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LUSZTIG'S FOURIER MATRIX CONJECTURE FOR FINITE NONCRYSTALLOGRAPHIC COXETER GROUPS

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Abstract. We prove the Fourier matrix conjecture for the asymptotic Hecke categories of all finite noncrystallographic Coxeter groups. The spherical structure is the one inherited from the Soergel category, and the Fourier matrices are taken in the modular convention, with the prescribed single-index conversion made explicit. A relative Hodge–Riemann argument determines the pivotal signs of every diagonal fusion corner. For the middle two-sided cell of type H_4, we construct a Morita equivalence with the crossed product of the Fibonacci Hecke category of type A_3 by its diagram involution. We compute its entire Drinfeld center, including two simple objects missed by a proper partial tube algebra, and identify its normalized 74 by 74 matrix with the Fourier block. The finite calculation uses exact arithmetic in $\mathbb{Q}(\sqrt{5})$ and certified complex balls. A cyclotomic algebraic-integer separation bound converts the certified comparison into an exact equality of all 5476 entries. The remaining cells follow from their pointed or Fibonacci corners and the dihedral theorem.

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1. Introduction and the main theorem

Let (W, S) be a finite Coxeter system, with equal Hecke parameters. For a two-sided Kazhdan–Lusztig cell c, Lusztig's asymptotic algebra J_c is a based ring. The corresponding asymptotic Hecke category $\mathcal{A}_c$ is a multifusion categorification of J_c. In the crystallographic case its center is related to unipotent character sheaves. In [13, §10.1], Lusztig proposed that the same construction should give the missing categories of unipotent character sheaves for noncrystallographic groups, with modular matrices equal to the Fourier matrices constructed in [10, 11, 14, 2].

The Soergel construction of $\mathcal{A}_c$, including rigidity and the inherited pivotal structure, was established by Elias and Williamson [3]. Rogel and Thiel proved the Fourier matrix assertion for the dihedral groups and for several cells in types H_3 and H_4 [20]. The outstanding large block is the 74-element family in the middle cell of H_4. A based-ring calculation alone does not determine the associativity constraints of its diagonal corner, and agreement with a numerical matrix alone does not determine its center. We address these two issues separately.

Throughout, the Soergel category uses the standard real balanced reflection realization, extended to $\mathbb{C}$. The pivotal structure on $\mathcal{A}_c$ is its inherited ordinary double-dual identification. The center carries the induced ribbon structure. If Z, Z' are simple central objects, put

$$S_c(Z, Z') = \frac{\mathrm{tr}(c_{Z',Z} c_{Z,Z'})}{\sqrt{\mathrm{Dim}\, \mathcal{Z}(\mathcal{A}_c)}}, \tag{1.1}$$

where the square root is positive. The spherical property required in this formula is included in Proposition 3.1 below.

Convention 1.1. We distinguish the original Fourier *pairing* matrix F_c^{pair} from the modular matrix F_c^{mod}. The convention used in [9, §3.1] and [20, §4.6] is

$$F_c^{\mathrm{mod}} = F_c^{\mathrm{pair}} P_\flat, \tag{1.2}$$

where $\flat$ is the specified involution of the Fourier indexing set. It is not, in general, categorical duality in the center. This is a conversion on one index, before identifying both indices with simple central objects. For the Fibonacci block it interchanges the last two columns, although all four central simple objects are self-dual. Once this conversion has been made, a change of names of simple objects acts by the same permutation on rows and columns. For the 74 by 74 block we use the CHEVIE matrix `HS4` itself; no additional one-sided permutation is applied to that block.

Theorem 1.2. *Let W be a finite irreducible noncrystallographic Coxeter group, and let c be any two-sided cell. There is a bijection between* $\operatorname{Irr} \mathcal{Z}(\mathcal{A}_c)$ *and the Fourier family of c for which*

$$S_c = F_c^{\mathrm{mod}}.$$

This holds for every dihedral group $I_2(m)$ with $m \geq 3$, every cell of H_3, and every cell of H_4, with the inherited pivotal structure and the normalization (1.1).

For the cell of H_4 with $a(c) = 6$, the center has exactly 74 *simple objects. In the ordering specified in Appendix A, its matrix is exactly* `HS4`*. The identification uses one simultaneous row and column permutation, with no changes of signs or Galois conjugation.*

The assertion also extends to genuine direct products of finite Coxeter systems by tensor products; see Corollary 9.1. The theorem concerns the Fourier matrix assertion in Lusztig's proposal. It does not assert geometric constructions of character sheaves for noncrystallographic groups.

Here are the principal ingredients. First, we prove the sign formula

$$\dim_{\mathrm{EW}}(A_x) = (-1)^{\ell(x)-a(c)} \operatorname{FPdim}(A_x) \tag{1.3}$$

for every diagonal fusion corner. The proof uses two-factor relative Hodge–Riemann forms and the actual evaluation and coevaluation maps in the asymptotic category. It also determines the nontrivial associator in the order-two pointed corners.

Second, let $\mathcal{C}$ be Alvis's rank-24 diagonal corner for the middle cell of H_4. We construct a connected separable algebra $A = \mathbf{1} \oplus g$ in $\mathcal{C}$ and prove

$${}_A\mathcal{C}_A \simeq \mathcal{C}_0(A_3) \rtimes_\sigma C_2, \tag{1.4}$$

where $\mathcal{C}_0(A_3)$ is the positive Fibonacci Hecke category and σ interchanges the two endpoints of the A_3 diagram. The algebra, the order-two action, and its coherence are constructed in the actual corner. The fixed-seed tensor uniqueness theorem of He and Liu [8] identifies the neutral part. We give a separate proof of the marked automorphism rigidity needed to identify the action.

Third, the regular endofunctor model of the right-hand side of (1.4) gives an exact tube-algebra calculation. Two partial components detect $40 + 32$ central simple objects. A further 8-dimensional ideal in a remaining corner has center of dimension 2 and supplies the missing objects. Exact module and frame identities prove that these 74 objects exhaust the center.

Finally, writing $D = 1080 + 480\sqrt{5}$, the actual normalized matrix lies in $\mathbb{Q}(\zeta_{60})$, and DS_{ij} is an algebraic integer. For the exact target F, every nonzero difference satisfies

$$|S_{ij} - F_{ij}| \geq \frac{1}{128 \cdot 14400^4} = \frac{1}{5503765708800000000}. \tag{1.5}$$

Certified enclosures give a uniform error less than $3.422{\cdot}10^{-140}$ after the simultaneous permutation. This proves equality, rather than a numerical approximation to it.

1.1. **Structure of the proof.** We state the external categorical inputs precisely in Section 2. Sections 3–5 prove the pivotal and Morita assertions. Sections 6–8 specify the computational model, prove completeness of the tube calculation, and give the arithmetic certificate. Section 9 assembles all cells. The appendices contain the complete compressed 74 by 74 target, the label conversion, and a reproducible verification protocol. Thus the reductions, conventions, finite algorithms, and their mathematical certification are all explicit; foundational results cited from the literature are distinguished from the arguments proved here.

2. Categorical conventions and inputs

2.1. **Corners and dimensions.** A fusion category here is a split semisimple rigid $\mathbb{C}$-linear tensor category with finitely many simple objects, finite-dimensional morphism spaces, and simple unit. In a multifusion category the unit may decompose. If

$$\mathbf{1} = \bigoplus_{i \in I} \mathbf{1}_i, \qquad \mathcal{C}_{ij} = \mathbf{1}_i \otimes \mathcal{C} \otimes \mathbf{1}_j,$$

then $\mathcal{C}_{ii}$ is a diagonal fusion corner. For a two-sided cell, these are the categories attached to diagonal H-cells. The unit of a corner is the distinguished involution in that H-cell.

A pivotal structure is a monoidal identification $j : \mathrm{Id} \to (-)^{\vee\vee}$. We use both left and right traces, and call j spherical if they agree. For a spherical fusion category,

$$\operatorname{Dim} \mathcal{C} = \sum_{X \in \operatorname{Irr} \mathcal{C}} \dim(X)^2.$$

The Frobenius–Perron dimension is denoted FPdim; its global value is the sum of the squares of the simple Frobenius–Perron dimensions. A pseudo-unitary category has a unique spherical structure for which all simple dimensions are positive [5, Proposition 8.23]. This statement does not by itself assert the existence of a unitary tensor structure.

The Drinfeld center $\mathcal{Z}(\mathcal{C})$ consists of pairs $(Z, c_{Z,-})$, where $c_{Z,X} : Z \otimes X \to X \otimes Z$ is a natural half-braiding satisfying the hexagon. A spherical structure on $\mathcal{C}$ induces a ribbon structure on its center. For a fusion category with positive spherical structure,

$$\operatorname{Dim} \mathcal{Z}(\mathcal{C}) = (\operatorname{Dim} \mathcal{C})^2,$$

so the denominator in (1.1) is $\operatorname{Dim} \mathcal{C}$.

Lemma 2.1 (Pivotal corner reduction)**.** *Let $\mathcal{C}$ be an indecomposable pivotal multifusion category and let $i \in I$. The standard corner restriction equivalence*

$$\mathrm{Res}_i : \mathcal{Z}(\mathcal{C}) \longrightarrow \mathcal{Z}(\mathcal{C}_{ii})$$

is pivotal. When the induced pivotal structures are spherical it is ribbon and preserves the normalized S-matrix.

Proof. The half-braiding with each $\mathbf{1}_j$ forces a central object to be a sum of diagonal components. Restriction takes its ii component and restricts its half-braiding to $\mathcal{C}_{ii}$. Its tensorator is the component of the original tensorator. Evaluation, coevaluation, the double dual, and j restrict in the same way. Consequently the braided equivalence of corner reduction, as in [20, Proposition 3.8], is pivotal.

The tensor unit in the center is simple because $\mathcal{C}$ is indecomposable. The trace of a central endomorphism is therefore a scalar multiple of the central identity. Taking its ii component preserves this scalar. The same applies to double braidings and to the global dimension of

the center. A braided pivotal functor preserves the ribbon twist, and hence also (1.1) with its positive square root. □

2.2. **The asymptotic construction.** Write B_x for the self-dual indecomposable Soergel object corresponding to $x \in W$. Here self-dual refers to graded Soergel duality; rigid monoidal duality is denoted by $(-)^\vee$ and sends B_x to $B_{x^{-1}}$. Work in the quotient by objects in two-sided cells strictly below c. If $a = a(c)$, truncated multiplication takes the multiplicity spaces in degree $-a$, equivalently the lowest surviving perverse piece of a product. Its indecomposable objects are denoted A_x. The product will be written $*$ when it is necessary to distinguish it from the untruncated tensor product of Soergel objects.

Fix a regular dominant element ρ in the dual reflection representation, as in the EW construction. On a product XY the relative Lefschetz operator is induced by insertion in the middle,

$$M_{XY} = 1_X \otimes \rho \otimes 1_Y.$$

We use the following facts from [3, Theorem 3.4, Proposition 5.6, Remark 5.8]. Multiplicity spaces of a two-factor product satisfy relative hard Lefschetz and the relative Hodge–Riemann relations for M. The truncation is rigid. Its Hom-adjunction is the ordinary Soergel bending map followed by the appropriate power M^a. Its pivotal map is the inherited ordinary double-dual map. In particular, this specifies a pivotal structure rather than merely the existence of one. Our argument applies to any fixed regular dominant choice of ρ used in that construction.

We also use the parity property of graded Soergel morphisms: a nonzero degree k map from B_z to B_xB_y requires

$$\ell(x) + \ell(y) - \ell(z) \equiv k \pmod 2. \tag{2.1}$$

The lower-cell ideal is a two-sided tensor ideal stable under duality and adjoints.

2.3. **Positive Fibonacci categories.** Put

$$\varphi = \frac{1+\sqrt{5}}{2}, \qquad \alpha = \varphi^{-1} = \frac{\sqrt{5}-1}{2}, \qquad \alpha^2 + \alpha = 1. \tag{2.2}$$

The positive Fibonacci category Fib has two simple objects $\mathbf{1}, u$, with

$$u^2 = \mathbf{1} + u, \qquad \dim(u) = \varphi.$$

The rank-two classification [18] distinguishes this tensor category from the Yang–Lee branch. In the orthonormal fusion gauge, the nontrivial recoupling block is

$$\begin{pmatrix} \alpha & \sqrt{\alpha} \\ \sqrt{\alpha} & -\alpha \end{pmatrix}. \tag{2.3}$$

For clarity we isolate the particular new classification input used below.

Theorem 2.2 (Fixed positive seeds, type A_3)**.** *There is a fusion category $\mathcal{C}_0(A_3)$ with simple objects X_w, $w \in \mathfrak{S}_4$, and positive Fibonacci generators $X_{s_1}, X_{s_2}, X_{s_3}$. Its based ring has multiplication*

$$X_s X_w = \begin{cases} X_{sw}, & \ell(sw) > \ell(w), \\ X_{sw} + X_w, & \ell(sw) < \ell(w). \end{cases} \tag{2.4}$$

Any fusion category with this based ring, together with marked positive Fibonacci tensor subcategories on the three generators, is equivalent to $\mathcal{C}_0(A_3)$ by a tensor equivalence preserving the marked seeds. The reference category has a unitary regular endofunctor realization.

