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CROSS-ENERGY OF DIRICHLET PARTIAL SUMS. PART 16: HOW MANY POSITIVE DIRECTIONS DOES THE ARCHIMEDEAN PLACE HAVE? A COUNTING LAW WITH A BOUNDARY PHASE

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Abstract. Restrict the archimedean part of Weil's quadratic form, for ζ or for a primitive real Dirichlet L-function of conductor q and parity κ, to test functions supported in an interval of length a, and let $n_+(a)$ be the number of its positive directions, the directions that the pole and the primes must absorb. A Szegő-type argument gives $n_+(a) = t_0 a/\pi + o(a)$, where t_0 is the zero of ϑ'. We observe that the form is $\Lambda - \psi_\kappa(-\Delta)$ with ψ_κ a complete Bernstein function (proved) and, using the boundary phase of Kaleta, Kwaśnicki and Małecki, we state, as a conjecture, the asymptotic law

$$n_+(a) = \Big\lfloor \frac{t_0 a}{\pi} + \frac{2\varpi_\kappa(t_0)}{\pi} \Big\rfloor,$$

with an explicit phase ϖ_κ that depends only on the parity, the conductor entering only through t_0. We test it in seven cases, ζ and $\chi_{-3}, \chi_{-4}, \chi_5, \chi_{-7}, \chi_8, \chi_{-8}$. On the finest mesh the shifts of the jumps $m = 2, \dots, 12$ bracket the prediction $2\varpi_\kappa(t_0)/\pi$, which ranges from 0.18 to 0.59, to within 1 to $3 \cdot 10^{-3}$ for the characters and $3 \cdot 10^{-2}$ for ζ. A second, independent implementation extrapolates in the mesh on three to nine jumps per case; the extrapolated shifts lie above the prediction by 0.2 to $2.3 \cdot 10^{-4}$ in every case, a one-sided offset of the size of the extrapolation error, which we have checked to be a bias in one case only. The first jump is an exception, and for ζ the shift reaches its limit only after about ten jumps. A smoothed count removes the integer part and leaves "degree plus constant" with the boundary phase as constant; on a circle the constant is carried by the end points and cancels between an arc and its complement; and treating $-\log q$ as a degree, $q \leftrightarrow 1/q$ acts on the band edge as $t_0 \mapsto 4\pi^2/t_0$ to leading order, with ζ at the self-dual point $t_0 \approx 2\pi$. We record what the law implies for the signature of Weil's form and where it stands with respect to the Riemann–Roch formulas for $\overline{\operatorname{Spec}\mathbb{Z}}$. The law is a conjecture supported by these computations. The Bernstein structure is proved here; the first-order count is quoted from Part 14 and its proof is not repeated.

1. The question

For ζ the explicit formula writes Weil's quadratic form, on real test functions g, as

$$Q(g) = W(g) - E(g), \qquad W(g) = 2\widehat{g}(\tfrac{i}{2})\widehat{g}(-\tfrac{i}{2}),$$
$$E(g) = A(g) + \sum_n \frac{\Lambda(n)}{\sqrt{n}} \int g(x)\big(g(x+\log n) + g(x-\log n)\big)\,dx,$$

Date: November 2026 (note, v1.0); renumbered 2 November 2026; v1.1, 10 November 2026 (the law stated as a conjecture, the smooth constant identified with Widom's second term) (formerly a standalone note). Part 16 of the book-series *Cross-Energy of Dirichlet Partial Sums* (forty-seven parts, Parts 0–46; block III: The margin and the constant $\widehat{B}$). Part 0 [CE0] is the guide. In book version 1 (2 November 2026) this was Part 15; the present numbering is that of book version 2 (10 November 2026).

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with $\widehat{g}(t) = \int g(x)e^{-itx}dx$ and the archimedean part

$$A(g) = -\frac{1}{\pi}\int_{-\infty}^{\infty} |\widehat{g}(t)|^2\, \vartheta'(t)\, dt, \qquad \vartheta'(t) = \tfrac{1}{2}\big(\operatorname{Re}\psi(\tfrac{1}{4}+\tfrac{it}{2}) - \log\pi\big),$$

ϑ the Riemann–Siegel theta function [Bo, CC]. The pole W is a hyperbolic plane, with one positive and one negative direction, and RH is $Q \geq 0$ [We, Bo]. Following Weil's proof for curves one reads W as the product of two degrees and E as a self-intersection: RH implies that E has at most one positive direction on every space of test functions (Lemma 4), as the Hodge index theorem forces for curves [CE14]. The archimedean place alone has many positive directions, and the primes must remove all of them but the pole's. This note is about how many there are.

