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THE INDEPENDENCE NUMBER OF THE GENERALIZED JOHNSON GRAPH $J(13, 5, 3)$

MUSTAFA STANIKZAI

Abstract. Let $J(13, 5, 3)$ be the graph on the 5-element subsets of a 13-element set, two subsets being adjacent when they share exactly 3 elements. We prove that $\alpha(J(13, 5, 3)) = 33$. The graph is an explicit unit-distance graph in $\mathbb{R}^{12}$, so $\chi(\mathbb{R}^{12}) \geq \lceil 1287/33 \rceil = 39$; earlier bounds from explicit finite graphs give at most 38. A recent preprint of Davies gives much larger bounds, none from an explicit finite graph. An explicit independent set gives $\alpha \geq 33$. For the upper bound, a symmetry reduction splits the problem into 200 clique subproblems, and a certifying solver refutes each one with a proof log that the checker VeriPB accepts. An uncertified computation reproduces the value $\alpha(J(13, 6, 4)) = 46$ reported by Exoo and Ismailescu, which gives $\chi(\mathbb{R}^{12}) \geq 38$.

1. Introduction

For $n \geq 1$ the chromatic number $\chi(\mathbb{R}^n)$ is the least number of colors needed to color the points of $\mathbb{R}^n$ so that no two points at Euclidean distance exactly 1 receive the same color.

In a fixed dimension n, finite unit-distance graphs give lower bounds on $\chi(\mathbb{R}^n)$: a finite graph G drawn in $\mathbb{R}^n$ with every edge of length 1 gives $\chi(\mathbb{R}^n) \geq \chi(G) \geq \lceil |V(G)|/\alpha(G) \rceil$, where $\alpha(G)$ is the independence number of G.

Theorem 1.1. *Let $J(13, 5, 3)$ be the graph whose vertices are the 5-element subsets of $\{1, \ldots, 13\}$. Two of its vertices are adjacent when they share exactly 3 elements. Then $\alpha(J(13, 5, 3)) = 33$. Consequently $\chi(\mathbb{R}^{12}) \geq \lceil 1287/33 \rceil = 39$.*

For $n = 12$, Kahle and Taha [13] obtained $\chi(\mathbb{R}^{12}) \geq 32$ by computation. Exoo and Ismailescu [11] computed the independence number of the generalized Johnson graph $J(13, 6, 4)$ (Definition 2.1) as 46 with two independent programs; this value gives 38 through (1), although they state the bound as 36 (Section 4.2). The largest bound stated in the literature from an explicit finite unit-distance graph is 37, which Cherkashin et al. [7] proved from a closed form for the independence number of a $\{-1, 0, 1\}$ distance graph. Theorem 1.1 raises this bound to 39. In dimensions 13 and 14 the theorems of [7] give 46 and 52, above the bound 39 that Theorem 1.1 gives there by monotonicity.

In a recent preprint, Davies [8] proves that for $n \geq 4$ the chromatic number of $\mathbb{R}^n$ equals its measurable chromatic number. With the lower bounds of Bachoc et al. [2] on the measurable chromatic number, the theorem gives $\chi(\mathbb{R}^{12}) \geq 164$, and with the density certificates of [8] it gives $\chi(\mathbb{R}^n) \geq 2516$, 5616 and 12 688 for $n = 12, 13, 14$. By the de Bruijn–Erdős theorem [6], each of these bounds implies that a finite unit-distance graph with at least that chromatic number exists; none

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has been exhibited. If these bounds stand, the contribution of Theorem 1.1 is the exact value of $\alpha(J(13,5,3))$ and an explicit 1287-vertex unit-distance graph of chromatic number at least 39.

A literature search in October 2026, including Brouwer's tables of constant-weight codes, the OEIS and the literature on codes with forbidden distances, found no published value of $\alpha(J(13,5,3))$. In coding terms, $\alpha(J(13,5,3))$ is the largest number of binary words of length 13 and constant weight 5 with no two at Hamming distance exactly 4. In this quantity one distance is forbidden, whereas in the quantity $A(n,d,w)$ of the standard tables of constant-weight codes every distance below d is forbidden. Bachoc et al. [2] tabulate bounds on $\alpha(J(n,k,t))$, which they write $\alpha(J(n,w,i))$; their table has rows for $J(13,5,2)$ and $J(13,6,2)$, none for the two graphs studied here, and its exact values end at $\alpha(J(12,5,2)) = 57$. Balogh et al. [3] determine the asymptotic order of the independence number of the graph on $\{0,1\}^n$ whose edges join the pairs at one fixed Hamming distance, in a setting that covers the constant-weight quantity above; their results give no value at $n = 13$. The Frankl–Wilson-type inequality stated in the survey [21] gives $\alpha(J(13,5,3)) \le 93$ and $\alpha(J(13,6,4)) \le 157$, and the Delsarte linear programs, solved exactly, give 58 and 78 (Proposition D.1).

We call a computational result *certified* if it comes with a certificate, an output that a separate program can check, and *uncertified* if it rests on a solver verdict alone. The proof of Theorem 1.1 uses certified results only; Section 3.5 states the limits of the tools behind it.

The lower bound $\alpha(J(13,5,3)) \ge 33$ comes from an explicit independent set (Appendix A). For the upper bound, vertex-transitivity and orbital branching reduce $\alpha \le 33$ to 200 clique subproblems (Section 2), and a certifying solver refutes each one with a proof log that the proof checker VeriPB accepts (Section 3). Section 4 gives $\alpha(J(13,6,4)) = 46$, conditional on uncertified solver verdicts, and shows that, up to isomorphism, $J(13,5,3)$ and $J(13,6,4)$ are the only generalized Johnson graphs with ground-set size 13 whose counting bound (1) can exceed 33. Section 5 lists open questions.

2. Generalized Johnson graphs and the reduction to clique subproblems

Definition 2.1. For integers $n \ge k > t \ge 0$, the *generalized Johnson graph* $J(n,k,t)$ has as vertices the k-element subsets of $[n] = \{1,\dots,n\}$, and two vertices A and B are adjacent if and only if $|A \cap B| = t$.

