At least one of $\beta(4), \beta(6), \ldots, \beta(32)$ is irrational

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Abstract

For the Dirichlet beta function, we prove that at least one of $\beta(4), \beta(6), \ldots, \beta(32)$ is irrational. This reduces Krisman's seventeen-value set $\beta(4), \beta(6), \ldots, \beta(36)$ [1] to fifteen values. The improvement comes from a sharper denominator estimate for his construction.

1 The result and the denominator improvement

For $s > 1$, let

$$\beta(s) = \sum_{t=0}^{\infty} \frac{(-1)^t}{(2t+1)^s}$$

be the Dirichlet beta function. Its values at positive odd integers are nonzero rational multiples of powers of π. The arithmetic of its even values is much less understood. We prove the following finite-set result.

Theorem 1.1. *At least one of $\beta(4), \beta(6), \ldots, \beta(32)$ is irrational.*

Krisman [1, Theorem 1.1] proves the corresponding assertion through $\beta(36)$, with seventeen candidate values. We reduce the number of candidates to fifteen by improving the denominator estimate in his construction and then choosing new parameters.

We construct a sequence of linear forms in $1, \beta(4), \beta(6), \ldots, \beta(32)$ with rational coefficients. If all fifteen beta values were rational, one fixed integer would clear their denominators. Multiplying the forms by that integer and by factors clearing their coefficients would produce a sequence of integers. We show that this sequence tends to zero but has infinitely many nonzero terms, which is impossible.

1.1 Where the saving occurs

Krisman's rational functions have integer zeros and half-integer poles. Their second derivatives produce sequences of linear forms in even beta values beginning at $\beta(4)$. The denominator-clearing factors must be small enough for the cleared forms still to tend to zero.

We call the integer a the address of the pole $-a - \frac{1}{2}$. At an odd prime p, a local denominator comes from differences between addresses that agree modulo p. If an address is alone in its residue class, none of its principal-part coefficients has a denominator divisible by p. The rational constant of the linear form also contains evaluation denominators. At a fixed evaluation point, however, only one residue class can have a denominator divisible by p. A zero identity expresses the same

contribution using poles on either side of that point. When the troublesome class contains just one pole, we choose the side that avoids it.

Thus only classes containing at least two pole addresses enter the large-prime cost. Krisman's argument already removes primes larger than the pole span [1, Section 4.3]; retaining this information also for smaller primes provides the saving used here. A related exceptional-group cancellation appears in [4, Section 3.2]. The refinement here is the use of isolated pole classes in the explicit denominator estimate.

1.2 Construction, nonvanishing, and sources

The rational kernel and its local valuation framework follow [1, Sections 2–6]. The second-derivative device adapts Rivoal's construction for odd zeta values [3, Section 2]. The role of zeros throughout the pole region in controlling rational constants is also present in Zudilin's work [6, proof of Lemma 1].

Smallness of the forms does not ensure that they are nonzero. We use the prime-dilation argument of [5, Section 4], which implements the auxiliary-prime principle of Lai and Sprang [2, Lemma 2.3]. At indices equal to a fixed base index times a large prime, a rescaling makes every beta coefficient vanish modulo that prime while the rational constant reduces to a fixed, nonzero seed constant. Section 5 gives the calculation for the present parameters.

We use v_p for the valuation on $\mathbb{Q}$ normalized by $v_p(p) = 1$, with $v_p(0) = +\infty$. A rational number is *integral at* p if its denominator is prime to p, and is a *unit at* p if its numerator is also prime to p. Congruences are taken in the ring $\mathbb{Z}_{(p)}$ of rationals integral at p. The proof otherwise uses partial fractions, Stirling's formula, and the prime number theorem.

2 A sequence of linear forms

2.1 Zeros, poles, and the choice of parameters

The integer zeros have two tasks: they remove initial terms from the infinite sum and give identities for the rational correction terms. The half-integer poles produce beta values. Varying their orders across the pole interval improves the balance between denominator growth and analytic decay.

Two differentiations raise each partial-fraction exponent by two. Reflection will remove the odd beta indices, so a largest pole order of 31 leaves 32 as the largest even index. The other parameters below are chosen so that the decay exceeds the denominator cost, as verified in Section 4.

Fix

$$S = 31, \qquad r = \frac{3}{4}, \qquad N = 96, \qquad \delta_j = \frac{j}{96} \quad (0 \le j \le 30), \tag{2.1}$$

and let n range over positive multiples of N. Define the blocks and their combined pole orders by

$$I_j(n) = [\delta_j n, (1-\delta_j)n] \cap \mathbb{Z}, \qquad B_a(n) = \#\{j : a \in I_j(n)\} \quad (0 \le a \le n).$$

The block $I_0(n)$ contains every address. At $n = 96$, the successive blocks are $[0, 96], [1, 95], \dots, [30, 66]$, so the orders rise from one at the ends to 31 in the middle. The base $N = 96$ makes these endpoints and rn integral. We often suppress n in $B_a(n)$.

