hexagon:2611.00355v1

The Maximum of per$(I - A)$ in Odd Order

Yair Lavi

8 August 2026

Abstract

Let Ω_n denote the set of $n \times n$ doubly stochastic matrices. Kim and Roush conjectured in 1981 that, for $n = 2k + 1 > 1$,

$$\max_{A \in \Omega_{2k+1}} \operatorname{per}(I - A) = 3 \cdot 2^{k-2}.$$

They proposed the following block construction as an extremizer:

$$A_\star = \frac{1}{2}(J_3 - I_3) \oplus P_2^{\oplus(k-1)}, \qquad P_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Here J_3 is the 3×3 all-ones matrix. They did not claim uniqueness. We fully prove their conjecture and classify equality: the maximizers are exactly the simultaneous-permutation conjugates of $A_\star$.

Keywords. Permanent; doubly stochastic matrix; functional digraph; hafnian; extremal matrix problem.
2020 Mathematics Subject Classification. 15A15, 05C20, 05A05.

1 Introduction

For an $n \times n$ matrix $D = (d_{ij})$, its permanent is

$$\operatorname{per} D = \sum_{\sigma \in S_n} \prod_{i=1}^{n} d_{i,\sigma(i)}.$$

We write Ω_n for the Birkhoff polytope of nonnegative doubly stochastic $n \times n$ matrices. The problem considered here is to maximize per$(I - A)$ over $A \in \Omega_n$. Although $I - A$ generally has negative off-diagonal entries, stochasticity gives this permanent the positive functional-digraph expansion proved in Proposition 3.1.

Kim and Roush [1] proved that for nonnegative row-stochastic matrices

$$\max_{A} \operatorname{per}(I - A) = 2^{\lfloor n/2 \rfloor}. \tag{1.1}$$

Their proof established the functional-digraph identity used below. For even $n = 2m$, the row-stochastic maximum is attained by a fixed-point-free involution, which is doubly stochastic, and hence the doubly stochastic maximum is 2^m. For odd $n = 2k + 1$, (1.1) gives the upper bound 2^k. Kim and Roush exhibited the doubly stochastic matrix

$$A_\star = \frac{1}{2}(J_3 - I_3) \oplus P_2^{\oplus(k-1)}, \qquad P_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{1.2}$$

and conjectured that its value $3 \cdot 2^{k-2}$ is maximal. They proved the case $n = 3$, where the value is $3/2$ [1, Proposition 5]. Thus the open range in their paper began at $n = 5$.

Minc later added the conjecture to his catalogue as Conjecture 35 [2, p. 133]. In their subsequent survey of Minc's list, Cheon and Wanless [3] reported no progress. Troanca [4] reproduced the exact construction and stated in 2008 that no partial results were known to the author. Chen and Cao [5] subsequently solved a neighboring problem with fixed total matrix mass. In their notation, for row- and doubly substochastic matrices of total mass $s = \sum_{i,j} a_{ij}$, they obtained the exact common maximum

$$2^{e/2}\left[1 + \left(\frac{s-e}{2}\right)^2\right], \tag{1.3}$$

where e is the largest even integer not exceeding s, whenever n is even or n is odd and $s \le n-1$. For odd n and $n-1 < s \le n$, they determined the row-substochastic maximum but left the doubly substochastic case open [5, Theorems 3.1, 3.13, and 4.1]. The endpoint $s = n$ is exactly the odd doubly stochastic problem. They also gave an independent proof of the $n = 3$ value [5, Lemma 4.2].

Our main result settles the conjecture in full and classifies all equality cases.

Theorem 1.1 (Kim–Roush conjecture)**.** *For every integer $k \ge 1$,*

$$\boxed{\max_{A \in \Omega_{2k+1}} \operatorname{per}(I - A) = 3 \cdot 2^{k-2}.} \tag{1.4}$$

The matrix $A_\star$ in (1.2) attains equality. Moreover, equality holds for $A \in \Omega_{2k+1}$ if and only if

$$A = P A_\star P^{\mathsf{T}} \tag{1.5}$$

for a permutation matrix P. Thus the number of distinct maximizers is $\binom{2k+1}{3}(2k-3)!!$, with the convention $(-1)!! = 1$.

