Cross-Energy of Dirichlet Partial Sums. Part 10: The Caceres Prime Function

A square-root damped prime weight, its exact Prime Zeta structure, and a dual spectrum of the Riemann zeros

Pedro Caceres

Universidad Europea de Valencia

pedrojesus.caceres@universidadeuropea.es

October 2026; book edition, 2 November 2026.

Part 10 of the book-series *Cross-Energy of Dirichlet Partial Sums* (forty-seven parts, Parts 0–46; block II: Primes and Euler products), formerly a standalone note. Part 0 is the guide and Part 1 [CE1] gives the plan of the series.

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Abstract

We study the *Caceres Prime Function*

$$\mathcal{C}(s) = \sum_{n\geq 2} \frac{a(n)}{n^s}, \qquad a(n) = \sum_{p^k \,\|\, n} \frac{\log p}{\sqrt{p^k}},$$

the Dirichlet series of the additive function that weights each prime factor p of n by $\log p$ divided by the square root of the exact power of p dividing n.

Structure. We prove that $\mathcal{C} = \zeta\mathcal{K}$, where the *prime kernel*, $\mathcal{K}(s) = \sum_p \log p\,(1 - p^{-s})/(p^{s+1/2} - 1)$, is an exact telescoping series of shifted derivatives of the Prime Zeta function. So $\mathcal{K}$ is meromorphic in $\operatorname{Re} s > -\frac{1}{4}$, and the line $\operatorname{Re} s = -\frac{1}{4}$ is a natural boundary. Every zero $\frac{1}{2} + i\gamma$ of ζ produces two singular points of $\mathcal{K}$, at $i\gamma$ and $i\gamma/2$, which are poles unless 2γ is again an ordinate with twice the multiplicity (respectively $\gamma/2$ an ordinate with half the multiplicity, for $i\gamma/2$); the exponent $\frac{1}{2}$ is the normalization that puts both families on the imaginary axis. More precisely, $\mathcal{K}(s) = -\frac{\zeta'}{\zeta}(s + \frac{1}{2}) + \frac{\zeta'}{\zeta}(2s + \frac{1}{2}) - \frac{1}{12s} + H(s)$, with H holomorphic in a neighbourhood of $\operatorname{Re} s \geq 0$.

RH criteria. RH is equivalent to the holomorphy of $\mathcal{K}(s) - \frac{1}{s-1/2} + \frac{1}{2(s-1/4)}$ in $\operatorname{Re} s > 0$. It is also equivalent to a bound $O(x^{\varepsilon})$, or even a one-sided bound, for the remainder R_b of the kernel's partial sums. Both are exact reformulations of RH, not proofs.

Mean value and explicit formula. Unconditionally, $\sum_{n\leq x} a(n) = \mathcal{K}(1)x + 2\zeta(\frac{1}{2})\sqrt{x} + O(\sqrt{x}\, e^{-c_*\sqrt{\log x}})$, with the explicit bound $5.51\sqrt{x}$ for the error after the first term when $x \geq 10^6$. Under RH, the Riesz mean of a has an explicit formula in which every zero contributes the pure phases $x^{i\gamma}$ and $x^{i\gamma/2}$ at constant amplitude.

The Cesàro margin. The kernel is Suzuki's one-variable margin M seen at two scales. Unconditionally, $R_b(x) = \alpha(x) - \alpha(\sqrt{x}) + o(1)$, where α is the slope of the margin. Under RH, the kernel's Riesz margin is $M(N) - 2M(\sqrt{N})$. Its fluctuation has energy $2(\log\xi)''(\frac{1}{2})$, reduced by $2\gamma^{-2}$ for every pair of ordinates with $\gamma' = 2\gamma$; the near pair $\gamma_{14} \approx 2\gamma_4$ is visible in the data.

Zeros of $\mathcal{C}$. A computer-assisted proof shows that $\mathcal{C}$ has a zero near $1.379 + 8.639\,i$, in the half-plane of absolute convergence, and Bohr's theory gives $\gg T$ such zeros up to height T. So $\mathcal{C}$ is not an L-function.

The theorems are proved here, except that Section 10 uses the formula for the margin and the almost periodicity of the slope from Part 11. Uncertified computations are labelled as numerical observations.

The Caceres Prime Function in five equations

$$a(n) = \sum_{p^k \| n} \frac{\log p}{\sqrt{p^k}} \tag{E1}$$

$$\mathcal{C}(s) = \sum_{n\geq 2} \frac{a(n)}{n^s} = \zeta(s)\,\mathcal{K}(s) \tag{E2}$$

$$\mathcal{K}(s) = \sum_{p} \log p\, \frac{1-p^{-s}}{p^{s+1/2}-1} \tag{E3}$$

$$\mathcal{K}(s) = \sum_{k\geq 1} \Big[P'\big(k(s+\tfrac12)+s\big) - P'\big(k(s+\tfrac12)\big)\Big], \qquad P(s) = \sum_p p^{-s} \tag{E4}$$

$$\textstyle\sum_{n\leq x} a(n) = \mathcal{K}(1)\, x + 2\zeta(\tfrac12)\sqrt{x} + o(\sqrt{x}), \qquad \mathcal{K}(1) = 1.16046\ldots \tag{E5}$$

$$\mathcal{K}(s) = -\frac{\zeta'}{\zeta}\big(s+\tfrac12\big) + \frac{\zeta'}{\zeta}\big(2s+\tfrac12\big) - \frac{1}{12s} + H(s) \tag{E6}$$

1 Introduction

Three classical arithmetic functions measure the prime content of an integer $n = \prod p^{k_p}$:

$$\log n = \sum_{p^k \| n} k \log p, \qquad \log \mathrm{rad}(n) = \sum_{p\mid n} \log p, \qquad \omega(n) = \sum_{p\mid n} 1.$$

Their Dirichlet series are $-\zeta'(s) = \zeta(s)\big(-\frac{\zeta'}{\zeta}(s)\big)$, $-\zeta(s)P'(s)$ and $\zeta(s)P(s)$, where $P(s) = \sum_p p^{-s}$ is the Prime Zeta function. In each case $\zeta(s)$ appears multiplied by a function that is either ζ'/ζ or a Prime Zeta object.

This paper studies the function obtained by *damping* the contribution of each prime by the square root of its exact power,

$$a(n) = \sum_{p^k \| n} \frac{\log p}{p^{k/2}}. \tag{1}$$

Thus $a(p) = \log p/\sqrt{p}$, $a(p^2) = \log p/p$, $a(6) = \frac{\log 2}{\sqrt{2}} + \frac{\log 3}{\sqrt{3}}$ and $a(12) = \frac{\log 2}{2} + \frac{\log 3}{\sqrt{3}}$. The damping makes a distinguish primes, prime powers and composite numbers, while keeping the additive structure of log rad.

Definition 1 (Caceres Prime Function)**.** The *Caceres Prime Function* is the Dirichlet series

$$\mathcal{C}(s) = \sum_{n\geq 2} \frac{a(n)}{n^s}, \qquad \mathrm{Re}\, s > 1,$$

and its *prime kernel* is $\mathcal{K}(s) = \mathcal{C}(s)/\zeta(s)$. We also write $A(x) = \sum_{n\leq x} a(n)$ and $F(x) = \sum_{n\leq x} a(n)\log(x/n) = \int_1^x A(t)\,\frac{dt}{t}$.

Main results.

1. **Structure** (Theorems 3, 5, 8, 9). $\mathcal{C} = \zeta\mathcal{K}$; $\mathcal{K}$ is the Prime Zeta telescope (E4); it is meromorphic in $\mathrm{Re}\, s > -\frac14$, with natural boundary $\mathrm{Re}\, s = -\frac14$; each zero ρ gives poles at $\rho - \frac12$ and $\frac12(\rho - \frac12)$.
2. **Critical damping** (Proposition 10). $\frac12$ is the only exponent that puts both images of the critical line on the imaginary axis.
3. **RH criteria** (Theorems 11, 20, 27). RH $\Leftrightarrow$ $\mathcal{K}$ has no poles in $\mathrm{Re}\, s > 0$ other than $\frac12, \frac14$ $\Leftrightarrow$ $R_b(x) = O(x^\varepsilon)$ $\Leftrightarrow$ $R_b(x) \geq -C_\varepsilon x^\varepsilon$.

4. **Mean value and explicit formula** (Theorems 13, 15, 17). An explicit $\sqrt{x}$ bound, the second term $2\zeta(\frac{1}{2})\sqrt{x}$, and, under RH, a constant-amplitude explicit formula for the Riesz mean.
5. **The Cesàro margin** (Section 10). The decomposition (E6); the bridge $R_b(x) = \alpha(x) - \alpha(\sqrt{x}) + o(1)$; the kernel's margin $M(N) - 2M(\sqrt{N})$ under RH; its energy, and the pairs $\gamma' = 2\gamma$ that reduce it.
6. **Zeros of the kernel** (Theorem 36, Corollary 37). $\mathcal{C}$ vanishes in its half-plane of absolute convergence, $\gg T$ times up to height T.

What the paper does not claim. Theorem 11 is an exact reformulation of RH, not a proof. Every singularity of $\mathcal{K}$ in $\operatorname{Re} s > 0$ comes from ζ'/ζ, so the construction cannot locate the zeros of ζ better than ζ'/ζ itself. Its value is that it turns the zeros into a clean, explicit, computable signal, which we exploit numerically. In particular, Section 10 shows that the kernel carries exactly the information of Λ, at two scales. All results labelled Theorem, Proposition, Lemma or Corollary are proved in full; Theorem 36 is computer-assisted, with every error term bounded explicitly. Results labelled *Numerical observation* are computations without certified error bounds. Parts 11 and 37 of the series [CE11, CE37] treat Suzuki's margin and the $6k \pm 1$ race.

Place in the book. This is Part 10 of the series, in block II (Primes and Euler products). It builds on Part 9 [CE9] (the prime zeta function $P(s)$, whose shifted derivatives make up the kernel $\mathcal{K}$), on Part 1 [CE1] (the C-transform, used in Section 9) and on Part 11 [CE11] (the Cesàro margin, used in Section 10). It is used by Part 11, which reads the margin through the kernel, and by Part 37 [CE37] (the $6k \pm 1$ race).

2 The arithmetic function

Lemma 2 (Elementary properties)**.**

(i) a *is additive:* $a(mn) = a(m) + a(n)$ *whenever* $\gcd(m, n) = 1$.

(ii) $0 < a(n) \le c_0\, \omega(n)$ *for* $n \ge 2$, *where* $c_0 = \max_p \frac{\log p}{\sqrt{p}} = \frac{\log 7}{\sqrt{7}} = 0.7355\ldots$

(iii) $a = 1 * b$, *where* b *is supported on prime powers and*

$$b(p) = \frac{\log p}{\sqrt{p}}, \qquad b(p^k) = \log p\left(p^{-k/2} - p^{-(k-1)/2}\right) \quad (k \ge 2).$$

Equivalently $b = \mu * a$, *and for every* $x \ge 1$, $\sum_{n \le x} a(n) = \sum_{q \le x} b(q)\lfloor x/q \rfloor$, *the sum over prime powers* q.

Proof. (i) and (ii) follow directly from (1), since every term satisfies $\log p\, p^{-k/2} \le \log p\, p^{-1/2}$, and the function $t \mapsto \log t/\sqrt{t}$ is maximal at $t = e^2$. For (iii), fix p and sum $b(p^j)$ over $1 \le j \le k$. The sum telescopes to $\log p\, p^{-k/2}$, which is exactly the contribution of p to $a(n)$ when $p^k \| n$. Summing over $p \mid n$ gives $\sum_{d \mid n} b(d) = a(n)$. Möbius inversion gives $b = \mu * a$, and summing $a = 1 * b$ over $n \le x$ gives the last identity. □

3 The factorization theorem

Theorem 3 (Factorization)**.** *For* $\operatorname{Re} s > 1$,

$$\boxed{\mathcal{C}(s) = \zeta(s)\,\mathcal{K}(s), \qquad \mathcal{K}(s) = \sum_p \log p\, \frac{1 - p^{-s}}{p^{s+1/2} - 1}}$$

and the series for $\mathcal{K}$ *converges absolutely for* $\operatorname{Re} s > \frac{1}{2}$.

Proof. Write each n with $p^k \| n$ as $n = p^k m$ with $p \nmid m$. Grouping the terms of $\mathcal{C}$ by the prime p and the exponent k gives

$$\mathcal{C}(s) = \sum_p \log p \sum_{k\geq 1} p^{-k/2}\, p^{-ks} \sum_{p\nmid m} m^{-s} = \zeta(s) \sum_p \log p\,(1-p^{-s}) \sum_{k\geq 1} p^{-k(s+1/2)}.$$

The inner geometric series equals $p^{-(s+1/2)}/(1-p^{-(s+1/2)}) = 1/(p^{s+1/2}-1)$. Rearranging is legitimate because all series converge absolutely for $\operatorname{Re} s > 1$. For $\operatorname{Re} s > \frac{1}{2}$, the general term of $\mathcal{K}$ is $O(\log p\, p^{-\operatorname{Re} s - 1/2})$, which is summable. □

Remark 4. Equivalently, $\mathcal{K}(s) = \sum_n b(n) n^{-s}$ with b as in Lemma 2(iii). The factor $1-p^{-s}$ is essential: it removes the multiples of p already counted by $\zeta(s)$. A naive guess with an s-independent denominator, such as $\sum_p \log p\, p^{-(s+1/2)}/(1-p^{-1/2})$, ignores the exponent $v_p(n)$ and is off by a factor of about 3. With all $n \leq 10^8$ and the tail $\mathcal{K}(1)N^{1-s}/(s-1) + \zeta(\frac{1}{2})N^{1/2-s}/(s-\frac{1}{2})$, the direct series agrees with $\zeta(s)\mathcal{K}(s)$ to relative error $3\cdot 10^{-11}$, $1\cdot 10^{-12}$ and $9\cdot 10^{-13}$ at $s = 1.5, 2, 3$.

4 Exact relation with the Prime Zeta function

Let $P(s) = \sum_p p^{-s}$, so that $P'(s) = -\sum_p \log p\, p^{-s}$. It is natural to embed a in a one-parameter family:

$$a_\alpha(n) = \sum_{p^k \| n} \frac{\log p}{p^{\alpha k}}, \qquad \mathcal{C}_\alpha(s) = \sum_{n\geq 2} \frac{a_\alpha(n)}{n^s}, \qquad \mathcal{K}_\alpha = \mathcal{C}_\alpha/\zeta,$$

so that $a = a_{1/2}$, $\mathcal{C} = \mathcal{C}_{1/2}$ and $\mathcal{K} = \mathcal{K}_{1/2}$.

