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CROSS-ENERGY OF DIRICHLET PARTIAL SUMS. PART 38: THE ENERGY OF THE CHEBYSHEV RACE: AN EXPLICIT UNCONDITIONAL FLOOR FOR THE PRIME RACE BETWEEN THE LANES $6k \pm 1$

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ABSTRACT. Let $R(x) = \pi(x;3,2) - \pi(x;3,1)$ be the Chebyshev race between the two prime lanes $6k\mp1$ (the prime 2 counted, by the congruence $2 \equiv -1 \bmod 3$, in the $6k-1$ lane). We study the race as an *energy* and establish three things (the energy law uses two estimates imported from the ζ case; see Proposition 4.1). First, an elementary ledger identity: the entire bias of the race is the bookkeeping of prime squares, which all land in the lane $6k+1$; subtracting the ledger $\frac{1}{2}\pi(\sqrt{x})$ turns a walk with *zero* sign changes up to 10^8 into one with 15,329 (21,329 if every passage between $D > 0$ and $D \le 0$ at a prime is counted). Second, the mean energy of the normalized race signal $e(u) = e^{-u/2}\psi(e^u, \chi_3)$ admits an explicit unconditional lower bound for its upper limit, which we call the floor,

$$\limsup_{U\to\infty} \frac{1}{U}\int_0^U e(u)^2\,du \;\ge\; \kappa_3 := \sum_\rho \frac{1}{\rho(1-\rho)} = \log\frac{3}{\pi} - \gamma_E + 2\,\frac{L'}{L}(1,\chi_3) = 0.113229969857\ldots,$$

the sum running over the nontrivial zeros of $L(s,\chi_3)$ with multiplicity, and the limit exists and attains the floor if and only if the Riemann hypothesis holds for $L(s,\chi_3)$ with all zeros simple. The constant is known: under GRH it is the variance of the limiting distribution of Rubinstein and Sarnak for $(\log x/\sqrt{x})R(x)$ (four times smaller than in their normalization $E(x;3,2) - E(x;3,1)$), and its closed form is that of Fiorilli and Martin (there under GRH); it is also the variance V_χ of Part 37, on the $6k \pm 1$ race, where the prime-square mechanism of the bias is identified; what is added here is the unconditional floor with its equality case and exponential-growth law, the ledger correction of the counting race, and the exact measurement. The constant κ_3 is the exact analogue, for the prime race, of the Báez-Duarte constant $2 + \gamma_E - \log 4\pi$ of the Nyman–Beurling criterion. Third, we measure: from all prime powers up to 10^8, by exact piecewise integration, the cumulative mean energy is 0.1201 at $x = 10^8$ and lies 4 to 12% above κ_3 for $10^4 \le x \le 10^8$. This is consistent with the theorem and does not test it: the floor is a statement about an upper limit, obtained from the dichotomy between GRH and its failure, and means over single decades fall below κ_3 (0.102 on $[10^4, 10^5]$, 0.113 on $[10^7, 10^8]$). The second-order term is $(\text{mean} - \kappa_3)\cdot U \approx 0.11$ (between 0.09 and 0.13), the race analogue of the second-order constant $K = 2.5124797\ldots$ of the prime energy. The proofs of the energy law transplant the C_2 energy law for ζ to $L(s,\chi_3)$, where the absence of a pole simplifies every step.

1. Introduction: the race, the ledger, and the constant

Among the primes other than 3, every one lies in one of the two residue lanes mod 3 — for $p > 3$ these are the familiar lanes $6k-1$ and $6k+1$, and the prime 2 joins the first by $2 \equiv -1 \pmod 3$. The two lanes are not symmetric in the data: writing

$$R(x) = \pi(x;3,2) - \pi(x;3,1) = -\sum_{p\le x} \chi_3(p),$$

where χ_3 is the nontrivial character mod 3, the lane $6k-1$ leads almost always. This is Chebyshev's bias [Ch] in its modulus-three form — exactly the race of Bays and Hudson [BH]. The lead is overwhelming in any computable range: in our data $R(x) \geq 1$ throughout $2 \leq x \leq 10^8$ — zero sign changes — and the first reversal is known to occur only at $x = 608{,}981{,}813{,}029$ [BH]. Yet the lead changes sign infinitely often [Li14], and under the Riemann hypothesis for $L(s, \chi_3)$ together with linear independence of the zeros, the logarithmic density of the set $\{R(x) > 0\}$ is $0.9990\ldots$ [RS].

