A Fibonacci Analogue of the Fifteen Theorem

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Abstract

We prove that a positive-definite quadratic form with integral Gram matrix represents every Fibonacci number if and only if it represents the first forty-two of them. The unique minimal criterion set consists of ten Fibonacci numbers, the largest of which is $F_{42} = 267914296$.

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1 Introduction

The Conway–Schneeberger Fifteen Theorem [8, 3] states that a positive-definite quadratic form with integral Gram matrix represents every positive integer if and only if it represents

$$\{1, 2, 3, 5, 6, 7, 10, 14, 15\}.$$

We ask the corresponding question for the Fibonacci numbers. Put $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$, and write $\mathcal{F} = \{F_n : n \geq 1\}$. A form representing every element of $\mathcal{F}$ is called *Fibonacci-universal*.

For a set $S \subseteq \mathbb{Z}_{>0}$, an *$S$-criterion set* is a finite subset $D \subseteq S$ such that every positive-definite quadratic form with integral Gram matrix representing every element of D also represents every element of S. An $\mathcal{F}$-criterion set is called a *Fibonacci criterion set*.

Every $S \subseteq \mathbb{Z}_{>0}$ has a finite criterion set [14], and its minimal criterion set is unique [15, Theorem 2.2]. We identify this set explicitly for the Fibonacci numbers $\mathcal{F}$.

Theorem 1.1 (Fibonacci criterion)**.** *A positive-definite quadratic form with integral Gram matrix is Fibonacci-universal if and only if it represents every element of*

$$\begin{aligned} C &= \{1, 2, 3, 5, 21, 55, 610, 987, 832040, 267914296\} \\ &= \{F_1, F_3, F_4, F_5, F_8, F_{10}, F_{15}, F_{16}, F_{30}, F_{42}\}. \end{aligned}$$

Moreover, every Fibonacci criterion set contains C. Thus C is the unique minimal Fibonacci criterion set with respect to inclusion.

Corollary 1.2 (The index 42)**.** *A positive-definite quadratic form with integral Gram matrix is Fibonacci-universal if and only if it represents $F_1, \ldots, F_{42}$. The index 42 cannot be lowered: the form $x^2 + y^2 + 2z^2$ represents $F_1, \ldots, F_{41}$ but not F_{42}.*

The size of the endpoint reflects a local obstruction: $x^2 + y^2 + 2z^2$ omits exactly the positive integers of the form $4^a(16b + 14)$, with $a, b \geq 0$, and the first Fibonacci number in that family is F_{42}.

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Compared with the integers, the odd integers, or the primes, the Fibonacci numbers are sparse: as $X \to \infty$, Binet's formula gives

$$\#(\mathcal{F} \cap [1, X]) = \frac{\log X}{\log \phi} + O(1), \qquad \phi = \frac{1+\sqrt{5}}{2}.$$

Sparsity alone need not make finding a criterion set difficult: for example, representing 1 gives every square, and representing 1 and 2 gives every power of 2, simply by scaling vectors. But the Fibonacci criterion is not of this kind. It forces, for example, representation of $F_{13} = 233$, although 233 is not an integer square multiple of any member of C.

We have formalized our results in Lean 4, with no unproven hypotheses; the Lean development closely follows the argument in the sequel.

The argument

Escalation. We use Conway's *escalation* method, as developed in Bhargava's proof of the Fifteen Theorem [3]. The *Fibonacci truant* of a lattice is the least Fibonacci number it does not represent. Throughout the paper, *truant* means Fibonacci truant unless explicitly qualified as ordinary. We *escalate* a lattice by adjoining a vector whose norm is its truant; the resulting lattice is an *escalation* of the original lattice. Starting from the zero lattice and making every possible such choice gives the *Fibonacci escalators*. Thus an escalator is a lattice in this construction, whereas an escalation specifies its relation to the preceding lattice. Up to integral isometry, there is one at rank 1, two at rank 2, and nine at rank 3. The truants encountered through rank 2 are $1, 2, 3, 5$; the ternaries contribute six further distinct values, giving exactly the ten elements of C.

A useful coincidence connects our argument with the Fifteen Theorem: the ordinary truants (least omitted positive integers) at ranks 0, 1, and 2 are $1, 2, 3$, and 5, all Fibonacci numbers. Thus ordinary and Fibonacci escalation produce exactly the same ternary lattices. At rank 3, however, their truants differ. For example, $\langle 1, 1, 1\rangle$ first omits the positive integer 7, but its first omitted Fibonacci number is 55. The next Fibonacci escalation therefore adjoins a vector of norm 55, rather than 7. We can consequently reuse the initial ternary classification, but must analyze different quaternary extensions.

Reduction to quaternary extensions. Suppose a form represents every element of C. Starting with the zero lattice inside it, we can adjoin a vector whose norm is the current truant because each truant encountered through rank 3 belongs to C. We will prove that each step increases the lattice rank (Lemma 3.6). After four steps, the form therefore contains a quaternary escalator obtained by escalating one of the nine ternaries. To prove sufficiency, it is enough to show that every such quaternary escalator is Fibonacci-universal.

We prove that every quaternary extension $M = T + \mathbb{Z}v$ of a ternary escalator T is Fibonacci-universal if it represents the truant of T. Here $v \notin T \otimes \mathbb{Q}$; its norm need not itself be the truant. Such an extension contains an orthogonal sublattice $T \perp \langle q\rangle$ with q bounded in terms of T alone. Since $F_n = (F_n - qk^2) + qk^2$, it is enough to choose an integer k for which the difference

$$F_n - qk^2$$

is nonnegative and represented by T.

The difference need not itself be a Fibonacci number—it only needs to satisfy the representation criterion for T. A positive-definite integral form is *regular* if it represents every positive integer represented locally at every prime. Eight of our nine ternaries are regular; the ninth is not. This

distinction determines how we choose k. In the regular cases, valuation formulas locate the omitted Fibonacci numbers and bound their prime-power parts by a constant times the index. Exponential growth of F_n then makes their unit parts large enough for the required subtraction. In the irregular case, congruences give a uniform bound on the subtraction, so growth leaves only a finite initial range to check. In both cases, periodic Fibonacci residues supply the needed congruence information; sparsity alone supplies neither these congruences nor the size bounds.

The regular escalators. For each of the eight regular ternaries, the omitted positive integers are described by a valuation and a residue condition at one prime. In most cases, subtracting a suitable qk^2 removes this local obstruction, and Fibonacci growth keeps the difference positive. For three ternaries we also use vectors outside the orthogonal sublattice: $\langle 1, 2, 2\rangle$, $\langle 1, 1, 3\rangle$, and $\langle 1, 2, 5\rangle$. For the latter two, quadratic polynomials derived from their representation criteria represent every nonnegative integer. These arguments establish Fibonacci-universality of the extensions without exhaustive searches over quaternary forms.

The irregular escalator f. The ninth escalator,

$$f = x^2 + 2y^2 + 2yz + 4z^2,$$

is not regular, so a local condition does not guarantee representation by f alone. Call a positive integer *eligible* if its 7-adic valuation is even or its 7-adic unit part is a square modulo 7. Two auxiliary-prime constructions show that every eligible integer is represented by f or by $h = x^2 + y^2 + 7z^2$ (Proposition 3.3). To pass from h back to f, we use explicit integral matrices B_p, for a finite set of primes $p \neq 7$, satisfying

$$f(B_p v) = p^2 h(v).$$

Division by p^2 preserves eligibility. Consequently an eligible positive difference divisible by p^2 is represented by f: if its quotient is represented by f, scale the vector by p; if it is represented by h, apply B_p. The square p^2 must divide the difference, not F_n: it is created by the choice of k, so the argument applies equally when F_n is squarefree.

The form f represents every Fibonacci number through F_{47} except its truant 21, which the extension represents by hypothesis. For the later terms, we treat the possible values of q. The form $f \perp \langle 7\rangle$ is universal by a short argument using the transfer maps for $p = 2, 3$. For the other values of q, small subtractions give eligible differences outside $7, 10 \pmod{12}$, which are represented directly by f. A finite periodic *congruence cover* supplies square factors for the remaining residue pairs. The Chinese remainder theorem bounds the subtraction, and growth reduces the argument to a short finite check.

Context

For a prescribed set $S \subseteq \mathbb{Z}_{>0}$, Kim, Lee, and Oh [15, Theorem 2.2] identify the minimal criterion set with the values that occur as the least element of S omitted by an integral lattice; their Remark 2.3 credits an earlier unpublished result of Bhargava. Kala, Krásenský, and Romeo [11, Theorem 3.3] develop the corresponding theory over totally real number fields. Our minimality argument follows their extension principle [11, Proposition 3.1], using Kim–Lee–Oh's four-square construction [15, Proposition 2.1] for the explicit Fibonacci escalators. The contributions here are the identification of the criterion and the proof that every quaternary extension arising in the escalation is Fibonacci-universal: by local arguments for the eight regular ternary escalators and, for the irregular escalator

f, by square transfers and a finite congruence cover, without analytic estimates for representations by the irregular ternary.

Related explicit criteria include the 290-theorem for integer-valued quadratic forms [4] and Rouse's 451-theorem for forms representing all odd positive integers [22]. Ju, Kim, Kim, Kim, and Oh prove a 67-theorem for diagonal forms representing all primes [10, Theorem 1.1]. Studies of individual Fibonacci numbers represented by binary quadratic forms include [2, 1].

The irregular form f belongs to Kaplansky's first nontrivial ternary genus [12, 13]. Analytic treatments of this genus [21, 6] use modular forms and bounds for L-functions, following the approach of Ono–Soundararajan [18]. For example, under GRH for the relevant L-functions, Chandee [6, §4] proves that both genus classes represent every locally represented squarefree integer $N \geq 2 \cdot 10^8$. For the Fibonacci extensions, our square transfers and congruence cover supply the needed differences unconditionally. The representation input is the elementary statement in Proposition 3.3; we do not need to classify all integers omitted by f.

Organization

The paper follows the reductions outlined above. Section 2 collects the Fibonacci arithmetic, §3 determines the ternary escalators, and §4 constructs and bounds orthogonal sublattices of their extensions. Sections 5 and 6 prove that extensions representing the corresponding truant are Fibonacci-universal. In §7, an elementary four-square construction shows that none of the ten truants can be omitted from any Fibonacci criterion set. Section 8 states the general transfer principle and discusses its application to other sequences. After the main text, we briefly describe the Lean formalization and some history of our argument.

Notation and conventions

By a *quadratic form* (or just *form*) we mean an expression $Q(x) = x^{\mathsf{T}}Gx$ with G integral, symmetric, and positive definite. Thus all cross-terms have even coefficients. A lattice is a free abelian group of finite rank carrying such a form. The *norm* of a vector x is $Q(x)$, and the determinant of a lattice is the determinant of a Gram matrix. For a lattice L, write Q_L for its quadratic form and β_L for its associated bilinear pairing; in a basis with Gram matrix G, these are $Q_L(x) = x^{\mathsf{T}}Gx$ and $\beta_L(x, y) = x^{\mathsf{T}}Gy$. We identify forms with their based lattices, and say that a form *represents* N when $Q(x) = N$ for an integral vector x. A form is *universal* if it represents every positive integer. The notation $\langle a_1, \ldots, a_r \rangle$ denotes the diagonal lattice $a_1x_1^2 + \cdots + a_rx_r^2$, and $L \perp K$ denotes an orthogonal direct sum.

Two lattices are in the same *genus* if they are isometric over $\mathbb{R}$ and over every $\mathbb{Z}_p$. Write $L_p = L \otimes \mathbb{Z}_p$. A positive integer is *locally represented* by L if every L_p represents it. A form is *regular* if it represents every locally represented positive integer, and *irregular* otherwise.

For a prime p and a positive integer N, let $v_p(N)$ denote its p-adic valuation and $u_p(N) = N/p^{v_p(N)}$ its *p-adic unit part*, or prime-to-p part. For $p = 2$, we also call this the *odd part*. We use the same valuation on $\mathbb{Q}_p$, with $v_p(0) = +\infty$. For positive odd m, (a/m) denotes the Jacobi symbol, including the convention $(a/1) = 1$; when m is prime, this is the Legendre symbol. We write T for a ternary escalator and M for a quaternary extension. Finally, $E(L) \subseteq \mathbb{Z}$ denotes the integers not represented by L; it includes all negative integers and excludes 0. This convention lets us say that a difference lies outside $E(L)$ without separately requiring nonnegativity. The *Fibonacci exceptions* of L are $\mathcal{F} \cap E(L)$.

2 Fibonacci arithmetic preliminaries

We use the classical valuation formulas of Lengyel [16, Lemmas 1–2 and the theorem in §3] and the modular periodicity studied by Wall [24]. For completeness, we recall the formulas and their specializations here, together with the short Fibonacci recurrence calculations needed in the proof.

The representation criteria in §3 depend on two pieces of information about an integer N: the valuation $v_p(N)$ at one prime, and the residue of its unit part $u_p(N)$. We begin with a growth estimate, which turns a bound at one index into a bound at every later index, and then record the Fibonacci identities that determine these two quantities.

Lemma 2.1 (Monotonicity of F_n/n)**.** *The sequence F_n/n is strictly increasing for $n \geq 2$.*

Proof. The inequality $nF_{n+1} > (n+1)F_n$ is equivalent to $nF_{n-1} > F_n$. It holds directly for $n = 2$; for $n \geq 3$, use $F_n = F_{n-1} + F_{n-2} \leq 2F_{n-1} < nF_{n-1}$. □

For a prime p, its *rank of apparition* $\rho_F(p)$ is the least positive index at which p divides a Fibonacci number. The standard Fibonacci divisibility property gives $p \mid F_n$ if and only if $\rho_F(p) \mid n$; see [16, Theorem A in §3], citing [24, Theorem 3].

Lemma 2.2 (Fibonacci valuations)**.** *For every $n \geq 1$,*

$$v_2(F_n) = \begin{cases} 0, & 3 \nmid n, \\ 1, & n \equiv 3 \pmod 6, \\ v_2(n) + 2, & 6 \mid n. \end{cases}$$

The ranks of apparition at 3 and 5 are $\rho_F(3) = 4$ and $\rho_F(5) = 5$, and

$$v_3(F_n) = \begin{cases} 1 + v_3(n), & 4 \mid n, \\ 0, & 4 \nmid n, \end{cases} \qquad v_5(F_n) = v_5(n).$$

Proof. The values $F_4 = 3$ and $F_5 = 5$, together with the preceding terms, give the two ranks of apparition. For an odd prime $p \neq 5$, the Lucas-sequence lifting formula at $m = \rho_F(p)$, for every $k \geq 1$, is

$$v_p(F_{mk}) = v_p(F_m) + v_p(k).$$

It gives the assertion at 3. At the discriminant prime 5, the corresponding formula is $v_5(F_n) = v_5(n)$. These formulas and the formula at 2 are given by Lengyel [16, Lemmas 1–2 and the theorem in §3]. The extra term at 2 when $6 \mid n$ can be understood from $F_{2r} = F_r L_r$, where $L_0 = 2$, $L_1 = 1$, and $L_{r+2} = L_{r+1} + L_r$ are the Lucas numbers. Starting at an index $3s$ with s positive and odd, one has $v_2(F_{3s}) = 1$. The first doubling of the index adds 2, because $v_2(L_r) = 2$ for $r \equiv 3 \pmod 6$; subsequent doublings add 1, because $v_2(L_r) = 1$ for $r \equiv 0 \pmod 6$. This gives $v_2(F_n) = v_2(n) + 2$ when $6 \mid n$. □

At each of these fixed primes, the prime-power part of F_n is at most a constant times its index. We will repeatedly use the bounds

$$2^{v_2(F_n)} \leq 4n, \qquad 3^{v_3(F_n)} \leq 3n, \qquad 5^{v_5(F_n)} \leq n. \tag{1}$$

Dividing F_n by the relevant bound gives a lower bound for $u_p(F_n)$.

Residues at a fixed modulus are equally explicit. The *Pisano period* $\pi(m)$ is the least positive period of the Fibonacci sequence modulo m.

Lemma 2.3 (Pisano periods)**.** *The Fibonacci sequence is periodic modulo every integer $m \geq 2$. In particular,*

m	5	7	8	12	16	49
$\pi(m)$	20	16	12	24	24	112.

