{Map}_{\mathcal{Z}}(K, L_a).\end{aligned}$$

The second equivalence uses local compactness, and the third commutes a filtered colimit with a finite homotopy limit. Thus K is compact, proving (ii).

For (i), apply (ii) to $\mathcal{O}_{\mathcal{Z}}$. In the affine case $\operatorname{AnSpec}(R, R^+)^{\square}$, the unit is the measure module associated with the one-point set and is compact [And21, Theorem 3.28 and Proposition 2.11]. A finite disjoint union gives a finite product of mapping spectra, so compactness persists. Empty intersections contribute zero. This proves the final assertion. □

B.3. **Integral subrings and bounded charts.** Fix a complete Tate Huber pair (R, R^+). For any R-algebra B, let B^+ be the integral closure of the image of R^+. Define its algebraic realization by

$$B_R^{\mathrm{alg}} = (R, R^+)^{\square} \otimes^{\mathrm{AnRing}}_{(R_{\mathrm{disc}}, R^+_{\mathrm{disc}})^{\square}} (B_{\mathrm{disc}}, B^+_{\mathrm{disc}})^{\square}.$$

Write $(\operatorname{Spec} B)^{\mathrm{alg}} = \operatorname{AnSpec} B_R^{\mathrm{alg}}$. When B is finitely presented, this is the construction of [LH26, A.9]. Its analytic locus is the union of the bounded charts associated with a pseudo-uniformizer and finitely many generators. In that case the constructions extend by Zariski gluing and agree for proper schemes [LH26, A.9–A.13].

We begin with the discrete pairs. The ring B determines the module action, while B^+ determines which of those modules are complete.

Lemma B.7. *Let $R \to B$ be a ring map, let $R^+ \subset R$ be integrally closed, and let B^+ be the integral closure of the image of R^+ in B. Put*

$$\begin{aligned}\mathcal{R}_0 &= (R_{\mathrm{disc}}, R^+_{\mathrm{disc}})^{\square}, \\ \mathcal{B}_0 &= (B_{\mathrm{disc}}, B^+_{\mathrm{disc}})^{\square}.\end{aligned}$$

Then B is $\mathcal{R}_0$-complete, its induced analytic structure is $\mathcal{B}_0$, and restriction gives a canonical symmetric monoidal equivalence

$$\mathsf{D}(\mathcal{B}_0) \simeq \operatorname{Mod}_B\big(\mathsf{D}(\mathcal{R}_0)\big).$$

Both sides consist of condensed B-complexes complete over $\mathcal{R}_0$. These identifications are functorial in maps of integral pairs.

Proof. Let N be a condensed B-complex. Put $C^+ = \operatorname{im}(R^+ \to B)$ and factor the map of plus rings as

$$R^+ \twoheadrightarrow C^+ \longrightarrow B^+.$$

Both arrows are integral, the first being a quotient map. Completeness may be tested on the plus ring [And21, Proposition 3.21], and integral extension does not change this condition

[And21, Proposition 3.22]. We therefore have

$$\begin{aligned} N \text{ is } \mathcal{B}_0\text{-complete} &\iff N|_{B^+} \text{ is } (B^+_{\mathrm{disc}})^{\square}\text{-complete} \\ &\iff N|_{C^+} \text{ is } (C^+_{\mathrm{disc}})^{\square}\text{-complete} \\ &\iff N|_{R^+} \text{ is } (R^+_{\mathrm{disc}})^{\square}\text{-complete} \\ &\iff N|_{R} \text{ is } \mathcal{R}_0\text{-complete.} \end{aligned}$$

In other words, N is $\mathcal{B}_0$-complete exactly when its restriction to R is $\mathcal{R}_0$-complete.

Apply this criterion to $B[0]$. It is $\mathcal{B}_0$-complete by [And21, Lemma 3.24], and hence $\mathcal{R}_0$-complete. The usual multiplication and unit therefore make B an algebra in $\mathsf{D}(\mathcal{R}_0)$.

Let $\mathcal{B}_{\mathrm{ind}}$ be the analytic structure induced along $R \to B$. By [Sch26a, Proposition 12.8], its module description is

$$\mathsf{D}(\mathcal{B}_{\mathrm{ind}}) \simeq \mathrm{Mod}_B(\mathsf{D}(\mathcal{R}_0))$$

[LH26, Lemma A.1 and A.3]. Its underlying ring is the $\mathcal{R}_0$-completion of B, hence B itself. Its modules are precisely the condensed B-complexes whose restrictions are $\mathcal{R}_0$-complete, as in the displayed chain.

