hexagon:2611.00339v4

AN EFFECTIVE ANDRÉ–OORT THEOREM FOR SHIMURA VARIETIES

MANVIR JASWAL

Abstract. We prove an effective André–Oort theorem for every Shimura variety. Let Z be a subvariety defined over a number field. For every special subvariety maximal in Z we bound its degree, and the discriminant and level invariant that Klingler and Yafaev attach to it, by effectively computable constants, polynomial in the degree of Z and that of its field of definition. The proof follows Edixhoven, Klingler, Ullmo and Yafaev through Hecke correspondences. For the discriminant and level bounds, a zero-free half-plane for one Dedekind zeta function suffices in place of the generalized Riemann hypothesis, and Cureton and Thorner prove such a half-plane for every number field. In Mumford's moduli spaces of abelian varieties an algorithm lists the special subvarieties maximal in Z from equations for its closure.

Contents

1. Introduction 2

Part I. Bounds 7
2. Setting, notation and the complexity 7
3. Split primes and Galois orbits under a zero-free half-plane 11
4. Point separation by the Baily–Borel bundle 14
5. Bézout's inequality uniformly in the level 15
6. The Hecke step with constants uniform in the level 16
7. Degrees of maximal special subvarieties 27
8. The main theorem for adjoint groups 31
9. Every Shimura datum 38
10. Orders and Galois orbits in $\mathcal{A}_g$ 48

Part II. Listing 52
11. Listing in Mumford's moduli spaces 52

Appendix A. The smoothed explicit formula 69
Appendix B. Fixers in Bruhat–Tits buildings 70
Appendix C. Two index bounds 71
Declarations 76
References 76

Date: November 27, 2026.
2020 *Mathematics Subject Classification.* Primary 11G18; Secondary 14G35, 11R42, 11F46.
Key words and phrases. André–Oort conjecture, Shimura varieties, special subvarieties, Hecke correspondences, zero-free regions.

1. Introduction

Let S be a Shimura variety and $Z \subset S$ a closed subvariety. A special subvariety $V \subset Z$ is *maximal in* Z if no special subvariety V' with $V \subsetneq V' \subset Z$ exists. The André–Oort conjecture [3, 76] says that Z contains only finitely many maximal special subvarieties. By the o-minimal strategy of Pila and Zannier [80], Tsimerman proved it for $\mathcal{A}_g$ [100], and Pila, Shankar and Tsimerman for every Shimura variety, by bounding the heights of special points [78, Thm. 1.2], to which Binyamini, Schmidt and Yafaev had reduced it [16, Thms. 1 and 2]. Earlier, Klingler and Yafaev [63] and Ullmo and Yafaev [102] proved it under the generalized Riemann hypothesis (GRH) for CM fields, along a route through Hecke correspondences and Galois orbits that goes back to Edixhoven [32, 33], Edixhoven and Yafaev [34] and Yafaev [106].

Neither proof bounds the maximal special subvarieties of Z effectively in terms of Z. Given the theorem, Scanlon's automatic uniformity shows that their number and their degrees are bounded in terms of the degree of Z [92, Thms. 2.4 and 4.2], without a way to compute the bounds. In the o-minimal proofs the counting theorem of Pila and Wilkie [79] for sets definable in $\mathbb{R}_{\mathrm{an,exp}}$ is not effective in general [10]; the uniformizations of Shimura varieties are definable in Binyamini's effectively o-minimal structure $\mathbb{R}_{\mathrm{LN,exp}}$, whose counting theorem is effective once their format is computed [12]. The lower bounds for Galois orbits of special points rest on the Brauer–Siegel theorem [16, Prop. 6], [100], whose constants are ineffective [93] unless Siegel zeros are excluded [95]. In the proof under GRH the existence of a small prime that splits completely in the splitting field of a torus is proved for the members of a Zariski-dense set of special subvarieties of Z, after the set is replaced by a subset with the same Zariski closure [63, Prop. 10.2.3], not for each special subvariety. The other half of the proof is equidistribution, through Ratner's theorem [21, 102], which gives no bound; Daw replaced it in that proof by Hecke correspondences and degree bounds [26, Thm. 10.1].

We give effective bounds along the route through Hecke correspondences, for every Shimura variety and polynomial in the degree of Z and the degree of a field of definition: for the degrees with no hypothesis, and for the discriminants and level invariants that Klingler and Yafaev attach to a special subvariety, with GRH replaced by a zero-free half-plane for one Dedekind zeta function. In Mumford's moduli spaces the bounds give an algorithm that lists the maximal special subvarieties of Z from equations for the closure of Z, and so their number (Theorem 1.5).

1.1. **Results.** Let (G, X) be a Shimura datum and $K = \prod_p K_p \subset G(\mathbb{A}_f)$ a neat compact open subgroup, and fix a faithful representation $\rho\colon G \to \mathrm{GL}_{n_\rho}$ with $\rho(K) \subset \mathrm{GL}_{n_\rho}(\widehat{\mathbb{Z}})$. Degrees on $\mathrm{Sh}_K(G, X)$ are taken for the Baily–Borel bundle $\mathcal{L}_K$, the extension of the bundle of top-degree holomorphic forms to the Baily–Borel compactification (Section 2). A special subvariety V has a generic Mumford–Tate group H_V; let T_V be the connected center of its image in the adjoint group G^{ad}, let L_V be the splitting field of T_V and d_{L_V} its absolute discriminant, and let $\alpha_V \geq 1$ be the invariant of Klingler and Yafaev that measures how far the level is from maximal on T_V (Definition 2.1); in Theorems 1.1 and 1.4 it is taken on the adjoint Shimura variety, for the adjoint representation. For $\delta \in (0, 1/2]$ and a number field L we write

$$\mathrm{QRH}(\delta; L)\colon \quad \zeta_L(s) \neq 0 \quad \text{for } \operatorname{Re} s > 1 - \delta,\ s \neq 1,$$

a quasi-Riemann hypothesis for ζ_L.

Theorem 1.1. *Let (G, X), K and ρ be as above, and let $\delta \in (0, 1/2]$. For all integers $D, d \geq 1$ there are numbers $c_{\deg}(D)$ and $c(\delta, D, d)$, effectively computable from (G, X, K, ρ), D, d, δ and fixed representatives of the connected components of the adjoint Shimura variety, with the following property. Let $F \supset E(G, X)$ be a number field with $[F : \mathbb{Q}] = d$, let $Z \subset \mathrm{Sh}_K(G, X)$ be a closed subvariety that is geometrically irreducible, defined over F and of degree at most D, and let V be a special subvariety maximal in Z. Then*

(i) $\deg V \leq c_{\deg}(D)$;

(ii) if $\mathrm{QRH}(\delta; L_V)$ *holds, then* $d_{L_V} \leq c(\delta, D, d)$ *and* $\alpha_V \leq c(\delta, D, d)$.

Moreover, for every $\varepsilon > 0$ there are C'_ε, effectively computable from (G, X, K, ρ), ε and the same representatives, and C_ε, effectively computable from these and δ, such that $c_{\deg}(D) \leq C'_\varepsilon D^{4^n+\varepsilon}$ and $c(\delta, D, d) \leq C_\varepsilon\, d^{2^n/\mu+\varepsilon} D^{3\cdot 4^n/\mu+\varepsilon}$, where $n := \dim \mathrm{Sh}_K(G, X)$ and $\mu \leq 1/32$ is the exponent of d_{L_V} in the lower bound for Galois orbits (Corollary 3.4), formed for the adjoint datum (Theorem 9.5); μ does not depend on δ.

The only analytic hypothesis is the zero-free half-plane for the one Dedekind zeta function ζ_{L_V} of the subvariety in question. It is used only to produce primes that split completely in L_V, so no lower bound for class numbers is needed. Two choices keep the bounds of (ii) polynomial: the lower bound for the Galois orbit of V is used as a power of d_{L_V}, where Klingler and Yafaev use a power of its logarithm, and the Hecke step is applied to the union of the Galois conjugates of Z, so that d does not enter the exponent of the prime l (Section 1.2). Cureton and Thorner prove $\mathrm{QRH}(1/32; L)$ for every number field L [23, Theorem 1.1].

Corollary 1.2 (with [23, Theorem 1.1])**.** *In the setting of Theorem 1.1,* $\deg V \leq c_{\deg}(D)$, $d_{L_V} \leq c(1/32, D, d)$ *and* $\alpha_V \leq c(1/32, D, d)$ *for every special subvariety V maximal in Z.*

Hu's zero-free half-plane $\mathrm{Re}\, s > 7/8$ for Hecke L-functions [57, Theorem 1.2] gives $\mathrm{QRH}(1/8; L)$ for every L, and with it Corollary 1.2 holds with $c(1/8, D, d)$.

The hypothesis that Z be geometrically irreducible can be dropped at the cost of replacing d by dD, and part (i) of Theorem 1.1 holds for every closed $Z \subset \mathrm{Sh}_K(G, X)_{\mathbb{C}}$ of degree at most D, with $c_{\deg}$ evaluated at an explicit power of D (Corollary 9.7).

We also prove lower bounds for Galois orbits of special points. With $\delta = 1/32$ the second bound of Theorem 1.3(a) is Daw's Conjecture 22.1 [25] and Theorem 1.3(b) is Edixhoven's form [100, Thm. 5.1], each with effective constants.

Theorem 1.3. *Let (G, X) be a Shimura datum with G adjoint and $G \neq 1$, K a neat product level, ρ as above and $\delta \in (0, 1/2]$.*

(a) There are $A, \varepsilon > 0$, effectively computable from (G, X, K, ρ), δ and the representatives of the connected components, such that for every special point $x \in \mathrm{Sh}_K(G, X)$ for which $\mathrm{QRH}(\delta; L)$ holds for the splitting field L of its Mumford–Tate group T, and every number field $F \supset E(G, X)$, $[F(x) : F] \geq A\,\mathrm{disc}(x)^\varepsilon / [F : \mathbb{Q}]$, with $\mathrm{disc}(x)$ the discriminant of [16, Def. 1]. For x in the image of $X^+ \times \{1\}$, also $|\mathrm{Gal}(\overline{\mathbb{Q}}/L) \cdot x| \geq c_1 B_1^{i(T)} [K_T^m : K_T]\, d_L^\mu$, with μ of Table 1 for ρ, in the form of [103, Thm. 6.1], with $c_1, B_1 > 0$ effectively computable from (G, X, K, ρ) and δ.

(b) Let $g \geq 1$. For every CM point x of $\mathcal{A}_{g,1}$, or of $\mathcal{A}_{g,K(N)}$ with $N \geq 1$, for which $\mathrm{QRH}(\delta; L)$ holds for the splitting field L of its Mumford–Tate group, $|\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x| \geq c_g(\delta)\,\mathrm{Disc}(R_x)^{\mu_g}$, with R_x the center of $\mathrm{End}(A_x)$ and $c_g(\delta), \mu_g > 0$ effectively computable from g and δ.

By [23, Theorem 1.1]*, (a) and (b) hold with* $\delta = 1/32$ *for every special point.*

Theorem 1.3(a) is Corollary 8.10, with its second bound from Corollaries 3.5 and 3.6, and Theorem 1.3(b) is Theorem 10.3. Daw states his conjecture for G adjoint, K a neat product level and special points in the image of $X^+ \times \{1\}$.

For a special point $V = \{x\}$ and G adjoint, d_{L_V} is the d_L in the discriminant $\mathrm{disc}(x) = [K_T^m : K_T]\, d_L$ of [16, Def. 1], the quantity in [78, Thm. 1.2]. For x maximal in Z, under $\mathrm{QRH}(\delta; L_V)$, both $\mathrm{disc}(x)$, at every neat product level and on every component, and $[F(x) : F]$ are bounded by polynomials in D and d (Corollaries 8.9, 8.10 and 9.8), and for $\mathcal{A}_g$ so is $\mathrm{Disc}(R_x)$ (Corollary 10.4).

For the Siegel modular varieties the constants simplify.

Theorem 1.4. *Let* $g \geq 1$*,* $N \geq 3$*, and let* $\mathcal{A}_{g,K(N)}$ *be the Shimura variety of* $(\mathrm{GSp}_{2g}, \mathcal{H}_g^{\pm})$ *at the principal congruence level* $K(N)$*.*

(i) Theorem 1.1 and Corollary 1.2 hold for $\mathcal{A}_{g,K(N)}$*, with* L_V *the splitting field of the connected center of* H_V *itself,* α_V *computed at the image of* $K(N)$ *in* $\mathrm{PGSp}_{2g}(\mathbb{A}_f)$*, and with constants given in Theorem 9.9(a).*

(ii) The same holds for $\mathcal{A}_{g,1} = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g$*, whose level is not neat, with degrees defined through the cover* $\mathcal{A}_{g,K(3)}$ *(Lemma 9.10),* α_V *computed at the image of* $\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$ *in* $\mathrm{PGSp}_{2g}(\mathbb{A}_f)$*, and constants given in Theorem 9.9(b).*

In genus 4 the closure of the Jacobian locus is the zero set of one modular form of weight 8, so its degree has an explicit upper bound, and Theorem 1.4(ii) bounds $\deg V$, d_{L_V} and α_V for its maximal special subvarieties V (Corollary 9.11).

The bounds are polynomial in D and d (Corollary 9.6). With $z := \dim Z$, for adjoint G the bound of Theorem 8.1 is at most $(dD)^{2^z/\mu + o(1)}$ as $dD \to \infty$, with μ of Table 1 for ρ (Remark 8.2).

Turning these bounds into a list needs, in addition, an explicit model of the Shimura variety with a universal family of abelian varieties. No explicit model is known in general. Let $g \geq 2$, let d_Υ be a positive multiple of 8 and $\Upsilon := (d_\Upsilon, \dots, d_\Upsilon)$, with g entries. Mumford's moduli space M_Υ of principally polarized abelian varieties with an Υ-marking is open in a closed subscheme of a projective space given by explicit equations over $\mathbb{Q}$, and $M_\Upsilon \otimes \mathbb{C}$ is a connected Shimura variety for GSp_{2g}, at a level K_Υ (Section 11.1).

Theorem 1.5 (with [23, Theorem 1.1])**.** *Let* $g \geq 2$*,* $8 \mid d_\Upsilon$*, and let* $Z \subset M_\Upsilon \otimes \overline{\mathbb{Q}}$ *be closed, geometrically irreducible and defined over a number field. Then Algorithm 11.13 computes, from equations in Mumford's coordinates for the closure of* Z *in Mumford's projective space, the special subvarieties maximal in* Z*. In particular, if* Z *is a curve that is not special, it computes the special points of* Z*.*

Algorithm 11.13 is built on Urbanik's algorithm [104, Cor. 6.3] (Section 1.2) and is run with the bounds that Theorem 1.4(i) gives at the level $K(2d_\Upsilon)$, which lies in the level of M_Υ (Lemmas 11.1 and 11.2), with $\delta = 1/32$ (Corollary 1.2). These bounds depend on Z through $\deg_{\mathcal{L}_{K_\Upsilon}} Z$ and the degree of its field of definition (Lemma 11.2), and $\deg_{\mathcal{L}_{K_\Upsilon}} Z$ is $(2(g+1))^{\dim Z}$ times the degree of the closure of Z in Mumford's projective space (Lemma 11.8).

Most effective results concern products of modular curves: curves in $\mathbb{C}^2$ [65, 9, 37], subvarieties of $Y(1)^n$ [10, 11, 77], linear subvarieties [8] and linear relations among powers of singular moduli [38, 39]; beyond them, non-compact curves in Hilbert modular varieties [15]

and fiber powers of the Legendre family [14]. In [10] Binyamini's degree bounds are effective and his discriminant bounds are effective up to one possible exceptional imaginary quadratic field, and [11, Thm. 7] has one constant from Siegel's theorem. Both exceptions disappear with the zero-free half-plane of [23], as Bhattacharya observed with one for Dirichlet L-functions [7, Thms. 35.2 and 35.3]. For fixed n, the bounds of [11, Thm. 7], and the number of steps of its algorithm that lists the maximal special subvarieties, are polynomial in the degree and logarithmic height of Z and the degree of its field of definition. As in [10], the height of Z does not enter the constants of Theorem 8.1.

Daw [26] bounded the degrees of strongly special subvarieties from below; Daw, Javanpeykar and Kühne extended this bound to all special subvarieties of positive dimension [27, Thm. 6.1] and bounded, with no arithmetic hypothesis, the degrees of maximal special subvarieties that are not of facteur type [27, Thm. 1.3], a class defined by Ullmo [101]. Binyamini and Daw bounded the degree of every weakly optimal subvariety of Z, on a connected Shimura variety $\Gamma\backslash X^+$ with Γ neat and $Z(G)(\mathbb{R})$ compact, hence of every special subvariety of positive dimension maximal in Z, by a polynomial in the degree of Z; the polynomial depends on equations for a Baily–Borel embedding and for the canonical connection [13, Thm. 5 and p. 2157].

1.2. **The method.** We follow the route of Edixhoven, Klingler, Ullmo and Yafaev. For a special subvariety $V \subset Z$ that is not strongly special, their Hecke step says that Z contains a larger special subvariety once a prime l as in [63, Thm. 9.2.1, (9.1)] exists. For a suitable prime l that splits completely in L_V and a Hecke correspondence T_m at l, the Galois orbit of V lies in $Z \cap T_m Z$; if $Z \not\subset T_m Z$, an upper bound for the degree of this intersection contradicts a lower bound for the Galois orbit, and if $Z \subset T_m Z$, then Z contains a larger special subvariety [63, Thm. 7.2.1]. Two inputs make this work: a lower bound for the Galois orbit of V of the form $C(N)\alpha_V(\log d_{L_V})^N$ [102, Thm. 2.19], and the existence of a split prime l smaller than a fixed power of $\alpha_V \log d_{L_V}$ [63, Prop. 10.2.3]. GRH enters only here, both times to find primes that split completely in L_V [63, Prop. 10.1.2], [102, Prop. 2.13]. We make each step explicit and replace GRH by $\mathrm{QRH}(\delta; L_V)$.

Split primes. Under $\mathrm{QRH}(\delta; L)$ an explicit formula with a smoothing that removes the constants at $s = 0$ and $s = -1$ gives primes that split completely in L in every interval $(x/4, x]$ with x a fixed power of $\log d_L$, the power being $1/\delta$ instead of the $2 + o(1)$ of GRH (Lemmas 3.1 and 3.2). Frei, Sofos and Widmer obtain, under the same hypothesis, primes of every Frobenius class in $(x/2, x]$ for $x \geq c(\delta)(\log d_L)^{1/\delta}(\log\log d_L)^{2/\delta}$, with a constant $c(\delta)$ that is not made explicit [40, Cor. 1.2]. Hu deduces from his half-plane, with no hypothesis, a prime of every Frobenius class of norm $\ll (\log(e\, d_L) + [L : \mathbb{Q}])^8$ [57, Cor. 9.6]. We need only the completely split case, but with explicit constants. For the Galois bound we rerun the argument of Ullmo and Yafaev for the torus T_V [103, §6], with the split primes taken up to a fixed power of the discriminant, as Tsimerman does under GRH for CM points of $\mathcal{A}_g$ [99, §7.3, Thm. 7.6] (Ullmo and Yafaev take them up to about $|H| \log |H|$, with $|H|$ the class-group factor of the orbit [103, pp. 33 and 35]). So any $\delta > 0$ suffices, and the exponent μ in the bound $\gg \alpha_V d_{L_V}^{\mu}$ does not depend on δ (Proposition 3.3 and Corollary 3.4). Klingler and Yafaev use this bound in the weaker form $\gg \alpha_V(\log d_{L_V})^N$; we use the power of d_{L_V} itself. Then the prime l of the Hecke step, of size a power of $\log d_{L_V}$ and $\log \alpha_V$, is set against an orbit of size $\alpha_V d_{L_V}^{\mu}$, so any $\delta > 0$ works there too, and for adjoint G the bounds for α_V and d_{L_V} are at most a power of $[F : \mathbb{Q}] \deg Z$ (Lemma 6.12 and Remark 8.2).

Uniformity in the level. The Hecke step works at an Iwahori level I_l that varies with the prime l. Every constant must therefore be uniform in the level, including the constant in

Bézout's inequality. The proof of [63, Thm. 9.2.1] bounds the degree of $Z \cap T_m Z$, taken at an Iwahori level, by Bézout's inequality with constant 1 [63, (9.5)], and the proof of [27, Prop. 10.1] does the same at its Iwahori level. For an ample bundle this inequality can fail (Example 5.1); we use the constant c_B of Lemma 5.2, which does not affect the main theorem of [63]: the choice of l in [63, §10.2] absorbs it. We show that the complete linear system of $\mathcal{L}_J^{\otimes e}$ is base-point free and separates points for every $e \geq e^*(n)$, at every neat level J of every Shimura datum of dimension at most n (Theorem 1.6, proved in Section 4). So Bézout's inequality holds at every neat level with the constant $e^*(n)^{2n}$ (Lemma 5.2), and the Hecke step runs as in [63], with degrees taken for the Baily–Borel bundle at each level and the exponent 2^r, r the codimension of V in Z.

Theorem 1.6. *Let (G, X) be a Shimura datum, K neat, and $n := \dim \mathrm{Sh}_K(G, X)$. Put $e^*(n) := \lfloor \sum_{k=1}^{n} k\, 2^{1/k} \rfloor + 2$. For every integer $e \geq e^*(n)$ the complete linear system $|\mathcal{L}_K^{\otimes e}|$ on the Baily–Borel compactification is base-point free and separates points. Moreover $e^*(n) \leq n(n+3)/2 + 2$.*

The proof applies Fujino's Angehrn–Siu theorem for log canonical pairs [42] to the Baily–Borel compactification, which is log canonical with log canonical bundle $\mathcal{L}_K$ by Mumford's description of the canonical extension [75, Prop. 3.4]. For ball quotients, G. Di Cerbo and L. F. Di Cerbo show that on a smooth toroidal compactification $\widetilde{Y}$ of dimension $n \geq 2$ with boundary $\partial\widetilde{Y}$ the divisor $m(K_{\widetilde{Y}} + \partial\widetilde{Y})$ is base-point free for $m \geq \binom{n}{2} + 1$ and separates the points of $\widetilde{Y} \smallsetminus \partial\widetilde{Y}$ for $m \geq \binom{n}{2} + 2$ [31, Cor. 3.3 and 3.4].

Degrees. Daw, Javanpeykar and Kühne bound the degree of a maximal special subvariety that is not of facteur type by an iteration of Hecke correspondences [27, §10], which builds on the one Daw ran for strongly special subvarieties [26, Thm. 9.3] after [63, Thm. 9.2.1]. For adjoint G, the iteration of [27], with the uniform Bézout step and a fiber-saturation lemma, also bounds every positive-dimensional one by an effectively computable function of D whose constants depend only on (G, X, K, ρ). The bound of [13, Thm. 5] holds for every weakly optimal subvariety of Z and is polynomial in D of unspecified degree; Theorem 1.7, proved in Section 7, treats only the positive-dimensional special subvarieties maximal in Z, for adjoint G and a neat product level, and there gives $D^{2^{\dim Z} + o(1)}$.

Theorem 1.7. *Let (G, X) be a Shimura datum with G adjoint and K a neat product level, let $Z \subset \mathrm{Sh}_K(G, X)_{\mathbb{C}}$ be closed and irreducible, of degree D, and let V be a special subvariety maximal in Z with $\dim V \geq 1$. Then $\deg V$ is at most the largest of the constants $c^{\mathrm{ad}}_{\deg}(\dim Z, D)$ of Theorem 7.2 at the connected components of $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ (Section 2.2), a number that depends only on (G, X, K, ρ), $\dim Z$ and D, is effectively computable given representatives of the components, and is $D^{2^{\dim Z} + o(1)}$ as $D \to \infty$. No arithmetic hypothesis is used.*

Assembly. Call the smallest special subvariety containing Z its *special closure*. Maximal special subvarieties fall into three classes: strongly special ones, which Theorem 1.7 bounds; those that are not strongly special in the special closure of Z, which the Hecke step bounds; and those whose torus is the center of the generic Mumford–Tate group of Z, which are strongly special in the special closure but not in S. The last class needs a separate argument (Lemmas 8.5 and 8.8). So that $[F : \mathbb{Q}]$ does not enter the exponent of l in the Hecke step, we apply it to the union of the Galois conjugates of the lift of Z to its special closure, over the reflex field of the closure; if the larger special subvariety lies in another conjugate, V lies in the intersection of Z with that conjugate, which has smaller dimension, and we repeat

until V is in the last class relative to the intersection (proof of Theorem 8.1). For a general Shimura datum, the map to the adjoint Shimura variety need not be injective on connected components, and maximality can be lost under it; a recursion on dimension at the fixed level handles this (Section 9).

Listing. For $\mathcal{A}_g$ in Mumford's model, the bounds determine finitely many PEL data, each with an explicit locus of abelian varieties with the given endomorphisms (Lemma 10.1). Every special subvariety V is rigid in the normalization of one of these loci (Lemma 11.7), so it lies in the finite set of weakly special subvarieties of bounded degree, containing the rigid ones, that Urbanik's algorithm computes on the normalization [104, Cor. 6.3]; a test on the algebraic monodromy at one point, computed on a certified curve (Lemmas 11.10 and 11.11 and Proposition 11.12), decides which outputs are rigid, and rigid ones are special.

1.3. **Plan.** Part I proves the bounds: Theorem 8.1 in Section 8 for adjoint groups, and from it Theorem 1.1 in Section 9. Sections 3 and 4 are independent of each other, and Section 6 uses both. Part II proves Theorem 1.5; it uses Part I only through Theorem 9.9 and its constants, Lemma 10.1 and the argument of Lemma 2.2, and can be read after Section 2 and Lemma 10.1. Appendices A and B prove Lemmas 3.1 and 6.5, and Appendix C proves Lemma C.2 and the bound for h_K in Corollary C.8.

Part I. Bounds

2. Setting, notation and the complexity

2.1. **Shimura varieties and degrees.** A Shimura datum (G, X) satisfies Deligne's axioms; $E(G, X)$ is its reflex field and X^+ a connected component of X. For a compact open subgroup $K \subset G(\mathbb{A}_f)$,

$$\mathrm{Sh}_K(G, X)_{\mathbb{C}} = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K,$$

with its canonical model over $E(G, X)$. We say that K is a *product level* if $K = \prod_p K_p$ with $K_p \subset G(\mathbb{Q}_p)$, and *neat* in the sense of Pink [81, 0.6]. We write $G(\mathbb{Q})_+ := G(\mathbb{Q}) \cap \lambda^{-1}(G^{\mathrm{ad}}(\mathbb{R})^+)$, where $\lambda\colon G \to G^{\mathrm{ad}}$ is the adjoint map and $G^{\mathrm{ad}}(\mathbb{R})^+$ the identity component, and $G^{\mathrm{ad}}(\mathbb{Q})_+ := G^{\mathrm{ad}}(\mathbb{Q}) \cap G^{\mathrm{ad}}(\mathbb{R})^+$. For neat K every connected component of $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ is a quotient $\Gamma\backslash X^+$, with $\Gamma \subset G^{\mathrm{ad}}(\mathbb{Q})_+$ the image of $G(\mathbb{Q})_+ \cap gKg^{-1}$ for some $g \in G(\mathbb{A}_f)$ [63, §4.1], which is neat [81, 0.5] and arithmetic [69, Prop. 3.2], as $G(\mathbb{Q})_+ \cap gKg^{-1}$ has finite index in the congruence subgroup $G(\mathbb{Q}) \cap gKg^{-1}$, $G^{\mathrm{ad}}(\mathbb{R})$ having finitely many components [69, p. 54]. We fix a faithful representation $\rho\colon G \to \mathrm{GL}_{n_\rho}$ with $\rho(K) \subset \mathrm{GL}_{n_\rho}(\widehat{\mathbb{Z}})$, and write $n := \dim \mathrm{Sh}_K(G, X)$.

Let $\overline{\mathrm{Sh}}_K(G, X)$ be the Baily–Borel compactification. For neat K the line bundle of holomorphic forms of top degree on $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ extends to an ample line bundle $\mathcal{L}_K$ on $\overline{\mathrm{Sh}}_K(G, X)$, given at the generic points of the boundary divisors by forms with logarithmic poles and defined over $E(G, X)$; for neat $K_2 \subset gK_1g^{-1}$ the morphism $\mathrm{Sh}_{K_2} \to \mathrm{Sh}_{K_1}$ induced by g extends to the compactifications and pulls $\mathcal{L}_{K_1}$ back to $\mathcal{L}_{K_2}$ [63, Prop. 5.3.2]. For a closed subvariety $Y \subset \mathrm{Sh}_K(G, X)_{\mathbb{C}}$ we put

$$\deg Y := \sum_{Y'} \deg_{\mathcal{L}_K} \overline{Y'},$$

the sum over the irreducible components Y' of Y of the degrees of their closures [63, §5.1 and Def. 5.3.3]. We use the projection formula in the form $\deg_{f^*\mathcal{L}} Y = (\deg f) \deg_{\mathcal{L}} f(Y)$ for f generically finite and proper [63, §5.1].

2.2. **Special subvarieties.** A *special subvariety* of $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ is an irreducible component of the image of $\mathrm{Sh}_{K_H}(H,X_H)_{\mathbb{C}} \to \mathrm{Sh}_K(G,X)_{\mathbb{C}}$, $[x,h] \mapsto [x,hg]$, for a Shimura subdatum $(H,X_H) \subset (G,X)$, an element $g \in G(\mathbb{A}_f)$ and $K_H := H(\mathbb{A}_f) \cap gKg^{-1}$. We may and do take H to be the generic Mumford–Tate group on X_H, and write H_V for it. Irreducible components of preimages and images of special subvarieties under the morphisms of Shimura varieties used below are special [63, p. 14]. An irreducible subvariety is *Hodge generic* if it is not contained in a special subvariety of smaller dimension than the connected component containing it, and a subvariety defined and irreducible over a number field is Hodge generic if its geometric components are [63, Def. 7.1.1, Lemma 7.1.2]. The reflex field of a Shimura subdatum contains that of the datum, $E(G,X) \subset E(H,X_H)$ [69, Remark 12.3(c)], and a morphism of Shimura varieties induced by $(H,X_H) \to (G,X)$ is defined over $E(H,X_H)$ [69, Remark 13.8].

Let V be special with generic Mumford–Tate group H_V. In Sections 2 to 8 the group G is adjoint wherever T_V appears (except in Lemma 2.2, where T_V is the torus of the introduction), and we write T_V for the connected center of H_V; in Section 9 we pass to the adjoint group first, so that T_V is always the torus of the introduction. Let L_V be the splitting field of T_V, $d_{L_V} := |\operatorname{disc} L_V|$, and

$$\beta_V := \log d_{L_V}.$$

The subvariety V is *strongly special* if $T_V = 1$ [63, Def. 2.1.5]. More generally, a special subvariety of $\mathrm{Sh}_{K_P}(P,X_P)_{\mathbb{C}}$ for a Shimura datum (P,X_P) is strongly special there if the connected center of the image of its generic Mumford–Tate group in P^{ad} is trivial [63, Def. 2.1.5]. If $T_V \neq 1$, then $T_V(\mathbb{R})$ is compact and T_V is not split over $\mathbb{Q}$ [63, Lemma 2.4.3], so $L_V \neq \mathbb{Q}$ and $\beta_V \geq \log 3$ by Minkowski's theorem.

Let $K_{T_V}^m$ be the maximal compact open subgroup of $T_V(\mathbb{A}_f)$ and, for V given by (H_V, X_{H_V}) and g as above, $K_{T_V} := T_V(\mathbb{A}_f) \cap gKg^{-1}$. Replacing (H_V, X_{H_V}, g) by $(qH_Vq^{-1}, qX_{H_V}, qgk)$ with $q \in G(\mathbb{Q})$ and $k \in K$ conjugates $K_{T_V}^m$ and K_{T_V} by q, and replacing g by hg with $h \in H_V(\mathbb{A}_f)$ changes neither, as T_V is central in H_V; so the indices $|K_{T_V,p}^m / K_{T_V,p}|$ depend only on V (compare [63, Remark 2.4.2]).

Definition 2.1 ([63, Def. 2.5.1])**.** For V special with $T_V \neq 1$,

$$\alpha_V := \prod_{p:\, K_{T_V,p}^m \neq K_{T_V,p}} \max\bigl(1, B\,|K_{T_V,p}^m / K_{T_V,p}|\bigr),$$

with the constant B of Table 1. If $T_V = 1$, we put $\alpha_V := 1$ and $d_{L_V} := 1$. The *complexity* of V is

$$\Delta(V) := \max\bigl(\deg V,\ d_{L_V},\ \alpha_V\bigr).$$

Right translation $[x,a] \mapsto [x, ag^{-1}]$ is an isomorphism $\mathrm{Sh}_K(G,X) \to \mathrm{Sh}_{gKg^{-1}}(G,X)$, defined over $E(G,X)$ [69, Thm. 13.6] and pulling $\mathcal{L}_{gKg^{-1}}$ back to $\mathcal{L}_K$ (Section 2.1), so it preserves degrees and fields of definition containing $E(G,X)$; it maps special subvarieties to special subvarieties with the same T_V, L_V and indices, and the image of $X^+ \times \{g\}$ onto $\mathrm{Sh}_{gKg^{-1}}(G,X^+)_{\mathbb{C}}$, the image of $X^+ \times \{1\}$. The level gKg^{-1} is again a neat product level, and if K satisfies (Lv′) or (Lv) of Section 2.3, so does gKg^{-1} for the lattice $\rho(g)\widehat{\mathbb{Z}}^{n_\rho}$.

So in Sections 6 to 8 we assume, as [63, p. 14] and [27, §9.1] do, that Z lies in $\mathrm{Sh}_K(G,X^+)_{\mathbb{C}}$. A special subvariety V of $\mathrm{Sh}_K(G,X^+)_{\mathbb{C}}$ is then the image of $X_{H_V}^+ \times \{1\}$ with $X_{H_V}^+ \subset X^+$ [102, Lemma 2.1], so that $K_{T_V} = T_V(\mathbb{A}_f) \cap K$; the same convention is used for a Shimura subdatum (P,X_P) with $X_P^+ \subset X^+$ in place of (G,X), and for an F-irreducible Z (Section 6)

it applies to a geometric component of Z containing V; when a proof chooses a component of a preimage in $\mathrm{Sh}_J(P, X_P)_{\mathbb{C}}$, for $J \subset K \cap P(\mathbb{A}_f)$ compact open, it is chosen in the image of $X_P^+ \times \{1\}$.

For Z in the image of $X^+ \times \{g_i\}$, with g_i in a fixed finite set of representatives of $G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K$, Theorems 7.2 and 8.1 hold with the constants of the level $g_i K g_i^{-1}$ and the lattice $\rho(g_i)\widehat{\mathbb{Z}}^{n_\rho}$ in place of those of K and $\widehat{\mathbb{Z}}^{n_\rho}$. The symbols $c^{\mathrm{ad}}_{\deg}(z, D)$ and $c^\sharp(\delta, z, D, d)$ below are the constants of the level K; Section 9 uses the maxima $\bar{c}^{\mathrm{ad}}_{\deg}$ of $c^{\mathrm{ad}}_{\deg}$ and $\bar{c}^\sharp$ of $c^\sharp$ over the representatives. These maxima are effectively computable from (G, X, K, ρ) and the chosen representatives; we do not discuss how to compute a set of representatives in general (for $\mathcal{A}_g$ none is needed, proof of Theorem 9.9).

2.3. **The constants of the datum.** We use the following normalization of the level:

(Lv′) K is a neat product level, $\rho(K) \subset \mathrm{GL}_{n_\rho}(\widehat{\mathbb{Z}})$, and K_{p_0} is neat for some prime p_0.

Here a subgroup of $G(\mathbb{Q}_p)$ is neat if, for each k in it, the subgroup of $\overline{\mathbb{Q}}_p^\times$ generated by the eigenvalues of $\rho(k)$ contains no root of unity other than 1, and a product level with one neat factor is neat [63, §4.1.4]. Klingler and Yafaev's normalization [63, Situation 9.1],

(Lv) K is a neat product level, and K_3 lies in the principal congruence subgroup of level 3 of $\mathrm{GL}_{n_\rho}(\mathbb{Z}_3)$,

implies (Lv′) with $p_0 = 3$, since that subgroup is neat [81, 0.6]; Lemma 2.2 passes from any neat product level to one satisfying (Lv), at an explicit cost. Ullmo and Yafaev assume (Lv) in [102, §2.2] only to make the level neat [102, p. 20 and Thm. 2.19]. The weaker condition serves $\mathcal{A}_g$, since at the level $K(N)$ the adjoint level $\lambda(K(N))$ has a neat factor at a prime dividing N, which need not be 3 (proof of Theorem 9.9(a)).

