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On the local geometry of the tight 39-point code in the octonionic projective plane

Robert R. MacGregor

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Abstract

Cohn, Kumar and Minton constructed a four-parameter family of tight 39-point codes in the octonionic projective plane. We show that near each member outside a closed nowhere-dense set of resonant parameters, the labeled tight codes modulo F_4 form exactly four ten-dimensional families, any two meeting in the four-parameter family, although the first-order deformations of the member form a 28-dimensional space modulo F_4. The originally constructed code is resonant, and we do not determine the tight codes near it; but every neighborhood of it in the space of unlabeled tight 39-point codes modulo F_4 contains a nonempty open subset homeomorphic to an open subset of $\mathbb{R}^{10}$, so any neighborhood of it that is a manifold has dimension ten.

1 Introduction

The points of the octonionic projective plane $\mathbb{O}P^2$ are the trace-one idempotents of the Albert algebra $J = H_3(\mathbb{O})$. Two points have an *overlap* $\langle p, q \rangle = \operatorname{tr}(p \circ q)$, the inner product of Cohn, Kumar and Minton[1], or $1 - \rho^2$ for the chordal distance ρ of p and q. A *code* is a finite set of points. By a linear programming bound, Lemma 2.7 of Cohn, Kumar and Minton, a code in which distinct points have overlap at most $1/3$ has at most 39 points; a code attaining this number is a *tight 39-point code*. These authors record that Hoggar[2] conjectured that such a code exists, and in their Theorem 8.3 they constructed one[1]: thirteen triples of mutually orthogonal points, with overlap $1/3$ between points of different triples. They show that it is tight and universally optimal[3], that is, it minimizes potential energy for every completely monotonic function of squared chordal distance.

Four unit complex numbers can be inserted into its defining formula[1], giving a four-parameter family of tight codes that remains four-dimensional modulo the isometry group F_4 of $\mathbb{O}P^2$. Write $P(\xi)$ for the member with parameters $\xi \in (S^1)^4$; the code they constructed is $P(1)$, the *original code*.

This paper describes the local geometry of the space of tight 39-point codes around this family. Outside a closed nowhere-dense set of *resonant* parameters, the tight codes near $P(\xi)$, with their points labeled, are, modulo F_4, exactly the

members of four ten-dimensional families, any two of which meet exactly in the four-parameter family (Theorem 1.1). The original code is resonant, and our methods do not determine the codes near it. But nonresonant members accumulate at it, and it follows that any neighborhood of it in the space of unlabeled tight 39-point codes modulo F_4 that is a manifold has dimension ten (Theorem 1.3, Corollary 1.4).

1.1 Setting

The Albert algebra J is the exceptional Euclidean, or formally real, Jordan algebra. It has dimension 27 and a unit e of trace three. Its product is $x \circ y = (xy + yx)/2$, and its trace form is $\langle x, y\rangle = \mathrm{tr}(x \circ y)$. The automorphism group of J is F_4; it has dimension 52 and preserves the trace form. Points are the primitive idempotents[4]. A *Jordan frame* is a triple of points with sum e and pairwise zero Jordan products. Two frames are *mutually unbiased*, as for bases of $\mathbb{C}^d$, when all nine overlaps between their points equal $1/3$.

The spaces studied are

$$\begin{gathered} C = \{(p_k) \in J^{39} : p_k \text{ a point, } \langle p_k, p_m\rangle \le 1/3 \ (k \ne m)\}, \\ Y = C/F_4, \qquad \mathcal{M} = Y/\mathfrak{S}_{39}, \end{gathered} \tag{1}$$

where the symmetric group $\mathfrak{S}_{39}$ permutes the labels and the quotients carry their quotient topologies. The *unlabeled space* $\mathcal{M}$ is the space of tight 39-point codes modulo F_4 in the sense of Cohn, Kumar and Minton.

Every element of C is a union of thirteen pairwise mutually unbiased Jordan frames in exactly one way, and the partition into frames is locally constant on C (Proposition 3.2, Lemma 3.3). Near a given code, C is therefore the real zero set $\mathcal{Z}$ of one polynomial map F of degree two on J^{39}, the residual of the *frame equations* (6).

The family rests on four subalgebras of J (Lemma 4.1). Let *frame* 0 be the frame of diagonal matrix units and D its span. There are subalgebras $A_0, \dots, A_3$, each a copy of $H_3(\mathbb{C})$ containing D, that are otherwise orthogonal to one another:

$$J = D \oplus N_0 \oplus N_1 \oplus N_2 \oplus N_3, \qquad A_i = D \oplus N_i.$$

A point that has overlap $1/3$ with the points of frame 0 has component $e/3$ in D, so two such points in different subalgebras have overlap $1/3$. The code $P(\xi)$, written out in (7), is frame 0 with three further frames in each subalgebra. These form with frame 0 a *complex quartet*: a complete set of four mutually unbiased bases of $\mathbb{C}^3$ (Figure 1). The parameters enter through diagonal conjugations within the subalgebras.

A parameter ξ is *nonresonant* if the linearized frame equations at $P(\xi)$ have a solution space of dimension 80, the least that occurs on the family. By Proposition 5.8 the resonant parameters are those at which one of twelve nonconstant monomials $\pm\xi_1^{a_1}\xi_2^{a_2}\xi_3^{a_3}\xi_4^{a_4}$, listed in Section 4, has cube one. The set $\mathcal{N} \subset (S^1)^4$ of nonresonant parameters is therefore open and dense.

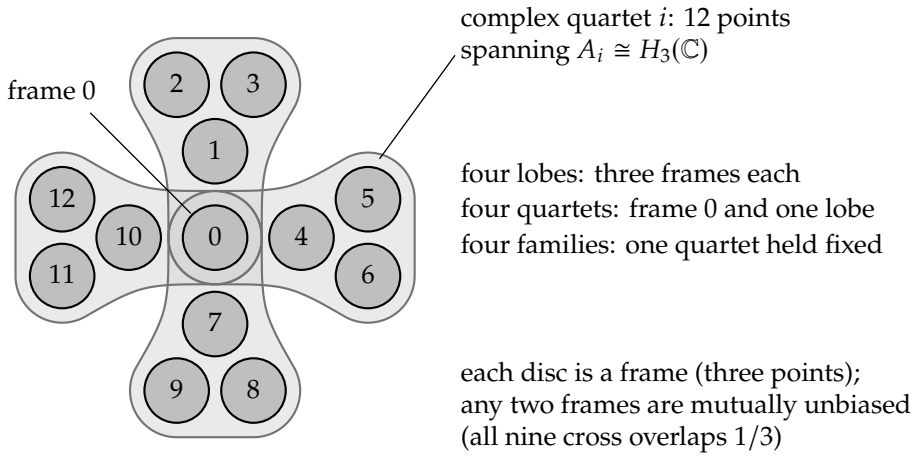


Figure 1: Each disc is a Jordan frame. Frames $3i+1$, $3i+2$, $3i+3$ make up *lobe* i; with frame 0 they form complex quartet i, which spans A_i.

1.2 Results

The local model is the *germ* at the origin of

$$\mathbb{R}^4 \times \bigcup_{i=0}^{3} W_i \;\subset\; \mathbb{R}^4 \times \mathbb{R}^{24}, \qquad W_i = \mathbb{R}^6_i \subset \bigoplus_{j=0}^{3} \mathbb{R}^6_j = \mathbb{R}^{24}, \tag{2}$$

four copies of $\mathbb{R}^{10}$ that meet pairwise in $\mathbb{R}^4 \times \{0\}$ and whose *transverse spaces* W_i are linearly independent.

Theorem 1.1 (Local solutions at nonresonant parameters)**.** *Let $\mathcal{Z}$ be the real zero set of the frame equations. At every $\xi \in \mathcal{N}$, the germ of $\mathcal{Z}$ at $P(\xi)$ in a local slice transverse to the F_4-orbit is homeomorphic to the germ of* (2) *at the origin. Its four branches are the families $M_0, \dots, M_3$ constructed in Sections 4 and 5, one for each complex quartet. Their common part is the slice image of the four-parameter family $\xi' \mapsto P(\xi')$ near ξ.*

Lean: `real germ`

The proof gives more: near $P(\xi)$ the slice has analytic coordinates in which its intersection with $\mathcal{Z}$ is, near the origin, the product of (2) with the origin of $\mathbb{R}^{973}$ (Propositions 5.11 and 6.8). The registered statement of Theorem 1.1 gives such a chart as a homeomorphism only; the analytic chart of Proposition 5.11, in which the families are coordinate subspaces, is registered separately.

In the family M_i the twelve points of quartet i stay fixed, up to an F_4 motion, while the other nine frames move. So every labeled tight code near $P(\xi)$ contains, up to F_4, one of the four complex quartets of $P(\xi)$.

The slice identifies its zero set with a neighborhood of $[P(\xi)]$ in Y (Section 8), so the theorem describes all labeled tight codes modulo F_4 near $P(\xi)$. The orbit $F_4 P(\xi)$ has dimension 52, so the *first-order deformations* (solutions of the linearized equations; infinitesimal flexes in rigidity theory) form modulo F_4 a space of dimension $80 - 52 = 28 = 4 + 4 \cdot 6$, the sum of the tangent spaces of the four families.

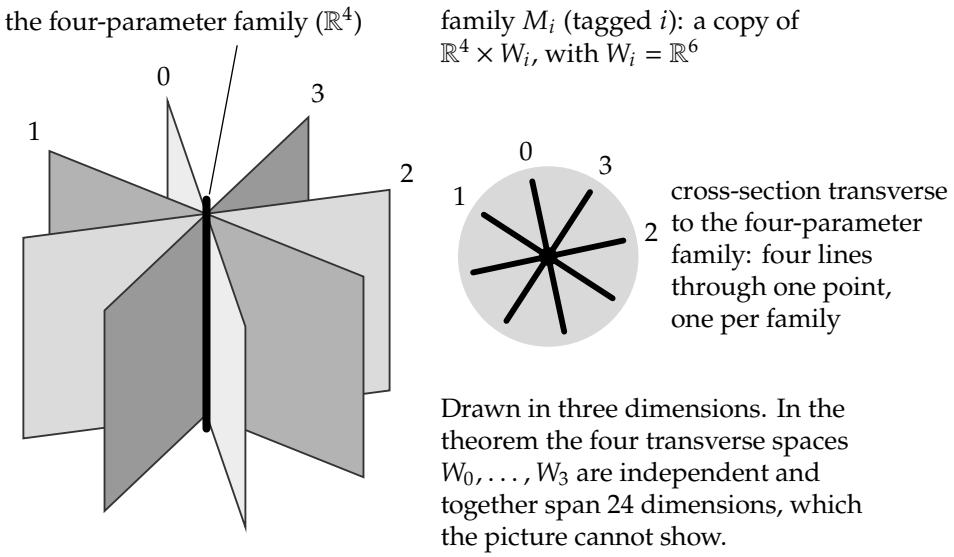


Figure 2: The local model (2): four ten-dimensional families meet along the four-parameter family, with independent transverse spaces.

The quadratic part of the equations obstructs all of them except those tangent to a single family (Figure 2).

In the next theorem the norms on J^{39} and on the residuals are any fixed norms; they affect only the constants.

Theorem 1.2 (Distance bound)**.** *Let $\xi \in \mathcal{N}$.*

(a) *There are an open neighborhood U_ξ of the orbit $F_4P(\xi)$ in J^{39} and a constant $C_\xi > 0$ such that*

$$\begin{aligned} \mathcal{Z} \cap U_\xi &= F_4(M_0 \cup \cdots \cup M_3) \cap U_\xi, \\ \operatorname{dist}(y, \mathcal{Z})^2 &\le C_\xi \|F(y)\| \qquad (y \in U_\xi). \end{aligned} \tag{3}$$

(b) *The exponent cannot be improved: for no $\alpha > 1/2$ does a bound* $\operatorname{dist}(y, \mathcal{Z}) \le C\|F(y)\|^\alpha$ *hold on a neighborhood of $P(\xi)$ in J^{39}.*

Part (b) holds because the linearization at $P(\xi)$ of any C^2 equations with zero set $\mathcal{Z}$ near $P(\xi)$ has solutions outside the *tangent cone* of $\mathcal{Z}$, the cone of limits of secant directions, and this excludes every exponent above $1/2$ (Lemma 7.1). Even on configurations whose thirteen triples are Jordan frames, no exponent above $1/2$ is possible (Section 7; proved in the text, not formalized).

Write $c_0 = [P(1)]$ for the class of the original code in $\mathcal{M}$.

Theorem 1.3 (Open patches near the original code)**.** *Every open neighborhood Ω of c_0 in $\mathcal{M}$ contains a nonempty open subset V of $\mathcal{M}$ homeomorphic to a nonempty open subset of $\mathbb{R}^{10}$.*

`Lean: open ten-patches`

The patch V need not contain c_0. A ten-dimensional family can lie inside a manifold of larger dimension (Figure 3(a)); a nonempty open set homeomorphic to an open subset of $\mathbb{R}^{10}$ cannot, by invariance of dimension.

Corollary 1.4 (Manifold neighborhoods)**.** *If a neighborhood of c_0 in $\mathcal{M}$ is homeomorphic to an open subset of $\mathbb{R}^n$, then $n = 10$. In particular the original code has no twenty-four-dimensional manifold neighborhood in $\mathcal{M}$.*

The last assertion is contrary to the neighborhood statement of Conjecture 8.4 of Cohn, Kumar and Minton[1].

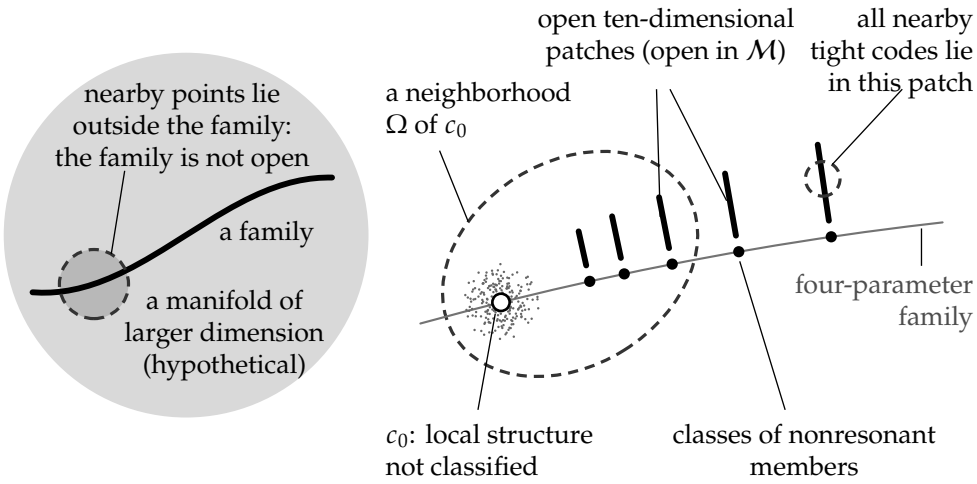


Figure 3: (a) A family inside a manifold of larger dimension is not open in it. (b) Every neighborhood Ω of c_0 in $\mathcal{M}$ contains a ten-dimensional patch open in $\mathcal{M}$ (heavy segments). Dots mark nonresonant members. The stippled shape at the original code is not determined. Schematic.

1.3 What is open

By (21) the linearized equations at the original code have a solution space of dimension 92, that is 40 modulo F_4. The four families can account for at most 28 of these dimensions, whereas the argument of Section 6 starts from coordinates in which they account for every first-order deformation. Along $\xi(t) = (e^{it}, e^{-it}, 1, e^{-it})$, with $t \in \mathbb{R}/2\pi\mathbb{Z}$, the dimension is also 92, except at the three values of t with $e^{3it} = -1$, where it is 104. So the resonance at the original code is not isolated in the family. The following questions are open.

(a) The germ of $\mathcal{M}$ at c_0, and at the other resonant members of the family. In particular the local dimension at c_0 is not determined: solution sets of dimension greater than ten passing through resonant members, away from the nonresonant ones, are not excluded.

(b) The finite group in the germ of $\mathcal{M}$ at a nonresonant member. That germ is the model (2) modulo an action of the group $\mathrm{Aut}(P(\xi))$ of automorphisms of J that permute the code (Remark 8.3); we determine neither the group nor its action.

(c) Whether the constants C_ξ stay bounded as ξ approaches a resonant parameter (Section 7).

(d) Tight 39-point codes away from the family. We make no claim that every tight 39-point code lies in one of the families described here.

1.4 Method and verification

Theorem 1.1, Theorem 1.2 in an equivalent form, and Corollary 1.4 for $n = 24$ are formalized in Lean 4 and registered (Section 9); gray notes in the margin link to the formal statements. The printed proof relies on two machine computations, both part of the formal development. The properties of the stresses (linear functionals on the residuals that annihilate the linearized equations) of Proposition 6.5 are finite polynomial identities over $\mathbb{Q}(i, \sqrt{3})$ with Laurent dependence on the parameters (Appendix C), checked exactly. The quartic identity (26) is checked by direct expansion. Section 9 says which further checks were made outside the formal development. Section 2 outlines the proof.

1.5 Related work

For spherical codes, Cohn, Jiao, Kumar and Torquato remark that there seems to be no simple method to turn an infinitesimal unjamming into an actual one[5]; here, at a nonresonant parameter, the first-order deformations modulo F_4 form a space of dimension 28, but each family through $P(\xi)$ has dimension ten. For complex Hadamard matrices the first-order count is the defect of Tadej and Życzkowski, which by their Theorem 4.7 is an upper bound for the dimension of a smooth family[6]. Barros e Sá and Bengtsson found perturbative evidence for two 13-dimensional families of Hadamard matrices of order 12 crossing along a nine-dimensional one, with defect $17 = 13 + 13 - 9$, on pp. 4 and 24–26 of their paper[7]. Theorem 1.1 describes a crossing of the same kind and proves that nothing else passes through the point.

In rigidity theory, Connelly's second-order rigidity asks for an equilibrium stress for each nontrivial first-order flex, and the prestress stability of Connelly and Whiteley (their Proposition 3.4.2 and Theorem 4.4.1) for one stress serving them all[8]. Lemma 6.1 asks for one stress field for each limiting direction along a family, in a situation those results do not cover: several exact families that cross.

For analytic equations some bound $\operatorname{dist} \le C\|F\|^\alpha$ is automatic by Łojasiewicz's inequality[9]. The theory of 2-regular maps[10] gives the exponent $1/2$ under a surjectivity condition in every critical direction (Theorem 4 of Izmailov and Solodov)[11], which makes the solution set a manifold away from the base point and so cannot hold where the zero set is locally (2).

2 Outline of the proof

Fix a nonresonant parameter ξ and write $P = P(\xi)$.

2.1 Step 1: one family for each quartet

Each quartet i gives a family M_i of tight codes through P with ten parameters modulo F_4, containing the four-parameter family.

Let θ_i be the automorphism of order three that fixes A_i and multiplies its orthogonal complement $M_3(\mathbb{C})$ by a primitive cube root of unity. Points having overlap $1/3$ with every point of quartet i are given by matrices in $SU(3)$, their frames are the orbits of θ_i, and two frames are mutually unbiased exactly when their matrices U, V are orthogonal, $\operatorname{tr}(U^*V) = 0$ (Lemma 4.2). So the other nine frames are nine pairwise orthogonal matrices in $SU(3)$. The shift-and-multiply construction[12] supplies such matrices with $8 + 8 + 2$ parameters; eight of them are automorphisms of J (Lemma 4.3), which leaves ten. Up to an automorphism, each $P(\xi')$ with ξ' near ξ is a member of this family (Lemma 5.10), so the family contains the four-parameter family.

2.2 Step 2: first order

Modulo F_4, the first-order deformations of P form a space $\overline{\mathcal{T}}$ of dimension 28, the sum of the tangent spaces of the four families.

With quartet i held fixed, the linearized problem is solved by a Fourier transform over $\mathbb{Z}/3$ (Lemma 5.1). Every solution is tangent to the family of quartet i unless an entry of the triple $d_i(\xi)$ of (16) has cube one, which makes ξ resonant.

The θ_i preserve the code and generate a group $K \simeq (\mathbb{Z}/3)^2$ (Lemma 5.2). The deformations fixed by θ_i are those just found, and counting characters of K gives

$$\overline{\mathcal{T}} = \overline{\mathcal{T}}^K \oplus W_0 \oplus W_1 \oplus W_2 \oplus W_3, \qquad 28 = 4 + 4 \cdot 6$$

(Proposition 5.8, Table 1). The first summand is tangent to the four-parameter family. Each W_i is a complex three-space; with the first summand it makes up the tangent space of M_i.

A slice transverse to the F_4-orbit of P has analytic coordinates

$$(\lambda, u, \eta) \in \mathbb{R}^4 \times (\mathbb{C}^3)^4 \times \mathbb{R}^{973}, \qquad u = (u_0, u_1, u_2, u_3),$$

in which M_i is the subspace where only λ and the triple u_i may be nonzero (Proposition 5.11).

2.3 Step 3: second order

Once η is removed, the quadratic part of the equations vanishes on a first-order deformation only if at most one u_i is nonzero.

The equations

$$xy = yz = zx = 0$$

have zero derivative at the origin, yet their solution set is the three coordinate axes (Figure 4), the analogues of the four families.