Theorem 5 (Prime Zeta structure)**.** *Let $\alpha \geq 0$ and $w = s + \alpha$. For $\operatorname{Re} s > 1$,*

$$\boxed{\mathcal{C}_\alpha(s) = \zeta(s) \sum_{k=1}^{\infty} \Big[P'(kw+s) - P'(kw) \Big]}$$

In particular, for the Caceres Prime Function,

$$\mathcal{K}(s) = \sum_{k\geq 1} \Big[P'\big(k(s+\tfrac{1}{2}) + s\big) - P'\big(k(s+\tfrac{1}{2})\big) \Big]. \tag{2}$$

Proof. The proof of Theorem 3 with $p^{-k/2}$ replaced by $p^{-\alpha k}$ gives

$$\mathcal{K}_\alpha(s) = \sum_p \log p\,(1-p^{-s}) \sum_{k\geq 1} p^{-kw} = \sum_{k\geq 1} \sum_p \log p\, \big(p^{-kw} - p^{-kw-s}\big).$$

The inner sum is $-P'(kw) + P'(kw+s)$. The interchange of sums is justified by absolute convergence for $\operatorname{Re} s > 1$. □

Corollary 6 (Telescoping and the undamped case)**.**

(i) $\mathcal{K}_\alpha(s) = \displaystyle\sum_{k\geq 1} \int_0^s P''(kw+v)\, dv$*: the kernel adds up the increments of P' along the lattice $\{kw\}$ with step s.*

(ii) For $\alpha = 0$ we have $w = s$, so $P'(ks+s) = P'((k+1)s)$ and the series telescopes completely, since $P'(\sigma) \to 0$ as $\sigma \to +\infty$:

$$\mathcal{C}_0(s) = -\zeta(s)\, P'(s), \qquad a_0(n) = \log \operatorname{rad}(n).$$

So the Caceres Prime Function is the $\sqrt{}$-damped form of $\log \operatorname{rad}(n)$. Setting $\alpha = 0$ collapses (E4) to a single term, while $\alpha = \frac{1}{2}$ leaves a genuinely infinite telescope, whose first two terms carry all the singularities (Theorem 8).

Corollary 7 (Form in terms of ζ only)**.** *Using* $P'(s) = \sum_{m\geq 1} \mu(m)\, \frac{\zeta'}{\zeta}(ms)$,

$$\mathcal{C}_\alpha(s) = \zeta(s) \sum_{k\geq 1} \sum_{m\geq 1} \mu(m) \left[\frac{\zeta'}{\zeta}\big(m(kw+s)\big) - \frac{\zeta'}{\zeta}\big(mkw\big) \right].$$

5 Analytic continuation and the dual spectrum

Recall that $P'(w)$ is meromorphic in $\operatorname{Re} w > 0$, where the line $\operatorname{Re} w = 0$ is a natural boundary (Landau–Walfisz [LW]). By Corollary 7 with $k = 1$, its poles in $\operatorname{Re} w > 0$ are at $w = 1/m$ for squarefree m and at $w = \rho/m$ for the nontrivial zeros ρ of ζ. Moreover, $\frac{\zeta'}{\zeta}$ has residue -1 at 1 and residue m_ρ (the multiplicity) at ρ.

Theorem 8 (Continuation and poles)**.** $\mathcal{K}$ *extends meromorphically to* $\operatorname{Re} s > -\frac{1}{4}$. *In* $\operatorname{Re} s > 0$,

$$\mathcal{K}(s) = -P'\big(s+\tfrac{1}{2}\big) + P'\big(2s+\tfrac{1}{2}\big) + G(s), \tag{3}$$

where G *is holomorphic in* $\operatorname{Re} s > 0$. *Consequently, the poles of* $\mathcal{K}$ *in* $\operatorname{Re} s > 0$ *are among the following:*

origin	*pole*	*residue*
pole of ζ *at* 1, *first term*	$s = \frac{1}{2}$	$+1$
pole of ζ *at* 1, *second term*	$s = \frac{1}{4}$	$-\frac{1}{2}$
zero $\rho = \beta + i\gamma$, *first term*	$s = \rho - \frac{1}{2}$	$-m_\rho$
zero $\rho = \beta + i\gamma$, *second term*	$s = \frac{1}{2}(\rho - \frac{1}{2})$	$+\frac{1}{2}m_\rho$

The poles in the last two rows lie in $\operatorname{Re} s > 0$ *only if* $\beta > \frac{1}{2}$. *Two poles in this list can coincide only if* $\rho - \frac{1}{2} = \frac{1}{2}(\rho' - \frac{1}{2})$ *for two zeros* ρ, ρ', *that is, if* $\rho' = 2\rho - \frac{1}{2}$.

Proof. In (2), the k-th pair of terms involves P' at arguments with real parts $k(\sigma + \frac{1}{2})$ and $(k+1)\sigma + \frac{k}{2}$, where $\sigma = \operatorname{Re} s$. Both are positive as soon as $\sigma > -\frac{1}{4}$, and they grow linearly in k. Since $P'(w) = O(2^{-\operatorname{Re} w})$ as $\operatorname{Re} w \to \infty$, the series converges locally uniformly away from the poles, which gives the continuation.

For $\sigma > 0$, write $\mathcal{K}(s) = \sum_p \log p\, (1 - p^{-s}) \sum_k p^{-k(s+1/2)}$ and separate the two terms $p^{-(s+1/2)}$ and $-p^{-(2s+1/2)}$ (both from $k = 1$). Every remaining term $p^{-k(s+1/2)}$ or $p^{-s-k(s+1/2)}$ with $k \geq 2$ has exponent with real part at least $\min(2\sigma+1, 3\sigma+1) > 1$. So the remaining sum $G(s)$ converges absolutely and locally uniformly, and G is holomorphic in $\sigma > 0$. This proves (3).

For the poles, apply Corollary 7 to $P'(s + \frac{1}{2})$ and $P'(2s + \frac{1}{2})$. Their arguments have real part greater than $\frac{1}{2}$ when $\sigma > 0$. Hence only the $m = 1$ term can be singular: for $m \geq 2$, the points ρ/m and $1/m$ have real part at most $\frac{1}{2}$. The table follows by evaluating the residues of $\pm\frac{\zeta'}{\zeta}$ at 1 and at ρ, divided by the derivative of the inner argument (1 for the first term, 2 for the second). □

Theorem 9 (Natural boundary)**.** *The line* $\operatorname{Re} s = -\frac{1}{4}$ *is a natural boundary of* $\mathcal{K}$ *and of* $\mathcal{C}$: *neither function has a meromorphic continuation to any disc centred at a point of this line.*

Proof. Split (2) as $\mathcal{K}(s) = P'(2s + \frac{1}{2}) + H(s)$, where H collects the terms $-P'(k(s + \frac{1}{2}))$ for $k \geq 1$ and $P'((k+1)s + \frac{k}{2})$ for $k \geq 2$. Their arguments have real parts $k(\sigma + \frac{1}{2})$ and $k(\sigma + \frac{1}{2}) + \sigma$. For $\sigma > -\frac{1}{3}$ and $k \geq 2$, both are at least $\min\big(\frac{1}{3},\ 3\sigma + 1\big) > 0$, and for $k = 1$ the first is $\sigma + \frac{1}{2} > \frac{1}{6}$. All these real parts grow linearly in k. As in the proof of Theorem 8, H is therefore meromorphic in $\operatorname{Re} s > -\frac{1}{3}$.

Suppose $\mathcal{K}$ were meromorphic in a disc D centred at s_* with $\operatorname{Re} s_* = -\frac{1}{4}$. Shrinking D, we may assume $D \subset \{\operatorname{Re} s > -\frac{1}{3}\}$. Then $P'(2s+\frac{1}{2}) = \mathcal{K}(s) - H(s)$ would be meromorphic in D, so P' would be meromorphic in a disc D' centred at $w_* = 2s_* + \frac{1}{2}$, a point of $\operatorname{Re} w = 0$. Since P' has only finitely many poles in D', there is a point $w_1 \in D'$ with $\operatorname{Re} w_1 = 0$ and a disc $D_1 \subset D'$ around w_1 on which P' is holomorphic. On D_1 the function $P(w_2) + \int_{w_2}^{w} P'(u)\,du$, with $w_2 \in D_1$, $\operatorname{Re} w_2 > 0$, is holomorphic and agrees with P on $D_1 \cap \{\operatorname{Re} w > 0\}$. So P would continue analytically across the line at w_1. This contradicts the theorem of Landau and Walfisz [LW] that $\operatorname{Re} w = 0$ is a natural boundary of P.

Finally, if $\mathcal{C}$ were meromorphic in such a disc, so would be $\mathcal{K} = \mathcal{C}/\zeta$, because ζ is meromorphic and not identically zero. □

Proposition 10 (Critical damping)**.** *Let $0 < \alpha < 1$, and assume RH unless $\alpha = \frac{1}{2}$. In the expansion $\mathcal{K}_\alpha(s) = \sum_{k\geq 1}\big[P'(k(s+\alpha)+s) - P'(k(s+\alpha))\big]$, a zero $\rho = \frac{1}{2} + i\gamma_\rho$ on the critical line makes the two terms $-P'(s+\alpha)$ and $P'(2s+\alpha)$ singular at*

$$s = \big(\tfrac{1}{2} - \alpha\big) + i\gamma_\rho \qquad \text{and} \qquad s = \tfrac{1}{2}\big(\tfrac{1}{2} - \alpha\big) + \tfrac{i\gamma_\rho}{2}.$$

For $0 < \alpha < \frac{3}{4}$, and in particular for $\alpha = \frac{1}{2}$, no term other than these two families is singular at these points. For $\alpha = \frac{1}{2}$ the point $i\gamma$ of the first family coincides with a point of the second exactly when 2γ is an ordinate; the residue there is $-m_\gamma + \frac{1}{2}m_{2\gamma}$. So $i\gamma$ is a pole unless $m_{2\gamma} = 2m_\gamma$, and $i\gamma/2$ is a pole unless $\gamma/2$ is an ordinate with $m_{\gamma/2} = \frac{1}{2}m_\gamma$. This case is unconditional: a term of the expansion with indices (m,k) is singular at a point of the imaginary axis only if ζ has a zero of real part $mk/2$, which is ≥ 1 unless $m = k = 1$; so a zero off the line cannot produce or cancel a singularity at these points. For $\alpha \neq \frac{1}{2}$, under RH, the two families lie on the different lines $\operatorname{Re} s = \frac{1}{2} - \alpha$ and $\operatorname{Re} s = \frac{1}{2}(\frac{1}{2} - \alpha)$, they are disjoint, and all their points are poles, with residues $-m_\rho$ and $\frac{1}{2}m_\rho$. Without RH a hypothetical zero off the line could cancel one of them (a zero at $1 - \alpha + 2i\gamma_\rho$ would cancel the first-family point through $P'(2s+\alpha)$, and a zero at $\frac{1}{4} + \frac{\alpha}{2} + \frac{i\gamma_\rho}{2}$ the second-family point through $-P'(s+\alpha)$), which is why RH is assumed. Both families lie on the imaginary axis if and only if $\alpha = \frac{1}{2}$; this is the statement $\frac{1}{2} - \alpha = 0$, the usual normalization $\Lambda(n)/\sqrt{n}$.

Proof. By Corollary 7, the singularities of a term $\pm\mu(m)\frac{\zeta'}{\zeta}(m(as+b))$ coming from zeros $\rho' = \frac{1}{2} + i\gamma'$ lie on the vertical line $\operatorname{Re} s = (\frac{1}{2m} - b)/a$. Here $(a,b) = (k, k\alpha)$ for $-P'(k(s+\alpha))$, and $(a,b) = (k+1, k\alpha)$ for $P'(k(s+\alpha)+s)$. The two families in the statement come from $m = 1$ with $(a,b) = (1,\alpha)$ and $(2,\alpha)$, and lie on $\operatorname{Re} s = \frac{1}{2} - \alpha$ and $\operatorname{Re} s = \frac{1}{2}(\frac{1}{2} - \alpha)$. We check every other (m,k).

Terms $-P'(k(s+\alpha))$. The real part is $\frac{1}{2mk} - \alpha$.

- It equals $\frac{1}{2} - \alpha$ only if $mk = 1$, which is the first family itself.
- It equals $\frac{1}{2}(\frac{1}{2} - \alpha)$ only if $mk = 2/(1+2\alpha) < 2$, that is, $mk = 1$. But then $\alpha = \frac{1}{2}$ and this is again the first family.

Terms $P'((k+1)s + k\alpha)$. The real part is $(\frac{1}{2m} - k\alpha)/(k+1)$.

- It equals $\frac{1}{2} - \alpha$ iff $\alpha = \frac{k+1}{2} - \frac{1}{2m}$. For $k \geq 2$ this forces $\alpha \geq 1$. For $k = 1$ it forces $m \geq 2$ and $\alpha = 1 - \frac{1}{2m} \geq \frac{3}{4}$.
- It equals $\frac{1}{2}(\frac{1}{2} - \alpha)$ iff $\alpha(1-k) = \frac{k+1}{2} - \frac{1}{m}$. For $k \geq 2$ the left side is ≤ 0 and the right side is positive. For $k = 1$ this forces $m = 1$, which is the second family itself.

Singularities coming from the pole of ζ and from the trivial zeros are real. So for $0 < \alpha < \frac{3}{4}$ only the two dominant terms are singular at the stated points, and the residues follow as in Theorem 8. Finally, $\frac{1}{2} - \alpha = 0$ iff $\alpha = \frac{1}{2}$. □

Proposition 10 explains why $\frac{1}{2}$ is the natural exponent. With $\alpha = \frac{1}{2}$, the critical line of ζ is mapped onto the imaginary axis of $\mathcal{K}$, and the ordinates appear unchanged (γ) and halved ($\gamma/2$). We call $\{\gamma_n\} \cup \{\gamma_n/2\}$ the *dual spectrum* of the Caceres Prime Function.

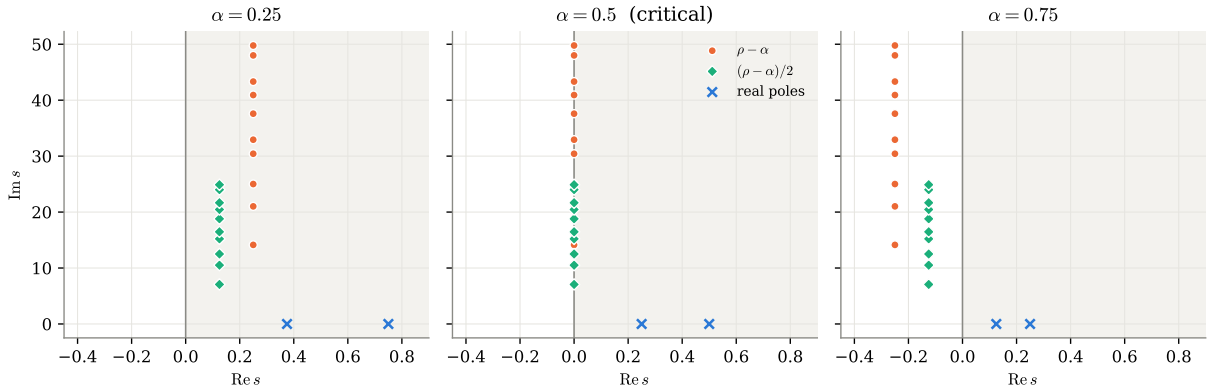


Figure 1: Where the zeros of ζ (first 10 ordinates, assuming RH) land as poles of $\mathcal{K}_\alpha$. Shading marks $\operatorname{Re} s > 0$. For $\alpha = \frac{1}{2}$ both images, $i\gamma$ and $i\gamma/2$, lie exactly on the boundary $\operatorname{Re} s = 0$, and the half-plane is free of zero-induced poles. Crosses mark the poles $1 - \alpha$ and $\frac{1-\alpha}{2}$ inherited from the pole of ζ.

6 A criterion for the Riemann Hypothesis

Theorem 11. *Let*

$$\mathcal{E}(s) = \mathcal{K}(s) - \frac{1}{s - \frac{1}{2}} + \frac{1}{2\,(s - \frac{1}{4})}.$$

The Riemann Hypothesis holds if and only if $\mathcal{E}$ is holomorphic in $\operatorname{Re} s > 0$.

Proof. ($\Rightarrow$) Under RH every zero has $\beta = \frac{1}{2}$, so by Theorem 8 the only poles of $\mathcal{K}$ in $\operatorname{Re} s > 0$ are $\frac{1}{2}$ (residue 1) and $\frac{1}{4}$ (residue $-\frac{1}{2}$). Subtracting their principal parts leaves a holomorphic function.

