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# SOME POSITIVITY PROPERTIES OF RIEMANN Ξ UNDER SMOOTHING

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Abstract. We show that smoothing the zeros of the Riemann Ξ-function makes it easy to establish positivity properties that are otherwise hard (as they are equivalent to the Riemann hypothesis). Specifically, we smooth each zero with a small density centered at it, and then test the impact of this smoothing on complete monotonicity (CM) of the heat sum $H(t) = \sum_\rho e^{-tz_\rho^2}$ over the zeros $\frac{1}{2} + iz_\rho$. A sufficiently wide density quickly implies CM of the smoothed heat sum. We bound the width (i.e., half-variance) the density needs using increasingly precise facts about the zeros. If we only use that they lie in the critical strip, half-variance $1/8$ suffices; the primes lower this width to $\pi^2/288$; and zeros verified through height $T$ further lower it to $\pi^2/(8\log^2 T)$ asymptotically. Ultimately, however, smoothing seems to be of limited utility: despite the CM result, the smoothed reciprocal laws of Ξ are not even log-concave, unlike the unsmoothed ones.

## 1. Introduction

This paper is a sequel to our first work (Sra, 2026) that studied positivity properties (such as complete monotonicity and Pólya frequency) of the Riemann Ξ function, wherein exploiting verified zeta zeros was important. We take a different route based on smoothing, where positivity holds for all the zeros at once. The (unsmoothed) verified zeros now arise more indirectly: while smoothing yields positivity directly, the more verified zeros we have, the less smoothing we need.

More concretely, we replace each zero $z_\rho$ of Ξ by a small density centered at it. Then, we ask which positivity properties hold for the smoothed zeros and how wide the density must be. We test this idea on complete monotonicity (CM) of the *heat sum* $H(t) = \sum_\rho e^{-tz_\rho^2}$ over the zeta zeros. CM of $H$ is equivalent to RH. Smoothing helps because every zero is displaced from the critical line by less than $1/2$. A density wide enough to cover this displacement makes the density of the smoothed zeros positive, and that positivity is CM of the smoothed heat sum. The narrower the density, the stronger the conclusion, and the more we must know about the zeros. The critical strip alone already yields a finite width, the primes lower it, verified zeros lower it further, and for Gaussian smoothing it vanishes exactly when RH holds.

Despite smoothing's impact on positivity, ultimately our conclusion is negative: *while smoothing helps us unravel some structure of Ξ, it seems to be of limited utility.* It ensures CM of a smoothed heat sum, but the width needed for it is very sensitive to what we know about the zeros, and the subtler positivity properties of Ξ remain out of reach. The reciprocal laws of Ξ are the densities Θ and Λ with Laplace transforms $1/\xi(\frac{1}{2} + \sqrt{s})$ and $1/\Xi$. In (Sra, 2026) we studied them, and used verified zeros to prove for them large (but obviously finite) Pólya frequency (PF) orders. The PF property is delicate. It depends on the exact position of the zeros and smoothing destroys it—for every smoothing considered here and whatever the width assumed we show that the smoothed reciprocal laws fail to be even log-concave. For CM, smoothing width and verified height are thus interchangeable but for the PF property they are not, which reveals a key limitation of smoothing.

*Key words and phrases.* Riemann Xi; zeta function; complete monotonicity; Pólya frequency; Gaussian smoothing.

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1.1. **Main contributions.** A quick word about notation. Let $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ and $\Xi(z) = \xi(\frac{1}{2}+iz)$ be as usual. Let $\pm z_\rho$ be the zeros of $\Xi$, and $H(t) := \sum_{\rho\in\mathcal{Z}_+} e^{-tz_\rho^2}$ be the *heat sum* over these zeros. Here $\mathcal{Z}_+$ is the multiset of zeros $\rho = \beta + i\gamma$ with $\gamma > 0$, counted with multiplicity, and $z_\rho = \gamma - i\delta_\rho$ with $\delta_\rho = \beta - \frac{1}{2}$. For an admissible density $K$ (see Def. 2.1) we define

$$p_K(x) := \tfrac{1}{2}\sum_{\rho\in\mathcal{Z}_+}\big(K(x-z_\rho)+K(x+z_\rho)\big), \qquad H_K(t) := \int_{\mathbb{R}} e^{-tx^2}p_K(x)\,\mathrm{d}x. \tag{1.1}$$

The symmetries of the zeros make $p_K$ real and even on $\mathbb{R}$, but its terms need not be positive. We call the amount of smoothing its *width*, and measure it by the half-variance $\frac{1}{2}\operatorname{Var}(K)$.

***Our main results*** are the following. Items marked * overlap in part with earlier or parallel work; the precise relation is described in Section 5.

- **(1)*** *CM without using zeros.* We show that the Poisson kernel of the critical strip, of half-variance $\frac{1}{8}$, makes $H_K$ and $H_K/t$ completely monotone, and that no narrower density works from the strip alone (Theorem 2.3). With a computational certificate from the primes we lower the width for $H_K/t$ to $\pi^2/288$ for a band-limited density (one with compactly supported Fourier transform), without appealing to any verified zero heights (Theorem 2.5).
- **(2)*** *CM from verified zeros.* If the zeros up to a height $T$ lie on the critical line, we show that a band-limited density of half-variance asymptotic to $\pi^2/(8\log^2 T)$ makes $H_K/t$ completely monotone (Theorem 3.4), and that our method cannot improve this rate (Theorem 3.5). A Gaussian of half-variance $1/(16\log T)$ also suffices (Theorem 3.3).
- **(3)** *Loss of $PF_2$ (log-concavity).* We prove that for Gaussian smoothing of any width, and for the other kernels of Theorem 4.1, the smoothed reciprocal laws do not have a log-concave (that is $\mathrm{PF}_2$) density. In contrast, in (Sra, 2026, Theorem 1.2) we used verified zeros to prove that the unsmoothed reciprocal laws are PF of order 1 810 540 931 075.

Finally, we also study the generalized Laguerre inequalities $L_k[\Xi](x) \geq 0$. Using the critical-line proportion of Alpöge and Furman (2026), we show that $L_k[\Xi](x) \geq 0$ holds for every $k \geq 0$ on a set of heights of dyadic density exceeding 89.46% (Theorem E.1). We also show that smoothing the Jacobi theta kernel by convolution preserves this common positivity set (Proposition E.2).

## 2. CM from the critical strip and the primes

Our first main result certifies complete monotonicity of the smoothed heat sums from the critical strip alone, and then from the primes. Both certificates rest on one reduction: Laplace representations turn complete monotonicity of $H_K/t$ and of $H_K$ into sign tests on the cumulative profile $P_K(x) = 2\int_0^x p_K(u)\,\mathrm{d}u$ and on $p_K$ (Proposition 2.2), with Bernstein consequences for each sign (Proposition A.7). Zeros on the critical line contribute positive terms to both profiles; a zero off the line contributes a term of either sign, and the smoothing must be wide enough for what we know about the zeros to control those terms. Throughout, we use the strip $|\delta_\rho| < 1/2$, an explicit zero-counting bound, a real zero $\gamma_* \in (14, 15)$, and, where specified, verification through $T_* = 3\cdot 10^{12}$; Appendix A records these inputs and their sources.

A smooth function $f$ on $(0,\infty)$ is completely monotone (CM) if its alternating derivatives $(-1)^n f^{(n)}$ are nonnegative, and a nonnegative one is Bernstein (BF) if its derivative is CM.

2.1. **Admissible densities and the Laplace representation.** The profile $p_K$ sums $K$ over the complex centers $\pm z_\rho$, so $K$ must extend off the real line and decay along horizontal lines fast enough for the sums over the zeros to converge; this is all that admissibility asks. Appendix A.2 gives the counting and normal-convergence details.

**Definition 2.1.** An even probability density $K$ on $\mathbb{R}$ is *admissible* if it extends to an entire function, real on $\mathbb{R}$, such that for every $Y > 0$ there is a constant $C_K(Y)$ with

$$|K(x+iy)| \le \frac{C_K(Y)}{(1+x^2)^2} \qquad (x \in \mathbb{R},\ |y| \le Y). \tag{2.1}$$

By Bernstein's theorem, a function is CM exactly when it is the Laplace transform of a nonnegative measure, and the substitution $r = x^2$ writes $H_K$ and $H_K/t$ as Laplace transforms in $r$. For the profile $p_K$ of (1.1) and its cumulative profile $P_K$, we therefore set (for all $t, r > 0$):

$$\ell_K(t) = \frac{H_K(t)}{t}, \qquad q_K(r) = \frac{p_K(\sqrt{r})}{\sqrt{r}}, \qquad Q_K(r) = \int_0^r q_K(u)\,\mathrm{d}u = P_K(\sqrt{r}). \tag{2.2}$$

**Proposition 2.2** (Laplace representation and the exact CM tests)**.** *Let $K$ be admissible. For $t > 0$,*

$$H_K(t) = \int_0^\infty e^{-tr} q_K(r)\,\mathrm{d}r, \qquad \ell_K(t) = \int_0^\infty e^{-tr} Q_K(r)\,\mathrm{d}r = \int_0^\infty e^{-tr} P_K(\sqrt{r})\,\mathrm{d}r, \tag{2.3}$$

*both integrals converging absolutely. Consequently*

$$H_K \in \mathrm{CM} \iff p_K(x) \ge 0\ (x \in \mathbb{R}), \qquad \ell_K \in \mathrm{CM} \iff P_K(x) \ge 0\ (x > 0). \tag{2.4}$$

Proof: Appendix A.2.

The signs in (2.4) carry further consequences, which we state through the exponent

$$\mathcal{B}_K(s) = \int_0^\infty (1 - e^{-st})\ell_K(t)\,\mathrm{d}t \qquad (s \ge 0).$$

The signs $H_K \ge 0$, $P_K \ge 0$ on $(0,\infty)$, and $p_K \ge 0$ are three successive levels of positivity: they hold exactly when $\mathcal{B}_K$ is a Bernstein, a complete Bernstein, and a Thorin–Bernstein function (Proposition A.7). When $P_K > 0$, the measure $Q_K(r)\,\mathrm{d}r$ representing $\ell_K$ charges all of $(0,\infty)$, so the derivative signs, moment matrices and ordered minors of $\ell_K$ are all strict (Proposition A.6).

2.2. **Width from the critical strip.** With the sign tests in place, we begin with the least information about the zeros, the critical strip itself. If the real part of a density stays positive throughout the strip, each symmetric pair of zeros $\rho$, $1-\bar{\rho}$ contributes positively, and the strip Poisson kernel has this property. We ask what variance this requirement imposes on any kernel that must work for every pair separately. For $a > 0$, put

$$K_a(z) = \frac{1}{2a}\operatorname{sech}\frac{\pi z}{2a}.$$

This is the hyperbolic-secant density (Ding, 2014) scaled to variance $a^2$, holomorphic on $|\operatorname{Im} z| < a$ with poles at $\pm ia$; it is the Poisson kernel of that strip (Carvalho Corso, 2026, Appendix B).

**Theorem 2.3** (Universal baseline)**.** (i) *Let $a \ge \frac{1}{2}$. Then $p_{K_a} > 0$ on $\mathbb{R}$, so $H_{K_a}$ and $H_{K_a}/t$ are completely monotone, and the Bernstein and Thorin representations of Proposition A.6 hold.*

(ii) *Let $h > 0$, and let $K$ be an even probability density holomorphic on $|\operatorname{Im} z| < h$ and $O((1+|x|)^{-4})$ on every closed substrip. Then $H_K/t$ is CM for every symmetric pair in the strip if and only if $K = K_h * \nu$ for an even probability measure $\nu$. Then $\mathrm{Var}(K) = h^2 + \mathrm{Var}(\nu) \ge h^2$, with equality only for $K = K_h$. For the zeros, $h = \frac{1}{2}$ and the least half-variance is $\frac{1}{8}$.*

The bound 1/8 is exact when each symmetric pair is treated separately. The narrower certificates of Sections 2.3 and 3.2 therefore use information about the zeros as a whole.

Both parts of the theorem rest on one property of $K_a$. For $|d| < a$ let $k_{a,d}(x) = \operatorname{Re} K_a(x+id)$ be its real part on a horizontal line inside the strip; then $k_{a,d}$ is itself a positive probability density.

**Lemma 2.4** (The shifted density)**.** *For $|d| < a$, $k_{a,d}$ is an even strictly positive probability density of variance $a^2 - d^2$: with $U = \pi x/(2a)$ and $V = \pi d/(2a)$,*

$$k_{a,d}(x) = \frac{\cosh U \cos V}{2a(\sinh^2 U + \cos^2 V)} > 0, \tag{2.5}$$

*and, with $\widehat{f}(\omega) = \int_{\mathbb{R}} e^{-i\omega x} f(x)\,\mathrm{d}x$,*

$$\widehat{K_a}(\omega) = \operatorname{sech}(a\omega), \qquad \widehat{k}_{a,d}(\omega) = \frac{\cosh(d\omega)}{\cosh(a\omega)}. \tag{2.6}$$

Proof: Appendix A.6.

For the zeta zeros we take $a = 1/2$. Conjugation pairs each complex translate with its conjugate, so Lemma 2.4 makes every paired contribution to $p_{K_{1/2}}$ strictly positive. The kernel is not entire, so the proof of part (i) in Appendix A.3 checks the convergence, the Laplace representation and the integrability directly, uniformly as the imaginary parts approach the strip boundary.

The converse starts with CM of $H_K/t$ for each symmetric pair, a condition on a Laplace transform, and recovers positivity of the kernel throughout the strip.

*Proof of Theorem 2.3(ii).* Let $K$ satisfy the assumptions of part (ii), with half-width $h$ (the zeta case is $h = 1/2$).

***Positivity in the strip.*** Suppose first that $H_K/t$ is CM for every symmetric pair; concentration at the origin then detects any negative value. If $\operatorname{Re} K(\gamma + id) < 0$ for some $\gamma > 0$, then the corresponding pair profile $p_{\gamma,d}$ is negative at zero. Gaussian concentration gives

$$\sqrt{t}\int_{\mathbb{R}} e^{-tx^2} p_{\gamma,d}(x)\,\mathrm{d}x \longrightarrow \sqrt{\pi}\, p_{\gamma,d}(0) < 0,$$

contradicting the positivity required by CM. Evenness and continuity therefore force $k_d(x) := \operatorname{Re} K(x + id) \geq 0$ throughout the strip.

***The convolution factor.*** We let the imaginary shift approach the strip boundary; the limiting probability law supplies the convolution factor. Contour translation gives

$$\int k_d = 1, \qquad \int x^2 k_d(x)\,\mathrm{d}x = \operatorname{Var}(K) - d^2, \qquad \widehat{k}_d(\omega) = \cosh(d\omega)\widehat{K}(\omega).$$

Thus the $k_d$ are even probability densities with uniformly bounded second moments. Along $d \uparrow h$ they have a weakly convergent subsequence with limit $\nu$, an even probability measure, and $\widehat{\nu} = \cosh(h\omega)\widehat{K}$. By (2.6), this identity says exactly that $K = K_h * \nu$. Conversely such a convolution has positive real part on every interior horizontal line, since it averages the strictly positive functions $k_{h,d}$. It therefore works for every symmetric pair.

***The variance.*** The convolution factor can only add variance. We write the convolution as $X + Y$ with independent $X \sim K_h$ and $Y \sim \nu$. The inequality $Y^2 \leq 2(X + Y)^2 + 2X^2$ gives a finite second moment for $\nu$, and evenness makes the means zero. Hence the variance splits as

$$\operatorname{Var}(K) = h^2 + \operatorname{Var}(\nu) \geq h^2,$$

with equality exactly when $\nu = \delta_0$, which proves part (ii). □

2.3. **Width from the primes.** To reduce the universal width without a verified zero height, we need other information about the zeros, and the explicit formula provides it through the primes. For a density of compact Fourier support the prime side is a finite sum, and we retain its signs rather than bounding it in absolute value.

Let $\kappa_A$ be the normalized square of the cosine amplitude $\psi_A(x) = \cos(Ax)/(1 - (2Ax/\pi)^2)$,

$$\kappa_A(x) = \frac{8A}{\pi^3}\left(\frac{\cos(Ax)}{1 - (2Ax/\pi)^2}\right)^2, \qquad \kappa_6(x) = \frac{48}{\pi^3}\left(\frac{\cos 6x}{1 - (12x/\pi)^2}\right)^2,$$

with $A > 0$ and removable singularities; it has Fourier support $[-2A, 2A]$ and half-variance $\pi^2/(8A^2)$ (Proposition D.1), and it is the building block of the band-limited family of Section 3.2.

**Theorem 2.5** (A certificate from the primes)**.** *For the density $\kappa_6$, unconditionally,*

$$\tfrac{1}{2}\operatorname{Var}(\kappa_6) = \frac{\pi^2}{288} \approx 0.03427, \qquad P_{\kappa_6}(x) \geq 10^{-8}\min(x,1) > 0 \quad (x > 0). \tag{2.7}$$

*Consequently $H_{\kappa_6}/t$ is CM, with the strict derivative and Hankel consequences of Proposition A.6.*

Proof: Appendix B.

The Guinand–Weil formula expresses $P_{\kappa_6}$ through the prime powers below $e^{12}$, the gamma factor and the pole term (Appendix B.1). Directed-rounding quadrature and Taylor bounds certify (2.7) on $0 < x \leq 60$ (Appendix B.2); an analytic estimate gives $P_{\kappa_6}(x) > 3/4$ for $x \geq 60$ (Appendix B.3). The proof is computer-assisted, and it certifies the sign of $P_{\kappa_6}$, the second level of Proposition A.7.

## 3. CM from verified zeros

So far we have used no verified zero height. Our second main result converts zeros verified on the critical line through a height $T$ into smaller widths. Such zeros contribute positive terms to $p_K$ and $P_K$, so the width need only control the unverified zeros, and it shrinks as $T$ grows. For Gaussian smoothing the least such width is tied to $H$ itself, and it vanishes exactly when RH holds. A band-limited density concentrates its Fourier mass better and reaches a width of order $1/\log^2 T$, the best rate that our certificates can give.

### 3.1. Gaussian smoothing and the RH endpoint.

Gaussian smoothing acts on $H$ by an explicit change of time, and the Gaussian profiles form a heat semigroup. Together, these two facts tie the least Gaussian width that gives complete monotonicity to $H$ itself.

For a variance parameter $\sigma > 0$, define the normalized Gaussian

$$G_\sigma(w) = \frac{1}{\sqrt{4\pi\sigma}} \exp\left(-\frac{w^2}{4\sigma}\right), \qquad w \in \mathbb{C}.$$

On $\mathbb{R}$ the Gaussian $G_\sigma$ is a probability density of variance $2\sigma$, and $|G_\sigma(x+iy)| = e^{y^2/(4\sigma)} G_\sigma(x)$, so $G_\sigma$ is admissible. Its real part at a displaced center is

$$\operatorname{Re} G_\sigma(u+id) = G_\sigma(u) e^{d^2/(4\sigma)} \cos\left(\frac{ud}{2\sigma}\right).$$

For $d \neq 0$ the cosine changes sign, and narrowing the Gaussian increases the factor $e^{d^2/(4\sigma)}$. Thus positivity of $G_\sigma$ on the real axis supplies no termwise positivity for the sum over the zeros. We write $p_\sigma, P_\sigma, H_\sigma, \ell_\sigma$ for the functions (1.1)–(2.2) with $K = G_\sigma$, and $\ell(t) = H(t)/t$. For $\sigma, \eta > 0$ the Gaussian profiles form a convolution semigroup,

$$p_{\sigma+\eta} = G_\eta * p_\sigma, \qquad (f*g)(x) := \int_{\mathbb{R}} f(x-u) g(u)\, \mathrm{d}u : \tag{3.1}$$

the identity $G_\eta * G_\sigma(\,\cdot - z) = G_{\sigma+\eta}(\,\cdot - z)$ holds for real $z$ by completing the square and for complex $z$ by the identity theorem, and the absolute bounds of Lemma A.4 permit its summation over the zeros and justify Fubini's theorem.