Proof. The state transition $(a, b) \mapsto (b, a + b)$ is invertible modulo m, so its orbit from $(F_0, F_1) = (0, 1)$ is periodic. Iterating until this state first returns gives the displayed periods. See Wall [24] for the systematic theory. □

Periodicity and valuations serve different purposes. A Pisano period lets us compute F_n modulo a fixed modulus from finitely many indices. For $n, j \geq 1$, to find the unit part $u_p(F_n)$ modulo p^j, we must also know $a = v_p(F_n)$: if r is the least nonnegative residue of F_n modulo p^{a+j}, then

$$u_p(F_n) \equiv r/p^a \pmod{p^j}.$$

Here r is divisible by p^a. A fixed Pisano period does not by itself determine these unit residues when the valuation a is not known. For the finite calculations below, we use the valuation formulas to select the possible indices and the Fibonacci recurrence to compute the required unit residues. For the later estimates valid at all sufficiently large indices, we instead use (1) to bound the prime-power factors removed from F_n and Lemma 2.1 to bound what remains.

The six families in the next lemma are the local obstructions that will occur for our ternary forms. The middle column characterizes membership by the parity of a valuation and the residue of the corresponding unit part. We will use the first Fibonacci member of each family to determine the truants of the ternary escalators.

Lemma 2.4 (First Fibonacci members of the exceptional families)**.** *Let N be a positive integer. In each row, N belongs to the family in the first column if and only if it satisfies the condition in the second. The rightmost column gives the least Fibonacci number in that family. Here $a, b \geq 0$ are integers.*

family	*condition on N*	*first Fibonacci member*
$4^a(8b+7)$	$v_2(N)$ *even*, $u_2(N) \equiv 7 \pmod 8$	$F_{10} = 55$
$4^a(16b+14)$	$v_2(N)$ *odd*, $u_2(N) \equiv 7 \pmod 8$	$F_{42} = 267914296$
$4^a(16b+10)$	$v_2(N)$ *odd*, $u_2(N) \equiv 5 \pmod 8$	$F_{30} = 832040$
$9^a(9b+6)$	$v_3(N)$ *odd*, $u_3(N) \equiv 2 \pmod 3$	$F_{16} = 987$
$5 \cdot 25^a(5b+r)$, $r \in \{2,3\}$	$v_5(N)$ *odd*, $u_5(N) \equiv 2, 3 \pmod 5$	$F_{15} = 610$
$7 \cdot 49^a(7b+r)$, $r \in \{3,5,6\}$	$v_7(N)$ *odd*, $u_7(N) \equiv 3, 5, 6 \pmod 7$	$F_8 = 21$.

Proof. The equivalence in each row follows by factoring out the full prime-power part of N. To find the first Fibonacci member, the valuation formulas reduce the search to a short list of indices; the Fibonacci recurrence then gives the unit residues. We record the calculations.

For the two families with odd 2-adic valuation, the relevant indices through 42 and their residues are

n	3	6	9	15	18	21	24	27	30	33	39	42
$v_2(F_n)$	1	3	1	1	3	1	5	1	3	1	1	3
$u_2(F_n) \bmod 8$	1	1	1	1	3	1	1	1	5	1	1	7.

Among Fibonacci numbers of odd 2-adic valuation, the first unit residue 5 occurs at F_{30} and the first unit residue 7 at F_{42}. For $4^a(8b + 7)$, the indices below 10 with even 2-adic valuation are $1, 2, 4, 5, 7, 8$. Their odd parts are $1, 1, 3, 5, 5, 5$ modulo 8, whereas $F_{10} = 55 \equiv 7 \pmod 8$.

For the 3-adic family, the only indices below 16 with odd positive valuation are 4 and 8, and both unit parts are 1 (mod 3). At 16, one has $F_{16} = 3 \cdot 329$ and $329 \equiv 2 \pmod 3$. For the 5-adic

family, the only possible earlier indices are 5 and 10, again with square unit part 1 (mod 5); the first nonsquare unit part is $F_{15}/5 = 122 \equiv 2 \pmod 5$. Finally, $7 \mid F_n$ first occurs at $F_8 = 21 = 7 \cdot 3$, and 3 is a nonsquare modulo 7. All these residue calculations use exact Fibonacci recurrence values, or equivalently the Fibonacci recurrence modulo the corresponding prime powers. □

3 Escalation through rank 3

We classify the escalators through rank 3 and compute their truants, which give exactly the ten numbers in C. We then show that none of these escalators represents its truant over $\mathbb{Q}$, so adjoining a vector of truant norm raises the rank. Sufficiency will follow once we prove that the resulting quaternary escalations are Fibonacci-universal.

Figure 1 displays the escalations through rank 3 and the ternary truants that determine the next step.

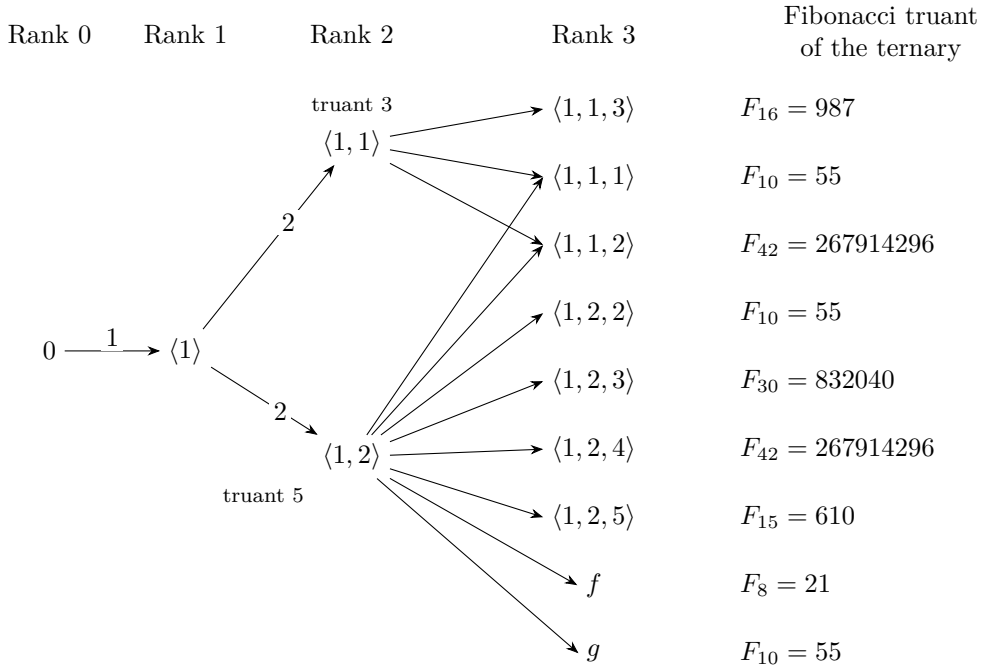


Figure 1: Fibonacci escalation through rank 3, with one vertex for each integral isometry class. Each arrow adjoins a vector whose norm is the truant of its source: 1, 2, 3, or 5. These are also ordinary truants, so the same diagram describes ordinary escalation through rank 3. Both $\langle 1, 1, 1\rangle$ and $\langle 1, 1, 2\rangle$ arise from each binary lattice; f and g are defined in Proposition 3.1. The rightmost column records the Fibonacci truants proven in Propositions 3.2 and 3.5; these determine the subsequent quaternary escalations, which are not drawn.

3.1 The nine ternary escalators

Recall that escalation adjoins a vector of truant norm. Formally, if L has Fibonacci truant τ, an escalation is a positive-definite integral lattice $M \supseteq L$ with $M = L + \mathbb{Z}u$ for some vector u of norm τ. A Fibonacci-universal lattice has no truant and is not escalated. A lattice obtained by successive escalations from the zero lattice is a *Fibonacci escalator*. We classify these lattices up to integral

isometry. In computing their Gram matrices, we may replace the new generator u by $u + x$ with $x \in L$ without changing the lattice.

The zero lattice has truant 1 and escalates to $\langle 1 \rangle$. Its truant is 2, and a vector of norm 2 cannot lie in its rational span because 2 is not a rational square. A binary escalation has Gram matrix

$$\begin{pmatrix} 1 & a \\ a & 2 \end{pmatrix}, \qquad a = 0, \pm 1.$$

The case $a = 0$ gives $\langle 1, 2 \rangle$; subtracting the norm-1 vector from the new vector when $a = 1$, or adding it when $a = -1$, gives $\langle 1, 1 \rangle$. Checking $1, 2, 3, 5$ gives the binary truants

$$\langle 1, 1 \rangle : \ 3, \qquad \langle 1, 2 \rangle : \ 5.$$

Neither binary truant is represented by the rational quadratic space of its escalator. Indeed, a rational representation would give $x^2 + y^2 = 3d^2$ or $x^2 + 2y^2 = 5d^2$ after clearing denominators. Reduction modulo 3 or 5, respectively, forces that prime to divide x, y, and then d. Dividing all three by the prime gives an infinite descent. Thus adjoining a vector of either truant norm produces a ternary lattice.

Proposition 3.1 (The nine ternaries)**.** *The escalations of the two binary escalators are, up to integral isometry, the seven diagonal lattices*

$$\begin{gathered} \langle 1, 1, 1 \rangle, \quad \langle 1, 1, 2 \rangle, \quad \langle 1, 1, 3 \rangle, \\ \langle 1, 2, 2 \rangle, \quad \langle 1, 2, 3 \rangle, \quad \langle 1, 2, 4 \rangle, \quad \langle 1, 2, 5 \rangle, \end{gathered}$$

together with $f = x^2 + 2y^2 + 2yz + 4z^2$ *and* $g = x^2 + 2y^2 + 2yz + 5z^2$*, whose Gram matrices are*

$$G_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix}, \qquad G_g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 5 \end{pmatrix}.$$

Proof. Since $\langle 1, 1 \rangle$ is unimodular, subtracting integral multiples of the two old basis vectors makes the new vector orthogonal to them. Its norm becomes $3 - r$, where r is a sum of two squares with $0 \le r < 3$. The possibilities $r = 0, 1, 2$ give $\langle 1, 1, 3 \rangle$, $\langle 1, 1, 2 \rangle$, and $\langle 1, 1, 1 \rangle$.

For $\langle 1, 2 \rangle$, first remove the pairing with the norm-1 vector. Subtracting multiples of the norm-2 vector reduces the remaining pairing modulo 2, so we may take $b \in \{0, 1\}$. The reduced form is

$$Q(x, y, z) = x^2 + 2y^2 + 2byz + mz^2.$$

The original vector of norm 5 still has new coordinate $z = 1$. Since $2y^2 + 2by \ge 0$ for integral y and $b \in \{0, 1\}$, its representation gives $m \le 5$; positive definiteness gives $m \ge 1$.

If $b = 0$, the five values of m give $\langle 1, 1, 2 \rangle$ and the four forms $\langle 1, 2, m \rangle$ with $2 \le m \le 5$. If $b = 1$ and $m = 1$, the identity

$$x^2 + 2y^2 + 2yz + z^2 = x^2 + y^2 + (y + z)^2$$

gives $\langle 1, 1, 1 \rangle$. The cases $m = 2$ and $m = 3$ do not represent 5: the values at most 5 of their binary parts are, respectively, $\{0, 2\}$ and $\{0, 2, 3\}$; subtracting any of these values from 5 never gives a square. Finally, $m = 4$ gives f, with $5 = f(1, 0, 1)$, and $m = 5$ gives g, with $5 = g(0, 0, 1)$. Each of the diagonal possibilities also has a representation of the truant with new coordinate 1, so all listed lattices do occur as escalations.

Their determinants, in the displayed order, are $1, 2, 3, 4, 6, 8, 10, 7, 9$. They are therefore pairwise inequivalent. □

3.2 The regular escalators

Eight ternary escalators are regular: the positive integers they omit are exactly those excluded by a local obstruction. The table groups forms with the same exceptional set. Comparing these sets with Lemma 2.4 gives their Fibonacci truants; later, the same criteria will determine whether a difference $F_n - qk^2$ is represented.

Proposition 3.2 (Regular ternaries)**.** *The eight forms below are regular, with the indicated positive exceptional sets and Fibonacci truants. Here $a, b \geq 0$ are integers.*

T	$E(T) \cap \mathbb{Z}_{>0}$	*Fibonacci truant*
$\langle 1,1,1\rangle,\ \langle 1,2,2\rangle,\ g$	$4^a(8b+7)$	$F_{10} = 55$
$\langle 1,1,2\rangle,\ \langle 1,2,4\rangle$	$4^a(16b+14)$	$F_{42} = 267914296$
$\langle 1,2,3\rangle$	$4^a(16b+10)$	$F_{30} = 832040$
$\langle 1,1,3\rangle$	$9^a(9b+6)$	$F_{16} = 987$
$\langle 1,2,5\rangle$	$\substack{5\cdot 25^a(5b+r) \\ r\in\{2,3\}}$	$F_{15} = 610.$

Proof. Ramanujan [20, §3, formulas (3.11)–(3.71)] records the exceptional sets of the seven diagonal forms, without giving the ternary proofs. For $\langle 1,1,3\rangle$, $\langle 1,2,3\rangle$, and $\langle 1,2,5\rangle$, Jones and Pall [9, Table I, p. 190] establish sufficiency. Again for completeness, we prove the obstructions and derive the other rows from the Gauss–Legendre three-square theorem.

The Gauss–Legendre three-square theorem gives the criterion for $\langle 1,1,1\rangle$. Over $\mathbb{Z}_2$, a sum of three squares is never 7 modulo 8, and divisibility by 4 forces all three coordinates to be even. Hence no integer $4^a(8b+7)$ is a sum of three squares over $\mathbb{Z}_2$, so $\langle 1,1,1\rangle$ is regular. The identity

$$x^2 + 2y^2 + 2z^2 = x^2 + (y+z)^2 + (y-z)^2$$

shows that every value of $\langle 1,2,2\rangle$ is a sum of three squares. Conversely, two coordinates in any three-square representation have the same parity; they can therefore be written as $y+z$ and $y-z$. Thus these two forms represent exactly the same integers.

The same holds for g, since

$$g(x,y,z) = x^2 + (y+2z)^2 + (y-z)^2.$$

Given a sum of three squares, choose two coordinates r_1, r_2 that are congruent modulo 3, changing a sign if necessary. This is always possible: either at least two coordinates are divisible by 3, or at least two are units. Then

$$y = (r_1 + 2r_2)/3, \qquad z = (r_1 - r_2)/3$$

are integers, and $r_1 = y + 2z$, $r_2 = y - z$ give a representation by g. Both displayed embeddings into three squares hold over every $\mathbb{Z}_p$. Hence any positive integer locally represented by $\langle 1,2,2\rangle$ or g is locally represented by three squares and therefore represented integrally by all three forms. This also proves their regularity.

For the next row, let N be a positive integer. If $N = x^2 + y^2 + 2z^2$, then $2N = (x+y)^2 + (x-y)^2 + (2z)^2$. Conversely, if $2N = r_1^2 + r_2^2 + r_3^2$, then the number of odd coordinates is even; after a permutation, r_1 and r_2 have the same parity and r_3 is even, so

$$N = \left(\frac{r_1+r_2}{2}\right)^2 + \left(\frac{r_1-r_2}{2}\right)^2 + 2\left(\frac{r_3}{2}\right)^2.$$

Thus $\langle 1,1,2\rangle$ represents N exactly when $2N$ is a sum of three squares, that is, when N is not of the form $4^a(16b+14)$. Every value of $\langle 1,2,4\rangle$ is a value of $\langle 1,1,2\rangle$, since $x^2+2y^2+4z^2 = x^2+(2z)^2+2y^2$.

For the converse, first let N be odd. In a representation $N = a^2+b^2+2c^2$, one of a, b is even, which gives a representation by $\langle 1,2,4\rangle$. Now let N be even. A three-square representation of $2N \equiv 0 \pmod 4$ has all coordinates even, so by the identity above $\langle 1,1,2\rangle$ represents N exactly when $N/2$ is a sum of three squares. In a representation $N = x^2+2y^2+4z^2$ the coordinate x is even and $N/2 = y^2+2(x/2)^2+2z^2$; so $\langle 1,2,4\rangle$ represents N exactly when $\langle 1,2,2\rangle$ represents $N/2$, that is, when $N/2$ is a sum of three squares. Thus $\langle 1,1,2\rangle$ and $\langle 1,2,4\rangle$ represent the same integers.

These criteria also prove regularity: a local representation by either form gives a local three-square representation of $2N$ through the integral identities above. The three-square obstruction at 2 therefore excludes precisely the displayed exceptional set.