The survey [21] denotes $J(n,k,t)$ by $G(n,k,t)$.

Lemma 2.2. *Let $n \ge k > t \ge 0$, and for a k-element subset A of $[n]$ let $\mathbf{1}_A \in \{0,1\}^n$ be its indicator vector. Then the points $x_A = \mathbf{1}_A/\sqrt{2(k-t)}$ lie in the affine hyperplane $\{y \in \mathbb{R}^n : \sum_i y_i = k/\sqrt{2(k-t)}\}$. This hyperplane is isometric to $\mathbb{R}^{n-1}$, and $\|x_A - x_B\| = 1$ whenever $|A \cap B| = t$. Hence every proper coloring of $\mathbb{R}^{n-1}$ restricts to a proper coloring of $J(n,k,t)$.*

Proof. Each $\mathbf{1}_A$ has coordinate sum k, and $\|\mathbf{1}_A - \mathbf{1}_B\|^2 = |A \setminus B| + |B \setminus A| = 2(k - |A \cap B|)$. □

Lemma 2.3. *Every finite graph G satisfies $\chi(G) \geq \lceil |V(G)|/\alpha(G) \rceil$. Moreover, for $n \geq k > t \geq 0$,*

$$\chi(\mathbb{R}^{n-1}) \geq \left\lceil \frac{\binom{n}{k}}{\alpha(J(n,k,t))} \right\rceil. \tag{1}$$

Proof. Each color class of a proper coloring is an independent set, which gives the first inequality; the second follows from it and Lemma 2.2. □

The graph $J(13,5,3)$ has $\binom{13}{5} = 1287$ vertices and $J(13,6,4)$ has $\binom{13}{6} = 1716$.

For a finite graph H and an integer $\tau \geq 1$, the *clique subproblem* (H,τ) asks whether H contains a clique on τ vertices; it is *UNSAT* if the answer is negative and *SAT* otherwise.

2.1. **Fixing one vertex**

Lemma 2.4. *For $n \geq k > t \geq 0$ the graph $J(n,k,t)$ is vertex-transitive.*

Proof. Every permutation of $[n]$ preserves intersection sizes and so acts on $J(n,k,t)$ as an automorphism. The permutations of $[n]$ act transitively on the k-element subsets of $[n]$. □

Proposition 2.5. *Let $n \geq k > t \geq 0$, let $J = J(n,k,t)$ and $v_1 = \{1,\dots,k\}$, let $\bar{N}(v_1) = V(J) \setminus (N_J(v_1) \cup \{v_1\})$ be the set of vertices that are distinct from v_1 and not adjacent to it, let G_0 be the complement of the induced subgraph $J[\bar{N}(v_1)]$, and let ω denote the clique number. Then*

$$\alpha(J) = 1 + \omega(G_0). \tag{2}$$

Proof. If C is a clique of G_0, then $C \cup \{v_1\}$ is independent in J, so $\alpha(J) \geq 1+\omega(G_0)$. Conversely, by Lemma 2.4 some maximum independent set I of J contains v_1, and $I \setminus \{v_1\} \subseteq \bar{N}(v_1)$ is then a clique of G_0. □

2.2. **The orbital branching tree**

Definition 2.6. Let G be a finite simple graph and $s \geq 1$ an integer. A *node* is a triple $\nu = (F, X, \tau)$ with F a clique of G (the forced clique), $X \subseteq V(G)$ (the excluded set) and $\tau = s - |F|$ (the local threshold). Its *residual set* is

$$R(\nu) = \Big(\bigcap_{f \in F} N_G(f) \Big) \setminus X, \qquad \text{reading the intersection as } V(G) \text{ when } F = \emptyset,$$

its *residual graph* is $H(\nu) = G[R(\nu)]$, and its *subproblem* is the clique subproblem $(H(\nu), \tau)$. The *root* is $(\emptyset, \emptyset, s)$.

Orbital branching, which originates with Ostrowski et al. [19], expands a node ν as follows. A subgroup Γ of the automorphism group of $H(\nu)$ is fixed, a vertex $r \in R(\nu)$ is chosen, and O denotes the Γ-orbit of r. The node gets two children: the include child $(F \cup \{r\}, X, \tau - 1)$ with residual set $R(\nu) \cap N_G(r)$, and the exclude child $(F, X \cup O, \tau)$ with residual set $R(\nu) \setminus O$. The set $F \cup \{r\}$ is again a clique of G, since r lies in the common neighborhood of F. Repeated expansion from the root gives a finite binary tree, whose leaf subproblems are decided by computation.

Lemma 2.7 (orbital branching)**.** *Let H be a finite simple graph, let $\Gamma \leq \mathrm{Aut}(H)$ be any subgroup, let $O \subseteq V(H)$ be a Γ-orbit, let $r \in O$, and let τ be an integer. Then H has a clique of size at least τ if and only if either H has a clique of size at least τ containing r or $H - O$ has a clique of size at least τ.*

Proof. Cliques containing r and cliques of the induced subgraph $H - O$ are cliques of H. Conversely, let C be a clique of H with $|C| \geq \tau$. If C misses O, it is a clique of $H - O$. Otherwise pick $x \in C \cap O$; since O is one orbit, some $\gamma \in \Gamma$ maps x to r, and $\gamma(C)$ is a clique of size $|C|$ containing r. □

Proposition 2.8. *Let G be a finite simple graph, let $s \geq 1$ be an integer, and let T be a finite tree obtained from the root $(\emptyset, \emptyset, s)$ by expansions as defined above. Suppose that the expansion at each internal node ν of T uses a subgroup $\Gamma_\nu \leq \mathrm{Aut}(H(\nu))$ and the Γ_ν-orbit of the chosen representative. Then $\omega(G) \geq s$ if and only if $\omega(H(\ell)) \geq \tau_\ell$ for some leaf ℓ of T, where τ_ℓ is the local threshold of ℓ.*