The two analytic structures have the same underlying ring and complete connective modules. By uniqueness [Sch26a, Proposition 12.20],

$$\mathcal{B}_{\mathrm{ind}} \simeq \mathcal{B}_0.$$

The equivalence identifies the module categories and their forgetful functors. On both sides, tensor product is tensor product over B followed by the same analytic completion, so the equivalence is symmetric monoidal.

All these identifications use the given ring maps and restriction functors. The left adjoint to the inclusion of complete connective modules is unique up to contractible choice. This compares the constructions for a map of integral pairs and supplies the composition homotopies for successive maps. □

For a Tor-independent base change, the derived tensor product is the ordinary ring $B \otimes_R R'$. We identify its analytic structure by comparing the complete modules over this ring.

Lemma B.8. *Let $R \to R'$ be a ring map carrying an integrally closed subring $R^+ \subset R$ into an integrally closed subring $R'^+ \subset R'$. Let B be an R-algebra with $\mathrm{Tor}_i^R(B, R') = 0$ for $i > 0$, and put $B' = B \otimes_R R'$. Write B^+, B'^+ for the integral closures of their images in B, B', respectively. There is a canonical equivalence of complete analytic rings*

$$(R'_{\mathrm{disc}}, R'^+_{\mathrm{disc}})^{\square} \otimes^{\mathrm{AnRing}}_{(R_{\mathrm{disc}}, R^+_{\mathrm{disc}})^{\square}} (B_{\mathrm{disc}}, B^+_{\mathrm{disc}})^{\square} \simeq (B'_{\mathrm{disc}}, B'^+_{\mathrm{disc}})^{\square}.$$

It is compatible with composition under the same Tor-vanishing for successive base changes, in particular for chains of flat maps.

Proof. Write $\mathcal{R}_0, \mathcal{B}_0, \mathcal{R}'_0, \mathcal{B}'_0$ for the analytic rings of the four discrete pairs, and set

$$\mathcal{P} = \mathcal{R}'_0 \otimes^{\mathrm{AnRing}}_{\mathcal{R}_0} \mathcal{B}_0.$$

A monic equation for $b \in B^+$ remains monic over the image of R'^+ in B', so b maps into B'^+. This map of pairs and the map from (R', R'^+) define $\mathcal{P} \to \mathcal{B}'_0$ by the pushout property.

We compute its underlying ring before analytic completion:

$$D = B_{\mathrm{disc}} \otimes^{\mathbf{L}}_{R_{\mathrm{disc}}} R'_{\mathrm{disc}}.$$

Discrete condensification commutes with derived tensor product, so the Tor-vanishing assumption gives

$$D \simeq (B \otimes_R R')_{\mathrm{disc}}[0] = B'_{\mathrm{disc}}[0].$$

By [Sch26a, Proposition 12.12], a connective condensed B'-complex N is complete for the pushout precisely when $N|_{R'}$ is $\mathcal{R}'_0$-complete and $N|_B$ is $\mathcal{B}_0$-complete. The first condition implies the second by restriction to R and Lemma B.7. Applying the same lemma to $R' \to B'$ gives

$$\begin{aligned} N \text{ is } \mathcal{P}\text{-complete} &\iff N|_{R'} \text{ is } \mathcal{R}'_0\text{-complete} \\ &\iff N \text{ is } \mathcal{B}'_0\text{-complete}. \end{aligned}$$

The unit $B'[0]$ is complete by Lemma B.7, so analytic completion leaves $D \simeq B'_{\mathrm{disc}}$ unchanged. The source and target have the same ring and complete connective modules; [Sch26a, Proposition 12.20] therefore makes the comparison an equivalence.

For $R' \to R''$ with $\mathrm{Tor}_i^{R'}(B', R'') = 0$ for $i > 0$, use

$$(B \otimes^{\mathbf{L}}_R R') \otimes^{\mathbf{L}}_{R'} R'' \simeq B \otimes^{\mathbf{L}}_R R''.$$

Tor-vanishing identifies this derived tensor product with $B \otimes_R R''$. Both analytic pushouts impose completeness over $(R''_{\mathrm{disc}}, R''^{+}_{\mathrm{disc}})^{\square}$, so the displayed associativity map compares them. Uniqueness of analytic completion gives the compatibilities for longer chains. □

Rational localizations of strongly noetherian Tate pairs are flat [KL19, Lemma 2.2.6], so the preceding calculation applies to the base changes in the main text. The bounded charts also commute with Tor-independent base change. In the following lemma the realizations and fiber products are taken in analytic stacks; for $B = R[T]$, its charts impose the bounds $|T| \le |\varpi|^{-n}$.