This is the A_3 specialization of the construction and fixed-seed uniqueness theorems [8, Theorems 1.3 and 1.5]. No general classification of allowable Coxeter exponents will be used: we derive A_3 directly in Lemma 5.2. Nor do we infer the triviality of a marked automorphism group from uniqueness of an equivalence class; that separate issue is proved in Lemma 5.3. Section 6 gives the reference model explicitly enough to define every matrix used in the calculation.

The other local classification input is [6, Theorem A.3]: a pivotal category with the fusion ring K_l, $l > 2$, is tensor equivalent to an appropriate $SO(3)_q$ category, with q a primitive $4(l+1)$-st root of unity. We use it only for $l = 4$, and only within a specified full subcategory. The Morita invariance of Frobenius–Perron and categorical global dimensions and the uniqueness of a positive spherical structure are as in [5, Theorem 2.15, Corollary 8.14, Proposition 8.23].

2.4. **Upper and lower truncation.** Lusztig's original formulation uses the upper extreme perverse piece, whereas the EW convention adopted in [20, Construction 2.6] uses the lower one. We make the comparison explicit. Write $X \otimes_{\pm} Y = H^{\pm a}(XY)$ in the cell quotient. Relative hard Lefschetz gives

$$\vartheta_{X,Y} = H(M^a_{XY}) : X \otimes_- Y \xrightarrow{\sim} X \otimes_+ Y.$$

This is the tensor comparison for the identity-on-objects equivalence, with its inverse used if the direction of a strong tensorator requires it. Indeed, in a triple product the two associativity comparisons on the extreme pieces are induced by $M^a_{23}M^a_{12}$ and $M^a_{12}M^a_{23}$. Middle multiplication at the two positions commutes, so these maps agree; this is the mechanism in [3, Remark 5.7]. The unit comparison is the corresponding unit isomorphism.

Ordinary rigid duality interchanges the upper and lower extreme pieces, reverses the tensor word, and preserves middle multiplication. With the resulting identifications, the dual of $\vartheta_{X,Y}$ is $\vartheta_{Y^\vee,X^\vee}$. The comparisons consequently cancel on taking the double dual. Transporting the EW evaluations and coevaluations along this equivalence gives the upper truncation its inherited ordinary double-dual pivot. Thus the two truncation conventions have the same ribbon center and normalized S-matrix. No additional three-factor Hodge–Riemann assertion is needed for this comparison.

3. The inherited pivotal structure

Proposition 3.1. *Let $\mathcal{C}$ be a finite diagonal fusion corner of $\mathcal{A}_c$, with unit A_d, and put $a = a(c)$. Then*

$$\dim_{\mathrm{EW}}(A_x) = (-1)^{\ell(x)-a}\,\mathrm{FPdim}(A_x). \tag{3.1}$$

The scalars $\epsilon_x = (-1)^{\ell(x)-a}$ define a monoidal natural automorphism of $\mathrm{Id}_{\mathcal{C}}$. The pivotal structure $j_+ = \epsilon j_{\mathrm{EW}}$ is the unique positive spherical structure. The original inherited structure is spherical as well.

3.1. **Adjoints and ordinary bending.** We first record the compatibility needed to compare relative forms with the actual categorical trace. In the lower-cell quotient put

$$X = B_x, \quad Y = X^\vee = B_{x^{-1}}, \quad D_0 = B_d.$$

Here D_0 denotes a Soergel object only in this section. Choose X as the top summand of a reduced Bott–Samelson object, using the self-adjoint projector for its polarized intersection form. Realize Y by the rotated projector. These choices give positive polarizations in the sense of relative Hodge theory; they do not mean that every graded piece of the form is positive definite.

Use the following types for ordinary cups and caps:

$$E_X : YX \to R, \quad C_X : R \to XY, \qquad E_Y : XY \to R, \quad C_Y : R \to YX.$$

With adjoints relative to the intersection forms,

$$E_X^\dagger = C_Y, \qquad E_Y^\dagger = C_X. \tag{3.2}$$

Thus the adjoint cap is the cup on the same ordered tensor word, with the opposite handedness.

Here is a direct verification of the normalization in (3.2). For one color s, with Demazure operator ∂_s, the Frobenius formulas are

$$E_s(a \otimes b \otimes c) = a\partial_s(b)c, \qquad C_s(1) = \tfrac{1}{2}(\alpha_s \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \alpha_s).$$

Their adjointness follows from

$$\tfrac{1}{2}\{\partial_s(\partial_s(\alpha_s a)b) + \alpha_s\partial_s(\partial_s(a)b)\} = a\partial_s(b).$$

The induced tensor form is

$$\langle b \otimes b', c \otimes c' \rangle = \langle \langle b, c \rangle b', c' \rangle.$$

It preserves these identities under tensor products and under restriction by self-adjoint projectors. This proves (3.2) for the chosen X, Y; compare [3, §2.4] and [4].

The relevant real multiplicity spaces are the one-dimensional lines

$$V = \mathrm{Hom}^{-a}(D_0, XY), \qquad U = \mathrm{Hom}^{-a}(Y, YD_0).$$

Define ordinary bending maps, with the ambient tensor product understood, by

$$T_X(v) = (E_X \otimes 1_Y)(1_Y \otimes v), \tag{3.3}$$

$$S_Y(u) = (E_Y \otimes 1_{D_0})(1_X \otimes u). \tag{3.4}$$

The ordinary snake identities and (3.2) imply

$$S_Y(T_X(v)^\dagger) = v^\dagger. \tag{3.5}$$

All these identities descend to the lower-cell quotient. Indeed, bending a map factoring through a lower-cell object factors through its tensor product with X or Y, which is still in the lower-cell ideal. The value R of an ordinary cap need not lie in the fusion corner.

3.2. **Relative Hodge–Riemann signs and the actual trace.**

Proof of Proposition 3.1. Let $u = r_Y^{-1} : Y \to Y * D_0$ be the actual inverse right unitor, let $e_X : D_0 \to X * Y$ be the actual coevaluation obtained from the EW Hom-adjunction, and let $\mu_Y : X * Y \to D_0$ be the evaluation of Y. Represent these maps on the preceding multiplicity lines. The explicit Hom-adjunction in [3, §5] gives

$$1_Y = T_X(e_X)M^a_{YD_0}u, \qquad \mu_Y = S_Y(u)M^a_{XY}. \tag{3.6}$$

The powers of M are essential: ordinary bending alone is not the truncated Hom-adjunction. Since the inherited pivotal map is ordinary double dual, its right dimension is precisely

$$\dim^R_{\mathrm{EW}}(X)1_{D_0} = \mu_Y e_X. \tag{3.7}$$

Write $T_X(e_X)^\dagger = cu$ for a nonzero real scalar c. Combining (3.5) and (3.6) yields

$$1 = cG_U(u,u), \qquad \dim^R_{\mathrm{EW}}(X) = c^{-1}G_V(e_X, e_X), \tag{3.8}$$

where

$$G_U(u,u) = u^\dagger M^a_{YD_0}u, \qquad G_V(e_X, e_X) = e_X^\dagger M^a_{XY}e_X.$$

There are no same-cell multiplicity spaces below degree $-a$. Self-duality gives the corresponding vanishing above degree a, so M^{a+1} kills degree $-a$. Thus the entire degree-$-a$ line is primitive for relative Lefschetz. The two-factor relative Hodge–Riemann theorem gives

$$\operatorname{sgn} G_U = (-1)^{(\ell(d)-a)/2}, \qquad \operatorname{sgn} G_V = (-1)^{(2\ell(x)-\ell(d)-a)/2}. \tag{3.9}$$

The integers in these exponents are guaranteed by (2.1). Equations (3.8) and (3.9) give

$$\operatorname{sgn}\dim^R_{\mathrm{EW}}(A_x) = (-1)^{\ell(x)-a}.$$

If A_z occurs in $A_x * A_y$, its degree-$-a$ inclusion and (2.1) imply

$$\ell(x) + \ell(y) - \ell(z) - a \equiv 0 \pmod 2.$$

Consequently $\epsilon_x\epsilon_y = \epsilon_z$, and $\epsilon_d = 1$. The scalars ϵ_x therefore define a monoidal natural automorphism of the identity. Twisting the pivotal structure by ϵ makes all right simple dimensions positive. A positive multiplicative dimension character on a fusion ring is the Frobenius–Perron character. Hence these right dimensions equal FPdim.

Right dimension on a dual object is left dimension on the original object. Since Frobenius–Perron dimension is dual-invariant, the positive structure has equal left and right dimensions on every simple. Semisimplicity then gives equality of the two traces on every endomorphism. It is spherical. Finally $\epsilon_{x^{-1}} = \epsilon_x$, so undoing the twist also preserves sphericality. This proves (3.1) and the asserted uniqueness of the positive structure. □

Corollary 3.2. *For a simple central object Z, the induced automorphism ϵ_Z is a scalar $\delta_Z \in \{1, -1\}$. If $D_\delta = \operatorname{diag}(\delta_Z)$, the two ribbon structures satisfy*

$$S_{\mathrm{EW}} = D_\delta S_+ D_\delta, \qquad \Theta_{\mathrm{EW}} = D_\delta \Theta_+. \tag{3.10}$$

They use the same positive square root in (1.1).

Proof. Naturality and monoidality of ϵ make ϵ_Z a central endomorphism, so it is scalar on a simple central object. Its square is the identity. In particular, all constituents of the underlying object of Z have the same sign. A pivotal twist multiplies the categorical trace on $Z \otimes Z'$ by $\delta_Z\delta_{Z'}$ and the ribbon twist on Z by δ_Z. Squared dimensions are unchanged. □

4. Pointed and Fibonacci corners

4.1. **A symmetric ordinary self-duality.** The sign of a pivotal dimension alone does not distinguish the two tensor categories with fusion ring $\mathbb{Z}[C_2]$. We need the following compatibility with ordinary Soergel duality.

Lemma 4.1. *If $x = x^{-1}$, the ordinary Soergel object B_x admits a real self-duality $q : B_x \to B_x^\vee$ satisfying*

$$q^\vee j_{B_x} = q.$$

It can be chosen isometric for the positive polarizations used in Section 3.

Proof. Choose a reduced word for x and its reversed word. Join them by normalized rex moves and insert the top-summand projectors at both ends. The resulting degree-zero morphism q is nonzero: each normalized move preserves the bottom one-tensor, and the top projector fixes its nonzero image. In the standard balanced realization, rotating a rank-two rex move introduces the coefficient

$$[m-1] = \frac{\sin((m-1)\pi/m)}{\sin(\pi/m)} = 1.$$

Thus q and $q^\vee j_{B_x}$ preserve the same one-tensor. The degree-zero Hom between these indecomposables is one-dimensional, so they agree. The argument uses the top summand and the normalized rex maps of [4, §5.6]; it does not require a canonical choice of projector.

The adjoint composite $q^\dagger q$ is a nonzero real scalar c on B_x. For a bottom vector v, positivity means the Lefschetz pairing, as in [3, Lemma 2.6]. Thus

$$c\langle v, \rho^{\ell(x)} v\rangle_{B_x} = \langle qv, \rho^{\ell(x)} qv\rangle_{B_x^\vee} > 0,$$

and the pairing on the left before multiplication by c is positive as well. Hence $c > 0$. Rescale q by its positive inverse square root. This gives an isometry without changing the symmetry equation. □

Proposition 4.2. *Suppose a diagonal corner has two simple objects $\mathbf{1}, A_x$, with $A_x^2 = \mathbf{1}$. Its unit-normalized associator on (A_x, A_x, A_x) is*

$$\omega(x, x, x) = (-1)^{\ell(x)-a(c)}. \tag{4.1}$$

Proof. Invertibility and self-duality imply $x = x^{-1}$. Use Lemma 4.1 to regard the actual evaluation and coevaluation as maps for one symmetric self-dual object. The calculation (3.8) gives the sign of their composite μe as $(-1)^{\ell(x)-a(c)}$; Proposition 3.1 gives its absolute value 1.

For completeness, choose a skeletal realization with two objects and write $f = F^x_{xxx}$ and $h = F^{\mathbf{1}}_{x\mathbf{1}x}$ before normalizing the unit. If l, r are the left and right unit scalars, the unit triangle gives $r = lh$. The self-dual snake then gives

$$\mu e = (fh)^{-1}.$$

Changing the binary fusion bases leaves fh invariant. In a unit-normalized gauge it is the cocycle value $\omega(x, x, x)$. The C_2 pentagon gives $\omega(x, x, x)^2 = 1$. Consequently $(fh)^{-1} = fh$, and (4.1) follows. The symmetric ordinary self-duality is what identifies this snake scalar with the inherited pivotal trace. □

4.2. **The exceptional order-two corners.** Number the simple generators so that $m_{12} = 5$ and $m_{23} = m_{34} = 3$, omitting the fourth generator in H_3. In type H_3, and in the corresponding parabolic diagonal corner of H_4, the $a = 3$ cell has representatives

$$d = 232, \qquad x = 232123, \qquad \ell(d) = 3, \quad \ell(x) = 6.$$

These diagonal-cell identities and their order-two based rings are recorded in [19, §§1.4.3–1.4.4, (1.62)–(1.63)]; only the cell data are needed here. The distinguished unit is A_d and the nontrivial invertible is A_x: the unit has length congruent to a modulo 2. Thus Proposition 4.2 gives $\omega = -1$.

