EXACT PROJECTOR-RANK THRESHOLDS FOR STATE-INDEPENDENT CONTEXTUALITY

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ABSTRACT. For every integer $r \geq 1$ we construct a finite exclusivity graph F_r with the following two properties. First, F_r has a projective representation by pairwise distinct projectors of rank r that exhibits state-independent contextuality. Second, no projective representation of F_r by projectors of any common rank $s < r$, in any finite dimension, exhibits state-independent contextuality; in every such representation the maximally mixed state is noncontextual. This establishes, for every r, the "rank efficient" scenarios sought by Höhn, Xu and Kleinmann, whose Section 7 reports numerical evidence for specific examples at $r = 2, 3$. The graph is the disjunctive product of an odd cycle C_{2r+1} with the orthogonality structure of a rank-one contextuality set whose fractional chromatic number exceeds its dimension by an arbitrarily small amount. Such sets are obtained from the infinite family of Kochen–Specker parity proofs in $\mathbb{R}^4$ of Elford and Lisoněk, together with a coning construction that raises the dimension by one.

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1. INTRODUCTION

A finite family of projectors exhibits *state-independent contextuality* (SIC) if, for every quantum state, the outcome probabilities are incompatible with every noncontextual hidden-variable model [13, 1]. In the graph-theoretic approach [4, 9, 3] the relevant datum is the exclusivity graph, which records which projectors are orthogonal. Most known SIC sets consist of rank-one projectors. Xu, Yu and Kleinmann [12] showed that there are exclusivity graphs for which SIC is possible with projectors of rank two but impossible with rank one.

Höhn, Xu and Kleinmann [6] gave a systematic construction, based on the disjunctive graph product with odd cycles, of graphs that admit SIC at some rank but at no rank up to a prescribed r, with explicit examples for $r \leq 5$, and conjectured that such examples exist for every r. This conjecture was recently proved by Westberg [11], using the arc graphs from Kohlmeyer and Kruer's solution of Erdős problem 108 [7, 8]. The prescribed-exact-rank question is separate from this arbitrary-threshold result. In [6, Section 7] the authors seek graphs that are *rank efficient*, "in the sense that rank r is sufficient for SIC, while for $r - 1$ no representation with SIC can be found", and report numerical evidence for $r = 2$ and $r = 3$. Westberg states that his theorem does not "determine a prescribed exact minimum rank" [11, Section 6]. The arbitrary-threshold constructions do not identify a prescribed first successful rank. Westberg also excludes unequal ranks bounded by the threshold and supplies a faithful successful representation; our exact-rank result below concerns uniform ranks and does not assert faithfulness.

We settle the question for every r.

Theorem 1. *For every integer $r \geq 1$ there is a finite graph F_r such that:*

(i) *F_r has a projective representation $(\Pi_v)_v$ on some $\mathbb{C}^p$ by pairwise distinct projectors of rank r, and weights $W_v \geq 0$, with*

$$\sum_v W_v \Pi_v = \lambda \mathbf{1} \qquad \text{and} \qquad \alpha(F_r, W) < \lambda.$$

In particular this representation exhibits state-independent contextuality.

(ii) *For every $1 \leq s < r$ and every projective representation of F_r by projectors of rank s on a finite-dimensional Hilbert space, the probability vector of the maximally mixed state lies in $\mathrm{STAB}(F_r)$. In particular no such representation exhibits state-independent contextuality.*

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For $r \geq 2$ one may take $F_r = B \vee C_{2r+1}$, where B is described in Section 4; then $p = D(2r+1)$, where $D \geq 4$ is the least integer coprime to r, and the SIC ratio of the representation in (i) is at least $1 + \frac{4}{D(N-1)}$ for a parameter N of order r^2.

Westberg excludes low ranks through a large chromatic number, but his result does not establish prescribed exact thresholds. The present proof does not use the Kohlmeyer–Kruer theorem. We use the rigidity of projective representations of odd cycles and the integrality of the dimension. The idea is as follows. If B contains a clique of size D, any rank-s representation of $B \vee C_{2r+1}$ contains a rank-sD representation of C_{2r+1}, which forces the dimension to be at least $2sD + \lceil sD/r \rceil$. If $\gcd(D, r) = 1$ and $s < r$ the rounding gains at least $1/r$. If moreover the fractional chromatic number of B exceeds D by a sufficiently small amount, this gain pushes the dimension to at least $s\,\chi_f(B \vee C_{2r+1})$, and then the maximally mixed state is noncontextual. At rank r there is no rounding and a tensor-product representation is contextual. The same template underlies [6, Theorem 3]; the new ingredients are the use of the rounding term to pin the threshold at r itself, and the supply of rank-one SIC sets B with $\chi_f(B) - D$ arbitrarily small in every dimension $D \geq 4$.