This note looks at the race through two elementary moves and one constant.

The ledger. The bias is not a mystery about primes; it is bookkeeping about their powers. The square of every prime lands in the lane $6k+1$, because $\chi_3(p^2) = \chi_3(p)^2 = 1$. In the Chebyshev–von Mangoldt accounting, where the natural object is $\psi(x, \chi_3) = \sum_{n \leq x} \chi_3(n)\Lambda(n)$, the lane $6k+1$ therefore carries the full weight of the prime squares, and the *prime* race inherits the debt with the opposite sign. Subtracting the ledger — the explicit term $\frac{1}{2}\pi(\sqrt{x})$ — removes the entire first-order bias: the corrected race

$$D(x) = R(x) - \tfrac{1}{2}\pi(\sqrt{x})$$

changes sign 15,329 times up to 10^8 (Section 2, Figure 1, left). The operation "subtract the primes, add their powers" is exactly the operation that makes the race fair.

The energy. Once the drift is understood, the fluctuation remains, and the right way to measure a fluctuation driven by zeros is its mean energy. Let

$$e(u) = e^{-u/2}\,\psi(e^u, \chi_3), \qquad u = \log x.$$

Because $L(s, \chi_3)$ has no pole, $e(u)$ is pure oscillation: a superposition of the modes $e^{(\rho-1/2)u}/\rho$, one per nontrivial zero. Its mean energy obeys an exact law with an explicit, unconditional lower bound for its upper limit.

The constant. Define

$$\kappa_3 \;=\; \sum_{\rho} \frac{1}{\rho(1-\rho)},$$

the sum over the nontrivial zeros of $L(s, \chi_3)$, counted with multiplicity; the series converges absolutely, with no hypothesis. In Section 3 we evaluate it in closed form,

$$\kappa_3 = \log\frac{3}{\pi} - \gamma_E + 2\,\frac{L'}{L}(1, \chi_3) = 0.113229969857\ldots, \tag{1}$$

using $L(1, \chi_3) = \pi/(3\sqrt{3})$ and $L'(1, \chi_3) = 0.2226629870\ldots$ The constant κ_3 plays, for the race, precisely the role that the Báez-Duarte constant $2 + \gamma_E - \log 4\pi = \sum_\rho 1/(\rho(1-\rho))$ (zeros of ζ) plays in the Nyman–Beurling criterion and in Cramér's mean square of the prime-counting error [BD, BDBLS, Cr]: it is an unconditional lower bound for the upper limit of a mean energy, attained as a limit exactly at "RH and simple zeros". The constant is not

new: under GRH it is the variance $\sum_\gamma (\frac{1}{4}+\gamma^2)^{-1}$ of the limiting distribution of the race [RS], and the closed form (1) is the case $q=3$ of the formula of Fiorilli and Martin for that variance [FM, Sect. 3].

Theorem 1.1 (race energy law; proved in Section 4)**.** *With $e(u)$ as above and $F(U)=\int_0^U e(u)^2\,du$:*

(i) *Unconditionally,* $\limsup_{U\to\infty} \dfrac{F(U)}{U} \;\geq\; \kappa_3$.

(ii) *The limit $\lim_{U\to\infty} F(U)/U$ exists and equals κ_3 if and only if all nontrivial zeros of $L(s,\chi_3)$ lie on the critical line and are simple.*

(iii) *If $\Theta_3=\sup_\rho \operatorname{Re}\rho>\frac{1}{2}$, then $\limsup_U \frac{1}{U}\log F(U)=2\Theta_3-1$: a single off-line zero makes the energy grow exponentially in u.*

Remark 1.2 (what the floor is). Part (i) is proved by a dichotomy. If GRH holds for $L(s,\chi_3)$, the limit exists and is at least κ_3 (Proposition 4.1); if it fails, the energy grows exponentially (Proposition 4.2). So (i) is a statement about the upper limit only. It gives no inequality $F(U)/U\geq\kappa_3$ at any finite U, and no finite computation can confirm or refute it. Under GRH with simple zeros the mean tends to κ_3, and nothing keeps means over windows above κ_3; in the data they fall below it (Section 5).