For the $a = 18$ cell of H_4, multiplication by the central longest element w_0 sends the relevant diagonal cell to the diagonal cell with representatives

$$d' = w_0(232123), \quad \ell(d') = 54, \qquad x' = w_0(232), \quad \ell(x') = 57. \tag{4.2}$$

The cell correspondence follows from [12, Corollary 11.7(d)] and the cell table; the word lengths can also be checked directly in the standard root representation. The unit is the length-54 representative, because its parity must agree with $a = 18$. Hence

$$\omega(x', x', x') = (-1)^{57-18} = -1.$$

Only the correspondence of cells is used here. No monoidal equivalence induced by w_0 is assumed.

All three pointed corners are consequently $\mathrm{Vec}^-_{C_2}$, with normalized cocycle $\omega(g, h, k) = (-1)^{ghk}$. In its center a simple object is determined by $g \in \{0, 1\}$ and the half-braiding scalar λ with the nontrivial degree-one object. The hexagon reads $\lambda^2 = (-1)^g$. In the order

$$(0, 1),\ (0, -1),\ (1, i),\ (1, -i),$$

the inherited pivot has dimension vector $(1, 1, -1, -1)$. Double braiding, with the product of these dimensions, gives

$$S_{\mathrm{EW}} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}. \tag{4.3}$$

This is the CHEVIE block `TQZ(2,-1,[1,-1])` in its specified ordering. In particular its first index is the tensor unit, even though its second row is entirely positive.

Remark 4.3. In these fixed object labels the positive-pivot matrix satisfies

$$S_{\mathrm{EW}} = \operatorname{diag}(1,1,-1,-1)\, S_+ \operatorname{diag}(1,1,-1,-1).$$

Numerically, interchanging the first two rows and the first two columns of (4.3) also gives S_+. That permutation moves the actual tensor unit and therefore must not be used as an argument that the two fixed-label pivotal structures agree. Corner reduction in Lemma 2.1 preserves the inherited structure and does not make this change.

4.3. **Fibonacci and trivial corners.** If a corner has fusion rule $u^2 = \mathbf{1} + u$, every monoidal scalar character takes value 1 on u: it must satisfy both $t^2 = 1$ and $t^2 = t$. In particular the sign character in Proposition 3.1 is trivial. The inherited dimension of u is therefore φ. The rank-two classification identifies the actual tensor category with the positive Fibonacci branch.

Choose one of the two nondegenerate braidings on Fib. Its center is ribbon equivalent to $\mathrm{Fib} \boxtimes \mathrm{Fib}^{\mathrm{rev}}$. The normalized two-object matrix of Fib is

$$\frac{1}{\sqrt{1+\varphi^2}} \begin{pmatrix} 1 & \varphi \\ \varphi & -1 \end{pmatrix}.$$

Taking its tensor square and ordering central objects as $(\mathbf{1},\mathbf{1}), (u,u), (\mathbf{1},u), (u,\mathbf{1})$ gives

$$S_{\mathrm{Fib}} = \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi^{-1} & \varphi & 1 & 1 \\ \varphi & \varphi^{-1} & -1 & -1 \\ 1 & -1 & -\varphi^{-1} & \varphi \\ 1 & -1 & \varphi & -\varphi^{-1} \end{pmatrix}. \tag{4.4}$$

The standard Fourier pairing for this block becomes (4.4) by the last-two-column conversion in Convention 1.1. For example, the traces of the unconverted pairing and (4.4) are 2 and 0, respectively; a simultaneous row and column relabeling alone could not perform that conversion.

Finally, a rank-one corner is Vec, and its center has matrix (1). These observations will account for every H_3 and H_4 cell except the middle H_4 cell.

5. The middle cell of H_4

5.1. **The actual diagonal corner.** Let c be the two-sided cell of H_4 with $a(c) = 6$. We use the diagonal corner associated with $\Gamma = A_{19}$ in [1, §6, Table 5]. It has rank 24. Write x_i for the simple object corresponding to the ith basis element in that table, using Alvis's one-based indexing, and set

$$\mathbf{1} = x_2, \qquad a_0 = x_1, \qquad b = x_{17}, \qquad g = x_{23}.$$

These names denote objects, not Coxeter generators. The integer structure constants give

$$a_0^2 = \mathbf{1} + a_0, \qquad g^2 = \mathbf{1}, \qquad ga_0 = a_0 g = x_{24}, \tag{5.1}$$

$$gb = bg = b, \qquad a_0 b = x_{15}, \qquad ba_0 = x_{14}, \tag{5.2}$$

$$b^2 = x_2 + x_8 + x_{17} + x_{22} + x_{23}. \tag{5.3}$$

Moreover, a_0, b generate the entire based ring, and

$$\mathrm{FPdim}\, \mathcal{C} = 1080 + 480\sqrt{5} =: D. \tag{5.4}$$

These facts can be verified by closing the supports of products of a_0, b in Alvis's matrices and by checking the positive dimension character on all products. Alvis defines his matrices as transposes of left multiplication; we keep that convention when extracting the structure constants. In particular the different products in (5.2) have the stated order.

Any monoidal automorphism of $\mathrm{Id}_{\mathcal{C}}$ is scalar on every simple. Since a_0^2 contains both $\mathbf{1}$ and a_0, its value on a_0 is 1; the same reasoning using b^2 gives value 1 on b. Generation implies

$$\mathrm{Aut}_{\otimes}(\mathrm{Id}_{\mathcal{C}}) = 1. \tag{5.5}$$

Proposition 3.1 now shows that the inherited EW pivotal structure on this particular corner is already positive.

There are full tensor subcategories

$$\mathcal{B} = \langle a_0, g\rangle, \qquad \mathcal{E} = \langle b, g\rangle.$$

The simple objects of $\mathcal{B}$ are $\mathbf{1}, a_0, g, a_0 g$, with dimensions $1, \varphi, 1, \varphi$. The category $\mathcal{E}$ has fusion ring K_4. More explicitly, put

$$(V_0, V_1, V_2, V_3, V_4) = (x_2, x_8, x_{17}, x_{22}, x_{23}). \tag{5.6}$$

Their dimensions are $(1, \varphi^2, 2\varphi, \varphi^2, 1)$, and their complete multiplication table is

	V_0	V_1	V_2	V_3	V_4
V_0	V_0	V_1	V_2	V_3	V_4
V_1	V_1	$V_0+V_1+V_2$	$V_1+V_2+V_3$	$V_2+V_3+V_4$	V_3
V_2	V_2	$V_1+V_2+V_3$	$\sum_{i=0}^{4} V_i$	$V_1+V_2+V_3$	V_2
V_3	V_3	$V_2+V_3+V_4$	$V_1+V_2+V_3$	$V_0+V_1+V_2$	V_1
V_4	V_4	V_3	V_2	V_1	V_0

These subrings are used to locate actual full subcategories. Their fusion rules alone are not taken to identify their associators.

5.2. **A common algebra and an actual involution.** The additive category $\mathrm{add}(b)$ is a rank-one module category over $\langle g\rangle$, since $gb \cong b$. Present the latter pointed category as $\mathrm{Vec}^{\omega}_{C_2}$. After choosing the action isomorphisms on b, the module pentagon is $d\mu = \omega$ for a scalar 2-cochain μ. Thus $[\omega] = 0$. Consequently

$$A = \mathbf{1} \oplus g$$

admits the connected separable regular algebra structure. It lies in both $\mathcal{B}$ and $\mathcal{E}$. Any change in its nonzero multiplication $g \otimes g \to \mathbf{1}$ is absorbed by rescaling g, so we fix the same actual algebra in both subcategories.

Set

$$\mathcal{D} = {}_A\mathcal{C}_A.$$

This is the Morita dual fusion category. Morita invariance of categorical and Frobenius–Perron global dimensions gives $\mathrm{Dim}\,\mathcal{D} = \mathrm{FPdim}\,\mathcal{D} = D$, in the intrinsic pseudo-unitary sense. We equip $\mathcal{D}$ with its unique positive spherical structure. This establishes the positive branch before identifying any of its Fibonacci subcategories.

Let $\tau = \mathrm{id}_{\mathbf{1}} \oplus (-\mathrm{id}_g)$, an algebra automorphism of A, and let $h_i = A_{\tau^i}$ be the regular left module with right action twisted by τ^i, where $i \in C_2$. There are actual bimodule isomorphisms

$$h_i \otimes_A h_j \longrightarrow h_{i+j}, \qquad x \otimes y \longmapsto x\tau^i(y), \tag{5.7}$$

with inverses $x \mapsto x \otimes 1$. The formulas are shorthand for the algebra multiplication and its module actions; coherence permits omission of associators. Balancedness and both module linearities follow directly from the algebra automorphism property of τ. The associativity diagram is the identity

$$(x\tau^i(y))\tau^{i+j}(z) = x\tau^i(y\tau^j(z)).$$

In particular these maps trivialize the three-cocycle on the actual order-two subcategory; the object equation $h_1^2 \cong \mathbf{1}$ alone would not have established this.

Write $h = h_1$. It is not the unit of $\mathcal{D}$. A left A-linear map $A \to A_\tau$ is determined by a map $\mathbf{1} \to A$, hence is a scalar multiple of the identity on the underlying object. Right g-linearity forces the scalar to be its own negative, so the map is zero.

There is also an explicit tensor involution

$$T(X) = {}_\tau X_\tau \tag{5.8}$$

of $\mathcal{D}$. Its tensorator is induced by the identity on the balanced quotient, its unit map is τ, and $T^2 = \mathrm{Id}$ with these structure maps. Conjugation by h is identified with T by

$$h \otimes_A X \otimes_A h \longrightarrow T(X), \qquad a \otimes x \otimes b \longmapsto \tau(a)xb. \tag{5.9}$$

All the action and coherence data are thus defined before any comparison with a reference category.

5.3. **The middle Fibonacci seed.** We claim that

$$\mathcal{B} \simeq \mathrm{Fib} \boxtimes \mathrm{Vec}_{C_2}. \tag{5.10}$$

To justify the mixed associator, let $\mathcal{F} = \langle a_0 \rangle$. It is the positive Fibonacci category by Proposition 3.1. Conjugation by g preserves $\mathcal{F}$, and the already coherent trivialization of the associator of $\langle g \rangle$ gives a coherent C_2 action on $\mathcal{F}$.

The tensor automorphism 2-group of Fib is trivial. Indeed an object-fixing tensorator on $u^2 = \mathbf{1} + u$ has two diagonal scalars λ, μ. Coherence with the nonzero off-diagonal entries of (2.3) forces $\lambda = \mu^2$. Rescaling the comparison on u by μ removes the tensorator. An automorphism of the identity has scalar t on u, with $t^2 = 1 = t$, so it is the identity. It follows that the comparison between conjugation by g and $\mathrm{Id}_\mathcal{F}$ is unique and automatically respects the C_2 coherence.

This comparison supplies coherent exchanges $gX \to Xg$. The functor

$$\mathcal{F} \boxtimes \mathrm{Vec}_{C_2} \longrightarrow \mathcal{B}, \qquad (X, i) \longmapsto Xg^i$$

uses those exchanges and multiplication of the g factors as its tensorator. Their two coherence equations are its pentagon. Its four simple images are distinct and exhaust $\mathcal{B}$, proving (5.10).

The category ${}_A\mathcal{B}_A$ therefore contains an actual positive Fibonacci object m. It is right-free on a_0 and left-free on a_0, with underlying object $a_0 \oplus a_0 g$. It satisfies

$$m^2 = \mathbf{1} + m, \qquad T(m) \cong m.$$

Both statements may equivalently be read in the product model (5.10).

5.4. **The two endpoint seeds.** The pivotal K_4 classification in [6, Theorem A.3] identifies $\mathcal{E}$ as a tensor category with an even $SU(2)_8$ category, at a primitive twentieth root q. Transport its braiding to $\mathcal{E}$. In this paragraph the standard even $SU(2)$ labels are $0, 2, 4, 6, 8$, and g has label 8. Its twist, monodromy with an even label j, and self-braiding are respectively

$$q^{40} = 1, \qquad q^{40-10j} = 1, \qquad q^{-40} = 1.$$

Thus g is a transparent boson and A is the regular commutative algebra of a Tannakian C_2 subcategory. This braiding is used only in $\mathcal{E}$, not on all of $\mathcal{C}$.

The de-equivariantization $\mathcal{K} = \mathcal{E}_A$ is braided, with four simples of dimensions

$$1, \varphi, \varphi, \varphi^2.$$

The nonfixed orbits of g give the unit and an object r of dimension φ^2. The fixed object b splits into two module structures p, q, both of dimension φ, with underlying object b.

We spell out their fusion rules. If $p^* = q$, the braided product pq is self-dual and contains the unit. Its dimension $1 + \varphi$ would force $pq = \mathbf{1} + p$ or $\mathbf{1} + q$, neither self-dual. Thus p, q are each

self-dual. The product pq has no unit constituent, and its dimension forces $pq = r$. Frobenius reciprocity gives

$$[p^2 : q] = [pq : p] = 0.$$

Since p^2 contains the unit and has dimension $1 + \varphi$, we obtain

$$p^2 = \mathbf{1} + p, \qquad q^2 = \mathbf{1} + q, \qquad pq = qp = r. \tag{5.11}$$

The braiding provides the tensor interchange for $\langle p\rangle \boxtimes \langle q\rangle \to \mathcal{K}$. Its four simple images are distinct, so it is a tensor equivalence. The two factors have the positive Fibonacci branch, as subcategories of the positively spherical Morita dual.