Put

$$\Lambda_n = [-(1+r)n - 1, rn] \cap \mathbb{Z}, \qquad R_n(t) = \frac{(2t+n+1)\prod_{\ell\in\Lambda_n}(t-\ell)^3}{\prod_{a=0}^{n}(t+a+\frac{1}{2})^{B_a}}.$$

Two degree parameters will track the factorial normalization:

$$\gamma = \sum_{j=0}^{30}(1-2\delta_j) = \frac{341}{16}, \qquad G = \gamma - 3(1+2r) = \frac{221}{16}. \tag{2.2}$$

There are $\gamma n + 31$ denominator factors and $3\#\Lambda_n = \frac{15}{2}n + 6$ integer zero factors, counted with multiplicity. Including the linear prefactor gives $\deg R_n = -Gn - 24$, so R_n is proper and decays rapidly at infinity. The prefactor cancels one order at the central pole; nominal pole orders remain convenient, with the corresponding leading coefficient equal to zero.

Reflection about $-(n+1)/2$ permutes the zeros and the pole blocks. Since γn and $3\#\Lambda_n$ are even, the prefactor and denominator both change sign. Hence

$$R_n(-n-1-t) = R_n(t). \tag{2.3}$$

2.2 The beta expansion

Write the partial fractions as

$$R_n(t) = \sum_{a=0}^{n}\sum_{i=1}^{B_a}\frac{a_{i,a}}{(t+a+\frac{1}{2})^i},$$

and set absent coefficients to zero. Let

$$(i)_2 = i(i+1), \qquad \sigma_a(s) = \sum_{h=0}^{a-1}\frac{(-1)^h}{(2h+1)^s}.$$

Differentiating twice and shifting each beta series gives

$$L_n = \sum_{t=0}^{\infty}(-1)^t R_n''(t) = \rho_n + \sum_{i=1}^{31}\tau_{i,n}\beta(i+2), \tag{2.4}$$

$$\tau_{i,n} = 2^{i+2}(i)_2\sum_{a=0}^{n}(-1)^a a_{i,a}, \tag{2.5}$$

$$\rho_n = -\sum_{a=0}^{n}\sum_{i=1}^{B_a}(-1)^a 2^{i+2}(i)_2 a_{i,a}\sigma_a(i+2). \tag{2.6}$$

All exponents in the shifted series are at least three, so the rearrangements are absolutely convergent. Uniqueness of partial fractions in (2.3) gives $a_{i,n-a} = (-1)^i a_{i,a}$. Since n is even, every $\tau_{i,n}$ with odd i vanishes. Thus

$$L_n \in \mathbb{Q} + \mathbb{Q}\beta(4) + \mathbb{Q}\beta(6) + \cdots + \mathbb{Q}\beta(32). \tag{2.7}$$

The triple zeros give two identities we shall use separately:

$$R_n''(t) = 0 \quad (0 \le t \le rn), \qquad R_n''(-\ell) = 0 \quad (1 \le \ell \le n), \tag{2.8}$$

where the arguments are integers. The first starts the analytic sum beyond rn. The second controls the denominators of ρ_n.

3 Only repeated pole classes contribute

3.1 Two expressions for each rational correction

The finite harmonic sums in (2.6) introduce denominators larger than the distances between poles. Bounding their terms separately would lose useful cancellation. We first group them at the integer zeros from (2.8).

For $1 \le \ell \le n$, put

$$T(\ell, a) = \sum_i (i)_2 a_{i,a} (a - \ell + \tfrac{1}{2})^{-i-2}.$$

Changing variables from h to $\ell = a - h$ in (2.6) gives

$$\rho_n = -\sum_{\ell=1}^{n} (-1)^\ell \sum_{a \ge \ell} T(\ell, a). \tag{3.1}$$

On the other hand, $R_n''(-\ell) = 0$ says

$$\sum_{a \ge \ell} T(\ell, a) = -\sum_{a < \ell} T(\ell, a). \tag{3.2}$$

Thus each group in (3.1) can be represented using poles on either side of $-\ell$.

Fix an odd prime p. An evaluation denominator is divisible by p exactly when $2(a - \ell) + 1 \equiv 0 \pmod p$. The troublesome addresses therefore form one residue class. If this class contains only one address, one side of (3.2) avoids it entirely.

Example 3.1. Consider addresses $0, \dots, 12$ modulo 11. The only repeated classes are $\{0, 11\}$ and $\{1, 12\}$; addresses $2, \dots, 10$ are isolated. For $\ell = 3$, the evaluation denominator is divisible by 11 only at $a = 8$. Choosing the side $a < 3$ in (3.2) removes that denominator. To make this useful, we must also control the coefficients of the terms that remain.

3.2 Local coefficients at isolated and repeated poles

Let p be odd and $p^2 > 7n$. For an address a, define

$$\begin{aligned} N_{p,j}(a) &= \#\{b \in I_j(n) : b \equiv a \pmod p\}, \\ c_p(a) &= 3\#\{\ell \in \Lambda_n : p \mid 2a + 1 + 2\ell\} + \mathbf{1}_{p \mid n-2a}, \\ F_p(a) &= \sum_{j=0}^{30} N_{p,j}(a) - c_p(a). \end{aligned}$$

The value of $F_p(a)$ depends only on the residue class of a.

Lemma 3.2. *For $1 \le i \le B_a$,*

$$v_p(a_{i,a}) \ge i - F_p(a). \tag{3.3}$$

If a is the only address in its class modulo p, every $a_{i,a}$ is integral at p. This latter assertion holds for every odd p.

Proof. At $t = -a - \frac{1}{2} + u$, remove the nominal pole. The remaining series is

$$(n - 2a + 2u) \prod_{\ell \in \Lambda_n} (-a - \tfrac{1}{2} - \ell + u)^3 \prod_{b \ne a} (b - a + u)^{-B_b}. \tag{3.4}$$

Its coefficient of u^{B_a-i} is $a_{i,a}$.

