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CROSS-ENERGY OF DIRICHLET PARTIAL SUMS. PART 37: THE $6k \pm 1$ RACE: SIEVE STRUCTURE, CHEBYSHEV'S BIAS AND A GRH-EQUIVALENT INEQUALITY

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Abstract. Every prime $p > 3$ is $6k+1$ or $6k-1$. We collect what can be proved about the two resulting sequences and add three results. The races are those modulo 3 (the prime 2 counts for the class 2); the race modulo 6 differs by that one prime. (i) In the weighting $n^{-1/2}\log(N/n)$ of the Cesàro margin, the race between the prime powers $\equiv 2$ and $\equiv 1 \pmod 3$ is an exact identity. The $6k-1$ side leads by $2\frac{L'}{L}(\frac{1}{2},\chi_{-3})\log N - \frac{1}{2}\psi'(\frac{3}{4}) = 1.1320\log N - 1.2709$ plus the margin $M_\chi(N)$ of $L(s,\chi_{-3})$. The generalized Riemann hypothesis (GRH) for $L(s,\chi_{-3})$ is equivalent to $M_\chi \geq 0$ for all large N, and under GRH the weighted race is never lost after $N = 2$. This is a criterion of the type of Hardy–Littlewood and Landau (1918) for Chebyshev's assertion, here with the Cesàro weight and with explicit constants. (ii) We verify $0.1024 \leq M_\chi(N) \leq 0.3549$ for $2 \leq N \leq 2^{27}$, inside the window $[0, 0.4536]$ that GRH predicts. We also check the formula against the first 271 zeros of $L(s,\chi_{-3})$. (iii) For the unweighted race, the bias is $1/\sqrt{V_\chi} = 2.97$ standard deviations, with $V_\chi = \sum_\rho \frac{1}{\rho(1-\rho)} = 0.113230$ (equal to $\sum_\rho |\rho|^{-2}$ under GRH); under GRH and linear independence, a recomputation with 271 zeros gives the density 0.99906, in agreement with the 0.9990 of Rubinstein and Sarnak. Along the way we prove the sieve characterization of $6k \pm 1$ and of twin primes, explain why Legendre-type counting cannot give the error term, compare twin primes and Goldbach representations modulo 6 with the Hardy–Littlewood predictions, and prove the formula for $\gcd_p(p^{2n}-1)$ (the sequence $24, 240, 504, 480, \dots$). Version 2 adds the generating function $\frac{1}{2}[\zeta_6(s)^2 \pm L(s,\chi_6)^2]$ of the three forms $6xy \pm x \pm y$, and a duality theorem: averaged over the zeros of ζ, the cross-energy $\mathcal{C}_2(N, \frac{1}{2}, \gamma)$ of Part 1 returns $\log N/N$; averaged over the zeros of $L(s,\chi_{-3})$ it returns $\frac{1}{N}\sum_{d|N}\chi_{-3}(d)\Lambda(d)$; and Möbius inversion separates the primes of the two lanes. Version 3 makes this exact at finite height and without RH: with a Gaussian weight on the zeros, the cross-energy satisfies an explicit formula whose pole term is $\mathcal{C}_2$ itself at $s = 0$, whose prime term is $\log N$ exactly plus a nonnegative remainder from the near coincidences $nj \approx N$, and which returns every prime power up to 60 from the 1517 zeros below 2000. Nothing here proves RH, GRH, or the twin prime or Goldbach conjectures.

1. Introduction

The integers prime to 6 are exactly the numbers $6k \pm 1$. So every prime $p > 3$ lies in one of the two progressions

$$P^+ = \{6k+1\}, \qquad P^- = \{6k-1\}.$$

In the author's notes [Ca20], these intertwined sequences are the starting point for primality tests, counting methods and several conjectures. One observation there is that P^- is almost always ahead of P^+ [Ca20, Idea 7]. This paper puts these observations on a firm footing. It separates what is proved, what is conditional and what is open.

For $p > 3$, $p \equiv \pm 1 \pmod 6$ exactly when $p \equiv \pm 1 \pmod 3$. So the two series are distinguished by the real character $\chi = \chi_{-3}$ of conductor 3 ($\chi(n) = 0, 1, -1$ for $n \equiv 0, 1, 2 \pmod 3$). The

Date: November 2026 (formerly a standalone note). Part 37 of the book-series *Cross-Energy of Dirichlet Partial Sums* (forty-seven parts, Parts 0–46; block VII: Möbius cancellation and prime races). Part 0 is the guide. Version 2, 2 November 2026: Section 11. Version 3, 2 November 2026: Section 11.3. Version 4, 10 November 2026: Hardy–Littlewood integrals recomputed, earlier work cited, title changed. In book version 1 (2 November 2026) this was Part 34; the present numbering is that of book version 2 (10 November 2026).

race between them is governed by $L(s,\chi)$, and the Dedekind zeta function of $\mathbb{Q}(\sqrt{-3})$ factors as $\zeta(s)L(s,\chi)$. Our main tool is the Cesàro margin of Part 11 [CE11], transferred from ζ to $L(s,\chi)$.

Earlier work. Chebyshev asserted in 1853 that there are more primes $4n+3$ than $4n+1$, in the sense that $\sum_p (-1)^{(p-1)/2} e^{-p/x} \to -\infty$ as $x \to \infty$ [Che]. Hardy and Littlewood proved this under GRH for the corresponding L-function [HL18], and Landau proved that the assertion is equivalent to that hypothesis [Lan18]. Littlewood showed that the unweighted differences change sign infinitely often [Li14]; Knapowski and Turán began the systematic study of such races [KT], and Rubinstein and Sarnak computed their logarithmic densities under GRH and LI [RuSa]. Theorems 5.2 and 6.1 below belong to the Hardy–Littlewood–Landau line: a smoothed race whose one-sided behaviour is equivalent to GRH. What is added here is the choice of weight $n^{-1/2}\log(N/n)$, for which the race is an identity with an explicit line $1.1320\log N - 1.2709$, an explicit window $[0, 0.4536]$ for the margin, and a verification to 2^{27}. The inequality is exact as an identity; as an inequality it is equivalent to GRH and is not proved.

Which race. The character χ_{-3} compares the classes 2 and 1 modulo 3, and the prime 2 belongs to the class 2. All statements about $\pi(x;3,\cdot)$, $\psi(x,\chi)$, C_1, C_2, M_χ and the densities are for this race modulo 3. The race modulo 6 between P^- and P^+ omits the prime 2: $\pi(x;6,5) = \pi(x;3,2) - 1$ and $\pi(x;6,1) = \pi(x;3,1)$ for $x \geq 2$, so its lead is smaller by exactly one, and in the weighted race by the terms of the powers of 2. The tables of counts (Table 2) are for primes > 3, that is, modulo 6.

Results. Section 2 proves the sieve characterization of $P^\pm$ and of twin primes, and shows why the error terms of Legendre-type counting are of the same order as the main term. Section 4 recalls Chebyshev's bias and computes it from the zeros of $L(s,\chi)$. Section 5 builds the margin M_χ, proves that GRH for $L(s,\chi)$ is equivalent to $M_\chi \geq 0$ (Theorem 5.2), and verifies it to 2^{27}. Section 6 contains the exact form of the weighted race (Theorem 6.1) and two-sided bounds for each series. Section 7 treats consecutive primes and runs, and Sections 8–10 treat twin primes, Goldbach modulo 6 and the even powers of primes. Section 11 (new in version 2) writes the three forms as a Dirichlet series and proves that the cross-energy $\mathcal{C}_2$ of Part 1, averaged over the zeros of ζ or of $L(s,\chi_{-3})$, returns the primes of each lane (Theorem 11.2); Section 11.3 (new in version 3) gives the exact finite form of this duality, Proposition 11.3, and explains term by term how the cross-energy hears the primes through the zeros. Section 12 summarizes the status.

Place in the book. This is Part 37 of the series, in block VII (Möbius cancellation and prime races). It builds on Part 11 (the Cesàro margin, transferred here from ζ to $L(s,\chi_{-3})$) and on the author's earlier notes on the primes $6k \pm 1$. It is used by Part 38 (the energy of the same race), and it is cited by Parts 10 and 11. It also uses Part 10 [CE10]. It is also used by Part 11 [CE11]. Section 11 uses the cross-energy of Part 1 [CE1], and it is the dual of the zero detection by that cross-energy studied there; the localized Weil windows of Part 20 [CE20] use the same functions X_N in the other direction.

2. The sieve structure of $6k \pm 1$

Lemma 2.1 (characterization)**.** *Let $k \geq 1$ and x, y range over the positive integers.*

(a) $6k+1$ *is composite if and only if* $k = 6xy + x + y$ *or* $k = 6xy - x - y$ *for some* x, y.
(b) $6k-1$ *is composite if and only if* $k = 6xy + x - y$ *for some* x, y.
(c) $(6k-1, 6k+1)$ *is a pair of twin primes if and only if* k *is of none of the three forms.*

Proof. If $n = 6k \pm 1$ is composite, write $n = ab$ with $1 < a \leq b$. Since $\gcd(n,6) = 1$, both factors are $\equiv \pm 1 \pmod 6$. If $n \equiv 1$, then $(a,b) \equiv (1,1)$ or $(-1,-1)$, that is, $n = (6x+1)(6y+1) = 6(6xy+x+y)+1$ or $n = (6x-1)(6y-1) = 6(6xy-x-y)+1$, with $x, y \geq 1$ because $a, b > 1$. If $n \equiv -1$, one factor is $\equiv 1$ and the other $\equiv -1$, and $(6x-1)(6y+1) = 6(6xy+x-y) - 1$. Conversely, each form gives a nontrivial factorization. Part (c) combines (a) and (b). □

Status: elementary and complete; checked for $k \le 10^5$. The three forms are the matrices A_n, B_n, C_n of [Ca20], and (c) is [Ca20, Idea 8]. The "duplicates" counted there have an exact meaning:

Proposition 2.2 (multiplicities)**.** *The number of ordered pairs (x, y) with $6xy + x + y = k$ equals the number of ordered factorizations $6k+1 = ab$ with $a, b > 1$ and $a \equiv b \equiv 1 \pmod 6$; similarly for the other two forms. In particular, the two forms $6xy+x+y$ and $6xy-x-y$ together represent k in exactly $\tau(6k+1)-2$ ordered ways, and the form $6xy+x-y$ represents k in $(\tau(6k-1)-2)/2$ ways. How the first number splits between the two forms depends on the residues of the prime factors modulo 6, not only on how many there are. For example, $343 = 7^3$ has three prime factors, and $k = 57$ has only the two representations $(1, 8)$, $(8, 1)$ by $6xy + x + y$ and none by $6xy - x - y$; and $175 = 5^2 \cdot 7$ gives $k = 29$ two representations by each form, from $7 \cdot 25$ and from $5 \cdot 35$.*

Proof. The map $(x, y) \mapsto (6x + 1, 6y + 1)$ is a bijection onto the factor pairs of the stated type. □

So a count of the composites in $P^{\pm}$ by "elements minus duplicates" is the inclusion–exclusion principle. Done exactly, it is Legendre's identity: for $z \ge 5$ and $P(z) = \prod_{p \le z} p$,

$$\pi(N) - \pi(\sqrt{N}) + 1 = \sum_{d \mid P(\sqrt{N})} \mu(d) \Big\lfloor \frac{N}{d} \Big\rfloor, \tag{2.1}$$

and the restriction to $P^{\pm}$ only removes the factors 2 and 3 from $P(z)$. The difficulty is not in the identity. It is in the error terms.

Proposition 2.3 (Legendre's main term is wrong by a constant factor)**.** *Write $\lfloor N/d \rfloor = N/d - \{N/d\}$ in (2.1). The main term is $N \prod_{p \le \sqrt{N}} (1 - \frac{1}{p}) \sim 2e^{-\gamma_E} \frac{N}{\log N}$, while $\pi(N) \sim \frac{N}{\log N}$. Hence the $2^{\pi(\sqrt{N})}$ fractional parts contribute $-(2e^{-\gamma_E} - 1 + o(1)) \frac{N}{\log N}$, a fixed proportion of the answer.*

Proof. Mertens' third theorem gives $\prod_{p \le z} (1 - \frac{1}{p}) \sim e^{-\gamma_E} / \log z$ with $z = \sqrt{N}$. Compare with the prime number theorem. □

Table 1 shows the effect. The main term exceeds $\pi(N) - \pi(\sqrt{N}) + 1$ by 1.3%, 4.8% and 5.7% at $N = 10^5$, 10^7 and 10^8, and the excess tends to $2e^{-\gamma_E} - 1 = 12.3\%$, slowly because of the factor $(\log N - 1)/\log N$.

TABLE 1. Legendre's main term $N \prod_{p \le \sqrt{N}} (1 - \frac{1}{p})$ against the true count.

N	main term	$\pi(N) - \pi(\sqrt{N}) + 1$	ratio
10^4	1 203.2	1 205	0.998
10^5	9 651.9	9 528	1.013
10^6	80 965.3	78 331	1.034
10^7	696 014.0	664 134	1.048
10^8	6 088 469.2	5 760 227	1.057

This is why regressions for the duplicates, as in [Ca20, §9–10], fit well on finite ranges but cannot give an asymptotic error term. The fractional parts $\sum_d \mu(d)\{N/d\}$ are the Möbius-weighted sawtooth sums that Part 11 [CE11] identifies as the core of the problem.