Here is the proof in outline. Independently for each $i \in [n]$, choose $F(i) = j$ with probability a_{ij}. These choices define a random map $F : [n] \to [n]$. Let m_c be the total product mass of maps whose functional digraphs have only even cycles and exactly c cycles, and put $M = \sum_c m_c$. The Kim–Roush expansion and a layer estimate give

$$\operatorname{per}(I - A) = \sum_{c=1}^{k} 2^c m_c \le 2^{k-1}(M + m_k). \tag{1.6}$$

Let Q_3 be the mass of permutation maps with cycle type $(3, 2, \dots, 2)$. The key estimate is the sharp triangle-charge inequality

$$m_k - Q_3 \le \frac{1}{2}. \tag{1.7}$$

Every map counted by Q_3 has an odd cycle, so $Q_3 \le 1 - M$. Consequently (1.7) implies $M + m_k \le 3/2$, which inserted into (1.6) proves the upper bound in (1.4).

The proof of (1.7) has two exact ingredients. First, averaging a matrix with its transpose does not decrease the difference $m_k - Q_3$; the local loss is a sum of three squares. Second, for symmetric

weights, releasing the terminal row of an ordered spanning path on three vertices yields a mass B satisfying $2m_k \le B$ and

$$B = m_k + 3Q_3 + O,$$

where O is a disjoint residual-map mass. Since $O \le 1 - m_k - Q_3$, inequality (1.7) follows. The factor 3 is the number of ways to open a directed triangle into a directed path.

Sections 2–5 develop the map expansion and prove the triangle-charge inequality. Section 6 establishes the maximum, and Section 7 classifies equality.

2 Maps, masses, and matchings

Fix $n = 2k+1$ with $k \ge 1$, and let $A = (a_{ij}) \in \Omega_n$. Write $[n] = \{1, \dots, n\}$. For a map $f : [n] \to [n]$, define its product mass by

$$\mu_A(f) = \prod_{i=1}^{n} a_{i,f(i)}. \tag{2.1}$$

The row sums of A are one, and therefore

$$\sum_{f:[n]\to[n]} \mu_A(f) = \prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = 1. \tag{2.2}$$

The directed graph of f has the edge $i \to f(i)$ from every vertex. Each weak component contains one directed cycle. Let $c(f)$ denote the number of these directed cycles.

For $1 \le c \le k$, let $\mathcal{E}_c$ be the maps all of whose directed cycles have even length and for which $c(f) = c$. Set

$$m_c = m_c(A) = \sum_{f\in\mathcal{E}_c} \mu_A(f), \qquad M = M(A) = \sum_{c=1}^{k} m_c. \tag{2.3}$$

The families $\mathcal{E}_c$ are disjoint subsets of all maps, so $0 \le M \le 1$.

For a finite set U of even cardinality, let $\mathrm{PM}(U)$ denote the set of perfect matchings of U. If $W = (w_{ij})_{i,j\in U}$ is symmetric, write

$$\operatorname{haf} W = \sum_{\nu\in\mathrm{PM}(U)} \prod_{\{r,s\}\in\nu} w_{rs}. \tag{2.4}$$

We use the convention that the hafnian of the empty matrix equals one. Put

$$C(A) = A \circ A^{\mathsf{T}}, \qquad C(A)_{ij} = a_{ij}a_{ji}, \tag{2.5}$$

where $\circ$ denotes the entrywise (Hadamard) product. For $U \subseteq [n]$, let $W[U]$ denote the principal submatrix of W indexed by U. Whenever $[n] \setminus S$ has even cardinality, define

$$H_S(A) = \operatorname{haf}\big(C(A)[[n] \setminus S]\big). \tag{2.6}$$

Thus $H_S(A)$ is the total weight of mutual-pair matchings outside S.

3 The positive expansion and the top layer

We first give a self-contained proof of the expansion from [1].

Proposition 3.1 (even-cycle expansion)**.** *If A is nonnegative and row-stochastic, then*

$$\operatorname{per}(I-A)=\sum_{\substack{f:[n]\to[n]\\ \text{every cycle of } f \text{ is even}}} 2^{c(f)}\mu_A(f). \tag{3.1}$$