The proof transplants the C_2 energy law of the author's cross-energy series — the growth-rate theorem and the energy law for ζ, [CE5, Theorem 2.2, Theorem B and Corollary 2.3], consolidated in [CP, §3] — from ζ to $L(s,\chi_3)$. The transplant is a simplification, not a generalization: the pole of ζ, which there had to be subtracted, is absent here, and every step shortens.

Finally we measure. From all prime powers up to 10^8, with the integral $F(U)$ computed *exactly* between consecutive prime-power jumps, the mean energy is

$$0.1255,\quad 0.1208,\quad 0.1209,\quad 0.1212,\quad 0.1201 \qquad \text{at } x=10^4,10^5,10^6,10^7,10^8,$$

against $\kappa_3=0.11323$: above it by 6–11% (nothing forces this at finite U; see Remark 1.2), and with the excess times U hovering between 0.09 and 0.13 — the race analogue of the second-order constant $K=2.512479790006\ldots$ determined for ζ in [CE6, Theorem 5 and Corollary 9] (closed form without zeros, certified under RH; see also [CP, §3]) (Section 5).

Relation to the author's earlier work. This part continues two earlier parts of the series, and repeats part of each. Part 37, on the $6k\pm1$ race [CE37], already identifies the prime squares as the cause of the bias (in the normalized θ-race they shift the $6k+1$ side by exactly 1, [CE37, Prop. 4.2]), evaluates the constant of this part in closed form — there it is the variance $V_\chi=\sum_\rho 1/(\rho(1-\rho))=0.113230$, [CE37, Prop. 4.3], with the same Hadamard proof as our Proposition 3.1 — states that under GRH with simple zeros it is the mean square of $\psi(x,\chi_3)/\sqrt{x}$, and measures that mean square on $[10^4,2^{27}]$. The energy law that we transplant, with its unconditional growth-rate part, is the author's law for ζ [CE5, Theorem 2.2, Theorem B,

Corollary 2.3], and the second-order constant we compare with is that of [CE6, Theorem 5, Corollary 9]. The ζ-row of the same instrument is the energy gauge, now a section of Part 6 [CE6].

What is new, and what is not. The bias mechanism (the prime squares) is classical, implicit in every treatment since Chebyshev and explicit in the framework of Rubinstein and Sarnak [RS]; the first-order debiasing is folklore, and the mechanism is already in Part 37 [CE37]. The distribution theory of races under GRH and linear independence is [RS]; we assume neither. The constant κ_3 and its closed form (1) are known [RS, FM]; they are also [CE37, Prop. 4.3]. What we believe is new here is: the *unconditional* floor with the equality characterisation and the exponential-growth law (Theorem 1.1), which in Part 37 [CE37] appear only under GRH and for a different signal; the ledger correction of the *counting* race, $D(x) = R(x) - \frac{1}{2}\pi(\sqrt{x})$, with its sign-change count; and the exact piecewise measurement of the cumulative race energy from $u = 0$ against the floor, with the measured second-order constant. Unconditional lower bounds of comparable strength for related mean values go back to the comparative theory of Knapowski and Turán and to Pintz's mean-value theorems; we have not found the floor-with-equality characterization stated outside the author's series. Theorem 1.1 is a reformulation that locates GRH for $L(s, \chi_3)$ as a statement about one number attached to one signal.

Place in the book. This is Part 38 of the series, in block VII (Möbius cancellation and prime races). It builds on Part 5 (the growth-rate theorem and the energy law, transplanted here from ζ to $L(s, \chi_3)$), Part 6 (the second-order constant K and the energy gauge, which is the ζ-row of the same instrument) and Part 37 (the prime squares as the mechanism of the bias, and the constant κ_3). It is used by Part 6, whose energy-gauge section is its ζ-analogue.

2. The ledger identity and the fair race

Everything in this section is elementary and unconditional. Write $\theta(x, \chi) = \sum_{p \le x} \chi(p) \log p$, with all characters mod 3, so that $\theta(x) = \theta(x, \chi_0) + \log 3$ for $x \ge 3$.