($\Leftarrow$) Suppose there is a zero ρ_0 with $\beta_0 > \frac{1}{2}$. Define $\rho_{j+1} = 2\rho_j - \frac{1}{2}$ for as long as ρ_{j+1} is again a zero of ζ. Then $\operatorname{Re} \rho_j = \frac{1}{2} + 2^j(\beta_0 - \frac{1}{2})$, which grows without bound, while every zero satisfies $\operatorname{Re} \rho < 1$. So the chain stops at some ρ_J, and $2\rho_J - \frac{1}{2}$ is not a zero. Look at the point $s_* = \rho_J - \frac{1}{2}$. It satisfies $\operatorname{Re} s_* > 0$ and $\operatorname{Im} s_* \neq 0$, so $s_* \neq \frac{1}{2}, \frac{1}{4}$. By Theorem 8, the first term contributes a pole with residue $-m_{\rho_J} \neq 0$ at s_*. The only pole that could cancel it would come from the second term with $\rho' = 2\rho_J - \frac{1}{2}$, and that point is not a zero. Hence $\mathcal{K}$, and therefore $\mathcal{E}$, has a pole at s_*. □

Remark 12. Theorem 11 is a reformulation of RH in the same family as the classical statement that ζ'/ζ is holomorphic in $\operatorname{Re} s > \frac{1}{2}$ except at $s = 1$. Its interest lies in two features. First, the criterion is expressed through a function with explicit non-negative coefficients, $\mathcal{C}(s) = \zeta(s)\mathcal{K}(s)$. Second, the critical line becomes the imaginary axis, so the boundary of the criterion is the natural spectral line $s = it$.

7 The mean value of $a(n)$

Throughout we use the explicit Chebyshev-type bounds of Rosser and Schoenfeld [RS]:

$$\theta(u) < 1.01624\,u \quad (u > 0), \qquad \left|\theta(u) - u\right| < \frac{u}{2\log u} \quad (u \geq 563), \qquad \sum_{p \leq y} \frac{\log p}{p} < \log y \quad (y > 1). \tag{4}$$

Theorem 13 (Mean value with an explicit constant)**.** *For every $x \geq 2$,*

$$\left|A(x) - \mathcal{K}(1)\,x\right| \leq B(x) := 5.0813\sqrt{x} + 3.7263\,x^{1/4} + \frac{\log^2 x}{\log 2} + 1.955\log x + 2,$$

where

$$\mathcal{K}(1) = \sum_{p} \frac{\log p\,(1-p^{-1})}{p^{3/2}-1}.$$

In particular $|A(x) - \mathcal{K}(1)x| \le 5.51\sqrt{x}$ *for all* $x \ge 10^6$.

Proof. By Lemma 2(iii), $A(x) = \sum_{d\le x} b(d)\lfloor x/d\rfloor$. Writing $\lfloor x/d\rfloor = x/d - \{x/d\}$ and using $\sum_d b(d)/d = \mathcal{K}(1)$ (an absolutely convergent series), we get

$$R(x) := A(x) - \mathcal{K}(1)x = -x\sum_{d>x}\frac{b(d)}{d} - \sum_{d\le x} b(d)\Big\{\frac{x}{d}\Big\}, \qquad |R(x)| \le x\sum_{d>x}\frac{|b(d)|}{d} + \sum_{d\le x}|b(d)|. \quad (5)$$

We use $|b(p)| = \log p\, p^{-1/2}$, $|b(p^2)| \le \log p\, p^{-1/2}$ and $|b(p^k)| \le \log p\, p^{-(k-1)/2} \le \log p\, p^{-1}$ for $k \ge 3$.

Head. By partial summation and (4),

$$\sum_{p\le y}\frac{\log p}{\sqrt{p}} = \frac{\theta(y)}{\sqrt{y}} + \frac{1}{2}\int_2^y \frac{\theta(u)}{u^{3/2}}\,du \le 2.0325\sqrt{y}.$$

The primes therefore give at most $2.0325\sqrt{x}$ and the prime squares at most $2.0325\,x^{1/4}$. For each $k \ge 3$ with $2^k \le x$, the powers $p^k \le x$ give at most $\sum_{p\le x^{1/k}} \log p/p < \log x$, and there are fewer than $\log x/\log 2$ such k.

Tail. By partial summation and $\theta(u) < 1.01624u$,

$$x\sum_{p>x}\frac{\log p}{p^{3/2}} \le x\cdot\tfrac{3}{2}\cdot 1.01624\int_x^\infty u^{-3/2}du = 3.0488\sqrt{x},$$

and in the same way

$$x\sum_{p>\sqrt{x}}\frac{\log p}{p^{5/2}} \le 1.6938\,x^{1/4}.$$

For $k \ge 3$, put $y = x^{1/k}$. If $p > y$ then $x/p^k \le (y/p)^3$, so the powers $p^k > x$ with $k \le \log x/\log 2$ contribute at most

$$\sum_k y^3 \sum_{p>y}\log p\, p^{-4} \le \sum_k \tfrac{4}{3}\cdot 1.01624 = 1.355\cdot\frac{\log x}{\log 2} = 1.955\log x.$$

For $k > \log x/\log 2$ every prime satisfies $p^k > x$. Using $p^{-(3k-1)/2} \le 2^{-(3k-1)/2}(2/p)^4$ and $\sum_p \log p\, p^{-4} \le -\zeta'(4) < 0.069$, these k contribute at most $2.42\,x^{-1/2} \le 2$.

Adding the head and the tail gives $B(x)$. The last claim follows because $B(x)/\sqrt{x}$ is decreasing for $x \ge e^4$ and $B(10^6) = 5.5035\cdot 10^3$. □

Proposition 14 (Rigorous value of $\mathcal{K}(1)$)**.**

$$1.16045671 \le \mathcal{K}(1) \le 1.16047301.$$

The value obtained from the Prime Zeta representation (2)*,*

$$\mathcal{K}(1) = \sum_{k\ge1}\Big[P'\big(\tfrac{3k}{2}+1\big) - P'\big(\tfrac{3k}{2}\big)\Big] = 1.1604648419\ldots,$$

lies in this interval. This value of the series is computed numerically and is not certified.

Proof. Write $\mathcal{K}(1) = \sum_{p\le P} T(p) + \sum_{p>P} T(p)$ with $T(p) = \log p\,(1-p^{-1})/(p^{3/2}-1)$ and $P = 10^8$. For $p > P$ we have $T(p) = \log p\, p^{-3/2}\phi(p)$ with $1 - P^{-1} \le \phi(p) \le 1$. Moreover

$$\sum_{p>P}\log p\, p^{-3/2} = -\theta(P)P^{-3/2} + \tfrac{3}{2}\int_P^\infty \theta(u)u^{-5/2}du,$$

which lies in

$$-\theta(P)P^{-3/2} + 3P^{-1/2}\,[1-\eta,\,1+\eta], \qquad \eta = \frac{1}{2\log P},$$

by (4). The finite sum, of 5 761 455 positive terms with total 1.160265, and $\theta(10^8) = 99\,987\,730.018\ldots$ were computed in double precision with sorted summation. The rounding error of the finite sum is at most $n\,u\sum|T| \le 5.8\cdot 10^6\cdot 2^{-52}\cdot 1.17 < 2\cdot 10^{-9}$, including the relative error $\le 2^{-52}$ of each logarithm and power. The error in $\theta(10^8)$ is at most 0.13, which changes $\theta(P)P^{-3/2}$ by less than 10^{-12}. Both are absorbed by widening the interval by 10^{-8}, which the stated rounded endpoints already include. □

Theorem 15 (The second term)**.** *There is an absolute constant $c_* > 0$ such that, unconditionally,*

$$\boxed{A(x) = \mathcal{K}(1)\,x + 2\zeta(\tfrac12)\sqrt{x} + O\big(\sqrt{x}\,e^{-c_*\sqrt{\log x}}\big)}$$

Under RH, the error term can be replaced by $O_\varepsilon(x^{1/3+\varepsilon})$.

Proof. Unconditional part. In (5), the prime powers p^k with $k \ge 2$ contribute $O(x^{1/4})$, as shown in the proof of Theorem 13. Hence

$$R(x) = -x\,S(x) - U(x) + O(x^{1/4}), \qquad S(x) = \sum_{p>x}\frac{\log p}{p^{3/2}}, \qquad U(x) = \sum_{p\le x}\frac{\log p}{\sqrt{p}}\Big\{\frac{x}{p}\Big\}.$$

By the prime number theorem with the de la Vallée Poussin error term, $\theta(u) = u + E(u)$ with $E(u) \ll u\,e^{-c_2\sqrt{\log u}}$. Then

$$S(x) = \int_x^\infty u^{-3/2}\,du + \int_x^\infty u^{-3/2}\,dE(u) = 2x^{-1/2} + O\big(x^{-1/2}e^{-c_2\sqrt{\log x}}\big).$$

For $U(x)$, fix an integer $J \ge 2$. The primes $p \le x/J$ contribute at most $2.0325\sqrt{x/J}$. On each interval $x/(m+1) < u \le x/m$ with $1 \le m < J$, we have $\{x/u\} = x/u - m$, so the corresponding part of U is $\int g_m\,d\theta$ with the smooth function $g_m(u) = xu^{-3/2} - mu^{-1/2}$. On that interval $0 \le g_m(u) \le ((m+1)/x)^{1/2}$ and $\int|g_m'| \ll m^{1/2}x^{-1/2}$. Integrating by parts against E, each interval contributes $\int g_m\,du + O\big(\sqrt{x}\,m^{-1/2}e^{-c_2\sqrt{\log(x/J)}}\big)$. Summing over $m < J$,

$$U(x) = \int_{x/J}^x \frac{\{x/u\}}{\sqrt{u}}\,du + O\big(\sqrt{x/J} + \sqrt{xJ}\,e^{-c_2\sqrt{\log(x/J)}}\big).$$

With $u = x/w$, the integral equals

$$\sqrt{x}\int_1^J \{w\}w^{-3/2}\,dw = \sqrt{x}\,I + O(\sqrt{x/J}), \qquad I = \int_1^\infty \{w\}w^{-3/2}\,dw.$$

Taking $J = \lfloor e^{(c_2/2)\sqrt{\log x}}\rfloor$ makes all error terms $O(\sqrt{x}\,e^{-c_*\sqrt{\log x}})$. Hence $R(x) = -(2+I)\sqrt{x} + O(\sqrt{x}\,e^{-c_*\sqrt{\log x}})$.

Finally, the classical formula $\zeta(s) = \frac{s}{s-1} - s\int_1^\infty\{w\}w^{-s-1}dw$, valid for $\operatorname{Re}s > 0$ [Tit, (2.1.5)], gives at $s = \frac12$ the identity $\zeta(\frac12) = -1 - \frac12 I$. Therefore $-(2+I) = 2\zeta(\frac12)$.

Under RH. Let $c_x = 1 + 1/\log x$ and $x^{1/2} \le T \le x$. Since $0 < a(n) \le c_0\,\omega(n) \le c_0\log n/\log 2$ and $\mathcal{C}(\sigma) \ll (\sigma-1)^{-1}$ as $\sigma \to 1^+$, the truncated Perron formula [Tit, Lemma 3.12] gives

$$A(x) = \frac{1}{2\pi i}\int_{c_x - iT}^{c_x+iT}\mathcal{C}(s)\,x^s\frac{ds}{s} + O\Big(\frac{x\log^2 x}{T}\Big).$$

Fix $\delta \in (0, \frac14)$. Under RH, $\mathcal{K}$ is holomorphic in $\operatorname{Re}s > 0$ apart from the poles $\frac12$ and $\frac14$ (Theorem 8). Moreover, every argument of ζ'/ζ appearing in Corollary 7 has real part at least $\frac12 + \delta$ when $\operatorname{Re}s \ge \delta$.

Under RH, the partial fraction formula $\frac{\zeta'}{\zeta}(u) = \sum_{|\operatorname{Im} u-\gamma|<1} \frac{1}{u-\rho} + O(\log|\operatorname{Im} u|)$ [Dav, Ch. 15] gives $\frac{\zeta'}{\zeta}(u) \ll_\delta \log(|\operatorname{Im} u|+2)$ for $\frac12+\delta \le \operatorname{Re} u \le 2$, since there $|u-\rho| \ge \delta$. For $\operatorname{Re} u \ge 2$, $\zeta'/\zeta(u)$ is bounded by $\sum_n \Lambda(n)n^{-2}$. Hence $\mathcal{K}(s) \ll_\delta \log(|t|+2)$ for $\operatorname{Re} s \ge \delta$. RH also implies the Lindelöf hypothesis [Tit, §14.2], so $\zeta(\sigma+it) \ll |t|^{\max(0,\frac12-\sigma)+\varepsilon}$ for $\sigma \ge \delta$.

Move the segment to $\operatorname{Re} s = \delta$. The residues at $s = 1, \frac12, \frac14$ are $\mathcal{K}(1)x$, $2\zeta(\frac12)\sqrt{x}$ and $-2\zeta(\frac14)x^{1/4}$. The new vertical segment contributes $\ll x^\delta T^{1/2-\delta+2\varepsilon}$, and the horizontal segments contribute $\ll x^{1+\varepsilon}/T$. With $T = x^{2/3}$ and $\delta = \varepsilon$, all error terms, including $x^{1/4}$, are $O(x^{1/3+3\varepsilon})$. □

Table 1 illustrates Theorem 15: the ratio $(A(x)-\mathcal{K}(1)x)/\sqrt{x}$ approaches $2\zeta(\frac12) = -2.92071$, and numerically, over $10 \le x \le 10^8$, the ratio $|R(x)|/B(x)$ never exceeds 0.562.

Table 1: Mean value of $a(n)$. The last column is $R_2(x) = A(x) - \mathcal{K}(1)x - 2\zeta(\frac12)\sqrt{x} + 2\zeta(\frac14)x^{1/4}$.

x	$A(x)$	$(A(x)-\mathcal{K}(1)x)/\sqrt{x}$	$R_2(x)$
10^3	1 077.489	−2.6239	0.238
10^4	11 330.017	−2.7463	1.174
10^5	115 150.436	−2.8336	−1.363
10^6	1 157 597.587	−2.8673	2.018
10^7	11 595 503.533	−2.8919	−0.261
10^8	116 017 439.601	−2.9045	−0.154

8 The explicit formula: the zeros at constant amplitude

We now prove the explicit formula for the Riesz mean $F(x) = \sum_{n\le x} a(n)\log(x/n)$ under RH. For $c > 1$, Perron's formula

$$F(x) = \frac{1}{2\pi i}\int_{(c)} \mathcal{C}(s)\, x^s\, \frac{ds}{s^2}$$

holds with an absolutely convergent integral, since $\frac{1}{2\pi i}\int_{(c)} y^s s^{-2} ds$ equals $\log y$ for $y \ge 1$ and 0 for $0 < y < 1$.

Lemma 16 (Poles of $\mathcal{K}$ near the imaginary axis)**.** *Assume RH. The poles of $\mathcal{K}$ in $\operatorname{Re} s > -\frac{1}{12}$ are contained in the set*

$$\tfrac12, \quad \tfrac14, \quad 0, \quad \pm i\gamma, \quad \pm\tfrac{i\gamma}{2},$$

where γ runs over the positive ordinates of the zeros. All of them are simple. The residues are 1 at $\frac12$, $-\frac12$ at $\frac14$, $-\frac{1}{12}$ at 0, $-m_\gamma$ from the first family at $i\gamma$, and $+\frac12 m_\gamma$ from the second family at $\frac{i\gamma}{2}$. Where the two families meet, at $\gamma = \gamma'/2$, the residues add, and the point is a pole unless $-m_\gamma + \frac12 m_{\gamma'} = 0$. That cannot happen if the zeros are simple. The points $\frac12, \frac14, 0$ are always poles.