In particular, for $n=2k+1$,

$$\operatorname{per}(I-A)=\sum_{c=1}^{k}2^c m_c. \tag{3.2}$$

Proof. For $S\subseteq[n]$, let $\mathfrak{S}(S)$ denote the set of permutations of S. Expand the permanent according to the rows in which the term from $-A$ is selected. This gives

$$\operatorname{per}(I-A)=\sum_{S\subseteq[n]}(-1)^{|S|}\sum_{\pi\in\mathfrak{S}(S)}\prod_{i\in S}a_{i,\pi(i)}, \tag{3.3}$$

where the empty product is one. For a fixed pair (S,π), row-stochasticity gives

$$\prod_{i\in S}a_{i,\pi(i)}=\sum_{\substack{f:[n]\to[n]\\ f|_S=\pi}}\mu_A(f). \tag{3.4}$$

After interchanging the finite sums, fix a map f. The subsets S for which $f|_S$ is a permutation are exactly the unions of directed cycles of f. Hence the coefficient of $\mu_A(f)$ is

$$\prod_{\gamma\text{ a cycle of }f}\left(1+(-1)^{|\gamma|}\right). \tag{3.5}$$

This coefficient vanishes if f has an odd cycle and equals $2^{c(f)}$ otherwise. Since every finite map has a cycle, and c even cycles use at least $2c$ vertices, in odd order $1\le c\le k$. This proves both formulas. □

The next reduction isolates the only layer requiring sharp control.

Lemma 3.2 (layer reduction)**.** *For $A\in\Omega_{2k+1}$,*

$$\operatorname{per}(I-A)\le 2^{k-1}(M+m_k). \tag{3.6}$$

Proof. In (3.2), bound $2^c\le 2^{k-1}$ for $c\le k-1$, while retaining the coefficient 2^k of the top layer. Thus

$$\operatorname{per}(I-A)\le 2^{k-1}(M-m_k)+2^k m_k=2^{k-1}(M+m_k).$$

For $k=1$ the first sum is empty and the displayed relation is an identity. □

Call a map in $\mathcal{E}_k$ a **top map**. Such a map has a particularly rigid form.

Lemma 3.3 (top-layer structure)**.** *One has*

$$m_k = \sum_{v=1}^{n}(1-a_{vv})H_{\{v\}}(A) = \sum_{\substack{v,u,w\in[n]\\ v,u,w \text{ distinct}}} a_{vu}a_{uw}a_{wu}H_{\{v,u,w\}}(A). \tag{3.7}$$

Proof. A map in $\mathcal{E}_k$ has k even cycles on $2k+1$ vertices. It must therefore have k transpositions covering $2k$ vertices and one vertex v not lying on a directed cycle (hence one nonperiodic vertex). The vertex v maps to some $u \neq v$, while the remaining vertices form a perfect matching of mutual pairs. For fixed v and fixed matching, the sum over the target of v is $\sum_{u\neq v} a_{vu} = 1 - a_{vv}$, giving the first formula.

For the second formula, let w be the mate of the chosen target u. The distinguished part of the map has weight $a_{vu}a_{uw}a_{wu}$, and the remaining mutual pairs contribute $H_{\{v,u,w\}}(A)$. The tail vertex, its target, and the target's mate are uniquely determined by the map. □

4 Triangle charge and transpose symmetrization

Let $\mathcal{Q}_3$ be the set of permutation maps whose cycle type is $(3, 2, \ldots, 2)$: one directed 3-cycle and $k-1$ transpositions. Define

$$Q_3 = Q_3(A) = \sum_{f\in\mathcal{Q}_3} \mu_A(f). \tag{4.1}$$

Fix a three-element set $T = \{i, j, \ell\}$. The vertices outside T must be paired into transpositions, whose total weight is $H_T(A)$, while the directed 3-cycle on T has exactly two orientations. Therefore

$$Q_3(A) = \sum_{|T|=3} (a_{ij}a_{j\ell}a_{\ell i} + a_{i\ell}a_{\ell j}a_{ji})\, H_T(A). \tag{4.2}$$

Each map in $\mathcal{Q}_3$ contains an odd cycle, whereas all maps counted by M have only even cycles. These map families are disjoint, so (2.2) gives

$$Q_3 \leq 1 - M. \tag{4.3}$$

We call $m_k - Q_3$ the **triangle charge**. To express it locally, group (3.7) and (4.2) by their distinguished three-element set and define

$$K_T(A) = \sum_{\substack{v,u,w\in T\\ v,u,w \text{ distinct}}} a_{vu}a_{uw}a_{wu} - \frac{1}{3}\sum_{\substack{v,u,w\in T\\ v,u,w \text{ distinct}}} a_{vu}a_{uw}a_{wv}. \tag{4.4}$$