**Proposition 3.1** (Exact transformation of $H$)**.** *For $t, \sigma > 0$, put $v_\sigma(t) = t/(1+4\sigma t)$. Then*

$$H_\sigma(t) := \int_{\mathbb{R}} e^{-tx^2} p_\sigma(x)\, \mathrm{d}x = \frac{H(v_\sigma(t))}{\sqrt{1+4\sigma t}}, \tag{3.2}$$

$$\ell_\sigma(t) := \frac{H_\sigma(t)}{t} = \frac{\ell(v_\sigma(t))}{(1+4\sigma t)^{3/2}}. \tag{3.3}$$

Proof: Appendix C.1, by completing the square in each summand.

Both sign tests survive further Gaussian smoothing. The semigroup (3.1) preserves $p_\sigma \geq 0$, and it carries $P_\sigma \geq 0$ on the half-line to $P_{\sigma+\eta} > 0$ there, because it acts on the odd function $P_\sigma$ through the positive Dirichlet heat kernel (Proposition C.1). We therefore define the thresholds

$$\sigma_H = \inf\{\sigma > 0 : H_\sigma \in \mathrm{CM}\}, \qquad \sigma_\ell = \inf\{\sigma > 0 : \ell_\sigma \in \mathrm{CM}\}.$$

They correspond to the two tests in (2.4). Since the pointwise test is stronger and Theorem C.7 gives $p_\sigma > 0$ for $\sigma \geq 100$, we have $0 \leq \sigma_\ell \leq \sigma_H \leq 100$. Because CM is closed under limits and the time change (3.2) recovers $H$ as $\sigma \to 0$, either threshold vanishes exactly when RH holds.

**Proposition 3.2** (The zero-smoothing endpoint)**.** *The following statements are equivalent:*

$$\mathrm{RH}, \qquad \sigma_H = 0, \qquad \sigma_\ell = 0.$$

*If either infimum is positive, its own endpoint belongs to the corresponding CM set.*

*Proof.* Under RH, the centers $\pm z_\rho$ are real. Every summand in (1.1) is then positive, for every $\sigma > 0$, so both infima vanish.

Conversely, if one infimum vanishes, choose parameters $\sigma_n \downarrow 0$ with $H_{\sigma_n}$, or $\ell_{\sigma_n}$, completely monotone. Equations (3.2)–(3.3) show locally uniform convergence with every fixed $t$-derivative to $H$ or $\ell$. Thus the limit is CM, and the equivalence (A.1) gives RH. The same derivative-continuity argument at a positive limiting parameter proves endpoint inclusion. □

3.2. **Width from verified zeros.** We saw that the Gaussian thresholds vanish only under RH. Short of that endpoint, we use verified zeros to obtain small explicit widths. The input is the statement that the zeros through a height $T$ lie on the critical line, which we write as

$$\mathsf{V}(T): \qquad \rho \in \mathcal{Z}_+,\ 0 < \gamma \leq T \implies \delta_\rho = 0. \tag{3.4}$$

At $T_* = 3 \cdot 10^{12}$ the statement $\mathsf{V}(T_*)$ is the theorem of Platt and Trudgian (2021); above $T_*$ it is a hypothesis. For Gaussian smoothing, this input gives an explicit bound on $\sigma_\ell$.

**Theorem 3.3** (Gaussian certificate at a variable height)**.** *Put $\sigma_T = 1/(16 \log T)$. For every $T \geq 10^{11}$ for which $\mathsf{V}(T)$ holds,*

$$P_\sigma(x) > 0 \quad (x > 0), \qquad \ell_\sigma \in \mathrm{CM} \qquad (\sigma \geq \sigma_T),$$

*with strictly positive alternating derivatives of every order and the conclusions of Proposition A.6. In particular, $\mathsf{V}(T)$ implies $\sigma_\ell \leq 1/(16 \log T)$, and (A.5) gives $\sigma_\ell < 0.00218$.*

Proof: Appendix C.5.

A band-limited density permits smaller variance because its growth in the strip is slower at the same spatial scale. For $T \geq T_*$, let $A = A(T) > 40$ solve $e^A/A^4 = T/20$, a relation explained after Theorem 3.4, and put $b_A = \pi/(2A)$ and $\varepsilon = A^{-2}$. With the density $\kappa_A$ of Section 2.3, we set

$$\mathcal{K}_T(x) = (1 - \varepsilon)\kappa_A(x) + \frac{\varepsilon}{2}(\kappa_A(x - b_A) + \kappa_A(x + b_A)). \tag{3.5}$$

The cosine amplitude $\psi_A$ minimizes variance at its bandwidth by the Dirichlet Poincaré inequality (Proposition D.1); the small translate mixture fills the real zeros of its square, makes $\mathcal{K}_T$ strictly positive on the real line, and increases the variance only by the factor $1 + A^{-2}$.

**Theorem 3.4** (Complete monotonicity after band-limited smoothing)**.** *Let $T \ge T_*$ and assume $\mathsf{V}(T)$. Then the density $\mathcal{K}_T$ of (3.5) satisfies $P_{\mathcal{K}_T}(x) > 0$ for $x > 0$. Consequently $H_{\mathcal{K}_T}/t$ is completely monotone, with the strict derivative, Hankel, and complete Bernstein conclusions of Proposition A.6. The density is strictly positive, has Fourier support $[-2A, 2A]$, and*

$$\tfrac{1}{2}\operatorname{Var}(\mathcal{K}_T) = \frac{\pi^2(1+A^{-2})}{8A^2}, \qquad A(T) = \log T + 4\log\log T - \log 20 + O\Big(\frac{\log\log T}{\log T}\Big). \tag{3.6}$$

*At $T = T_*$, where $\mathsf{V}(T_*)$ is a theorem, $A(T_*) \approx 40.543$ and $\frac{1}{2}\operatorname{Var}(\mathcal{K}_{T_*}) \approx 0.000751$.*

Proof: Appendix D.3; the uniform estimates are in Appendix D.

Both certificates compare the cumulative profile $P_K$ with the positive function obtained by replacing each complex center $z_\rho$ by its real ordinate $\gamma$. The error of this replacement is an integral of $K$ along the vertical segments between the real and the complex centers, so it comes mainly from the zeros near $x$, while the positive function grows like $x\log x$. Below the verified height, distance suppresses the error of the unverified zeros; near the origin, where both functions vanish, one real zero supplies a pointwise lower bound on $p_K$.

For the band-limited family, the relation $e^A/A^4 = T/20$ balances the positive function against the growth of the density in the strip, and it gives half-variance $\pi^2/(8\log^2 T)(1+o(1))$. For the Gaussian, the factor $e^{1/(16\sigma)}$ by which $G_\sigma$ grows at the edge of the strip equals $T$ at $\sigma = 1/(16\log T)$, which gives that scale. The two widths are spreads of two different densities, and only the Gaussian one bounds the threshold $\sigma_\ell$.

3.3. **Optimality of the verified-height rate.** Theorem 3.4 reaches half-variance $\pi^2/(8\log^2 T)$ from verified height $T$, while the Gaussian of Theorem 3.3 reaches only $1/(16\log T)$. We show here that the first rate is the best that any certificate of our type can give, and we explain the gap. Both certificates bound the complex displacement by integrals of $|K|$ over vertical segments in the strip, and an uncertainty estimate governs such integrals: small spatial variance requires a spread of Fourier mass, whereas a moderate size in the strip confines it to a short interval. We state the bound for finite sums of real squares whose strip budget, defined below, grows polynomially.

We consider a centered probability density with a chosen representation

$$K(x) = \sum_{j=1}^{m} g_j(x)^2, \qquad g_j(x) = \int_{\mathbb{R}} e^{ix\omega} f_j(\omega)\,\mathrm{d}\omega, \tag{3.7}$$

where the $g_j$ are real on the real axis and $f_j \in H^1(\mathbb{R})$; the number of components may vary between kernels. We assume that the weighted frequency integral in (3.10) below is finite, which gives analytic $L^2$ continuation into $|\operatorname{Im} z| < 1/2$, and we define

$$\mathfrak{S}(K; g_1, \dots, g_m) = \sum_{j=1}^{m} \int_0^{1/2} \int_{\mathbb{R}} |g_j(x+iv)|^2\,\mathrm{d}x\,\mathrm{d}v. \tag{3.8}$$

We call this quantity the *strip budget* of the chosen representation and abbreviate it to $\mathfrak{S}$. It majorizes $\int_0^{1/2}\int_{\mathbb{R}} |K(x+iv)|\,\mathrm{d}x\,\mathrm{d}v$, the continuation integral in (D.18), because it discards the cancellation between the squares. The bound below concerns this budget as a function of a parameter $R$.

**Theorem 3.5** (Variance under a polynomial strip budget)**.** *Let $K_R$ be centered probability densities with representations (3.7) and corresponding budgets $\mathfrak{S}_R$. If $\log \mathfrak{S}_R \le (p+o(1))\log R$ as $R \to \infty$ for a fixed $p > 0$, then*

$$\liminf_{R\to\infty} (\log R)^2\, \tfrac{1}{2}\operatorname{Var}(K_R) \ge \frac{\pi^2}{8p^2}. \tag{3.9}$$

*In particular, $\mathfrak{S}_R \le R(\log R)^M$ for fixed $M$ forces the leading constant $\pi^2/8$.*

*Proof.* We normalize the vector $f = (f_1, \dots, f_m)$ in $L^2(\mathbb{R})^m$ and call it $F$. Reality of the $g_j$ gives $f_j(-\omega) = \overline{f_j(\omega)}$, so $|F(\omega)|^2$ is even. Plancherel, Tonelli, and centering give

(3.10)
$$\|F\|_2 = 1, \qquad \mathrm{Var}(K) = \|F'\|_2^2, \qquad \mathfrak{S} = \int_{\mathbb{R}} m(\omega)|F(\omega)|^2 \,\mathrm{d}\omega, \quad m(\omega) = \frac{\sinh \omega}{2\omega}, \quad m(0) = \tfrac{1}{2}.$$

For $|\omega| \geq 1$, the even weight $m$ increases in $|\omega|$ and

$$m(\omega) \geq \frac{e^{|\omega|}}{5|\omega|}.$$

Fix $\delta > 0$ and set $R_0 = (p + \delta) \log R$. By the weight bound and the budget, for all large $R$,

$$e_R := \int_{|\omega| > R_0} |F_R(\omega)|^2 \,\mathrm{d}\omega \leq 5R_0 e^{-R_0} \mathfrak{S}_R \leq 5R_0 R^{-\delta/2},$$

and $(\log R)\sqrt{e_R} \to 0$. Let $\chi$ equal one on $[-R_0, R_0]$, vanish outside $[-R_0 - 1, R_0 + 1]$, and be affine on the intervening intervals. Then

$$\|\chi F_R\|_2 \geq \sqrt{1 - e_R}, \qquad \|(\chi F_R)'\|_2 \leq \|F_R'\|_2 + \sqrt{e_R}.$$

The Dirichlet Poincaré inequality on $(-R_0 - 1, R_0 + 1)$, applied componentwise, gives

$$\|F_R'\|_2 \geq \frac{\pi}{2(R_0 + 1)} \sqrt{1 - e_R} - \sqrt{e_R}.$$

Multiplying by $\log R$, passing to the lower limit, and letting $\delta \downarrow 0$ gives $\liminf (\log R)\|F_R'\|_2 \geq \pi/(2p)$, and (3.10) proves (3.9). □

The cosine amplitude attains this bound, including its leading constant. We write $\kappa_A = c_A \psi_A^2$ with $c_A = 8A/\pi^3$ and the cosine amplitude $\psi_A$ of Section 2.3. For the base density we use the single square $\sqrt{c_A}\,\psi_A$; for (3.5), we use $\sqrt{(1-\varepsilon)c_A}\,\psi_A$ and $\sqrt{\varepsilon c_A/2}\,\psi_A(\cdot \pm b_A)$.

**Proposition 3.6** (The cosine amplitude attains the bound)**.** *For the base density and the three-square representation of its translate mixture, the strip budget is*

$$\mathfrak{S}_A = \frac{1}{A} \int_0^A \cos^2\!\left(\frac{\pi\omega}{2A}\right) \frac{\sinh \omega}{\omega} \,\mathrm{d}\omega \sim \frac{\pi^2}{4} \frac{e^A}{A^4} \qquad (A \to \infty).$$

*Consequently the family $\mathcal{K}_T$ satisfies*

$$\mathfrak{S}_T \sim \frac{\pi^2}{80} T, \qquad (\log T)^2 \tfrac{1}{2} \,\mathrm{Var}(\mathcal{K}_T) \longrightarrow \frac{\pi^2}{8}. \tag{3.11}$$

*Thus the constant of Theorem 3.5 with $R = T$ and $p = 1$ is attained.*

Proof: Appendix D.4.

The Gaussian has a larger strip budget at a fixed small variance, which explains its different rate under the same polynomial growth. The budget of the square root of $G_\sigma$ is $e^{1/(16\sigma)}$ up to a power of $\sigma$, so a budget $\log \mathfrak{S} \leq (p + o(1)) \log R$ forces $\liminf_{R\to\infty} \sigma_R \log R \geq 1/(16p)$ (Proposition D.3). The Gaussian certificate attains the resulting scale $1/(16 \log T)$, while $\mathcal{K}_T$ attains the sharp leading constant for the broader sum-of-squares class. Both rates limit certificates with the specified strip budget, and the least smoothing that the zeta zeros require remains open. The arithmetic certificate of Theorem 2.5 lies outside this class, since it rests on the explicit formula.

**Remark 3.7** (A narrower zero strip)**.** *Suppose that $|\delta_\rho| \leq h$ for every zero, with $0 < h < \frac{1}{2}$. Then the universal width and the verified-height widths scale by $(2h)^2$, as do the bounds of Theorem 3.5 and Proposition D.3 when the strip budget is taken over $|\operatorname{Im} z| \leq h$ (Appendix D.5). The zero-free half-plane $\operatorname{Re} s > 7/8$ claimed by OpenAI (2026) gives $h = 3/8$, hence the universal infimum $9/128 \approx 0.0703$, the Gaussian width $9/(256 \log T)$, below $0.00123$ at $T_*$, and the band-limited half-variance $0.000422\ldots$ at $T_*$.*

## 4. Loss of $\mathrm{PF}_2$ (log-concavity)

The certificates above establish complete monotonicity of $H_K/t$ or of $H_K$. The unsmoothed reciprocal densities $\Theta$ and $\Lambda$ have more: they are Pólya frequency functions of a large finite order (Sra, 2026), and of every order under RH. In this section we show that smoothing keeps none of this property, not even log-concavity, at any variance. The reason is that the smoothed profile is analytic. Its first nonzero Taylor coefficient at the origin gives the exponent $\mathcal{B}_K$ of the smoothed laws a fractional-power singularity at $s = 0$, so these laws have no exponential moment, whereas a log-concave density decays exponentially.

A nonnegative integrable function $f$ on $\mathbb{R}$ is a Pólya frequency function of order $k$, written $f \in \mathrm{PF}_k$, if for every $1 \le r \le k$ and all $x_1 < \cdots < x_r$, $y_1 < \cdots < y_r$,

$$\det[f(x_i - y_j)]_{i,j=1}^{r} \ge 0.$$

One-sided densities are extended by zero to the negative half-line, with the value zero at a possible jump at the origin. Then $\mathrm{PF}_{k+1} \subseteq \mathrm{PF}_k$, and $\mathrm{PF}_2$ is log-concavity.

Let $K$ be admissible with $H_K \ge 0$ on $(0,\infty)$, or let $K = K_a$ with $a \ge \frac{1}{2}$. By Proposition A.7, and by Theorem 2.3(i) for $K_a$, the exponent $\mathcal{B}_K$ is Bernstein without killing or drift, with Lévy density $\ell_K = H_K/t \ge 0$. Thus $e^{-\mathcal{B}_K}$ defines an infinitely divisible law on $[0,\infty)$ (Schilling et al., 2012, Chapter 5). The *reciprocal laws* of $\mathcal{B}_K$ are the law of the random variable $S_K \ge 0$ with $\mathbb{E}e^{-sS_K} = e^{-\mathcal{B}_K(s)}$ and the law $\mu_K$ of $\sqrt{2S_K}\,G$, with $G$ standard normal and independent of $S_K$; conditioning on $S_K$ gives the characteristic function

$$\int_{\mathbb{R}} e^{i\omega x}\mu_K(\,\mathrm{d}x) = e^{-\mathcal{B}_K(\omega^2)}.$$

**Theorem 4.1** (No Pólya frequency property of order two)**.** *Let $K$ be admissible with $H_K \ge 0$ on $(0,\infty)$ and $H_K \not\equiv 0$, or let $K = K_a$ with $a \ge \frac{1}{2}$ be the strip Poisson kernel of Theorem 2.3, applied to the zeros of $\zeta$. Then the reciprocal laws of $\mathcal{B}_K$ satisfy*

$$\mathbb{E}e^{cS_K} = \infty, \qquad \int_{\mathbb{R}} e^{c|x|}\mu_K(\,\mathrm{d}x) = \infty \qquad (c > 0).$$

*Consequently neither law has a log-concave density, and so neither has a $\mathrm{PF}_m$ density for any $m \ge 2$. Every Gaussian width $\sigma > 0$ satisfies the hypotheses, with $p_\sigma(0) > 0$.*

*Proof.* For admissible $K$, Lemma A.4 makes the profile entire and even, and $H_K \not\equiv 0$ implies $p_K \not\equiv 0$. Let $c_K x^{2m}$ be its first nonzero Taylor term at the origin. For $K = K_a$, Theorem 2.3 gives $p_{K_a}(0) > 0$, so take $m = 0$ and $c_K = p_{K_a}(0)$. Laplace's method gives

$$\ell_K(t) \sim c_K \Gamma(m + \tfrac{1}{2})\, t^{-m-3/2} \qquad (t \to \infty). \tag{4.1}$$

Indeed, the Taylor expansion controls $H_K(t) = \int_{\mathbb{R}} e^{-tx^2} p_K(x)\,\mathrm{d}x$ near zero, and the profile bounds make the remaining integral exponentially small. Nonnegativity of $H_K$ forces $c_K > 0$.