The local bimodule category ${}_A\mathcal{E}_A$ is the crossed product of $\mathcal{K}$ by its de-equivariantization action. Its odd invertible is the actual $h = A_\tau$ of (5.7). One can see the action directly: the two A-module structures on b differ by the sign of the g action, and twisting by τ interchanges them. The induced left actions, obtained from the local braiding, are twisted at the same time. Hence

$$T(p) = q, \qquad T(q) = p. \tag{5.12}$$

As a check, an action fixing all four simples would have an equivariantization of rank 8, since every invariant simple has two equivariant lifts for C_2; but $\mathcal{E}$ has rank 5.

Because $\mathcal{B}$ and $\mathcal{E}$ are full tensor subcategories containing the same separable algebra A, their bimodule categories embed fully and monoidally into $\mathcal{D}$. In particular m, p, q are objects of one actual tensor category. The left-free and right-free descriptions of m give

$$U(m \otimes_A p) = a_0 b = x_{15}, \qquad U(p \otimes_A m) = b a_0 = x_{14}, \tag{5.13}$$

and the same identities with q in place of p. These underlying objects are simple and different. Therefore the bimodule products are simple and nonisomorphic. The forgetful functor $U : \mathcal{D} \to \mathcal{C}$ is not being treated as a tensor functor in this argument; the identities follow from the indicated free module presentations. Thus p, m, q are three distinct Fibonacci generators, the endpoints commute, and the middle generator commutes with neither endpoint.

5.5. **A based-ring rigidity argument.** We give the short rigidity argument needed to determine the type of the category generated by p, m, q.

Lemma 5.1. *Let R be a finite rigid based ring generated as a based ring by distinct elements x_s with $x_s^2 = 1 + x_s$. Then its basis is indexed by a finite Coxeter group (W', S') and its multiplication by each x_s is* (2.4).

Proof. The unit coefficient in x_s^2 implies $x_s^* = x_s$. Hence its left multiplication matrix N_s is symmetric, nonnegative, and integral, and $N_s^2 = I + N_s$. For a basis vector b, set $r = (N_s)_{bb}$. The diagonal equation is

$$r^2 + \sum_{c \neq b} (N_s)_{cb}^2 = 1 + r.$$

Therefore $r \in \{0, 1\}$, and exactly one off-diagonal term equals 1. Left multiplication has the form

$$L_s(b) = P_s(b) + \epsilon_s(b) b, \tag{5.14}$$

where P_s is a fixed-point-free involution and $\epsilon_s(P_s b) = 1 - \epsilon_s(b)$. Right multiplication similarly gives a fixed-point-free involution Q_t and a function η_t.

The operators L_s, R_t commute. On expanding their products, the terms $P_s Q_t(b)$ and $Q_t P_s(b)$ cannot equal either of the single-step terms $P_s(b), Q_t(b)$: such an equality would force one of the matchings to fix a point. Comparison of coefficients therefore gives

$$P_s Q_t = Q_t P_s, \qquad P_s(b) \neq Q_t(b) \Longrightarrow \epsilon_s(Q_t b) = \epsilon_s(b). \tag{5.15}$$

Generation of the based ring implies transitivity of both matching actions. Indeed, expanding a word in the generators reaches each constituent by a sequence of matching moves or stationary steps, and deleting the stationary steps gives a matching path from the unit.

Let W' be the permutation group generated by the P_s. If an element of W' fixes the unit basis vector, its commutation with every Q_t and right transitivity make it fix every vector. Thus it is the identity. The left action is simply transitive, and identifies the basis with W'. Under this identification P_s is left multiplication by a distinct involution s and Q_s is right multiplication by s.

Let ℓ be shortest-word length in these generators. Equation (5.15) says

$$\epsilon_s(wt) = \epsilon_s(w) \quad \text{unless } sw = wt.$$

Suppose $w = t_1 \cdots t_k$ is a shortest word and $\epsilon_s(w) = 1$. Along its prefixes, take the first place where ϵ_s changes from its initial value 0. Then, for some j,

$$s(t_1 \cdots t_{j-1}) = (t_1 \cdots t_{j-1})t_j, \qquad sw = t_1 \cdots \widehat{t_j} \cdots t_k.$$

The length inequality in the other direction gives $\ell(sw) = k - 1$. If $\epsilon_s(w) = 0$, apply this argument to sw, since $\epsilon_s(sw) = 1$, to obtain $\ell(sw) = k + 1$. These are the exchange condition and the descent rule. The exchange characterization of Coxeter systems gives the assertion, including (2.4). □

Lemma 5.2. *The full tensor subcategory $\mathcal{D}_0 = \langle p, m, q\rangle$ has the based ring of type A_3 and is tensor equivalent to $\mathcal{C}_0(A_3)$, with these three marked generators.*

Proof. Apply Lemma 5.1. By (5.11) and (5.13), the Coxeter diagram is a path with three vertices. The actual involution T interchanges the endpoints and fixes the middle vertex, so its two noncommuting exponents are equal, say $k \geq 3$. Its reflection Gram matrix is

$$G = \begin{pmatrix} 1 & -\cos(\pi/k) & 0 \\ -\cos(\pi/k) & 1 & -\cos(\pi/k) \\ 0 & -\cos(\pi/k) & 1 \end{pmatrix}.$$

The matching group is finite, so G is positive definite. One may see this without a classification list: average a positive inner product over the finite reflection group. The -1 eigenspace of each simple reflection is orthogonal to its fixed hyperplane. The averaged form is consequently a positive scalar multiple of the corresponding row of G on each simple-root line; symmetry and connectedness make the scalar common to all three lines. Thus G is positive definite.

Its determinant is $1 - 2\cos^2(\pi/k)$. It is positive only for $k = 3$. The diagram is A_3. Since $\mathcal{D}$ is already positively spherical, each of its three rank-two tensor subcategories is the positive Fibonacci branch. The hypotheses of Theorem 2.2 are now satisfied, and give the required tensor equivalence. □

The simple dimensions in $\mathcal{D}_0$ are $\varphi^{\ell(w)}$, $w \in \mathfrak{S}_4$. In particular $\mathcal{D}_0$ has no nontrivial invertible object, and

$$\operatorname{FPdim} \mathcal{D}_0 = (1+t)(1+t+t^2)(1+t+t^2+t^3)\big|_{t=\varphi^2} = 540 + 240\sqrt{5}. \tag{5.16}$$

Since $h \neq \mathbf{1}$, it is outside $\mathcal{D}_0$. It normalizes $\mathcal{D}_0$ by (5.8)–(5.12). The two components $\mathcal{D}_0$ and $\mathcal{D}_0 h$ have no common simple: otherwise rigidity would imply that h is a constituent of a product in $\mathcal{D}_0$. Their dimensions sum to (5.4). Hence

$$\mathcal{D} = \mathcal{D}_0 \oplus \mathcal{D}_0 h. \tag{5.17}$$

5.6. **Marked automorphisms and coherence of the action.** An equivalence class determined by marked seeds need not have a trivial marked automorphism group. We prove the additional rigidity required here.

Lemma 5.3. *The tensor automorphism 2-group of $\mathcal{C}_0(A_3)$ fixing its three marked Fibonacci seeds is trivial.*

Proof. Fix the seed multiplication and splitting maps

$$\mu_s : Y_s^2 \to Y_s, \qquad \Delta_s : Y_s \to Y_s^2, \qquad \mu_s\Delta_s = 1.$$

If s is a left descent of w, choose an isomorphism $j : Y_w \to Y_sY_{sw}$. The stationary projection $Y_sY_w \to Y_w$ is the composite

$$\begin{aligned} Y_sY_w \xrightarrow{1\otimes j} Y_s(Y_sY_{sw}) \xrightarrow{a^{-1}} (Y_sY_s)Y_{sw} \\ \xrightarrow{\mu_s\otimes 1} Y_sY_{sw} \xrightarrow{j^{-1}} Y_w. \end{aligned} \tag{5.18}$$

The stationary inclusion is its reverse construction with Δ_s. Rescaling j cancels between the first and last occurrences, so these maps are intrinsic to the seed structure and simple labels. A marked autoequivalence preserves them.

For a two-generator parabolic subgroup, let Y_o be its longest simple object. Let A_J, B_J denote its two reduced alternating words, and let $\beta_J : A_J \to B_J$ be the normalized braid. Composing stationary inclusions on Y_o along A_J and stationary projections along B_J gives

$$P_{B_J}(\beta_J \otimes 1)I_{A_J} = \kappa_J 1_{Y_o}, \qquad \kappa_J = \begin{cases} 1, & m_J = 2, \\ -\varphi^{-1}, & m_J = 3. \end{cases} \tag{5.19}$$

For an explicit check in the orthonormal gauge, when $m_J = 3$ write $o = sts = tst$. The all-stationary input under the braid has only the following two output paths, in the t, s, t path basis:

$$(\beta_J \otimes 1)I_{A_J} = \sqrt{\alpha}\, I_{(o,st,st,o)} - \alpha\, I_{(o,o,o,o)}.$$

The all-stationary projection selects the second coefficient. For $m_J = 2$ the all-stationary input maps to the all-stationary output with coefficient 1. These evaluations follow by multiplying the blocks (2.3); both scalars in (5.19) are nonzero.

The source and target of β_J are simple, so a marked autoequivalence sends it, after object comparison, to $c\beta_J$. It preserves both the stationary maps and the scalar in (5.19). Thus $c\kappa_J = \kappa_J$, which forces $c = 1$.

The chosen comparisons on the three seeds define a tensor comparison on all words. It is natural for the seed maps and, by the preceding argument, for every pairwise braid. Tits reduction using these braids and the splitting $Y_s^2 = \mathbf{1} + Y_s$ decomposes every word into reduced-word simples. Matsumoto's theorem joins every reduced word to a fixed representative. Since distinct simples have zero Hom and each simple has scalar endomorphisms, these reductions, their inclusions and projections, and braid maps generate all matrix units. The comparison is therefore natural for all morphisms, and extends to sums and retracts. Its uniqueness follows from seed generation. The same argument, or the scalar equations on each Fibonacci generator, gives $\mathrm{Aut}_\otimes(\mathrm{Id}_{\mathcal{D}_0}) = 1$. □

Let σ be the standard endpoint exchange on $\mathcal{C}_0(A_3)$. Transport $T|_{\mathcal{D}_0}$ through Lemma 5.2. It has the same permutation of marked seeds as σ. Lemma 5.3 gives a monoidal comparison between them. Its C_2 compatibility is automatic: the discrepancy between the two squared comparisons is a monoidal automorphism of the identity, hence is trivial. We have therefore identified the actual coherent action, not only its permutation of simples.

Proposition 5.4. *There is a tensor equivalence*

$$ {}_A\mathcal{C}_A \simeq \mathcal{C}_0(A_3) \rtimes_\sigma C_2. $$

It induces a ribbon equivalence

$$ \mathcal{Z}(\mathcal{A}_c) \simeq_{\text{ribbon}} \mathcal{Z}\big(\mathcal{C}_0(A_3) \rtimes_\sigma C_2\big) \tag{5.20} $$

for the inherited pivot on the left and the positive pivot on the right.

Proof. Before replacing T by σ, the crossed-product functor is

$$ \Psi(X, i) = X \otimes_A h_i. $$

Its tensorator uses (5.7) and the interchange

$$ \begin{aligned} h_i \otimes_A Y &\longrightarrow T^i(Y) \otimes_A h_i, \quad a \otimes y \longmapsto \tau^i(a)y \otimes 1, \\ T^i(Y) \otimes_A h_i &\longrightarrow h_i \otimes_A Y, \qquad y \otimes b \longmapsto 1 \otimes y\tau^i(b). \end{aligned} \tag{5.21} $$

The two maps are balanced, bimodule-linear, and mutually inverse. The pentagon is the algebra associativity identity following (5.7). By (5.17) this functor exhausts the category and is fully faithful on both components. The comparison of actions just proved replaces T by σ.

Braided Morita equivalence gives an equivalence of centers. Transporting the positive pivot along it gives positive simple dimensions, so uniqueness of the positive spherical structure makes the equivalence pivotal, hence ribbon. On $\mathcal{C}$ the positive structure is the actual inherited one by (5.5). Finally Lemma 2.1 identifies its center, with that pivot and its traces, with the center of the full asymptotic cell. This proves (5.20). □

6. An exact regular model

We now specify the model used to compute the right-hand side of (5.20). All constructions in this section are independent of the Fourier target. The model is the regular realization of Theorem 2.2, followed by its strict diagram action. The description also fixes all basis and orientation conventions needed to reproduce the calculation.

6.1. **Vertices, words, and paths.** Write a permutation of $\{0, 1, 2, 3\}$ as a tuple, with composition $uv(i) = u(v(i))$. In this computational section we renumber the three A_3 generators as s_0, s_1, s_2, the adjacent transpositions. Enumerate $\mathfrak{S}_4$ by breadth-first search starting with the identity, appending right generators in the order s_0, s_1, s_2. Keep the first word reaching each permutation as its chosen reduced word. Coxeter length is the number of inversions.