Proof. By Corollary 7, $\mathcal{K}$ is a locally uniformly convergent sum of terms $\pm\mu(m)\frac{\zeta'}{\zeta}\big(m(\alpha s+\beta)\big)$, with $(\alpha,\beta) = (k, \frac{k}{2})$ (sign $-$) or $(k+1, \frac{k}{2})$ (sign $+$). Each term has simple poles where $m(\alpha s+\beta) \in \{1\} \cup \{\rho\} \cup \{-2,-4,\dots\}$, that is, at

$$s = \frac{1/m-\beta}{\alpha}, \qquad s = \frac{\rho/m-\beta}{\alpha}, \qquad s = \frac{-2j/m-\beta}{\alpha}.$$

Under RH $\operatorname{Re}\rho = \frac12$. A direct check over (m,k) shows the following. The real parts of these points that are $> -\frac{1}{12}$ are $\frac12$ (only $m=k=1$, first type), $\frac14$ (only $m=k=1$, second type) and 0. The value 0 arises from the pole of ζ for $mk = 2$, and from the zeros only for $m=k=1$, at $i\gamma$ and $i\gamma/2$. The next values are $-\frac{1}{12}$ (real, $m=3$, $k=1$, second type), then $-\frac18$, $-\frac16$, and so on. The trivial zeros give real parts $\le -\frac14$.

The residue of $\frac{\zeta'}{\zeta}(m(\alpha s+\beta))$ is $-1/(m\alpha)$ at a pole coming from 1, and $m_\rho/(m\alpha)$ at a pole coming from ρ. At $s=0$ the four contributions are $-\frac12+\frac14+\frac12-\frac13=-\frac{1}{12}$, from $(m,k)=(2,1)$ and $(1,2)$ of each type. □

Theorem 17 (Explicit formula for F under RH)**.** *Assume RH and let $-\frac{1}{12}<\sigma_0<0$. For $x\ge 1$,*

$$F(x)=\mathcal{K}(1)\,x+4\zeta(\tfrac12)\sqrt{x}-8\zeta(\tfrac14)\,x^{1/4}+\tfrac{1}{48}\log^2 x+c_1\log x+c_0 \\ +2\,\mathrm{Re}\sum_{\gamma>0}\frac{m_\gamma}{\gamma^2}\Big[\zeta(i\gamma)\,x^{i\gamma}-2\,\zeta\big(\tfrac{i\gamma}{2}\big)\,x^{i\gamma/2}\Big]+O_{\sigma_0}\big(x^{\sigma_0}\big).$$

Here c_1 and c_0 are the coefficients in the Laurent expansion $\mathcal{C}(s)=\frac{1}{24s}+c_1+c_0 s+O(s^2)$ at $s=0$, and the series over the zeros converges absolutely. In particular, $F(x)$ minus its smooth part is a uniformly almost periodic function of $\log x$, with frequencies in $\{\pm\gamma\}\cup\{\pm\gamma/2\}$, up to $O(x^{-1/12+\varepsilon})$.

Proof. Step 1: bounds on vertical lines. Let $\sigma=\sigma_0$. In (2) the arguments of P' have real parts

$$\tfrac12+\sigma_0\in(\tfrac{5}{12},\tfrac12),\quad \tfrac12+2\sigma_0\in(\tfrac13,\tfrac12),\quad 1+2\sigma_0\in(\tfrac56,1),\quad 1+3\sigma_0\in(\tfrac34,1),$$

and greater than 1 for the terms with $k\ge 3$. Expanding each P' by Corollary 7, the terms whose argument has real part greater than 1 are bounded uniformly, since $|\frac{\zeta'}{\zeta}(u)|\le|\frac{\zeta'}{\zeta}(\mathrm{Re}\,u)|$ there. They are summable in m and k because $|\frac{\zeta'}{\zeta}(u)|\le C\,2^{-\mathrm{Re}\,u}$ for $\mathrm{Re}\,u\ge 2$. Only finitely many terms remain: $\frac{\zeta'}{\zeta}$ at real parts $\frac12+\sigma_0$, $\frac12+2\sigma_0$, $1+2\sigma_0$, $1+3\sigma_0$ and $1+4\sigma_0$, none of which equals $\frac12$.

For a real part $\sigma_u\in(\frac12,2]$, RH and the partial fraction formula give $\frac{\zeta'}{\zeta}(u)\ll\log|\mathrm{Im}\,u|/(\sigma_u-\frac12)$. For $\sigma_u<\frac12$, the functional equation gives $\frac{\zeta'}{\zeta}(u)=-\frac{\zeta'}{\zeta}(1-u)+O(\log|\mathrm{Im}\,u|)$, which reduces to the previous case. Hence

$$\mathcal{K}(\sigma_0+it)\ll_{\sigma_0}\log(|t|+2).$$

By convexity [Tit, §5.1], $\zeta(\sigma+it)\ll(|t|+2)^{1/2-\sigma_0}\log(|t|+2)$ uniformly for $\sigma_0\le\sigma\le 2$.

Step 2: horizontal lines. Choose $T_j\in[j,j+1]$ with $|T_j-\gamma|\ge c/\log j$ and $|2T_j-\gamma|\ge c/\log j$ for every ordinate γ. This is possible for a small absolute $c>0$, since $[j-1,j+3]$ and $[2j-2,2j+4]$ contain $O(\log j)$ ordinates. On $\mathrm{Im}\,s=T_j$, $\sigma_0\le\mathrm{Re}\,s\le 2$, only the two arguments $s+\frac12$ and $2s+\frac12$ cross the critical line. For them the partial fraction formula gives $\frac{\zeta'}{\zeta}\ll\log^2 T_j$. All other arguments stay at distance at least $\frac16$ from $\mathrm{Re}\,u=\frac12$, which gives $\ll\log T_j$. Hence $\mathcal{K}(\sigma+iT_j)\ll\log^2 T_j$.

Step 3: residues. Apply the residue theorem to the rectangle with vertices $\sigma_0\pm iT_j$ and $2\pm iT_j$. By Lemma 16 the poles of $\mathcal{C}(s)x^s/s^2$ inside are as follows. The first family is cut at $\gamma<T_j$ and the second at $\gamma<2T_j$; both cutoffs tend to infinity.

- $s=1$, residue $\mathcal{K}(1)x$;
- $s=\frac12$ and $s=\frac14$, residues $\zeta(\frac12)\cdot 1\cdot x^{1/2}/\frac14=4\zeta(\frac12)\sqrt{x}$ and $\zeta(\frac14)(-\frac12)x^{1/4}/\frac{1}{16}=-8\zeta(\frac14)x^{1/4}$;
- the triple pole at $s=0$, where $\mathcal{C}(s)=\frac{\zeta(0)\cdot(-1/12)}{s}+\cdots=\frac{1}{24s}+c_1+c_0 s+\cdots$, with residue $\frac{1}{48}\log^2 x+c_1\log x+c_0$;
- $s=\pm i\gamma$, with residue $\zeta(i\gamma)(-m_\gamma)x^{i\gamma}/(i\gamma)^2=m_\gamma\zeta(i\gamma)x^{i\gamma}/\gamma^2$ and its conjugate;
- $s=\pm\frac{i\gamma}{2}$, with residue $\zeta(\frac{i\gamma}{2})\frac{m_\gamma}{2}x^{i\gamma/2}/(\frac{i\gamma}{2})^2=-2m_\gamma\zeta(\frac{i\gamma}{2})x^{i\gamma/2}/\gamma^2$ and its conjugate.

Here $\zeta(\pm i\gamma)\ne 0$, since ζ has no zeros on $\mathrm{Re}\,s=0$.

Step 4: limits. On the horizontal sides, the integrand is $\ll x^2\,T_j^{1/2-\sigma_0}\log^3 T_j/T_j^2\to 0$. The left side converges absolutely as $T_j\to\infty$, because $\int(|t|+2)^{1/2-\sigma_0-2}\log^2(|t|+2)\,dt<\infty$, and it is $\ll x^{\sigma_0}$. Finally, $|\zeta(i\gamma)|=|\chi(i\gamma)||\zeta(1-i\gamma)|\ll\gamma^{1/2}\log\gamma$, and likewise $|\zeta(i\gamma/2)|\ll\gamma^{1/2}\log\gamma$. Since $\sum_\gamma m_\gamma\gamma^{-3/2}\log\gamma<\infty$, the residue sums converge absolutely to the full series. □

The formula for A. Formally differentiating, or repeating the argument with $1/s$ in place of $1/s^2$, gives the corresponding expansion for $A(x)$:

$$A(x) \approx \mathcal{K}(1)x + 2\zeta(\tfrac{1}{2})\sqrt{x} - 2\zeta(\tfrac{1}{4})x^{1/4} + \tfrac{1}{24}\log x + c_1 + 2\,\mathrm{Re}\sum_{\gamma>0} m_\gamma \Big[-\frac{\zeta(i\gamma)}{i\gamma}\,x^{i\gamma} + \frac{\zeta(i\gamma/2)}{i\gamma}\,x^{i\gamma/2}\Big], \quad (6)$$

where the sum is not absolutely convergent. We use (6) only as the model in the spectral comparison below. Rigorous statements about A are Theorems 13 and 15.

Why this matters. In the explicit formula for $\psi(x) = \sum_{n\le x}\Lambda(n)$, a zero contributes $x^\rho/\rho = x^{1/2+i\gamma}/\rho$, an oscillation whose amplitude grows like $\sqrt{x}$. The square-root damping in (E1) removes exactly this growth. In Theorem 17 every zero contributes a *pure phase* $x^{i\gamma}$, and a second copy $x^{i\gamma/2}$, at constant amplitude. By contrast, a zero off the line would make $\mathcal{K}$ have a pole in $\mathrm{Re}\,s > 0$ (Theorem 11), and the corresponding term would grow like a positive power of x.

Numerical verification

We computed $a(n)$ for all $n \le 10^8$ and used the 96 zeros with $\gamma < 230$ (Appendix A). After removing the smooth terms:

- the fitted coefficients of $\log x$ in A and of $\log^2 x$ in F are 0.0459 and 0.0226, about 10% above the proved values $\frac{1}{24} = 0.0417$ and $\frac{1}{48} = 0.0208$ (no error bars);
- the residual of F follows the zero sum of Theorem 17 with correlation 0.96–0.99 in every decade between 10^3 and 10^8;
- the windowed Fourier transform $\Phi(t)$ of the residual of A in the variable $\log x$ reproduces the predicted spectrum with correlation 0.985 over $5 \le t \le 60$, and the main peaks agree to within a few percent (Figure 2, Table 2).

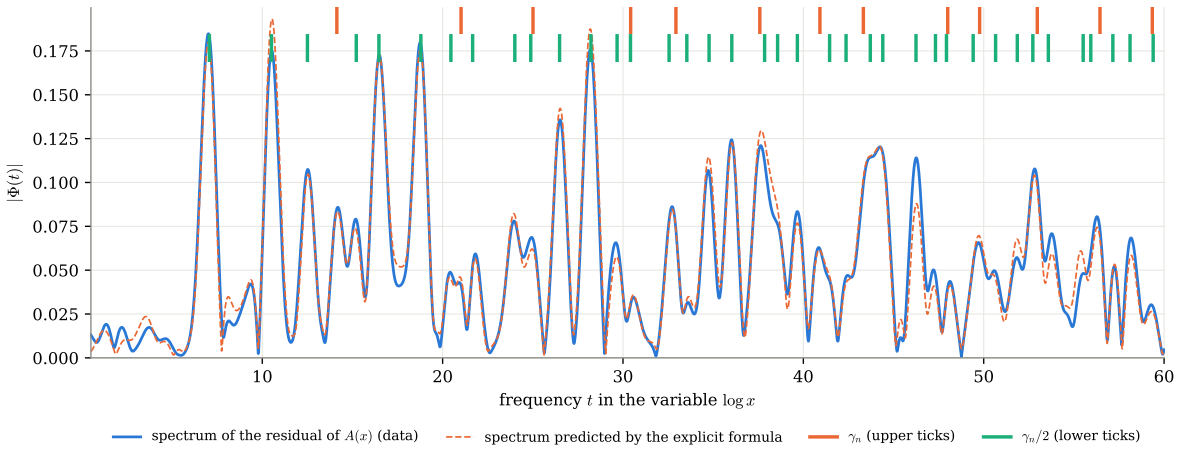


Figure 2: Spectrum of the arithmetic oscillation of $A(x) = \sum_{n\le x} a(n)$ in the variable $\log x$, computed with all $n \le 10^8$ (solid), against the spectrum of the explicit formula (6) with 96 zeros (dashed). Upper ticks mark γ_n and lower ticks mark $\gamma_n/2$.

Table 2: Largest spectral peaks of the residual of A: measured height against the height predicted by (6). The last column gives the nearest frequency of the dual spectrum. With a frequency resolution of about 0.9, nearby frequencies merge.

t	measured	predicted	difference	nearest
7.02	0.1848	0.1798	2.8%	$\gamma_1/2 = 7.07$
12.53	0.1075	0.1047	2.6%	$\gamma_3/2 = 12.51$
16.50	0.1726	0.1727	0.1%	$\gamma_5/2 = 16.47$
18.75	0.1798	0.1759	2.2%	$\gamma_6/2 = 18.79$
26.52	0.1358	0.1422	4.6%	$\gamma_{11}/2 = 26.49$
36.02	0.1244	0.1224	1.6%	$\gamma_{18}/2 = 36.03$
37.63	0.1211	0.1293	6.4%	$\gamma_6 = 37.59$
44.25	0.1204	0.1205	0.1%	$\gamma_{25}/2 = 44.40$

9 The C-transformation and the Laplace transform of the kernel

9.1 The C-transformation

The C-transformation [CP] measures the difference between a discrete sum and its continuous counterpart. For a weight w with continuous density m_w and a test function f,

$$C_N(f;w) = \sum_{n\le N} w(n)f(n) - \int_1^N m_w(x)f(x)\,dx.$$

For $w=1$, $m_w=1$ and $f(x)=x^{-s}$, its limit is $\zeta(s)+\frac{1}{1-s}$.

We apply it with $f(x)=x^{-s}$ to the kernel weight b (Lemma 2) and to a. For each weight we use as density the function whose Mellin transform is the principal part of $\mathcal{K}$ (respectively $\mathcal{C}$) at its real poles $\frac12,\frac14,0$ (respectively $1,\frac12,\frac14,0$):

$$m_b(x) = x^{-1/2} - \tfrac12 x^{-3/4} - \tfrac{1}{12}x^{-1}, \qquad m_a(x) = \mathcal{K}(1) + \zeta(\tfrac12)x^{-1/2} - \tfrac12\zeta(\tfrac14)x^{-3/4} + \tfrac{1}{24}x^{-1}.$$

The three terms of m_b correspond to the poles of $\mathcal{K}$ at $\frac12,\frac14,0$, with residues $1,-\frac12,-\frac{1}{12}$ (Theorem 8 and Proposition 24 below). We write

$$B(x)=\sum_{n\le x} b(n), \qquad M_b(x)=\int_1^x m_b = 2\sqrt{x} - 2x^{1/4} - \tfrac{1}{12}\log x, \qquad R_b(x)=B(x)-M_b(x),$$

and similarly $M_a(x)=\int_1^x m_a$ and $R_a(x)=A(x)-M_a(x)$.