Lemma 2.1 (the ledger)**.** *For $x \ge 2$,*

$$\theta(x, \chi_3) \;=\; \psi(x, \chi_3) - \theta(\sqrt{x}, \chi_0) - \theta(x^{1/3}, \chi_3) - \theta(x^{1/4}, \chi_0) - \cdots, \qquad \theta(y, \chi_0) = \theta(y) - \log 3 \;\; (y \ge 3),$$

the sum having at most $\log_2 x$ terms, the even-index terms carrying the principal-character weight ($\chi_3(p^{2k}) = 1$ for $p \ne 3$) and the odd-index terms the character weight. Consequently

$$\theta(x, \chi_3) = \psi(x, \chi_3) - \theta(\sqrt{x}) + O\big(x^{1/3}\big),$$

and, by partial summation, with $\Pi_\chi(x) = \sum_{p^k \le x} \chi_3(p^k)/k$,

$$-R(x) = \sum_{p \le x} \chi_3(p) = \Pi_\chi(x) - \tfrac{1}{2}\pi(\sqrt{x}) + O\big(x^{1/3}\big), \qquad \Pi_\chi(x) = \frac{\psi(x, \chi_3)}{\log x} + \int_2^x \frac{\psi(t, \chi_3)}{t \log^2 t}\, dt.$$

Proof. $\psi(x,\chi) = \sum_{k\ge1} \theta(x^{1/k}, \chi^k)$ by grouping prime powers, and χ_3^k is principal for even k, equal to χ_3 for odd k; the principal-character θ differs from θ by the single prime 3. The second display keeps the $k=2$ term and absorbs $k \ge 3$ into the error. The third is partial summation against $d\psi$, each prime power p^k receiving the weight $1/\log(p^k) = 1/(k\log p)$, which is the definition of Π_χ; replacing $\Pi_\chi - \sum_{p\le x} \chi_3(p)$ by its $k=2$ term costs $O(\pi(x^{1/3}))$. □

The content is the sign pattern: $\psi(x,\chi_3)$ is driftless (it is the pure zero oscillation; see (2) below), while the race R carries the drift $+\frac{1}{2}\pi(\sqrt{x})$ inherited from the squares. The data agree with the accounting to the hilt:

up to $x = 10^8$	sign changes	size at 10^8
raw race $R(x)$	0	+420
ledger $\frac{1}{2}\pi(\sqrt{x})$	—	614.5
fair race $D(x) = R(x) - \frac{1}{2}\pi(\sqrt{x})$	15,329	−194.5

The raw race never trails in a hundred million integers — consistent with the first known reversal at $6.09\cdot10^{11}$ [BH] — and the corrected race changes sign fifteen thousand times (Figure 1, left). The count depends on the convention. D jumps at the primes and at the squares of primes and takes the value 0 often (at 10,820 primes). Counting the changes of sign of D over all its jump points, with the zeros skipped, gives 15,329. Counting every passage between $D > 0$ and $D \le 0$ at the primes, so that a touch of zero counts twice, gives 21,329, the number printed in the first version of this part. One subtraction, dictated by Lemma 2.1, is the difference between a loaded coin and a fair one.

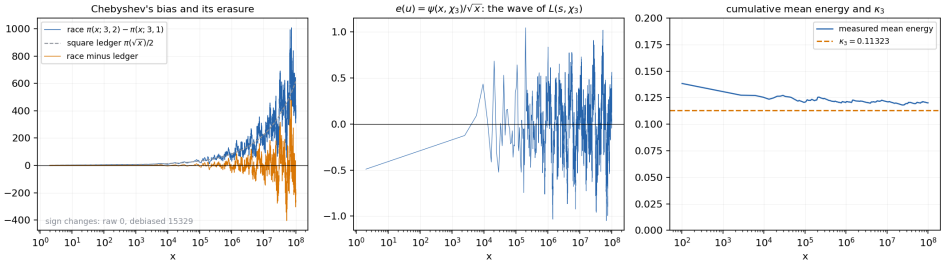


FIGURE 1. Left: the race $R(x)$, the ledger $\frac{1}{2}\pi(\sqrt{x})$, and the fair race $D(x)$, to $x = 10^8$. Centre: the normalized signal $e(u) = \psi(x,\chi_3)/\sqrt{x}$, the superposition of the modes of the zeros of $L(s,\chi_3)$. Right: the running mean energy $F(U)/U$ against the floor κ_3 of Theorem 1.1.