Proof. Let ν be an internal node with local threshold τ, representative r and orbit O, and apply Lemma 2.7 to $H = H(\nu)$ and $\Gamma = \Gamma_\nu$. Deleting r maps the cliques of size at least τ through r onto the cliques of size at least $\tau - 1$ in the include child's residual graph, and the cliques of $H(\nu)$ avoiding O are those of the exclude child's residual graph. So $\omega(H(\nu)) \geq \tau$ if and only if some child μ of ν has $\omega(H(\mu)) \geq \tau_\mu$, where τ_μ is its local threshold. Induction on the height of T gives the claim, since the root has residual graph G and local threshold s. □

Let G_0 be the graph of Proposition 2.5 for $J = J(n,k,t)$, and let T be a tree for G_0 and s as in Proposition 2.8. Since $\alpha(J) \leq s$ exactly when $\omega(G_0) < s$ by (2), Proposition 2.8 gives

$$\alpha(J) \leq s \iff \omega(H(\ell)) < \tau_\ell \text{ for every leaf } \ell \text{ of } T. \tag{3}$$

3. Certified computation for $J(13,5,3)$

This section proves Theorem 1.1. We build an orbital branching tree for the graph G_0 of $J(13,5,3)$ at $s = 33$, certify its 200 leaf subproblems, and check the decomposition.

3.1. Construction and decomposition

All computations on graphs, permutations and witnesses use exact integer arithmetic. We carry out the following steps for $J(13,5,3)$ at $s = 33$, and steps 1 to 4 also for $J(13,6,4)$ at $s = 46$ (Section 4.2).

(1) Build $J(n,k,t)$ from Definition 2.1 by two implementations that share no code, *implementation A* and *implementation B*, and compare their edge lists edge for edge.

(2) Build G_0, the graph of Proposition 2.5, with a map from its vertices back to k-element subsets. The k-element subsets are numbered in lexicographic order, so that v_1 is vertex 1, and G_0 numbers its vertices in the same order.

(3) Branch. Starting from the root $(\emptyset, \emptyset, s)$, which is node 1, repeatedly expand the frontier node with the largest residual vertex count until the tree has 200 leaves; break ties by the smallest node identifier. An expansion gives its include child and then its exclude child the next two free identifiers. Each expansion of a node ν runs four substeps.

(a) Each node carries a partition of [13]. The root partition has the blocks $\{1,\dots,k\}$ and $\{k{+}1,\dots,13\}$, and an include step that forces a vertex u (a k-element subset of [13]) splits every block B into $B \cap u$ and $B \setminus u$. For each block $\{b_1 < \cdots < b_m\}$ of the partition of ν, take the transposition $(b_1\, b_2)$ if $m \geq 2$ and the cycle $(b_1\, b_2 \cdots b_m)$ if $m \geq 3$. These generate the Young subgroup of the partition; at the root it is $\mathrm{Sym}(\{1,\dots,k\}) \times \mathrm{Sym}(\{k{+}1,\dots,13\})$, of order $5!\,8! = 4\,838\,400$ for $J(13,5,3)$ and $6!\,7! = 3\,628\,800$ for $J(13,6,4)$. A permutation of [13] that maps $\{1,\dots,k\}$ onto itself fixes v_1 and preserves intersection sizes, so it acts as an automorphism of G_0. The Young subgroup of the partition of ν fixes every forced vertex and lies in the group of every ancestor of ν; since the excluded set of ν is a union of orbits of the groups of its ancestors, the subgroup maps $R(\nu)$ onto itself.

(b) Check each generator by three tests, so that the soundness of the expansion does not depend on the group the generators are meant to generate; any failure aborts the construction, and no generator failed.

(G1) The permutation of [13] induces a permutation of $V(G_0)$.

(G2) The induced permutation maps the edge list of G_0 onto itself.

(G3) It maps $R(\nu)$ onto itself and preserves the residual edge list.

(c) Compute orbits as forward closures under the checked generators; on a finite set the forward closure of a vertex is its orbit under the generated group, since the inverse of each generator is one of its powers. Choose the largest orbit O; among orbits of equal size, choose the one whose least vertex identifier is smallest. Take the vertex with the smallest identifier in O as the representative r, and check one more test.

(G4) The orbit satisfies $r \in O \subseteq R(\nu)$.

(d) Form the include child and the exclude child.

A node is a leaf when it lies at depth 12, when every orbit of its group on $R(\nu)$ is a single vertex, or when the tree reaches 200 leaves. These rules affect only the shape of the tree: by Lemma 2.7, every choice of orbit and representative gives a sound expansion. Record the rule, the generators, every leaf's (F, X, τ) and every residual vertex count in a *manifest*, written before any leaf subproblem is decided and not changed afterwards.

(4) Write each leaf's residual graph, with a vertex map, to its own file, the *leaf file*, numbering the leaves in increasing order of node identifier.

(5) Certify each leaf subproblem (Section 3.3).

(6) Check the decomposition (Section 3.4).

Call the resulting tree T_{33}. Each leaf ℓ of T_{33} has forced clique F_ℓ, residual set R_ℓ, residual graph H_ℓ and local threshold $\tau_\ell = 33 - |F_\ell|$. The vertex v_1 of $J(13,5,3)$ has $\binom{5}{3}\binom{8}{2} = 280$ neighbors, so G_0 has $1287 - 1 - 280 = 1006$ vertices; it has 391 135 edges (density 0.774). Of the 200 leaves of T_{33}, 66 lie at depth 12, 84 have only single-vertex orbits, and 50 were on the frontier when the tree reached 200 leaves. The tree uses 21 distinct generators and has no leaf with an empty residual set. The margin $|R_\ell| - \tau_\ell$ is at least 63 on every leaf, so no leaf subproblem is UNSAT for lack of vertices. Over the 200 leaves, $|R_\ell|$ runs from 96 to 466 with median 205.5,

and τ_ℓ runs from 29 to 33: it equals 29 on 140 leaves, 30 on 47, 31 on 10, 32 on 2 and 33 on 1.