Proposition 18 ($\mathcal{E}$ is the C-transform of the kernel)**.** *For every $N\ge 1$ and every s,*

$$C_N(x^{-s};b) = R_b(N)N^{-s} + s\int_1^N R_b(x)\,x^{-s-1}dx.$$

For $\operatorname{Re} s>1$*, the limit as $N\to\infty$ exists and*

$$\boxed{\lim_{N\to\infty} C_N(x^{-s};b) = \mathcal{E}(s) + \frac{1}{12s} = s\int_1^\infty R_b(x)\,x^{-s-1}\,dx}$$

Proof. Since $B(1)=M_b(1)=0$, Stieltjes integration by parts gives

$$C_N = \int_1^N x^{-s}\,d(B-M_b)(x) = R_b(N)N^{-s} + s\int_1^N R_b x^{-s-1}dx.$$

For $\operatorname{Re} s > 1$, $\sum b(n)n^{-s} = \mathcal{K}(s)$ and

$$\int_1^\infty m_b(x)x^{-s}dx = \frac{1}{s-\frac{1}{2}} - \frac{1}{2(s-\frac{1}{4})} - \frac{1}{12s},$$

and the difference is $\mathcal{E}(s) + \frac{1}{12s}$. □

Lemma 19. *Under RH, $R_b(x) = O(\log^3 x)$.*

Proof. Under RH, $\theta(u) = u + \Delta(u)$ with $\Delta(u) \ll \sqrt{u}\log^2 u$ [Dav, Ch. 18]. Hence

$$\sum_{p\le x}\frac{\log p}{\sqrt{p}} = \int_{2^-}^{x} u^{-1/2}\,d\theta(u) = 2\sqrt{x} + O(1) + \frac{\Delta(x)}{\sqrt{x}} + \frac{1}{2}\int_2^x \frac{\Delta(u)}{u^{3/2}}\,du = 2\sqrt{x} + O(\log^3 x).$$

The prime squares contribute $\sum_{p\le\sqrt{x}} \log p\,(p^{-1} - p^{-1/2})$. By Mertens' theorem and the previous line with x replaced by $\sqrt{x}$, this equals $-2x^{1/4} + O(\log^3 x)$. The powers p^k with $k \ge 3$ contribute $O(\log^2 x)$, as in the proof of Theorem 13. So $B(x) = 2\sqrt{x} - 2x^{1/4} + O(\log^3 x)$, and the term $\frac{1}{12}\log x$ is absorbed in the error. □

Theorem 20 (RH as a constant-scale statement)**.** *The following are equivalent:*

(i) the Riemann Hypothesis;

(ii) $C_N(x^{-s}; b)$ converges as $N \to \infty$ for every s with $\operatorname{Re} s > 0$;

(iii) $R_b(x) = O_\varepsilon(x^\varepsilon)$ for every $\varepsilon > 0$.

Proof. (i)⇒(iii) is Lemma 19.

(iii)⇒(ii): by Proposition 18, $R_b(N)N^{-s} \to 0$ and the integral converges absolutely when $\operatorname{Re} s > \varepsilon$.

(ii)⇒(i): suppose $C_N(x^{-\varepsilon}; b)$ converges, and put $G(y) = \int_1^y x^{-\varepsilon} d(B - M_b)$. Then

$$R_b(N) = \int_1^N x^\varepsilon\,dG(x) = N^\varepsilon\big(G(N) - G(\infty)\big) - \varepsilon\int_1^N x^{\varepsilon-1}\big(G(x) - G(\infty)\big)dx + O(1) = o(N^\varepsilon).$$

So (iii) holds, and the integral of Proposition 18 converges locally uniformly in $\operatorname{Re} s > 0$. It therefore defines a holomorphic function there, which agrees with $\mathcal{E}(s) + \frac{1}{12s}$ for $\operatorname{Re} s > 1$. Hence $\mathcal{E}$ is holomorphic in $\operatorname{Re} s > 0$, and Theorem 11 gives RH. □

Remark 21. By partial summation,

$$\sum_{p\le x}\frac{\log p}{\sqrt{p}} - 2\sqrt{x} = \frac{\theta(x) - x}{\sqrt{x}} + \frac{1}{2}\int_2^x \frac{\theta(u) - u}{u^{3/2}}\,du + O(1).$$

Moreover $R_b(x) = D(x) - D(\sqrt{x}) + O(\log^2 x)$ with $D(y) = \sum_{p\le y} \log p/\sqrt{p} - 2\sqrt{y}$, and iterating this in $x \mapsto \sqrt{x}$ recovers D from R_b. So Theorem 20 is von Koch's criterion $\theta(x) - x = O(x^{1/2+\varepsilon})$ written in normalized form. The $\sqrt{}$-damping of $\mathcal{C}$ moves the RH scale from $x^{1/2}$ to the constant scale: RH says that the discrete-minus-continuous difference of the prime kernel stays $O(x^\varepsilon)$.

Proposition 22 (C-transform of $\mathcal{C}$)**.** *$C_N(x^{-s}; a)$ converges unconditionally for $\operatorname{Re} s \ge \frac{1}{2}$, and under RH for $\operatorname{Re} s > \frac{1}{3}$. The limit is*

$$\widehat{\mathcal{C}}(s) = \mathcal{C}(s) - \frac{\mathcal{K}(1)}{s-1} - \frac{\zeta(\frac{1}{2})}{s-\frac{1}{2}} + \frac{\zeta(\frac{1}{4})}{2(s-\frac{1}{4})} - \frac{1}{24s}.$$

If $\mathcal{K}$ has a pole on $\operatorname{Re} s = 0$ (in particular, unconditionally, at $s = i\gamma_1/2$, where no first-family pole can cancel it because γ_1 is the least ordinate), the C-transform of $\mathcal{C}$ converges in no half-plane $\operatorname{Re} s > \sigma$ with $\sigma < 0$.

Proof. As in Proposition 18, $C_N(x^{-s};a) = R_a(N)N^{-s} + s\int_1^N R_a x^{-s-1}dx$. By Theorem 15, $R_a(x) = O(\sqrt{x}\,e^{-c_*\sqrt{\log x}})$ unconditionally, with the constant $c_* > 0$ of that theorem. Since $\int^\infty e^{-c_*\sqrt{\log x}}\frac{dx}{x} < \infty$, the integral converges absolutely for $\operatorname{Re} s \geq \frac{1}{2}$. Under RH, $R_a(x) = O(x^{1/3+\varepsilon})$. The value of the limit follows by computing $\int_1^\infty m_a x^{-s}dx$ for $\operatorname{Re} s > 1$, as before.

Finally, a convergent C-transform defines a holomorphic function in its half-plane. But $\widehat{\mathcal{C}} = \zeta\mathcal{K} - \cdots$ has a pole wherever $\mathcal{K}$ does on $\operatorname{Re} s = 0$, because $\zeta(i\gamma) \neq 0$. □

Numerical observation 23. With all $n \leq 10^8$:

- $R_b(x)$ stays in $[-1.56, 0.77]$ for $10 \leq x \leq 10^8$ (infimum -1.557 as $x \to 11897^-$, supremum 0.768 at $x = 113$; left and right limits at every prime power); its maximum modulus per decade stays below 1.56 and shows no systematic growth (Table 3, Figure 3).
- The C-transform of the kernel at $s = 0.3$ gives -0.20291, -0.19737 and -0.19913 at $N = 10^6$, 10^7 and 10^8, against $\mathcal{E}(0.3) + \frac{1}{3.6} = -0.19737$ computed by analytic continuation: agreement to about $3 \cdot 10^{-3}$ (the five-digit match at 10^7 is accidental). At $s = 0.2 + 10i$ and $N = 10^8$ it gives $0.230 + 1.158i$, against $0.223 + 1.134i$.
- In contrast, R_a grows slowly: its rms over a decade goes from 0.98 to 2.64, and its maximum from 2.2 to 9.5, between 10^2 and 10^8.

The growth of R_a comes from the integer layer. Indeed, $A(x) = \sum_{m \leq x} B(x/m)$, so R_a contains hyperbola sums such as $\sum_p \frac{\log p}{\sqrt{p}}(\{\frac{x}{p}\} - \frac{1}{2})$, which behave like the error term of a divisor problem. Such sums are independent of the zeros of ζ.

Table 3: Size of the two C-transform remainders, decade by decade.

decade ending at	10^2	10^3	10^4	10^5	10^6	10^7	10^8
$\max\lvert R_b\rvert$	1.00	1.36	1.54	1.56	1.40	1.49	1.31
$\max\lvert R_a\rvert$	2.20	3.08	3.74	6.10	6.50	7.74	9.47
rms R_a	0.98	1.13	1.37	1.57	1.97	2.19	2.64

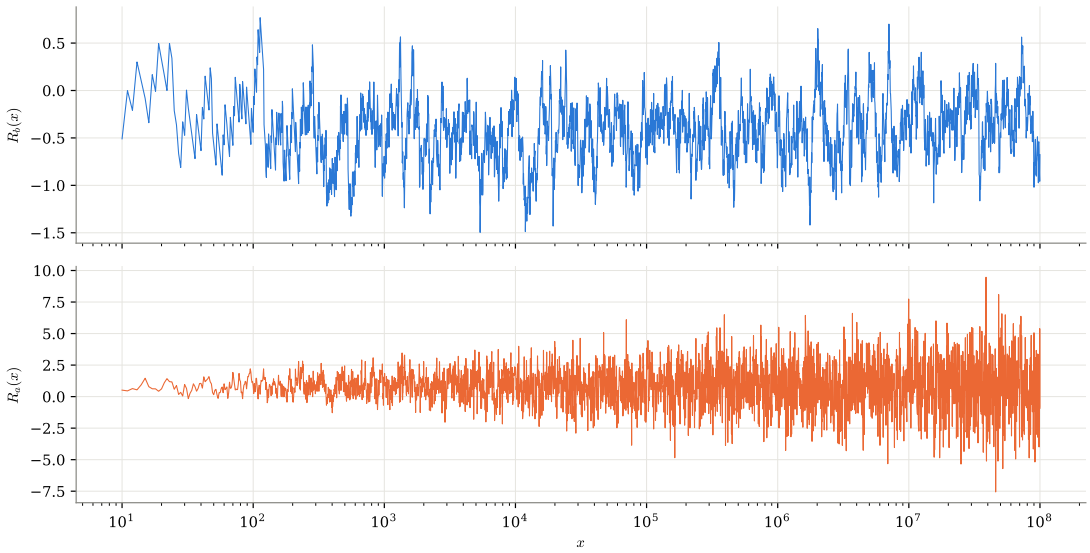


Figure 3: The C-transform remainders $R_b(x)$ of the kernel (top), which is bounded on the computed range, and $R_a(x)$ of $\mathcal{C}$ (bottom), which grows slowly through the divisor-type integer layer.

9.2 The Laplace transform

In the variable $u = \log x$ the identities above are Laplace transforms. With the notation $\mathcal{L}[f](s) = \int_0^\infty f(u)e^{-su}du$ and $f_b(u) = R_b(e^u)$, partial summation gives, for $\operatorname{Re} s > 1$,

$$\mathcal{C}(s) = s\,\mathcal{L}[A(e^u)](s), \qquad \mathcal{K}(s) = s\,\mathcal{L}[B(e^u)](s), \qquad \mathcal{E}(s) + \frac{1}{12s} = s\,\mathcal{L}[f_b](s).$$

The poles of $\mathcal{L}[f_b] = (\mathcal{E} + \frac{1}{12s})/s$ on $\operatorname{Re} s = 0$ are the possible frequencies of f_b, and the residues there are its Bohr–Fourier coefficients whenever these exist.

Proposition 24 (The kernel fluctuation has mean zero)**.**

$$\lim_{s\to 0}\Big(\mathcal{K}(s) + \frac{1}{12s}\Big) = 0.$$

Consequently, $\mathcal{E}(s) + \frac{1}{12s}$ is holomorphic at $s = 0$ and vanishes there, and $\mathcal{L}[f_b]$ has no pole at $s = 0$. If f_b has a mean value $\lim_U \frac{1}{U}\int_0^U f_b$, that mean is 0. (The existence of such a mean, or merely $\int_0^U f_b = O(U)$, already forces $\mathcal{L}[f_b]$ to converge in $\operatorname{Re} s > 0$, hence RH; so this statement has content only under RH.)

Proof. Near $w = 1$, $P'(w) = -\frac{1}{w-1} + h_1(w)$. Near $w = \frac{1}{2}$, $P'(w) = \frac{1/2}{w-1/2} + h_{1/2}(w)$, the pole coming from the term $\mu(2)\frac{\zeta'}{\zeta}(2w)$. Here h_1 and $h_{1/2}$ are holomorphic, since no other term of Corollary 7 is singular at these points.

In (2), only $k = 1$ and $k = 2$ have arguments tending to a pole of P' as $s \to 0$:

$$P'(2s + \tfrac{1}{2}) - P'(s + \tfrac{1}{2}) = \frac{1}{4s} - \frac{1}{2s} + o(1), \qquad P'(3s + 1) - P'(2s + 1) = -\frac{1}{3s} + \frac{1}{2s} + o(1).$$

For $k \geq 3$, both arguments tend to $\frac{k}{2} \geq \frac{3}{2}$, where P' is holomorphic, so each bracket is holomorphic near 0 and vanishes at $s = 0$. The series of these brackets converges uniformly near 0, so its sum is holomorphic and vanishes at 0. Altogether $\mathcal{K}(s) = -\frac{1}{12s} + o(1)$.

At $s = 0$, the terms $-\frac{1}{s-1/2}$ and $\frac{1}{2(s-1/4)}$ equal 2 and -2, so $\mathcal{E}(s) + \frac{1}{12s} \to 0$. The last claim is the Abelian theorem for Laplace transforms [Wid]: if the mean c exists, then $\sigma\mathcal{L}[f_b](\sigma) \to c$ as $\sigma \to 0^+$. □

Numerically, with the exact Euler–Maclaurin derivative of ζ, $\mathcal{K}(s)+\frac{1}{12s}$ extrapolates to $-8\cdot 10^{-8}$ at $s = 0$. The negative offset of about -0.4 visible in Figures 3 and 4 is not a mean. It is the real pole of $\mathcal{K}$ at $-\frac{1}{12}$, with residue $\frac{1}{6}$, which contributes $\frac{1/6}{-1/12}x^{-1/12} = -2x^{-1/12}$ to R_b and decays very slowly.