For $\langle 1,2,3\rangle$, squares modulo 16 lie in $\{0,1,4,9\}$, so $x^2+2y^2+3z^2 \not\equiv 10 \pmod{16}$. If $N = x^2+2y^2+3z^2$ is divisible by 4, then y is even and $x \equiv z \pmod 2$. If x and z are even, all three coordinates are already even. When x and z are odd, replace z by $-z$ if necessary so that $x \equiv z \pmod 4$; then $X = (x+3z)/2$ and $Z = (x-z)/2$ are even and $X^2+3Z^2 = x^2+3z^2$. In either case all three coordinates can be taken even, so $N/4$ is represented. Repeating this, a representation of $4^a(16b+10)$ would give one of $16b+10$, which is impossible modulo 16.

For the odd-prime rows, write the form as $x^2+\alpha y^2+pz^2$, with $(\alpha, p) = (1,3)$ or $(2,5)$. The binary form $x^2+\alpha y^2$ is anisotropic modulo p, meaning that its only zero over $\mathbb{F}_p$ is $(0,0)$. Consequently, if a represented integer N is divisible by p, then $x = pX$, $y = pY$, and

$$N = p(pX^2+\alpha pY^2+z^2).$$

At valuation 1, the unit part of N must therefore be a nonzero square modulo p. If $p^2 \mid N$, the same identity forces $p \mid z$; dividing all coordinates by p divides the represented value by p^2. Repeated division therefore excludes every integer with odd valuation and nonsquare unit part. The nonsquares are $\{2\}$ modulo 3 and $\{2,3\}$ modulo 5, giving the last two rows. The classical representation criteria supply sufficiency. The congruence and descent arguments also hold over $\mathbb{Z}_2$, $\mathbb{Z}_3$, or $\mathbb{Z}_5$, respectively, so the excluded integers are locally obstructed. This proves regularity for $\langle 1,2,3\rangle$, $\langle 1,1,3\rangle$, and $\langle 1,2,5\rangle$. The Fibonacci truants now follow from Lemma 2.4. □

3.3 The determinant-7 ternary

For the ninth ternary $f = x^2+2y^2+2yz+4z^2$, local conditions do not characterize representation by f alone. We first characterize representation by at least one of f and $h = \langle 1,1,7\rangle$ using a condition at 7, then give two changes of variables that convert many h-representations into f-representations. Together with a few explicit witnesses, these determine the first two Fibonacci exceptions of f. To prove Fibonacci-universality of the extensions in §6, we combine this representation input with square transfers.

Proposition 3.3 (Representation by f or h)**.** *Put $h = \langle 1,1,7\rangle$. Call a positive integer N* eligible *if*

$$v_7(N) \text{ is even} \quad \text{or} \quad u_7(N) \equiv 1,2,4 \pmod 7.$$

Then N is represented by f or by h if and only if N is eligible.

Proof. We first reduce to three isometry classes and distinguish one of them by a residue invariant. Sufficiency will then follow by constructing a positive-definite Gram matrix of determinant 7 that represents N and has the invariant of f and h.

The three classes. A positive-definite integral ternary lattice of determinant 7 has a Minkowski-reduced Gram matrix

$$G = \begin{pmatrix} a & r & s \\ r & b & t \\ s & t & c \end{pmatrix}, \qquad 1 \le a \le b \le c, \quad 2|r|, 2|s| \le a, \quad 2|t| \le b.$$

The classical ternary reduction bound gives $abc \le 2 \det G = 14$; see Mahler [17, p. 193, (3)]. Hence $a \in \{1, 2\}$. If $a = 1$, then $r = s = 0$ and $bc - t^2 = 7$. Since $b \le c$ and $|t| \le b/2$, we have $3b^2 \le 28$, so $b \le 3$. The only possibilities are $(b, c, t) = (1, 7, 0)$ and $(2, 4, \pm 1)$, giving h and f.

If $a = 2$, then $b = 2$ and $c \in \{2, 3\}$, with $|r|, |s|, |t| \le 1$. The case $c = 2$ has even determinant: modulo 2, a symmetric matrix of odd size with zero diagonal is alternating and singular. For $c = 3$, the determinant equation gives

$$r^2 = 1, \qquad s^2 + t^2 - rst = 1.$$

If one of s, t vanishes, changing signs and interchanging the first two basis vectors gives the Gram matrix of a third lattice K_7:

$$G_{K_7} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}.$$

Otherwise $rst = 1$; changing signs makes $r = s = t = 1$, and the basis $(e_1, -e_2, e_3 - e_1)$, where e_i are the standard basis vectors, again gives G_{K_7}. Thus every positive-definite integral ternary lattice of determinant 7 is isometric to f, h, or K_7.

The residue invariant. A Gram matrix of determinant 7 has rank 2 modulo 7: rank at most 1 would force its determinant to be divisible by 49. Its form over $\mathbb{F}_7$ therefore has a one-dimensional radical, the subspace pairing to zero with every vector. The induced form on the quotient by this radical is a nonsingular binary form. The determinant of its Gram matrix has a well-defined square class: changing basis multiplies it by the square of the basis-change determinant. Any nonsingular principal 2×2 minor lies in this square class: the corresponding coordinate plane meets the radical trivially and maps isomorphically onto the quotient. For f, h, and K_7, the leading principal minors of order 2 are 2, 1, and 3, respectively. Thus the invariant is a square for f and h and a nonsquare for K_7.

Necessity. Modulo 7, we have $f(x, y, z) \equiv x^2 + 2(y + 4z)^2$. Since -2 is a nonsquare modulo 7, $7 \mid f(x, y, z)$ forces $x = 7X$ and $y = -4z + 7Y$, giving

$$f(x, y, z) = 7\big(7X^2 + 14Y^2 - 14Yz + 4z^2\big).$$

At valuation 1 the unit part is congruent to $4z^2$ modulo 7, a nonzero square. If 49 divides the value, then $7 \mid z$, hence $7 \mid x, y, z$, and the value divided by 49 is again represented. For h, the same descent applies: since -1 is a nonsquare modulo 7, $7 \mid x^2 + y^2 + 7z^2$ forces $x = 7X$, $y = 7Y$, and $h(x, y, z) = 7(7X^2 + 7Y^2 + z^2)$. Hence every value of f or h is eligible whenever it is positive.

Sufficiency. Scaling a vector by 7 multiplies its value by 49 for both f and h. Since division by 49 preserves eligibility, it suffices to consider an eligible N with $49 \nmid N$. We construct the required Gram matrix separately according as $7 \nmid N$ or $N = 7u$. In each case, we put N in the first diagonal entry, choose a leading binary minor using an auxiliary prime, and solve the determinant equation by a quadratic congruence. We choose the auxiliary prime so that this congruence is solvable and the resulting matrix has the required square class modulo 7.

Suppose first that $7 \nmid N$, and write $N = 2^e m$ with m odd. By the Chinese remainder theorem and Dirichlet's theorem choose a prime $\ell > 7N$ with

$$\ell \equiv -1 \pmod{8N}, \qquad \ell \equiv 1 \pmod{7}.$$

The two moduli are coprime, and the prescribed class is prime to both. Also, $(2/\ell) = 1$ because $\ell \equiv 7 \pmod 8$, and $(-7/\ell) = (\ell/7) = 1$. Since $\ell \equiv 3 \pmod 4$ and $\ell \equiv -1 \pmod m$, Jacobi reciprocity gives $(m/\ell) = (-1/m)(\ell/m) = (-1/m)^2 = 1$. Hence $(-7N/\ell) = 1$; choose λ with

$\lambda^2 \equiv -7N \pmod{\ell}$. Put $a_0 = (\ell+1)/N$ and $c_0 = (7 + a_0\lambda^2)/\ell$. The congruence for ℓ makes a_0 integral, and c_0 is integral because $a_0 N \equiv 1 \pmod{\ell}$ gives $a_0\lambda^2 \equiv -7 \pmod{\ell}$. The matrix

$$G = \begin{pmatrix} N & 1 & \lambda \\ 1 & a_0 & 0 \\ \lambda & 0 & c_0 \end{pmatrix}$$

has $\det G = c_0(a_0N - 1) - a_0\lambda^2 = c_0\ell - a_0\lambda^2 = 7$ and leading minors N and ℓ. Thus G is positive definite of determinant 7 and represents N. Its leading binary minor is $\ell \equiv 1 \pmod{7}$, so its residue invariant is a square.

Suppose next that $N = 7u$ with $u \equiv 1, 2, 4 \pmod{7}$, and write $u = 2^e m$ with m odd. Since $7 \nmid u$, Dirichlet's theorem supplies a prime $\ell > 7u$ with $\ell \equiv -7 \pmod{8u}$. Then $\ell \equiv 1 \pmod{8}$, so

$$\left(\frac{-u}{\ell}\right) = \left(\frac{m}{\ell}\right) = \left(\frac{\ell}{m}\right) = \left(\frac{-7}{m}\right) = \left(\frac{m}{7}\right) = \left(\frac{u}{7}\right) = 1.$$

Since $\ell > 7$, choose η with $49\eta^2 \equiv -u \pmod{\ell}$, and put $a_0 = (\ell+7)/u$ and $c_0 = (1 + 7a_0\eta^2)/\ell$. Since $a_0 u \equiv 7 \pmod{\ell}$ and $49\eta^2 \equiv -u \pmod{\ell}$, we have $7(1 + 7a_0\eta^2) \equiv 7 - a_0 u \equiv 0 \pmod{\ell}$. As $\ell \neq 7$, this proves that c_0 is integral. The matrix

$$G = \begin{pmatrix} 7u & 7 & -7\eta \\ 7 & a_0 & 0 \\ -7\eta & 0 & c_0 \end{pmatrix}$$

has $\det G = 7(\ell c_0 - 7a_0\eta^2) = 7$ and leading minors $7u$ and 7ℓ. These are positive, so G is positive definite; its first diagonal entry shows that it represents N. Modulo 7 it is $\operatorname{diag}(0, a_0, c_0)$. Since $a_0 u \equiv \ell$ and $\ell c_0 \equiv 1 \pmod{7}$, its nonzero principal minor satisfies $a_0 c_0 \equiv u^{-1} \pmod{7}$, a square.

In both cases the invariant excludes K_7, so G is isometric to f or to h, and N is represented by f or by h. □

The example $7 = h(0, 0, 1)$ shows that eligibility does not imply representation by f alone. Indeed, $2y^2 + 2yz + 4z^2 = 2\left(y + \frac{z}{2}\right)^2 + \frac{7z^2}{2}$ takes only the values $0, 2, 4$ below 8 at integral (y, z), and 7, 5, and 3 are not squares. Nevertheless, 7 is locally represented by f: for every odd prime p, the vector $(0, -2, 1/2) \in \mathbb{Z}_p^3$ has value 7; over $\mathbb{Z}_2$, choose θ with $\theta^2 = -15$, which exists because $-15 \equiv 1 \pmod{8}$, and use $f(\theta, 1, 2) = 7$. Thus f is irregular (see also [12, 13]). The next lemma converts an h-representation into an f-representation in most residue classes modulo 12. Six additional witnesses will then determine the first two Fibonacci exceptions of f.

Lemma 3.4 (From h to f modulo 12)**.** *Let $N \in \mathbb{Z}$. If N is represented by f or h and $N \not\equiv 7, 10 \pmod{12}$, then N is represented by f.*

Proof. Only an h-representation $N = a^2 + b^2 + 7c^2$ needs consideration. The first change of variables below is integral under a parity condition; the second is integral under a condition modulo 3. Direct calculation gives

$$f\left(-b, 2c, -\frac{a+c}{2}\right) = a^2 + b^2 + 7c^2,$$
$$f\left(\frac{-a-b+7c}{3}, \frac{-a+2b+c}{3}, \frac{-a-b-2c}{3}\right) = a^2 + b^2 + 7c^2.$$

If $N \equiv 0, 1 \pmod{4}$, one of a, b has the same parity as c: otherwise both have parity opposite to c, giving $N \equiv 2$ or $3 \pmod{4}$. Interchange a, b if necessary and use the first identity.

If $N \not\equiv 1 \pmod 3$, the number of nonzero coordinates among a, b, c modulo 3 is zero, two, or three. Change their signs so that $-a - b + c \equiv 0 \pmod 3$. This is possible in each of those three cases; all numerators in the second identity are then divisible by 3.

The only residue classes modulo 12 satisfying neither condition are 7 and 10. □

The modulo-12 sufficient condition in Lemma 3.4 also appears, in terms of local representability, in Bhargava's proof of the Fifteen Theorem [3, §4, p. 30]. The two identities in our proof are the maps $B_2/2$ and $B_3/3$ from our Table 3 in §A.

Proposition 3.5 (First Fibonacci exceptions)**.** *The Fibonacci truant of f is $F_8 = 21$, and its next Fibonacci exception is $F_{48} = 4807526976$.*

Proof. The representations

N	1	2	3	5	8	13
(x, y, z)	$(1, 0, 0)$	$(0, 1, 0)$	$(1, 1, 0)$	$(1, 0, 1)$	$(0, 2, 0)$	$(3, 0, 1)$

cover $F_1, \ldots, F_7$, while $21 = 7 \cdot 3$ is locally obstructed. For $9 \le n \le 47$, the Fibonacci recurrence modulo 7 shows that the only multiples of 7 occur at $n = 16, 24, 32, 40$. Their residues modulo 49 are $7, 14, 14, 7$, respectively, so every term in the interval is eligible. Modulo 12, the only terms congruent to 7 or 10 occur at $n = 9, 10, 15, 33, 34, 39$. Thus Proposition 3.3 and Lemma 3.4 show that f represents F_n at all other indices in the interval. The six remaining representations are

n	(x, y, z)
9	$(4, 3, 0)$
10	$(3, 3, 2)$
15	$(12, 7, 8)$
33	$(12, 1051, 368)$
34	$(9, 1473, 322)$
39	$(12, 5289, 568)$

and are verified by substitution into f.

Finally, $F_{48} \equiv 21 \pmod{49}$, so it has valuation 1 at 7 and nonsquare unit part $3 \pmod 7$; Proposition 3.3 excludes it. □

The truants encountered through rank 3 are exactly the elements of C. Table 1 collects one escalator for each value. These same escalators will provide the omission examples in the minimality proof.

3.4 Each truant raises the rank

For the unary and binary escalators we already checked that adjoining a vector of truant norm raises the rank. The local criteria give the same conclusion for all nine ternaries. This ensures that an escalation inside an ambient lattice reaches rank 4, rather than merely enlarging a sublattice of the same rational space.

Lemma 3.6 (No same-rank escalation)**.** *Let T be a Fibonacci escalator of rank at most 3, with truant τ_T, and let L be an integral lattice containing T. Then τ_T is not represented by $T \otimes \mathbb{Q}$. In particular, if $u \in L$ has norm τ_T, then $u \notin T \otimes \mathbb{Q}$, so $T + \mathbb{Z}u$ has rank one larger than T.*

Table 1: The criterion values and escalators having them as truants. The last two columns give an independent congruence check: the values of P_t modulo m_t omit t mod m_t.

Truant t	Escalator P_t	m_t	$t \bmod m_t$
1	zero lattice	2	1
2	$\langle 1 \rangle$	4	2
3	$\langle 1, 1 \rangle$	4	3
5	$\langle 1, 2 \rangle$	8	5
21	f	49	21
55	$\langle 1, 1, 1 \rangle$	8	7
610	$\langle 1, 2, 5 \rangle$	25	10
987	$\langle 1, 1, 3 \rangle$	9	6
832040	$\langle 1, 2, 3 \rangle$	64	40
267914296	$\langle 1, 1, 2 \rangle$	64	56

Proof. The zero, unary, and binary cases were established above. For a ternary escalator, its truant lies in the local obstruction described in Propositions 3.2 and 3.3. Each obstruction is unchanged by multiplication by a nonzero integer square: its valuation changes by an even integer, and its unit part changes by a square. At 2, an odd square is $1 \pmod 8$; at the odd primes occurring in the obstructions, square multiplication preserves whether a unit is a square or nonsquare. Thus every nonzero integer-square multiple of the truant is still locally obstructed. Clearing denominators in a rational representation would produce an integral representation of one of these multiples, a contradiction. □

4 Extensions of the ternary escalators

To prove sufficiency, it remains to show that every positive-definite integral quaternary lattice containing one of the nine ternary escalators and representing its Fibonacci truant is Fibonacci-universal. Sections 5 and 6 prove this for the regular and irregular ternaries, respectively. This section supplies the common reduction to orthogonal sublattices and bounds the norm of the adjoined orthogonal vector.