We split the Lévy measure at $t = 1$. Then $S_K$ is the sum of two independent nonnegative variables, one of which, $J$, is compound Poisson with finite rate $\int_1^\infty \ell_K(t)\,\mathrm{d}t$ and jump density proportional to $\ell_K$ on $(1,\infty)$. By (4.1), for every $c > 0$,

$$\mathbb{E}e^{cS_K} \ge \mathbb{E}e^{cJ} = \exp\Big(\int_1^\infty (e^{ct} - 1)\ell_K(t)\,\mathrm{d}t\Big) = \infty.$$

Conditioning on $S_K$ gives $\int e^{cx}\mu_K(\,\mathrm{d}x) = \mathbb{E}e^{c^2 S_K} = \infty$, and hence $\int e^{c|x|}\mu_K(\,\mathrm{d}x) = \infty$. A log-concave integrable function is bounded by $Ce^{-c|x|}$ for some $c > 0$, so no such density represents either law. Since $\mathrm{PF}_2$ functions are log-concave (Karlin, 1968) and $\mathrm{PF}_m \subseteq \mathrm{PF}_2$, the PF conclusions follow.

For the Gaussian, (3.2) and $H > 0$ (Sra, 2026, Theorem 1.1) give $H_\sigma > 0$ and $p_\sigma(0) = H(1/(4\sigma))/\sqrt{4\pi\sigma} > 0$. □

When $p_K(0) > 0$, the characteristic function has a Cauchy cusp at the origin and $\mu_K$ has infinite first absolute moment (Proposition A.8); the same appendix records when $\mu_K$ has a density.

## 5. Related work

The results marked * in Section 1 overlap with earlier and parallel work, and this section records the exact relation. The reciprocal law was constructed under RH by Roynette and Yor (2005, Théorème V.3.2) and unconditionally by Polson (2026b, v24, Theorem A), who also proves its self-decomposability; the equivalence of RH with the generalized gamma convolution property is Polson (2026b, v24, Theorem 5.6) and Polson (2026a, v9, Theorem 26). Our first work (Sra, 2026) records the exact overlap and proves the Pólya frequency orders of $\Theta$ and $\Lambda$.

*Universal strip smoothing.* The closest antecedent is Polson (2026c, v25, Theorem 9.8(i)). For $\varepsilon = \alpha - \frac{1}{2} > 0$, the density $\rho_\varepsilon(x) = \pi^{-1}\operatorname{Re}(\xi'/\xi)(\alpha + ix)$ is the sum over the zeros of the real parts of Cauchy kernels of width $\varepsilon$ centered at the complex points $\pm z_\rho$, and $\operatorname{Re} C_\varepsilon(u + id) = \frac{1}{2}(C_{\varepsilon-d}(u) + C_{\varepsilon+d}(u))$ for $|d| < \varepsilon$, where $C_\varepsilon(u) = \varepsilon/(\pi(\varepsilon^2 + u^2))$. Hence $\rho_\varepsilon > 0$ as soon as $\varepsilon \geq \frac{1}{2}$, because every zero has $|\delta_\rho| < \frac{1}{2}$, and the Thorin measure of $-\log F_\alpha$, $F_\alpha(s) = \xi(\alpha)/\xi(\alpha + \sqrt{s})$, is then positive. The mechanism is the same as in Theorem 2.3(i), where each symmetric pair contributes positively once the width reaches the strip half-width. The kernels differ, since the Cauchy kernel has infinite variance and the strip Poisson kernel has variance $a^2$. The classification in part (ii) and its exact width $\frac{1}{8}$ have no counterpart there.[1]

*The RH endpoint.* Theorem 9.8(iv) of Polson (2026c) formulates RH as positivity of $\rho_\varepsilon$ for every $0 < \varepsilon < \frac{1}{2}$, that is, as the Thorin property of $-\log F_\alpha$ down to the critical line; Proposition 3.2 is the Gaussian form of the same statement, at the third positivity level of Proposition A.7. That theorem also phrases the criterion as complete monotonicity of $F_\alpha$; this weaker property holds for every $\alpha \geq \frac{1}{2}$ unconditionally, since $B(s) = \log(\xi(\frac{1}{2}+\sqrt{s})/\xi(\frac{1}{2}))$ is Bernstein and $-\log F_\alpha(s) = B((\varepsilon+\sqrt{s})^2) - B(\varepsilon^2)$ is the composition of two Bernstein functions. Only the Thorin-level formulation is an RH criterion. Nakamura (2015) works on the primal side, $\xi(\sigma - it)/\xi(\sigma)$ as a characteristic function, where RH becomes a signed Lévy–Khintchine representation; see Sra (2026, Section 5).

## AI usage

The results in this paper were developed through extensive use of an AI assistant (OpenAI's GPT Pro). None of the claims were obtained zero-shot: they emerged through sustained back-and-forth discussion, including failed attempts, counterexamples, audits, and repairs. Anthropic's Claude (Fable 5.1 and 5.2) helped with the writing and with verifying some of the mathematics.

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## Appendix A. Analytic inputs, convergence and transforms

The main constructions use the complex zeros throughout. We record the external information used to control the sums, derive the counting bounds needed by the continuation estimates, and prove the convergence statements and the Bernstein consequences that Sections 2 and 4 use.

A.1. **External inputs.** We collect what we take from the literature about the zeros: the equivalence of RH with complete monotonicity of $H$, the critical strip, an explicit counting bound, one real zero, and the verified height. The full zero multiset of $\Xi$ is $\{z_\rho, -z_\rho : \rho \in \mathcal{Z}_+\}$. By its conjugation and sign symmetries, the sums below are real for real arguments.

We recall the elementary equivalences

$$\ell \in \mathrm{CM} \iff H \in \mathrm{CM} \iff \mathrm{RH} \tag{A.1}$$

(see Sra, 2026, Proposition A.1), where $\ell(t) = H(t)/t$ and RH means that every $\rho$ has $\beta = 1/2$.

Write $N(Y) = \#\{\rho \in \mathcal{Z}_+ : \gamma \le Y\}$, with multiplicity, and

$$M(Y) = \frac{Y}{2\pi}\left(\log\frac{Y}{2\pi} - 1\right) \quad (Y > 0), \qquad M(0) = 0.$$

We use the following inputs:

$$|\delta_\rho| < \tfrac{1}{2}, \tag{A.2}$$

$$|N(Y) - M(Y)| \le \tfrac{1}{8}\log(Y+2) + 4 \quad (Y \ge 0), \tag{A.3}$$

$$\text{there is a real zero } \gamma_* \in (14, 15) \text{ of } \Xi. \tag{A.4}$$

The counting bound follows directly from Bellotti and Wong (2025, Theorem 1.1), as recorded below. The real zero is the first zero, $\gamma_1 \approx 14.1347$. The fourth input is the finite verification of RH,

$$T_* := 3 \cdot 10^{12}, \qquad 0 < \gamma \le T_* \implies \delta_\rho = 0, \tag{A.5}$$

which is Theorem 1 of Platt and Trudgian (2021). It is the statement $\mathsf{V}(T_*)$ of (3.4), and it gives the values at $T_*$ in Theorems 3.3 and 3.4.

**Lemma A.1** (The counting bound)**.** *The bound* (A.3) *holds.*

*Proof.* Bellotti and Wong (2025, Theorem 1.1) prove, for $Y \ge e$,

$$|N(Y) - M(Y)| \le a_2 \log Y + b_2 \log\log Y + c_2, \qquad (a_2, b_2, c_2) = (0.11200, 0.12567, 3.77417). \tag{A.6}$$

Put $u = \log Y \ge 1$. The excess over $(\log Y)/8$ is $0.12567 \log u - 0.013u + 3.77417$, whose maximum at $u = 0.12567/0.013$ is less than $3.934 < 4$. At $Y = e$ the same estimate gives $N(e) < 4$, so on $[0, e]$ one has $N(Y) \le 3$ and $-1 < M(Y) \le 0$. □

We use two elementary consequences of (A.3) repeatedly: a bound on the zeros in a unit interval, for the convergence of the sums over the zeros and the profile at $\sigma \geq 100$, and a bound on the zeros in a centered interval, for the envelopes of Appendices C and D.

**Lemma A.2** (A unit-interval upper bound)**.** *For $y \geq 0$, the number of ordinates in any interval of length at most one contained in $[y, y+1]$, with either endpoint included, is at most $9\log(y+3)$.*

*Proof.* By (A.3) and its one-sided limits, $N(y+1)-N(y-) \leq M(y+1)-M(y)+\frac{1}{4}\log(y+3)+8$. For $y = 0$, use $N(0-) = N(0) = 0$. Since $M'(u) = \log(u/(2\pi))/(2\pi)$ is increasing, $M(y+1) - M(y) \leq \frac{1}{2\pi}\log(y+3)$. The logarithmic singularity of $M'$ at zero is integrable. Finally, $\frac{1}{2\pi} + \frac{1}{4} + \frac{8}{\log 3} < 9$, which proves the result. □

**Lemma A.3** (A local counting bound)**.** *For $y \geq 0$ and $r \geq 0$, let $n_y(r)$ count the full real centers $\{\pm\gamma : \rho \in \mathcal{Z}_+\}$ in $[y-r, y+r]$, with multiplicity. Then*

$$n_y(r) \leq \left(\frac{r}{\pi} + \frac{1}{4}\right)\log(y+r+2) + 8. \tag{A.7}$$

*Proof.* For $r < y$, subtract the two values of $N$ at $y+r$ and $y-r$; for $r \geq y$, add the counts at $y+r$ and $r-y$. In both cases, integrate the upper bound $M'(u) \leq \log(y+r+2)/(2\pi)$ over total length at most $2r$, and add the two error envelopes of (A.3). □

A.2. **Convergence details for Section 2.** We now justify the sums over the zeros in Section 2. The decay hypothesis of Definition 2.1, summed over unit intervals with Lemma A.2, gives their convergence and growth, and the Laplace representation then follows.

**Lemma A.4** (Convergence and growth)**.** *Let $K$ be admissible. The series (1.1) converges locally normally on $\mathbb{C}$, so $p_K$ is entire, and each of its derivatives is the locally normally convergent sum of the termwise derivatives; $p_K$ is real and even on $\mathbb{R}$, and $P_K$ is odd and entire. There is a constant $C$, depending on $K$, with*

$$\sum_{\rho\in\mathcal{Z}_+} \left(|K(x-z_\rho)|+|K(x+z_\rho)|\right) \leq C\log(|x|+2), \qquad |P_K(x)| \leq C|x|\log(|x|+2) \qquad (x \in \mathbb{R}), \tag{A.8}$$

*and $P_K(x) = 2p_K(0)x + O(x^3)$ as $x \to 0$.*

*Proof of Lemma A.4.* On a compact set of $w \in \mathbb{C}$ with $|\operatorname{Re} w| \leq R$ and $|\operatorname{Im} w| \leq Y - \frac{1}{2}$, the bound (2.1) and (A.2) give $|K(w \mp z_\rho)| \leq C_K(Y)(1+(\gamma-R)^2)^{-2}$ for $\gamma > R$, and $\sum_\rho (1+\gamma)^{-4} < \infty$ by (A.3). The series therefore converges normally, and Weierstrass' theorem gives the entire extension and the termwise derivatives. Reality and evenness follow from the symmetries $z_\rho \mapsto \overline{z_\rho}$ and $x \mapsto -x$ of the zero multiset, and the statement about $P_K$ follows by integration.

For real $x$, partition the centers $c \in \{\pm\gamma\}$ into the shells $k \leq |x-c| < k+1$, $k \in \mathbb{N}_0$. After reflection onto positive ordinates, the $k$th shell is covered by at most four intervals of length one with endpoints at most $|x|+k+1$, so by Lemma A.2 it contains at most $36\log(|x|+k+4)$ centers. With (2.1) at $Y = 1/2$, the left side of (A.8) is at most

$$36C_K(\tfrac{1}{2})\sum_{k\geq 0}\frac{\log(|x|+k+4)}{(1+k^2)^2} \leq 36C_K(\tfrac{1}{2})\sum_{k\geq 0}\frac{\log(|x|+2)+\log(k+2)}{(1+k^2)^2},$$

since $\log(|x|+k+4) \leq \log(|x|+2)+\log(k+2)$ for $|x|+2 \geq 2$ and $k+2 \geq 2$. The series converges, which proves the first bound; integrating it gives the second. The expansion at zero is evenness and analyticity of $p_K$. □

The growth bounds (A.8) are what the change of variables and the integration by parts in Proposition 2.2 need.

*Proof of Proposition 2.2.* By evenness, $H_K(t) = 2\int_0^\infty e^{-tx^2} p_K(x)\,\mathrm{d}x$; the substitution $r = x^2$ gives the first formula, absolutely convergent by (A.8). Since $Q_K' = q_K$ on $(0,\infty)$, $Q_K(0) = 0$ and $Q_K(r) = O(\sqrt{r}\log(r+2))$, integration by parts gives $\int_0^\infty e^{-tr} q_K(r)\,\mathrm{d}r = t\int_0^\infty e^{-tr} Q_K(r)\,\mathrm{d}r$, which is the second formula. If $p_K \geq 0$, or $P_K \geq 0$ on $(0,\infty)$, the corresponding function is the Laplace transform of a nonnegative measure and hence CM. Conversely, if $H_K \in \mathrm{CM}$, Bernstein's theorem provides a nonnegative representing measure; weighting the signed density $q_K(r)\,\mathrm{d}r$ by $e^{-t_0 r}$ with $t_0 > 0$ makes it a finite signed measure, so ordinary uniqueness of Laplace transforms identifies it with the (weighted) representing measure, and continuity of $q_K$ on $(0,\infty)$ gives $q_K \geq 0$, hence $p_K \geq 0$. The same argument applies to $\ell_K$ and $Q_K$. □

A.3. **Convergence for the strip kernel.** Theorem 2.3(i) concerns a kernel that is not entire, so Lemma A.4 does not apply to it; we prove the convergence and the Laplace representation directly, for any multiset of centers in a strip, the form that Appendix D.5 needs for a narrower strip.

**Proposition A.5** (The strip kernel for a multiset of centers)**.** *Let $h > 0$ and $a \geq h$, and let $\mathcal{Z}$ be a nonempty multiset of pairs $(\gamma, d)$ with $\gamma > 0$ and $|d| < h$, invariant with multiplicity under $d \mapsto -d$, with $\#\{(\gamma,d) \in \mathcal{Z} : \gamma \leq R\} \leq C(1+R)^M$. Then the profile*

$$p_a(x) = \frac{1}{2}\sum_{(\gamma,d)\in\mathcal{Z}} \big(K_a(x-\gamma+id) + K_a(x+\gamma-id)\big)$$

*is strictly positive on $\mathbb{R}$, and $H_a(t) = \int_{\mathbb{R}} e^{-tx^2} p_a(x)\,\mathrm{d}x$ and $H_a/t$ are completely monotone with strictly positive alternating derivatives of every order. If moreover*

$$\sum_{(\gamma,d)\in\mathcal{Z}} \frac{1}{1+\gamma^2} < \infty, \tag{A.9}$$

*the complete Bernstein exponent and the Thorin representation of Proposition A.6 hold as well. For the zeros of $\zeta$, $h = \frac{1}{2}$ and $\mathcal{Z} = \{(\gamma, \delta_\rho) : \rho \in \mathcal{Z}_+\}$, so that $p_a = p_{K_a}$; the identification uses only $0 < \beta < 1$, the symmetry $\rho \mapsto 1-\bar\rho$ and the order of growth of the counting bound.*

*Proof of Proposition A.5 and of Theorem 2.3(i).* By the symmetry of $\mathcal{Z}$ the summands pair into $k_{a,d}(x-\gamma) + k_{a,d}(x+\gamma)$, each strictly positive by Lemma 2.4, so $p_a > 0$ once the series converges. For $|r| \geq 1$ and every $|d| < a$, $|K_a(r+id)| \leq 1/(2a|\sinh(\pi r/(2a))|) \leq C_a e^{-\pi|r|/(2a)}$, with the same bound for each horizontal derivative, so the counting hypothesis gives locally uniform convergence of the series and of its derivatives; the finitely many terms with $|x \mp \gamma| < 1$ are finite since $|d| < h \leq a$. For the moments fix $t > 0$, $n \geq 0$ and $\gamma \geq 2$, and split $\int x^{2n} e^{-tx^2} k_{a,d}(x-\gamma)\,\mathrm{d}x$ at $|x-\gamma| = \gamma/2$: on the inner part use the unit mass of $k_{a,d}$ and $x^{2n}e^{-tx^2} \leq C_{n,t} e^{-t\gamma^2/8}$, on the outer part $k_{a,d}(x-\gamma) \leq C_a e^{-\pi\gamma/(4a)}$. The resulting bound $C(e^{-t\gamma^2/8} + e^{-\pi\gamma/(4a)})$ is uniform in $|d| < h$ and summable over $\mathcal{Z}$, locally uniformly in $t$; so $H_a$ is smooth, $(-1)^n H_a^{(n)}(t) = \int x^{2n} e^{-tx^2} p_a > 0$, and $r = x^2$ gives the first Laplace formula. The function $q_a(r) = p_a(\sqrt{r})/\sqrt{r}$ is positive and locally integrable with $\int_0^r q_a = P_a(\sqrt{r})$, and Tonelli gives $\int_0^\infty e^{-tr} P_a(\sqrt{r})\,\mathrm{d}r = t^{-1}\int_0^\infty e^{-ts} q_a(s)\,\mathrm{d}s = H_a(t)/t$.

For the Bernstein and Thorin assertions fix $s > 0$. The same splitting at $|x-\gamma| = \gamma/2$, now using the integrable function $\log(1+s/x^2)$, gives, uniformly in $|d| < h$ and $\gamma \geq 2$,

$$\int_{\mathbb{R}} k_{a,d}(x-\gamma)\log(1+s/x^2)\,\mathrm{d}x \leq C_{a,s}(\gamma^{-2} + e^{-\pi\gamma/(4a)}).$$

The inner part uses unit mass and $|x| \geq \gamma/2$; on the outer part use the exponential bound above. The finitely many smaller centers give finite integrals as well. Hence (A.9) makes $\int_{\mathbb{R}} p_a(x)\log(1+s/x^2)\,\mathrm{d}x$ finite. Tonelli gives the Bernstein and Thorin representations exactly as in Proposition A.6. For $\zeta$, $|\delta_\rho| < \frac{1}{2}$ is the absence of zeros on the lines $\operatorname{Re} s \in \{0,1\}$, the symmetry is the functional equation, and $N(R) = O(R\log R)$ from (A.3) implies (A.9) by partial summation. Finally $\mathrm{Var}(K_a) = a^2$ by (2.6) at $d = 0$. □

A.4. **Consequences of a positive cumulative profile.** We state and prove here the all-order consequences of the strict sign $P_K > 0$ summarized in Section 2.1. We read each of them off the Laplace representation (2.3), whose density $Q_K$ is then strictly positive. They are stated in the following classes. A complete Bernstein function (CBF) has a CM Lévy density; a Thorin–Bernstein function (TBF) has a Stieltjes derivative, equivalently a positive Thorin measure. A kernel is $\mathrm{STP}_\infty$ if all its minors at strictly increasing row and column arguments are positive. The equivalences and uniqueness statements we use are in Schilling et al. (2012, Chapters 2, 6, 8).