For a primitive real character χ modulo q of parity $\kappa \in \{0,1\}$ the completed function $(q/\pi)^{(s+\kappa)/2}\Gamma(\frac{s+\kappa}{2})L(s,\chi)$ has no pole, and the archimedean part is the same with

$$\vartheta'_\chi(t) = \tfrac{1}{2}\big(\operatorname{Re}\psi(\tfrac{1}{4}+\tfrac{\kappa}{2}+\tfrac{it}{2}) + \log\tfrac{q}{\pi}\big);$$

ζ is $\kappa = 0$, $q = 1$. For $a > 0$ let $n_+(a)$ be the largest dimension of a subspace of real $g \in H^1_0([0,a])$ on which A is positive definite.

Theorem 1 (first order [CE14])**.** *If $\vartheta'_\chi(0) < 0$, let t_0 be the positive zero of ϑ'_χ. Then $n_+(a) = t_0 a/\pi + o(a)$. If $\vartheta'_\chi(0) \geq 0$, $n_+(a) = 0$ for all a.*

The symbol $\sigma = -\vartheta'_\chi/\pi$ is even, strictly decreasing in $|t|$, positive exactly on $|t| < t_0$ and $\sim -\frac{1}{2\pi}\log|t|$, unbounded below, so Landau's eigenvalue distribution theorem [La] does not apply directly. The proof in [CE14] truncates it: for the upper bound, $A \leq 2\pi W_a(\max(\sigma,-M))$ and Landau's theorem applies to the bounded symbol $\max(\sigma,-M)+M$ with compact support, counting eigenvalues above M; for the lower bound, the prolate spheroidal functions of the band $|t| < c$, $c < t_0$, cut off smoothly to the window, span a space of dimension $c\,a/\pi + O(\log a)$ [LW] on which A is positive. If $\vartheta'_\chi(0) \geq 0$ then $\sigma < 0$ for $t \neq 0$ and A is negative definite. The theorem is imported: this part does not prove it. In particular the lower bound needs the positivity of A on the cut-off prolate space in spite of the unbounded negative tail of σ, that is, a bound for the energy that the cut-off functions leave outside the band; that estimate is in [CE14] and is not re-derived or re-checked here. Nothing below depends on more than the leading term $t_0 a/\pi$. For ζ, $t_0 = 6.289836$, so $n_+ \approx 2.0021\log N$ in the window $[0,\log N]$. The condition $\vartheta'_\chi(0) < 0$ holds for odd χ with $q < \pi e^{-\psi(3/4)} = 9.31$ and even χ with $q < \pi e^{-\psi(1/4)} = 215.3$.

Place in the book. This is Part 16 of the series, the last chapter of block III (The margin and the constant $\widehat{B}$). It builds on Part 14 [CE14], where the margin is written as $W - E$, the first-order count $t_0 a/\pi$ of the positive directions of the archimedean place is proved, and the law below is first recorded; this part states that law with its boundary phase and tests it for ζ and six real Dirichlet L-functions. No later part cites it.

2. The structure of the symbol

Lemma 2 (complete Bernstein structure)**.** *Put $c = \frac{1}{4} + \frac{\kappa}{2}$ and $\psi_\kappa(\xi) = \vartheta'_\chi(\sqrt{\xi}) - \vartheta'_\chi(0)$, $\xi \geq 0$. Then*

$$\psi_\kappa(\xi) = \frac{1}{2}\sum_{n\geq 0} \frac{1}{n+c}\,\frac{\xi}{\xi + 4(n+c)^2},$$

a complete Bernstein function, independent of q, with $\psi_\kappa(\xi) = \frac{1}{4}\log\xi + O(1)$. Hence on the window

$$A(g) = 2\big\langle(\Lambda - \psi_\kappa(-\Delta))g, g\big\rangle, \qquad \Lambda = -\vartheta'_\chi(0) = \psi_\kappa(t_0^2),$$

with $\langle\psi(-\Delta)g,g\rangle = \frac{1}{2\pi}\int|\widehat{g}(t)|^2\psi(t^2)\,dt$, and $n_+(a)$ is the number of eigenvalues of $\psi_\kappa(-\Delta)$ on an interval of length a, with zero exterior condition, strictly below the level Λ (H^1_0 is a form core, so the counts agree).

Proof. From $\psi(z) = -\gamma_E + \sum_{n\ge0}(\frac{1}{n+1} - \frac{1}{n+z})$ and $\operatorname{Re}\frac{1}{n+c+it/2} = \frac{n+c}{(n+c)^2+t^2/4}$, $\operatorname{Re}\psi(c+\frac{it}{2}) - \psi(c) = \sum_n \left(\frac{1}{n+c} - \frac{n+c}{(n+c)^2+t^2/4}\right) = \sum_n \frac{1}{n+c}\frac{t^2}{t^2+4(n+c)^2}$. Each $\xi \mapsto \xi/(\xi+z)$, $z>0$, is complete Bernstein, the weights are positive and $\sum_n (n+c)^{-3} < \infty$. The rest is Plancherel. □

So the conductor does not change the operator, only the level: ζ, χ_5, χ_8 share ψ_0, and $\chi_{-3}, \chi_{-4}, \chi_{-7}, \chi_{-8}$ share ψ_1.