The normalization $1/3$ removes a simple overcount; this correction is needed for identity (4.5). In the first sum, the ordered triple (v, u, w) records a top map's nonperiodic vertex v, its target u, and the mate w of u. In the second sum, $a_{vu}a_{uw}a_{wv}$ is the weight of the directed triangle $v \to u \to w \to v$. Each directed triangle on T occurs three times in that sum, once for each cyclic choice of the first vertex v. Thus the second sum, after multiplication by $1/3$, is exactly the sum of the two orientation weights in (4.2). It follows that

$$m_k(A) - Q_3(A) = \sum_{|T|=3} K_T(A)H_T(A). \tag{4.5}$$

Although writing $T = \{i, j, \ell\}$ in (4.2) requires an arbitrary ordering, the sum of the two orientation weights in (4.2), the quantity $K_T(A)$, and identity (4.5) depend only on the set T.

For nonnegative x, y, z, put

$$\begin{aligned}\Phi(x, y, z) &= xy(x + y) + yz(y + z) + zx(z + x) - 2xyz \\ &= (x + y)(y + z)(z + x) - 4xyz \\ &= x(y - z)^2 + y(z - x)^2 + z(x - y)^2 + 4xyz.\end{aligned} \tag{4.6}$$

In particular, $\Phi(x, y, z) \geq 0$.

Lemma 4.1 (local transpose identity)**.** *Let $X = (A + A^{\mathsf{T}})/2$. Orient $T = \{i, j, \ell\}$ cyclically as $i \to j \to \ell \to i$, and write*

$$\begin{aligned} a_{ij} &= x + p, \quad a_{ji} = x - p, \quad a_{j\ell} = y + q, \\ a_{\ell j} &= y - q, \quad a_{\ell i} = z + r, \quad a_{i\ell} = z - r. \end{aligned} \tag{4.7}$$

Then

$$\frac{K_T(A) + K_T(A^{\mathsf{T}})}{2} = \Phi(x, y, z) - D_T, \tag{4.8}$$

where

$$D_T = y(p + r)^2 + z(p + q)^2 + x(q + r)^2 \geq 0. \tag{4.9}$$

Proof. Substitute the six expressions in (4.7) into (4.4), add the expression obtained after changing (p, q, r) to $(-p, -q, -r)$, and collect the even terms. The terms independent of (p, q, r) give (4.6), and the remaining even terms are the negative of (4.9). □

For the variables attached to T in (4.7), abbreviate $\Phi_T = \Phi(x, y, z)$.

Theorem 4.2 (transpose symmetrization)**.** *For every $A \in \Omega_{2k+1}$ and $X = (A + A^{\mathsf{T}})/2$,*

$$m_k(A) - Q_3(A) \leq m_k(X) - Q_3(X). \tag{4.10}$$

Proof. Here the column-sum condition enters the upper-bound argument: because A is doubly stochastic, A^{T} is row-stochastic and X is symmetric and doubly stochastic. The quantities m_k and Q_3 are transpose invariant. For m_k, apply the first formula in (3.7) to both A and A^{T} and use $C(A) = C(A^{\mathsf{T}})$; for Q_3, transposition exchanges the two orientations in (4.2). Moreover $H_T(A) = H_T(A^{\mathsf{T}})$. We may therefore replace $K_T(A)$ in (4.5) by its transpose average.

For an edge $\{r, s\}$ outside T, write

$$x_{rs} = \frac{a_{rs} + a_{sr}}{2}, \qquad \kappa_{rs} = \frac{a_{rs} - a_{sr}}{2}.$$

Then

$$a_{rs}a_{sr} = x_{rs}^2 - \kappa_{rs}^2, \qquad 0 \leq x_{rs}^2 - \kappa_{rs}^2 \leq x_{rs}^2. \tag{4.11}$$

Expanding the hafnian in (4.5) over perfect matchings and using Lemma 4.1 expresses the charge as a sum of terms

$$(\Phi_T - D_T) \prod_{\{r,s\} \in \nu} (x_{rs}^2 - \kappa_{rs}^2). \tag{4.12}$$