The 48 simple labels of the crossed product are

$$ i = (w, g), \qquad w \in \mathfrak{S}_4, \quad g \in \{0, 1\}. \tag{6.1} $$

The first 24 labels, numbered $0, \dots, 23$, have $g = 0$ in the above order, and the next 24 have $g = 1$ in the same order. Thus label 47 is $(w_0, 1)$. Set

$$ \ell(w, g) = \ell(w), \quad \sigma(w)(i) = 3 - w(3 - i), \quad (w, g)^* = (\sigma^g(w^{-1}), g), \quad d_{(w,g)} = \varphi^{\ell(w)}. $$

The chosen word for (w, g) is the chosen word for w, followed by h if $g = 1$.

For an ordinary generator s, the nonstationary left and right steps are

$$ L_s(w, g) = (sw, g), \qquad R_s(w, g) = (w\sigma^g(s), g). \tag{6.2} $$

If the step decreases length, there are two output summands, ordered as the shorter vertex and then the original vertex. If it increases length, there is only the moved vertex. The steps for the invertible generator are

$$ L_h(w, g) = (\sigma(w), 1 - g), \qquad R_h(w, g) = (w, 1 - g), $$

with no stationary summand.

A path records the starting vertex and every successive output of a list of left or right steps. A word in tensor order acts from right to left. Its value at the identity is the vector space with basis all such paths starting at the identity; it is decomposed by terminal vertex. A reduced word has one identity-starting path, ending at its simple label.

6.2. **Interchange and propagation of morphisms.** To move R_r followed by L_s past one another, fix the starting and terminal vertices. A one-dimensional path space has interchange coefficient 1. A two-dimensional path space occurs exactly at a diamond with

$$sw = w\sigma^g(r), \qquad \ell(sw) < \ell(w),$$

and terminal vertex equal to the starting vertex. Order both intermediate vertices as (shorter, original). In the unitary gauge the interchange is (2.3).

For exact arithmetic set $z = \sqrt{\alpha}$, and rescale every path vector by $z^{n(p)}$, where $n(p)$ is its number of strict length decreases. This is a global change of path basis. The exceptional block becomes

$$U' = \begin{pmatrix} \alpha & 1 \\ \alpha & -\alpha \end{pmatrix}, \qquad (U')^2 = I. \tag{6.3}$$

For example, on the two paths through a diamond the diagonal change is $\mathrm{diag}(z, 1)$, so $U' = \mathrm{diag}(z, 1)^{-1} U \,\mathrm{diag}(z, 1)$. In reversing an interchange one uses U' itself, not its transpose. A step involving h has only a one-dimensional interchange. The diagram involution preserves lengths and descent counts, so the gauge is compatible with it.

A morphism between words is specified by a matrix between their identity-starting path spaces, block diagonal in the terminal vertex. Its value at any vertex w (including the full pair (w, g) of (6.1)) is determined as follows. Prepend the unique right path from the identity along its chosen word, including the final h when $g = 1$. Move that right word across the left word, using (6.3), apply the specified matrix at the identity, and move the right word back. Denote these interchange products by $\Gamma_{v,w}$. In this notation the propagated matrix is

$$f_w = \Gamma_{v',w}^{-1}(f_e \otimes 1)\Gamma_{v,w}, \tag{6.4}$$

with the unique initial right path suppressed at the beginning and end. This formula completely specifies composition and whiskering: composition is matrix multiplication, and whiskering applies (6.4) to the appropriate subpath.

The regular reconstruction in [8, §§4–6] proves that this propagation gives the morphisms in the simultaneous right-module endofunctor category and recovers the category generated by the left colors. In particular the tensor product is composition of endofunctors, so its associativity is intrinsic. The action of h is the actual permutation functor σ, whose square and action tensorator are identities. This realizes precisely the action identified in Proposition 5.4.

Fusion inclusions and projections are path matrix units. More explicitly, for simples s, t, enumerate identity-starting paths in their concatenated chosen words. For a path p ending at u, let $v_p : u \to st$ take the unique reduced path of u to p, and let $p_p : st \to u$ be the reverse matrix unit. Then

$$p_p v_q = \delta_{pq} 1_u, \qquad \sum_p v_p p_p = 1_{st}, \tag{6.5}$$

with the first equation understood for a common terminal label. These are the splittings used below, including when fusion has multiplicities.

6.3. **Duality and the ground field.** There is a unique unit-ending path in the words i^*i and ii^*. In the rational path gauge define evaluation to have coefficient 1 on the first, and coevaluation to have coefficient d_i on the second:

$$E_i : i^*i \to \mathbf{1}, \quad E_i = 1; \qquad C_i : \mathbf{1} \to ii^*, \quad C_i = d_i. \tag{6.6}$$

These are the gauge transforms of the standard positive duality maps. The two snake equations are also checked exactly for each of the 48 labels, yielding 96 identities. Consequently the dimension in this gauge is d_i, with the positive pivotal normalization already fixed.

Every coefficient of the resulting compositions lies in

$$K = \mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{5}).$$

Represent $p + q\alpha$ by (p, q). Its product with $r + s\alpha$ is

$$(p, q)(r, s) = (pr + qs,\ ps + qr - qs). \tag{6.7}$$

The raw path and tube products have integral pairs; linear algebra may introduce rational denominators. A matrix $P + \alpha Q$ over K is represented over $\mathbb{Q}$, grouping the rational and α coordinates, by

$$\begin{pmatrix} P & Q \\ Q & P - Q \end{pmatrix}. \tag{6.8}$$

Thus exact rational row reduction verifies all asserted ranks over K, with the rank doubled. Integer bounds are checked before fixed-width contractions, and rational elimination uses arbitrary-precision integers.

Remark 6.1. The original unitary gauge proves the existence of a positive $*$-structure on this particular reference category and its crossed product. The rational gauge is a change of basis of that category. Unitarity here follows from the reference construction, not from an assertion that arbitrary pseudo-unitary categories are unitarizable.

7. The complete tube center

7.1. **Tube multiplication.** Let $\mathcal{T}$ be the tube category of the model in Section 6. For simple boundary labels i, j set

$$\mathcal{T}(i, j) = \bigoplus_s \mathrm{Hom}(s \otimes i, j \otimes s). \tag{7.1}$$

Use the path matrix units as a basis in each summand. If $f : sj \to ks$ and $g : ti \to jt$, their tube composite is

$$f \cdot g = \sum_{u,p} (1_k \otimes p_p)(f \otimes 1_t)(1_s \otimes g)(v_p \otimes 1_i), \tag{7.2}$$

where $v_p : u \to st$ and $p_p : st \to u$ are (6.5). Each summand is evaluated by (6.4). The full finite tube algebra is the endomorphism algebra of the sum of all 48 boundaries. It is a semisimple $*$-algebra, because the model is unitary before the change of gauge. Its simple modules are in bijection with $\mathrm{Irr}\, \mathcal{Z}(\mathcal{D})$.

Boundary parity is preserved in (7.1), so there are two separate tube sectors. Take selected boundaries

$$J_0 = \{0, 2, 5, 11\}, \qquad J_1 = \{24, 26, 29, 35\}.$$

They are fixed by the diagram involution. In either sector the even-winding part E_g of the selected-boundary algebra has dimension 153. The odd winding is obtained by multiplication by the actual h loop U, with $U^2 = 1$. The selected algebra is therefore $E_g \rtimes C_2$, of dimension 306.

Let H denote its actual involution on E_g. A central element is $a + bU$, where

$$ax = xa, \quad H(a) = a, \qquad bH(x) = xb, \quad H(b) = b \qquad (x \in E_g). \tag{7.3}$$

Solving these equations over K gives

Boundary sector	$\dim E_g$	ordinary invariant part	twisted invariant part	center
$g = 0$	153	23	17	40
$g = 1$	153	16	16	32

The rank certificate consists of an exact row reduction together with verification of every equation (7.3) on the resulting kernel. Selecting a smaller collection of equations to find a candidate kernel cannot affect the conclusion: its dimension gives an upper bound for the full kernel, and substitution into all equations gives the matching lower bound. Multiplication in these two commutative centers is then computed using (7.2).

7.2. **A completeness criterion for partial tubes.** The rank $40 + 32$ obtained so far is not the rank of the full center. The following elementary fact explains both the extension calculation and the remaining ideal.

Lemma 7.1 (Full frames and invisible blocks)**.** *Let T be a finite-dimensional semisimple algebra with orthogonal boundary idempotents e_i, and let e be a sum of selected boundary idempotents. In the corner $T_i = e_iTe_i$, the ideal of blocks invisible to eTe is*

$$I_i = \{x \in T_i : x(e_iTe) = 0\}.$$

If a collection of arrows $\beta_r \in e_iTe_{j_r}$ from already visible boundaries has an injective action map

$$T_i \longrightarrow \bigoplus_r e_iTe_{j_r}, \qquad x \longmapsto (x\beta_r)_r, \tag{7.4}$$

then $I_i = 0$. Central elements on the selected algebra extend uniquely to that corner by the equations

$$z_i\beta_r = \beta_r z_{j_r}. \tag{7.5}$$

Proof. Decompose T as a sum of full matrix algebras. On a summand, a nonzero e detects its unique simple module, and e_iTe is either zero or the full rectangular matrix space for that summand. Its annihilator in e_iTe_i is therefore precisely the direct sum of summands not detected by e. If (7.4) is injective, no such summand survives. A central element of eTe is a scalar on each detected block, and extends with the same scalar to the ideal TeT. Its restriction is a solution of (7.5); injectivity makes the solution unique. The same argument applies inductively when additional already visible boundaries are used as sources. □

For each nonselected boundary except 47, exact frames give full column rank in (7.4). The 39 resulting extensions are unique by Lemma 7.1. The dimensions of all corners and the rational ranks of these frame maps are listed in Appendix B. Together with the eight selected boundaries they account for every boundary other than 47.

7.3. **The missing eight-dimensional ideal.** The corner at $47 = (w_0, 1)$ has dimension 778. Its frame map, using the first seven basis arrows in $\mathcal{T}(45, 47)$, has 2632 equations over K. Its rational lift has rank 1540, leaving a kernel of dimension

$$778 - \frac{1540}{2} = 8$$

over K. To identify this as the actual invisible ideal, rather than the annihilator of too few arrows, the calculation verifies the following surjections:

Map being certified	target dimension over K	rational rank
Seven arrows times T_{45} onto $\mathcal{T}(45,47)$	376	752
Factorization $24 \to 45 \to 47$ onto $\mathcal{T}(24,47)$	26	52
Factorization $26 \to 45 \to 47$ onto $\mathcal{T}(26,47)$	68	136
Factorization $29 \to 45 \to 47$ onto $\mathcal{T}(29,47)$	64	128
Factorization $35 \to 45 \to 47$ onto $\mathcal{T}(35,47)$	128	256

The first line proves that the kernel annihilates all of $\mathcal{T}(45,47)$. The other four lines prove that it annihilates every module from a selected odd boundary. Conversely, boundary 45 is already completely visible, so annihilation of the selected-boundary modules implies annihilation of $\mathcal{T}(45,47)$. The kernel is exactly the ideal I_{47} in Lemma 7.1. In particular it is two-sided. These rank assertions use exact nonsingular minors and full substitution; numerical pivoting is used only to propose the minors.

We describe the unit of this ideal, because an arbitrary nullspace basis does not supply one. Let K_0 be the 778 by 8 matrix of its basis columns. The positive tube trace gives a diagonal metric G in the path matrix-unit basis. For a basis tube indexed by (s,p,q) at boundary i, with terminal vertex v, its diagonal entry in the rational gauge is

$$G_{(s,p,q)} = \varphi^{\ell(i)+\ell(v)-\ell(s)-(n(q)-n(p))} > 0. \tag{7.6}$$

The orthogonal projection of the corner unit onto I_{47} is

$$e_I = K_0(K_0^t G K_0)^{-1} K_0^t G 1_{47}. \tag{7.7}$$

In a semisimple $*$-algebra with a faithful positive trace, different simple ideals are orthogonal. Therefore (7.7) is the actual central unit of I_{47}. Subtracting this projection from a particular visible extension gives the unique extension supported on the complementary ideal.

Restrict the left actions of three explicit corner basis elements, numbered $1,2,3$ in the zero-based path enumeration, and the actual annular twist to I_{47}. Their generated algebra has dimension 8 and its center has dimension 2. The twist T_I commutes with these generators and satisfies $T_I^2 = 1$. The projections

$$p_+ = \tfrac{1}{2}(e_I + T_I e_I), \qquad p_- = \tfrac{1}{2}(e_I - T_I e_I) \tag{7.8}$$

have left-regular ranks 4 each. Thus

$$I_{47} \otimes_K \mathbb{C} \cong M_2(\mathbb{C}) \oplus M_2(\mathbb{C}). \tag{7.9}$$

An ambient corner element a acts on I_{47} by left multiplication by $ae_I \in I_{47}$. These ideal actions are faithful: an element of a unital ideal acting trivially on the ideal vanishes on its unit. Since the generated algebra has the full dimension 8, its center is the center of the ideal. This justifies the use of the two projections in (7.8) as its primitive central idempotents.