Lemma B.9. *Let $(R, R^+) \to (R', R'^+)$ be a morphism of complete Tate Huber pairs, and let B be finitely presented over R with $\mathrm{Tor}_i^R(B, R') = 0$ for $i > 0$. Choose a topologically nilpotent unit $\varpi \in R$ and generators $\mathbf{b} = (b_1, \dots, b_m)$, and put $B' = B \otimes_R R'$. Use primes for their images after base change. Then:*

(i) *There is a canonical equivalence*

$$B^{\mathrm{alg}}_R \otimes^{\mathrm{AnRing}}_{(R,R^+)^{\square}} (R', R'^+)^{\square} \simeq (B')^{\mathrm{alg}}_{R'}.$$

(ii) *The chart $|\varpi^n b_i| \le 1$ base changes to $|\varpi'^n b'_i| \le 1$, compatibly with the transitions for $n \ge 0$.*

(iii) *Their increasing union is $(\mathrm{Spec}\, B)^{\mathrm{an}}$, and*

$$(\mathrm{Spec}\, B)^{\mathrm{an}} \times_{\mathrm{AnSpec}(R,R^+)^{\square}} \mathrm{AnSpec}(R', R'^+)^{\square} \simeq (\mathrm{Spec}\, B')^{\mathrm{an}}.$$

This is functorial in affine schemes, coherent under successive base changes satisfying the same Tor-vanishing condition, and independent of ϖ and $\mathbf{b}$.

Proof. We compute the algebraic realization using Lemma B.8. Put

$$\begin{aligned} \mathcal{R}_0 &= (R_{\mathrm{disc}}, R^+_{\mathrm{disc}})^{\square}, \\ \mathcal{B}_0 &= (B_{\mathrm{disc}}, B^+_{\mathrm{disc}})^{\square}, \end{aligned}$$

and $\mathcal{R} = (R, R^+)^\square$, with primes after base change. By [LH26, A.9],

$$B_R^{\mathrm{alg}} = \mathcal{R} \otimes_{\mathcal{R}_0}^{\mathrm{AnRing}} \mathcal{B}_0.$$

Reassociating the pushouts and applying Lemma B.8 gives

$$\begin{aligned} B_R^{\mathrm{alg}} \otimes_{\mathcal{R}}^{\mathrm{AnRing}} \mathcal{R}' &\simeq \mathcal{R}' \otimes_{\mathcal{R}_0'}^{\mathrm{AnRing}} \left(\mathcal{R}_0' \otimes_{\mathcal{R}_0}^{\mathrm{AnRing}} \mathcal{B}_0 \right) \\ &\simeq \mathcal{R}' \otimes_{\mathcal{R}_0'}^{\mathrm{AnRing}} \mathcal{B}_0' \\ &= (B')_{R'}^{\mathrm{alg}}. \end{aligned}$$

Here Lemma B.8 computes the inner pushout in the first line. This proves (i).

To include the bounds at level n, form the pushout

$$\begin{array}{ccc} (\mathbf{Z}[T_1, \ldots, T_m], \mathbf{Z})^\square & \longrightarrow & \mathbf{Z}[T_1, \ldots, T_m]^\square \\ {\scriptstyle T_i \mapsto \varpi^n b_i}\downarrow & & \downarrow \\ B_R^{\mathrm{alg}} & \longrightarrow & B^{\mathrm{univ}}_{|\varpi^n \mathbf{b}| \leq 1}. \end{array}$$

The upper map imposes integral bounds on the variables. Thus a map from the lower right ring is a map from B_R^{alg} for which the chosen elements satisfy those bounds. The upper map is a steady localization [LH26, A.9]; its base change therefore defines a subobject of $\operatorname{AnSpec} B_R^{\mathrm{alg}}$, with a contractible space of factorizations whenever one exists.

The upper row does not depend on R. After base change to $\mathcal{R}'$, the lower left ring becomes $(B')_{R'}^{\mathrm{alg}}$ and the bounded elements become $\varpi'^n b_i'$. The pushout consequently gives

$$B^{\mathrm{univ}}_{|\varpi^n \mathbf{b}| \leq 1} \otimes_{\mathcal{R}}^{\mathrm{AnRing}} \mathcal{R}' \simeq (B')^{\mathrm{univ}}_{|\varpi'^n \mathbf{b}'| \leq 1}.$$

The transition to level $n+1$ uses

$$\varpi^{n+1} b_i = \varpi(\varpi^n b_i).$$

Since $\varpi \in R^+$, a bound at level n implies the bound at level $n+1$. Thus the charts increase, with their ring maps going in the opposite direction. The same identity holds over R', so the comparison respects these transitions. This proves (ii).