The implicit function theorem removes η. The remaining equations vanish on the four families and have no linear term. A *stress* at a solution is a linear functional

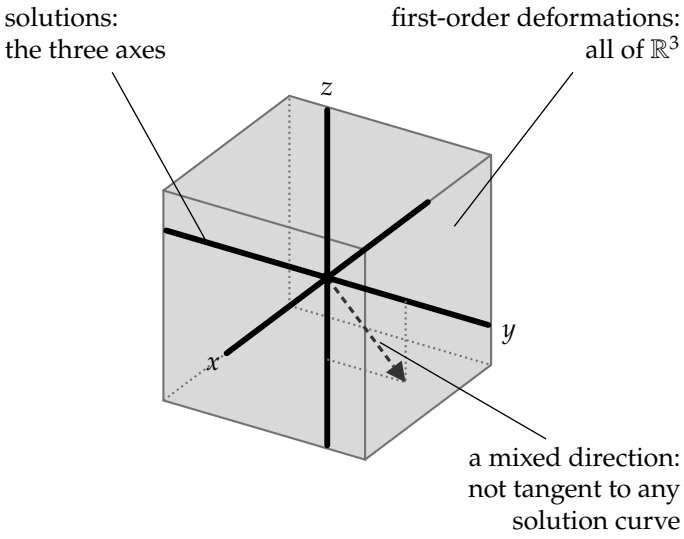


Figure 4: The model $xy = yz = zx = 0$; the dashed direction is tangent to no solution curve.

on the residuals of the frame equations that annihilates their linearization there. Composed with the frame equations it gives a function, its *energy*, which is a quadratic form in the displacement because the frame equations have degree two (Definition 6.3).

Proposition 6.5 gives twenty-four complex stresses at P. On transverse directions their energies are the components of quadratic maps, two for each quartet (Appendix C). Every monomial of these maps multiplies entries of two different triples u_j. The quartic identity (26), verified by expansion, writes the sum of the six products $\|u_i\|^2\|u_j\|^2$ as a sum of terms each containing such a component, as

$$x^2y^2 + y^2z^2 + z^2x^2 = (xy)^2 + (yz)^2 + (zx)^2$$

does for the model. So these energies all vanish only if at most one triple is nonzero, and the tangent cone of the solution set is the model (2). Part of (26) is the multiplicativity of the octonion norm: up to signs and conjugations, and after adding products of entries of one triple, the first maps of the four quartets together give the off-diagonal entries of the square of an element of J with zero diagonal, as computer algebra verifies (Appendix C).

2.4 Step 4: all orders

Near P in the slice the solutions are exactly the points of the four families (*Proposition 6.8*)*; this proves Theorem 1.1.*

A tangent cone does not determine a solution set: $y(y-x^2) = 0$ has the branches $y = 0$ and $y = x^2$, but its lowest-order term y^2 vanishes only on their common tangent (Figure 5). Solutions could likewise leave a family at higher order.

Lemma 6.1 excludes this. It asks for a stress at every point of each family, continuous in the point. Some directions leaving the family are excluded already

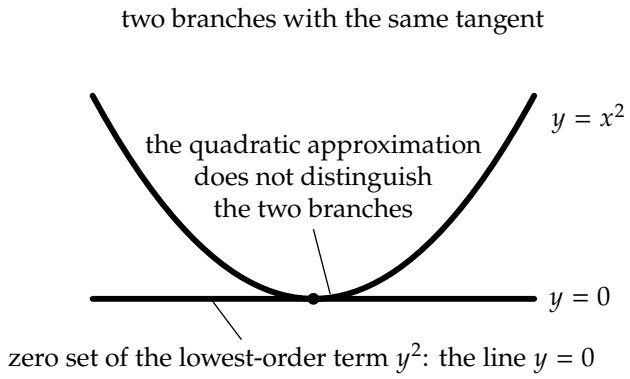


Figure 5: The curve $y(y - x^2) = 0$ and the zero set of its lowest-order term y^2.

by the mixed second-order terms; on the others, the energy of the stress at P must be positive. The proof compares a supposed further solution with the nearest point of the nearest family and rescales the difference to unit size; the stress there removes the linear term and leaves a positive energy, a contradiction.

Such stresses exist because, for each quartet, three of the twenty-four have energies involving only its twelve points, fixed along its family; the positivity is (30), a consequence of (26). The toy curve has none: the derivative leaving $y = 0$ is $-x^2$, so a continuous multiplier annihilating it vanishes at the origin.

The same argument bounds the distance to the solution set by the square root of the residual (Theorem 1.2).

2.5 Step 5: the unlabeled space and the original code

Every neighborhood of the original code in $\mathcal{M}$ contains an open ten-dimensional manifold (Theorem 1.3), whence Corollary 1.4 by invariance of dimension.

By Theorem 1.1, points of M_i off the four-parameter family have ten-dimensional manifold neighborhoods in Y. Forgetting the labels divides by $\mathfrak{S}_{39}$; this is a local homeomorphism where the stabilizer is locally constant, which gives open patches in $\mathcal{M}$ (Lemma 8.1). Nonresonant parameters tend to $\xi = 1$.

3 Recovering frames from tight codes

A reader who takes the frame equations (6) as given can go on to Section 4.

The trace form on J is positive definite and associative:

$$\langle x \circ y, z \rangle = \langle x, y \circ z \rangle.$$

In this section $\|x\|^2 = \langle x, x \rangle$. The trace-zero subspace J_0 has dimension 26. For a point p, idempotency gives $\|p\|^2 = 1$ and $\langle p, e \rangle = 1$.

The linear programming bound used here is the one formulated by Cohn, Kumar and Minton[1] in their Section 2.3 (Proposition 2.5 and Lemma 2.7). In the

overlap variable $t = \langle p, q \rangle$ its degree-two certificate (the auxiliary polynomial) is

$$t\,(t - 1/3).$$

For the equality case, Lemma 2.7 of Cohn, Kumar and Minton applied to the certificate shows that a tight 39-point code has all overlaps in $\{0, 1/3\}$ and is a projective 2-design[1]. Such a code carries a two-class association scheme[13], so its graph of orthogonal pairs is strongly regular. Cohn, Kumar and Minton report that Hoggar[2] classified the admissible parameters of such graphs.

We give a direct proof from the Gram matrix, of the kind that gives the Welch bound.

Lemma 3.1. *If p, q are idempotents, then $\langle p, q \rangle \geq 0$.*

Lean: `positive overlap`

Proof. Write $L_p x = p \circ x$. Associativity of the trace form makes L_p self-adjoint. The polarized Jordan identity, specialized to $p \circ p = p$, gives

$$2L_p^3 - 3L_p^2 + L_p = 0.$$

Hence L_p is positive semidefinite, with eigenvalues in $\{0, 1/2, 1\}$, the Peirce eigenvalues. Since $q \circ q = q$,

$$\langle p, q \rangle = \langle p, q \circ q \rangle = \langle L_p q, q \rangle \geq 0. \qquad \square$$

Proposition 3.2 (The 39-point bound and frame decomposition)**.** *Let $p_1, \ldots, p_N$ be points such that $\langle p_i, p_j \rangle \leq 1/3$ for $i \neq j$. Then $N \leq 39$. If $N = 39$, the points are in exactly one way the union of thirteen pairwise mutually unbiased Jordan frames. Conversely, the points of thirteen pairwise mutually unbiased Jordan frames satisfy the hypothesis with $N = 39$.*

Lean: `frame equality`

Proof. The bound. Center the points, $p_i^\circ = p_i - e/3$, and let

$$G^\circ = \big(\langle p_i^\circ, p_j^\circ \rangle\big)_{i,j=1}^{N}$$

be their Gram matrix. The p_i° lie in J_0, so G° is positive semidefinite of rank at most 26. Its diagonal entries are $2/3$, and Lemma 3.1 puts its off-diagonal entries in $[-1/3, 0]$. Let s be the sum of all its entries:

$$s = \mathbf{1}^{\mathsf{T}} G^\circ \mathbf{1} = \Big\| \sum_i p_i^\circ \Big\|^2 \geq 0.$$

Cauchy–Schwarz for the at most 26 nonzero eigenvalues of G° gives

$$(\operatorname{tr} G^\circ)^2 \leq 26 \operatorname{tr}\big((G^\circ)^2\big).$$

Here $\operatorname{tr} G^\circ = 2N/3$. The square of each off-diagonal entry is at most $-1/3$ times the entry, and

$$\begin{aligned} (2N/3)^2 &\leq 26 \sum_{i,j} (G_{ij}^\circ)^2 \qquad (4) \\ &\leq 26 \left(\frac{4N}{9} - \frac{1}{3} \sum_{i \neq j} G_{ij}^\circ \right) = 26 \left(\frac{2N}{3} - \frac{s}{3} \right) \leq \frac{52N}{3}. \end{aligned}$$

This yields $N \le \frac{3}{2} \cdot 26 = 39$.

Equality. At $N = 39$ all inequalities are equalities. Each off-diagonal entry of G° is 0 or $-1/3$, and $s = 0$. Consequently

$$\langle p_i, p_j \rangle \in \{0, 1/3\} \ (i \neq j), \qquad \sum_i p_i = 13e. \tag{5}$$

The Gram matrix $G = (\langle p_i, p_j \rangle)$ therefore has row sums 13: besides the diagonal entry 1, each row has thirty-six entries $1/3$ and two zeros.

Thirteen triangles. Join two distinct labels when their overlap is zero. This simple graph is 2-regular. Let Δ be its graph Laplacian. Comparing entries gives

$$3G = \Delta + \mathbf{1}\mathbf{1}^{\mathsf{T}}.$$

If the graph has k components, the kernel of Δ consists of the vectors constant on each component. Both matrices on the right are positive semidefinite, so the kernel of their sum is the intersection of their kernels, of dimension $k - 1$. Thus

$$\operatorname{rank} G = 40 - k.$$

The p_i lie in J, of dimension 27, so $\operatorname{rank} G \le 27$ and $k \ge 13$. Each component has at least three vertices; hence $k \le 13$. All thirteen components are triangles.

Frames. For the points p, q, r of one triangle, expansion gives

$$\|p + q + r - e\|^2 = 3 + 3 - 6 = 0,$$

so $p+q+r = e$. Then $e-p-q$ is the idempotent r; expanding $(e-p-q)\circ(e-p-q) = e - p - q$ leaves $2p \circ q = 0$. The other pairs follow by permutation, and points of distinct components have overlap $1/3$.

Uniqueness and converse. The three points of any Jordan frame contained in the code have pairwise overlap zero, so they form a triangle of the graph, and its only triangles are its components. The converse is immediate. □

The codes with $N = 39$ in Proposition 3.2 are exactly the tight 39-point codes of Cohn, Kumar and Minton[1]. Such a code attains the bound of the certificate, so it is tight in their sense (their Definition 2.6). Conversely, a tight code is optimal (their Proposition 2.8): no code of its size has smaller maximal overlap. Their construction has maximal overlap $1/3$ (their Theorem 8.3), so a tight 39-point code has no overlap above $1/3$. Proposition 3.2 does not show that an arbitrary tight 39-point code contains a complex quartet.

Lemma 3.3 (Local constancy of the frame partition)**.** *The partition in Proposition 3.2 is locally constant on C.* Lean: `partition`

Proof. On a small neighborhood each pair overlap changes by less than $1/6$. On C it can only be 0 or $1/3$, so it is constant there, and so are the zero-overlap graph and the partition. □

Locally on C the points can therefore be indexed by a frame $f \in \{0, \dots, 12\}$ and a position $t \in \{0, 1, 2\}$ in it, and studied through the equations

$$\begin{gathered} p_{f,t} \circ p_{f,t} = p_{f,t}, \qquad \operatorname{tr} p_{f,t} = 1, \qquad \sum_{t} p_{f,t} = e, \\ \langle p_{f,t}, p_{f',t'} \rangle = \begin{cases} 0, & f = f',\ t \neq t', \\ 1/3, & f \neq f'. \end{cases} \end{gathered} \tag{6}$$

Their real solution set is the set $\mathcal{Z}$ of the introduction. It is exactly the set of codes in C whose frames are the triples $(p_{f,0}, p_{f,1}, p_{f,2})$: the equations give overlaps 0 within frames and $1/3$ across them, and Proposition 3.2 gives the equations for such a code. By Lemma 3.3 it is open and closed in C, so near a point of $\mathcal{Z}$ every code of C lies in $\mathcal{Z}$. The residual

$$F : J^{39} \longrightarrow \mathbb{R}^{2184}$$

consists of $1053 = 39 \cdot 27$ idempotency coordinates, 39 trace equations, 351 frame-sum coordinates and 741 distinct-pair overlap equations. Some equations are redundant; all are retained in Theorem 1.2.

Lemma 3.4 (Spanning)**.** *The thirty-nine points of a tight* 39*-point code span* J*.*

Proof. In the proof of Proposition 3.2, $\operatorname{rank} G = 40 - 13 = 27 = \dim J$. □

4 Constructing the four families

The matrices placing $P(\xi)$ in each quartet's coordinates are needed only for verification (Appendix A).

4.1 The family and its four subalgebras

Fix the octonion basis $(1, i, j, k, \ell, i\ell, j\ell, k\ell)$, put $n = j\ell$, and use left complex coefficients, so $in = -k\ell$. For two unit octonions u, w put

$$p(1, u, w) = \tfrac{1}{3}\, vv^*, \qquad v = (1, u, w)^{\mathsf{T}}.$$

Artin's theorem makes the subalgebra generated by u, w associative, so this is a point. With $\omega = e^{2\pi i/3}$, the four-parameter family of Cohn, Kumar and Minton is

frame	point index t
0	the coordinate idempotent E_{tt}
$1 + s$	$p(1, \omega^t, \omega^{s-t})$
$4 + s$	$p(1, \omega^t j, \omega^{s+t} \ell)$
$7 + s$	$p(1, \xi_1 \omega^t \ell, \xi_2 \omega^{s+t} n)$
$10 + s$	$p(1, \xi_3 \omega^t n, \xi_4 \omega^{s+t} j)$.

(7)

Here $s, t \in \{0, 1, 2\}$. The parameters $\xi = (\xi_1, \dots, \xi_4)$ are four unit complex numbers, those on p. 49 of Cohn, Kumar and Minton[1], and $\xi = 1$ gives their code (8.2). In

their notation $(1, \omega^a u, \omega^b w)$ the frames of the first type are the classes of $a + b$ modulo 3, and those of the three octonionic types the classes of $b - a$; the proof of their Theorem 8.3 describes all four by $a + b$.

Each later row of (7) holds three frames: for $i = 0, 1, 2, 3$ the frames $1 + 3i + s$ form *lobe* i, and frame 0 with lobe i forms *quartet* i.

Left multiplication by $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$ splits the octonions into four mutually orthogonal complex lines,

$$\mathbb{O} = \mathbb{C} \oplus \mathbb{C}j \oplus \mathbb{C}\ell \oplus \mathbb{C}n,$$

each closed under conjugation. The table

i	$(1,2)$	$(1,3)$	$(2,3)$
0	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$
1	$\mathbb{C}j$	$\mathbb{C}\ell$	$\mathbb{C}n$
2	$\mathbb{C}\ell$	$\mathbb{C}n$	$\mathbb{C}j$
3	$\mathbb{C}n$	$\mathbb{C}j$	$\mathbb{C}\ell$

(8)

assigns to each quartet one line for each of the three positions above the diagonal. Let $A_i \subset J$ consist of the matrices whose entries in these positions lie in the lines of row i. Let $D \subset J$ be the space of real diagonal matrices, the span of frame 0, and let N_i be the matrices in A_i with zero diagonal, so that $A_i = D \oplus N_i$. Row 0 gives $A_0 = H_3(\mathbb{C})$.

Lemma 4.1 (Four subalgebras)**.** (a) *Each A_i is a subalgebra of J of dimension nine, isomorphic to $H_3(\mathbb{C})$ and containing D. The sum*

$$J = D \oplus N_0 \oplus N_1 \oplus N_2 \oplus N_3 \tag{9}$$

is direct and orthogonal for the trace form. In particular $A_i \cap A_{i'} = D$ for $i \neq i'$.

(b) *Let two points lie in two different subalgebras A_i and $A_{i'}$, and let each have overlap* $1/3$ *with the three points of frame* 0*. Then the two points have overlap* $1/3$*.*

(c) *For every ξ the points of lobe i of $P(\xi)$ lie in A_i. Lobes* 0 *and* 1 *do not depend on ξ. Lobes* 2 *and* 3 *of $P(\xi)$ are the images of the same lobes of $P(1)$ under the automorphisms of the algebras A_2 and A_3 that fix the diagonal and multiply the entries in positions* $(1,2)$, $(1,3)$, $(2,3)$ *on the left, respectively, by*

$$\xi_1,\ \xi_2,\ \overline{\xi_1\xi_2} \quad \text{in } A_2, \qquad\qquad \xi_3,\ \xi_4,\ \overline{\xi_3\xi_4} \quad \text{in } A_3.$$

Proof. Part (b) follows from part (a). A point having overlap $1/3$ with the three points of frame 0 has diagonal entries $1/3$. So the two points are $p = e/3 + p'$ and $q = e/3 + q'$ with $p' \in N_i$ and $q' \in N_{i'}$, and by (9)

$$\langle p, q \rangle = \langle e/3, e/3 \rangle + \langle p', q' \rangle = 1/3.$$

Parts (a) and (c) are verified entry by entry in Appendix A.3. In (8), each column contains each of the four lines once, so the spaces N_i are mutually orthogonal and with the diagonal fill J. In each row the product of the lines in two positions lies in the line of the third, as closure of A_i under squaring requires. □

With the tightness of $P(1)$ (Theorem 8.3 of Cohn, Kumar and Minton[1]), parts (b) and (c) show again that every $P(\xi)$ is tight, and that each quartet i is a complex quartet lying in A_i.

Quartet 0 is the standard complex quartet[14], the eigenbases of Z, X, XZ and XZ^2, with X, Z as defined before (14).

4.2 Coordinates relative to a quartet

By (9) the orthogonal complement of $A_0 = H_3(\mathbb{C})$ in J is $N_1 \oplus N_2 \oplus N_3$. An element of this complement has three entries above the diagonal, each in $\mathbb{C}j \oplus \mathbb{C}\ell \oplus \mathbb{C}n$; its nine complex coordinates are arranged in a matrix $B \in M_3(\mathbb{C})$. This gives a real-linear isomorphism

$$J \simeq H_3(\mathbb{C}) \oplus M_3(\mathbb{C}), \qquad x \longmapsto (a, B),$$

written out and verified in Appendix A. It is the second Tits construction for $M_3(\mathbb{C})$ with conjugate transposition, in the case that gives the Euclidean algebra J; Yokota writes out the same decomposition with its product in his Section 2.12[15].

In these coordinates the unit is $(I, 0)$, the trace of (a, B) is the trace of a, and the trace form is

$$\langle (a, B), (a', B') \rangle = \operatorname{Re}\operatorname{tr}(aa') + 2\operatorname{Re}\operatorname{tr}(BB'^*). \tag{10}$$

The square formula is

$$(a, B)^2 = \big(a^2 - BB^* + \operatorname{tr}(BB^*)I,\ ((\operatorname{tr} a)I - a)B + \operatorname{adj}(B)^*\big). \tag{11}$$

Polarization recovers the product of J. The formulas hold for every complex matrix B.

Lean: `Jordan product`

Appendix A gives, for $i = 1, 2, 3$, an automorphism ι_i of J that maps A_0 onto A_i; let ι_0 be the identity. The *coordinates of quartet i* of an element x are the pair (a, B) of $\iota_i^{-1}x$. In them A_i is the set of pairs $(a, 0)$.

Let θ_i be the map of J that in the coordinates of quartet i is

$$\theta_i \colon (a, B) \longmapsto (a, \omega B).$$

It preserves squares, by (11) and $\operatorname{adj}(\omega B)^* = \omega \operatorname{adj}(B)^*$. So it is an automorphism of J of order three. Appendix A gives θ_i in octonion coordinates; up to the conventions for the octonion basis, θ_0 is the automorphism w of Yokota's Section 2.12, or its inverse[15].

The twelve points of a complex quartet in A_i span A_i. Subtract $e/3$ from the three points of a frame: the results span a plane, because their Gram matrix $I - \mathbf{1}\mathbf{1}^{\mathsf{T}}/3$ has rank two. The four planes are mutually orthogonal because the frames are mutually unbiased, and with the unit they span $4 \cdot 2 + 1 = 9$ dimensions. Consequently a point having overlap $1/3$ with all twelve has component $e/3$ in A_i: once $e/3$ is subtracted, it is orthogonal to their span.

Lemma 4.2 (Points and frames outside a quartet)**.** *Let a complex quartet lie in A_i. Write $\theta = \theta_i$ and use the coordinates (a, B) of quartet i.*

(i) *For a point p the following are equivalent: p has overlap $1/3$ with all twelve points of the quartet; $p + \theta p + \theta^2 p = e$; and*

Lean: points off a quartet

$$p = p(U) = (I/3, U/3) \qquad \text{with } U \in SU(3).$$

(ii) *An ordered frame of such points is either $(p, \theta p, \theta^2 p)$ or $(p, \theta^2 p, \theta p)$; here $\theta p(U) = p(\omega U)$. The frames consisting of such points are thus the orbits of θ on them.*

Lean: ordered frames

(iii) *Two such frames, the orbits of $p(U)$ and $p(V)$, are mutually unbiased if and only if* $\operatorname{tr}(U^*V) = 0$.