Remark 2.4 (on primality tests and conjectures). Characterizations like Lemma 2.1 are exact, but they do not by themselves prove statements about gaps or representations, such as Legendre's, Grimm's or Goldbach's conjectures. Those require control of the error terms above. A related caution concerns the test "$\gcd(2^n + n, 3^n + n, n + 1) = n + 1$ iff $n + 1$ is prime" [Ca20,

Idea 17]. The "if" direction follows from Fermat's little theorem. The converse fails for the Fermat pseudoprimes to bases 2 and 3, which include the Carmichael numbers prime to 6: $n+1 = 1105, 1729, 2465, 2701, 2821, 6601, 8911, \ldots$

3. A smooth indicator and the near-factorization energy

Lemma 2.1 can be packaged into a pair of smooth functions on the integers. Put, for $k \geq 1$,

(3.1)
$$F_+(k) = \prod_{x,y\geq 1} \tanh^2(k-6xy-x-y)\,\tanh^2(k-6xy+x+y), \qquad F_-(k) = \prod_{x,y\geq 1} \tanh^2(k-6xy+x-y),$$

with the convention $\tanh^2(0) = 0$; the infinite products converge, since $-\log\tanh^2 n \leq 4e^{-2n}/(1-e^{-2})$ and the number of lattice points with $6xy \pm (\cdots) = m$ is $O(\tau(6m \pm 1))$, and truncating at $x, y \leq k$, as one would compute them, changes nothing beyond e^{-10k}.

Proposition 3.1 (indicator and energy)**.** (1) *$F_+(k) > 0$ if and only if $6k+1$ is prime, and $F_-(k) > 0$ if and only if $6k-1$ is prime.*

(2) *Exactly, with $w(d) = -2\log\tanh|d|$,*

(3.2)
$$-\log F_\pm(k) = \sum_{m\geq 1,\, m\neq k} r_\pm(m)\, w(k-m), \quad r_+(m) = \tau(6m+1) - 2, \quad r_-(m) = \frac{\tau(6m-1)-2}{2}.$$

Proof. (1) is Lemma 2.1: a factor vanishes exactly at a representation. (2) By Proposition 2.2, the number of pairs (x, y) with $6xy + x + y = m$ or $6xy - x - y = m$ is the number of ordered factorizations $6m+1 = ab$, $a, b > 1$, which is $\tau(6m+1) - 2$, both factors being automatically in matching classes mod 6. For $6m-1 = ab$ the classes are forced to be opposite, $a \equiv 1$, $b \equiv -1$ or conversely, and the single form $6xy - x + y = m$ picks one orientation from each unordered pair, giving $(\tau(6m-1)-2)/2$ representations ($6m-1$ is never a square). Group the factors of (3.1) by the value m and take logarithms. □

Since $w(d) = 4e^{-2|d|}(1+O(e^{-4|d|}))$, identity (3.2) says what F is: an *exponentially localized count of the factorizations of the neighbours.* $F(k)$ is not a useful primality test – it is Lemma 2.1 in disguise, at cost k^2 against $\sqrt{6k}$ for trial division, and the smoothness of tanh gives no analytic leverage, the arguments being integers. Its interest is the value it assigns to each *prime*: a measure of how crowded the prime is by near-composites of its own progression. The nearest misses dominate: $F_+(3) = 0.5799$ against $\tanh^2(1) = 0.5800$, the single neighbour $k = 4$ ($25 = 5^2$) accounting for all of it.

Numerically, with (3.2) checked against the defining product to $3\cdot 10^{-10}$ and the divisor function sieved to $6\cdot 10^6$ (`tanh_f.py`): over the primes with $k \leq 10^6$, the mean of $-\log F_+$ grows like $0.47\log k$, and the mean of $-\log F_-$ like half of that – the factor 2 of (3.2), one representation per unordered factor pair instead of one per ordered pair, not any asymmetry of the progressions: indeed $\sum_{m\leq 10^6}\tau(6m+1) = 6\,082\,216$ against $\sum_{m\leq 10^6}\tau(6m-1) = 6\,082\,200$, equal to $2.6\cdot 10^{-6}$. The divisor mass is blind, at this order, to the Chebyshev bias of Section 4, which lives at the scale $\sqrt{x}/\log x$ of Proposition 4.3.

The extremes of F over the primes have arithmetic meaning (Figure 1(c)). The maximum on $100 \leq k \leq 10^6$ is $F_+ = 0.8246$ at $k = 443\,567$, $p = 2\,661\,403$: there $p-12$, $p-6$, p and $p+6$ are all prime, a run in the progression, and the nearest factorization is three steps away. The minimum is $F_+ = 1.1\cdot 10^{-20}$ at $k = 846\,678$, $p = 5\,080\,069$: its neighbour $6(k+1)+1 = 5\,080\,075 = 5^2\cdot 7^2\cdot 11\cdot 13\cdot 29$ has $\tau = 72$, and the 70 nontrivial ordered factorizations at distance 1 each contribute $w(1) = 0.5447$. So $\max F$ marks prime clusters and $\min F$ marks primes next to a highly divisible member of the progression: a graded structure on top of Lemma 2.1, with the hyperbola picture of Figure 1(b) classifying the points $e^{2\theta} = 6k \pm 1$ of $X^2 - Y^2 = 1$ by it.

Two further remarks. *A real variable.* Replacing the lattice product (3.1) by

$$G_{\pm}(x) = \prod_{m \geq 1} \tanh^{2r_{\pm}(m)}(x - m), \qquad x \in \mathbb{R},$$

gives smooth functions with the same values at the integers up to the truncation noted above, positive except for a zero of multiplicity $2r_{\pm}(m)$ at each composite index m; the exponents are even because $6m - 1$ is never a square and r_+ counts ordered pairs, so no sign ambiguity arises. This is a clean interpolation, but it carries exactly the information of τ on the progressions and no more.

Energy, gaps and twins. Over the 205 960 primes $6k + 1$ with $10^3 < k \leq 10^6$, the energy correlates only weakly with the local prime gaps of its own progression: after removing the $0.47 \log k$ trend, $\mathrm{Corr}(-\log F_+,$ gap to the next prime of the family$) = 0.05$, and 0.07 for the sum of the two adjacent gaps – larger gaps mean composite neighbours, hence a little more energy. The twin condition does better: at the 37 773 values of k where $6k - 1$ is also prime, the detrended energy $-\log F_+$ is higher by 0.91, which is 0.25 standard deviations (45σ significance at this sample size), and symmetrically $-\log F_-$ is higher by 0.26 standard deviations at the twins. This is the Hardy–Littlewood repulsion seen from the composite side: conditioning $6k \pm 1$ to avoid every small prime p leaves k in $p-2$ residue classes instead of $p-1$, so each neighbouring $6(k \pm d) \pm 1$ is *more* likely to be divisible by p, and the twins sit in measurably more crowded surroundings. The energy measures the factorizations of the neighbours, not the gaps themselves, and these correlations say nothing beyond that.

Status: elementary and complete; the correlations are empirical. The function bears on the fine structure of the progressions, not on primality testing nor on the L-function questions of the later sections.

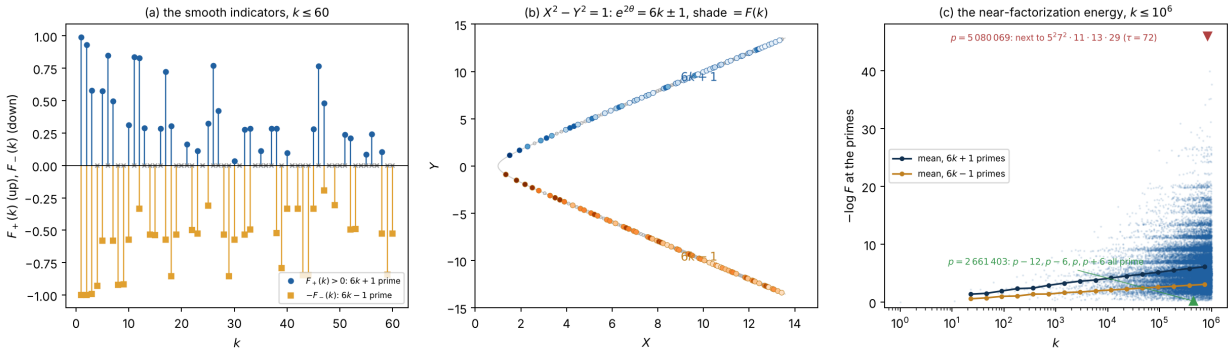


FIGURE 1. The smooth indicators (3.1). (a) $F_+(k)$ (up) and $F_-(k)$ (down) for $k \leq 60$; grey crosses at 0 are the composites. (b) The points $e^{2\theta} = 6k \pm 1$ on $X^2 - Y^2 = 1$, upper branch half for $6k + 1$, lower for $6k - 1$, shaded by $F(k)$; open grey points are composites. (c) $-\log F$ at the primes to $k = 10^6$: dyadic means for the two families, the most isolated prime (2 661 403, green) and the most crowded one (5 080 069, red).

4. Equidistribution and Chebyshev's bias

Theorem 4.1 (Dirichlet, de la Vallée Poussin)**.** $\pi(x; 3, 1) \sim \pi(x; 3, 2) \sim \frac{1}{2}\,\mathrm{li}(x)$. *Equivalently, P^+ and P^- each contain half of the primes asymptotically.*

The difference in Table 2 is positive throughout. It is not a fluctuation of the counts: it has a cause, the prime squares. Let $\psi(x, \chi) = \sum_{n \leq x} \chi(n)\Lambda(n)$ and $\theta(x, \chi) = \sum_{p \leq x} \chi(p) \log p = \theta(x; 3, 1) - \theta(x; 3, 2)$.

TABLE 2. Primes in P^+ and P^- up to 10^k (primes > 3).

x	$\pi(x;6,1)$	$\pi(x;6,5)$	difference
10^3	80	86	6
10^4	611	616	5
10^5	4 784	4 806	22
10^6	39 231	39 265	34
10^7	332 194	332 383	189
10^8	2 880 517	2 880 936	419

Proposition 4.2 (the explicit formula for the race)**.** *Since χ is primitive and odd,*

$$\psi(x,\chi) = -\sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{L'}{L}(0,\chi) + \sum_{m\geq 1} \frac{x^{1-2m}}{2m-1}$$

(sum over the nontrivial zeros, symmetrically), and

$$\frac{\theta(x;3,2)-\theta(x;3,1)}{\sqrt{x}} = 1 - \frac{\psi(x,\chi)}{\sqrt{x}} + o(1).$$

Proof. The first formula is [Da, Ch. 19]. For the second, $\psi(x,\chi)-\theta(x,\chi) = \sum_{p^2\leq x}\chi(p^2)\log p + O(x^{1/3}\log x)$, and $\chi(p^2)=1$ for $p\neq 3$. So the difference is $\theta(\sqrt{x}) + O(x^{1/3}\log x) = \sqrt{x}(1+o(1))$ by the prime number theorem. □

Every square of a prime $p>3$ is $\equiv 1 \pmod 3$. In the normalized race, the squares therefore shift the $6k+1$ side down by exactly 1. Under GRH, the remaining term $-\psi(x,\chi)/\sqrt{x} = \sum_\rho x^{i\gamma}/\rho + o(1)$ is a mean-zero almost periodic function in $\log x$ [RuSa].

Proposition 4.3 (the size of the fluctuation)**.** *Unconditionally,*

$$V_\chi := \sum_{\rho} \frac{1}{\rho(1-\rho)} = 2\frac{\xi_\chi'}{\xi_\chi}(1) = \log\frac{3}{\pi} - \gamma_E + 2\frac{L'}{L}(1,\chi) = 0.113230,$$

where $\xi_\chi(s) = (3/\pi)^{(s+1)/2}\Gamma(\frac{s+1}{2})L(s,\chi)$. Under GRH, $V_\chi = \sum_\rho |\rho|^{-2}$, and for simple zeros it is the mean square of $\psi(x,\chi)/\sqrt{x}$ in $\log x$.

Proof. ξ_χ is entire of order 1 with $\xi_\chi(s) = \xi_\chi(1-s)$. Its zeros are symmetric under $\rho\mapsto\bar\rho$ and $\rho\mapsto 1-\rho$, so Hadamard's product gives $\frac{\xi_\chi'}{\xi_\chi}(s) = \sum_\rho \frac{1}{s-\rho}$, summed symmetrically. Since $\frac{1}{\rho(1-\rho)} = \frac{1}{\rho} + \frac{1}{1-\rho}$ and the zeros are symmetric under $\rho\mapsto 1-\rho$, $V_\chi = 2\sum_\rho \frac{1}{1-\rho} = 2\frac{\xi_\chi'}{\xi_\chi}(1)$. On the line, $1-\rho=\bar\rho$, so $\frac{1}{\rho(1-\rho)} = \frac{1}{|\rho|^2}$. The value uses $\frac{L'}{L}(1,\chi) = 0.368282$, computed from the Hurwitz zeta function. The mean square follows from Parseval in the Besicovitch space B^2. □

The 271 zeros below height 400 give $\sum 1/|\rho|^2 = 0.108257$. Since the mean is 1 and the spread is $\sqrt{V_\chi} = 0.3365$, the bias is 2.97 standard deviations.

Theorem 4.4 (Rubinstein–Sarnak [RuSa])**.** *Assume GRH and the linear independence (LI) of the ordinates of $L(s,\chi)$. Then the set of x with $\pi(x;3,2) > \pi(x;3,1)$ has logarithmic density $\delta(3;2,1) = 0.9990\ldots$*

Under GRH and LI, $\psi(x,\chi)/\sqrt{x}$ has the limiting distribution of $X = \sum_{\gamma>0}\frac{2}{|\rho|}\cos\vartheta_\gamma$ with independent uniform phases. So the density is $P(X<1) = \frac12 + \frac{1}{\pi}\int_0^\infty \frac{\sin t}{t}\prod_{\gamma>0} J_0\big(\frac{2t}{|\rho|}\big)\,dt$. With the 271 zeros and a Gaussian factor for the tail of V_χ, we obtain 0.999063, in agreement with the value 0.9990 printed in [RuSa].