Separate bounds on the m generators give an $\mathbf{N}^m$-indexed system. Each tuple is dominated by its largest entry, so the diagonal $\mathbf{N} \to \mathbf{N}^m$ is cofinal. By [LH26, A.11–A.15],

$$(\operatorname{Spec} B)^{\mathrm{an}} \simeq \operatorname*{colim}_{n \geq 0} \operatorname{AnSpec} B^{\mathrm{univ}}_{|\varpi^n \mathbf{b}| \leq 1}.$$

Pullback preserves this colimit, so taking unions proves (iii). Successive Tor-independent base changes reassociate the same pushouts, compatibly with the transitions in n.

To compare the choices of generators and pseudo-uniformizer, write

$$U_{\varpi,n}(\mathbf{b}) = \operatorname{AnSpec} B^{\mathrm{univ}}_{|\varpi^n \mathbf{b}| \leq 1}.$$

Every topologically nilpotent element lies in R^+: a sufficiently large power lies in this open subring, so integral closedness gives the claim. We also use the universal bounded-locus maps for sums and products from [LH26, A.9 and the proof of Proposition A.12(2)], which turn the following polynomial identities into maps of bounded charts.

For two generating tuples, enlarge both to their union $\mathbf{c}$. It is enough to compare one tuple $\mathbf{b}$ with this union. Write each added generator as

$$c_j = \sum_{\nu} a_{j\nu}\mathbf{b}^{\nu}.$$

Fix a level n. Only finitely many coefficients occur, so openness of R^+ and topological nilpotence of ϖ let us choose $m \geq n$ such that

$$\varpi^{m-n|\nu|}a_{j\nu} \in R^+$$

for every coefficient. On $U_{\varpi,n}(\mathbf{b})$, this choice gives

$$\varpi^m c_j = \sum_{\nu} \big(\varpi^{m-n|\nu|}a_{j\nu}\big)(\varpi^n\mathbf{b})^{\nu}.$$

Each factor on the right is bounded, so the sum and product maps give the required bound on $\varpi^m c_j$. The original generators remain bounded at level m because $m \geq n$. Hence

$$U_{\varpi,n}(\mathbf{b}) \subset U_{\varpi,m}(\mathbf{c}).$$

Forgetting the added bounds gives the reverse comparison

$$U_{\varpi,n}(\mathbf{c}) \subset U_{\varpi,n}(\mathbf{b}).$$

We may choose m nondecreasing with n. The two comparisons then identify the unions of the chart systems.

Now change the topologically nilpotent unit to $\widetilde{\varpi}$. Choose $r, s > 0$ with

$$\widetilde{\varpi}^r/\varpi \in R^+,$$
$$\varpi^s/\widetilde{\varpi} \in R^+.$$

Such choices exist because the corresponding sequences of quotients tend to zero. On $U_{\varpi,n}(\mathbf{c})$, the identity

$$\widetilde{\varpi}^{rn}c_i = \left(\frac{\widetilde{\varpi}^r}{\varpi}\right)^n (\varpi^n c_i)$$

expresses the new coordinate as a product of bounded elements. Reversing the roles of the two units gives

$$U_{\varpi,n}(\mathbf{c}) \subset U_{\widetilde{\varpi},rn}(\mathbf{c}),$$
$$U_{\widetilde{\varpi},n}(\mathbf{c}) \subset U_{\varpi,sn}(\mathbf{c}).$$

Thus these unions agree as well. On their common refinements the base-change map is still obtained from the same pushout, so it is independent of both choices.

For an R-algebra map $B \to C$, express the images of the b_i in generators of C. The same coefficient bound sends each chart for C into a chart for B; taking unions gives the map of analytic loci. Enlarging bounds compares choices. Contractibility of factorizations through localization subobjects supplies compatibility with identities, composition, the stated base changes, and higher homotopies. □

B.4. **Comparison over a Tate affinoid.** The module description used in the algebraization argument follows by applying condensification to an algebra and its modules. Recall the formula

$$\mathrm{Cond}_R(M) = \underline{R} \otimes^{\mathbf{L}}_{R_{\mathrm{disc}}} M_{\mathrm{disc}}$$

for $M \in \mathrm{D}(R)$, with values in complete modules [And21, Theorem 5.9].

