A UNIMODULAR CONTINUED FRACTION FOR CUBIC IRRATIONALS

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Abstract. We construct a single algorithm that assigns to every real number a recoverable sequence of positive integers which is eventually periodic exactly when the input is a cubic irrational. Its matrix digits are positive unimodular transformations on the two-dimensional simplex. Their products determine nested rational triangles whose diameters decrease by at least a factor of two at each step. An effectively specified partition combines Euclidean reduction with the dominant eigenvectors of cubic units. In both cubic signatures the matrix expansion eventually repeats a single matrix, whose Perron eigenvalue is a unit generating the input field.

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1. Introduction

In 1848, Hermite asked Jacobi for a characterization of cubic irrationals by eventual periodicity, analogous to Lagrange's theorem for ordinary continued fractions [1, 2]. Karpenkov's periodicity theorem covers the totally real case and lists the case of a real cubic with two nonreal conjugates as Problem 4 [2, § 9]. Tagnon gives exact periodicity certificates for a collection of inputs in this latter signature [4].

We use the general projective formulation of multidimensional continued fractions [3, § 2.1]. A fixed, countable partition selects an integer matrix from the current vector. Here the partition is specified by finite searches, including an enumeration of integer matrices. By an exact comparison on a real input we mean a test of the sign, including equality, of a rational polynomial evaluated at that input.

Theorem 1.1 (Hermite's problem)**.** *There is a deterministic rule assigning to each real number α an infinite sequence of positive integers which determines α. Every digit can be determined in finitely many steps, and all input-dependent comparisons have degree at most two in α. The sequence is eventually periodic if and only if $[\mathbb{Q}(\alpha):\mathbb{Q}]=3$. In the cubic case the matrix digits are eventually constant, and the Perron eigenvalue of the repeating matrix is a unit generating $\mathbb{Q}(\alpha)$.*

2. Matrix digits and convergence

Let $\Delta=\{x\in\mathbb{R}^3_{\geq 0}:\sum_i x_i=1\}$, let $S=\operatorname{relint}\Delta$, and write $\pi(v)=v/\sum_i v_i$ for a nonzero nonnegative vector v. A matrix M acts as a digit through the map

$$F_M(x)=\pi(M^{-1}x)$$

on the set of $x\in S$ for which $M^{-1}x$ has strictly positive coordinates. We use matrices $M\in \mathrm{SL}_3(\mathbb{Z})$ with strictly positive entries satisfying

$$M_{ij}M_{k\ell}\leq 2M_{i\ell}M_{kj}\qquad (1\leq i,k,j,\ell\leq 3). \tag{2.1}$$

For any nonnegative invertible matrix Q, let $K(Q)=\pi(Q\Delta)$. This is the convex hull of the normalized columns of Q, since normalizing a nonnegative linear combination of the columns gives a convex combination of their normalizations, and conversely.

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Lemma 2.1. *For nonnegative invertible Q and any matrix digit M satisfying (2.1),*

$$K(QM) \subseteq K(Q) \quad \textit{and} \quad \operatorname{diam}_\infty K(QM) \le \tfrac{1}{2} \operatorname{diam}_\infty K(Q).$$

Proof. Let $c_i = \pi(Qe_i)$ and let s_i denote the sum of the entries of column i of Q. Column j of QM, after normalization, is $\sum_i p_i^{(j)} c_i$, where

$$p_i^{(j)} = \frac{s_i M_{ij}}{\sum_h s_h M_{hj}}.$$

This proves the inclusion. For fixed columns j, ℓ, set $r_i = M_{ij}/M_{i\ell}$. Condition (2.1) gives $\max_i r_i \le 2 \min_i r_i$, so

$$\frac{p_i^{(j)}}{p_i^{(\ell)}} = \frac{r_i}{\sum_h p_h^{(\ell)} r_h} \ge \tfrac{1}{2}.$$

Thus $p^{(j)}$ is the average of $p^{(\ell)}$ and another probability vector. The normalized columns j, ℓ of QM are at distance at most half the diameter of $K(Q)$. The diameter of a convex hull is the largest distance between two of its vertices, which proves the claim. □

Suppose $M_0, M_1, \ldots$ are successive digits of x, and set $Q_n = M_0 \cdots M_{n-1}$ with $Q_0 = I$. If $x^{(n)}$ is the state after n digits, then $x = \pi(Q_n x^{(n)})$. In an infinite expansion the triangles $K(Q_n)$ contain x and have diameter at most 2^{-n}. They recover x as their unique common point, with explicit rational enclosures from their vertices.