Proof. (i) Write $p = (a, B)$. The three cube roots of unity sum to zero, so

$$p + \theta p + \theta^2 p = (3a, 0),$$

and the second condition says $a = I/3$. The first condition implies this by the paragraph before the lemma. Conversely, if $a = I/3$, then by (10) the overlap of p with a point $(a', 0)$ of the quartet is $\operatorname{tr}(a')/3 = 1/3$.

Now let $a = I/3$ and put $U = 3B$. By (11) the two components of the idempotency equation are

$$BB^* = \big(\operatorname{tr}(BB^*) - 2/9\big)I, \qquad \operatorname{adj}(B)^* = B/3.$$

Taking traces in the first gives $\operatorname{tr}(BB^*) = 1/3$, so together they say

$$UU^* = I, \qquad \operatorname{adj}(U)^* = U.$$

The adjugate identity $U \operatorname{adj}(U) = \det(U)I$ shows that these conditions are equivalent to $U \in SU(3)$. Substitution proves the converse.

(ii) Let three such points, with matrices U, V, V', form a frame. Their sum is the unit, so

$$U + V + V' = 0, \qquad \text{that is,} \qquad V' = -U(I + \Theta) \quad \text{with } \Theta = U^*V.$$

Since V' and Θ are unitary, this gives

$$\Theta + \Theta^* = -I, \qquad \text{whence} \qquad \Theta^2 + \Theta + I = 0.$$

So each eigenvalue of Θ is a primitive cube root of unity. If m of them equal ω, the determinant is ω^{6-m}; it equals one, which forces $m = 0$ or $m = 3$. Being unitary, Θ is diagonalizable, hence scalar. So (V, V') is $(\omega U, \omega^2 U)$ or $(\omega^2 U, \omega U)$. Conversely, the three points of an orbit of θ sum to the unit, by (i); their Jordan orthogonality follows as in Proposition 3.2.

(iii) By (10), for $k, l \in \mathbb{Z}/3$,

$$\langle p(\omega^k U), p(\omega^l V)\rangle = \frac{1}{3} + \frac{2}{9}\operatorname{Re}\big(\omega^{k-l}\operatorname{tr}(UV^*)\big). \tag{12}$$

The nine cross overlaps equal $1/3$ precisely when $\operatorname{tr}(UV^*)$ and its product with ω both have real part zero, that is, when it vanishes; and $\operatorname{tr}(U^*V)$ is its complex conjugate. □

4.3 Nine trace-orthogonal unitary matrices

By Lemma 4.2, a tight code that contains the quartet is given by nine matrices in $SU(3)$, pairwise orthogonal for $\operatorname{tr}(U^*V)$ and each determined up to a cube root of unity. They form an orthogonal basis of $M_3(\mathbb{C})$ consisting of unitary matrices, that is, a unitary error basis[16].

Lemma 4.3 (Pointwise stabilizer of a complex quartet)**.** *The automorphisms of J fixing a complex quartet in A_i pointwise are, in the coordinates of quartet i, exactly the maps*

Lean: `pointwise stabilizer`

$$(a, B) \longmapsto (a, Bh), \qquad h \in SU(3).$$

Proof. Such an automorphism fixes the span A_i of the quartet pointwise and, preserving the trace form, preserves the orthogonal complement.

Its action there commutes with left multiplication by every Hermitian matrix, because

$$(a, 0) \circ (0, B) = \left(0, \frac{(\operatorname{tr} a)I - a}{2} B\right),$$

and every Hermitian matrix b occurs as the multiplier: take $a = (\operatorname{tr} b)I - 2b$. Hermitian matrices generate $M_3(\mathbb{C})$ as a real algebra, so a commuting real-linear map is right multiplication by a matrix h, the image of I.

Preservation of the trace form gives $hh^* = I$, and preservation of (11) at $(0, I)$ gives $\operatorname{adj}(h)^* = h$. Thus $h \in SU(3)$. Conversely, substitution in the product obtained from (11) shows that right multiplication of B by any matrix in $SU(3)$ is an automorphism. □

These right multiplications form the factor $SU(3)$ that fixes $H_3(\mathbb{C})$ pointwise in Yokota's description[15] of the subgroup $(SU(3)\times SU(3))/(\mathbb{Z}/3)$ (his Proposition 2.12.1 and Theorem 2.12.2). His left action on his matrix M is our right action, since B corresponds to M^* up to sign and the identification of bases.

Let (U_r), $r = 0, \ldots, 8$, be a labeled array of representatives of mutually unbiased frames. The right multiplication of Lemma 4.3 is removed by

$$U_r \longmapsto U_r U_0^*. \tag{13}$$

The result is unchanged when all nine matrices are multiplied on the right by the same matrix, and its first entry is I.

Let e_0, e_1, e_2 be the standard basis of $\mathbb{C}^3$, and let

$$Xe_k = e_{k+1}, \qquad Ze_k = \omega^k e_k,$$

with indices modulo three (the shift and clock matrices). The triples $d = (d_0, d_1, d_2)$ of unit complex numbers with product one form a two-torus, written T^2. For $d \in T^2$ set

$$E_{ab}(d) = X^b D_b Z^a, \qquad D_0 = D_1 = I, \quad D_2 = \operatorname{diag}(d). \tag{14}$$

These nine matrices lie in $SU(3)$ and form a unitary error basis: different powers of X have disjoint supports, and for a fixed power the orthogonality is that of

the characters $k \mapsto \omega^{ak}$ of $\mathbb{Z}/3$. This is an instance of the shift-and-multiply construction of Vollbrecht and Werner[12] (their eq. (3.6)). Up to the equivalence of unitary error bases, Svegborn, Pauwels, Gisin and Pozas-Kerstjens[17] classify a one-parameter family of bases of this form (their Theorem 1). That equivalence allows unitary factors on both sides, phases and relabeling. Lemma 4.4 divides only by the common right factor of Lemma 4.3, which leaves the eight parameters of g; up to conjugation and phases, their family lies on a circle in T^2.

Lemma 4.4 (The normalized family)**.** *Applying* (13) *to the array* $gE_{ab}(d)h^*$*, with* $g, h \in SU(3)$*, gives*

$$U_{ab}(g, d) = gE_{ab}(d)g^*, \qquad ([g], d) \in PSU(3) \times T^2. \tag{15}$$

The map sending $([g], d)$ *to this array is an embedding of a manifold of dimension* $8+2 = 10$*. By Lemma 4.2 its image is a ten-parameter family of labeled solutions of* (6) *in which the quartet is held fixed.*

Lean: `embedded family`

Proof. The entries gZg^* and gXg^* of the array determine g up to the center, because a matrix commuting with both X and Z is scalar; then the entry $gX^2 \operatorname{diag}(d)g^*$ determines d. Likewise a tangent variation fixing the first two of these entries is scalar, hence zero in $\mathfrak{su}(3)$, and then the variation of d is zero. So the map is an injective immersion, and an embedding because its domain is compact. The last assertion is Lemma 4.2 applied to the nine matrices of the array, which are special unitary and pairwise trace-orthogonal. □

4.4 Applying the construction to each quartet

For the quartets $i = 0, 1, 2, 3$, Lemma 4.5 below uses the triples

$$\begin{aligned} d_0(\xi) &= (\xi_1\xi_2/\xi_3, (\xi_1\xi_4)^{-1}, \xi_3\xi_4/\xi_2), & d_1(\xi) &= (\xi_1\xi_3\xi_4, -\xi_2/\xi_3, -(\xi_1\xi_2\xi_4)^{-1}), \\ d_2(\xi) &= (-\xi_3\xi_4, \xi_3^{-1}, -\xi_4^{-1}), & d_3(\xi) &= (-\xi_1^{-1}, -\xi_2^{-1}, \xi_1\xi_2). \end{aligned} \tag{16}$$

Each triple has product one, so it lies in T^2. The subscript of $d_i(\xi)$ is the quartet, whereas d_0, d_1, d_2 without an argument are the entries of a triple d. Each of the twelve entries is a sign times a nontrivial character of the torus $(S^1)^4$. Hence the parameters at which some entry has cube one form a finite union of cosets of closed subgroups of codimension one; by Proposition 5.8 these are the resonant parameters. At $\xi = 1$ six of the twelve entries equal 1, so the original code is resonant.

Lemma 4.5 (The four families with one quartet held fixed)**.** *Fix* ξ *and* i*. In the coordinates of quartet* i *the nine frames of* $P(\xi)$ *outside quartet* i *are represented by the matrices*

$$\zeta_r L_i E_{s_i(r)}(d_i(\xi))R_i, \qquad r = 0, \ldots, 8.$$

Here L_i *and* R_i *are unitary with* $\det L_i \det R_i = 1$*, each* ζ_r *is a cube root of unity, and* s_i *is a bijection from the nine frames onto the nine labels* (a, b) *of* (14)*; all four are written out in Appendix A. Keep the quartet and replace these matrices by*

$$\zeta_r g L_i E_{s_i(r)}(d)R_i h^*, \qquad g, h \in SU(3), \quad d \in T^2, \tag{17}$$

Table 1: Real dimensions; r_i is the number of entries of $d_i(\xi)$ with cube one, and every $r_i = 0$ at a nonresonant parameter.

dimension	space	count	proved in
27, 26	J, J_0	$27 = 9 + 18$	Section 4
1053	configurations J^{39}	$39 \cdot 27$	(6)
2184	residual coordinates of F	$1053 + 39 + 351 + 741$	(6)
52	F_4; orbit tangent space $\mathcal{O}$		Lemma 5.7
8	pointwise stabilizer of a quartet	$SU(3)$	Section 4
18	parameters of (17)	$8 + 8 + 2$	Section 4
$18 + 2r$	first-order matrix solutions	$6 + 12 + 2r$	Lemma 5.1
8	first-order deformations of a quartet inside $H_3(\mathbb{C})$	$\mathfrak{su}(3)$	Lemma 5.4
$26 + 2r_i$	$\mathcal{T}^{K_i}$	$8 + (18 + 2r_i)$	Lemma 5.5
8	$\mathcal{T}^{K}$	$2 + 6$	Lemma 5.6
16, 4	$\mathcal{O}^{K_i}$, $\mathcal{O}^{K}$	$8 + 8$, $2 + 2$	Lemma 5.7
$80 + 2\sum r_i$	$\mathcal{T} = \ker DF_{P(\xi)}$	$\sum_i (26 + 2r_i) - 24$	(21)
1001	slice S	$1053 - 52$	Lemma 5.9
28	$\overline{\mathcal{T}} = \mathcal{T}/\mathcal{O}$, nonresonant	$80 - 52 = 4 + 4 \cdot 6$	(22)
4, 10, 6	$\overline{\mathcal{T}}^{K}$, $\overline{\mathcal{T}}^{K_i}$, W_i, nonresonant	$8 - 4$, $26 - 16$, $10 - 4$	(22)
62	F_4-orbits of one family	$52 + 18 - 8$	Lemma 5.10
10, 4	M_i, M_* in the slice	$10 = 4 + 6$	Lemma 5.10
973	complement of $\overline{\mathcal{T}}$ in the slice	$1001 - 28$	Prop. 5.11
92, 40	$\mathcal{T}$, $\overline{\mathcal{T}}$ at the original code, $(r_i) = (3, 1, 1, 1)$	$80 + 2 \cdot 6$, $92 - 52$	(21)

then return to the original coordinates and labels. Every code so obtained is a labeled solution of (6)*, and $g = h = I$, $d = d_i(\xi)$ gives $P(\xi)$. Changing h moves the code by an automorphism of J (Lemma 4.3).*

Proof. The matrices (17) have determinant $\zeta_r^3 \det L_i \det R_i = 1$ and are pairwise trace-orthogonal, because the $E_{ab}(d)$ are and gL_i, $R_i h^*$ are unitary. The selected quartet stays fixed, and Lemma 4.2 proves all equations for the other nine frames. The base point is (38), and Lemma 4.3 absorbs h. □

5 First-order deformations and local coordinates

Fix a parameter ξ and write $P = P(\xi)$, with points $p_{f,t}$ labeled as in (7). The first-order deformations of P form the space

$$\mathcal{T} = \ker DF_P \subset J^{39}.$$

It contains the tangent space $\mathcal{O}$ of the orbit $F_4 P$, and $\overline{\mathcal{T}} = \mathcal{T}/\mathcal{O}$ consists of the first-order deformations modulo F_4. Table 1 collects the dimensions.

5.1 Solving the linearized matrix equations

Hold one quartet fixed. By Lemma 4.2 the nine frames outside it are nine special unitary matrices U_{ab} with

$$\operatorname{tr}(U_{ab}^{*}U_{a'b'}) = 0 \qquad \big((a,b) \neq (a',b')\big).$$

We linearize these conditions at the matrices $E_{ab}(d)$ of (14). Every first-order variation of $E_{ab}(d)$ inside $SU(3)$ has the unique form

$$\dot{U}_{ab} = X^{b} D_b M_{ab} Z^{a}, \qquad M_{ab}^{*} = -M_{ab}, \quad \operatorname{tr} M_{ab} = 0.$$

Differentiating the trace conditions gives

$$\operatorname{tr}\Big(Z^{a'-a}[W_{bb'}M_{a'b'} - M_{ab}W_{bb'}]\Big) = 0, \qquad W_{bb'} = D_b^{*} X^{b'-b} D_{b'}. \tag{18}$$

Both real and imaginary parts are required. The matrices $W_{bb'}$ occur only here and in Appendix B; they are not the transverse spaces W_i.

Lemma 5.1 (First-order solutions of the matrix problem)**.** *Let $d \in T^2$. The real vector space of all solutions (M_{ab}) of* (18)*, with no normalization imposed, is isomorphic to*

Lean: unrestricted kernel

$$\mathbb{R}^6 \oplus \mathbb{C}^6 \oplus \mathbb{C}^{\{j : d_j^3 = 1\}}.$$

In particular its dimension is $18 + 2r$, where r is the number of entries of d with cube one; this number is 0, 1 or 3.

Proof. The matrix W_{bb} is the identity, and $W_{bb'}$ has zero diagonal for $b' \neq b$. So the equations with $b = b'$ involve only the diagonal entries of the matrices M_{ab}, and the others only the off-diagonal entries.

Diagonal entries. The equations with $b = b'$ say that the purely imaginary diagonal of M_{ab} does not depend on a: two real parameters for each b.

Off-diagonal entries. Write the off-diagonal part of M_{ab} uniquely as

$$Y_{ab}X^{-1} - (Y_{ab}X^{-1})^{*}, \qquad Y_{ab} = \operatorname{diag}(y_{ab,k}),$$

and take the Fourier transform in a,

$$\widehat{y}_{b,k}(\mu) = \tfrac{1}{3}\sum_{a} y_{ab,k}\,\omega^{-\mu a}.$$

The equations with $b' = b + 1$ become a three-step scalar recurrence in b. By Appendix B, going once round it multiplies the initial value $\widehat{y}_{0,k}(\mu)$ by 1, d_{k+1}^3, 1 for $\mu = 0, 1, 2$. Hence six complex initial values are free, and the remaining one for index k is free exactly when its multiplier is one.

The inverse Fourier transform and the displayed reconstruction of M_{ab} give both directions of the isomorphism. Two entries of d with cube one force the third, because the product of the three entries is one. □

The exact solutions $gE_{ab}(d)h^{*}$ of Section 4 have $18 = 8 + 8 + 2$ parameters, with injective derivative (Lemma 4.4; Lemma 5.10(a)). So when no entry of d has cube one, every first-order solution with one quartet held fixed is tangent to this family.

5.2 The finite symmetry

The calculations for the four quartets are combined by a group $K \simeq (\mathbb{Z}/3)^2$ of automorphisms of J that map P to itself, each followed by a relabeling within frames that keeps the labeled configuration fixed. The relabeling cannot be omitted, because the thirty-nine points span J (Lemma 3.4), so only the identity fixes them all.

Lemma 5.2 (The symmetry K)**.** *For $i = 0, 1, 2, 3$ let θ_i be the map of Section 4, which is $(a, B) \mapsto (a, \omega B)$ in the coordinates of quartet i. It does not depend on ξ.*

Lean: `action in quartet coordinates`

(a) *Each θ_i is an automorphism of J of order three. It fixes the twelve points of quartet i and permutes the three points of every other frame of P cyclically.*

Lean: `finite symmetry`

(b) *The four maps commute and generate a group $K \simeq (\mathbb{Z}/3)^2$, with*

$$\theta_2 = \theta_0^2 \theta_1^2, \qquad \theta_3 = \theta_0^2 \theta_1.$$

Its four subgroups of order three are $K_i = \langle \theta_i \rangle$. The maps θ_0 and θ_1 fix the points of frame 0; on the other frames they send $p_{f,t}$ to $p_{f,t-m}$, with indices modulo three, where m depends only on the lobe containing frame f:

lobe	0	1	2	3
m for θ_0	0	1	1	1
m for θ_1	1	0	2	1

(c) *For $k \in K$ let $\tilde{k}$ be the map of J^{39} that applies k to each point and then undoes its permutation of the labels:*

$$(\tilde{k}y)_{f,t'} = k\, y_{f,t} \qquad \text{when } k p_{f,t} = p_{f,t'}.$$

Then $\tilde{k}$ fixes the labeled configuration P, is orthogonal for the trace form, and satisfies

$$\tilde{k}(gy) = (kgk^{-1})(\tilde{k}y) \qquad (g \in F_4,\ y \in J^{39}).$$

It preserves the space $\mathcal{T}$ of first-order deformations, the orbit tangent space $\mathcal{O}$ and its orthogonal complement $\mathcal{O}^{\perp}$.

Proof. (a) The map θ_i is right multiplication of B by the scalar matrix ωI, which lies in $SU(3)$. By Lemma 4.3 it is an automorphism of J fixing quartet i pointwise. By Lemma 4.2 every other frame of P is one of its orbits.

(b) By (a), every element of the group generated by the four maps permutes the points of each frame cyclically. Cyclic permutations of three objects commute, so a commutator fixes all thirty-nine points and, as these span J (Lemma 3.4), is the identity; likewise an element is determined by the permutations it induces. Substitution in (7) gives the table, and shows that the two sides of each displayed relation induce the same permutations. The group has order nine, since θ_1 moves points of quartet 0 and so is not a power of θ_0. Its four subgroups of order three are the K_i, distinct because θ_i fixes the points of quartet i and no others.

(c) By construction $\tilde{k}$ fixes P. Automorphisms preserve the trace form, and the relabeling permutes the equations (6), so $F \circ \tilde{k}$ is F followed by an invertible linear map; differentiating at P shows that the first-order deformations are preserved. The displayed identity holds because the relabeling commutes with the action of F_4. Differentiated, it gives $\tilde{k}(\delta P) = (k\delta k^{-1})P$ for every derivation δ of J. So the orbit tangent space is preserved and, by orthogonality, so is its orthogonal complement. □

From now on K acts on J^{39} through the maps $\tilde{k}$, and so on the first-order deformations, on the orbit tangent space and on the quotient $\overline{\mathcal{T}}$.

5.3 Combining the four linearized calculations

Lemma 5.3 (Real representations of K)**.** *Let V be a real vector space of finite dimension on which K acts linearly. Write V^K and V^{K_i} for the vectors fixed by K and by K_i, and $V_{[i]}$ for the vectors fixed by K_i whose average over K is zero.*

(a) Splitting. *The space V and its fixed spaces split as*

$$V = V^K \oplus V_{[0]} \oplus V_{[1]} \oplus V_{[2]} \oplus V_{[3]}, \qquad V^{K_i} = V^K \oplus V_{[i]}, \tag{19}$$

and consequently

$$\dim V = \sum_{i=0}^{3} \dim V^{K_i} - 3 \dim V^K. \tag{20}$$

(b) Quotients. *The averages π_i and π_K over K_i and over K are the projections onto V^{K_i} and V^K, and $V_{[i]}$ is the image of $\pi_i - \pi_K$. For an invariant subspace V',*

$$(V/V')^{K_i} = V^{K_i}/V'^{K_i}, \qquad (V/V')^K = V^K/V'^K.$$

(c) Complex structure. *The group K acts on $V_{[i]}$ through $K/K_i \simeq \mathbb{Z}/3$, and a generator θ of this quotient fixes no nonzero vector of $V_{[i]}$. So $\theta^2 + \theta + 1 = 0$ on $V_{[i]}$, which is thereby a complex vector space with θ as multiplication by ω (the other generator gives the conjugate structure).*

Proof. (a) This is the isotypic decomposition: each of the eight nontrivial characters $K \to \{1, \omega, \omega^2\}$ has one of the four subgroups K_i as kernel, each K_i is the kernel of a conjugate pair, and the two isotypic components of the pair in $V \otimes \mathbb{C}$ form the complexification of $V_{[i]}$. Taking dimensions gives (20).

(b) Averaging over a finite group projects onto its fixed vectors, and it commutes with equivariant maps, in particular with the quotient map from V to V/V'.