3. The law

For $\psi(-\Delta)$ on an interval, ψ complete Bernstein, Kwaśnicki found the generalized eigenfunctions on the half-line, $\sin(\mu x + \varpi_\mu) - G_\mu(x)$ with G_μ completely monotone [Kw1], and the eigenvalues of the fractional Laplacian on an interval [Kw2]; Kaleta, Kwaśnicki and Małecki proved, when $\xi\psi'(\xi) \to \infty$, that on an interval of length a [KKM]

$$\lambda_n = \psi(\mu_n^2) + O(\tfrac{1}{n}), \qquad \mu_n a = n\pi - 2\varpi_{\mu_n}, \qquad \varpi_\mu = \frac{1}{\pi}\int_0^\infty \frac{\mu}{r^2-\mu^2}\log\frac{\psi'(\mu^2)(\mu^2-r^2)}{\psi(\mu^2)-\psi(r^2)}\,dr \in [0, \tfrac{\pi}{2}).$$

For $\psi(\xi) = \xi^{\alpha/2}$, $\varpi_\mu = (2-\alpha)\pi/8$. (We write ϖ for this boundary phase, which [Kw1, KKM] call ϑ_μ, to keep ϑ for the Riemann–Siegel theta function.) Our ψ_κ grows like a logarithm, $\xi\psi_\kappa'(\xi) \to \frac14$, just outside that hypothesis, which excludes $\log(1+\xi)$ explicitly. Our question is at a fixed level on a growing interval, and there an error $O(1/n)$ in λ_n is an error $O(1/a)$ in μ_n, that is $O(1)$ in $t_0 a/\pi$, the size of the shift to be predicted. So even a transfer of [KKM] would give only $n_+(a) = t_0 a/\pi + O(1)$; the law below needs the eigenvalue formula at the level $\Lambda = \psi_\kappa(t_0^2)$ with error $o(1/a)$, smaller than the gap between eigenvalues, which is stronger than what [KKM] prove under their hypothesis. If it holds, the eigenvalues below the level are those with $\mu_n < t_0$:

Conjecture 3 (the counting law; asymptotic)**.** *Let $\varpi_\kappa(\mu)$ be the phase of ψ_κ. Then $n_+(a) = \lfloor t_0 a/\pi + 2\varpi_\kappa(t_0)/\pi \rfloor$ for a outside neighbourhoods of the jumps that shrink as $a \to \infty$. Equivalently, the count reaches m at a_m with $t_0 a_m/\pi = m - \delta_m$ and $\delta_m \to \delta = 2\varpi_\kappa(t_0)/\pi$ as $m \to \infty$.*

The phase is a universal function of the level for each parity (Figure 1(a)). Both phases vanish at $\mu = 0$ and tend to $\pi/4$ as $\mu \to \infty$, the logarithmic case, the limit $\alpha \to 0$ of $(2-\alpha)\pi/8$. The odd phase increases monotonically to $\pi/4$; the even phase rises above it, to a maximum $2\varpi/\pi \approx 0.59$ near $\mu \approx 1.2$, and then decreases to $\pi/4$. We computed it with two quadratures (adaptive, and a trapezoid rule in $\log r$ with $4\cdot10^6$ nodes), which agree to better than 10^{-6} in $2\varpi/\pi$, and checked the routine on $\xi^{\alpha/2}$, $\alpha = 2, 1, \frac12$.

4. The test

Method. With the $k-1$ interior hat functions of step h on $[0,a]$, $a = kh$, the Gram matrix of A is the Toeplitz matrix $T_{k-1}(f_h)$ of the symbol

$$f_h(v) = \sum_{j\in\mathbb{Z}} \sigma\Big(\frac{v+2\pi j}{h}\Big)\operatorname{sinc}^4\Big(\frac{v+2\pi j}{2}\Big), \qquad \sigma = -\vartheta_\chi'/\pi,$$

up to a positive factor ($\operatorname{sinc} x = \sin x/x$), so by Sylvester's law the number of positive directions on the hat space is the number of positive pivots of $T_{k-1}(f_h)$, and the Levinson recursion gives the pivots of all sections at once. The hat space is a subspace of H_0^1, so the discrete count is a lower bound for $n_+(a)$, its jumps come late, and the measured δ increases to its limit as $h \to 0$. The Toeplitz entries were computed by FFT on 2^{12} to 2^{18} points with 241 aliasing terms; the Levinson counts agree with full eigenvalue counts in all cases checked (n up to 4000; smallest |eigenvalue| $\ge 3.7\cdot10^{-3}$).