On $10^4 \le x \le 2^{27}$, sampled uniformly in $\log x$, the normalized race has mean 0.967 and variance 0.1124, against 1 and $V_\chi = 0.1132$. Its histogram follows the limiting density (Figure 6(a)). On the computed range, $\pi(x;3,2) > \pi(x;3,1)$ for all $2 \le x \le 2^{27}$. The first reversal of this race modulo 3 is at $x = 608\,981\,813\,029$ [BH]. For the race modulo 6 the lead is one less: $\pi(x;6,5) \ge \pi(x;6,1)$ on the whole range; for $x \ge 5$ equality holds exactly from each of the primes $7, 13, 19, 37, 43, 79, 163, 223, 229$ up to the next prime, and the inequality is strict for $233 \le x \le 2^{27}$. Its first reversal is strictly earlier than the value of [BH], because a tie of the race modulo 3 precedes its first reversal. The θ-race fails on 0.13% of $[10^4, 2^{27}]$ in logarithmic measure (Figure 2). For prime races in general see [GM].

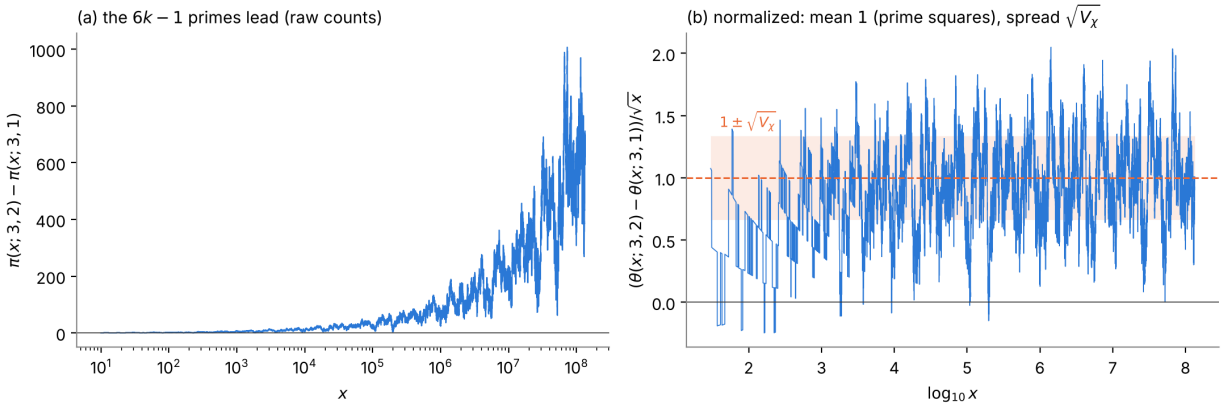


FIGURE 2. The race between P^- and P^+ up to 2^{27}. (a) Raw counts. (b) The normalized θ-race of Proposition 4.2: mean 1 from the prime squares, band $1 \pm \sqrt{V_\chi}$.

5. The Cesàro margin of $L(s, \chi_{-3})$

For ζ, Part 11 [CE11] uses Suzuki's criterion [Su1]: with

$$C(N) = 2 \sum_{n<N} \frac{\Lambda(n)}{\sqrt{n}} \log \frac{N}{n}$$

and an explicit $R(N)$, RH holds if and only if $M = R - C \ge 0$ for large N. Under RH, $0 \le M \le 4(\log \xi)''(\frac{1}{2}) = 0.18484$. We transfer this to χ.

χ is primitive and odd. $L(s,\chi)$ has no pole, its trivial zeros are $-1, -3, -5, \dots$, and $L(0,\chi) = \frac{1}{3} \ne 0$. Write $t_\rho := -i(\rho - \frac{1}{2})$ ($t_\rho = \gamma_\rho$ for a zero $\rho = \frac{1}{2} + i\gamma_\rho$ on the line, $t_\rho = \gamma - i\delta$ for a zero $\rho = \beta + i\gamma$ off it, $\delta = \beta - \frac{1}{2}$); the zeros come in pairs $\pm t_\rho$ because χ is real. Define

$$C_\chi(N) = 2 \sum_{n<N} \frac{\chi(n)\Lambda(n)}{\sqrt{n}} \log \frac{N}{n}, \qquad R_\chi(N) = -2\frac{L'}{L}(\tfrac{1}{2},\chi) \log N + 2 \sum_{m \ge 1} \frac{1 - N^{-(2m-\frac{1}{2})}}{(2m-\frac{1}{2})^2}, \tag{5.1}$$

and $M_\chi = R_\chi - C_\chi$. The constants are $L(\frac{1}{2},\chi) = 0.4808675577$, $\frac{L'}{L}(\frac{1}{2},\chi) = 0.5659892385$ and $2\sum_{m\ge1}(2m-\frac{1}{2})^{-2} = \frac{1}{2}\psi'(\frac{3}{4}) = \frac{1}{2}(\pi^2 - 8G) = 1.2709398$, with G Catalan's constant.

Proposition 5.1 (the margin over the zeros)**.** *For every $N > 1$, unconditionally,*

$$M_\chi(N) = \sum_\rho \frac{2\,(1 - N^{it_\rho})}{t_\rho^2} = \sum_\rho F_{\log N}(t_\rho), \qquad F_L(z) = \frac{2 - e^{iLz} - e^{-iLz}}{z^2},$$

the sum over the nontrivial zeros with multiplicity, absolutely convergent. Under GRH,

$$M_\chi(N) = 4\sum_{\gamma>0} \frac{1-\cos(\gamma \log N)}{\gamma^2} \in \big[0,\, 4(\log \xi_\chi)''(\tfrac12)\big],$$

where $(\log \xi_\chi)''(\frac12) = \frac14\psi'(\frac34) + (\frac{L'}{L})'(\frac12,\chi) = 0.1134022.$

Proof. For $c > \frac12$, Perron's formula with the kernel N^s/s^2 gives $C_\chi(N) = \frac{1}{2\pi i}\int_{(c)} 2(-\frac{L'}{L})(s+\frac12,\chi)N^s\frac{ds}{s^2}$. Move the contour to the left; $\frac{L'}{L}$ grows at most like $\log^2|t|$ on suitable horizontal segments, and the factor s^{-2} makes the integrals converge. The residues are as follows. The double pole at $s=0$ gives $-2\frac{L'}{L}(\frac12)\log N - 2(\frac{L'}{L})'(\frac12)$. A zero ρ gives $-2N^{\rho-1/2}/(\rho-\frac12)^2 = 2N^{it_\rho}/t_\rho^2$. The trivial zero $-(2m-1)$ gives $-2N^{-(2m-1/2)}/(2m-\frac12)^2$. There is no pole term. By Hadamard's product and the symmetry of ξ_χ, $\sum_\rho t_\rho^{-2} = (\log\xi_\chi)''(\frac12) = \frac14\psi'(\frac34) + (\frac{L'}{L})'(\frac12)$. Adding $\sum_\rho 2(1-N^{it_\rho})/t_\rho^2$ to C_χ therefore gives $\frac12\psi'(\frac34) - 2\frac{L'}{L}(\frac12)\log N - 2\sum_m N^{-(2m-1/2)}/(2m-\frac12)^2 = R_\chi(N)$. As a check, both sides vanish at $N=1$. Under GRH, pair $\pm\gamma$ and use $0 \le 1-\cos \le 2$. □

Theorem 5.2 (GRH for χ_{-3} in one variable)**.** *The following are equivalent:*

1. *GRH for $L(s,\chi_{-3})$;*
2. *$M_\chi(N) \ge 0$ for all large N;*
3. *$M_\chi(N) \le B$ for some B and all large N;*
4. *for every $N>1$, $R_\chi(N) - 0.45361 \le C_\chi(N) \le R_\chi(N)$.*

Proof. (1)⇒(4) is Proposition 5.1, and (4)⇒(2),(3) is immediate. For (2)⇒(1): for $\operatorname{Re} s > \frac12$, Fubini gives

$$\int_0^\infty M_\chi(e^L)e^{-sL}dL = \sum_\rho \frac{2}{s(s^2+t_\rho^2)} =: \mathcal{G}_\chi(s),$$

meromorphic with poles at 0 and at $\pm it_\rho$. A zero $\frac12+\delta+i\gamma_0$ with $\delta>0$ gives a pole at $\delta+i\gamma_0$, with nonzero residue (only $\pm\gamma$ meet at a given pole). It lies off the real axis because $L(\sigma,\chi_{-3})\neq 0$ for $0<\sigma<1$; indeed, for $\sigma>0$ the series converges (its coefficients have bounded partial sums), its value is the limit of the partial sums up to $3k+2$, and so, pairing consecutive terms, $L(\sigma,\chi_{-3}) = \sum_{k\ge0}\big[(3k+1)^{-\sigma} - (3k+2)^{-\sigma}\big] > 0$ (the values at the ends are $L(0)=\frac13$ and $L(1) = \frac{\pi}{3\sqrt3}$). So $\mathcal{G}_\chi$ is analytic on $(0,\infty)$. By Landau's theorem [Wi], the Laplace integral of a function that is eventually ≥ 0 has a singularity at the real point of its abscissa of convergence. Hence the abscissa is ≤ 0, and there is no pole with $\operatorname{Re} s > 0$. For (3)⇒(1), apply the same argument to $B - M_\chi$. □

Why this margin is well conditioned. For ζ, the two sides R and C are of size $8\sqrt{N}$ and must stay ordered to $O(1)$. For χ there is no pole, and $R_\chi, C_\chi = O(\log N)$.

Lemma 5.3 (concavity)**.** *In $L=\log N$, M_χ is continuous, smooth and strictly concave between consecutive prime powers. At a prime power q its slope jumps by $-2\chi(q)\Lambda(q)/\sqrt{q}$. So the local minima of M_χ lie at the prime powers with $\chi(q)=-1$, namely $p\equiv 2 \pmod 3$ and its odd powers.*

Proof. Between prime powers, C_χ is linear in L, and $R_\chi''(L) = -2\sum_m e^{-(2m-1/2)L} < 0$. The jump is the derivative of $2\chi(q)\Lambda(q)q^{-1/2}(L-\log q)_+$. □

Proposition 5.4 (verification)**.** *For $2 \le N \le 2^{27}$,*

$$0.10242 \le M_\chi(N) \le 0.35486,$$

inside the window $[0, 0.45361]$ of Theorem 5.2, and $M_\chi(N) = R_\chi(N) > 0$ for $1<N<2$. The least value is at $q = 30\,561\,677 \equiv 2 \pmod 3$, the largest at $q = 17\,509$.

Proof. By Lemma 5.3, it suffices to evaluate M_χ at the 7 605 117 prime powers $q \le 2^{27}$ with $3 \nmid q$, and to check every gap for an interior stationary point; there are none. The quantities compared are $O(\log N) \le 40$, so the double-precision rounding error is below 10^{-12}. □

Status: a floating-point computation, not certified with interval arithmetic; the distance 0.10 to the bound is 10^{11} times the rounding error. Unlike for ζ, the margin stays far from 0 on the computed range.

The zeros. $Z_\chi(t) = e^{i\vartheta_\chi(t)} L(\frac{1}{2} + it, \chi)$, with $\vartheta_\chi(t) = \frac{t}{2}\log\frac{3}{\pi} + \operatorname{Im}\log\Gamma(\frac{3/2+it}{2})$, is real. Its sign changes give 271 zeros on the line with $0 < \gamma < 400$: $8.0397, 11.2492, 15.7046, 18.2620, 20.4558, \ldots$ The counts agree with $\vartheta_\chi(T)/\pi + 1$ to within 1.5, and $2\sum_{\gamma<400}\gamma^{-2} = 0.108429$ plus the tail estimate 0.004983 gives 0.113412, against $(\log\xi_\chi)''(\frac{1}{2}) = 0.113402$. With the tail replaced by its mean, $M_\chi(N)$ from these zeros agrees with the value from the primes to $3\cdot10^{-4}$ for $10^2 \le N \le 10^8$ (Figure 4).

Proposition 5.5 (the slope of χ_{-3})**.** *Let $\alpha_\chi(x) = \sum_{n<x}\chi(n)\Lambda(n)n^{-1/2} + \frac{L'}{L}(\frac{1}{2},\chi)$. Under GRH, $u \mapsto \alpha_\chi(e^u)$ lies in B^2, with mean 0 and coefficients $\frac{im_\gamma}{\gamma}$ at the ordinates. The margin $M_\chi(e^L)$ is at most the variogram $\langle|\alpha_\chi(e^{u+L}) - \alpha_\chi(e^u)|^2\rangle$, with equality for simple zeros, and $\langle\alpha_\chi^2\rangle = 2\sum_{\gamma>0} m_\gamma^2/\gamma^2 \ge (\log\xi_\chi)''(\frac{1}{2})$. Unconditionally, GRH holds if and only if $\int_1^X \alpha_\chi(x)^2\frac{dx}{x} = X^{o(1)}$.*

Proof. The Laplace transform of $\alpha_\chi(e^u)$ is $\frac{1}{s}\big(\frac{L'}{L}(\frac{1}{2},\chi) - \frac{L'}{L}(s+\frac{1}{2},\chi)\big)$. It is regular at $s=0$, so the mean is 0 by the Abelian theorem, and its residue at $i\gamma$ is $-\frac{m_\gamma}{i\gamma}$. Membership in B^2 follows from the explicit formula, as for ζ [RuSa, ANS]. The variogram identity is Parseval, and the comparison uses $m^2 \ge m$. For the energy criterion, if the energy is $X^{o(1)}$, Cauchy–Schwarz shows that the Laplace integral converges absolutely in $\operatorname{Re} s > 0$, so $\frac{L'}{L}(w,\chi)$ is holomorphic in $\operatorname{Re} w > \frac{1}{2}$, exactly as for ζ in [CE11]. □

The block energies of α_χ on $[2^k, 2^{k+1})$, $12 \le k \le 26$, have mean 0.110, against 0.1134. The amplitude spectrum of α_χ has its peaks at the zeros of $L(s,\chi)$ with heights $2/\gamma$, and nothing at the zeros of ζ (Figure 3).