(c) A vector of $V_{[i]}$ fixed by θ is fixed by K, so it equals its average over K, which is zero. The image of $\theta^2 + \theta + 1$ consists of such vectors, as $\theta^3 = 1$. □

Lemma 5.4 (The complex quartet)**.** *The first-order deformations of a complex quartet inside its subalgebra $H_3(\mathbb{C})$ are exactly the infinitesimal conjugations $q \mapsto [N, q]$ with $N \in \mathfrak{su}(3)$. Each complex quartet thus contributes exactly eight tangent directions.*

The proof is in Appendix B.1.

Lemma 5.5 (First-order deformations fixed by one subgroup)**.** *In the coordinates of quartet i, the fixed space $\mathcal{T}^{K_i}$ is the direct sum of the first-order deformations of the quartet inside $H_3(\mathbb{C})$ and the first-order solutions of the matrix problem for the nine remaining frames. Consequently*

$$\dim \mathcal{T}^{K_i} = 26 + 2r_i,$$

where r_i is the number of entries of the triple $d_i(\xi)$ with cube one.

Proof. Let U_r, for $r = 0, \dots, 8$, be the matrices of Lemma 4.5 representing the nine frames outside quartet i; the points of frame r are $p(\omega^t U_r)$, permuted cyclically by θ_i. Let v be a first-order deformation fixed by $\tilde{\theta}_i$. Its components at the quartet are fixed by θ_i, so lie in $H_3(\mathbb{C})$; its components at frame r are carried to one another by θ_i, so they are

$$(\dot{a}_r,\ \omega^t V_r/3), \qquad t = 0, 1, 2.$$

The proof of Lemma 4.2 holds at first order, step by step.

(1) The frame sums give $3\dot{a}_r = 0$.

(2) By (11), linearized idempotency is then the linearized condition for U_r to lie in $SU(3)$:

$$U_r V_r^* + V_r U_r^* = 0, \qquad \operatorname{tr}(U_r^* V_r) = 0.$$

(3) By (12), the linearized overlap equations between two different frames r and s say that the real parts of z and ωz vanish, where $z = \operatorname{tr}(V_r U_s^* + U_r V_s^*)$. So $z = 0$, that is,

$$\operatorname{tr}(V_r^* U_s + U_r^* V_s) = 0.$$

The overlap equations inside one frame follow from step (2).

(4) The overlap of a quartet point with a point outside the quartet varies by one third of the trace of the variation of the quartet point; this vanishes by the trace equation, so the two kinds of variation are not coupled.

The only divisions are by 3 and $\omega - 1$, so no solution is lost. The equations of steps (2) and (3) are the linearized matrix problem at the matrices U_r, the rest are the linearized frame equations of the quartet inside $H_3(\mathbb{C})$, and conversely solutions of the two reconstruct a first-order deformation fixed by $\tilde{\theta}_i$.

The first summand has dimension eight by Lemma 5.4. For the second, the map $V \mapsto \zeta_r L_i V R_i$ carries the linearized matrix problem at the matrices $E_{ab}(d_i(\xi))$ to the one at the matrices U_r, so Lemma 5.1 gives dimension $18 + 2r_i$. □

Lemma 5.6 (First-order deformations fixed by K)**.** $\dim \mathcal{T}^K = 8$.

Proof. Work in the coordinates of quartet 0. By Lemmas 5.5 and 5.4 a first-order deformation fixed by K_0 consists of an infinitesimal conjugation of the quartet by some $N \in \mathfrak{su}(3)$ and a first-order solution (V_r) of the matrix problem. It remains to impose invariance under $\tilde{\theta}_1$. In these coordinates

$$\theta_1(a, B) = (gag^*, gBh^*), \qquad g = \operatorname{diag}(1, \omega^2, \omega), \quad h = \operatorname{diag}(\omega, \omega^2, 1).$$

This is a finite check. By Appendix A, in each off-diagonal entry $\theta_1 = \iota_1\theta_0\iota_1^{-1}$ rotates the planes $(1,i)$, (j,k), $(\ell, i\ell)$, $(n, k\ell)$ through multiples of $2\pi/3$, and the coordinate map turns these rotations into the factors g and h.

On the quartet, $\tilde{\theta}_1$ replaces N by gNg^*. As g has three distinct entries, the fixed N are diagonal, which leaves two dimensions.

On the other nine frames, write the solution in the form of Lemma 5.1:

$$V_r = \zeta_r L_0\, X^b D_b M_{ab} Z^a\, R_0, \qquad (a,b) = s_0(r).$$

The automorphism θ_1 maps each of the nine frames to itself, so $gU_rh^* = \omega^m U_r$ for some m depending on r. Hence θ_1 sends the point $p(\omega^t U_r)$ to $p(\omega^{t+m}U_r)$ and the variation $(0, \omega^t V_r/3)$ at it to $(0, \omega^t gV_rh^*/3)$. The relabeling in $\tilde{\theta}_1$ cancels the factor ω^m, so $\tilde{\theta}_1$ replaces V_r by $\omega^{-m}gV_rh^*$. Since Z^aR_0 and h are diagonal, they commute, and the identity for U_r becomes

$$g\,\zeta_r L_0 X^b D_b\, h^* = \omega^m \zeta_r L_0 X^b D_b.$$

Therefore

$$\omega^{-m} gV_rh^* = \omega^{-m}\big(g\,\zeta_r L_0 X^b D_b\, h^*\big)\big(hM_{ab}h^*\big)Z^aR_0 = \zeta_r L_0\, X^b D_b\,(hM_{ab}h^*)Z^a\, R_0,$$

and $\tilde{\theta}_1$ replaces M_{ab} by $hM_{ab}h^*$. As h has three distinct entries, the fixed solutions have every M_{ab} diagonal, a space of dimension six by the proof of Lemma 5.1. The dimension is $2 + 6 = 8$. □

Lemma 5.7 (The orbit tangent space)**.** *The labeled stabilizer of $P(\xi)$ in F_4, that is, the group of automorphisms fixing each of the thirty-nine points, is trivial, and*

$$\dim \mathcal{O} = 52, \qquad \dim \mathcal{O}^{K_i} = 16, \qquad \dim \mathcal{O}^{K} = 4.$$

Proof. The thirty-nine points span J (Lemma 3.4), so the stabilizer is trivial. Hence $\delta \mapsto \delta P$ is an isomorphism from the derivations of J onto $\mathcal{O}$, and $\dim \mathcal{O} = 52$. By Lemma 5.2(c) it carries conjugation by k to $\tilde{k}$, so $\mathcal{O}^{K_i}$ and $\mathcal{O}^K$ correspond to the Lie algebras of the centralizers of θ_i and of K.

In the coordinates of quartet i, the maps

$$(a, B) \longmapsto (gag^*, gBh^*), \qquad g, h \in SU(3),$$

preserve (11), because $\operatorname{adj}(gBh^*)^* = g \operatorname{adj}(B)^* h^*$. They are therefore automorphisms of J; they commute with θ_i and form a connected group of dimension sixteen, the identity component of the centralizer, because θ_i is conjugate in F_4 to θ_0, which up to the conventions for the octonion basis is Yokota's w or its inverse (Section 4), and the centralizer of w in F_4 is $(SU(3) \times SU(3))/(\mathbb{Z}/3)$ by Theorem 2.12.2 of Yokota [15]. Thus $\mathcal{O}^{K_i}$ has dimension sixteen.

For $\mathcal{O}^K$ take $i = 0$. By the proof of Lemma 5.6, θ_1 is one of these maps, with both matrices diagonal with distinct entries. A derivation $(a, B) \mapsto ([N, a], NB - BN')$ in the Lie algebra of this group commutes with θ_1 exactly when N commutes with the first of them and N' with the second, that is, when both are diagonal. So $\mathcal{O}^K$ has dimension $2 + 2 = 4$. □

Proposition 5.8 (First-order deformations and their splitting)**.** *At every parameter ξ,*

$$\dim \mathcal{T} = \sum_i \dim \mathcal{T}^{K_i} - 3 \dim \mathcal{T}^K = 80 + 2 \sum_i r_i. \tag{21}$$

In particular ξ is nonresonant, that is $\dim \mathcal{T} = 80$*, exactly when every r_i is zero: when no entry of the four triples* (16) *has cube one. At nonresonant parameters, $\overline{\mathcal{T}} = \mathcal{T}/\mathcal{O}$ satisfies*

$$\overline{\mathcal{T}} = \overline{\mathcal{T}}^K \oplus \bigoplus_i W_i, \qquad \overline{\mathcal{T}}^{K_i} = \overline{\mathcal{T}}^K \oplus W_i, \qquad \dim \overline{\mathcal{T}}^K = 4, \quad \dim W_i = 6. \tag{22}$$

Here $W_i = \overline{\mathcal{T}}_{[i]}$ is the ith isotypic part in the notation of Lemma 5.3, the image of $\pi_i - \pi_K$. The sum is orthogonal for every K-invariant inner product, and each W_i is a complex vector space of dimension three, on which K acts through K/K_i by cube roots of unity.

Proof. Apply Lemma 5.3(a). For $V = \mathcal{T}$, formula (20) is the first equality of (21), and Lemmas 5.5 and 5.6 give the second: $\sum_i (26 + 2r_i) - 3 \cdot 8 = 80 + 2 \sum_i r_i$. For $V = \overline{\mathcal{T}}$, (19) is the splitting in (22). At a nonresonant parameter the left side of (21) is 80, so every r_i is zero, and by Lemma 5.3(b) and Lemma 5.7

$$\dim \overline{\mathcal{T}}^{K_i} = 26 - 16 = 10, \qquad \dim \overline{\mathcal{T}}^K = 8 - 4 = 4,$$

so each W_i has dimension six. Distinct isotypic parts are orthogonal for an invariant inner product, and the complex structure is that of Lemma 5.3(c). □

5.4 Coordinates for the four families

The next lemma is the slice theorem for compact group actions at a point with trivial stabilizer; Appendix A.7 has a proof.

Lemma 5.9 (The slice)**.** *Let $S = P + \mathcal{O}^\perp$ and, for $\varepsilon > 0$, let S_ε be the set of $s \in S$ with $\|s - P\| < \varepsilon$ in the norm of the trace form. For ε small enough the map $(g, s) \mapsto gs$ is an equivariant analytic diffeomorphism from $F_4 \times S_\varepsilon$ onto an open invariant neighborhood U_ε of the orbit of P in J^{39}; orbits, invariance and equivariance are those of F_4. In particular:*

(a) *two points of S_ε lie in the same orbit only if they are equal;*

(b) *every invariant subset Σ of U_ε equals $F_4(\Sigma \cap S_\varepsilon)$, and $s \mapsto F_4 s$ is a homeomorphism from $\Sigma \cap S_\varepsilon$ onto Σ/F_4.*

A point of U_ε is gs for a unique point s of the slice S_ε, its *slice representative*. The slice representative depends analytically on the point, and the derivative of this dependence at P is the orthogonal projection Π of J^{39} onto $\mathcal{O}^\perp$. This projection identifies $\overline{\mathcal{T}}$ with $\mathcal{T} \cap \mathcal{O}^\perp$, compatibly with the action of K (Lemma 5.2(c)).

Lemma 5.10 (The four families in the slice)**.** (a) *The eighteen-parameter map* (17) *has injective derivative at $(I, I, d_i(\xi))$, and the velocities in the eight parameters h are exactly the velocities of the family that lie in $\mathcal{O}$* (*Lemma 4.3*)*. Consequently:*

- *the slice representatives of the members with $h = I$ and (g, d) near $(I, d_i(\xi))$ form a ten-dimensional analytic submanifold M_i of S through P, the family of quartet i;*
- *the union of the F_4-orbits of the points of M_i is a submanifold of dimension $62 = 52 + 18 - 8$.*

(b) *The four derivatives $\partial P/\partial \xi_j$ are independent modulo $\mathcal{O}$, and the slice representatives of the codes $P(\xi')$, ξ' near ξ, form a four-dimensional submanifold M_*. Moreover $M_* \subset M_i$ for every i.*

(c) *Every point of M_i is fixed by the subgroup K_i of K. At nonresonant parameters the tangent space of M_i at P is $\overline{\mathcal{T}}^{K_i}$, and that of M_* is $\overline{\mathcal{T}}^{K}$.*

Proof. (a) The injectivity is the argument with the entries I, X, Z used for Lemma 4.4; it is written out in Appendix A.5. For the velocities in h: the family does not move the quartet, so a velocity of it in $\mathcal{O}$ is δP for a derivation δ annihilating the quartet, and by Lemma 4.3, differentiated, these are exactly the velocities in h. Hence the map sending (g, d) to the slice representative of the member with $h = I$ has injective derivative at $(I, d_i(\xi))$, and the image of a small neighborhood is M_i. By Lemma 5.9 the union of the orbits of its points is the image of $F_4 \times M_i$, of dimension $52 + 10$.

(b) Suppose that a real combination of the four derivatives lies in the orbit tangent space, so that it is δP for a derivation δ. The parameters do not enter the first seven frames of (7), so δ annihilates quartet 0; by Lemma 4.3, differentiated, it is then $(a, B) \mapsto (0, BN)$ for some N in $\mathfrak{su}(3)$. It also annihilates a point $p(U)$ of frame 4, so $UN = 0$ and $N = 0$. Thus the combination is zero. Its coefficients then vanish, because the first two parameters occur in different entries of frame 7, and the last two in different entries of frame 10. So the map sending a parameter ξ' to the slice representative of $P(\xi')$ has injective derivative at ξ, and its image near P is M_*.

By Lemma 4.1(c) a change of the parameter moves the last two lobes by automorphisms of their subalgebras that multiply the off-diagonal entries by unit complex numbers. For ξ' near ξ, Appendix A.6 constructs diagonal matrices g and h in $SU(3)$, analytic in ξ' and equal to the identity at $\xi' = \xi$, and an automorphism of J that carries the member with parameters $(g, h, d_i(\xi'))$ to $P(\xi')$. Replacing h by the identity moves that member by an automorphism (Lemma 4.3). So $P(\xi')$ lies in the orbit of the member with parameters $(g, I, d_i(\xi'))$, and these are near the parameters of P; hence the slice representative of $P(\xi')$ lies in M_i.

(c) Every member of (17) is fixed by $\tilde{\theta}_i$: the quartet lies in the subalgebra fixed by θ_i, and each other frame is one of its orbits, labeled as at P. Let s be the slice representative of such a member y. By Lemma 5.2(c), $\tilde{\theta}_i$ maps S_ε to itself and maps s into the orbit of $\tilde{\theta}_i y = y$; by Lemma 5.9(a) it fixes s. So K_i fixes every point of M_i.

The points of M_i are zeros of F, so the tangent space of M_i at P lies in $\mathcal{T} \cap \mathcal{O}^\perp$ and is fixed by K_i; it is contained in $\overline{\mathcal{T}}^{K_i}$, and both have dimension ten at a nonresonant parameter. By (b) the points of M_* are fixed by two of the K_i, hence by K, and the same argument with dimension four applies. □

Proposition 5.11 (Straightening the four families)**.** *At every nonresonant parameter there are analytic coordinates*

Lean: simultaneous chart

$$(\lambda, u_0, u_1, u_2, u_3, \eta) \in \mathbb{R}^4 \times (\mathbb{C}^3)^4 \times \mathbb{R}^{973}$$

on a neighborhood of P in the slice, each $\mathbb{C}^3$ being regarded as a real vector space of dimension six, in which each family M_i is the coordinate subspace $\eta = 0$, $u_j = 0$ for $j \neq i$. The four-parameter family M_* is $u_0 = \cdots = u_3 = 0$, $\eta = 0$; it is the intersection of any two of the M_i.

Proof. Choose analytic parametrizations $\varphi_*(\lambda)$ of M_* and $\varphi_i(\lambda, u_i)$ of M_i, with $\lambda \in \mathbb{R}^4$ and $u_i \in \mathbb{C}^3$, such that $\varphi_i(\lambda, 0) = \varphi_*(\lambda)$ (take a linear complement to the tangent space of M_* in that of M_i and apply the inverse function theorem). In the affine space S define

$$\Phi(\lambda, u_0, u_1, u_2, u_3) = \sum_i \varphi_i(\lambda, u_i) - 3\varphi_*(\lambda).$$

Its derivative at the origin is injective by (22). Add a linear coordinate $\eta \in \mathbb{R}^{973}$ on a complement of the image of this derivative in $O^\perp$. The resulting map is an analytic chart of the 1001-dimensional slice near P. On the coordinate subspace named in the proposition it equals φ_i, so M_i is that subspace. Because the chart is injective, any two of the four families meet exactly in M_*. □

6 No further solutions

This section proves Proposition 6.8, which gives Theorem 1.1.

6.1 A stress criterion

The criterion assumes the following.

(S1) $E = W_0 \perp \cdots \perp W_{m-1}$ is a finite-dimensional Euclidean space, $\mathcal{A} = \bigcup_i W_i$, and $\Lambda, \mathcal{R}$ are finite-dimensional real normed spaces.

(S2) $f : \Lambda \times E \to \mathcal{R}$ is C^2 near zero and vanishes near zero on each branch $\Lambda \times W_i$.

(S3) The quadratic part $q(u) = \frac{1}{2} D^2 f(0)[(0, u), (0, u)]$ has zero set exactly $\mathcal{A}$.

Write β for the polar form of q, the symmetric bilinear map with $\beta(u, u) = q(u)$.

Lemma 6.1 (A stress criterion for the distance bound)**.** *Assume* (S1)–(S3). *Suppose that for each i and each unit vector $w \in W_i$ there is a continuous field of covectors $\ell(\lambda, a) \in \mathcal{R}^*$, defined for (λ, a) near zero in $\Lambda \times W_i$ and allowed to depend on w, such that*

Lean: supported stresses

$$\ell(\lambda, a) D f(\lambda, a)(0, n) = 0 \qquad (n \in W_i^\perp), \tag{23}$$

$$\ell(0, 0) q(v) > 0 \qquad (0 \neq v \in W_i^\perp,\ \beta(w, v) = 0). \tag{24}$$

Then there are $c > 0$ and a neighborhood of zero in $\Lambda \times E$ on which

$$\operatorname{dist}(u, \mathcal{A})^2 \leq c \|f(\lambda, u)\|.$$

In particular $f(\lambda, u) = 0$ there if and only if $u \in \mathcal{A}$.

Proof. Each $f(\lambda, \cdot)$ vanishes on every W_j, and these subspaces span E. Hence $D_u f(\lambda, 0) = 0$ for small λ.

A violating sequence. Suppose the bound fails for every constant and every neighborhood. Then there is a sequence with

$$(\lambda_k, u_k) \to 0, \qquad \|f(\lambda_k, u_k)\| = o(\delta_k^2), \qquad \delta_k = \operatorname{dist}(u_k, \mathcal{A}) > 0.$$

Each u_k has a nearest subspace W_i, and one index i occurs infinitely often; pass to that subsequence. Put

$$a_k = \operatorname{proj}_{W_i} u_k, \qquad n_k = u_k - a_k \in W_i^{\perp}, \qquad \text{so that } \|n_k\| = \delta_k.$$

The nearest subspace is the one in which u_k has its largest component, so $a_k \neq 0$ and $\|n_k\|^2 \leq (m-1)\|a_k\|^2$. A further subsequence has

$$\frac{a_k}{\|a_k\|} \to w \in W_i, \qquad \frac{n_k}{\|n_k\|} \to v \in W_i^{\perp}, \qquad \frac{\|n_k\|}{\|a_k\|} \to \kappa \in [0, \infty),$$

with unit vectors w and v.

One expansion. Expand f about its zero (λ_k, a_k). The second derivative is continuous at the origin, so

$$f(\lambda_k, u_k) = D_u f(\lambda_k, a_k) n_k + q(n_k) + o(\|n_k\|^2). \tag{$*$}$$

The derivative in u vanishes at $(\lambda_k, 0)$, and integrating the second derivative along the segment to (λ_k, a_k) gives

$$D_u f(\lambda_k, a_k) n_k = 2\beta(a_k, n_k) + o(\|a_k\|\|n_k\|).$$

Divide $(*)$ by $\|a_k\|\|n_k\|$. The left side tends to zero, by the choice of the sequence and because the ratio $\|n_k\|/\|a_k\|$ is bounded. In the limit

$$2\beta(w, v) + \kappa\, q(v) = 0.$$

The limit ratio is zero. Suppose $\kappa > 0$. Since $q(w) = 0$,

$$q(w + \kappa v) = q(w) + \kappa\big(2\beta(w, v) + \kappa\, q(v)\big) = 0,$$

so $w + \kappa v$ lies in $\mathcal{A}$ by (S3). But this vector has nonzero components both in W_i and in its orthogonal complement, so it lies in none of the subspaces. Therefore $\kappa = 0$, and then $\beta(w, v) = 0$.

Contradiction. Take the field attached to this i and w, and apply its value at (λ_k, a_k) to $(*)$. The first term on the right vanishes exactly, by (23). Divide by $\|n_k\|^2$ and pass to the limit. The left side tends to zero, by the choice of the sequence and because the field is bounded near zero. By continuity of the field the right side tends to $\ell(0, 0) q(v)$, which is therefore zero. This contradicts (24).