Let M be such a lattice, containing the ternary escalator T and a vector w of truant norm. By Lemma 3.6, $T + \mathbb{Z}w$ has rank 4, and M represents every integer that this sublattice represents. It is therefore enough to treat lattices of the form $T + \mathbb{Z}v$; no primitivity assumption on T is needed. The generator v need not be perpendicular to T. We construct an orthogonal sublattice $T \perp \langle q \rangle$ of $T + \mathbb{Z}v$ and bound q; representing $F_n - qk^2$ by T then gives a representation of F_n by the extension.

4.1 Extensions and their orthogonal sublattices

Let T be a ternary escalator with basis b_1, b_2, b_3, Gram matrix G_T, and Fibonacci truant τ_T. An *extension* of T is a positive-definite integral lattice of rank 4 of the form

$$M = T + \mathbb{Z}v, \qquad \mathrm{Gram}(M) = \begin{pmatrix} G_T & c \\ c^{\mathsf{T}} & m \end{pmatrix}, \qquad c_i = \beta_M(v, b_i), \quad m = Q_M(v).$$

We must prove Fibonacci-universality for the extensions that represent τ_T. An extension need not satisfy this representation condition unless it is explicitly imposed.

Every escalation of T is an extension that represents τ_T. For an arbitrary extension representing τ_T, however, a vector w of that norm need not generate M together with T: the sublattice $T + \mathbb{Z}w$ may be proper because the coordinate of w along v need not be ± 1.

Identify vectors of T with their coordinate columns in this basis. Replacing v by $v + x$ with $x \in T$ changes c to $c + G_T x$ without changing M. We may therefore choose c from a finite set of representatives for $\mathbb{Z}^3/G_T\mathbb{Z}^3$. The chosen generator v need not have norm τ_T, even when M was originally obtained by escalation.

The representatives are chosen as follows. If T is diagonal, with $G_T = \operatorname{diag}(1, a_2, a_3)$, then the quotient $\mathbb{Z}^3/G_T\mathbb{Z}^3$ is the product of the coordinate quotients $\mathbb{Z}/a_2\mathbb{Z}$ and $\mathbb{Z}/a_3\mathbb{Z}$. Let e_i denote the standard basis vectors of $\mathbb{Z}^3$. For f, the columns of G_f give $[e_1] = 0$, $2[e_2] + [e_3] = 0$, and $[e_2] + 4[e_3] = 0$ in $\mathbb{Z}^3/G_f\mathbb{Z}^3$. Thus $[e_2] = -4[e_3]$ and $7[e_3] = 0$. Since the quotient has order $\det G_f = 7$, it is cyclic of order 7, generated by $[e_3]$. For g, the same computation gives $[e_2] = -5[e_3]$ and $9[e_3] = 0$, so the quotient is cyclic of order 9, generated by $[e_3]$. Table 2 lists the resulting representatives for the nine escalators.

Extend the bilinear pairing to $M \otimes \mathbb{Q}$. The orthogonal projection of v onto $T \otimes \mathbb{Q}$ has coordinate column $G_T^{-1}c$. Hence

$$v_\perp = v - \sum_{i=1}^{3} (G_T^{-1}c)_i b_i$$

is perpendicular to T, and its norm is the Schur complement

$$\delta = Q_M(v_\perp) = m - c^{\mathsf{T}} G_T^{-1} c > 0.$$

Let d be the least positive integer for which $dG_T^{-1}c \in \mathbb{Z}^3$, that is, the least common denominator of the entries of $G_T^{-1}c$, and put $q = d^2\delta$. Thus m is the norm of the chosen generator, δ is the norm of its rational perpendicular component, and q is the norm of the integral vector obtained by clearing denominators. It is q, rather than the diagonal entry m of M, that appears in the orthogonal summand $\langle q \rangle$. Table 2 gives $G_T^{-1}c$ and $c^{\mathsf{T}}G_T^{-1}c$ for each representative. For example, for $T = \langle 1, 2, 2 \rangle$ and $c = (0, 1, 1)$ it gives $d = 2$ and $q = 4(m - 1)$.

Table 2: Representatives $c = (0, c_2, c_3)$ for $\mathbb{Z}^3/G_T\mathbb{Z}^3$ and the quantities used to compute d and q.

T	representatives	$G_T^{-1}c$	$c^{\mathsf{T}}G_T^{-1}c$
$\langle 1,1,1 \rangle$	$c = 0$	0	0
$\langle 1,1,2 \rangle$	$c_2 = 0$, $c_3 \in \{0,1\}$	$(0, 0, c_3/2)$	$c_3^2/2$
$\langle 1,1,3 \rangle$	$c_2 = 0$, $c_3 \in \{0,1,2\}$	$(0, 0, c_3/3)$	$c_3^2/3$
$\langle 1,2,2 \rangle$	$c_2, c_3 \in \{0,1\}$	$(0, c_2/2, c_3/2)$	$(c_2^2 + c_3^2)/2$
$\langle 1,2,3 \rangle$	$c_2 \in \{0,1\}$, $c_3 \in \{0,1,2\}$	$(0, c_2/2, c_3/3)$	$c_2^2/2 + c_3^2/3$
$\langle 1,2,4 \rangle$	$c_2 \in \{0,1\}$, $c_3 \in \{0,1,2,3\}$	$(0, c_2/2, c_3/4)$	$c_2^2/2 + c_3^2/4$
$\langle 1,2,5 \rangle$	$c_2 \in \{0,1\}$, $c_3 \in \{-2,\dots,2\}$	$(0, c_2/2, c_3/5)$	$c_2^2/2 + c_3^2/5$
f	$c_2 = 0$, $c_3 \in \{0,\dots,6\}$	$(0, -c_3/7, 2c_3/7)$	$2c_3^2/7$
g	$c_2 = 0$, $c_3 \in \{0,\dots,8\}$	$(0, -c_3/9, 2c_3/9)$	$2c_3^2/9$

Lemma 4.1 (The orthogonal sublattice)**.** *Let $M = T + \mathbb{Z}v$ be an extension of T.*

(i) *q is a positive integer, and M contains $T \perp \langle q \rangle$ with index d.*

(ii) *For every $x \in T$ and every $k \in \mathbb{Z}$,*

$$d^2\, Q_M(x + kv) = Q_T(dx + k\, dG_T^{-1}c) + qk^2. \tag{2}$$

(iii) *For every integer N, if there is an integer $k \geq 0$ such that $N - qk^2 \notin E(T)$, then M represents N.*

Proof. Multiplying $v_\perp$ by d gives the integral vector

$$w_\perp = dv_\perp = dv - \sum_{i=1}^{3} (dG_T^{-1}c)_i b_i \in M.$$

This vector is orthogonal to T and has norm $d^2\delta = q$. Since M has integral Gram matrix, $q \in \mathbb{Z}_{>0}$. Moreover, $T + \mathbb{Z}w_\perp$ has index d in $T + \mathbb{Z}v$, as is seen from the coefficient of v. This proves (i). For (ii), write

$$d(x + kv) = \big(dx + k\, dG_T^{-1}c\big) + kw_\perp,$$

where the first summand lies in T and the second is orthogonal to T, and take norms. For (iii), if $Q_T(x) = N - qk^2$, then $x + kw_\perp$ represents N in M. □

The condition in part (iii) is sufficient but not necessary: the sublattice $T \perp \mathbb{Z}w_\perp$ has index d in M, and vectors of M outside it can represent integers that the sublattice omits. We use such vectors in Propositions 5.7, 5.11, and 5.12. In the other direction, part (ii) shows that $T \perp \langle q \rangle$ represents d^2N whenever M represents N. Applied to the truant, this bounds q.

Lemma 4.2 (Extensions representing the truant)**.** *Let M be an extension of T. Then d divides $\det G_T$. If M represents τ_T, then there are a vector $y \in T$ and an integer $k \geq 1$ such that*

$$d^2\tau_T = Q_T(y) + qk^2. \tag{3}$$

In that case

$$\delta \leq \tau_T \tag{4}$$

and

$$q \leq \tau_T(\det G_T)^2. \tag{5}$$

Proof. The matrix $(\det G_T)G_T^{-1}$ is the adjugate of G_T and is integral, so $(\det G_T)G_T^{-1}c \in \mathbb{Z}^3$ and $d \mid \det G_T$. Suppose that $Q_M(x + kv) = \tau_T$ with $x \in T$ and $k \in \mathbb{Z}$. Then $k \neq 0$, since T omits τ_T, and after replacing the vector by its negative we may assume $k \geq 1$. Now (2) gives (3) with $y = dx + k\, dG_T^{-1}c$. Since $Q_T(y) \geq 0$ and $k \geq 1$, we obtain $q \leq d^2\tau_T$, which is (4); and (5) follows because $d \leq \det G_T$. □

For an extension representing τ_T and a fixed representative c, (4) gives $m \leq \lfloor \tau_T + c^{\mathsf{T}} G_T^{-1} c \rfloor$. Over all representatives, this gives $m \leq 56$ for $\langle 1, 2, 2 \rangle$, $m \leq 988$ for $\langle 1, 1, 3 \rangle$, $m \leq 611$ for $\langle 1, 2, 5 \rangle$, and $m \leq 31$ for f.

4.2 Values of q for $\langle 1,1,3\rangle$, $\langle 1,2,5\rangle$, and f

The subtraction arguments also require divisibility information about q. For extensions of $\langle 1,1,3\rangle$ and $\langle 1,2,5\rangle$ representing the truant, we prove $v_3(q) \le 1$ and $v_5(q) \le 1$, respectively. For f, we determine a finite set containing every possible q; §6 treats this set.

Lemma 4.3 (Valuation bound)**.** *Let T be $\langle 1,1,3\rangle$ or $\langle 1,2,5\rangle$, and let p be 3 or 5, respectively. Every extension of T that represents τ_T satisfies $v_p(q) \le 1$.*

Proof. Use the representatives for c in Table 2. First suppose $c_3 \neq 0$. For $\langle 1,1,3\rangle$ we have $d = 3$ and

$$q = 9m - 3c_3^2,$$

so $v_3(q) = 1$. For $\langle 1,2,5\rangle$, we have $d \in \{5, 10\}$ and

$$q = d^2 m - \frac{d^2 c_2^2}{2} - \frac{d^2 c_3^2}{5}.$$

The first two terms are divisible by 25 (when $d = 5$, we have $c_2 = 0$), and the last has valuation 1 at 5, so $v_5(q) = 1$.

Now suppose $c_3 = 0$, so that $p \nmid d$, and suppose that $p^2 \mid q$. By Lemma 4.2 there are $y \in T$ and $k \ge 1$ with

$$Q_T(y) = d^2\tau_T - qk^2 \equiv d^2\tau_T \pmod{p^2}.$$

The truant has valuation 1 at p and nonsquare unit part modulo p:

$$987 = 3 \cdot 329, \quad 329 \equiv 2 \pmod 3; \qquad 610 = 5 \cdot 122, \quad 122 \equiv 2 \pmod 5.$$

Since $p \nmid d$, the integer $Q_T(y)$ therefore has valuation 1 at p, and its unit part is $d^2 u_p(\tau_T)$ modulo p, a nonsquare. By Proposition 3.2, such an integer is not represented by T, a contradiction. Hence $v_p(q) \le 1$. □

Lemma 4.4 (Possible values of q for f)**.** *Every quaternary extension of f representing 21 contains $f \perp \langle q\rangle$ with q in the set*

$$\begin{aligned} \mathcal{Q}_7 = {} & \{q \in \mathbb{Z} : 1 \le q \le 21,\ 7 \nmid q\} \\ & \cup \{7\} \\ & \cup \{7s : s \in \mathbb{Z},\ 1 \le s \le 147,\ s \equiv 3, 5, 6 \pmod 7\}. \end{aligned} \tag{6}$$

The set $\mathcal{Q}_7$ has 82 elements, and its largest element is 1022.

Proof. By Table 2, $c = (0, 0, c_3)$ with $0 \le c_3 \le 6$. If $c_3 \neq 0$, then $d = 7$ and $q = 49m - 14c_3^2 = 7s$ with $s = 7m - 2c_3^2$. The bound (5) is $q \le 21 \cdot 7^2 = 1029$, so $s \le 147$, and $s \equiv -2c_3^2 \equiv 3, 5, 6 \pmod 7$. If $c_3 = 0$, then the extension is $f \perp \langle m\rangle$, and $q = m \le 21$ by (4). Among the multiples of 7 in this range, the values 7 and 21 belong to the displayed set. The value 14 is impossible: a representation of 21 by $f \perp \langle 14\rangle$ would require f to represent 21 or 7. Both values are omitted by f, as shown in §3.3. The three sets in (6) are disjoint and have 18, 1, and 63 elements; the largest element is $7 \cdot 146 = 1022$. □

5 Extensions of the regular ternaries

Let M be a quaternary extension of one of the eight regular ternaries T, and suppose that M represents the truant of T. Any Fibonacci number already represented by T is also represented by M, so only the Fibonacci exceptions of T require attention. We usually represent such an exception F_n by choosing k so that $F_n - qk^2$ is nonnegative and lies outside the exceptional set in Proposition 3.2. When the orthogonal sublattice does not suffice, we use vectors of M outside it. We treat the six ternaries with obstructions at 2 together, and then $\langle 1,1,3\rangle$ and $\langle 1,2,5\rangle$.

5.1 Ternaries with obstructions at 2

The six ternaries in the first three rows of Proposition 3.2 have obstructions at 2. We treat their extensions together.

Proposition 5.1 (Fibonacci-universality in the 2-adic cases)**.** *Let T be one of the six ternaries with obstructions at 2. Every quaternary extension of T representing τ_T is Fibonacci-universal.*

The proof occupies this subsection. Throughout, $T \perp \langle q\rangle$ is the orthogonal sublattice supplied by Lemma 4.1, and the argument divides according to $v_2(q)$. If $v_2(q) \le v_2(\tau_T) + 2$, then subtracting a suitable qk^2 from a Fibonacci exception larger than τ_T leaves an integer represented by T (Lemmas 5.2 and 5.3); the one pair (g, F_{22}) that this leaves open is treated in Lemma 5.5. If $v_2(q) \ge v_2(\tau_T)+3$, the orthogonal sublattice does not represent the truant. In this case Lemmas 5.4 and 5.6 and the hypothesis on the truant leave exactly fourteen extensions, all of $\langle 1,2,2\rangle$; Proposition 5.7 lists them and constructs representations of the odd Fibonacci exceptions of $\langle 1,2,2\rangle$ using vectors outside their orthogonal sublattices.

For $\varepsilon \in \{0,1\}$ and odd r, put

$$\mathcal{E}_{\varepsilon,r} = \{N \in \mathbb{Z}_{>0} : v_2(N) \equiv \varepsilon \pmod 2,\ u_2(N) \equiv r \pmod 8\}.$$

By Proposition 3.2 and Lemma 2.4, the positive integers omitted by each of these six ternaries T form the family $\mathcal{E}_{\varepsilon,r}$ with

T	(ε, r)
$\langle 1,1,1\rangle,\ \langle 1,2,2\rangle,\ g$	$(0,7)$
$\langle 1,1,2\rangle,\ \langle 1,2,4\rangle$	$(1,7)$
$\langle 1,2,3\rangle$	$(1,5)$.

Subtracting qk^2 can leave such a family in either of two ways: it can change the parity of the valuation, or change the odd part modulo 8. The next lemma gives explicit choices of k and identifies when the subtraction cannot remove the obstruction.

Lemma 5.2 (2-adic subtraction)**.** *Let $N = 2^a u \in \mathcal{E}_{\varepsilon,r}$ and $q = 2^s\omega \in \mathbb{Z}_{>0}$, where a, s are nonnegative integers and u, ω are odd integers.*

(i) *If $s \ge a+3$, then for every integer k,*

$$v_2(N - qk^2) = a, \qquad 2^{-a}(N - qk^2) \equiv r \pmod 8.$$

In particular, every positive difference $N - qk^2$ remains in $\mathcal{E}_{\varepsilon,r}$.

(ii) *If $s \le a+2$ and $N \ge 2^{\max(4,a)}q$, then there is an integer $k \ge 1$ such that*

$$0 \le N - qk^2 \quad \text{and} \quad N - qk^2 \notin \mathcal{E}_{\varepsilon,r}.$$

Proof. For (i), qk^2 is divisible by 2^{a+3}, so subtraction preserves the valuation and the odd part modulo 8.