Lemma 2.2. *Let K be a neat product level, $K'_p := K_p$ for $p \neq 3$, $K'_3 := K_3 \cap \ker(\mathrm{GL}_{n_\rho}(\mathbb{Z}_3) \to \mathrm{GL}_{n_\rho}(\mathbb{F}_3))$, $K' := \prod_p K'_p$ and $g_3 := |\mathrm{GL}_{n_\rho}(\mathbb{F}_3)|$. Then K' satisfies (Lv), $[K : K'] \leq g_3$, and $\pi\colon \mathrm{Sh}_{K'}(G, X) \to \mathrm{Sh}_K(G, X)$ is finite étale of degree at most $[K : K']$ with $\pi^*\mathcal{L}_K = \mathcal{L}_{K'}$. Let $Z \subset \mathrm{Sh}_K(G, X)$ be closed and geometrically irreducible, defined over $F \supset E(G, X)$, of degree D, and V special and maximal in Z. Then there are a closed, geometrically irreducible $Z' \subset \mathrm{Sh}_{K'}(G, X)$, defined over a field of degree at most $\deg \pi \cdot [F : \mathbb{Q}]$, with $\deg_{\mathcal{L}_{K'}} Z' \leq \deg \pi \cdot D$, and a special V' maximal in Z' with $\pi(V') = V$, $\deg V \leq \deg_{\mathcal{L}_{K'}} V'$ and $T_{V'} = T_V$.*

Proof. K/K' acts transitively on the fibers of π, and for neat K the stabilizers are the image of $Z(G)(\mathbb{Q}) \cap K$, which acts trivially; so π is finite étale of degree at most $[K : K']$, and $\pi^*\mathcal{L}_K = \mathcal{L}_{K'}$ [63, Prop. 5.3.2(1)]. Choose a component V' of $\pi^{-1}(V)$ and a component Z' of $\pi^{-1}(Z)$ containing it. The set $\pi^{-1}(Z)$ is defined over F and has at most $\deg \pi$ components, which gives the field of Z'; and $\deg_{\mathcal{L}_{K'}} Z' \leq \deg_{\mathcal{L}_{K'}} \pi^{-1}(Z) = \deg \pi \cdot D$ by the projection formula. If $V' \subsetneq V'' \subset Z'$ with V'' special, $\pi(V'')$ would be a special subvariety of Z of dimension larger than $\dim V$. □

Since $gK'g^{-1} \cap T(\mathbb{A}_f) \subset gKg^{-1} \cap T(\mathbb{A}_f)$ for every torus T and every $g \in G(\mathbb{A}_f)$, each index $|K^m_{T,p}/K_{T,p}|$ can only grow when K is replaced by K', so α computed at K' is at least α computed at K.

Put $c_U := 1/(2C_\Phi)$, where C_Φ bounds the orders of the component groups of the stabilizers in [34, Lemma 4.4.1] in the following form. For a special V let $T := T_V$, let $F_T \supset L_T$ and $r_T\colon \mathrm{Res}_{F_T/\mathbb{Q}}\mathbb{G}_{\mathrm{m}} \to T$ be the field and surjection of [102, p. 42], and for $d \geq 1$ and a vector

v of the d-th exterior power of $\overline{\mathbb{Q}}^{n_\rho}$ let N be the subgroup of $X^*(\mathrm{Res}_{F_T/\mathbb{Q}}\mathbb{G}_\mathrm{m})$ generated by the differences $x - x'$ of the weights x, x' of $\mathrm{Res}_{F_T/\mathbb{Q}}\mathbb{G}_\mathrm{m}$, acting through $\rho \circ r_T$, in the support of v. The bound C_Φ satisfies

$$\left|\left(X^*(\mathrm{Res}_{F_T/\mathbb{Q}}\mathbb{G}_\mathrm{m})/N\right)_{\mathrm{tors}}\right| \;\le\; C_\Phi \qquad \text{for all special } V,\ d \ge 1 \text{ and } v \in \textstyle\bigwedge^d \overline{\mathbb{Q}}^{n_\rho}.$$

These come from the finitely many characters allowed by [102, Lemmas 2.5–2.6], so C_Φ, and with it c_U, is computable from (G, ρ).

TABLE 1. Constants attached to (G, X, K, ρ), for G adjoint.

Constant	Meaning	Source
n_T	a bound for $[L_V : \mathbb{Q}]$ over all special V	[102, proof of Lemma 2.5]
R_G	a bound for the degree of the Galois closure of $E(H_V, X_{H_V})$ over all special V	[102, Lemma 2.5], [63, Def. 2.5.1]
C_0	every prime $l > C_0$ has $K_l = G(\mathbb{Z}_l)$ hyperspecial	[63, p. 53]
a, A, r_U	a bounds the indices of the auxiliary levels; A an integer as in [102, Prop. 2.9]; r_U bounds the number of cyclic factors of $K^m_{T_V,p}/K_{T_V,p}$	[102, pp. 20–22, 29]
B	$B = 1/(A^{r_U}(a!)^{r_U})$	[102, p. 31]
c_U	$\lvert K^m_{T_V,p}/K_{T_V,p}\rvert \ge c_U p$ for $p > C_0$ unramified in L_V with $K^m_{T_V,p} \ne K_{T_V,p}$	[102, Prop. 3.15], [34, §4.4], Lemma 6.4
k_1, f	$k_1 = 2An_\rho^3 B_\rho$; $f := (\dim G + 1)^2$	[34, Prop. 7.4.3], [63, (8.6)]
t_ρ, s_ρ	s_ρ, for instance $s_\rho = n_\rho$, bounds the number of characters of the representation of H_V/H_V^{der} induced by $t \mapsto \rho(t)^m$ (m the degree of $T_V \to H_V/H_V^{\mathrm{der}}$); t_ρ is a positive integer bounding the absolute values of their Hodge weights	[102, proof of Lemma 2.6]
μ	$\mu = 1/(16\, t_\rho s_\rho\, n_T(n_T - 1))$	Proposition 3.3

The constants in Table 1 depend only on (G, X, K, ρ) and, except μ and f, are the ones in [63, 102, 34]. We write $B' := Bc_U$. In Table 1, a bounds the indices $[K_H^m : K_H^{m'}]$ and $[K_H : K_H']$ of the auxiliary levels of [102, pp. 21–22], for instance $a = |\mathrm{GL}_{n_\rho}(\mathbb{F}_3)|$; A, with the property of [102, Prop. 2.9], is fixed once and for all; and r_U is a bound over all special V and primes p. In k_1, B_ρ is the bound B of [34, Prop. 7.4.3] for $\rho|_{T_V}$, uniform in V by [102, Lemma 2.6] as in [63, §8.3.1] ([106, Prop. 2.6] for special points), and f bounds the Iwahori indices of [63, (8.8)] as $< l^f$ (Lemma 6.5). The bound n_T and the field degrees R_G are bounds over all special subvarieties at once. Each constant except C_0 is given by an explicit formula (B, c_U, k_1, f, μ, s_ρ) or is bounded by the proof in its source through finiteness arguments that can be carried out effectively. For n_T and R_G they concern the finitely many conjugacy classes of finite subgroups of $\mathrm{GL}_r(\mathbb{Z})$, $r \le \operatorname{rank} G$ [102, proof of Lemma 2.5], and both are at most $|\mathrm{GL}_{\operatorname{rank} G}(\mathbb{F}_3)|$. For r_U it is the local unit theorem, and one may take $r_U = 2 \operatorname{rank} G$. For A, t_ρ, B_ρ and C_Φ they concern root systems, minuscule coweights and the Hodge types of ρ [102, Lemmas 2.4–2.8]. The constant C_0 is any integer larger than every prime l at which the schematic closure of $\rho(G)$ in $\mathrm{GL}_{n_\rho,\mathbb{Z}}$ is not reductive over $\mathbb{Z}_l$ or $K_l \ne G(\mathbb{Z}_l)$. The primes of the first kind divide the integer M_0 of Section 9, which

is found from equations for $\rho(G)$ by making effective the steps used there (the finiteness of the non-smooth locus and [22, Props. 3.1.9 and 3.1.12]), and those of the second kind are finitely many and are read off from K. None of the constants involves an analytic input. When G is adjoint we assume $G \neq 1$. Then a special point x has $\mathrm{MT}(x) \neq 1$, so $n_T \geq 2$ (Section 2.2) and $s_\rho, t_\rho \geq 1$, and μ is defined.

For B_ρ we supply one step. The coordinates of the characters of T_V in ρ, read in $X^*(\mathrm{Res}_{F_T/\mathbb{Q}}\mathbb{G}_{\mathrm{m}})$ through r_T (above), are bounded by [102, Lemma 2.6], and in an LLL-reduced basis of the lattice $X^*(T_V) \subset X^*(\mathrm{Res}_{F_T/\mathbb{Q}}\mathbb{G}_{\mathrm{m}})$ the coordinates of a vector are at most a constant depending only on $\dim T_V$ times its length, since nonzero integer vectors have length at least 1.

2.4. **The analytic hypothesis.** For a number field L and $\delta \in (0, 1/2]$, $\mathrm{QRH}(\delta; L)$ is the statement that $\zeta_L(s) \neq 0$ for $\mathrm{Re}\, s > 1 - \delta$, $s \neq 1$. By [23, Theorem 1.1], $\mathrm{QRH}(1/32; L)$ holds for every number field L, and by [57, Theorem 1.2], $\mathrm{QRH}(1/8; L)$ does; so every statement below made under $\mathrm{QRH}(\delta; L)$ holds with no hypothesis at $\delta = 1/32$ and at $\delta = 1/8$. We put

$$\kappa(\delta) := \frac{2-\delta}{\delta(1+\delta)}, \qquad w(\delta) := (1 + 2^{-1-\delta})(1 + 2^{-\delta}) < 3.$$

3. Split primes and Galois orbits under a zero-free half-plane

This section proves the two analytic inputs of the Hecke step for one Galois field L under $\mathrm{QRH}(\delta; L)$: a count of primes that split completely in L from x a power of $\log d_L$ on, and the lower bound of Ullmo and Yafaev [103, §6] for Galois orbits with an explicit constant. The first is an explicit formula; the second uses it.

3.1. **Split primes.** Let $L/\mathbb{Q}$ be Galois of degree n_L and signature (r_1, r_2), $d_L := |\operatorname{disc} L|$, and put

$$A_L := \tfrac{3}{2} + \tfrac{1}{2}\log d_L, \qquad \vartheta_L(x) := \sum_{\substack{x/4 < p \leq x \\ p \text{ splits completely in } L}} \log p.$$

Lemma 3.1. *Let $\delta \in (0, 1/2]$ and assume $\mathrm{QRH}(\delta; L)$. For every $x \geq 8$,*

$$\vartheta_L(x) \;\geq\; \frac{3x}{8n_L}\Big(1 - \tfrac{16}{3}w(\delta)\kappa(\delta)A_L\, x^{-\delta}\Big) - 1.4262\sqrt{x} - \epsilon_L(x)\log \mathrm{rad}(d_L) - 0.694 - \frac{1.922}{x},$$

where $\epsilon_L(x) = 1$ if some prime dividing d_L lies in $(x/4, x]$ and $\epsilon_L(x) = 0$ otherwise.

Lemma 3.1 is proved in Appendix A.

Lemma 3.2. *Let $\delta \in (0, 1/2]$ and assume $\mathrm{QRH}(\delta; L)$. Put*

$$x_1(L, \delta) := \max\Big(\big(\tfrac{32}{3}w(\delta)\kappa(\delta)A_L\big)^{1/\delta},\ (48n_L)^2,\ (\log d_L + 1)^2\Big).$$

For every $x \geq x_1(L, \delta)$ we have $\vartheta_L(x) \geq x/(8n_L)$, so at least $x/(8n_L \log x)$ primes in $(x/4, x]$ split completely in L. If $\beta := \log d_L \geq 1$, the condition $x \geq x_1(L, \delta)$ holds as soon as

$$x^\delta \geq 64\kappa(\delta)\beta, \qquad x \geq (48n_L)^2, \qquad \sqrt{x} \geq \beta + 1.$$

Proof. As $x \geq 8$, Lemma 3.1 applies, and $x^\delta \geq \frac{32}{3}w\kappa A_L$ makes the bracket at least $1/2$, so the main term is at least $3x/(16n_L)$. Bound $\epsilon_L \log \mathrm{rad}(d_L) \leq \log d_L$ and $0.694 + 1.922/x < 1$. As $\sqrt{x} \geq 48n_L$, $x/(16n_L) \geq 3\sqrt{x}$, and as $\sqrt{x} \geq \log d_L + 1$, $3\sqrt{x} \geq 1.4262\sqrt{x} + \log d_L + 1$. So $\vartheta_L(x) \geq 3x/(16n_L) - x/(16n_L) = x/(8n_L)$, and each prime counted contributes at most $\log x$. For the last assertion, $w(\delta) < 3$ and $A_L \leq 2\beta$ when $\beta \geq 1$, so $\frac{32}{3}w(\delta)\kappa(\delta)A_L \leq 64\kappa(\delta)\beta$. □

3.2. **The Galois bound.** Let (G, X) be a Shimura datum with G adjoint, $(H, X_H) \subset (G, X)$ a Shimura subdatum with H the generic Mumford–Tate group on X_H, and $H^{\mathrm{ab}} := H/H^{\mathrm{der}}$, which is isogenous to the connected center T of H [102, Remark 2.20]; assume $T \neq 1$. Fix $x \in X_H$, let $(H^{\mathrm{ab}}, \{\bar{x}\})$ be the induced Shimura datum of CM type, $E := E(H^{\mathrm{ab}}, \{\bar{x}\})$, L the splitting field of H^{ab} (that of T), and $r_L := r \circ N_{L/E} \colon \mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_{\mathrm{m}} \to H^{\mathrm{ab}}$, where r is the reciprocity morphism, which is surjective [102, proof of Prop. 2.14]. Let $K^m_{H^{\mathrm{ab}}}$ be the maximal compact open subgroup of $H^{\mathrm{ab}}(\mathbb{A}_f)$ and

$$\mathrm{Im}_L := \text{the image of } r_L\big((\mathbb{A}_f \otimes L)^\times\big) \text{ in } H^{\mathrm{ab}}(\mathbb{Q})\backslash H^{\mathrm{ab}}(\mathbb{A}_f)/K^m_{H^{\mathrm{ab}}}.$$

Let $\chi_1, \chi_2, \dots$ be the characters, at most s_ρ in number, of the representation of H^{ab} induced by $t \mapsto \rho(t)^m$, with m the degree of the isogeny $T \to H^{\mathrm{ab}}$ (which kills its kernel) [102, proof of Lemma 2.6]; they span $X^*(H^{\mathrm{ab}}) \otimes \mathbb{Q}$, and their coordinates in the canonical basis $(\chi_\sigma)_{\sigma \in \mathrm{Gal}(L/\mathbb{Q})}$ of $X^*(\mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_{\mathrm{m}})$ are Hodge weights, of absolute value at most t_ρ [102, proof of Lemma 2.6]. Put $\mathfrak{d} := 2t_\rho s_\rho n_T(n_T - 1)$, $\theta := 1/(4\mathfrak{d}) = 2\mu$, and

$$\beta_0(\delta) := \max\Big(\frac{128\kappa(\delta)}{\theta^2\delta^2},\ \Big(\frac{96}{\theta^3}\Big)^{1/2},\ 3n_T \log n_T + 1,\ \frac{64n_T}{\theta}\Big), \qquad c_*(\delta) := \exp(-\mu\beta_0(\delta)).$$

Proposition 3.3. *If* $\mathrm{QRH}(\delta; L)$ *holds, then* $|\mathrm{Im}_L| \geq c_*(\delta)\, d_L^{\mu}$. *The exponent* $\mu = 1/(16t_\rho s_\rho n_T(n_T - 1))$ *does not depend on* δ.

This is the bound to which Ullmo and Yafaev reduce their theorem [103, Thm. 6.1 and p. 33], which assumes GRH, with the split primes supplied by Lemma 3.2; their argument goes back to Amoroso and Dvornicich [2] and Yafaev [106, Thm. 2.15]. We write the proof out, with the primes taken up to a power of the discriminant, $x = d_L^\theta$, as in [99, proof of Thm. 7.6].

Proof. Let $p_1, \dots, p_\ell \leq x$ be distinct primes that split completely in L. Fix places $v_j \mid p_j$ and let $P_j \in (\mathbb{A}_f \otimes L)^\times$ be p_j at v_j and 1 elsewhere.

(1) Suppose $a \in \mathbb{Z}^\ell \smallsetminus \{0\}$ and $r_L(\prod_j P_j^{a_j}) = \pi k$ with $\pi \in H^{\mathrm{ab}}(\mathbb{Q})$ and $k \in K^m_{H^{\mathrm{ab}}}$. Write $\chi_i \circ r_L = \sum_\sigma c_{i,\sigma}\chi_\sigma$ with $|c_{i,\sigma}| \leq t_\rho$. Since χ_σ acts on $(\mathbb{A}_f \otimes L)^\times$ through $1 \otimes \sigma$ and the decomposition group at v_j is trivial, $\pi_i := \chi_i(\pi) \in L^\times$ has valuation of absolute value at most $t_\rho|a_j|$ at the places above p_j and valuation 0 at every other finite place.

(2) As G is adjoint, $H^{\mathrm{ab}}(\mathbb{R})$ is compact, complex conjugation acts by -1 on $X^*(H^{\mathrm{ab}})$ and L is CM, so $|\sigma(\pi_i)| = 1$ for every embedding σ. With $Q := \prod_j p_j^{|a_j|}$, the numbers $\pi_i' := Q^{t_\rho}\pi_i$ are algebraic integers with $|\sigma(\pi_i')| = Q^{t_\rho}$.

(3) The π_i generate L [106, Lemma 2.16]: if $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ fixes every π_i, comparing valuations at the places above some p_j with $a_j \neq 0$ shows that σ fixes each $\chi_i \circ r_L$ in the regular module $X^*(\mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_{\mathrm{m}})$. As r_L is surjective, σ acts trivially on $X^*(H^{\mathrm{ab}}) \otimes \mathbb{Q}$, so $\sigma = 1$.

(4) The monomials $\prod_i \pi_i'^{e_i}$ with $0 \leq e_i \leq n_T - 1$ span L over $\mathbb{Q}$; a basis $b_1, \dots, b_{n_L}$ among them satisfies $|\sigma(b_k)| \leq Q^{t_\rho s_\rho (n_T - 1)}$, and Hadamard's inequality gives $|\det(\mathrm{Tr}\, b_k b_{k'})| \leq n_T^{3n_T/2} Q^{\mathfrak{d}}$. This is a multiple of d_L, so $\log d_L \leq \frac{3}{2} n_T \log n_T + \mathfrak{d} \sum_j |a_j| \log x$.

(5) If $\ell \geq |\mathrm{Im}_L|$, two of the classes of the $r_L(P_j)$ coincide or one is trivial, which gives a as in (1) with $0 < \sum |a_j| \leq 2$. So if $\log d_L - \frac{3}{2} n_T \log n_T > 2\mathfrak{d} \log x$, then $\ell < |\mathrm{Im}_L|$ [103, p. 33].

(6) Let $\beta := \log d_L \geq \beta_0(\delta)$ and $x := d_L^\theta$. Then $x^\delta = e^{\theta\delta\beta} \geq (\theta\delta\beta)^2/2 \geq 64\kappa(\delta)\beta$; $\theta\beta \geq 64n_T \geq 2\log(48n_T)$; and $\sqrt{x} = e^{\theta\beta/2} \geq (\theta\beta/2)^3/6 \geq 2\beta \geq \beta + 1$. By Lemma 3.2 there are at least $x/(8n_T \log x)$ such primes in $(x/4, x]$. Also $\beta - \frac{3}{2} n_T \log n_T > \beta/2 = 2\mathfrak{d} \log x$. By

(5), $|\mathrm{Im}_L| > x/(8n_T \log x) = e^{2y}/(16n_T y)$ with $y := \theta\beta/2 \geq 32n_T$, and $e^y \geq y^2/2 \geq 16n_T y$, so $|\mathrm{Im}_L| \geq e^y = d_L^\mu$. If $\beta < \beta_0(\delta)$, then $|\mathrm{Im}_L| \geq 1 \geq c_*(\delta) d_L^\mu$. □

For a special V with $T_V \neq 1$ let F_V be the Galois closure of $E(H_V, X_{H_V})$; it contains L_V [102, proof of Prop. 2.14] and $[F_V : \mathbb{Q}] \leq R_G$. Put

$$C_* := \frac{c_*(\delta)}{R_G\, a\, (a!)^{r_U}}, \qquad \Omega_V := C_*\, \alpha_V\, d_{L_V}^\mu.$$

Corollary 3.4. *Let G be adjoint and K satisfy* (Lv′). *Let $(H, X_H) \subset (G, X)$ be a Shimura subdatum with H the generic Mumford–Tate group on X_H and $T := Z(H)^\circ \neq 1$, $K_H := K \cap H(\mathbb{A}_f)$, V_H a geometric component of $\mathrm{Sh}_{K_H}(H, X_H)_{\mathbb{C}}$ and V its image in $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$, so that $T_V = T$. Let $F \supset E(H, X_H)$ be a number field whose Galois closure F^c has degree at most R_G (for instance $F = E(H', X_{H'})$ for a subdatum $(H', X_{H'}) \subset (H, X_H)$, or its Galois closure). Assume $\mathrm{QRH}(\delta; L_V)$. Then:*

(i) $\deg_{\mathcal{L}_{K_H}}(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot V_H) \geq \Omega_V$, *and hence* $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot V) \geq \Omega_V$ *in* $\mathrm{Sh}_K(G, X)$;

(ii) if $Y \subset V_H$ is closed, geometrically irreducible and defined over $\overline{\mathbb{Q}}$, $K_H^m := K_T^m K_H$, $\pi\colon \mathrm{Sh}_{K_H}(H, X_H) \to \mathrm{Sh}_{K_H^m}(H, X_H)$ is the natural map, $\Theta_A \subset K_T^m/K_T$ is the image of $x \mapsto x^A$, with A the integer of Table 1, and $\Theta_A \cdot Y \subset \mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y \cap \pi^{-1}\pi(Y)$, then $\deg_{\mathcal{L}_{K_H}}(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot Y) \geq \Omega_V$.

The same bounds hold for the orbits under any Galois group containing $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$.

Proof. The field F^c contains the Galois closure of $E(H, X_H)$, hence L_V [102, proof of Prop. 2.14]. We first prove (i) for $F = F_V$, the Galois closure of $E(H, X_H)$. The first bound of (i) is then [102, Thm. 2.19, (6)] (which is [63, Thm. 2.4.4]) with GRH replaced by $\mathrm{QRH}(\delta; L_V)$, and with $d_{L_V}^\mu$ in place of $(\log d_{L_V})^N$. In its proof GRH is used only in [102, Prop. 2.13] (see [102, pp. 30 and 33]), which bounds $|\mathrm{Im}_L|$ from below for $L = L_V$; this bound enters the proof of [102, Thm. 2.19] only as a factor, the number of Galois conjugates in [102, Prop. 2.14]. We replace it by Proposition 3.3. The image of $r_{(H^{\mathrm{ab}}, \bar{x})}((\mathbb{A}_f \otimes F_V)^\times) = r_L(N_{F_V/L}((\mathbb{A}_f \otimes F_V)^\times))$ has index at most $[F_V : L] \leq R_G$ in Im_L, because r_L maps $L^\times$ into $H^{\mathrm{ab}}(\mathbb{Q})$ and the unit ideles into $K_{H^{\mathrm{ab}}}^m$, so factors through an idele class group of L, where the norm from F_V has image of index at most $[F_V : L]$ by class field theory. So the number of conjugates in [102, Prop. 2.14] is at least $c_*(\delta) d_L^\mu / R_G$. By [102, Remark 2.20] the constant of [102, Thm. 2.19] is that of Prop. 2.14 divided by $a\,(a!)^{r_U}$, and the factor α_V comes from [102, Lemmas 2.15–2.18], which use no GRH. This gives (6) of [102, Thm. 2.19], which is the first bound of (i); the second follows from the first by [102, Lemma 2.2] and [63, Cor. 5.3.10], as $Z(H)(\mathbb{R})$ is compact.

Form (7) of [102, Thm. 2.19] is proved in the same way for Y in place of V_H [102, pp. 29–31], at the auxiliary levels $K_H' \subset K_H^{m'}$ of [102, pp. 21–22], for components V' and Y' over V_H and Y and the map $\pi'\colon \mathrm{Sh}_{K_H'}(H, X_H) \to \mathrm{Sh}_{K_H^{m'}}(H, X_H)$ [102, Lemma 2.11]. By [102, Lemma 2.12] the degree of the Galois orbit of Y' is at least the degree of the union of its irreducible components mapping onto $\pi'(Y')$ times the number of Galois conjugates of $\pi'(Y')$. GRH enters only in this number, through [102, Prop. 2.14], whose assertion for Y' follows from that for V' because the Galois conjugates of $\pi'(V')$ are disjoint.

For general F we prove (ii), and then (i), by running this argument over F itself: [102, (5) and Lemmas 2.12, 2.16–2.18] use only that the map $\mathrm{Sh}_{K_H'}(H, X_H) \to \mathrm{Sh}_{K_H}(H, X_H)$, π and π' are defined over F, and the hypothesis on Θ_A is used with F; the remaining

factor, the number of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-conjugates of $\pi'(Y')$, is at least the number of $\mathrm{Gal}(\overline{\mathbb{Q}}/F^c)$-conjugates, which the argument above bounds with $F^c \supset L_V$ in place of F_V and index at most $[F^c : L_V] \le R_G$. This is (ii). For (i), $\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot V_H \supset \mathrm{Gal}(\overline{\mathbb{Q}}/F^c)\cdot V_H$, and the argument above applies verbatim to F^c. □

Corollary 3.5. *Let (G,X) be a Shimura datum with G adjoint and $G \neq 1$, let K satisfy* (Lv′)*, and let $\delta \in (0,1/2]$. Let $x \in X^+$ be such that $[x,1]$ is a special point, $T := \mathrm{MT}(x)$, L the splitting field of T, $K_T := K \cap T(\mathbb{A}_f)$, K_T^m the maximal compact open subgroup of $T(\mathbb{A}_f)$, and $i(T)$ the number of primes p with $K_{T,p} \neq K_{T,p}^m$. If $\mathrm{QRH}(\delta; L)$ holds, then*

$$\left|\mathrm{Gal}(\overline{\mathbb{Q}}/L)\cdot[x,1]\right| \;\ge\; C_*\,\alpha_{[x,1]}\,d_L^{\mu} \;\ge\; C_*\,B^{i(T)}\,[K_T^m : K_T]\,d_L^{\mu},$$

with $C_* = c_*(\delta)/(R_G\,a\,(a!)^{r_U})$ and B, μ of Table 1; all are effectively computable from (G,X,K,ρ) and δ, and μ does not depend on δ.

Proof. As $G \neq 1$ is adjoint, $T \neq 1$ (Section 2.3). The subdatum $(T,\{x\})$ has T as its generic Mumford–Tate group and $Z(T)^\circ = T$, and $V := \{[x,1]\}$ is the image of a point of $\mathrm{Sh}_{K_T}(T,\{x\})_{\mathbb{C}}$, with $T_V = T$, $L_V = L$ and $K_{T_V} = K_T$ (Section 2.2). By Corollary 3.4(i) for $F = F_V$, which contains L, and its last sentence, $|\mathrm{Gal}(\overline{\mathbb{Q}}/L)\cdot[x,1]| \ge \Omega_V = C_*\alpha_V d_L^{\mu}$, as a finite set of points has degree its number of points. As K is a product level, $[K_T^m : K_T] = \prod_p |K_{T,p}^m/K_{T,p}|$, and each factor $\max(1, B|K_{T,p}^m/K_{T,p}|)$ of α_V is at least $B|K_{T,p}^m/K_{T,p}|$, so $\alpha_V \ge B^{i(T)}[K_T^m : K_T]$. □

This is the form of [103, Thm. 6.1].

Corollary 3.6. *Let (G,X) and δ be as in Corollary 3.5, let K be a neat product level, and let x, T, L, K_T, K_T^m and $i(T)$ be as there. If $\mathrm{QRH}(\delta; L)$ holds, then*

$$\left|\mathrm{Gal}(\overline{\mathbb{Q}}/L)\cdot[x,1]\right| \;\ge\; \frac{C_*' B_{K'}}{g_3}\,B_{K'}^{i(T)}\,[K_T^m : K_T]\,d_L^{\mu},$$

where K' and g_3 are as in Lemma 2.2 and C_*', $B_{K'}$ are the constants C_* and B at K'.

Proof. This follows from Corollary 3.5 at K', which satisfies (Lv): the map $\mathrm{Sh}_{K'}(G,X) \to \mathrm{Sh}_K(G,X)$ is defined over $E(G,X) \subset L$, its fibers have at most g_3 points, and $K_T' \subset K_T$ differs from K_T only at 3. □

With $\delta = 1/32$ this is [25, Conj. 22.1], with any $c_1 < C_*$, $B_1 = B$ and $\mu_1 = \mu$ under (Lv′), and with any $c_1 < C_*' B_{K'}/g_3$, $B_1 = B_{K'}$ and $\mu_1 = \mu$ at every neat product level; the argument is that of [103, §6]. Under (Lv′), with Lemma 6.9, the first bound of Corollary 3.5 gives a lower bound for $[F([x,1]) : F]$ by a power of the discriminant of [16, Def. 1] (Corollary 6.10).

4. Point separation by the Baily–Borel bundle

The Bézout inequality of Lemma 5.2 needs one exponent e for which the complete linear system of $\mathcal{L}_K^{\otimes e}$ on the Baily–Borel compactification is base-point free and separates points; very ampleness is not needed. This section gives such an exponent for every Shimura variety, depending only on the dimension.

Proof of Theorem 1.6. Reduction to one component. $\overline{\mathrm{Sh}}_K(G,X)$ is the disjoint union of the Baily–Borel compactifications of its connected components $\Gamma_i\backslash X^+$, with Γ_i neat arithmetic (Section 2.1); as sections over a disjoint union are tuples of sections, it suffices to treat one

component $Y := \overline{\Gamma\backslash X^+}$, with interior $Y^\circ = \Gamma\backslash X^+$. For $n = 0$ there is nothing to prove, so let $n \geq 1$.

Y is a log canonical pair with log canonical bundle $\mathcal{L}_K|_Y$. Let $\pi\colon \widetilde{Y} \to Y$ be a smooth toroidal compactification of Y° with reduced normal crossings boundary $\partial\widetilde{Y}$, which exists as Γ is neat [5]; π is an isomorphism over Y°. That Y with its boundary is log canonical is known [1, Thm. 3.2 and Cor. 3.3]; we recall the argument, which also identifies the bundle. By Mumford's description of the canonical extensions [75, Prop. 3.4(a), (b)], the extension of the cotangent bundle is $\Omega^1_{\widetilde{Y}}(\log \partial\widetilde{Y})$ and the extension of the canonical bundle is the pull-back of an ample bundle $\mathcal{O}_Y(1)$ on Y, whose high powers give the Baily–Borel embedding; taking top exterior powers,

$$K_{\widetilde{Y}} + \partial\widetilde{Y} \cong \pi^*\mathcal{O}_Y(1).$$

The bundles $\mathcal{O}_Y(1)$ and $\mathcal{L}_K|_Y$ agree on Y° and at the generic point of each boundary divisor of Y (both are given there by forms with logarithmic poles [63, Prop. 5.3.2(1)]), so they are isomorphic on the normal variety Y. Let $\Delta := \pi_*\partial\widetilde{Y}$, the reduced sum of the boundary divisors of Y ($\Delta = 0$ if the boundary has codimension at least 2). Then $K_Y + \Delta = \pi_*(K_{\widetilde{Y}} + \partial\widetilde{Y})$ is linearly equivalent to $\mathcal{L}_K|_Y$, hence Cartier, and $K_{\widetilde{Y}} + \partial\widetilde{Y} - \pi^*(K_Y + \Delta)$ is a principal divisor with zero push-forward, hence zero, since a rational function with trivial divisor on the normal Y is a unit. So $K_{\widetilde{Y}} + \partial\widetilde{Y} = \pi^*(K_Y + \Delta)$. The pair $(\widetilde{Y}, \partial\widetilde{Y})$ is log canonical [64, Cor. 3.12], and therefore so is (Y, Δ) [64, Lemma 3.10].

Angehrn–Siu. Let $e \geq e^*(n)$ and $\sigma(n) := \sum_{k=1}^n k\, 2^{1/k}$. Then $\mathcal{L}_K^{\otimes e}|_Y \equiv K_Y + \Delta + N$ with $N := (e-1)\mathcal{L}_K|_Y$ ample. For every irreducible $W \subset Y$ of dimension $k \geq 1$, $(N^k \cdot W) = (e-1)^k \deg_{\mathcal{L}_K} W \geq (e-1)^k > \sigma(n)^k$, as $\deg_{\mathcal{L}_K} W$ is a positive integer and $e - 1 \geq \lfloor \sigma(n) \rfloor + 1$. With the constant choice $c(k) := \sigma(n)$ we have $\sum_{k \leq n} 2^{1/k} k/c(k) = 1$ and $\sum_{k \leq n} k/c(k) \leq 1$. Fujino's theorems [42, Thms. 1.1 and 1.2] now give, for every point of Y, a section of $\mathcal{L}_K^{\otimes e}|_Y$ not vanishing there, and, for every two points, sections separating them. Finally $k\, 2^{1/k} \leq k + 1$ because $(1 + 1/k)^k \geq 2$, so $\sigma(n) \leq n(n+3)/2$. □

The values are $e^*(1) = 4$, $e^*(2) = 6$, $e^*(3) = 10$. In the compact case $\Delta = 0$ and $\mathcal{L}_K$ is the canonical bundle, and Theorem 1.6 is an Angehrn–Siu bound for it.

5. Bézout's inequality uniformly in the level

In this section (G, X) is a Shimura datum and $K \subset G(\mathbb{A}_f)$ a neat compact open subgroup. The Hecke step works on Shimura varieties of subdata at levels that vary with a prime.

5.1. **An example.**

Example 5.1. Let C be a hyperelliptic curve of genus $g \geq 2$ with hyperelliptic involution ι, let $p \in C$, and let $\mathcal{L} := \mathcal{O}(p) \boxtimes \mathcal{O}(p)$ on $C \times C$, an ample line bundle. The diagonal Δ_C and the graph Γ_ι of ι both have $\mathcal{L}$-degree 2, but $\Delta_C \cap \Gamma_\iota$ consists of the $2g + 2$ fixed points of ι. So $\deg(V \cap W) \leq \deg V \cdot \deg W$, which holds for a very ample bundle [43, Example 8.4.6], fails for an ample one.

5.2. **The inequality at every neat level.**

Lemma 5.2. *(a) Let $A, B \subset \mathbb{P}^N_{\mathbb{C}}$ be closed; the degree of a closed set is the sum of the degrees of its irreducible components, and* Irr *denotes the set of irreducible components, of all dimensions. Then $\sum_{C \in \mathrm{Irr}(A \cap B)} \deg C \leq \deg A \cdot \deg B$.*

(b) Let (H, X_H) be a Shimura datum, $J \subset H(\mathbb{A}_f)$ a neat compact open subgroup, $n_J := \dim \mathrm{Sh}_J(H, X_H)$ and $e \geq e^*(n_J)$ an integer. For closed $A, B \subset \overline{\mathrm{Sh}}_J(H, X_H)$, of dimensions a and b (maxima over components),

$$\sum_{C \in \mathrm{Irr}(A \cap B)} \deg_{\mathcal{L}_J} C \;\leq\; e^{a+b} \deg_{\mathcal{L}_J} A \cdot \deg_{\mathcal{L}_J} B.$$

The same holds for closed $A, B \subset \mathrm{Sh}_J(H, X_H)_{\mathbb{C}}$, with degrees of closures and C running over $\mathrm{Irr}(A \cap B)$.

(c) In particular, if (H, X_H) is a Shimura subdatum of (G, X) and $n := \dim \mathrm{Sh}_K(G, X)$, then (b) holds with $e := e^*(n)$ at every neat $J \subset H(\mathbb{A}_f)$, so with the constant $c_B := e^*(n)^{2n}$, which does not depend on (H, X_H) or J.

Proof. (a) is [44, Cor. 2, pp. 98–99] for irreducible A, B; in general each component of $A \cap B$ is a component of some $A_i \cap B_j$, and we sum over pairs of components.