3. The constant κ_3 in closed form

Proposition 3.1. *Let $\kappa_3 = \sum_\rho 1/(\rho(1-\rho))$, over the nontrivial zeros of $L(s,\chi_3)$ with multiplicity. The series converges absolutely, and*

$$\kappa_3 \;=\; 2\,\frac{\xi'}{\xi}(1,\chi_3) \;=\; \log\frac{3}{\pi} - \gamma_E + 2\,\frac{L'}{L}(1,\chi_3) \;=\; 0.113229969857\ldots$$

where $\xi(s,\chi_3) = (3/\pi)^{(s+1)/2}\,\Gamma\!\left(\frac{s+1}{2}\right) L(s,\chi_3)$ is the completed L-function.

This is [CE37, Prop. 4.3]; we repeat the proof, adding the absolute convergence and the absence of real zeros in $(0,1]$.

Proof. Absolute convergence: $|\rho(1-\rho)|^{-1} \asymp \gamma^{-2}$ and the zero-counting function of $L(s,\chi_3)$ grows like $\frac{T}{\pi}\log\frac{T}{2\pi e/3}$, so the sum over $|\gamma| \geq 1$ converges; there are no zeros with $\gamma = 0$ since $L(s,\chi_3) > 0$ on $(0,1]$ (indeed $L(\sigma,\chi_3) = \sum_{k\geq 0}\big((3k+1)^{-\sigma} - (3k+2)^{-\sigma}\big)$ is a sum of positive terms).

χ_3 is odd and primitive, so $\xi(s,\chi_3)$ is entire of order one, its zeros are exactly the nontrivial zeros of L, and Hadamard's factorization gives

$$\frac{\xi'}{\xi}(s) = B + \sum_\rho \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big),$$

the sum absolutely and locally uniformly convergent in the paired sense. At $s = 0$ the summands vanish termwise, so $\xi'/\xi(0) = B$. At $s = 1$,

$$\frac{\xi'}{\xi}(1) = B + \sum_\rho \Big(\frac{1}{1-\rho} + \frac{1}{\rho}\Big) = B + \sum_\rho \frac{1}{\rho(1-\rho)} = B + \kappa_3.$$

The functional equation $\xi(1-s,\overline{\chi_3}) = \epsilon\,\xi(s,\chi_3)$ with $|\epsilon| = 1$ and $\overline{\chi_3} = \chi_3$ gives, on taking logarithmic derivatives at $s = 1$, $\xi'/\xi(0) = -\xi'/\xi(1)$. Hence $B = -\xi'/\xi(1)$ and $\kappa_3 = 2\,\xi'/\xi(1)$. Finally

$$\frac{\xi'}{\xi}(1) = \tfrac{1}{2}\log\tfrac{3}{\pi} + \tfrac{1}{2}\psi\big(1\big) + \frac{L'}{L}(1,\chi_3) = \tfrac{1}{2}\log\tfrac{3}{\pi} - \tfrac{\gamma_E}{2} + \frac{L'}{L}(1,\chi_3),$$

since $\psi(1) = -\gamma_E$, which is the stated formula.

For the numerical value: $L(1,\chi_3) = \pi/(3\sqrt{3}) = 0.604599788078\ldots$ is classical (Dirichlet). We computed $L'(1,\chi_3) = -\sum_{n\geq 1}\chi_3(n)\log n/n$ by pairing consecutive terms $n = 3k+1, 3k+2$ — after pairing the series converges absolutely, the general term being $O(\log k/k^2)$ — summing 10^7 pairs and correcting with the integral tail; the same scheme reproduces $L(1,\chi_3)$ to 16 digits, and gives

$$L'(1,\chi_3) = 0.22266298697\ldots, \qquad 2\,\frac{L'}{L}(1,\chi_3) = 0.73656323194\ldots,$$

whence $\kappa_3 = -0.04611759718 - 0.57721566490 + 0.73656323194 = 0.11322996986$, reliable to the printed digits (paired tails of first order at 10^7 pairs). □

Remark 3.2. A closed form for $L'(1,\chi_3)$ in terms of $\log\Gamma(1/3)$ follows from the Lerch–Kronecker limit formulas; we do not need it. We record one curiosity with a warning: $\log(3/\pi) = -0.0461176\ldots$ while the Báez-Duarte constant for ζ is $2+\gamma_E-\log 4\pi = +0.0461914\ldots$; the near-coincidence of digits is numerical accident, not identity — the two differ in the fourth decimal.