Proposition 25 (Bohr spectrum of the kernel)**.** *Assume RH. Then $\mathcal{L}[f_b]$ is holomorphic in $\operatorname{Re} s > 0$ and meromorphic in $\operatorname{Re} s > -\frac{1}{4}$. Its poles on $\operatorname{Re} s = 0$ lie at $\pm i\gamma$ and $\pm\frac{i\gamma}{2}$, with residues*

$$\operatorname*{Res}_{s=i\gamma}\mathcal{L}[f_b] = \frac{-m_\gamma}{i\gamma} = \frac{i\,m_\gamma}{\gamma}, \qquad \operatorname*{Res}_{s=i\gamma/2}\mathcal{L}[f_b] = \frac{m_\gamma/2}{i\gamma/2} = -\frac{i\,m_\gamma}{\gamma},$$

the two being added where the families meet. Consequently, whenever the Bohr–Fourier coefficient $c(\omega) = \lim_U \frac{1}{U}\int_0^U f_b(u)e^{-i\omega u}du$ exists at $\omega = \gamma$ or $\omega = \gamma/2$, it equals the corresponding residue. Formally,

$$\boxed{R_b(x) \;\sim\; \sum_{\gamma>0} \frac{2m_\gamma}{\gamma}\Big[\sin\big(\tfrac{\gamma}{2}\log x\big) - \sin\big(\gamma\log x\big)\Big] + \textit{(decaying terms).}}$$

Proof. The residues of $\mathcal{K}$ at $i\gamma$ and $i\gamma/2$ are $-m_\gamma$ and $\frac{1}{2}m_\gamma$ (Lemma 16). The terms $-\frac{1}{s-1/2}+\frac{1}{2(s-1/4)}+\frac{1}{12s}$ add no poles on $\operatorname{Re} s=0$ apart from 0, which is removable by Proposition 24. Dividing by s gives the residues. If $c(\omega)$ exists, the Abelian theorem gives $\sigma\mathcal{L}[f_b](\sigma+i\omega)\to c(\omega)$, and near a simple pole $\sigma\mathcal{L}[f_b](\sigma+i\omega)\to\text{Res}$. Finally, $2\operatorname{Re}[\frac{i}{\gamma}e^{i\gamma u}]=-\frac{2}{\gamma}\sin\gamma u$ and $2\operatorname{Re}[-\frac{i}{\gamma}e^{i\gamma u/2}]=\frac{2}{\gamma}\sin\frac{\gamma u}{2}$. □

Numerical observation 26. We fitted $f_b(u)$, sampled at 2^{16} points on $[\log 10^2,\log 10^8]$ and with the analytically known decaying terms removed, by least squares on $\{\cos,\sin\}(\omega u)$ for the 192 frequencies $\omega\in\{\gamma_j,\gamma_j/2\}$ with $\gamma_j<230$. The sine coefficients agree with $\mp 2/\gamma$ and the cosine coefficients are close to 0 (Table 4). The reconstruction with 96 zeros has correlation 0.91 with R_b (Figure 4).

The visible misses (γ_4, γ_9, $\gamma_{10}/2$) are explained by near-coincident frequencies of the two families that a window of length 13.8 cannot separate. The closest is $\gamma_{14}=60.8318$ against $2\gamma_4=60.8498$, a gap of 0.018; Proposition 32 predicts the merged coefficient $-2(\frac{1}{\gamma_4}-\frac{1}{\gamma_{14}})=-0.0329$ at γ_4.

Table 4: Bohr coefficients of the kernel fluctuation $f_b(u)=R_b(e^u)$: fitted sine coefficients against the predictions of Proposition 25.

j	γ_j	sin at γ_j	predicted $-2/\gamma_j$	sin at $\gamma_j/2$	predicted $+2/\gamma_j$
1	14.1347	−0.1418	−0.1415	+0.1438	+0.1415
2	21.0220	−0.0973	−0.0951	+0.0906	+0.0951
3	25.0109	−0.0807	−0.0800	+0.0807	+0.0800
4	30.4249	−0.0362	−0.0657	+0.0717	+0.0657
5	32.9351	−0.0615	−0.0607	+0.0618	+0.0607
6	37.5862	−0.0544	−0.0532	+0.0561	+0.0532
7	40.9187	−0.0456	−0.0489	+0.0482	+0.0489
8	43.3271	−0.0446	−0.0462	+0.0462	+0.0462
9	48.0052	−0.0326	−0.0417	+0.0419	+0.0417
10	49.7738	−0.0412	−0.0402	+0.0295	+0.0402

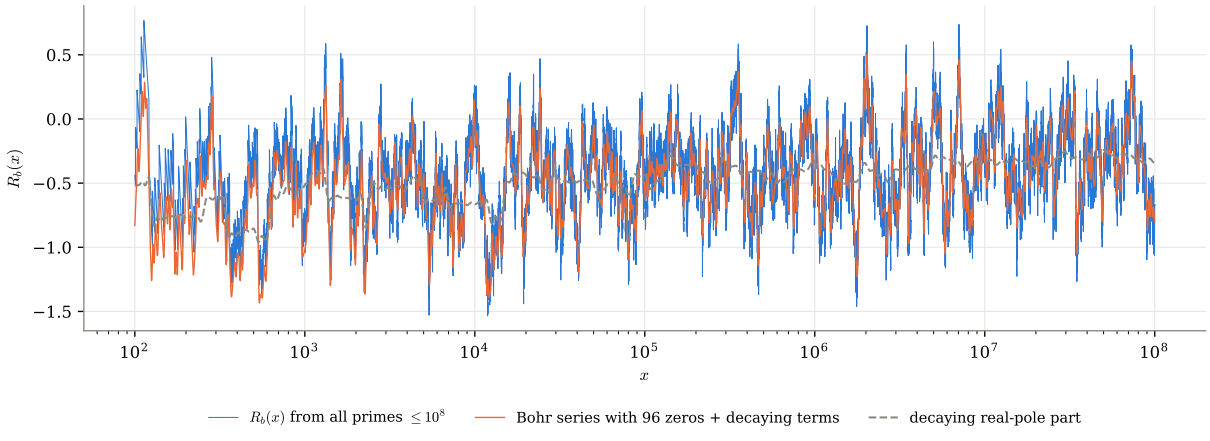


Figure 4: The kernel fluctuation $R_b(x)$ computed from all primes up to 10^8 (blue), against its Bohr series of Proposition 25 with 96 zeros plus the analytically known decaying terms (orange). The dashed curve is the slowly decaying real-pole part, dominated by $-2x^{-1/12}$.

Theorem 27 (A one-sided criterion)**.** *The Riemann Hypothesis holds if and only if, for every $\varepsilon > 0$, there is C_ε with*

$$R_b(x) \geq -C_\varepsilon\, x^\varepsilon \qquad (x \geq 1).$$

The same holds with the upper bound $R_b(x) \leq C_\varepsilon x^\varepsilon$ in place of the lower bound.

Proof. If RH holds, both bounds follow from Lemma 19.

Conversely, fix $\varepsilon > 0$ and put $g(u) = f_b(u) + C_\varepsilon e^{\varepsilon u} \geq 0$. For $\operatorname{Re} s > 1$,

$$\mathcal{L}[g](s) = \frac{\mathcal{E}(s) + \frac{1}{12s}}{s} + \frac{C_\varepsilon}{s-\varepsilon} =: \Phi(s).$$

By Theorem 8, unconditionally, $\mathcal{K}$ has no real poles in $\operatorname{Re} s > 0$ other than $\frac{1}{2}$ and $\frac{1}{4}$, because zeros of ζ are not real. Hence $\mathcal{E}$, and so Φ, is holomorphic on a neighbourhood of every real point $\sigma > \varepsilon$.

By Landau's theorem for Laplace transforms of nonnegative functions [Wid, Ch. II, Thm. 5b], the real point σ_g of the abscissa of convergence of $\mathcal{L}[g]$ is a singularity of the function it defines. That function coincides with Φ wherever the integral converges, so $\sigma_g \leq \varepsilon$. Thus $\mathcal{L}[g]$ converges in $\operatorname{Re} s > \varepsilon$, and $\mathcal{E}$ is holomorphic there. Since ε is arbitrary, Theorem 11 gives RH. For the upper bound, use $g = C_\varepsilon e^{\varepsilon u} - f_b$. □

Remark 28. Theorem 27 is the normalized analogue of Landau's classical one-sided theorem for $\psi(x) - x$. It is again a reformulation of RH. The case $\varepsilon = 0$, a uniform one-sided bound, is not expected to hold: Littlewood's $\psi(x) - x = \Omega_\pm(\sqrt{x}\log\log\log x)$ suggests that R_b is unbounded in both directions, though extremely slowly. On the computed range R_b is asymmetric, with $\inf R_b = -1.56$ and $\sup R_b = 0.77$, mainly because of the slowly decaying term $-2x^{-1/12}$.

For $\mathcal{C}$ itself the Laplace picture is analogous, with Bohr coefficients, whenever they exist, $-m_\gamma\zeta(i\gamma)/(i\gamma)$ and $m_\gamma\zeta(i\gamma/2)/(i\gamma)$; here $\mathcal{L}[R_a(e^u)]$ is only known to converge in $\operatorname{Re} s > \frac{1}{3}$ under RH. These are the amplitudes of the model (6), whose spectrum matches that of the residual of A in Table 2. A one-sided criterion does not follow for $\mathcal{C}$, because zeros of ζ in $\mathcal{C} = \zeta\mathcal{K}$ could in principle cancel poles of $\mathcal{K}$.

10 The kernel and the Cesàro margin

This section connects the kernel with Suzuki's one-variable form of RH, studied as the *Cesàro margin* in Part 11 [CE11]. We recall the objects. For $N > 1$ and $L = \log N$, let

$$C(N) = 2\sum_{n<N} \frac{\Lambda(n)}{\sqrt{n}} \log\frac{N}{n}, \qquad M = R - C,$$

$$R(N) = 8(\sqrt{N} - 1) - c_0 \log N + 2\sum_{m\geq 1} \frac{1 - N^{-(2m+\frac{1}{2})}}{(2m+\frac{1}{2})^2},$$

with $c_0 = 2\frac{\zeta'}{\zeta}(\frac{1}{2}) = 5.3721834$. Then, unconditionally, $M(N) = 2\sum_\rho (1 - N^{it_\rho})/t_\rho^2$ with $t_\rho = (\rho - \frac{1}{2})/i$, summed over the zeros with multiplicity [CE11]. RH holds if and only if $M(N) \geq 0$ for all large N (Suzuki [Su1]). Under RH, $M(N) = 4\sum_{\gamma>0} m_\gamma(1 - \cos\gamma L)/\gamma^2 \in [0, 4(\log\xi)''(\frac{1}{2})]$, where $(\log\xi)''(\frac{1}{2}) = \sum_\rho t_\rho^{-2} = 0.0462100$. The *slope* of the margin is

$$\alpha(x) = \sum_{n\leq x} \frac{\Lambda(n)}{\sqrt{n}} - 2\sqrt{x} + \frac{c_0}{2}.$$

Under RH, $u \mapsto \alpha(e^u)$ is a Besicovitch almost periodic function with mean 0 and Fourier coefficients $\frac{im_\gamma}{\gamma}$ at the frequencies γ [RuSa, ANS, CE11].

10.1 The decomposition of the kernel

By Lemma 2(iii), $b(n) = \Lambda(n)n^{-1/2} + q(n)$, where q lives on the prime powers p^j, $j \geq 2$, with $q(p^j) = -\log p\, p^{-(j-1)/2}$. So

$$\mathcal{K}(s) = -\frac{\zeta'}{\zeta}\big(s+\tfrac{1}{2}\big) + Q(s), \qquad Q(s) = \sum_n q(n)n^{-s} = \sum_{k\geq 1} P'\big((k+1)s + \tfrac{k}{2}\big). \tag{7}$$

Proposition 29 (the echo term)**.** *For* $\operatorname{Re} s > 0$*,*

$$\mathcal{K}(s) = -\frac{\zeta'}{\zeta}\big(s+\tfrac{1}{2}\big) + \frac{\zeta'}{\zeta}\big(2s+\tfrac{1}{2}\big) - \frac{1}{12s} + H(s),$$

where H *extends holomorphically to a neighbourhood of the closed half-plane* $\operatorname{Re} s \geq 0$*, and* $H(0) = 0$*. Under RH,* H *is holomorphic in* $\operatorname{Re} s > -\frac{1}{12}$*, and its first singularity is a simple pole at* $s = -\frac{1}{12}$ *with residue* $\frac{1}{6}$*.*

Proof. Use $-\frac{\zeta'}{\zeta}(w) = -\sum_{m\geq 1} P'(mw)$. In (7), the term $k = 1$ of Q is $P'(2s + \frac{1}{2})$, which is the term $m = 1$ of $-\frac{\zeta'}{\zeta}(2s + \frac{1}{2})$ with the opposite sign. Hence

$$D(s) := Q(s) - \frac{\zeta'}{\zeta}\big(2s+\tfrac{1}{2}\big) = \sum_{k\geq 2} P'\big((k+1)s + \tfrac{k}{2}\big) - \sum_{m\geq 2} P'\big(m(2s+\tfrac{1}{2})\big).$$

At $s = 0$ the arguments are $\frac{k}{2} \geq 1$ and $\frac{m}{2} \geq 1$. By Corollary 7, P' is singular only at $w = 1/j$ and $w = \rho/j$, j squarefree. An argument equal to 1 occurs only for $k = 2$ $(3s + 1)$ and $m = 2$ $(4s+1)$. Since $P'(w) = -\frac{1}{w-1} + O(1)$ near 1, we get $D(s) = -\frac{1}{3s} + \frac{1}{4s} + O(1) = -\frac{1}{12s} + O(1)$. Put $H = D + \frac{1}{12s}$.

A zero $\rho = \beta + i\gamma$ makes a term singular where its argument equals ρ/j. For an argument $as+b$ with $b \geq 1$, this happens at $\operatorname{Re} s = (\beta/j - b)/a < 0$, since $\beta < 1$. Near any point of $\operatorname{Re} s = 0$ only finitely many terms are involved, so H is holomorphic in a neighbourhood of $\operatorname{Re} s \geq 0$. $H(0) = 0$ is Proposition 24, because the two $\frac{\zeta'}{\zeta}$ terms cancel at $s = 0$. Under RH, the zero-induced singularities have real parts $(\frac{1}{2j} - b)/a \leq -\frac{1}{8}$. The real ones come from $as + b = 1/j$. The largest of these below 0 is $6s + \frac{3}{2} = 1$, from $m = 3$, at $s = -\frac{1}{12}$; there $-P'(6s + \frac{3}{2}) \sim \frac{1}{6s+1/2}$, which has residue $\frac{1}{6}$. □

So the kernel is $-\frac{\zeta'}{\zeta}$ at two scales, $s + \frac{1}{2}$ and $2s + \frac{1}{2}$, with opposite signs and weights 1 and $\frac{1}{2}$ for the residues. Everything else is either the explicit pole $-\frac{1}{12s}$, which produces the term $-\frac{1}{12}\log x$ of R_b and the $\frac{1}{48}\log^2 x$ of Theorem 17, or holomorphic across the imaginary axis.

10.2 The bridge

Proposition 30 (the kernel's remainder is the slope at two scales)**.** *Without any hypothesis,*

$$R_b(x) = \alpha(x) - \alpha(\sqrt{x}) + o(1) \qquad (x \to \infty).$$

Proof. Let $A_\Lambda(x) = \sum_{n\leq x} \Lambda(n)n^{-1/2}$. By (7), $B(x) = A_\Lambda(x) + \sum_{n\leq x} q(n)$. The prime squares in $\sum q$ give $-\sum_{p\leq\sqrt{x}} \log p/\sqrt{p}$, which is $-A_\Lambda(\sqrt{x})$ up to the prime powers counted in $A_\Lambda(\sqrt{x})$. Collecting terms,

$$R_b(x) - \big(\alpha(x) - \alpha(\sqrt{x})\big) = \sum_{i\geq 2}\Big(\sum_{p^{2i}\leq x} - \sum_{p^{i+1}\leq x}\Big)\frac{\log p}{p^{i/2}} + \frac{1}{12}\log x.$$

For $i = 2$ the bracket is $-\sum_{x^{1/4}<p\leq x^{1/3}} \log p/p = -\frac{1}{12}\log x + o(1)$ by the prime number theorem in the form $\sum_{p\leq y} \frac{\log p}{p} = \log y + E + o(1)$. For $i \geq 3$ it is minus a sum over $x^{1/(2i)} < p \leq x^{1/(i+1)}$ of terms of the convergent series $\sum_p \log p\, p^{-i/2}$. These tails tend to 0, and so does their sum over $i \geq 3$, dominated by $\sum_{i\geq 3}\sum_{p>x^{1/(2i)}} \log p\, p^{-i/2}$. □

With all prime powers below 10^{12}, the difference in Proposition 30 is at most 0.73 per decade up to 10^5, and decreases to 0.20 on $[10^{11}, 10^{12}]$. The slow decay is the term $-2x^{-1/12}$ of Proposition 29, followed by terms of order $x^{-1/8}$ ($O(x^{-1/8}\log^2 x)$ under RH), (Figure 5(a)).