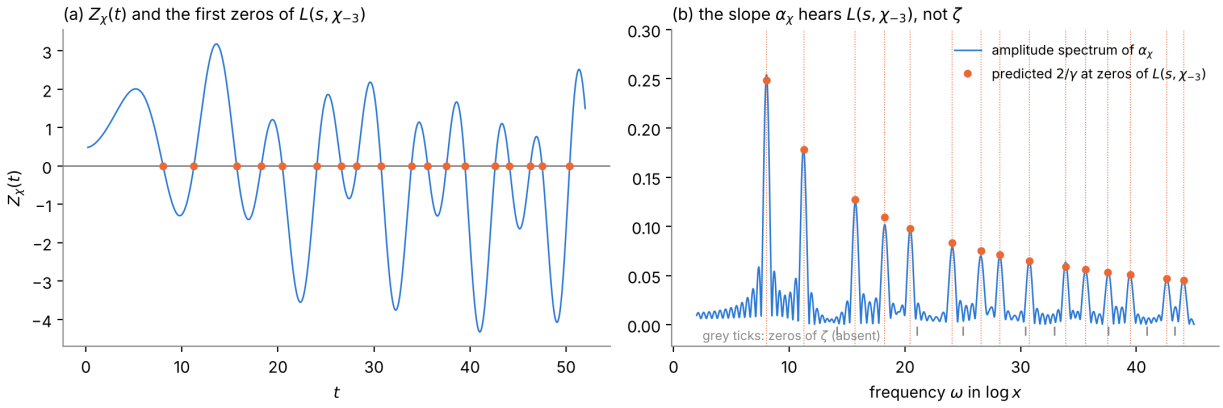


FIGURE 3. (a) $Z_\chi(t)$ and the zeros of $L(s,\chi_{-3})$ below 52. (b) The amplitude spectrum of $\alpha_\chi(e^u)$ on $2^8 \le x \le 2^{27}$: peaks at the ordinates of $L(s,\chi_{-3})$ with the predicted heights $2/\gamma$; the ordinates of ζ (grey ticks) are absent.

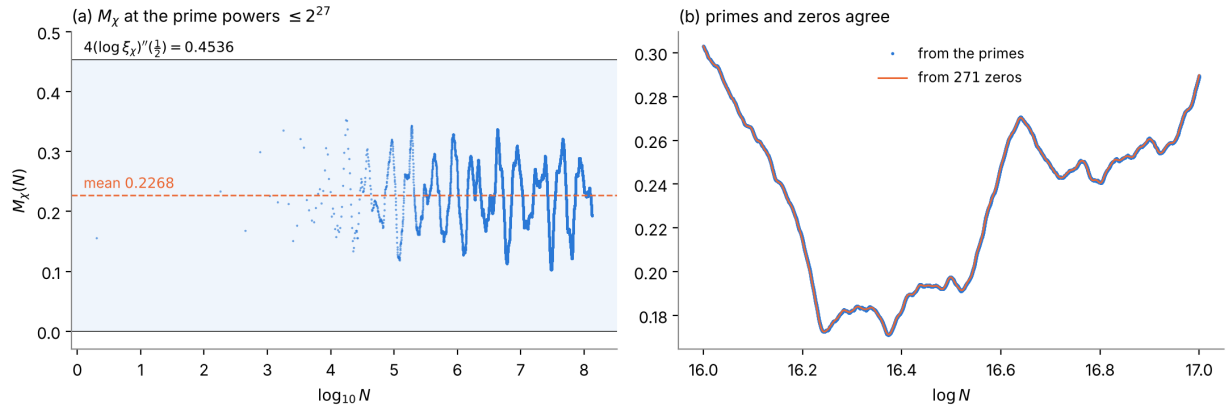


FIGURE 4. (a) M_χ at every 50th prime power up to 2^{27}, inside the window $[0, 4(\log \xi_\chi)''(\frac{1}{2})]$, around its mean $2(\log \xi_\chi)''(\frac{1}{2})$. (b) M_χ from the primes and from the 271 zeros, $16 \le \log N \le 17$.

6. The weighted race

For $a = 1, 2$ let

$$C_a(N) = 2 \sum_{\substack{n<N \\ n\equiv a \ (3)}} \frac{\Lambda(n)}{\sqrt{n}} \log \frac{N}{n}.$$

C_1 collects the prime powers p^k with $p \equiv 1$, or $p \equiv 2$ and k even; C_2 those with $p \equiv 2$ and k odd. For $p > 3$ these are the series P^+ and P^-. Since $\chi = +1$ on the first class and -1 on the second, $C_2 - C_1 = -C_\chi$.

Theorem 6.1 (the weighted $6k \pm 1$ race)**.** *For every $N > 1$, unconditionally,*

$$C_2(N) - C_1(N) = 2\frac{L'}{L}(\tfrac{1}{2}, \chi_{-3}) \log N - \tfrac{1}{2}\psi'(\tfrac{3}{4}) + 2 \sum_{m\ge 1} \frac{N^{-(2m-\frac{1}{2})}}{(2m-\frac{1}{2})^2} + M_\chi(N). \tag{6.1}$$

Hence GRH for $L(s, \chi_{-3})$ holds if and only if, for all large N,

$$C_2(N) - C_1(N) \ \ge \ 2\frac{L'}{L}(\tfrac{1}{2}, \chi_{-3}) \log N - \tfrac{1}{2}\psi'(\tfrac{3}{4}),$$

where $2\frac{L'}{L}(\frac{1}{2}, \chi_{-3}) = 1.1319784770\ldots$ and $\frac{1}{2}\psi'(\frac{3}{4}) = 1.2709398\ldots$ The equivalence is for the exact constants: a line with a rounded slope differs from this one by an unbounded multiple of $\log N$. Under GRH, $C_2(N) > C_1(N)$ for every $N > 2$, and the lead exceeds the line by at most 0.45361 plus the trivial-zero terms. Unconditionally, $C_2(N) > C_1(N)$ for $2 < N \le 2^{27}$.

Proof. (6.1) is $-C_\chi = M_\chi - R_\chi$ with (5.1). Write $T(N) = 2\sum_{m\ge1} N^{-(2m-\frac{1}{2})}/(2m-\frac{1}{2})^2$ for the trivial-zero terms; $T > 0$ and $T = O(N^{-3/2})$. The inequality of the statement says $M_\chi + T \ge 0$ for all large N. GRH gives $M_\chi \ge 0$ (Theorem 5.2), hence the inequality. Conversely, $\int_0^\infty T(e^L)e^{-sL}dL = 2\sum_{m\ge1}(2m-\frac{1}{2})^{-2}(s+2m-\frac{1}{2})^{-1}$ is analytic in $\operatorname{Re} s > -\frac{3}{2}$, so $M_\chi + T$ has the same Laplace singularities in $\operatorname{Re} s > 0$ as M_χ, and the Landau argument in the proof of Theorem 5.2, (2)⇒(1), applies to it without change. Under GRH, $M_\chi \ge 0$, and the lead is positive once $1.1319785 \log N > 1.2709398$, that is, for $N > 3.0733$. For $2 < N \le 3.0733$ only the prime $2 \equiv 2 \pmod 3$ has entered, and the lead is $2\frac{\log 2}{\sqrt{2}} \log \frac{N}{2} > 0$. The unconditional range is Proposition 5.4. □

This is the precise form of the observation of [Ca20, Idea 7], for the weighted counts. The $6k-1$ prime powers lead at the constant rate $2\frac{L'}{L}(\frac{1}{2}, \chi_{-3}) = 1.132$ per unit of $\log N$. The deviation from the line is M_χ, which GRH confines to a window of width 0.454 (Figure 5). The weight is what makes the bias total. It is the Fejér smoothing of Part 11 [CE11], which turns every zero into a nonnegative term. For the raw counts there is no such smoothing, and the bias holds only 99.9% of the time (Theorem 4.4).

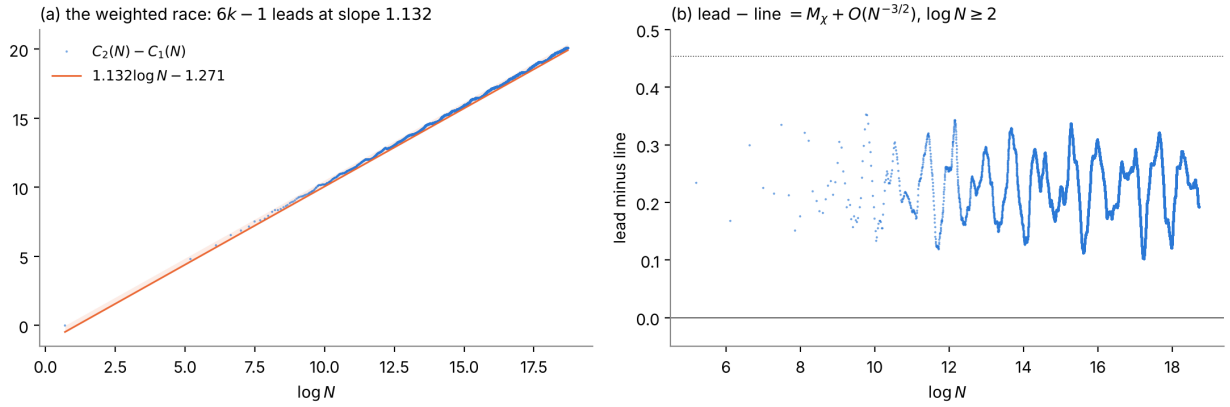


FIGURE 5. The weighted race of Theorem 6.1 at every 50th prime power up to 2^{27}. (a) $C_2 - C_1$ and the line $1.132 \log N - 1.271$; the shaded band is the window allowed by GRH. (b) The deviation from the line, which is M_χ up to the trivial-zero terms.

Both series separately. Adding the margins of ζ and of χ gives the margin of the Dedekind zeta function of $\mathbb{Q}(\sqrt{-3})$. Let $C_{3\bullet}(N)$ be the contribution of the powers of 3 to C. Then $C + C_\chi = 2C_1 + C_{3\bullet}$ and $C - C_\chi = 2C_2 + C_{3\bullet}$.

Corollary 6.2 (a sandwich for each series)**.** *With*

$$R(N) = 8(\sqrt{N} - 1) - 2\frac{\zeta'}{\zeta}(\tfrac{1}{2}) \log N + 2 \sum_{m \geq 1} \frac{1 - N^{-(2m+1/2)}}{(2m + \frac{1}{2})^2}$$

as in [CE11]*, for every* $N > 1$*,*

$$C_1 = \tfrac{1}{2}\big(R + R_\chi - C_{3\bullet}\big) - \tfrac{1}{2}(M + M_\chi), \qquad C_2 = \tfrac{1}{2}\big(R - R_\chi - C_{3\bullet}\big) - \tfrac{1}{2}(M - M_\chi).$$

Under RH and GRH for χ_{-3}*,* C_1 *lies in a window of width* $\frac{1}{2}(0.18484 + 0.45361) = 0.31923$ *below* $\frac{1}{2}(R + R_\chi - C_{3\bullet})$*, and* $C_2 - \frac{1}{2}(R - R_\chi - C_{3\bullet}) \in [-0.09242, 0.22681]$.

Proof. Linear algebra on the two identities, with $M \in [0, 0.18484]$ [Su1, CE11] and $M_\chi \in [0, 0.45361]$. □

7. CONSECUTIVE PRIMES AND RUNS

The bias of Section 4 concerns totals. A different and stronger effect concerns neighbours. Let $p_n < p_{n+1}$ be consecutive primes > 3, and count the transitions between the series.

A prime is followed by one in the same series only 44.0% of the time below 2^{27}, not 50% (Figure 6(b)). Lemke Oliver and Soundararajan [LOS] discovered this phenomenon for general moduli and explained it with the Hardy–Littlewood k-tuple conjecture. In their model the deviation from independence decays only like $\log\log x / \log x$, which matches the slow drift towards $\frac{1}{2}$ in the figure. The small excess of $5 \to 5$ over $1 \to 1$ is the bias of Section 4 again.

TABLE 3. Transitions between consecutive primes $p_n \to p_{n+1}$ ($p_{n+1} \le x$), by series ($1 = P^+$, $5 = P^-$).

x	$1 \to 1$	$1 \to 5$	$5 \to 1$	$5 \to 5$	same/different
10^4	233	377	378	238	0.624
10^6	16 394	22 837	22 837	16 427	0.719
10^8	1 264 047	1 616 470	1 616 470	1 264 465	0.782
2^{27}	1 671 812	2 129 608	2 129 608	1 672 522	0.785

Runs behave as the transitions suggest. Below 2^{27} the longest strings of consecutive primes in one series have length 15 in P^- and 16 in P^+, and there are 171 387 and 171 605 maximal strings of length ≥ 4. The sequences A296011 and A296055 of [Ca20] list the starting generators k of four consecutive primes $6k-1, \dots, 6k+17$ in P^- (resp. $6k+1, \dots, 6k+19$ in P^+) with no prime of the other series between them; our computation reproduces $42, 897, 1052, 2107, 2242, 2457, 2632, 2912, \dots$

Theorem 7.1 (Shiu [Sh]). *For every m, each of P^+ and P^- contains m consecutive primes.*

Shiu's theorem holds for every residue class $a \bmod q$ with $\gcd(a, q) = 1$, and it is unconditional. Its proof uses the Maier matrix method, not the sieve structure of Lemma 2.1. It does not settle A296011 and A296055. Those sequences require the four primes to sit at consecutive generators, that is, four consecutive primes in arithmetic progression with difference 6. Their infinitude follows from the Hardy–Littlewood prime k-tuples conjecture and is open.