For (ii), put $e = s - a$. All choices below satisfy $k^2 \le \max(16, 2^a)$, so the size hypothesis gives $N - qk^2 \ge 0$. The displayed odd factors exclude zero, making the differences positive and their unit parts well defined. When $-2 \le e \le 2$, the following choices suffice. Write $R = N - qk^2$ in the table. The final column identifies the condition defining $\mathcal{E}_{\varepsilon,r}$ that fails.

e	k	$N - qk^2$	change in the difference
2	1	$2^a(u - 4\omega)$	$u_2(R) \equiv r - 4 \pmod 8$
1	1	$2^a(u - 2\omega)$	$u_2(R) \equiv r - 2\omega \not\equiv r \pmod 8$
0	2	$2^a(u - 4\omega)$	$u_2(R) \equiv r - 4 \pmod 8$
-1	1	$2^{a-1}(2u - \omega)$	$v_2(R) = a - 1 \not\equiv \varepsilon \pmod 2$
-2	4	$2^a(u - 4\omega)$	$u_2(R) \equiv r - 4 \pmod 8$.

Every parenthesized factor is odd, giving the displayed valuations and residues of the odd parts. In the second row, $2\omega \equiv 2$ or $6 \pmod 8$.

For $e \le -3$, write $j = a - s \ge 3$. If j is odd, take $k = 1$:

$$N - q = 2^s(2^j u - \omega), \qquad v_2(N - q) = s \not\equiv a \pmod 2.$$

If j is even, compare $k = 1$ and $k_0 = 2^{(j-2)/2}$:

k	$v_2(N - qk^2)$	$u_2(N - qk^2) \bmod 8$
1	s	$-\omega$
k_0	$a - 2$	$4 - \omega$.

The valuations have the same parity as a, but the residues of the odd parts differ by 4 modulo 8, so at least one is not r.

The stated bound on k^2 holds for the fixed choices $1, 2, 4$; for k_0, it follows from $k_0^2 = 2^{a-s-2} \le 2^a$, since $s \ge 0$. □

The next lemma supplies the valuation and size bounds needed to apply Lemma 5.2 (ii) to Fibonacci exceptions. We write $\overline{q}_T$ for the uniform upper bound on q in (5).

Lemma 5.3 (Fibonacci exceptions for obstructions at 2)**.** *Let T be one of the six ternaries with obstructions at 2, put*

$$\overline{q}_T = \tau_T(\det G_T)^2,$$

and let N be a Fibonacci exception of T.

(i) *We have $v_2(N) \ge v_2(\tau_T)$.*

(ii) *If $N > \tau_T$, then*

$$N \ge 2^{\max(4, v_2(N))}\overline{q}_T,$$

except when $(T, N) = (g, F_{22})$.

Proof. Write $N = F_n$. For (i), if $(\varepsilon, r) = (0, 7)$, the truant 55 has valuation 0, so $v_2(N) \ge v_2(55)$. In the other two cases an exception has odd valuation, and the truant has valuation 3 by Lemma 2.2. Valuation 1 occurs only at $n \equiv 3 \pmod 6$, where the Fibonacci recurrence modulo 16 gives $F_n \equiv 2 \pmod{16}$ and hence unit part $1 \pmod 8$, so that F_n is not an exception.

For (ii), we first note that, for $n \geq 4$, (1) gives

$$2^{\max(4,v_2(F_n))} \leq 4n.$$

Thus an inequality $F_n \geq 4n\overline{q}_T$ at one index proves the desired bound at every later index, by monotonicity of F_n/n. For the family $\mathcal{E}_{0,7}$, the first two exceptions after F_{10} are F_{22} and F_{34}, by Lemmas 2.2 and 2.3. Indeed, modulo 8 the period is 12, and $F_n \equiv 7 \pmod 8$ exactly when $n \equiv 10 \pmod{12}$. Among $1 \leq n \leq 34$, the only index with positive even 2-adic valuation is $n = 12$, where $u_2(F_{12}) = 9 \equiv 1 \pmod 8$. The bounds $\overline{q}_T$ for $\langle 1,1,1\rangle$, $\langle 1,2,2\rangle$, and g are 55, 880, and 4455, respectively. We have

$$F_{22} > 4 \cdot 22 \cdot 55, \qquad F_{34} > 4 \cdot 34 \cdot 4455.$$

The second inequality proves the required bound for all three ternaries from index 34 onward. At F_{22}, the first proves it for $\langle 1,1,1\rangle$, and $F_{22} > 16 \cdot 880$ proves it for $\langle 1,2,2\rangle$, since $v_2(F_{22}) = 0$. This proves the bound for that family apart from (g, F_{22}).

For the other two families, exceptions have odd valuation at least 3, by part (i). For these three escalators, $\overline{q}_T \leq 64F_{42}$, and

$$F_{64} = 10610209857723 > 4 \cdot 64 \cdot (64F_{42}).$$

Thus the required bound holds for every $n \geq 64$. By Lemma 2.2, the only indices $30 < n < 64$ with odd valuation at least 3 are 42 and 54, both of valuation 3; their unit parts are 7 and 1 modulo 8, respectively. The value F_{42} is the truant for the forms omitting $\mathcal{E}_{1,7}$, and neither F_{42} nor F_{54} belongs to $\mathcal{E}_{1,5}$. There are therefore no further exceptions below the cutoff. □

Consequently, if $v_2(q) \leq v_2(\tau_T) + 2$, then the extension M represents every Fibonacci exception $N > \tau_T$, apart from the possible case (g, F_{22}). Indeed, $v_2(q) \leq v_2(N) + 2$ by Lemma 5.3 (i), and $q \leq \overline{q}_T$ by (5), so Lemma 5.3 (ii) gives the size hypothesis of Lemma 5.2 (ii); Lemma 4.1 then applies to the resulting difference.

To prove representation in the case (g, F_{22}) and classify the extensions with $v_2(q) \geq v_2(\tau_T) + 3$, we first show that such an extension cannot represent the truant when d is odd.

Lemma 5.4 (The obstruction for odd d)**.** *Let T be one of the six ternaries with obstructions at 2. If an extension of T has odd d and $v_2(q) \geq v_2(\tau_T) + 3$, then it does not represent τ_T. In particular, an extension with $c = 0$ and $v_2(q) \geq v_2(\tau_T) + 3$ does not represent τ_T.*

Proof. Suppose that it represents τ_T. By Lemma 4.2 there are $y \in T$ and $k \geq 1$ with $Q_T(y) = d^2\tau_T - qk^2$. Since d is odd, $d^2 \equiv 1 \pmod 8$, so $d^2\tau_T$ has the same valuation as τ_T and the same odd part modulo 8; thus $d^2\tau_T \in \mathcal{E}_{\varepsilon,r}$. As $v_2(q) \geq v_2(d^2\tau_T) + 3$, part (i) of Lemma 5.2 shows that $d^2\tau_T - qk^2$ has the same valuation and the same odd part modulo 8. Since it equals $Q_T(y) \geq 0$ and has finite valuation, it is positive and thus lies in $\mathcal{E}_{\varepsilon,r}$. This contradicts its representation by T. When $c = 0$ we have $d = 1$. □

Lemma 5.5 (Representation of F_{22} by extensions of g)**.** *Every quaternary extension of g representing 55, the truant of g, also represents $F_{22} = 17711$.*

Proof. By Table 2, $c = (0, 0, c_3)$ with $0 \leq c_3 \leq 8$, and d divides 9; by (4), $\delta \leq 55$. If $d \leq 3$, then $q \leq 495$. If $d = 9$, then $3 \nmid c_3$ and

$$q = 81m - 18c_3^2.$$

Thus $9 \mid q$ when $d = 9$. In every case, $q = d^2\delta \leq 81 \cdot 55 = 4455$.

Since d is odd, Lemma 5.4 gives $v_2(q) \leq 2$.

Put $N = 17711 \equiv 7 \pmod 8$. If q is even, its valuation is 1 or 2, and $N - q > 0$ is odd with residue different from 7 modulo 8. If q is odd and $q \le N/4$, then $N - 4q > 0$ has residue 3 modulo 8. Each difference is represented by g. The only remaining case is odd $q > N/4$. It requires $d = 9$, and the only odd multiples of 9 in $(17711/4, 4455]$ are 4437 and 4455. The corresponding differences $N - q$ are

$$17711 - 4437 = 2 \cdot 6637, \qquad 17711 - 4455 = 2^3 \cdot 1657.$$

Both have odd 2-adic valuation and are represented by g. Apply Lemma 4.1 in each case. □

It remains to classify the extensions with $v_2(q) \ge v_2(\tau_T) + 3$. Such an extension can require vectors outside the orthogonal sublattice. For example, the extension of $T = \langle 1, 2, 2\rangle$ with $c = (0, 1, 1)$ and $m = 55$ has $q = 216$, and

$$F_{22} - 216k^2 \equiv 7 \pmod 8$$

for every integer k. This extension represents 55, the truant of T, as Proposition 5.7 will show, but its orthogonal sublattice omits F_{22}.

For $T = \langle 1, 2, 4\rangle$, the remaining extensions have even d and require a parity argument.

Lemma 5.6 (The remaining extensions for $\langle 1, 2, 4\rangle$)**.** *For $T = \langle 1, 2, 4\rangle$, no quaternary extension of T representing τ_T has $v_2(q) \ge v_2(\tau_T) + 3$.*

Proof. By Table 2, $c = (0, c_2, c_3)$ with $c_2 \in \{0, 1\}$ and $c_3 \in \{0, 1, 2, 3\}$. For odd c_3 we have $d = 4$ and $q = 16m - 8c_2^2 - 4c_3^2$, so $v_2(q) = 2$. For even c_3 and $c_2 = 1$ we have $d = 2$ and $q = 4m - 2 - c_3^2$, so $v_2(q) = 1$. After the case $c = 0$ is excluded by Lemma 5.4, the only possibility with $v_2(q) \ge 6$ is therefore $c = (0, 0, 2)$. In this case $q = 4(m - 1)$, so $m \equiv 1 \pmod{16}$, and

$$Q(x, y, z, w) = x^2 + 2y^2 + (2z + w)^2 + (m - 1)w^2.$$

The truant satisfies $F_{42} \equiv 8 \pmod{16}$. If w is odd, then $(2z + w)^2$ is 1 or 9 modulo 16 and $(m - 1)w^2 \equiv 0 \pmod{16}$, whereas $x^2 + 2y^2$ is never 7 or 15 modulo 16. Thus no vector with odd w represents F_{42}. If $w = 2w'$, write instead

$$Q = x^2 + 2y^2 + 4(z + w')^2 + 4(m - 1)w'^2.$$

The final term is divisible by 64. Since $v_2(F_{42}) = 3$, subtracting it preserves the valuation and the odd part modulo 8. In a representation of F_{42}, the difference would equal $x^2+2y^2+4(z+w')^2 \ge 0$; its valuation would force it to be positive. It would therefore lie in $\mathcal{E}_{1,7}$, contradicting the representation criterion for $\langle 1, 2, 4\rangle$. Neither parity of w permits a representation of F_{42}, contrary to the hypothesis on the extension. □

Proposition 5.7 (Extensions with large $v_2(q)$)**.** *Among the six ternaries T with obstructions at 2, only $\langle 1, 2, 2\rangle$ has quaternary extensions that represent the truant and have $v_2(q) \ge v_2(\tau_T) + 3$. With the representatives of Table 2, the fourteen extensions are given by*

$$c = (0, 1, 1), \qquad m \equiv 3 \pmod 4, \qquad 3 \le m \le 55.$$

Each of them is Fibonacci-universal.

Proof. We use the representatives of Table 2 throughout, and first show that the other five ternaries have no such extensions. For $\langle 1, 1, 1\rangle$ and g, the integer d is always odd, so Lemma 5.4 applies. For $\langle 1, 1, 2\rangle$, the only nonzero representative for c is $c = (0, 0, 1)$; it gives $q = 4m - 2$ and $v_2(q) = 1$.

Extensions with $c = 0$ and the stated valuation bound are excluded by Lemma 5.4, and Lemma 5.6 excludes such extensions of $\langle 1, 2, 4\rangle$.

For $\langle 1, 2, 3\rangle$, if $c_2 = 1$ then $d \in \{2, 6\}$ and $q = d^2m - d^2/2 - d^2c_3^2/3$ has valuation 1. The remaining nonzero representatives for c are $(0, 0, 1)$ and $(0, 0, 2)$, with $d = 3$ and respective values

$$q = 3(3m - 1), \qquad q = 3(3m - 4).$$

Since d is odd for these and for $c = 0$, Lemma 5.4 excludes every extension with $v_2(q) \geq v_2(\tau_T) + 3$.

Now let $T = \langle 1, 2, 2\rangle$. A representative c with exactly one of c_2, c_3 nonzero gives $q = 4m - 2$, of valuation 1; extensions with $c = 0$ are already excluded. Thus only $c = (0, 1, 1)$ remains, with

$$q = 4(m - 1), \qquad \delta = m - 1.$$

The bounds $0 < \delta \leq 55$ give $2 \leq m \leq 56$, and $v_2(q) \geq 3$ is equivalent to m being odd.

The quadratic form of the extension is $Q = x^2 + 2y^2 + 2z^2 + 2(y + z)w + mw^2$. If $m \equiv 1 \pmod 4$, it omits 55. For even $w = 2r$, completing squares in y, z gives

$$Q = x^2 + 2(y + r)^2 + 2(z + r)^2 + 4(m - 1)r^2,$$

which cannot be $7 \pmod 8$: the first three terms form a value of $\langle 1, 2, 2\rangle$, and $4(m - 1)r^2 \equiv 0 \pmod 8$ because m is odd. For odd w, use $w^2 \equiv 1 \pmod 8$ to obtain

$$Q \equiv x^2 + m + 2y(y + w) + 2z(z + w) \pmod 8.$$

Each of the last two terms is 0 or $4 \pmod 8$; since $m \equiv 1$ or $5 \pmod 8$, the sum again cannot be $7 \pmod 8$.

For $m \equiv 3 \pmod 4$, $3 \leq m \leq 55$, a single construction proves both that the extension represents 55 and that it represents all odd Fibonacci exceptions of T. At $w = 1$ its value is

$$Q(x, y, z, 1) = x^2 + (y + z + 1)^2 + (y - z)^2 + m - 1.$$

If $N \geq 55$ and $N \equiv 7 \pmod 8$, then $N - (m - 1) > 0$ is 1 or $5 \pmod 8$, so it is a sum of three squares. Since it is also $1 \pmod 4$, any such representation has one odd coordinate and two even coordinates. Place an even coordinate last and write

$$N - (m - 1) = X^2 + Y^2 + 4Z^2.$$

The integers X, Y have opposite parity. Assign $2Z$ to x and solve $y + z + 1 = X$, $y - z = Y$. This gives the integral representing vector

$$x = 2Z, \qquad y = \frac{X + Y - 1}{2}, \qquad z = \frac{X - Y - 1}{2}, \qquad w = 1.$$

Taking $N = 55$ proves that all fourteen extensions represent the truant of T. These vectors have $w = 1$, whereas vectors in the orthogonal sublattice have even w; the vectors outside the sublattice supply the missing representations.

It remains to represent the even Fibonacci exceptions of T. For each of the fourteen extensions, $m - 1 \equiv 2 \pmod 4$, hence $v_2(q) = 3$. An exception of $\langle 1, 2, 2\rangle$ that is even has even valuation, and Lemma 2.2 rules out valuation 2 for a Fibonacci number, so its valuation is at least 4. It therefore satisfies the valuation hypothesis of Lemma 5.2 (ii), and Lemma 5.3 (ii), with (5), gives the size hypothesis. By Lemma 5.2 (ii) and Lemma 4.1, the extension represents these even exceptions as well. Together with the odd exceptions just represented, these exhaust the Fibonacci exceptions of $\langle 1, 2, 2\rangle$, so the extension is Fibonacci-universal. □

Proof of Proposition 5.1. The truant is represented by hypothesis, and every Fibonacci number that is not an exception is represented by the escalator. If $v_2(q) \leq v_2(\tau_T) + 2$, the paragraph following Lemma 5.3, together with Lemma 5.5, shows that the extension represents every Fibonacci exception larger than the truant. If $v_2(q) \geq v_2(\tau_T) + 3$, the extension is one of those in Proposition 5.7. □

5.2 Tools for obstructions at odd primes

For $T = \langle 1, 1, 3\rangle$ and $\langle 1, 2, 5\rangle$, extensions with $c_3 \neq 0$ are treated by fixing the coordinate w along the new generator: $w = 1$ for $\langle 1, 1, 3\rangle$ and $w = 2$ for $\langle 1, 2, 5\rangle$, where the even value makes the completion of the square in y integral. The extension's form then becomes a constant plus a polynomial $\Phi_{p,a}$ as in the next lemma. Since $\Phi_{p,a}$ represents every nonnegative integer, the extension represents every integer at least that constant.