Say a series has bound (e,λ) at p if its degree-d coefficient has valuation at least $e-d\lambda$. Products add the first parameters. A linear numerator factor divisible by p has bound $(1,1)$; a reciprocal factor whose constant term has valuation one has bound $(-1,1)$. Unit denominator factors have integral series. These are the local-series estimates of [1, Section 4.1], directly from the geometric series.

All nonzero integer differences involved have absolute value below $7n < p^2$. The numerator in (3.4) contributes $c_p(a)$, and the other poles contribute $-\sum_j N_{p,j}(a) + B_a$. Extraction of degree $B_a - i$ then gives (3.3). At the central pole the prefactor has zero constant term; the bound $(1,1)$ remains valid since its linear coefficient is a p-unit.

If a is isolated, every difference $b-a$ with $b \neq a$ is a p-unit. All factors in (3.4) then have integral series at p, since p is odd. No Taylor-degree loss is needed. □

Proposition 3.3 (The isolated-pole refinement)**.** *For an odd prime $p \le n$ with $p^2 > 7n$, let M_p be the maximum of $F_p(a)$ over addresses in residue classes containing at least two addresses. Set*

$$e_p = \max(0, M_p + 2). \tag{3.5}$$

Then $p^{e_p}\rho_n$ and every $p^{e_p}\tau_{i,n}$ are integral at p. For every prime $p > n$, these coefficients are integral at p without a multiplier.

Proof. At a repeated class, Lemma 3.2 bounds a coefficient below by $1 - M_p$; at an isolated class it is integral. This already controls the beta coefficients.

In a group (3.2), terms with unit evaluation denominators have valuation at least $\min(0, 1-M_p) \ge -e_p$. If the troublesome class is isolated, choose the side that avoids its address. If it is repeated, its evaluation denominators have valuation one, so each corresponding term has valuation at least

$$i - F_p(a) - (i+2) \ge -M_p - 2 \ge -e_p.$$

In either case the group has valuation at least $-e_p$. Summing (3.1) proves the assertion for ρ_n.

If $p > n$, all pole classes are isolated. Their coefficients are integral, and at least one side of (3.2) has only unit evaluation denominators. This proves the last assertion even when a denominator on the other side has higher valuation. □

The complement identity is already used for $p > n$ in [1, Lemma 4.6]. The extra information in Lemma 3.2 allows the same reasoning to discard isolated classes when $p \le n$. In the important range $n/2 < p < n$, only classes with least representative at most $n - p$ are repeated.

3.3 A common clearing factor

Appendix A defines clearing exponents at the remaining primes, namely 2 and the odd primes with $p^2 \le 7n$. Together with (3.5), and $e_p = 0$ for $p > n$, they give a positive integer

$$D_n = \prod_{p \le n} p^{e_p}, \qquad D_n\rho_n,\ D_n\tau_{i,n} \in \mathbb{Z}. \tag{3.6}$$

The estimates proved there are

$$e_p - Gv_p(n!) \le 128(\lambda_p + 1), \qquad \lambda_p = \lfloor \log_p(7n) \rfloor \qquad (p \text{ odd},\ p^2 \le 7n), \tag{3.7}$$

$$e_2 - Gv_2(n!) \le 15n + 36 + 107(\lambda_2 + 1), \qquad \lambda_2 = \lfloor \log_2 n \rfloor. \tag{3.8}$$

The small odd primes therefore cost $O(\sqrt{n}\log n) = o(n)$ in $\log D_n - G\log(n!)$, while 2 contributes at most $15n\log 2 + o(n)$. The next section estimates the large primes and compares the total cost with the decay of L_n.

4 Decay after clearing denominators

To prove that $D_n L_n \to 0$, we compare denominator growth with decay of the forms. The factorial growth of D_n is matched by the factorial decay of L_n. After these contributions cancel, we will bound the remaining denominator growth rate by $8.2 + 15\log 2$ and the decay rate from below by 18.83. Since

$$8.2 + 15\log 2 - 18.83 < -0.23,$$

the cleared forms will tend to zero exponentially. The two denominator terms come from the large odd primes and from 2, respectively. The small odd primes contribute only $o(n)$ after factorial normalization.

4.1 The arithmetic envelope

For a large prime write $x = n/p$. If $1 < x < 2$, a repeated address class has its least representative between 0 and $n - p$. Dividing by p restricts the corresponding variable y to $[0, x-1]$. For $x \geq 2$, every residue class is repeated and we use a full period.

For real $x \geq 1$, define

$$\begin{aligned} N_j(x,y) &= \#\{h \in \mathbb{Z} : \delta_j x \leq y + h \leq (1-\delta_j)x\}, \\ C(x,y) &= \lfloor y + rx + \tfrac{1}{2} \rfloor - \lfloor y - (1+r)x + \tfrac{1}{2} \rfloor, \\ f(x,y) &= 2 + \sum_{j=0}^{30} N_j(x,y) - 3C(x,y), \end{aligned}$$

$$J_x = \begin{cases} [0, x-1], & 1 \leq x < 2, \\ [0,1], & x \geq 2, \end{cases} \qquad e(x) = \max\left(0, \max_{y \in J_x} f(x,y)\right) - G\lfloor x \rfloor.$$

The subtraction $G\lfloor x \rfloor$ removes the factorial contribution: for $x = n/p$ and $p^2 > 7n$, Legendre's formula gives $Gv_p(n!) = G\lfloor n/p \rfloor$. At $x = n/p$ and $y = a/p$, $N_j(x,y) = N_{p,j}(a)$. Since p is odd and the interval endpoints are integers, $C(x,y)$ counts the $\ell \in [-(1+r)n, rn - 1] \cap \mathbb{Z}$ for which $p \mid 2a + 1 + 2\ell$. Thus $3C(x,y)$ omits the two endpoint contributions and the prefactor indicator from $c_p(a)$. Hence Proposition 3.3 gives

$$e_p - G\lfloor n/p \rfloor \leq e(n/p). \tag{4.1}$$

The restriction to J_x for $x < 2$ is where isolated pole classes lower the estimate.

