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Finite-length permutation modules for free-amalgamation structures of arbitrary arity

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Abstract

Let $\mathbb{A}$ be the countable homogeneous limit of a free-amalgamation class in a finite relational language, expanded by a generic linear order. We prove that its tuple permutation modules have finite composition length over every field, also after naming finitely many points. More precisely, each orbit of increasing d-tuples disjoint from the named points has a permutation module of length 2^d. This extends the theorem of Yang–Bojańczyk–Klin from unary and binary relations to relations of arbitrary arity. The main step shows that every finitely supported array whose sums over each single coordinate vanish is a linear combination of alternating sums obtained by independently doubling its coordinates. Its proof combines a Ramsey separation theorem with a construction that moves finite configurations into prescribed order intervals. Over a finite field, we also obtain finitely many closed invariant linear subspaces of $\mathbb{F}^{A^k}$; each is generated, in the product topology, by functions definable using at most k additional parameters.

1 Introduction

A permutation group acting on an infinite set gives rise to an infinite-dimensional vector space with a distinguished basis. The subspaces preserved by the group can nevertheless have a finite and rigid structure. For example, under the full symmetric group on an infinite set X, the space $\mathbb{F}[X]$ of finite formal linear combinations over a field $\mathbb{F}$ has exactly one nonzero proper invariant subspace: the vectors whose coefficients sum to zero. Indeed, swapping a point with nonzero coefficient with a point outside the support and subtracting shows that every nonzero invariant subspace contains all differences of basis vectors; these span the coefficient-sum kernel, which has codimension one, so $\mathbb{F}[X]$ has composition length two [7, Example 4.2].

For an automorphism group of a homogeneous structure, tuples provide a natural supply of permutation modules. Their invariant subspaces encode linear relations compatible with the structure's symmetries. Finite composition length asks whether the number of strict steps in a chain of such subspaces is bounded. This question has arisen both in the theory of infinite permutation groups and in the study of computation over infinite alphabets; see Evans [11, Section 1.1] and Bojańczyk–Fijalkow–Klin–Moerman [7]. The latter authors prove finite length over arbitrary fields for the pure countable set and the countable dense linear order, including finite parameter expansions [7, Theorems 4.8, 4.10 and Corollary 4.9].

Yang–Bojańczyk–Klin [25, Theorem 7.3] extend these results to homogeneous structures obtained from free-amalgamation classes in finite unary and binary relational languages. Their theorem allows a generic linear order and finitely many named parameters, and it holds over every field. It covers, for instance, the random graph and the Henson graphs,

as well as their generic ordered expansions. They also determine the exact length of an increasing tuple orbit and describe invariant subspaces by coordinate sums [25, Theorem 8.14 and Corollary 8.16]. They leave the removal of the restriction on relation arity open [25, Remark 7.4]. The purpose of this paper is to remove that restriction; see Theorem 1.1.

For background on homogeneous structures, ω-categoricity, automorphism groups and structural Ramsey theory, we refer to Bodirsky's monograph [2, Chapters 2, 4, 5, 9 and 11].

1.1 Setting and main results

Let L be a finite relational language and $\mathcal{K}$ a Fraïssé class of finite L-structures with free amalgamation. Here free amalgamation means the following. If $B, C \in \mathcal{K}$ have a common induced substructure D, their union over D belongs to $\mathcal{K}$ when its basic relations are exactly those already present in B or C. In particular, no relation tuple is added whose entries meet both $B \setminus D$ and $C \setminus D$. For graphs, this is the familiar operation of gluing over an induced subgraph and adding no edges between the two new parts. The same definition applies to relations of any arity.

Let $\mathcal{K}^<$ consist of all expansions of members of $\mathcal{K}$ by a new linear order relation $<$, and let $\mathbb{A}$ be its countable Fraïssé limit, with domain A. We call this the *generically ordered limit*. The word "generic" means that every linear order on every finite member of $\mathcal{K}$ is admitted.

Free amalgamation is required only for the original language L; the added order relation $<$ is treated separately. After forming the free L-amalgam of two ordered structures over their common ordered substructure, we equip it with a linear order extending the orders on both pieces. Every such extension is allowed. For example, points from opposite pieces lying in the same gap of the common ordered substructure may be interleaved, provided the order within each piece is preserved. The resulting amalgam has order comparisons between the two new parts, but no basic L-relation instance meeting both parts. Choosing such an order in a particular construction places no further restriction on the class $\mathcal{K}^<$.

No bound on the arity of the symbols of L is assumed, and relation tuples may have repeated entries. The underlying unordered limit will be denoted by $\mathbb{A}_0$.

We will repeatedly use the *finite extension property* of the Fraïssé limit; see [2]. If B is a finite induced substructure of $\mathbb{A}$ and $C \in \mathcal{K}^<$ contains B as an induced substructure, then there is an embedding $C \to \mathbb{A}$ fixing B pointwise. Indeed, universality first gives an embedding of C into $\mathbb{A}$. Homogeneity extends the inverse of its restriction to B to an automorphism of $\mathbb{A}$; composing with that automorphism gives an embedding fixing B.

Let $\mathbb{F}$ be a field. For a finite set $S \subseteq A$, write

$$H_S = \{g \in \mathrm{Aut}(\mathbb{A}) : g(s) = s \text{ for every } s \in S\}.$$

For any H_S-set X, the *permutation module* $\mathbb{F}[X]$ is the vector space with basis $\{[x] : x \in X\}$ and action $g[x] = [g(x)]$, extended linearly. Its *composition length* $\mathrm{length}_{H_S} \mathbb{F}[X]$ is the supremum of the set of integers n for which there is a chain of invariant linear subspaces

$$\{0\} = W_0 < W_1 < \cdots < W_n = \mathbb{F}[X].$$

An *increasing orbit of degree d over S* is an H_S-orbit $\mathcal{O} \subseteq A^d$ consisting of tuples $a_1 < \cdots < a_d$ with all $a_i \notin S$. The degree-zero orbit consists of the empty tuple.

Theorem 1.1. *Let $\mathbb{A}$ be as above. For every field $\mathbb{F}$, every finite set $S \subseteq A$, and every increasing orbit $\mathcal{O}$ of degree d over S,*

$$\mathrm{length}_{H_S} \mathbb{F}[\mathcal{O}] = 2^d.$$

Consequently $\mathbb{F}[A^k]$ has finite composition length for every k, also after naming finitely many parameters. The same finite-length conclusion holds for the pointwise stabilizer $\mathrm{Aut}(\mathbb{A}_0)_{(S)}$ acting on $\mathbb{F}[A^k]$.

The theorem applies, in particular, to generic ordered r-uniform hypergraphs for every finite r. In degree two it gives length four; in degree three it gives length eight. The independence of the answer from the relation arities is part of the conclusion. Different tuple orbits can have different relational diagrams, but each increasing orbit of the same degree has the same length.

There is also a dual formulation over a finite field. Give $\mathbb{F}^{A^k}$, the space of all functions $A^k \to \mathbb{F}$, the product topology, with $\mathbb{F}$ discrete. The group action is $(gf)(a) = f(g^{-1}a)$. A closed invariant linear subspace of this space is often called an *invariant linear code*. A function is *fixed over* $S \cup T$ if it is invariant under $H_{S\cup T}$. There are finitely many tuple orbits over any finite parameter set, each defined by its finite relational diagram. For finite $\mathbb{F}$, invariance is therefore equivalent to definability of the fibres using parameters from $S \cup T$.

Corollary 1.2. *For every finite set $S \subseteq A$, every finite field $\mathbb{F}$, and every integer $k \geq 0$, there are only finitely many closed H_S-invariant linear subspaces of $\mathbb{F}^{A^k}$. Each such subspace is the closed linear span of its functions fixed over $S \cup T$, where $T \subseteq A$ ranges over sets of size at most k.*

The bound is on the number of additional parameters: different spanning functions may use different sets T. The field, the structure, S, and k are fixed in the finiteness assertion.

This is a consequence of the marginal description of [25, Theorem 8.14], transferred to arbitrary arity below, together with the duality in [11, Lemma 2.2].

1.2 Method and relation to earlier work

The proof begins with coefficient sums. If v is a finite linear combination of tuples, summing its coefficients over a deleted coordinate produces a vector on shorter tuples. Call v *balanced* if all these sums vanish. In degree two, this says that a finite matrix has zero row sums and zero column sums. Alternating sums around suitable doubled configurations are elementary examples of balanced vectors. The terminology of balanced vectors and *cogs* is used for ordered atoms in [7, Section 4.1] and generalized to the free-amalgamation setting with unary and binary relations in [25, Section 8]. Section 2 gives the definitions used here and illustrates them in degrees one and two.

The central step is to prove that these cogs span all balanced vectors, in every degree. For binary structures this is [25, Theorem 8.12]. Our proof adapts the slice decomposition and the separation of coordinate positions in [25, Appendix E]. We extend these steps to relations of arbitrary arity and obtain replacement points in prescribed compatible order cuts. We combine this construction with Evans's Ramsey separation theorem [11, Corollary 1.5]. The latter provides a separating linear functional invariant over finitely many parameters. We move the support of a hypothetical counterexample into fixed order intervals over those parameters, without changing its class modulo cogs. Free amalgamation then makes the separator's value a sum of functions of proper coordinate subtuples. Every such function annihilates a balanced vector, giving a contradiction.

Sections 3–5 prove this extension of cog generation. Section 6 then checks the higher-arity compatibility needed to transfer YBK's reconstruction proof [25, Theorem 8.14 and Appendix F.1], and states the resulting marginal description. The exact-length argument is also inherited from [25, Corollary 8.16 and Appendix F.2]; Section 7 cites that argument and derives the closed-function consequences.