Lemma B.10. *Let (R, R^+) be a complete Tate Huber pair and let A be an R-algebra. There is a canonical symmetric monoidal equivalence*

$$\mathcal{C}_\square((\operatorname{Spec} A)^{\mathrm{alg}}) \simeq \operatorname{Mod}_{\operatorname{Cond}_R(A)}\big(\mathsf{D}_\square(R, R^+)\big).$$

For the structural map q_A^{alg}, $\mathbf{R}q_{A,*}^{\mathrm{alg}}$ is the forgetful functor. The comparison α_A^{alg} is induced by Cond_R on modules and is fully faithful, with essential image detected on underlying objects in $\operatorname{Cond}_R(\mathrm{D}(R))$.

Proof. Let A^+ be the integral closure of the image of R^+ in A. Put

$$\mathcal{R}_0 = (R_{\mathrm{disc}}, R^+_{\mathrm{disc}})^\square,$$
$$\mathcal{A}_0 = (A_{\mathrm{disc}}, A^+_{\mathrm{disc}})^\square.$$

Set $\mathcal{R} = (R, R^+)^\square$. By the definition above,

$$A_R^{\mathrm{alg}} = \mathcal{R} \otimes^{\mathrm{AnRing}}_{\mathcal{R}_0} \mathcal{A}_0.$$

Pass to complete-module categories and apply Lemma B.7:

$$\begin{aligned}\mathsf{D}(A_R^{\mathrm{alg}}) &\simeq \mathsf{D}(\mathcal{R}) \otimes_{\mathsf{D}(\mathcal{R}_0)} \mathsf{D}(\mathcal{A}_0)\\ &\simeq \mathsf{D}(\mathcal{R}) \otimes_{\mathsf{D}(\mathcal{R}_0)} \operatorname{Mod}_A(\mathsf{D}(\mathcal{R}_0))\\ &\simeq \operatorname{Mod}_{\operatorname{Cond}_R(A)}(\mathsf{D}_\square(R, R^+)).\end{aligned}$$

The first equivalence is the pushout formula [LH26, Lemma A.1]; the second tests completeness after restriction to $\mathcal{R}_0$. The last extends the module category to $\mathsf{D}(\mathcal{R})$ [Lur17, Section 4.8.5], carrying A to $\operatorname{Cond}_R(A)$. All three respect tensor products, so the equivalence is symmetric monoidal.

Under this equivalence, structural pullback is the free-module functor

$$K \longmapsto \operatorname{Cond}_R(A) \otimes K.$$

Maps from this free module to N are maps from K to its underlying object in $\mathsf{D}_\square(R, R^+)$. Thus the right adjoint $\mathbf{R}q_{A,*}^{\mathrm{alg}}$ forgets the $\operatorname{Cond}_R(A)$-action.

An ordinary A-complex is an A-module in $\mathrm{D}(R)$. Condensifying the object and its action defines the comparison. Since Cond_R is fully faithful, exact, and colimit-preserving [And21, Theorem 5.9 and Lemma 5.10], Lemma B.3 gives full faithfulness on modules.

If N has underlying object $\operatorname{Cond}_R(M)$, the same lemma lifts its action to $A \otimes^{\mathbf{L}}_R M \to M$, including the unit, associativity, and higher relations. Hence N comes from an ordinary A-complex. Conversely, an object in the image has underlying complex in $\operatorname{Cond}_R(\mathrm{D}(R))$ by construction. □

Zariski descent glues the affine comparisons, with their scalar-extension identifications.

Proposition B.11. *Let (R, R^+) be a complete Tate Huber pair and let S be a separated finitely presented R-scheme. Scalar extension over the Tate base induces a fully faithful colimit-preserving strong symmetric monoidal functor $\alpha_S^{\mathrm{alg}} : \mathrm{D}_{\mathrm{qc}}(S) \longrightarrow \mathcal{C}_\square(S^{\mathrm{alg}})$. It is functorial in S and compatible with pullback.*

Proof. Choose a finite affine cover (S_a) of S. Separatedness makes all multiple intersections affine, so Lemma B.10 gives a fully faithful comparison on each.