Result. For each case and $h = 0.02, 0.01, 0.005, 0.0025$ we located the jumps $m = 1, \dots, 12$; a jump is located to $t_0 h/\pi$ in δ. With $h = 0.0025$:

	κ	q	$\Lambda = -\vartheta'_\chi(0)$	t_0	predicted $2\varpi_\kappa(t_0)/\pi$	measured δ, $m = 2, \dots, 12$
ζ	0	1	2.686092	6.289836	0.50902	0.483 to 0.522; 0.5089 (extrapolated)
χ_{-3}	1	3	0.565989	2.106205	0.44053	0.4379 to 0.4411
χ_{-4}	1	4	0.422148	1.563882	0.39707	0.3953 to 0.3976
χ_5	0	5	1.881373	1.349504	0.58649	0.5846 to 0.5869
χ_{-7}	1	7	0.142340	0.719570	0.24007	0.2392 to 0.2403
χ_8	0	8	1.646371	0.965814	0.58471	0.5835 to 0.5850
χ_{-8}	1	8	0.075575	0.500953	0.17551	0.1750 to 0.1756

The ranges are the union over the eleven jumps of the brackets; for ζ, whose t_0 is largest, the bracket is 0.005 wide and the range is wider; bisection in a for the tenth jump gives 0.4742, 0.4904, 0.4991, 0.5037 for $h = 0.04$, 0.02, 0.01, 0.005, with geometric extrapolation 0.5089. For the characters the shift does not depend on m for $m \geq 2$; for ζ it oscillates and settles (see below). In every case it converges from below as h decreases; for χ_5 the brackets at the tenth jump are $[0.575, 0.584]$, $[0.584, 0.588]$, $[0.584, 0.586]$, $[0.585, 0.586]$ for $h = 0.02$ to 0.0025. A second discretization, by the sine basis $\sin(j\pi x/a)$, gives for the sixth jump of χ_{-4} the lower bounds 0.363, 0.378, 0.384 with $4a$, $8a$, $12a$ functions, converging more slowly, from below, towards 0.397.

Independent check. The computation was re-implemented from scratch, with a different derivation of the Gram matrix: in x-space, from Gauss's integral for ψ, $A(g) = -\big[\int_0^\infty \big(\frac{e^{-u}}{u}R(0) - \frac{e^{-cu}}{1-e^{-u}}R(\frac{u}{2})\big)du + \log\frac{q}{\pi}\,R(0)\big]$ with R the autocorrelation of g (for hats, h times the cubic B-spline), its own trigamma for ϑ'', the jumps located as zeros of the m-th eigenvalue in a, meshes $ht_0 = 0.12$, 0.06, 0.03, 0.015 (and 0.0075 for some jumps), and Aitken extrapolation (first-order convergence, ratio 1.92–1.95). Its Toeplitz entries agree with a direct t-integral to 10^{-9} and its phases with ours to better than 10^{-6}. Extrapolated shifts against the prediction:

	predicted	extrapolated δ_m	jumps	first jump δ_1
ζ	0.509024	0.50905 to 0.50921	$m = 13, 14, 20$	0.571
χ_{-3}	0.440532	0.44062 to 0.44066	$m = 2, 3, 5, 8, 10$	0.4449
χ_{-4}	0.397071	0.39709 to 0.39721	$m = 2, \dots, 10$	0.3984
χ_5	0.586486	0.58651 to 0.58660	$m = 3, \dots, 10$	0.6086
χ_{-7}	0.240069	0.24019 to 0.24024	$m = 2, 3, 5, 8, 10$	0.2401
χ_8	0.584709	0.58473 to 0.58479	$m = 2, 3, 5, 8, 10$	0.5905
χ_{-8}	0.175515	0.17569 to 0.17574	$m = 2, 3, 5$	0.1757

All the extrapolated values lie above the prediction, by 0.2 to $2.3 \cdot 10^{-4}$; we attribute this to a bias of the Aitken extrapolation, checked in one case: for χ_{-4} a finer mesh moves it to 0.39709. Two departures. The first jump does not follow the law in five cases, always with δ_1 above the limit (by 0.062 for ζ, 0.022 for χ_5, 0.006 for χ_8, 0.004 for χ_{-3}, 0.001 for χ_{-4}), so the first positive direction appears earlier than the law says. For ζ the shift alternates around its limit before settling, $\delta_m - \delta = -0.019$, $+0.012$, -0.007, $+0.005$, -0.003 for $m = 2, \dots, 6$, -0.0009 at $m = 8$, -0.0003 at $m = 10$ and below $2 \cdot 10^{-4}$ for $m = 13$ to 20, an even–odd effect; for the characters, whose t_0 is smaller, it is flat from $m = 2$ on. A plausible reason, not tested, is that for ζ the first jumps occur in windows that are short compared with the decay length of the kernel in x. None of this changes the integer count for $m \geq 2$, since the deviations are far below the distance between jumps.