For those consequences, we use the standard duality recorded in [11, Lemma 2.2] and the finite-dimensional separation method in the proof of [11, Corollary 2.3]. Applied to YBK's marginal evaluations, this method gives generating functions fixed over the entries of a projected tuple, hence over at most k additional parameters. This bound and the finite-field

finiteness assertion already follow from YBK's marginal description in the binary setting. Their extension here rests on the arbitrary-arity cog-generation theorem; the duality and marginal arguments are inherited.

The model-theoretic motivation concerns first-order reducts. A *first-order reduct* of a structure M is a relational structure on the same domain whose basic relations are definable without parameters in M. Two structures are *first-order interdefinable* if each is a first-order reduct of the other. Thomas's conjecture [22, p. 177] asserts that a countable structure homogeneous in a finite relational language has only finitely many first-order reducts up to this equivalence.

Several complete classifications concern examples within the present setting. Cameron showed that $(\mathbb{Q}, <)$ has five reducts [9], and Thomas proved the corresponding five-reduct theorem for the random graph [22]. Thomas subsequently classified the reducts of the random r-uniform hypergraph for each fixed $r \geq 2$; see [23]. For the random ordered graph, Bodirsky–Pinsker–Pongrácz classified all reducts, finding 42 proper nontrivial ones, besides the original structure and its equality reduct [4]. Named parameters also lead to nontrivial finite lattices: Junker–Ziegler classified the reducts of $(\mathbb{Q}, <)$ expanded by finitely many named constants [15], and Pongrácz classified the reducts of the Henson graphs with one named vertex [21]. All these classifications are up to first-order interdefinability.

General finiteness theorems are also known for broad classes of structures. To state one such result, let $o_n^{\neq}(M)$ be the number of $\mathrm{Aut}(M)$-orbits on n-tuples of pairwise distinct elements. Bodirsky–Bodor [3, Theorem 6.37] proved that a countable structure M has finitely many first-order reducts up to interdefinability if

$$o_n^{\neq}(M) \leq c\, n^{\alpha n} \qquad (n \geq 1)$$

for some constants $c > 0$ and $\alpha < 1$. Bodor [6, Theorem 5.11] extends this finiteness result to all countable ω-categorical monadically stable structures. These results establish the conjectured finiteness for entire classes, without a separate classification of the reducts of each individual structure.

The connection with permutation modules is explained by Evans: failure of finite length over a finite field for a structure homogeneous in a finite relational language would yield a counterexample to Thomas's conjecture [11, Remark 1.3]. Our result establishes this finite-length condition throughout the free-amalgamation setting. Determining all reducts of these structures requires additional arguments.

A second motivation concerns closed normal subgroups. Macpherson [17, Question 2.2.7(4)] asks whether the automorphism group of every countable structure homogeneous in a finite relational language has only finitely many closed normal subgroups, where closure is taken in the topology of pointwise convergence. Evans observes that a failure of finite composition length over a finite field would also yield a counterexample to this question [11, Remark 1.3]. Classical positive cases include the random graph, whose automorphism group was proved simple by Truss [24]. For $(\mathbb{Q}, <)$, the work of Higman [13] and Holland [14], as stated in [10, Theorem 1], gives exactly three proper nontrivial normal subgroups: the automorphisms fixing some final interval pointwise, those fixing some initial interval pointwise, and their intersection. All three are dense, so $\mathrm{Aut}(\mathbb{Q}, <)$ is topologically simple, although it is not simple as an abstract group. Broader simplicity results apply to free-amalgamation limits. Macpherson–Tent [18, Theorem 1.1] prove that $\mathrm{Aut}(M)$ is simple when M is countably infinite and homogeneous, its age has free amalgamation, and $\mathrm{Aut}(M)$ is transitive on M but is not the full symmetric group $\mathrm{Sym}(M)$. Calderoni–Kwiatkowska–Tent [8, Theorem 1.3] prove simplicity for the generic linear order expansions of these structures. These results also cover the Henson graphs and generic r-uniform hypergraphs for $r \geq 2$, together with their generic ordered expansions. In all these cases the only closed normal subgroups are the

identity subgroup and the whole automorphism group.

2 Coordinate sums and doubled configurations

We fix a finite set $S \subseteq A$, a field $\mathbb{F}$, and an increasing H_S-orbit $\mathcal{O} \subseteq A^d$ of degree $d \geq 0$. For a tuple a, let $\underline{a}$ be the set of its entries. For $v = \sum_{a \in \mathcal{O}} v(a)[a] \in \mathbb{F}[\mathcal{O}]$, distinguish the *tuple support* and the *vertex support*:

$$\operatorname{supp}(v) = \{a : v(a) \neq 0\}, \qquad \operatorname{vsupp}(v) = \bigcup_{a \in \operatorname{supp}(v)} \underline{a}.$$

Both are finite. Homogeneity implies that two tuples are in the same orbit over S precisely when the map between their coordinates, together with the identity on S, is a finite partial isomorphism. In particular, the full relational diagram over S is fixed on $\mathcal{O}$.

2.1 Marginals and balanced vectors

Write $[d] = \{1, \ldots, d\}$, with $[0] = \varnothing$. For $J \subseteq [d]$ and $a \in \mathcal{O}$, let $p_J(a)$ be the subtuple in the positions indexed by J, in increasing order of indices. Its image $p_J(\mathcal{O})$ is again an increasing orbit over S. We also write

$$p_J : \mathbb{F}[\mathcal{O}] \longrightarrow \mathbb{F}[p_J(\mathcal{O})]$$

for the linear extension. For $v \in \mathbb{F}[\mathcal{O}]$ and $b \in p_J(\mathcal{O})$, it is given by

$$p_J(v) = \sum_{a \in \mathcal{O}} v(a)[p_J(a)], \qquad (p_J v)(b) = \sum_{\substack{a \in \mathcal{O} \\ p_J(a) = b}} v(a). \tag{1}$$

Thus coefficients with the same projected tuple are added. We call $p_J v$ a *coordinate marginal*. The empty marginal is identified with the scalar $p_\varnothing(v) = \sum_a v(a)$. Put $p_{-i} = p_{[d] \setminus \{i\}}$.

Definition 2.1. For $d > 0$, the space of balanced vectors is

$$\operatorname{Bal}(\mathcal{O}) = \bigcap_{i=1}^{d} \ker p_{-i}.$$

For ordered atoms, this is the balancedness condition of [7, Section 4.1, equation (4)]; its formulation on tuple orbits follows [25, Section 8.1].

Every proper marginal of a balanced vector vanishes: a projection onto a proper subset of positions factors through at least one of the one-coordinate deletions. All these maps commute with H_S, so $\operatorname{Bal}(\mathcal{O})$ is invariant.

Example 2.2. For $d = 1$, balanced means simply that the coefficient sum is zero.

For $d = 2$, we can arrange the coefficients of a vector

$$v = \sum_{(x,y) \in \mathcal{O}} \lambda_{x,y}[(x, y)]$$

in a table: put $\lambda_{x,y}$ in row x, column y, and put zero in every position $(x, y) \notin \mathcal{O}$. Denote the resulting matrix by M. Its rows and columns are indexed by A, but only finitely many entries are nonzero. Thus a finite portion of the matrix contains all the nonzero coefficients.

The projection $p_{\{1\}}$ keeps the first coordinate and forgets the second. In particular, both $[(x, y)]$ and $[(x, z)]$ are sent to $[x]$, so their coefficients are added. More generally,

$$p_{\{1\}} v = \sum_{x \in p_{\{1\}}(\mathcal{O})} \left(\sum_{y \in A} M_{x,y} \right) [x].$$

The coefficient of $[x]$ is the sum of row x. Since distinct basis vectors are linearly independent, $p_{\{1\}}v = 0$ precisely when every row sum is zero. Similarly, $p_{\{2\}}$ keeps the second coordinate, and the coefficient of $[y]$ in $p_{\{2\}}v$ is the column sum $\sum_{x\in A} M_{x,y}$. Hence v is balanced exactly when every row and every column separately sums to zero. Merely requiring the sum of all coefficients to vanish would be weaker.

For a concrete example, take $\mathbb{A} = (\mathbb{Q}, <)$, $S = \varnothing$, and the orbit $\mathcal{O} = \{(x, y) \in \mathbb{Q}^2 : x < y\}$. The degree-two cog described in [7, p. 11] has the following numerical instance:

$$v = [(1,3)] - [(1,4)] - [(2,3)] + [(2,4)].$$

The part of its coefficient matrix containing all nonzero entries is

	$y = 3$	$y = 4$	row sum
$x = 1$	1	-1	0
$x = 2$	-1	1	0
column sum	0	0	

and every omitted entry is zero. Thus v is a nonzero balanced vector.

The orbit specifies which matrix positions are allowed to carry nonzero coefficients. In this example, all positions with $x \geq y$ must be zero: the pair $(3, 1)$, for instance, does not belong to $\mathcal{O}$. If instead $\mathcal{O}$ is the orbit of increasing edges in a random ordered graph, then an entry at (x, y) is allowed only when both $x < y$ and $E(x, y)$ hold. The row and column conditions are imposed on the coefficients in these allowed positions, with zeros in all other positions.

2.2 Independent doubling and cogs

For a finite $B \subseteq A$, a *cut over* B is a maximal convex subset of $A \setminus B$, including the two unbounded intervals. Two points outside B have the same cut if they have the same order comparisons with every element of B.