The distance bound follows. The zero-set assertion follows from it, because $\mathcal{A}$ is closed and f vanishes on $\Lambda \times \mathcal{A}$. □

6.2 A quartic identity for the four transverse spaces

In the application the quadratic part is known through two quadratic maps H and T.

Appendix C fixes coordinates on the transverse spaces, a complex triple on each:

$$u = (u_0, u_1, u_2, u_3) \in \mathbb{C}^{12}, \qquad u_i = (u_{i0}, u_{i1}, u_{i2}) \in \mathbb{C}^3.$$

Write W_i also for the set where the other three triples vanish. Norms are Euclidean, and $\langle x, y\rangle = \sum_a \overline{x_a} y_a$. The maps are

$$H = (H_0, \dots, H_3), \quad T = (T_0, \dots, T_3), \qquad H_i, T_i : \mathbb{C}^{12} \to \mathbb{C}^3.$$

A third map involves one u_i at a time:

$$\sigma(u_i) = (u_{i1}u_{i2},\ u_{i2}u_{i0},\ u_{i0}u_{i1}), \qquad \sigma(u) = (\sigma(u_0), \dots, \sigma(u_3)).$$

The array. Arrange the twelve coordinates in four rows and three columns:

	x_1	x_2	x_3
u_0	u_{00}	u_{02}	u_{01}
u_1	u_{10}	u_{11}	u_{12}
u_2	u_{21}	u_{22}	u_{20}
u_3	u_{32}	u_{30}	u_{31}

(25)

Row i holds the entries of u_i, in the order shown. The columns are three vectors x_1, x_2, x_3 in $\mathbb{C}^4$, with entries $x_{r,i}$. Each monomial of σ multiplies two entries of one row; each monomial of H multiplies two entries, or their conjugates, from different rows and different columns; each monomial of T, two entries from different rows of one column.

Lemma 6.2 (Quartic identity)**.** *For all $u \in \mathbb{C}^{12}$,*

Lean: quartic identity

$$\sum_{i<j} \|u_i\|^2\|u_j\|^2 = \sum_i \|H_i(u)\|^2 + \tfrac{1}{2}\sum_i \|T_i(u)\|^2 + 3\,\mathrm{Re}\sum_i \langle \sigma(u_i), H_i(u)\rangle. \tag{26}$$

Proof. The formal development proves (26) by direct expansion: the left side has 54 monomials, and every other monomial on the right cancels.

The identity also follows from three others, which were verified by expanding both sides in computer algebra, with H and T as printed in Appendix C, and are not formalized:

$$\|\sigma(u) + H(u)\|^2 = \sum_{r<s} \|x_r\|^2\|x_s\|^2, \tag{27}$$

$$\tfrac{1}{2}\|T(u)\|^2 + \mathrm{Re}\sum_i \langle \sigma(u_i), H_i(u)\rangle = \sum_r \sum_{i<j} |x_{r,i}|^2 |x_{r,j}|^2, \tag{28}$$

$$\sum_{i<j} \|u_i\|^2\|u_j\|^2 = \sum_{r<s} \|x_r\|^2\|x_s\|^2 - \|\sigma(u)\|^2 + \sum_r \sum_{i<j} |x_{r,i}|^2 |x_{r,j}|^2. \tag{29}$$

Expanding the square in (27) and substituting it and (28) in (29) gives (26). Identity (27) is the multiplicativity of the octonion norm; the details are in Appendix C.2. □

Positivity. Let w lie in W_i and n in its orthogonal complement, where row i is zero, and suppose that both polar terms vanish:

$$H(w+n) - H(w) - H(n) = 0, \qquad T(w+n) - T(w) - T(n) = 0.$$

Both maps vanish at w, so they take the same values at $w+n$ and at n, and these two points differ only in row i. Subtract (26) at the second point from (26) at the first. All terms on the right cancel except the pairing with $\sigma(w_i)$. Thus

$$3\,\mathrm{Re}\langle \sigma(w_i), H_i(n)\rangle = \|w_i\|^2 \sum_j \|n_j\|^2. \tag{30}$$

6.3 Applying the criterion to the frame equations

Write $P = P(\xi)$. The residual F is a polynomial map of degree two. Let Q be its homogeneous quadratic part and $\mathcal{B}$ the polar form of Q. For every configuration x and every $v \in J^{39}$,

$$F(x+v) = F(x) + DF_x v + Q(v), \qquad DF_{x+v} = DF_x + 2\mathcal{B}(v, \cdot\,).$$

In particular

$$F(P+v) = DF_P v + Q(v). \tag{31}$$

For an automorphism g of J, let ρ_g act by g on the idempotency and frame-sum residuals and trivially on the scalar trace and overlap residuals. Then $F(gx) = \rho_g F(x)$ for every configuration x. For a derivation δ of J write ρ_δ for the derivative of $g \mapsto \rho_g$ at the identity in the direction δ.

Definition 6.3 (Stress and its energy)**.** Let x be a zero of F. A *stress* at x is a linear functional ℓ on the residual space with $\ell\, DF_x = 0$. Its *energy* is the function $E_\ell = \ell \circ F$ on J^{39}.

These are the equilibrium stresses, or self-stresses, of rigidity theory (Definition 2.3.1 of Connelly and Whiteley[8]), with the frame equations in the place of the edge-length equations of a framework.

Lemma 6.4 (Calculus of stresses)**.** *Let ℓ be a stress at a zero x of F.*

(a) *$E_\ell(x+v) = \ell Q(v)$ for every v.*

(b) *If $v \in \ker DF_x$ and δ is a derivation of J, then $\ell\mathcal{B}(v, \delta x) = 0$. Hence adding an orbit velocity δx to such a v does not change $\ell Q(v)$.*

(c) *For $g \in F_4$ the functional $\ell \circ \rho_g^{-1}$ is a stress at gx. If x' is a zero of F and $\ell\mathcal{B}(x'-x, \cdot\,) = 0$, then ℓ is a stress at x'.*

(d) *If v is the velocity at x of a C^2 curve of zeros of F, then $\ell Q(v) = 0$. Hence $\ell\mathcal{B}(v, v') = 0$ for any two vectors v, v' tangent at x to one submanifold consisting of zeros of F.*

The proof is in Appendix C.6.

The machine-checked identities at P. For quartet i, let $P_i'(g, h, d)$ be the code (17), and let $\mathfrak{n}_i(u_i)$ be the matrix in $\mathfrak{su}(3)$ with zero diagonal whose off-diagonal coordinates are those of row i of the table of signed coordinates in Appendix C.4. Then $V_\xi u$ is the sum of four velocities of simultaneous conjugation, one in the family of each quartet:

Lean: transverse normalization

$$V_\xi u = \sum_{i=0}^{3} \frac{d}{dt}\bigg|_{t=0} P_i'\big(e^{t\mathfrak{n}_i(u_i)},\, e^{t\mathfrak{n}_i(u_i)},\, d_i(\xi)\big), \qquad u \in \mathbb{C}^{12}.$$

It is a first-order deformation of P at every parameter. The velocity in the right parameter h is an orbit velocity (Lemma 5.10(a)), so taking $h = g$ rather than $h = I$ changes neither the projection of $V_\xi u$ to $O^\perp$ nor, by Lemma 6.4(b), any energy of a stress at P. At nonresonant parameters this projection identifies the four complex triples u_i with the four transverse spaces W_i of Proposition 5.8 (Appendix C.4).

Lean: Hessian identities

Proposition 6.5 (Finite identities)**.** *For every $\xi \in (S^1)^4$ there are twelve complex covectors h_{ia} and twelve t_{ia} on the residual space $\mathbb{R}^{2184}$ $(0 \le i \le 3,\ 0 \le a \le 2)$, specified in Appendix C, such that*

$$h_{ia} DF_P = t_{ia} DF_P = 0, \tag{32}$$

$$h_{ia} Q(V_\xi u) = H_i(u)_a, \qquad t_{ia} Q(V_\xi u) = T_i(u)_a \tag{33}$$

for all $u \in \mathbb{C}^{12}$, and such that $h_{ia}Q$ is supported on the twelve points of quartet i: it depends on no other coordinates.

By (32) the real and imaginary parts of these covectors are stresses at P. The proposition does not determine the covectors. The three points of frame 0 do not move in $V_\xi u$, so any covector on the idempotency equations of one of them that annihilates their linearization may be added to each of the twenty-four; for each of the three points these covectors form a space of dimension sixteen. The ancillary file `native-data.json` (Appendix C) gives one choice, with coefficients in $\mathbb{Q}(\sqrt{-1}, \sqrt{3})$: each t_{ia} with entries that are Laurent polynomials in ξ, and each h_{ia} at one parameter ξ_*, whose coordinates lie in $\mathbb{Q}(\sqrt{-1})$, carried to every ξ by a diagonal automorphism of J. In this choice each h_{ia} gives weight only to equations among the twelve points of quartet i: their idempotency equations, the frame-sum equations of its four frames, and the overlaps of points in two different frames of the quartet. Each t_{ia} gives weight only to the same kinds of equation among the twenty-seven points outside quartet i, with overlaps only between frames in different lobes. The identities were verified by exact computation as part of the formal development (Section 9); Appendix C lists the checks and describes the coefficient file.

Stresses along each family. By the support condition of Proposition 6.5, $h_{ia}\mathcal{B}(v, \cdot)$ vanishes whenever the components of v at the twelve points of quartet i vanish. Let P' be a code of the form (17) for this quartet. It is a zero of F with the same twelve points as P, so h_{ia} is a stress at P' by Lemma 6.4(c).

The points of M_i are the slice representatives of such codes (Lemma 5.10(a)). The point with parameter p (a parameter of the family, not a point of $\mathbb{O}P^2$) is $g(p)^{-1}P'(p)$, where $P'(p)$ has the form (17) and $g(p) \in F_4$ depends analytically on p (Lemma 5.9). By Lemma 6.4(c) again,

$$h_{ia}(p) = h_{ia} \circ \rho_{g(p)}$$

is a stress at that point. This is an analytic field of stresses along the family, equal to h_{ia} at P.

Lean: `along-family field`

Reduction to the kernel coordinates. In Lemma 6.1 take $\Lambda = \mathbb{R}^4$, the space of λ, $E = \mathbb{C}^{12}$ and $\mathcal{R} = \mathbb{R}^{2184}$, the residual space.

Lemma 6.6 (The reduced equations)**.** *At a nonresonant parameter, the chart of Proposition 5.11 can be chosen so that its transverse coordinates are those of* (25)*, with the norm of $\mathbb{C}^{12}$ on $E = \bigoplus_i W_i$. There is then an analytic map b into $\mathcal{O}^\perp$, defined near zero in $\Lambda \times E$ and zero at the origin, for which the map*

$$f(\lambda, u) = F(P + b(\lambda, u)),$$

with all residual rows retained, has three properties.

(i) *The zeros of F near P in the slice are exactly the points $P + b(\lambda, u)$ at which f vanishes.*

Lean: `Lyapunov-Schmidt reduction`

(ii) *For $u \in W_i$, the point $P + b(\lambda, u)$ is the point of M_i with coordinates (λ, u); so f vanishes on $\Lambda \times \bigcup_i W_i$.*

(iii) *$D_u b(0) = \Pi V_\xi$, where Π is the orthogonal projection onto $\mathcal{O}^\perp$.*

The proof is a Lyapunov–Schmidt reduction, given in Appendix C.5.

Lemma 6.7 (Reduced identities)**.** *Put $q_f(u) = \frac{1}{2}D^2 f(0)[(0,u),(0,u)]$. Then*

$$h_{ia} q_f(u) = H_i(u)_a, \qquad t_{ia} q_f(u) = T_i(u)_a,$$

and the zero set of q_f is the union of the four transverse spaces.

Proof. By Lemma 6.4(a), $h_{ia} f(\lambda, u) = h_{ia} Q(b(\lambda, u))$ exactly. By Lemma 6.6(iii), and since b vanishes at the origin, the part of second order in u at $\lambda = 0$ is

$$h_{ia} q_f(u) = h_{ia} Q(\Pi V_\xi u) = h_{ia} Q(V_\xi u) = H_i(u)_a.$$

The second equality is Lemma 6.4(b): the vector $V_\xi u$ lies in $\ker DF_P$, and its projection differs from it by an orbit velocity. The third is (33). The same argument applies to t_{ia}.

The form q_f vanishes on the four transverse spaces because f does (Lemma 6.6(ii)). Conversely, where q_f vanishes so do H and T, and (26) leaves at most one u_i nonzero. □

Proposition 6.8 (No further solutions)**.** *At every nonresonant parameter, the real zero set of the frame equations in a sufficiently small neighborhood of $P(\xi)$ in the slice S is exactly the union of the four families M_i. In the reduced coordinates, for some $c > 0$ and all (λ, u) near zero,*

Lean: stress criterion

$$\operatorname{dist}\big(u, \textstyle\bigcup_i W_i\big)^2 \le c\|f(\lambda, u)\|.$$

Proof. We check the hypotheses of Lemma 6.1 for the map f of Lemma 6.6. Conditions (S1) and (S2) hold by that lemma, and (S3) is Lemma 6.7.

For the stresses, fix i and a unit vector $w \in W_i$. By Lemma 6.6(ii) the point $P + b(\lambda, u)$ with $u \in W_i$ lies in M_i; let p be its parameter and take the real field

$$\ell_{i,w}(p) = 3\,\mathrm{Re} \sum_{a=0}^{2} \overline{\sigma(w_i)_a}\, h_{ia}(p),$$

with $h_{ia}(p)$ the field along the family constructed above. It is continuous and a stress at that point. By the chain rule $Df = DF \circ Db$ it annihilates the derivative of f there, which gives (23).

Let $v \in W_i^{\perp}$ satisfy $\beta_f(w, v) = 0$, where β_f is the polar form of q_f. Then both polar terms of H and T vanish at this pair, by Lemma 6.7, so (30) applies with $n = v$. Since w_i is a unit vector, it gives

$$\ell_{i,w}(0) q_f(v) = 3\,\mathrm{Re}\langle \sigma(w_i), H_i(v)\rangle = \|v\|^2,$$

which is (24).

Lemma 6.1 therefore gives the bound and shows that f vanishes near zero only where u lies in one of the transverse spaces. By Lemma 6.6(i) the zeros of F near P in the slice are then the points $P + b(\lambda, u)$ with such u, and by Lemma 6.6(ii) these are the points of the four families. □

Proof of Theorem 1.1. In the coordinates of Proposition 5.11 each family is a coordinate subspace. So by Proposition 6.8 the zero set of F in a small slice is a neighborhood of the origin in $\mathbb{R}^4 \times \bigcup_i \mathbb{R}^6_i \times \{0\}$, the model (2), with the families as branches. Their common part, where u and η vanish, is the family M_* of Lemma 5.10(b), the slice image of the four-parameter family. □

The constants and neighborhoods may depend on ξ.

7 A sharp bound on distance to the solution set

This section proves Theorem 1.2. Part (a) is the bound of Proposition 6.8 in the slice, carried to a neighborhood of the orbit by Lemma 5.9. Fix a nonresonant ξ and write $P = P(\xi)$. Distances in J^{39} are measured in the norm of the trace form, for which F_4 acts by isometries.

7.1 The bound near the orbit

In the slice. Let s be a point of the slice near P, with coordinates (λ, u, η) in the chart of Proposition 5.11, and let $s' = P + b(\lambda, u)$ have the same (λ, u). The Lyapunov–Schmidt reduction (Appendix C.5) replaces the last coordinate by one that vanishes exactly at such points s', the component of $F(s)$ in the image of DF_P. This change of coordinates and F are locally Lipschitz, so for some constant c

$$\|s - s'\| \le c\|F(s)\|, \qquad \|F(s')\| \le c\|F(s)\|.$$

By definition $F(s') = f(\lambda, u)$. A point u' of $\bigcup_i W_i$ nearest to u is an orthogonal projection of it, so no farther from the origin, and $P + b(\lambda, u')$ lies in a family, hence in $\mathcal{Z}$. Since b is locally Lipschitz, the reduced bound of Proposition 6.8 gives a constant c' with

$$\operatorname{dist}(s', \mathcal{Z})^2 \le c'\|F(s')\|.$$

For s so close to P that $\|F(s)\| \le 1$, the two combine:

$$\operatorname{dist}(s, \mathcal{Z})^2 \le 2\|s - s'\|^2 + 2\operatorname{dist}(s', \mathcal{Z})^2 \le (2c^2 + 2cc')\,\|F(s)\|.$$

The whole orbit. Choose ε in Lemma 5.9 so small that this estimate holds on S_ε and that the zeros of F there are the points of the four families (Proposition 6.8). Let U_ξ be the tube $U_\varepsilon = F_4 S_\varepsilon$, an open neighborhood of the orbit. Every point of the tube is $y = gs$ with $g \in F_4$ and $s \in S_\varepsilon$. The isometry g preserves the zero set, and $F(gs) = \rho_g F(s)$ by the equivariance of Section 6, so

$$\operatorname{dist}(y, \mathcal{Z}) = \operatorname{dist}(s, \mathcal{Z}), \qquad \|F(s)\| \le c''\|F(y)\|.$$

Here c'' bounds the operator norms of the ρ_g^{-1}; it is finite because F_4 is compact. Hence

$$\operatorname{dist}(y, \mathcal{Z})^2 \le C_\xi\|F(y)\| \quad (y \in U_\xi), \qquad C_\xi = (2c^2 + 2cc')\,c''.$$

An F_4-invariant subset of the tube is F_4 times its part in the slice (Lemma 5.9(b)). For the zeros in the tube that part is the union of the four families in S_ε, by the choice of ε. Therefore

$$\mathcal{Z} \cap U_\xi = F_4\big((M_0 \cup \cdots \cup M_3) \cap S_\varepsilon\big) \subset F_4(M_0 \cup \cdots \cup M_3) \cap U_\xi \subset \mathcal{Z} \cap U_\xi.$$

The last inclusion holds because the families consist of zeros and $\mathcal{Z}$ is invariant. This proves part (a). The formal statement has the union of the families in place of the zero set; that form, and its equivalence with part (a), is item (1) of Section 9.

7.2 Optimality of the square-root exponent

Lemma 7.1 (Kernel larger than the tangent cone)**.** *Let G be a C^2 map between finite-dimensional normed spaces with $G(p) = 0$, and let $v \in \ker DG_p$. Suppose that v is not in the tangent cone of $G^{-1}(0)$ at p: it is not a limit of vectors $(x_k - p)/t_k$ with $G(x_k) = 0$ and $t_k \downarrow 0$. Then there is $c > 0$ such that, for all small $t > 0$,*

$$\operatorname{dist}\big(p + tv, G^{-1}(0)\big) \ge ct, \qquad \textit{whereas} \qquad \|G(p + tv)\| = O(t^2).$$

In particular no bound $\operatorname{dist}(y, G^{-1}(0)) \le C\|G(y)\|^{\alpha}$ *with* $\alpha > 1/2$ *holds on a neighborhood of* p.

Proof. If there were no such c, the distance would be $o(t_k)$ along some sequence $t_k \downarrow 0$. Zeros x_k of G with $\|x_k - p - t_k v\| = o(t_k)$ would then satisfy $(x_k - p)/t_k \to v$, against the hypothesis. The second estimate is Taylor's formula, since $G(p) = 0$ and $DG_p v = 0$. A bound with exponent α would give $ct \le C' t^{2\alpha}$ for all small $t > 0$, which is impossible when $2\alpha > 1$. □

The witness. For the frame equations, let every u_i be $(1, 1, 1)$ and put $v = V_\xi u$, a sum of velocities of the four families and so a first-order deformation. Let $h = h_{00}$. It is a complex-valued stress at P by (32), and its energy (Definition 6.3) is given by Lemma 6.4(a):

$$E_h(P + w) = h\boldsymbol{Q}(w).$$

The energy vanishes at every point of $\mathcal{Z}$, whereas $E_h(P + v) = H_0(u)_0 = 1$, by (33) and the formula for H_0 in Appendix C. The form $h\boldsymbol{Q}$ is continuous, so it has no zero in some ball of radius $c > 0$ about v; it is homogeneous, so it has none in the ball of radius ct about tv. Hence

Lean: sharpness witness

$$\operatorname{dist}(P + tv, \Sigma) \ge ct \quad (t > 0)$$

for every nonempty $\Sigma \subset \mathcal{Z}$. In particular v is not in the tangent cone of $\mathcal{Z}$ at P, and Lemma 7.1, applied to F at P, gives part (b).

The same inequality excludes an exponent above $1/2$ for the distance to any nonempty subset of the zero set. Lemma 7.1 applies equally to any C^2 map vanishing exactly on $\mathcal{Z}$ near P, since its derivative at P vanishes on the orbit tangent space and on the tangent spaces of the families, which together span $\mathcal{T}$ (Proposition 5.8, Lemma 5.10(c)).