5. A smooth count

The integer part in Conjecture 3 disappears if the count is smoothed. For a smooth F with $F \to 0$ below the level and $F \to 1$ above, put

$$h_F^0(a) = \operatorname{Tr} F\big(W_a(\sigma)\big), \qquad \sigma = -\vartheta'_\chi/\pi,$$

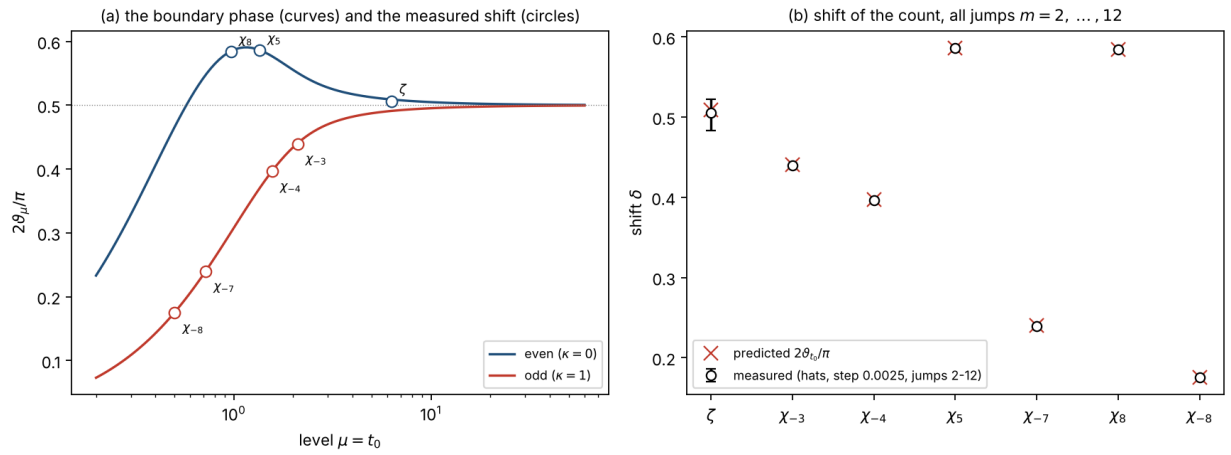


FIGURE 1. (a) The phases $2\varpi_\kappa(\mu)/\pi$ of ψ_0 (even) and ψ_1 (odd) as functions of the level, and the measured shifts at $\mu = t_0(\chi)$. (b) Predicted and measured shifts in the seven cases (hats of step 0.0025, jumps 2 to 12).

$W_a(\sigma)$ the operator on $L^2(0,a)$ with $\langle W_a(\sigma)g, g\rangle = \frac{1}{2\pi}\int |\widehat{g}|^2\sigma$, so that $A(g) = 2\pi\langle W_a(\sigma)g, g\rangle$. If the eigenvalues near the level are $\sigma(\mu_n)$ with $\mu_n a = n\pi - 2\varpi_\kappa(\mu_n)$, the Euler–Maclaurin formula, with $\varpi_\kappa(0) = 0$, gives

$$h_F^0(a) = \frac{a}{\pi}\int_0^\infty F(\sigma(\mu))\,d\mu + \chi_F + o(1), \qquad \chi_F = \frac{2}{\pi}\int_0^\infty F(\sigma(\mu))\,d\varpi_\kappa(\mu) - \frac{1}{2}F(\sigma(0)), \tag{1}$$

and $\chi_F \to 2\varpi_\kappa(t_0)/\pi - \frac{1}{2}$ as F tends to the step at 0, the mean of $\lfloor x+\delta\rfloor - x$. This is a "degree plus constant" formula with real-valued dimension, of the shape of the van der Geer–Schoof h^0 [GS], and its constant is the boundary phase. For $F(x) = \frac{1}{2}(1+\tanh(x/\varepsilon))$, for which $F(\sigma(\mu)) \sim \mu^{-1/(\pi\varepsilon)}$ and the trace converges if $\varepsilon < 1/\pi$, we computed $h_F^0(a)$ from all the generalized eigenvalues (with respect to the mass matrix) of the hat discretization (`traza_suave.py`, `traza_pocos.py`; χ_F by integration by parts, `chi_pred.py`):