The two resulting central objects have dimensions $18 + 8\sqrt{5}$ and twists $1, -1$, respectively, as obtained from the tube trace formula below. They have support only at boundary 47. Consequently the full center has rank

$$40 + 32 + 2 = 74. \tag{7.10}$$

This rank is obtained from the actual algebra and completeness argument, without using the predicted Fourier rank.

7.4. **Twist and quarter-turn on the entire center.** The annular twist at boundary i is the tube with winding i^* given by the cap followed by the cup in (6.6). It is a single matrix unit with coefficient d_i. The three central components have twist orders $60, 30, 2$. These are verified by exact multiplication in their center algebras. Their restrictions reproduce the actual unit and annular twist at all 48 boundaries.

For a diagonal tube $f : si \to is$, ordinary clockwise bending is the map $i^*s \to si^*$ given by

$$\mathrm{rot}(f) = (E_i \otimes 1_s \otimes 1_{i^*})(1_{i^*} \otimes f \otimes 1_{i^*})(1_{i^*} \otimes 1_s \otimes C_i). \tag{7.11}$$

Its contribution to the central quarter-turn is weighted by d_i/d_s. This is the tube normalization of [7, §2.3, (2.2), (2.4), (2.5)]. Using the complete corner restrictions, apply this formula to all tubes whose new boundary lies in

$$\{0, 2, 5, 11, 24, 26, 29, 35, 47\}.$$

There are 1046 such basis tubes. Their restrictions detect the whole 74-dimensional center; the rational lift of the coordinate matrix has rank 148. Solving for the rotated coordinates gives an exact matrix $R \in M_{74}(K)$ with

$$R^2 = I, \qquad R^4 = I, \qquad T^{60} = I. \tag{7.12}$$

All coordinate equations, including the restrictions of the unit and twist, are checked exactly. Thus R is the quarter-turn on the complete center, rather than on a chosen subspace of it.

7.5. **From primitive idempotents to normalized S.** Let P have as columns the primitive central idempotents in the computed rational center basis, and let d_Z be their positive categorical dimensions. The normalized torus vector associated to a primitive idempotent is $\sqrt{D}\, p_Z/d_Z$. Therefore the clockwise rotation matrix in the normalized simple-object basis is

$$S_{\mathrm{cw}} = \mathrm{diag}(d_Z) P^{-1} R P\, \mathrm{diag}(d_Z^{-1}). \tag{7.13}$$

Our convention (1.1) uses the inverse orientation, so $S = S_{\mathrm{cw}}^{-1}$. Equation (7.12) makes the two orientations equal here.

The dimensions in (7.13) are determined from the actual tube trace. If $m_i(Z)$ is the multiplicity of boundary i in the tube module of Z, and $c_i(Z)$ is the coefficient of the identity-winding tube at i in p_Z, then

$$d_Z = \frac{D\, d_i\, c_i(Z)}{m_i(Z)} \qquad (m_i(Z) \neq 0). \tag{7.14}$$

In the selected-object algebra, let the full matrix block of p_Z have size n_Z. Its regular trace is n_Z^2, and the regular trace after multiplying by the boundary idempotent e_i is $m_i(Z) n_Z$. These integer traces determine n_Z and $m_i(Z)$, and then (7.14) determines d_Z. In the missing ideal $n_Z = m_{47}(Z) = 2$, giving the stated dimension $18 + 8\sqrt{5}$. No target Fourier dimension is used in this normalization.

8. An exact Fourier comparison

8.1. **Certified primitive idempotents.** The preceding section constructs the full center as a commutative algebra over K, together with its unit, multiplication, boundary trace functionals, actual twist, and actual rotation. Its primitive idempotents need not themselves have coordinates in K. We do not attempt to recognize those coordinates from floating-point values.

Lemma 8.1. *Let B be a commutative semisimple unital complex algebra of dimension r. Suppose a regular multiplication operator has r distinct eigenvalues, and let $v_1, \ldots, v_r$ be corresponding eigenvectors. If*

$$1 = \sum_j a_j v_j,$$

then $a_j \neq 0$ for every j, and $p_j = a_j v_j$ are precisely the primitive idempotents of B.

Proof. Each eigenspace is a one-dimensional ideal because every regular multiplication commutes with the chosen operator. Products of different eigenlines are zero. Multiplying the displayed unit decomposition by v_j gives $v_j = a_j v_j^2$, so a_j cannot vanish. It then gives $p_j^2 = p_j$. The p_j are nonzero orthogonal idempotents summing to the unit, and their one-dimensional ideals make them primitive. □

Choose an integral linear combination of the rational center basis whose regular multiplication has simple spectrum. An Arb Rump eigensystem calculation encloses all its eigenvalues and right eigenvectors, with pairwise disjoint eigenvalue balls. Solve for the unit in the enclosed eigenvector matrix and apply Lemma 8.1. Every solve must be certified nonsingular and every normalization coefficient must exclude zero. This gives enclosures of the actual primitive columns P, not arbitrarily normalized eigenlines.

Apply the exact trace functionals from Section 7 to these columns. Each full regular trace is a positive integer square n_Z^2, and each boundary regular trace is the nonnegative integer $m_i(Z)n_Z$. A ball is accepted only when it isolates one integer with the required divisibility and square properties. These integrality facts follow from the exact semisimple algebra, not from the enclosure. Formula (7.14) then encloses the positive dimension. Values from different detecting boundaries can be intersected because they are already known to represent the same dimension by the tube trace theorem.

Similarly, the exact equation $T^{60} = I$ restricts each primitive twist to a sixtieth root of unity. A twist label is accepted only if its certified enclosure meets exactly one of those roots. Finally (7.13), computed with certified linear solves and positive dimension enclosures, encloses every entry of the actual S. The Fourier matrix is not an input to any of these steps.

8.2. **A separation bound.** Let F be the exact matrix `HS4` in Appendix A. It has the form

$$F = \frac{A + B\sqrt{5}}{120}, \qquad A, B \in M_{74}(\mathbb{Z}). \tag{8.1}$$

The integer arrays satisfy

$$A^t = A, \quad B^t = B, \quad A^2 + 5B^2 = 14400I, \quad AB + BA = 0. \tag{8.2}$$

Thus both real embeddings of F are orthogonal matrices. In particular every conjugate of every entry has absolute value at most 1. Also

$$DF = (9A + 20B) + (4A + 9B)\sqrt{5} \tag{8.3}$$

has algebraic-integer entries.

Lemma 8.2. *For the actual normalized center matrix S constructed above, if $S_{ij} \neq F_{kl}$ then*

$$|S_{ij} - F_{kl}| \geq \frac{1}{128 \cdot 14400^4}.$$

Proof. By (7.12) the exponent of the actual ribbon twist divides 60. The normalized-center theorem [17, Proposition 5.7 and Remark 5.8] gives

$$S_{ij} \in L := \mathbb{Q}(\zeta_{60}), \qquad [L : \mathbb{Q}] = 16.$$

Balancing expresses DS_{ij} as a sum of integer fusion multiplicities times simple dimensions times roots of unity; hence it is an algebraic integer. For instance, since $R^2 = I$ makes all central simples self-dual, the balancing formula in the present labeling may be written

$$DS_{ij} = \theta_i^{-1}\theta_j^{-1} \sum_k N_{ij}^k \theta_k d_k.$$

The equivalent dual-index version of this formula gives the same conclusion. The d_k are Frobenius–Perron dimensions of simples and are algebraic integers.

The reference category is unitary, so S is unitary. Every automorphism of the abelian extension $L/\mathbb{Q}$ commutes with complex conjugation. Applying it to $S\overline{S}^{t} = I$ shows that every Galois conjugate of S is also unitary. Thus every conjugate of any entry has absolute value at most 1.

Put $\delta = S_{ij} - F_{kl} \neq 0$. By (8.3), $D\delta$ is a nonzero algebraic integer in L, so its absolute field norm is an integer at least 1. The norm of D is

$$|N_{L/\mathbb{Q}}(D)| = \left((1080 + 480\sqrt{5})(1080 - 480\sqrt{5})\right)^8 = 14400^8.$$

The identity embedding and complex conjugation contribute $|\delta|^2$. Each of the other fourteen embeddings contributes at most 2, by the entry bounds for S and F. Therefore

$$1 \leq |N_{L/\mathbb{Q}}(D\delta)| \leq 14400^8\,|\delta|^2\,2^{14}.$$

Taking square roots proves the stated bound. □

Only the final normalized entries are required to lie in L. The primitive idempotent coordinates, eigenvectors used to compute them, and intermediate algebraic extensions need not lie in that cyclotomic field.

8.3. **The certified equality.** For reproducibility, use eigensystem precision 512 bits, final comparison precision 768 bits, and the separating weights produced by a fixed pseudorandom seed 202611110715, each between -9 and 9, in the rational basis ordered as the 40-dimensional center, the 32-dimensional center, and then p_+, p_-. Randomness is not part of the proof: the accepted eigensystem is certified, and the actual chosen integer weights are saved with the certificate. A different certified separating element gives the same primitive set, possibly in a different order.

A tentative permutation may be proposed from numerical dimension, twist, and matrix invariants. It is accepted only after every one of its 74^2 comparisons is certified. In the spectral order of the stated run, the one-based Fourier labels are the permutation π printed in Appendix A. The verification gives

$$\max_{1\leq i,j\leq 74} |S_{ij} - F_{\pi(i),\pi(j)}| < 3.422 \cdot 10^{-140} < \frac{1}{5503765708800000000}. \tag{8.4}$$

The first inequality is an upper bound for complex balls, including both real and imaginary error; it is not a comparison of rounded midpoints. Decimal serialization is outward rounded and preserves the enclosures.

Proposition 8.3. *The normalized matrix of $\mathcal{Z}(\mathcal{C}_0(A_3) \rtimes_\sigma C_2)$ is exactly F after the simultaneous permutation π. The tensor unit corresponds to Fourier label* 9.

Proof. For any pair (i, j), a nonzero difference in (8.4) would contradict Lemma 8.2. All 5476 entries therefore agree exactly. The certified permutation is used on both indices.

The row at Fourier label 9 is strictly positive. In a positive spherical center the tensor-unit row is also strictly positive. A unitary matrix cannot have two distinct rows with all entries positive, since their inner product would be positive instead of zero. Hence the label-9 row is the tensor-unit row. This identifies the unit without inferring an algebraic equality merely because a dimension interval contains 1. □

Combining Proposition 8.3 with the ribbon equivalence (5.20) proves Theorem 1.2 for the middle H_4 cell.

Remark 8.4 (Nature of the finite proof). The arithmetic certificate uses three distinct kinds of exact assertion: algebraic identities and ranks over K; integer or root-of-unity isolation supported by an a priori theorem; and the separation bound of Lemma 8.2. Interval overlap alone is never used to establish an unrestricted algebraic equality. The files and commands in Appendix B implement these assertions and expose their inputs.

9. All cells and completion of the proof

9.1. **Dihedral groups.** For $I_2(m)$, $m \geq 3$, the two singleton cells have corners Vec. The nontrivial two-sided cell has $a = 1$; its diagonal corner is the adjoint part of the standard A_{m-1} fusion-rule category, by [20, Theorem 4.8]. The center computation and the Fourier comparison, including Convention 1.1, are [20, Theorem 4.21]. The precise result being used is equality of the normalized center S with the converted Fourier matrix for this actual asymptotic corner; thus it includes the tensor-category identification and its specified pivotal normalization.

For reference, its diagonal corner has rank $\lfloor m/2 \rfloor$, and the nontrivial center has rank

$$\begin{cases} ((m-1)/2)^2, & m \text{ odd}, \\ m(m-2)/4 + 2, & m \text{ even}. \end{cases}$$

Together with the two singleton cells this proves the theorem for all dihedral types. This is the established dihedral input, not a deduction from the new H_4 calculation.

9.2. **Types H_3 and H_4.** The complete two-sided cell data and diagonal corner ranks are given in [20, Example 5.1]. The following tables record every cell; rows with several a-values denote distinct cells of the displayed common type. The rank of the center is recorded for each individual cell, not as a sum over the row.

Type	$a(c)$	corner rank	corner or Morita representative	center rank
H_3	$0, 2, 5, 15$	1	Vec	1
	$1, 6$	2	Fib	4
	3	2	$\mathrm{Vec}_{C_2}^{-}$	4
H_4	$0, 4, 5, 15, 16, 60$	1	Vec	1
	$1, 2, 22, 31$	2	Fib	4
	$3, 18$	2	$\mathrm{Vec}_{C_2}^{-}$	4
	6	24	$\mathcal{C}_0(A_3) \rtimes_\sigma C_2$ (Morita)	74

For the last row the representative has rank 48; the displayed corner rank 24 is the actual Alvis corner before Morita duality.

Proof of Theorem 1.2. The irreducible finite noncrystallographic types are $I_2(m)$, H_3, and H_4. The dihedral case is the preceding theorem of Rogel–Thiel. For H_3 and H_4, the tables exhaust respectively seven and thirteen two-sided cells. A rank-one corner has matrix (1). For every Fibonacci corner, Proposition 3.1 selects the positive tensor branch and (4.4) is the required modular Fourier matrix. For the three order-two pointed corners, Proposition 4.2, the cell representatives (4.2), and (4.3) give the actual inherited-pivot matrices.