3.2. **Lower-bound witness**

Proposition 3.1. *The 33 five-element subsets of* [13] *listed in Appendix A are pairwise non-adjacent in* $J(13,5,3)$. *Hence* $\alpha(J(13,5,3)) \geq 33$.

Proof. Check the 528 pairs; Appendix A reduces the check to pairs from different blocks. □

The set was found by randomized local search.

3.3. **Certified leaves**

Each leaf subproblem is encoded as a *pseudo-Boolean formula* (a system of linear inequalities over 0/1 variables). For a leaf ℓ with n_ℓ residual vertices, m_ℓ residual edges and local threshold τ_ℓ, the formula has one variable x_w for each vertex w of the residual graph and two families of constraints:

(C) one cardinality constraint $\sum_w x_w \geq \tau_\ell$;

(N) one constraint $x_u + x_w \leq 1$ for each of the $\binom{n_\ell}{2} - m_\ell$ non-adjacent pairs $\{u, w\}$.

The 0/1 solutions are exactly the indicator vectors of cliques on at least τ_ℓ vertices, so the formula has no solution if and only if the leaf subproblem is UNSAT.

Each leaf was decided in three steps.

(1) The clique solver of the Glasgow Subgraph Solver [17], run in decision mode at τ_ℓ with the proof logging of Gocht et al. [12], wrote its own pseudo-Boolean formula for the leaf. It refuted the formula and wrote a VeriPB proof log of the refutation [1].

(2) A program separate from the solver, the *encoding checker*, rebuilt the expected formula from the leaf file and from $\tau_\ell = 33 - |F_\ell|$, and required the solver's formula to equal it (the *encoding check*) up to constraint labels, the order of constraints and of terms, the rewriting of a negated literal $\bar{x}$ as $1 - x$, and division of a constraint by a common factor of its coefficients with the right-hand side rounded up. None of these changes the 0/1 solutions. The expected formula has the variables $x_1, \ldots, x_{n_\ell}$, one constraint of family (C), which carries all n_ℓ variables with coefficient 1 and has right-hand side τ_ℓ, and the constraints of family (N) for exactly the non-adjacent pairs; it contains nothing else. The encoding checker shares no code with the solver or with the script that runs the solver.

(3) The proof checker VeriPB checked the proof log against the formula. A proof log is a sequence of linear inequalities, each derived from the formula and earlier lines by sound rules and the last a contradiction, so VeriPB's acceptance shows that the formula has no 0/1 solution.

The procedure was first tested on $J(8,3,1)$, $J(9,3,1)$, $J(10,5,3)$ and $J(11,5,3)$, whose independence numbers are known. VeriPB accepted the proof logs of all 200 leaves of T_{33}. The leaf verdicts thus rest on the encoding checker and VeriPB alone. Table 5 in Appendix C gives proof-log sizes and times for nine of the leaves.

Five of the leaves (16, 169, 170, 173 and 176) also have a proper coloring of the residual graph with fewer colors than the local threshold (Table 4), found by a

randomized DSATUR search [5]. Since the vertices of a clique receive pairwise different colors, each such coloring certifies its leaf by itself. A program separate from the search checks every coloring edge by edge and matches it to its leaf file and local threshold.

3.4. **Checks of the decomposition**

One program ran the ten steps of Table 1, from the construction of $J(13,5,3)$ to the bound 39, and all ten passed. Steps 1 to 6 run implementation B, three programs that share no code with implementation A or with each other; they recompute the graphs, the tree and its residual sets without reading any recorded check result, and compare them with the stored graph, manifest and leaf files. The rule and generators recorded in the manifest fix every branching decision of the replay. Steps 7 and 8 read the VeriPB acceptances and encoding-check results recorded when each leaf was certified. Implementation A ran tests (G1) to (G4) during construction and replayed the tree with code separate from its tree construction. As a test of the generator checks, a generator composed with a transposition of two vertices of equal degree was rejected.

TABLE 1. The end-to-end check for $J(13,5,3)$

Step	Content	Check
1	$J(13,5,3)$ rebuilt from its definition, 1287 vertices, degree 280, edge list matched to the recorded one	recomputed
2	G_0 rebuilt on $1287-1-280=1006$ vertices, its file reproduced byte for byte	recomputed
3	21 generators: permutations of all 1006 vertices, edge preserving, each mapping its node's residual set onto itself	recomputed
4	tree replayed from its rule: 200 leaves, 0 dropped, residual sets recomputed	recomputed
5	the 200 leaf files equal G_0 induced on the recomputed residual sets	recomputed
6	$\tau_\ell = 33 - \lvert F_\ell \rvert$ on 200 of 200 leaves	recomputed
7	5 colorings re-verified; proof-log acceptances and encoding-check records read for all 200 leaves	record read
8	$\omega(G_0) \le 32$, from steps 3, 4 and 7	record read
9	$\alpha(J(13,5,3)) = 1+\omega(G_0) \le 33$ from step 8; vertex-transitivity recomputed (orbit of vertex 1 under the 12 adjacent transpositions of [13] is all 1287 vertices)	recomputed; rests on step 8
10	$\lceil 1287/33 \rceil = 39$; indicator vectors in the hyperplane $x_1+\cdots+x_{13}=5$, adjacent ones at distance 2, so the points $\mathbf{1}_A/2$ are at distance 1	recomputed

3.5. **Limitations**

The proof logs were checked with VeriPB, which is not formally verified; the formally verified checker CakePB [1], which checks VeriPB proof logs compiled into a restricted kernel format, was not run on them because of its expected running time. The encoding checker and the programs that check the decomposition are not formally verified either.