3. Euclidean reduction in small cones

Lemma 3.1. *Let $x \in S$, and let $D \subset S$ be a closed triangle with rational vertices and $x \in \operatorname{relint} D$. In finitely many steps one can find a positive $B \in \mathrm{SL}_3(\mathbb{Z})$ with $K(B) \subseteq D$ and $B^{-1}x > 0$, or produce a nonzero rational row c with $cx = 0$. The tests involving x are signs of rational linear forms.*

Proof. The cone over D is the intersection of three homogeneous half-spaces with integer defining rows. Start at $B = I$ and impose these inequalities one at a time on its columns. For the current inequality $\ell v \ge 0$, write $b_i = Be_i$, $a_i = \ell b_i$, and $t = B^{-1}x$. Maintain $t > 0$.

If the a_i have both signs, choose indices i, j with $a_i = P > 0$ maximal and $a_j = -N < 0$ minimal. If $t_i > t_j$, replace b_j by $b_j + b_i$. This replaces t_i by $t_i - t_j$ and leaves the other coordinates of t unchanged. If $t_j > t_i$, replace b_i by $b_i + b_j$. If $t_i = t_j$, the difference of the two corresponding rows of B^{-1} gives the desired relation.

A replacement changes one extreme value to $P - N$. As long as the signs stay mixed, P and N never increase. If their sum is unchanged, the number of entries equal to P or $-N$ decreases. Hence the pair consisting of their sum and this count decreases lexicographically. The loop terminates. Since $\sum_i a_i t_i = \ell x > 0$, its successful termination leaves all $a_i \ge 0$. Column additions preserve the inequalities already imposed, as well as determinant one. After the third inequality, every column is in the cone over D. Since $D \subset S$, every column is strictly positive. □

Lemma 3.2. *Under the hypotheses of Lemma 3.1, suppose that each ratio v_i/v_k varies by a factor of at most two on D. Given any rational $\eta > 0$, there is a finite procedure returning either a nonzero rational relation on x or a matrix digit M satisfying (2.1) and*

$$K(M) \subseteq D, \qquad M^{-1}x > 0, \qquad \tau(F_M(x)) < \eta, \qquad \tau(v) = v_1 + v_2. \tag{3.1}$$

Proof. Use Lemma 3.1 to obtain B or a relation. In the former case apply Euclidean division to the first two coordinates of $B^{-1}x$, keeping the third fixed. A division replaces the larger coordinate by its remainder modulo the smaller coordinate. The update is effected by the inverse of a nonnegative unimodular elementary matrix. A zero remainder gives a nonzero rational linear relation on x, since the coordinate change is invertible and rational.

If no remainder vanishes, the successive remainders tend to zero. Indeed, their decreasing limit L satisfies $L \geq 2L$ by the division recurrence. Since the third coordinate stays positive, finitely many divisions yield (3.1) for $M = BE$, where E is the product of the elementary matrices. Each column of M is a nonzero nonnegative combination of columns of B, so M is positive and $K(M) \subseteq K(B)$. The ratio bound on D applied to the normalized columns of M gives (2.1). □

In applying Lemma 3.2 one can choose D with any prescribed positive upper bound on its diameter. One can also require that it lie in a specified open half-space containing x. Sufficiently small triangles about x satisfy the ratio condition by continuity, and their vertices can be chosen rational. Enumerating rational triangles in height order finds one in finitely many steps. The ratio and diameter bounds can be tested on the vertices. The linear-fractional ratio v_i/v_k is bounded between its extreme vertex values, since its value on a convex combination is a weighted average of the vertex ratios with weights proportional to the denominator coordinates. The half-space tests needed below have algebraic coefficients fixed during the construction.

4. Cubic directions and fixed points

Call $x \in S$ a *cubic direction* if $1, x_2/x_1, x_3/x_1$ form a $\mathbb{Q}$-basis of a real cubic field. This property is invariant under invertible rational projective changes of coordinates whenever the new direction lies in S. The change of coordinates replaces the three entries of a field basis by another basis, and normalization divides them by one nonzero element of the same field.

Lemma 4.1. *The cubic directions are exactly the positive eigenvector directions of matrices $A \in \mathrm{SL}_3(\mathbb{Z})$ with irreducible characteristic polynomial and an eigenvalue $\lambda > 1$ strictly dominant in modulus. In this statement the positive eigenvector is for λ. Such pairs (x, A), with $x \in S$, admit an effective enumeration.*

Proof. Given a cubic direction, put $b = (1, x_2/x_1, x_3/x_1)^{\mathsf{T}}$, $K = \mathbb{Q}(b_1, b_2, b_3)$, and $L = \sum_i \mathbb{Z} b_i$. By Dirichlet's unit theorem, the logarithmic vectors of the units of $\mathcal{O}_K$ form a full lattice in the weighted sum-zero hyperplane. The open cone in this hyperplane with a positive component at the chosen real embedding and negative components at all other places contains a lattice point. To see this, follow a ray in the interior of the cone and approximate it to bounded distance by lattice points. Squaring the resulting unit makes its real value positive and its norm one.