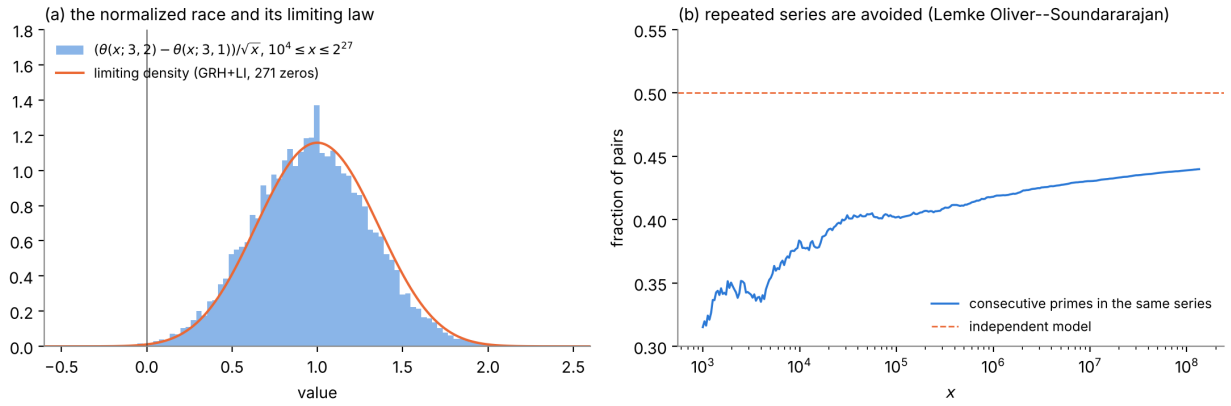


FIGURE 6. (a) Distribution of the normalized θ-race over $10^4 \le x \le 2^{27}$ (uniform in $\log x$) and the limiting density of Theorem 4.4, computed from the 271 zeros by Fourier inversion of $\prod J_0(2t/|\rho|)$. (b) Fraction of consecutive pairs of primes in the same series.

8. TWIN PRIMES

By Lemma 2.1(c), every twin pair other than $(3, 5)$ is $(6k-1, 6k+1)$, with k a twin generator (OEIS A002822). Twin centres $6k$ form A014574, and the representations of a twin centre as a sum of two smaller ones form A305825 [Ca20].

Conjecture 8.1 (Hardy–Littlewood [HL]). $\pi_2(x) \sim 2C_2 \int_2^x \frac{dt}{\log^2 t}$, *with* $C_2 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) = 0.6601618158\dots$

TABLE 4. Twin prime pairs up to x against Conjecture 8.1. The integral is computed by Gauss–Legendre quadrature in $\log t$; at 10^8 and 2^{27} the count differs from it by 56 and by 3.

x	$\pi_2(x)$	$2C_2 \int_2^x dt/\log^2 t$	ratio
10^4	205	214.2	0.957
10^5	1 224	1 248.7	0.980
10^6	8 169	8 248.0	0.990
10^7	58 980	58 753.8	1.004
10^8	440 312	440 367.8	1.000
2^{27}	571 313	571 316.0	1.000

Brun proved that $\sum(\frac{1}{p} + \frac{1}{p+2})$ over twin primes converges [Br]. Its value $B_2 \approx 1.90216$ is an extrapolation; rigorous bounds are much wider [PTb]. Our partial sum to 2^{27} is 1.7611. Whether there are infinitely many twin primes, that is, infinitely many k avoiding the three forms of Lemma 2.1, is open.

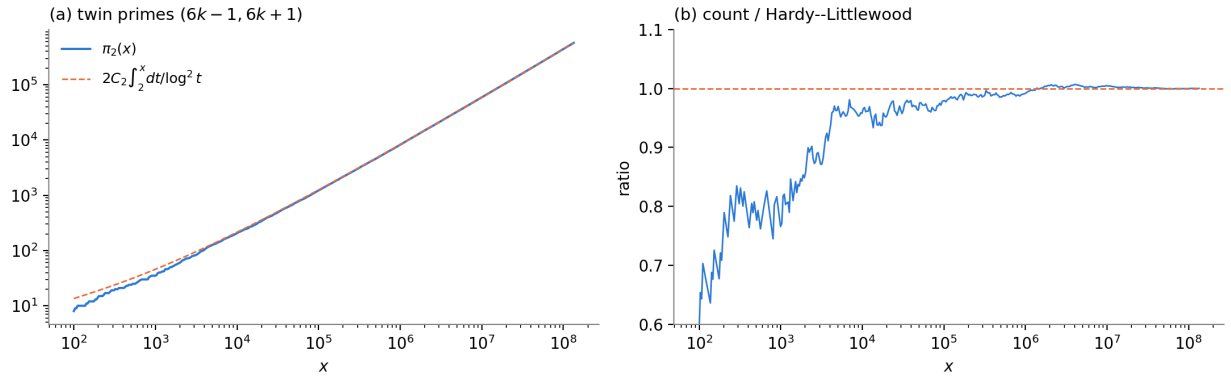


FIGURE 7. Twin primes $(6k-1, 6k+1)$ up to 2^{27} against the Hardy–Littlewood prediction.

9. GOLDBACH MODULO 6

Lemma 9.1 (residue classes)**.** *Let N be even and $N = p_1 + p_2$ with primes $p_1, p_2 > 3$. If $N \equiv 0 \pmod 6$, one prime is in P^+ and the other in P^-. If $N \equiv 2$, both are in P^+. If $N \equiv 4$, both are in P^-. Correspondingly, $N/6$, $(N-2)/6$ or $(N+2)/6$ is a sum of two generators $k_1 + k_2$* [Ca20, Idea 30].

Proof. $(\pm 1) + (\pm 1) \pmod 6$. □

Let $r(N)$ be the number of ordered representations $N = p_1 + p_2$. The observation behind A322921 [Ca20] is that among three consecutive even numbers, the multiple of 6 usually has the most representations. The Hardy–Littlewood singular series explains it.

Conjecture 9.2 (Hardy–Littlewood [HL])**.** *For even N,*

$$r(N) \sim 2C_2 \prod_{\substack{p \mid N \\ p>2}} \frac{p-1}{p-2} \int_2^{N-2} \frac{dt}{\log t \, \log(N-t)}.$$

The factor for $p = 3$ is 2. So a multiple of 6 expects twice as many representations as a neighbour with the same other prime factors. On average over $N \le 2 \cdot 10^5$, $r(N)\log^2 N/N$ is 3.63 for $N \equiv 0$, and 1.80 and 1.83 for $N \equiv 2, 4 \pmod 6$. The ratios are $2:1:1$, not the factors $e:1:\sqrt{2}$ suggested in [Ca20, Figs. 26–28]. Divided by the prediction, the three classes give distributions of the same shape with slightly different centres: for $N > 2 \cdot 10^4$ the means are 0.991, 0.984 and 0.998 for $N \equiv 0, 2, 4 \pmod 6$, with standard deviations 0.013, 0.020 and 0.019 (Figure 8). The offsets are real, and they are Chebyshev's bias again. The Hardy–Littlewood main term is natural for prime powers p^k counted with weight $1/k$, and the squares of primes are all $\equiv 1 \pmod 3$: a representation $N = p^2 + q$ with $q > 3$ prime exists only for $N \equiv 2$ or $0 \pmod 6$, never for $N \equiv 4$. Counting primes only therefore leaves the class 4 almost untouched and removes a part of order $N^{-1/2}$ from the classes 2 and 0 (twice as much from the class 2): the mean for $N \equiv 2$ is 0.976, 0.985, 0.988 on $(2 \cdot 10^4, 6 \cdot 10^4]$, $(6 \cdot 10^4, 1.2 \cdot 10^5]$, $(1.2 \cdot 10^5, 2 \cdot 10^5]$. When prime powers are counted with weight $1/k$, the three means become 0.9987, 0.9999 and 1.0002 (`goldbach_powers_10nov.py`). That the multiple of 6 has the most representations among $6m, 6m+2, 6m+4$ holds in 33 322 of the 33 325 triples with $48 \le 6m < 2 \cdot 10^5$, but not always. In the sliding window $(80078, 80080, 80082)$ (the aligned triple is $80076, 80078, 80080$), $r = 1005, 2012, 2010$, and $80080 \equiv 4 \pmod 6$ wins. This is because $80080 = 2^4 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ has singular series 1.94, against 2.00 for $80082 = 2 \cdot 3^3 \cdot 1483$. There are three such events below $2 \cdot 10^5$, near 80080, 170170 and 190190 ($170170 = 2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$ and $190190 = 2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 19$); they make seven of the sliding windows of three consecutive even numbers exceptional. Goldbach's conjecture itself is verified to $4 \cdot 10^{18}$ [OHP] and is open.

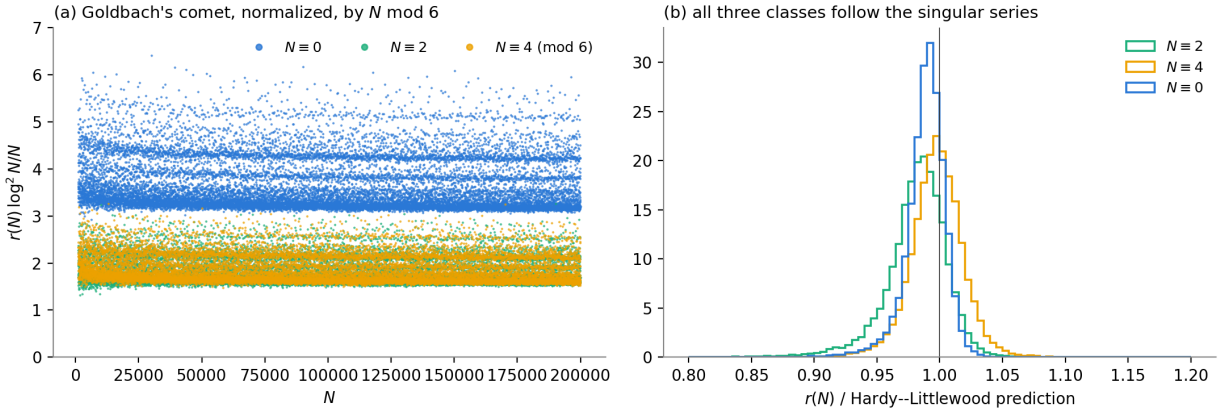


FIGURE 8. Goldbach representations for even $N \le 2 \cdot 10^5$. (a) $r(N)\log^2 N/N$ by N mod 6: the multiples of 6 form a band at twice the height. (b) $r(N)$ divided by the Hardy–Littlewood prediction, by class.

10. Even powers of primes

Every prime $p \ge 5$ satisfies $p^2 \equiv 1 \pmod{24}$. The table of [Ca20, Idea 32] lists the largest such modulus for each even exponent: $24, 240, 504, 480, 264, 65520, 24, 16320, \ldots$

Theorem 10.1. *For $n \ge 1$, let $g_n = \gcd\{p^{2n} - 1 : \ p \text{ prime}, \ p > 2n+1\}$. Then*

$$g_n = 2^{v_2(2n)+2} \prod_{\substack{\ell \text{ odd prime} \\ (\ell-1) \mid 2n}} \ell^{\,v_\ell(2n)+1}.$$

Proof. Fix a prime power ℓ^k. By Dirichlet's theorem, the primes $p > 2n+1$ meet every residue class mod ℓ^k prime to ℓ. So $\ell^k \mid g_n$ if and only if $a^{2n} \equiv 1 \pmod{\ell^k}$ for every unit a, that is, if and

only if the exponent of $(\mathbb{Z}/\ell^k)^\times$ divides $2n$. This requires $\ell \neq p$ for all the p involved; that holds whenever the condition below is met, since then $\ell \leq 2n+1$.

For odd ℓ the group is cyclic of order $\ell^{k-1}(\ell-1)$, so the condition is $(\ell-1) \mid 2n$ and $k-1 \leq v_\ell(2n)$. If $(\ell-1) \nmid 2n$, not even $\ell \mid g_n$.

For $\ell = 2$ and $k \geq 3$, $(\mathbb{Z}/2^k)^\times \cong C_2 \times C_{2^{k-2}}$ has exponent 2^{k-2}, so the condition is $k \leq v_2(2n)+2$; for $k \leq 2$ it always holds. □

TABLE 5. g_n of Theorem 10.1 for $n \leq 12$. The direct gcd over the first 400 admissible primes, the formula, and the denominator of $B_{2n}/(4n)$ agree in every case.

$n=1$	2	3	4	5	6	7	8	9	10	11	12
24	240	504	480	264	65520	24	16320	28728	13200	552	131040

The formula is the denominator of $B_{2n}/(4n)$, where B_{2n} are the Bernoulli numbers. This is a classical consequence of the von Staudt–Clausen theorem and its refinement for $B_{2n}/2n$ [IR, Ad]. Equivalently, it is the denominator of $\frac{1}{2}\zeta(1-2n) = -B_{2n}/(4n)$, the order of the image of the J-homomorphism in Adams' work [Ad]. The case $n=1$ is $p^2 \equiv 1 \pmod{24}$, and $n=6$ gives $p^{12} \equiv 1 \pmod{65520}$ for $p > 13$.

11. The cross-energy as a two-way detector: the zeros return the two lanes

This section is new in version 2 (2 November 2026). It connects the sieve structure of Section 2 with the zeros of ζ and of $L(s,\chi_{-3})$ through the cross-energy $\mathcal{C}_2$ of Part 1 [CE1]. Part 1 uses $\mathcal{C}_2$ to detect zeros by running over the integers. Here the roles are reversed: averaged over the zeros, $\mathcal{C}_2$ returns $\log N$, and a Möbius inversion returns each prime of each lane.

11.1. **The three forms as a Dirichlet series.** Let χ_6 be the character modulo 6 induced by χ_{-3} ($\chi_6(n) = 1, -1, 0$ for $n \equiv 1, 5$ and otherwise), so that $L(s,\chi_6) = (1+2^{-s})L(s,\chi_{-3})$, and let $\zeta_6(s) = (1-2^{-s})(1-3^{-s})\zeta(s)$. Write $r_+(k)$ for the number of ordered pairs (x,y), $x,y \geq 1$, with $k = 6xy+x+y$ or $k = 6xy-x-y$, and $r_-(k)$ for the number with $k = 6xy+x-y$.