2. Definitions

All graphs are finite and simple; all Hilbert spaces are finite-dimensional and complex. We follow [6].

A *projective representation* (PR) of a graph G on $\mathbb{C}^d$ assigns to each vertex v a nonzero orthogonal projector Π_v such that $\Pi_u \Pi_v = 0$ for every edge uv. It has *rank* s if $\operatorname{rank} \Pi_v = s$ for all v. Let $\mathrm{STAB}(G) \subset \mathbb{R}^{V(G)}$ be the convex hull of the indicator vectors of the independent sets of G, and for $w \colon V(G) \to \mathbb{R}_{\geq 0}$ let $\alpha(G, w)$ be the maximum of $\sum_{v \in S} w_v$ over independent sets S. A PR *exhibits SIC* if for no density matrix ρ the vector $(\operatorname{tr} \rho \Pi_v)_v$ lies in $\mathrm{STAB}(G)$. If $\sum_v W_v \Pi_v \succeq \lambda \mathbf{1}$ with $\lambda > \alpha(G, W)$, then every state violates the inequality $\sum_v W_v p_v \leq \alpha(G, W)$, which is valid on $\mathrm{STAB}(G)$, so the PR exhibits SIC.

The fractional chromatic number $\chi_f(G)$ is the least total weight of a fractional colouring, that is, of a function $y \geq 0$ on independent sets with $\sum_{S \ni v} y_S \geq 1$ for every v. The *disjunctive product* $G \vee H$ has vertex set $V(G) \times V(H)$, with $(u, v) \sim (u', v')$ if $u \sim u'$ or $v \sim v'$. The cycle C_n has vertex set $\mathbb{Z}_n$ and edges $\{j, j+1\}$.

3. Lemmas

Lemma 2 (Odd cycles)**.** *Let $(Q_j)_{j \in \mathbb{Z}_{2r+1}}$ be projectors of rank m on $\mathbb{C}^d$ with $Q_j Q_{j+1} = 0$ for all j. Then $d \geq 2m + \lceil m/r \rceil$.*

Proof. Let V_j be the range of Q_j and $T_j = V_0 \cap V_2 \cap \cdots \cap V_{2j}$ for $0 \leq j \leq r$. Both $T_j \subseteq V_{2j}$ and V_{2j+2} are contained in $V_{2j+1}^{\perp}$, which has dimension $d - m$. Hence $\dim T_{j+1} \geq \dim T_j + m - (d - m)$, and so $\dim T_r \geq m - r(d - 2m)$. Since $2r$ and 0 are adjacent, $T_r \subseteq V_0 \cap V_{2r} = \{0\}$. Thus $d \geq 2m + m/r$, and d is an integer. □

This is the dimension count behind the value $2 + 1/r$ of the projective rank of C_{2r+1} [10, Theorem 6.15.3, p. 140], credited there to joint work with Laura Mančinska; it sharpens [6, Lemma 4].

Lemma 3 ([11, Lemma 2.2])**.** *If $0 \leq t \leq 1/\chi_f(G)$, then $t\,\mathbf{1} \in \mathrm{STAB}(G)$.*

Proof. Normalising an optimal fractional colouring gives a random independent set containing each vertex v with probability $m_v \geq 1/\chi_f(G)$. Retain each of its vertices independently with probability t/m_v. The result is a random independent set that contains each vertex with probability t. □

Lemma 4 (Products, cf. [6, Theorem 1])**.** (a) $\alpha(G \vee H, w \otimes a) \leq \alpha(G, w)\,\alpha(H, a)$ *for weights* $w, a \geq 0$. (b) $\chi_f(G \vee H) \leq \chi_f(G)\,\chi_f(H)$.