Assume $d > 0$ in this subsection, and start with a tuple $a = (a_1, \ldots, a_d) \in \mathcal{O}$. We want to replace each coordinate a_i by either itself or a new point b_i, so that all 2^d choices give tuples in $\mathcal{O}$. We prescribe more than this: the whole finite configuration is an independent doubling of the original one.

Definition 2.3. An *independent doubling* of a over S consists of a tuple $b = (b_1, \ldots, b_d)$ of pairwise distinct points outside $S \cup \underline{a}$, with

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_d < b_d, \tag{2}$$

such that a_i, b_i have the same cut over S and the basic L-relations on $S \cup \underline{a} \cup \underline{b}$ are given as follows. Let $r : S \cup \underline{a} \cup \underline{b} \longrightarrow S \cup \underline{a}$ fix $S \cup \underline{a}$ and send b_i to a_i for each $i \in [d]$. A basic L-relation instance whose arguments contain both a_i and b_i for some i is false. For every other instance $R(c_1, \ldots, c_m)$, its truth value is that of $R(r(c_1), \ldots, r(c_m))$ in the original structure on $S \cup \underline{a}$.

This rule covers unary relations, nonsymmetric relations, and repeated arguments. The order is governed separately by (2).

Independent doublings can be constructed by successive free amalgamations, adapting [25, Lemmas 8.7–8.9] to the rule for relations of arbitrary arity in Definition 2.3. Let C_0 be the induced structure on $S \cup \underline{a}$. At stage $i \in [d]$, take a second copy of the entire current structure C_{i-1} over $C_{i-1} \setminus \{a_i\}$, with the copied point named b_i. Form the free amalgam in the language L, and place b_i immediately after a_i in the resulting finite order. Denote

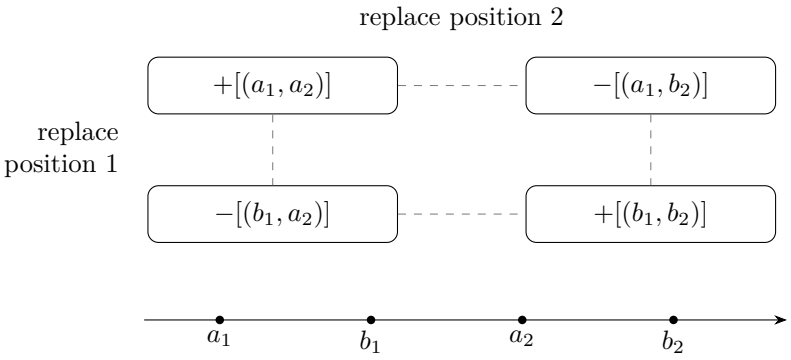


Figure 1: A schematic of the degree-two cog described in [7, p. 11]. The four boxes represent basis tuples, not vertices of the underlying structure. Each row and column of the square cancels under the corresponding marginal. The strip below records the order of the four actual vertices. The dashed lines indicate coordinate replacement, not basic relations.

the resulting ordered structure by C_i. All twins introduced at earlier stages belong to the common part of this amalgamation.

An induction on i verifies the relation rule in Definition 2.3. Instances containing both a_i and b_i are false by free amalgamation. For an instance containing b_i but not a_i, its truth value is copied from the corresponding instance with a_i in place of b_i. Instances involving only old points are unchanged. Thus the conditions for earlier doubled pairs are preserved, and the new pair satisfies the same rule. This also determines relations involving several new twins. The order choice preserves the required interleaving and the comparisons with S. After d stages, the extension property of the Fraïssé limit realizes C_d in $\mathbb{A}$, fixing C_0.

For $J \subseteq [d]$, let a^J denote the tuple whose ith entry is b_i if $i \in J$ and a_i otherwise. Each a^J has the same induced diagram over S as a, and hence belongs to $\mathcal{O}$.

Definition 2.4. The *cog* associated with the independent doubling (a, b) is

$$c(a, b) = \sum_{J \subseteq [d]} (-1)^{|J|} [a^J]. \tag{3}$$

Let $\mathrm{Cog}(\mathcal{O})$ be the linear span of all such cogs.

This extends the duo and cog definitions of [25, Definitions 8.3 and 8.5]. The doubled configurations in Definition 2.3 all have the same diagram over S, and thus form one H_S-orbit. Every automorphism fixing S sends an independent doubling to an independent doubling and its cog to the corresponding cog. Consequently $\mathrm{Cog}(\mathcal{O})$ is H_S-invariant.

Example 2.5. In degree one a cog is $[a_1] - [b_1]$, where b_1 is a free copy of a_1 over S lying to its right in the same S-cut. In degree two it is

$$[(a_1, a_2)] - [(b_1, a_2)] - [(a_1, b_2)] + [(b_1, b_2)]. \tag{4}$$

For ordered atoms, this is the degree-two cog of [7, p. 11]; the same expression applies to any orbit admitting an independent doubling as defined here. In the matrix interpretation, these are the four corners of a rectangle. Each row and each column has two cancelling coefficients; see Figure 1. For example, projection onto the first coordinate gives $[a_1] - [b_1] - [a_1] + [b_1] = 0$.

In general, deleting position i identifies the term indexed by J with the term indexed by $J \triangle \{i\}$, and their coefficients are opposite. Therefore

$$\mathrm{Cog}(\mathcal{O}) \subseteq \mathrm{Bal}(\mathcal{O}). \tag{5}$$

The reverse inclusion is the substantive statement.

Theorem 2.6 (Generation by cogs)**.** *For every finite $S \subseteq A$, every positive integer d, and every increasing orbit $\mathcal{O}$ of degree d over S,*

$$\mathrm{Bal}(\mathcal{O}) = \mathrm{Cog}(\mathcal{O}).$$

2.3 The degree-one argument

It is useful to prove the first case explicitly. Although every zero-sum vector is a linear combination of differences of points, an arbitrary difference $[x] - [y]$ need not itself be a cog: the points might have a basic relation between them, or might occur in the wrong order. A fresh point provides the needed bridge.

Proof of Theorem 2.6 in degree one. Let $v = \sum_{x \in F} \lambda_x [x]$ with $F \subseteq \mathcal{O}$ finite and $\sum_x \lambda_x = 0$. All points of $\mathcal{O}$ have the same cut and the same relational type over S. Freely amalgamate the structure on $S \cup F$ with one new copy b of a point of $\mathcal{O}$ over S. Put b to the right of every point of F, still in their common S-cut.

For each $x \in F$, the pair (x, b) is an independent doubling over S: its two one-point diagrams agree over S, and every basic L-relation instance on $S \cup F \cup \{b\}$ that uses both x and b is false. Hence $[x] - [b]$ is a cog. Now

$$\sum_{x \in F} \lambda_x ([x] - [b]) = \sum_{x \in F} \lambda_x [x] - \left(\sum_{x \in F} \lambda_x \right) [b] = v.$$

The auxiliary coefficient at b disappears because v is balanced. □

The higher-degree proof uses the same principle: auxiliary points allow a controlled replacement, and their unwanted contributions cancel. We must perform these replacements without disturbing the coordinate sums.

3 Ramsey separation and the proof strategy

A class of finite ordered structures is *Ramsey* if, for every B, C in the class and every positive integer r, some member D has the following property: every colouring of the copies of B in D with r colours has a copy of C all of whose copies of B receive the same colour. A homogeneous ordered structure is called Ramsey when its age, the class of its finite induced substructures up to isomorphism, has this property.

The Ramsey theorem for ordered relational structures originates in the work of Nešetřil–Rödl [19], with further development for classes of set systems in [20]. For unrestricted finite ordered relational structures, it was also proved independently by Abramson–Harrington [1].

The free-amalgamation form of the Nešetřil–Rödl theorem states that $\mathcal{K}^<$ is Ramsey; see [12, Theorem 1.2] for a modern formulation covering the relational languages used here. Naming finitely many constants preserves the Ramsey property: see the elegant proof in [5, Proposition 10], based on the theorem of Kechris–Pestov–Todorcevic [16] describing Ramsey structures via a dynamical property.

Homogeneity in a finite relational language also implies that H_S has finitely many orbits on A^n for every n: only finitely many induced diagrams on n entries and the fixed set S can occur. By the Ryll–Nardzewski theorem, $(\mathbb{A}, S)$ is ω-categorical; see [2, Chapter 4] and [11, Section 1.3] for this background. These facts permit the following application of Evans's theorem.

Lemma 3.1 (Evans's separation theorem)**.** *Let $U \leq \mathbb{F}[\mathcal{O}]$ be H_S-invariant and $v \in \mathbb{F}[\mathcal{O}] \setminus U$. Set $T = S \cup \mathrm{vsupp}(v)$. There is a function $f : \mathcal{O} \to \mathbb{F}$ invariant under H_T such that*

$$\sum_{a \in \mathcal{O}} u(a)f(a) = 0 \quad (u \in U), \qquad \sum_{a \in \mathcal{O}} v(a)f(a) \neq 0.$$

Proof. If $d = 0$, then $\mathcal{O}$ consists of the empty tuple and $\mathbb{F}[\mathcal{O}]$ is one-dimensional. Since $v \notin U$, necessarily $U = 0$ and $v \neq 0$, so the function taking value 1 on the empty tuple has the required properties. Henceforth assume $d \geq 1$.

We apply [11, Corollary 1.5], identifying its structure, group, module, and support explicitly. Let

$$M = (\mathbb{A}, (s)_{s \in S})$$

be the expansion obtained by naming each element of S as a constant. Its automorphism group is $G = \mathrm{Aut}(M) = H_S$: an automorphism of M is exactly an automorphism of $\mathbb{A}$ fixing every element of S. The preceding discussion shows that M is both ω-categorical and Ramsey, as required by Evans's corollary.