4. The race energy law

We now prove Theorem 1.1, following the C_2 energy law of [CP, §3] step by step; we keep the proofs short where the transplant is verbatim and spell out what changes. Throughout, $\rho=\beta+i\gamma$ runs over the nontrivial zeros of $L(s,\chi_3)$ with multiplicity $m_\rho \ll \log(|\gamma|+2)$, and $\Theta_3=\sup_\rho \beta$.

The explicit formula for $\psi(x,\chi_3)$ has no main term, because $L(s,\chi_3)$ has no pole [Dav, §19–20]: for $x\geq 2$,

$$\psi(x,\chi_3) = -\sum_{|\gamma|\leq T}\frac{x^\rho}{\rho} + O\Big(\frac{x\log^2(xT)}{T}+\log x\Big), \tag{2}$$

and in the limit $e(u)=e^{-u/2}\psi(e^u,\chi_3) = -\sum_\rho e^{(\rho-1/2)u}/\rho + O(ue^{-u/2})$ in the symmetric sense. On the critical line each zero contributes a unimodular mode $e^{i\gamma u}/\rho$ of energy $1/|\rho|^2 = 1/(\rho(1-\rho))$.

Proposition 4.1. *If all nontrivial zeros of $L(s,\chi_3)$ lie on the critical line, then*

$$\lim_{U\to\infty}\frac{1}{U}\int_0^U e(u)^2\,du = \sum_{\rho\ \text{distinct}}\frac{m_\rho^2}{|\rho|^2} \ \in\ [\kappa_3,\infty),$$

and the value equals κ_3 if and only if all zeros are simple.

Proof. This is Cramér's argument [Cr] in the form of [CE5, Prop. 2.1] and [CP, Prop. 3.1(a)] (for $L(s,\chi_3)$ and simple zeros the statement is in [CE37, Prop. 4.3]), with one simplification: no pole term has to be removed from $e(u)$. Group the zeros by height: $e(u) = -\sum_\gamma c_\gamma e^{i\gamma u}+o(1)$ in mean square, where for each distinct ordinate the coefficient collects the multiplicity, $|c_\gamma| = m_\rho/|\rho|$. The system $\{e^{i\gamma u}\}$ is orthogonal in mean on growing intervals and $\sum|c_\gamma|^2$ converges or diverges with $\sum m_\rho^2/|\rho|^2$; the standard almost-periodic mean-value theorem gives the limit $\sum_{\text{distinct}} m_\rho^2/|\rho|^2$. Two quantitative inputs are imported from [CP, §3], where they are established for ζ using only $N(T)\asymp \frac{T}{\pi}\log T$, $m_\rho\ll\log|\gamma|$ and $\sum|\rho|^{-2}<\infty$ — all three available for $L(s,\chi_3)$: the mean-square truncation of the explicit formula, and the off-diagonal bound of Montgomery–Vaughan [MV]. The sum is finite, because $m_\rho\ll\log(|\gamma|+2)$ and $\sum_\rho\log(|\gamma|+2)/|\rho|^2<\infty$. On the line, $|\rho|^2=\rho(1-\rho)$, so

$$\sum_{\text{distinct}}\frac{m_\rho^2}{|\rho|^2} \ \geq\ \sum_{\text{distinct}}\frac{m_\rho}{|\rho|^2} = \sum_\rho\frac{1}{\rho(1-\rho)} = \kappa_3,$$

by $m^2 \geq m$, with equality precisely when every $m_\rho = 1$. *Status.* The proof is by import: the two quantitative inputs are proved in [CP, §3] for ζ, and we assert, without rewriting those proofs, that they use only the three properties listed. A reader who wants a self-contained proof must check that transfer. □

Proposition 4.2. *Unconditionally, with* $F(U) = \int_0^U e(u)^2 du$ *and* $I(\delta) = \int_0^\infty e(u)^2 e^{-2\delta u} du$*:*

(a) *If* $I(\delta) < \infty$ *for every* $\delta > 0$*, then* $L(s, \chi_3)$ *has no zeros in* $\operatorname{Re} s > \frac{1}{2}$.
(b) *If* $\Theta_3 > \frac{1}{2}$*, then* $\limsup_U \frac{1}{U} \log F(U) = 2\Theta_3 - 1$*; in particular* $\limsup_U F(U)/U = \infty$.