Proof. (a) The coordinate projections S_G, S_H of an independent set S of $G \vee H$ are independent and $S \subseteq S_G \times S_H$. (b) If I, J are independent then so is $I \times J$; take the product of two fractional colourings. □

Lemma 5 (The cycle factor, [6, Section 4])**.** *Index the standard basis of $\mathbb{C}^{2r+1}$ by $\mathbb{Z}_{2r+1}$ and let E_j be the projector onto the coordinates $jr, jr+1, \dots, jr+r-1$. Then (E_j) is a PR of C_{2r+1} of rank r by pairwise distinct projectors, $\sum_j E_j = r\,\mathbf{1}$, $\alpha(C_{2r+1}) = r$ and $\chi_f(C_{2r+1}) \le (2r+1)/r$.*

Proof. The coordinate blocks of E_j and E_{j+1} are consecutive of total length $2r < 2r+1$, hence disjoint. As $\gcd(r, 2r+1) = 1$, the blocks are pairwise distinct and each coordinate c lies in exactly r of them. The set I_c of those j is independent, because adjacent blocks are disjoint. The $2r+1$ sets I_c cover each vertex r times, so $\chi_f \le (2r+1)/r$. Finally $\alpha(C_{2r+1}) = r$. □

Definition 6. A *datum of dimension D with parameters (λ, α_0, T)* consists of a graph B, unit vectors $x_b \in \mathbb{C}^D$ ($b \in V(B)$) spanning pairwise distinct rays, with $P_b = |x_b\rangle\langle x_b|$, and weights $w_b \ge 0$, such that

(D1) $x_b \perp x_{b'}$ for every edge bb' of B;
(D2) $\sum_b w_b P_b = \lambda\,\mathbf{1}$;
(D3) $\alpha(B, w) \le \alpha_0 < \lambda$;
(D4) B contains a clique of size D;
(D5) $\chi_f(B) \le T$.

By (D1)–(D3) a datum is a rank-one SIC set. By (D4), $\chi_f(B) \ge D$; we need data with $T - D$ small.

Lemma 7 (Base data)**.** *Let $p, q \ge 3$ be odd and coprime and $N = pq$. There is a datum of dimension 4 with parameters $\lambda = N/2$, $\alpha_0 = (N-1)/2$ and $T = 4 + \frac{4}{N-1}$.*

Proof. By Elford and Lisoněk [5, Theorem 1 and Proposition 1] there are an orthogonal matrix M of order N on $\mathbb{R}^4$, vectors $a, b \in \mathbb{R}^4$, and an integer ρ with $\rho \equiv 1 \pmod p$ and $\rho \equiv -1 \pmod q$, such that for every $i \in \mathbb{Z}_N$

$$\beta_i = \{M^i a,\ M^{i-1}a,\ M^i b,\ M^{i-\rho}b\}$$

is an orthogonal basis of $\mathbb{R}^4$, and the $2N$ vectors $M^i a$, $M^i b$ ($i \in \mathbb{Z}_N$) are pairwise linearly independent. Thus $M^i a$ lies in exactly the bases β_i, β_{i+1} and $M^i b$ in exactly $\beta_i, \beta_{i+\rho}$. Let Γ be the graph on $\mathbb{Z}_N$ with edges $\{i, i+1\}$ and $\{i, i+\rho\}$; it is simple and 4-regular because $\rho \not\equiv \pm 1 \pmod N$ and N is odd. Identify the edge $\{i, i+1\}$ with the ray of $M^i a$ and $\{i, i+\rho\}$ with the ray of $M^i b$; then the four edges at the vertex i are the four rays of β_i. Let $B = L(\Gamma)$ be the line graph, x_e the normalised vector of the edge e, and $w \equiv 1$.

(D1) Adjacent edges lie in a common basis. (D2) Each ray lies in exactly two bases, so $\sum_e P_e = \frac{1}{2}\sum_i \sum_{e \ni i} P_e = \frac{N}{2}\mathbf{1}$. (D3) An independent set of $L(\Gamma)$ is a matching of Γ, which has at most $(N-1)/2$ edges since N is odd. (D4) The four edges at a vertex. (D5) Since $\gcd(\rho, N) = 1$, each of the two edge classes $\{i, i+1\}$ and $\{i, i+\rho\}$ is a Hamiltonian cycle of odd length N. A cycle of length N has a matching with $(N-1)/2$ edges, and its N rotations cover every edge of the cycle exactly $(N-1)/2$ times. Giving weight $2/(N-1)$ to each of these $2N$ matchings yields a fractional colouring of B of total weight $4N/(N-1)$. □

Remark 8. The 18-ray set of Cabello, Estebaranz and García-Alcaine [2] has $N = 9$ bases and the same parameters; its basis graph is $K_3 \,\square\, K_3$. Pulling its rays back along the covering maps $C_{3a} \,\square\, C_{3a} \to C_3 \,\square\, C_3$ (a odd) gives an alternative proof of Theorem 1 without the distinctness of the projectors. The family of [5] gives distinct rays.