(b) By Theorem 1.6 for (H, X_H) and J, the complete linear system $|\mathcal{L}_J^{\otimes e}|$ is base-point free and separates points of $\overline{\mathrm{Sh}}_J$. It defines $\varphi\colon \overline{\mathrm{Sh}}_J \to \mathbb{P}^N$ with $\varphi^*\mathcal{O}(1) \cong \mathcal{L}_J^{\otimes e}$; φ is finite, since $\mathcal{L}_J$ is ample and so φ contracts no curve, and injective. A finite injective morphism is a homeomorphism onto its closed image, so φ maps $\mathrm{Irr}(A \cap B)$ bijectively onto $\mathrm{Irr}(\varphi(A) \cap \varphi(B))$, and $\varphi(A \cap B) = \varphi(A) \cap \varphi(B)$. In characteristic 0, φ restricted to an irreducible closed Y is birational onto $\varphi(Y)$, so the projection formula gives $\deg \varphi(Y) = \deg_{\mathcal{L}_J^{\otimes e}} Y = e^{\dim Y} \deg_{\mathcal{L}_J} Y$. By (a),

$$\sum_C \deg_{\mathcal{L}_J} C \leq \sum_C e^{\dim C} \deg_{\mathcal{L}_J} C = \sum_C \deg \varphi(C)$$
$$\leq \deg \varphi(A) \cdot \deg \varphi(B) \leq e^{a+b} \deg_{\mathcal{L}_J} A \cdot \deg_{\mathcal{L}_J} B,$$

using $e \geq 1$ and, for reducible A, B, the sum over pairs of components. For A, B in the open part, the components of $\overline{A} \cap \overline{B}$ that meet Sh_J are the closures of the components of $A \cap B$, and the others only add to the left side.

(c) As $X_H \subset X$, $n_J \leq n$ for every neat J, so $e^*(n) \geq e^*(n_J)$, and $a, b \leq n_J \leq n$. □

Lemma 5.3. *(a) Let (H, X_H) be a Shimura datum, $I \subset H(\mathbb{A}_f)$ neat compact open, $m \in H(\mathbb{Q}_l)$, $I_m := I \cap mIm^{-1}$, and $T_m := \pi_2 \circ \pi_1^{-1}$ on $\mathrm{Sh}_I(H, X_H)$, with π_1 natural and π_2 induced by m. Then $\overline{\pi}_1^* \mathcal{L}_I \cong \mathcal{L}_{I_m} \cong \overline{\pi}_2^* \mathcal{L}_I$, and $\deg_{\mathcal{L}_I} T_m Y \leq [I : I_m] \deg_{\mathcal{L}_I} Y$ for every closed Y.*

(b) If $(H, X_H) \subset (G, X)$, $K_H := K \cap H(\mathbb{A}_f)$ and $t \in Z(H)(\mathbb{A}_f)$, then $\theta_t\colon [x, h] \mapsto [x, ht]$ satisfies $\overline{\theta}_t^* \mathcal{L}_{K_H} \cong \mathcal{L}_{K_H}$.

Proof. [63, Prop. 5.3.2(1)] (Section 2.1) for the morphisms induced by 1, by m and by t; then the projection formula, π_1 being finite étale of degree at most $[I : I_m]$. □

6. The Hecke step with constants uniform in the level

Under $\mathrm{QRH}(\delta; L_V)$, a special subvariety $V \subset Z$ that is not strongly special lies in a larger special subvariety of Z once Ω_V exceeds an explicit bound in terms of $\deg Z$, the degree of the field of definition of Z, $\log d_{L_V}$ and $\log \alpha_V$ (Theorem 6.11); for maximal V the inequality fails, which bounds $\alpha_V d_{L_V}^{\mu}$ (Lemma 6.12). The argument is that of [63, §9], with constants uniform in the level of the Hecke correspondence and with Galois groups over the field of definition of Z: a saturation lemma (Lemma 6.1), a criterion (Theorem 6.3), then a prime that meets it (Proposition 6.8).

6.1. **Saturation.** In this subsection G is adjoint and K is a neat product level. We do not assume (Lv′), as Section 7 uses both lemmas at every neat product level. Let $(P, X_P) \subset (G, X)$ be a Shimura subdatum with P the generic Mumford–Tate group on X_P, and put $K(P) := K \cap P(\mathbb{A}_f)$, $K(P)_\ell := K_\ell \cap P(\mathbb{Q}_\ell)$ and $K(P)^p := \prod_{\ell \neq p} K(P)_\ell$ for primes ℓ and p. Let $P^{\mathrm{ad}} = P_1 \times \cdots \times P_s$ be the decomposition into $\mathbb{Q}$-simple factors, with projections π_i, and $X_P^+ = X_1^+ \times \cdots \times X_s^+$ the corresponding decomposition. For $\mathcal{A} \subset \{1, \dots, s\}$ write $P_{\mathcal{A}} := \prod_{i \in \mathcal{A}} P_i$, $X_{\mathcal{A}}^+ := \prod_{i \in \mathcal{A}} X_i^+$ and $\mathcal{A}^c$ for the complement. A subset $\widehat{Y} \subset X_P^+$ is *$X_{\mathcal{A}}^+$-saturated* if $\widehat{Y} = X_{\mathcal{A}}^+ \times \mathrm{pr}_{\mathcal{A}^c}(\widehat{Y})$. For $x \in X_P^+$ we write $x_{\neq i} := \mathrm{pr}_{\{i\}^c}(x)$.

Lemma 6.1. *Let $Y \subset \mathrm{Sh}_{K(P)}(P, X_P^+)_{\mathbb{C}}$ be closed, irreducible and Hodge generic, containing a special point, and let $\widehat{Y} \subset X_P^+$ be its preimage. Let p be a prime, $J_p \subset K(P)_p$ compact open, $J := K(P)^p J_p$, $\widetilde{Y}$ a component of the preimage of Y in $\mathrm{Sh}_J(P, X_P)_{\mathbb{C}}$, $\alpha \in P(\mathbb{Q}_p)$ and $\mathcal{I} \subset \{1, \dots, s\}$. Suppose that $\widetilde{Y} \subset T_\alpha \widetilde{Y}$, and that $\pi_i(u_1 \alpha u_2)$ generates an unbounded subgroup of $P_i(\mathbb{Q}_p)$ for all $i \in \mathcal{I}$ and $u_1, u_2 \in J_p$. Then $\widehat{Y}$ is $X_{\mathcal{I}}^+$-saturated.*

Proof. This is the proof of [63, Thm. 7.2.1], applied to $\widetilde{Y}$ itself in place of the preimage of a special subvariety, with the monodromy step of [63, p. 33] argued from a special point of Y (Step 1) and the last step (Step 2) carried out on X_P^+ rather than at the adjoint product level of [63, Lemma 7.2.8].

Step 0. Let $\widetilde{Y}^\infty$ be a component of the preimage of $\widetilde{Y}$ in the pro-étale cover of $\mathrm{Sh}_J(P, X_P)$ with group J_p [63, §4.1.3]. As Y is geometrically irreducible, the argument of [63, (7.2)] gives $u_1, u_2 \in J_p$ with $\widetilde{Y}^\infty = \widetilde{Y}^\infty \cdot (u_1 \alpha u_2)$; the Galois step of [63, Lemma 7.2.4] is not needed. Let $\widehat{Y}'$ be an analytic component of $\widehat{Y}$.

Step 1 (the monodromy). Let M_Y be the identity component of the Zariski closure of the monodromy of the variation of Hodge structure over the smooth locus Y^{sm} of Y. Let $r\colon Y' \to Y$ be a resolution of singularities that is an isomorphism over Y^{sm}. The pull-back to Y' of the variation on $\mathrm{Sh}_{K(P)}(P, X_P^+)_{\mathbb{C}}$ is polarizable; its Mumford–Tate group is P at the points over the Hodge generic points of Y, and a torus at the points over a special point of Y. Hence its connected monodromy group is P^{der} [4, Prop. 2]; this group is M_Y, as $\pi_1(Y^{\mathrm{sm}}) \to \pi_1(Y')$ is onto. So $M_Y = P^{\mathrm{der}}$, and the closure K'_p in $P(\mathbb{Q}_p)$ of the monodromy of $\widetilde{Y}$ at the lift $\widehat{Y}'$ contains an open subgroup of $M_Y(\mathbb{Q}_p)$ [63, Prop. 4.2.1]. (Without a special point this fails: for $N \geq 3$ and an algebraic non-CM point $\sigma \in Y(N)$, the curve $Y(N) \times \{\sigma\} \subset Y(N)^2$ is Hodge generic with connected monodromy $\mathrm{PGL}_2 \times 1$ in the adjoint variation.)

Step 2. Steps 0 and 1 hold for every choice of $\widetilde{Y}^\infty$ and $\widehat{Y}'$; translating by some $k \in K(P)_p$, which replaces J_p and α by $k^{-1} J_p k$ and $k^{-1} \alpha k$ and keeps the hypotheses, we may assume that $\widetilde{Y}$ is the image of $\widehat{Y}' \times \{1\}$, and we take for $\widetilde{Y}^\infty$ the component containing the image of $\widehat{Y}' \times \{1\}$; then K'_p is the closure of the stabilizer of $\widehat{Y}'$ in $P(\mathbb{Q})_+ \cap J$. By Steps 0 and 1, $\widetilde{Y}^\infty$ is stabilized by $U_p := \langle K'_p, u_1 \alpha u_2 \rangle$ [63, Cor. 7.2.5], and $M_Y = P^{\mathrm{der}}$, so K'_p contains an open subgroup of $P^{\mathrm{der}}(\mathbb{Q}_p)$ and the image $\bar{U}_p$ of U_p in $P^{\mathrm{ad}}(\mathbb{Q}_p)$ is open. Fix $i \in \mathcal{I}$. Let $P_i^\natural$ be the $\mathbb{Q}$-simple factor of P^{der} over P_i, $\varpi\colon \widetilde{P}_i \to P_i^\natural$ its simply connected cover, H a $\mathbb{Q}_p$-simple factor of $P_{i,\mathbb{Q}_p}$ in which the image of $u_1 \alpha u_2$ is unbounded, and $\widetilde{\mathsf{H}}$ the factor of $\widetilde{P}_{i,\mathbb{Q}_p}$ over H. By [63, Lemma 7.2.6] for $\bar{U}_p$, $\varpi(\widetilde{\mathsf{H}}(\mathbb{Q}_p)) \subset U_p\, Z(P)(\mathbb{Q}_p)$, so commutators of elements of $\varpi(\widetilde{\mathsf{H}}(\mathbb{Q}_p))$ lie in U_p. The group $\widetilde{\mathsf{H}}(\mathbb{Q}_p)$ is perfect: it is $\widetilde{\mathsf{H}}'(\mathsf{k})$ for a finite $\mathsf{k}/\mathbb{Q}_p$ and $\widetilde{\mathsf{H}}'$ simply connected and absolutely almost simple, isotropic because $\mathsf{H}(\mathbb{Q}_p)$ is not compact [85], so $\widetilde{\mathsf{H}}'(\mathsf{k}) = \widetilde{\mathsf{H}}'(\mathsf{k})^+$ [82], [83, Thm. 7.6], which is perfect [45, Fait 4.1]. Hence $\varpi(\widetilde{\mathsf{H}}(\mathbb{Q}_p)) \subset U_p$.

Let $\Gamma_i := \{\gamma \in P_i^\natural(\mathbb{Q}) \cap P(\mathbb{Q})_+ \cap K(P)^p : \gamma_p \in K_p'\}$, an arithmetic subgroup of $P_i^\natural$ with closure $\overline{\Gamma}_i \subset K_p'$ in $P(\mathbb{Q}_p)$, and fix $g \in \widetilde{\mathsf{H}}(\mathbb{Q}_p)$ lying in no compact subgroup. By strong approximation for $\widetilde{P}_i$ with respect to the infinite place ($P_i(\mathbb{R})$ is not compact), $\widetilde{P}_i(\mathbb{Q})$ is dense in $\widetilde{P}_i(\mathbb{A}_f)$ [69, Thm. 4.16]. So $\overline{\Gamma}_i$ contains $\varpi(\widetilde{C}_p)$, with $\widetilde{C}_p$ the preimage of K_p' in $\widetilde{P}_i(\mathbb{Q}_p)$ (the elements of $\widetilde{P}_i(\mathbb{Q})$ in $\widetilde{C}_p$ whose image lies in $K(P)^p$ are dense in $\widetilde{C}_p$, and their images lie in Γ_i), and is therefore open in $P_i^\natural(\mathbb{Q}_p)$; and there is $q' \in \varpi(\widetilde{P}_i(\mathbb{Q}))$ that lies in $K(P)_\ell$ for every prime $\ell \neq p$ and has $q_p' \in \varpi(g)\overline{\Gamma}_i$; so $q' \in P(\mathbb{Q})_+$ and $q_p' \in U_p$. The closure of $\Gamma_i' := \langle \Gamma_i, q' \rangle$ in $P(\mathbb{Q}_p)$ contains $\varpi(g)$, so Γ_i has infinite index in Γ_i'. The image $\pi_i(\Gamma_i')$ is dense in $P_i(\mathbb{R})^+$. Indeed, let C be its closure in $P_i(\mathbb{R})$ and $\mathfrak{c}$ the Lie algebra of C. The group $\pi_i(\Gamma_i)$ is arithmetic in P_i [69, Prop. 3.2], hence a lattice in $P_i(\mathbb{R})$ and Zariski dense in P_i [69, Thms. 3.3 and 3.19], and it normalizes C; so $\mathfrak{c}$ is an ideal of $\operatorname{Lie} P_i(\mathbb{R})$, and $P_i(\mathbb{R}) = \mathsf{N} \times \mathsf{N}'$ with N, N' products of $\mathbb{R}$-simple factors and $\operatorname{Lie} \mathsf{N} = \mathfrak{c}$. Suppose $\mathsf{N}' \neq 1$. As C is closed with the identity component of N as its identity component, and N has finitely many components, the projection of C to N' is discrete; and $P_i(\mathbb{Q}) \to \mathsf{N}'$ is injective, as P_i is $\mathbb{Q}$-simple. Hence a sequence in $\pi_i(\Gamma_i')$ that tends to 1 is eventually 1, and $\pi_i(\Gamma_i')$ is discrete; it then contains the lattice $\pi_i(\Gamma_i)$ with finite index, and $[\Gamma_i' : \Gamma_i]$ is finite, as $P_i^\natural \to P_i$ has finite kernel. Thus $\mathsf{N}' = 1$ and $\mathsf{C} \supset P_i(\mathbb{R})^+$.

Each $\gamma \in \Gamma_i'$ lies in $P(\mathbb{Q})_+ \cap K(P)^p$, has $\gamma_p \in U_p$ and moves only the i-th coordinate of X_P^+; for $x \in \widehat{Y}'$, $[\gamma x, 1] = [x, \gamma_p^{-1}] \in \widetilde{Y}^\infty U_p = \widetilde{Y}^\infty$, so $\gamma x \in \widehat{Y}$. As $\widehat{Y}$ is closed, $X_i^+ \times \{x_{\neq i}\} \subset \widehat{Y}$ for every $x \in \widehat{Y}'$, hence for every $x \in \widehat{Y} = (K(P) \cap P(\mathbb{Q})_+) \cdot \widehat{Y}'$, since $P(\mathbb{Q})_+$ acts factorwise. Applying this for each $i \in \mathcal{I}$ in turn gives the lemma. □

Lemma 6.2. *In Lemma 6.1, let $V \subset Y$ be a special subvariety defined by (H, X_H), and suppose that the image of H^{der} in P^{ad} is $P_{\mathcal{I}}$. Then $\deg_{\mathcal{L}_{K(P)}} V \leq \deg_{\mathcal{L}_{K(P)}} Y$.*

Proof. Let $\Gamma \subset P^{\mathrm{ad}}(\mathbb{Q})$ be the image of $K(P) \cap P(\mathbb{Q})_+$, so that $\mathrm{Sh}_{K(P)}(P, X_P^+)_{\mathbb{C}} = \Gamma \backslash X_P^+$, and put $\Gamma_{\mathcal{I}} := \pi_{\mathcal{I}}(\Gamma)$ and $\Gamma_c := \pi_{\mathcal{I}^c}(\Gamma)$. These are arithmetic, as images of an arithmetic group under surjective homomorphisms [69, Prop. 3.2], and neat [81, 0.5], [63, §4.1.4], so Γ has finite index e in $\Gamma_{\mathcal{I}} \times \Gamma_c$, and $f \colon \Gamma \backslash X_P^+ \to S' := \Gamma_{\mathcal{I}} \backslash X_{\mathcal{I}}^+ \times \Gamma_c \backslash X_{\mathcal{I}^c}^+$ is finite étale of degree e. The Baily–Borel compactification of S' is the product of those of the factors, and $f^*(\mathcal{L}_{\mathcal{I}} \boxtimes \mathcal{L}_c) = \mathcal{L}_{K(P)}$, where $\mathcal{L}_{\mathcal{I}}$, $\mathcal{L}_c$ are the extensions of the canonical bundles of the factors. Since $Z(H)(\mathbb{R})$ fixes the points of X_H and H^{der} maps onto $P_{\mathcal{I}}$, $X_H^+ = X_{\mathcal{I}}^+ \times \{x_2\}$, so $f(V) = S_{\mathcal{I}}' \times \{w_0\}$ with $S_{\mathcal{I}}' := \Gamma_{\mathcal{I}} \backslash X_{\mathcal{I}}^+$. By Lemma 6.1, $f(Y) = S_{\mathcal{I}}' \times W$ with W closed and irreducible, of dimension b.

Every fiber $f^{-1}(S_{\mathcal{I}}' \times \{w\})$ is irreducible: for $w = \Gamma_c y$ it is the image of $X_{\mathcal{I}}^+ \times \Gamma_c y$, and Γ permutes the slices $X_{\mathcal{I}}^+ \times \{\gamma y\}$ transitively, as $\pi_{\mathcal{I}^c}(\Gamma) = \Gamma_c$. Every component of $f^{-1}(S_{\mathcal{I}}' \times W)$ maps onto $S_{\mathcal{I}}' \times W$, as f is finite and flat, so it contains the fiber over a general point of W; hence $f^{-1}(S_{\mathcal{I}}' \times W)$ is irreducible, and $Y = f^{-1}(S_{\mathcal{I}}' \times W)$, $V = f^{-1}(S_{\mathcal{I}}' \times \{w_0\})$, each of degree e over its image. With $a := \dim X_{\mathcal{I}}^+$ the projection formula gives $\deg_{\mathcal{L}_{K(P)}} V = e \deg_{\mathcal{L}_{\mathcal{I}}} S_{\mathcal{I}}'$ and $\deg_{\mathcal{L}_{K(P)}} Y = e\binom{a+b}{a} \deg_{\mathcal{L}_{\mathcal{I}}} S_{\mathcal{I}}' \deg_{\mathcal{L}_c} W \geq \deg_{\mathcal{L}_{K(P)}} V$, since $\deg_{\mathcal{L}_c} W \geq 1$. □

The choice of the groups $\Gamma_{\mathcal{I}}$, Γ_c rather than the projections of $K(P)$ is what makes the fibers irreducible: with $\pi_{\mathcal{I}^c}(K(P)) \cap P_{\mathcal{I}^c}(\mathbb{Q})_+$ in place of Γ_c the fiber would have as many components as the index of Γ_c in that group.

6.2. **The criterion.** In the rest of this section G is adjoint and K satisfies (Lv′), and we put $n := \dim \mathrm{Sh}_K(G, X)$ and $c_B := e^*(n)^{2n}$ (Lemma 5.2(c)). Let $(P, X_P) \subset (G, X)$ be a

Shimura subdatum, $K_P := K \cap P(\mathbb{A}_f)$, $F \supset E(P, X_P)$ a number field and $\bar{d} \geq 1$ an integer. For a special subvariety V put $E_V := E(H_V, X_{H_V})$; a special subvariety V of $\mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$ and its image in $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$, which is special, have the same α_V, β_V and reflex field E_V [63, Remark 9.1.1]. All degrees on $\mathrm{Sh}_J(P, X_P)$ are taken for its own Baily–Borel bundle $\mathcal{L}_J$. Put $k_F := k_1 \bar{d} + f$ and, for $D' \geq 1$ and a prime l,

$$\Psi(D') := c_B\, D'^{\,2}, \qquad \Psi_l(D') := l^{k_F+2f}\, \Psi(D') = c_B\, l^{k_F+2f} D'^{\,2}.$$

Theorem 6.3. *Let G be adjoint and K satisfy* (Lv′)*, and let (P, X_P) and F be as above. Let $Z \subset \mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$ be Hodge generic and F-irreducible, $D_Z := \deg_{\mathcal{L}_{K_P}} Z$, and $V \subset Z$ special with $r := \dim Z - \dim V > 0$. Let $\bar{d} \geq [F E_V : E_V]$ be an integer and assume* $\mathrm{QRH}(\delta; L_V)$*. Suppose that*

(1) V is not strongly special in $\mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$, and

(2) there is a prime $l > C_0$ that splits completely in L_V, with $l > 1/c_U$ and $K_{T_V,l} = K^m_{T_V,l}$, such that

$$\left(c_B\, l^{k_F+2f}\, \bar{d}\, D_Z\right)^{2^r} \;<\; \Omega_V. \tag{6.1}$$

Then Z contains a special subvariety V' with $V \subsetneq V'$, in a geometric component of Z containing V.

Lemma 6.4. *Let V be special, $T := T_V$, and let $p > C_0$ be a prime unramified in L_V with $K^m_{T,p} \neq K_{T,p}$. Then $|K^m_{T,p}/K_{T,p}| \geq (p-1)/C_\Phi \geq c_U p$.*

Proof. This is [102, Prop. 3.15] at the primes $p > C_0$, with an explicit constant; we derive it from [34, §4.4]. As p is unramified in L_V, $T_{\mathbb{Q}_p}$ is the generic fiber of a torus $\mathcal{T}$ over $\mathbb{Z}_p$ [34, §3.3], and $\mathcal{T}(\mathbb{Z}_p) = K^m_{T,p}$, as a compact subgroup of $T(\mathbb{Q}_p)$ lies in $\mathcal{T}(\mathcal{O}_{\mathsf{k}}) \cap T(\mathbb{Q}_p) = \mathcal{T}(\mathbb{Z}_p)$ for an unramified splitting field k. As $K_p = G(\mathbb{Q}_p) \cap \mathrm{GL}_{n_\rho}(\mathbb{Z}_p)$ for $p > C_0$, $K_{T,p}$ is the stabilizer of $W := \mathbb{Z}_p^{n_\rho}$ in $\mathcal{T}(\mathbb{Z}_p)$, so $|K^m_{T,p}/K_{T,p}| = |\mathcal{T}(\mathbb{Z}_p) \cdot W|$, and $\mathcal{T}$ does not fix W if $K_{T,p} \neq K^m_{T,p}$. The argument of [34, §4.4] gives $|\mathcal{T}(\mathbb{Z}_p) \cdot W| \geq (p-1)/|\Phi|$, where Φ is the group of components of the stabilizer in $\mathcal{T}_{\overline{\mathbb{F}}_p}$ of a subspace; if the subspace has dimension d, the stabilizer is $\bigcap \ker(x - x')$ over the weights x, x' in the support of one vector of its d-th exterior power, which are weights of T under $X^*(\mathcal{T}_{\overline{\mathbb{F}}_p}) = X^*(T_{\overline{\mathbb{Q}}_p})$ [34, proof of Lemma 4.4.1]; so $|\Phi|$ is at most the order of the torsion of $X^*(T)/N$, with N the subgroup generated by these differences. As the map $r_T^*\colon X^*(T) \to X^*(\mathrm{Res}_{F_T/\mathbb{Q}}\mathbb{G}_{\mathrm{m}})$ dual to the surjection r_T of [102, p. 42] is injective, $X^*(T)/N$ embeds in $X^*(\mathrm{Res}_{F_T/\mathbb{Q}}\mathbb{G}_{\mathrm{m}})/r_T^* N$, whose torsion has order at most C_Φ. So $|K^m_{T,p}/K_{T,p}| \geq (p-1)/C_\Phi \geq c_U p$. □

Remark. The condition $l > 1/c_U$ in (2) of Theorem 6.3 is needed for $(T_V)_{\mathbb{F}_l}$ to be a torus, as [63, Thm. 8.1] requires. A split torus whose level is maximal need not have a torus as its closure. Let $\mathbb{G}_{\mathrm{m}}$ act on $\mathbb{Q}^3$ by $t \mapsto \mathrm{diag}(1, t^2, t)$ with the lattice $\Lambda_0 := \mathbb{Z}(e_1+e_2) + 3\mathbb{Z}e_2 + \mathbb{Z}e_3$. In the basis $e_1+e_2, 3e_2, e_3$ of Λ_0, t sends e_1+e_2 to $(e_1+e_2) + \frac{t^2-1}{3}\, 3e_2$, so $\mathbb{G}_{\mathrm{m}}(\mathbb{Q}_3) \cap \mathrm{GL}(\Lambda_0 \otimes \mathbb{Z}_3) = \mathbb{Z}_3^\times$ is maximal, while the closure is $\operatorname{Spec} \mathbb{Z}_3[t^{\pm 1}, (t^2-1)/3]$, whose special fiber is $\mathbb{Z}/2 \times \mathbb{G}_{\mathrm{a}}$. By the argument of [34, §4.4], maximality does give a torus once $p - 1 > C_\Phi$; here the analogue of C_Φ is 2, $p = 3$, and $1/c_U = 4 > 3$. The CM elliptic curve with endomorphism ring $\mathbb{Z}[\sqrt{-7}]$ gives the same phenomenon at $l = 2$ inside $(\mathrm{GL}_2, \mathcal{H}^{\pm})$.

Compared with [63, (9.1)], which reads $l^{(k+2f)2^r} (\deg Z)^{2^r} < C(N) \alpha_V \beta_V^N$, the base gains the factor $c_B\, \bar{d}$ and has k_F in place of k and the f of Table 1 in place of theirs, the

exponent stays 2^r, and the right side is the power form Ω_V of Corollary 3.4. As $\Psi_l^{\circ r}(x) = (c_B l^{k_F+2f})^{2^r-1}x^{2^r}$, (6.1) implies

$$\Omega_V \ > \ \Psi_l^{\circ r}(\bar{d}D_Z), \tag{6.2}$$

the form the proof uses; also $\bar{d}\,\Psi_l(D') \le \Psi_l(\bar{d}D')$ and $\Psi_l(x) \ge x$ for $x \ge 1$. The proof follows that of [63, Thm. 9.2.1], with the following differences. The Galois groups are those over F, which need not lie in E_V, at the cost of the factor $\bar{d}$ (Step 1, Lemma 6.6). The Galois lower bound is that of Corollary 3.4. Bézout's inequality, at [63, (9.5)] and in [63, Sublemma 9.2.5], has the constant of Lemma 5.2. In Step 3, as P need not be adjoint, the proof of [63, Thm. 7.2.1] is run on the image of $\widetilde{Z}$ (the union of the conjugates of a component of the preimage of Z at the Iwahori level, Step 2) in the Shimura variety $\mathrm{Sh}_{I^{\mathrm{ad}}}(P^{\mathrm{ad}}, X_P^{\mathrm{ad}})$ of the adjoint group at an adjoint level I^{ad}. Finally, $\widetilde{Y}_H$, the lift of the union $\widetilde{Y}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-conjugate components of $\widetilde{Z} \cap T_m\widetilde{Z}$ (Step 4) to the Shimura variety of the smallest special subvariety containing it, is chosen so as to be Hodge generic of dimension $\dim \widetilde{Y}$ (Lemma 6.7).

Step 1 of the proof of Theorem 6.3 follows [63, §8.3.2], with the following lemma in place of [63, Lemma 8.1.6]. It writes every maximal compact subgroup of $P(\mathbb{Q}_l)$ that contains the level as the fixer of a point in the apartment of a maximal split torus containing T_V, using a condition on weights that (2) gives, and it bounds the Iwahori indices without a hyperspecial subgroup, which $P(\mathbb{Q}_l)$ need not have, as (2) does not make P unramified at l. For a connected reductive $\mathbb{Q}_l$-group H we write $\mathcal{B}(H)$ for its extended Bruhat–Tits building [98, 1.2 and 2.1] and $H(\mathbb{Q}_l)^{\Omega}$ for the fixer of $\Omega \subset \mathcal{B}(H)$ in $H(\mathbb{Q}_l)$; as in [63, §8.1.1 and Remark 8.1.1], the fixers of chambers are called Iwahori subgroups.

Lemma 6.5. *Let l be a prime and H a connected reductive $\mathbb{Q}_l$-group, and put $d := \dim H$.*

(a) Let $T \subset H$ be a split torus such that no nonzero weight of T on $\mathrm{Lie}\, H$ lies in $(l-1)X^*(T)$, and let $\Theta \subset T(\mathbb{Q}_l)$ be the subgroup of elements of order dividing $l-1$. Then every point of $\mathcal{B}(H)$ fixed by Θ lies in the apartment of a maximal split torus $S \supset T$ of H, and the maximal compact subgroup of $S(\mathbb{Q}_l)$ fixes that apartment pointwise.

(b) If a point $y \in \mathcal{B}(H)$ lies in the closure of a chamber C, then $[H(\mathbb{Q}_l)^y : H(\mathbb{Q}_l)^C] \le l^{d(d+1)/2}$.

(c) Let (P, X_P), V and l be as in Theorem 6.3, with (2), and $K_{P,l} := K_l \cap P(\mathbb{Q}_l)$. Every maximal compact subgroup $K_l^1 \supset K_{P,l}$ of $P(\mathbb{Q}_l)$ is $P(\mathbb{Q}_l)^y$ for a point y of the apartment of a maximal split torus $S \supset T_{V,\mathbb{Q}_l}$ of $P_{\mathbb{Q}_l}$. Let C^1 be a chamber of this apartment with y in its closure, x_0 a special point in the closure of C^1, and C^2 a chamber of the apartment with x_0 in its closure. Then $I_l^1 := P(\mathbb{Q}_l)^{C^1} \subset K_l^1$ and $I_l^2 := P(\mathbb{Q}_l)^{C^2}$ are in good position with respect to S, $I_l := I_l^1 \cap I_l^2 \cap K_{P,l}$ is in good position with respect to T_V, and $[K_{P,l} : I_l] < l^f$ with $f := (\dim G + 1)^2$.

Lemma 6.5 is proved in Appendix B. The condition on weights in (a) cannot be dropped: for SL_2 over $\mathbb{Q}_3$ and the diagonal torus T, a maximal split torus, the roots ± 2 lie in $2X^*(T)$, and $T(\mathbb{Z}_3)$, which contains $\Theta = \{\pm 1\}$, fixes every vertex at distance 1 from the apartment of T; none of these lies in it.

Lemma 6.6 (Lower bound at every level)**.** *Let $(P, X_P) \subset (G, X)$ be a Shimura subdatum, $K_P := K \cap P(\mathbb{A}_f)$, $J \subset K_P$ compact open, $V \subset \mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$ special with $T_V \neq 1$, defined by $(H_V, X_{H_V}) \subset (P, X_P)$, and $\widetilde{V}$ a component of the preimage of V in $\mathrm{Sh}_J(P, X_P)_{\mathbb{C}}$. If $\mathrm{QRH}(\delta; L_V)$ holds, then, with Ω_V as in Corollary 3.4,*

$$\deg_{\mathcal{L}_J}(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)\cdot\widetilde{V}) \;\geq\; \deg_{\mathcal{L}_{K_P}}(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)\cdot V) \;\geq\; \Omega_V,$$

and $\deg_{\mathcal{L}_J}(\mathrm{Gal}(\overline{\mathbb{Q}}/FE_V)\cdot\widetilde{V}) \geq \Omega_V/\bar{d}$ when $\bar{d} \geq [FE_V : E_V]$.

Proof. The first inequality is the projection formula for the finite étale $\tau\colon \mathrm{Sh}_J \to \mathrm{Sh}_{K_P}$, defined over $E(P, X_P) \subset E_V$, with $\tau^*\mathcal{L}_{K_P} = \mathcal{L}_J$ [63, Prop. 5.3.2(1)]: each component C of the orbit has $\deg_{\mathcal{L}_J} C = \deg(\tau|_C)\deg_{\mathcal{L}_{K_P}}\tau(C)$, and τ maps the orbit of $\widetilde{V}$ onto that of V. For the second, let V_H be a geometric component of $\mathrm{Sh}_{K_{H_V}}(H_V, X_{H_V})$, $K_{H_V} := K_P \cap H_V(\mathbb{A}_f) = K \cap H_V(\mathbb{A}_f)$, mapping onto V. The natural map q is defined over E_V, injective on the Hodge generic locus [102, Lemma 2.2], and the components of $\mathrm{Sh}_{K_{H_V}}(H_V, X_{H_V})$ are Hodge generic, so distinct conjugates of V_H have distinct images and [63, Cor. 5.3.10] ($Z(H_V)(\mathbb{R})$ is compact) gives $\deg_{\mathcal{L}_{K_P}}(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)\cdot V) \geq \deg_{\mathcal{L}_{K_{H_V}}}(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)\cdot V_H)$, which is $\geq \Omega_V$ by Corollary 3.4(i) with $F = E_V$. For the last assertion, $\mathcal{L}_J$ and $\overline{\mathrm{Sh}}_J(P, X_P)$ are defined over $E(P, X_P) \subset E_V$, so $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$-conjugate subvarieties of $\mathrm{Sh}_J(P, X_P)_{\mathbb{C}}$ have equal degrees; and $\mathrm{Gal}(\overline{\mathbb{Q}}/FE_V)$ has index $[FE_V : E_V] \leq \bar{d}$ in $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$, so the $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$-orbit of $\widetilde{V}$ is the union of at most $\bar{d}$ conjugates of $\mathrm{Gal}(\overline{\mathbb{Q}}/FE_V)\cdot\widetilde{V}$. □

Descent. Let Z, V and $\bar{d}$ be as in Theorem 6.3, with (1) and $\mathrm{QRH}(\delta; L_V)$, and let $J \subset K_P$ be compact open, $\tau\colon \mathrm{Sh}_J(P, X_P) \to \mathrm{Sh}_{K_P}(P, X_P)$ the natural map, $\widetilde{V}$ a component of $\tau^{-1}(V)$ in the image of $X_P^+ \times \{1\}$, and $\widetilde{Y} \subset \mathrm{Sh}_J(P, X_P)_{\mathbb{C}}$ closed and F-irreducible with $\widetilde{V} \subset \widetilde{Y}$, $\tau(\widetilde{Y}) \subset Z$ and $r_Y := \dim\widetilde{Y} - \dim\widetilde{V} > 0$. Let $\widetilde{Y}^{(1)}$ be a geometric component of $\widetilde{Y}$ containing $\widetilde{V}$, and S_Y the smallest special subvariety of $\mathrm{Sh}_J(P, X_P)_{\mathbb{C}}$ containing $\widetilde{Y}^{(1)}$. It is the image of $X_H^+ \times \{1\}$ for a Shimura subdatum $(H, X_H) \subset (P, X_P)$ with H the generic Mumford–Tate group on X_H and $X_H^+ \subset X_P^+$ [102, Lemma 2.1]. Put $K_H := K_P \cap H(\mathbb{A}_f)$, $J_H := J \cap H(\mathbb{A}_f)$ and $F_H := E(H, X_H)$; then $F_H \subset E_V$, because a Hodge generic point of $\widetilde{V}$ lifts to X_H^+, so that (H, X_H) contains a $P(\mathbb{Q})$-conjugate of (H_V, X_{H_V}). Let q denote the natural maps $\mathrm{Sh}_{J_H}(H, X_H) \to \mathrm{Sh}_J(P, X_P)$ and $\mathrm{Sh}_{K_H}(H, X_H) \to \mathrm{Sh}_{K_P}(P, X_P)$, and τ also the map $\mathrm{Sh}_{J_H}(H, X_H) \to \mathrm{Sh}_{K_H}(H, X_H)$; these maps are finite and defined over F_H, and $q\tau = \tau q$. Let S_H be the image of $X_H^+ \times \{1\}$ in $\mathrm{Sh}_{J_H}(H, X_H)_{\mathbb{C}}$, so that $q(S_H) = S_Y$. As q is finite, some component C_1 of $q^{-1}(\widetilde{Y}^{(1)}) \cap S_H$ maps onto $\widetilde{Y}^{(1)}$, and some component $\widetilde{V}_H$ of $q^{-1}(\widetilde{V}) \cap C_1$ maps onto $\widetilde{V}$; so $\dim C_1 = \dim\widetilde{Y}$ and $\dim\widetilde{V}_H = \dim\widetilde{V}$. Let $\widetilde{Y}_H$ be the union of the $\mathrm{Gal}(\overline{\mathbb{Q}}/FF_H)$-conjugates of C_1, and put $V_H := \tau(\widetilde{V}_H)$ and $Y_H := \tau(\widetilde{Y}_H) \subset \mathrm{Sh}_{K_H}(H, X_H)_{\mathbb{C}}$.