Choose $m \geq 1$ with $m\mathcal{O}_K L \subseteq L$ by clearing the denominators of the products of an integral basis of $\mathcal{O}_K$ with the b_i, expressed in the basis b. A further positive power u of the unit is 1 modulo $m\mathcal{O}_K$. The same is true of u^{-1}, so multiplication by u preserves L bijectively. Define A by $ub_i = \sum_j A_{ij} b_j$. It is integral, has determinant $\mathrm{Norm}_{K/\mathbb{Q}}(u) = 1$, and satisfies $Ab = ub$. Its eigenvalues are the three conjugates of u. The element $u > 1$ is a unit and cannot be rational, so it has degree three and the characteristic polynomial is irreducible.

Conversely, for a matrix A as stated, choose an eigenvector for λ over $\mathbb{Q}(\lambda)$. Its images under the three field embeddings are eigenvectors for distinct eigenvalues of A and are linearly independent over $\mathbb{C}$. Any rational linear relation on the coordinates of the first vector would hold on all three, which is impossible. Dividing by its first coordinate gives a basis of $\mathbb{Q}(\lambda)$, and hence the normalized positive direction is cubic.

Enumerate the integer matrices in order of increasing maximum entry in absolute value, using a fixed tie order. Determinant, polynomial irreducibility, strict spectral dominance, and the sign pattern of an eigenvector are finite tests in exact algebraic arithmetic. Normalize the positive eigenvectors to obtain the list. □

Lemma 4.2. *Let $x \in S$ be a cubic direction with matrix A from Lemma 4.1. For any rational $\eta > 0$ one can find two matrix digits M, P, both satisfying (2.1), and a point $y \in S$ with*

$$y = F_M(x), \qquad \tau(y) < \eta, \qquad F_P(y) = y.$$

Proof. Take a left eigenvector ℓ of A for λ with $\ell x = 1$. Such a normalization exists because the dominant eigenvalue is simple. Apply Lemma 3.2 with D contained in $\ell v > 0$. Rational independence rules out a relation, and every entry of ℓM is positive. Set $H = M^{-1}AM$. Its normalized powers satisfy

$$\lambda^{-k} H^k \longrightarrow (M^{-1}x)(\ell M).$$

The limit is positive and has rank one. Therefore H^k is positive and satisfies (2.1) for all sufficiently large k. Choose the first such power P by exact integer tests. It is integral and unimodular, and $y = \pi(M^{-1}x)$ is its positive eigenvector direction. □

5. The partition of the simplex

Fix the enumeration in Lemma 4.1 and a deterministic ordering for every subsequent finite search. Starting with $\eta_0 = 1$, define rational numbers $0 < \eta_{s+1} < \eta_s/2$ and a rule on the bands

$$\mathcal{B}_s = \{x \in S : \eta_{s+1} \le \tau(x) < \eta_s\}, \qquad s = 0, 1, 2, \ldots.$$

A band has an ordinary branch and may have two special branches. The ordinary branch moves to $\tau < \eta_{s+1}$ or returns a rational relation. The special branches stay in the band.

At stage s, take the s-th listed direction, counting from zero. Use the already constructed bands to follow this direction until it reaches $\tau < \eta_s$ or a previously specified fixed point. The finite residence property proved below ensures that the simulation terminates. If it reaches a fixed point, leave the two special branches empty and set $\eta_{s+1} = \eta_s/3$.

Otherwise let x be the entry point into $\tau < \eta_s$. Carry its matrix A along the simulation by conjugation, replacing A by $N^{-1}AN$ at each step with digit N. Apply Lemma 4.2 with a rational $\eta < \tau(x)/2$ to get M, P, y. Choose open rational triangles U, V in S with $x \in U$, $y \in V$, and the following properties.

(1) F_M is defined on $\overline{U}$, and $F_M(\overline{U}) \subset V$.
(2) F_P is defined on $\overline{V}$, and $F_P(\overline{V}) \cap \overline{U} = \varnothing$.
(3) $\overline{U}, \overline{V}, F_P(\overline{V})$ are compact subsets of $\{v \in S : \tau(v) < \eta_s\}$, with $\overline{U} \cap \overline{V} = \varnothing$.

To make these choices, use $x \neq y$ and $F_P(y) = y$. First take V small enough that both its closure and its image avoid a neighborhood of x. Then take U small enough that $F_M(\overline{U}) \subset V$. Both searches use rational triangles and finite algebraic tests. A projective image of a closed triangle on the positive domain of either map is the triangle spanned by the images of the vertices. Choose η_{s+1} less than the minimum of τ on the three compact sets and less than $\eta_s/2$.