**Proposition A.6** (All-order consequences of a positive cumulative profile)**.** *Let $K$ be admissible with $P_K(x) > 0$ for every $x > 0$. Then:*

(i) $(-1)^n \ell_K^{(n)}(t) = \int_0^\infty r^n e^{-tr} Q_K(r)\,\mathrm{d}r > 0$ *for every $n \in \mathbb{N}_0$ and $t > 0$.*

(ii) *For every $j \in \mathbb{N}_0$, the Hankel kernel $(u, v) \mapsto (-1)^j \ell_K^{(j)}(u + v)$ is $\mathrm{STP}_\infty$ on $(0, \infty)^2$. In differential form, for every $m \in \mathbb{N}$ and $t > 0$,*

$$\left[(-1)^{a+b+j} \ell_K^{(a+b+j)}(t)\right]_{a,b=0}^{m-1} \succ 0,$$

*where $\succ 0$ denotes positive definiteness of a real symmetric matrix.*

(iii) *The function*

$$\mathcal{B}_K(s) = \int_0^\infty (1 - e^{-st}) \ell_K(t)\,\mathrm{d}t = \int_0^\infty \frac{s}{s+r} \frac{Q_K(r)}{r}\,\mathrm{d}r \qquad (s \geq 0) \tag{A.10}$$

*is a complete Bernstein function with Lévy density $\ell_K$, and without killing or drift terms.*

*If moreover $p_K(x) > 0$ for every $x \in \mathbb{R}$, then (i) and (ii) hold with $H_K$ in place of $\ell_K$, and $\mathcal{B}_K$ has the logarithmic, or Thorin, representation*

$$\mathcal{B}_K'(s) = \int_0^\infty \frac{q_K(r)}{s+r}\,\mathrm{d}r, \qquad \mathcal{B}_K(s) = \int_0^\infty q_K(r) \log(1 + s/r)\,\mathrm{d}r; \tag{A.11}$$

*in particular $\mathcal{B}_K'$ is a Stieltjes function, and $\exp(-a\mathcal{B}_K(s))$ is the Laplace transform of a generalized gamma convolution law for each $a > 0$.*

*Proof of Proposition A.6.* By Lemma A.4, $Q_K(r) = 2p_K(0)\sqrt{r} + O(r^{3/2})$ as $r \downarrow 0$ and $Q_K(r) = O(\sqrt{r}\log(r+2))$ as $r \to \infty$. These bounds justify differentiation under the integral in (2.3), which gives (i) with a strictly positive integrand on the open half-line.

For (ii), let $w$ be a positive continuous function on $(0, \infty)$ with $\int_0^\infty e^{-tr} w(r)\,\mathrm{d}r < \infty$ for $t > 0$, and $f(t) = \int e^{-tr} w(r)\,\mathrm{d}r$. The continuous Cauchy–Binet formula is

$$\det[f(s_i + t_j)]_{i,j=1}^m = \int_{0<r_1<\cdots<r_m} \det[e^{-s_i r_k}]_{i,k=1}^m \det[e^{-t_j r_k}]_{j,k=1}^m \prod_{k=1}^m w(r_k)\,\mathrm{d}r_k. \tag{A.12}$$

For strictly increasing $s_i$ and strictly increasing $r_k$, the determinant $\det[e^{-s_i r_k}]$ has the sign $(-1)^{m(m-1)/2}$ and is nonzero; the sign follows by induction, dividing by the first positive row and writing consecutive row differences as integrals of derivatives, or from the Wronskian $e^{-x\sum_k r_k} \prod_{a<b}(r_a - r_b)$ of the functions $e^{-r_k x}$. The two determinants in (A.12) thus have the same sign, their product is strictly positive on the ordered region, and so is the weight. The integral is therefore strictly positive, and the choice $w(r) = r^j Q_K(r)$ gives the $\mathrm{STP}_\infty$ claim. The differential form is the positive definiteness of the Hankel moment matrix of the measure $r^j e^{-tr} Q_K(r)\,\mathrm{d}r$, whose support is infinite.

For (iii), the bounds on $Q_K$ give $\int_0^\infty Q_K(r)\, r^{-1}(1+r)^{-1}\,\mathrm{d}r < \infty$. Tonelli's theorem and $\int_0^\infty (1 - e^{-st}) e^{-rt}\,\mathrm{d}t = s/(r(s+r))$ prove (A.10), which is the Stieltjes representation of a complete Bernstein function with representing measure $Q_K(r) r^{-1}\,\mathrm{d}r$ (Schilling et al., 2012, Theorem 6.2); the absence of killing and drift is read off from it. Equivalently, (2.3) and the bounds on $Q_K$ give $\ell_K(t) =$

$O(t^{-3/2}\log(1/t))$ as $t \downarrow 0$ and $\ell_K(t) = O(t^{-3/2})$ as $t \to \infty$, so $\int_0^\infty \min(1,t)\ell_K(t)\,\mathrm{d}t < \infty$, and the Lévy density $\ell_K$ is CM by (i).

If also $p_K > 0$, then $q_K > 0$ on $(0,\infty)$ and the same arguments apply to $H_K = \int e^{-tr} q_K$; Tonelli applied to $\mathcal{B}_K'(s) = \int_0^\infty te^{-st}\ell_K(t)\,\mathrm{d}t = \int_0^\infty e^{-st}H_K(t)\,\mathrm{d}t$ gives the first formula in (A.11), and integrating from zero to $s$ gives the second; the integrals converge because $q_K(r) = O(r^{-1/2})$ near zero and $q_K(r) = O(r^{-1/2}\log(r+2))$ at infinity. The last statement is the probabilistic reading of a positive Thorin representation: approximate its measure by finite atomic measures, obtaining products of gamma Laplace transforms, and pass to the weak limit; see (Schilling et al., 2012, Chapter 8) for the terminology. □

A.5. **Thorin measures and the three positivity levels.** Proposition A.6 concerns admissible kernels, but the strip Poisson kernel of Section 2.2 is not entire. We therefore state the three positivity levels of Section 2.1 for any profile integrable against $(1+x^2)^{-1}$, which also gives the converse at the Thorin level. Let $p$ be real, even and locally bounded on $\mathbb{R}$ with $\int |p(x)|(1+x^2)^{-1}\,\mathrm{d}x < \infty$, and set

$$P(x) = 2\int_0^x p(u)\,\mathrm{d}u, \qquad H(t) = \int_{\mathbb{R}} e^{-tx^2}p(x)\,\mathrm{d}x, \qquad L(s) = \int_{\mathbb{R}} p(x)\log(1+s/x^2)\,\mathrm{d}x.$$

The singularity of $\log(1+s/x^2)$ at $x=0$ is integrable, and the weighted integral controls all three formulas at infinity. Substituting $\lambda = x^2$ and integrating by parts gives

(A.13)
$$H(t) = \int_0^\infty e^{-t\lambda}\frac{p(\sqrt{\lambda})}{\sqrt{\lambda}}\,\mathrm{d}\lambda, \qquad \frac{H(t)}{t} = \int_0^\infty e^{-t\lambda}P(\sqrt{\lambda})\,\mathrm{d}\lambda, \qquad L(s) = \int_0^\infty \frac{s}{s+\lambda}\frac{P(\sqrt{\lambda})}{\lambda}\,\mathrm{d}\lambda;$$

the boundary term at infinity vanishes because $P(x) = o(x^2)$, which the weighted integrability of $p$ gives once $\int_0^x p$ is split at a large fixed cutoff. For an admissible $K$ and $p = p_K$, these functions are $P_K$, $H_K$ and $L = \mathcal{B}_K$; the last identity is the signed Lévy representation used in the proof below.

**Proposition A.7** (Three positivity levels)**.** *Under these hypotheses,*

$$\begin{aligned} L \in \mathrm{BF} &\iff H \ge 0, \\ L \in \mathrm{CBF} &\iff H/t \in \mathrm{CM} \iff P \ge 0 \text{ on } (0,\infty), \\ L \in \mathrm{TBF} &\iff H \in \mathrm{CM} \iff p \ge 0, \end{aligned}$$

*where the signs of $p$ and $P$ are understood almost everywhere, and pointwise for continuous $p$.*

The proof of Proposition A.7 rests on two identities for the Thorin exponent of a measure on $(0,\infty)$. Let $\tau$ be a nonzero positive measure on $(0,\infty)$ with

$$\int_{(0,\infty)} \log(1+1/\lambda)\,\tau(\mathrm{d}\lambda) < \infty, \tag{A.14}$$

and put

$$L_\tau(s) = \int_{(0,\infty)} \log(1+s/\lambda)\,\tau(\mathrm{d}\lambda) \qquad (s \ge 0).$$

Then $L_\tau$ is a Thorin–Bernstein function with Thorin measure $\tau$, without drift or killing, and $e^{-L_\tau}$ is the Laplace transform of the generalized gamma convolution with Thorin measure $\tau$ (Bondesson, 1992; Schilling et al., 2012). For $\tau$ as in (A.14),

$$H_\tau(t) = \int e^{-t\lambda}\,\tau(\mathrm{d}\lambda),$$

$$R_\tau(s) = L_\tau'(s) = \int \frac{\tau(\mathrm{d}\lambda)}{s+\lambda}, \tag{A.15}$$

$$L_\tau(s) = \int_0^\infty (1-e^{-st})\frac{H_\tau(t)}{t}\,\mathrm{d}t. \tag{A.16}$$

Tonelli's theorem and Frullani's integral prove these identities, and (A.14) is exactly the integrability of (A.16) at $s = 1$, since $1 - e^{-t}$ is comparable with $\min(1, t)$.

*Proof of Proposition A.7.* The signed Lévy representation (A.16) converges absolutely in the Lévy norm, by Tonelli applied to $|p|$ at $s = 1$. Uniqueness of Lévy representations gives the first line and identifies the Lévy density $H/t$ in the second; uniqueness of Laplace transforms in (A.13) gives $H/t \in \mathrm{CM} \iff P \geq 0$ and $H \in \mathrm{CM} \iff p \geq 0$; integrating $H$ in $t$ gives the Stieltjes derivative (A.15), and uniqueness of Stieltjes representations gives the converse in the third line. No drift or killing is present since $L(0+) = 0$ and the derivatives are as displayed. □

A.6. **The shifted strip density.** We prove Lemma 2.4, the shifted form of the strip density on which Theorem 2.3 rests. Positivity is a direct computation, and the Fourier transforms come from a contour integral that uses $K_a(x + 2ia) = -K_a(x)$.

*Proof of Lemma 2.4.* Since $1/\cosh(U+iV) = (\cosh U \cos V - i \sinh U \sin V)/(\sinh^2 U + \cos^2 V)$, the bound $|V| < \pi/2$ makes both factors in (2.5) positive. For the transform integrate $e^{i\omega z} K_a(z)$, $\omega \geq 0$, over the rectangle with vertices $\pm R$ and $\pm R + 2ia$: the only pole inside is $z = ia$, with residue $e^{-a\omega}/(i\pi)$; the vertical sides vanish as $R \to \infty$; and since $K_a(x + 2ia) = -K_a(x)$ the horizontal sides give $(1 + e^{-2a\omega})\widehat{K_a}(\omega) = 2e^{-a\omega}$. Evenness extends this formula to $\omega < 0$. Translating the contour by $\pm id$ gives $\int_{\mathbb{R}} e^{-i\omega x} K_a(x \pm id)\,\mathrm{d}x = e^{\mp d\omega}\widehat{K_a}(\omega)$, and averaging proves the second formula; its value 1 at $\omega = 0$ is the unit mass and its expansion $1 - \frac{1}{2}(a^2 - d^2)\omega^2 + O(\omega^4)$ the variance. □

A.7. **The Cauchy cusp.** Theorem 4.1 uses the first nonzero Taylor term of $p_K$ to exclude exponential moments. A nonzero constant term also excludes the first absolute moment.

**Proposition A.8.** *Under the hypotheses of Theorem 4.1, if $p_K(0) > 0$, then*

$$\mathcal{B}_K(s) = 2\pi p_K(0)\sqrt{s} + O(s) \qquad (s \downarrow 0),$$

*and the characteristic function of $\mu_K$ is $1 - 2\pi p_K(0)|\omega| + O(\omega^2)$ as $\omega \to 0$. Consequently $\int_{\mathbb{R}} |x| \mu_K(\mathrm{d}x) = \infty$. The leading cusp agrees with that of the Cauchy law of scale $2\pi p_K(0)$.*

*Proof.* Proposition 2.2 and Fubini's theorem give

$$\mathcal{B}_K'(s) = \int_0^\infty e^{-st} H_K(t)\,\mathrm{d}t = \int_0^\infty \frac{q_K(r)}{s + r}\,\mathrm{d}r.$$

The integral converges absolutely by (A.8), since $q_K(r) = p_K(\sqrt{r})/\sqrt{r}$; for $K_a$ it is the Thorin representation of Theorem 2.3. Write $q_K(r) = p_K(0) r^{-1/2} + (p_K(\sqrt{r}) - p_K(0)) r^{-1/2}$. The second term is $O(\sqrt{r})$ on $(0, 1]$, because $p_K$ is even and $C^2$, so it contributes $O(1)$ to the integral over $(0, 1]$. The integral over $(1, \infty)$ is also $O(1)$, while $\int_0^1 \frac{r^{-1/2}}{s+r}\,\mathrm{d}r = \pi s^{-1/2} + O(1)$. Hence $\mathcal{B}_K'(s) = \pi p_K(0) s^{-1/2} + O(1)$, and integration from 0 to $s$ gives the expansion of $\mathcal{B}_K$. Substituting $s = \omega^2$ gives $\varphi(\omega) = e^{-\mathcal{B}_K(\omega^2)} = e^{-2\pi p_K(0)|\omega|} e^{O(\omega^2)}$. If $\int |x| \mu_K(\mathrm{d}x)$ were finite, $\varphi$ would be continuously differentiable, whereas its one-sided derivatives at 0 are $\mp 2\pi p_K(0) \neq 0$. □

Strict positivity at the origin is not automatic. The admissible density proportional to $e^{-x^2}\Xi(x)^2$ vanishes at every $\pm z_\rho$ and has $p_K(0) = 0$.

We also record when $\mu_K$ has a density, for $K$ as in Theorem 4.1. When $\int_0^\infty H_K(t) t^{-1}\,\mathrm{d}t = \infty$, $S_K > 0$ almost surely and $\mu_K$ has an even, strictly positive density $\Lambda_K$. Indeed, monotone convergence gives

$$\mathbb{P}(S_K = 0) = \lim_{s \to \infty} e^{-\mathcal{B}_K(s)} = \exp\Big(-\int_0^\infty \ell_K(t)\,\mathrm{d}t\Big).$$

When the integral diverges, conditioning on $S_K$ writes $\mu_K$ as a mixture of centered Gaussian densities, which is even and strictly positive.

## Appendix B. The arithmetic certificate and its tail estimates

This appendix proves Theorem 2.5. On $0 < x \le 60$ we certify the signed prime sum cell by cell, with quadrature and rounding errors included; for $x \ge 60$ an analytic estimate suffices, because the positive background grows. Throughout, $K = \kappa_6$ and $P = P_K$. The normalization, admissibility and variance follow from Proposition D.1 and (D.4). Evaluating the convolution of Proposition D.1, in the Fourier convention of Appendix D.1, gives for $A > 0$

$$m_A(v) := \widehat{\kappa}_A(v) = \begin{cases} (1 - |v|/(2A))\cos(\pi|v|/(2A)) + \pi^{-1}\sin(\pi|v|/(2A)), & |v| \le 2A, \\ 0, & |v| > 2A. \end{cases}$$

Write $m = m_6$, so $0 \le m \le 1$, and write $\Lambda_{\mathrm{ar}}$ for the von Mangoldt function.

B.1. **A finite prime formula.** Compact Fourier support truncates the prime side of the explicit formula. We separate the prime powers, gamma factor, and pole term so that each contribution has its own computable remainder bound.

We define

$$a_n = \frac{\Lambda_{\mathrm{ar}}(n)}{\sqrt{n}} m(\log n), \qquad C = \gamma_E + \log \pi.$$

Every sum below is restricted to $2 \le n < e^{12}$.

**Proposition B.1** (Prime-side cumulative identity)**.** *For real $x$, the cumulative profile $P_K$ of Section 2 satisfies*

$$\begin{aligned} \pi P_K(x) = &-\frac{C}{2}x - \frac{x}{2}\log(1 - e^{-24}) \\ &+ \int_0^{12} \left\{ \frac{xe^{-2v} - m(v)e^{-v/2}\sin(xv)/v}{1 - e^{-2v}} + 2m(v)\cosh(v/2)\frac{\sin(xv)}{v} \right\} \mathrm{d}v \\ &- \sum_{2 \le n < e^{12}} a_n \frac{\sin(x \log n)}{\log n}. \end{aligned} \tag{B.1}$$

*The singularity of the displayed integrand at $v = 0$ is removable.*

*Proof.* We apply the Guinand–Weil formula (Carneiro et al., 2017, Lemma 8) to the even function

$$h_x(z) = \int_{-x}^{x} K(u - z)\,\mathrm{d}u, \qquad \widehat{h}_x(v) = 2m(v)\frac{\sin(xv)}{v}.$$

Its sum over all zeros, divided by 2, is $P_K(x)$. The pole contribution is

$$h_x(i/2) = \frac{2}{\pi}\int_0^{12} m(v)\cosh(v/2)\frac{\sin(xv)}{v}\,\mathrm{d}v.$$

The constant term is $-x\log\pi/(2\pi)$, and the prime term is $-\pi^{-1}\sum a_n \sin(x\log n)/\log n$.

To handle the gamma factor, use

$$\psi(z) = -\gamma_E + \int_0^{\infty} \frac{e^{-t} - e^{-zt}}{1 - e^{-t}}\,\mathrm{d}t \qquad (\mathrm{Re}\, z > 0).$$

The gamma contribution, after substituting $t = 2v$, is

$$-\frac{\gamma_E x}{2\pi} + \frac{1}{\pi}\int_0^{\infty} \frac{xe^{-2v} - m(v)e^{-v/2}\sin(xv)/v}{1 - e^{-2v}}\,\mathrm{d}v.$$

For $v > 12$, $m(v) = 0$, and the remaining integral is $-\frac{x}{2}\log(1 - e^{-24})$. Combining the terms proves (B.1). The conditions of the explicit formula hold because $K$, and hence $h_x$, are entire and decay as $O((1 + |\mathrm{Re}\, z|)^{-4})$ on every fixed horizontal strip. □

B.2. **The finite arithmetic certificate.** On $0 < x \le 60$ we prove positivity by a finite computation with rigorous error bounds; the coefficient estimate below also enters the tail argument. Appendix F.1 gives the sieve and the derivative and quadrature bounds that the computation uses.

**Lemma B.2** (Finite prime coefficient bound)**.** *The zeta coefficients satisfy*

$$\frac{1}{\pi}\sum_{2\le n<e^{12}}\frac{a_n}{\log n} < \frac{31}{10}. \tag{B.2}$$

*Proof.* The directed interval sum of Appendix F.1 lies in

$$[3.059960235329535, 3.059960235329538] \subset (0, 31/10).$$

Its terms come from the integer prime powers below $e^{12}$. □

With the coefficients bounded, we cover $(0, 60]$ by cells and bound $P_K$ on each by Taylor expansion.

**Proposition B.3** (Certified compact interval)**.** *For every $0 < x \le 60$,*

$$P_K(x) \ge 10^{-8}\min(x, 1). \tag{B.3}$$

*Proof.* Oddness allows the origin interval $0 < x \le 1/10$ to be treated by a Taylor lower bound for $P_K(x)/x$, namely (F.1); the certified bound exceeds $4.30 \cdot 10^{-7}$. Cover the rest by 599 intervals $I_i = [i/10, (i+1)/10]$ with centers $c_i = (2i+1)/20$ and radius $r = 1/20$. On each cell Taylor's theorem gives

$$P_K(x) \ge P_K(c_i) - \sum_{j=1}^{16}\frac{|P_K^{(j)}(c_i)|r^j}{j!} - \frac{10^6 12^{15} r^{17}}{17!}.$$

Lemmas F.1 and F.2 justify every term and the remainder. Directed interval evaluation makes each right side greater than $10^{-8}\min((i+1)/10, 1)$, which proves (B.3) throughout each cell. □

B.3. **The unbounded tail.** Beyond the finite covering we replace numerical evaluation by an analytic lower bound. We keep a positive main term, which comes from the growth of the Riemann–Siegel theta function, and we bound the prime and pole terms uniformly in $x$.