Lemma 6.7. *In this situation Y_H is FF_H-irreducible and Hodge generic, $q(V_H) = V$ and V_H has torus T_V, and $\dim Y_H - \dim V_H = r_Y$* [63, pp. 48–49]*. Moreover:*

- *(a) $\deg_{\mathcal{L}_{K_H}} Y_H \leq \deg_{\mathcal{L}_J}\widetilde{Y}$;*
- *(b) if $\Psi^{\circ r_Y}(\bar{d}\deg_{\mathcal{L}_{K_H}} Y_H) < \Omega_V$, then V_H is not strongly special in $\mathrm{Sh}_{K_H}(H, X_H)_{\mathbb{C}}$;*
- *(c) if Y_H contains a special $V_H' \supsetneq V_H$, then Z contains a special $V' \supsetneq V$ in a geometric component of Z containing V.*

Proof. A special subvariety $S' \subsetneq S_H$ containing C_1 would give the special subvariety $q(S') \supset \widetilde{Y}^{(1)}$, of dimension less than $\dim S_Y$; so C_1 is Hodge generic in $\mathrm{Sh}_{J_H}(H, X_H)_{\mathbb{C}}$. Galois

conjugates of special subvarieties are special [63, proof of Lemma 7.1.2], so every component $\sigma(C_1)$ of $\widetilde{Y}_H$ is Hodge generic, of dimension $\dim \widetilde{Y}$, and $\widetilde{Y}_H$ is closed and FF_H-irreducible. Hence Y_H is closed, FF_H-irreducible and Hodge generic. As τ and q are finite, $\dim Y_H - \dim V_H = \dim \widetilde{Y} - \dim \widetilde{V} = r_Y$. The irreducible closed set $\widetilde{V}_H$ has the dimension of $q^{-1}(\widetilde{V})$, so it is a component of it, hence special, and so is V_H; and $q(V_H) = \tau(q(\widetilde{V}_H)) = \tau(\widetilde{V}) = V$. As V_H lies in the image of $X_H^+ \times \{1\}$, it is the image of $X_{H''}^+ \times \{1\}$ for a subdatum $(H'', X_{H''}) \subset (H, X_H)$ with H'' the generic Mumford–Tate group on $X_{H''}$ and $X_{H''}^+ \subset X_H^+$ [102, Lemma 2.1], and then so is $V = q(V_H)$ in $\mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$. So $X_{H''}^+ = \gamma X_{H_V}^+$ for some $\gamma \in P(\mathbb{Q})_+ \cap K_P$, and we may take $(H'', X_{H''})$ for (H_V, X_{H_V}) (Section 2.2): V_H and V have the same torus T_V and the same $K_{T_V} = T_V(\mathbb{A}_f) \cap K$, hence the same L_V, E_V, α_V and Ω_V (compare [63, Remark 9.1.1]).

(a) Each geometric component of Y_H is $\tau(C)$ for a component C of $\widetilde{Y}_H$, and $\deg_{\mathcal{L}_{J_H}} C = \deg(\tau|_C) \deg_{\mathcal{L}_{K_H}} \tau(C)$ by the projection formula, so $\deg_{\mathcal{L}_{K_H}} Y_H \le \deg_{\mathcal{L}_{J_H}} \widetilde{Y}_H$. For a component $C = \sigma(C_1)$ of $\widetilde{Y}_H$, $q(C) = \sigma(\widetilde{Y}^{(1)})$ is a geometric component of $\widetilde{Y}$, because q is defined over F_H and $\widetilde{Y}$ over F. Distinct C give distinct components, as q is injective on the Hodge generic locus [102, Lemma 2.2], and $\deg_{\mathcal{L}_{J_H}} C \le \deg_{\mathcal{L}_J} q(C)$ [63, Cor. 5.3.10], as $Z(H)(\mathbb{R})$ is compact. So $\deg_{\mathcal{L}_{J_H}} \widetilde{Y}_H \le \deg_{\mathcal{L}_J} \widetilde{Y}$.

(c) $V' := q(V'_H)$ is special; it lies in $q(Y_H) = \tau(q(\widetilde{Y}_H)) \subset \tau(\widetilde{Y}) \subset Z$ and contains $q(V_H) = V$; and $\dim V' = \dim V'_H > \dim V_H = \dim V$, as q is finite. Being irreducible, V' lies in a geometric component of Z, which contains V.

(b) As in [63, Lemma 9.2.3], suppose V_H is strongly special in $\mathrm{Sh}_{K_H}(H, X_H)_{\mathbb{C}}$, that is, $T_V \subset Z(H)$. Then $T_V = Z(H)^\circ$ [63, Lemma 6.2]. Put $K_H^m := K_{T_V}^m K_H$ and let $\pi\colon \mathrm{Sh}_{K_H}(H, X_H) \to \mathrm{Sh}_{K_H^m}(H, X_H)$ be the natural map. The group $K_{T_V}^m / K_{T_V}$ acts on $\mathrm{Sh}_{K_H}(H, X_H)$ by the maps θ_t of Lemma 5.3(b), which are defined over F_H and so commute with $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$, and its orbits are the fibers of π. Let $\Theta_A \subset K_{T_V}^m / K_{T_V}$ be the group of Corollary 3.4(ii). Then $\Theta_A \cdot V_H \subset \mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot V_H$ [63, Sublemma 9.2.4]. Write deg for $\deg_{\mathcal{L}_{K_H}}$ and $\mathcal{Y} := \mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot Y$ for geometrically irreducible Y; $\deg \theta\mathcal{Y} = \deg \mathcal{Y}$ by Lemma 5.3(b), and $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$-conjugate subvarieties have equal degrees, as $\mathcal{L}_{K_H}$ is defined over $F_H \subset E_V$. Let Y_1 be a geometric component of Y_H containing V_H. As Y_H is defined over $FF_H \subset FE_V$, it contains $\mathrm{Gal}(\overline{\mathbb{Q}}/FE_V) \cdot Y_1$, and $\mathcal{Y}_1$ is the union of at most $\bar{d}$ conjugates of this set; so $\deg \mathcal{Y}_1 \le D_1 := \bar{d} \deg Y_H$.

Suppose Y_j is geometrically irreducible, $V_H \subset Y_j$, $\dim Y_j > \dim V_H$ and $\deg \mathcal{Y}_j \le D_j := \Psi^{\circ(j-1)}(D_1)$. If $\Theta_A \cdot Y_j \subset \mathcal{Y}_j$, stop. Otherwise choose $\theta \in \Theta_A$ with $\theta Y_j \not\subset \mathcal{Y}_j$. The sets $\mathcal{Y}_j$ and $\theta\mathcal{Y}_j$ are of pure dimension $\dim Y_j$, and they have no common component: the components of $\theta\mathcal{Y}_j$ are the $\sigma\theta Y_j$, and $\sigma\theta Y_j = \sigma' Y_j$ would give $\theta Y_j \subset \mathcal{Y}_j$. Also $V_H \subset \mathcal{Y}_j \cap \theta\mathcal{Y}_j$: as Θ_A is a group, $\theta^{-1} V_H = \sigma V_H \subset \sigma Y_j$ for some $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$. Let Y_{j+1} be a geometric component of $\mathcal{Y}_j \cap \theta\mathcal{Y}_j$ containing V_H. This set is $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$-stable, so the conjugates of Y_{j+1} are among its components; $\dim Y_{j+1} < \dim Y_j$; and Lemma 5.2(c) at the neat level K_H gives

$$\deg \mathcal{Y}_{j+1} \le c_B D_j^2 = D_{j+1}.$$

The dimension drops at each step, so $j \le r_Y$ throughout, and $D_j \le D_{j+1} \le \Psi^{\circ r_Y}(D_1) < \Omega_V$, as $\Psi(x) \ge x$. If $Y_{j+1} = V_H$, then $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot V_H) \le D_{j+1} < \Omega_V$, contradicting Lemma 6.6 for V_H at the level K_H. Otherwise we continue. So the process stops at some Y' with $\Theta_A \cdot Y' \subset \mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot Y'$ and $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot Y') < \Omega_V$. As the orbits of Θ_A lie in the fibers of π, also $\Theta_A \cdot Y' \subset \pi^{-1}\pi(Y')$, and Y' lies in the geometric component of

$\mathrm{Sh}_{K_H}(H,X_H)_{\mathbb{C}}$ containing V_H; so Corollary 3.4(ii), applied to this component, whose image has torus $Z(H)^\circ = T_V$ and hence the same L_V, α_V and Ω_V, with $F = E_V \supset E(H,X_H)$ and $Y = Y'$, gives $\deg(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)\cdot Y') \geq \Omega_V$: a contradiction. □

Proof of Theorem 6.3. We prove the theorem with (6.2) in place of (6.1), by induction on r, for all subdata (P,X_P), fields F, and Z, V, $\bar{d}$ and l at once. Recall that V is the image of $X^+_{H_V}\times\{1\}$ with $X^+_{H_V}\subset X^+_P$ (Section 2.2).

Step 1 (the level I, the element m and σ). Put $K_{P,l} := K_l \cap P(\mathbb{Q}_l)$. As $l > C_0$, $K_l = G(\mathbb{Z}_l)$ is hyperspecial, and l splits T_V. Under (2) the Zariski closure of T_V in $\mathrm{GL}_{n_\rho,\mathbb{Z}_l}$ is a torus, by [34, Lemma 3.3.1 and §4.4] applied to the split torus model of $T_{V,\mathbb{Q}_l}$ and the lattice $\mathbb{Z}_l^{n_\rho}$, so $(T_V)_{\mathbb{F}_l}$ is a torus, as [63, Thm. 8.1] requires. So (P,X_P), V and l are as in [63, Thm. 8.1], whose proof [63, §8.3] we follow. By (1) the image of T_V in P^{ad} is not trivial, and [34, Prop. 7.4.3] gives $m_0\in T_V(\mathbb{Q}_l)$ whose image in $P^{\mathrm{ad}}(\mathbb{Q}_l)$ lies in no compact subgroup, with $[K_{P,l} : K_{P,l}\cap m_0^{Aj}K_{P,l}m_0^{-Aj}] < l^{k_1 j}$ for every integer $j\geq 1$: for $j=1$ this is [63, (8.5), (8.6)], and the proof of [34, Prop. 7.4.3], which bounds this index for the powers m_0^{Aj} of its element (m^e there, with $m=m_0$, $e=Aj$), chooses m_0 independently of j and gives an exponent linear in j. There is a compact open subgroup $I_l\subset K_{P,l}$, the intersection of $K_{P,l}$ with two Iwahori subgroups, one chosen with respect to m_0 [63, Def. 8.3.1], such that $[K_{P,l} : I_l]\leq l^f$ [63, (8.8)] and, for every $j\geq 1$ and all $u_1,u_2\in I_l$, the image of $u_1 m_0^{Aj}u_2$ in $P^{\mathrm{ad}}(\mathbb{Q}_l)$ generates an unbounded subgroup [63, (8.7), Prop. 8.1.4 and §8.3.3]. Here [63, §8.3.2] is applied with the torus S, the chambers C^1, C^2 and the groups I_l^1, I_l^2, I_l of Lemma 6.5(c), with $K_l^m := P(\mathbb{Q}_l)^{x_0}$ and C^2 the chamber that [63, §8.3.2] then chooses with respect to m_0; this also gives [63, (8.8)], whether or not $P(\mathbb{Q}_l)$ has hyperspecial subgroups. Put $I := K_P^l I_l$, with $K_P^l := \prod_{p\neq l}(K_p\cap P(\mathbb{Q}_p))$, and let $\tau\colon \mathrm{Sh}_I(P,X_P)\to\mathrm{Sh}_{K_P}(P,X_P)$ be the natural map, which is finite étale of degree at most $[K_{P,l} : I_l]\leq l^f$ and defined over $E(P,X_P)\subset F$. Let $\widetilde{V}$ be the image of $X^+_{H_V}\times\{1\}$ in $\mathrm{Sh}_I(P,X_P)_{\mathbb{C}}$, a component of $\tau^{-1}(V)$.

The reciprocity map $r_{(H_V,X_{H_V})}$ is a homomorphism from $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$ to an abelian quotient of $H_V(\mathbb{A})$ [102, §2.1], and by [102, Prop. 2.9] the class of m_0^A there is $r_{(H_V,X_{H_V})}(\sigma_0)$ for some $\sigma_0\in\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$. The image of $\mathrm{Gal}(\overline{\mathbb{Q}}/FE_V)$ has index $j\leq [FE_V : E_V]\leq \bar{d}$ in that of $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$, so the class of $m := m_0^{Aj}$ is $r_{(H_V,X_{H_V})}(\sigma)$ for some $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/FE_V)$. Then $\sigma(\widetilde{V})$ is the image of $X^+_{H_V}\times\{m\}$ [63, proof of Prop. 8.2.1], so $\sigma(\widetilde{V})\subset T_m\widetilde{V}$, with T_m the correspondence of Lemma 5.3(a) on $\mathrm{Sh}_I(P,X_P)$. As in [63, (8.9)],

$$[I_l : I_l\cap mI_lm^{-1}] \;\leq\; [K_{P,l} : K_{P,l}\cap mK_{P,l}m^{-1}]\cdot[K_{P,l} : I_l] \;<\; l^{k_1 j+f} \;\leq\; l^{k_F}.$$

So [63, Thm. 8.1] holds with $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/FE_V)$ and k_F in place of k_1+f.

Step 2 (the intersection $\widetilde{Z}\cap T_m\widetilde{Z}$). Let $\widetilde{Z}$ be the union of the $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-conjugates of a geometric component of $\tau^{-1}(Z)$ containing $\widetilde{V}$. Then $\widetilde{Z}$ is F-irreducible and $\tau(\widetilde{Z})=Z$; it is Hodge generic, as Z is; and the projection formula and Lemma 5.3(a) give, as in [63, (9.4)],

$$\deg_{\mathcal{L}_I}\widetilde{Z}\leq l^f D_Z,\qquad \deg_{\mathcal{L}_I} T_m\widetilde{Z}\leq [I_l : I_l\cap mI_lm^{-1}]\deg_{\mathcal{L}_I}\widetilde{Z} < l^{k_F+f}D_Z.$$

The set $T_m\widetilde{Z}$ is closed and defined over F. By Step 1, $\sigma\widetilde{V}\subset T_m\widetilde{V}\subset T_m\widetilde{Z}$, and σ fixes F, so $\widetilde{V}\subset T_m\widetilde{Z}$. Hence $\widetilde{V}\subset\widetilde{Z}\cap T_m\widetilde{Z}$, and as this set is defined over F, it contains $\mathrm{Gal}(\overline{\mathbb{Q}}/FE_V)\cdot\widetilde{V}$. The level I is neat, as $I\subset K_P$, so Lemma 5.2(c) applies on $\mathrm{Sh}_I(P,X_P)$:

$$\sum_{C\in\mathrm{Irr}(\widetilde{Z}\cap T_m\widetilde{Z})}\deg_{\mathcal{L}_I} C \;\leq\; c_B\cdot l^f D_Z\cdot l^{k_F+f}D_Z \;=\; \Psi_l(D_Z). \tag{6.3}$$

Step 3 (the case $\widetilde{Z} \subset T_m\widetilde{Z}$). We use the notation of Section 6.1, with $\mathrm{pr}_{\neq i} := \mathrm{pr}_{\{i\}^c}$ and X_i^+-saturated meaning $X_{\{i\}}^+$-saturated. We obtain the conclusion of [63, Thm. 7.2.1] for (P, X_P) at the level I, with p_0 the prime of (Lv′), by its proof run as described below. Here $l \neq p_0$, as the hyperspecial K_l is not neat, so the factor of I at p_0 is $K_{p_0} \cap P(\mathbb{Q}_{p_0})$, which is neat as a subgroup of K_{p_0}. Like V, $\widetilde{V}$ is defined by (H_V, X_{H_V}) and is not strongly special, and the unboundedness condition is that of Step 1. Three points of the proof of [63, Thm. 7.2.1] need care here, as P need not be adjoint and $\widetilde{Z}$ is only F-irreducible: the element q, the adjoint image and saturation.

Step 3a. The element q and the group Γ can be taken in the level. In the proof of [63, Lemma 7.2.8], in its notation, the element q and every element of the group $\Gamma = \mathcal{K}_1 \cap G_1(\mathbb{Q})_+$ must lie in the level group at every prime other than l and have l-component in U_l, so that $[\gamma x, 1] = [x, \gamma_l^{-1}]$ lies in the image of $\widetilde{V} \cdot U_l$ for all $\gamma \in \Gamma_q$; q is found by strong approximation and Γ is replaced by its subgroup of finite index $\{\gamma : \gamma_l \in U_l\}$, as in Step 2 of the proof of Lemma 6.1.

Step 3b. The image of $\widetilde{Z}$ in the adjoint Shimura variety has Zariski dense monodromy. The reduction to the adjoint group in [63, Lemma 7.2.2] gives V' only in $\jmath^{-1}(\jmath(\widetilde{Z}))$, for the map $\jmath$ below, and [63, Lemmas 7.2.3–7.2.4] cannot be run on P itself, since the closure of the monodromy is not open in $K_{P,l}$ when $Z(P)^\circ \neq 1$. Let $\widetilde{Z}^{(1)}$ be a geometric component of $\widetilde{Z}$ containing $\widetilde{V}$, and $\jmath\colon \mathrm{Sh}_I(P, X_P) \to \mathrm{Sh}_{I^{\mathrm{ad}}}(P^{\mathrm{ad}}, X_P^{\mathrm{ad}})$ the finite morphism of the proof of [63, Lemma 7.2.2], where, for the adjoint map $\lambda\colon P \to P^{\mathrm{ad}}$, $I_{p_0}^{\mathrm{ad}} = \lambda(I_{p_0})$, $I_l^{\mathrm{ad}} = \lambda(I_l)$ and I_p^{ad} is a maximal compact subgroup containing $\lambda(I_p)$ for $p \neq p_0, l$. The factor of I^{ad} at p_0, and those of its projections to the $\mathbb{Q}$-simple factors of P^{ad}, are images of a subgroup of the neat group K_{p_0}, so these levels are neat, by the argument of [81, 0.6]. Moreover $\jmath(\widetilde{Z})$ is F-irreducible and Hodge generic, $\jmath(\widetilde{Z}) \subset T_{\lambda(m)}\jmath(\widetilde{Z})$, and the unboundedness condition holds for $\lambda(I_l)$ [63, pp. 32–33]. We therefore run [63, Lemmas 7.2.3–7.2.6] on $\jmath(\widetilde{Z}) \subset \mathrm{Sh}_{I^{\mathrm{ad}}}(P^{\mathrm{ad}}, X_P^{\mathrm{ad}})$, where the monodromy of $\jmath(\widetilde{Z}^{(1)})$, a geometric component of $\jmath(\widetilde{Z})$ containing $\jmath(\widetilde{V})$, is Zariski dense by the argument of Step 1 of the proof of Lemma 6.1.

Step 3c. $\widetilde{Z}$ contains a special subvariety $\widetilde{V}' \supsetneq \widetilde{V}$. The argument of Step 2 of the proof of Lemma 6.1, with $U_l := \langle K_l', (u_1\lambda(m)u_2)^t \rangle$ of [63, Cor. 7.2.5] in place of the group of Step 0, run on $X_P^{\mathrm{ad},+} = X_P^+$, shows that the preimage $\widehat{A} \subset X_P^+$ of $\jmath(\widetilde{Z}^{(1)})$ is X_i^+-saturated for a factor P_i with $\pi_i(T_V) \neq 1$. Let $\widehat{Z}'$ be an irreducible component of the preimage of $\widetilde{Z}^{(1)}$ containing $X_{H_V}^+$; as both preimages have pure dimension $\dim \widetilde{Z}^{(1)}$, $\widehat{Z}'$ is an irreducible component of $\widehat{A}$. A connected complex submanifold contained in a locally finite union of irreducible analytic sets lies in one of them (Baire and the identity theorem); applied to $X_i^+ \times \{x_{\neq i}\}$ at the points x of $\widehat{Z}'$ on no other component of $\widehat{A}$, and by closedness at all points of $\widehat{Z}'$, this shows that $\widehat{Z}'$ is X_i^+-saturated. So $\widetilde{Z}$ contains the image $\widetilde{V}'$ of $X_i^+ \times \mathrm{pr}_{\neq i}(X_{H_V}^+)$, which is special, being the image of $X_{M'}^+ \times \{1\}$ for the Shimura subdatum defined by the identity component M' of the preimage in P of $P_i \times \mathrm{pr}_{\neq i}(\lambda(H_V))$, and contains $\widetilde{V}$ properly [63, Lemma 7.2.9]. Then $V' := \tau(\widetilde{V}')$ is special, $V \subset V' \subset \tau(\widetilde{Z}) = Z$, and $\dim V' = \dim \widetilde{V}' > \dim V$; being irreducible, V' lies in a geometric component of Z, which contains V.

Step 4 (the case $\widetilde{Z} \not\subset T_m\widetilde{Z}$). As $T_m\widetilde{Z}$ is defined over F and $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ permutes the geometric components of $\widetilde{Z}$ transitively, no geometric component of $\widetilde{Z}$ lies in $T_m\widetilde{Z}$; so every component of $\widetilde{Z} \cap T_m\widetilde{Z}$ has dimension less than $\dim Z$. Suppose $\widetilde{V}$ is one of them, as it must be if $r = 1$. Then so are its $\mathrm{Gal}(\overline{\mathbb{Q}}/FE_V)$-conjugates, by Step 2, and Lemma 6.6 at $J = I$ and (6.3) give $\Omega_V/\bar{d} \leq \deg_{\mathcal{L}_I}(\mathrm{Gal}(\overline{\mathbb{Q}}/FE_V) \cdot \widetilde{V}) \leq \Psi_l(D_Z)$, against $\bar{d}\,\Psi_l(D_Z) \leq \Psi_l(\bar{d}D_Z) \leq$

$\Psi_l^{\circ r}(\bar{d}D_Z) < \Omega_V$. So this case does not occur for $r = 1$, and for $r \geq 2$ a geometric component $\widetilde{Y}^{(1)}$ of $\widetilde{Z} \cap T_m\widetilde{Z}$ contains $\widetilde{V}$ properly. Let $\widetilde{Y}$ be the union of the $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-conjugates of $\widetilde{Y}^{(1)}$, which are components of $\widetilde{Z} \cap T_m\widetilde{Z}$. It is closed and F-irreducible, $\tau(\widetilde{Y}) \subset Z$, $\deg_{\mathcal{L}_I} \widetilde{Y} \leq \Psi_l(D_Z)$ by (6.3), and $0 < r_Y := \dim \widetilde{Y} - \dim \widetilde{V} < r$.

Step 5 (descent and induction). Apply Lemma 6.7 with $J = I$, and let (H, X_H), F_H, Y_H and V_H be as there. By Lemma 6.7(a), $\deg_{\mathcal{L}_{K_H}} Y_H \leq \Psi_l(D_Z)$, so, as $\Psi \leq \Psi_l$, Ψ_l is increasing and $\bar{d}\,\Psi_l(D_Z) \leq \Psi_l(\bar{d}D_Z)$,

$$\Psi^{\circ r_Y}(\bar{d}\deg_{\mathcal{L}_{K_H}} Y_H) \;\leq\; \Psi_l^{\circ r_Y}(\bar{d}\deg_{\mathcal{L}_{K_H}} Y_H) \;\leq\; \Psi_l^{\circ(r_Y+1)}(\bar{d}D_Z) \;\leq\; \Psi_l^{\circ r}(\bar{d}D_Z) \;<\; \Omega_V.$$

So the subdatum (H, X_H), the field $FF_H \supset E(H, X_H)$, and $V_H \subset Y_H$ in $\mathrm{Sh}_{K_H}(H, X_H)_{\mathbb{C}}$ satisfy the hypotheses of the theorem with $r_Y < r$ in place of r: Y_H is Hodge generic and FF_H-irreducible; (1) holds by Lemma 6.7(b); and (6.2) holds by the display for the same l and the same $\bar{d}$, since the conditions on l concern only K and T_V, and $F_H \subset E_V$ gives $FF_HE_V = FE_V$. By induction Y_H contains a special $V'_H \supsetneq V_H$, and Lemma 6.7(c) gives V'. □

The uniformity comes from two places: Lemma 5.2 has a constant that depends only on n, by Theorem 1.6 at every neat level; and the varying level enters only through $[K_{P,l} : I_l] \leq l^f$ and $[I_l : I_l \cap mI_lm^{-1}] < l^{k_F}$.

6.3. **The prime.** The criterion (6.1) reads $l^{b_r}\Delta_r(D_Z) < \Omega_V$, where, for $r \geq 0$ and $D' \geq 1$,

$$b_r := (k_F + 2f)\,2^r, \qquad \Delta_r(D') := \left(c_B\,\bar{d}\,D'\right)^{2^r}.$$

For a special subvariety V with $T_V \neq 1$ put

$$x_V := \max\Big(\big(64\kappa(\delta)\beta_V\big)^{1/\delta},\ (\beta_V+1)^2,\ (48n_T)^2,\ 4C_0,\ 64n_T^2\big(\max(e^2/B', \log\alpha_V)+2\big)^2\Big).$$

Proposition 6.8. *Let V be special with $T_V \neq 1$, and assume* $\mathrm{QRH}(\delta; L_V)$. *Then there is a prime l with $x_V/4 < l \leq x_V$ and $l > C_0$ that splits completely in L_V and has $K_{T_V,l} = K^m_{T_V,l}$. Moreover $l > x_V/4 > 1/c_U$.*

Proof. We follow the proof of [63, Prop. 10.2.3]. (1) As $T_V \neq 1$, $\beta_V \geq \log 3 > 1$ (Section 2.2), and $[L_V : \mathbb{Q}] \leq n_T$; so the first three terms of x_V give the three conditions of Lemma 3.2 at $x = x_V$ for $L = L_V$. Hence at least $x_V/(8n_T \log x_V) \geq \sqrt{x_V}/(8n_T)$ primes in $(x_V/4, x_V]$ split completely in L_V, as $x/\log x \geq \sqrt{x}$ for $x > 1$. By the last term of x_V there are at least $\max(e^2/B', \log\alpha_V) + 2$ of them, and by the fourth they exceed C_0.

(2) We count as in [63, Lemma 10.2.5], over the primes $p > C_0$ only. Let i_V be the number of primes $p > C_0$, unramified in L_V, with $K^m_{T_V,p} \neq K_{T_V,p}$; for these $|K^m_{T_V,p}/K_{T_V,p}| \geq c_U p$ by Lemma 6.4. As i_V distinct primes have product at least $i_V!$, $\alpha_V \geq B'^{i_V} i_V! \geq (B'i_V/e)^{i_V}$, which exceeds e^{i_V} if $i_V > e^2/B'$. Hence $i_V \leq \max(e^2/B', \log\alpha_V)$.

(3) The primes of (1) are unramified in L_V and exceed C_0, and there are more than i_V of them; so one of them has $K_{T_V,l} = K^m_{T_V,l}$.

(4) $l > x_V/4 > 1/c_U$, since $x_V/4 \geq 16n_T^2(e^2/B' + 2)^2 > 16e^4/c_U^2 > 1/c_U$, as $B' = Bc_U \leq c_U \leq 1/2$. □

Step (2) of the proof of Proposition 6.8 also bounds the index of K_{T_V} in $K_{T_V}^m$ from above.

Lemma 6.9. *Let V be special with $T_V \neq 1$. Then*

$$\alpha_V \;\leq\; [K_{T_V}^m : K_{T_V}] \;\leq\; B^{-(C_0+e^2/B')}\,\alpha_V^{1+\log(1/B)}\,d_{L_V}^{\log_2(1/B)},$$

with B, $B' = Bc_U$ and C_0 of Table 1, which depend only on (G, X, K, ρ).

Proof. Let $\mathcal{P}_V$ be the set of primes p with $K_{T_V,p}^m \neq K_{T_V,p}$. As K is a product level, $[K_{T_V}^m : K_{T_V}] = \prod_{p\in\mathcal{P}_V} |K_{T_V,p}^m/K_{T_V,p}|$, and as $0 < B \leq 1$, each factor is at least its factor $\max(1, B|K_{T_V,p}^m/K_{T_V,p}|)$ in Definition 2.1 and at most B^{-1} times it. A prime of $\mathcal{P}_V$ is at most C_0, or ramified in L_V (at most $\log_2 d_{L_V}$ primes), or one of the $i_V \leq \max(e^2/B', \log\alpha_V)$ primes of step (2) of the proof of Proposition 6.8; so $|\mathcal{P}_V| \leq C_0 + \log_2 d_{L_V} + e^2/B' + \log\alpha_V$, and

$$\alpha_V \;\leq\; [K_{T_V}^m : K_{T_V}] \;\leq\; B^{-|\mathcal{P}_V|}\,\alpha_V \;\leq\; B^{-(C_0+e^2/B')}\,\alpha_V^{1+\log(1/B)}\,d_{L_V}^{\log_2(1/B)}.$$

□

For $V = \{x\}$ a special point and G adjoint, Binyamini, Schmidt and Yafaev [16, Def. 1] take x in the image of $X^+ \times \{1\}$, where $K_T = K \cap T(\mathbb{A}_f)$, for the Mumford–Tate group $T = T_V$ of x, is the group K_{T_V} of Section 2.2, so that $[K_T^m : K_T] = [K_{T_V}^m : K_{T_V}]$ and $\operatorname{disc}(x) = [K_{T_V}^m : K_{T_V}]\,d_{L_V}$. With Theorem 8.1, Lemma 6.9 bounds $\operatorname{disc}(x)$ for x maximal in Z, with Z as in that theorem (Corollary 8.9).

Corollary 6.10. *Let (G, X), K, δ, x and L be as in Corollary 3.5. If $\mathrm{QRH}(\delta; L)$ holds, then for every number field $F \supset E(G, X)$*

$$[F([x,1]) : F] \;\geq\; \frac{C_*\,B^{\varepsilon(C_0+e^2/B')}}{[F:\mathbb{Q}]}\,\operatorname{disc}([x,1])^{\varepsilon},$$

with $\varepsilon := \min(1/(1+\log(1/B)), \mu/(1+\log_2(1/B)))$, disc as in [16, Def. 1], C_* as in Corollary 3.5, and B, μ, C_0 and $B' = Bc_U$ of Table 1.

Proof. Put $\alpha := \alpha_{[x,1]}$. By Corollary 3.5, $|\operatorname{Gal}(\overline{\mathbb{Q}}/L)\cdot[x,1]| \geq C_*\alpha\,d_L^{\mu}$. By the paragraph before Corollary 6.10, $\operatorname{disc}([x,1]) = [K_T^m : K_T]\,d_L$, and by Lemma 6.9 this is at most $B^{-(C_0+e^2/B')}\alpha^{1+\log(1/B)}d_L^{1+\log_2(1/B)}$. As $\alpha, d_L \geq 1$, $\varepsilon(1+\log(1/B)) \leq 1$ and $\varepsilon(1+\log_2(1/B)) \leq \mu$, this gives $\operatorname{disc}([x,1])^{\varepsilon} \leq B^{-\varepsilon(C_0+e^2/B')}\alpha\,d_L^{\mu}$. Finally $\operatorname{Gal}(\overline{\mathbb{Q}}/FL)$ lies in $\operatorname{Gal}(\overline{\mathbb{Q}}/F)$ and has index $[FL : L] \leq [F : \mathbb{Q}]$ in $\operatorname{Gal}(\overline{\mathbb{Q}}/L)$, so $[F([x,1]) : F] \geq |\operatorname{Gal}(\overline{\mathbb{Q}}/FL)\cdot[x,1]| \geq |\operatorname{Gal}(\overline{\mathbb{Q}}/L)\cdot[x,1]|/[F:\mathbb{Q}]$. □

6.4. **The Hecke step.**

Theorem 6.11. *Let G be adjoint and K satisfy (Lv'), let $(P, X_P) \subset (G, X)$ be a Shimura subdatum and $F \supset E(P, X_P)$ a number field. Let $Z \subset \operatorname{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$ be Hodge generic and F-irreducible, of degree D_Z as in Theorem 6.3, and let $V \subset Z$ be special and not strongly special in $\operatorname{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$, with $r := \dim Z - \dim V > 0$. Let $\bar{d} \geq [FE_V : E_V]$ be an integer and assume $\mathrm{QRH}(\delta; L_V)$. If*

$$\Omega_V \;>\; \Delta_r(D_Z)\,x_V^{b_r},$$

then Z contains a special subvariety $V' \supsetneq V$ in a geometric component of Z containing V.

Proof. The prime l of Proposition 6.8 satisfies $l^{b_r}\Delta_r(D_Z) \leq x_V^{b_r}\Delta_r(D_Z) < \Omega_V$, which is (6.1); so Theorem 6.3 gives V'. □

For maximal V the inequality of Theorem 6.11 fails, and the following lemma turns that into bounds for α_V and d_{L_V}. Put

$$\varsigma := \max\Big(\frac{64\kappa(\delta)+1}{\mu} + 8n_T,\ 1 + 48n_T + 4C_0 + 8n_T\big(e^2/B' + 2\big)\Big),$$

and for $\Lambda \geq 1$ and $b \geq 1$

$$\mathcal{W}(\Lambda, b) := 2\log\Lambda + 1 + \frac{2b}{\delta}\log\frac{2\varsigma b}{e\,\delta}, \qquad \mathcal{B}(\Lambda, b) := \Lambda\left(\varsigma\left(\mathcal{W}(\Lambda, b) + 1\right)\right)^{b/\delta}.$$

Lemma 6.12. *Let $\Lambda \geq 1$ and $b \geq 1$, and let V be special with $T_V \neq 1$. If $\alpha_V d_{L_V}^{\mu} \leq \Lambda\, x_V^b$, then $\alpha_V d_{L_V}^{\mu} \leq \mathcal{B}(\Lambda, b)$. The function $\mathcal{B}$ is non-decreasing in Λ and in b.*

Proof. Put $\omega := \log(\alpha_V d_{L_V}^{\mu}) = \log\alpha_V + \mu\beta_V \geq 0$ and $S := (64\kappa(\delta)+1)\beta_V + 8n_T\log\alpha_V + 1 + 48n_T + 4C_0 + 8n_T(e^2/B' + 2)$. Since $S \geq 1$ and $1/\delta \geq 2$, each of the five terms of x_V is at most $S^{1/\delta}$, as $64\kappa(\delta)\beta_V$, $\beta_V + 1$, $48n_T$, $4C_0$ and $8n_T(\max(e^2/B', \log\alpha_V) + 2)$ are at most S. As $\beta_V \leq \omega/\mu$ and $\log\alpha_V \leq \omega$, we have $S \leq \varsigma(\omega + 1)$. So the hypothesis gives

$$\omega \ \leq\ \log\Lambda + \frac{b}{\delta}\log\big(\varsigma(\omega+1)\big).$$

With $\log y \leq y/s + \log s - 1$ for $y = \omega + 1$ and $s = 2b/\delta$, this gives $\omega \leq \mathcal{W}(\Lambda, b)$; putting this back into the right side gives $\omega \leq \log\mathcal{B}(\Lambda, b)$. Both $\mathcal{W}$ and $\mathcal{B}$ are non-decreasing in Λ, and in b because $2\varsigma b/(e\delta) > 1$. □

7. Degrees of maximal special subvarieties

In this section G is adjoint, K is a neat product level, $n := \dim \mathrm{Sh}_K(G, X)$ and $c_B := e^*(n)^{2n}$ (Lemma 5.2(c)); (Lv′) is not assumed and no analytic hypothesis is used. The iteration of [27, §10] cuts Z by a Hecke translate at an Iwahori level and bounds the intersection by Bézout's inequality with constant 1 at that level [27, proof of Prop. 10.1, p. 24]. We run it with Lemma 5.2 at the level of the Hecke translate and at primes $p \geq N_K$, with the inputs (DJK1)–(DJK3) in the form given below, and treat the facteur case, which Daw, Javanpeykar and Kühne exclude [27, p. 2], with a saturation lemma (Lemma 6.1).

7.1. **Inputs.** Let $\mathcal{G}_{\mathbb{Z}_p}$ be the Zariski closure of $\rho(G)$ in $\mathrm{GL}_{n_\rho,\mathbb{Z}_p}$, and let $N_K > 2D_{\mathrm{sp}} + 1$ be an integer such that $K_p = \mathcal{G}_{\mathbb{Z}_p}(\mathbb{Z}_p)$ is hyperspecial for every prime $p \geq N_K$, where $D_{\mathrm{sp}} \geq 1$ is an integer for which [27, Lemma 7.5] holds. For a reductive $\mathbb{Q}$-group H and a product level $K_H \subset H(\mathbb{A}_f)$, let $\Sigma(H, K_H)$ be the set of primes p at which $H_{\mathbb{Q}_p}$ is not quasi-split, or does not split over an unramified extension, or $K_{H,p}$ is not hyperspecial, and $\Pi(H, K_H)$ the product of these primes (1 if there are none) [27, §2.5]. For a Shimura subdatum $(P, X_P) \subset (G, X)$ write $K(P) := K \cap P(\mathbb{A}_f)$ (the K_P of Sections 6 and 8), and for a special subvariety V of $\mathrm{Sh}_K(G, X^+)_{\mathbb{C}}$, the image of $X_H^+ \times \{1\}$, put $\Sigma_V := \Sigma(H^{\mathrm{der}}, K(H^{\mathrm{der}}))$ and $\Pi_V := \Pi(H^{\mathrm{der}}, K(H^{\mathrm{der}}))$; these depend only on V [27, Def. 3.5], and are the same for V in $\mathrm{Sh}_{K(P)}(P, X_P)$ and for its image in $\mathrm{Sh}_K(G, X)$, since $K(P) \cap H^{\mathrm{der}}(\mathbb{A}_f) = K \cap H^{\mathrm{der}}(\mathbb{A}_f)$. We use the following results of [27], all with effectively computable constants depending only on (G, X, K, ρ).