For the set W in that corollary, take $W = \mathcal{O}$. Every orbit on finite tuples under the automorphism group of an ω-categorical structure is definable without parameters. Thus $\mathcal{O}$, being a G-orbit, is definable without parameters in M: the elements of S have already been named in its language. To check the sort hypothesis, recall that Evans permits sorts D/E, where D is a parameter-free definable set and E is a parameter-free definable equivalence relation on D [11, Section 1.3.2]. Taking $D = \mathcal{O}$ and E to be equality identifies D/E with $\mathcal{O}$.

The action of G on $W = \mathcal{O}$ is coordinatewise:

$$g(a_1, \ldots, a_d) = (g(a_1), \ldots, g(a_d)) \qquad (g \in G,\ (a_1, \ldots, a_d) \in \mathcal{O}).$$

The free vector space denoted by FW in Evans's corollary is therefore exactly $\mathbb{F}[\mathcal{O}]$. We take his field F to be our $\mathbb{F}$, his invariant subspace V to be U, and his vector x to be v. The hypotheses that V is G-invariant and $x \notin V$ are precisely the assumptions that U is H_S-invariant and $v \notin U$.

The support used in Evans's conclusion is the support of x as a vector with basis W. In our application this is

$$\Sigma = \mathrm{supp}(v) = \{a \in \mathcal{O} : v(a) \neq 0\}.$$

Thus Σ is a finite set of tuples, whereas S and $T = S \cup \mathrm{vsupp}(v)$ are finite sets of vertices. We use the letter Σ because the support is denoted by S in Evans's statement, with a different meaning from our parameter set S.

Evans's corollary gives invariance under $G_{(\Sigma)}$, the subgroup fixing each tuple in Σ individually. Fixing a tuple under the coordinatewise action is equivalent to fixing each of its entries. Since every element of $G = H_S$ already fixes S pointwise, we obtain

$$\begin{aligned} G_{(\Sigma)} &= \{g \in H_S : g(a) = a \text{ for every } a \in \Sigma\} \\ &= H_{S \cup \bigcup_{a \in \Sigma} \underline{a}} = H_{S \cup \mathrm{vsupp}(v)} = H_T. \end{aligned}$$

Here it is essential that the tuples are fixed individually: merely permuting the tuples in Σ would not suffice.

With these choices, the corollary supplies a function $f : \mathcal{O} \to \mathbb{F}$ satisfying

$$f(g(a)) = f(a) \qquad (g \in H_T,\ a \in \mathcal{O})$$

and

$$\sum_{a \in \mathcal{O}} u(a)f(a) = 0 \quad (u \in U), \qquad \sum_{a \in \mathcal{O}} v(a)f(a) \neq 0.$$

These sums express Evans's separation conclusion: the linear functional obtained by summing against f vanishes on every vector in U and does not vanish on v. All sums are finite because the vectors have finite support. Together with the H_T-invariance, these are exactly the conclusions of the lemma. □

We use the same symbol f for the function on tuples and its linear extension to $\mathbb{F}[\mathcal{O}]$: for $w = \sum_{a\in\mathcal{O}} w(a)[a]$, we set

$$f(w) = \sum_{a\in\mathcal{O}} w(a)f(a).$$

The sum is finite because w has finite support. Ordinary linear algebra gives a functional separating v from U; the additional content of Lemma 3.1 is invariance under H_T. In particular, the value of f on a tuple depends only on its induced diagram over T.

3.1 Proof outline for cog generation

Our goal is Theorem 2.6, $\mathrm{Bal}(\mathcal{O}) = \mathrm{Cog}(\mathcal{O})$. The inclusion $\mathrm{Cog}(\mathcal{O}) \subseteq \mathrm{Bal}(\mathcal{O})$ was established in (5), and the degree-one case was proved in Section 2. We outline the inductive step for $d \geq 2$, assuming the theorem in degree $d-1$ for every finite parameter set. The constructions needed for this step are proved in Sections 4 and 5; here we describe what they must achieve and how they yield a contradiction.

Suppose that $v \in \mathrm{Bal}(\mathcal{O}) \setminus \mathrm{Cog}(\mathcal{O})$, and put $U = \mathrm{Cog}(\mathcal{O})$ and $T = S \cup \mathrm{vsupp}(v)$. Applying Lemma 3.1 and rescaling the resulting function gives an H_T-invariant $f : \mathcal{O} \to \mathbb{F}$ whose linear extension vanishes on U and satisfies $f(v) = 1$.

The first construction, proved in Lemma 4.3 in Section 4, produces a balanced vector $u \in \mathbb{F}[\mathcal{O}]$ such that

$$u - v \in U, \qquad \mathrm{vsupp}(u) \cap T = \varnothing,$$

and, for each position i, all ith coordinates of tuples in $\mathrm{supp}(u)$ lie in one fixed cut over T. The construction repeatedly replaces all occurrences of one vertex in one coordinate position. After that coordinate is deleted, the chosen slice is balanced in degree $d-1$ over S together with the fixed vertex. The induction hypothesis decomposes this slice into smaller cogs. Lemma 4.1 constructs a replacement point so that each smaller cog, with the old and new points inserted, gives a degree-d cog. Consequently each replacement changes the vector by an element of U. Lemma 4.3 specifies the order of replacements and their destination cuts, proving that finitely many replacements give all the stated support conditions. These are the constructions to be justified in Section 4. Once u has been obtained, it satisfies

$$f(u) = f(v) + f(u - v) = 1.$$

The second construction concerns the individual tuples occurring in u. Let $a = (a_1, \ldots, a_d)$ range over $\mathrm{supp}(u)$, so that

$$u = \sum_{a\in\mathrm{supp}(u)} u(a)[a].$$

We introduce a tuple $b(a)$ to complete a to a cog. Since f vanishes on every cog, this will give an identity expressing $f(a)$ through its values on mixed tuples. The choice of $b(a)$ must also ensure that these other values depend only on proper subtuples of a. For each such a, Section 5 constructs a tuple $b = b(a) = (b_1, \ldots, b_d)$ of fresh points outside $T \cup \underline{a}$ forming an independent doubling of a over S (Definition 2.3). Each b_i lies in the same T-cut as a_i. In the induced structure on $T \cup \underline{a} \cup \underline{b}$, every basic L-relation instance meeting both $\underline{b}$ and $T \setminus S$ is false. The tuple $b(a)$ is chosen separately for each a. For $J \subseteq [d]$, recall that a^J is obtained

from a by replacing a_i with b_i exactly when $i \in J$. In particular, $a^{\varnothing} = a$ and $a^{[d]} = b(a)$. The independent doubling gives $c(a, b(a)) \in U$, so applying f yields the connecting identity

$$0 = f(c(a, b(a))) = f(a) + \sum_{\varnothing \neq J \subseteq [d]} (-1)^{|J|} f(a^J) \qquad (a \in \mathrm{supp}(u)).$$

Thus the auxiliary tuple $b(a)$ supplies the mixed tuples whose values under f determine $f(a)$.

The additional conditions imposed on $b(a)$ relative to T ensure dependence on retained coordinates. More precisely, Section 5 proves the following assertion. For each nonempty $J \subseteq [d]$, put $K = [d] \setminus J$. There is a well-defined function

$$g_J : p_K(\mathrm{supp}(u)) \longrightarrow \mathbb{F} \quad \text{such that} \quad f(a^J) = g_J(p_K(a)) \qquad (a \in \mathrm{supp}(u)).$$

Thus, if two supported tuples have the same retained subtuple, their mixed tuples have the same value under f. The later proof establishes this by comparing the induced diagrams over T and using the H_T-invariance of f.

Substituting these values into the displayed cog identity expresses $f(a)$ as a signed sum of functions of proper subtuples. When multiplied by $u(a)$ and summed over $a \in \mathrm{supp}(u)$, each such term vanishes because the corresponding proper marginal of u is zero. Hence $f(u) = 0$, contrary to $f(u) = 1$. The detailed summation is given in Section 5, after the constructions and the dependence on retained coordinates have been proved.

4 Moving a balanced vector into fixed intervals

We now prove the first construction used in the outline of Section 3.1. Given a balanced vector v and a finite set $T \supseteq S$, we will construct a balanced vector u with no support vertex in T, with each coordinate position confined to a specified cut over T, and with $v - u \in \mathrm{Cog}(\mathcal{O})$. Thus the construction changes the representative of the coset $v + \mathrm{Cog}(\mathcal{O})$ while controlling where its vertices lie.

Throughout this section $d \geq 2$. We assume Theorem 2.6 in degree $d - 1$ for every finite parameter set and every increasing orbit of that degree. This simultaneous induction is needed because fixing one coordinate at a vertex z leaves a vector of degree $d - 1$ over the enlarged parameter set $S \cup \{z\}$.

The local operation replaces all terms having a specified vertex in a specified position. Lemma 4.1 proves that a suitable replacement changes the vector by a sum of cogs. Lemma 4.3 then arranges a finite sequence of these replacements to reach the required cuts.

4.1 Replacing one coordinate slice

The slice-decomposition method comes from [25, proofs of Claim E.5 and Theorem E.6]. The proof below extends its lifting step to relations of arbitrary arity and establishes the freedom to prescribe a cut for the replacement point. We give the construction explicitly, including the placement of the auxiliary vertices used in the decomposition.