The following sign distinction is essential. If $\Phi_T - D_T \le 0$, the term in (4.12) is nonpositive and hence at most $\Phi_T \prod_{\{r,s\}\in\nu} x_{rs}^2$. If $\Phi_T - D_T > 0$, first discard D_T and then use (4.11), obtaining the same upper bound. Summing over triples and outside perfect matchings gives

$$m_k(A) - Q_3(A) \le \sum_{|T|=3} \Phi_T H_T(X) = m_k(X) - Q_3(X),$$

where the last equality is (4.5) specialized to the symmetric matrix X. □

Remark 4.3 (why the transpose average is necessary). The unaveraged local inequality suggested by (4.8) is false. For example, on a cyclically oriented triple take

$$(a_{12}, a_{21}, a_{13}, a_{31}, a_{23}, a_{32}) = \left(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0\right). \tag{4.13}$$

The ungrouped comparison fails by exactly $1/8$. These six entries have row and column subsums at most one and extend to a doubly stochastic matrix. The odd-in-skew remainder cancels only after pairing A with A^{T}. Thus the transpose invariance used in Theorem 4.2 is a structural part of the proof, not a cosmetic symmetrization.

5 The symmetric charge inequality

We now prove the sharp estimate for a symmetric doubly stochastic matrix $X = (x_{ij})$ of order $2k+1$. All weights below are entries of X.

For distinct vertices v, u, w and a perfect matching ν of the other $2k-2$ vertices, define the released-path weight

$$h(v, u, w; \nu) = x_{vu} x_{uw} \prod_{\{a,b\}\in\nu} x_{ab}^2. \tag{5.1}$$

It fixes the choices

$$v \longmapsto u, \qquad u \longmapsto w, \qquad a \longleftrightarrow b \quad (\{a, b\} \in \nu), \tag{5.2}$$

and releases the terminal row w. Indeed, summing $w \mapsto t$ over all t multiplies (5.1) by $\sum_t x_{wt} = 1$. Let B be the total released-path mass:

$$B = \sum_{\substack{v,u,w \text{ distinct} \\ \nu\in \mathrm{PM}([n]\setminus\{v,u,w\})}} h(v, u, w; \nu). \tag{5.3}$$

Lemma 5.1 (released mass dominates the top layer)**.** *One has*

$$2m_k(X) \le B. \tag{5.4}$$

Proof. Pair the paths (v, u, w) and (w, u, v) while fixing the center u and the outside matching ν. Put

$$a = x_{vu}, \qquad b = x_{uw}, \qquad H = \prod_{\{r,s\}\in\nu} x_{rs}^2.$$

Their combined released mass is $2abH$. Closing their terminal rows back to u produces two top maps of total mass

$$ab^2H + a^2bH = ab(a+b)H.$$

Since the center row is stochastic,

$$a + b \le \sum_{z \neq u} x_{uz} = 1 - x_{uu} \le 1.$$

Thus twice the mass of the two top completions is at most their released mass. Every top map has a unique tail vertex v, tail target u, mate w of u, and outside matching ν, so summing proves (5.4). Equivalently, the slack is the manifestly nonnegative sum

$$B - 2m_k = \sum_{u,\{v,w\},\nu} 2x_{vu}x_{uw}(1 - x_{vu} - x_{uw}) \prod_{\{a,b\}\in\nu} x_{ab}^2. \tag{5.5}$$

Here the sum is over centers u, unordered endpoint pairs $\{v, w\}$ disjoint from u, and matchings $\nu \in \mathrm{PM}([n] \setminus \{u, v, w\})$. The right-hand side is nonnegative. □

We next classify the maps obtained when the released row is completed.

Lemma 5.2 (completion identity)**.** *There is a nonnegative map mass O, disjoint from the masses m_k and Q_3, such that*

$$B = m_k + 3Q_3 + O \tag{5.6}$$

and

$$O \le 1 - m_k - Q_3. \tag{5.7}$$

Proof. In a released configuration (5.2), classify the completion by the choice $w \mapsto t$.

If $t = u$, the terminal row closes the mutual pair $u \leftrightarrow w$ and produces a top map. Its representation is unique, so every top map occurs once.

If $t = v$, the path closes into the directed triangle $v \to u \to w \to v$. Every map counted by Q_3 occurs three times, once for each cyclic choice of where its directed triangle is opened.