The interval defining N_j is closed, so its exact expression is

$$N_j(x,y) = \lfloor y - \delta_j x \rfloor - \lceil y - (1-\delta_j)x \rceil + 1.$$

It is at most $(1 - 2\delta_j)x + 1$, while $C(x,y) \geq (1+2r)x - 1$. Consequently

$$e(x) \leq 36 + G\{x\}, \qquad \{x\} = x - \lfloor x \rfloor. \tag{4.2}$$

This bound also holds after replacing $e(x)$ by $\max(e(x), 0)$.

4.2 A finite partition and the prime sum

To turn the prime costs into an exponential bound for D_n, we use the prime number theorem. For fixed $1 \leq a < b$, it gives

$$\sum_{a < n/p < b} \log p = n(1/a - 1/b) + o(n).$$

The factor $1/a - 1/b = \int_a^b dx/x^2$ explains why the total prime cost is controlled by $\int_1^\infty e(x)\,dx/x^2$. We compute the integral on $[1, 400]$ by a finite partition and bound the remaining tail.

The dependence on y changes only at the affine offsets

$$\delta_j x, \quad (1-\delta_j)x \quad (0 \le j \le 30), \qquad -rx - \tfrac{1}{2}, \quad (1+r)x - \tfrac{1}{2}, \tag{4.3}$$

taken modulo one. On $[1, 400]$, include every integer and every point at which two distinct offsets coincide modulo one. For offsets $sx + t$ and $s'x + t'$ with $s \neq s'$, these points are $(t' - t + h)/(s - s')$, $h \in \mathbb{Z}$, whenever they lie in that interval. There are 38 372 distinct event points.

Between consecutive events, the offsets keep their order, and $\lfloor x \rfloor$ is unchanged. The moving boundary $x - 1$ for $x < 2$ is itself the offset x modulo one. Thus e is constant on each open event interval. At a non-event x, a breakpoint value in y equals one of its neighbouring cell values. At $y = 0$ the outer pole block takes its right-hand value; at $y = x - 1$ it takes its left-hand value. Checking the open cells inside J_x therefore also captures its boundary maxima.

For each event interval (a, b), its contribution to $\int_1^{400} e(x)\,dx/x^2$ is $e((a+b)/2)(1/a - 1/b)$. Exact rational computation, rounding each contribution upwards to a multiple of 2^{-64}, gives

$$A_{\mathrm{body}} \le \frac{73794151590486879261}{9223372036854775808}. \tag{4.4}$$

The rounding is upwards also for negative contributions. For an integer $X \ge 2$, (4.2) gives

$$\begin{aligned} \int_X^\infty \frac{\max(e(x), 0)}{x^2}\,dx &\le \frac{36}{X} + \frac{G}{2}\sum_{h=X}^{\infty}\frac{1}{h^2} \\ &\le \frac{36}{X} + \frac{G}{2(X-1)}. \end{aligned}$$

At $X = 400$ this is $34253/319200$. Together with (4.4), the certified upper bound is

$$A_{\mathrm{cert}} = \frac{745969417189431202995707}{9200313606762638868480 0} = 8.10808684435616\ldots < \frac{41}{5}. \tag{4.5}$$

The exact partition and two calculations of its cell maxima are supplied with the paper; Section 6 describes them.

Proposition 4.1. *As $n \to \infty$ through positive multiples of 96,*

$$\log D_n \le G\log(n!) + n\left(\frac{41}{5} + 15\log 2 + o(1)\right). \tag{4.6}$$

Proof. The prime-number-theorem estimate above, summed over the finite partition, yields its integral contribution. Each event point can equal n/p for at most one prime; by (4.2), all such primes together cost $O(\log n)$.

For the tail replace e by $\max(e, 0)$ on $[400, \infty)$. The omitted small primes eventually lie there and may now be inserted in an upper bound. On a fixed interval $[400, M]$, finite partitions and the prime number theorem give the integral estimate. The remaining primes $p \le n/M$ contribute at most $(36 + G)\vartheta(n/M) = O(n/M)$ by Chebyshev's bound, where $\vartheta(u) = \sum_{p \le u} \log p$. Letting M tend to infinity proves

$$\sum_{\substack{p \le n \\ p^2 > 7n}} (e_p - G\lfloor n/p \rfloor)\log p \le n(A_{\mathrm{cert}} + o(1)).$$

Add (3.7) and (3.8). The small odd primes cost $o(n)$, and 2 costs at most $15n\log 2 + o(n)$. Finally use $A_{\mathrm{cert}} < 41/5$. □

4.3 The real phase and its maximum

Write $H(u) = u \log u$ for $u > 0$, with $H(0) = 0$, and set

$$\Phi(x) = 3\big(H(x+1+r) - H(x-r)\big) - \sum_{j=0}^{30}\big(H(x+1-\delta_j) - H(x+\delta_j)\big), \quad x \geq r. \tag{4.7}$$

This phase records the exponential size of the numerator and pole products after their factorial normalization.