Proposition 11.1 (the forms as divisor counts)**.** *For $k \geq 1$,*

$$r_+(k) = d(6k+1) - 2, \qquad r_-(k) = \tfrac{1}{2}\,d(6k-1) - 1,$$

and

$$\sum_{n \equiv \pm 1 \,(6)} \frac{d(n)}{n^s} = \frac{1}{2}\Big[\zeta_6(s)^2 \pm L(s,\chi_6)^2\Big] \qquad (\operatorname{Re} s > 1).$$

In particular $6k+1$ is prime if and only if $r_+(k) = 0$, and $6k-1$ is prime if and only if $r_-(k) = 0$.

Proof. By Proposition 2.2, $r_+(k)$ counts the ordered factorizations $6k+1 = ab$ with $a,b > 1$ and $a \equiv b \pmod 6$. Every factorization of a number $\equiv 1 \pmod 6$ has $a \equiv b$, so $r_+(k) = d(6k+1) - 2$. For $n = 6k-1$ the divisors split equally between the classes 1 and 5, since $d \mapsto n/d$ exchanges them. The factorizations $(6x+1)(6y-1)$ with $x,y \geq 1$ correspond to the divisors $\equiv 1$ other than 1, which gives $\frac{1}{2}d(n)-1$. For the series, $\sum_{(n,6)=1} d(n)n^{-s} = \zeta_6(s)^2$ and $\sum \chi_6(n)d(n)n^{-s} = L(s,\chi_6)^2$, because χ_6 is completely multiplicative. Half the sum and half the difference select the two classes. □

Status: elementary; checked for $k \leq 20\,000$. The "operator" behind the forms $6xy \pm x \pm y$ is multiplication in the monoid of integers prime to 6, graded by χ_6: the rule $(+)(+) = (-)(-) = +$, $(+)(-) = -$ is exactly the factor $L(s,\chi_6)^2$. The diagonal $x = y$ of the first two forms consists of the squares $(6x \pm 1)^2$. It falls only in the lane $6k+1$, and it is the source of the bias of Proposition 4.2.

11.2. **Duality.** For $N \geq 2$ and $t \in \mathbb{R}$ let

$$\mathcal{C}_2(N, \tfrac{1}{2}, t) = \frac{2}{\sqrt{N}} \sum_{j<N} \frac{\cos\big(t \log(N/j)\big)}{\sqrt{j}} = |X_N(\tfrac{1}{2} + it)|^2 - |X_{N-1}(\tfrac{1}{2} + it)|^2 - \frac{1}{N},$$

the cross-energy of Part 1 [CE1] on the critical line. (Part 1 writes it C_2; we write $\mathcal{C}_2$ because C_2 already denotes a sum in Corollary 6.2.)

Theorem 11.2 (the zeros return the lanes)**.** *Assume RH for ζ and GRH for $L(s, \chi_{-3})$. For every fixed $N \geq 2$, with γ the ordinates of the zeros of ζ and γ_χ those of $L(s, \chi_{-3})$,*

$$A_\zeta(N) := -\lim_{T\to\infty} \frac{\pi}{T} \sum_{0<\gamma\leq T} \mathcal{C}_2(N, \tfrac{1}{2}, \gamma) = \frac{\log N}{N},$$

$$A_L(N) := -\lim_{T\to\infty} \frac{\pi}{T} \sum_{0<\gamma_\chi\leq T} \mathcal{C}_2(N, \tfrac{1}{2}, \gamma_\chi) = \frac{1}{N} \sum_{d|N} \chi_{-3}(d)\Lambda(d).$$

Consequently, for every N,

$$\Lambda(N)\, \frac{1 \pm \chi_{-3}(N)}{2} = \sum_{d|N} \mu\Big(\frac{N}{d}\Big)\, d\, \frac{A_\zeta(d) \pm A_L(d)}{2}.$$

Proof. For fixed $x > 1$, Landau's formula [Lan] and its analogue for $L(s, \chi)$ with χ real give

$$\sum_{0<\gamma\leq T} x^\rho = -\frac{T}{2\pi}\Lambda(x) + O_x(\log T), \qquad \sum_{0<\gamma_\chi\leq T} x^{\rho_\chi} = -\frac{T}{2\pi}\chi(x)\Lambda(x) + O_x(\log T),$$

with $\Lambda(x) = 0$ unless x is a prime power. Under RH and GRH, $x^\rho = \sqrt{x}\, x^{i\gamma}$, and taking real parts, $\sum_{0<\gamma\leq T} \cos(\gamma \log x) = -\frac{T}{2\pi}\Lambda(x)/\sqrt{x} + O_x(\log T)$. Insert $x = N/j$ for each $j < N$. Only the j dividing N survive. With $d = N/j$,

$$\sum_{0<\gamma\leq T} \mathcal{C}_2(N, \tfrac{1}{2}, \gamma) = \frac{2}{\sqrt{N}} \sum_{d|N,\, d>1} \sqrt{\frac{d}{N}}\Big(-\frac{T}{2\pi}\frac{\Lambda(d)}{\sqrt{d}}\Big) + O_N(\log T) = -\frac{T}{\pi N} \sum_{d|N} \Lambda(d) + O_N(\log T),$$

and $\sum_{d|N} \Lambda(d) = \log N$. The same computation with $\chi(d)\Lambda(d)$ gives A_L. For the last identity, $N(A_\zeta \pm A_L)/2 = \sum_{d|N} \Lambda(d) \frac{1\pm\chi(d)}{2}$ is the Dirichlet convolution of $\Lambda \frac{1\pm\chi}{2}$ with 1, and Möbius inversion undoes it. □

Status: proved under RH and GRH. Without them, the same proof gives the identity with $\mathcal{C}_2(N, \frac{1}{2}, \gamma)$ replaced by $\frac{2}{\sqrt{N}} \sum_j j^{-1/2} \operatorname{Re}(N/j)^{\rho-1/2}$. The two versions differ only through zeros off the line. The theorem is Landau's formula read through $\mathcal{C}_2$. What it adds is the reading: the same function $\mathcal{C}_2$ detects zeros as N runs over the integers (Part 1), and returns $\log N$ as γ runs over the zeros. The lane of a prime is decided only by which family of zeros is used.

Numerical check. We computed the zeros directly (Euler–Maclaurin and Hardy's Z-function). The counts below 2000 close against the Riemann–von Mangoldt formulas, and the first zero of $L(s, \chi_{-3})$ is at 8.0397. For $N = 2, 10, 30, 60$, $A_\zeta(N) = 0.3472,\ 0.2290,\ 0.1116,\ 0.0660$, against $\log N/N = 0.3466,\ 0.2303,\ 0.1134,\ 0.0682$. Table 6 shows the two detectors. Every prime $5 \leq p \leq 79$ appears in its lane and not in the other. The composites $35, 55, 77$ give values near 0. The squares $25 = 5^2$ and $49 = 7^2$ give $\log 5$ and $\log 7$ in the lane $6k + 1$, as they must, since $\Lambda(p^2) = \log p$. These are the diagonal terms $x = y$ of Proposition 11.1, the same squares that cause the bias. The largest error up to $N = 79$ is 0.57 (at $N = 67$). The error term is $O_N(\log T)/T$ with a constant that grows with N, so the table degrades slowly with N at fixed T.

TABLE 6. The two detectors of Theorem 11.2 with the 1517 zeros of ζ and the 1865 zeros of $L(s, \chi_{-3})$ below $T = 2000$; the last column is the value of $\Lambda(N)$ in its lane.

N	lane	detector $6k+1$	detector $6k-1$	expected
5	$-$	-0.002	1.600	$\log 5$
7	$+$	1.928	-0.013	$\log 7$
11	$-$	-0.030	2.368	$\log 11$
13	$+$	2.518	-0.020	$\log 13$
25	$+$	1.636	-0.076	$\log 5$
29	$-$	0.002	3.220	$\log 29$
35	$-$	-0.046	-0.083	0
43	$+$	3.652	0.001	$\log 43$
49	$+$	1.834	-0.155	$\log 7$
55	$+$	0.172	-0.331	0
59	$-$	-0.284	4.000	$\log 59$
77	$-$	0.061	-0.191	0
79	$+$	4.133	-0.285	$\log 79$

The densities as waves. Differentiating the explicit formula gives, for the density of each lane,

$$\frac{d}{dx}\psi(x; 6, \pm 1) = \frac{1}{2}\Big[1 - \frac{2}{\sqrt{x}}\sum_{\gamma>0}\cos(\gamma \log x) \mp \frac{2}{\sqrt{x}}\sum_{\gamma_\chi>0}\cos(\gamma_\chi \log x)\Big] + \dots$$

in the sense of distributions, where the dots are the contributions of the trivial zeros and of the powers of 2 and 3. Figure 9 shows the truncation at height 400 with Fejér weights $1-\gamma/400$. In the lane $6k-1$ every prime up to 200 has a peak, and no composite does. In the lane $6k+1$ the only peaks at composites are at 25, 49, 121 and 169, the squares of primes. The zeros of ζ contribute equally to both lanes. Only the zeros of $L(s, \chi_{-3})$ separate them, with opposite signs.

11.3. **The duality at finite height: an exact explicit formula for the cross-energy.** This subsection is new in version 3 (2 November 2026). Theorem 11.2 needs two things: a limit $T \to \infty$, and RH and GRH to turn x^ρ into a cosine. Here we remove both. A Gaussian weight on the zeros gives an exact identity at every N and every resolution. Its four terms (zeros, pole, Gamma factor and primes) can be computed separately, and each has a meaning in terms of $\mathcal{C}_2$. We first state the identity, then explain how the cross-energy hears the primes through the zeros.

On the line, $\mathcal{C}_2$ is the restriction of the reflected cross-energy of Part 3 [CE3],

$$\mathcal{C}_2^{\mathrm{FR}}(N, s) = N^{-s}X_{N-1}(1-s) + N^{s-1}X_{N-1}(s), \qquad \mathcal{C}_2^{\mathrm{FR}}(N, \tfrac{1}{2}+it) = \mathcal{C}_2(N, \tfrac{1}{2}, t),$$

an entire function of s, symmetric under $s \leftrightarrow 1-s$. For a zero ρ write $t_\rho = (\rho - \frac{1}{2})/i$. In this subsection $\sum_\rho$ runs over all nontrivial zeros, of both signs of the ordinate; under RH, $t_\rho = \gamma_\rho$ is real and $\mathcal{C}_2^{\mathrm{FR}}(N, \rho) = \mathcal{C}_2(N, \frac{1}{2}, \gamma_\rho)$.

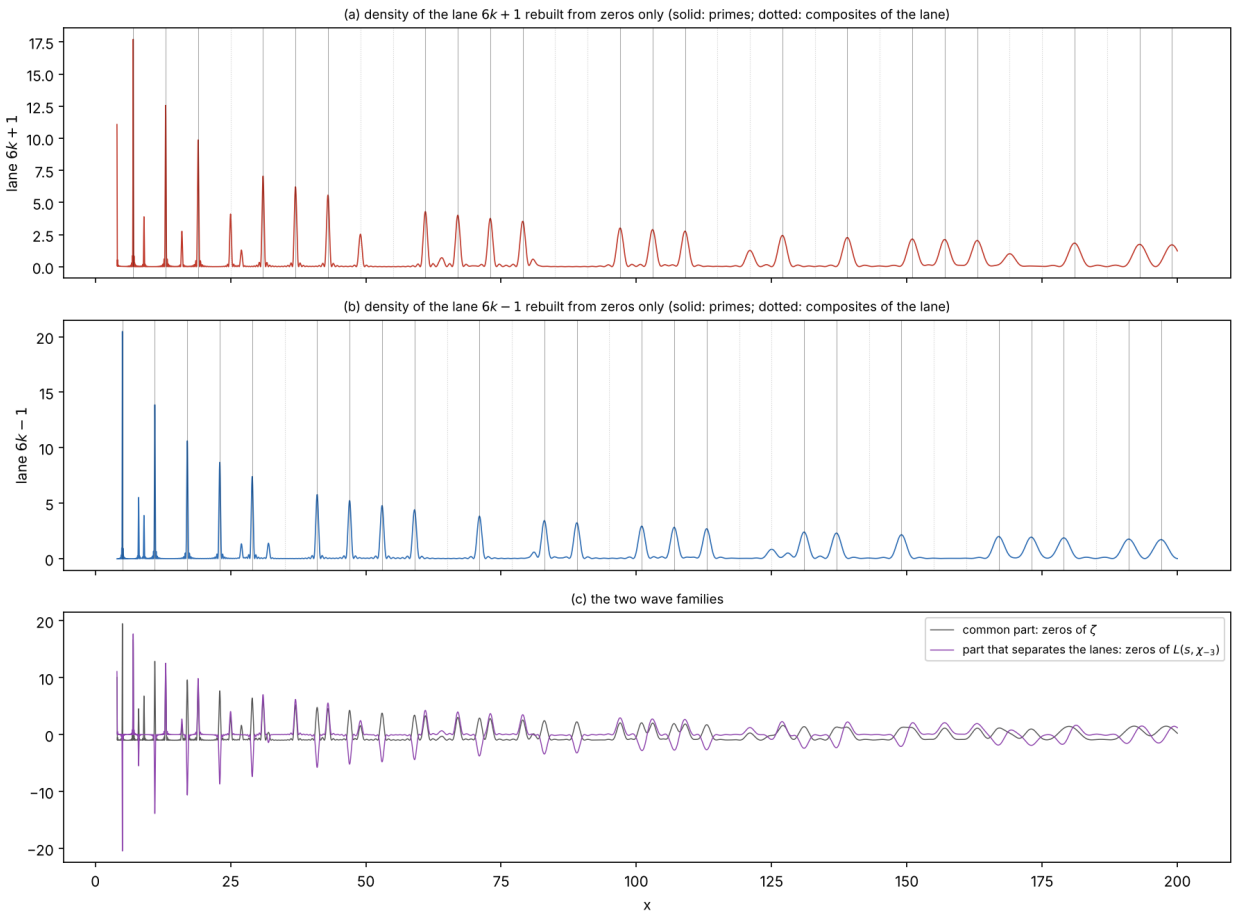


FIGURE 9. (a), (b) The prime densities of the two lanes rebuilt from the 201 zeros of ζ and the 271 zeros of $L(s,\chi_{-3})$ below 400, with Fejér weights; vertical lines at the integers of each lane (solid: primes, dotted: composites). (c) The two families of waves: the zeros of ζ (common to both lanes) and the zeros of $L(s,\chi_{-3})$ (opposite signs in the two lanes).

