A CONGRUENCE OBSTRUCTION TO IBN AL-KHAWWĀM'S ELEVENTH PROBLEM

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Abstract. We prove that there are no two rational squares whose sum is ten and such that adding its own square root to each gives another rational square. This is the existence question in Ibn al-Khawwām's eleventh arithmetic problem. The proof uses only the residues of squares modulo three and a divisibility argument.

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1. The problem and its solution

Ibn al-Khawwām's eleventh problem asks for a decomposition of ten into two rational squares, each remaining a rational square after addition of its positive square root. The problem occurs in the unsolved-problems section of his thirteenth-century arithmetic treatise. Its Arabic text and equations are reproduced by Fazlıoğlu [2, pp. 103–104, 123–124]. It also appears in Derouiche's list [1, §5.4]. Writing the original squares as x^2 and y^2 gives

$$x^2 + y^2 = 10, \qquad x^2 + x = u^2, \qquad y^2 + y = v^2. \tag{1.1}$$

These are the same equations used by Rashed [3, p. 143]. The following theorem settles the existence question even when all four variables are allowed to have either sign.

Theorem 1.1. *The system* (1.1) *has no solution in* $\mathbb{Q}^4$.

Proof. Suppose that a rational solution exists. Write

$$x = \frac{a}{c}, \qquad y = \frac{b}{c}, \qquad u = \frac{d}{c}, \qquad v = \frac{e}{c}.$$

with $a, b, c, d, e \in \mathbb{Z}$, $c > 0$, and $\gcd(a, b, c, d, e) = 1$. Clearing denominators gives

$$a^2 + b^2 = 10c^2, \qquad d^2 = a(a + c), \qquad e^2 = b(b + c). \tag{1.2}$$

First, $3 \nmid c$. Indeed, if $3 \mid c$, then the first equation of (1.2), together with the fact that a square is congruent to 0 or 1 modulo 3, implies $3 \mid a$ and $3 \mid b$. The remaining equations then imply $3 \mid d$ and $3 \mid e$. This contradicts $\gcd(a, b, c, d, e) = 1$.

Consequently $c^2 \equiv 1 \pmod 3$, and the first equation of (1.2) implies that exactly one of a, b is divisible by 3. Interchanging (a, d) and (b, e) if necessary, assume $3 \mid b$. Then $3 \mid e$, so $9 \mid e^2 = b(b + c)$. Since $3 \nmid b + c$, it follows that

$$9 \mid b. \tag{1.3}$$

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Both a and c are nonzero modulo 3. If $a \equiv c \pmod 3$, then $d^2 = a(a+c) \equiv 2c^2 \equiv 2 \pmod 3$, which is impossible. Therefore

$$a \equiv -c \pmod 3, \qquad 3 \mid d. \tag{1.4}$$

Now multiply $d^2 = a(a+c)$ by $a-c$ and use the first equation of (1.2). This yields the identity

$$(a-c)d^2 = a(a^2-c^2) = a(9c^2-b^2). \tag{1.5}$$

By (1.3) and (1.4), write $b = 9k$ and $d = 3\ell$ with $k, \ell \in \mathbb{Z}$. Dividing (1.5) by 9 gives

$$(a-c)\ell^2 = a(c^2 - 9k^2).$$

Modulo 3, the congruence $a \equiv -c$ implies $a - c \equiv c$. Thus

$$c\ell^2 \equiv -c^3 \pmod 3.$$

Since $3 \nmid c$, cancellation gives

$$\ell^2 \equiv -c^2 \equiv -1 \equiv 2 \pmod 3,$$

contradicting the possible residues of an integer square. □

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References

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