The continuous Riemann–Siegel theta function is

$$\theta(t) = \operatorname{Im}\log\Gamma(1/4 + it/2) - \frac{t}{2}\log\pi, \qquad \theta(0) = 0.$$

The branch is continuous in the right half-plane, and $\theta$ is odd. The real-variable explicit formula is

$$P_K(x) = Q(x) + \frac{1}{\pi}\int_{\mathbb{R}} K(y)\theta(x-y)\,\mathrm{d}y - \frac{1}{\pi}\sum a_n\frac{\sin(x\log n)}{\log n}, \tag{B.4}$$

where $Q(x) = \int_{-x}^{x} K(u + i/2)\,\mathrm{d}u$ is real. The identity follows either from Proposition B.1 or directly by integrating the gamma term of the Guinand–Weil formula. The lower bound for the integral in (B.4) needs the following bounds on $\theta$.

**Lemma B.4** (Elementary theta bounds)**.** *For $t \ge 0$,*

$$\theta(t) \ge -4, \qquad |\theta(t)| \le t\log(t+3) + 4. \tag{B.5}$$

*For $t \ge 39$ we have $\theta(t) \ge t/4$.*

*Proof.* The Euler–Maclaurin remainder for Stirling's formula can be written, for $\operatorname{Re} z > 0$, as

$$\log\Gamma(z) = (z - \tfrac{1}{2})\log z - z + \tfrac{1}{2}\log(2\pi) + R(z), \qquad R(z) = \frac{1}{12z} - \frac{1}{2}\int_0^\infty \frac{B_2(\{u\})}{(z+u)^2}\,\mathrm{d}u.$$

The periodic Bernoulli function satisfies $|B_2(\{u\})| \le 1/6$; see also the remainder discussion in (NIST, 2026, Section 5.11(ii)). For $z = 1/4 + it/2$, $t > 0$, this bound gives $|R(z)| \le \frac{\pi+2}{12t}$. Consequently, for $t \ge 1$,

$$\theta(t) \ge \frac{t}{2}\log\frac{t}{2\pi e} - \frac{\pi}{8} - \frac{\pi+2}{12t}. \tag{B.6}$$

The first term has minimum $-\pi$ on $(0,\infty)$, so the right side is greater than $-4$ for $t \ge 1$. For $0 \le t \le 1$, the digamma series implies $\theta'(t) \ge \theta'(0)$, and the value $\psi(1/4) = -\gamma_E - \pi/2 - 3\log 2$ gives $\theta'(0) > -4$, proving the same lower bound. For $t \ge 1$, Stirling's formula gives $\theta(t) \le \frac{t}{2}\log(t+1) + \frac{\pi+2}{12t} < t\log(t+3) + 4$. On $[0,1]$, the $n = 0$ term in the digamma series is at most 1, and the sum of the terms $n \ge 1$ is at most $\frac{1}{4}\sum_{n\ge1} n^{-3} < \frac{1}{2}$. Thus $\operatorname{Re}\psi(1/4 + it/2) < 3/2$, proving the required upper estimate also there. Combining this upper estimate with $\theta(t) \ge -4$ proves the absolute-value bound in (B.5). Finally, if $t \ge 39$, then $t/(2\pi e) > 2$ and $\log 2 > 2/3$. Equation (B.6) gives $\theta(t) \ge t/3 - \pi/8 - (\pi+2)/(12t) \ge t/4$. □

**Proposition B.5** (Unbounded tail)**.** *For every $x \ge 60$,*

$$P_K(x) > \frac{3}{4}. \tag{B.7}$$

*Proof.* The variance in (2.7) and Chebyshev's inequality imply $\int_{-1}^{1} K(y)\,dy \ge 1 - \frac{\pi^2}{144} > \frac{11}{12}$. For $x \ge 60$ and $|y| \le 1$, Lemma B.4 yields $\theta(x-y) \ge (x-1)/4$. Outside this interval but with $y \le x$, use $\theta(x-y) \ge -4$. For $y \ge 1$, the explicit kernel satisfies $K(y) \le \frac{1}{10y^4}$. Hence the potentially negative contribution from $y > x$ is bounded by

$$\int_x^\infty K(y)|\theta(y-x)|\,dy \le \frac{1}{10}\int_x^\infty \frac{y\log(y+3)+4}{y^4}\,dy < \frac{1}{100}.$$

It follows that

$$\int_{\mathbb{R}} K(y)\theta(x-y)\,dy \ge \frac{11(x-1)}{48} - 4 - \frac{1}{100}. \tag{B.8}$$

Contour shifting gives $\int_{\mathbb{R}} K(u+i/2)\,du = 1$. For $u \ge 60$ we have $|K(u+i/2)| \le \frac{12}{u^4}$, using $|\cos(6(u+i/2))|^2 \le \cosh^2 3 < e^6$ and the denominator in (D.4). Therefore

$$Q(x) \ge 1 - 2\int_x^\infty |K(u+i/2)|\,du > \frac{99}{100}. \tag{B.9}$$

Combining (B.4), (B.2), (B.8), and (B.9) gives

$$P_K(x) > \frac{99}{100} - \frac{31}{10} + \frac{1}{\pi}\left(\frac{11(x-1)}{48} - \frac{401}{100}\right).$$

This expression is increasing in $x$ and exceeds $3/4$ at $x = 60$. Thus (B.7) holds on the entire unbounded range. □

Propositions B.3 and B.5 give (2.7). The Laplace criterion and all-order consequences in Propositions 2.2 and A.6 complete the proof of Theorem 2.5.

Every larger half-variance is available by Gaussian convolution. For $r > 0$ let $G_r$ be the Gaussian density of variance $2r$ (Section 3.1). The density $\kappa_6 * G_r$ has half-variance $\pi^2/288 + r$, and

$$P_{\kappa_6 * G_r}(x) = \int_0^\infty [G_r(x-y) - G_r(x+y)]P_{\kappa_6}(y)\,dy > 0 \qquad (x > 0).$$

The identity follows by extending $P_{\kappa_6}$ oddly and pairing the two halves of the convolution, as in Proposition C.1, and the bracket is positive for $x, y > 0$.

## Appendix C. Gaussian certificates

This appendix proves the Gaussian statements of Section 3: the transformation and persistence used in Section 3.1, Theorem 3.3, and the pointwise certificate at $\sigma \geq 100$. By the transform (3.2), each conclusion is a statement about $H$ at a Möbius-transformed time.

C.1. **The Gaussian transform and the half-line evolution.** We prove Proposition 3.1 and the persistence of $P_\sigma \geq 0$ under further Gaussian smoothing, both used in Section 3.1.

*Proof of Proposition 3.1.* Completing the square gives, first for real $z$ and then for complex $z$,

$$\int_{\mathbb{R}} e^{-tx^2} G_\sigma(x-z)\,\mathrm{d}x = \frac{1}{\sqrt{1+4\sigma t}} \exp\left(-\frac{tz^2}{1+4\sigma t}\right).$$

Each integrand is bounded in absolute value by $e^{1/(16\sigma)} e^{-tx^2} G_\sigma(x - \operatorname{Re} z)$, whose integral is a Gaussian in $\operatorname{Re} z$, so the sum over the zeros converges absolutely. The two signs of $z_\rho$ give the same value, which cancels the factor $1/2$ in (1.1). Dividing by $t$ and using $v_\sigma(t)/t = (1+4\sigma t)^{-1}$ gives (3.3). □

A cumulative certificate at one width should survive further Gaussian smoothing, even where the density $p_\sigma$ changes sign. Because $P_\sigma$ is odd, we test it on the positive half-line, and pairing the two halves of its convolution gives the positive Dirichlet heat kernel below.

**Proposition C.1** (A positive half-line evolution)**.** *For $\sigma, \eta > 0$ and $x > 0$,*

$$P_{\sigma+\eta}(x) = \int_0^\infty D_\eta(x,u) P_\sigma(u)\,\mathrm{d}u, \qquad D_\eta(x,u) = G_\eta(x-u) - G_\eta(x+u). \tag{C.1}$$

*The Dirichlet heat kernel $D_\eta$ is strictly positive on $(0,\infty)^2$, and $P_\sigma$ solves the heat equation*

$$\partial_\sigma P_\sigma = \partial_x^2 P_\sigma, \qquad P_\sigma(0) = 0. \tag{C.2}$$

*Proof.* We differentiate $G_\eta * P_\sigma$ with respect to $x$, integrating by parts with Gaussian decay and the growth bound of Lemma A.4; the derivative is $2G_\eta * p_\sigma = 2p_{\sigma+\eta} = P'_{\sigma+\eta}$. Both functions are odd, so their constant difference vanishes. Splitting their ordinary convolution into its positive and negative halves gives (C.1). Differentiation with respect to the Gaussian parameter gives (C.2).

For $x, u > 0$, $(x-u)^2 < (x+u)^2$ gives $D_\eta(x,u) > 0$. □

Once $P_\sigma \geq 0$ on the half-line, further Gaussian smoothing makes it strictly positive there.

C.2. **The cumulative comparison.** For the Gaussian, the comparison of Section 3.2 takes an explicit form, because the antiderivative of $G_\sigma$ is an error function. Throughout, $T$ is a height through which the zeros are verified.

The entire antiderivative of $G_\sigma$ and the positive displacement amplitude are

$$F_\sigma(w) := \frac{1}{2}\operatorname{erf}\left(\frac{w}{2\sqrt{\sigma}}\right), \qquad \operatorname{erf}(w) := \frac{2}{\sqrt{\pi}}\int_0^w e^{-u^2}\,\mathrm{d}u,$$

$$A_\sigma(d) := \frac{1}{\sqrt{4\pi\sigma}}\int_0^d e^{v^2/(4\sigma)}\,\mathrm{d}v = \frac{1}{2}\operatorname{erfi}\left(\frac{d}{2\sqrt{\sigma}}\right), \qquad \operatorname{erfi}(w) := \frac{2}{\sqrt{\pi}}\int_0^w e^{u^2}\,\mathrm{d}u. \tag{C.3}$$

In (C.3), $d \geq 0$ is real. Since $F'_\sigma = G_\sigma$, normal convergence and oddness give

$$P_\sigma(x) = \sum_{\rho \in \mathcal{Z}_+} \big(F_\sigma(x - z_\rho) + F_\sigma(x + z_\rho)\big). \tag{C.4}$$

The two terms of each summand are grouped before summation. We compare $P_\sigma$ with

$$P_\sigma^{\mathrm{re}}(x) := \sum_{\rho \in \mathcal{Z}_+} \big(F_\sigma(x-\gamma) + F_\sigma(x+\gamma)\big).$$

For $x \geq 0$ each summand is nonnegative, since $F_\sigma$ is increasing and odd on the real line. The next lemma bounds what the replacement of $z_\rho$ by $\gamma$ changes.

**Lemma C.2** (Vertical displacement)**.** *For real $u, b$, $\sigma > 0$,*

$$|\operatorname{Re} F_\sigma(u - ib) - F_\sigma(u)| \leq A_\sigma(|b|)e^{-u^2/(4\sigma)}. \tag{C.5}$$

*Hence $|P_\sigma - P_\sigma^{\mathrm{re}}|$ is at most the sum of these Gaussian errors over the unverified zeros $\gamma > T$.*

*Proof.* Integrate $F'_\sigma$ along the vertical segment. At a point $u - iv$ we have $|G_\sigma(u - iv)| = \frac{1}{\sqrt{4\pi\sigma}} e^{-u^2/(4\sigma)} e^{v^2/(4\sigma)}$. The triangle inequality proves (C.5). Since the full sum is real, we may sum these real-part estimates over the zeros, and a verified zero has $b = 0$ and contributes no error. □

C.3. **A uniform Gaussian bound from zero counting.** The displacement bound of Lemma C.2 weights each ordinate by a Gaussian factor, so we bound the sum of these factors by zero counting. For $x \geq 0$, we define

$$S_\sigma(x) := \sum_{\rho \in \mathcal{Z}_+} \left( e^{-(x-\gamma)^2/(4\sigma)} + e^{-(x+\gamma)^2/(4\sigma)} \right),$$

$$C_\sigma(x) := \left( \sqrt{\frac{\sigma}{\pi}} + \frac{1}{4} \right) \log(x+2) + 8 + \frac{4\sigma/\pi + \sqrt{\pi\sigma}/4}{x+2}. \tag{C.6}$$

The superscript "far, $d$" on $S$ restricts the sum to centers at distance at least $d$ from $x$.

**Lemma C.3** (Global and far-tail envelopes)**.** *For $x \geq 0$, $\sigma > 0$, and $d \geq 0$,*

$$S_\sigma(x) \leq C_\sigma(x), \qquad S_\sigma^{\mathrm{far},d}(x) \leq e^{-d^2/(8\sigma)} C_{2\sigma}(x).$$

*Proof.* Let $n_x(r)$ count the full real centers $\{\pm\gamma\}$ in $[x-r, x+r]$, so that $n_x(r) \leq (r/\pi + 1/4)\log(x + r + 2) + 8$ by Lemma A.3. The nonnegative layer-cake identity is

$$S_\sigma(x) = \int_0^\infty \frac{r}{2\sigma} e^{-r^2/(4\sigma)} n_x(r)\, \mathrm{d}r.$$

Use $\log(x + r + 2) \leq \log(x+2) + r/(x+2)$. The measure $(r/(2\sigma))e^{-r^2/(4\sigma)}\,\mathrm{d}r$ has mass one, first moment $\sqrt{\pi\sigma}$, and second moment $4\sigma$, which proves the first estimate. For a distance $d_j \geq d$, $e^{-d_j^2/(4\sigma)} \leq e^{-d^2/(8\sigma)} e^{-d_j^2/(8\sigma)}$; summing proves the second. □

C.4. **Counting comparison and the tail envelope.** In the range $x \geq T/2$ of the proof of Theorem 3.3 we compare $P_\sigma$ with the zero count $N$, using only the strip and the counting bound. This subsection proves that comparison and the explicit tail envelope it needs.

*Comparison with the counting function.* The comparison uses the envelope $C_\sigma(x)$ of (C.6).

**Proposition C.4** (Uniform counting comparison)**.** *For every $x \geq 0$ and $\sigma > 0$,*

$$|P_\sigma(x) - N(x)| \leq \left( \frac{1}{2} + A_\sigma(1/2) \right) C_\sigma(x). \tag{C.7}$$

*Proof.* For $y \geq 0$, $\operatorname{erfc}(y) \leq e^{-y^2}$. Hence the cumulative contribution of a real center satisfies

$$\left| F_\sigma(x - \gamma) + F_\sigma(x + \gamma) - \mathbf{1}_{\{\gamma \leq x\}} \right| \leq \frac{1}{2} \left( e^{-(x-\gamma)^2/(4\sigma)} + e^{-(x+\gamma)^2/(4\sigma)} \right).$$

Summing gives $|P_\sigma^{\mathrm{re}}(x) - N(x)| \leq S_\sigma(x)/2$. Apply Lemma C.2 to *all* zeros, followed by $S_\sigma \leq C_\sigma$, which proves (C.7). □

The amplitude $A_\sigma(d)$ in (C.7) grows like $e^{d^2/(4\sigma)}$; we record the bounds used for Theorem 3.3.

**Lemma C.5** (Two useful amplitude bounds)**.** *For $\sigma > 0$ and $d > 0$,*

$$A_\sigma(d) \le \min\left\{\frac{d}{2\sqrt{\pi\sigma}}, \frac{2\sqrt{\sigma}}{d\sqrt{\pi}}\right\} e^{d^2/(4\sigma)}.$$

*For fixed $d > 0$,*

$$A_\sigma(d) \sim \frac{\sqrt{\sigma}}{d\sqrt{\pi}} e^{d^2/(4\sigma)} \qquad (\sigma \downarrow 0).$$

*Proof.* For the first bound, bound the integrand in (C.3) by its endpoint value. For the second, use $d^2 - v^2 \ge d(d-v)$ for $0 \le v \le d$ and integrate the resulting exponential. For the asymptotic, set $v = d - 2\sigma w/d$. The integral defining $A_\sigma(d)$ becomes

$$\frac{\sqrt{\sigma}}{d\sqrt{\pi}} e^{d^2/(4\sigma)} \int_0^{d^2/(2\sigma)} e^{-w+\sigma w^2/d^2}\, \mathrm{d}w.$$

The integrand, extended by zero past its upper endpoint, is bounded by $e^{-w/2}$ and tends to $e^{-w}$. Dominated convergence gives a limiting integral equal to one. □

*Explicit counting constants.* The threshold height of Theorem 3.3 needs the published counting constants $(a_2, b_2, c_2)$ of (A.6), which are sharper than the rounded bound (A.3).

For $x \ge 100$ and $L_x = \log(x+2)$, we define

$$C_\sigma^{(2)}(x) := (\sqrt{\sigma/\pi} + 2a_2)L_x + 2b_2 \log L_x + 2c_2 + \frac{4\sigma/\pi + (2a_2 + 2b_2/L_x)\sqrt{\pi\sigma}}{x+2}. \tag{C.8}$$

**Lemma C.6.** *For $x \ge 100$, $\sigma > 0$, $S_\sigma(x) \le C_\sigma^{(2)}(x)$, so (C.7) holds with $C_\sigma^{(2)}$ in place of $C_\sigma$.*

*Proof.* In the proof of Lemma C.3, the error terms at the two count endpoints are each at most $a_2 \log(x+r+2) + b_2 \log\log(x+r+2) + c_2$. If an endpoint is below $e$, use instead $|N(y) - M(y)| \le 4 + \log(e+2)/8 < 4.195$. The Bellotti–Wong envelope at $x + r + 2 \ge 102$ exceeds $4.483 > 4.195$. Thus the same error bound covers an endpoint below $e$. The main term is bounded as in (A.7). Integrate with the Rayleigh measure used there and use

$$\log(x+r+2) \le L_x + \frac{r}{x+2}, \qquad \log\log(x+r+2) \le \log L_x + \frac{r}{(x+2)L_x}.$$

Its first two moments give (C.8). □

C.5. **The Gaussian certificate at a variable height.** The certificate of Theorem 3.3 is the Gaussian counterpart of Theorem 3.4. It uses the verified height in the same way, as a positive core, and above it the unconditional comparison of $P_\sigma$ with the counting function proved in Appendix C.4, with the explicit envelope of Bellotti–Wong fixing the threshold height. The amplitude $A_\sigma(d)$, the antiderivative $F_\sigma$ and the counting envelopes $C_\sigma$, $C_\sigma^{(2)}$ are those of (C.3), (C.6) and (C.8), and $(a_2, b_2, c_2) = (0.11200, 0.12567, 3.77417)$ are the constants of (A.6).