10.3 The kernel's margin

For a weight w, write $F_w(N) = \sum_{n<N} w(n)\log\frac{N}{n}$, so that $C = 2F_{\Lambda/\sqrt{\cdot}}$ and $2F_b = C + 2F_q$.

Theorem 31 (the kernel's margin). *Assume RH and let $-\frac{1}{12} < \sigma_0 < 0$. Then*

$$F_q(N)+C(\sqrt{N}) = -\tfrac{1}{24}\log^2 N+d_1+O(N^{\sigma_0}), \qquad d_1 = H'(0) = \frac{d}{ds}\Big(\mathcal{K}(s)+\frac{1}{12s}\Big)\Big|_{s=0} - \Big(\frac{\zeta'}{\zeta}\Big)'(\tfrac{1}{2}) = -10.2541\ldots,$$

and consequently the kernel's Riesz margin

$$M_b(N) := R(N) - 2R(\sqrt{N}) - \tfrac{1}{12}\log^2 N + 2d_1 - 2F_b(N)$$

satisfies

$$M_b(N) = M(N) - 2M(\sqrt{N}) + O(N^{\sigma_0}) = 16\sum_{\gamma>0}\frac{m_\gamma \sin^2\frac{\gamma L}{4}\cos\frac{\gamma L}{2}}{\gamma^2} + O(N^{\sigma_0})$$
$$\in [-0.3697,\, 0.1848] + O(N^{\sigma_0}).$$

Proof. For $c > 1$, Perron's formula with the kernel N^t/t^2 gives $F_q(N) = \frac{1}{2\pi i}\int_{(c)} Q(t)N^t\frac{dt}{t^2}$. The substitution $s = 2t$ in the Perron integral of $C(X)$ at $X = \sqrt{N}$ gives $C(\sqrt{N}) = \frac{1}{2\pi i}\int_{(c)}\big(-\frac{\zeta'}{\zeta}\big)(2t+\frac{1}{2})N^t\frac{dt}{t^2}$. Hence $F_q(N) + C(\sqrt{N}) = \frac{1}{2\pi i}\int_{(c)} D(t)N^t\frac{dt}{t^2}$, with $D = -\frac{1}{12t} + H$ as in the proof of Proposition 29.

Under RH, every argument of $\frac{\zeta'}{\zeta}$ in the expansion of D stays at distance at least $\frac{1}{6}$ from the critical line when $\operatorname{Re} t \ge \sigma_0$. So $D(t) \ll \log(|t|+2)$ on $\operatorname{Re} t = \sigma_0$ and on horizontal segments, and the contour can be moved to $\operatorname{Re} t = \sigma_0$ with error $O(N^{\sigma_0})$. The only pole crossed is $t = 0$, where $D(t)N^t t^{-2}$ has residue $-\frac{1}{24}\log^2 N + H(0)\log N + H'(0)$; and $H(0) = 0$. Since $D(t) = \mathcal{K}(t) + \frac{\zeta'}{\zeta}(t+\frac{1}{2}) - \frac{\zeta'}{\zeta}(2t+\frac{1}{2})$, $D + \frac{1}{12t}$ has derivative $H'(0)$ at 0, and the stated d_1 is $\frac{d}{dt}(\mathcal{K}(t)+\frac{1}{12t})|_0 - (\frac{\zeta'}{\zeta})'(\frac{1}{2})$.

For the margin, substitute $C(N) = R(N) - M(N)$ and $C(\sqrt{N}) = R(\sqrt{N}) - M(\sqrt{N})$ in $2F_b = C + 2F_q$. The bounds follow from $0 \le M \le 0.18484$ at N and at $\sqrt{N}$. The last form uses $(1-\cos\theta) - 2(1-\cos\frac{\theta}{2}) = 4\sin^2\frac{\theta}{4}\cos\frac{\theta}{2}$. □

With the prime powers below 10^8, a smooth fit gives $d_1 \approx -10.34$, against -10.254 (here $(\mathcal{K}+\frac{1}{12s})'(0) = -6.5072$ and $(\zeta'/\zeta)'(\frac{1}{2}) = 3.7469$). Removing $-C(\sqrt{N})$ from F_q removes the echo: the correlation of the residual with $-\sum_\gamma \frac{4}{\gamma^2}\cos\frac{\gamma\log N}{2}$ drops from 0.83 to 0.12. Unlike M, the kernel's margin changes sign term by term, since $\cos\frac{\gamma L}{2}$ does. With the first 96 zeros, $M(N) - 2M(\sqrt{N})$ lies in $[-0.183, -0.005]$ for $1.65 \le N \le e^{25}$. This loss of positivity is why the one-sided criterion of Theorem 27 needs the factor x^ε.

10.4 Energy, and pairs of zeros with $\gamma' = 2\gamma$

Proposition 32 (the kernel's energy). *Assume RH and that all zeros are simple. Then $u \mapsto R_b(e^u)$ is Besicovitch almost periodic, and*

$$\big\langle (R_b - \langle R_b\rangle)^2\big\rangle = 2(\log\xi)''(\tfrac{1}{2}) - 2\sum_{\substack{\gamma>0\\ 2\gamma \text{ is an ordinate}}}\frac{1}{\gamma^2}.$$

In particular, if no two ordinates satisfy $\gamma' = 2\gamma$ (as the linear independence hypothesis implies), the energy is $2(\log\xi)''(\frac{1}{2}) = 0.092420$, and the variogram is $\langle |R_b(e^{u+L}) - R_b(e^u)|^2\rangle = M(e^L) + M(e^{L/2})$. On a window of length U in $\log x$, a pair with $|\gamma' - 2\gamma| \ll 1/U$ is not resolved, and the amplitude seen at γ is $2(\frac{1}{\gamma} - \frac{1}{\gamma'})$ instead of $\frac{2}{\gamma}$.

Proof. By Proposition 30, $R_b(e^u) = \alpha(e^u) - \alpha(e^{u/2}) + o(1)$. The first term has the coefficients $\frac{i}{\gamma}$ at the frequencies $\pm\gamma$. The second has $-\frac{i}{\gamma'}$ at $\pm\frac{\gamma'}{2}$. If the frequency sets are disjoint, Parseval adds $2\sum_{\gamma>0}\gamma^{-2} = (\log\xi)''(\frac{1}{2})$ twice. If $\gamma' = 2\gamma$, the frequency γ receives $\frac{i}{\gamma} - \frac{i}{\gamma'}$, and $|\frac{1}{\gamma} - \frac{1}{\gamma'}|^2 = \frac{1}{\gamma^2} + \frac{1}{\gamma'^2} - \frac{2}{\gamma\gamma'}$. Counting both signs, the defect is $\frac{4}{\gamma\gamma'} = \frac{2}{\gamma^2}$. The variogram is computed the same way, using the formula for M at L and at $L/2$. The finite-window statement is the same computation for two frequencies closer than the resolution $2\pi/U$. □

The linear independence hypothesis excludes exact pairs, but near pairs are unavoidable. Among the first zeros, $\gamma_{14} = 60.8318$ and $2\gamma_4 = 60.8498$ differ by 0.018, far below the resolution $2\pi/U \approx 0.5$ of the range $2^8 \le x \le 2^{26}$. Proposition 32 predicts the amplitude 0.0329 at γ_4 instead of $\frac{2}{\gamma_4} = 0.0657$. The fit finds 0.024, while the amplitudes at $\gamma_1, \gamma_2, \gamma_3$ and at their halves are within 10% of $\frac{2}{\gamma}$ (Figure 5(b); compare Table 4). The block energies of R_b have mean 0.106 for $2^{12} \le x \le 2^{26}$. The excess over 0.0924 matches the energy of the slowly decaying terms of Proposition 29 (Figure 5(c)). So the Caceres kernel is sensitive to additive relations $\gamma' \approx 2\gamma$ among the ordinates, which α and M do not see.

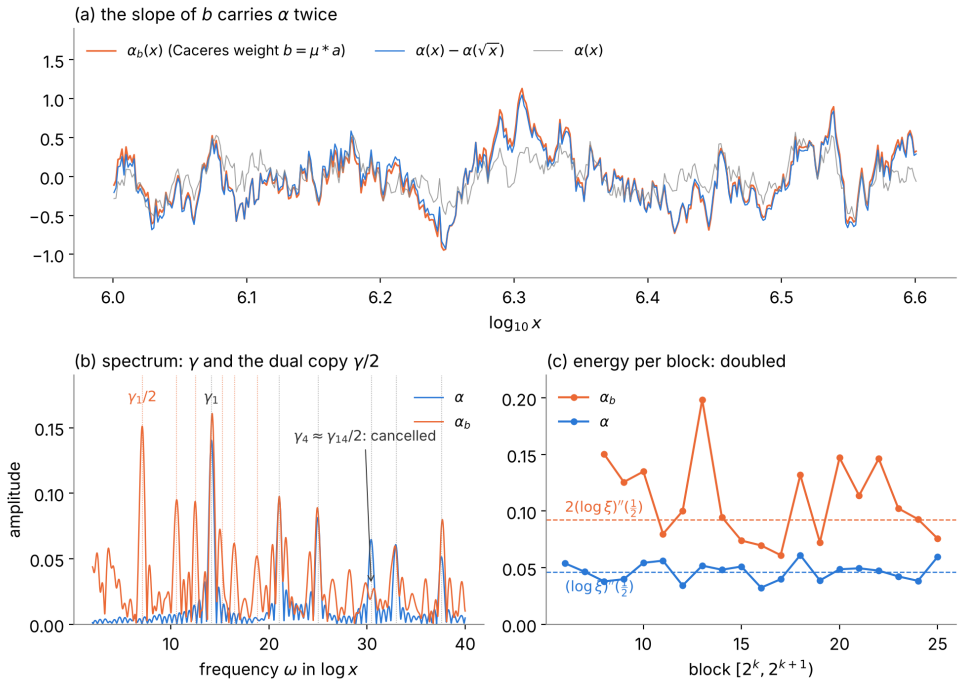


Figure 5: The kernel and the margin. (a) $\alpha_b = R_b - \langle R_b\rangle$ against $\alpha(x) - \alpha(\sqrt{x})$ (Proposition 30) and $\alpha(x)$, $10^6 \le x \le 4\cdot 10^6$. (b) Amplitude spectrum in $\log x$ on $2^8 \le x \le 2^{26}$: α has peaks at γ; α_b at γ and $\gamma/2$ (dotted), except at γ_4, which the dual copy of $\gamma_{14} \approx 2\gamma_4$ cancels. (c) Energy per dyadic block: α_b around $2(\log\xi)''(\frac{1}{2})$ and α around $(\log\xi)''(\frac{1}{2})$.

Remark 33 (which object is rigid, and what the kernel does not add)**.** The coefficients of R_b, $\pm\frac{i}{\gamma}$, depend only on the ordinates. Those of $\sum_{n\le x} a(n)$ do not. Since $\mathcal{C} = \zeta\mathcal{K}$, the residue of $\mathcal{C}(s)x^s/s$ at $i\gamma$ is $-m_\gamma\zeta(i\gamma)x^{i\gamma}/(i\gamma)$. By the functional equation, $|\zeta(i\gamma)| = (\gamma/2\pi)^{1/2}|\zeta(1+i\gamma)|$, and $|\zeta(1+i\gamma)| \gg 1/\log\gamma$. So these terms decay only like $\gamma^{-1/2}$, up to that factor. They depend on ζ at the edge of the strip, and the series over the zeros is not absolutely convergent. This is why Theorem 17 works with the Riesz mean F. Summing over all integers multiplies by ζ and destroys rigidity; the rigid object is $b = \mu * a$, which lives on the prime powers.

Since $b(p) = \Lambda(p)/\sqrt{p}$ and $b(p^j)$ is a fixed function of p, the kernel carries exactly the information of Λ. A positive perturbation of Λ that plants an off-line zero [CE11] plants it in b and

a as well. A true off-line zero $\frac{1}{2}+\delta+i\gamma_0$ would appear in R_b twice, as x^{δ} at γ_0 and as $-x^{\delta/2}$ at $\frac{\gamma_0}{2}$. These are distinct frequencies, so the two copies can neither cancel nor reinforce into a contradiction. Status: Propositions 29, 30, 32 and Theorem 31 are proved, the last two under RH. They give a dictionary between the kernel and the margin, not a new route to RH.

11 The critical line and the zeros of the kernel

11.1 On the critical line

Theorem 8 lets us evaluate $\mathcal{C}(\frac{1}{2}+it) = \zeta(\frac{1}{2}+it)\mathcal{K}(\frac{1}{2}+it)$ by analytic continuation. As an independent check, we used the smoothed Dirichlet sum

$$\sum_n a(n)\, n^{-1/2-it} e^{-n/X} = \mathcal{C}(\tfrac{1}{2}+it) + \mathcal{K}(1)\Gamma(\tfrac{1}{2}-it)X^{1/2-it} + \zeta(\tfrac{1}{2})\Gamma(-it)X^{-it} - \tfrac{1}{2}\zeta(\tfrac{1}{4})\Gamma(-\tfrac{1}{4}-it)X^{-1/4-it} + O(X^{-1/2}),$$

with $X = 2\cdot 10^5$. This expansion comes from the Mellin–Barnes formula $\sum a(n)n^{-w}e^{-n/X} = \frac{1}{2\pi i}\int \mathcal{C}(w+z)\Gamma(z)X^z\,dz$ by moving the contour past the poles of $\mathcal{C}$ at $1, \frac{1}{2}, \frac{1}{4}$. We use it only as an independent numerical check, and do not claim it with a proved error term.

Numerical observation 34. On $5 \le t \le 65$ (step 0.1), the two computations differ by 0.0065 (median) and 0.028 (maximum), on values of order 1 (Figure 6). $\mathcal{C}(\frac{1}{2}+it)$ vanishes at every γ_n. On this grid $|\mathcal{K}(\frac{1}{2}+it)| \ge 0.005$, so $\mathcal{C}$ shows no other zeros on the line there. The minimum, at $t \approx 45.30$, is explained by a zero of $\mathcal{K}$ at $0.4967 + 45.3037\,i$, just off the line.

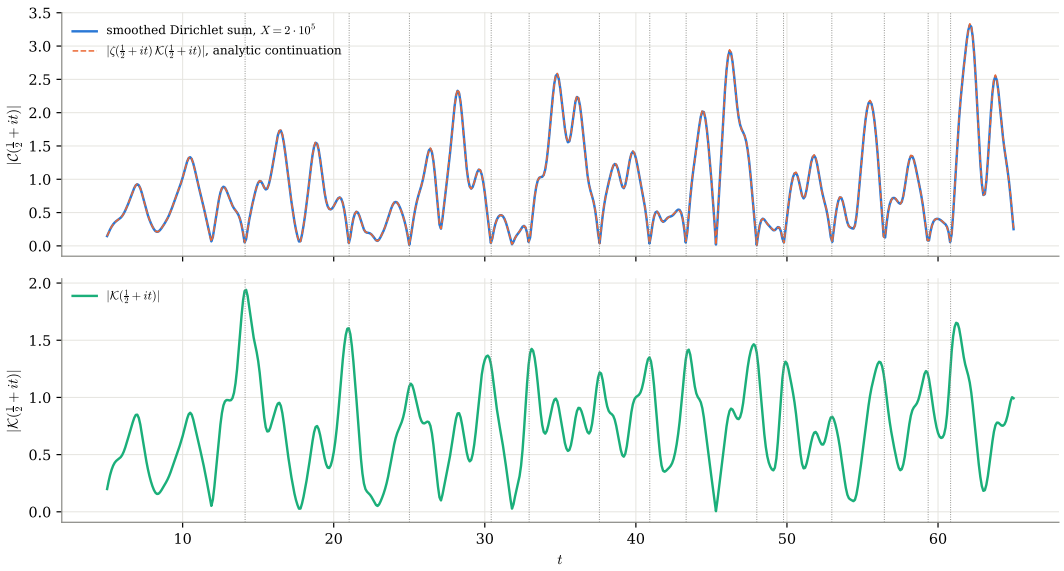


Figure 6: Top: $|\mathcal{C}(\frac{1}{2}+it)|$ from the smoothed Dirichlet sum (solid) and from $\zeta\mathcal{K}$ by analytic continuation (dashed). Bottom: $|\mathcal{K}(\frac{1}{2}+it)|$. Dotted lines mark the ordinates γ_n.