Lemma 4.1 (Slice replacement)**.** *Let $v \in \mathrm{Bal}(\mathcal{O})$, let $i \in [d]$, and let $z \in A \setminus S$ occur in position i in some tuple in* $\mathrm{supp}(v)$*. Define*

$$w = \sum_{\substack{a \in \mathcal{O} \\ a_i = z}} v(a)[p_{-i}(a)], \qquad P = \mathrm{vsupp}(w). \tag{6}$$

The vector w is the slice at $a_i = z$*, with its fixed coordinate deleted. Restoring z in position i recovers each original tuple, so deletion is injective on this slice and no coefficients combine.*

The set P consists of the vertices in the other positions of the supported tuples in this slice; we call these vertices the partners of z in position i. *In particular, $z \notin P$.*

Let $D \subseteq A$ be finite with $S \cup \mathrm{vsupp}(v) \subseteq D$. Let J be a cut over D such that

$$p < y \quad \Longleftrightarrow \quad p < z \qquad (p \in S \cup P,\ y \in J).$$

Thus moving z into J preserves its order comparisons with the permanent parameters and all partners in this slice. There is a point $z' \in J$ such that replacing z by z' only in position i gives a vector $v' \in \mathrm{Bal}(\mathcal{O})$ with

$$v - v' \in \mathrm{Cog}(\mathcal{O}), \qquad \mathrm{vsupp}(v') \subseteq \mathrm{vsupp}(v) \cup \{z'\}.$$

In particular, $z' \notin D$.

Proof. We first decompose the slice using the induction hypothesis. We then construct one replacement point that works for every cog in this decomposition. Finally, we compute the difference between the original vector and its replacement.

Step 1: the slice is balanced in degree $d-1$. Put

$$\Theta = \{p_{-i}(a) : a \in \mathcal{O},\ a_i = z\}.$$

This is one increasing orbit of degree $d-1$ over $S \cup \{z\}$. Indeed, if $a, \widetilde{a} \in \mathcal{O}$ both have z in position i, an element of H_S taking a to $\widetilde{a}$ must fix z. Their remaining coordinates avoid $S \cup \{z\}$ because all tuples in $\mathcal{O}$ have distinct entries outside S. Thus $w \in \mathbb{F}[\Theta]$.

We spell out the balanced-slice observation of [25, Lemma E.1]. Temporarily label the coordinates of w by their original positions $[d] \setminus \{i\}$. Fix $j \neq i$ and put $K = [d] \setminus \{i, j\}$. Let t be any K-subtuple occurring in this slice. Let b be the tuple whose positions are indexed, in increasing order, by $[d] \setminus \{j\}$, with entry z in original position i and entries t in positions K. The coefficient of $[t]$ in the marginal of w deleting original position j is

$$\sum_{\substack{a \in \mathcal{O} \\ a_i = z,\ p_K(a) = t}} v(a) = (p_{-j}v)(b) = 0.$$

The equality holds because $p_{-j}(a) = b$ specifies exactly $a_i = z$ and $p_K(a) = t$. The last expression vanishes by balancedness of v. This proves that every one-coordinate marginal of w is zero.

The induction hypothesis, applied to Θ over $S \cup \{z\}$, now gives a finite expression

$$w = \sum_{r=1}^{m} \alpha_r c_r, \tag{7}$$

where $\alpha_r \in \mathbb{F}$ and every c_r is a degree-$(d-1)$ cog over $S \cup \{z\}$. This use of cog generation is in smaller degree.

Step 2: place the auxiliary configurations so that z can be replaced in the prescribed cut. Let $C_0 \subseteq A$ be a finite set containing

$$B = S \cup \{z\} \cup P$$

and all vertices of the doubled configurations defining $c_1, \ldots, c_m$. We use the same letters for these sets and their induced structures. The vertices in $C_0 \setminus B$ are auxiliary: all coefficients of tuples using them cancel in (7), since w is supported on tuples with entries in P.

We will place a copy C of C_0 over B so that no vertex of $C \setminus \{z\}$ separates z from the eventual replacement point. This ensures that doubling z will preserve the order in every cog

configuration. The vertices of B must remain fixed, but the auxiliary vertices can be copied elsewhere.

Here is the finite construction. Introduce a formal marker $*$, placed in the prescribed cut J of D. It is used only to specify an order position, not as a vertex of an L-structure. Take a copy of C_0 over B, with its new vertices disjoint from D, and form its free L-amalgam with D over B. Thus the copy, denoted by C, satisfies $C \cap D = B$.

We must extend the orders on $D \cup \{*\}$ and C so that, for every copied auxiliary vertex, its side of z is also its side of $*$. This is possible as follows. By the hypothesis on J, the points z and $*$ lie in the same gap of $B \setminus \{z\} = S \cup P$. In this gap, place the copied auxiliary vertices that originally lay below z below both z and $*$, and those that originally lay above z above both. Preserve their order within each of these two lists. In every other gap of $B \setminus \{z\}$, preserve the order of the copied vertices and interleave them arbitrarily with the vertices of D. All these choices preserve the order on C_0 over B and the order on $D \cup \{*\}$.

Remove the marker, remembering its position as a cut over $D \cup C$. This cut is contained in J. Since every linear order on a member of $\mathcal{K}$ is allowed in $\mathcal{K}^<$, the ordered free amalgam belongs to $\mathcal{K}^<$. The finite extension property of $\mathbb{A}$ realizes it over the actual structure D. Let $h : C_0 \to C$ be the resulting copying isomorphism. It fixes B pointwise, so it fixes w. Applying h to (7) therefore gives the same decomposition of w by the transported cogs $h(c_r)$. We henceforth write c_r for these transported cogs.

Step 3: add the new doubled coordinate. Take a second copy of C over $C \setminus \{z\}$, naming the copy of z by z'. Freely amalgamate this copy with the structure on $D \cup C$ over $C \setminus \{z\}$. The only new vertex is z'. Place it in the cut recorded by the marker. The order construction in Step 2 ensures that z and z' have the same comparisons with every point of $C \setminus \{z\}$, so this order extends the order on the second copy of C. The extension property again realizes the finite ordered amalgam in $\mathbb{A}$, fixing $D \cup C$.

Consequently $z' \in J$ and $z' \notin D \cup C$. Replacing z by z' induces an isomorphism from C onto its second copy. In the induced structure on $D \cup C \cup \{z'\}$, every basic L-relation instance involving both z and z' is false. All configurations used below lie in this finite structure, so these statements specify the relations of all arities needed for the lifting step.

Let $\Xi \subseteq \Theta$ be the finite set of tuples appearing in $c_1, \dots, c_m$; it contains $\mathrm{supp}(w)$. For $t \in \Xi$, let $I_z(t)$ be obtained by inserting z in position i, and define $I_{z'}(t)$ by inserting z' there. The first tuple belongs to $\mathcal{O}$ by the definition of Θ. The second has the same diagram over S, by the copied relations and order, and hence also belongs to $\mathcal{O}$. Extend these assignments linearly:

$$I_z, I_{z'} : \mathbb{F}[\Xi] \longrightarrow \mathbb{F}[\mathcal{O}].$$

For each r we claim that

$$I_z(c_r) - I_{z'}(c_r)$$

is a degree-d cog, up to its overall sign. This is the lifting step of [25, Lemma 8.8], with the following verification for arbitrary relation arity. The proposed doubled pairs are the $d-1$ pairs defining c_r, together with (z, z'). Both entries of each older pair lie on the same side of z, because that pair has the same cut over $S \cup \{z\}$. Step 2 places them on the same side of z' as well. Thus the new pair has position i and the pairs have the interleaving required in Definition 2.3.

To verify the relation rule, consider any basic L-relation instance on these vertices and S. If it does not contain z', its truth value is already governed by the cog configuration over $S \cup \{z\}$. If it contains both z and z', it is false by free amalgamation. If it contains z' but not z, its truth value is obtained by replacing every occurrence of z' by z, since the second copy of C is isomorphic to the first. In particular, instances containing both entries of any older pair remain false. This verifies the rule even when an instance uses several pairs or

repeats an argument. If $z' < z$, orient the new pair as (z', z); reversing this pair changes the alternating sum by its overall sign only. Expanding the alternating sum with respect to the choice of z or z' gives precisely $I_z(c_r) - I_{z'}(c_r)$, with this possible sign.

Step 4: compute the replacement vector. Set

$$v' = v - I_z(w) + I_{z'}(w).$$

The vector $I_z(w)$ is exactly the part of v whose ith coordinate equals z. Thus this formula replaces that entire slice and leaves all other terms unchanged, including terms where z occurs in a different position. By linearity and (7),

$$\begin{aligned} v - v' &= I_z(w) - I_{z'}(w) \\ &= \sum_{r=1}^{m} \alpha_r \big(I_z(c_r) - I_{z'}(c_r)\big) \in \mathrm{Cog}(\mathcal{O}). \end{aligned} \tag{8}$$

The formula for v' uses w itself, so the auxiliary tuples in its cog decomposition cancel before and after insertion. Only the new point z' can enter the vertex support. Moreover, replacing one coordinate by a fresh point maps the supported tuples injectively to supported tuples: replaced tuples remain distinct and cannot coincide with unchanged tuples, none of which contains z'. Finally, every cog is balanced by (5). Hence $v' = v - (v - v')$ is balanced, as required. □

Example 4.2 (The replacement identity in degree two)**.** Take $d = 2$ and replace the first-coordinate slice at z. Then

$$w = \sum_{\substack{y \in A \\ (z,y) \in \mathcal{O}}} v(z,y)[y], \qquad \sum_{\substack{y \in A \\ (z,y) \in \mathcal{O}}} v(z,y) = 0.$$

The second equality is the zero row-sum condition for v. Choose a finite $D \supseteq S \cup \mathrm{vsupp}(v)$ and prescribe its cut immediately to the right of z. This cut is admissible, since it has the same comparisons as z with every other point of D. Steps 2 and 3 of the lemma's proof give one point $z' > z$ that works for the entire cog decomposition of w. Each transported degree-one cog has the form $[b] - [c]$, where (b, c) is an independent doubling over $S \cup \{z\}$ and $b < c$. Its lift is

$$I_z([b] - [c]) - I_{z'}([b] - [c]) = [(z, b)] - [(z, c)] - [(z', b)] + [(z', c)].$$

This is the rectangle cog of Example 2.5, with doubled pairs (z, z') and (b, c) and order $z < z' < b < c$. Subtracting this rectangle cog transfers the coefficient pattern $[b] - [c]$ from the row indexed by z to the row indexed by z'. Subtracting the linear combination of these rectangles with the coefficients in the decomposition of w transfers the whole row. The auxiliary column coefficients cancel because their total contribution to w is zero.