Finally, suppose $t \notin \{u, v\}$, allowing either $t = w$ or t in an outside pair. Call the resulting map residual. Its mutual pairs are exactly the pairs of ν: none of $\{v, u\}$, $\{u, w\}$, or $\{w, t\}$ is mutual. After removing those pairs, the three remaining vertices contain the unique directed path $v \to u \to w$; the vertex v is its unique vertex of in-degree zero, and w either loops or maps out to a removed pair. Hence v, u, w, and ν are uniquely reconstructed from the residual map. Every residual map therefore occurs once.

These three classes are disjoint: top maps have exactly one nonperiodic vertex, Q_3 maps are permutations with a directed odd cycle, and residual maps have at least two nonperiodic vertices. If O is the total product mass of the residual maps, the multiplicities are exactly $1, 3, 1$, proving (5.6). All three classes lie inside the map space of total mass one in (2.2), which proves (5.7). □

Theorem 5.3 (symmetric triangle charge)**.** *For every symmetric $X \in \Omega_{2k+1}$,*

$$m_k(X) - Q_3(X) \le \frac{1}{2}. \tag{5.8}$$

Proof. Lemmas 5.1 and 5.2 give

$$2m_k \le B = m_k + 3Q_3 + O \le m_k + 3Q_3 + (1 - m_k - Q_3) = 1 + 2Q_3.$$

Rearranging proves (5.8). □

Combining Theorems 4.2 and 5.3 yields the promised universal estimate.

Corollary 5.4 (triangle-charge inequality)**.** *For every $A \in \Omega_{2k+1}$,*

$$\boxed{m_k(A) - Q_3(A) \le \frac{1}{2}.} \tag{5.9}$$

6 Proof of the main theorem

We now assemble the short final implication. By (4.3) and (5.9),

$$M + m_k \le 1 - Q_3 + m_k \le \frac{3}{2}. \tag{6.1}$$

Lemma 3.2 therefore gives

$$\operatorname{per}(I - A) \le 2^{k-1}(M + m_k) \le 2^{k-1} \cdot \frac{3}{2} = 3 \cdot 2^{k-2}. \tag{6.2}$$

It remains to check sharpness. For the matrix $A_\star$ in (1.2), block multiplicativity of the permanent gives

$$\begin{aligned} \operatorname{per}(I - A_\star) &= \operatorname{per}\Big(I_3 - \frac{1}{2}(J_3 - I_3)\Big) \operatorname{per}(I_2 - P_2)^{k-1} \\ &= \frac{3}{2}\, 2^{k-1} = 3 \cdot 2^{k-2}. \end{aligned} \tag{6.3}$$

Thus equality is attained, proving the maximum-value part of Theorem 1.1. The argument includes $k = 1$: empty outside matchings have weight one, Q_3 is then the mass of the two oriented 3-cycles, and (6.3) gives 3/2. □

At $A_\star$, the map masses are

$$(M, m_k, Q_3) = \left(\frac{3}{4}, \frac{3}{4}, \frac{1}{4}\right). \tag{6.4}$$

Consequently both the triangle-charge constant 1/2 and the compensation constant 3/2 in (6.1) are sharp.

7 Classification of equality

We now prove the equality assertion in Theorem 1.1. Suppose that $A \in \Omega_{2k+1}$ attains the maximum. The proof above gives the monotone chain

$$\begin{aligned} \operatorname{per}(I - A) &\le 2^{k-1}(M + m_k) \\ &\le 2^{k-1}\left(M + Q_3 + \frac{1}{2}\right) \\ &\le 2^{k-1} \cdot \frac{3}{2}. \end{aligned} \tag{7.1}$$

Equality at the two ends forces equality at every step, and in particular

$$Q_3 = 1 - M. \tag{7.2}$$

The maps counted by M and Q_3 are disjoint subsets of the map space of total mass one in (2.2). Since every map mass is nonnegative, (7.2) has the following support consequence:

Every map f satisfying $a_{i,f(i)} > 0$ for every i either has only even cycles, or is a permutation of cycle type $(3, 2, \ldots, 2)$.

We call this property the support dichotomy. Let S be the support digraph of A, with a directed edge $i \to j$ exactly when $a_{ij} > 0$. Any partial assignment of supported outgoing edges extends to a full positive-mass map: in each unused row, simply choose any positive entry. We use this observation repeatedly.

First, S has no loop. Indeed, a map extending a supported loop has an odd 1-cycle, so it does not have only even cycles, and its fixed point prevents it from having type $(3, 2, \ldots, 2)$. The same extension argument shows that S has no directed odd cycle of length at least five.