3.6. **Proof of the main theorem**

Proposition 3.2. *For every leaf ℓ of T_{33}, the residual graph H_ℓ contains no clique on τ_ℓ vertices.*

Proof. By steps 4 to 6 of Table 1, the 200 leaf files hold the residual graphs H_ℓ of the leaves of T_{33}, whose local thresholds are $\tau_\ell = 33 - |F_\ell|$. By step 7, for each leaf VeriPB accepted a proof log refuting a formula that the encoding check matched to the leaf file and to τ_ℓ (Section 3.3), and the 0/1 solutions of that formula are the indicator vectors of the cliques of H_ℓ on at least τ_ℓ vertices. □

Proof of Theorem 1.1. Tests (G1) to (G4) of step 3 show that T_{33} satisfies the hypotheses of Proposition 2.8, and by Proposition 3.2 no leaf residual graph H_ℓ has a clique on τ_ℓ vertices. Hence $\alpha(J(13,5,3)) \le 33$ by (3), and Proposition 3.1 gives equality. Since $33 \cdot 39 = 1287$, Lemma 2.3 gives $\chi(\mathbb{R}^{12}) \ge 39$. □

Corollary 3.3. *The fractional chromatic number of* $J(13,5,3)$ *is* $\chi_f(J(13,5,3)) = 39$.

Proof. Since $J(13,5,3)$ is vertex-transitive (Lemma 2.4), Proposition 3.1.1 of [22] gives

$$\chi_f(J(13,5,3)) = \frac{1287}{\alpha(J(13,5,3))}.$$

Theorem 1.1 gives $\alpha(J(13,5,3)) = 33$, so $\chi_f(J(13,5,3)) = 1287/33 = 39$. □

4. Further computations

This section reports what Theorem 1.1 does not use: the uncertified MaxCliqueDyn search with its reruns, the computation for $J(13,6,4)$, and the other parameter pairs with ground-set size 13.

4.1. The MaxCliqueDyn search and reruns

The MaxCliqueDyn search used a build of the branch-and-bound solver MaxCliqueDyn [14] modified to solve the decision problem at a fixed threshold, with its search logic unchanged (Table 3). It returned UNSAT on all 200 leaves of each tree at their local thresholds. Both implementations checked that each leaf has exactly one such verdict, matched to its leaf file; implementation B also matched each verdict to its solver transcript.

The solvers MoMC [15] and cliquer [18] share no code with MaxCliqueDyn, were written by other authors, and use different bounding rules. MoMC reran every leaf of T_{33} and returned UNSAT on all 200, so three solvers agree on all 200 leaves of T_{33}: MaxCliqueDyn, MoMC and the certifying solver of Section 3.3. On the tree for $J(13,6,4)$, MoMC and cliquer reran a sample of 14 of the 200 leaves, fixed before the reruns: the three leaves other than leaf 1 with the longest MaxCliqueDyn times and, in each of the 11 nonempty classes defined by local threshold and tercile of MaxCliqueDyn time, the lowest-numbered of the remaining leaves. MoMC returned UNSAT on 13 of the 14 leaves; cliquer, run on the 8 with MaxCliqueDyn times up to 600 s, returned UNSAT on 6. The other three runs, one of MoMC and two of cliquer, reached their time cap of 600 s. Each of the 14 leaves was returned UNSAT by at least one of the two solvers, and no rerun verdict contradicted MaxCliqueDyn. Leaf 1 of that tree, the largest, with 761 residual vertices and a MaxCliqueDyn time of 20.6 h, was left out of the sample for its projected cost. Every solver read the same leaf files, so the reruns check neither the construction of the leaves nor the coverage of the tree.

4.2. The independence number of $J(13,6,4)$

The computation for $J(13,6,4)$ followed steps 1 to 4 of Section 3.1 at $s = 46$, and the generator, tree and leaf-file checks of Section 3.4 were applied to it as well. Let G_0' be the graph of Proposition 2.5 for $J(13,6,4)$ and T_{46} its tree, with F_ℓ, R_ℓ and H_ℓ as for T_{33} and $\tau_\ell = 46 - |F_\ell|$. The vertex v_1 of $J(13,6,4)$ has $\binom{6}{4}\binom{7}{2} = 315$ neighbors, so G_0' has $1716 - 1 - 315 = 1400$ vertices; it has 793 135 edges (density 0.810). Like T_{33}, the tree T_{46} has 200 leaves and uses 21 distinct generators. On every leaf the margin $|R_\ell| - \tau_\ell$ is at least 129; over the 200 leaves, $|R_\ell|$ runs from 175 to 761 with median 394, and τ_ℓ runs from 42 to 46 with median 42.

Proposition 4.1. *The 46 six-element subsets of* $[13]$ *listed in Appendix B are pairwise non-adjacent in* $J(13,6,4)$. *Hence* $\alpha(J(13,6,4)) \geq 46$.

Proof. Check the 1035 pairs (Appendix B). □

The set was found by randomized local search.

Claim 4.2. *For every leaf* ℓ *of* T_{46}, *the residual graph* H_ℓ *contains no clique on* τ_ℓ *vertices.*

Leaves 11 and 170 are certified by colorings with fewer colors than their local thresholds (Table 4); a DSATUR search like that of Section 3.3, run earlier on 15 leaves of T_{46}, found the two colorings. The other 198 leaves rest on MaxCliqueDyn verdicts, and 12 of them were rerun (Section 4.1). No proof logging was attempted for $J(13,6,4)$. On its leaves other than leaf 1, the MaxCliqueDyn search visited about 120 times as many nodes as on all 200 leaves of T_{33}. Its verdict records name the same solver version as those of $J(13,5,3)$ but predate the recording of the binary's SHA-256 digest, which was linked to them after the run.

Proposition 4.3. *If Claim 4.2 holds, then* $\alpha(J(13,6,4)) = 46$.

Proof. The argument for $\alpha(J(13,5,3)) = 33$ in the proof of Theorem 1.1 applies with G_0', $s = 46$, T_{46}, Claim 4.2 and Proposition 4.1 in place of G_0, $s = 33$, T_{33}, Proposition 3.2 and Proposition 3.1. □

Under Claim 4.2, Lemma 2.3 gives $\chi(\mathbb{R}^{12}) \geq \lceil 1716/46 \rceil = 38$, which Theorem 1.1 supersedes.