(DJK1) [27, Thm. 6.1] There are $c_1, \eta_1 > 0$ such that $\deg V > c_1\Pi_V^{\eta_1}$ for every special subvariety V of $\mathrm{Sh}_K(G, X^+)_{\mathbb{C}}$ of positive dimension.

(DJK2) [27, Lemmas 4.1 and 4.2] There is an integer $d_G \geq 1$ such that for every Shimura subdatum (P, X_P) with a component $X_P^+ \subset X^+$, every closed irreducible $Y_P \subset$

$\mathrm{Sh}_{K(P)}(P, X_P^+)_{\mathbb{C}}$ and its image Y in $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$, we have $\deg Y \leq d_G \deg_{\mathcal{L}_{K(P)}} Y_P$; and $\deg_{\mathcal{L}_{K(P)}} Y_P \leq \deg Y$ for every Hodge generic closed irreducible $Y_P \subset \mathrm{Sh}_{K(P)}(P, X_P)_{\mathbb{C}}$.

(DJK3) [27, Thm. 7.1], for primes $p \geq N_K$. There are integers $k_2, f_2 \geq 1$ with the following property. Let (P, X_P) be a Shimura subdatum with a component $X_P^+ \subset X^+$, $P^{\mathrm{ad}} = P_1 \times \cdots \times P_s$ the decomposition into $\mathbb{Q}$-simple factors with projections π_i, $V \subset \mathrm{Sh}_{K(P)}(P, X_P^+)_{\mathbb{C}}$ a special subvariety of positive dimension, the image of $X_H^+ \times \{1\}$ for $(H, X_H) \subset (P, X_P)$ and a component $X_H^+ \subset X_P^+$, $\mathcal{I}$ the set of i with $\pi_i(H^{\mathrm{der}}) \neq 1$, and $p \geq N_K$ a prime with $p \notin \Sigma_V$. Then there are a compact open $J_p \subset K(P)_p := K_p \cap P(\mathbb{Q}_p)$ and $\alpha \in P(\mathbb{Q}_p)$ such that, with $J := K(P)^p J_p$:

(i) $[K(P)_p : J_p] \leq p^{f_2}$;
(ii) $[J_p : J_p \cap \alpha J_p \alpha^{-1}] \leq p^{k_2}$;
(iii) the image $\widetilde{V}$ of $X_H^+ \times \{1\}$ in $\mathrm{Sh}_J(P, X_P)_{\mathbb{C}}$ satisfies $\widetilde{V} \subset T_\alpha \widetilde{V}$;
(iv) $\pi_i(u_1 \alpha u_2)$ generates an unbounded subgroup of $P_i(\mathbb{Q}_p)$ for all $i \in \mathcal{I}$ and $u_1, u_2 \in J_p$.

The argument of the proof of [27, Lemma 4.2] gives $\deg Y \leq d^{\dim Y_P} \deg_{\mathcal{L}_{K(P)}} Y_P$ for the integer d of that proof, so (DJK2) holds with $d_G := d^{\dim \mathrm{Sh}_K(G,X)}$.

Lemma 7.1. *Let V be a special subvariety of $\mathrm{Sh}_K(G, X^+)_{\mathbb{C}}$, the image of $X_H^+ \times \{1\}$, let $p \geq 5$ be a prime with $K_p = \mathcal{G}_{\mathbb{Z}_p}(\mathbb{Z}_p)$, let $\mathcal{H}$ be a reductive model over $\mathbb{Z}_p$ of $H^{\mathrm{der}}_{\mathbb{Q}_p}$, and let K' be the stabilizer of $\mathbb{Z}_p^{n_\rho}$ in $\mathcal{H}(\mathbb{Z}_p)$. Then $K' = \mathcal{H}(\mathbb{Z}_p)$ or $[\mathcal{H}(\mathbb{Z}_p) : K'] \geq (p-1)/c$, with c the constant of* [27, p. 19].

Proof. Suppose $K' \neq \mathcal{H}(\mathbb{Z}_p)$. The kernel $\mathcal{H}_1$ of $\mathcal{H}(\mathbb{Z}_p) \to \mathcal{H}(\mathbb{F}_p)$ is a pro-p group; if $\mathcal{H}_1 \not\subset K'$, the $\mathcal{H}_1$-orbit of $\mathbb{Z}_p^{n_\rho}$ has more than one element and a power of p as size, so $[\mathcal{H}(\mathbb{Z}_p) : K'] \geq p$. Otherwise K' is the preimage of a proper subgroup of $\mathcal{H}(\mathbb{F}_p)$. As $p \geq 5$, $\mathcal{H}(\mathbb{F}_p)$ is generated by the groups $\bar{T}(\mathbb{F}_p)$ of its maximal tori $\bar{T}$ defined over $\mathbb{F}_p$: for $\mathcal{H}_{\mathbb{F}_p}$ simply connected because every semisimple element of $\mathcal{H}(\mathbb{F}_p)$ lies in such a group, so they generate a normal subgroup with quotient a p-group, and $\mathcal{H}(\mathbb{F}_p)$, a product of groups $\mathcal{H}_i(\mathbb{F}_{q_i})$ with $\mathcal{H}_i$ absolutely almost simple and simply connected, is perfect [68, Thm. 24.17], and in general because $\mathcal{H}(\mathbb{F}_p) = \bar{T}(\mathbb{F}_p)\, \pi(\widetilde{\mathcal{H}}(\mathbb{F}_p))$ for the simply connected cover $\pi\colon \widetilde{\mathcal{H}} \to \mathcal{H}_{\mathbb{F}_p}$, by Lang's theorem. So some $\bar{T}(\mathbb{F}_p)$ is not in the image of K', and $\bar{T}$ lifts to a maximal torus $\mathcal{T}$ of $\mathcal{H}$ [22, Thm. 3.2.6], by Hensel's lemma, which also makes $\mathcal{T}(\mathbb{Z}_p) \to \bar{T}(\mathbb{F}_p)$ onto; so $\mathcal{T}(\mathbb{Z}_p) \not\subset K'$. Then $\mathcal{T}$ does not fix $\mathbb{Z}_p^{n_\rho}$ in the sense of [34, §3.3], and the proof of [34, Prop. 4.3.9] in [34, §4.4], applied to $\mathcal{T}$, to the largest sublattice W' of $\mathbb{Z}_p^{n_\rho}$ that $\mathcal{T}$ fixes [34, proof of Lemma 3.3.1], and to $\mathbb{Z}_p^{n_\rho}$, gives $[\mathcal{T}(\mathbb{Z}_p) : \mathcal{T}(\mathbb{Z}_p) \cap K'] \geq (p-1)/|\pi_0(S)(\overline{\mathbb{F}}_p)|$, where $S \subset \mathcal{T}_{\mathbb{F}_p}$ is the stabilizer of the image L' of $p\mathbb{Z}_p^{n_\rho} \cap W'$ in W'/pW' [34, p. 633]. Over $\overline{\mathbb{F}}_p$, S is the common kernel of the differences $\chi - \chi'$ of the characters lying in one block of the finest partition of the weights of $W' \otimes \overline{\mathbb{F}}_p$ along which $L' \otimes \overline{\mathbb{F}}_p$ decomposes, and these characters are bounded by [27, Lemma 6.9]; so $|\pi_0(S)(\overline{\mathbb{F}}_p)| \leq c$, and $[\mathcal{H}(\mathbb{Z}_p) : K'] \geq [\mathcal{T}(\mathbb{Z}_p) : \mathcal{T}(\mathbb{Z}_p) \cap K'] \geq (p-1)/c$. □

Proof of (DJK1). We follow [27, §6], with Lemma 7.1 for the bound that [27, Lemmas 6.10 and 6.11] take from the proof of [26, Lemma 6.6]. Let V be the image of $X_H^+ \times \{1\}$, $\pi\colon \widetilde{H} \to H^{\mathrm{der}}$ the simply connected cover, $K(\widetilde{H}) := \pi^{-1}(K(H^{\mathrm{der}}))$, and $K(\widetilde{H})^m \supset K(\widetilde{H})$ a maximal compact open subgroup of $\widetilde{H}(\mathbb{A}_f)$, as in [27, proof of Thm. 6.3].

Let B_Z and b be the constants of [27, Lemmas 6.4 and 6.5], so that $|\ker \pi(\mathbb{C})| \leq B_Z$, let p_K be the largest prime with $K_p \neq \mathcal{G}_{\mathbb{Z}_p}(\mathbb{Z}_p)$ ($p_K := 1$ if there is none), and let $p \notin \Sigma(\widetilde{H}, K(\widetilde{H})^m)$

be a prime with $p > \max\{B_Z, p_K\}$, so that $K(\widetilde{H})_p^m$ is hyperspecial. Let $\widetilde{\mathcal{H}}$ be the reductive model of $\widetilde{H}_{\mathbb{Q}_p}$ with $\widetilde{\mathcal{H}}(\mathbb{Z}_p) = K(\widetilde{H})_p^m$, $\mathcal{H}$ the reductive model of $H^{\mathrm{der}}_{\mathbb{Q}_p}$ to which π extends, and $\mathcal{Z}$ the kernel of $\widetilde{\mathcal{H}} \to \mathcal{H}$, which is étale over $\mathbb{Z}_p$ as $p \nmid |\ker \pi(\mathbb{C})|$. The group $K(H^{\mathrm{der}})_p$ need not lie in $\mathcal{H}(\mathbb{Z}_p)$, and we put $K' := K(H^{\mathrm{der}})_p \cap \mathcal{H}(\mathbb{Z}_p)$. As $K(\widetilde{H})_p = \pi^{-1}(K(H^{\mathrm{der}})_p)$ lies in $\widetilde{\mathcal{H}}(\mathbb{Z}_p)$ and contains $\ker\pi$, we have $[K(\widetilde{H})_p^m : K(\widetilde{H})_p] \geq [\mathcal{H}(\mathbb{Z}_p) : K']/|H^1(\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p), \mathcal{Z}(\mathbb{Z}_p^{\mathrm{un}}))|$, and $K' \neq \mathcal{H}(\mathbb{Z}_p)$ if $K(\widetilde{H})_p \neq K(\widetilde{H})_p^m$, since otherwise $K(\widetilde{H})_p \supset \widetilde{\mathcal{H}}(\mathbb{Z}_p) = K(\widetilde{H})_p^m$. This H^1 has order at most B_Z^{2b+2} [27, proof of Lemma 6.10], and K' is the group of Lemma 7.1. So, by Lemma 7.1, if $p > c_3 := \max\{4, B_Z, p_K\}$ and $K(\widetilde{H})_p \neq K(\widetilde{H})_p^m$, then

$$[K(\widetilde{H})_p^m : K(\widetilde{H})_p] \;\geq\; \frac{p-1}{c\, B_Z^{2b+2}} \;>\; c_4\, p, \qquad c_4 := (3cB_Z^{2b+2})^{-1},$$

with c the constant of Lemma 7.1. This is [27, Lemma 6.10] with these c_3 and c_4, and the conclusion of [27, Lemma 6.11] with $\Sigma(H^{\mathrm{der}}, K(H^{\mathrm{der}}))$ in place of its printed $\Sigma(H, K(H))$ follows, as in its proof, from Lemma 7.1 and the bound for this H^1, with $c_5 := \max\{c_3, c_4^{-1} B_Z^{2b+2}\}$.

We now write in the notation of Lemma C.2 (Appendix C.1). In that notation $\widetilde{H} = \prod_i \mathrm{Res}_{K_i/\mathbb{Q}} H_i'$, with H_i' absolutely almost simple and simply connected over a number field K_i; ℓ_i and s_i are the field and the integer of [86, §§0.2 and 0.4] for the quasi-split inner form of H_i', $N_i = d_{\ell_i}/d_{K_i}^{[\ell_i:K_i]}$, $\varpi_i = d_{K_i}^{\dim H_i'/2} N_i^{s_i/2}$, r_G is the absolute rank of G, and c_2^* is the constant of the lemma. Its bound carries $\prod_i \varpi_i^{1/2}$, where $\prod_i \varpi_i$ is the discriminant factor of Prasad's formula [86, Thm. 3.7], from which [27, proof of Thm. 6.3] draw the lower bound for the volume of $\Gamma(\widetilde{H})^m \backslash \widetilde{H}(\mathbb{R})$; in that bound the remaining factor $\prod_i \varpi_i^{1/2}$ suffices.

Prasad's theorem applies to each H_i', with S the set of archimedean places of K_i, because $H_i'(K_i \otimes \mathbb{R})$ is not compact. Indeed, as H is the generic Mumford–Tate group on X_H, the projection of $h \in X_H$ to each $\mathbb{Q}$-simple factor of H^{ad} is non-trivial, so that factor has non-compact real points [69, Lemma 4.7]. As $\dim H_i' \geq 3$, and $s_i \geq 5$ when $\ell_i \neq K_i$, we have $\varpi_i^{1/2} \geq (d_{K_i} N_i)^{1/2}$, which pays the primes of $\Sigma(\widetilde{H}, K(\widetilde{H})^m)$ that divide some $d_{K_i} N_i$.

At every other prime p of that set, some H_i' at some place $v \mid p$ of K_i is not quasi-split, or splits over the maximal unramified extension of $K_{i,v}$ while $K(\widetilde{H})^m$ is not hyperspecial at v (otherwise, as K_i and ℓ_i are unramified above p, $\widetilde{H}_{\mathbb{Q}_p}$ would be quasi-split and split over an unramified extension, and $K(\widetilde{H})_p^m$ would be hyperspecial). Then the factor of Prasad's Euler product at v is at least $q_v^{r_v+1}/(q_v+1) \geq q_v^2/(q_v+1)$, where q_v is the order of the residue field and $r_v \geq 1$ the rank of H_i' over the maximal unramified extension of $K_{i,v}$ [86, §2.5 and Prop. 2.10(iv)]; this is at least $\sqrt{p}$ for $p \geq 3$ and $\frac{4}{3} = \frac{2\sqrt{2}}{3}\sqrt{2}$ for $p = 2$, and all the factors are greater than 1 [86, Rem. 3.10]. So $\prod_i \varpi_i^{1/2}$ times the Euler product is at least $\frac{2\sqrt{2}}{3}\, \Pi(\widetilde{H}, K(\widetilde{H})^m)^{1/2}$; dividing by the bound of Lemma C.2 as in [27, proof of Thm. 6.3], the factors $\prod_i \varpi_i^{1/2}$ cancel, and [27, (7)] holds with $C := B_Z^{3r_G}$ and $c_2 := 3c_2^*/(2\sqrt{2})$. As in [27, §6], (DJK1) follows from [27, (7)], [27, Lemmas 6.10 and 6.11] and [27, Thm. 6.2]. □

Proof of (DJK3). The proof of [27, Thm. 7.1] follows that of [26, Thm. 7.1], which takes $J_p \subset P(\mathbb{Q}_p)$ from [63, §8.3.2] and bounds its index by [63, Lemma 8.1.6(b)] applied in $P(\mathbb{Q}_p)$ [26, p. 98]; [27, p. 22] makes this bound explicit by the proof of [63, Lemma 8.1.6(b)], with the Weyl group of $H^{\mathrm{der}}_{\mathbb{Q}_p}$. That proof first reduces to a hyperspecial subgroup of the group containing the Iwahori subgroups, here $P(\mathbb{Q}_p)$, which need not have one, as $p \notin \Sigma_V$ constrains only H^{der}; the proof goes through with Lemma 6.5 in place of [63, Lemma 8.1.6].

Let T be a maximal split torus of $H^{\mathrm{der}}_{\mathbb{Q}_p}$ whose apartment contains the point fixed by the hyperspecial subgroup $K_p \cap H^{\mathrm{der}}(\mathbb{Q}_p)$ [98, 3.8.1]; the maximal compact subgroup of $T(\mathbb{Q}_p)$ fixes that apartment pointwise (proof of Lemma 6.5), so it lies in $K_p \cap H^{\mathrm{der}}(\mathbb{Q}_p) \subset K(P)_p$. Let $\Theta \subset T(\mathbb{Q}_p)$ be the subgroup of elements of order dividing $p-1$. Two distinct characters χ, χ' of T intervening in ρ have coordinates of absolute value at most D_{sp} in the basis of [27, Lemma 7.5], so $\chi - \chi' \notin (p-1)X^*(T)$, as $p - 1 > 2D_{\mathrm{sp}}$, and $\chi \neq \chi'$ on Θ. As $\Theta \subset \mathrm{GL}_{n_\rho}(\mathbb{Z}_p)$ has order prime to p and its characters have values in $\mathbb{Z}_p^\times$, the projectors onto the weight spaces of T preserve $\mathbb{Z}_p^{n_\rho}$; so the Zariski closure $T_{\mathbb{Z}_p}$ of T in $\mathrm{GL}_{n_\rho,\mathbb{Z}_p}$ is a torus [34, Lemma 3.3.1], as [34, Prop. 7.4.3] requires. A nonzero weight of T on $\mathrm{Lie}\, P \subset \mathfrak{gl}_{n_\rho}$ is such a difference $\chi - \chi'$, so it does not lie in $(p-1)X^*(T)$; and $\Theta \subset K(P)_p$. So Lemma 6.5(a), with $H = P_{\mathbb{Q}_p}$, and Lemma 6.5(b) give the torus, chambers and groups of Lemma 6.5(c) with p, T and $K(P)_p$ in place of l, $T_{V,\mathbb{Q}_l}$ and $K_{P,l}$, the chamber C^2 being chosen with respect to α as in [63, §8.3.2]; hence (i) with $f_2 := (\dim G + 1)^2$; then (ii) holds with $k_2 := k' + f_2$, for the integer k' of [27, p. 21], and (iv) follows from [63, Prop. 8.1.4] in each $P_{i,\mathbb{Q}_p}$, $i \in \mathcal{I}$. □

7.2. **The degree bound.** For $\Pi \geq 1$ put $y(\Pi) := \max(6,\, N_K + \log_2 \Pi + 1)$ and $p^*(\Pi) := 2y(\Pi)\log y(\Pi)$. For integers $0 \leq z \leq n$ and $D \geq 1$ put

$$R_{z,D}(\Pi) := d_G\big(d_G\, c_B\, p^*(\Pi)^{k_2+2f_2}\big)^{2^z-1} D^{2^z},$$

let $\Pi_0(z,D) := \sup\{\Pi \geq 1 : c_1\Pi^{\eta_1} \leq R_{z,D}(\Pi)\}$, which is finite because $R_{z,D}$ grows like a power of $\log \Pi$ ($\Pi_0 := 1$ if the set is empty), and

$$c^{\mathrm{ad}}_{\mathrm{deg}}(z,D) := R_{z,D}(\Pi_0(z,D)).$$

Theorem 7.2. *Let $Z \subset \mathrm{Sh}_K(G,X)_{\mathbb{C}}$ be closed and irreducible, $D := \deg Z$, and V a special subvariety maximal in Z with $\dim V \geq 1$. Put $r := \dim Z - \dim V$. For every prime $p \geq N_K$ with $p \notin \Sigma_V$,*

$$\deg V \;\leq\; d_G\big(d_G\, c_B\, p^{k_2+2f_2}\big)^{2^r-1} D^{2^r}, \qquad c_B = e^*(n)^{2n}. \tag{7.1}$$

Consequently $\deg V \leq c^{\mathrm{ad}}_{\mathrm{deg}}(\dim Z, D)$.

Proof. Fix p. We build closed irreducible $Y_0 := Z \supset Y_1 \supset \ldots$, all containing V, with $\dim Y_{j+1} < \dim Y_j$ and $\deg Y_j \leq D_j$, where $D_0 := D$ and $D_{j+1} := d_G\, c_B\, p^{k_2+2f_2} D_j^2$. Since $Y_j \subset Z$, V is maximal in every Y_j.

If $Y_j = V$, we stop. Otherwise Y_j is not special. Let (P, X_P) be the Shimura subdatum, with P the generic Mumford–Tate group on X_P, of the smallest special subvariety containing Y_j, let $\psi\colon \mathrm{Sh}_{K(P)}(P,X_P)_{\mathbb{C}} \to \mathrm{Sh}_K(G,X)_{\mathbb{C}}$ be the natural map, Y_P a component of $\psi^{-1}(Y_j)$ with $\psi(Y_P) = Y_j$, which is Hodge generic, and $V_P \subset Y_P$ a component of $\psi^{-1}(V)$ with $\psi(V_P) = V$ [27, proof of Thm. 1.3]. Then V_P is special, of positive dimension and maximal in Y_P, since ψ is finite and maps special subvarieties to special subvarieties, and $\Sigma_{V_P} = \Sigma_V$. By (DJK2), $\deg_{\mathcal{L}_{K(P)}} Y_P \leq \deg Y_j \leq D_j$. Let J_p, α and $\mathcal{I}$ be as in (DJK3) for V_P and p, $J := K(P)^p J_p$, $\psi_J\colon \mathrm{Sh}_J(P,X_P)_{\mathbb{C}} \to \mathrm{Sh}_K(G,X)_{\mathbb{C}}$ the natural map, $\widetilde{V}$ the component of the preimage of V_P given by (DJK3)(iii), and $\widetilde{Y}$ a component of the preimage of Y_P containing $\widetilde{V}$. By the projection formula and Lemma 5.3(a),

$$\deg_{\mathcal{L}_J} \widetilde{Y} \leq p^{f_2} D_j, \qquad \deg_{\mathcal{L}_J} T_\alpha \widetilde{Y} \leq p^{k_2+f_2} D_j.$$

Cut case: $\widetilde{Y} \not\subset T_\alpha \widetilde{Y}$. By (DJK3)(iii), $\widetilde{V} \subset T_\alpha \widetilde{V} \subset T_\alpha \widetilde{Y}$, so $\widetilde{V} \subset \widetilde{Y} \cap T_\alpha \widetilde{Y}$, a proper closed subset of the irreducible $\widetilde{Y}$. Let C be a component of $\widetilde{Y} \cap T_\alpha \widetilde{Y}$ containing $\widetilde{V}$ and $Y_{j+1} := \psi_J(C)$, closed and irreducible as ψ_J is finite, containing V, with $\dim Y_{j+1} = \dim C < \dim Y_j$.

The level $J \subset K(P)$ is neat, so Lemma 5.2(c) gives $\deg_{\mathcal{L}_J} C \le c_B\, p^{k_2+2f_2} D_j^2$, and by (DJK2) and the projection formula for $\mathrm{Sh}_J \to \mathrm{Sh}_{K(P)}$, $\deg Y_{j+1} \le d_G \deg_{\mathcal{L}_J} C \le D_{j+1}$.

Stable case: $\widetilde{Y} \subset T_\alpha \widetilde{Y}$. The variety Y_P contains a special point, as the special subvariety $V_P \subset Y_P$ does [69, Lemma 13.3]. By (DJK3)(iv) and Lemma 6.1, the preimage of Y_P in X_P^+ is $X_{\mathcal{I}}^+$-saturated. By saturation, Y_P contains the image V' of $X_{\mathcal{I}}^+ \times \mathrm{pr}_{\mathcal{I}^c}(X_H^+)$, which contains V_P; the second factor is a point, as $Z(H)(\mathbb{R})$ fixes the points of X_H and $\pi_i(H^{\mathrm{der}}) = 1$ for $i \notin \mathcal{I}$, and V' is special, as in Step 3 of the proof of Theorem 6.3 with $P_{\mathcal{I}}$ and $\mathcal{I}^c$ in place of P_i and $\{i\}^c$. If $V' = V_P$, then $X_H^+ = X_{\mathcal{I}}^+ \times \mathrm{pr}_{\mathcal{I}^c}(X_H^+)$ by dimension, so H is the generic Mumford–Tate group on this set; its image in P^{ad} lies in the product of $P_{\mathcal{I}}$ with the image of H in $P_{\mathcal{I}^c}$, a torus, and projects onto $P_{\mathcal{I}}$, as P is the generic Mumford–Tate group on X_P; so its derived group, the image of H^{der}, is $P_{\mathcal{I}}$. Otherwise V' strictly contains V_P, which the maximality of V_P excludes (this is the dichotomy of [27, Thm. 8.1]). So Lemma 6.2 and (DJK2) give

$$\deg V \le d_G \deg_{\mathcal{L}_{K(P)}} V_P \le d_G \deg_{\mathcal{L}_{K(P)}} Y_P \le d_G \deg Y_j \le d_G D_j,$$

and we stop.

Each cut lowers the dimension, so there are at most r cuts. The recursion gives

$$D_j = (d_G\, c_B\, p^{k_2+2f_2})^{2^j-1} D^{2^j},$$

which is increasing in j, so every D_j that occurs is at most D_r. With the factor d_G of the stable case this is (7.1).

For the last assertion let p be the least prime $p \ge N_K$ not dividing Π_V; then $p \notin \Sigma_V$. At most $\log_2 \Pi_V$ primes divide Π_V. With $y := y(\Pi_V)$ and $x := p^*(\Pi_V) \ge 17$ we have $x/\log x \ge y$, since $\log(2y \log y) \le 2 \log y$, and $\pi(x) \ge x/\log x$ [88, (3.5)]; so $[N_K, x]$ contains at least $y - N_K + 1 > \log_2 \Pi_V$ primes, and $p \le p^*(\Pi_V)$. As the bases in (7.1) are at least 1 and $r \le \dim Z$, (DJK1) and (7.1) give $c_1 \Pi_V^{\eta_1} < \deg V \le R_{\dim Z, D}(\Pi_V)$. Hence $\Pi_V \le \Pi_0(\dim Z, D)$, and $\deg V \le c_{\deg}^{\mathrm{ad}}(\dim Z, D)$, as $R_{z,D}$ is non-decreasing. □

Theorem 1.7 is Theorem 7.2, applied at the level $g_i K g_i^{-1}$, with the lattice $\rho(g_i)\widehat{\mathbb{Z}}^{n_\rho}$, of the component containing Z (Section 2.2), with $c_B = e^*(n)^{2n}$ (Lemma 5.2(c)). So, in the setting of Theorem 1.7, (7.1) holds with d_G, k_2, f_2 and N_K the maxima over the connected components of the constants of Section 7.1 at the levels $g_i K g_i^{-1}$ with the lattices $\rho(g_i)\widehat{\mathbb{Z}}^{n_\rho}$, and Σ_V that of the translate of V. The bound holds for facteur and non-facteur V alike; for non-facteur V and neat K it gives [27, Thm. 1.3], with the constant effectively computable given the g_i. Facteur special subvarieties of bounded degree come in infinite families [27, p. 2], so a bound for the degree alone leaves infinitely many candidates; Theorem 8.1 bounds the other two terms d_{L_V} and α_V of $\Delta(V)$.

8. The main theorem for adjoint groups

In this section G is adjoint, K satisfies (Lv′), $n := \dim \mathrm{Sh}_K(G, X)$ and $e_0 := e^*(n)$. Let $F \supset E(G, X)$ be a number field, $d := [F : \mathbb{Q}]$, and $Z \subset \mathrm{Sh}_K(G, X)$ a closed subvariety, geometrically irreducible and defined over F, with $D := \deg Z$ and $z := \dim Z$. Let $(P, X_P) \subset (G, X)$ be the Shimura subdatum, with P the generic Mumford–Tate group on X_P, that defines the smallest special subvariety containing Z [63, p. 14], the special closure of Z; put $K_P := K \cap P(\mathbb{A}_f)$, $E_P := E(P, X_P)$, $T_Z := Z(P)^\circ$, and let $\psi\colon \mathrm{Sh}_{K_P}(P, X_P) \to \mathrm{Sh}_K(G, X)$ be the natural morphism, defined over E_P [69, Remark 13.8].

8.1. **Statement.** A special subvariety V maximal in Z belongs to exactly one of three classes:

(S) $T_V = 1$;
(N) $T_V \neq 1$ and $T_V \not\subset Z(P)$;
(C) $T_V \neq 1$ and $T_V \subset Z(P)$.

Here V is regarded inside the special subvariety defined by (P, X_P) (Lemma 8.3). Class (N) says that V is not strongly special in $\mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$, since the image of T_V in P^{ad} is the connected center of the image of H_V; so the Hecke step applies to it. Class (C) consists of special subvarieties that are strongly special in $\mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$ but not in $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$, and the Hecke step says nothing about them; the bounds come instead from the Galois orbit of the component containing the lift Z_P of Z (Lemma 8.3), for d_{L_V} (Lemma 8.5), and from the translates of Z_P by the center, for α_V (Lemma 8.8).

With b_r, Δ_r of Section 6 and $\mathcal{B}$ of Lemma 6.12, computed with $\bar{d} = d$, and $c^{\mathrm{ad}}_{\deg}$ of Section 7, put

$$c_{\mathrm{N}}(\delta, z, D, d) := \mathcal{B}\big(\Delta_z(D)/C_*,\ b_z\big),$$

and

$$\begin{aligned} c^{\sharp}_{\mathrm{N}}(\delta, z, D, d) &:= \mathcal{B}\big((c_B dD)^{2^z}/C_*,\ (k_1 + 3f)2^z\big), \\ c^{\sharp}(\delta, z, D, d) &:= \max\Big(c^{\mathrm{ad}}_{\deg}(z, D),\ c^{\sharp}_{\mathrm{N}}(\delta, z, D, d)^{1/\mu}, \\ &\qquad e^{\beta_0(\delta)}\big(R_G^2 (c_B dD)^{2^z}\big)^{1/\mu},\ a^4 (a!)^{r_U} R_G\, e_0^{2z(z+1)} (c_B dD)^{2^{z+1}}\Big). \end{aligned}$$

The arguments of $\mathcal{B}$ in $c^{\sharp}_{\mathrm{N}}$ are $\Delta_z(dD)/C_*$ and b_z computed with $\bar{d} = 1$.

Theorem 8.1. *In this setting let V be a special subvariety maximal in Z.*

(a) $\deg V \le c^{\mathrm{ad}}_{\deg}(z, D)$.

(b) If V is of class (S), then $\Delta(V) = \deg V \le c^{\mathrm{ad}}_{\deg}(z, D)$.

(c) If V is of class (N) and $\mathrm{QRH}(\delta; L_V)$ *holds, then* $\alpha_V\, d_{L_V}^{\mu} \le c_{\mathrm{N}}(\delta, z, D, d)$; *hence* $\alpha_V \le c_{\mathrm{N}}(\delta, z, D, d)$ *and* $d_{L_V} \le c_{\mathrm{N}}(\delta, z, D, d)^{1/\mu}$. *Moreover, either* $\alpha_V\, d_{L_V}^{\mu} \le c^{\sharp}_{\mathrm{N}}(\delta, z, D, d)$, *or* $d_{L_V} \le e^{\beta_0(\delta)}\big(R_G^2 (c_B dD)^{2^z}\big)^{1/\mu}$ *and* $\alpha_V \le a^4 (a!)^{r_U} R_G\, e_0^{2z(z+1)} (c_B dD)^{2^{z+1}}$.

(d) If V is of class (C), then $T_V = T_Z$ and $\alpha_V \le a^4 (a!)^{r_U}\, d\, (e_0^z D)^{2(z+1)}$; *if moreover* $\mathrm{QRH}(\delta; L_V)$ *holds, then* $d_{L_V} \le e^{\beta_0(\delta)} (R_G d)^{1/\mu}$.

In particular, if $\mathrm{QRH}(\delta; L_V)$ *holds, then* $\Delta(V) \le c^{\sharp}(\delta, z, D, d)$. *No hypothesis is used in (a), (b) and in the bound for α_V in (d).*

The constants are non-decreasing in z, D and d (by their definitions, with Lemma 6.12 for $\mathcal{B}$).

Remark 8.2. For adjoint G the constant $c^{\sharp}(\delta, z, D, d)$ is at most $(dD)^{2^z/\mu + o(1)}$ as $dD \to \infty$, the $o(1)$ depending on (G, X, K, ρ), z and δ. Indeed, let $t := dD$, $\Lambda := (c_B t)^{2^z}/C_*$, $b := (k_1 + 3f)2^z$ and $p := b/\delta$. Then $\mathcal{W}(\Lambda, b) + 1 = 2^{z+1} \log t + C_1$ with $C_1 := 2^{z+1} \log c_B + 2\log(1/C_*) + 2 + 2p \log(2\varsigma p/e)$, which depends only on (G, X, K, ρ), z and δ, so

$$c^{\sharp}_{\mathrm{N}}(\delta, z, D, d) \;=\; \frac{(c_B t)^{2^z}}{C_*}\,\big(\varsigma\,(2^{z+1} \log t + C_1)\big)^p \;=\; O\big(t^{2^z}(1 + \log t)^p\big) \;=\; t^{2^z + o(1)},$$

effectively, since for every $\eta > 0$ the maximum C_η of $(1+u)^p e^{-\eta u}$ on $u \ge 0$ gives $(1 + \log t)^p \le C_\eta t^\eta$; the last two terms of $c^{\sharp}$ are $O(t^{2^z/\mu})$ and $O(t^{2^{z+1}})$, and $2^{z+1} < 2^z/\mu$ as $1/\mu \ge 32$;

and as $\log \Pi_0(z, D) = O(\log D)$ (Theorem 7.2), $c^{\mathrm{ad}}_{\deg}(z, D)$ is at most $D^{2^z+o(1)}$. In c_{N} the exponent $b_z = (k_1 d + 3f)2^z$ grows with d, but for fixed d the same computation gives, for V of class (N), $\alpha_V \le D^{2^z+o(1)}$ and $d_{L_V} \le D^{2^z/\mu+o(1)}$ as $D \to \infty$. For α_V this is better than the second assertion of Theorem 8.1(c), whose bound for α_V, from class (C) at a later stage of the recursion, is of order $D^{2^{z+1}}$ for fixed d; so (c) keeps both. With the form $C(N)\alpha_V(\log d_{L_V})^N$ of the Galois bound used in [63] and a fixed $N > b/\delta$, the same argument bounds only $\alpha_V(\log d_{L_V})^{N-b/\delta}$, so d_{L_V} would be exponential in a power of dD.

At $\delta = 1/32$ the factor $1/C_*$ is large. As $\kappa(\delta)/\delta^2 \ge 8$ and $1/\theta \ge 8n_T \ge 16$, the first term of $\beta_0(\delta)$ is the largest, so $\log(1/c_*(\delta)) = \mu\beta_0(\delta) = 32\kappa(\delta)/(\mu\delta^2)$, about $2.0 \cdot 10^6/\mu$.

8.2. **The lift to the special closure.**

Lemma 8.3. *(a) Exactly one irreducible component Z_P of $\psi^{-1}(Z)$ satisfies $\psi(Z_P) = Z$. It is Hodge generic, geometrically irreducible and defined over FE_P, $\psi|_{Z_P} \colon Z_P \to Z$ is finite and birational, and $\deg_{\mathcal{L}_{K_P}} Z_P \le D$.*

(b) If $V \subset Z$ is special and maximal in Z, some component V_P of $\psi^{-1}(V) \cap Z_P$ has $\psi(V_P) = V$. It is special in $\mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$, maximal among the special subvarieties of Z_P, and has the same T_V, α_V, β_V and reflex field as V.

Proof. (a) Let $z_0 = [h_0, 1] \in Z$, $h_0 \in X_P^+$, have Mumford–Tate group P; such points form the complement in Z of a countable union of proper closed subsets. If $[h, g] \in \psi^{-1}(z_0)$, then $h = qh_0$ for some $q \in G(\mathbb{Q})$, so the Mumford–Tate group of h is qPq^{-1}; it is contained in P, as $h \in X_P$, hence equal to P. So every point over z_0 is Hodge generic, and $\psi^{-1}(z_0)$ is one point by the injectivity of ψ on the Hodge generic locus [102, Lemma 2.2]. As ψ is finite and Z lies in its image, the smallest special subvariety containing Z being the image of $X_P^+ \times \{1\}$ (Section 2), some component of $\psi^{-1}(Z)$ maps onto Z. Two such components C_1, C_2 both contain the points over the Hodge generic points of Z, which are dense in each by Baire's theorem, as $\psi|_{C_i}$ is finite onto Z; so $C_1 = C_2$. The group $\mathrm{Gal}(\overline{\mathbb{Q}}/FE_P)$ permutes the components of $\psi^{-1}(Z)$ that map onto Z, so it fixes Z_P. Finally $\psi|_{Z_P}$ is finite and injective on a dense set, so of degree 1, and $\deg_{\mathcal{L}_{K_P}} Z_P \le \deg Z = D$ by [63, Cor. 5.3.10], as Z_P is Hodge generic and $Z(P)(\mathbb{R})$ is compact.