*Proof of Theorem 3.3.* Let $\sigma = \sigma_T$ and $L = \log T \ge \log 10^{11} > 25.3$, so that $e^{1/(16\sigma)} = T$. Then Lemma C.5 and, since $\sigma \le 1$, the envelope (C.6) give

$$A_\sigma(1/2) \le \frac{4\sqrt{\sigma}}{\sqrt{\pi}}\, e^{1/(16\sigma)} = \frac{T}{\sqrt{\pi L}}, \qquad C_\sigma(x) \le \left(\sqrt{\sigma/\pi} + \tfrac{1}{4}\right)\log(x+2) + 9.$$

We treat three ranges of $x$ and check the inequalities in interval arithmetic (Appendix F.2).

***The range*** $x \ge T/2$**.** This range needs no hypothesis on the zeros. For $x \ge 100$, Proposition C.4 with the envelope of Lemma C.6, and the counting bound (A.6), give

$$P_\sigma(x) \ge N(x) - (\tfrac{1}{2} + A_\sigma(1/2))C_\sigma^{(2)}(x), \qquad N(x) \ge M(x) - (a_2 \log x + b_2 \log\log x + c_2).$$

We put $L_x = \log(x+2)$ and $L_0 = \log(T/2+2) \le L$, and we divide by $L_x$. The quotient $M(x)/L_x$ increases with $x$, while for $x \ge T/2$

$$\frac{a_2 \log x + b_2 \log\log x + c_2}{L_x} \le a_2 + \frac{b_2 \log L_0}{L_0} + \frac{c_2}{L_0},$$
$$\frac{C_\sigma^{(2)}(x)}{L_x} \le \sqrt{\sigma/\pi} + 2a_2 + \frac{2b_2 \log L_0}{L_0} + \frac{2c_2}{L_0} + \frac{3e^{-L}}{L_0},$$

because $\log L_x/L_x$ and $1/L_x$ decrease for $L_x \ge e$ and the last term of (C.8) is at most $\frac14/(x+2) \le \frac12 e^{-L}$. Hence $P_\sigma(x) \ge (L_x/L_0)(T/2)\,\varphi(L)$ for every $x \ge T/2$, where

$$\varphi(L) = \frac{L - \log(4\pi) - 1}{2\pi} - 2e^{-L}(a_2 L + b_2 \log L + c_2) - \Big(e^{-L} + \frac{2}{\sqrt{\pi L}}\Big)\widehat{C}(L),$$
$$\widehat{C}(L) = \frac{\sqrt{L}}{4\sqrt{\pi}} + 2a_2 L + 2b_2 \log L + 2c_2 + 3e^{-L}.$$

Here we used $M(T/2) = \frac{T}{4\pi}(L - \log(4\pi) - 1)$, $\sqrt{\sigma/\pi}\,L_0 \le \sqrt{L}/(4\sqrt{\pi})$, and $L_0 \le L$ in the subtracted terms. The function $\varphi$ increases on $[e^2, \infty)$: the first term has slope $1/(2\pi)$, the second and $e^{-L}\widehat{C}(L)$ decrease, and $(2/\sqrt{\pi L})\widehat{C}(L)$ has derivative at most $2a_2/\sqrt{\pi L} < 1/(2\pi)$ once $L \ge e^2$, since $b_2(2 - \log L) \le 0$ there. Interval arithmetic gives $\varphi(\log 10^{11}) > 0.16$. Hence $P_\sigma(x) > 0$ for all $x \ge T/2$ and $T \ge 10^{11}$.

***The range*** $16 \le x \le T/2$**.** We assume $\mathsf{V}(T)$. In (C.4) the zeros with $\gamma \le T$ are then real. Each of them contributes a nonnegative term, and the anchor $\gamma_* \in (14, 15)$ contributes almost a full unit:

$$F_\sigma(x-\gamma) + F_\sigma(x+\gamma) = \int_{\gamma - x}^{\gamma + x} G_\sigma \ge 0,$$
$$\int_{-x-\gamma_*}^{x-\gamma_*} G_\sigma \ge \int_{-1}^{1} G_\sigma \ge 1 - e^{-1/(4\sigma)} = 1 - T^{-4}.$$

For $\gamma > T$ both $\gamma - x$ and $\gamma + x$ are at least $T/2$. By Lemma C.2 and $|F_\sigma(x-\gamma) + F_\sigma(x+\gamma)| \le 2xG_\sigma(\gamma - x)$, the contribution of such a zero to $P_\sigma(x)$ is at most

$$(T(4\pi\sigma)^{-1/2} + A_\sigma(1/2))\big(e^{-(x-\gamma)^2/(4\sigma)} + e^{-(x+\gamma)^2/(4\sigma)}\big),$$

and the far-tail envelope of Lemma C.3 bounds the sum over $\gamma > T$ by

$$(T(4\pi\sigma)^{-1/2} + A_\sigma(1/2))e^{-T^2/(32\sigma)}C_{2\sigma}(x).$$

Since $e^{-T^2/(32\sigma)} = T^{-T^2/2}$, this tail bound is below $\frac12 T^{-4}$ for $T \ge 10^{11}$ (the ratio decreases in $T$; at $10^{11}$ its logarithm is below $-10^{23}$), and $P_\sigma(x) > 0$.

***The range*** $0 < x \le 16$**.** Near the origin we bound the profile $p_\sigma$ itself. Each verified zero contributes $\frac12(G_\sigma(x-\gamma) + G_\sigma(x+\gamma)) > 0$ to $p_\sigma(x)$, and the anchor contributes at least $\frac12 G_\sigma(31) = \frac12(4\pi\sigma)^{-1/2}T^{-3844}$. Each zero with $\gamma > T$ contributes at most

$$e^{1/(16\sigma)}(4\pi\sigma)^{-1/2}\big(e^{-(x-\gamma)^2/(4\sigma)} + e^{-(x+\gamma)^2/(4\sigma)}\big)$$

in absolute value, so their sum is at most $T(4\pi\sigma)^{-1/2}T^{-T^2/2}C_{2\sigma}(x)$, again below half the anchor term for $T \ge 10^{11}$. Hence $p_\sigma > 0$ on $[0, 16]$ and $P_\sigma > 0$ on $(0, 16]$.

Thus $P_{\sigma_T} > 0$ on $(0, \infty)$; Propositions 2.2 and A.6 give $\ell_{\sigma_T} \in \mathrm{CM}$ with all its consequences, and Proposition C.1 carries positivity to every $\sigma \ge \sigma_T$. □

The threshold $10^{11}$ is what the Bellotti–Wong envelope gives at the scale $1/(16\log T)$ exactly.

C.6. **The profile at $\sigma \geq 100$.** The following certificate gives the bound $\sigma_H \leq 100$ of Section 3.1.

**Theorem C.7** (Explicit global positivity)**.** *Under* (A.2)–(A.4)*,*

$$p_\sigma(x) > 0 \qquad (x \in \mathbb{R},\ \sigma \geq 100). \tag{C.9}$$

The main estimate uses the bounded imaginary parts to delay any negative cosine contribution until the Gaussian is extremely small. For $x \geq 100$ the positive contribution comes from the zeros within distance 20 of $x$, and the next lemma shows that there are enough of them.

**Lemma C.8** (A supply of nearby zeros)**.** *For every $x \geq 100$,*

$$N(x+20) - N(x-20) \geq \log(x+2).$$

*Proof.* The left side is at least $\frac{20}{\pi}\log\frac{x-20}{2\pi} - \frac{1}{4}\log(x+22) - 8$. For $x \geq 100$, use $x-20 \geq (4/5)x$, $x+22 \leq (61/50)x$, and $x+2 \leq (51/50)x$. Subtracting $\log(x+2)$ gives a lower bound affine in $\log x$, with positive slope $20/\pi - 5/4$. Its value at $x = 100$ is $\frac{20}{\pi}\log\frac{40}{\pi} - \frac{1}{4}\log 122 - 8 - \log 102 > 2.37058 > 0$. Thus the lower bound is positive throughout the required range. □

*Proof of Theorem C.7.* It suffices first to take $\sigma = 100$ and $x \geq 0$, and we write

$$D(x) = 2\sqrt{400\pi}\, p_{100}(x) = \sum_{z \in \mathcal{Z}_\Xi} \exp(-(x-z)^2/400),$$

where $\mathcal{Z}_\Xi = \{\pm z_\rho : \rho \in \mathcal{Z}_+\}$ is the full multiset of xi zeros. For $z = a + ib$, with $|b| \leq 1/2$,

$$\operatorname{Re} e^{-(x-z)^2/400} = e^{b^2/400} e^{-(x-a)^2/400} \cos\left(\frac{(x-a)b}{200}\right). \tag{C.10}$$

If $|x-a| \leq 600$, the cosine argument has absolute value at most $3/2 < \pi/2$, so every such contribution is nonnegative, and a negative contribution must have $|x-a| > 600$.

Let $\Omega(x)$ be the sum of the absolute values of all negative real contributions in (C.10). For an integer $k \geq 600$, the centers with $k \leq |x-a| < k+1$ are covered, after reflection onto positive ordinates if necessary, by at most four intervals of length one whose endpoints are at most $x+k+1$. Lemma A.2 gives the count $36\log(x+k+4)$. Hence

$$\Omega(x) \leq 36 e^{1/1600} \sum_{k=600}^{\infty} \log(x+k+4) e^{-k^2/400}.$$

For $x \geq 0$ and $k \geq 600$ we have $\log(x+k+4) \leq \log(x+2) + \log(k+2) \leq (k+1)\log(x+2)$. The second inequality follows from $\log(x+2) \geq \log 2$ and $k+2 \leq 2^k$. Set $r = e^{-3}$. Since $(600+j)^2/400 \geq 900 + 3j$,

$$\begin{aligned}\sum_{k=600}^{\infty} (k+1) e^{-k^2/400} &\leq e^{-900}\left(\frac{601}{1-r} + \frac{r}{(1-r)^2}\right) \\ &< 1000 e^{-900}.\end{aligned}$$

Consequently, the negative total satisfies

$$\Omega(x) < 10^{-300}\log(x+2). \tag{C.11}$$

If $x \geq 100$, retain only the zeros with $|\gamma - x| \leq 20$. Each has real contribution at least $e^{-1}\cos(1/20) > \frac{1}{3}$. Lemma C.8 and (C.11) imply $D(x) \geq (\frac{1}{3} - 10^{-300})\log(x+2) > 0$. If $0 \leq x \leq 100$, retain just the real zero $\gamma_* \in (14, 15)$. Its contribution is at least $e^{-25}$ because $|x - \gamma_*| < 100$. The discarded negative contributions total at most $10^{-300}\log 102$. Thus $D(x) \geq e^{-25} - 10^{-300}\log 102 > 0$. Evenness handles negative $x$. Finally, if $\sigma > 100$, (3.1) gives $p_\sigma = G_{\sigma-100} * p_{100}$. Both factors are strictly positive on $\mathbb{R}$, and the integral is finite, which proves (C.9). □

## Appendix D. Estimates for the band-limited density

The density $\mathcal{K}_T$ and its parameters are those of Section 3.2; here we derive its properties from its frequency-space form and prove Theorem 3.4 uniformly over its range of heights.

Throughout, $T_* = 3 \cdot 10^{12}$, $T \geq T_*$, and $\mathsf{V}(T)$ is the verified-height hypothesis (3.4), so that (A.5) is $\mathsf{V}(T_*)$, the theorem of Platt and Trudgian (2021). The half-bandwidth $A = A(T)$ uniquely solves

$$\frac{e^A}{A^4} = \frac{T}{20}, \qquad b_A = \frac{\pi}{2A}, \qquad \varepsilon = A^{-2}, \qquad X = \frac{99}{100}T, \tag{D.1}$$

which exists and exceeds 40 because $A \mapsto e^A/A^4$ increases for $A > 4$ and $20e^{40}/40^4 < T_*$; in particular $2b_A < 0.1$. The inputs are (A.2), (A.4), $\mathsf{V}(T)$, and the counting bound in the weaker form

$$|N(Y) - M(Y)| \leq \tfrac{1}{8}\log(Y+2) + \tfrac{33}{8} \qquad (Y \geq 0), \tag{D.2}$$

which follows from Lemma A.1; the weaker constant matches the interval checks. Every function of $A$ below is monotone on $A \geq 40$, so we check the inequalities at $A = 40$, at $X = \frac{99}{100}T_*$, and on a bracket for $A(T_*)$, in outward-rounded interval arithmetic (Appendix F.2).

D.1. **The density in frequency space.** We choose the density in frequency space: squaring a cosine amplitude makes it nonnegative on $\mathbb{R}$ and minimizes its second moment at the chosen bandwidth. After computing its normalization and variance we add the translates that make it strictly positive.

For $A > 0$ we define the probability density in the frequency variable

$$w_A(\omega) = \frac{\pi}{4A}\cos\left(\frac{\pi\omega}{2A}\right)\mathbf{1}_{[-A,A]}(\omega). \tag{D.3}$$

Its characteristic function is the real even entire function

$$\psi_A(z) = \int_{-A}^{A} e^{iz\omega} w_A(\omega)\,\mathrm{d}\omega = \frac{\cos(Az)}{1-(2Az/\pi)^2} = \frac{\cos(Az)}{1-(z/b_A)^2}.$$

The singularities at $z = \pm b_A$ are removable, with $\psi_A(\pm b_A) = \pi/4$. Put

$$c_A = \frac{8A}{\pi^3}, \qquad C_A = \frac{\pi}{2A^3}, \qquad \kappa_A(z) = c_A\psi_A(z)^2 = C_A\frac{\cos^2(Az)}{(z-b_A)^2(z+b_A)^2}. \tag{D.4}$$

**Proposition D.1** (Mass, variance, and a sharp variational normalization)**.** *The function $\kappa_A$ is a nonnegative even probability density with Fourier transform $\widehat{\kappa}_A = 2\pi c_A(w_A * w_A)$, supported in $[-2A, 2A]$ and strictly positive on its interior, and*

$$\int_{\mathbb{R}} x^2\kappa_A(x)\,\mathrm{d}x = \frac{\pi^2}{4A^2}. \tag{D.5}$$

*Moreover, if $g(x) = \int_{-A}^{A} e^{ix\omega} f(\omega)\,\mathrm{d}\omega$ with $0 \neq f \in H_0^1(-A, A)$, then*

$$\frac{\int_{\mathbb{R}} x^2|g(x)|^2\,\mathrm{d}x}{\int_{\mathbb{R}}|g(x)|^2\,\mathrm{d}x} \geq \frac{\pi^2}{4A^2}, \tag{D.6}$$

*with equality precisely for constant complex multiples of $f(\omega) = \cos(\pi\omega/(2A))$. Thus $\kappa_A$ has the least variance among normalized squares $|g|^2$ of amplitudes with $H_0^1$ frequency profile on $[-A, A]$.*

*Proof.* With $\widehat{f}(\omega) = \int_{\mathbb{R}} e^{-i\omega x} f(x)\,\mathrm{d}x$, Plancherel gives $\int|g|^2 = 2\pi\int|f|^2$ and $\int x^2|g|^2 = 2\pi\int|f'|^2$. Expanding $f$ in the Dirichlet sine basis of $(-A, A)$,

$$\int_{-A}^{A}|f'|^2\,\mathrm{d}\omega \geq \left(\frac{\pi}{2A}\right)^2\int_{-A}^{A}|f|^2\,\mathrm{d}\omega,$$

with equality exactly at the first eigenfunction $\cos(\pi\omega/(2A))$. This eigenvalue bound proves (D.6) and its equality case. For (D.3), $\int w_A^2 = \pi^2/(16A)$, so $\int\psi_A^2 = \pi^3/(8A)$ and $c_A$ is exactly the normalizing constant; the same ratio proves (D.5). Finally $\widehat{\psi_A^2} = 2\pi(w_A * w_A)$ is nonnegative, supported in $[-2A, 2A]$ and strictly positive inside. □

This variational problem explains the choice of density, and Section 3.3 proves that the choice is also optimal for the continuation estimates.

The density $\kappa_A$ vanishes at the odd multiples of $b_A$ other than $\pm b_A$. We fill these real zeros by the small symmetric translate mixture

$$\mathcal{K}_{A,\varepsilon}(z) = (1-\varepsilon)\kappa_A(z) + \frac{\varepsilon}{2}\big(\kappa_A(z-b_A) + \kappa_A(z+b_A)\big), \qquad 0 < \varepsilon < 1/2.$$

It is an even probability density with Fourier support $[-2A, 2A]$, since its Fourier transform is $(1-\varepsilon+\varepsilon\cos(b_A\omega))\widehat{\kappa}_A(\omega)$, and since $b_A^2 = \pi^2/(4A^2)$,

$$\operatorname{Var}(\mathcal{K}_{A,\varepsilon}) = \frac{\pi^2(1+\varepsilon)}{4A^2}. \tag{D.7}$$

With (D.1) it is the density $\mathcal{K}_T = \mathcal{K}_{A(T),A(T)^{-2}}$ of (3.5); we abbreviate $\mathcal{K}_T$ to $\mathcal{K}$ within this appendix, and its half-variance is (3.6).

We record how this half-variance depends on the height.

**Proposition D.2** (The width as a function of the height)**.** *As $T \to \infty$,*

$$A(T) = \log T + 4\log\log T - \log 20 + O\Big(\frac{\log\log T}{\log T}\Big), \qquad \tfrac{1}{2}\operatorname{Var}(\mathcal{K}_T) \sim \frac{\pi^2}{8\log^2 T}. \tag{D.8}$$

*Proof.* Taking logarithms in (D.1), $A - 4\log A = \log T - \log 20$. First $A \sim \log T$; substituting back, $A - \log T = O(\log\log T)$, so $\log A = \log\log T + O(\log\log T/\log T)$, which gives the expansion. The exact identity (D.7) with $\varepsilon = A^{-2}$ gives the second statement. The term $4\log\log T$ comes from the factor $A^{-4}$ in (D.1). □

D.2. **Envelopes and counting.** The proof of Theorem 3.4 in Appendix D.3 rests on the following bounds for $\mathcal{K}$ and for the zero count.

Real lower bound. The anchor's contribution near the origin uses, for real $x$,

$$\mathcal{K}(x) \geq \frac{\varepsilon c_A}{[1 + (|x|/b_A + 1)^2]^2} > 0. \tag{D.9}$$

Indeed $\kappa_A(x) \geq c_A\cos^2(Ax)/[1+(x/b_A)^2]^2$ since $|1-u^2| \leq 1+u^2$, and $\cos^2(A(x \pm b_A)) = \sin^2(Ax)$. Bounding all three denominators by the one in (D.9), the numerator is at least $c_A[(1-\varepsilon)\cos^2(Ax) + \varepsilon\sin^2(Ax)] \geq \varepsilon c_A$. In particular $\mathcal{K}$ is strictly positive on $\mathbb{R}$.