Exoo and Ismailescu [11] report $\alpha(J(13,6,4)) = 46$: "both mcqd and our specialized program were able to determine that the independence number is 46" (mcqd is MaxCliqueDyn [14]). They give the resulting bound as $\lceil 1716/46 \rceil = 36$. The ceiling is 38, since $46 \cdot 37 = 1702 < 1716 \leq 1748 = 46 \cdot 38$, so their stated bound $\chi(\mathbb{R}^{12}) \geq 36$ is valid and two below what their value gives. The introduction of [7] lists $\chi(\mathbb{R}^{12}) \geq 36$ as an announced improvement.

The present computation reproduces the value 46, uncertified, with its own reduction, decomposition and leaf files. Its verdicts come from MaxCliqueDyn, which Exoo and Ismailescu [11] also ran; their second program repeated the full search, while MoMC and cliquer reran 14 of the 200 leaves here. The present work adds the decomposition proof, implementation B and the two certified leaves.

4.3. **Other generalized Johnson graphs with ground-set size 13**

Up to isomorphism, $J(13,5,3)$ and $J(13,6,4)$ are the only generalized Johnson graphs with ground-set size 13 whose counting bound (1) can exceed 33. The pairs $(8,6)$ and $(7,5)$ excluded below give their complements $J(13,8,6) \cong J(13,5,3)$ and $J(13,7,5) \cong J(13,6,4)$.

Proposition 4.4. *Let (k,t) range over the 91 pairs with $1 \le k \le 13$ and $0 \le t < k$. For each of the 87 pairs outside $\{(5,3),(8,6),(6,4),(7,5)\}$ there is an independent set of size a in $J(13,k,t)$ with $\lceil \binom{13}{k}/a \rceil \le 33$; hence the counting bound (1) for these graphs is at most 33.*

Proof. Two k-element subsets of $[13]$ share at least $2k-13$ elements, so the 49 graphs with $t < 2k-13$ are edgeless; for them $a = \binom{13}{k}$ and the ceiling is 1. The graphs $J(13,1,0)$ and $J(13,12,11)$ are complete on 13 vertices, so $a = 1$ and the ceiling is 13. Table 2 gives a and the ceiling for the 18 pairs with $2 \le k \le 6$. For $t = 0$ the witness is the set of k-element subsets containing the element 1, of size $\binom{12}{k-1}$, a maximum independent set by the Erdős–Ko–Rado theorem [10] since $13 \ge 2k$. The other 13 witnesses were found by randomized local search and give lower bounds only. All pairs of sets in each witness were checked by machine. The remaining 18 pairs have $k \ge 7$ and follow through the isomorphism $A \mapsto [13] \setminus A$ from $J(13,k,t)$ onto $J(13,13-k,13-2k+t)$. The largest ceiling, 33, occurs at $J(13,4,2)$ and $J(13,9,7)$. □

TABLE 2. Witness size a and ceiling $\lceil \binom{13}{k}/a \rceil$, written a (ceiling), for the 18 pairs (k,t) of Proposition 4.4 with $2 \le k \le 6$; the excluded pairs $(5,3)$ and $(6,4)$ are blank

k	$\binom{13}{k}$	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
2	78	12 (7)	6 (13)				
3	286	66 (5)	12 (24)	26 (11)			
4	715	220 (4)	55 (13)	22 (33)	65 (11)		
5	1287	495 (3)	165 (8)	62 (21)		122 (11)	
6	1716	792 (3)	372 (5)	148 (12)	86 (20)		156 (11)

The proposition bounds only the right-hand side of (1). Up to isomorphism, the independence and chromatic numbers of the 87 graphs are determined here only for the edgeless graphs, for $J(13,1,0)$, and for the Kneser graphs $J(13,k,0)$ with $2 \le k \le 6$, where $13 \ge 2k$, so that the Erdős–Ko–Rado theorem gives $\alpha = \binom{12}{k-1}$ and Lovász's theorem [16] gives $\chi = 15 - 2k$.

5. OPEN QUESTIONS

Question 5.1. Is $\chi(J(13,5,3)) = 39$? The fractional chromatic number is 39 (Corollary 3.3), and since $\alpha(J(13,5,3)) = 33$ and $39 \cdot 33 = 1287$, a 39-coloring would partition the vertices into 39 maximum independent sets. A negative answer would give $\chi(\mathbb{R}^{12}) \ge 40$ from the same graph.

Question 5.2. Are all maximum independent sets of $J(13,5,3)$ equivalent under $\mathrm{Sym}(13)$?

Question 5.3. Does some $J(n,k,t)$ with $n \in \{14,15\}$ give, through (1), a bound above 46 on $\chi(\mathbb{R}^{13})$ or above 52 on $\chi(\mathbb{R}^{14})$?

Use of AI tools

I directed the work, ran the computations, and reviewed the manuscript, the code and the reported computations. I used Claude Code (Anthropic) with the Claude Fable 5, Fable 5.1, Fable 5.2 and Opus 5.5 models to plan, draft, revise and review this paper and to write its code, including the programs for the computation and the verification programs. I used OpenAI Codex with the GPT-5.6 Sol model to evaluate plans and, on some occasions, to review drafts of the paper.

Data availability

The following are on Zenodo, https://doi.org/10.5281/zenodo.24145863, each with its SHA-256 digest: the programs (the verification programs, the MaxCliqueDyn patch with its tests, the pinned solver revisions and the MaxCliqueDyn binary); the certificates and proof logs (the lower-bound witnesses, the Delsarte and coloring certificates, and the 200 proof logs for $J(13, 5, 3)$ with their pseudo-Boolean formulas and checker transcripts); and the run records (the run manifests, the 400 leaf files, the per-leaf verdict records, the result records and the records of the checks, the per-leaf rerun table, the coloring attempts and the 91-pair parameter table of Section 4.3). The proof logs are compressed with the PPMd method of 7-Zip 26.04; the archives total 40.6 GB.