11.2 The kernel has zeros of its own

Since $\mathcal{C} = \zeta\mathcal{K}$, the zeros of $\mathcal{C}$ are the zeros of ζ together with the zeros of $\mathcal{K}$. Newton's method applied to the analytic continuation of $\mathcal{K}$ locates the zeros in Table 6.

Lemma 35 (Rigorous tail)**.** *Let* $N \ge 10^6$ *and* $\operatorname{Re} s = \sigma > 1$. *Then*

$$\Big|\mathcal{C}(s) - \sum_{n\le N}\frac{a(n)}{n^s} - \mathcal{K}(1)\frac{N^{1-s}}{s-1}\Big| \le 5.51\, N^{1/2-\sigma}\Big(1+\frac{|s|}{\sigma-\frac{1}{2}}\Big).$$

Proof. By partial summation,

$$\sum_{n>N} a(n)n^{-s} = -A(N)N^{-s} + s\int_N^\infty A(t)t^{-s-1}dt.$$

Inserting $A(t) = \mathcal{K}(1)t + R(t)$, the main part equals $\mathcal{K}(1)N^{1-s}/(s-1)$. By Theorem 13, $|R(t)| \le 5.51\sqrt{t}$ for $t \ge N$. Hence the remainder $-R(N)N^{-s} + s\int_N^\infty R(t)t^{-s-1}dt$ is at most the stated bound. □

Theorem 36 (A zero of $\mathcal{C}$ in the half-plane of absolute convergence)**.** *Let $s_0 = 1.379034 + 8.639155\,i$. The function $\mathcal{C}$, and hence $\mathcal{K}$, has exactly one zero in the disc $|s - s_0| < 0.01$, and exactly one zero in $|s - s_0| < 0.1$.*

Proof (computer-assisted). Let $N = 10^6$ and

$$f(s) = \sum_{n\le N} a(n)n^{-s} + \kappa\,\frac{N^{1-s}}{s-1},$$

where κ is the midpoint of the interval of Proposition 14. By Lemma 35 and Proposition 14, on a circle $|s - s_0| = r$ we have $|\mathcal{C}(s) - f(s)| \le E_r$, where

$$E_r = 8.2\cdot 10^{-6}\,\frac{N^{1-\sigma_r}}{|s_0-1|-r} + 5.51\,N^{1/2-\sigma_r}\Big(1 + \frac{|s_0|+r}{\sigma_r - \frac12}\Big), \qquad \sigma_r = \operatorname{Re} s_0 - r.$$

Let L_r be the explicit bound for $|f'|$ on the closed disc given by $\sum_{n\le N} a(n)\log n\, n^{-\sigma_r}$ plus the derivative of the last term. We evaluated f at M equally spaced points z_j of the circle. Let h be the chord between consecutive points. The checks were:

$$\min_j |f(z_j)| - L_r h > E_r \qquad \text{and} \qquad \frac{1}{2\pi}\sum_j \arg\frac{f(z_{j+1})}{f(z_j)} = 1.$$

The first inequality gives $|f| > E_r \ge |\mathcal{C} - f|$ on the whole circle, since $|f(z)| \ge |f(z_j)| - L_r h$ on the arc after z_j. It also gives $|f(z) - f(z_j)| < |f(z_j)|$ there, so each increment of the argument is determined by its endpoints and the sampled winding number equals the true one. By the argument principle f has exactly one zero inside, and by Rouché's theorem so does $\mathcal{C}$. Since $\zeta(s) \ne 0$ for $\operatorname{Re} s > 1$, it is a zero of $\mathcal{K}$.

The certified numbers are listed in Table 5. Every $f(z_j)$ was computed in double precision as a sum of at most 10^6 terms whose absolute values add up to less than 2.61, since $\sum_{n\le N} a(n)n^{-1.279} = 2.6008$. There are three sources of floating-point error.

- *The values $a(n)$.* Each is a sum of at most 20 terms $b(p^k)$ with correctly rounded logarithms and powers, so its relative error is below $50\cdot 2^{-52}$.
- *Each term $a(n)e^{-s\log n}$.* The argument satisfies $|s\log n| \le 9.9\cdot 13.9 < 140$, and exp, cos and sin are accurate to a few ulps. So the relative error of each term is below $140\cdot 8\cdot 2^{-52} < 3\cdot 10^{-13}$.
- *The summation.* The rounding error is at most $10^6\cdot 2^{-52}\cdot 2.61 < 6\cdot 10^{-10}$.

In total, every computed $f(z_j)$ is within 10^{-9} of its exact value. This is far below the smallest margin, $8.6\cdot 10^{-4}$, and L_r and E_r are increased by an equally negligible amount. □

Corollary 37 (Infinitely many zeros with $\operatorname{Re} s > 1$)**.** *There are constants $c > 0$ and T_0 such that, for $T \ge T_0$, $\mathcal{C}$ has at least cT zeros in the region $|\operatorname{Re} s - \operatorname{Re} s_0| < 0.1$, $0 < \operatorname{Im} s < T$.*

Table 5: Data of the computer-assisted proof of Theorem 36, with $N = 10^6$.

r	M	$\min_j \lvert f(z_j)\rvert$	L_r	$L_r h$	E_r	winding
0.01	4000	$1.3374 \cdot 10^{-3}$	6.67	$1.05 \cdot 10^{-4}$	$3.73 \cdot 10^{-4}$	1
0.1	6000	$1.2134 \cdot 10^{-2}$	11.46	$1.20 \cdot 10^{-3}$	$1.44 \cdot 10^{-3}$	1

Proof. Let D be the closed disc $|s - s_0| \le 0.1$. It lies in $\operatorname{Re} s \ge \sigma_1 := 1.279$. By Table 5, on ∂D we have $|\mathcal{C}| \ge \mu := 1.2134 \cdot 10^{-2} - 1.20 \cdot 10^{-3} - 1.44 \cdot 10^{-3} > 9 \cdot 10^{-3}$.

Choose M_0 with $\sum_{n>M_0} a(n)n^{-\sigma_1} < \mu/4$, and put $Q(\tau) = \sum_{n \le M_0} w_n n^{-i\tau}$ with $w_n = a(n)n^{-\sigma_1}$. Q is a trigonometric polynomial with distinct frequencies $\log n$, so it is almost periodic, and for every $\eta > 0$ its η-almost periods form a relatively dense set [Bes].

If τ is such an almost period, then for each $n \le M_0$

$$w_n |n^{-i\tau} - 1| = \Big| \lim_{U\to\infty} \frac{1}{2U} \int_{-U}^{U} \big(Q(t+\tau) - Q(t)\big) n^{it}\, dt \Big| \le \eta.$$

Taking $\eta = \mu/(4M_0)$, for all $s \in D$

$$|\mathcal{C}(s + i\tau) - \mathcal{C}(s)| \le \sum_{n \le M_0} w_n |n^{-i\tau} - 1| + 2 \sum_{n > M_0} a(n) n^{-\sigma_1} < \tfrac{\mu}{4} + \tfrac{\mu}{2} < \mu \le |\mathcal{C}(s)| \quad (s \in \partial D).$$

By Rouché's theorem, $\mathcal{C}$ has a zero in $D + i\tau$. Relative density gives such τ in every interval of some fixed length ℓ. Choosing one in each of the intervals $[k(\ell+1), k(\ell+1)+\ell]$, $k \ge 0$, gives pairwise disjoint discs in $\operatorname{Im} s > 8.5$, hence at least $T/(\ell+1) - 2$ zeros with $0 < \operatorname{Im} s < T$. □

A Dirichlet series with an Euler product cannot vanish in its half-plane of absolute convergence. $\mathcal{C}$ has no Euler product, since a is additive rather than multiplicative, and Theorem 36 shows that it really does vanish there. The same phenomenon occurs for the Davenport–Heilbronn functions [DH]. Newton's method applied to the analytic continuation of $\mathcal{K}$ locates further zeros (Table 6, Figure 7), scattered on both sides of $\operatorname{Re} s = \frac{1}{2}$.

Numerical observation 38. The points in Table 6 with $\operatorname{Re} s < 1$ are zeros of $\mathcal{K}$ to the precision shown: Newton's method, with ζ'/ζ computed exactly by term-wise differentiated Euler–Maclaurin summation, converges to them with $|\mathcal{K}| < 10^{-14}$. They are not certified.

So $\mathcal{C}$ has no "Riemann Hypothesis" of its own. Its zeros on the critical line coincide with those of ζ on the range computed (Observation 34) only because $\mathcal{K}$ does not happen to vanish there.

Table 6: Zeros of $\mathcal{K}$ located by Newton's method in $0 < \operatorname{Im} s < 65$. The list is not exhaustive.

$\operatorname{Re} s$	$\operatorname{Im} s$	$\operatorname{Re} s$	$\operatorname{Im} s$
1.379034	8.639155	0.474351	31.820298
0.449090	11.912069	0.496746	45.303688
0.548725	17.752431	0.339194	54.463500
0.359085	22.892907	0.295984	63.025159
0.405149	27.072657		

12 Discussion and open problems

The Caceres Prime Function is the $\sqrt{}$-damped analogue of $\log \operatorname{rad}(n)$, and the damping exponent $\frac{1}{2}$ is critical: it is the only one that sends the critical line of ζ onto the imaginary axis (Proposition 10). At this exponent three things happen together. The kernel becomes an exact Prime Zeta telescope

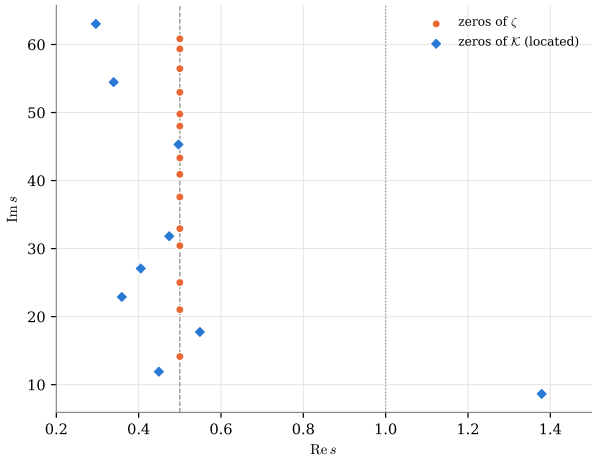


Figure 7: Zeros of $\mathcal{C}(s) = \zeta(s)\mathcal{K}(s)$ with $0 < \operatorname{Im} s < 65$: the zeros of ζ on $\operatorname{Re} s = \frac{1}{2}$ and the located zeros of $\mathcal{K}$.

(E4). The zeros of ζ appear as a dual spectrum $\{\gamma_n\} \cup \{\gamma_n/2\}$ on the imaginary axis. And the arithmetic fluctuation of $\sum_{n \le x} a(n)$ carries these frequencies at constant amplitude.

At the same time, every singularity of $\mathcal{K}$ comes from ζ'/ζ, and $\mathcal{C}$ has zeros of its own. The construction is therefore a precise and computable *instrument* for observing the zeros through the integers, not a mechanism that constrains them. We list some natural questions.

1. **Zeros of the kernel.** How many zeros does $\mathcal{K}$ have with $0 < \operatorname{Im} s < T$, and how are they distributed in $\operatorname{Re} s$? Does $\mathcal{K}(\frac{1}{2} + it)$ ever vanish?
2. **Unconditional explicit formula.** Is there a version of Theorem 17 without RH, with the off-line zeros contributing growing terms? Can (6) be proved in a summability sense?
3. **Certification.** Certify Observation 34 and the zeros of Table 6 with $\operatorname{Re} s < 1$ by interval arithmetic.
4. **The integer layer.** Determine the true order of $R_a(x)$ (Observation 23).
5. **Pairs $\gamma' \approx 2\gamma$.** How small can $|\gamma' - 2\gamma|$ be? Proposition 32 turns this into a statement about the energy of R_b on finite windows.
6. **Two scales.** With the first 96 zeros, $M(N) - 2M(\sqrt{N})$ lies in $[-0.183, -0.005]$ for $1.65 \le N \le e^{25}$; the omitted zeros contribute $-4\sum_{\gamma > \gamma_{96}} \gamma^{-2} = -0.013$ on average, so the true upper end is nearer -0.018. It cannot stay below a negative constant: under RH it is $\sum_\gamma \frac{8m_\gamma}{\gamma^2} c_\gamma(1 - c_\gamma)$ with $c_\gamma = \cos\frac{\gamma L}{2}$, a uniformly almost periodic function of L that vanishes at $L = 0$, so its upper limit is ≥ 0; and if the first 649 ordinates are linearly independent over $\mathbb{Q}$, Kronecker's theorem gives arbitrarily large L where it comes arbitrarily close to, or exceeds, $2\sum_{\gamma<1000} \gamma^{-2} - 16\sum_{\gamma>1000} \gamma^{-2} = 0.0443 - 0.0155 > 0$. What is the first N at which the kernel's margin is positive?

A Numerical methods

$a(n)$ for $n \le 10^8$ was computed by sieving, adding $b(p^k)$ to every multiple of p^k (Lemma 2(iii)); it was checked against factorization for $n \le 2 \cdot 10^4$ (error $\le 9 \cdot 10^{-16}$), and cumulative sums were kept in 80-bit precision. ζ and ζ' were evaluated by Euler–Maclaurin summation with 12 Bernoulli corrections, differentiated term by term. The 96 zeros with $\gamma < 230$ come from sign changes of Hardy's $Z(t)$, refined by Brent's method; their number agrees with $N(230) \approx 96.06$, and the first 30 agree with tabulated values to $5 \cdot 10^{-10}$. For high-precision fits we used $\mathcal{K}(1) = 1.1604648419$

from (E4). The spectrum is $\Phi(t) = \frac{2}{\sum_j w_j} |\sum_j R(e^{y_j}) e^{-ity_j} w_j|$ with a Hann window on 2^{17} points $y_j \in [\log 10^2, \log 10^8]$.

Data and code

All numbers in this paper are produced by the Python (NumPy, SciPy, Matplotlib) scripts in the ancillary archive: `core.py`, `explicit.py`, `zeros.py`, the `step*` scripts (mean value, explicit formula, critical line, zeros of $\mathcal{K}$), `rigor_meanvalue.py` and `rigor_rouche.py` (Proposition 14, Theorem 36), `ctransform.py` and `laplace.py` (Section 9), `bridge_M.py`, `bridge_M2.py`, `bridge_alpha.py`, `variogram.py`, `slope_b.py` and `fig_kernelspec.py` (Section 10), and `paper_figures.py`.

Use of AI assistance

The function, the research programme and the questions are the author's. Claude (Anthropic) was used as a research and writing aid: to derive and check proofs, to write and run the computations, to search the literature, and to draft and edit text and figures. The author checked all results and is responsible for the content.

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