Lemma 4.2. *For $x \geq r$, $\Phi(x) < -18.83$. For $x \geq 3$, $\Phi'(x) \leq -3/x$.*

Proof. For $x > r$, the sign of $\Phi'(x)$ is the sign of

$$(x+1+r)^3 \prod_{j=0}^{30}(x+\delta_j) - (x-r)^3 \prod_{j=0}^{30}(x+1-\delta_j).$$

Put $x = r + z$ and clear positive denominators. The resulting polynomial can be taken as

$$P(z) = (2z+5)^3 \prod_{j=0}^{30}(96z+72+j) - 8z^3 \prod_{j=0}^{30}(96z+168-j). \tag{4.8}$$

Its degree is 33. Exact multiplication shows that its coefficients in degrees $0, 1, 2$ are positive and all remaining coefficients are negative. Descartes' rule of signs, together with $P(0) > 0$ and its negative leading coefficient, therefore gives exactly one positive zero. Thus Φ increases to a unique global maximum and then decreases.

Exact evaluations of P bracket the maximum point x_* by

$$\frac{378667493}{500000000} < x_* < \frac{757334987}{1000000000}.$$

Rational logarithm enclosures on this interval give

$$\Phi(x_*) \leq -\frac{18835112315041}{10^{12}} < -18.83. \tag{4.9}$$

These finite checks are reproducible directly from (4.8). For the logarithms, reduce the argument to $1 \leq y \leq 2$ and use

$$\log y = 2\sum_{j=0}^{47} \frac{w^{2j+1}}{2j+1} + R, \qquad w = \frac{y-1}{y+1}, \qquad 0 \leq R \leq \frac{2w^{97}}{97(1-w^2)}.$$

The same enclosure supplies $\log 2$ when rescaling the argument.

To control the infinite tail, use $\log(1+u) \leq u$ and

$$\log \frac{x+1-\delta_j}{x+\delta_j} = \int_{x+\delta_j}^{x+1-\delta_j} \frac{dt}{t} \geq \frac{1-2\delta_j}{x+1}.$$

For $x \geq 3$ these give

$$x\Phi'(x) \leq 3(1+2r)\frac{3}{3-r} - \frac{3\gamma}{4} = -\frac{383}{64} < -3.$$

□

4.4 Uniform estimates and the infinite sum

For real $t \geq rn+1$, every factor of $R_n(t)$ is positive. Its gamma-product expression is

$$R_n(t) = (2t+n+1)\left(\frac{\Gamma(t+(1+r)n+2)}{\Gamma(t-rn)}\right)^3 \prod_{j=0}^{30} \frac{\Gamma(t+\delta_j n+\frac{1}{2})}{\Gamma(t+(1-\delta_j)n+\frac{3}{2})}.$$

Stirling's formula, uniformly for real arguments at least one, yields

$$\log R_n(t) = -G\log(n!) + n\Phi(t/n) + O(\log(n+t)). \tag{4.10}$$

The potentially small argument $t-rn$ is still at least one, even when $(t-rn)/n \to 0$. Its Stirling remainder and the logarithmic term omitted from $u\log u - u$ are therefore $O(\log(n+t))$. Every other argument is bounded below by a positive multiple of n. The $n\log n$ and linear terms give $-G\log(n!)$, up to $O(\log n)$. This explains the uniformity of (4.10) down to $t = rn+1$.

Differentiation costs only a power of $\log n$. Indeed,

$$\frac{R_n'(t)}{R_n(t)} = \frac{2}{2t+n+1} + 3\sum_{\ell\in\Lambda_n}\frac{1}{t-\ell} - \sum_{a=0}^{n}\frac{B_a}{t+a+\frac{1}{2}}.$$

The reciprocals of the consecutive zero distances sum to $O(\log n)$, and their squares sum to $O(1)$, uniformly for $t \geq rn+1$. The corresponding pole sums are bounded since $r > 0$. Differentiating this identity gives

$$|R_n''(t)| \leq C(1+\log n)^2 R_n(t) \tag{4.11}$$

for a fixed C.

Only terms $t \geq rn+1$ remain in L_n by (2.8). For $t \leq 3n$, equations (4.10)–(4.11) and Lemma 4.2 bound the $O(n)$ terms by a polynomial in n times $(n!)^{-G}e^{-18.83n}$. For $t > 3n$, the derivative bound for Φ gives

$$\Phi(t/n) \leq \Phi(3) - 3\log(t/(3n)).$$

The resulting power tail dominates the fixed factor $(1+t/n)^C$ from the error in (4.10); its sum adds only a polynomial factor in n. We obtain

$$|L_n| \leq (n!)^{-G}\exp(n(-18.83+o(1))). \tag{4.12}$$

Combining this with Proposition 4.1,

$$\limsup_{\substack{n\to\infty \\ 96\mid n}} \frac{1}{n}\log|D_nL_n| \leq \frac{41}{5} + 15\log 2 - \frac{1883}{100} = -0.2327922916\ldots < -0.23. \tag{4.13}$$

Thus the forms tend to zero after their coefficient denominators are cleared. It remains to ensure that infinitely many are nonzero under the rationality hypothesis.