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Appendix A. The 33-vertex witness for $J(13,5,3)$

The 33 sets of Proposition 3.1 fall into five blocks. Each of the first three blocks consists of the six 5-element subsets of one 6-element set; the three 6-element sets are $\{1,\dots,6\}$, $\{3,6,9,10,12,13\}$ and $\{4,5,7,8,9,10\}$. The fourth block consists of the six sets $\{1,7,11,12,x\}$ with $x \in \{3,4,5,6,9,10\}$, and the fifth of the nine sets $\{2,8,11,13,x\}$ with $x \in \{1,3,4,5,6,7,9,10,12\}$. Two sets in the same block meet in exactly 4 elements, so only pairs from different blocks need checking. Block by block, the 33 sets are:

$\{1,2,3,4,5\}$	$\{1,2,3,4,6\}$	$\{1,2,3,5,6\}$
$\{1,2,4,5,6\}$	$\{1,3,4,5,6\}$	$\{2,3,4,5,6\}$
$\{3,6,9,10,12\}$	$\{3,6,9,10,13\}$	$\{3,6,9,12,13\}$
$\{3,6,10,12,13\}$	$\{3,9,10,12,13\}$	$\{6,9,10,12,13\}$
$\{4,5,7,8,9\}$	$\{4,5,7,8,10\}$	$\{4,5,7,9,10\}$
$\{4,5,8,9,10\}$	$\{4,7,8,9,10\}$	$\{5,7,8,9,10\}$
$\{1,3,7,11,12\}$	$\{1,4,7,11,12\}$	$\{1,5,7,11,12\}$
$\{1,6,7,11,12\}$	$\{1,7,9,11,12\}$	$\{1,7,10,11,12\}$
$\{1,2,8,11,13\}$	$\{2,3,8,11,13\}$	$\{2,4,8,11,13\}$
$\{2,5,8,11,13\}$	$\{2,6,8,11,13\}$	$\{2,7,8,11,13\}$
$\{2,8,9,11,13\}$	$\{2,8,10,11,13\}$	$\{2,8,11,12,13\}$

No two of the 33 subsets intersect in exactly 3 elements (all $\binom{33}{2} = 528$ pairs checked by implementations A and B), so the set is independent in $J(13,5,3)$.

Appendix B. The 46-vertex witness for $J(13,6,4)$

As 6-element subsets of $\{1,\dots,13\}$, the 46 sets of Proposition 4.1 are:

$\{1,2,3,4,5,6\}$	$\{1,2,3,4,5,7\}$	$\{1,2,3,4,6,7\}$	$\{1,2,3,5,6,7\}$
$\{1,2,3,8,9,10\}$	$\{1,2,4,5,6,7\}$	$\{1,2,4,8,9,10\}$	$\{1,2,5,8,12,13\}$
$\{1,2,6,10,11,12\}$	$\{1,2,7,9,11,13\}$	$\{1,3,4,5,6,7\}$	$\{1,3,4,8,9,10\}$
$\{1,3,5,8,12,13\}$	$\{1,3,6,10,11,12\}$	$\{1,3,7,9,11,13\}$	$\{1,4,5,10,11,13\}$
$\{1,4,6,9,12,13\}$	$\{1,4,7,8,11,12\}$	$\{1,5,6,8,9,11\}$	$\{1,5,7,9,10,12\}$
$\{1,6,7,8,10,13\}$	$\{2,3,4,5,6,7\}$	$\{2,3,4,8,9,10\}$	$\{2,3,5,8,12,13\}$
$\{2,3,6,10,11,12\}$	$\{2,3,7,9,11,13\}$	$\{2,4,5,10,11,13\}$	$\{2,4,6,9,12,13\}$
$\{2,4,7,8,11,12\}$	$\{2,5,6,8,9,11\}$	$\{2,5,7,9,10,12\}$	$\{2,6,7,8,10,13\}$
$\{3,4,5,10,11,13\}$	$\{3,4,6,9,12,13\}$	$\{3,4,7,8,11,12\}$	$\{3,5,6,8,9,11\}$
$\{3,5,7,9,10,12\}$	$\{3,6,7,8,10,13\}$	$\{4,5,6,8,9,11\}$	$\{4,5,7,10,11,13\}$
$\{4,6,7,9,12,13\}$	$\{4,6,9,10,12,13\}$	$\{5,6,7,8,9,11\}$	$\{5,6,8,9,10,11\}$
$\{5,6,8,9,11,12\}$	$\{5,6,8,9,11,13\}$		

No two of the 46 subsets intersect in exactly 4 elements (all $\binom{46}{2} = 1035$ pairs checked by implementations A and B), so the set is independent in $J(13,6,4)$.

Appendix C. Software and run data

The MaxCliqueDyn searches and the certification ran on the same Apple M1 machine, on which the solver passed all 30 tests of its own proof test suite. The reruns ran on a Linux machine. No build log was recorded for the MaxCliqueDyn binary.

Each coloring in Table 4 was re-verified as a proper coloring of the leaf's residual graph with fewer colors than its local threshold. In Table 5, family (N) counts the at-most-one constraints of the leaf's formula, search nodes are those of the Glasgow solver, and the solve and check columns give wall-clock times of the solver and of VeriPB on the certification machine, rounded to 0.1 s.