Proof. (a) For $\operatorname{Re} s > 1$,

$$\int_1^\infty \psi(x, \chi_3)\, x^{-s-1}\, dx = -\frac{1}{s}\,\frac{L'}{L}(s, \chi_3),$$

with *no* pole term on the right — this is the transplant's one structural simplification. Writing $x = e^u$, the left side is the Laplace transform $\int_0^\infty e(u)\, e^{-(s-\frac{1}{2})u}\, du$. If $I(\delta) < \infty$, Cauchy–Schwarz bounds the transform for $\operatorname{Re} s > \frac{1}{2} + \delta$ by $\big(I(\delta)\big)^{1/2}\big(2(\operatorname{Re} s - \frac{1}{2} - \delta)\big)^{-1/2}$, locally uniformly, so the left side is analytic there; hence $L'/L(s, \chi_3)$ continues analytically to $\operatorname{Re} s > \frac{1}{2} + \delta$ and L has no zeros there. Letting $\delta \to 0$ gives (a).

(b) is the Laplace–Stieltjes abscissa argument of [CE5, Theorem 2.2] ([CP, Thm. 3.2]), unchanged: the abscissa δ_c of I satisfies $\limsup \frac{1}{U} \log F = 2\delta_c$; the bound $\psi(x, \chi_3) \ll x^{\Theta_3} \log^2 x$ gives $\delta_c \leq \Theta_3 - \frac{1}{2}$, and $\delta_c < \Theta_3 - \frac{1}{2}$ would continue L'/L past a zero by the argument of (a). □

Proof of Theorem 1.1. (iii) is Proposition 4.2(b). For (i) and (ii): if $\Theta_3 > \frac{1}{2}$, the energy grows exponentially, so the lim sup in (i) is $+\infty > \kappa_3$ and the limit in (ii) does not exist finitely — and on the right side of (ii) a zero off the line exists, so both sides of the equivalence fail. If $\Theta_3 = \frac{1}{2}$, all zeros are on the line (the zeros are symmetric under $\rho \mapsto 1 - \bar{\rho}$), and Proposition 4.1 applies: the limit exists, equals $\sum m_\rho^2/|\rho|^2 \geq \kappa_3$, with equality iff all zeros are simple. This proves (ii), and (i) in the remaining case. □

Remark 4.3. As with the C_2 energy law for ζ, the theorem is a *reformulation*: proving that the race energy is bounded is proving the Riemann hypothesis for $L(s, \chi_3)$. Its value is the same as there — it converts a sign problem into a size problem with an explicit constant, and the constant can be measured against data (Section 5).

Remark 4.4 (general modulus). Nothing is special to $q = 3$ except the tidiness of the lanes. For a primitive character χ mod q with parity $a \in \{0, 1\}$ ($\chi(-1) = (-1)^a$), the same Hadamard argument gives, for the energy of

$|e_\chi(u)|^2$, $e_\chi(u) = e^{-u/2}\psi(e^u, \chi)$, the floor

$$\kappa_\chi = \sum_\rho \frac{1}{\rho(1-\rho)} = \log\frac{q}{\pi} + \psi\Big(\frac{1+a}{2}\Big) + 2\ \mathrm{Re}\,\frac{L'}{L}(1,\chi),$$

with $\psi(\frac{1}{2}) = -\gamma_E - 2\log 2$ in the even case, and Theorem 1.1 holds verbatim with κ_χ and GRH for $L(s,\chi)$. The ledger generalizes too: $\chi(p^2)$ is the square character, so the biased lanes are the squares modulo q, which is Chebyshev's bias in the generality of [RS]. For ζ itself, with the pole subtracted, the same instrument is the energy gauge of Part 6 [CE6], whose slope constant is $2\widehat{B} = 2 + \gamma_E - \log 4\pi$.