(b) Some component of $\psi^{-1}(V) \cap Z_P$ maps onto V, as $\psi|_{Z_P}$ is finite onto Z; it has dimension $\dim V$, so it is a component of $\psi^{-1}(V)$, hence special [63, p. 14]. A special V' with $V_P \subsetneq V' \subset Z_P$ would give the special subvariety $\psi(V') \subset Z$ of dimension larger than $\dim V$. Finally V_P and V are defined by the same Shimura datum $(H, X_H) \subset (P, X_P)$, V_P being the image of $X_H^+ \times \{1\}$ (Section 2), and $K_P \cap T(\mathbb{A}_f) = K \cap T(\mathbb{A}_f)$ for every torus $T \subset P$. □

Lemma 8.4. *If V is special and maximal in Z, with $T_V \ne 1$ and $T_V \subset Z(P)$, then $T_V = T_Z$. If moreover V is a point, then $Z = V$.*

Proof. Let $\nu \colon P \to P^{\mathrm{ab}} := P/P^{\mathrm{der}}$. All $h \in X_P$ have the same image $h^{\mathrm{ab}} := \nu \circ h$, as X_P is one $P(\mathbb{R})$-conjugacy class and P^{ab} is commutative. The Mumford–Tate group of h^{ab} is the image of that of a Hodge generic h, so it is P^{ab}. For $h \in X_{H_V}$ the Mumford–Tate group of h lies in H_V and maps onto that of h^{ab}, so $\nu(H_V) = P^{\mathrm{ab}}$; as $H_V = T_V \cdot H_V^{\mathrm{der}}$ with $H_V^{\mathrm{der}} \subset P^{\mathrm{der}}$, $\nu(T_V) = P^{\mathrm{ab}}$. If $T_V \subset Z(P)$, then $T_V \subset Z(P)^\circ$, and $Z(P)^\circ \to P^{\mathrm{ab}}$ is an isogeny, so $\dim T_V \ge \dim Z(P)^\circ$ and $T_V = T_Z$. If V is a point h, then $H_V = T_V$ is central in P, so $X_P = \{h\}$ and $Z = V$. □

8.3. **Class (C): the discriminant.**

Lemma 8.5. *Suppose $T_Z \neq 1$ and let L be its splitting field. If $\mathrm{QRH}(\delta; L)$ holds, then $c_*(\delta) d_L^{\mu} \leq R_G d$, that is, $d_L \leq e^{\beta_0(\delta)} (R_G d)^{1/\mu}$.*

Proof. Let S_P^+ be the connected component of $\mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$ containing Z_P. Every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/FE_P)$ fixes Z_P (Lemma 8.3(a)), hence S_P^+; so the $\mathrm{Gal}(\overline{\mathbb{Q}}/E_P)$-orbit of S_P^+ has at most $[FE_P : E_P] \leq d$ members.

Let $C := P^{\mathrm{ab}}$, $\bar{x} := h^{\mathrm{ab}}$, $E_C := E(C, \bar{x}) \subset E_P$, K_C^m the maximal compact open subgroup of $C(\mathbb{A}_f)$ and $\mathrm{Cl}_C := C(\mathbb{Q}) \backslash C(\mathbb{A}_f) / K_C^m$. The map $\pi_0(\mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}) \to \mathrm{Cl}_C$, $[x, g] \mapsto [\nu(g)]$, is well defined, as $\nu(K_P) \subset K_C^m$, and it is $\mathrm{Gal}(\overline{\mathbb{Q}}/E_P)$-equivariant when σ acts on Cl_C by multiplication by $r_C(N_{E_P/E_C}(s_\sigma))$, where s_σ is a finite idele of E_P with Artin symbol $\sigma|_{E_P^{\mathrm{ab}}}$ and r_C is the reciprocity morphism of $(C, \bar{x})$. This follows from Shimura reciprocity at a special point x of X_P and $\nu \circ r_x = r_C \circ N_{E(x)/E_C} = r_C \circ N_{E_P/E_C} \circ N_{E(x)/E_P}$, as in [69, §13, (64) and footnote 73]; that argument does not use that P^{der} is simply connected. Hence the orbit of S_P^+ has at least $|\mathrm{Im}_{E_P}|$ members, where Im_{E_P} is the image of $r_C \circ N_{E_P/E_C}$ in Cl_C.

The splitting field of C is L, as C and T_Z are isogenous [102, Remark 2.20], and $E_C \subset L$ [69, Remark 12.3(a)]. Let $L' := LE_P$. Since $N_{E_P/E_C} \circ N_{L'/E_P} = N_{L/E_C} \circ N_{L'/L}$, Im_{E_P} contains the image of $N_{L'/L}((\mathbb{A}_f \otimes L')^\times)$ under $r_L := r_C \circ N_{L/E_C}$. As in the proof of Corollary 3.4, this image has index at most $[L' : L] \leq [E_P : \mathbb{Q}] \leq R_G$ in Im_L. Proposition 3.3, applied to $(H, X_H) = (P, X_P)$, gives $|\mathrm{Im}_L| \geq c_*(\delta) d_L^{\mu}$. So $c_*(\delta) d_L^{\mu} / R_G \leq d$. □

8.4. **Class (C): the level.** For the next two lemmas compare [55]; we include short proofs, which give the constants used.

Lemma 8.6. *Let $Y \subset \mathbb{P}^N_{\mathbb{C}}$ be closed, reduced and of pure dimension, of degree e (the sum of the degrees of its components). Then Y is the common zero set of the forms of degree e vanishing on Y.*

Proof. Let $Y = Y_1 \cup \cdots \cup Y_t$, with $\deg Y_i = a_i$ and $\dim Y =: m < N$ (the case $Y = \mathbb{P}^N$ is trivial). Let $y \notin Y$; it suffices to find forms P_i of degree at most a_i vanishing on Y_i with $P_i(y) \neq 0$, and to multiply them and a power of a linear form not vanishing at y. If $m = N - 1$, take P_i an equation of Y_i. If $m \leq N - 2$, a general $(N - m - 1)$-plane Π through y misses Y_i, since projection from y maps Y_i onto an m-dimensional subvariety of $\mathbb{P}^{N-1}$. Let $\Lambda \subset \Pi$ be a hyperplane of Π not containing y. The projection $\pi_\Lambda \colon \mathbb{P}^N \smallsetminus \Lambda \to \mathbb{P}^{m+1}$ is finite on Y_i, its image is a hypersurface $V(F_i)$ of degree at most a_i, and the fiber of π_Λ through y is $\Pi \smallsetminus \Lambda$, which misses Y_i; take $P_i := F_i \circ \pi_\Lambda$. □

Lemma 8.7. *Let $X_0 \subset \mathbb{P}^N$ be closed and irreducible, of dimension m and degree at most Δ, and let $(Y_i)_{i \in I}$ be any family of closed subsets of $\mathbb{P}^N$, each of pure dimension and of degree at most Δ. Then the irreducible components C of $X_0 \cap \bigcap_i Y_i$ satisfy $\sum_C \deg C \leq \Delta^{m+1}$.*

Proof. Since degrees are integers and the bound increases with Δ, we may first replace Δ by $\lfloor \Delta \rfloor$. By Lemma 8.6, each Y_i is the common zero set of forms of degree $\deg Y_i \leq \Delta$; multiplying a form of degree $\Delta' < \Delta$ by $x_0^{\Delta - \Delta'}, \ldots, x_N^{\Delta - \Delta'}$ does not change the zero set. So $X_0 \cap \bigcap_i Y_i = X_0 \cap V(f_1, \ldots, f_s)$ for finitely many forms f_k of degree Δ. Build a tree with root X_0: a node X at stage k has the single child X if $X \subset V(f_{k+1})$, and otherwise the components of $X \cap V(f_{k+1})$, of dimension $\dim X - 1$ and total degree at most $\Delta \deg X$ (Lemma 5.2(a)). Along each path the dimension drops at most m times, so the leaves after stage s have total degree at most $\Delta^m \deg X_0 \leq \Delta^{m+1}$. They cover $X_0 \cap V(f_1, \ldots, f_s)$ and lie in it, so its components are among them. □

Let a, A and r_U be as in Table 1. For a torus T with $K_T := K \cap T(\mathbb{A}_f)$ and maximal compact open $K_T^m \subset T(\mathbb{A}_f)$, let $\Theta_A \subset K_T^m/K_T$ be the image of $x \mapsto x^A$.

Lemma 8.8. *If V is a special subvariety maximal in Z of class (C), then*

$$\alpha_V \le a^4 (a!)^{r_U}\, d\, (e_0^z D)^{2(z+1)}.$$

Proof. Put $\Delta := e_0^z D$ and $T := T_Z$, so $T_V = T$ by Lemma 8.4. Let $H := H_V$, $K_H := K \cap H(\mathbb{A}_f)$, and let V_P be as in Lemma 8.3(b), the image of a component V_H of $\mathrm{Sh}_{K_H}(H, X_H)_{\mathbb{C}}$.

Step 1 (translates stay in Z_P). Let F_V be the Galois closure of $E_V = E(H, X_H)$, of degree at most R_G. By [102, Lemma 2.15], with A from [102, Prop. 2.9], neither of which uses GRH, $\Theta_A \cdot V_H \subset \mathrm{Gal}(\overline{\mathbb{Q}}/F_V) \cdot V_H$. Translation by $\theta \in T(\mathbb{A}_f)$ is defined on $\mathrm{Sh}_{K_H}(H, X_H)$ and on $\mathrm{Sh}_{K_P}(P, X_P)$, as T is central in H and in P; it commutes with the natural map and with $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$. So Θ_A permutes the at most $[FF_V : F_V] \le d$ orbits of $\mathrm{Gal}(\overline{\mathbb{Q}}/FF_V)$ on $\mathrm{Gal}(\overline{\mathbb{Q}}/F_V) \cdot V_H$, and the stabilizer $\Theta^{(F)} \subset \Theta_A$ of the orbit of V_H has index at most d. For $\theta \in \Theta^{(F)}$, $\theta V_H = \sigma V_H$ with $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/FF_V)$. As $E_P \subset E_V$ [69, Remark 12.3(c)], σ fixes FE_P, so $\sigma Z_P = Z_P$ and $\theta V_P = \sigma V_P \subset Z_P$.

Step 2 (cutting by all translates). Each $\theta \in \Theta^{(F)}$ has $V_P \subset \theta^{-1} Z_P \cap Z_P$, so θ stabilizes S_P^+, the component containing Z_P. We have $S_P^+ = \Gamma \backslash X_P^+$ with Γ the image in $P^{\mathrm{ad}}(\mathbb{Q})$ of $P(\mathbb{Q})_+ \cap g K_P g^{-1}$ for some g, which is arithmetic [69, Prop. 3.2] and neat [81, 0.6], [63, §4.1.4]. Translation by θ extends to the Baily–Borel compactification $\overline{S}_P^+$ and preserves $\mathcal{L} := \mathcal{L}_{K_P}|_{\overline{S}_P^+}$, the extension of the canonical bundle. By Theorem 1.6 for (P, X_P) and K_P, restricted to the component $\overline{S}_P^+$, and as e^* is increasing and $\dim S_P^+ \le n$, $|\mathcal{L}^{\otimes e_0}|$ defines a finite injective $\varphi\colon \overline{S}_P^+ \to \mathbb{P}^N$ with $\varphi^* \mathcal{O}(1) = \mathcal{L}^{\otimes e_0}$, birational on irreducible closed subsets, as in the proof of Lemma 5.2(b). Each $Y_\theta := \varphi(\theta^{-1} \overline{Z}_P)$ is irreducible of dimension z and degree $e_0^z \deg_{\mathcal{L}} Z_P \le \Delta$, since $\deg_{\mathcal{L}} Z_P \le D$ (Lemma 8.3(a)). As φ is injective, $\varphi(\bigcap_\theta \theta^{-1} \overline{Z}_P) = \bigcap_\theta Y_\theta$, and Lemma 8.7 with $X_0 = Y_1$ shows that $\overline{W} := \bigcap_{\theta \in \Theta^{(F)}} \theta^{-1} \overline{Z}_P$ has at most Δ^{z+1} components, each of $\mathcal{L}$-degree at most Δ^{z+1}. The set $\overline{W}$ contains V_P and is $\Theta^{(F)}$-stable. Let W_0 be a component of $\overline{W} \cap S_P^+$ containing V_P and Θ_1 its stabilizer in $\Theta^{(F)}$; then $[\Theta^{(F)} : \Theta_1] \le \Delta^{z+1}$ and $\deg_{\mathcal{L}_{K_P}} W_0 \le \Delta^{z+1}$.

Step 3 (a stable subvariety is at least as large as its stabilizer). We use the level change of [102, pp. 21–22] for P in place of H. Put $K_P^m := K_T^m K_P$, compact open as T is central; choose $\gamma_3 \in \mathrm{GL}_{n_\rho}(\mathbb{Q}_3)$ with $K_{P,3}^m \subset \gamma_3 \mathrm{GL}_{n_\rho}(\mathbb{Z}_3) \gamma_3^{-1}$, let $K_P^{m'} := (K_{P,3}^m \cap \gamma_3 K(3)_3 \gamma_3^{-1}) \times \prod_{p \ne 3} K_{P,p}^m$, with $K(3)_3$ the level-3 congruence subgroup of $\mathrm{GL}_{n_\rho}(\mathbb{Z}_3)$, and $K_P' := K_P \cap K_P^{m'}$. Then $K_P^{m'}$ is neat, K_P' is normal in $K_P^{m'}$, and $[K_P^m : K_P^{m'}]$, $[K_P : K_P'] \le a$. The map $\pi'\colon \mathrm{Sh}_{K_P'}(P, X_P) \to \mathrm{Sh}_{K_P^{m'}}(P, X_P)$ is finite étale, and $\mathfrak{K} := K_P^{m'}/K_P'$ acts simply transitively on its fibers [102, Lemma 2.11 and its proof]; $f\colon \mathrm{Sh}_{K_P'}(P, X_P) \to \mathrm{Sh}_{K_P}(P, X_P)$ is finite étale of degree at most a; both pull the Baily–Borel bundle back to the Baily–Borel bundle [63, Prop. 5.3.2]. Let $\widetilde{\Theta}_1 \subset K_T^m$ be the preimage of Θ_1 and $\widetilde{\Theta}_1' := \widetilde{\Theta}_1 \cap K_P^{m'}$, of index at most a. It acts on $\mathrm{Sh}_{K_P'}(P, X_P)$ by central translations commuting with f, so it permutes the at most a components of $f^{-1}(W_0)$. The stabilizer $\widetilde{\mathfrak{S}}$ of one of them, W', has index at most a in $\widetilde{\Theta}_1'$, so its image in K_T^m/K_T has at least $|\Theta_1|/a^2$ elements, and since $K_P' \cap T(\mathbb{A}_f) \subset K_T$, so does its image $\mathfrak{S}$ in $\mathfrak{K}$. As $\mathfrak{S}$ acts freely on W', $\deg(\pi'|_{W'}) \ge |\mathfrak{S}|$, and the projection formula gives

$$|\Theta_1|/a^2 \le |\mathfrak{S}| \le \deg_{\mathcal{L}_{K_P'}} W' \le \deg_{\mathcal{L}_{K_P'}} f^{-1}(W_0) \le a \deg_{\mathcal{L}_{K_P}} W_0 \le a \Delta^{z+1}.$$

Step 4 (from Θ_A to α_V). Steps 1–3 give $|\Theta_A| \le d \cdot \Delta^{z+1} \cdot a^3\Delta^{z+1}$. Let $K_H^{m'}$ and K'_H be the levels of [102, pp. 21–22] for H, $K_T^{m'} := K_H^{m'} \cap T(\mathbb{A}_f)$, $K'_T := K'_H \cap T(\mathbb{A}_f)$, and Θ'_e the image of $x \mapsto x^e$ on $K_T^{m'}/K'_T$. By [102, Lemma 2.18] with $e = a!$, $\alpha_V \le (a!)^{r_U} |\Theta'_{A\cdot a!}|$. The map $K_T^{m'}/K'_T \to K_T^m/K_T$ sends $\Theta'_{A\cdot a!}$ into Θ_A, with kernel $(K_T^{m'} \cap K_T)/K'_T$ of order at most $[K_H : K'_H] \le a$. Hence $\alpha_V \le (a!)^{r_U} \cdot a \cdot |\Theta_A| \le a^4 (a!)^{r_U} d\Delta^{2(z+1)}$. □

8.5. **Proof of Theorem 8.1.** (a) If $\dim V \ge 1$, Theorem 7.2 applied to Z gives $\deg V \le c^{\mathrm{ad}}_{\deg}(z, D)$; points have degree $1 \le c^{\mathrm{ad}}_{\deg}(z, D)$.

(b) For class (S), $\alpha_V = d_{L_V} = 1$ by Definition 2.1.

(c) Let Z_P and V_P be as in Lemma 8.3. Class (N) says that V_P is not strongly special in $\mathrm{Sh}_{K_P}(P, X_P)_{\mathbb{C}}$, and then $r := z - \dim V \ge 1$: if $V = Z$, then $P = H_V$ and $T_V = Z(P)^\circ$. Apply Theorem 6.11 to (P, X_P), the field $F_1 := FE_P \supset E(P, X_P)$, the variety Z_P, which is Hodge generic, geometrically irreducible and defined over F_1 with $D_Z \le D$ (Lemma 8.3(a)), and to V_P. Since $E_P \subset E_V$ [69, Remark 12.3(c)], $[F_1E_V : E_V] = [FE_V : E_V] \le d$, so $\bar{d} = d$ is allowed. If $\Omega_V > \Delta_r(D) x_V^{b_r}$, then $\Omega_V > \Delta_r(D_Z) x_V^{b_r}$, as Δ_r is increasing, and Theorem 6.11 gives a special V' with $V_P \subsetneq V' \subset Z_P$, and $\psi(V') \subset Z$ is special of dimension larger than $\dim V$, against maximality. So $\alpha_V d_{L_V}^{\mu} \le (\Delta_r(D)/C_*)\, x_V^{b_r}$, and Lemma 6.12 gives $\alpha_V d_{L_V}^{\mu} \le \mathcal{B}(\Delta_r(D)/C_*, b_r) \le c_{\mathrm{N}}(\delta, z, D, d)$, as $r \le z$, $\Delta_r(D)/C_* \ge 1$ and b_r, $\Delta_r(D)$ are non-decreasing in r. As $\alpha_V \ge 1$ and $d_{L_V} \ge 1$, both bounds follow.

For the second assertion we apply Theorem 6.11 with $\bar{d} = 1$, to the Galois orbit of a lift over the reflex field of a special closure, as Klingler and Yafaev do with the conjugates of Z over $E(G, X)$ [63, §10.2]; then d enters only through the degree of the orbit, not through the exponent b_r. The larger special subvariety may then lie in another conjugate, and we pass to an intersection of two conjugates, of smaller dimension. Put $E := E(G, X)$. For a closed subvariety $Y \subset Z$ (so Y lies in the image of $X^+ \times \{1\}$), geometrically irreducible and defined over a number field, let $M(Y)$ be the number of its $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$-conjugates, and F_Y the fixed field of its stabilizer in $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$, over which Y is defined, with $[F_Y : E] = M(Y)$. As $\mathcal{L}_K$ and the Baily–Borel compactification are defined over E, conjugates have the same degree, and the orbit of Y has degree

$$\mathcal{D}(Y) := \deg(\mathrm{Gal}(\overline{\mathbb{Q}}/E) \cdot Y) = M(Y) \deg Y.$$

So $M(Y) \le \mathcal{D}(Y)$, and $\mathcal{D}(Z) \le [F : E]\, D \le dD$. The setting of this section applies to Y over F_Y in place of Z over F. We write (P_Y, X_{P_Y}), K_{P_Y}, E_{P_Y} and ψ_Y for the objects attached there to Y, Y_{P_Y} and V_{P_Y} for the lifts of Lemma 8.3, and speak of the class of V relative to Y.

Step 1 (one Hecke step). Let V be maximal in Y and of class (N) relative to Y, put $r := \dim Y - \dim V$, and suppose that $\Omega_V > (c_B \mathcal{D}(Y))^{2^r} x_V^{(k_1+3f)2^r}$. Then there is $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E_{P_Y})$ with $\sigma Y \ne Y$ and $V \subset Y \cap \sigma Y$. Indeed, $r \ge 1$, as $Y = V$ would put V in class (C). Let $\mathcal{Y} := \mathrm{Gal}(\overline{\mathbb{Q}}/E_{P_Y}) \cdot Y_{P_Y}$, a closed and E_{P_Y}-irreducible subvariety of $\mathrm{Sh}_{K_{P_Y}}(P_Y, X_{P_Y})_{\mathbb{C}}$; it is Hodge generic, as Y_{P_Y} is and Galois conjugates of special subvarieties are special [63, proof of Lemma 7.1.2]. If $\sigma Y = \sigma' Y$ for $\sigma, \sigma' \in \mathrm{Gal}(\overline{\mathbb{Q}}/E_{P_Y})$, then $\tau := \sigma'^{-1}\sigma$ fixes Y, and τY_{P_Y} is a component of $\psi_Y^{-1}(Y)$ mapping onto Y, as ψ_Y is defined over E_{P_Y}; so $\tau Y_{P_Y} = Y_{P_Y}$ by Lemma 8.3(a). Hence distinct components σY_{P_Y} of $\mathcal{Y}$ have distinct images σY, so there are at most $M(Y)$ of them, each of degree $\deg_{\mathcal{L}_{K_{P_Y}}} Y_{P_Y} \le \deg Y$, and $\deg \mathcal{Y} \le \mathcal{D}(Y)$. As above, $E_{P_Y} \subset E_V$, so Theorem 6.11 applies to (P_Y, X_{P_Y}), the field E_{P_Y}, $\mathcal{Y}$ and V_{P_Y} with $\bar{d} = 1$, for which $b_r = (k_1 + 3f)2^r$ and $\Delta_r(\deg \mathcal{Y}) \le (c_B \mathcal{D}(Y))^{2^r}$. It gives a

special $V' \supsetneq V_{P_Y}$ in a component σY_{P_Y} of $\mathcal{Y}$. As V_{P_Y} is maximal in Y_{P_Y} (Lemma 8.3(b)), $\sigma Y_{P_Y} \neq Y_{P_Y}$; so $\sigma Y \neq Y$, and $V = \psi_Y(V_{P_Y}) \subset \psi_Y(\sigma Y_{P_Y}) = \sigma Y$.

Step 2 (intersections). Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E)$ with $\sigma Y \neq Y$, and let Y' be a component of $Y \cap \sigma Y$. Then $\dim Y' < \dim Y$, and Y' is defined over a number field, as $\mathrm{Gal}(\overline{\mathbb{Q}}/F_Y\sigma(F_Y))$ permutes the components of $Y \cap \sigma Y$. Moreover $\mathcal{D}(Y') \le c_B\,\mathcal{D}(Y)^2$: each conjugate $\tau Y'$ is a component of $\tau Y \cap \tau\sigma Y$, an intersection of two distinct conjugates of Y, so Lemma 5.2(c) at the level K gives

$$\mathcal{D}(Y') \;\le\; \sum_{A \neq B} \sum_{C \in \mathrm{Irr}(A \cap B)} \deg C \;\le\; c_B \sum_{A \neq B} \deg A \deg B \;\le\; c_B\,\mathcal{D}(Y)^2,$$

the sums running over ordered pairs of distinct conjugates of Y.

Step 3 (the recursion). For V of class (N) put $b^\sharp := (k_1 + 3f)2^z$. If $\Omega_V \le (c_B dD)^{2^z} x_V^{b^\sharp}$, then Lemma 6.12 with $\Lambda = (c_B dD)^{2^z}/C_* \ge 1$ and $b = b^\sharp$ gives $\alpha_V d_{L_V}^{\mu} \le c_{\mathrm{N}}^\sharp(\delta, z, D, d)$. Otherwise we build $Z = Y_0 \supsetneq Y_1 \supsetneq \cdots$, closed, geometrically irreducible and containing V, with $\dim Y_j \le z - j$ and $\mathcal{D}(Y_j) \le c_B^{2^j-1}(dD)^{2^j}$; V is maximal in each Y_j, as it is in Z. If V is of class (C) relative to Y_j, we stop. Otherwise $r_j := \dim Y_j - \dim V$ satisfies $1 \le r_j \le z - j$, so

$$\big(c_B\mathcal{D}(Y_j)\big)^{2^{r_j}} \le (c_B dD)^{2^{j+r_j}} \le (c_B dD)^{2^z}, \qquad x_V^{(k_1+3f)2^{r_j}} \le x_V^{b^\sharp},$$

and Step 1 gives σ with $\sigma Y_j \neq Y_j$ and $V \subset Y_j \cap \sigma Y_j$. Let Y_{j+1} be a component of $Y_j \cap \sigma Y_j$ containing V; by Step 2, $\mathcal{D}(Y_{j+1}) \le c_B\mathcal{D}(Y_j)^2 \le c_B^{2^{j+1}-1}(dD)^{2^{j+1}}$. The dimension drops at each step, and a stage of dimension $\dim V$ is V, of class (C) relative to itself; so we stop at some Y_s with $s \le z$, of dimension $z_s \le z - s$. By Lemma 8.4 for Y_s, L_V is the splitting field of T_{Y_s}. The field F_{Y_s} has degree $[E : \mathbb{Q}]M(Y_s) \le R_G M(Y_s)$, as $E \subset E_V$, and $M(Y_s) \le \mathcal{D}(Y_s) \le (c_B dD)^{2^s}$. So Lemmas 8.5 and 8.8 for Y_s over F_{Y_s} give $d_{L_V} \le e^{\beta_0(\delta)}\big(R_G^2(c_B dD)^{2^z}\big)^{1/\mu}$ and

$$\begin{aligned}\alpha_V &\le a^4(a!)^{r_U} R_G\, M(Y_s)\big(e_0^{z_s}\deg Y_s\big)^{2(z_s+1)} \le a^4(a!)^{r_U} R_G\, e_0^{2z(z+1)}\mathcal{D}(Y_s)^{2(z-s+1)} \\ &\le a^4(a!)^{r_U} R_G\, e_0^{2z(z+1)}(c_B dD)^{2^{z+1}},\end{aligned}$$

as $M(Y_s) \ge 1$ and $z - s + 1 \le 2^{z-s}$.

(d) $T_V = T_Z$ by Lemma 8.4, so L_V is the splitting field of T_Z; Lemmas 8.5 and 8.8 give the bounds.

The last assertion follows from (a)–(d), since the three classes exhaust the maximal V; in class (N), $\alpha_V \le c_{\mathrm{N}}^\sharp \le (c_{\mathrm{N}}^\sharp)^{1/\mu}$ when $\alpha_V d_{L_V}^{\mu} \le c_{\mathrm{N}}^\sharp$, as $c_{\mathrm{N}}^\sharp \ge 1$; in class (C) the bounds of (d) are at most the last two terms of $c^\sharp$, as $R_G, c_B \ge 1$ and $2(z+1) \le 2^{z+1}$. □

Corollary 8.9. *Let G be adjoint, let K satisfy* (Lv′)*, let F, d, Z, D and z be as in Theorem 8.1, and let x be a special point maximal in Z and in the image of $X^+ \times \{1\}$. If* $\mathrm{QRH}(\delta; L_V)$ *holds for $V := \{x\}$, then*

$$\mathrm{disc}(x) \;\le\; B^{-(C_0+e^2/B')}\, c^\sharp(\delta, z, D, d)^{2+\log(1/B)+\log_2(1/B)},$$

with disc *as in* [16, Def. 1] *and B, C_0 and $B' = Bc_U$ of Table 1.*

Proof. As $G \neq 1$ is adjoint, $T_V \neq 1$ (Section 2.3), and $\mathrm{disc}(x) = [K_{T_V}^m : K_{T_V}]\, d_{L_V}$ (the paragraph before Corollary 6.10). By Lemma 6.9,

$$\mathrm{disc}(x) \;\le\; B^{-(C_0+e^2/B')}\alpha_V^{1+\log(1/B)} d_{L_V}^{1+\log_2(1/B)},$$

and Theorem 8.1 gives $\alpha_V, d_{L_V} \le \Delta(V) \le c^\sharp(\delta, z, D, d)$. □

Corollaries 6.10 and 8.9 extend to every neat product level K and every connected component. For such K and each representative g_i of Section 2.2, let $J_i \subset g_i K g_i^{-1}$ be the level of Lemma 2.2 for the lattice $\rho(g_i)\widehat{\mathbb{Z}}^{n_\rho}$, which satisfies (Lv), and let g_3 be as in Lemma 2.2.

Corollary 8.10. *Let $G \neq 1$ be adjoint, K a neat product level and $\delta \in (0, 1/2]$. There are $A, \varepsilon > 0$, effectively computable from (G, X, K, ρ), δ and the representatives g_i, such that for every special point $x = [h, g_i]$ with $h \in X^+$, $T := \mathrm{MT}(h)$ with splitting field L satisfying $\mathrm{QRH}(\delta; L)$, and every number field $F \supset E(G, X)$, $[F(x) : F] \geq A \operatorname{disc}(x)^{\varepsilon}/[F : \mathbb{Q}]$, with $\operatorname{disc}(x) := [K_T^m : K_T]\, d_L$ and $K_T := T(\mathbb{A}_f) \cap g_i K g_i^{-1}$. If moreover x is maximal in a closed geometrically irreducible $Z \subset \mathrm{Sh}_K(G, X)$ of dimension z and degree at most D, defined over F with $[F : \mathbb{Q}] = d$, then $\operatorname{disc}(x)$ is at most the right side of Corollary 8.9 at $(\delta, z, g_3 D, g_3 d)$ for the level J_i.*

Proof. Here $\operatorname{disc}(x)$ is the discriminant of [16, Def. 1] of the translate $[h, 1]$ of x at the level $g_i K g_i^{-1}$ (Section 2.2). Let $x' := [h, 1]$ at J_i. The map to the level $g_i K g_i^{-1}$ is defined over $E(G, X)$, has fibers of at most g_3 points and maps x' to the translate of x, and $\operatorname{disc}(x') \geq \operatorname{disc}(x)$, as T and L are unchanged and $T(\mathbb{A}_f) \cap J_i \subset K_T$. So under $\mathrm{QRH}(\delta; L)$, Corollary 6.10 at J_i gives, for every number field $F \supset E(G, X)$,

$$[F(x) : F] \;\geq\; \frac{A_i}{g_3 [F : \mathbb{Q}]} \operatorname{disc}(x)^{\varepsilon_i},$$

with $A_i := C_* B^{\varepsilon_i (C_0 + e^2/B')}$ and ε_i formed at J_i. If moreover x is maximal in Z as in the statement, then a component of the preimage of the translate of Z through x' has degree at most $g_3 D$, is defined over a field of degree at most $g_3 d$ and contains x' as a maximal special point (Lemma 2.2 and its proof), so Corollary 8.9 at J_i bounds $\operatorname{disc}(x) \leq \operatorname{disc}(x')$ by its right side at $(\delta, z, g_3 D, g_3 d)$. As $\operatorname{disc}(x) \geq 1$, the least A_i/g_3 and the least ε_i over the finitely many i give A and ε. □

9. Every Shimura datum

Let (G, X) be any Shimura datum, $K = \prod_p K_p$ a neat product level, $\rho\colon G \to \mathrm{GL}_{n_\rho}$ faithful with $\rho(K) \subset \mathrm{GL}_{n_\rho}(\widehat{\mathbb{Z}})$, $\lambda\colon G \to G^{\mathrm{ad}}$ the adjoint map and $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ the adjoint datum. The morphism $f\colon \mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$, for a level $K^{\mathrm{ad}} \supset \lambda(K)$, maps special subvarieties to special subvarieties, and a subvariety is special if and only if its image is [34, Prop. 2.2]. But f need not be injective on connected components, and then a special subvariety maximal in Z can have an image that is not maximal in $f(Z)$. We choose an adjoint level for which the restriction of f to each component is the quotient by a finite group of automorphisms, and we recurse on the dimension at that fixed level.

9.1. **The level.** For the rest of this section we replace K by the level K' of Lemma 2.2, so that K_3 lies in the level-3 congruence subgroup for ρ.

An integral model. Let $\mathcal{G}$ be the schematic closure of $\rho(G)$ in $\mathrm{GL}_{n_\rho, \mathbb{Z}}$. For some integer M_0 it is reductive over $\mathbb{Z}[1/M_0]$: its non-smooth locus lies over finitely many primes, as $\mathcal{G} \to \operatorname{Spec} \mathbb{Z}$ is of finite presentation with smooth generic fiber [97, Tag 054K], and then [22, Props. 3.1.9 and 3.1.12] apply. Let $\mathcal{S}$ be the finite set of primes consisting of 3, the primes dividing M_0, and the p with $K_p \neq \mathcal{G}(\mathbb{Z}_p)$. For $p \notin \mathcal{S}$ put $\mathcal{G}_p := \mathcal{G} \otimes \mathbb{Z}_p$, let $\mathcal{Z}_p$ be its center, a group of multiplicative type with generic fiber $Z(G)$ [22, Thm. 3.3.4], and $\mathcal{G}_p^{\mathrm{ad}} := \mathcal{G}_p/\mathcal{Z}_p$, an adjoint semisimple $\mathbb{Z}_p$-group with generic fiber G^{ad}; the sequence $1 \to \mathcal{Z}_p \to \mathcal{G}_p \to \mathcal{G}_p^{\mathrm{ad}} \to 1$ is exact as fppf sheaves [22, Lemma 3.3.1(2), Cor. 3.3.5].

Lemma 9.1. *Let R be a ring and $1 \to \mathcal{Z} \to \mathcal{G} \to \mathcal{G}' \to 1$ an exact sequence of fppf sheaves of groups over R with $\mathcal{Z}$ central. Then $\mathcal{G}(R) \to \mathcal{G}'(R) \xrightarrow{\partial} H^1_{\mathrm{fppf}}(R, \mathcal{Z})$ is exact, and ∂ is a homomorphism.*

Proof. For $y \in \mathcal{G}'(R)$ the sheaf P_y of local lifts of y is a $\mathcal{Z}$-torsor: $\mathcal{Z}$ acts simply transitively, and P_y has sections locally, as $\mathcal{G} \to \mathcal{G}'$ is an epimorphism. Let $\partial(y)$ be its class [97, Tag 03AJ]; it vanishes if and only if $P_y(R) \neq \emptyset$. The map $(a, b) \mapsto ab$ identifies $P_y \wedge^{\mathcal{Z}} P_{y'}$ with $P_{yy'}$, which is well defined because $\mathcal{Z}$ is central, and the class of a contracted product is the sum of the classes: the group law on H^1 is the contracted product, since on a cover trivializing both torsors the contracted product has the product Čech cocycle. □

The adjoint level. Put

$$K^{\mathrm{ad}} := \prod_{p \in \mathcal{S}} \lambda(K_p) \times \prod_{p \notin \mathcal{S}} \mathcal{G}_p^{\mathrm{ad}}(\mathbb{Z}_p) \ \subset G^{\mathrm{ad}}(\mathbb{A}_f).$$

When $Z(G)$ is not connected, $\lambda(K)$ itself need not be open: for $W = M_2(\mathbb{Q})$ with the quadratic form det, $\mathrm{SO}(W) = \{(A, B) \in \mathrm{GL}_2^2 : \det A = \det B\}/\mathbb{G}_{\mathrm{m}}$ has center μ_2 and adjoint group PGL_2^2, and the image of $\mathrm{SO}(W)(\mathbb{Z}_p)$ has index 2 in $\mathrm{PGL}_2(\mathbb{Z}_p)^2$ for odd p.

Lemma 9.2. *(i) K^{ad} is a compact open subgroup of $G^{\mathrm{ad}}(\mathbb{A}_f)$ containing $\lambda(K)$, and it normalizes K for the action of G^{ad} on G by inner automorphisms.*

(ii) Let $\rho^{\mathrm{ad}} := \mathrm{Ad}\colon G^{\mathrm{ad}} \to \mathrm{GL}(\mathrm{Lie}\, G^{\mathrm{der}})$ and $\Lambda := \mathrm{Lie}\, G^{\mathrm{der}} \cap \mathrm{End}(\mathbb{Z}^{n_\rho})$, via $d\rho$. Then $\rho^{\mathrm{ad}}(K^{\mathrm{ad}})$ preserves $\Lambda \otimes \widehat{\mathbb{Z}}$, K_3^{ad} lies in the level-3 congruence subgroup for $(\rho^{\mathrm{ad}}, \Lambda)$, and K^{ad} is neat. So K^{ad} satisfies (Lv) for $(\rho^{\mathrm{ad}}, \Lambda)$.

(iii) $\lambda^{-1}(K^{\mathrm{ad}}) = Z(G)(\mathbb{A}_f) \cdot K$.