Lemma 5.8 (Representation by quadratic polynomials)**.** *Let (p, α) be $(3, 1)$ or $(5, 2)$, and let $a \in \mathbb{Z}$ with $p \nmid a$. Then*

$$\Phi_{p,a}(x, y, z) = x^2 + \alpha y^2 + pz^2 + 2az$$

represents every nonnegative integer.

Proof. For an integer $r \geq 0$, the integer $p^2 r + pa^2$ has valuation 1 at p and square unit part modulo p. By Proposition 3.2, write

$$p^2 r + pa^2 = X^2 + \alpha Y^2 + pZ^2.$$

In both cases, $X^2 + \alpha Y^2 \equiv 0 \pmod{p}$ forces $p \mid X, Y$. Put $X = px$ and $Y = py$ and divide by p:

$$pr + a^2 = px^2 + \alpha py^2 + Z^2.$$

Thus $Z \equiv \pm a \pmod{p}$. After changing its sign, write $Z = pz + a$. Substitution and division by p give $r = \Phi_{p,a}(x, y, z)$. □

When $c_3 = 0$, we instead subtract qk^2 for a suitable k. The next lemma is used to choose k when $p = 5$; we will also use it with $p = 7$ in §6.

Lemma 5.9 (A quadratic congruence over $\mathbb{F}_p$)**.** *Let $p \geq 5$ be prime and let $u_0, q_0 \in \mathbb{F}_p^\times$. There is an integer t, with $|t| \leq (p-1)/2$, such that $u_0 - q_0 t^2$ is a nonzero square modulo p. If u_0 is a nonsquare, then $t \neq 0$.*

Proof. Write χ for the quadratic character of $\mathbb{F}_p$, extended by $\chi(0) = 0$. For $\xi \neq 0$, the equation $y^2 = t^2 - \xi$ is equivalent to $(t - y)(t + y) = \xi$, which has $p - 1$ solutions. Counting instead for each t gives $\sum_t (1 + \chi(t^2 - \xi)) = p - 1$, hence $\sum_t \chi(t^2 - \xi) = -1$. Taking $\xi = u_0/q_0$ shows that the conic $x^2 + q_0 t^2 = u_0$ has

$$p + \sum_{t \in \mathbb{F}_p} \chi(u_0 - q_0 t^2) = p - \chi(-q_0) \geq p - 1$$

points. At most two have $x = 0$. Since $p - 1 \geq 4$, some point therefore has $x \neq 0$. Choose the least absolute representative of its t-coordinate. If u_0 is a nonsquare, the equation rules out $t = 0$. □

Lemma 5.10 (Removing an odd-prime obstruction)**.** *Let $p \in \{3, 5\}$, and let T be $\langle 1, 1, 3\rangle$ or $\langle 1, 2, 5\rangle$, respectively. Let a, u, b be positive integers, with a odd, $p \nmid ub$, and u a nonsquare modulo p. Put $N = p^a u$ and $q = pb$. If $u > 4b$, then $T \perp \langle q\rangle$ represents N.*

Proof. We choose $t \in \{1, 2\}$ to remove the obstruction. Putting $k = p^{(a-1)/2}t$, we have $N - qk^2 = p^a(u - bt^2) > 0$. For $p = 5$, Lemma 5.9 supplies t with $u - bt^2$ a nonzero square modulo 5 (replace t by its absolute value). For $p = 3$, we have $u \equiv 2 \pmod 3$. If $b \equiv 1 \pmod 3$, take $t = 1$. If $b \equiv 2 \pmod 3$, both $u - b$ and $u - 4b$ are divisible by 3, but their difference $3b$ is not divisible by 9; choose t so that $v_3(u - bt^2) = 1$. In this last case, $v_3(N - qk^2) = a + 1$ is even. In every case $N - qk^2$ lies outside the exceptional set of T in Proposition 3.2, proving the assertion. □

5.3 Extensions of $\langle 1, 1, 3\rangle$

The escalator $\langle 1, 1, 3\rangle$ omits a positive integer precisely when its 3-adic valuation is odd and its unit part is $2 \pmod 3$. For extensions with $c_3 \neq 0$, Lemma 5.8 gives representations with coordinate 1 along the new generator. For those with $c_3 = 0$, which are orthogonal sums $\langle 1, 1, 3\rangle \perp \langle m\rangle$, a small subtraction changes the parity of the valuation or the class of the unit part.

Proposition 5.11 (Fibonacci-universality of extensions of $\langle 1, 1, 3\rangle$)**.** *Every quaternary extension of* $\langle 1, 1, 3\rangle$ *representing* $F_{16} = 987$ *is Fibonacci-universal.*

Proof. By Table 2, the extension has the form

$$Q = x^2 + y^2 + 3z^2 + 2c_3zw + mw^2, \qquad c_3 \in \{0, 1, 2\}.$$

It represents the truant $F_{16} = 987$ of $\langle 1, 1, 3\rangle$ by assumption. Every later Fibonacci exception has index divisible by 4, so it is at least $F_{20} = 6765$. The bound (4) gives $m \leq 987 + c_3^2/3$, hence $m \leq 988$. If $c_3 \neq 0$, then

$$Q(x, y, z, 1) = m + \Phi_{3,c_3}(x, y, z).$$

By Lemma 5.8, the polynomial $Q(x, y, z, 1)$ represents every integer at least m, and therefore every remaining Fibonacci exception.

Suppose $c_3 = 0$. Then $q = m \leq 987$ and $v_3(m) \leq 1$ by Lemma 4.3. If $3 \nmid m$, subtract m from each remaining exception; the difference is positive and prime to 3, so the escalator represents it. Adding the unary coordinate 1 represents the exception by the extension.

It remains to consider $m = 3b$, where $3 \nmid b$ and $b \leq 329$. Write a remaining exception as $N = F_n = 3^a u$, with a odd and $u \equiv 2 \pmod 3$. At $N = F_{20}$ we have $u = 2255 > 4 \cdot 329$. Every other remaining exception has index $n \geq 28$, since its index is divisible by 4 and F_{24} has even 3-adic valuation. For $n \geq 28$, (1) and Lemma 2.1 give

$$u \geq \frac{F_n}{3n} \geq \frac{F_{28}}{84} > 4 \cdot 329 \geq 4b.$$

Thus $u > 4b$ in every case, and by Lemma 5.10 the form $\langle 1, 1, 3\rangle \perp \langle m\rangle$ represents each remaining exception. □

5.4 Extensions of $\langle 1, 2, 5\rangle$

For $\langle 1, 2, 5\rangle$, the obstruction is an odd 5-adic valuation with nonsquare unit part. We use Lemma 5.8 when $c_3 \neq 0$. For the remaining extensions, we subtract qk^2 to change the valuation or the unit part. Only F_{20} requires more: there we use again the hypothesis that the extension represents the truant.

Proposition 5.12 (Fibonacci-universality of extensions of $\langle 1, 2, 5\rangle$)**.** *Every quaternary extension of* $\langle 1, 2, 5\rangle$ *representing* $F_{15} = 610$ *is Fibonacci-universal.*

Proof. Let M be an extension of $\langle 1,2,5\rangle$ representing 610. By Table 2, its form is

$$Q(x,y,z,w) = x^2 + 2y^2 + 5z^2 + 2c_2yw + 2c_3zw + mw^2,$$

where $c_2 \in \{0,1\}$ and $c_3 \in \{-2,-1,0,1,2\}$. The extension represents the truant $F_{15} = 610$ of $\langle 1,2,5\rangle$, so (4) gives $m \le 611$. Every Fibonacci exception has index divisible by 5, so every later one is at least $F_{20} = 6765$. Let $N = F_n$ be such an exception.

If $c_3 \neq 0$, set $w = 2$. Completing the square in y gives

$$Q(x,y,z,2) = \Phi_{5,2c_3}(x, y + c_2, z) + 4m - 2c_2^2.$$

Here $5 \nmid 2c_3$, so Lemma 5.8 shows that $\Phi_{5,2c_3}$ represents $N - (4m - 2c_2^2) > 0$, since $4m - 2c_2^2 \le 2444 < 6765$. Thus M represents N.

Suppose now that $c_3 = 0$. The bound $\delta \le 610$ gives $m \le 610$, and the norm q of the orthogonal vector in Lemma 4.1 is

$$q = \begin{cases} m, & c_2 = 0, \\ 4m - 2, & c_2 = 1. \end{cases}$$

In particular, $0 < q < 2440$, and Lemma 4.3 gives $v_5(q) \le 1$. If $5 \nmid q$, the positive difference $N - q$ is prime to 5 and is represented by the escalator. Hence M represents N by Lemma 4.1.

We may therefore assume $q = 5b$, where $5 \nmid b$ and $b < 488$. Write $N = 5^a u$, where a is odd and $u \equiv 2$ or $3 \pmod 5$. Whenever $u > 4b$, Lemma 5.10 shows that $\langle 1,2,5\rangle \perp \langle q\rangle \subseteq M$ represents N.

For $N > F_{20}$, its index is at least 25, and (1) and Lemma 2.1 give

$$u \ge \frac{F_n}{n} \ge \frac{F_{25}}{25} > 4 \cdot 488 > 4b.$$

For $N = F_{20}$ we have $a = 1$ and $u = 1353$. If $b \le 338$, then $u > 4b$, so Lemma 5.10 again proves representation.

It remains to treat $N = F_{20} = 6765$ with $338 < b < 488$. Here the representation of the truant restricts b enough to ensure that $\langle 1,2,5\rangle$ represents $6765 - 5b$. In this range $c_2 = 1$, since $c_2 = 0$ would give $q = m \le 610$. Thus $q = 4m - 2 = 5b$, and $b \equiv 2 \pmod 4$. Choose a representation $Q(x,y,z,w) = 610$. The identity

$$4Q(x,y,z,w) = (2x)^2 + 2(2y + w)^2 + 5(2z)^2 + qw^2$$

forces $w = \pm 1$: the escalator omits 610, so $w \neq 0$, while $q > 610$ rules out $|w| \ge 2$. Consequently the escalator represents

$$2440 - q = 5(488 - b).$$

This excludes $b \equiv 1 \pmod 5$, since the displayed value would then have valuation 1 and nonsquare unit part $2 \pmod 5$. Together with $5 \nmid b$, this leaves $b \equiv 2, 3, 4 \pmod 5$.

We claim that $\langle 1,2,5\rangle$ represents $N - q$, so that Lemma 4.1 applies with $k = 1$. The difference is $R = 6765 - 5b = 5(1353 - b) > 0$. If $b \equiv 2$ or $4 \pmod 5$, its unit part is a nonzero square modulo 5. If $b \equiv 3 \pmod 5$ but $b \not\equiv 3 \pmod{25}$, then $v_5(R) = 2$. Finally, if $b \equiv 3 \pmod{25}$, the condition $b \equiv 2 \pmod 4$ gives $b \equiv 78 \pmod{100}$. The range $338 < b < 488$ leaves only $b = 378$ and $b = 478$, giving respectively

$$R = 4875 = 5^3 \cdot 39, \qquad R = 4375 = 5^4 \cdot 7.$$

The first has square unit part $39 \equiv 4 \pmod 5$, and the second has even valuation. Thus the escalator represents R in every case, and M represents N by Lemma 4.1. □

6 Extensions of f

Finally, we consider quaternary extensions of f.

Proposition 6.1 (Fibonacci-universality of extensions of f)**.** *Every quaternary extension of f representing* 21 *is Fibonacci-universal.*

By Lemma 4.4, such an extension contains $f \perp \langle q \rangle$ for some $q \in \mathcal{Q}_7$. By Proposition 3.5, it represents $F_1, \ldots, F_{47}$: f represents all these values except 21, which is represented by hypothesis. It therefore suffices to prove that, for every $q \in \mathcal{Q}_7$, the form $f \perp \langle q \rangle$ represents F_n whenever $n \geq 48$. We first treat $q = 7$. For the other values of q, small subtractions prove representation except for four pairs ($q \bmod 12, F_n \bmod 12$); square transfers and a congruence argument prove representation in those cases from index 61 onward. Between indices 48 and 60, only 57 and 58 require the small-prime table at the end of the section. The auxiliary form h enters through Proposition 3.3: after arranging that $F_n - qk^2 = p^2 r > 0$ with r eligible, we represent r by f or h. Scaling in the first case, or a square transfer in the second, gives a representation of $F_n - qk^2$ by f.

6.1 The universal form $f \perp \langle 7 \rangle$

The case $q = 7$ requires separate treatment. For a positive integer N divisible by 49, if $7 \nmid k$, then $N - 7k^2$ has valuation 1 and nonsquare unit part $-k^2 \pmod 7$; if $7 \mid k$, the difference remains divisible by 49. Thus the construction used for the other values of q, which produces eligible differences of valuation 0 or 1, does not apply. Instead we use induction and scaling by 7.

Proposition 6.2 (Universality of $f \perp \langle 7 \rangle$)**.** *The form $Q(x, y, z, w) = f(x, y, z) + 7w^2$ is universal.*

Proof. Use strong induction on the positive integer N. If $49 \mid N$, the induction hypothesis gives a representation of $N/49$ by Q; multiplying every coordinate by 7 gives a representation of N. Hence suppose $49 \nmid N$.

If $7 \nmid N$, take $w = 0$ when $N \not\equiv 7, 10 \pmod{12}$ and $w = 1$ otherwise. The difference $R = N - 7w^2$ is positive and prime to 7, and $R \not\equiv 7, 10 \pmod{12}$. Indeed, in the second case its residue modulo 12 is 0 or 3, and it is positive because the least N prime to 7 in the classes 7 and 10 modulo 12 are 19 and 10, respectively. By Proposition 3.3 and Lemma 3.4, f represents R, so Q represents N.

It remains to consider $N = 7u$ with $7 \nmid u$. We seek w such that $u - w^2$ is a nonzero square modulo 7 and is not 1 or 10 modulo 12. When $u > w^2$, these conditions make $7(u - w^2)$ eligible and exclude the residues 7 and 10 modulo 12. Choose w from the following table: use the value in the second column unless u mod 12 is one of the residues named in the third.

$u \bmod 7$	w	Exceptions, by $u \bmod 12$
1	0	$w = 2$ if $u \equiv 1, 10$
2	0	$w = 1$ if $u \equiv 1, 10$
3	1	$w = 3$ if $u \equiv 2, 11$
4	0	$w = 3$ if $u \equiv 1$; $w = 4$ if $u \equiv 10$
5	1	$w = 6$ if $u \equiv 2$; $w = 2$ if $u \equiv 11$
6	2	$w = 3$ if $u \equiv 2, 5$

Each row gives

$$u - w^2 \equiv 1, 2, 4 \pmod 7, \qquad u - w^2 \not\equiv 1, 10 \pmod{12}.$$

Since $w \leq 6$, we have $u - w^2 > 0$ for $u > 36$. Checking the prescribed choices for $1 \leq u \leq 36$ with $7 \nmid u$ leaves only $u = 1, 26$ for which $u \leq w^2$. For every other u, the positive difference

$$R = N - 7w^2 = 7(u - w^2)$$

is eligible and is not 7 or 10 modulo 12. Again Proposition 3.3 and Lemma 3.4 show that f represents R, and adjoining the coordinate w gives a representation of N by Q. Finally, the two remaining values $N = 7$ and $N = 182$ have explicit representations:

$$Q(0,0,0,1) = 7, \qquad Q(7,3,-6,1) = 182. \qquad \square$$

Remark. Universality also follows from the Conway–Schneeberger Fifteen Theorem [8, 3], since Q represents $1, 2, 3, 5, 6, 7, 10, 14, 15$. We include the short proof for completeness; it uses only Proposition 3.3 and Lemma 3.4.

6.2 Small subtractions

The small-subtraction and square-transfer arguments both choose k in a residue class modulo 7 for which every positive difference $N - qk^2$ is eligible.