5. The measurement

We computed every prime power $p^k \le 10^8$ with its weight $\chi_3(p^k)\log p$ (5,762,859 events), the running $\psi(x,\chi_3)$, and the energy integral *exactly* between events: on a maximal interval $[x_i, x_{i+1})$ where $\psi(\cdot,\chi_3) = C_i$ is constant,

$$\int_{\log x_i}^{\log x_{i+1}} e(u)^2\,du = C_i^2\Big(\frac{1}{x_i} - \frac{1}{x_{i+1}}\Big),$$

so $F(U)$ is a finite exact sum (in double precision) with no quadrature error; the integral starts at $u = 0$, and $\psi(x,\chi_3) = 0$ on $[1,2)$. The results:

x	10^4	10^5	10^6	10^7	10^8
mean energy $F(U)/U$	0.12553	0.12080	0.12086	0.12121	0.12015
excess $\times U$: $(F(U)/U - \kappa_3)\,U$	0.113	0.087	0.105	0.129	0.127

Three remarks, in the spirit of strict bookkeeping. First, the cumulative mean $F(U)/U$ lies above $\kappa_3 = 0.113230$ for every $10^4 \le x \le 10^8$, between 0.1181 and 0.1273. This is consistent with Theorem 1.1 and is not a test of it: part (i) concerns the upper limit and holds whatever the data to 10^8 are (Remark 1.2). Means over windows are not bounded below by κ_3: $(F(U_2) - F(U_1))/(U_2 - U_1)$ is 0.1019, 0.1211, 0.1233, 0.1127 on the four decades from 10^4 to 10^8, and 0.1148 on $[10^4, 10^8]$. What the data can test is the closed form (1) in the conditional sense: under GRH with simple zeros the mean tends to κ_3, and the window $[10^4, 10^8]$ is within 1.4% of it. (The table uses $U = \log x$ and includes the last interval up to x; the first version stopped at the last prime power below x, which gave 0.12544 at 10^4.) Second, the excess times U is roughly stable, between 0.09 and 0.13 over four decades: this is the signature of a finite-U expansion $F(U) = \kappa_3 U + K_3 + \Phi_3(U) + o(1)$ with a second-order constant $K_3 \approx 0.11$ and an almost-periodic interference term Φ_3, exactly parallel to the expansion $\int_1^X (\psi(x) - x)^2 x^{-2} dx = C_B \log X + K + \Phi(\log X) + o(1)$, $K = 2.512479790006\ldots$, established for ζ under RH, simple zeros and a spacing condition in [CE6, Theorem 5] (closed form without zeros and certified value in [CE6, Prop. 8, Cor. 9]; see also [CP, §3]); we record

K_3 as *measured*, with the conditional derivation deferred. Third, K_3 is much smaller than K; we have no explanation for this.

For comparison, [CE37, §4] measures the variance of the normalized θ-race, sampled uniformly in $\log x$ on $[10^4, 2^{27}]$, as 0.1124; the present measurement is a different functional (the cumulative energy from $u = 0$, computed exactly), and both place the race energy within about 10% of κ_3.

The sign-change counts of Section 2 (0 raw, 15,329 debiased, to 10^8) and Figure 1 complete the picture: drift explained by the ledger, fluctuation measured by the energy, energy measured against κ_3.

6. What this does and does not do

The theorem converts "GRH for $L(s, \chi_3)$ with simple zeros" into the statement that one number attached to the prime race equals one computable constant. It does not make the number easier to bound: as in the ζ case [CP, §3], proving $F(U) = O(U)$ *is* the Riemann hypothesis for this L-function, and nothing in our measurement to 10^8 — four decades within 12% of κ_3 — constitutes evidence that can distinguish GRH from a zero of real part 0.500001. The distributional theory of races (densities, comparisons between moduli) remains that of Rubinstein–Sarnak [RS] and its descendants [GM], which assume GRH and linear independence; our floor assumes nothing, and characterises rather than estimates. We do not prove the Riemann hypothesis for $L(s, \chi_3)$.

Data and code

All computations are reproducible from the ancillary scripts: `race.py` (sieve to 10^8, the race, the ledger, the sign-change counts in both conventions, the exact piecewise energy and the window means; revised 10 November 2026), `kappa3.py` (the paired series for $L(1, \chi_3)$, $L'(1, \chi_3)$ and κ_3), `racefig.py` (Figure 1); `recheck_10nov.py` (the sign-change counts in the two conventions, the energy with exact end points, the window means). Sums are accumulated in double precision over exact integer event positions; the series for L and L' are validated against the closed form $L(1, \chi_3) = \pi/(3\sqrt{3})$ to 16 digits.

Use of AI assistance

The research programme and the choice of objects are the author's. Claude (Anthropic) was used as a research and writing aid: to derive and check the lemmas, to write and run the computations, and to draft text and figures. The author checked all results and is responsible for the content.

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