The band rule emits M on U and P on V, and applies the corresponding map. On every other point of the band apply Lemma 3.2 with target η_{s+1}. In this ordinary branch also require

$$\operatorname{diam}_\infty D < \frac{1}{[3(s+1)]^2}. \tag{5.1}$$

Any matrix digit returned by the ordinary branch then has an entry greater than $s+1$. If all its entries were at most $s+1$, every normalized column would have denominator at most $3(s+1)$. Two distinct normalized columns would differ in some coordinate by at least $1/[3(s+1)]^2$. Invertibility makes the normalized columns distinct, and their containment in D contradicts (5.1).

Proposition 5.1. *Every stage of the partition is constructed in finitely many steps. The resulting rule depends only on the current point of S. Every cubic direction has an infinite, eventually constant matrix expansion. For a noncubic direction the rule either returns a rational relation after finitely many digits or has an infinite expansion with unbounded matrix entries.*

Proof. In a completed band, the branch on U sends a point into V. The branch on V stays in the band and cannot send a point into U. A point that stays in V forever lies in every $K(P^n)$. By Lemma 2.1, their intersection is the single fixed point y. A point in V other than y thus leaves V in finite time and takes the ordinary branch. Consequently an orbit leaves a completed band in finite time unless it halts or reaches the fixed point.

When constructing stage s, the simulated direction is cubic throughout its orbit, so rational independence rules out a halt. The simulation visits the earlier bands in strictly increasing order as it leaves them. It therefore reaches a known fixed point or $\tau < \eta_s$ in finite time. Exact algebraic equality suffices to recognize the known fixed points in this simulation. This proves the induction, including the first stage, for which there are no earlier bands.

For an arbitrary current point, only finitely many bands must be constructed to locate it because $\tau(x) > 0$ and $\eta_s \to 0$. The data of those bands depend solely on the fixed enumeration. Evaluating the rule on the point uses rational linear tests, including the floor tests in Euclidean division. Each finite sequence of these tests specifies a piece by homogeneous rational linear conditions. This gives the countable partition underlying the matrix expansion.

Every cubic direction occurs in the enumeration. At its stage it either reaches a previous fixed point or is sent by the new branch from x to y, where its digit remains P. A noncubic direction cannot reach a cubic fixed point, by invariance under the invertible rational moves. An infinite orbit of such a direction must take ordinary branches in bands of arbitrarily large index. The entry bound from (5.1) proves the last claim. □

6. Expansion of a real number

Proof of Theorem 1.1. For $\alpha \neq 0$, record its sign. Set $b = |\alpha|$ and initialize the rule at $x = \pi((1, b, b^2)^{\mathsf{T}})$. Fix disjoint, effective positive-integer codes for signs, matrix digits, and terminal records. If the rule runs forever, the nested triangles recover x, and the sign and x_2/x_1 recover α.

The initial direction is cubic exactly when b has degree three. For degree three, $1, b, b^2$ form a basis of $\mathbb{Q}(b)$. Conversely, if these three elements form a basis of a cubic field K, then b generates K because its powers span K. Proposition 5.1 now gives a constant matrix tail in the cubic case and unbounded entries in every other infinite case. An injective code for matrices preserves the distinction, since an eventually periodic sequence uses only finitely many digits.

If a rational relation $cx^{(n)} = 0$ is detected after the prefix product Q_n, then b is a root of the polynomial

$$p(T) = cQ_n^{-1}(1, T, T^2)^{\mathsf{T}}.$$

The polynomial is nonzero because $c \neq 0$ and Q_n is invertible. Encode p and the index of b among its distinct real roots as a terminal record. The root index is obtained by comparisons with rational separators between the roots. Append the sequence $1, 2, 3, \ldots$ after the terminal record. Use a dedicated terminal record for zero with the same aperiodic tail. This makes the expansion infinite and recoverable on all inputs, including those that halt.

At any step, the sign of a rational linear form on $x^{(n)}$ is the sign of the corresponding rational linear form on $Q_n^{-1}(1, b, b^2)^{\mathsf{T}}$, since the normalization factor is positive. After fixing the sign of α, these are polynomial comparisons of degree at most two in α. Each stage and each digit has already been shown to require only finitely many operations.

Finally, let Q be the prefix before the constant tail and let P be its repeating matrix. If w is the positive Perron eigenvector of P, then the recovered direction is $\pi(Qw)$. Both P and P^{-1} are integral, so the Perron eigenvalue λ_P and its inverse are algebraic integers. The direction w can be represented over $\mathbb{Q}(\lambda_P)$, and consequently $\alpha \in \mathbb{Q}(\lambda_P)$. Since α has degree three and P is 3×3, this gives $\mathbb{Q}(\lambda_P) = \mathbb{Q}(\alpha)$. □

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