The witness can be moved onto configurations whose thirteen triples are Jordan frames. On them only the overlaps between distinct frames can fail. Fix a frame r. If it lies in quartet i, the family of quartet i does not move it. Otherwise, in the coordinates of quartet i, frame r is the orbit of θ_i on a point $p(U)$ (Lemma 4.2), and with $N = \mathfrak{n}_i(u_i)$ the family moves it by right multiplication with $U^* e^{tN} U e^{-tN}$. This is an automorphism of J by Lemma 4.3; let $\delta_{i,r}$ be its derivative at $t = 0$, a derivation. With δ_r the sum of the $\delta_{i,r}$ over the families that move frame r, the component of v at frame r is δ_r applied to the points of that frame. Let $\gamma(t)$ have frame r equal to $\exp(t\delta_r)$ applied to frame r of P. Its triples are Jordan frames, and $\gamma(t) = P + tv + O(t^2)$. Since $DF_P v = 0$, $\|F(\gamma(t))\| = O(t^2)$, while $\operatorname{dist}(\gamma(t), \mathcal{Z}) \ge ct - O(t^2)$. So no exponent above $1/2$ is possible even on configurations whose triples are Jordan frames. This consequence is proved here only; it is not part of the formal development.

8 Open patches in the unlabeled space $\mathcal{M}$

This section proves Theorem 1.3 and Corollary 1.4. Only nonresonant members of the family enter, so the germ of $\mathcal{M}$ at the original code is not determined.

8.1 Open patches modulo F_4

The canonical partition of a tight code into thirteen frames is locally constant by Lemma 3.3, so the zero set $\mathcal{Z}$ of F is open in C (Section 3). It is F_4-invariant, so its quotient $\mathcal{Z}/F_4$ is open in Y.

Lean: code/frame topology

Now fix ξ, and let S_ε and U_ε be the slice and the tube of Lemma 5.9 at $P(\xi)$. By part (b) of that lemma, applied to the zeros in the tube, sending a point to its orbit is a homeomorphism from the zero set $\mathcal{Z} \cap S_\varepsilon$ in the slice onto $(\mathcal{Z} \cap U_\varepsilon)/F_4$. The latter is open in $\mathcal{Z}/F_4$ and hence in Y.

Lean: open labeled slice

Let ξ be nonresonant. In the model (2) the set $\mathbb{R}^4 \times (W_i \setminus \{0\})$ is open, because its complement, the union of the other three branches, is closed; it is a ten-dimensional manifold, and the origin lies in its closure. By Theorem 1.1 and the homeomorphism just described, every neighborhood of the class of $P(\xi)$ in Y contains a nonempty open subset of Y homeomorphic to an open subset of $\mathbb{R}^{10}$.

8.2 Passing to the unlabeled quotient

The symmetric group $\Gamma = \mathfrak{S}_{39}$ acts on Y by homeomorphisms, and $Y/\Gamma = \mathcal{M}$. It need not preserve $\mathcal{Z}/F_4$, since permuting points can change the chosen frame partition.

Lemma 8.1 (Quotient by a finite group)**.** *Let a finite group Γ act by homeomorphisms on a Hausdorff space Y, with quotient map $q\colon Y \to Y/\Gamma$, and write Γ_y for the stabilizer of y.*

(a) *Every $y \in Y$ has an open neighborhood Ω_y with $\gamma\Omega_y \cap \Omega_y = \varnothing$ for all $\gamma \notin \Gamma_y$. In particular the stabilizer of every point of Ω_y is contained in Γ_y.*

(b) *Let Ω be an open subset of Ω_y all of whose points have stabilizer Γ_y. Then q restricts to a homeomorphism from Ω onto an open subset of Y/Γ.*

(c) *If $\Omega \subset Y$ is a nonempty open set locally homeomorphic to $\mathbb{R}^n$, then its image in Y/Γ contains a nonempty open subset homeomorphic to a nonempty open subset of $\mathbb{R}^n$.*

Proof. (a) For each $\gamma \notin \Gamma_y$ choose disjoint open sets $\Omega'_\gamma \ni y$ and $\Omega''_\gamma \ni \gamma y$. Let Ω_y be the intersection of the finitely many sets $\Omega'_\gamma \cap \gamma^{-1}\Omega''_\gamma$. Then $\Omega_y \subset \Omega'_\gamma$ and $\gamma\Omega_y \subset \Omega''_\gamma$ are disjoint. An element fixing a point z of Ω_y does not move Ω_y off itself, so it lies in Γ_y.

(b) The elements of Γ_y fix Ω pointwise, and the others move it off itself, so q is injective on Ω. The image of an open subset Ω' of Ω is open, because its preimage $\bigcup_{\gamma\in\Gamma} \gamma\Omega'$ is open. Thus the restriction is an open continuous injection.

(c) Choose $y \in \Omega$ whose stabilizer has the least order among points of Ω. By (a) the stabilizers of the points of $\Omega \cap \Omega_y$ are subgroups of Γ_y, and by minimality they equal Γ_y. Apply (b) to a coordinate ball about y in $\Omega \cap \Omega_y$. □

The space Y is Hausdorff because F_4 is compact: its quotient topology is induced by the metric

$$d_Y([x],[x']) = \min_{g\in F_4} \|x - gx'\|$$

in the norm of the trace form. As in the lemma, $q\colon Y \to \mathcal{M}$ is the quotient map.

Proposition 8.2. *Let ξ be nonresonant. Every open neighborhood Ω of the class of $P(\xi)$ in $\mathcal{M}$ contains a nonempty open subset of $\mathcal{M}$ homeomorphic to a nonempty open subset of $\mathbb{R}^{10}$.*

Proof. The open set $q^{-1}(\Omega) \ni [P(\xi)]$ contains, by the end of the previous subsection, a nonempty open subset Ω' of Y homeomorphic to an open subset of $\mathbb{R}^{10}$. Lemma 8.1(c), applied to Ω', gives the required subset of $\mathcal{M}$ inside $q(\Omega') \subset \Omega$. □

Remark 8.3 (The unlabeled germ). Let $\mathrm{Aut}(P) \subset F_4$ be the group of automorphisms of J that permute the points of $P = P(\xi)$; it contains K. For $h \in \mathrm{Aut}(P)$ let $\sigma_h \in \mathfrak{S}_{39}$ be the relabeling with $\sigma_h h P = P$, and let $\tilde{h} = \sigma_h h$: it applies h to each point and then restores the labels of P, as in Lemma 5.2(c). The map $h \mapsto \sigma_h$ is a homomorphism onto the stabilizer Γ_y of $y = [P]$ in $\mathfrak{S}_{39}$: if $\sigma P = gP$ with $g \in F_4$, then g^{-1} permutes the points of P and $\sigma = \sigma_{g^{-1}}$. It is injective because only the identity fixes every point (Lemma 5.7), so $\mathrm{Aut}(P)$ is finite.

Since h preserves overlaps, σ_h maps frames to frames, so σ_h and $\tilde{h}$ preserve $\mathcal{Z}$. The map $\tilde{h}$ is orthogonal, fixes P and satisfies $\tilde{h}(gx) = (hgh^{-1})\tilde{h}x$ for $g \in F_4$. Hence it preserves O, $O^\perp$, S_ε and $U_\varepsilon = F_4 S_\varepsilon$, and $\mathrm{Aut}(P)$ acts on $\mathcal{Z} \cap S_\varepsilon$. The maps $\tilde{h}$ and σ_h induce the same map of Y, so the homeomorphism $s \mapsto [s]$ from $\mathcal{Z} \cap S_\varepsilon$ onto $V = (\mathcal{Z} \cap U_\varepsilon)/F_4$ carries $\tilde{h}$ to σ_h.

Let Ω be the intersection of the sets $\gamma(\Omega_y \cap V)$ with $\gamma \in \Gamma_y$, where Ω_y is as in Lemma 8.1(a). It is a Γ_y-invariant open neighborhood of y, and every $\gamma \notin \Gamma_y$ moves it off itself. So q induces a continuous injection of Ω/Γ_y into $\mathcal{M}$, which is open by the argument for part (b) of that lemma. Thus a neighborhood of the class of P in $\mathcal{M}$ is homeomorphic to the quotient by $\mathrm{Aut}(P)$ of an invariant open neighborhood of P in $\mathcal{Z} \cap S_\varepsilon$. For nonresonant ξ this is, by Theorem 1.1, a neighborhood of the origin in the model (2) modulo an action of $\mathrm{Aut}(P)$ by homeomorphisms fixing the origin. We determine neither the group nor the action.

Modulo F_4 and relabeling, each member of M_i is quartet 0 together with the nine frames of $kE_{ab}(d)k^*$, where $k \in SU(3)$ is a scalar multiple of $G_i^* g L_i$ (Appendix A): conjugation by the diagonal matrix G_i^* in the coordinates of quartet i is an automorphism of J taking that quartet to quartet 0 (Appendix A.6 and (38)), and Lemma 4.3 removes the right factor. So, for nonresonant ξ, near the class of $P(\xi)$ in $\mathcal{M}$ the four families are four pieces of the image of the one map of Lemma 4.4, near the four points $([G_i^* L_i], d_i(\xi))$ of $PSU(3) \times T^2$, and by Theorem 1.1 they make up all of $\mathcal{M}$ there. In Y any two of them meet only in the four-parameter family; whether they are identified further in $\mathcal{M}$ depends on $\mathrm{Aut}(P)$. This remark is not part of the formal development.

8.3 Patches near the original code

Nonresonant parameters approach $\xi = 1$, since the resonant ones form a finite union of cosets of closed subgroups of codimension one (Section 4).

By continuity of the configuration and quotient maps, every open neighborhood of c_0 contains the class of such a member, and Proposition 8.2 at that parameter gives the patch. This proves Theorem 1.3.

Proof of Corollary 1.4. Let a neighborhood of c_0 in $\mathcal{M}$ be homeomorphic to an open subset of $\mathbb{R}^n$. It contains an open neighborhood Ω of c_0, again homeomorphic to an open subset of $\mathbb{R}^n$. By Theorem 1.3, Ω contains a nonempty open subset of $\mathcal{M}$ homeomorphic to an open subset of $\mathbb{R}^{10}$. The image of that subset in the chart is open in $\mathbb{R}^n$, so $n = 10$ by invariance of dimension. □

The case $n = 24$ is the formally verified statement. In the labeled space Y more is true: no neighborhood of $[P(1)]$ is homeomorphic to an open subset of any $\mathbb{R}^n$. Such a neighborhood would contain the class of a nonresonant $P(\xi)$ as an interior point. By Theorem 1.1 and the slice homeomorphism of the first subsection, Y near that class is the model (2) near the origin. Write the model as $\mathbb{R}^4 \times C$, where C is the union of four 6-planes meeting only at the origin. Since $C \setminus 0$ is homotopy equivalent to four disjoint 5-spheres, $H_*(C, C \setminus 0)$ is $\mathbb{Z}^3$ in degree 1 and $\mathbb{Z}^4$ in degree 6, and by the Künneth formula the local homology of the model at the origin is $\mathbb{Z}^3$ in degree 5 and $\mathbb{Z}^4$ in degree 10, which is not that of a manifold point. The argument does not apply to $\mathcal{M}$, whose germ there is the model modulo the finite group of Remark 8.3: a quotient of the model by a finite group that permutes its branches can be a manifold.

9 Formal verification and data

A development of the results in the Lean 4 theorem prover[18], built on Mathlib[19], is available at

`https://github.com/macgregor-sudo/tight-39-code`

at revision `46f01c1`. It is registered as `PALOMAR-2026-10-29-000034`, version 1, in the Palomar registry[20]. The registry's record for that revision reports that the 29 submitted theorems were accepted by the Lean kernel and, independently, by the external checker NanoDa, with no axioms beyond `propext`, `Quot.sound` and `Classical.choice`. It includes a Comparator configuration listing the 29 targets. A third checker, con-ron, was run and stopped at its resource ceiling; no acceptance by it is claimed. Registration is not acceptance by a human referee.

What was checked is stated in one file: `Challenge.lean` (993 lines) imports only Mathlib and contains the 29 statements with every definition they use. The registry also checks that `Solution.lean`, which assembles the proofs in the directory `Proofs` (956 Lean files, about 151,000 lines), states the same theorems with the same definitions.

The formal development uses the product of J, the frame equations, the automorphism action and the quotient $\mathcal{M}$ as defined in this paper. Its predicate `IsGoodPhase` says that no entry of the four triples (16) has cube one, which by Proposition 5.8 is nonresonance. The dimension formula (21) that gives this

equivalence is itself a registered target, `finrank_original_derivative_kernel`, at every ξ.

Table 2 gives the status of every numbered theorem, proposition, lemma and corollary. *Registered* means that it is stated by one of the 29 registered targets, *compiled* that a declaration of the development at the revision above states it but was not submitted as a target, and *text* that its proof is in this paper only. Where the formal statement is narrower than the printed one, the last column says how.

Three of the differences in Table 2 need a comment.

(1) *The distance bound.* For nonresonant ξ write $P = P(\xi)$, let S_0 be the union of suitable neighborhoods of P in the four families, and let $S_\xi = F_4 S_0$. `native_real_endpoint` asserts that some open neighborhood U of $F_4 P$ and some constant C satisfy

Lean: whole-orbit estimate

$$\operatorname{dist}(y, S_\xi)^2 \le C\|F(y)\|, \qquad F(y) = 0 \iff y \in S_\xi \quad (y \in U), \tag{34}$$

for both the coordinate norm and the norm of the trace form, and that every exponent $\alpha > 1/2$ fails for S_ξ. This form is equivalent to Theorem 1.2. It implies part (a) with $U_\xi = U$, since $S_\xi \subset \mathcal{Z}$. Conversely, in the notation of Section 7, for $S_0 = (M_0 \cup \cdots \cup M_3) \cap S_\varepsilon$ the proof given there shows that $\mathcal{Z} \cap U_\xi = S_\xi$. If the $2r$-neighborhood of the orbit lies in U_ξ and y is within r of the orbit, the zeros within r of y include an orbit point and lie in $U_\xi \cap \mathcal{Z} = S_\xi$; so $\operatorname{dist}(y, S_\xi) = \operatorname{dist}(y, \mathcal{Z})$ and (34) holds there with $C = C_\xi$. The lower bound for an arbitrary nonempty $\Sigma \subset \mathcal{Z}$ is the compiled theorem `native_zero_set_lower_bound`, which was not submitted as a separate target.

(2) *The open patches.* The open-patch theorem is compiled, and the formal proof of the registered corollary rests on the same statement, but it was not submitted as a separate target.

(3) *The frames.* The formal statement corresponding to Proposition 3.2 asserts that a partition into frames exists and not that it is unique.

Lean: formal patch endpoint

The formal proof of the open-patch theorem uses a nonzero Jacobian minor and complex local isolation on a dense subset of $\mathcal{N}$, where Section 8 combines Theorem 1.1 with Lemma 8.1. The isolation rests on a bound for the Hilbert–Samuel function of the complexified equations at $P(\xi)$, obtained from the stresses of Proposition 6.5. The formal route thus avoids Theorem 1.1, Lemma 6.1 and Proposition 6.8, but not Proposition 6.5. The printed proof of Lemma 6.1 obtains both of its limits from one expansion, where the formal proof treats the two limits separately; the statement is the same. The formal check covers the Lean statements, not the prose of this paper.

The finite identities of Appendix C and the quartic identity (26) are part of the formal development. The coefficients of the former are supplied with this paper as the ancillary file `anc/native-data.json`, exported from the definitions at the revision above. What was replayed from the file, and in what arithmetic, is stated at the end of Appendix C. The re-expressions of (26) in the proof of Lemma 6.2,

Table 2: Status of the numbered statements. Declarations are named without their namespaces.

Statement	Status	Lean declarations; difference from the printed statement
Thm. 1.1	registered	`normal_form_exists`; `native_A5_geometry` for the common part
Thm. 1.2	registered	`native_real_endpoint`; stated for S_ξ, item (1) below
Thm. 1.3	compiled	`originalCode_has_ten_patches`; item (2) below
Cor. 1.4	registered	`originalCode_no_open24_chart`, `originalCode_no_charted24_neighborhood`; only $n = 24$, the general case is text
Lem. 3.1	compiled	`overlap_nonneg`
Prop. 3.2	registered	`code_bound`, `code_bound_sharp`, `tight_code_iff_frames`; uniqueness of the partition is compiled only (`tight_code_frames`), item (3) below
Lem. 3.3	compiled	`partition_native_nhds`
Lem. 3.4	text	
Lem. 4.1	text	Appendix A.3
Lem. 4.2	registered	(i) `native_idempotent_iff_scaled_specialUnitary_of_literal_quartet_readouts`, `nativeComplementaryPoint_scaled_hidden_inverse`, `state_equivalence`; (ii) `native_complementary_frame_iff_exists_central_order`; (iii) `native_ordered_central_triples_overlap_iff_hilbertSchmidt`; also `nine_frame_array`. Only for quartet 0 of P; the condition $p + \theta p + \theta^2 p = e$ is not stated
Lem. 4.3	registered	`quartet_fixer_classification`, `quartet_fixer_converse`; only for quartet 0 of P
Lem. 4.4	registered	`full_A2`; only for quartet 0 of P
Lem. 4.5	text	`native_A5_geometry` states, at nonresonant ξ, that the members with $h = I$, moved by automorphisms, are zeros of F
Lem. 5.1	registered	`unrestricted_matrix_normal_form`, `firstOrder_dimension`, `resonance_count_cases`
Lem. 5.2	registered	`full_A4`: θ_0 and θ_1, of order three and commuting, their permutation table, and the action of K on $\mathcal{T}$ and O; the rest of (b) and (c) is text
Lem. 5.3	text	
Lem. 5.4	text	Appendix B.1; the dimension 8 is compiled for the standard quartet (`finrank_quartetFirstOrder_std`)
Lem. 5.5	compiled	`finrank_anchorFixed`, the dimension only
Lem. 5.6	compiled	`finrank_commonFixed`
Lem. 5.7	registered	`full_A4`: the three dimensions and injectivity of $\delta \mapsto \delta P$; triviality of the stabilizer is text
Prop. 5.8	registered	`finrank_original_derivative_kernel` for (21); `full_A4` for the action of K on $\overline{\mathcal{T}}$; both at every ξ
Lem. 5.9	text	Appendix A.7; `native_A5_geometry` gives, at nonresonant ξ, an analytic chart near P and the open embedding of the slice zeros into $\mathcal{Z}/F_4$; the open embedding is compiled at every ξ (`exists_sliceToOrbit_isOpenEmbedding`)
Lem. 5.10	registered	`native_A5_geometry`, at nonresonant ξ only: the two consequences listed in part (a) and part (b); the first sentence of (a) and part (c) are text
Prop. 5.11	registered	`native_A5_geometry`
Lem. 6.1	registered	`supported_stress_local_criterion`
Lem. 6.2	compiled	`quartic_identity_values`; the three identities of its proof are computer algebra, not formalized
Lem. 6.4	text	Appendix C.6
Prop. 6.5	compiled	`H_equilibrium`, `T_equilibrium`, `H_reproduction_phase`, `T_reproduction_phase`; support: `hasAnchorSupport`
Lem. 6.6	compiled	part (i): `originalF_zero_iff_coordinates`; Appendix C.5
Lem. 6.7	text	
Prop. 6.8	compiled	`local_reduced_bound`
Lem. 7.1	text	
Lem. 8.1	text	
Prop. 8.2	text	compiled on a dense subset of $\mathcal{N}$ (`native_code_ten_patches`)

and the matrix form (40) of the quadratic maps, were verified by computer algebra and are not formalized.

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Use of AI. Except for attributed third-party material, all text and code of this work was generated by AI systems under the author's direction: Anthropic's Claude Fable 5.2 and OpenAI's Astra GPT 6. The work has not had independent human peer review. The author is responsible for the work; AI systems are not authors. Robert is an amateur who welcomes collaboration (macgregor@gmail.com).

A Coordinates and changes of quartet

Appendix A.3 and Appendices A.5–A.7 contain the rest of the proofs of Lemmas 4.1, 5.10 and 5.9. The rest of this appendix is verification by direct computation; a reader who accepts the displayed identities does not need the tables.

A.1 The octonions and the coordinate map

Write an octonion as $x = c + z_1 j + z_2 \ell + z_3 n = (c, z)$, with $c, z_r \in \mathbb{C}$ on the left. Its eight real coordinates in the basis $(1, i, j, k, \ell, i\ell, j\ell, k\ell)$ are

$$(\operatorname{Re} c, \operatorname{Im} c, \operatorname{Re} z_1, \operatorname{Im} z_1, \operatorname{Re} z_2, \operatorname{Im} z_2, \operatorname{Re} z_3, -\operatorname{Im} z_3).$$

In this splitting $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ the octonion multiplication becomes

$$(c, z)(c', z') = (cc' - z \cdot \bar{z}',\ cz' + \bar{c}'z + \overline{z \times z'}), \tag{35}$$

where dot and cross products are complex bilinear, with orientation $(1, 2, 3)$. This follows by multiplying the eight basis units; it fixes the sign of the last coordinate. Up to the sign of the vector part it is the multiplication in Section 1.5 of Yokota[15].

Write an element of J as the Hermitian matrix

$$y = \begin{pmatrix} \alpha_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \alpha_2 & x_1 \\ x_2 & \bar{x}_1 & \alpha_3 \end{pmatrix}, \qquad x_r = (c_r, z_r).$$

Put $b_r = -\bar{z}_r \in \mathbb{C}^3$. The coordinate map sends y to the pair (a, B), where

$$a = \begin{pmatrix} \alpha_1 & c_3 & \bar{c}_2 \\ \bar{c}_3 & \alpha_2 & c_1 \\ c_2 & \bar{c}_1 & \alpha_3 \end{pmatrix}$$

and B is the matrix with rows b_1, b_2, b_3. This real-linear bijection visibly preserves unit and trace; expanding the trace form gives (10).