(iv) If $Z(G)$ is connected, then $K^{\mathrm{ad}} = \lambda(K)$.

Proof. (i) Each $\lambda(K_p)$ is compact open, as λ is smooth, and for $p \notin \mathcal{S}$ the factor is $\mathcal{G}^{\mathrm{ad}}(\mathbb{Z}_p)$ for the model $\mathcal{G}/Z(\mathcal{G})$ of G^{ad} over $\mathbb{Z}[1/M_0]$; so K^{ad} is compact open. At $p \notin \mathcal{S}$, $\lambda(\mathcal{G}_p(\mathbb{Z}_p)) \subset \mathcal{G}_p^{\mathrm{ad}}(\mathbb{Z}_p)$. At $p \in \mathcal{S}$, $\mathrm{Int}(\lambda(k)) = \mathrm{Int}(k)$ preserves K_p; at $p \notin \mathcal{S}$ the conjugation action of $\mathcal{G}_p$ on itself factors through $\mathcal{G}_p^{\mathrm{ad}}$ [22, proof of Cor. 3.3.5], so $\mathcal{G}_p^{\mathrm{ad}}(\mathbb{Z}_p)$ preserves $\mathcal{G}_p(\mathbb{Z}_p) = K_p$.

(ii) At $p \in \mathcal{S}$, $\mathrm{Ad}(\lambda(k))$ is conjugation by $\rho(k) \in \mathrm{GL}_{n_\rho}(\mathbb{Z}_p)$, which preserves $\mathrm{End}(\mathbb{Z}_p^{n_\rho})$ and $\mathrm{Lie}\, G^{\mathrm{der}} \otimes \mathbb{Q}_p$. At $p \notin \mathcal{S}$, $\Lambda \otimes \mathbb{Z}_p$ is the Lie algebra of the derived group $\mathcal{G}_p^{\mathrm{der}}$ [22, Thm. 5.3.1], a smooth closed subgroup of $\mathrm{GL}_{n_\rho, \mathbb{Z}_p}$, whose Lie algebra is saturated in $\mathrm{End}(\mathbb{Z}_p^{n_\rho})$; and $\mathcal{G}_p^{\mathrm{ad}}$ acts on $\mathcal{G}_p^{\mathrm{der}}$ by (i). For $k \in K_3$ write $\rho(k) = 1 + 3M$ and $\rho(k)^{-1} = 1 + 3M'$ with M, M' integral; then $\rho(k) Y \rho(k)^{-1} - Y = 3(MY + YM' + 3MYM')$ lies in $3\,\mathrm{End}(\mathbb{Z}_3^{n_\rho}) \cap (\mathrm{Lie}\, G^{\mathrm{der}} \otimes \mathbb{Q}_3) = 3(\Lambda \otimes \mathbb{Z}_3)$. For neatness: the eigenvalues of $\mathrm{Ad}(\lambda(k_3))$ are ratios of eigenvalues of $\rho(k_3)$, which are $\equiv 1$ modulo 3 in $\overline{\mathbb{Z}}_3$, so every element ξ of the group they generate satisfies $\xi \equiv 1 \bmod 3$, and a root of unity $\equiv 1 \bmod 3$ is 1 [81, p. 5]. So every element of K^{ad} is neat in the sense of [81, 0.6], and so is every conjugate of K^{ad} under $G^{\mathrm{ad}}(\mathbb{A}_f)$.

(iii) "$\supset$" is clear, and at $p \in \mathcal{S}$ the kernel of λ on $G(\mathbb{Q}_p)$ is $Z(G)(\mathbb{Q}_p)$. Let $p \notin \mathcal{S}$, $x \in G(\mathbb{Q}_p)$ with $y := \lambda(x) \in \mathcal{G}_p^{\mathrm{ad}}(\mathbb{Z}_p)$. By Lemma 9.1, applied over $\mathbb{Z}_p$ and $\mathbb{Q}_p$, the class of y in $H^1(\mathbb{Z}_p, \mathcal{Z}_p)$ maps to 0 in $H^1(\mathbb{Q}_p, Z(G))$, as y lifts to x. We show that $H^1(\mathbb{Z}_p, \mathcal{Z}_p) \to H^1(\mathbb{Q}_p, Z(G))$ is injective; then $y = \lambda(k)$ with $k \in \mathcal{G}_p(\mathbb{Z}_p)$, and $x \in Z(G)(\mathbb{Q}_p) K_p$. Let $\mathcal{T} \subset \mathcal{G}_p$ be a maximal torus over $\mathbb{Z}_p$; it exists by [22, Thm. 3.2.6], Lang's theorem and Hensel's lemma, and contains $\mathcal{Z}_p$ [22, Cor. 3.3.5]. Let $\mathcal{Z}_p^\circ$ be the maximal subtorus of $\mathcal{Z}_p$ [22, App. B], $\mathcal{T}_1 := \mathcal{T}/\mathcal{Z}_p^\circ$ and $\mathcal{T}^{\mathrm{ad}} := \mathcal{T}/\mathcal{Z}_p$, so that $\lambda|_{\mathcal{T}} = \varkappa \circ \pi$ with $\varkappa\colon \mathcal{T}_1 \to \mathcal{T}^{\mathrm{ad}}$ an isogeny. Torsors under a smooth affine $\mathbb{Z}_p$-group with connected special fiber are trivial (Lang's theorem and Hensel's

lemma), so $H^1(\mathbb{Z}_p, \mathcal{T}) = H^1(\mathbb{Z}_p, \mathcal{Z}_p^\circ) = 0$ and $H^1(\mathbb{Z}_p, \mathcal{Z}_p) = \mathcal{T}^{\mathrm{ad}}(\mathbb{Z}_p)/\lambda(\mathcal{T}(\mathbb{Z}_p))$, which maps to $T^{\mathrm{ad}}(\mathbb{Q}_p)/\lambda(T(\mathbb{Q}_p)) \subset H^1(\mathbb{Q}_p, Z(G))$. Let $u \in \mathcal{T}^{\mathrm{ad}}(\mathbb{Z}_p)$ with $u = \lambda(t)$, $t \in T(\mathbb{Q}_p)$. For every character χ of $\mathcal{T}_1$ over an unramified splitting field, some multiple $m\chi$ is a character χ' of $\mathcal{T}^{\mathrm{ad}}$, so $|\chi(\pi(t))|^m = |\chi'(u)| = 1$; hence $\pi(t) \in \mathcal{T}_1(\mathbb{Z}_p)$. As $H^1(\mathbb{Z}_p, \mathcal{Z}_p^\circ) = 0$, $\pi(t) = \pi(t')$ with $t' \in \mathcal{T}(\mathbb{Z}_p)$, so $t'^{-1}t \in Z(G)(\mathbb{Q}_p)$ and $u = \lambda(t') \in \lambda(\mathcal{T}(\mathbb{Z}_p))$. Adelically, if $x = (z_p k_p)_p$, then $z_p \in Z(G)(\mathbb{Q}_p) \cap \mathcal{G}_p(\mathbb{Z}_p)$ for almost all p, so $(z_p) \in Z(G)(\mathbb{A}_f)$.

(iv) If $Z(G)$ is connected, $\mathcal{Z}_p$ is a torus for $p \notin \mathcal{S}$, $H^1(\mathbb{Z}_p, \mathcal{Z}_p) = 0$, and $\lambda(\mathcal{G}_p(\mathbb{Z}_p)) = \mathcal{G}_p^{\mathrm{ad}}(\mathbb{Z}_p)$ by Lemma 9.1. □

9.2. **The adjoint map on one component.** Let $S_i = [X^+ \times g_i]$ be a connected component of $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$, $\Gamma_i := G(\mathbb{Q})_+ \cap g_i K g_i^{-1}$, so that $S_i = \lambda(\Gamma_i)\backslash X^+$, and $\Gamma_i' := G^{\mathrm{ad}}(\mathbb{Q})_+ \cap \lambda(g_i)K^{\mathrm{ad}}\lambda(g_i)^{-1}$, so that the component of $\mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})_{\mathbb{C}}$ containing $f(S_i)$ is $\Gamma_i'\backslash X^+$. Put $\mathfrak{G}_i := \Gamma_i'/\lambda(\Gamma_i)$,

$$h_K := \max_i |\mathfrak{G}_i|,$$

and let $\mathrm{Cl}(K) := Z(G)(\mathbb{Q})\backslash Z(G)(\mathbb{A}_f)/(Z(G)(\mathbb{A}_f) \cap K)$, which is finite.

Proposition 9.3. *(a) $\lambda(\Gamma_i)$ is a normal subgroup of finite index of Γ_i', both are neat and arithmetic, and $f|_{S_i} \colon S_i \to \Gamma_i'\backslash X^+$ is the quotient by $\mathfrak{G}_i$, which acts freely. So h_K is finite.*

(b) Each element of $\mathfrak{G}_i$ is an automorphism of S_i that extends to the Baily–Borel compactification and preserves $\mathcal{L}_K$. If S_i is defined over $F \supset E(G, X)$, all these automorphisms are defined over one extension $F_{\mathfrak{G}}$ of F of degree at most $h_K!$. Each $\bar\gamma \in \Gamma_i' \cap \lambda(G(\mathbb{Q}))$ acts as right translation by an element of $Z(G)(\mathbb{A}_f)$, which is defined over $E(G, X)$; so $F_{\mathfrak{G}} = F$ if $\Gamma_i' \subset \lambda(G(\mathbb{Q}))$ for every i.

(c) $f^*\mathcal{L}_{K^{\mathrm{ad}}} = \mathcal{L}_K$ on the compactification of S_i. Hence, for every closed irreducible $Y \subset S_i$, we have $\deg_{\mathcal{L}_{K^{\mathrm{ad}}}} f(Y) \le \deg Y$ and $\deg Y \le h_K \deg_{\mathcal{L}_{K^{\mathrm{ad}}}} f(Y)$.

(d) Let $\mathrm{Sel}_{\mathcal{S}}(Z(G))$ be the set of $c \in H^1(\mathbb{Q}, Z(G))$ with $c_\infty = 0$, $c_p = 0$ for $p \in \mathcal{S}$, and c_p in the image of $H^1(\mathbb{Z}_p, \mathcal{Z}_p)$ for $p \notin \mathcal{S}$. Then $|\mathfrak{G}_i| \le |\mathrm{Cl}(K)| \cdot |\mathrm{Sel}_{\mathcal{S}}(Z(G))|$. If $\mathrm{Sel}_{\mathcal{S}}(Z(G)) = 0$, which holds if $H^1(\mathbb{Q}, Z(G)) = 0$ or if $Z(G)$ is split, then $\Gamma_i' \subset \lambda(G(\mathbb{Q}))$ and $|\mathfrak{G}_i| \le |\mathrm{Cl}(K)|$.

Proof. Here $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap \lambda^{-1}(G^{\mathrm{ad}}(\mathbb{R})^+)$, as in Section 2.1.

(a) $\lambda(\Gamma_i) \subset \Gamma_i'$ by Lemma 9.2(i). The group Γ_i' has finite index in the congruence subgroup $G^{\mathrm{ad}}(\mathbb{Q}) \cap \lambda(g_i)K^{\mathrm{ad}}\lambda(g_i)^{-1}$, as $G^{\mathrm{ad}}(\mathbb{R})^+$ has finite index in $G^{\mathrm{ad}}(\mathbb{R})$ [69, p. 54], and $\lambda(\Gamma_i)$ is arithmetic [69, Prop. 3.2]; two arithmetic subgroups are commensurable, so $[\Gamma_i' : \lambda(\Gamma_i)]$ is finite. They are neat by Lemma 9.2(ii) and [81, 0.5–0.6]. Let $\bar\gamma \in \Gamma_i'$, $\bar\gamma = \lambda(g_i)\kappa\lambda(g_i)^{-1}$ with $\kappa \in K^{\mathrm{ad}}$. On $G(\mathbb{A}_f)$, $\mathrm{Int}(\bar\gamma)$ preserves $g_i K g_i^{-1}$ by Lemma 9.2(i); it preserves $G(\mathbb{Q})$, and $G(\mathbb{Q})_+$ because $\lambda(\mathrm{Int}(\bar\gamma)\gamma') = \bar\gamma\lambda(\gamma')\bar\gamma^{-1}$. So $\mathrm{Int}(\bar\gamma)\Gamma_i = \Gamma_i$, and $\lambda(\Gamma_i)$ is normal in Γ_i'. As Γ_i' is neat, it acts freely on X^+, and $f|_{S_i}$ is the Galois covering with group $\mathfrak{G}_i$.

(b) Each $\bar\gamma \in \Gamma_i'$ normalizes $\lambda(\Gamma_i)$, so it induces an automorphism of S_i, which extends to the compactification and preserves the extension of the canonical bundle. $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ acts on the finite group $\mathfrak{G}_i = \mathrm{Aut}(S_i/f(S_i))$ by conjugation, and the kernel of this action has index at most $|\mathrm{Aut}\,\mathfrak{G}_i| \le h_K!$. If $\bar\gamma = \lambda(\gamma)$ with $\gamma \in G(\mathbb{Q})$, then $\gamma \in G(\mathbb{Q})_+$, and $\gamma \in g_i\lambda^{-1}(K^{\mathrm{ad}})g_i^{-1} = Z(G)(\mathbb{A}_f)g_i K g_i^{-1}$ by Lemma 9.2(iii); writing $\gamma = zk$ we get $[\bar\gamma^{-1}x, g_i] = [x, \gamma g_i] = [x, g_i z k']$ with $k' \in K$, as z is central. So $\bar\gamma$ acts as right translation by z^{-1}, a Hecke translation, defined over $E(G, X)$.

(c) $f|_{S_i}$ is étale between quotients of X^+ by neat groups, so it pulls top-degree forms back to top-degree forms; the two extensions agree in codimension one on the normal compactification, as logarithmic forms pull back to logarithmic forms, and so are isomorphic (compare [63, Prop. 5.3.2]). The projection formula and $\deg(f|_Y) \le |\mathfrak{G}_i|$ give the inequalities.

(d) Let $\partial\colon G^{\mathrm{ad}}(\mathbb{Q}) \to H^1(\mathbb{Q}, Z(G))$ be the connecting map of Lemma 9.1, a homomorphism with kernel $\lambda(G(\mathbb{Q}))$. For each place v, $\partial_v(\bar\gamma) = \partial_v(\kappa_v)$, since ∂_v kills $\lambda(G(\mathbb{Q}_v)) \ni \lambda(g_{i,v})$. So $\partial_p(\bar\gamma) = 0$ for $p \in \mathcal{S}$, as $\kappa_p \in \lambda(K_p)$; $\partial_p(\bar\gamma)$ comes from $H^1(\mathbb{Z}_p, \mathcal{Z}_p)$ for $p \notin \mathcal{S}$, as $\kappa_p \in \mathcal{G}_p^{\mathrm{ad}}(\mathbb{Z}_p)$; and $\partial_\infty(\bar\gamma) = 0$, as $G(\mathbb{R})^+ \to G^{\mathrm{ad}}(\mathbb{R})^+$ is onto. Hence $\Gamma_i'/(\Gamma_i' \cap \lambda(G(\mathbb{Q})))$ embeds in $\mathrm{Sel}_{\mathcal{S}}(Z(G))$. Next, $\bar\gamma = \lambda(zk) \mapsto [z]$, with z, k as in (b), is a well-defined homomorphism $(\Gamma_i' \cap \lambda(G(\mathbb{Q})))/\lambda(\Gamma_i) \to \mathrm{Cl}(K)$: γ is determined up to $Z(G)(\mathbb{Q})$, and then z up to $Z(G)(\mathbb{A}_f) \cap K$. It is injective: if $z = rw$ with $r \in Z(G)(\mathbb{Q})$ and $w \in Z(G)(\mathbb{A}_f) \cap K$, then $r^{-1}\gamma = wk \in \Gamma_i$. If $Z(G)$ is split, a product of copies of $\mathbb{G}_{\mathrm{m}}$ and of groups μ_m, then $H^1(\mathbb{Q}, \mathbb{G}_{\mathrm{m}}) = 0$ and $H^1(\mathbb{Q}, \mu_m) = \mathbb{Q}^\times/\mathbb{Q}^{\times m}$, and a class a in $\mathrm{Sel}_{\mathcal{S}}$ has $v_p(a) \equiv 0 \bmod m$ for $p \notin \mathcal{S}$ and is an m-th power in $\mathbb{Q}_p$ for $p \in \mathcal{S}$, so $a \in \pm\mathbb{Q}^{\times m}$; for m odd -1 is an m-th power, and for m even $c_\infty = 0$ excludes -1. Thus $\mathrm{Sel}_{\mathcal{S}}(Z(G)) = 0$. □

For example, for $\mathrm{SO}(W)$ with W of signature $(2, m)$, $m \ge 2$ even, $Z(G) = \mu_2$ is split, and $h_K \le 2^{|\mathcal{S}|-1}$, since $-1 \notin K_3$.

9.3. **Where maximality is lost.**

Lemma 9.4. *Let $Z \subset S_i$ be closed and irreducible and $V \subset Z$ special and maximal in Z. If $f(V)$ is not maximal in $f(Z)$, there is $\theta \in \mathfrak{G}_i$ with $\theta Z \ne Z$ and $V \subset Z \cap \theta Z$.*

Proof. Let $V^\sharp$ be special with $f(V) \subsetneq V^\sharp \subset f(Z)$, and $\widetilde{V}$ a component of $f^{-1}(V^\sharp)$ containing V. It is special [63, p. 14], of dimension $\dim V^\sharp > \dim V$ as f is finite, and lies in S_i, being connected. By Proposition 9.3(a), $\widetilde{V} \subset f^{-1}(f(Z)) \cap S_i = \bigcup_{\theta \in \mathfrak{G}_i} \theta Z$, so $\widetilde{V} \subset \theta Z$ for some θ. If $\theta Z = Z$, $\widetilde{V}$ would contradict the maximality of V. □

9.4. **The theorem.** For a special subvariety V of $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$, $f(V)$ is special with torus T_V, the connected center of $\lambda(H_V)$, as in the introduction. We put

$$\alpha_V := \alpha_{f(V)} \text{ in } \mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \text{ (Definition 2.1)}, \qquad \Delta(V) := \max\bigl(\deg V,\ d_{L_V},\ \alpha_V\bigr),$$

with $\deg V$ taken for $\mathcal{L}_K$ at the original level. For V at the original level, where $\lambda(K)$ need not lie in K^{ad}, $\alpha_V := \alpha_{f(V')}$ for a component V' of the preimage of V in $\mathrm{Sh}_{K'}(G, X)$; as K' is normal in K, these components are the $V'k$, $k \in K$, and $f(V'k) = f(V')\lambda(k)$ with $\lambda(k)$ normalizing K^{ad}, so $\alpha_{f(V')}$ does not depend on V'. If G is adjoint and K satisfies (Lv), then $K^{\mathrm{ad}} = K$ and these are the invariants of Section 2 for the representation ρ^{ad} and the lattice Λ; they are those for ρ formed with the same constant B, as the groups $K_{T_V,p}$ and $K^m_{T_V,p}$ do not depend on the representation; this B is admissible for ρ as well if a also bounds the indices of [102, pp. 21–22] for ρ, as $a = |\mathrm{GL}_{\dim G}(\mathbb{F}_3)|$ does when $n_\rho \le \dim G$. At any compact open product level containing K^{ad}, $\alpha_{f(V)}$ is at most this α_V.

Let $\bar g_1, \dots, \bar g_m$ represent $G^{\mathrm{ad}}(\mathbb{Q})_+ \backslash G^{\mathrm{ad}}(\mathbb{A}_f)/K^{\mathrm{ad}}$, a finite set [69, Lemma 5.12], and let $\bar c^\sharp(\delta, z, D, d)$ and $\bar c^{\mathrm{ad}}_{\deg}(z, D)$ be the maxima over t of the constants $c^\sharp$ and $c^{\mathrm{ad}}_{\deg}$ of Sections 7 and 8 for $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ at the level $\bar g_t K^{\mathrm{ad}} \bar g_t^{-1}$, with ρ^{ad} and the lattice $\mathrm{Ad}(\bar g_t)(\Lambda \otimes \widehat{\mathbb{Z}})$ (Section 2), and with $n := \dim \mathrm{Sh}_K(G, X)$; put $e_0 := e^*(n)$. The constant B of Table 1 is the same at all these levels (we take $a := |\mathrm{GL}_{\dim G^{\mathrm{der}}}(\mathbb{F}_3)|$, which bounds the indices of [102, pp. 21–22]

for every lattice $\mathrm{Ad}(\bar{g}_t)(\Lambda \otimes \widehat{\mathbb{Z}})$). Put $h_{\mathfrak{G}} := 1$ if $\Gamma_i' \subset \lambda(G(\mathbb{Q}))$ for every i, and $h_{\mathfrak{G}} := h_K!$ otherwise, and define

$$D_0 := g_3 D, \quad d_0 := g_3 d\, h_{\mathfrak{G}}, \qquad D_{j+1} := e_0^{2n} D_j^2, \quad d_{j+1} := e_0^{2n} D_j^2\, d_j.$$

Theorem 9.5. *Let (G, X) be a Shimura datum with $G^{\mathrm{ad}} \neq 1$, K a neat product level and ρ as above, $F \supset E(G, X)$ a number field with $[F : \mathbb{Q}] = d$, $Z \subset \mathrm{Sh}_K(G, X)$ closed and geometrically irreducible, defined over F, of degree D and dimension z, and V a special subvariety maximal in Z. Let K' be the level of Lemma 2.2, and K^{ad}, h_K, $h_{\mathfrak{G}}$ as above, built from K'; Corollary C.8 bounds h_K in closed form, and $h_{\mathfrak{G}} \leq h_K!$.*

(a) $\deg V \leq h_K\, \bar{c}^{\mathrm{ad}}_{\deg}(z, D_z)$, and if $\mathrm{QRH}(\delta; L_V)$ holds, then $\Delta(V) \leq h_K\, \bar{c}^{\sharp}(\delta, z, D_z, d_z)$.

(b) If $\lambda(K)$ is open in $G^{\mathrm{ad}}(\mathbb{A}_f)$, for instance if $Z(G)$ is connected, the same holds with $K^{\mathrm{ad}} := \lambda(K)$ and h_K, $h_{\mathfrak{G}}$ computed at the level K itself, with $D_0 := D$, $d_0 := d\, h_{\mathfrak{G}}$, and with $\bar{c}^{\sharp}$, $\bar{c}^{\mathrm{ad}}_{\deg}$ the maxima for $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ at the level that Lemma 2.2 makes from $\lambda(K)$ for $(\rho^{\mathrm{ad}}, \Lambda)$, evaluated at $(g_3^{\mathrm{ad}} D_z, g_3^{\mathrm{ad}} d_z)$, where $g_3^{\mathrm{ad}} := |\mathrm{GL}_{\dim G^{\mathrm{der}}}(\mathbb{F}_3)|$. If $\lambda(K)$ satisfies (Lv′) for $(\rho^{\mathrm{ad}}, \Lambda)$, the constants are those at the level $\lambda(K)$ itself, evaluated at (D_z, d_z) (Theorem 8.1).

In either part, if $h_K = 1$, the bounds hold with (D_0, d_0) in place of (D_z, d_z).

If $Z(G)$ is connected, $K^{\mathrm{ad}} = \lambda(K') \subset \lambda(K)$ (Lemma 9.2(iv)), so the bound for α_V holds with α_V computed at $\lambda(K)$; for $\mathcal{A}_g$ at $K(N)$, $N \geq 3$, this is the level of Theorem 1.4(i). If G is adjoint, $K^{\mathrm{ad}} = K' \subset K$, so α_V may be computed at K.

Proof. (a) Lemma 2.2 gives $Z' \subset \mathrm{Sh}_{K'}(G, X)$, geometrically irreducible over a field F' of degree at most $g_3 d$, with $\deg_{\mathcal{L}_{K'}} Z' \leq g_3 D$, and V' maximal in Z' with $\pi(V') = V$, $\deg V \leq \deg_{\mathcal{L}_{K'}} V'$ and $T_{V'} = T_V$. Let S_i be the component containing Z'. Put $Z^{(0)} := Z'$, defined over $F_0 := F' F_{\mathfrak{G}}$, of degree at most d_0 over $\mathbb{Q}$.

Given $Z^{(j)} \supset V'$ closed and irreducible, defined over F_j with $[F_j : \mathbb{Q}] \leq d_j$, and of degree at most D_j: V' is maximal in $Z^{(j)} \subset Z'$. If $f(V')$ is maximal in $f(Z^{(j)})$, stop. Otherwise Lemma 9.4 gives $\theta \in \mathfrak{G}_i$ with $\theta Z^{(j)} \neq Z^{(j)}$ and $V' \subset Z^{(j)} \cap \theta Z^{(j)}$, and we let $Z^{(j+1)}$ be a component of this intersection containing V'; $\dim Z^{(j+1)} < \dim Z^{(j)}$. Put $\mathcal{L} := \mathcal{L}_{K'}|_{\overline{S}_i}$. As θ preserves $\mathcal{L}$ (Proposition 9.3(b)), $\theta Z^{(j)}$ has $\mathcal{L}$-degree at most D_j, and by Lemma 5.2(b) at the neat level K', with $e = e_0$, the components of $Z^{(j)} \cap \theta Z^{(j)}$ have total $\mathcal{L}$-degree at most $e_0^{2n} D_j^2$. So $\deg_{\mathcal{L}} Z^{(j+1)} \leq D_{j+1}$, and $Z^{(j)} \cap \theta Z^{(j)}$ has at most $e_0^{2n} D_j^2$ components. It is defined over F_j, since θ is defined over $F_{\mathfrak{G}} \subset F_j$, so $Z^{(j+1)}$ is defined over an extension F_{j+1} of F_j of degree at most $e_0^{2n} D_j^2$. The recursion stops after $j^* \leq z - \dim V \leq z$ steps, since $Z^{(j)} = V'$ once $\dim Z^{(j)} = \dim V$. If $h_K = 1$, every $\mathfrak{G}_i$ is trivial, so by Lemma 9.4 it stops at $j^* = 0$.

Put $W := f(Z^{(j^*)})$; then $f(V')$ is maximal in W. Let $\bar{g}_t$ be the representative for which W lies in the image of $X^+ \times \{\bar{g}_t\}$ in $\mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})_{\mathbb{C}}$. The level K^{ad} satisfies (Lv) for $(\rho^{\mathrm{ad}}, \Lambda)$ (Lemma 9.2(ii)), so $\bar{g}_t K^{\mathrm{ad}} \bar{g}_t^{-1}$ satisfies it for the lattice $\mathrm{Ad}(\bar{g}_t)(\Lambda \otimes \widehat{\mathbb{Z}})$, and we apply Theorem 8.1 at that level to the translates of W and $f(V')$ by $\bar{g}_t^{-1}$ (Section 2); the translation changes no degree, field of definition, T_V, L_V or α_V. The variety W is geometrically irreducible, defined over $F_{j^*} \supset E(G, X) \supset E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ [30, 2.2.1], of dimension at most z and degree at most D_{j^*} by Proposition 9.3(c). This bounds $\deg f(V')$, d_{L_V} and α_V, and $\deg V \leq \deg_{\mathcal{L}_{K'}} V' \leq h_K \deg f(V')$ by Proposition 9.3(c). The constants

are non-decreasing in z, D and d, and D_j, d_j increase with j, so the values at $j = z$ bound all cases, and those at $j = 0$ if $h_K = 1$.

(b) When $\lambda(K)$ is open, the argument runs at the level K with $K^{\mathrm{ad}} := \lambda(K)$: Proposition 9.3(a)–(d) and Lemma 9.4 hold with the same proofs, $\lambda(K)$ being neat [81, 0.6] and normalizing K, and $\lambda^{-1}(\lambda(K)) = Z(G)(\mathbb{A}_f)K$. Theorem 8.1 is applied at the level $\lambda(K)$ through Lemma 2.2 for $(G^{\mathrm{ad}}, X^{\mathrm{ad}}, \lambda(K), \rho^{\mathrm{ad}})$, which costs the factor g_3^{ad} and can only increase α. Here too the constants at a level K'' are the maxima over the levels $\bar{g}_t K'' \bar{g}_t^{-1}$, with the lattices $\mathrm{Ad}(\bar{g}_t)(\Lambda \otimes \widehat{\mathbb{Z}})$, for $\bar{g}_t$ representing $G^{\mathrm{ad}}(\mathbb{Q})_+ \backslash G^{\mathrm{ad}}(\mathbb{A}_f)/K''$. □

Proof of Theorem 1.1. If $G^{\mathrm{ad}} = 1$, then G is a torus, $\mathrm{Sh}_K(G, X)$ is finite and $V = Z$ is a point with $T_V = 1$, so $\deg V = d_{L_V} = \alpha_V = 1$, and we put $c_{\deg}(D) = c(\delta, D, d) := 1$. Otherwise a special point x has $\lambda(\mathrm{MT}(x)) \neq 1$, so $n_T \geq 2$ (Section 2.2) and $s_\rho, t_\rho \geq 1$ for $(G^{\mathrm{ad}}, X^{\mathrm{ad}}, \rho^{\mathrm{ad}})$, and Theorem 9.5(a) with $z \leq n$ and the monotonicity of the constants give the bounds of Theorem 1.1 with $h_K\, \bar{c}^{\mathrm{ad}}_{\deg}(n, D_n)$ and $h_K\, \bar{c}^{\sharp}(\delta, n, D_n, d_n)$. Let H_K be the integer part of the right side of Corollary C.8 for the level K' of Lemma 2.2, and let d_n^* be d_n computed from $d_0 := g_3 d\, H_K!$. As $h_K \leq H_K$, $h_{\mathfrak{G}} \leq h_K! \leq H_K!$ and the constants are non-decreasing in d, Theorem 1.1 holds with

$$c_{\deg}(D) := H_K\, \bar{c}^{\mathrm{ad}}_{\deg}(n, D_n), \qquad c(\delta, D, d) := H_K\, \bar{c}^{\sharp}(\delta, n, D_n, d_n^*),$$

and these are effectively computable, H_K being read off the character group of $Z(G)$, ρ and the level. Smaller bounds for h_K and $h_{\mathfrak{G}}$, such as those in Theorem 9.9, may be used in place of H_K and $H_K!$. The bounds in the last sentence of the theorem are Corollary 9.6(a). □

Corollary 9.6. *Let (G, X), K, ρ, δ, n and μ be as in Theorem 1.1.*

(a) For every $\varepsilon > 0$ the constants of Theorem 1.1 satisfy

$$c_{\deg}(D) \;\leq\; C'_\varepsilon\, D^{4^n+\varepsilon}, \qquad c(\delta, D, d) \;\leq\; C_\varepsilon\, d^{2^n/\mu+\varepsilon} D^{3\cdot 4^n/\mu+\varepsilon},$$

with C'_ε effectively computable from (G, X, K, ρ), ε and the representatives of Section 9, and C_ε from these and δ.

(b) If $G^{\mathrm{ad}} \neq 1$ and $h_K = 1$, then Theorem 1.1 holds with $\bar{c}^{\mathrm{ad}}_{\deg}(n, D_0)$ and $\bar{c}^{\sharp}(\delta, n, D_0, d_0)$, $D_0 = g_3 D$, $d_0 = g_3 d$, in place of $c_{\deg}(D)$ and $c(\delta, D, d)$, and for every $\varepsilon > 0$ these constants are at most $C'_\varepsilon D^{2^n+\varepsilon}$ and $C_\varepsilon (dD)^{2^n/\mu+\varepsilon}$.

Proof. For a general Shimura datum with $G^{\mathrm{ad}} \neq 1$ (for a torus the constants are 1, proof of Theorem 1.1), Theorem 9.5 evaluates the constants of Remark 8.2 at (D_n, d_n), where $D_n \leq (e_0^{2n} D_0)^{2^n}$ and $d_n \leq d_0 (e_0^{2n} D_0)^{2^{n+1}}$. With H_K and d_n^* as in the proof of Theorem 1.1, $d_n^* D_n \leq g_3 H_K!\, d\, (e_0^{2n} g_3 D)^{3\cdot 2^n}$, so for every $\varepsilon > 0$, by Remark 8.2 with exponent slack $\varepsilon/(3 \cdot 2^n)$, $c(\delta, D, d)$ satisfies the bound of (a). By Remark 8.2, $\bar{c}^{\mathrm{ad}}_{\deg}(n, D) \leq D^{2^n + o(1)}$ as $D \to \infty$, the maximum being over finitely many levels, and $c^{\mathrm{ad}}_{\deg}$ involves no δ; with $c_{\deg}(D) = H_K\, \bar{c}^{\mathrm{ad}}_{\deg}(n, D_n)$ and $D_n \leq (e_0^{2n} g_3 D)^{2^n}$, exponent slack $\varepsilon/2^n$ gives the bound of (a) for $c_{\deg}(D)$. If $h_K = 1$, then $h_{\mathfrak{G}} = 1$ and the recursion of Theorem 9.5(a) stops at $j^* = 0$, so Theorem 1.1 holds with $\bar{c}^{\mathrm{ad}}_{\deg}(n, D_0)$ and $\bar{c}^{\sharp}(\delta, n, D_0, d_0)$, $D_0 = g_3 D$, $d_0 = g_3 d$, in place of $c_{\deg}(D)$ and $c(\delta, D, d)$ (the last sentence of Theorem 9.5); by Remark 8.2 at these inputs, these constants are at most $C'_\varepsilon D^{2^n+\varepsilon}$ and $C_\varepsilon (dD)^{2^n/\mu+\varepsilon}$. □

Corollary 9.7. *Let (G,X), K, ρ and δ be as in Theorem 1.1, $n := \dim \mathrm{Sh}_K(G,X)$, $D, d \geq 1$, and V a special subvariety maximal in a closed subvariety Z.*

(a) If $Z \subset \mathrm{Sh}_K(G,X)$ is defined over a number field $F \supset E(G,X)$ with $[F:\mathbb{Q}] = d$ and has degree at most D, then $\deg V \leq c_{\deg}(D)$, and $d_{L_V}, \alpha_V \leq c(\delta, D, dD)$ if $\mathrm{QRH}(\delta; L_V)$ holds.

(b) If $Z \subset \mathrm{Sh}_K(G,X)_{\mathbb{C}}$ has degree at most D, then $\deg V \leq c_{\deg}(c_B^{2^n-1} D^{2^n})$, with c_B as in Lemma 5.2(c).

Proof. Let W be an irreducible component of $Z_{\mathbb{C}}$ containing V; then V is maximal in W, and $\deg W \leq \deg Z$, the degree being the sum over the components (Section 2.1).

(a) $Z_{\mathbb{C}}$ has at most D components, which $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ permutes, so W is defined over an extension of F of degree at most D, and Theorem 1.1 for W gives the bounds, its constants being non-decreasing in d (proof of Theorem 9.5).

(b) Put $W_0 := W$. If $W_j \supset V$ is irreducible and not defined over $\overline{\mathbb{Q}}$, some $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ has $\sigma W_j \neq W_j$. As V is defined over $\overline{\mathbb{Q}}$ (Section 2.2) and $\mathcal{L}_K$ over $E(G,X)$, $V = \sigma V \subset W_j \cap \sigma W_j$ and $\deg \sigma W_j = \deg W_j$. Let W_{j+1} be a component of $W_j \cap \sigma W_j$ containing V; then $\dim W_{j+1} < \dim W_j$, V is maximal in W_{j+1}, and $\deg W_{j+1} \leq c_B(\deg W_j)^2$ by Lemma 5.2(b),(c). A stage of dimension $\dim V$ is V, so after $j \leq n$ steps W_j is geometrically irreducible and defined over a number field containing $E(G,X)$, with $\deg W_j \leq c_B^{2^j-1} D^{2^j} \leq c_B^{2^n-1} D^{2^n}$, and Theorem 1.1(i) for W_j gives the bound, as $c_{\deg}$ is non-decreasing. □

Corollary 9.8. *In the setting of Theorem 1.1, let $G^{\mathrm{ad}} \neq 1$, let x be a special point maximal in Z, $V := \{x\}$, and assume $\mathrm{QRH}(\delta; L_V)$. Then*

$$[F(x):F] \;\leq\; g_3\, H_K\, n_T\, C_{\mathrm{ind}}\, h_{L_V}\, c(\delta,D,d)^{1+\log(1/B)+\log_2(1/B)},$$

with g_3 as in Lemma 2.2, H_K as in the proof of Theorem 1.1, h_{L_V} the class number of L_V, which Lemma C.1(a) bounds through $d_{L_V} \leq c(\delta,D,d)$, and n_T, B and $C_{\mathrm{ind}} := \max_t B^{-(C_0+e^2/B')}$ formed with the constants of Table 1 for $(G^{\mathrm{ad}}, X^{\mathrm{ad}}, \rho^{\mathrm{ad}})$ at the levels $\bar{g}_t K^{\mathrm{ad}} \bar{g}_t^{-1}$ of Section 9, each with the lattice $\mathrm{Ad}(\bar{g}_t)(\Lambda \otimes \widehat{\mathbb{Z}})$ and the common B. By Corollary 9.6, the right side is at most a polynomial in D and d.