By the Birkhoff–von Neumann theorem, S contains the edges of a permutation. This permutation has no fixed point. Its cycle lengths sum to the odd number $2k + 1$, so one of them is odd; by the preceding paragraph it must have length three. Thus S contains a directed triangle γ on a vertex triple T.

No supported edge enters T from outside. Otherwise, extend γ together with an edge $z \to t$, where $z \notin T$ and $t \in T$. The resulting map has the odd cycle γ, but it is not a permutation because t has two distinct preimages. This contradicts the support dichotomy. Each column in T therefore receives its full unit mass from rows in T. Those three column sums exhaust the total mass of the three rows in T, so no supported edge leaves T either. After simultaneous relabelling,

$$A = A_1 \oplus A_2, \qquad A_1 \in \Omega_3, \quad A_2 \in \Omega_{2k-2}, \tag{7.3}$$

and both blocks have zero diagonal.

Choose arbitrarily one supported outgoing edge from each row of A_2 and combine these choices with γ. The resulting map contains an odd triangle, so the support dichotomy forces it to be a $(3, 2, \ldots, 2)$-permutation. Consequently every such row choice in the complement is a fixed-point-free involution.

If a complementary row z had two supported targets $t_1 \neq t_2$, choose such an involution g with $g(z) = t_1$. Then $g(t_1) = z$, and $g(t_2) \neq z$. Changing only the choice in row z from t_1 to t_2 would produce another supported row choice, but its edge $z \to t_2$ would not be reciprocated. This contradicts the preceding universal involution property. Hence every row of A_2 has exactly one positive entry. After relabelling the complementary vertices, row stochasticity and the support dichotomy now give

$$A_2 = P_2^{\oplus(k-1)}. \tag{7.4}$$

Finally, A_1 is a zero-diagonal 3×3 doubly stochastic matrix. Let C be a cyclic permutation matrix. The only fixed-point-free permutations of three vertices are C and C^{T}, so Birkhoff–von Neumann gives

$$A_1 = xC + (1 - x)C^{\mathsf{T}}, \qquad 0 \leq x \leq 1, \tag{7.5}$$

Direct expansion yields

$$\operatorname{per}(I_3 - A_1) = 1 - x^3 - (1 - x)^3 + 3x(1 - x) = 6x(1 - x). \tag{7.6}$$

Using (7.4) and block multiplicativity,

$$\operatorname{per}(I - A) = 6x(1 - x)2^{k-1}. \tag{7.7}$$

Equality with $3 \cdot 2^{k-2}$ forces $x(1-x) = 1/4$, hence $x = 1/2$. Thus $A_1 = \frac{1}{2}(J_3 - I_3)$, proving (1.5). The converse is (6.3).

It remains to count the distinct matrices. Choose the three vertices of the half-weight triangle in $\binom{2k+1}{3}$ ways, then choose a perfect matching of the other $2k-2$ vertices in $(2k-3)!!$ ways. The triangle is symmetric, so there is no orientation factor, and distinct choices produce distinct matrices. With the convention $(-1)!! = 1$, this also covers $k = 1$. This completes the proof of Theorem 1.1. □

Acknowledgments

The proof of this conjecture was carried out by GPT-5.6-sol and Claude Fable 5, under the guidance of the author. The author has reviewed the resulting proof arguments. Responsibility for the final text rests with the author.

References

[1] K. H. Kim and F. W. Roush, Expressions for Certain Minors and Permanents, *Linear Algebra and its Applications* **41** (1981), 93–97. doi:10.1016/0024-3795(81)90090-2.

[2] H. Minc, Theory of Permanents 1982–1985, *Linear and Multilinear Algebra* **21** (1987), 109–148. doi:10.1080/03081088708817786.

[3] G.-S. Cheon and I. M. Wanless, An Update on Minc's Survey of Open Problems Involving Permanents, *Linear Algebra and its Applications* **403** (2005), 314–342. doi:10.1016/j.laa.2005.02.030.

[4] L. I. Troanca, *Permanents of Doubly Stochastic Matrices*, M.Sc. thesis, University of Manitoba, 2008. University repository PDF.

[5] Z. Chen and L. Cao, On the Maximum of the Permanent of $(I - A)$, *Linear Algebra and its Applications* **555** (2018), 412–431. doi:10.1016/j.laa.2018.06.031; arXiv:1801.00225.