Proposition 11.3 (the explicit formula for the cross-energy)**.** *For every integer $N \geq 2$ and every $\varepsilon > 0$, unconditionally,*

$$\underbrace{\sum_{\rho} e^{-\varepsilon^2 t_\rho^2}\, \mathcal{C}_2^{\mathrm{FR}}(N,\rho)}_{zeros} = \underbrace{2e^{\varepsilon^2/4}\Big(H_{N-1} + 1 - \frac{1}{N}\Big)}_{pole} + \underbrace{\frac{1}{2\pi}\int_{\mathbb{R}} e^{-\varepsilon^2 r^2}\, \mathcal{C}_2(N,\tfrac12,r)\Big(\operatorname{Re}\frac{\Gamma'}{\Gamma}\big(\tfrac14+\tfrac{ir}{2}\big) - \log\pi\Big)dr}_{Gamma} - \underbrace{\frac{1}{\sqrt{\pi}\,\varepsilon\sqrt{N}}\sum_{n\geq 2}\sum_{j<N}\frac{\Lambda(n)}{\sqrt{nj}}\Big(e^{-\log^2(nj/N)/4\varepsilon^2} + e^{-\log^2(nN/j)/4\varepsilon^2}\Big)}_{primes},$$

where $H_{N-1} = \sum_{k<N} 1/k$. *Consequently,*

$$A_\varepsilon(N) := \sqrt{\pi}\,\varepsilon N \Big[\mathit{pole} + \mathit{Gamma} - \mathit{zeros}\Big] = \log N + R_\varepsilon(N),$$

with

$$0 \le R_\varepsilon(N) \le \sqrt{N} \sum_{m\ge 2,\ m\ne N} \frac{\log m}{\sqrt{m}}\, e^{-\log^2(m/N)/4\varepsilon^2} + 2N \sum_{n\ge 2} \frac{\Lambda(n)}{\sqrt{n}}\, e^{-\log^2 n/4\varepsilon^2} \ \longrightarrow\ 0 \qquad (\varepsilon \to 0),$$

and $\sum_{d|N} \mu(N/d)\, A_\varepsilon(d) = \Lambda(N) + \sum_{d|N} \mu(N/d) R_\varepsilon(d)$.

Proof. Put $h(r) = e^{-\varepsilon^2 r^2}\, \mathcal{C}_2^{\mathrm{FR}}(N, \frac12 + ir)$. It is entire and even, and it has Gaussian decay in every horizontal strip, because $\mathcal{C}_2^{\mathrm{FR}}(N,\cdot)$ is a finite exponential sum. So Weil's explicit formula applies [CE3]:

$$\sum_\rho h(t_\rho) = h(\tfrac{i}{2}) + h(-\tfrac{i}{2}) + \frac{1}{2\pi}\int h(r)\Big(\operatorname{Re}\frac{\Gamma'}{\Gamma}\big(\tfrac14 + \tfrac{ir}{2}\big) - \log\pi\Big)dr - 2\sum_{n\ge 2}\frac{\Lambda(n)}{\sqrt{n}}\, g(\log n),$$

with $g(x) = \frac{1}{2\pi}\int h(r) e^{-irx} dr$. We compute the three terms on the right.

Pole. $h(\pm\frac{i}{2}) = e^{\varepsilon^2/4}\, \mathcal{C}_2^{\mathrm{FR}}(N,0)$ by the symmetry $s \leftrightarrow 1-s$, and $\mathcal{C}_2^{\mathrm{FR}}(N,0) = X_{N-1}(1) + N^{-1} X_{N-1}(0) = H_{N-1} + \frac{N-1}{N}$.

Gamma. On the line $\mathcal{C}_2^{\mathrm{FR}} = \mathcal{C}_2$, which gives the integral as stated.

Primes. On the line, $\mathcal{C}_2(N, \frac12, r) = \frac{2}{\sqrt{N}} \sum_{j<N} j^{-1/2} \cos(r\ell_j)$, with $\ell_j = \log(N/j)$. Since

$$\frac{1}{2\pi}\int e^{-\varepsilon^2 r^2} e^{irs}\, dr = \frac{1}{2\sqrt{\pi}\,\varepsilon}\, e^{-s^2/4\varepsilon^2},$$

we get

$$g(x) = \frac{1}{2\sqrt{\pi}\,\varepsilon\sqrt{N}} \sum_{j<N} \frac{e^{-(x-\ell_j)^2/4\varepsilon^2} + e^{-(x+\ell_j)^2/4\varepsilon^2}}{\sqrt{j}}.$$

At $x = \log n$ we have $x - \ell_j = \log(nj/N)$ and $x + \ell_j = \log(nN/j)$; multiplying by $2\Lambda(n)/\sqrt{n}$ and summing over n gives the prime term.

The consequence. Multiply by $\sqrt{\pi}\,\varepsilon N$. In the prime term, the exact coincidences $nj = N$ (so $n = d$ is a divisor of N, $d > 1$, $j = N/d$) have $\sqrt{nj} = \sqrt{N}$, and together they give exactly $\sum_{d|N} \Lambda(d) = \log N$. All other terms are nonnegative, so $R_\varepsilon(N) \ge 0$. For the upper bound, note that $\sqrt{\pi}\,\varepsilon N$ times the prime term is $\sqrt{N}\sum_{n,j} \Lambda(n)(nj)^{-1/2}(\dots)$. In the first family, group the terms with $nj \ne N$ by $m = nj$ and use $\sum_{nj=m} \Lambda(n) \le \sum_{d|m} \Lambda(d) = \log m$. In the second family, $j < N$ gives $\log(nN/j) > \log n$, and $\sum_{j<N} j^{-1/2} \le 2\sqrt{N}$. Both sums tend to 0 as $\varepsilon \to 0$ by dominated convergence, since every exponent is bounded away from 0 (by $\log^2(1 + \frac{1}{N})$ in the first family and by $\log^2 2$ in the second). Möbius inversion of $\log = \Lambda * 1$ gives the last statement. □

Status: proved, unconditionally. The pole term is the cross-energy itself at $s = 0$, and the main term $\log N$ is exact and positive: the error $R_\varepsilon(N)$ only adds. Under RH the left side is $\sum_\rho e^{-\varepsilon^2\gamma^2} \mathcal{C}_2(N, \frac12, \gamma)$, a sum of real numbers. The same proof with the explicit formula of $L(s, \chi_{-3})$ (Gamma factor $\Gamma(\frac{s+1}{2})$, conductor 3, no pole) gives the lane version, with $\chi_{-3}(n)\Lambda(n)$ in the prime term and $\sum_{d|N} \chi_{-3}(d)\Lambda(d)$ in place of $\log N$.

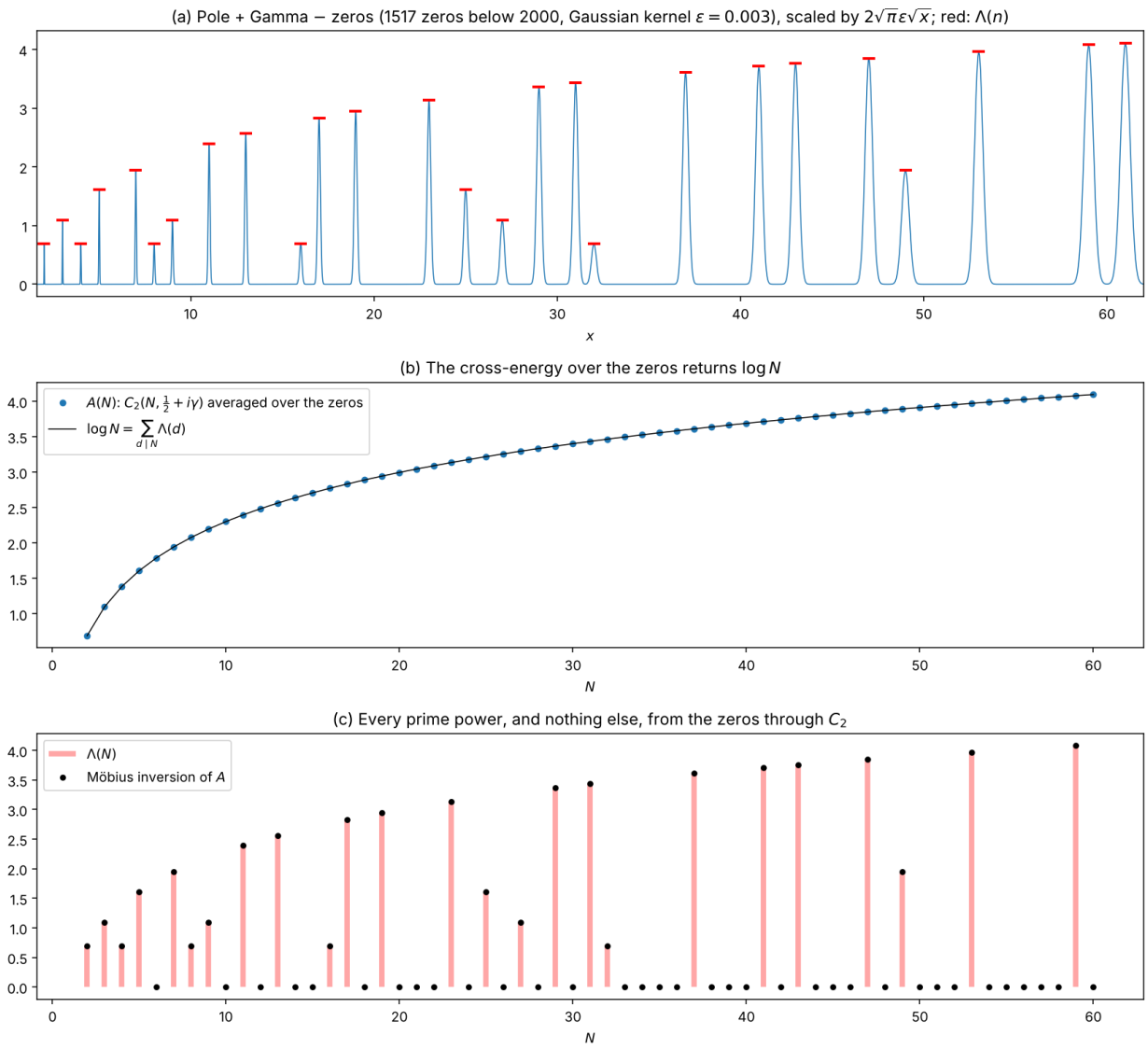


FIGURE 10. The primes from the 1517 zeros of ζ below 2000, $\varepsilon = 0.003$. (a) The single-frequency form of Proposition 11.3, pole + Gamma − zeros at $u = \log x$, scaled by $2\sqrt{\pi}\,\varepsilon\sqrt{x}$: a peak of height $\Lambda(n)$ (red bars) at every prime power $n \le 61$ and nothing elsewhere. This is Cramér's function [Cr] with a Gaussian in place of Abel's weight. (b) $A_\varepsilon(N)$ against $\log N$, $2 \le N \le 60$. (c) Möbius inversion of A_ε against $\Lambda(N)$.

How the cross-energy hears the primes through the zeros. We now read Proposition 11.3 term by term, with the numbers of Figure 11, and use $N = 12$ as a running example.

1. *A chord of $N-1$ intervals.* On the line,

$$\mathcal{C}_2(N, \tfrac{1}{2}, t) = \frac{2}{\sqrt{N}} \sum_{j<N} \frac{\cos\left(t \log \frac{N}{j}\right)}{\sqrt{j}}$$

is the interaction energy of the newest term N^{-s} of the partial sum with all the previous ones. Each earlier integer j enters through the interval N/j, with frequency $\log(N/j)$ and volume $j^{-1/2}$.