Complex envelopes. The displacement error needs bounds for $\kappa_A$ off the real axis. For real $x$ and $y \geq 0$, the rational form (D.4), $|\cos(A(x+iy))| \leq \cosh(Ay)$ and the AM–GM inequality give

$$|\kappa_A(x+iy)| \leq \frac{C_A\cosh^2(Ay)}{2}\left(\frac{1}{((x-b_A)^2+y^2)^2} + \frac{1}{((x+b_A)^2+y^2)^2}\right). \tag{D.10}$$

There is also a nonsingular envelope:

$$|\kappa_A(x+iy)| \leq 36c_A\frac{\cosh^2(Ay)}{(1+A^2x^2)^2} \qquad (x \in \mathbb{R},\ y \geq 0). \tag{D.11}$$

To prove it, the positive symmetric density (D.3) gives $|\psi_A(x+iy)| \leq \cosh(Ay)$. When $A^2x^2 > \pi^2/4$, the rational formula also gives $|\psi_A(x+iy)| \leq (\pi^2/4)\cosh(Ay)/(A^2x^2 - \pi^2/4)$. Combining the two bounds yields $|\psi_A(x+iy)| \leq (1+\pi^2/2)\cosh(Ay)/(1+A^2x^2)$; since $1+\pi^2/2 < 6$, squaring proves (D.11). The translated components obey the same bounds with $x$ shifted by $b_A$, and since $1 + A^2x^2 \leq 12(1 + A^2(x \mp b_A)^2)$, the density $\mathcal{K}$ satisfies (2.1) with $C_{\mathcal{K}}(Y)$ a constant multiple of $\cosh^2(AY)$: $\mathcal{K}$ is admissible, and Lemma A.4 and Propositions 2.2–A.6 apply to it. (The convergence lemma uses the counting bound only through its logarithmic order, so (D.2) suffices for it.)

Retained mass. The mass of $\mathcal{K}$ on $[-1,1]$ converts the zero count into a cumulative lower bound. Put $r_0 = 1 - b_A$. For $|x| \geq r_0$, (D.4) gives $\kappa_A(x) \leq C_A x^{-4}(1-(b_A/r_0)^2)^{-2}$, so $\int_{|u|\geq r_0}\kappa_A \leq \frac{2C_A}{3r_0^3}(1-(b_A/r_0)^2)^{-2}$, and each translate $\kappa_A(\cdot \mp b_A)$ has at most this mass outside $[-1,1]$. Hence the retained mass satisfies

$$\int_{-1}^{1}\mathcal{K}(x)\,\mathrm{d}x \geq w(A), \qquad w(A) := 1 - \frac{2C_A}{3r_0^3}\left(1-(b_A/r_0)^2\right)^{-2} > 0.9999 \qquad (A \geq 40), \tag{D.12}$$

since $w$ increases with $A$ and $w(40) > 0.99998$.

Counting consequences. We translate the global counting estimate into the local quantities used by the kernel bounds. Counts in centered intervals control the convolution error, while the lower bound for $N(x-1)$ supplies the positive cumulative mass. A reciprocal-square tail bound handles centers beyond the verified range.

With (D.2) in place of (A.3), the proof of Lemma A.3 gives, for $y, r \geq 0$,

$$n_y(r) \leq \left(\frac{r}{\pi} + \frac{1}{4}\right)\log(y+r+2) + \frac{33}{4}. \tag{D.13}$$

Let $L_x = \log(x+2)$. For $x \geq X$, $|s| \leq 0.1$ and all $r \geq 0$, (D.13) and $\log(x+s+r+2) \leq L_x + (r+0.1)/(x+2)$ give

$$n_{x+s}(r) \leq L_x(c_1 r + c_0' + c_2 r^2), \qquad c_1 = 0.32, \quad c_0' = 0.55, \quad c_2 = 10^{-14}. \tag{D.14}$$

After division by $L_x$, the constant, linear and quadratic coefficients are at most

$$\frac{1}{4} + \frac{33}{4L_x} + \frac{0.1}{4(x+2)L_x}, \qquad \frac{1}{\pi} + \frac{1/4 + 0.1/\pi}{(x+2)L_x}, \qquad \frac{1}{\pi(x+2)L_x},$$

which decrease as $x$ grows and at $x = \frac{99}{100}T_* \leq X$ are less than 0.538, 0.319 and $3.8\cdot 10^{-15}$. The same input gives, with $L_X = \log(X+2)$,

$$\frac{N(x-1)}{xL_x} \geq \frac{1-X^{-1}}{2\pi}\left(1 - \frac{\log(2\pi)+1+6/X}{L_X}\right) - \frac{1}{8X} - \frac{33}{8XL_X} > 0.14 \qquad (x \geq X): \tag{D.15}$$

use $x - 1 \geq (1-X^{-1})x$ and $\log(x-1) \geq L_x - 6/X$ in the lower bound of (D.2), and note that the right side increases with $X$ and exceeds 0.143 at $X = \frac{99}{100}T_*$. Finally, since (D.2) gives $N(y) \leq y\log y$ for $y \geq T$, partial summation yields

$$\sum_{\gamma > T}\gamma^{-2} \leq \frac{2(\log T + 1)}{T}. \tag{D.16}$$

D.3. **Proof of Theorem 3.4.** We prove $P_{\mathcal{K}} > 0$ on $(0,\infty)$ by the comparison with real ordinates of Section 3.2. The proof works with the cumulative profile because a bound on each density summand separately would miss the growth of the accumulated mass.

***The range*** $x \geq X$**.** The relation $e^A/A^4 = T/20$ is needed in this range, where it balances the complex continuation against the accumulated real mass. Let $P_{\mathcal{K}}^{\mathrm{re}}$ be the cumulative profile $P_{\mathcal{K}}$ with every zero projected onto its ordinate, that is, with $z_\rho$ replaced by $\gamma$. Since $\mathcal{K} \geq 0$, every projected zero contributes nonnegatively to $P_{\mathcal{K}}^{\mathrm{re}}(x) = \sum_\rho \int_{-x-\gamma}^{x-\gamma}\mathcal{K}(u)\,\mathrm{d}u$ for $x > 0$, and each zero with $\gamma \leq x - 1$ contributes at least $w(A)$ by (D.12); with (D.15),

$$P_{\mathcal{K}}^{\mathrm{re}}(x) \geq 0.14\, w(A)\, xL_x \qquad (x \geq X). \tag{D.17}$$

Integrating the entire antiderivative of $\mathcal{K}$ along vertical segments, and using the symmetries $\mathcal{K}(-w) = \mathcal{K}(w)$ and $\mathcal{K}(\overline{w}) = \overline{\mathcal{K}(w)}$,

$$|P_{\mathcal{K}}(x) - P_{\mathcal{K}}^{\mathrm{re}}(x)| \leq \int_0^{1/2}\sum_{\rho\in\mathcal{Z}_+}\left(|\mathcal{K}(x-\gamma+iy)| + |\mathcal{K}(x+\gamma+iy)|\right)\mathrm{d}y. \tag{D.18}$$

In (D.18) we have bounded the displacement of every zero, verified or not, by $1/2$, which enlarges the right side and simplifies the count.

For $x \ge X$ and $b_3 = c_1\pi/4$, integrating the bound (D.14) against the rational envelope gives

$$\sum_{c\in\{\pm\gamma\}} \frac{1}{((x+s-c)^2+y^2)^2} \le L_x\left(\frac{b_3}{y^3}+\frac{c_0'}{y^4}+\frac{c_2}{y^2}\right), \qquad |s|\le 0.1,\ y>0. \tag{D.19}$$

Indeed, by the layer-cake formula the left side is $\int_0^\infty 4r(r^2+y^2)^{-3}n_{x+s}(r)\,\mathrm{d}r$, and the three integrals $\int_0^\infty 4r^{k+1}(r^2+y^2)^{-3}\,\mathrm{d}r$ for $k=1,0,2$ equal $\pi/(4y^3)$, $y^{-4}$ and $y^{-2}$.

Use (D.11) for $0\le y\le 1/4$ and (D.10) for $1/4\le y\le 1/2$ in (D.18). For the first range, $(1+A^2u^2)^{-2} = A^{-4}(A^{-2}+u^2)^{-2}$ is (D.19) at $y=1/A$ times $A^{-4}$, and the interval has length $1/4$. For the second range, we bound the hyperbolic cosine and the powers of $y$ by

$$\cosh^2(Ay) \le \tfrac14(1+e^{-A/2})^2e^{2Ay}, \qquad y^{-k}\le 2^k e^{4k(1/2-y)} \quad (k=2,3,4,\ \tfrac14\le y\le\tfrac12),$$

and integrate the resulting exponentials. The two terms in (D.10) have total weight one, as do the three mixture components, whose center shifts are at most $2b_A<0.1$. The right side of (D.18) is therefore at most $(E_0+E_1)L_x$ with

$$E_0(A) = 9c_A\cosh^2(A/4)\left(\frac{b_3}{A}+c_0'+\frac{c_2}{A^2}\right),$$
$$E_1(A) = \frac{C_A(1+e^{-A/2})^2e^A}{4}\left(\frac{8b_3}{2A-12}+\frac{16c_0'}{2A-16}+\frac{4c_2}{2A-8}\right).$$

After the substitution $T=20e^A/A^4$ from (D.1), both $E_0/T$ and $E_1/T$ decrease on $A\ge 40$: the summands of the first are positive multiples of $A^je^{-A/2}(1+e^{-A/2})^2$ with $j\in\{3,4,5\}$, and the second equals $\frac{\pi}{160}(1+e^{-A/2})^2$ times a positive combination of $(2-c/A)^{-1}$ with $c\in\{8,12,16\}$. Evaluation at $A=40$ gives $(E_0(A)+E_1(A))/T<0.135$ for $A\ge40$, the value at $A=40$ being $0.13462\ldots$. With (D.17) and $x\ge X=\frac{99}{100}T$,

$$P_{\mathcal K}(x) \ge (0.14\cdot0.9999\,x - 0.135\,T)L_x > \left(0.14\cdot0.9999-\frac{0.135}{0.99}\right)xL_x > 0.0036\,xL_x > 0 \qquad (x\ge X). \tag{D.20}$$

***The unverified zeros below $X$.*** The two lower ranges use the separation between the argument and every possibly nonreal center. Let $0\le x\le X$. For $\gamma>T$ and each of the six translated components $\kappa_A(x\mp\gamma\pm i\delta_\rho+s)$, $s\in\{0,\pm b_A\}$, the two factors $z\mp b_A$ in the denominator of (D.4) have real part of modulus at least $\gamma-X-2b_A\ge\gamma/200$, and at least $\gamma-101-2b_A\ge\gamma/3$ when $x\le101$, while $|\cos(Az)|^2\le\cosh^2(A/2)\le e^A = TA^4/20$. Since $\log T = A-4\log A+\log20\le A-1$ for $A\ge40$, the tail bound (D.16) gives for the contribution of the unverified zeros to the profile

$$|p_{\mathcal K,>T}(x)| \le \begin{cases} D_A := 81C_A\cosh^2(A/2)\dfrac{2(\log T+1)}{T^3} \le \dfrac{81\pi A^2}{20T^2} & (0\le x\le 101),\\[2ex] D_A' := 200^4\,\dfrac{\pi A^2}{20T^2} & (0\le x\le X); \end{cases}$$

the two center signs cancel the $1/2$ in (1.1). The verified zeros are real by $\mathsf V(T)$, and each contributes $\frac12(\mathcal K(x-\gamma)+\mathcal K(x+\gamma))>0$ to $p_{\mathcal K}(x)$. Retain only the anchor $\gamma_*\in(14,15)$ of (A.4).

***The range $0\le x\le 101$.*** Near the origin the cumulative profile vanishes, so we bound $p_{\mathcal K}$ itself from below. For $0\le x\le101$, $|x-\gamma_*|\le101$, and (D.9) gives the anchor contribution

$$\frac{\varepsilon c_A}{2[1+(101/b_A+1)^2]^2} > \frac{1}{2\cdot10^8A^5} =: a_A,$$

using $202A/\pi+1<65A$, $\pi^3<32$ and $8\cdot4226^2<2\cdot10^8$. The ratio

$$\frac{D_A}{a_A} \le \frac{2\cdot10^8\cdot81\pi}{20^3}A^{15}e^{-2A} < 0.001 \qquad (A\ge40)$$

decreases on $A \geq 40$, its value at 40 being less than $1.3 \cdot 10^{-4}$. Hence

$$p_{\mathcal{K}}(x) \geq a_A - D_A > 0.999\, a_A > 0 \qquad (0 \leq x \leq 101), \tag{D.21}$$

so $p_{\mathcal{K}}(0) > 0$ and $P_{\mathcal{K}} > 0$ on $(0, 101]$.

***The range*** $101 \leq x \leq X$**.** Away from the origin the anchor has already contributed most of its mass to the cumulative profile, so a cruder bound on the unverified zeros suffices. For $101 \leq x \leq X$, the anchor's contribution $\int_{-x-\gamma_*}^{x-\gamma_*} \mathcal{K}$ to $P_{\mathcal{K}}(x)$ is at least $w(A)$ by (D.12), since $x - \gamma_* \geq 86$; every other verified zero contributes nonnegatively; and the integrated unverified contribution is at most

$$2XD'_A \leq \frac{1.98 \cdot 200^4 \pi}{20^2} A^6 e^{-A} < 0.44 \qquad (A \geq 40),$$

again a decreasing function, with value less than 0.434 at $A = 40$. Hence

$$P_{\mathcal{K}}(x) \geq w(A) - 2XD'_A > 0.9999 - 0.44 > 0.55 \qquad (101 \leq x \leq X). \tag{D.22}$$

The bounds (D.20), (D.21) and (D.22) prove $P_{\mathcal{K}_T} > 0$ on $(0, \infty)$. Propositions 2.2 and A.6 give the Laplace representation, every strict derivative sign and Hankel order, and the complete Bernstein exponent. Proposition D.1 and (D.7) give the mass, Fourier support, and variance. At $T = T_*$ the verified-height hypothesis is a theorem, and the equation for $A$ gives $40.54339240265939935 < A(T_*) < 40.54339240265939936$. Substitution in (D.7) gives $\frac{1}{2}\operatorname{Var}(\mathcal{K}_{T_*}) = 0.000750989241\ldots$, the value in Theorem 3.4, and the asymptotic assertion is (D.8) of Proposition D.2.

D.4. **Strip budgets of the cosine amplitude and the Gaussian.** This subsection proves Proposition 3.6 and computes the strip budget of the Gaussian, which explains its rate in Section 3.3.

*Proof of Proposition 3.6.* The normalized frequency density is $A^{-1}\cos^2(\pi\omega/(2A))\mathbf{1}_{[-A,A]}$, so (3.10) gives the integral. Its leading contribution comes from $\omega$ near $A$; the substitution $u = A - \omega$ gives $A^4 e^{-A}\mathfrak{S}_A \to (\pi^2/8)\int_0^\infty u^2 e^{-u}\,\mathrm{d}u = \pi^2/4$, where the following bounds justify the passage to the limit under the integral. On $[0, A/2]$, $\sinh\omega/\omega \leq e^\omega$ shows that this part of the integral, multiplied by $A^4e^{-A}$, tends to zero. On $[A/2, A]$ put $u = A - \omega$; then $\cos^2(\pi\omega/(2A)) = \sin^2(\pi u/(2A))$ and $\sinh(A-u) \sim \frac{1}{2}e^{A-u}$, so after multiplication by $A^4e^{-A}$ the integrand tends to $(\pi^2/8)u^2e^{-u}$ and is dominated by a constant multiple of $u^2e^{-u}$, because $A - u \geq A/2$ and $\sin v \leq v$. Real translations preserve the horizontal $L^2$ norms in (3.8), and the mixture weights sum to one. Substituting $e^A/A^4 = T/20$ and (3.6) proves (3.11). □

The Gaussian has a larger strip budget at a fixed small variance, which we now quantify.

**Proposition D.3** (The Gaussian strip budget)**.** *The square root* $g_\sigma(x) = (4\pi\sigma)^{-1/4}e^{-x^2/(8\sigma)}$ *of the Gaussian* $G_\sigma$ *has the strip budget*

$$\mathfrak{S}(G_\sigma; g_\sigma) = \int_0^{1/2} e^{v^2/(4\sigma)}\,\mathrm{d}v, \qquad 4\sigma e^{1/(16\sigma)-1+4\sigma} \leq \mathfrak{S}(G_\sigma; g_\sigma) \leq \tfrac{1}{2}e^{1/(16\sigma)} \quad (0 < \sigma \leq \tfrac{1}{8}).$$

*For fixed* $p > 0$*, a budget* $\log\mathfrak{S}(G_{\sigma_R}; g_{\sigma_R}) \leq (p + o(1))\log R$ *forces* $\liminf_{R\to\infty} \sigma_R \log R \geq 1/(16p)$.

*Proof.* Integrating $|g_\sigma(x + iv)|^2$ over $x$ gives the displayed integral. Its upper bound uses $v \leq 1/2$; restricting to $[1/2 - 4\sigma, 1/2]$ gives the lower bound. Hence $\log\mathfrak{S} = 1/(16\sigma) + O(\log(1/\sigma))$ as $\sigma \to 0$, which proves the limit along every subsequence with $\sigma_R \to 0$; along the others it is immediate. □

D.5. **Scaling under a narrower zero strip.** This subsection explains the scaling stated in Remark 3.7 by following the proofs with $h$ in place of $\frac{1}{2}$. Suppose that $|\delta_\rho| \leq h$ for every zero, with $0 < h < \frac{1}{2}$. Proposition A.5, applied with a strip half-width in $(h, a]$, gives $p_{K_a} > 0$ for every $a > h$, so the universal width drops to the infimum $h^2/2$, the least value that Theorem 2.3(ii) allows at half-width $h$. The proof of Theorem 3.3 applies at $\sigma = h^2/(4\log T)$, which is smaller than $1/(16\log T)$. There the displacement amplitude $A_\sigma(h)$ of (C.3) and the factor $e^{h^2/(4\sigma)}$ take the values that $A_\sigma(1/2)$ and $e^{1/(16\sigma)}$ take at $\sigma = 1/(16\log T)$, and every other comparison in that

proof still holds at the smaller $\sigma$. The proof of Theorem 3.4 applies to the compressed density $q^{-1}\mathcal{K}_T(x/q)$ with $q = 2h$, whose half-variance is $q^2$ times that of $\mathcal{K}_T$. The substitution $y = qv$ in the vertical integral of that proof shows that the displacement errors do not increase; the bounds on the unverified tail and on the anchor both acquire the factor $q^3$, and the mass retained on $[-1, 1]$ increases. By dilation, the bounds of Theorem 3.5 and Proposition D.3, with strip budget over $|\operatorname{Im} z| \le h$, scale alike.

## Appendix E. Generalized Laguerre inequalities and theta smoothing

The generalized Laguerre inequalities are the positivity property of Ξ that smoothing of the theta kernel reaches directly, and this appendix records what our methods give for them. They test an entire function through the coefficients of its squared modulus: for a real entire function $f$, define

$$L_k[f](x) = \frac{1}{(2k)!}\sum_{j=0}^{2k}(-1)^{k+j}\binom{2k}{j}f^{(j)}(x)f^{(2k-j)}(x), \qquad k \in \mathbb{N}_0.$$

Taylor expansion gives

$$f(x+iy)f(x-iy) = \sum_{k=0}^{\infty} L_k[f](x)y^{2k}, \qquad L_k[fg](x) = \sum_{j=0}^{k} L_j[f](x)L_{k-j}[g](x). \tag{E.1}$$

In particular, $L_0[f] = f^2$ and $L_1[f] = (f')^2 - ff''$. Functions in the Laguerre–Pólya class $\mathcal{LP}$ satisfy every $L_k[f] \ge 0$ on $\mathbb{R}$ (Patrick, 1973); see Csordas and Varga (1990, Theorem 2.9).