The lemma already allows us to remove all support vertices from any fixed finite set $T \supseteq S$. Let v_{current} denote the vector at the current stage, initially the given balanced vector. Whenever a vertex $z \in T \setminus S$ occurs in position i, take $D = T \cup \mathrm{vsupp}(v_{\mathrm{current}})$ and use its cut immediately to the right of z. This cut preserves comparisons with all partners and with S. The replacement removes the pair (z, i) from the occurrences of vertices of T in the support and introduces no new such pair, since $z' \notin T$. There are only finitely many such pairs, so this process terminates. The successive differences belong to $\mathrm{Cog}(\mathcal{O})$ by (8). To obtain the additional requirement that each position occupy one fixed cut over T, we use the ordering of replacements proved next.

4.2 Separating positions and moving from right to left

The following lemma supplies the full conclusion needed in Section 3.1. The first stage of its proof uses the removal of shared coordinate positions from [25, proof of Theorem E.6]. The second stage uses the prescribed-cut version of Lemma 4.1 to move all entries into the specified T-cuts.

Lemma 4.3 (Fixed-interval representative)**.** *Let $T \supseteq S$ be finite. For each $i \in [d]$, let I_i be the S-cut containing the ith coordinate of every tuple in $\mathcal{O}$, and let J_i be the rightmost T-cut contained in I_i. For every $v \in \mathrm{Bal}(\mathcal{O})$, there is $u \in \mathrm{Bal}(\mathcal{O})$ such that*

$$v - u \in \mathrm{Cog}(\mathcal{O}), \qquad \mathrm{vsupp}(u) \cap T = \varnothing,$$

and

$$a_i \in J_i \quad \text{for every } a \in \mathrm{supp}(u) \text{ and every } i \in [d].$$

The cut I_i is well-defined because the orbit $\mathcal{O}$ fixes all order comparisons of its ith coordinate with S. The cut J_i consists of the points in I_i lying above every point of $T \cap I_i$. If $T \cap I_i$ is empty, then $J_i = I_i$. Several coordinate positions may have the same S-cut and therefore the same destination T-cut; their relative order must still be preserved.

Proof. If $v = 0$, take $u = 0$. Otherwise, let q denote the current vector, initially v. Every operation below is an application of Lemma 4.1; after an operation producing q', replace q by q'. Thus every current vector is balanced, and every successive difference belongs to $\mathrm{Cog}(\mathcal{O})$. We must specify admissible cuts for the operations and show that finitely many of them achieve the stated support conditions.

Stage 1: give each support vertex a unique coordinate position. A vertex may occur in different positions in different tuples of $\mathrm{supp}(q)$. For $x \in \mathrm{vsupp}(q)$, define its set of positions by

$$R_q(x) = \{j \in [d] : a_j = x \text{ for some } a \in \mathrm{supp}(q)\}.$$

If $|R_q(x)| > 1$, choose $i \in R_q(x)$ and put $D = T \cup \mathrm{vsupp}(q)$. Choose the cut immediately to the right of x in the finite order on D. Every point of $D \setminus \{x\}$ has the same comparison with this cut as with x. In particular this holds for S and for all partners of x in the chosen slice, which do not include x itself. Lemma 4.1 therefore applies. It replaces all occurrences of x in position i by one fresh vertex x', leaving the other positions unchanged.

The replacement is injective on the supported tuples, as shown in the proof of Lemma 4.1. Thus x loses exactly position i, the new vertex x' occurs only in position i, and the position sets of every other vertex are unchanged. The nonnegative integer

$$E(q) = \sum_{x \in \mathrm{vsupp}(q)} (|R_q(x)| - 1)$$

therefore decreases by exactly one. After finitely many steps $E(q) = 0$, so every support vertex has a unique position. This property will persist through Stage 2, since each replacement point is fresh and is used in just one position.

Stage 2: move the positions into their destination cuts. Process the coordinate positions in the order $d, d-1, \ldots, 1$. Within a position, process its support vertices in any order. At the beginning of this stage, declare all support vertices unmoved. Let N and M denote, respectively, the sets of currently unmoved and moved support vertices; initially $N = \mathrm{vsupp}(q)$ and $M = \varnothing$. A fresh replacement is declared moved and is never processed again.

For each cut I over S, maintain the order condition

$$x < y \qquad (x \in (N \cap I) \cup (T \cap I),\ y \in M \cap I). \tag{9}$$

In words, all moved vertices in I lie above all unmoved vertices there and above all points of T there. The set T stays fixed throughout: a vertex of T remains an order constraint even after it disappears from the support. The condition holds initially because M is empty.

Suppose the next vertex is $z \in N$, whose unique position is i, and let $I = I_i$ be its S-cut. Put $D = T \cup \mathrm{vsupp}(q)$. We prescribe the cut over D in I lying above every point of

$$(N \cap I) \cup (T \cap I)$$

and below every point of $M \cap I$; see Figure 2. Such a cut exists: the two finite sets are separated in this order by (9), and every point of $D \cap I$ belongs to one of them. If $M \cap I$ is empty, take the gap after the lower set and before the right endpoint of I, if that endpoint exists. The lower set is nonempty because it contains z. In particular the new point will lie to the right of z.

We now verify the hypothesis of Lemma 4.1, which concerns the partners in tuples having z in position i. A partner in position $j < i$ is still unmoved, because its position has not been processed yet. It is below z in its tuple, and if it lies in I it is also below the prescribed cut. A partner in position $j > i$ has already moved, because that entire position has been processed. It is above z in its tuple, and if it lies in I it is also above the prescribed cut. The unique-position property from Stage 1 ensures that these statements about moved and unmoved vertices are unambiguous. Partners outside I retain their comparisons automatically, since z and its replacement stay in the same S-cut. The comparisons with S are preserved for the same reason. Thus the cut has exactly the required comparisons with S and all partners, and Lemma 4.1 supplies a replacement z' in it.

Replace the slice, remove z from N, and add z' to M. The vertex z disappears from the support because it had only position i. Every other old support vertex remains, and z' is fresh. The old moved vertices still satisfy (9); the new moved vertex z' satisfies it by the chosen cut. Hence the invariant is preserved. Each step decreases $|N|$ by one, so Stage 2 terminates after finitely many replacements.

Conclusion. Let u be the final vector. Every support vertex of u was introduced in Stage 2 outside a set D containing T, so $\mathrm{vsupp}(u) \cap T = \varnothing$. Every such vertex in position i lies in I_i and above every point of $T \cap I_i$, by (9). It therefore lies in J_i. Balancedness has been preserved at every step.

Finally, list all vectors in both stages as $q_0 = v, q_1, \ldots, q_\ell = u$, where ℓ is the total number of replacements. Lemma 4.1 gives $q_{k-1} - q_k \in \mathrm{Cog}(\mathcal{O})$ for each $k = 1, \ldots, \ell$. Since this is a vector subspace,

$$v - u = \sum_{k=1}^{\ell} (q_{k-1} - q_k) \in \mathrm{Cog}(\mathcal{O}).$$

This proves the remaining conclusion. □

For three coordinates lying in the same S-interval, this procedure first puts all third-coordinate points to the right of T, then places the second-coordinate points between T and those third coordinates, and finally places the first-coordinate points in the remaining gap. The fixed order of the coordinates is preserved throughout. Notice also that the induction always uses S as its permanent parameter set. The extra points of $T \setminus S$ may be crossed; they constrain the desired final cuts, not the allowed cog replacements.

5 Completion of the cog-generation proof

We now carry out the argument outlined in Section 3. The essential issue is to make the other corners of a doubled tuple depend only on the coordinates that were not replaced.

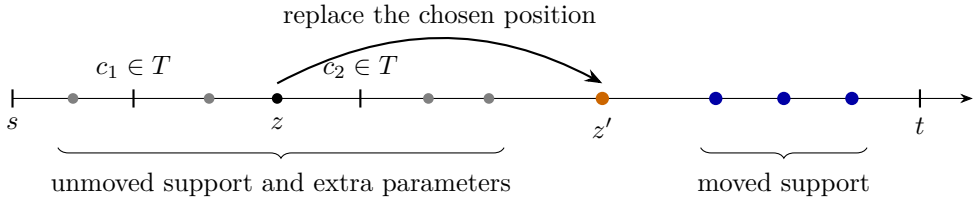


Figure 2: One step inside an S-interval $I = (s, t)$. The new point z' is placed after the unmoved support and the parameters in $T \cap I$, and before the moved support. Processing larger coordinate positions first ensures that every partner in a smaller position stays below z', while every partner in a larger position stays above it. The same construction works in unbounded S-intervals.