Lemma 9 (Coning)**.** *From a datum of dimension D with parameters (λ, α_0, T) one obtains a datum of dimension $D+1$ with parameters $(D\lambda + \alpha_0,\ (D+1)\alpha_0,\ T+1)$.*

Note that $(D\lambda + \alpha_0) - (D+1)\alpha_0 = D(\lambda - \alpha_0) > 0$ and that $T - D$ is unchanged.

Proof. Let $e_0, \dots, e_D$ be the standard basis of $\mathbb{C}^{D+1}$ and $J_i \colon \mathbb{C}^D \to e_i^\perp$ the coordinate embedding. Since unitary images of a datum are data, we may choose unitaries $U_0, \dots, U_D$ of $\mathbb{C}^D$ such that no vector $U_i x_b$ has a zero coordinate; the unitaries violating this for a given i, b and coordinate form a nowhere dense closed subset of the unitary group, and there are finitely many conditions.

The graph B' consists of $D+1$ disjoint copies of B, with vertices (i,b), and further vertices $a_0,\dots,a_D$. Its edges are the copies of the edges of B, the pairs $a_i \sim (i,b)$ for all b, and the pairs $a_i \sim a_j$ for $i \neq j$. Put $x'_{(i,b)} = J_i U_i x_b$, $x'_{a_i} = e_i$, $w'_{(i,b)} = w_b$ and $w'_{a_i} = \alpha_0$.

Distinct rays. The vector $x'_{(i,b)}$ vanishes in coordinate i and in no other coordinate. Hence rays from different copies differ, rays in the same copy differ because J_iU_i is injective, and no $x'_{(i,b)}$ is a multiple of some e_j because $D+1 \geq 3$.

(D1) J_iU_i is an isometry with range $e_i^\perp$. (D2) $\sum_i \big[\lambda(\mathbf{1} - |e_i\rangle\langle e_i|) + \alpha_0 |e_i\rangle\langle e_i|\big] = (D\lambda + \alpha_0)\mathbf{1}$. (D3) An independent set of B' contains at most one of the a_i. If it contains none, its weight is at most $(D+1)\alpha_0$. If it contains a_i, it contains no vertex of copy i, so its weight is at most $\alpha_0 + D\alpha_0$. (D4) A D-clique of copy i together with a_i.

(D5) We have $T \geq \chi_f(B) \geq D$. Regard a fractional colouring of B of total weight T as a piecewise constant map from $[0,T)$ to independent sets under which every vertex is present for total time at least 1. Divide $[0,T+1)$ into unit intervals $I_0,\dots,I_D$ and a final interval of length $T-D$. During I_i colour a_i. Copy j runs its schedule during the D intervals I_i, $i \neq j$, and during the final interval, which have total length T. At every time the coloured set is a union of independent sets of distinct copies together with at most one vertex a_i, whose copy i is then idle; it is therefore independent. Every vertex is coloured for time at least 1, so $\chi_f(B') \leq T+1$. □

4. Proof of Theorem 1

For $r=1$ any rank-one SIC set, for instance that of Lemma 7, satisfies (i), and (ii) is vacuous. Let $r \geq 2$. Let $D \geq 4$ be the least integer with $\gcd(D,r)=1$, and choose odd coprime $p,q \geq 3$ with

$$N = pq \geq 4(r-1)(2r+1)+1.$$

Apply Lemma 7, and then Lemma 9 $D-4$ times. This gives a datum B of dimension D with parameters (λ,α_0,T), where $\lambda > \alpha_0$ and

$$T - D = \frac{4}{N-1} \leq \frac{1}{(r-1)(2r+1)}. \tag{1}$$

Put $F_r = B \vee C_{2r+1}$.

Proof of (i). On $\mathbb{C}^D \otimes \mathbb{C}^{2r+1}$ let $\Pi_{(b,j)} = P_b \otimes E_j$, with E_j as in Lemma 5, and $W_{(b,j)} = w_b/r$. These projectors have rank r and are pairwise distinct, because the P_b are pairwise distinct and so are the E_j. If $(b,j) \sim (b',j')$ then $P_bP_{b'} = 0$ or $E_jE_{j'} = 0$, so (Π_v) is a PR of F_r. By (D2) and Lemma 5,

$$\sum_v W_v \Pi_v = \Big(\sum_b w_b P_b\Big) \otimes \Big(\frac{1}{r}\sum_j E_j\Big) = \lambda\,\mathbf{1}.$$

By Lemma 4(a), (D3) and $\alpha(C_{2r+1}) = r$ we get $\alpha(F_r, W) \leq \alpha_0 < \lambda$.