	ε	h_1	$h_1/2$	$h_1/4$	extrapolated	χ_F in (1)
χ_{-7}, $h_1 = 0.02$	0.02	−0.263096	−0.261988	−0.261420	−0.260823	−0.260834
ζ, $h_1 = 0.01$	0.05	0.000011	0.004776	0.007266	0.00999	0.009889

The difference $h_F^0(a) - \frac{a}{\pi}\int F(\sigma)$ is constant in a to six digits from $a = 25$ for χ_{-7} and from $a = 10$ for ζ; the mesh error is first order, and the geometric extrapolation agrees with (1) to $1.1\cdot 10^{-5}$ and 10^{-4}. The sharp limits are $\chi = -0.25993$ for χ_{-7} and $+0.00902$ for ζ. Status: (1) is numerical; for bounded symbols, two-term formulas of this kind are theorems of Widom and Sobolev [Wi, So], whose hypotheses exclude our logarithmic symbol.

Identification with Widom's second term. For a smooth F and a suitable bounded symbol (with $F\circ\sigma - F(0)$ integrable, which makes the trace finite), Widom's trace formula [Wi] reads

$$\operatorname{Tr} F\big(W_a(\sigma)\big) = \frac{a}{2\pi}\int F(\sigma(\xi))\,d\xi + \mathcal{B}(\sigma;F) + o(1), \qquad \mathcal{B}(\sigma;F) = \frac{1}{4\pi^2}\iint \frac{U\big(\sigma(\xi_1),\sigma(\xi_2);F\big)}{(\xi_1-\xi_2)^2}\,d\xi_1 d\xi_2,$$

with $U(x,y;F) = \int_0^1 \frac{F((1-t)x+ty)-(1-t)F(x)-tF(y)}{t(1-t)}\,dt$ (one half of $\mathcal{B}$ for each end of the interval). The constant χ_F of (1) is this second term for our symbol: a direct evaluation of the double integral (`widom_check.py`, written by the referee of this part and rerun by us; stable to 10^{-8} under changes of the quadrature) gives $\mathcal{B}(\sigma;F) = 0.00988902$ for ζ ($\varepsilon = 0.05$) and -0.26083400 for χ_{-7} ($\varepsilon = 0.02$), against $\chi_F = 0.009889$ and -0.260834 in the table. As a check of the constant $\frac{1}{4\pi^2}$: for $F(x) = x^2$

one has $U = -(x-y)^2$, and the formula reduces to $\operatorname{Tr} W_a^2 = a\int|k|^2 - \int|u||k(u)|^2du$, which is elementary. So (1) is Widom's formula extended, numerically, to a symbol of logarithmic growth, and the two expressions for the constant, one through the boundary phase ϖ_κ of [KKM] and one through $\mathcal{B}$, agree. In the step limit this gives an expression for the constant of Conjecture 3 that does not use the phase; we have not tested that limit. No proof of the identity $\chi_F = \mathcal{B}(\sigma; F)$ for complete Bernstein symbols is given here.

6. Complements on a circle

A first candidate for a dual count is the complement. Put the window on a circle of length L and let W^{S^1} be the archimedean operator there, $\langle W^{S^1}g, g\rangle = \frac{1}{L}\sum_k \sigma(\omega_k)|\widehat{g}(\omega_k)|^2$, $\omega_k = 2\pi k/L$, whose eigenvalues are exactly the $\sigma(\omega_k)$. Restrict it to an arc I of length a and to the complementary arc I^c of length $L-a$, and write $\deg I = \frac{|I|}{\pi}\int_0^\infty F(\sigma(\mu))\,d\mu$ for the volume term of (1). Numerically (`circulo.py`, hats of step h, the circle operator folded exactly):

	L, h, ε	a	$h^0_F(I)+h^0_F(I^c)-h^0_F(S^1)$	$h^0_F(I)-h^0_F(I^c)-(\deg I-\deg I^c)$
χ_{-7}	70, 0.02, 0.02	25, 30, 35	-0.526190, -0.526192, -0.526192	$2\cdot10^{-6}$, 0, 0
χ_5	60, 0.02, 0.05	20, 27	$+0.151729$, $+0.151729$	0, 0
ζ	30, 0.01, 0.05	10, 13	$+0.000015$, $+0.000016$	$7\cdot10^{-6}$, $3\cdot10^{-6}$