Lemma 6.3 (A class modulo 7)**.** *Let $q \in \mathcal{Q}_7 \setminus \{7\}$, and let N be a positive integer. There is a residue class modulo 7 such that, for every integer k in this class, the difference $N - qk^2$ is eligible whenever it is positive.*

Proof. We choose the class so that every positive difference $R = N - qk^2$ with k in the class satisfies

$$7 \nmid R \quad \text{or} \quad v_7(R) = 1 \text{ and } R/7 \equiv 1, 2, 4 \pmod 7;$$

such an R is eligible. If $7 \nmid q$, avoid the at most two roots of $qk^2 \equiv N \pmod 7$. If $7 \mid q$ but $7 \nmid N$, then $R \equiv N \not\equiv 0 \pmod 7$ for every k. Otherwise write $q = 7s$ and $N = 7u$; here $s \equiv 3, 5, 6 \pmod 7$ by (6). For $7 \nmid u$, Lemma 5.9 supplies a class with $u - sk^2$ a nonzero square modulo 7. For $7 \mid u$, take $k \equiv 1 \pmod 7$; then $u - sk^2 \equiv -s \equiv 1, 2, 4 \pmod 7$. $\square$

Lemma 6.4 (Small subtraction)**.** *Let $q \in \mathcal{Q}_7 \setminus \{7\}$ and let N be an integer with $N > 441q$. Then $f \perp \langle q \rangle$ represents N unless $(q \bmod 12, N \bmod 12)$ is one of*

$$(0,7), \quad (0,10), \quad (3,10), \quad (9,7). \tag{7}$$

Proof. If $N \not\equiv 7, 10 \pmod{12}$, impose $k \equiv 0 \pmod 6$, so that $N - qk^2 \equiv N \pmod{12}$. Otherwise impose $k \equiv 1 \pmod 6$; then $N - qk^2 \equiv N - q \pmod{12}$, which is not 7 or 10 except in the four displayed cases. Combine this class with the class modulo 7 of Lemma 6.3. The resulting class modulo 42 contains an integer k with $|k| \le 21$, so $N - qk^2 > 0$. This difference is eligible and is not 7 or 10 modulo 12; hence f represents it by Proposition 3.3 and Lemma 3.4. $\square$

Since $441 \cdot 1022 = 450702 < F_{48}$, the lemma applies to $N = F_n$ for every $q \in \mathcal{Q}_7 \setminus \{7\}$ and every $n \ge 48$. Sixty-one of these 81 values of q are not 0, 3, or 9 modulo 12, and for them the lemma shows that $f \perp \langle q \rangle$ represents every such F_n. For the other twenty values of q, it leaves only the indices n at which $(q \bmod 12, F_n \bmod 12)$ is one of the pairs in (7).

6.3 Subtractions using square transfers

Put

$$\mathcal{P}_7 = \{5, 11, 13, 17, 19, 31, 53, 83, 107, 109\}.$$

For each $p \in \{2, 3\} \cup \mathcal{P}_7$, Table 3 in §A gives an integral matrix B_p satisfying

$$B_p^{\mathsf{T}} G_f B_p = p^2 G_h, \qquad G_h = \operatorname{diag}(1, 1, 7). \tag{8}$$

Each identity is checked by matrix multiplication. Thus $f(B_p v) = p^2 h(v)$ for every integral vector v, so B_p turns a representation of an integer R by h into a representation of $p^2 R$ by f. For example, f does not represent $7 = h(0,0,1)$, but B_5 sends $(0,0,1)$ to $(7,3,-6)$, and $f(7,3,-6) = 175 = 5^2 \cdot 7$.

To represent a difference $N - qk^2$ using a square transfer, we require it to be positive, eligible, and divisible by p^2. The next lemma shows that, when N is large enough, a suitable k exists as soon as the congruence $qk^2 \equiv N \pmod{p}$ has a solution prime to p and $p \nmid q$.

Lemma 6.5 (Subtraction with a square transfer)**.** *Let $q \in \mathcal{Q}_7 \setminus \{7\}$ and $N \in \mathbb{Z}_{>0}$. Suppose that there are $p \in \mathcal{P}_7$ and $k_0 \in \mathbb{Z}$ with $qk_0^2 \equiv N \pmod{p}$ and $p \nmid qk_0$. If*

$$N > q \left\lfloor \frac{7p^2}{2} \right\rfloor^2,$$

then $f \perp \langle q \rangle$ represents N.

Proof. Since p is odd and $p \nmid qk_0$, Hensel's lemma lifts k_0 to a solution of $qk^2 \equiv N \pmod{p^2}$. Combine its class modulo p^2 with the class modulo 7 of Lemma 6.3, and choose k in the resulting class modulo $7p^2$ with $|k| \le \lfloor 7p^2/2 \rfloor$. Then $R = N - qk^2$ is positive, eligible, and divisible by p^2. Since $p \neq 7$, division by p^2 preserves eligibility, so R/p^2 is represented by f or by h (Proposition 3.3). In the first case, multiplying the representing vector by p represents R by f; in the second, applying B_p does so, by (8). With k as the last coordinate, this represents N by $f \perp \langle q \rangle$. □

It remains to find, for the pairs in (7) with $N = F_n$, a prime of $\mathcal{P}_7$ to which Lemma 6.5 applies once its size bound holds. The next lemma gives a *congruence cover*: every relevant pair (q, n) satisfies the required congruence for at least one prime in $\mathcal{P}_7$.

Lemma 6.6 (Congruence cover)**.** *Let $q \in \mathcal{Q}_7 \setminus \{7\}$ and let $n \ge 0$ be an integer. Suppose that $(q \bmod 12, F_n \bmod 12)$ is one of the pairs in (7). Then there is a prime $p \in \mathcal{P}_7$ with $p \nmid q$ and $F_n \bmod p \in q(\mathbb{F}_p^\times)^2$; that is, $F_n \equiv qk^2 \pmod{p}$ for some $k \not\equiv 0 \pmod{p}$.*

Proof. The Fibonacci recurrence modulo 12 gives

$$F_n \equiv 10 \pmod{12} \iff n \equiv 9, 15 \pmod{24}, \qquad F_n \equiv 7 \pmod{12} \iff n \equiv 10 \pmod{24}.$$

Thus the relevant odd indices are multiples of 3, and the relevant even indices are $10 \pmod{24}$. We group the twenty values of q in $f \perp \langle q \rangle$ that are congruent to 0, 3, or 9 modulo 12 by their quadratic characters at suitable primes, and use the fact that the quotient of two nonzero residues modulo p is a square precisely when their quadratic characters agree. In the following table, an entry $p : +$ says that each value of q in the row is a nonzero square modulo p, and an entry $p : -$ that each is a nonsquare.

Values of q	Parities of n treated	Characters of q
{3, 903}	odd	$5 : -;\ 11 : +$
{12, 168, 672}	odd and even	$5 : -;\ 11 : +;\ 31 : -$
{15, 231, 483, 819}	odd	$19 : -$
{21, 84, 189, 336, 525, 756}	odd and even	$17 : +;\ 19 : -$
{9, 777}	even	$13 : +;\ 83 : +$
315	odd	$107 : +$
609	even	$17 : -;\ 53 : -;\ 109 : +$
924	odd and even	$19 : -;\ 53 : -$

The table includes exactly the elements of $\mathcal{Q}_7 \setminus \{7\}$ congruent to $0, 3, 9 \pmod{12}$. For $q \equiv 0 \pmod{12}$, both the odd and even index classes above are needed. If $q \equiv 3 \pmod{12}$, the hypothesis requires $n \equiv 9, 15 \pmod{24}$; if $q \equiv 9 \pmod{12}$, it requires $n \equiv 10 \pmod{24}$. Some rows prove the congruence for both parities, although only one is needed for certain values of q. The character entries are found by reducing the values of q modulo the indicated primes and comparing with the squares of the nonzero residues.

It remains to find, in every index class that the row requires, a prime of the row at which F_n is nonzero and has the same character as the values of q in that row. The periods of the Fibonacci sequence modulo the primes of $\mathcal{P}_7$ are listed in Table 3; each is a period because $(F_{\pi_p}, F_{\pi_p+1}) \equiv (0, 1) \pmod{p}$ for the listed value π_p, and the Fibonacci recurrence gives the residues below. For each group we use only the periods of its own primes.

Values of q in $\{3, 903\}$ and $\{12, 168, 672\}$. For every odd n, at least one of $p = 5, 11$ gives $F_n \bmod p \in q(\mathbb{F}_p^\times)^2$ whenever q is a nonsquare modulo 5 and a nonzero square modulo 11. Indeed, if $n \equiv 3, 7 \pmod{10}$, then $F_n \equiv 2$ or $3 \pmod{5}$, both nonsquares. In the other odd classes $n \equiv 1, 5, 9 \pmod{10}$, the residues modulo 11 are $1, 5, 1$, all squares. This proves the required congruence for $q \in \{3, 903\}$ and, at odd indices, for $q \in \{12, 168, 672\}$. For the latter group, a relevant even index is $10, 34, 58, 82$, or 106 modulo 120. For the classes $34, 106$ modulo 120, the Fibonacci residues modulo 5 are $2, 3$; for the class 82, the residue modulo 11 is 1. For the classes $10, 58$, use 31: the Fibonacci residues are respectively $24, 30 \pmod{31}$, both nonsquares.

The two groups using $p = 19$. Consider the third and fourth rows of the table. For an odd multiple of 3, the index is $3, 9$, or 15 modulo 18. The Fibonacci residues modulo 19 are respectively $2, 15, 2$, all nonsquares. Thus $p = 19$ gives the required congruence at all relevant odd indices for both groups. For the latter group and $n \equiv 10 \pmod{24}$, split according to $n \equiv 10, 34, 58 \pmod{72}$. At $10, 34$, the residues modulo 17 are $4, 16$, both squares; at 58, the residue modulo 19 is 3, a nonsquare.

Values of q in $\{9, 777\}$. Here $n \equiv 10 \pmod{24}$, so

$$n \bmod 28 \in \{2, 6, 10, 14, 18, 22, 26\}.$$

In the classes $2, 10, 18, 26$ modulo 28, the Fibonacci residues modulo 13 are $1, 3, 10, 12$, all nonzero squares. Combining the remaining classes modulo 28 with $n \equiv 10 \pmod{24}$ gives $n \equiv 34, 106, 154 \pmod{168}$. Their Fibonacci residues modulo 83 are $40, 51, 38$, respectively, with square roots $17, 36, 11$.

The value $q = 315$. The relevant odd indices are $9, 15, 33, 39, 57, 63$ modulo 72. Their Fibonacci residues modulo 107 are

$$34, \quad 75, \quad 105, \quad 105, \quad 75, \quad 34,$$

respectively. The three distinct values 34, 75, and 105 have square roots 26, 17, and 31, respectively, modulo 107. Also $315 \equiv 101 \equiv 23^2 \pmod{107}$.

The value $q = 609$. Among the indices $n \equiv 10 \pmod{24}$, those with $n \equiv 58 \pmod{72}$ have $F_n \equiv 14 \pmod{17}$, a nonsquare. Since 609 is also a nonzero nonsquare modulo 17, the quotient

$F_n/609$ is a nonzero square modulo 17. For the other classes of $n \equiv 10 \pmod{24}$, the calculation modulo 53 is

$n \bmod 216$	10	34	82	106	154	178
$F_n \bmod 53$	2	34	30	52	21	20.

All entries except 52 are nonsquares. At the remaining class $n \equiv 106 \pmod{216}$, use 109: since $106 \equiv -2 \pmod{108}$,

$$F_n \equiv -1 \equiv 33^2 \pmod{109}, \qquad 609 \equiv 64 \equiv 8^2 \pmod{109}.$$

The residue $F_{108-2} \equiv -1 \pmod{109}$ follows by running the Fibonacci recurrence backward from $(F_{108}, F_{109}) \equiv (0, 1) \pmod{109}$.

The value $q = 924$. As above, $p = 19$ gives the required congruence at the relevant odd indices. It also does so for $n \equiv 34, 58 \pmod{72}$, where the Fibonacci residues are $18, 3$, both nonsquares. The remaining indices satisfy $n \equiv 10 \pmod{72}$, that is, $n \equiv 10, 82, 154 \pmod{216}$. The residues modulo 53 are $2, 30, 21$, all nonsquares.

In each case q and F_n are nonzero modulo p and have the same character. Hence F_n/q is a nonzero square modulo p, as required. □

Proof of Proposition 6.1. By the reduction at the start of the section, it suffices to show that $f \perp \langle q \rangle$ represents F_n for every $q \in \mathcal{Q}_7$ and every $n \geq 48$. For $q = 7$, this follows from Proposition 6.2. For every other value of q, Lemma 6.4 proves representation for every F_n whose pair ($q \bmod 12, F_n \bmod 12$) is outside (7). In the remaining cases, Lemma 6.6 supplies a prime $p \in \mathcal{P}_7$, and Lemma 6.5 applies to it as soon as $F_n > B_*$, where

$$\overline{k} = \left\lfloor \frac{7 \cdot 109^2}{2} \right\rfloor = 41583, \qquad B_* = 1022\overline{k}^2 = 1767187098558; \tag{9}$$

here 109 is the largest prime in $\mathcal{P}_7$ and 1022 the largest element of $\mathcal{Q}_7$. Since $F_{61} = 2504730781961 > B_*$, the required representation follows for every $n \geq 61$.

For $48 \leq n \leq 60$, we have $F_n \equiv 10 \pmod{12}$ only at $n = 57$ and $F_n \equiv 7 \pmod{12}$ only at $n = 58$. For these two indices, we apply Lemma 6.5 with $p \leq 19$. The following table supplies a prime satisfying its congruence hypothesis for every remaining pair (q, n):

p	q for $n = 57$	q for $n = 58$
5	{3, 12, 168, 483, 672, 903}	{9, 21, 84, 189, 336, 609, 756, 924}
11	{84, 315, 336, 756}	{525, 777}
13	{231, 924}	{12, 168, 672}
17	15	—
19	819	—

The two columns contain exactly the elements of $\mathcal{Q}_7$ with $q \equiv 0, 3 \pmod{12}$ and $q \equiv 0, 9 \pmod{12}$, respectively. In each entry, $p \nmid q$ and $F_n \bmod p \in q(\mathbb{F}_p^\times)^2$, as one checks by reducing q modulo p and using the residues

p	5	11	13	17	19
$F_{57} \bmod p$	2	2	1	15	2
$F_{58} \bmod p$	4	10	1	—	—

(the last two entries are not needed). Since $p \leq 19$ and

$$q \left\lfloor \frac{7p^2}{2} \right\rfloor^2 \leq 1022 \cdot 1263^2 = 1630262718 < F_{48},$$

Lemma 6.5 applies here as well. □

7 The criterion and its minimality

We can now complete the escalation argument. The Fibonacci-universality results prove sufficiency; a four-square construction then shows that every element of C is indispensable.

Proof of Theorem 1.1. Sufficiency. Suppose that Q represents every element of C, and let L_Q be its lattice. Put $L_0 = 0$. Given an escalator $L_i \subseteq L_Q$ of rank $i \leq 3$, its truant τ_i belongs to C by §3. Thus L_Q contains a vector u_i with $Q(u_i) = \tau_i$. Put $L_{i+1} = L_i + \mathbb{Z}u_i$. By Lemma 3.6, $u_i \notin L_i \otimes \mathbb{Q}$, so L_{i+1} has rank $i+1$. Consequently L_3 is isometric to one of the nine ternary escalators, and L_4 is a quaternary escalation of it. In particular, L_4 is an extension representing the truant, as required in §4.

Every such extension is Fibonacci-universal—this follows from Proposition 5.1 for the six ternaries with obstructions at 2, from Propositions 5.11 and 5.12 for the other two regular ternaries, and from Proposition 6.1 for f. Since the extension is contained in L_Q, the form Q is Fibonacci-universal. The converse is immediate because $C \subseteq \mathcal{F}$.

Minimality. The argument follows the extension principle of Kala–Krásenský–Romeo [11, Proposition 3.1], using the four-square construction of Kim–Lee–Oh [15, Proposition 2.1]. We include the proof for completeness. Fix $t \in C$ and take the escalator P_t with Fibonacci truant t listed in Table 1. We enlarge P_t to represent every integer greater than t while still omitting t. Set

$$V_t = P_t \perp \langle t+1, t+1, t+1, t+1 \rangle \perp \bigoplus_{j=t+1}^{2t+1} \langle j \rangle,$$

with all sums orthogonal. Every nonzero vector in the added summands has norm greater than t. Since P_t omits t, so does V_t.