5 A nonzero seed and prime dilation

5.1 The seed that survives dilation

We follow [5, Section 4]: at indices $n = Np$, with fixed $N = 96$ and p an increasing prime, a rescaling will separate the rational constant from all beta coefficients modulo p. The half-integer poles are easier to count in the centred coordinate $x = 2t+n+1$. Define

$$F_n(x) = \frac{x\prod_{\substack{v \text{ odd} \\ |v|\leq(1+2r)n+1}}(x-v)^3}{\prod_{a=0}^{n}(x-n+2a)^{B_a(n)}}.$$

Counting factors of two gives

$$R_n(t) = 2^{Gn+25} F_n(2t+n+1), \qquad L_n = 2^{Gn+27}\ell_n, \tag{5.1}$$

where ℓ_n is the alternating sum of $F_n''(2t+n+1)$. Write

$$F_n(x) = \sum_{a=0}^{n} \sum_{i=1}^{B_a(n)} \frac{c_{n;a,i}}{(x-n+2a)^i}.$$

The corresponding coefficient of R_n is $a_{i,a} = 2^{Gn+25-i} c_{n;a,i}$. In these coordinates the beta expansion is

$$\ell_n = \sum_{a,i} (-1)^a (i)_2 c_{n;a,i} \big(\beta(i+2) - \sigma_a(i+2)\big), \tag{5.2}$$

with rational constant

$$q_n = -\sum_{a,i} (-1)^a (i)_2 c_{n;a,i} \sigma_a(i+2). \tag{5.3}$$

Odd beta coefficients vanish by the same reflection as before.

At the base index put

$$L = (1+2r)N = 240, \quad W = \gamma N = 2046, \quad GN = 1326. \tag{5.4}$$

Take an odd prime $p > 337 = N + L + 1$. In $F_{Np}(pu)$, a numerator zero divisible by p is pv with v odd and $|pv| \le Lp + 1$. Since L is even, this leaves $|v| \le L - 1 = 239$. The selected pole addresses are pa, and $B_{pa}(Np) = B_a(N)$. The resulting seed is

$$\widetilde{F}(u) = \frac{u \prod_{\substack{v \text{ odd} \\ |v| \le 239}} (u-v)^3}{\prod_{a=0}^{96} (u-96+2a)^{B_a(96)}}. \tag{5.5}$$

In particular, its last zero is 239, rather than the untrimmed endpoint 241. The two outer zeros are lost when the divisible factors are selected.

The selected prefactor, zeros, and poles contribute respectively 1, $3L$, and $-(W+31)$ to the exponent of p. This determines

$$A_0 = W - 3L + 30 = 1356, \qquad \tau = A_0 + 2 = 1358. \tag{5.6}$$

The extra two in τ will match the two differentiations.

5.2 Selected and nonselected coefficients

A block $[pa, pb] \cap \mathbb{Z}$ contains $b-a$ representatives of each nonzero residue modulo p, and $b-a+1$ representatives of zero. Our blocks have $a = j$ and $b = 96 - j$. Summing over the 31 blocks therefore gives weighted counts W for every nonzero residue and $W+31$ for zero. These different counts account for the valuation separation between selected and nonselected poles.

Let $\widetilde{c}_{a,i}$ be the principal-part coefficients of $\widetilde{F}$.

Lemma 5.1. *For all sufficiently large odd primes p,*

$$p^{A_0-i} c_{Np;pa,i} \equiv -\widetilde{c}_{a,i} \pmod{p}, \tag{5.7}$$

$$v_p(c_{Np;a,i}) \ge -GN + i \qquad (p \nmid a). \tag{5.8}$$

The scaled coefficients in (5.7) are integral at p.

Proof. After the selected factors are removed from $F_{Np}(pu)$, all remaining factors are units whose reductions are independent of u. Their combined unit is -1. To see the sign, the Lp odd numbers from $-Lp+1$ to $Lp-1$ contain each residue class L times. Wilson's theorem gives nonzero product $(-1)^L = 1$. The extra endpoints $\pm(Lp+1)$ contribute -1, also after cubing. In the denominator, Fermat's theorem removes the factors 2^{p-1}, and the total denominator unit is $(-1)^W = 1$. Thus

$$p^{A_0} F_{Np}(pu) \equiv -\widetilde{F}(u) \pmod{p}. \tag{5.9}$$

For coefficient extraction, put $u = 96 - 2a + w$ and remove the nominal pole by multiplying by $w^{B_a(96)}$. The local series after scaling lies in $\mathbb{Z}_{(p)}[[w]]$: the other selected pole differences $2(b-a)$ are units, and every unselected denominator has a unit constant term and an integral geometric-series inverse. Reduction therefore commutes with extraction of degree $B_a - i$. That coefficient is $p^{A_0-i} c_{Np;pa,i}$, proving (5.7). Terms analytic at this pole begin in degree B_a after pole removal, so they do not affect the extraction.

At a nonselected pole address a, the prefactor is a p-unit. Each numerator residue class contains at least L zeros before cubing, whereas a nonzero pole-address class has weighted count W. After removing its pole of order B_a, the local series has bound $(3L - W + B_a, 1)$ in the terminology of Lemma 3.2. All nonzero differences have absolute value below p^2 since $p > 337$. Extracting degree $B_a - i$ yields $v_p(c_{Np;a,i}) \geq 3L - W + i = -GN + i$. □

5.3 Why the rational constant remains

Define the seed constant

$$\widetilde{q} = -\sum_{a=0}^{96} \sum_{i} (-1)^a (i)_2 \widetilde{c}_{a,i} \sigma_a(i+2). \tag{5.10}$$

Proposition 5.2. *Every beta coefficient of $p^\tau \ell_{Np}$ is divisible by p, and*

$$p^\tau q_{Np} \equiv -\chi_4(p)\widetilde{q} \pmod{p}, \qquad \chi_4(p) = (-1)^{(p-1)/2}. \tag{5.11}$$

All scaled quantities reduced here are integral at p.