Proof of Theorem 2.6. The degree-one case was proved in Section 2. Assume the theorem in degree $d-1$ for every finite parameter set. Suppose that $v \in \mathrm{Bal}(\mathcal{O}) \setminus \mathrm{Cog}(\mathcal{O})$. By Lemma 3.1, choose an H_T-invariant function $f : \mathcal{O} \to \mathbb{F}$ with $f(\mathrm{Cog}(\mathcal{O})) = 0$ and $f(v) = 1$, where $T = S \cup \mathrm{vsupp}(v)$. Lemma 4.3 gives a balanced u in the prescribed fixed T-cuts, disjoint from T, with $u - v \in \mathrm{Cog}(\mathcal{O})$. Hence $f(u) = 1$.

Let $Q \subseteq \mathcal{O}$ consist of the tuples avoiding T with these fixed cuts, so that $\mathrm{supp}(u) \subseteq Q$. For each $a = (a_1, \ldots, a_d) \in Q$, form an independent doubling over S and freely amalgamate it with the actual structure on $T \cup \underline{a}$ over $S \cup \underline{a}$; see Figure 3. Write $b = b(a) = (b_1, \ldots, b_d)$ for the tuple of new points in this amalgam. This choice is made separately for each a. Choose their order so that $a_i < b_i < a_{i+1}$ for $i < d$ and $a_d < b_d$, with b_i in the same T-cut as a_i. These conditions are compatible: a twin can be inserted immediately to the right of its original in the finite order. The finite extension is therefore realizable in $\mathbb{A}$.

In the induced structure on $T \cup \underline{a} \cup \underline{b}$, every basic L-relation instance containing both a new twin and a point of $T \setminus S$ is false. This includes instances with any number of additional arguments from this finite set. The statement uses the definition of free amalgamation and does not depend on the arity of the relation.

Fix a nonempty $J \subseteq [d]$ and put $K = [d] \setminus J$. We claim that the type of the mixed tuple a^J over T depends only on $p_K(a)$. To check this, consider a basic L-relation instance in entries of a^J and parameters from T.

(i) If it contains no new twin, it uses only the retained entries and T, whose actual induced structure is fixed by $p_K(a)$.

(ii) If it contains a new twin and a parameter in $T \setminus S$, it is false by the free amalgamation just performed.

(iii) If it contains a new twin but no parameter in $T \setminus S$, its truth value is prescribed by the fixed diagram of $\mathcal{O}$ over S, through the doubling rule.

Repeated arguments satisfy the same three-way distinction. Equalities are fixed because all the tuple entries and twins are distinct and outside T. Comparisons with T are fixed by the chosen cuts, and comparisons among tuple entries are fixed by their positions. These observations account for the entire finite diagram.

More explicitly, if $a, \widetilde{a} \in Q$ have $p_K(a) = p_K(\widetilde{a})$, the map fixing T and the retained entries and sending the new entries of a^J to those of $\widetilde{a}^J$ is a partial isomorphism. Homogeneity extends it to an element of H_T. Consequently there is a well-defined function $g_J : p_K(Q) \to \mathbb{F}$ with

$$f(a^J) = g_J(p_K(a)).$$

Its value does not depend on how the individual doubling was chosen.

We now obtain the contradiction by summing the cog identities. For every $a \in \mathrm{supp}(u)$, the vector $c(a, b(a))$ is a cog, so

$$0 = f(c(a, b(a))) = \sum_{J \subseteq [d]} (-1)^{|J|} f(a^J).$$

The term with $J = \varnothing$ is $f(a)$. Moving the remaining terms to the other side and substituting the functions g_J therefore gives, for every $a \in \mathrm{supp}(u)$,

$$f(a) = - \sum_{\varnothing \neq J \subseteq [d]} (-1)^{|J|} g_J(p_{[d]\setminus J}(a)). \tag{10}$$

To evaluate $f(u)$, multiply this identity by $u(a)$ and sum over all $a \in \mathrm{supp}(u)$. Fix one nonempty J and again set $K = [d] \setminus J$. Grouping the supported tuples according to their retained subtuple t gives

$$\begin{aligned} &\sum_{a \in \mathrm{supp}(u)} u(a) g_J(p_K(a)) \\ &\quad = \sum_{t \in p_K(\mathrm{supp}(u))} g_J(t) \left(\sum_{\substack{a \in \mathrm{supp}(u) \\ p_K(a) = t}} u(a) \right) \\ &\quad = \sum_{t \in p_K(\mathrm{supp}(u))} g_J(t) (p_K u)(t) = 0. \end{aligned}$$

The inner coefficient sum is precisely the marginal $(p_K u)(t)$. It vanishes because K is proper and u is balanced. Applying this to each nonempty J in (10) yields

$$\begin{aligned} f(u) &= \sum_{a \in \mathrm{supp}(u)} u(a) f(a) \\ &= - \sum_{\varnothing \neq J \subseteq [d]} (-1)^{|J|} \sum_{a \in \mathrm{supp}(u)} u(a) g_J(p_{[d]\setminus J}(a)) = 0. \end{aligned}$$

This contradicts $f(u) = 1$. The cancellation takes place in the weighted sum; it does not require $f(a) = 0$ for individual tuples.

This completes the simultaneous induction. □

With the notation of the preceding proof, when $d = 2$ the identity being summed is

$$f(a_1, a_2) = g_{\{1\}}(a_2) + g_{\{2\}}(a_1) - g_{\{1,2\}}(\varnothing), \qquad (a_1, a_2) \in \mathrm{supp}(u).$$

The contribution of $g_{\{1\}}(a_2)$ vanishes by the zero column sums of the coefficient matrix of u, and that of $g_{\{2\}}(a_1)$ vanishes by its zero row sums. The constant term contributes zero because the total coefficient sum is zero. This is the matrix calculation from Example 2.2.

The separation into the three cases above is where arbitrary arity is handled. A relation instance in this finite configuration can involve many retained entries and many parameters, but as soon as it uses a twin and an extra parameter, free amalgamation determines its value. The use of separate doubled configurations is sufficient because the separating function is invariant over T.

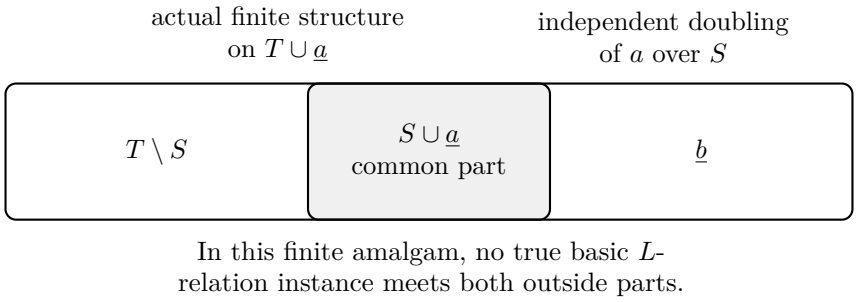


Figure 3: The free amalgam used for one tuple a. The two diagrams agree on $S \cup \underline{a}$. Within the displayed finite structure, relations between new twins and extra parameters are absent, including higher-arity relations also using vertices of the common part. A separate finite amalgam is chosen for each tuple; the proof compares only the resulting types over T.

6 Describing invariant subspaces by marginals

Theorem 2.6 supplies the cog-generation input to the reconstruction argument of Yang–Bojańczyk–Klin [25, Theorem 8.14 and Appendix F.1]. That argument describes an invariant subspace by the allowed joint values of its marginal coefficients. A strict inclusion of invariant subspaces must then change one of these finite-dimensional spaces of allowed values, giving a bound on chain length. We verify the remaining structural hypotheses in arbitrary arity and state the resulting description in the notation needed in Section 7. All numbered references to [25] refer to its expanded version.

6.1 The higher-arity compatibility check

The extraction arguments in [25, Theorem 8.10 and Claim F.5] use automorphisms that replace one coordinate by a twin while fixing every other relevant vertex. To obtain them here, fix a finite parameter set S, an increasing tuple $a = (a_1, \dots, a_r)$ of distinct points outside S, with $r > 0$, and any finite set $D \supseteq S \cup \underline{a}$. Put $B = D \setminus \underline{a}$ and apply Definition 2.3 over B. The resulting independent doubling (a, b) has each b_i immediately after a_i in the finite order on

$$C = D \cup \underline{b}.$$

For each $i \in [r]$, define

$$\sigma_i : C \setminus \{b_i\} \longrightarrow C \setminus \{a_i\}, \qquad \sigma_i(a_i) = b_i,$$

fixing every other point of its domain.

Each σ_i is a partial isomorphism. To check the L-relations, consider any relation instance in its domain. Whether it contains both members of another twin pair is unchanged by σ_i; if it does, the doubling rule makes both the instance and its image false. Otherwise, their truth values are read from the same tuple in D under the map fixing D and sending every b_j to a_j. This verifies preservation and reflection for every arity, including instances with several twins or repeated arguments. The order is preserved because a_i and b_i are consecutive in C. Homogeneity therefore extends σ_i to an automorphism $\pi_i \in H_S$. Since D was arbitrary, it may include the entire vertex support of a vector to which YBK's successive-difference argument is applied.

One further structural fact used at the start of [25, Appendix F.1] is the absence of algebraicity: every point outside a finite set D has an infinite orbit under its pointwise stabilizer. Indeed, for any $x \in A \setminus D$, free amalgamation produces arbitrarily many copies of $D \cup \{x\}$ over D, with their new points distinct and in the same D-cut as x. The

extension property and homogeneity realize them as images of x over D. Consequently, an H_S-equivariant bijection between increasing tuple orbits outside S must preserve the set of entries: each image entry is fixed by the stabilizer of the source tuple, and the inverse gives the reverse containment. Since the tuples are increasing, the bijection is the identity after coordinate reindexing.