Now let $N > t$. Choose the unique $j \in \{t+1, \ldots, 2t+1\}$ congruent to N modulo $t+1$. This is the least integer greater than t in its residue class, so $j \leq N$ and $N - j = (t+1)m$ with $m \geq 0$. Write $m = a_1^2 + \cdots + a_4^2$ by Lagrange's four-square theorem. The four equal summands then represent $(t+1)m$, and the unary summand $\langle j \rangle$ supplies j. Thus V_t represents N. Since P_t already represents every Fibonacci number below t, the lattice V_t represents every Fibonacci number except t.

If a Fibonacci criterion set omitted t, then V_t would represent every element of that set without being Fibonacci-universal. Hence every criterion set contains every $t \in C$. Together with sufficiency, this proves that C is the unique minimal Fibonacci criterion set. □

Proof of Corollary 1.2. The numbers $F_1, \ldots, F_{42}$ include every member of C, so Theorem 1.1 proves sufficiency. Proposition 3.2 shows that $\langle 1, 1, 2 \rangle$ has Fibonacci truant F_{42}, proving sharpness. □

8 Concluding remarks

The proof of Theorem 1.1 combines sequence-specific calculations with more general representation arguments. The truants and escalators, and the residue calculations used to prove universality of their extensions, depend on the Fibonacci sequence. Periodicity, growth, and the bounds (1) on prime-power parts make these calculations sufficient for all indices. We conclude by isolating the square-transfer principle and explaining what would be needed to apply the strategy to other sequences.

In §6, every eligible integer is represented by f or by h, and the maps B_p turn representations by h into representations by f after multiplication by p^2. More generally, let L, L' be positive-definite integral lattices, and let $\mathcal{A} \subseteq \mathbb{Z}_{>0}$ consist of integers represented by L or L'. Fix a finite nonempty set $\mathcal{P}$ of primes. For each $p \in \mathcal{P}$, suppose there is a $\mathbb{Z}$-linear map $B_p : L' \to L$ such that

$$Q_L(B_p v) = p^2 Q_{L'}(v) \qquad (v \in L'),$$

and suppose $R/p^2 \in \mathcal{A}$ whenever $R \in \mathcal{A}$ and $p^2 \mid R$. Let $\mu \geq 1$ be coprime to every prime in $\mathcal{P}$. In §6, these data are $L = f$, $L' = h$, the eligible integers $\mathcal{A}$, $\mathcal{P} = \mathcal{P}_7$, and $\mu = 7$. For $q \in \mathcal{Q}_7 \setminus \{7\}$, Lemma 6.3 supplies the class modulo μ required below. The proposition abstracts Lemma 6.5.

Proposition 8.1 (Representation using congruences)**.** *In the setting above, let $N, q \in \mathbb{Z}_{>0}$ and $p \in \mathcal{P}$. Suppose that $N \equiv qk_p^2 \pmod{p^2}$ for some integer k_p. Suppose also that there is a residue class k_μ modulo μ such that every positive difference $N - qk^2$ with $k \equiv k_\mu \pmod{\mu}$ belongs to $\mathcal{A}$. If*

$$N > q \left\lfloor \frac{\mu p^2}{2} \right\rfloor^2,$$

then $L \perp \langle q \rangle$ represents N.

Proof. As in Lemma 6.5, the Chinese remainder theorem gives $k \equiv k_p \pmod{p^2}$ and $k \equiv k_\mu \pmod{\mu}$ with $|k| \leq \lfloor \mu p^2/2 \rfloor$. Hence $R = N - qk^2$ is positive, belongs to $\mathcal{A}$, and is divisible by p^2. Since $R/p^2 \in \mathcal{A}$, represent it by L or L' and then scale by p or apply B_p, respectively. This represents R by L; the coordinate k supplies qk^2. □

When p is odd and $p \nmid qN$, the hypothesis modulo p^2 follows by Hensel's lemma from a solution of $N \equiv qk^2 \pmod{p}$ with $p \nmid k$; this is the form used in Lemma 6.5.

To turn this representation principle into a criterion for another sequence (s_n), one must first determine its escalators and prove universality of the relevant extensions. The stopping rank need not be four, or even the same on every branch; varying depths already occur in the Fifteen Theorem [3]. Moreover, unlike the Fibonacci case of Lemma 3.6, a truant may be represented over $\mathbb{Q}$ by its escalator. An escalation can then enlarge the lattice without raising its rank.

For steps that do raise the rank, the bound on q remains available. The proof of Lemma 4.2 uses neither the rank of T nor any property of the Fibonacci numbers: an extension of an escalator T by one dimension that represents its truant contains $T \perp \langle q \rangle$ with $q \leq \tau_T(\det G_T)^2$. Thus, for each such T, only finitely many values of q need be considered. What remains is to find a usable criterion for representation by T. In the regular cases of this paper, local conditions reduce the problem to residue calculations and lower bounds on the prime-to-p parts of the sequence terms. In the irregular case, representation by f or h and the square-transfer maps provide a substitute.

If (s_n) tends to infinity and is periodic modulo the relevant moduli, a congruence cover for fixed transfer data can be checked over a common period. Once that cover and the required classes modulo μ have been established, growth gives the size bound in Proposition 8.1 beyond a finite initial range. Periodicity does not itself guarantee that a cover exists, however, and the initial terms still require separate treatment.

Lucas sequences are natural candidates, because their arithmetic provides analogues of the periodicity, growth, and valuation bounds used here. Consider $t_0 = 0$, $t_1 = 1$, and $t_{n+2} = at_{n+1} + bt_n$, with coprime integers a, b, and suppose that the roots α, β of $X^2 - aX - b$ satisfy $\alpha > \max(1, |\beta|)$. For $n \geq 1$, the terms $t_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ are positive and grow exponentially. Modulo an integer m coprime to b, the pair recurrence is invertible on a finite set, so the sequence is periodic. For each fixed prime p, the valuation formulas of Sanna [23, Theorem 1.5] give $v_p(t_n) \leq v_p(n) + C_p$

for a constant C_p depending on the recurrence; if $p \mid b$, all terms of positive index are p-units. Thus $p^{v_p(t_n)} \le p^{C_p} n$, and the unit part $t_n/p^{v_p(t_n)}$ still grows exponentially, which is what the regular-case subtractions require.

The sequence-specific inputs, however, can change from the start. In the Fibonacci case, the ordinary and Fibonacci truants agree through rank 2, so the ternary escalators are the classical ones. For the Pell numbers ($a = 2$, $b = 1$), this fails already at rank 2: the least Pell number omitted by $\langle 1, 1\rangle$ is 12, whereas its ordinary and Fibonacci truants are 3. Pell escalation therefore produces different ternary lattices, and an explicit criterion would require their classification, along with associated representation results and the corresponding congruence arguments.

It would also be interesting to know whether the number of initial terms that suffices for universality with respect to the sequence can be bounded uniformly over these Lucas sequences. It is not clear whether this method can yield such a bound, because the escalation, including the rank at which it stops, and the arithmetic thresholds all depend on the recurrence.

In the meantime, the Fibonacci criterion gives one further instance in which 42 is the answer to everything.

> *"The Answer to the Great Question... Of Life, the Universe and Everything... Is... Forty-two"*
>
> —Douglas Adams, *The Hitchhiker's Guide to the Galaxy*

Lean formalization

Theorem 1.1 has been formalized in Lean 4. The formalization proves sufficiency, the indispensability of every member of C, the uniqueness of the minimal criterion set, and Corollary 1.2. As in this paper, the statement concerns positive-definite forms with integral symmetric Gram matrices, in arbitrary finite rank. The final theorem is `fibonacci_criterion` in `FibonacciCriterion/Criterion/Theorem.lean`. The complete formal development and instructions for reproducing its verification are available in the public accompanying repository at `https://github.com/scduko/Fibonacci-Universality-Criterion`. It has no unproven hypotheses: the classical representation criteria and the Fibonacci valuation formulas used in the proof are proven in the formal development or in the libraries on which it depends. The same file restates the criterion for an integer matrix that is positive definite in the sense of Lean's mathematical library Mathlib (`fibonacci_criterion_matrix`), so the statement can be read against standard definitions.

Organization. The formal proof uses the reductions, case divisions, representatives, and cutoffs of this paper, and its files are grouped by section: `Fibonacci/` for §2, `Escalation/` for §3, `Forms/` for §4, `Regular/` for §5, `Irregular/` for §6, and `Criterion/` for §7. The square-transfer matrices of Appendix A are included with the formalization of §6. The classical inputs are kept separate. The file `ClassicalInputs/Definitions.lean` collects them: the three-square theorem, the criteria of Proposition 3.2 for $\langle 1, 1, 3\rangle$, $\langle 1, 2, 3\rangle$, and $\langle 1, 2, 5\rangle$, the sufficiency direction of Proposition 3.3, and Lemma 2.2. `Criterion/Proof.lean` proves the criterion from these inputs, and `ClassicalInputs/Theorems.lean` proves the inputs. The file `docs/READING_GUIDE.md` maps the manuscript's results to the corresponding Lean modules.

Differences from the paper. The formalization follows the argument needed for the criterion, rather than formalizing every auxiliary assertion in the paper. The principal differences are as follows. The file `docs/PAPER_CORRESPONDENCE.md` gives the complete list.

- *Gram matrices in place of lattices.* Lean works with forms given by Gram matrices and does not construct the lattices L_i. The escalation of Proposition 3.1 produces three vectors whose Gram matrix is that of one of the nine ternary escalators, and an extension $T + \mathbb{Z}v$ is recorded by the pair (c, m) of §4. Lean proves the identity (2) and the containment of $T \perp \langle q \rangle$ in M, but not that the index is d, which the proof does not use. It also proves the truant equation (3), including $k \geq 1$, and derives the bounds of Lemma 4.2 from it.

- *The regular ternaries.* Proposition 3.2 is proven as a list of criteria for integral representation; regularity, as a local–global statement, is not formalized. The three-square theorem is taken from the library atlas-lean. For $\langle 1, 1, 3 \rangle$, $\langle 1, 2, 3 \rangle$, and $\langle 1, 2, 5 \rangle$, where this paper cites Jones–Pall [9], Lean proves sufficiency by the Mordell-type construction of Blackwell, Durham, Thompson, and Treece [5], together with Minkowski reduction and lists of the reduced ternary forms of determinants 3, 6, and 10. The determinant-10 argument includes the even-valued class among the four reduced classes; for even integers it chooses the parameters of the construction so that the resulting form represents an odd integer, which excludes that class.

- *The forms f and h.* Lean proves the direction of Proposition 3.3 that the criterion uses: every eligible positive integer is represented by f or by h. The construction is stated for integers not divisible by 49. This is all the applications need: the Fibonacci numbers of Proposition 3.5 and the differences of §6 are not divisible by 49, and Proposition 6.2 treats multiples of 49 by scaling by 7. This does not require 7-adic Fibonacci valuation estimates. Lean follows the two auxiliary-prime constructions of the proof, with Minkowski reduction from the library TauCeti, and it excludes reduced forms outside the classes of f and h by counting their zeros modulo 7 rather than by the square class of a binary determinant. Of the converse, it proves only the 7-adic obstruction for f.

- *Fibonacci arithmetic.* Lemma 2.2 is proven from the formulas for F_{2m}, F_{3m}, and F_{5m}. The 2-adic rows of Lemma 2.4 and the Fibonacci numbers needed in Lemma 5.3 are read from a table of $v_2(F_n)$ and $u_2(F_n)$ for $1 \leq n \leq 64$. Lean proves that the Fibonacci residues return after the periods it uses, but not that these periods are the least ones of Lemma 2.3.

- *The congruence cover.* For Lemma 6.6, Lean uses the same eight groups of values of q and the same ten primes. In place of the calculation with quadratic characters, it checks the residue classes required by (7) over a common period for each group, of length at most 216, and proves that this implies the statement for every index.

Finite checks. The principal finite assertions of the formal proof are the table and the residue checks just described, the twelve identities of Table 3, the witnesses and the eligibility checks of Proposition 3.5, the small-prime table for $n = 57, 58$ in the proof of Proposition 6.1, the congruences of Table 1, and the lists of reduced ternary forms; the file `docs/PAPER_CORRESPONDENCE.md` lists all of them. The residue and enumeration checks are evaluated by the Lean kernel, with soundness and coverage lemmas giving the statements needed for all integers or indices. The transfer identities are proven by `ring`, whose proofs are also checked by the kernel. The formal proof uses no external certificates, data files, or scripts.

Verification. The version described here is commit `499b60d88ab0` of the public accompanying repository. It uses Lean 4.35.0 and pinned versions of Mathlib, TauCeti, and atlas-lean. The verification run at this commit passed the complete build, an independent replay of every module

in the Lean kernel by `leanchecker`, and an audit of the axioms used by all 2,960 declarations. The audit allows only Lean's standard axioms `propext`, `Classical.choice`, and `Quot.sound`. The repository includes the full source and instructions for reproducing these checks.

Provenance of the argument

The first version of the argument established the Fibonacci criterion conditional on the Generalized Riemann Hypothesis (GRH), using an analytic representation result for the irregular ternary f. An attempt to remove that hypothesis by means of OpenAI's quasi-Riemann-hypothesis work [19] led instead to the square-transfer and congruence argument of §6, which uses no analytic representation estimates. Neither GRH nor the quasi-Riemann-hypothesis result is assumed in the present proof.

The paper and the formalization were developed together, and each shaped the other. Automated proof development in Lean suggested the direct construction for determinant 7 in Proposition 3.3, the congruence obstructions of Table 1, and the reductions to three squares in Proposition 3.2; after these arguments were checked in Lean and their mathematical content reviewed, they were adopted here.

Refinements of the manuscript subsequently informed the formalization. Comparing the paper with the formal proof led to the smaller set of possible values of q in Lemma 4.4 and the short proof of Proposition 6.2, which together removed the need for 7-adic Fibonacci valuation estimates. The case analysis in Lemma 6.6 replaced an exhaustive computer check of the congruence cover. These refinements were then incorporated into the Lean, so the formal proof follows the revised argument.

Following a preliminary draft of this paper, the elements of C were also recorded in the On-Line Encyclopedia of Integer Sequences as A401169.

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Use of Generative AI Technology

I used LLMs to assist with coding, computations, analysis, and revision, including GPT-6 Astra, GPT-5.2 Pro, GPT-5.5 Pro, GPT-5.6 Sol, and Claude Fable 5 (all accessed in part via Poe with the support of Quora, where I am an advisor). The Lean formalization was developed using Claude Opus 5.5, Claude Fable 5.2, and GPT-6 Astra. GPT-6 Astra identified a key step that made it possible to make my earlier conditional argument unconditional. The problem, methods, and final written form are my own, and any errors remain my responsibility.

A Square-transfer matrices

Kneser's neighbor construction gives a structural source of the square-transfer maps in (8); see [7]. If lattices L_f, L_h, isometric to f, h respectively, are p-neighbors in the same rational quadratic space, their intersection has index p in each. Hence $pL_h \subseteq L_f$, and writing this inclusion in integral

bases gives a matrix satisfying (8). For the chosen primes, the displayed matrix identities provide the complete verification needed here.

Table 3 gives the matrices in (8) by their rows. Periods are listed modulo p for the ten primes in $\mathcal{P}_7$. The rows $p = 2, 3$ supply the coordinate maps in Lemma 3.4 and are not used in Lemma 6.6. Each period and matrix identity can be checked independently of the others.

Table 3: Integral square transfers and Fibonacci periods.

p	Period	Row 1	Row 2	Row 3
2	—	$(0, -2, 0)$	$(0, 0, 4)$	$(-1, 0, -1)$
3	—	$(-1, -1, 7)$	$(-1, 2, 1)$	$(-1, -1, -2)$
5	20	$(-3, -3, 7)$	$(-2, 3, 3)$	$(-1, -1, -6)$
11	10	$(-3, -7, -21)$	$(-1, -6, 15)$	$(-5, 3, -2)$
13	28	$(-5, -9, -21)$	$(-6, -3, 19)$	$(-3, 5, -10)$
17	36	$(-1, -15, -21)$	$(-3, 6, -29)$	$(9, -1, 2)$
19	18	$(-3, -3, 49)$	$(-6, 13, 3)$	$(-7, -7, -6)$
31	30	$(-15, -13, 63)$	$(-2, 21, 27)$	$(-13, -3, -26)$
53	108	$(-21, -39, 77)$	$(-14, 27, 69)$	$(-19, -5, -54)$
83	168	$(-9, -9, 217)$	$(-26, 57, 9)$	$(-31, -31, -18)$
107	72	$(-35, -51, -231)$	$(-30, -59, 123)$	$(-39, 41, -22)$
109	108	$(-3, -105, -77)$	$(-41, -18, 183)$	$(-37, 13, -114)$

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