Proof. Write $s = i + 2$. At a selected pole, $\tau - s = A_0 - i$, so (5.7) says that the scaled coefficient $p^\tau (i)_2 c_{Np;pa,i}$ has valuation at least $s > 0$. Thus this contribution to the beta coefficient vanishes modulo p, while leaving enough powers of p to cancel a harmonic denominator of exponent s. At a nonselected pole, (5.8) gives valuation at least $\tau - GN + i = 32 + i > 0$ for the beta coefficient.

Every denominator $2h+1$ with $h < Np$ is below $2Np < p^2$. Nonselected terms in the rational constant therefore have scaled valuation at least $\tau - GN + i - (i+2) = 30 > 0$. At a selected address pa, the surviving terms in $p^s \sigma_{pa}(s)$ satisfy $2h + 1 = p(2v+1)$, so

$$h = pv + (p-1)/2, \quad 0 \leq v < a, \qquad p^s \sigma_{pa}(s) \equiv \chi_4(p)\sigma_a(s) \pmod{p}.$$

Combining this with $(-1)^{pa} = (-1)^a$ and (5.7) in (5.3) proves (5.11). □

5.4 A finite nonzero seed

We need only one nonzero reduction of $\widetilde{q}$. At $u_a = 96 - 2a$, form the truncated local product

$$J_a(w) = (u_a + w) \prod_{\substack{v \text{ odd} \\ |v| \leq 239}} (u_a - v + w)^3 \prod_{b \neq a} (2(b-a) + w)^{-B_b(96)} \pmod{w^{B_a(96)}}. \tag{5.12}$$

Then $\widetilde{c}_{a,i} = [w^{B_a - i}]J_a(w)$. There are $W + 31 = 2077$ coefficients. Two exact calculations give

$$\widetilde{q} \equiv 3738 \pmod{10007}, \qquad \widetilde{q} \equiv 5333 \pmod{10009}. \tag{5.13}$$

Both moduli are prime by trial division. Pole differences have absolute value at most 192, and harmonic denominators at most 191, so these are legitimate reductions of a rational number. Either residue proves $\widetilde{q} \neq 0$.

One calculation multiplies the truncated products in (5.12). The other expands their logarithmic derivatives and recovers Taylor coefficients recursively. At the central pole, its factor w is removed before inversion and restored afterwards. The second method inverts only nonzero numerator differences of absolute value at most 335 and Taylor indices below 31. Both methods agree on all 2077 coefficients at both primes.

5.5 Proof of Theorem 1.1

Suppose all fifteen beta values are rational, and let T be a positive integer clearing their denominators. By (3.6), TD_nL_n is an integer for every positive multiple n of 96. Equation (4.13) makes these integers tend to zero.

The fixed rational $\widetilde{q}$ is nonzero by (5.13). For every sufficiently large prime p outside the finite set dividing T or the numerator or denominator of $\widetilde{q}$, the beta values are integral at p. Proposition 5.2 makes their coefficients in $p^{\tau}\ell_{Np}$ divisible by p, while its constant is a unit. Thus $\ell_{Np} \neq 0$, and (5.1) gives $L_{Np} \neq 0$. Infinitely many nonzero integers $TD_{Np}L_{Np}$ therefore tend to zero, a contradiction. □

6 Computational certificates and review

The proof uses finite calculations at three points: the arithmetic integral, the maximum of the real phase, and the nonzero seed. Each is performed with exact integer or rational arithmetic. Decimal values in the text are illustrations of the certified bounds.

Calculation	Size	Certified conclusion
Arithmetic partition	38 371 intervals	$A_{\mathrm{cert}} < 41/5$
Critical polynomial	degree 33	$\max \Phi < -18.83$
Seed coefficients	2077 at each of two primes	$\widetilde{q} \neq 0$

The program `certify_fifteen.py` generates the rational event partition and computes its maxima by sweeping the jumps in y. A separate C++ program, `verify_fifteen_arithmetic.cpp`, recomputes every supplied midpoint maximum by direct floor evaluation on the cells. Thus the second calculation checks the maxima; completeness of the event partition follows from the affine-offset argument in Section 4 and its explicit enumeration in the first program.

The polynomial calculation in `certify_phase_polynomial.py` multiplies the factors in (4.8), checks every coefficient sign, brackets its positive root, and encloses the logarithms used in (4.9). The arithmetic program also checks the weaker phase bound -18.83 by a separate concavity argument.

For the seed, `seed_fifteen.py` uses truncated products over finite fields. The separate calculation `verify_fifteen_seed.py` uses the Taylor recurrence described after (5.13). It computes all coefficients before comparing them with the saved product calculation. Both give the residues in (5.13). The scripts, certificate data, and reproduction commands accompany the source in `checks/`.

The work was prepared with OpenAI Codex using GPT-6-Astra max. The isolated-pole refinement and the fifteen-value conclusion were obtained separately in two research sessions using the same public sources, without access to each other's unpublished work. This paper combines the subsequent comparison into one proof. The supplied calculations have been replayed for this manuscript.