and $h^0_F(S^1) = \deg S^1$ to the six digits printed in every case. So, to this accuracy, (i) the closed circle has no constant; (ii) the difference between an arc and its complement has no constant, $h^0_F(I)-h^0_F(I^c) = \deg I - \deg I^c$; and (iii) the sum carries $2\chi_F$, which is $-0.526192 = 2\times(-0.263096)$ for χ_{-7}, twice the value of the interval at the same mesh, $0.151729 \approx 2\chi_F$ for χ_5 (at this mesh) and ≈ 0 for ζ (the mesh offset of ζ at $h = 0.01$ cancels its χ_F). The constant is local: each of the four end points contributes $\chi_F/2$, which tends to $\varpi_\kappa(t_0)/\pi - \frac{1}{4}$ as F tends to the step. With $K = S^1$ and $K - D = I^c$, (ii) reads $h^0(D) - h^0(K-D) = \deg D - \deg(K-D)$, a duality that removes the boundary constant. But it is not Serre duality: $h^0(K-D)$ does not vanish for large D, the circle is not canonical (L is free), and (i)–(iii) are those of an Euler characteristic with boundary, of Mayer–Vietoris type, with the boundary phase as a local index at each end point. Status: numerical.

7. The conductor as a degree

Lemma 2 says that the conductor only moves the level: $\Lambda = -\frac{1}{2}(\psi(c) + \log\frac{q}{\pi})$. Treat $d = -\log q$ as a degree, letting $q > 0$ be real ($q < 1$ is formal), so that $\Lambda(d) = \Lambda(0) + \frac{d}{2}$, and consider the reflection $d \mapsto -d$, $q \mapsto 1/q$, whose fixed point is $q = 1$. Since $\vartheta_q'(t) = \frac{1}{2}\log\frac{qt}{2\pi} + O(t^{-2})$ (Stirling), the band edge is $t_0(q) \approx 2\pi/q$ and the reflection acts as

$$t_0 \longmapsto \frac{4\pi^2}{t_0},$$

the inversion $t \mapsto 1/t$ of the theta function, rescaled by 2π, that gives van der Geer and Schoof their Riemann–Roch formula [GS]. Its fixed point is $t_0 = 2\pi$, and ζ sits there: $t_0(\zeta) = 6.289836$ against $2\pi = 6.283185$. This is why the slope $t_0/\pi = 2.0021$ of ζ is so close to 2: ζ is the self-dual point, and 0.0021 is the Stirling correction. With $n_+ \approx t_0 a/\pi$, the logarithm of the count then satisfies, to leading order,

$$\log n_+(a; d) - \log n_+(a; -d) \approx 2d,$$

a "degree" relation of the shape of $h^0(D) - h^0(K-D) = \deg D$ for the logarithmic h^0 of [GS], with $n_+ = 0$ once d is so negative that $\Lambda \le 0$, as $h^0(D) = 0$ in negative degree. But the duality is not exact (`dualidad_q.py`):

$d = -\log q$	0.25	0.5	1	1.5	2	3
$t_0(d)\,t_0(-d)/4\pi^2$, $\kappa = 0$	1.0024	1.0033	1.0102	1.0531	1.1955	1.8759
$t_0(d)\,t_0(-d)/4\pi^2$, $\kappa = 1$	1.0024	1.0033	1.0060	0.9837	0.7474	—

It holds while both band edges are in the logarithmic range of ϑ' and fails beyond, where the full gamma factor departs from Stirling. The exact inversion lives one level up, in the Gaussian $e^{-\pi q x^2}$ and its Fourier transform, whose Mellin transform is the gamma factor; the count sees it only asymptotically. Status: elementary (Stirling) plus numerical; the real conductors are $q \geq 1$, so the dual side $q < 1$ is a formal extension.

8. What the law says about Weil's form

Lemma 4 (inertia [CE14])**.** *On any space of test functions, if $n_-(Q)$ and $n_+(E)$ count the negative directions of Q and the positive directions of $E = W - Q$, then $n_-(Q) - 1 \leq n_+(E) \leq n_-(Q) + 1$, and $n_+(E) = n_-(Q)$ on the hyperplane $\{\widehat{g}(-\frac{i}{2}) = 0\}$, where W vanishes.*

This is subadditivity of n_+ with $n_+(\pm W) \leq 1$. Applied to the form $Q_R = W - A$ of the pole and the archimedean place without primes, it gives $n_-(Q_R) \geq n_+(A) - 1$: the form without primes must be indefinite as soon as A has two positive directions. For ζ this happens at $\log N \approx 0.754$, $N \approx 2.13$ (the second jump), and numerically Q_R loses positivity at $\log N = 0.744$, $N = 2.106$ [CE14]; Yoshida proved positivity below $N = 2$ [Yo]. With all the primes, E has one positive direction in every window we computed, up to $N = 1097$ [CE14], as RH requires; so, numerically, the orbits of the prime powers turn $n_+(A)$ positive directions into one, that is $n_+(A) - 1$, which by Conjecture 3 is $\lfloor t_0 \log N/\pi + 0.509 \rfloor - 1$ for $\log N$ beyond the first jumps; in [CE14] they do so only all together and only with the prime powers (numerical). For the Davenport–Heilbronn function the same count holds for the archimedean place, and, numerically, E gains one pair of positive directions for each off-line zero that the window resolves [CE14]; Bombieri proved that if finitely many zeros are off the line, large truncations of Weil's form have half as many negative eigenvalues as off-line zeros [Bo], and the pairs are the window form of this count.