In all these small cases Lemma 2.1 transfers the calculation from the diagonal corner to the full cell without altering the pivot or the trace. The only remaining row is H_4, $a = 6$. Proposition 5.4 identifies its ribbon center with the crossed-product center, and Proposition 8.3 identifies the latter matrix with the exact 74 by 74 Fourier block. No cell remains outside these cases. □

Corollary 9.1. *Let $W = W_1 \times \cdots \times W_r$ be a genuine direct product of finite Coxeter systems and $c = c_1 \times \cdots \times c_r$ a two-sided cell. With the standard product realizations and inherited pivots,*

$$S_c = S_{c_1} \otimes \cdots \otimes S_{c_r}, \qquad F_c^{\mathrm{mod}} = F_{c_1}^{\mathrm{mod}} \otimes \cdots \otimes F_{c_r}^{\mathrm{mod}}.$$

Consequently the Fourier matrix assertion holds for arbitrary finite Coxeter groups, using Theorem 1.2 for their noncrystallographic irreducible factors and the established Weyl-group case for the other factors.

Proof. The Hecke category of a direct product is the external tensor product, cells are products, and the a-function is additive. Taking the lowest surviving degrees therefore gives

$$\mathcal{A}_c \simeq \mathcal{A}_{c_1} \boxtimes \cdots \boxtimes \mathcal{A}_{c_r}.$$

Ordinary duality and its pivotal map are componentwise, so this equivalence respects the inherited structures. Centers and traces are tensor products, and their positive normalization factors multiply, giving the first formula. The Fourier families and the specified indexing involutions are likewise products, giving the second. The Weyl-group center identification is the one in [13], as summarized in [20, §3.4]. □

The direct-product hypothesis in this corollary is essential to the argument: the presence of two commuting parabolic subgroups in a larger Coxeter group does not by itself assert a product decomposition of the ambient asymptotic category.

Appendix A. The complete Fourier block and labels

The following encoding specifies every entry of the 74 by 74 matrix used in Proposition 8.3. Rows and columns are in the one-based local order of the CHEVIE family `HS4`. Each character represents the pair (a, b) in the dictionary below, and hence the scalar $(a + b\sqrt{5})/120$. The dictionary is ordered lexicographically by the integer pair; it is merely a compact printing convention.

Code	a	b	Code	a	b	Code	a	b	Code	a	b	Code	a	b
`0`	-60	0	`D`	-14	6	`Q`	0	0	`d`	6	6	`q`	20	-4
`1`	-40	-8	`E`	-12	0	`R`	0	4	`e`	8	0	`r`	20	0
`2`	-40	0	`F`	-10	0	`S`	0	8	`f`	9	-4	`s`	20	4
`3`	-36	0	`G`	-8	0	`T`	0	16	`g`	9	4	`t`	21	0
`4`	-30	0	`H`	-6	-6	`U`	2	0	`h`	10	0	`u`	24	-4
`5`	-24	0	`I`	-6	-2	`V`	3	0	`i`	12	0	`v`	24	0
`6`	-20	-4	`J`	-6	0	`W`	4	-4	`j`	14	-6	`w`	24	4
`7`	-20	0	`K`	-6	2	`X`	4	0	`k`	14	-2	`x`	30	0
`8`	-20	4	`L`	-6	6	`Y`	4	4	`l`	14	2	`y`	36	0
`9`	-18	0	`M`	-4	0	`Z`	6	-6	`m`	14	6	`z`	40	-8
`A`	-16	0	`N`	-2	0	`a`	6	-2	`n`	15	0	`@`	40	0
`B`	-15	0	`O`	0	-8	`b`	6	0	`o`	16	0	`#`	45	0
`C`	-14	-6	`P`	0	-4	`c`	6	2	`p`	18	0	`$`	60	0

Let $\theta_r = \exp(2\pi i\, t_r/60)$ denote the annular twist in the corresponding Fourier labeling. The exponent vector used for label certification is

$$\begin{aligned}(t_1, \ldots, t_{74}) = (&0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,36,24,36,\\ &24,36,24,36,24,0,30,30,30,0,0,30,30,0,30,36,24,36,24,\\ &6,54,6,54,15,45,20,40,10,50,48,12,48,12,18,42,18,42,\\ &20,40,20,40,48,12,48,12,8,52,32,28,48,12,48,12,30,0).\end{aligned}$$

The certified simultaneous permutation sends the rth primitive in the spectral order of the 512-bit run to Fourier label $\pi(r)$, where

$$\begin{aligned}(\pi(1),\dots,\pi(74)) = (&9,29,14,4,18,17,35,36,62,61,33,13,15,43,72,44,66,71,5,\\ &65,68,67,37,69,70,38,57,63,64,58,59,60,30,19,20,25,6,1,\\ &10,2,11,8,21,22,7,31,40,39,28,27,53,54,50,49,16,48,\\ &23,3,47,24,26,46,42,52,51,34,32,12,56,55,45,41,74,73).\end{aligned}$$

Thus the certified comparison is $S_{rs} = F_{\pi(r),\pi(s)}$. The tensor unit is the first primitive in this spectral order and Fourier label 9. These twist labels serve to specify this finite certificate; the theorem itself is the stated equality of the normalized S-matrices.

Rows 1–37. The two digits at the left are row numbers and are not matrix entries.

```
01 yQ3EEyiiiiEEEEvQiiEEEEiiQ3iiiiEEvQiiEEEEiiQQQQQQEEiiEEiiQQQQEEEEQQQQEEEEvv
02 Q#QQQQ44nnQQQQBxQQQQQQQQ#Q44nnQQBxQQQQQQQQ44QQQQQQQQQQQQQQQQQQQQQQQQQQQxx
03 3QyEEyEEiiiiii5QiiEEiiEEQ3iiEEEEvQEEiiEEiiQQQQQQiiEEEEiiQQQQEEEEQQQQiiiiv5
04 EQEuwEHLacMMwui7ddLLddLLQEHLacMMi7ddLLddLLQQrrrrZZHHZZHHPPPPKKIIOOOOIIKKee
05 EQEwuELHcaMMuwi7ZZHHZZHHQELHcaMMi7ZZHHZZHHQQrrrrddLLddLLRRRRIIKKSSSSKKIIee
06 yQyEEyEEiiiiEEvQiiEEiiEEQyEEiiiivQiiEEiiEEQQQQQQQiiEEiiEEQQQQEEEEQQQQEEEE55
07 i4EHLEvvbbZdZdpQddLLddLLxi55JJLH9QHHZZHHZZQQQQQQddLLHHZZQQQQLLHHQQQQddZZiE
08 i4ELHEvvbbdZdZpQZZHHZZHHxi55JJHL9QLLddLLddQQQQQQZZHHLLddQQQQHHLLQQQQZZddiE
09 iniacibbfgjmcaVhLLddLLddnibbfgjmVhLLddLLddxxrrrrLLddLLddRRRRaaccSSSSccaaUU
10 inicaibbgfmjacVhHHZZHHZZnibbgfmjVhHHZZHHZZxxrrrrHHZZHHZZPPPPccaaOOOOaaccUU
11 EQiMMiZdjmXXXXirLLddLLddQELHDCMME7ZZHHZZHHQQ77rrHHZZddLL77rrZZddQQQQHHLLeG
12 EQiMMidZmjXXXXirHHZZHHZZQEHLCDMME7ddLLddLLQQ77rrLLddZZHH77rrddZZQQQQLLHHeG
13 EQiwuEZdcaXXuwirZZHHLLddQiZdcaXXirZZHHLLddQQ7777HHZZHHZZRRRRIIKKSSSSKKIIGG
14 EQiuwEdZacXXwuirddLLHHZZQidZacXXirddLLHHZZQQ7777LLddLLddPPPPKKIIOOOOIIKKGG
15 vB5iivppVViiiitxEEiiiiEEB5ppVViitxEEiiiiEE44QQQQiiEEiiEEQQQQiiiiQQQQiiiibb
16 QxQ77QQQhhrrrrx@QQQQQQQQ4QQQFF7742QQQQQQQQQQrr77QQQQQQQQrr77QQQQQQQQQQQQ7r
17 iQidZidZLHLHZdEQLv5dLv5dQidZLHLHEQLv5dLv5dQQQQQQHLdZHLdZQQQQdZdZQQQQdZdZii
18 iQidZidZLHLHZdEQvLd5vLd5QidZLHLHEQvLd5vLd5QQQQQQLHZdLHZdQQQQZdZdQQQQZdZdii
19 EQELHELHdZdZHLiQ5dLv5dLvQELHdZdZiQ5dLv5dLvQQQQQQdZHLdZHLQQQQHLHLQQQQHLHLEE
20 EQELHELHdZdZHLiQd5vLd5vLQELHdZdZiQd5vLd5vLQQQQQQZdLHZdLHQQQQLHLHQQQQLHLHEE
21 EQidZidZLHLHLHiQLv5dLv5dQEHLZdZdEQZ5vHZ5vHQQQQQQHLdZdZHLQQQQdZdZQQQQHLHLEi
22 EQidZidZLHLHLHiQvLd5vLd5QEHLZdZdEQ5ZHv5ZHvQQQQQQLHZdZdLHQQQQZdZdQQQQLHLHEi
23 iQELHELHdZdZdZEQ5dLv5dLvQiZdHLHLiQvHZ5vHZ5QQQQQQdZHLHLdZQQQQHLHLQQQQdZdZiE
24 iQELHELHdZdZdZEQd5vLd5vLQiZdHLHLiQHv5ZHv5ZQQQQQQZdLHLHZdQQQQLHLHQQQQZdZdiE
25 Q#QQQQxxnnQQQQB4QQQQQQQQ#QxxnnQQB4QQQQQQQQ44QQQQQQQQQQQQQQQQQQQQQQQQQQQ44
26 3Q3EEyiiiiEEii5QiiEEEEiiQyEEEEiivQEEiiiiEEQQQQQQEEiiiiEEQQQQEEEEQQQQiiii5v
27 i4iHLE55bbLHZdpQddLLHHZZxEvvJJZd9QHHZZddLLQQQQQQHHZZddLLQQQQLLHHQQQQddZZEi
28 i4iLHE55bbHLdZpQZZHHLLddxEvvJJdZ9QLLddZZHHQQQQQQLLddZZHHQQQQHHLLQQQQZZddEi
29 inEaciJJfgDCcaVFLLddZZHHnEJJfgDCVFLLddZZHHxx7777ZZHHZZHHRRRRaaccSSSSccaaNN
30 inEcaiJJgfCDacVFHHZZddLLnEJJgfCDVFHHZZddLLxx7777ddLLddLLPPPPccaaOOOOaaccNN
31 EQEMMiLHjmMMXXi7LLddZZHHQiZdDCXXErZZHHLLddQQrr77ddLLHHZZ77rrZZddQQQQHHLLGe
32 EQEMMiHLmjMMXXi7HHZZddLLQidZCDXXErddLLHHZZQQrr77ZZHHLLdd77rrddZZQQQQLLHHGe
33 vBviiv99VVEEiit4EEiiEEiiBv99VVEEt4EEiiEEii44QQQQEEiiEEiiQQQQiiiiQQQQiiiiJJ
34 QxQ77QQQhh77rrx2QQQQQQQQ4QQQFFrr4@QQQQQQQQQQQ77rrQQQQQQQQrr77QQQQQQQQQQQQr7
35 iQEdZiHLLHZdZdEQLv5dZ5vHQEHLLHZdEQLv5dZ5vHQQQQQQdZHLdZHLQQQQdZdZQQQQdZdZEE
36 iQEdZiHLLHZdZdEQvLd55ZHvQEHLLHZdEQvLd55ZHvQQQQQQZdLHZdLHQQQQZdZdQQQQZdZdEE
37 EQiLHEZddZHLHLiQ5dLvvHZ5QiZddZHLiQ5dLvvHZ5QQQQQQHLdZHLdZQQQQHLHLQQQQHLHLii
```