For $\operatorname{Spec} A' \subset \operatorname{Spec} A$, restriction is $M \mapsto A' \otimes^{\mathbf{L}}_{A} M$. Condensification commutes with scalar extension [And21, Lemma 5.10], identifying its image with analytic restriction. For successive inclusions, the comparisons are

$$A'' \otimes^{\mathbf{L}}_{A'} (A' \otimes^{\mathbf{L}}_{A} M) \simeq A'' \otimes^{\mathbf{L}}_{A} M.$$

Associativity includes the higher compatibilities, so these functors form a map of the full Čech diagrams.

Zariski descent computes $\mathrm{D}_{\mathrm{qc}}(S)$ as the ordinary limit. The presentation in [LH26, Theorem A.13] and Proposition B.1 compute $\mathcal{C}_{\square}(S^{\mathrm{alg}})$ as the analytic limit. Taking limits defines α_S^{alg}, and Lemma B.2 gives full faithfulness and colimit preservation. The affine tensor and unit comparisons agree under restriction, so they descend to the strong symmetric monoidal structure.

Common affine refinements identify different covers. For $S' \to S$, choose source affines mapping into target affines. Pullback compatibility there is the scalar-extension formula above; it commutes with restriction and hence glues. Tensor associativity compares the composites. □

References

[And21] G. Andreychev, *Pseudocoherent and Perfect Complexes and Vector Bundles on Analytic Adic Spaces*, arXiv:2105.12591v1 (2021).

[AM69] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison–Wesley Publishing Co., Reading, MA, 1969.

[CS26] D. Clausen and P. Scholze, *Condensed Mathematics and Complex Geometry*, arXiv:2605.11731v1 (2026).

[FF18] L. Fargues and J.-M. Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p-adique*, Astérisque **406** (2018), xiii+382 pp.

[FS24] L. Fargues and P. Scholze, *Geometrization of the local Langlands correspondence*, arXiv:2102.13459v4 (2024), accepted version.

[FK18] K. Fujiwara and F. Kato, *Foundations of rigid geometry. I*, EMS Monographs in Mathematics, European Mathematical Society, Zürich, 2018.

[Hal22] J. Hall, *GAGA Theorems*, arXiv:1804.01976v3 (2022).

[Hub94] R. Huber, *A generalization of formal schemes and rigid analytic varieties*, Math. Z. **217** (1994), 513–551.

[Ked17] K. S. Kedlaya, *Sheaves, stacks, and shtukas*, Arizona Winter School lecture notes, official version of 10 March 2017, `https://swc-math.github.io/aws/2017/2017KedlayaNotes.pdf`.

[Ked23] K. S. Kedlaya, *Analytification and GAGA over the Fargues–Fontaine curve*, seminar abstract, University of Maryland, 16 October 2023, `https://www-math.umd.edu/gcal_rss.php?html=&seminar_key=NTRT&year=2024`.

[Ked24] K. S. Kedlaya, *Questions I'm thinking about*, item "GAGA for proper varieties over Fargues–Fontaine curves," last updated 26 December 2024, `https://kskedlaya.org/questions.shtml`, accessed 8 October 2026.

[KL19] K. S. Kedlaya and R. Liu, *Relative p-adic Hodge theory, II: Imperfect period rings*, arXiv:1602.06899v3 (2019).

[LH26] S. D. Li–Huerta, *Courbes et fibrés vectoriels en théorie de Hodge z-adique globale*, arXiv:2602.04978v1 (2026).

[Lur09] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[Lur17] J. Lurie, *Higher Algebra*, available from the author's webpage, 2017.

[Lur11] J. Lurie, *Derived Algebraic Geometry VIII: Quasi-Coherent Sheaves and Tannaka Duality Theorems*, available from the author's webpage, 2011.

[Mac98] S. Mac Lane, *Categories for the Working Mathematician*, second edition, Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.
[RC24] J. E. Rodríguez Camargo, *The analytic de Rham stack in rigid geometry*, arXiv:2401.07738v1 (2024).
[Sch26a] P. Scholze, *Lectures on Analytic Geometry*, arXiv:2605.03655v1 (2026).
[Sch26b] P. Scholze, *Six-Functor Formalisms*, arXiv:2510.26269v2 (2026).
[Ser56] J.-P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier **6** (1956), 1–42.
[Stacks] The Stacks Project Authors, *The Stacks Project*, `https://stacks.math.columbia.edu`.
[Wan26] Q. Wang, *The Relative GAGA Theorem and an Application to the Analytic Mapping Stacks*, arXiv:2601.12299v1 (2026).
[Wei94] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.
[Zav25] B. Zavyalov, *Some Foundational Results in Adic Geometry*, arXiv:2409.15516v2 (2025).

Email address: `aniruddhv@berkeley.edu`