Proof of (ii). Let $1 \leq s < r$ and let (Π_v) be a PR of F_r of rank s on $\mathbb{C}^d$. Let K be a clique of size D in B and put $Q_j = \sum_{b\in K} \Pi_{(b,j)}$ for $j \in \mathbb{Z}_{2r+1}$. For fixed j the vertices (b,j), $b \in K$, are pairwise adjacent, so Q_j is a projector of rank sD. If $j \sim j'$ then every (b,j) is adjacent to every (b',j'), so $Q_jQ_{j'} = 0$. By Lemma 2,

$$d \geq 2sD + \Big\lceil \frac{sD}{r} \Big\rceil.$$

Since $\gcd(D,r)=1$ and $0<s<r$, the integer r does not divide sD, hence $\lceil sD/r \rceil \geq sD/r + 1/r$ and

$$d \geq \frac{sD(2r+1)}{r} + \frac{1}{r}.$$

On the other hand, by Lemma 4(b), Lemma 5 and (D5), $\chi_f(F_r) \leq T(2r+1)/r$, and by (1)

$$s\,\chi_f(F_r) \leq \frac{sD(2r+1)}{r} + \frac{s(T-D)(2r+1)}{r} \leq \frac{sD(2r+1)}{r} + \frac{1}{r} \leq d.$$

For the maximally mixed state every vertex has probability $s/d \leq 1/\chi_f(F_r)$. By Lemma 3 the probability vector lies in $\mathrm{STAB}(F_r)$. □

Remark 10 (Coprimality). If $\gcd(D, r) = g > 1$ then for $s = r/g < r$ the rounding term vanishes and the argument for (ii) fails. Parity proofs exist only in even dimension [5], so an odd D, needed for even r, is not available from Lemma 7 alone. This is the purpose of Lemma 9.

Remark 11 (Size). With $p = 3$ the smallest admissible parameters give, for $r = 2, 3, 4, 5, 6$: $N = 21, 57, 111, 177, 267$; $|V(F_r)| = 1075, 798, 10035, 3894, 34775$; Hilbert space dimension $25, 28, 45, 44, 65$; and SIC ratio at least 26/25, 57/56, 277/275, 177/176, 667/665. No attempt was made to optimise these.

5. Scope and open points

- *Uniform rank.* Part (ii) concerns representations in which all projectors have the same rank $s < r$, the setting of [6]. Whether a representation of F_r with unequal ranks, all smaller than r, can exhibit SIC is not decided here.
- *Faithfulness.* The projectors in (i) are pairwise distinct, but the representation is not claimed to be faithful: non-adjacent vertices of F_r may carry orthogonal projectors.
- *Ranks above r.* Nothing is claimed for ranks $s > r$ that are not multiples of r.
- *Optimality.* The number of projectors and the size of the violation are far from optimal.

Verification

A separate Python verification script was run for $r = 2, \ldots, 6$ and for the base pairs $(p, q) = (3, 5), (3, 7), (5, 7), (5, 9), (7, 11)$. It checks edge orthogonality with absolute tolerance 10^{-9}, the weighted matrix identity (D2) with absolute tolerance 10^{-7}, and distinctness by requiring every off-diagonal absolute inner product of normalised rays to be below $1 - 10^{-9}$. These are floating-point consistency checks, not exact certificates of the vector identities. In exact integer or rational arithmetic it checks (D4), the independence and covering property of every colour class in the explicit fractional colouring, and $2sD + \lceil sD/r \rceil \geq sT(2r + 1)/r$ for every $s < r$ in those instances. All checks passed. The script is retained separately from the Hexagon deposit.

The mathematical proof does not depend on these computations. In particular, condition (D3), the general lemmas and the all-ranks conclusion are justified by the proofs above, not by a finite test. This paper has not been formalised in a proof assistant or peer reviewed.

AI disclosure

The author reports that the argument was generated entirely by Claude Fable 5.2 (Anthropic), including the construction and its mathematical justification. The choice of problem, the replacement of the original base construction by the family of Elford and Lisoněk, and the adversarial checks of novelty and correctness were subsequently developed through further interaction with the model. The model also produced the verification code, the exposition and the LaTeX manuscript. This disclosure describes the provenance of the argument; it does not supersede the attribution of previously published mathematical methods cited in the paper or constitute independent verification of the proofs. The human author retains responsibility for the correctness of the mathematical claims, the accuracy of this disclosure, and the decision to submit or disseminate the work. OpenAI Codex subsequently assisted with source verification, editorial corrections, rerunning the numerical and combinatorial checks, compilation, and preparation of this submission package on 11 November 2026. This further AI-assisted review is not independent human verification.

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