9. Relation with Riemann–Roch

The law has the shape "dimension = degree × slope + constant". It is not one of the known Riemann–Roch formulas for $\overline{\operatorname{Spec}\mathbb{Z}}$. Connes and Consani's integer-valued dimension of H^0 of the divisor $\log N\{\infty\}$ is $\lceil \log(2\lfloor N \rfloor + 1)/\log 3 \rceil \approx 0.91 \log N$ [CoCo1, CoCo2]; the log-theta number of van der Geer and Schoof grows like $\log N$ [GS]. Both measure the integers in $[-N, N]$, the first by the dimension over $\mathbb{S}[\pm 1]$ of a minimal set of generators, the second by a Gaussian-weighted theta sum, $\log \sum_n e^{-\pi n^2/N^2} = \log N + o(1)$; our count is spectral, the number of degrees of freedom of the band $|t| < t_0$ in a window of length $\log N$. It is also not the spectral count of the prolate operator, which fixes the window ($\lambda = \sqrt{2}$) and counts by energy, reproducing the Riemann–von Mangoldt count of the zeros [CM, CCM]. A natural reading, which is a heuristic analogy and not a theorem, is the counting formula $N(T) = \vartheta(T)/\pi + 1 + S(T)$: the band $|t| < t_0$ is where ϑ decreases, the archimedean surplus counts its degrees of freedom, and the 1 is the pole, the one direction that survives. The smooth count (1) removes the integer part and has real-valued dimension, like the h^0 of van der Geer and Schoof, whose Riemann–Roch formula is the functional equation of the theta function, the same theta whose Mellin transform is the gamma factor behind ϑ'. What it still lacks to be a Riemann–Roch formula in the proper sense is a second term and a duality, $h^0 - h^1$ with $h^1(D) = h^0(K - D)$: we have only one count, and the negative directions of A are infinitely many, so h^1 cannot be their number. The slope $t_0/\pi = 2.0021$ is not an integer, and the constant comes from the boundary of the window, through ϖ_κ.

10. Open questions

(1) Prove $n_+(a) = t_0 a/\pi + O(1)$, and Conjecture 3: this needs the eigenvalues of $\psi_\kappa(-\Delta)$ on a growing interval at a fixed level with error $o(1/a)$, for a symbol of logarithmic growth, beyond what the half-line analysis of [Kw1, KKM] gives. (2) Prove (1), for instance by extending Widom's theorem to symbols of logarithmic growth and proving $\chi_F = \mathcal{B}(\sigma; F)$, and the complement laws of Section 6, and find a dual count h^1 that vanishes for large degree, with a canonical K, that would make $h^0_F - h^1_F$ a Riemann–Roch formula in the proper sense. (3) Explain why the phase separates the parities at small levels and what the curves of Figure 1(a) mean arithmetically. (4) Find an exact form of the conductor duality of Section 7, from the Gaussian rather than from Stirling, and see whether it combines with the complement duality of Section 6 into a single Riemann–Roch formula in which ζ is the self-dual point.

Data and code

The scripts (NumPy, SciPy) are in the ancillary files: `rr_common.py` (symbols, t_0, phase, Toeplitz entries, Levinson), `rr_test_fase.py` (t_0 and the predicted shifts), `rr_scan.py` (all windows and jumps), `fase_check.py` (second quadrature of the phase), `seno_check.py` (sine basis), `traza_suave.py`, `traza_pocos.py`, `chi_pred.py` (the smooth count; several minutes), `widom_check.py` (Widom's second term, ten seconds), `circulo.py` (complements on a circle), `dualidad_q.py` (conductor duality), `fig_rr.py`, with the log `rr_scan.jsonl`; the independent re-implementation is in the folder `verificacion_independiente`. File names are in Spanish (*fase* phase, *traza* trace, *seno* sine, *suave* smooth).

Use of AI assistance

The research programme is the author's. Claude (Anthropic) was used as a research and writing aid: to derive and check proofs, to write and run the computations, to search the literature, and to draft the text and figures. The main computation was re-implemented independently, from a separate derivation, as a check (Section 4). The author checked the results and is responsible for the content.

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