To check the product, it suffices to check squares. For cyclic (r, s, t), octonion matrix multiplication gives the entries α'_r, x'_r of y^2:

$$\alpha'_r = \alpha_r^2 + |x_s|^2 + |x_t|^2, \qquad x'_r = (\alpha_s + \alpha_t)x_r + \overline{x_s x_t}.$$

By (35), the scalar and vector components of x'_r are

$$\begin{aligned} c'_r &= (\alpha_s + \alpha_t)c_r + \bar{c}_s\bar{c}_t - \bar{z}_s \cdot z_t, \\ b'_r &= (\alpha_s + \alpha_t)b_r - \bar{c}_s b_t - c_t b_s + \overline{b_s \times b_t}. \end{aligned}$$

The first equation together with the diagonal formula is

$$a' = a^2 - BB^* + \operatorname{tr}(BB^*)I.$$

In the second, the conjugated cross products are the rows of $\operatorname{adj}(B)^*$ and the other terms those of $((\operatorname{tr} a)I - a)B$. This proves (11).

A.2 The automorphisms ι_i and θ_i

The maps $\iota_1, \iota_2, \iota_3$ are signed permutations of the octonion basis in each off-diagonal entry; they fix the diagonal entries. Let e_m, $m = 0, \dots, 7$, be the basis $(1, i, j, k, \ell, i\ell, j\ell, k\ell)$. On the entry x_r, the map ι_i sends

$$e_m \longmapsto \epsilon_{rm}\, e_{m \operatorname{xor} \mu_r},$$

with the masks μ_r and the signs ϵ_{rm} of the table.

	(μ_1, μ_2, μ_3)	signs on x_1	signs on x_2	signs on x_3
ι_1	$(6, 4, 2)$	$-+++{+}{-}{-}{-}$	$--+++{-}{-}{+}$	$++-+-+++$
ι_2	$(2, 6, 4)$	$-+--++{+}{-}$	$+++---{-}{+}$	$-+-++++$
ι_3	$(4, 2, 6)$	$++--+-{-}{+}$	$--+--+++$	$-+++-+++$

The masks 2, 4, 6 move the line $\mathbb{C}$ to $\mathbb{C}j$, $\mathbb{C}\ell$, $\mathbb{C}n$, so ι_i maps A_0 onto the space A_i of (8).

The ι_i are automorphisms because they preserve the cubic norm, which in octonion coordinates is

$$N(y) = \alpha_1\alpha_2\alpha_3 - \sum_r \alpha_r |x_r|^2 + 2\operatorname{Re}((x_1x_2)x_3).$$

Preservation of its mixed term reduces to the identity

$$\epsilon_{1m_1}\epsilon_{2m_2}\epsilon_{3m_3} \operatorname{Re}\big((e_{m_1 \operatorname{xor} \mu_1} e_{m_2 \operatorname{xor} \mu_2}) e_{m_3 \operatorname{xor} \mu_3}\big) = \operatorname{Re}((e_{m_1} e_{m_2}) e_{m_3}). \tag{36}$$

Both sides vanish unless $m_3 = m_1 \operatorname{xor} m_2$, because $\mu_1 \operatorname{xor} \mu_2 \operatorname{xor} \mu_3 = 0$. The remaining sixty-four identities per row follow by substituting the signs and (35).

So the ι_i preserve unit, norm and trace form. The adjoint (or sharp map) $y \mapsto y^\#$ is the gradient of the cubic norm for the trace form,

$$\langle y^\#, y' \rangle = \frac{d}{dt} N(y + t y') \Big|_{t=0},$$

so the ι_i commute with it. The sharp expression for the square in Section II.4.2.2 of McCrimmon[21],

$$y^2 = y^\# + \operatorname{tr}(y)\, y - \tfrac{1}{2}\big((\operatorname{tr} y)^2 - \langle y, y \rangle\big) e,$$

then proves square preservation, and polarization proves that the ι_i are automorphisms of J.

The automorphism θ_0 of Section 4 fixes the diagonal entries. In every off-diagonal octonion entry it rotates the four real planes

$$(1, i), \qquad (j, k), \qquad (\ell, i\ell), \qquad (n, k\ell)$$

through the multiples 0, 2, 2, 1 of $2\pi/3$, each from its first basis vector towards its second. This multiplies c by 1 and each z_r by ω^2, hence B by ω. For $i = 1, 2, 3$,

$$\theta_i = \iota_i \theta_0 \iota_i^{-1}.$$

All these maps fix the diagonal entries and act on each off-diagonal entry by a real 8×8 matrix, so the relations among the θ_i stated in Lemma 5.2 are a finite check: products of these rotation matrices and signed permutations, for each of the three entries.

A.3 The four subalgebras: proof of Lemma 4.1(a) and (c)

By the multiplication rule (35), the product of two different lines among $\mathbb{C}j$, $\mathbb{C}\ell$, $\mathbb{C}n$ lies in the third, in either order:

$$(\mathbb{C}j)(\mathbb{C}\ell) \subset \mathbb{C}n, \qquad (\mathbb{C}\ell)(\mathbb{C}n) \subset \mathbb{C}j, \qquad (\mathbb{C}n)(\mathbb{C}j) \subset \mathbb{C}\ell. \tag{37}$$

This product is conjugate-linear in each factor: for $c, c' \in \mathbb{C}$ and two different units x, y among j, ℓ, n,

$$(cx)(c'y) = \bar{c}\,\bar{c}'\,(xy).$$

Proof of Lemma 4.1(a). Each column of (8) contains each of the four lines once, so the N_i are mutually orthogonal and with D span J ($27 = 3 + 4 \cdot 6$). Let $y \in J$ have diagonal entries α_r and entries y_{12}, y_{13}, y_{23} above the diagonal. The entries of y^2 above the diagonal are

$$(\alpha_1 + \alpha_2) y_{12} + y_{13}\overline{y_{23}}, \qquad (\alpha_1 + \alpha_3) y_{13} + y_{12} y_{23}, \qquad (\alpha_2 + \alpha_3) y_{23} + \overline{y_{12}}\, y_{13}.$$

By (37), and in row 0 because $\mathbb{C}$ is closed under multiplication, each product lies in the line prescribed for its position, so A_i is closed under squaring, hence under the Jordan product. The automorphism ι_i maps A_0 onto A_i. □

Proof of Lemma 4.1(c). The point $p(1,u,w)$ has diagonal entries $1/3$ and entries $\bar{u}/3$, $\bar{w}/3$, $u\bar{w}/3$ above the diagonal. For the four rows of (7) after the first, these lie in the lines of rows $0,1,2,3$ of (8), by (37). In the row of lobe 2 the parameters multiply $\bar{u}$ by ξ_1, $\bar{w}$ by ξ_2 and, by conjugate-linearity, $u\bar{w}$ by $\overline{\xi_1\xi_2}$; likewise for lobe 3. The same rule, applied to the three displayed entries of y^2, shows that the map in (c) commutes with squaring on A_2 (the diagonal entries of y^2 involve only α_r and absolute values), so it is an automorphism of A_2; likewise for A_3. □

A.4 The code in the coordinates of each quartet

Let $q_{f,t} \in H_3(\mathbb{C})$, $0 \le f < 4$, be the points of the first four frames of (7), which do not depend on ξ. In the coordinates of quartet i, the points of quartet i are $G_i q_{f,t} G_i^*$, where

$$G_0 = I, \quad G_1 = \mathrm{diag}(1,-1,-1), \quad G_2 = \mathrm{diag}(1,\xi_1,\xi_2^{-1}), \quad G_3 = \mathrm{diag}(1,\xi_3^{-1},-\xi_4).$$

The frames are relabeled by the permutations φ_i, listed by their values at $0,\dots,12$:

$$\begin{aligned}
\varphi_0 &= (0,1,2,3,4,5,6,7,8,9,10,11,12),\\
\varphi_1 &= (0,4,5,6,1,2,3,7,8,9,10,11,12),\\
\varphi_2 &= (0,7,9,8,1,10,4,3,12,5,2,11,6),\\
\varphi_3 &= (0,10,11,12,1,2,3,4,5,6,7,8,9).
\end{aligned}$$

The point index t within frame f is changed to $\psi_i(f,t)$, with arithmetic modulo three:

$$\psi_0(f,t) = \begin{cases} t & f < 4, \\ -t & f \ge 4; \end{cases}$$

$$\psi_1(f,t) = \begin{cases} t & f = 0, \\ -t & 1 \le f \le 6, \\ t-(f-7) & 7 \le f \le 9, \\ (f-10)-t & 10 \le f \le 12; \end{cases}$$

$$\psi_2(f,t) = \begin{cases} -t & f \in \{5,8,11\}, \\ t & \text{otherwise}; \end{cases}$$

$$\psi_3(f,t) = \begin{cases} t & f \in \{0,7,8,9\}, \\ -t & \text{otherwise}. \end{cases}$$

The nine frames outside quartet i are labeled by $f = 4+r$, $r = 3a+b$, $a,b \in \{0,1,2\}$. With $d_i = d_i(\xi)$ as in (16), their matrices are

$$U_{ab}^{(i)} = \zeta_i(a,b)\, L_i\, E_{s_i(a,b)}(d_i)\, R_i.$$

These are the matrices of Lemma 4.5, with the fixed unitary matrices

$$L_0 = \begin{pmatrix} 0 & 0 & \xi_3 \\ 0 & -\xi_4^{-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad R_0 = \operatorname{diag}(1, \xi_4, \xi_3^{-1}),$$

$$L_1 = \begin{pmatrix} 0 & 0 & -\xi_1^{-1} \\ 0 & \xi_2 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad R_1 = \operatorname{diag}(-1, -\xi_2^{-1}, -\xi_1),$$

$$L_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -(\xi_3\xi_4)^{-1} \\ 0 & \xi_3^{-1} & 0 \end{pmatrix}, \qquad R_2 = \operatorname{diag}(-1, -\xi_3, -\xi_3\xi_4),$$

$$L_3 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad R_3 = I,$$

and the central phases ζ_i and label bijections s_i

$$\zeta_0(a,b) = \omega^{ab}, \qquad s_0(a,b) = (-b,-a),$$

$$\zeta_1(a,b) = 1, \qquad s_1(a,b) = \begin{cases} (-b,2) & a = 2, \\ (b,a) & a \neq 2; \end{cases}$$

$$\zeta_2(a,b) = \omega^{ab-a}, \qquad s_2(a,b) = (a,b),$$

$$\zeta_3(a,b) = \begin{cases} \omega^{b} & a = 0, \\ \omega^{-b} & a = 1, \\ 1 & a = 2, \end{cases} \qquad s_3(a,b) = \begin{cases} (-b,a) & a = 2, \\ (b,a) & a \neq 2. \end{cases}$$

The two indices in $s_i(a,b)$ are respectively the clock and shift indices of (14), not the frame label (a,b).

Define $\widehat{P}_i(f,t)$ by the quartet points $(G_i q_{f,t} G_i^*, 0)$ for $f < 4$, and by $p(\omega^t U_{ab}^{(i)})$ for $f = 4 + 3a + b$. The code $P(\xi)$ is recovered as

$$P(\xi)_{\varphi_i(f),\psi_i(f,t)} = \iota_i\, \widehat{P}_i(f,t). \tag{38}$$

It is checked by substituting (7) into the coordinate map, using (35), and it fixes the orientations and labels in (17).

A.5 The derivative of the family of a quartet

This subsection proves the first assertion of Lemma 5.10(a). The velocity of $U_r = \zeta_r L_i E_{s_i(r)}(d) R_i$ is

$$\dot{g} U_r - U_r \dot{h} + \zeta_r L_i \dot{E}_{s_i(r)} R_i,$$

with $\dot{E} = 0$ for $E = I, X, Z$. If all velocities vanish, then $L_i^* \dot{g} L_i = R_i \dot{h} R_i^*$ commutes with X and Z, so $\dot{g} = \dot{h} = 0$, and then $\dot{d} = 0$.

A.6 The four-parameter family in the family of a quartet

This subsection supplies the construction used in the proof of Lemma 5.10(b). Let ξ' be near ξ, let G_i, L_i, R_i be the matrices for quartet i, regarded as functions of the parameter, and put

$$v = G_i(\xi')G_i(\xi)^*.$$

Choose the scalar κ, analytic in ξ' and equal to one at $\xi' = \xi$, with

$$\kappa^3 = \det\big(v^* L_i(\xi')L_i(\xi)^*\big),$$

and set

$$g = \kappa^{-1} v^* L_i(\xi')L_i(\xi)^*, \qquad h^* = \kappa R_i(\xi)^* R_i(\xi')v.$$

Both are special unitary, since $\det L_i \det R_i = 1$, and diagonal, since v and R_i are diagonal and L_i is a constant permutation matrix times a diagonal one. By construction

$$\zeta_r \, g L_i(\xi) E_{s_i(r)}(d_i(\xi'))R_i(\xi)h^* = v^*\big(\zeta_r \, L_i(\xi')E_{s_i(r)}(d_i(\xi'))R_i(\xi')\big)v.$$

The left side is the member of (17) with parameters $(g, h, d_i(\xi'))$. The map $(a, B) \mapsto (vav^*, vBv^*)$ in the coordinates of quartet i preserves (11), so is an automorphism of J; it sends the quartet at ξ to the quartet at ξ' and, by the display, that member to $P(\xi')$. Here g, h are analytic in ξ' and equal to I at $\xi' = \xi$.

A.7 Proof of Lemma 5.9

Proof. The compact group F_4 acts linearly on J^{39} and preserves the trace form, and the stabilizer of P is trivial by Lemma 5.7. The derivative of the map at $(1, P)$ is $(\delta, v) \mapsto \delta P + v$, an isomorphism from $\mathrm{Der}(J) \times O^\perp$ onto $O \oplus O^\perp$. So it is invertible at $(1, s)$ for s in some S_{ε_0}, by equivariance at every (g, s), and the map is a local analytic diffeomorphism on $F_4 \times S_{\varepsilon_0}$.

It is injective on $F_4 \times S_\varepsilon$ for small ε. Otherwise there are $(g_n, s_n) \neq (g'_n, s'_n)$ with $g_n s_n = g'_n s'_n$ and $s_n, s'_n \to P$. Then $h_n = g_n'^{-1} g_n \neq 1$ satisfies $h_n s_n = s'_n$, and by compactness a subsequence converges to some h with $hP = P$, so $h = 1$; the distinct points (h_n, s_n) and $(1, s'_n)$ near $(1, P)$ then have the same image, contradicting local injectivity.

The image U_ε is open and invariant, and (a), (b) follow because the map carries the projection $F_4 \times S_\varepsilon \to S_\varepsilon$ to the quotient map of U_ε. □

B The Fourier recurrence and the deformations of a quartet

The notation is that of Lemma 5.1 and (18). All indices lie in $\mathbb{Z}/3$.

Write $D_b = \mathrm{diag}(d_{b,k})$, so that $d_{0,k} = d_{1,k} = 1$ and $d_{2,k} = d_k$. The matrix $W_{b,b+1}$ of (18) is then

$$W_{b,b+1} = \mathrm{diag}(w_{b,k})X, \qquad w_{b,k} = \bar{d}_{b,k}\, d_{b+1,k-1}.$$

Only the part $Y_{ab}X^{-1}$ of each M_{ab} contributes to the equations of the oriented pairs $(b, b+1)$, which read

$$\sum_k \omega^{(a'-a)k}\left(w_{b,k}\, y_{a',b+1,k-1} - w_{b,k+1}\, y_{a,b,k}\right) = 0. \tag{39}$$

The transform $\widehat{y}_{b,k}(\mu)$ of the lemma is inverted by

$$y_{a,b,k} = \sum_\mu \widehat{y}_{b,k}(\mu)\, \omega^{\mu a},$$

Substitute this in (39). The coefficient of $\omega^{-\beta a+(\beta+\mu)a'}$ is

$$w_{b,\beta}\, \widehat{y}_{b+1,\beta-1}(\mu) - w_{b,\beta+\mu+1}\, \widehat{y}_{b,\beta+\mu}(\mu).$$

The transform in a, a' is invertible, so the pair equations say that every such coefficient vanishes. With $k = \beta - 1$ this is the recurrence

$$\widehat{y}_{b+1,k}(\mu) = \nu_{b,k}\, \widehat{y}_{b,k+\mu+1}(\mu), \qquad \nu_{b,k} = \frac{w_{b,k+\mu+2}}{w_{b,k+1}}.$$

The numbers $\nu_{b,k}$ are nonzero; they depend on μ.

Two steps of the recurrence express everything through the values at $b = 0$:

$$\widehat{y}_{1,k}(\mu) = \nu_{0,k}\, \widehat{y}_{0,k+\mu+1}(\mu), \qquad \widehat{y}_{2,k}(\mu) = \nu_{1,k}\, \nu_{0,k+\mu+1}\, \widehat{y}_{0,k+2\mu+2}(\mu).$$

The third step returns to $b = 0$ and leaves the closure equation

$$\widehat{y}_{0,k}(\mu) = m_{k\mu}\, \widehat{y}_{0,k}(\mu),$$

with the multiplier

$$m_{k\mu} = \nu_{2,k}\, \nu_{1,k+\mu+1}\, \nu_{0,k+2\mu+2} = \frac{d_{k+1}\, d_{k+2\mu+2}}{d_{k+\mu+2}\, d_{k+\mu+1}}.$$

Using $d_0 d_1 d_2 = 1$, the multipliers for $\mu = 0, 1, 2$ are

$$1, \qquad d_{k+1}^3, \qquad 1.$$

Same-support pairs supply nine unordered equations; the three oriented different-support pairs supply the other twenty-seven. Reversing a pair conjugates its equation. These are all thirty-six unordered complex pair constraints.

B.1 Proof of Lemma 5.4

Proof. Conjugating by a unitary matrix, we may assume that the first frame is the coordinate basis and that the second contains the line spanned by $(1, 1, 1)$. The points of the other three frames are then the projections onto lines spanned by vectors $(1, x, y)$ with $|x| = |y| = 1$.

A first-order deformation of one frame is $q \mapsto [N, q]$ with N anti-Hermitian. Subtracting one, we may assume that the coordinate basis does not move; the conjugations still available have N diagonal. Each vector $(1, x, y)$ stays unbiased to the coordinate basis, so only the phases of x and y vary, at rates α and β say, and a diagonal conjugation shifts (α, β) equally for all vectors; so we may assume $\alpha = \beta = 0$ for $(1, 1, 1)$.

Another vector $(1, x, y)$ of the second frame (a row of the dephased Hadamard matrix) satisfies $1 + x + y = 0$, with linearization $x\alpha + y\beta = 0$; as x and y are not real multiples of each other, $\alpha = \beta = 0$. This step says that the Fourier matrix of order three has defect zero, in the sense of Sections 3 and 5.3 of Tadej and Życzkowski[22]. A vector $(1, x, y)$ of the third or fourth frame is unbiased to the second frame:

$$\bar{x} + x\bar{y} + y = 0.$$

These three unit numbers form an equilateral triangle, so a real linear relation between them has equal coefficients; the derivative has coefficients $-\alpha$, $\alpha - \beta$, β, equal only when $\alpha = \beta = 0$.

So the deformation is a conjugation, and conjugation by $N \in \mathfrak{su}(3)$ vanishes on the quartet only for $N = 0$, because the quartet spans $H_3(\mathbb{C})$. □

C The quadratic maps and their coefficients

Apart from Appendices C.4–C.6, this appendix is needed only to repeat the verification.

C.1 The maps in matrix form

The map H and the first component of T have a short description in the coordinates (a, B) of Section 4. From the twelve numbers u_{ia} form the pair

$$a = \begin{pmatrix} 0 & \bar{u}_{01} & u_{02} \\ u_{01} & 0 & \bar{u}_{00} \\ \bar{u}_{02} & u_{00} & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} u_{21} & u_{32} & u_{10} \\ -u_{11} & u_{22} & -u_{30} \\ u_{31} & u_{12} & u_{20} \end{pmatrix},$$

and put

$$a' = a^2 - BB^*, \qquad B' = -aB + \operatorname{adj}(B)^*.$$

Since $\operatorname{tr} a = 0$, the square formula (11) shows that B' is the second component of $(a, B)^2$ and that a' agrees with the first component off the diagonal. With σ as in Section 6,

$$\begin{aligned} H_0(u) + \sigma(u_0) &= \big(a'_{12},\ a'_{01},\ a'_{20}\big), \\ H_1(u) + \sigma(u_1) &= \big(-\overline{B'_{02}},\ \overline{B'_{10}},\ -\overline{B'_{21}}\big), \\ H_2(u) + \sigma(u_2) &= \big(\overline{B'_{22}},\ \overline{B'_{00}},\ \overline{B'_{11}}\big), \\ H_3(u) + \sigma(u_3) &= \big(\overline{B'_{12}},\ -\overline{B'_{20}},\ -\overline{B'_{01}}\big), \\ T_0(u) &= \big(-(B^*B)_{12},\ -(B^*B)_{02},\ (B^*B)_{10}\big), \end{aligned} \tag{40}$$

with matrix indices running over $0, 1, 2$.