An independent mathematical audit of the complete fourteen-page version, reported on 11 November 2026, found no mathematical error or gap affecting Theorem 1.1. It checked the general arguments and independently reproduced the arithmetic integral, the phase bound, and all 2077 seed coefficients at both primes. The audit accepts the stated classical inputs, including Stirling's formula and the prime number theorem; it is not formal verification. The accompanying audit record identifies the reviewed version and the verification scope.

A The remaining clearing exponents

The isolated-pole estimate handles the large odd primes. Here we supply the bounds at 2 and at odd primes with $p^2 \le 7n$. These estimates follow the local-series and Legendre bounds in [1, Sections 4.1 and 4.5]. Their purpose is to show that the small odd primes cost only $o(n)$ after factorial normalization, whereas the prime 2 contributes the term $15\log 2$.

A.1 Clearing the coefficients

Put $Z = 3\#\Lambda_n = \frac{15}{2}n + 6$. For odd p, define

$$\lambda_p = \lfloor \log_p(7n) \rfloor, \qquad \nu_p(a) = 3\sum_{\ell\in\Lambda_n} v_p(2a+1+2\ell) - \sum_{b\ne a} B_b v_p(b-a).$$

At 2, use

$$\lambda_2 = \lfloor \log_2 n \rfloor, \qquad \nu_2(a) = 1 - Z - \sum_{b\ne a} B_b v_2(b-a).$$

The factor $n - 2a + 2u$ in the local series (3.4) is integral at odd primes, so it can be omitted in a lower bound. At 2, its constant and linear terms are divisible by 2. Each half-integer numerator constant has valuation -1, giving the term $-Z$ in ν_2.

Every nonzero numerator integer or pole difference has absolute value below $7n$, and each pole difference has absolute value at most n. The series rule from Lemma 3.2, now with slope λ_p, therefore gives

$$v_p(a_{i,a}) \ge \nu_p(a) - \lambda_p(B_a - i). \tag{A.1}$$

The harmonic sums satisfy $v_p(\sigma_a(i+2)) \ge -(i+2)\lambda_p$ at odd p and are integral at 2. Hence the integer exponents

$$e_p = \max\left(0, \max_{0\le a\le n}(-\nu_p(a) + 33\lambda_p)\right) \qquad (p \text{ odd},\ p^2 \le 7n), \tag{A.2}$$

$$e_2 = \max\left(0, \max_{0\le a\le n}(-\nu_2(a) + 31\lambda_2)\right) \tag{A.3}$$

clear all beta coefficients and the rational constant in (2.4). The factors $2^{i+2}(i)_2$ only improve these valuation bounds.

A.2 Their cost after factorial normalization

Legendre's formula gives

$$v_p(n!) \geq \frac{n}{p-1} - (\lambda_p + 1). \tag{A.4}$$

For each pole block $I_j(n)$, split the differences $b-a$ at zero. Counting multiples of p^k on the resulting two intervals gives

$$\sum_{\substack{b \in I_j(n) \\ b \neq a}} v_p(b-a) \leq \frac{\#I_j(n)}{p-1} + 2(\lambda_p + 1).$$

After summing over the 31 blocks,

$$\sum_{b \neq a} B_b v_p(b-a) \leq \frac{\gamma n + 31}{p-1} + 62(\lambda_p + 1). \tag{A.5}$$

For odd p, the integers $2a+1+2\ell$ consist of a negative and a positive progression of odd integers, of lengths

$$A_- = (1+r)n - a + 1, \qquad A_+ = a + rn + 1.$$

The absolute values in either progression are $1, 3, \ldots, 2A-1$, with $A = A_-$ or $A = A_+$. Their product is $(2A)!/(2^A A!)$, so Legendre's formula gives

$$v_p\left(\frac{(2A)!}{2^A A!}\right) \geq \frac{A}{p-1} - (\lambda_p + 1),$$

since $2A < 7n$. The lengths satisfy $A_- + A_+ = Z/3$. Each product occurs with multiplicity three, so the two estimates give

$$3 \sum_{\ell \in \Lambda_n} v_p(2a+1+2\ell) \geq \frac{Z}{p-1} - 6(\lambda_p + 1).$$

Combining this with (A.5) yields

$$-\nu_p(a) \leq \frac{Gn + 25}{p-1} + 68(\lambda_p + 1).$$

Equations (A.2) and (A.4) now give

$$e_p - G v_p(n!) \leq \frac{25}{p-1} + (68 + 33 + G)(\lambda_p + 1) \leq 128(\lambda_p + 1).$$

The maximum with zero in (A.2) causes no problem: when it is attained at zero, the left side is nonpositive. There are $O(\sqrt{n})$ such primes, and $(\lambda_p + 1)\log p = O(\log n)$, so their total cost is $o(n)$.

At 2, equation (A.5) with λ_2 gives

$$-\nu_2(a) \leq Z - 1 + \gamma n + 31 + 62(\lambda_2 + 1) = (G + 15)n + 36 + 62(\lambda_2 + 1).$$

Using $v_2(n!) \geq n - \lambda_2 - 1$ and (A.3), we obtain

$$e_2 - G v_2(n!) \leq 15n + 36 + 107(\lambda_2 + 1).$$

Its contribution to $\log D_n - G\log(n!)$ is therefore at most $15n \log 2 + o(n)$, as used in Section 4.

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