6.2 The marginal description inherited from YBK

We retain the notation for joint marginals because the duality argument in Section 7 uses the individual coefficient evaluations. Fix a finite $S \subseteq A$ and $k \geq 0$. Let $\mathscr{O}$ be a finite collection of distinct increasing H_S-orbits of degree at most k. For each $\mathcal{O} \in \mathscr{O}$, let $E_{\mathcal{O}}$ be a finite-dimensional $\mathbb{F}$-vector space with trivial H_S-action, and put

$$V = \bigoplus_{\mathcal{O} \in \mathscr{O}} E_{\mathcal{O}} \otimes_{\mathbb{F}} \mathbb{F}[\mathcal{O}].$$

Thus $v \in V$ has coefficients $v_{\mathcal{O}}(a) \in E_{\mathcal{O}}$, only finitely many of which are nonzero. Choosing a basis of $E_{\mathcal{O}}$ identifies its summand with $\dim E_{\mathcal{O}}$ labelled copies of $\mathbb{F}[\mathcal{O}]$, as in [25, Remark 8.13].

Following [25, Definition F.1], group all projections with the same target orbit $\mathcal{P}$. Keep a separate coefficient slot for each source and projection label:

$$Z_{\mathcal{P}} = \bigoplus_{\substack{\mathcal{O} \in \mathscr{O},\ J \subseteq [\deg \mathcal{O}] \\ p_J(\mathcal{O}) = \mathcal{P}}} E_{\mathcal{O}}.$$

For $v \in V$ and $b \in \mathcal{P}$, the joint marginal coefficient is

$$\rho_{\mathcal{P}} : V \longrightarrow Z_{\mathcal{P}} \otimes \mathbb{F}[\mathcal{P}], \qquad \big(\rho_{\mathcal{P}}(v)(b)\big)_{(\mathcal{O},J)} = \sum_{\substack{a \in \mathcal{O} \\ p_J(a) = b}} v_{\mathcal{O}}(a). \tag{11}$$

The target orbits $\mathcal{P}$ range over all coordinate projections, including the identity and empty projections. Keeping the slots together records linear relations between different marginal coefficients evaluated at the same tuple.

Theorem 6.1 (Marginal description)**.** *For an H_S-invariant subspace $W \leq V$, set*

$$E_{\mathcal{P}}(W) = \{\rho_{\mathcal{P}}(w)(b) : w \in W\} \leq Z_{\mathcal{P}},$$

where $b \in \mathcal{P}$ is fixed. This subspace is independent of the choice of b, and

$$W = \{v \in V : \rho_{\mathcal{P}}(v)(b) \in E_{\mathcal{P}}(W) \text{ for every } \mathcal{P} \text{ and every } b \in \mathcal{P}\}. \tag{12}$$

Proof by transfer of the YBK argument. When $k = 0$, the statement is ordinary finite-dimensional linear algebra. For general k, linearity makes $E_{\mathcal{P}}(W)$ a subspace; transitivity on $\mathcal{P}$ and invariance of W make it independent of b. These are the coefficient spaces in [25, Theorem 8.14], and $\rho_{\mathcal{P}}$ is their grouped projection from Definition F.1, with the multiplicities recorded by the spaces $E_{\mathcal{O}}$. The preceding no-algebraicity observation justifies grouping by equality of increasing target orbits.

The proof in [25, Appendix F.1, Lemma F.3] reconstructs membership in W by successively removing assumptions that marginals vanish. Its two steps carry over as follows. The calculation in Claim F.4 uses only composition of coordinate projections and finite sums, so is unchanged. It produces a balanced vector with coefficients in a finite-dimensional subspace. Theorem 2.6, applied in each component of a basis of that coefficient space, supplies the cog decomposition used there from their Theorem 8.12. Claim F.5 then lifts these cogs to vectors

of W by successive differences; the automorphisms required for this step were constructed above. All these statements allow several labelled orbit components, as in [25, Remark 8.13 and Appendix F.1]. Thus the induction of Lemma F.3 proves (12) without any further change depending on relation arity. □

The exact-length argument of [25, Corollary 8.16 and Appendix F.2] now carries over as well. We give that calculation in Section 7, followed separately by the duality argument and the bound on additional parameters.

7 Composition lengths and closed function spaces

We apply the marginal description to obtain the length formula by YBK's argument, and then use Evans's formulation of standard duality to deduce the assertions about closed function spaces.

7.1 The exact length

Proof of Theorem 1.1. By Theorem 6.1, a strict inclusion of invariant subspaces strictly increases at least one of the spaces $E_{\mathcal{P}}(W)$. The dimension count of [25, Corollary F.6] therefore gives

$$\operatorname{length}_{H_S} V \le \sum_{\mathcal{P}} \dim Z_{\mathcal{P}} = \sum_{\mathcal{O}\in\mathscr{O}} (\dim E_{\mathcal{O}}) 2^{\deg \mathcal{O}}. \tag{13}$$

For a single increasing orbit $\mathcal{O}$ of degree $d > 0$, this is the upper bound 2^d. The matching lower-bound proof in [25, Corollary 8.16, Appendix F.2] constructs 2^d strict inclusions using partial alternating sums on an independent doubling. The configuration and automorphisms are available by Section 6; the remaining coordinate-projection identities are unchanged in every arity and characteristic. Thus $\operatorname{length}_{H_S} \mathbb{F}[\mathcal{O}] = 2^d$. In degree zero, the orbit module is one-dimensional, giving the same formula.

To pass from increasing orbits to A^k, partition A^k into its finitely many H_S-orbits. A tuple in one orbit is determined by the increasing list of its distinct entries outside S, together with a fixed pattern specifying repeated entries and entries from S. This identifies its orbit equivariantly with an increasing orbit of degree at most k. If several orbits give the same increasing orbit, they supply a finite coefficient multiplicity. Thus $\mathbb{F}[A^k]$ has the form of V above, and has finite length.

The unordered stabilizer $\operatorname{Aut}(\mathbb{A}_0)_{(S)}$ contains H_S. Every subspace invariant under that larger group is invariant under H_S, so its chains satisfy the same finite bound. This proves the unordered assertion as well. □

7.2 Duality and the parameter bound

The closed-function assertions follow by combining the marginal description with standard vector-space duality. We use [11, Lemma 2.2]: for the pairing

$$\langle v, f\rangle = \sum_{x\in A^k} v(x) f(x), \qquad v \in \mathbb{F}[A^k], \quad f \in \mathbb{F}^{A^k},$$

annihilators give an inclusion-reversing correspondence between invariant subspaces of $\mathbb{F}[A^k]$ and closed invariant subspaces of $\mathbb{F}^{A^k}$. Here the annihilator consists of the elements on the other side that pair to zero with the given subspace; the double annihilator of a function space is its closure.

The deduction below follows the finite-dimensional separation and pullback argument in the proof of [11, Corollary 2.3], using the joint marginals of [25, Theorem 8.14]. Evaluating a marginal at a tuple of length at most k yields the uniform parameter bound. The same deduction already applies to YBK's binary setting.

Proof of Corollary 1.2. Let $\mathbb{F}$ be finite and let $C \leq \mathbb{F}^{A^k}$ be closed and H_S-invariant. Let $\Psi : \mathbb{F}[A^k] \to V$ be the equivariant orbit decomposition from the proof of Theorem 1.1, and put $W = \Psi(C^\perp)$. For every projected orbit $\mathcal{P}$, every $b \in \mathcal{P}$, and every linear functional $\lambda : Z_\mathcal{P} \to \mathbb{F}$ vanishing on $E_\mathcal{P}(W)$, define

$$f_{\mathcal{P},b,\lambda}(x) = \lambda(\rho_\mathcal{P}(\Psi([x]))(b)), \qquad x \in A^k. \tag{14}$$

These functions annihilate $C^\perp$, hence belong to C. Equivariance fixes each of them over $S \cup \underline{b}$; since $\deg \mathcal{P} \leq k$, this uses at most k additional parameters. Theorem 6.1 and finite-dimensional separation in each $Z_\mathcal{P}$ show that their common annihilator is exactly $C^\perp$. Double-annihilator duality therefore gives C as their closed linear span.

Finally, the field and all the spaces $Z_\mathcal{P}$ are finite, and there are only finitely many projected orbits $\mathcal{P}$. There are consequently only finitely many choices for the subspaces $E_\mathcal{P}(W)$. These determine W by Theorem 6.1, and duality then determines C. □

There is an immediate quotient consequence. Under the same hypotheses, let $C_1 < C_2 \leq \mathbb{F}^{A^k}$ be closed invariant subspaces. The generators of C_2 from Corollary 1.2 cannot all lie in the closed subspace C_1. Thus some $f \in C_2 \setminus C_1$ is fixed over $S \cup T$ with $|T| \leq k$, and $f + C_1$ is a nonzero fixed element of C_2/C_1. This gives the uniform bound in the present setting for the separation conclusion of [11, Corollary 2.3].

8 Concluding remarks

The proof uses free amalgamation to build independent copies and to control the relations involving a new point and an extra parameter within each finite construction. It uses the generic order to place those copies in specified intervals. These are the two structural requirements behind the argument.

The result settles the arbitrary-arity extension of the free-amalgamation finite-length theorem in [25]. It leaves open the corresponding finite-length problem for general structures homogeneous in finite relational languages, as posed in [11, Question 1.2]. Extending the support-moving argument to classes whose amalgamation imposes additional relations would require a replacement for the free-amalgam type calculation in Section 5.

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