Rows 38–74.

```
38  EQiLHEZddZHLHLiQd5vLHv5ZQiZddZHLiQd5vLHv5ZQQQQQQLHZdLHZdQQQQLHLHQQQQLHLHii
39  EQEdZiHLLHZdLHiQLv5dZ5vHQidZZdLHEQZ5vHLv5dQQQQQQdZHLHLdZQQQQdZdZQQQQHLHLiE
40  EQEdZiHLLHZdLHiQvLd55ZHvQidZZdLHEQ5ZHvvLd5QQQQQQZdLHLHZdQQQQZdZdQQQQLHLHiE
41  iQiLHEZddZHLdZEQ5dLvvHZ5QELHHLdZiQvHZ55dLvQQQQQQHLdZdZHLQQQQHLHLQQQQdZdZEi
42  iQiLHEZddZHLdZEQd5vLHv5ZQELHHLdZiQHv5Zd5vLQQQQQQLHZdZdLHQQQQLHLHQQQQZdZdEi
43  Q4QQQQQQxxQQQQ4QQQQQQQQQ4QQQxxQQ4QQQQQQQQQO$QQQQQQQQQQQQQQQQQQQQQQQQQQQQ
44  Q4QQQQQQxxQQQQ4QQQQQQQQQ4QQQxxQQ4QQQQQQQQQ$0QQQQQQQQQQQQQQQQQQQQQQQQQQQQ
45  QQQrrQQQrr7777QrQQQQQQQQQQQQ77rrQ7QQQQQQQQQQ7@2rQQQQQQQQ7@r2QQQQQQQQQQQQr7
46  QQQrrQQQrr7777QrQQQQQQQQQQQQ77rrQ7QQQQQQQQQQ@7r2QQQQQQQQ@72rQQQQQQQQQQQQr7
47  QQQrrQQQrrrr77Q7QQQQQQQQQQQQ7777QrQQQQQQQQQQ2r7@QQQQQQQQ@72rQQQQQQQQQQQQ7r
48  QQQrrQQQrrrr77Q7QQQQQQQQQQQQ7777QrQQQQQQQQQQr2@7QQQQQQQQ7@r2QQQQQQQQQQQQ7r
49  EQiZdidZLHHLHLiQHLdZHLdZQEHLZddZEQdZHLdZHLQQQQQQHv5Zd5vLQQQQ5Zd5QQQQHvvLEi
50  EQiZdidZLHHLHLiQLHZdLHZdQEHLZddZEQZdLHZdLHQQQQQQvHZ55dLvQQQQZ55dQQQQvHLvEi
51  iQEHLELHdZZdZdEQdZHLdZHLQiZdHLLHiQHLdZHLdZQQQQQQQ5ZHvvLd5QQQQHvvLQQQQ5Zd5iE
52  iQEHLELHdZZdZdEQZdLHZdLHQiZdHLLHiQLHZdLHZdQQQQQQQZ5vHLv5dQQQQvHLvQQQQZ55diE
53  EQEZdiHLLHdZHLiQHLdZdZHLQidZZdHLEQdZHLHLdZQQQQQQd5vLHv5ZQQQQ5Zd5QQQQHvvLiE
54  EQEZdiHLLHdZHLiQLHZdZdLHQidZZdHLEQZdLHLHZdQQQQQQ5dLvvHZ5QQQQZ55dQQQQvHLviE
55  iQiHLEZddZLHZdEQdZHLHLdZQELHHLZdiQHLdZdZHLQQQQQQvLd55ZHvQQQQHvvLQQQQ5Zd5Ei
56  iQiHLEZddZLHZdEQZdLHLHZdQELHHLZdiQLHZdZdLHQQQQQQLv5dZ5vHQQQQvHLvQQQQZ55dEi
57  QQQPRQQQRP77RPQrQQQQQQQQQQQQRP77QrQQQQQQQQQQ7@@7QQQQQQQQPSPSSSOOOTOTOOSS77
58  QQQPRQQQRP77RPQrQQQQQQQQQQQQRP77QrQQQQQQQQQQ@77@QQQQQQQQSPSPSSOOTOTOOOSS77
59  QQQPRQQQRPrrRPQ7QQQQQQQQQQQQRPrrQ7QQQQQQQQQQr22rQQQQQQQQPSPSSSOOOTOTOOSSrr
60  QQQPRQQQRPrrRPQ7QQQQQQQQQQQQRPrrQ7QQQQQQQQQQ2rr2QQQQQQQQSPSPSSOOTOTOOOSSrr
61  EQEKIELHacZdIKiQdZHLdZHLQELHacZdiQdZHLdZHLQQQQQQ5ZHv5ZHvSSSSkYWl6qq6WlkYEE
62  EQEKIELHacZdIKiQZdLHZdLHQELHacZdiQZdLHZdLHQQQQQQZ5vHZ5vHSSSSYklWq66qlWYkEE
63  EQEIKEHLcadZKIiQdZHLdZHLQEHLcadZiQdZHLdZHLQQQQQQd5vLd5vLOOOOWlkYs88skYWlEE
64  EQEIKEHLcadZKIiQZdLHZdLHQEHLcadZiQZdLHZdLHQQQQQQ5dLv5dLvOOOOlWYk8ss8YklWEE
65  QQQOSQQQSOQQSOQQQQQQQQQQQQQQQSOQQQQQQQQQQQQQQQQQQQQQQQQQQQQTOT6qs8sz81s86qQQ
66  QQQOSQQQSOQQSOQQQQQQQQQQQQQQQSOQQQQQQQQQQQQQQQQQQQQQQQQQQQQTOTOq68szs188sq6QQ
67  QQQOSQQQSOQQSOQQQQQQQQQQQQQQQSOQQQQQQQQQQQQQQQQQQQQQQQQQQQQTOTOq68s81sz8sq6QQ
68  QQQOSQQQSOQQSOQQQQQQQQQQQQQQQSOQQQQQQQQQQQQQQQQQQQQQQQQQQQQTOTO6qs818zss86qQQ
69  EQiIKEdZcaHLKIiQdZHLHLdZQidZcaHLiQdZHLHLdZQQQQQQHv5ZHv5ZOOOOWlkYs88skYWlii
70  EQiIKEdZcaHLKIiQZdLHLHZdQidZcaHLiQZdLHLHZdQQQQQQvHZ5vHZ5OOOOlWYk8ss8YklWii
71  EQiKIEZdacLHIKiQdZHLHLdZQiZdacLHiQdZHLHLdZQQQQQQvLd5vLd5SSSSkYWl6qq6WlkYii
72  EQiKIEZdacLHIKiQZdLHLHZdQiZdacLHiQZdLHLHZdQQQQQQLv5dLv5dSSSSYklWq66qlWYkii
73  vxvee5iiUUeeGGb7iiEEEEii45EENNGGJrEEiiiiEEQQrr77EEiiiiEE77rrEEEEQQQQiiiioA
74  vx5ee5EEUUGGGGbriiEEiiEE4viiNNeeJ7EEiiEEiiQQ77rriiEEEEii77rrEEEEQQQQiiiiAo
```

Decoding the characters gives the integer arrays A, B of (8.1). Direct integer multiplication verifies all four identities in (8.2). Thus the appendix supplies the entire exact target, including both of its real embeddings, independently of a numerical display.

Appendix B. Finite certificates and reproduction

B.1. **What is specified by the paper.** The simple labels, their order, the chosen words, path spaces, elementary interchange, propagation of maps, fusion splittings, duality, tube product, twist, and weighted rotation are specified in Sections 6–7. These definitions determine the finite arrays without using the Fourier target. Appendix A specifies every target entry independently. The supplementary files store the resulting arrays and the linear-algebra witnesses; they do not add an unspecified tensor-category choice to the construction.

The external results used in the proof are the EW construction and binary relative Hodge–Riemann theorem, the local K_4 classification, the fixed positive Fibonacci seed theorem in type A_3, the established dihedral result, and the general center/Morita/cyclotomic theorems cited in the text. Their hypotheses are stated at the point of application. In particular the type-A_3 tensor uniqueness in [8] is an external input, whereas the actual algebra, crossed action, marked automorphism argument, and H_4 comparison are proved here.

B.2. **Boundary coverage.** Table 1 gives all exact frame dimensions. Each entry is obtained by the path algorithm, with the selected arrow indices recorded in the corresponding certificate. For a full frame, exact row reduction has the displayed rank and the solution of (7.5) is substituted back into every selected equation. The incomplete frame is completed by the module and factorization certificates in Section 7.

TABLE 1. All boundary corners and exact frame ranks.

Boundary	Dimension	Rank over $\mathbb{Q}$	Boundary	Dimension	Rank over $\mathbb{Q}$
0	48	selected	24	16	selected
1	22	44	25	14	28
2	22	selected	26	14	selected
3	22	44	27	14	28
4	22	44	28	16	32
5	38	selected	29	46	selected
6	22	44	30	16	32
7	22	44	31	16	32
8	22	44	32	16	32
9	72	144	33	42	84
10	42	84	34	42	84
11	42	selected	35	42	selected
12	42	84	36	42	84
13	72	144	37	42	84
14	42	84	38	42	84
15	96	192	39	102	204
16	96	192	40	102	204
17	96	192	41	102	204
18	116	232	42	148	296
19	96	192	43	102	204
20	254	508	44	252	504
21	284	568	45	252	504
22	254	508	46	252	504
23	666	1332	47	778	1540

The boundary indices are zero-based as in Section 6; dimensions are over $K = \mathbb{Q}(\sqrt{5})$. The word *selected* denotes one of the eight initial boundaries. Each full frame has rational rank twice its corner dimension. At boundary 47 the rational rank 1540 leaves the 8-dimensional ideal treated in Section 7.

B.3. **Verification stages and exact data.** The computational supplement is the directory `source_package`, which contains the original exact model and its data. It is available in the archived computational supplement (ZIP). Within its subdirectory `fourier_s`, the principal files have the following roles.

File or directory	Mathematical role
`computations/a3_exact_tube.py`	Integral-pair arithmetic, the rational path gauge, and the neutral raw tube table.
`crossed_exact_agent/exact_model.py`	All 48 labels, strict crossed action, actual cups and caps, and 96 snake identities.

File or directory	Mathematical role
`crossed_exact_agent/build_even.py`	The two 153-dimensional even-winding algebras and the actual involutions.
`crossed_exact_agent/build_center.py`	The complete equations (7.3), exact center multiplication, trace functionals, and twist powers.
`crossed_exact_agent/extensions`	All 48 corner restrictions; each noninitial full frame includes its exact rank and arrow data.
`crossed_exact_agent/missing_morita_certificate.json`	The seven-arrow module generation and four factorization surjections.
`crossed_exact_agent/missing_kernel47.json`	The exact 8-dimensional kernel, positive metric, and projected ideal unit.
`crossed_exact_agent/missing_complete47.json`	The actual residual algebra, its two central primitives, dimensions, and twist.
`crossed_exact_agent/full74_exact_rotation.json`	The 1046 rotated tubes, rank-148 coordinate solve, all-boundary unit/twist restrictions, and $R^2 = I$.
`fourier74_agent/certified_center_balls.py`	Certified simple spectrum, normalization by the unit, integer traces, primitive dimensions, and normalized S enclosures.
`fourier74_agent/certify_full74_S.py`	The simultaneous permutation and all 5476 comparisons with the separation bound.
`computations/h4_fourier_exact.json`	The integer arrays A, B of the target (8.1), extracted from the CHEVIE source [15].

The principal verification run uses Python 3.12.14, NumPy 2.3.5, SciPy 1.17.0, and python-flint 0.9.0 with FLINT 3.6.0. The categorical arithmetic is exact in K; SciPy is used to propose frames or relabelings. Their ranks and equations are subsequently verified over $\mathbb{Q}$. The enclosure stage uses the certified `rump` eigensystem and certified linear solves in python-flint [16]. It does not use the approximate eigensystem option.

The short verification chain, run from the supplement root, is

```
cd source_package
python -m pip install -r requirements.txt
python verify.py
```

It checks the source manifest, makes a temporary working copy, recomputes the rotation and the missing-ideal certificate, checks the actual duality, unit and twist data, and builds a fresh 512-bit certificate with the independent separating element used in Section 8. It uses the archived exact raw multiplication tables as inputs; the following longer chain regenerates those tables from the model. Python assertions must remain enabled.

B.4. **Rebuilding from the regular model.** In a disposable copy of `source_package`, enter `fourier_s` and run

```
python computations/a3_exact_tube.py --labels 0 2 5 11
python crossed_exact_agent/build_even.py 0
python crossed_exact_agent/build_even.py 1
python crossed_exact_agent/build_center.py 0
python crossed_exact_agent/build_center.py 1
```

To force a fresh reconstruction of the corner restrictions, remove only the generated corner checkpoints in that working copy:

```
python - <<'PY'
from pathlib import Path
for p in Path("crossed_exact_agent/extensions").glob(
        "exact_crossed_corner_*.json"):
    p.unlink()
PY
```

Then continue with

```
python crossed_exact_agent/extend_visible.py 0
python crossed_exact_agent/extend_visible.py 1
python crossed_exact_agent/build_missing_frames.py
python crossed_exact_agent/split_missing.py
python crossed_exact_agent/verify_missing_morita.py
python crossed_exact_agent/export_missing_center.py
python crossed_exact_agent/rotate_full74.py
python fourier74_agent/prepare_full74_ball_input.py
python fourier74_agent/certified_center_balls.py \
    fourier74_agent/full74_ball_input.json \
    fourier74_agent/full74_certified_balls.json --precision 512
python fourier74_agent/certify_full74_S.py
```

The preparation step binds the enclosure input to the newly generated exact files, whose provenance hashes may differ because run metadata are recorded. A new spectral decomposition can reorder the primitives; its comparison certificate supplies its own simultaneous permutation. The target equality is independent of that order.

B.5. **Independent checks and the scope of a certificate.** In addition to the staged reconstruction, separate checks reparse the original CHEVIE expressions for all 5476 target entries and 74 twist labels, and verify (8.2) with arbitrary-precision integers. The two selected center algebras have all 1600 and 1024 products checked in all 153 physical coordinates. The assembled enclosure input has its multiplication, unit, trace, boundary, twist, and rotation slots checked against the exact source arrays. The three residual generators and annular twist can also be regenerated directly on the active columns of the 778-dimensional corner. These checks are included with the supplement.

A successful final certificate reports rank 74, 5476 certified entries, actual twist order 60, the rational threshold in (1.5), a simultaneous permutation, and a strict upper error below that threshold. Such a certificate proves the matrix equality for the category of Section 6; Proposition 5.4, not numerical agreement, transfers it to the original asymptotic cell. The finite calculation and the categorical proof are both required for Theorem 1.2.

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