E.1. **A common set of positive heights.** An off-line zero can affect coefficient signs only near its ordinate. The resulting interval cover gives a density bound for all orders on the same set. The unshifted bound $1-\log(11/4)/(2\pi)$ is recorded by span_of_dilates (2026, Lemma 1 and Corollary 1); the all-order zero-exclusion principle also appears in Sra (2026, Lemma 4.2). We improve that bound by combining a right-shifted mollified mean with a critical-line zero proportion.

**Theorem E.1** (Simultaneous Laguerre positivity in density)**.** *Let*

$$\mathcal{G} = \{x \ge 0 : L_k[\Xi](x) \ge 0 \text{ for every } k \in \mathbb{N}_0\}, \qquad D = 1 - \frac{\log\left(1 + \dfrac{1-e^{-7/3}}{e^{4/3}-1}\right) + 917/2400}{2\pi}.$$

*Using the critical-line proportion of Alpöge and Furman (2026, Theorem A), one has*

$$\liminf_{T\to\infty}\frac{|\mathcal{G}\cap[T,2T]|}{T} \ge D, \qquad \liminf_{T\to\infty}\frac{|\mathcal{G}\cap[0,T]|}{T} \ge D, \qquad D > 0.8946,$$

*where $|\cdot|$ denotes Lebesgue measure.*

*Proof.* For every zero $\rho = \beta + i\gamma$ with $\beta > 1/2$, put $b_\rho = \beta - 1/2$, and form the open set

$$E = \bigcup_{\beta>1/2}(\gamma - b_\rho, \gamma + b_\rho).$$

The union includes both signs of the ordinate. A conjugate pair $\gamma \pm ib$ of zeros of Ξ contributes to (E.1), up to a positive constant, the polynomial

$$Y^2 + 2((x-\gamma)^2 - b^2)Y + ((x-\gamma)^2 + b^2)^2, \qquad Y = y^2.$$

Its coefficients are nonnegative when $|x-\gamma| \ge b$. Real zeros contribute factors $(x-r)^2+Y$, including $Y$ if $x = r$. In the conjugate-paired genus-one Hadamard product, the exponential factors contribute positive constants independent of $y$. Local uniform convergence passes each coefficient sign to the limit, so $[0,\infty)\setminus E \subseteq \mathcal{G}$.

To bound the total interval length, fix $0 < \theta < 4/7$, $R > 0$, $\delta = R/\log T$, and a real polynomial $P$ with $P(0) = 0$ and $P(1) = 1$. With the Möbius function $\mu$, set

$$M_T(s) = \sum_{n \le T^\theta} \mu(n) P\left(\frac{\log(T^\theta/n)}{\theta \log T}\right) n^{\delta - s}, \qquad F_T(s) = \zeta(s) M_T(s).$$

The shifted mollified mean of Conrey (1989, Theorem 2 and Section 5, Proposition) gives

(E.2)
$$\frac{1}{T}\int_T^{2T} |F_T(\tfrac{1}{2} + \delta + it)|^2 \, \mathrm{d}t \le C(P; R, \theta) + o(1), \qquad C(P; R, \theta) = 1 + \frac{1 - e^{-2R}}{2R\theta} \int_0^1 (P' - \theta R P)^2 \, \mathrm{d}u.$$

The positive shift is allowed by the bounded complex shifts in the cited proposition. For completeness, its Gaussian means are uniform in the center $w \in [T, 2T]$, with width $\Delta = T^{1-\varepsilon}$ for a sufficiently small fixed $\varepsilon > 0$. Integrating over $w$ bounds the mass on $[T + A\Delta, 2T - A\Delta]$ by $(C + o(1))T/(1 - \operatorname{erfc}(A))$. The endpoint Gaussian means bound the two remaining strips by $O_A(\Delta)$. Taking $T \to \infty$ and then $A \to \infty$ proves (E.2) on the stated interval.

Define the weighted zero counts, with multiplicities,

$$W_\delta(T) = \sum_{\substack{T < \gamma \le 2T \\ \beta > 1/2 + \delta}} (\beta - \tfrac{1}{2} - \delta), \qquad W_0(T) = \sum_{\substack{T < \gamma \le 2T \\ \beta > 1/2}} (\beta - \tfrac{1}{2}).$$

Uniformly on $\operatorname{Re} s = 2$, absolute convergence and $P(1) = 1$ give $F_T(s) = \frac{\zeta(s)}{\zeta(s-\delta)} + O(1/\log T) = 1 + O(1/\log T)$. Thus the right boundary term in Littlewood's lemma is $o(T)$. Choose horizontal boundaries $U \in (T, T+1)$ and $V \in (2T - 1, 2T)$ that avoid zeros. Their argument integrals are $O(\log T)$. Indeed, Jensen's formula applied to the reflected real part $\frac{1}{2}(F_T(z + iU) + \overline{F_T(\bar{z} + iU)})$ on fixed disks centered at a real point to the right of 2 bounds its real zeros by $O(\log T)$: the central value is $1 + o(1)$ and the maximum is $T^{O(1)}$. The same argument applies at $V$, and boundary zeros are handled by limiting contours. Changing the endpoints in the zero sums costs $O(\log T)$. Zeros of $M_T$ add nonnegative weights. Littlewood's lemma, Jensen's inequality for log, and (E.2) give

$$W_\delta(T) \le \frac{T}{4\pi} \log C(P; R, \theta) + o(T).$$

Let $N(T, 2T) = N(2T) - N(T)$ and let $N_0(T, 2T)$ count the critical-line zeros with multiplicity. Suppose $N_0(T, 2T) \ge (\kappa - o(1))N(T, 2T)$. Functional-equation pairing and zero counting imply

$$\begin{aligned} W_0(T) &\le W_\delta(T) + \frac{\delta}{2}\big(N(T, 2T) - N_0(T, 2T)\big), \\ |E \cap [T, 2T]| &\le 2W_0(T) + O(\log T) \\ &\le \frac{T}{2\pi}\big(\log C(P; R, \theta) + (1 - \kappa)R\big) + o(T). \end{aligned}$$

The endpoint error uses $b_\rho < 1/2$ and $N(t+1) - N(t) = O(\log t)$; the zero-count main term is $N(T, 2T) = T \log T/(2\pi) + O(T)$.

The minimum of $\int_0^1 (P' - \theta R P)^2$ with the given endpoints is attained in $H^1(0,1)$ by $P_*(u) = \sinh(\theta R u)/\sinh(\theta R)$. Its value gives

$$C_*(R, \theta) = 1 + \frac{1 - e^{-2R}}{e^{2\theta R} - 1}.$$

Approximate $P_*$ in $C^1$ by polynomials with the same endpoints, fixing each polynomial before taking $T \to \infty$, and then let $\theta \uparrow 4/7$. The cited critical-line theorem supplies the proportion $\frac{3}{2} - \frac{1}{\sqrt{2}} \cot \frac{1}{\sqrt{2}} > \frac{269}{400}$ on dyadic intervals; its count of simple critical-line zeros is at most $N_0(T, 2T)$. Taking $\kappa = 269/400$ and $R = 7/6$ gives $D$. The explicit expression satisfies $0.8946126 < D <$

0.8946130. Summing dyadic intervals, and discarding a bounded initial interval, gives the natural-density assertion. □

E.2. **Convolution of the Jacobi theta function.** To smooth the Jacobi theta function while preserving its transformation law, we convolve it in its logarithmic variable. We write

$$\vartheta(t)=\sum_{n\in\mathbb{Z}}e^{-\pi n^2t},\qquad \Psi(u)=e^{u/2}\vartheta(e^{2u}),\qquad \Phi(u)=\frac{1}{2}\left(\Psi''(u)-\frac{1}{4}\Psi(u)\right).$$

The theta transformation makes $\Psi$ even, and the classical Fourier formula is

$$\Xi(z)=\int_{\mathbb{R}}e^{izu}\Phi(u)\,\mathrm{d}u,\qquad \Phi(u)=\sum_{n\geq1}(4\pi^2n^4e^{9u/2}-6\pi n^2e^{5u/2})e^{-\pi n^2e^{2u}}.$$

See Titchmarsh (1986, Section 2.16) for the theta formula and Sra (2026, eq. (B.1)) for this normalization. The kernel $\Phi$ is positive, even, and decays faster than every exponential. For $\eta>0$, use the Gaussian $G_\eta$ of Section 3.1 to define

$$\vartheta_\eta(t)=e^{-\eta/4}\int_{\mathbb{R}}G_\eta(v)e^{-v/2}\vartheta(te^{-2v})\,\mathrm{d}v,\qquad \Phi_\eta=e^{-\eta/4}G_\eta*\Phi.$$

The normalization retains $\vartheta_\eta(t)\to1$ as $t\to\infty$. We now compute what this smoothing does to $\Xi$ and to its Laguerre coefficients.

**Proposition E.2** (Laguerre identities for smoothed Jacobi theta functions)**.** *For $\eta>0$ we have*

$$\vartheta_\eta(t)=t^{-1/2}\vartheta_\eta(1/t),\qquad \xi_\eta(s):=\frac{s(s-1)}{4}\int_0^\infty(\vartheta_\eta(t)-1)t^{s/2-1}\,\mathrm{d}t=e^{\eta s(s-1)}\xi(s)\quad(\operatorname{Re}s>1).$$

*Its entire continuation gives*

$$\Xi_\eta(z):=\xi_\eta(\tfrac{1}{2}+iz)=\int_{\mathbb{R}}e^{izu}\Phi_\eta(u)\,\mathrm{d}u=e^{-\eta(z^2+1/4)}\Xi(z).$$

*For every $\eta\geq0$, $x\in\mathbb{R}$, and $k\in\mathbb{N}_0$,*

$$L_k[\Xi_\eta](x)=e^{-2\eta(x^2+1/4)}\sum_{j=0}^{k}\frac{(2\eta)^{k-j}}{(k-j)!}L_j[\Xi](x),\tag{E.3}$$

*with $\Xi_0=\Xi$. Consequently every $x\in\mathcal{G}$ satisfies $L_k[\Xi_\eta](x)\geq0$ simultaneously for all $k$ and all $\eta\geq0$. If $\mathsf{V}(T)$ holds with $T>1/2$, the same inequalities hold on $|x|\leq T-1/2$. For any fixed $\eta\geq0$, their validity for all $k$ and all real $x$ is equivalent to RH.*

*Proof.* The function $e^{-\eta/4}G_\eta*\Psi$ is even and equals $e^{u/2}\vartheta_\eta(e^{2u})$, proving the transformation law. The bound $\vartheta(t)\leq1+t^{-1/2}$ and the Gaussian exponential moments justify the normalization. For $\operatorname{Re}s>1$, Tonelli's theorem at $\operatorname{Re}s$, the classical convergent Mellin integral of $\vartheta-1$, and the Gaussian exponential moments justify the Mellin interchange. The change of variables $t\mapsto te^{2v}$ multiplies the integrand by $e^{sv}$; its Gaussian average is $e^{-\eta/4}\int G_\eta(v)e^{(s-1/2)v}\,\mathrm{d}v=e^{\eta s(s-1)}$. Differentiation commutes with convolution, so $\frac{1}{2}(\partial_u^2-\frac{1}{4})(e^{-\eta/4}G_\eta*\Psi)=\Phi_\eta$. The exponential moments of both kernels justify Fourier transformation and give the asserted $\Xi_\eta$.

Multiplying the generating identity (E.1) by $e^{-2\eta(x^2+1/4)}e^{2\eta y^2}$ proves (E.3). The coefficients are nonnegative on $\mathcal{G}$. Under $\mathsf{V}(T)$, every interval $(\gamma-b_\rho,\gamma+b_\rho)$ of an off-line zero lies outside $[-T+1/2,T-1/2]$, so the same conclusion follows there. Under RH, $\Xi_\eta\in\mathcal{LP}$. Conversely, if $\Xi_\eta$ had a zero $x+iy$ with $y\neq0$, all nonnegative coefficients in (E.1) at $x$ could not sum to zero at $y^2>0$: at least one coefficient is nonzero, since $\Xi_\eta\not\equiv0$. The zeros of $\Xi_\eta$ are exactly those of $\Xi$, proving the equivalence. □

In particular,

$$L_1[\Xi_\eta](x) = e^{-2\eta(x^2+1/4)}\big(L_1[\Xi](x) + 2\eta\Xi(x)^2\big).$$

This correction can improve the first inequality at an individual height. The smoothing here convolves $\Phi$ in $u$, whereas the de Bruijn–Newman deformation multiplies $\Phi(u)$ by $e^{\eta u^2}$.

## Appendix F. Computer-assisted steps

The proofs of Theorems 2.5, 3.3 and 3.4 end in finitely many numerical inequalities. We record here how they are certified, and we prove the two error bounds that the arithmetic certificate of Appendix B.2 uses. Every comparison is made between interval endpoints computed with directed rounding, so an inconclusive comparison never counts as a certificate; the decimals that we print are for inspection.

F.1. **The arithmetic certificate.** In this subsection $K = \kappa_6$, and $m$ and $a_n$ are as in Appendix B.1. We evaluate the formula (B.1) and its first 16 derivatives in $x$ in interval arithmetic at 256 bits, with the correctly rounded operations of MPFR (The MPFR Team, 2025). The prime powers come from an integer sieve: directed rounding gives $162754 < e^{12} < 162755$, and the sieve finds 15040 prime powers below $e^{12}$. We compute each summand of (B.2) from its integer prime power, and the sum lies in the interval displayed in the proof of Lemma B.2.

The Taylor expansions in the proof of Proposition B.3 need a bound on the higher derivatives of $P_K$, for their remainders, and a bound on the error of the quadrature that evaluates the integral in (B.1). We prove both.

**Lemma F.1** (Derivative bound)**.** *For every real $x$ and every integer $j \ge 2$, we have $|P_K^{(j)}(x)| \le 10^6 \cdot 12^{j-2}$.*

*Proof.* Two derivatives remove the terms of (B.1) that are linear in $x$, and the integrand that remains is $m(v)W(v)\sin(xv)/v$, where

$$W(v) = 2\cosh(v/2) - \frac{e^{-v/2}}{1-e^{-2v}} = e^{v/2} - \frac{e^{-5v/2}}{1-e^{-2v}}.$$

Since $1 - e^{-2v} \ge 2ve^{-2v}$, we have $\int_0^{12} v|W(v)|\,\mathrm{d}v \le \int_0^{12} ve^{v/2}\,\mathrm{d}v + \frac{1}{2}\int_0^{12} e^{-v/2}\,\mathrm{d}v < 8100$. Since $a_n \log n \le 144\, n^{-1/2}$, we also have $\sum_n a_n \log n \le 144 \sum_{n=2}^{162754} n^{-1/2} < 117000$. Finally, $0 \le m \le 1$ and all frequencies are at most 12, so differentiating (B.1) $j$ times gives

$$|P_K^{(j)}(x)| \le \frac{12^{j-2}}{\pi}\Big(\int_0^{12} v|W(v)|\,\mathrm{d}v + \sum_n a_n \log n\Big) < 10^6 \cdot 12^{j-2}. \qquad \square$$

**Lemma F.2** (Quadrature error)**.** *Let $0 \le x \le 60$ and $0 \le j \le 16$. The composite 20-point Gauss–Legendre rule with 96 panels of width $1/8$ on $[0, 12]$ approximates the contribution of the integral to $P_K^{(j)}(x)$ with absolute error at most $10^{-14} \cdot 13^j$.*

*Proof.* Let $F_j(z)$ be the $j$th derivative in $x$ of the integrand of (B.1), with its canceling terms kept together at $z = 0$; here $m(z) = (1 - z/12)\cos(\pi z/12) + \pi^{-1}\sin(\pi z/12)$ is the entire continuation of $m$ from $[0, 12]$. Each $F_j$ is holomorphic on the closed $\frac{1}{2}$-neighborhood of $[0, 12]$, and we first show that $|F_j(z)| \le 10^{31} \cdot 13^j$ there, a deliberately loose bound. On this neighborhood $|z| < 13$, $|\operatorname{Im} z| \le \frac{1}{2}$, $|m(z)| < 4$ and $|m'(z)| < 2$. The function $D(z) = (1 - e^{-2z})/(2z) = \int_0^1 e^{-2tz}\,\mathrm{d}t$ has $\operatorname{Re} D(z) \ge (\cos 1)e^{-25}$, so $|D(z)|^{-1} < 2 \cdot 10^{11}$. For $0 \le x \le 60$,

$$\Big|\frac{\sin(xz)}{z}\Big| \le 60e^{30} < 7 \cdot 10^{14}, \qquad \Big|\Big(\frac{\sin(xz)}{z}\Big)'\Big| \le 1800e^{30} < 2 \cdot 10^{16}.$$

For $j = 0, 1$ the numerators over $1 - e^{-2z}$ vanish at $z = 0$, and we divide them by $z$ using their derivatives on the segment from 0 to $z$. For $j \geq 2$ we use $F_j(z) = m(z)W(z)z^{j-1}\sin(xz + j\pi/2)$ and $|zW(z)| < 3 \cdot 10^{11}$. In both cases the bounds above give $|F_j(z)| \leq 10^{31} \cdot 13^j$.

On a panel of width $h = \frac{1}{8}$, Cauchy's estimate on the radius $R = \frac{1}{2}$ and the Gauss–Legendre remainder formula bound the error by $M_j h\,(h/R)^{40}/(41\binom{40}{20}^2)$ with $M_j = 10^{31} \cdot 13^j$. Summing over the 96 panels gives at most $12 \cdot 10^{31} \cdot 13^j \cdot 4^{-40}/(41\binom{40}{20}^2) < 10^{-14} \cdot 13^j$, and the factor $1/\pi$ only reduces it. □

The quadrature nodes are enclosed by sign-bracketed bisection for the roots of the Legendre polynomial $P_{20}$, and the weights by the formula $2(1 - r^2)/(20^2 P_{19}(r)^2)$ at each node $r$; as a check, the rule encloses all moments through degree 39. The uncertainty in the nodes and weights is thus part of the interval evaluations, in addition to the error of Lemma F.2.

On the origin interval $0 < x \leq \frac{1}{10}$ we use that $P_K$ is odd. Taylor's theorem and Lemma F.1 give

$$\frac{P_K(x)}{x} \geq P_K'(0) - \sum_{\substack{3 \leq j \leq 15 \\ j \text{ odd}}} \frac{|P_K^{(j)}(0)|}{j!}\, 10^{-(j-1)} - \frac{10^6 \cdot 12^{15}}{17!}\, 10^{-16}, \tag{F.1}$$

and the interval evaluation of the right side exceeds $4.30 \cdot 10^{-7}$. On each of the 599 cells of Proposition B.3 we enclose $P_K^{(j)}(c_i)$ for $0 \leq j \leq 16$, and the resulting lower bound exceeds $10^{-8}\min((i+1)/10, 1)$.

F.2. **The verified-height certificates.** The certificates from verified zeros need only scalar inequalities, because monotonicity reduces each estimate in their proofs to an endpoint value. For Theorem 3.3 these are the endpoint and monotonicity inequalities of Appendix C.5, such as $\varphi(\log 10^{11}) > 0.16$, and the bounds on the counting envelopes of Appendix C.4. For Theorem 3.4 they are the inequalities of Appendix D at $A = 40$ and at $X = \frac{99}{100}T_*$, the bracket for $A(T_*)$, and the half-variance at $T_*$. We evaluate each of them in outward-rounded interval arithmetic, with exact rational or decimal-string inputs, and we accept an inequality only when the whole computed interval has the asserted sign. The inequalities in the proof of Theorem C.7 can be checked by hand.