The equations (40) were verified by expansion in computer algebra against the formulas written out below, which are the definition used in the formal development. The pair (a, B) is an element of J with zero diagonal, so $\sigma + H$ is, up to signs and conjugations, the off-diagonal part of its square. It is not claimed to be the first-order deformation with coordinates u. For T_1, T_2, T_3 we give only the written-out formulas.

C.2 The identities in the proof of Lemma 6.2

Identity (29) counts pairs of entries of the array: pairs in different rows on the left; on the right, pairs in different columns, less those in one row, plus those in one column. For (28) we have only the expansion.

Identity (27) is the multiplicativity of the octonion norm. By (40) the twelve components of $\sigma + H$ are, up to signs and conjugations, the off-diagonal coordinates of the square of an element of J with zero diagonal. In the coordinates of Appendix A the three octonion entries of that element have the columns x_1, x_2, x_3 as coordinates, again up to signs and conjugations, and the off-diagonal entries of its square are the conjugates of their pairwise products.

C.3 The maps written out

The formulas below fix all signs and conjugations in (26).

Lean: `quadratic maps`

$$
\begin{aligned}
H_0 &= \big(u_{30}\bar{u}_{20} + u_{11}\bar{u}_{31} - u_{22}\bar{u}_{12},\quad u_{10}\bar{u}_{30} + u_{21}\bar{u}_{11} - u_{32}\bar{u}_{22},\\
&\qquad\qquad - u_{20}\bar{u}_{10} - u_{31}\bar{u}_{21} - u_{12}\bar{u}_{32}\big),\\
H_1 &= \big(-u_{01}\bar{u}_{30} + \bar{u}_{02}\bar{u}_{20} + u_{31}u_{22},\quad -u_{00}\bar{u}_{31} - \bar{u}_{01}\bar{u}_{21} - u_{20}u_{32},\\
&\qquad\qquad \bar{u}_{00}\bar{u}_{22} + u_{02}\bar{u}_{32} - u_{30}u_{21}\big),\\
H_2 &= \big(\bar{u}_{00}\bar{u}_{30} - u_{02}\bar{u}_{10} + u_{32}u_{11},\quad u_{01}\bar{u}_{11} - \bar{u}_{02}\bar{u}_{31} + u_{30}u_{12},\\
&\qquad\qquad - u_{00}\bar{u}_{12} - \bar{u}_{01}\bar{u}_{32} - u_{10}u_{31}\big),\\
H_3 &= \big(-u_{00}\bar{u}_{20} - \bar{u}_{01}\bar{u}_{10} - u_{21}u_{12},\quad -\bar{u}_{00}\bar{u}_{11} + u_{02}\bar{u}_{21} + u_{10}u_{22},\\
&\qquad\qquad u_{01}\bar{u}_{22} + \bar{u}_{02}\bar{u}_{12} - u_{20}u_{11}\big).
\end{aligned}
$$

$$
\begin{aligned}
T_0 &= \big(-u_{10}\bar{u}_{32} + u_{30}\bar{u}_{22} - u_{20}\bar{u}_{12},\quad -u_{10}\bar{u}_{21} - u_{30}\bar{u}_{11} - u_{20}\bar{u}_{31},\\
&\qquad\qquad - u_{11}\bar{u}_{22} + u_{31}\bar{u}_{12} + u_{21}\bar{u}_{32}\big),\\
T_1 &= \big(u_{00}u_{21} + \bar{u}_{01}u_{31} - \bar{u}_{30}\bar{u}_{22},\quad -u_{00}\bar{u}_{32} - \bar{u}_{02}\bar{u}_{22} - u_{20}u_{31},\\
&\qquad\qquad u_{01}u_{20} - \bar{u}_{02}u_{30} + \bar{u}_{32}\bar{u}_{21}\big),\\
T_2 &= \big(u_{00}u_{32} + \bar{u}_{01}u_{12} + \bar{u}_{30}\bar{u}_{11},\quad -u_{00}\bar{u}_{10} + \bar{u}_{02}\bar{u}_{30} - u_{31}u_{12},\\
&\qquad\qquad u_{01}u_{31} - \bar{u}_{02}u_{11} + \bar{u}_{10}\bar{u}_{32}\big),\\
T_3 &= \big(u_{00}u_{10} + \bar{u}_{01}u_{20} - \bar{u}_{22}\bar{u}_{11},\quad -u_{00}\bar{u}_{21} + \bar{u}_{02}\bar{u}_{11} - u_{20}u_{12},\\
&\qquad\qquad u_{01}u_{12} + \bar{u}_{02}u_{22} + \bar{u}_{10}\bar{u}_{21}\big).
\end{aligned}
$$

C.4 Normalizing the transverse coordinates

Use the anti-Hermitian generators $\sqrt{-1}\,(E_{rs} + E_{sr})$ and $E_{rs} - E_{sr}$, in pair order 01, 02, 12. For quartet i, the infinitesimal simultaneous conjugation with parameter u_i has zero diagonal coordinates. Writing $u_{ia} = x_a + \sqrt{-1}\,y_a$ in row i, its six off-diagonal coordinates are

i	01 imaginary	01 real	02 imaginary	02 real	12 imaginary	12 real
0	y_0	$-x_0$	y_2	x_2	y_1	$-x_1$
1	y_0	x_0	y_1	$-x_1$	y_2	x_2
2	y_1	$-x_1$	y_2	x_2	y_0	$-x_0$
3	y_2	x_2	y_0	$-x_0$	y_1	x_1

By Section 6, $V_\xi u$ is the sum over i of the velocities of (17) with equal left and right parameters $g = h = e^{t\mathfrak{n}_i(u_i)}$, transported back to the original coordinates. A motion of the right parameter alone is tangent to the F_4-orbit (Lemma 4.3), so the projection to $\mathcal{O}^\perp$ is that of the derivative in the left parameter.

Each row of the table maps u_i invertibly and real-linearly onto the six off-diagonal parameters. In the family of quartet i the tangent space of M_* has only diagonal left parameters and parameters d (proof of Lemma 5.10(b), Appendix A.6), and the parametrization of M_i has injective derivative, so the image of row i meets the tangent space of M_* only at zero. Hence the projected image of row i is a complement of the tangent space of M_* in that of M_i. At nonresonant parameters these are $\overline{\mathcal{T}}^K$ and $\overline{\mathcal{T}}^{K_i}$ (Lemma 5.10(c)), and by (22) ($4 + 4 \cdot 6 = 28$) the four images and $\overline{\mathcal{T}}^K$ span $\overline{\mathcal{T}}$.

At nonresonant parameters the image of row i is the isotypic part W_i itself, although the proofs use only that it is a complement. For $j \neq i$, in the coordinates of quartet i, the map θ_j is $(a, B) \mapsto (g'ag'^*, g'Bh'^*)$ with g' and h' diagonal and the entries of g' distinct, as for θ_1 in the proof of Lemma 5.6. As in that proof, $\tilde{\theta}_j$ replaces the left parameter $\dot{g}$ by $g'\dot{g}g'^*$ modulo $\mathcal{O}$, and this multiplies each off-diagonal entry by a cube root of unity other than one. So the image, which lies in $\overline{\mathcal{T}}^{K_i}$, has average zero over K; hence it lies in W_i, and both have dimension six.

C.5 Proof of Lemma 6.6

Normalizing the chart. Let Π be the orthogonal projection of J^{39} onto $\mathcal{O}^\perp$. By the preceding subsection and Lemma 5.10(c), ΠV_ξ maps the ith factor $\mathbb{C}^3$ of $\mathbb{C}^{12}$ isomorphically onto a complement of the tangent space of M_* in the tangent space of M_i at P. A linear change of the parameters (λ, u_i) fixing the points with $u_i = 0$ makes the derivative in u_i of the parametrization of M_i in Proposition 5.11 equal to this map. The transverse coordinates of the chart are then the u_i of (25), with the norm of $\mathbb{C}^{12}$ on $E = \bigoplus_i W_i$.

Solving for η. Let $\Phi(\lambda, u, \eta)$ be the point of the slice with coordinates (λ, u, η) in this chart, and let $\pi_{\operatorname{im} DF_P}$ be the orthogonal projection of the residual space onto the image of DF_P. The map

$$(\lambda, u, \eta) \longmapsto \big(\lambda, u, \pi_{\operatorname{im} DF_P} F(\Phi(\lambda, u, \eta))\big)$$

has invertible derivative at the origin, since the directions (λ, u) span the kernel of DF_P in the slice and DF_P maps the 973 directions η isomorphically onto its image. By the inverse function theorem the equation $\pi_{\mathrm{im}\, DF_P} F(\Phi(\lambda, u, \eta)) = 0$ is solved near the origin by $\eta = \hat{\eta}(\lambda, u)$, with $\hat{\eta}$ analytic. Put

$$b(\lambda, u) = \Phi(\lambda, u, \hat{\eta}(\lambda, u)) - P, \qquad f(\lambda, u) = F(P + b(\lambda, u)),$$

Then b is analytic with values in $\mathcal{O}^\perp$, and $b(0) = 0$.

(i) A zero of F near P in the slice is a point $\Phi(\lambda, u, \eta)$ at which the last component of the map above vanishes. So $\eta = \hat{\eta}(\lambda, u)$, the point is $P + b(\lambda, u)$, and $f(\lambda, u) = 0$. The converse holds by the definition of f.

(ii) For $u \in W_i$ the point $\Phi(\lambda, u, 0)$ lies in M_i and is a zero of F, so $\hat{\eta}(\lambda, u) = 0$ by uniqueness. Thus $P + b(\lambda, u)$ is the point of M_i with coordinates (λ, u), and f vanishes on $\Lambda \times \bigcup_i W_i$.

(iii) $D\hat{\eta}(0) = 0$, because the directions (λ, u) lie in the kernel of DF_P. Hence $D_u b(0)$ is the derivative of the chart in u at the origin, which is ΠV_ξ by the normalization.

C.6 Proof of Lemma 6.4

(a) Apply ℓ to $F(x + v) = DF_x v + \mathcal{Q}(v)$.

(b) Differentiate $F(gy) = \rho_g F(y)$ in g, and then in y at x in the direction v:

$$DF_y(\delta y) = \rho_\delta F(y),$$
$$2\mathcal{B}(v, \delta x) + DF_x(\delta v) = \rho_\delta DF_x v = 0.$$

Apply ℓ to the second line. For the last assertion, the first line at $y = x$ shows that δx lies in $\ker DF_x$ as well; so the first assertion applies to the cross term and to the square of δx.

(c) The first assertion follows from $DF_{gx} \circ g = \rho_g \circ DF_x$, the second from $DF_{x'} = DF_x + 2\mathcal{B}(x' - x, \cdot)$.

(d) Let γ be the curve, with $\gamma(0) = x$ and $\gamma'(0) = v$. Differentiating $F(\gamma(t)) = 0$ twice at $t = 0$ gives $DF_x \gamma''(0) + 2\mathcal{Q}(\gamma'(0)) = 0$; apply ℓ. A quadratic form that vanishes on a linear subspace has vanishing polar form there.

C.7 Exact coefficient data

The ancillary file `native-data.json` holds the coefficients of V_ξ, h_{ia} and t_{ia} as exact integer tuples. All indices start at zero. The explanatory strings inside the file use an earlier notation: there a coefficient is written $[A, B, C, D, N]$, the parameters $\xi_1, \ldots, \xi_4$ are called A, B, C, D, and the file's z, y_j, u_j, K_j, R, S, $d_i(u)$, λ and d_0, d_1, d_2 are our u, $\tilde{u}_j$, U_j, $V^{(j)}$, $V'^{(j)}$, $V''^{(j)}$, $\tau_i(U)$, h^* and $\alpha_1, \alpha_2, \alpha_3$; the matrices $V'^{(j)}$ and $V''^{(j)}$ are stored under the keys `real_difference` and `imaginary_sum`, and "phase", "anchor" and "sheet" there mean parameter, quartet and family.

Encoding. A coefficient $[n_1, n_2, n_3, n_4, N]$ means

$$\big(n_1 + n_2\sqrt{-1} + n_3\sqrt{3} + n_4\sqrt{-1}\sqrt{3}\big)/N.$$

A sparse matrix entry $[r, s, n_1, n_2, n_3, n_4, N]$ adds this coefficient in row r, column s; unspecified entries are zero. An entry of a list for h is $[r, n_1, n_2, n_3, n_4, N]$, the coefficient of residual row r; apply its list's `conjugated` and `negated` flags to the coefficient, not its row index, before summing. An entry of a list for t appends four integer exponents to this six-tuple.

Coordinates and residual rows. A point of J has the 27 real coordinates $(\alpha_1, \alpha_2, \alpha_3, o_{01}, o_{02}, o_{12})$: the diagonal, then the three octonions above it, each in basis $(1, i, j, k, \ell, i\ell, j\ell, k\ell)$. The cyclic coordinates of Appendix A satisfy $o_{01} = x_3$, $o_{02} = \bar{x}_2$, $o_{12} = x_1$, and $n = j\ell$, $in = -k\ell$. Point $p = 3f + t$ has coordinates $27p + k$, $0 \le k < 27$. The rows of F are

row	equation
$27p + k$	$(P_p^2 - P_p)_k$
$1053 + p$	$\operatorname{tr} P_p - 1$
$1092 + 27f + k$	$(\sum_t P_{3f+t} - e)_k$
$1443 + p(77 - p)/2 + q - p - 1$	$\langle P_p, P_q\rangle - \gamma_{pq} \quad (p < q)$,

where $\gamma_{pq} = 0$ within a frame and $1/3$ otherwise.

The directions $V_\xi u$. Place the four triples u_0, u_1, u_2, u_3 in coordinates $(0, 1, 2)$, $(6, 9, 12)$, $(8, 11, 14)$, $(7, 10, 13)$ of $\tilde{u} \in \mathbb{C}^{24}$. The remaining coordinate in each ordered pair

$$(0, 3), (1, 5), (2, 4), (6, 15), (9, 21), (12, 18),$$
$$(8, 16), (11, 22), (14, 19), (7, 17), (10, 23), (13, 20)$$

is minus the conjugate of the first. For each pair (j, j'), the file gives two sparse real 27×27 matrices $V'^{(j)}$ and $V''^{(j)}$. Set

$$V^{(j)} = \big(V'^{(j)} - \sqrt{-1}\, V''^{(j)}\big)/2, \qquad V^{(j')} = -\big(V'^{(j)} + \sqrt{-1}\, V''^{(j)}\big)/2.$$

At every point p put

$$(V_\xi u)_p = \operatorname{Re} \sum_{j=0}^{23} \tilde{u}_j V^{(j)} P(\xi)_p.$$

Substituting the matrices G_i, L_i, R_i of Appendix A and the table of Appendix C.4 shows that this is the transported family velocity for every $\xi \in (S^1)^4$. The same matrix acts on every point; the $V^{(j)}$ are not asserted to be derivations of J. As a redundant check, the file also stores $V^{(j)}P(\xi_*)$, for the base parameter ξ_* below, as column `eigen_index[j]` of `seed_columns`, with common denominator 384540.

The covectors t_{ia}. Put

$$\xi_* = \big((3+4\sqrt{-1})/5,\ (5+12\sqrt{-1})/13,\ (15+8\sqrt{-1})/17,\ (21+20\sqrt{-1})/29\big)$$

and $U_j = \xi_{j+1}/\xi_{*,j+1}$ for $j = 0, \dots, 3$. The file stores each t_{ia} with Laurent monomial coefficients: an entry (r, c, ν) contributes $cU_0^{\nu_0} \cdots U_3^{\nu_3}$ to row r of t_{ia}; as the U_j are units, no further denominators arise.

The covectors h_{ia}. The file stores the twelve h_{ia} at ξ_*, written h^*_{ia}. The automorphism $\tau_i(U)$ fixes the first three diagonal coordinates. On each coordinate pair (x_j, y_j) in positions $(3+2j, 4+2j)$, $0 \le j < 12$, it multiplies $x_j - \sqrt{-1}\, y_j$ and $x_j + \sqrt{-1}\, y_j$ by χ_j, χ'_j, respectively. For $i = 0, 1$ it is the identity. For $i = 2$ put

$$(\varphi_0, \varphi_1, \varphi_2) = (U_0U_1, U_1^{-1}, U_0^{-1}), \qquad (\psi_0, \psi_1, \psi_2) = (1, 1, 1);$$

for $i = 3$ put

$$(\varphi_0, \varphi_1, \varphi_2) = (1, 1, 1), \qquad (\psi_0, \psi_1, \psi_2) = (U_3, (U_2U_3)^{-1}, U_2).$$

In either case the character table is

j	χ_j	χ'_j	j	χ_j	χ'_j
0	$\varphi_1^2\varphi_2$	$\varphi_0^2\varphi_2$	6	$\varphi_1\psi_0\psi_2$	$\varphi_0\varphi_2\psi_1$
1	$\varphi_2\psi_1\psi_2$	$\varphi_0\varphi_1\psi_0$	7	$\varphi_0\varphi_2\psi_2$	$\varphi_1\psi_0\psi_1$
2	$\varphi_2\psi_0\psi_2$	$\varphi_0\varphi_1\psi_1$	8	$\varphi_0\varphi_2^2$	$\varphi_0\varphi_1^2$
3	$\varphi_0\varphi_1\psi_2$	$\varphi_2\psi_0\psi_1$	9	$\varphi_0\psi_1\psi_2$	$\varphi_1\varphi_2\psi_0$
4	$\varphi_1\varphi_2^2$	$\varphi_0^2\varphi_1$	10	$\varphi_0\psi_0\psi_2$	$\varphi_1\varphi_2\psi_1$
5	$\varphi_1\psi_1\psi_2$	$\varphi_0\varphi_2\psi_0$	11	$\varphi_1\varphi_2\psi_2$	$\varphi_0\psi_0\psi_1$

That these maps are automorphisms of J follows by substitution in the product formula of Appendix A. On the twelve points of quartet i, $\tau_i(U)P(\xi_*) = P(\xi)$. Let

$$c_0 = c_1 = (1, 1, 1), \qquad c_2 = ((U_0U_1)^{-1}, U_0, U_1),$$
$$c_3 = (U_3, (U_2U_3)^{-1}, U_2),$$

with entries c_{ia} (these triples are not the class c_0 of the introduction). Then

$$h_{ia} = c_{ia}^{-1}\, h^*_{ia}\, \rho^{-1}_{\tau_i(U)},$$

with ρ as in Section 6.

That h_{ia} is a stress at $P(\xi)$ follows from Lemma 6.4(c) and the following: h^*_{ia} is a stress at $P(\xi_*)$; the quadratic form $h^*_{ia}Q$ is supported on the twelve points of quartet i; the configurations $\tau_i(U)P(\xi_*)$ and $P(\xi)$ agree at those twelve points; and F is equivariant. The normalization by c_{ia} is needed for reproduction, not for equilibrium.

The identities to be checked. The finite verification of (32)–(33) has three checks, with F formed by the Jordan square and trace-form formulas of Appendix A.

(i) *Equilibrium.* For each of the 24 covectors, the contraction with the 2184×1053 matrix DF_P is zero.

(ii) *Reproduction.* For each covector, substitute $V_\xi u$ into the homogeneous quadratic part of F, contract, and collect the 300 monomials of degree two in the 24 real and imaginary parts of the entries of u. The coefficients are those of the displayed component of H or T.

(iii) *Support.* For each of the twelve h_{ia}, the contraction with the homogeneous quadratic part has no monomial involving a point outside the twelve points of quartet i.

These are finite polynomial identities over $\mathbb{Q}(\sqrt{-1}, \sqrt{3})$, Laurent in the parameters ξ_j, with their conjugates replaced by inverses. Equality is coefficient equality, not agreement at sampled parameters. In the formal development each check reduces to finitely many equalities of integer-coded data, decided by evaluation in the Lean kernel. For t, the stored Laurent coefficients supply the identities for all ξ directly. For h, the formal development derives reproduction at every ξ from the identities for h^*_{ia} at ξ_*. Only the twelve points of quartet i enter, and there the transport multiplies each coordinate of u and of $\bar{u}$ by a monomial in U; every monomial of $H_i(u)_a$ then acquires the factor c_{ia}, which the normalization removes (`H_reproduction_phase`, the declaration behind the note on Proposition 6.5).

Lean: all-parameter T identity

Outside the formal development, the script `anc/finite-identities.py`, which uses only the Python standard library, replays checks (i)–(iii) from the file in exact arithmetic over $\mathbb{Q}(\sqrt{-1}, \sqrt{3})$. It compares Laurent coefficients in $U_0, \ldots, U_3$, so the checks hold for every $\xi \in (S^1)^4$. It forms $P(\xi)$, F, H and T from (7) and the formulas of this appendix. It also checks, identically in ξ, that $F(P(\xi)) = 0$, that $DF_P V_\xi u = 0$, and that the formula above for $V_\xi u$ is the sum of the four family velocities of Section 6, built from Appendix A and the table of Appendix C.4. It does not compare these formulas with the definitions of the formal development.

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