TABLE 3. Software that produced or checked a computational result

Tool	Version	Role
modified MaxCliqueDyn [14]	upstream revision of 2016-09-01; `clang++ -O3`, as reported by the binary	uncertified search
MoMC [15]	2019 revision of the 2016 source	reruns
cliquer [18]	1.21	reruns
Glasgow Subgraph Solver [17, 12]	upstream revision of 2026-06-19; Apple clang 21.0.0	certifying search
VeriPB [4]	3.0.2	proof checker
R [20]	4.6.1, package `gmp` 0.7.5.1	implementation A (construction, decomposition), Delsarte linear programs
DSATUR coloring heuristic [5]	randomized, in R	coloring certificates
Python	3.14.8 and 3.15.0	implementation B and checking programs

TABLE 4. The seven colorings that certify leaves, sorted by residual vertex count within each computation

Computation	Leaf	Residual vertices	Residual edges	Local threshold τ_ℓ	Colors used
$J(13,5,3)$	16	96	3160	33	17
$J(13,5,3)$	170	113	4524	32	26
$J(13,5,3)$	169	124	5861	31	30
$J(13,5,3)$	176	125	5769	31	30
$J(13,5,3)$	173	128	6230	30	29
$J(13,6,4)$	11	175	10 185	46	36
$J(13,6,4)$	170	216	16 962	45	42

TABLE 5. Proof-log statistics for nine of the 200 $J(13,5,3)$ leaves, among them the smallest and the largest, sorted by residual vertex count n_ℓ

Leaf	n_ℓ	Family (N)	τ_ℓ	Search nodes	Proof-log size	Solve (s)	Check (s)
16	96	1400	33	1	4.71 kB	< 0.1	< 0.1
170	113	1804	32	1	4.33 kB	< 0.1	0.1
169	124	1765	31	3	9.54 kB	< 0.1	< 0.1
176	125	1981	31	5	14.2 kB	< 0.1	< 0.1
77	208	5372	29	5916	12.3 MB	0.5	0.3
78	286	10 449	30	303 030	604 MB	10.1	16.1
25	348	15 914	29	5 655 159	9.87 GB	166.2	281.2
1	404	21 940	29	23 208 561	42.2 GB	664.5	1360.1
10	466	28 184	30	140 108 190	274 GB	4382.6	9062.6

Appendix D. Delsarte bounds

Proposition D.1. *The independence numbers of the two graphs satisfy*

$$\alpha(J(13,5,3)) \le 58 \quad \textit{and} \quad \alpha(J(13,6,4)) \le 78.$$

Proof. Class i of the Johnson scheme on the k-element subsets of $[13]$ holds the ordered pairs of subsets that share $k-i$ elements. Adjacency in $J(13,k,t)$ is therefore class $k-t$, and $k-t=2$ for both graphs here. Let $v=\binom{13}{k}$, and let P be the first eigenmatrix of the scheme, with rows indexed by eigenspaces and columns indexed by classes. Its entry in row i and column j,

$$P_j(i)=\sum_h (-1)^h \binom{i}{h}\binom{k-i}{j-h}\binom{13-k-i}{j-h},$$

is the eigenvalue of class j on the i-th eigenspace. The entries of the second eigenmatrix are $Q_m(i)=\mu_m P_i(m)/v_i$, where $v_i=\binom{k}{i}\binom{13-k}{i}$ is the valency of class i and, since $2k\le 13$, $\mu_m=\binom{13}{m}-\binom{13}{m-1}$ is the dimension of the m-th eigenspace [9]. For an independent set S of $J(13,k,t)$, the numbers $a_i=|\{(A,B)\in S^2 : (A,B) \text{ in class } i\}|/|S|$ satisfy $a_0=1$, $a_{k-t}=0$, $a_i\ge 0$ and $\sum_i a_i=|S|$. They also satisfy Delsarte's inequalities $\sum_i a_i Q_m(i)\ge 0$ for $m=0,\dots,k$ [9]. Maximizing $\sum_i a_i$ under these constraints is the Delsarte linear program. A dual certificate is a vector $u=(u_1,\dots,u_k)$ with $u_m\ge 0$ and

$$\sum_{m=1}^{k} u_m Q_m(i) \le -v \qquad \text{for every class } i\notin\{0,k-t\}.$$

Multiplying the m-th Delsarte inequality by u_m and summing over $m\ge 1$ gives $0\le \sum_m u_m\mu_m - v(|S|-1)$, so

$$\alpha(J(13,k,t)) \le 1+\frac{1}{v}\sum_{m=1}^{k} u_m\mu_m.$$

For $J(13,5,3)$ the certificate is $u=(9945/16, 117/2, 0, 0, 1755/16)$, and Table 6 lists the entries $Q_m(i)$ with $u_m\neq 0$.

Table 6. The dual certificate u for $J(13,5,3)$ and the entries $Q_m(i)$ of the second eigenmatrix of the Johnson scheme on the 5-element subsets of $[13]$, for the indices m with $u_m\neq 0$; the adjacency class $i=2$ is omitted, since $a_2=0$

m	u_m	$Q_m(0)=\mu_m$	$Q_m(1)$	$Q_m(3)$	$Q_m(4)$	$Q_m(5)$
1	9945/16	12	81/10	3/10	−18/5	−15/2
2	117/2	65	26	−169/28	13/14	325/14
5	1755/16	572	−143/2	−143/14	286/35	−143/14

The four constraint sums are -1287 at $i=1,3,4$ and $-70\,785/16$ at $i=5$, all at most -1287, and

$$1+\tfrac{1}{1287}\big(12\cdot\tfrac{9945}{16}+65\cdot\tfrac{117}{2}+572\cdot\tfrac{1755}{16}\big)=1+\tfrac{1}{1287}\cdot\tfrac{148\,005}{2}=1+\tfrac{115}{2}=\tfrac{117}{2},$$

so $\alpha(J(13,5,3))\le 58$, since α is an integer. For $J(13,6,4)$ the certificate is

$$u=(702, 936/5, 0, 0, 78, 156);$$

all five constraint sums equal -1716, and $1+132\,132/1716=78$, so $\alpha(J(13,6,4))\le 78$ with no rounding step. A separate program checked both certificates in exact rational arithmetic. □

The Delsarte linear programs, solved exactly, have the optima 117/2 and 78, so they give no bound below 58 and 78. Stronger relaxations, such as the semidefinite bound of Schrijver [23], were not computed.

Independent researcher

Email address: m.st87nk@gmail.com