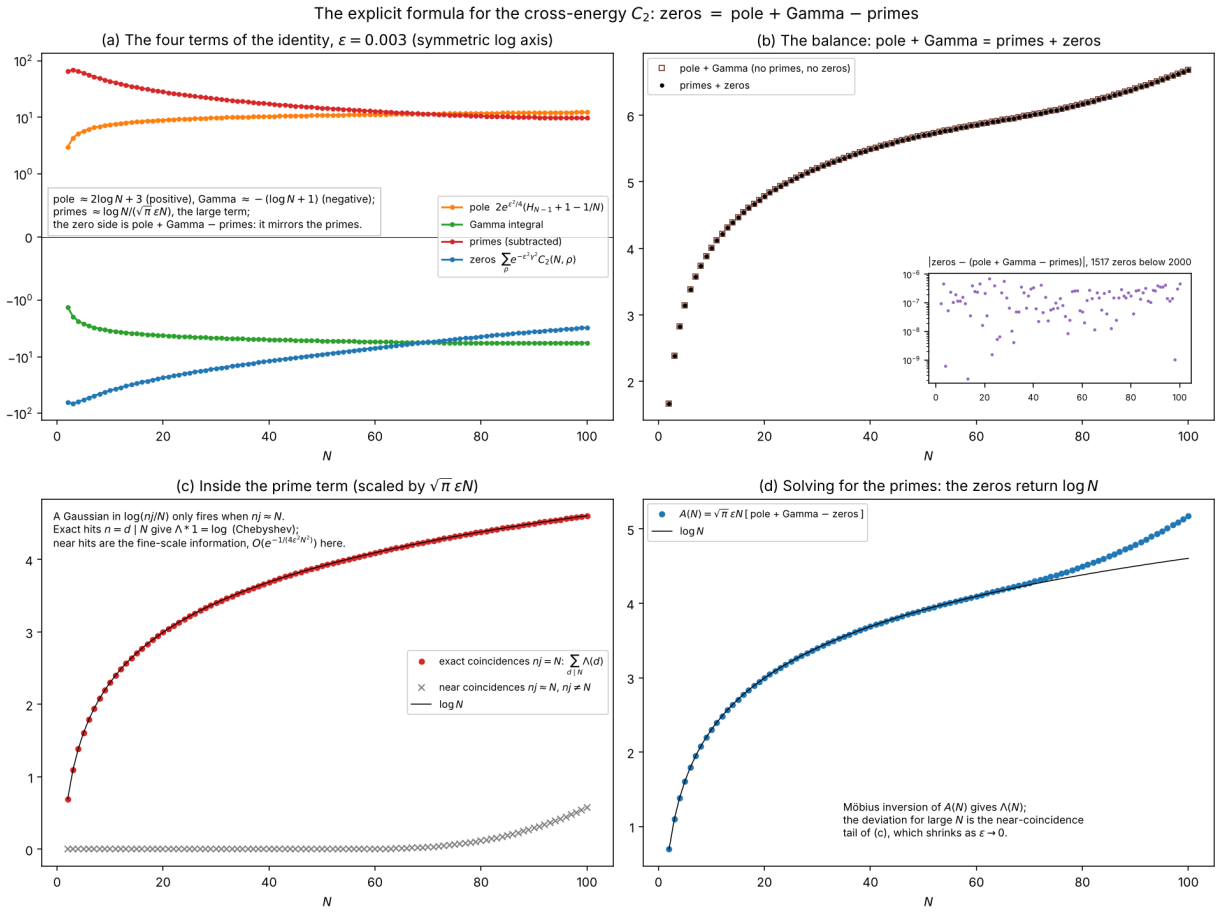


FIGURE 11. The four terms of Proposition 11.3 for $2 \le N \le 100$, $\varepsilon = 0.003$, computed independently (zeros: the 1517 zeros below 2000; Gamma: quadrature; primes: the double sum). (a) The four terms, symmetric logarithmic axis. (b) The balance pole + Gamma = primes + zeros; inset: the residual, below $7.3 \cdot 10^{-7}$. (c) The prime term split into exact coincidences $nj = N$, which give $\log N$, and near coincidences $nj \approx N$, $nj \neq N$, which give $R_\varepsilon(N)$. (d) $A_\varepsilon(N)$ against $\log N$.

For $N = 12$ the intervals are $12/1, 12/2, \ldots, 12/11$. Five of them are integers $(12, 6, 4, 3, 2)$; the others $(12/5,\ 12/7,\ \ldots)$ are not.

2. Two faces. With t fixed and N running, $\mathcal{C}_2$ is the zero detector of Part 1: by Theorem A of [CE1], it is a smooth part plus the wave $2\,\mathrm{Re}(\zeta(s)N^{-\bar{s}})$, which carries the factor $\zeta(s)$ and vanishes exactly at the zeros. With N fixed and t running, it is a chord, and Proposition 11.3 plays that chord at every zero.

3. Silence below, music above. At a zero the wave is absent, and Part 3 [CE3] gives $\mathcal{C}_2^{\mathrm{FR}}(N, \rho) = \mathcal{G}_N^*(\lambda_\rho)$, $\lambda_\rho = \rho(1-\rho)$. As long as N is large compared with the height, this is the smooth value $\frac{1}{\lambda} - \frac{1}{N} + O(N^{-2})$, and nothing oscillates. When the height passes the threshold of the approximate functional equation, $\gamma \approx 2\pi N$, the value oscillates with amplitude of order $N^{-1/2}$ (Table 7). For $N = 12$, at the first zero $\mathcal{C}_2(12, \frac{1}{2}, \gamma_1) = -0.079$, equal to $\frac{1}{\lambda_1} - \frac{1}{12}$ to $7 \cdot 10^{-4}$. At the 1499 zeros between 75 and 2000 its root mean square is 0.52. The low zeros contribute only their smooth value; the arithmetic is carried by the oscillation at the high ones.

TABLE 7. The cross-energy at the zeros of ζ below 2000, in three ranges of height: the largest deviation from the smooth value $\frac{1}{\lambda} - \frac{1}{N}$ and the root mean square of $\mathcal{C}_2(N, \frac{1}{2}, \gamma)$.

N	range of γ	zeros	$\max \lvert\mathcal{C}_2 - (\frac{1}{\lambda} - \frac{1}{N})\rvert$	r.m.s. of $\mathcal{C}_2$
12	$[0,\ \pi N/2) = [0, 19)$	1	$7.1 \cdot 10^{-4}$	0.079
	$[19,\ 75)$	17	0.131	0.114
	$[75,\ 2000)$	1499	2.05	0.517
60	$[0,\ 94)$	26	$3.3 \cdot 10^{-5}$	0.016
	$[94,\ 377)$	161	0.088	0.028
	$[377,\ 2000)$	1330	0.646	0.216

4. Why the zeros pick the divisors. Summed over many zeros, $\cos(\gamma \log x)$ cancels for almost every x. The exception is a prime power $x = p^m$, where the zeros add up to $-\Lambda(x)/\sqrt{x}$ per unit of density: this is Landau's formula, and it is how the zeros together know the primes. In $\mathcal{C}_2$ the frequencies are $x = N/j$, and N/j is an integer exactly when $d = N/j$ divides N. Of the divisors, only the prime powers resonate. For $N = 12$ these are 2, 3 and $4 = 2^2$; the intervals 6 and 12 are integers but not prime powers, and the fractional intervals resonate with nothing. Then the weights fit: the divisor d resonates with amplitude $\Lambda(d)/\sqrt{d}$, and $\mathcal{C}_2$ gives it the volume $j^{-1/2} = \sqrt{d/N}$, so

$$\frac{\Lambda(d)}{\sqrt{d}} \cdot \sqrt{\frac{d}{N}} = \frac{\Lambda(d)}{\sqrt{N}}.$$

The factor $\sqrt{d}$ cancels exactly, so every divisor counts by its Λ alone. The weights $j^{-1/2}$ in the definition of $\mathcal{C}_2$ are the ones that make this happen. Summing, $\Lambda(2) + \Lambda(3) + \Lambda(4) = \log 2 + \log 3 + \log 2 = \log 12$, and in general $\sum_{d|N} \Lambda(d) = \log N$: Chebyshev's identity $\log = \Lambda * 1$ is built into the cross-energy.

5. The four terms at $N = 12$. With $\varepsilon = 0.003$,

$$\underbrace{-34.724}_{\text{zeros}} = \underbrace{7.873}_{\text{pole}}\ \underbrace{-3.654}_{\text{Gamma}}\ \underbrace{-38.943}_{\text{primes}},$$

with a residual of 10^{-7}.

- *Pole.* $2e^{\varepsilon^2/4}(H_{11} + 1 - \frac{1}{12}) = 2(3.0199 + 0.9167)$. It comes from the pole of ζ at $s = 1$, the source of the main term x in $\psi(x) = x - \ldots$. In terms of the cross-energy it is $\mathcal{C}_2^{\mathrm{FR}}(12, 0) = X_{11}(1) + \frac{1}{12} X_{11}(0)$: the integers below 12 seen from 12 with all phases removed. It is the average harmony of the integers and contains no arithmetic.
- *Gamma.* It comes from the factor $\pi^{-s/2}\Gamma(s/2)$ of the functional equation, the place at infinity in Weil's language. It is a smooth integral, with no arithmetic, and it corrects for the growth $\frac{1}{2\pi} \log \frac{\gamma}{2\pi}$ of the mean density of zeros. Over $2 \le N \le 100$ it behaves like $-(\log N + 1)$, while the pole behaves like $2 \log N + 3$.
- *Primes.* Exactly $\log 12/(\sqrt{\pi}\,\varepsilon \cdot 12) = 2.4849/0.06381 = 38.94$: the three resonant divisors. The factor $1/(\sqrt{\pi}\,\varepsilon)$ is the height of a Gaussian peak; the finer the resolution, the taller and narrower the peak.
- *Zeros.* What is left: the chord of 12 played at the 1517 zeros. It is large and negative because it mirrors the prime term. The zeros carry the information of the primes with the opposite sign, over the common background of pole and Gamma.

6. The resolution and its cost. The parameter ε does two things at once. On the zeros, $e^{-\varepsilon^2\gamma^2}$ switches off the zeros above about $6/\varepsilon$ (here 2000, which is why 1517 zeros suffice). On the primes, it turns each resonance into a peak of width about ε on the logarithmic scale. The weight must

be Gaussian, or at least analytic: Cramér's weight $e^{-|\gamma|y}$ is not, and with it the pole and Gamma terms do not separate cleanly. To tell $\log N$ from $\log(N+1)$ one needs $\varepsilon \lesssim 1/(4N)$, hence zeros up to height about $24N$. Seeing the primes up to x through the zeros costs of the order of x zeros. The duality is exact, but it is not cheap.

7. From $\log N$ *to* $\Lambda(N)$. $A_\varepsilon(N)$ returns the sum over all divisors, and Möbius inversion keeps N alone: for $N = 12$, $\log 12 - \log 6 - \log 4 + \log 2 = 0$; for $N = 8$, $\log 8 - \log 4 = \log 2$; for $N = 7$, $\log 7$. With $\varepsilon = 0.003$ and $N \le 60$ every prime power appears with its value and every other integer gives 0, to $3.7 \cdot 10^{-3}$ (Figure 10).

8. Near coincidences: where the difficulty lives. Up to $N \approx 70$ the fractional intervals N/j fall far from every prime, and their peaks touch nothing: $R_\varepsilon(N)$ is negligible. Beyond that, some N/j comes close to a prime n without equalling it ($nj \approx N$, $nj \neq N$), and the Gaussian fires a little. This is the upward drift of A_ε in Figure 11(d), 0.58 at $N = 100$. These near coincidences carry the fine-scale information of the primes: how they sit against the fractions of integers. Part 3 [CE3] shows that the positivity of Weil's form at the windows of the zeros lives in exactly these terms. Controlling them requires information of the strength of pair correlation.

9. What is classical and what is the cross-energy's. The explicit formula, Cramér's function, Landau's formula, Chebyshev's identity and Möbius inversion are classical. What $\mathcal{C}_2$ contributes is the double reading. The same object detects the zeros as N runs over the integers, because its wave vanishes there, and returns $\log N$ as it runs over the zeros. The weights $j^{-1/2}$ of its definition are exactly those that make the divisor sum come out, and its pole term is the cross-energy itself at $s = 0$. The sum is linear and holomorphic in the zeros, so an off-line zero does not break it, and it does not see β. Seeing β requires the sign at the filter or the growth of a mean square (Part 3, Section 8).

12. What is proved, what is conditional, what is open

Proved:

- Lemma 2.1 and Proposition 2.2 (the sieve structure);
- Proposition 2.3 (Legendre's main term is off by the factor $2e^{-\gamma_E}$);
- Propositions 4.2 and 4.3 (the race and its variance);
- Proposition 5.1 and Theorem 5.2 (the margin of χ_{-3} and its criterion);
- Theorem 6.1 (the exact weighted race) and Corollary 6.2;
- Theorem 10.1;
- Proposition 11.1 (the three forms as a Dirichlet series);
- Proposition 11.3 (the explicit formula for the cross-energy; unconditional).

Proved by citation: Theorem 7.1 (Shiu). *Computed:* the four terms of Proposition 11.3 for $2 \le N \le 100$ (Figures 10 and 11, Table 7); Proposition 5.4, the transition statistics, the zeros of $L(s, \chi_{-3})$ below 400, the densities, and the twin-prime and Goldbach statistics.

Conditional: Theorem 11.2 (RH and GRH; the version with $x^{\rho-1/2}$ is unconditional); the conclusions of Theorem 6.1 under GRH; Theorem 4.4 (GRH and LI); the energy statements of Proposition 5.5 (GRH).

Open: GRH for $L(s, \chi_{-3})$, which is as hard as RH; the twin prime and Goldbach conjectures; Conjectures 8.1 and 9.2.

The $6k \pm 1$ structure organizes these questions clearly: the two series are the two characters modulo 3. It does not make any of them easier.

Data and code

All numbers and figures are produced by the Python scripts in the ancillary archive:

- `tanh_f.py`, `fig_tanh.py`: the smooth indicators, the energy (3.2) and Figure 1;

- `lfun.py`: $L(s,\chi_{-3})$ via the Hurwitz zeta function, derivatives by Cauchy's formula, and Z_χ;
- `zeros.py`: the zeros of $L(s,\chi_{-3})$;
- `margin_chi.py`: M_χ at all prime powers up to 2^{27};
- `bias.py`: V_χ and the density;
- `consec.py`: transitions, runs and the limiting density;
- `race_data.py`: the characterization check, Mertens' factor, the counts, twins, Goldbach and g_n; `fix_hl_10nov.py` recomputes the two Hardy–Littlewood integrals by Gauss–Legendre quadrature (the first version used a uniform trapezoid rule, which was 0.15% too high at 10^8 for twins and 0.5% too high for Goldbach) and redraws Figures 7 and 8;
- `figures.py`: the figures;
- (version 2, folder `v2_dual`) `ceros.py`: zeros of ζ and $L(s,\chi_{-3})$ below 2000; `dual.py`, `dualidad_C2.py`: Theorem 11.2 and Table 6; `reconstruir.py`: Figure 9;
- (version 3, folder `v3_explicit`) `cramer_C2.py`: Figure 10; `c2_explicit_terms.py`: Figure 11 and the numbers of Section 11.3; `c2_regimes.py`: Table 7.

All the scripts listed are in the archive; `c2_regimes.py` was rewritten on 10 November 2026 and reproduces Table 7 to the digits printed.

Use of AI assistance

The $6k\pm1$ framework and the questions are the author's [Ca20]. Claude (Anthropic) was used as a research and writing aid: to derive and check proofs, to write and run the computations, to search the literature, and to draft text and figures. The author checked the results and is responsible for the content.

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