Proof. By Theorem 1.1(ii), $d_{L_V}, \alpha_V \leq c := c(\delta,D,d)$, with $\alpha_V = \alpha_{f(V')}$ at the level K^{ad} built from the level K' of Lemma 2.2. By that lemma and its proof there are x' over x and a component Z' of $\pi^{-1}(Z)$ through x', defined over an extension F' of F of degree at most g_3, as $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ permutes the at most $\deg \pi \leq g_3$ components of $\pi^{-1}(Z)$. Let S_i be the connected component of $\mathrm{Sh}_{K'}(G,X)_{\mathbb{C}}$ containing Z', and $y := f(x')$. Every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F'(y))$ preserves Z', hence S_i, and maps x' into $S_i \cap f^{-1}(y)$, an orbit of $\mathfrak{G}_i$, which acts freely (Proposition 9.3(a)); so $[F'(x'):F'(y)] \leq h_K \leq H_K$. For the t with y in the image of $X^+ \times \{\bar{g}_t\}$, the right translation by $\bar{g}_t$ is an isomorphism from $\mathrm{Sh}_J(G^{\mathrm{ad}}, X^{\mathrm{ad}})$, $J := \bar{g}_t K^{\mathrm{ad}} \bar{g}_t^{-1}$, onto $\mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$, defined over $E(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \subset F'$ (Section 2.2); so we may write $y = [h,1]$ at the level J, which satisfies (Lv) for $\mathrm{Ad}(\bar{g}_t)(\Lambda \otimes \widehat{\mathbb{Z}})$ by Lemma 9.2(ii) and transport of the lattice, with $\mathrm{MT}(h) = T_V$ and $\alpha_y = \alpha_V$. As $\mathrm{Gal}(\overline{\mathbb{Q}}/L_V)$ acts on y through the reciprocity map r_{L_V}, which maps $L_V^\times$ into $T_V(\mathbb{Q})$ and the unit ideles into $K_{T_V}^m$ (proof of Corollary 3.4), the orbit of y maps to at most h_{L_V} points of $T_V(\mathbb{Q}) \backslash T_V(\mathbb{A}_f) / K_{T_V}^m$, with fibers of at most $[K_{T_V}^m : K_{T_V}]$ points, $K_{T_V} := T_V(\mathbb{A}_f) \cap J$. With Lemma 6.9 at J,

$$\begin{aligned}[F'(y):F'] \;\leq\; [F'L_V : F']\,|\,\mathrm{Gal}(\overline{\mathbb{Q}}/L_V)\cdot y| \;&\leq\; n_T\, h_{L_V}\, [K_{T_V}^m : K_{T_V}] \\ &\leq\; n_T\, h_{L_V}\, C_{\mathrm{ind}}\, c^{1+\log(1/B)+\log_2(1/B)}.\end{aligned}$$

Finally $[F(x):F] \le [F'(x'):F] = [F':F]\,[F'(x'):F'(y)]\,[F'(y):F']$, as $x=\pi(x')$ and $y=f(x')$ with π and f defined over $E(G,X)\subset F$. □

9.5. **Siegel modular varieties.** In this subsection $\nu\colon \mathrm{GSp}_{2g}\to\mathbb{G}_{\mathrm{m}}$ is the similitude character.

Theorem 9.9. *Let $g\ge 1$, $(G,X)=(\mathrm{GSp}_{2g},\mathcal{H}_g^{\pm})$ with its standard representation, $n=g(g+1)/2$, $e_0=e^*(n)$, and $s_g:=|\mathrm{Sp}_{2g}(\mathbb{F}_3)|=3^{g^2}\prod_{i=1}^{g}(3^{2i}-1)$ [96, p. 70]. Let $c^{\sharp}(\delta,z,D,d)$ and $c^{\mathrm{ad}}_{\deg}(z,D)$ be the constants of Section 8 for $(\mathrm{PGSp}_{2g},X^{\mathrm{ad}})$ with $(\mathrm{Ad},\mathfrak{sp}_{2g}(\widehat{\mathbb{Z}}))$ at the level named below. Let F, d, Z, D, z and V be as in Theorem 9.5, and define D_j, d_j by $D_{j+1}:=e_0^{2n}D_j^2$, $d_{j+1}:=e_0^{2n}D_j^2d_j$ from the initial values given in (a).*

(a) Let $N\ge 3$, $K=K(N)$ the principal congruence subgroup of level N, $D_0:=D$ and $d_0:=d$. Then h_K, computed at the level $K(N)$ as in Theorem 9.5(b), is at most $\#\{u\in(\mathbb{Z}/N)^{\times}:u^2=1\}/2$, $h_{\mathfrak{G}}=1$, and

$$\deg V\le h_K\,c^{\mathrm{ad}}_{\deg}(z,D_z),\qquad \Delta(V)\le h_K\,c^{\sharp}(\delta,z,D_z,d_z)\ \text{ if } \mathrm{QRH}(\delta;L_V)\text{ holds},$$

with the constants, and α_V, at the level $\lambda(K(N))$. If ±1 are the only square roots of 1 in $(\mathbb{Z}/N)^{\times}$, that is, if $N=4$, p^k or $2p^k$ for an odd prime p, then $h_K=1$ and these bounds hold with (D,d) in place of (D_z,d_z).

(b) Let $K(1):=\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$, so that $\mathrm{Sh}_{K(1)}(G,X)_{\mathbb{C}}=\mathcal{A}_{g,1}$, and take degrees on $\mathcal{A}_{g,1}$ as in Lemma 9.10. Then

$$\deg V\le c^{\mathrm{ad}}_{\deg}(z,(s_g/2)D),$$

$$\max(\deg V,d_{L_V},\alpha_V)\le c^{\sharp}(\delta,z,(s_g/2)D,s_gd)\ \text{ if } \mathrm{QRH}(\delta;L_V)\text{ holds},$$

with the constants at the level $\lambda(K(3))$ and α_V computed at the level $\lambda(K(1))$.

In both cases d_{L_V} is also the discriminant of the splitting field of the connected center of H_V.

For $\mathcal{A}_{g,1}$ we need a degree. Let $\pi\colon \mathrm{Sh}_{K(3)}(G,X)\to\mathrm{Sh}_{K(1)}(G,X)$ be the natural map. The group $K(1)/K(3)=\mathrm{GSp}_{2g}(\mathbb{F}_3)$ acts on $\mathrm{Sh}_{K(3)}(G,X)$ by right translations, which preserve $\mathcal{L}_{K(3)}$, through $\overline{G}:=\mathrm{GSp}_{2g}(\mathbb{F}_3)/\{\pm1\}$, the image of $Z(G)(\mathbb{Q})\cap K(1)=\{\pm1\}$ acting trivially; π is the quotient by $\overline{G}$, and $\deg\pi=|\overline{G}|=s_g$, since a general point of $\mathcal{H}_g$ has stabilizer $\{\pm1\}$ in $\mathrm{Sp}_{2g}(\mathbb{Z})$. The two geometric components of $\mathrm{Sh}_{K(3)}(G,X)$ correspond to $\mathbb{F}_3^{\times}$, and k moves them by $\nu(k)$, so the stabilizer of a component has order $s_g/2$.

Lemma 9.10. *For $Y\subset\mathrm{Sh}_{K(1)}(G,X)_{\mathbb{C}}$ closed and irreducible and C an irreducible component of $\pi^{-1}(Y)\subset\mathrm{Sh}_{K(3)}(G,X)_{\mathbb{C}}$, both over $\mathbb{C}$, put $\deg Y:=\deg_{\mathcal{L}_{K(3)}}C/[\kappa(C):\kappa(Y)]$.*

(a) This does not depend on C.

(b) Let $G_C\subset\overline{G}$ be the stabilizer of C and $I_C\subset G_C$ the subgroup fixing C pointwise. Then $[\kappa(C):\kappa(Y)]=|G_C/I_C|\le s_g/2$, and $\deg_{\mathcal{L}_{K(3)}}\pi^{-1}(Y)/\deg\pi=\deg Y/|I_C|$, with $1\le|I_C|\le s_g/2$.

(c) If $\mathcal{M}$ is a $\mathbb{Q}$-line bundle on the Baily–Borel compactification of $\mathcal{A}_{g,1}$ with $\pi^*\mathcal{M}\cong\mathcal{L}_{K(3)}$, then $\deg Y=\deg_{\mathcal{M}}\overline{Y}$.

(d) Replacing $K(3)$ by any neat K'' normal in $K(1)$ gives the same number.

Proof. (a) $\overline{G}$ permutes the components of $\pi^{-1}(Y)$ transitively; each $\sigma\in\overline{G}$ preserves $\mathcal{L}_{K(3)}$, so $\deg\sigma C=\deg C$, and $\sigma\colon C\to\sigma C$ is an isomorphism over Y. (b) There are $|\overline{G}|/|G_C|$ components, all of degree $\deg_{\mathcal{L}_{K(3)}}C$. Take $x\in C$ off the proper closed subsets $C\cap\sigma C$

($\sigma \notin G_C$) and C^σ ($\sigma \in G_C \smallsetminus I_C$); the fiber of $C \to Y$ through x is $G_C \cdot x$, with $|G_C/I_C|$ points, and a finite dominant morphism of complex varieties has $[\kappa(C) : \kappa(Y)]$ points in its general fiber. So $\deg_{\mathcal{L}_{K(3)}} \pi^{-1}(Y) = (|\overline{G}|/|G_C|)\,|G_C/I_C| \deg Y$, and we divide by $\deg \pi = |\overline{G}|$. Finally G_C stabilizes a geometric component, so $|G_C| \le s_g/2$. (c) The projection formula for the finite surjective map $\overline{C} \to \overline{Y}$, applied to a multiple of $\mathcal{M}$ that is a line bundle, gives $\deg_{\pi^*\mathcal{M}} \overline{C} = [\kappa(C) : \kappa(Y)] \deg_{\mathcal{M}} \overline{Y}$. (d) For neat $K''' \subset K''$, both normal in $K(1)$, the map $\mathrm{Sh}_{K'''} \to \mathrm{Sh}_{K''}$ is finite étale and pulls $\mathcal{L}_{K''}$ back to $\mathcal{L}_{K'''}$ [63, Prop. 5.3.2]; for a component C''' over C, the projection formula gives $\deg C''' = [\kappa(C''') : \kappa(C)] \deg C$, and the field degrees multiply. Two such subgroups compare through their intersection. □

Proof of Theorem 9.9. (a) $K(N)$ is a neat product level for $N \ge 3$. The center $Z(G) = \mathbb{G}_{\mathrm{m}}$ is connected and $H^1(\mathbb{Q}, \mathbb{G}_{\mathrm{m}}) = 0$, so $\lambda(K)$ is open and Theorem 9.5(b) applies; $\Gamma_i' \subset \lambda(G(\mathbb{Q}))$, so $h_{\mathfrak{G}} = 1$ and $d_0 = d$; and Proposition 9.3(d) gives $h_K \le |\mathbb{Q}^\times \backslash \mathbb{A}_f^\times / \ker(\widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/N)^\times)| = |\widehat{\mathbb{Z}}^\times / \{\pm 1\} \ker(\widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/N)^\times)| = \varphi(N)/2$. More precisely, the class of $\bar\gamma = \lambda(zk)$ in $\mathrm{Cl}(K)$ is that of z, and $\nu(zk) = z^2\nu(k) \in \mathbb{Q}_{>0}$ with $\nu(k) \in \ker(\widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/N)^\times)$, so writing $z = qu$ with $q \in \mathbb{Q}^\times$, $u \in \widehat{\mathbb{Z}}^\times$, we get $u^2 \in \ker(\widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/N)^\times)$. Hence $h_K \le \#\{u \in (\mathbb{Z}/N)^\times : u^2 = 1\}/2$.

Let p_0 be an odd prime dividing N, or $p_0 = 2$ if N is a power of 2, and $v := v_{p_0}(N)$, so $v \ge 2$ if $p_0 = 2$. Every eigenvalue ξ of an element of $K(N)_{p_0} \subset 1 + p_0^v M_{2g}(\mathbb{Z}_{p_0})$ has $v_{p_0}(\xi - 1) \ge v$; the elements with this property form a group; and a root of unity $\zeta \ne 1$ has $v_{p_0}(\zeta - 1) \le 1/(p_0 - 1) < v$. So $K(N)_{p_0}$ is neat, hence so is $\lambda(K(N))_{p_0}$ [81, 0.6]; and $\mathrm{Ad}(\lambda(K(N)))$ preserves $\mathfrak{sp}_{2g}(\widehat{\mathbb{Z}})$, as in the proof of Lemma 9.2(ii). Thus $\lambda(K(N))$ satisfies (Lv′), and Theorem 8.1 applies at the level $\lambda(K(N))$ with no factor.

For PGSp_{2g} the representatives $\bar g_t$ can be taken in $\lambda(\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}}))$. Indeed $\lambda\colon \mathrm{GSp}_{2g}(\mathbb{A}_f) \to \mathrm{PGSp}_{2g}(\mathbb{A}_f)$ is onto, as $H^1(\mathbb{Q}_p, \mathbb{G}_{\mathrm{m}}) = H^1(\mathbb{Z}_p, \mathbb{G}_{\mathrm{m}}) = 0$, and $\mathrm{GSp}_{2g}(\mathbb{A}_f) = \mathrm{GSp}_{2g}(\mathbb{Q})_+ \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$: for $x \in \mathrm{GSp}_{2g}(\mathbb{A}_f)$ write $\nu(x) = qu$ with $q \in \mathbb{Q}_{>0}$ and $u \in \widehat{\mathbb{Z}}^\times$, and take $\gamma \in \mathrm{GSp}_{2g}(\mathbb{Q})$ with $\nu(\gamma) = q$ and $k \in \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$ with $\nu(k) = u$; then $\gamma^{-1} x k^{-1} \in \mathrm{Sp}_{2g}(\mathbb{A}_f) = \mathrm{Sp}_{2g}(\mathbb{Q}) \mathrm{Sp}_{2g}(\widehat{\mathbb{Z}})$ by strong approximation for Sp_{2g}, and $\nu > 0$ on $\gamma \mathrm{Sp}_{2g}(\mathbb{Q})$. The group $\lambda(\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}}))$ normalizes $\lambda(K(M))$ for every M, and the level that Lemma 2.2 makes from it, and preserves $\mathfrak{sp}_{2g}(\widehat{\mathbb{Z}})$; so the maxima of Theorem 9.5 are the constants at the level named. If ± 1 are the only square roots of 1 in $(\mathbb{Z}/N)^\times$, then $h_K = 1$, and Theorem 9.5 gives the bounds at $(D_0, d_0) = (D, d)$.

(b) Choose a component V' of $\pi^{-1}(V)$ and a component Z' of $\pi^{-1}(Z)$ containing it. As in Lemma 2.2, V' is special and maximal in Z', with $T_{V'} = T_V$, and Z' is defined over a field of degree at most $s_g d$, since $\pi^{-1}(Z)$ has at most $|\overline{G}| = s_g$ components. By Lemma 9.10(b), $\deg_{\mathcal{L}_{K(3)}} Z' = [\kappa(Z') : \kappa(Z)]\, D \le (s_g/2)D$. Part (a) at $N = 3$, where $h_{K(3)} = 1$, bounds $\deg_{\mathcal{L}_{K(3)}} V'$ and the torus data at $\lambda(K(3))$, with the constants at $((s_g/2)D, s_g d)$, as they are non-decreasing in D and d. Finally $\deg V = \deg_{\mathcal{L}_{K(3)}} V'/[\kappa(V') : \kappa(V)] \le \deg_{\mathcal{L}_{K(3)}} V'$, and since $\lambda(K(3)) \subset \lambda(K(1))$, each index $|K^m_{T_V,p}/K_{T_V,p}|$ of Definition 2.1, and hence α_V (the constant B does not depend on the level), is at least as large at $\lambda(K(3))$ as at $\lambda(K(1))$.

In both cases the connected center of H_V contains the central $\mathbb{G}_{\mathrm{m}}$ (the weight) and is isogenous to $\mathbb{G}_{\mathrm{m}} \times T_V$, so the two splitting fields coincide. □

For $\mathcal{A}_{g,1}$, and for $\mathcal{A}_{g,K(N)}$ with $N = 4$, p^k or $2p^k$ for an odd prime p, where $h_K = 1$, Theorem 9.9 gives bounds of the forms $C'_\varepsilon D^{2^n+\varepsilon}$ and $C_\varepsilon (dD)^{2^n/\mu+\varepsilon}$ of Corollary 9.6(b). For every $N \ge 3$, Theorem 9.9(a) evaluates $c^{\mathrm{ad}}_{\deg}$ at D_z and $c^\sharp$ at (D_z, d_z), and from

$D_{j+1} = e_0^{2n} D_j^2$, $d_{j+1} = e_0^{2n} D_j^2 d_j$ one gets $D_z \le (e_0^{2n} D)^{2^z}$ and $d_z D_z \le d\,(e_0^{2n} D)^{3\cdot 2^z}$; so, by Remark 8.2 with exponent slack $\varepsilon/2^z$ and $\varepsilon/(3\cdot 2^z)$ and as $z \le n$, $h_K\, c^{\mathrm{ad}}_{\deg}(z, D_z) \le C'_\varepsilon D^{4^n+\varepsilon}$ and $h_K\, c^\sharp(\delta, z, D_z, d_z) \le C_\varepsilon\, d^{2^n/\mu+\varepsilon} D^{3\cdot 4^n/\mu+\varepsilon}$, the bounds of Theorem 1.1, with C'_ε and C_ε depending also on N.

The bound for h_K in Theorem 9.9(a) is 1 unless $(\mathbb{Z}/N)^\times$ has a square root of 1 other than ± 1, for instance for $N = 8$, 12 or 15; only then can f fail to be injective on a component, and only then is Lemma 9.4 needed.

In genus 4 the closure of the Jacobian locus is a hypersurface with an explicit equation. Let $\overline{J}_4 \subset \mathcal{A}_{4,1}$ be the Zariski closure of the locus of Jacobians of smooth projective curves of genus 4, the closure of the image of the Torelli map; it is defined over $\mathbb{Q}$, irreducible and of dimension 9. Let $S_3 := \Gamma(3)\backslash\mathcal{H}_4$, with $\Gamma(3) \subset \mathrm{Sp}_8(\mathbb{Z})$ the principal congruence subgroup; as $K(3)$ is normal in $\mathrm{GSp}_8(\widehat{\mathbb{Z}})$, both geometric components of $\mathrm{Sh}_{K(3)}(\mathrm{GSp}_8, \mathcal{H}_4^{\pm})$ are copies of S_3. Let $s_4 = |\mathrm{Sp}_8(\mathbb{F}_3)|$ be as in Theorem 9.9.

Corollary 9.11. *Put $D_4 := \frac{8}{5}\deg_{\mathcal{L}_{K(3)}} S_3$. Let V be a special subvariety of $\mathcal{A}_{4,1}$ maximal in $\overline{J}_4$, with $\deg V$ as in Lemma 9.10 and α_V computed at $\lambda(K(1))$. Then $\deg V \le c^{\mathrm{ad}}_{\deg}(9, D_4)$, and if $\mathrm{QRH}(\delta; L_V)$ holds, then*

$$\max\bigl(\deg V, d_{L_V}, \alpha_V\bigr) \le c^\sharp(\delta, 9, D_4, s_4),$$

with the constants of Theorem 9.9 at the level $\lambda(K(3))$.

Proof. For $g \ge 1$ let $F_g := 2^g \sum_m \theta_m^{16} - \bigl(\sum_m \theta_m^8\bigr)^2$, with m running over the even characteristics in $(\frac{1}{2}\mathbb{Z}/\mathbb{Z})^{2g}$ and θ_m the theta constant of genus g with characteristic m. Then F_4 is a modular form of weight 8 for $\mathrm{Sp}_8(\mathbb{Z})$, not identically zero, that vanishes on the Jacobian locus [51, Thm. 3.8 and eq. (9), p. 10]. It is a cusp form: Siegel's operator Φ takes $\theta[(\varepsilon', \varepsilon_4), (\delta', \delta_4)]$ to $\theta[\varepsilon', \delta']$ if $\varepsilon_4 = 0$ and to 0 otherwise, so, as each even characteristic of genus 3 arises for both values of δ_4, $\Phi F_4 = 2^5 \sum_{m'} \theta_{m'}^{16} - 4\bigl(\sum_{m'} \theta_{m'}^8\bigr)^2 = 4F_3$, which vanishes identically [51, Thm. 3.8]. By Freitag's theorem [41, Satz 1, p. 256], the zero divisor of a cusp form of weight at most 8 for $\mathrm{Sp}_8(\mathbb{Z})$ that is not identically zero is irreducible; as $\overline{J}_4$ is irreducible of dimension 9 and lies in the zero set of F_4, that zero set is $\overline{J}_4$. This is Igusa's theorem [59], which we use in Freitag's form.

Let $\mathcal{L} := \mathcal{L}_{K(3)}$ and let $Z_3 \subset S_3$ be the preimage of $\overline{J}_4$. On S_3, $\mathcal{L} = \omega^{\otimes 5}$ [58, §II.1], with ω the bundle of modular forms of weight 1, so F_4^5 is a section of $\mathcal{L}^{\otimes 8}$ on S_3. It extends to S_3^{BB}, which is normal with boundary of codimension 4, as in the proof of Lemma 11.8. Its divisor is $5\sum_i m_i \overline{Z}_i$, with Z_i the components of Z_3, $\overline{Z}_i$ their closures and integers $m_i \ge 1$, and intersecting with $c_1(\mathcal{L})^9$ gives $5\sum_i m_i \deg_{\mathcal{L}} Z_i = 8 \deg_{\mathcal{L}} S_3$. So $\deg_{\mathcal{L}} Z_3 \le D_4$.

Let $\pi\colon \mathrm{Sh}_{K(3)}(G, X) \to \mathrm{Sh}_{K(1)}(G, X)$ be as in Lemma 9.10, V' a component of $\pi^{-1}(V)$ and Z' a component of $\pi^{-1}(\overline{J}_4)$ containing V'. Then Z' is a component of the preimage of $\overline{J}_4$ in a copy of S_3, so $\deg_{\mathcal{L}} Z' \le D_4$, and $\dim Z' = 9$. As in the proof of Theorem 9.9(b), V' is special and maximal in Z', with $T_{V'} = T_V$, and Z' is defined over a field of degree at most s_4. Theorem 9.9(a) at $N = 3$, where $h_{K(3)} = 1$, bounds $\deg_{\mathcal{L}} V'$ and the torus data at $\lambda(K(3))$ with the constants at (D_4, s_4), as they are non-decreasing in D and d, and the last step of the proof of Theorem 9.9(b) passes to V. □

By Hirzebruch–Mumford proportionality in van der Geer's form [105, Thm. (1.13), p. 8, and p. 9], ω^{10} has degree $1/1814400$ on the stack $[\mathrm{Sp}_8(\mathbb{Z})\backslash\mathcal{H}_4]$, which S_3 covers with degree s_4. So $\deg_{\mathcal{L}_{K(3)}} S_3 = 5^{10} s_4/1814400 = 2^8\cdot 3^{12}\cdot 5^{10}\cdot 13\cdot 41$, and $D_4 = 2^{11}\cdot 3^{12}\cdot 5^9\cdot 13\cdot 41 \approx 1.13\cdot 10^{18}$.

Remark 9.12. In Corollary 9.11, "maximal in" cannot be weakened to "contained in". Let k be an imaginary quadratic field, $x \in \mathcal{A}_{1,1}$ a point with complex multiplication by an order of k, and V_k the image of $\{x\} \times \mathcal{A}_{3,1}$ under $(\tau_1, \tau_3) \mapsto \tau_1 \oplus \tau_3$. Each theta constant of $\tau_1 \oplus \tau_3$ is the product of one of τ_1 and one of τ_3, and the odd ones vanish, so $F_4(\tau_1 \oplus \tau_3) = 16A_{16}B_{16} - A_8^2 B_8^2$, where A_j and B_j are the sums of the j-th powers of the even theta constants of τ_1 and of τ_3; and $2A_{16} = A_8^2$ and $8B_{16} = B_8^2$, as F_1 and F_3 vanish identically [51, Thm. 3.8]. So the image of $\mathcal{A}_{1,1} \times \mathcal{A}_{3,1}$, a special subvariety, lies in $\overline{J}_4$, and so does V_k. The variety V_k is special of dimension 6, with generic Mumford–Tate group $\mathrm{Res}_{k/\mathbb{Q}}\mathbb{G}_{\mathrm{m}} \times_{\mathbb{G}_{\mathrm{m}}} \mathrm{GSp}_6$, the fiber product over the norm and the similitude; its connected center is split exactly by k, so $d_{L_{V_k}}$ is the absolute discriminant of k, which is unbounded. The closure of the genus-4 Jacobian locus also contains infinitely many Shimura curves whose general points parametrize Jacobians of smooth curves [52]. By a remark of Daw, Javanpeykar and Kühne [27, p. 3], explicit bounds for the degrees of the maximal special subvarieties of the closure of the Jacobian locus that are not of facteur type follow from their theorem and an effective bound for the degree of that closure. For $g \geq 5$ the closure has codimension $(g-2)(g-3)/2 \geq 3$, so it is not cut out by one modular form, and Theorem 9.9(b) needs an explicit bound for its degree.

10. Orders and Galois orbits in $\mathcal{A}_g$

In this section $G = \mathrm{GSp}(W, \psi)$ with $W = \mathbb{Q}^{2g}$, $W_{\mathbb{Z}} = \mathbb{Z}^{2g}$ and ψ perfect on $W_{\mathbb{Z}}$, $X = \mathcal{H}_g^{\pm}$, and $\lambda\colon G \to G^{\mathrm{ad}}$. For a special subvariety V we write $T := T_V^G$ for the connected center of H_V itself, which contains the scalars, and $\bar{T} := \lambda(T)$ for the torus of Section 9.

Let $K = \prod_p K_p$ be compact open with $K(N) \subset K \subset G(\widehat{\mathbb{Z}})$ for some $N \geq 1$, and let $V \subset \mathrm{Sh}_K(G, X)_{\mathbb{C}}$ be special, defined by (H, X_H), the image of $X_H^+ \times \{\gamma\}$ with $\gamma \in G(\mathbb{A}_f)$. Put $\mathcal{E} := \mathbb{Q}[T]$, the $\mathbb{Q}$-span of $T(\mathbb{Q})$ in $\mathrm{End}(W)$, $n_{\mathcal{E}} := \dim_{\mathbb{Q}} \mathcal{E}$, $\Lambda_V \subset W$ the lattice with $\Lambda_{V,p} = \gamma_p W_{\mathbb{Z}_p}$, and $R := \mathcal{E} \cap \mathrm{End}(\Lambda_V)$. For a prime p let $\iota_p := [K_{T,p}^m : T(\mathbb{Q}_p) \cap \gamma_p K_p \gamma_p^{-1}]$, and $\bar{\iota}_p$ the same index for $\bar{T}$ and $\lambda(\gamma_p K_p \gamma_p^{-1})$; so $\alpha_V = \prod_{\bar{\iota}_p > 1} \max(1, \bar{B}\bar{\iota}_p)$ with $\bar{B}$ the constant B of the adjoint datum. Let m_V be the least common multiple of the dimensions of the irreducible summands of $W \otimes \overline{\mathbb{Q}}$ under H_V; so $m_V = 1$ if V is a point, and $m_V \leq \mathrm{lcm}(1, \ldots, 2g)$.

Lemma 10.1. *Suppose $d_{L_V} \leq D_1$ and $\alpha_V \leq A_1$ with $A_1 \geq 1$. Put $A_1' := A_1 \max(1, 1/\bar{B})$ and $P_1 := 1 + m_V A_1'$.*

(a) $\mathcal{E} = \prod_i F_i$ with each $F_i \subset L_V$ a totally real or CM field, $d_{F_i} \leq D_1$ and $n_{\mathcal{E}} \leq 2g$. R is an order of $\mathcal{E}$ and $R = \mathcal{E} \cap \mathrm{End}(A_x)$ for the abelian variety A_x of the point $[x, \gamma]$, for every $x \in X_H^+$ (so $H_1(A_x, \mathbb{Z}) = \Lambda_V$); if V is a point, R is the center of $\mathrm{End}(A_x)$.

(b) For every prime $p \geq m_V + 2$ unramified in L_V, either $R_p = \mathcal{O}_{\mathcal{E},p}$ or $\iota_p \geq (p-1)/m_V$.

(c) $R_p = \mathcal{O}_{\mathcal{E},p}$ for every prime $p > P_1$ with $p \nmid N d_{L_V}$.

(d) Both as written and with $n_{\mathcal{E}}$ replaced by $2g$,

$$[\mathcal{O}_{\mathcal{E}} : R] \leq \left(2 \cdot 4^{P_1} N^2 D_1 {A_1'}^{P_1 + \log_2(N D_1)}\right)^{n_{\mathcal{E}}(n_{\mathcal{E}}-1)/2} \left(2 m_V n_{\mathcal{E}} (n_{\mathcal{E}} - 1)(n_{\mathcal{E}}!)^2 D_1^{1/2}\right)^{n_{\mathcal{E}}(n_{\mathcal{E}}-1)/2}.$$

Parts (c) and (d) hold with $\bar{B}$ replaced in A_1' by any $\bar{B}_0 \in (0, \bar{B}]$.

Proof. Part (b) is an explicit form of [28, Prop. 4.11] combined with [102, Prop. 3.15], and the proof of (d) follows [99, Lemmas 7.4 and 7.5], [28, Props. 4.8 and 4.12].

Weights and the involution. Let Φ be the set of weights of T on $W \otimes \overline{\mathbb{Q}}$. As $T(\mathbb{Q})$ is Zariski dense in T, $\mathcal{E} \otimes \overline{\mathbb{Q}}$ is the algebra of operators that are scalar on each weight space; so

$\mathcal{E}$ is étale of dimension $|\Phi|$, with embeddings φ_χ, $\chi \in \Phi$, taking values in L_V. The ψ-adjoint of $t \in T(\mathbb{Q})$ is $\nu(t)t^{-1} \in T(\mathbb{Q})$, with ν the similitude character, so the adjoint preserves $\mathcal{E}$; on $\mathcal{E} \subset \mathrm{End}^0(A_x)$ it is the Rosati involution, which is positive, so it preserves each factor of $\mathcal{E}$ and is the identity or complex conjugation on it. As $d_{F_i}^{[L_V:F_i]}$ divides d_{L_V}, $d_{F_i} \le D_1$; and $n_\mathcal{E} \le \dim W$. For a point, $\mathrm{End}^0(A_x) = \mathrm{End}_\mathcal{E}(W)$ has center $\mathcal{E}$. This gives (a).

Let $\mu_c \in X_*(T)\otimes\mathbb{Q}$ be the preimage, under the isogeny $T \to H^{\mathrm{ab}}$, of the Hodge cocharacter of H^{ab}, which is the same for all $x \in X_H$. If T acts on an irreducible $H_{\overline{\mathbb{Q}}}$-summand W_j by χ, then $\det_{W_j}$ is trivial on H^{der} and equals $\dim(W_j)\chi$ on T, so $\langle\chi,\mu_c\rangle = a_j/\dim W_j \in [0,1] \cap m_V^{-1}\mathbb{Z}$, with a_j the multiplicity of the weight 1 of a Hodge generic cocharacter on W_j. Let Y be the Galois orbit of $m_V\mu_c$. Then $\langle\chi,y\rangle \in \{0,\dots,m_V\}$ for $y \in Y$ and $\chi \in \Phi$, so $Y \subset X_*(T)$; and Y spans $X_*(T)\otimes\mathbb{Q}$, since otherwise the Hodge cocharacter would factor through a proper subtorus and h_x through a proper $\mathbb{Q}$-subgroup of H. For $\chi \ne \chi'$ in Φ some $y \in Y$ has $\langle\chi-\chi',y\rangle \ne 0$, and $|\langle\chi-\chi',y\rangle| \le m_V$ for all $y \in Y$.

(b) Let $p \ge m_V+2$ be unramified in $L := L_V$, $\mathfrak{P} \mid p$, f the degree and σ the Frobenius of $L_\mathfrak{P}/\mathbb{Q}_p$, $q := p^f$, and ζ a generator of the roots of unity of order $q-1$ in $L_\mathfrak{P}$. For $y \in Y$ the element $t := \sum_{k<f} \sigma^k y \otimes \zeta^{p^k}$ of $X_*(T)\otimes L_\mathfrak{P}^\times$ is σ-invariant, so $t \in T(\mathbb{Q}_p)$, and $t \in K_{T,p}^m$ as it has finite order. For $\chi \ne \chi'$ choose y with $b_0 := \langle\chi-\chi',y\rangle \ne 0$; then $(\chi/\chi')(t) = \zeta^a$ with $a = \sum_{k<f} p^k\langle\chi-\chi',\sigma^k y\rangle \equiv b_0 \not\equiv 0 \bmod p$ and $|a| \le m_V(q-1)/(p-1) < q-1$, so ζ^a has order at least $(p-1)/m_V$. If $R_p \ne \mathcal{O}_{\mathcal{E},p}$, the image of R_p in the étale algebra $\mathcal{O}_{\mathcal{E},p}/p$ is a proper subalgebra by Nakayama's lemma, so two embeddings $\varphi_\chi \ne \varphi_{\chi'}$ agree modulo the maximal ideal on R_p. As $T(\mathbb{Q}_p) \cap \gamma_p K_p \gamma_p^{-1} \subset R_p^\times$, every element s of $\langle t\rangle$ in this group has $\varphi_\chi(s) \equiv \varphi_{\chi'}(s)$, and both are roots of unity of order prime to p, so they are equal. Hence $\iota_p \ge [\langle t\rangle : \ker(\chi/\chi')] \ge (p-1)/m_V$.

(c) For every p, $\iota_p \le [\mathbb{Z}_p^\times : \mathbb{Z}_p^\times \cap K_p]\,\bar\iota_p \le p^{v_p(N)}\bar\iota_p$: the map $T(\mathbb{Q}_p) \to \bar T(\mathbb{Q}_p)$ is onto by Hilbert's Theorem 90, it maps the level subgroup of T onto that of $\bar T$ and $K_{T,p}^m$ into $K_{\bar T,p}^m$, not necessarily onto, which is all the inequality needs, and its kernel on $K_{T,p}^m$ is $\mathbb{Z}_p^\times$. If $p > P_1$, $p \nmid N d_{L_V}$ and $R_p \ne \mathcal{O}_{\mathcal{E},p}$, then $p \ge m_V+2$ and (b) gives $\bar\iota_p \ge \iota_p \ge (p-1)/m_V > A_1' \ge 1$, so $\alpha_V \ge \bar B\bar\iota_p > A_1$, a contradiction.

(d) Let $\mathfrak{t} := \mathrm{Lie}\,T \subset \mathcal{E}$, $k_p := 1$ for odd p and $k_p := 2$ for $p = 2$. For $y_0 \in \mathfrak{t} \cap \mathcal{O}_\mathcal{E}$ with $\Delta_0 := \mathrm{disc}(y_0) \ne 0$, the element $u := \exp(p^{k_p}y_0)^{\iota_p}$ lies in $(K_{T,p}^m)^{\iota_p} \subset R_p^\times$, since exp converges on $p^{k_p}\mathcal{O}_{\mathcal{E},p}$ and maps $p^{k_p}\mathcal{O}_{\mathcal{E},p} \cap (\mathfrak{t}\otimes\mathbb{Q}_p)$ into $T(\mathbb{Q}_p) \cap \mathcal{O}_{\mathcal{E},p}^\times = K_{T,p}^m$. As $v_p(e^z-1) = v_p(z)$ for $v_p(z) > 1/(p-1)$, $v_p(\mathrm{disc}\,u) = n_\mathcal{E}(n_\mathcal{E}-1)(v_p(\iota_p)+k_p) + v_p(\Delta_0)$, so $\mathbb{Z}_p[u] \subset R_p$ gives $[\mathcal{O}_{\mathcal{E},p} : R_p] \le p^{(n_\mathcal{E}(n_\mathcal{E}-1)(v_p(\iota_p)+k_p)+v_p(\Delta_0))/2}$. To choose y_0, identify $\mathcal{E}\otimes L = L^\Phi$, let $w := (\langle\chi, m_V\mu_c\rangle)_\chi$ and $y_\ell := (\mathrm{id}\otimes\mathrm{Tr}_{L/\mathbb{Q}})(w\ell) \in \mathfrak{t}\cap\mathcal{O}_\mathcal{E}$ for $\ell \in \mathcal{O}_L$. The $\mathbb{Q}$-linear map $L \to \mathfrak{t}$, $\ell \mapsto y_\ell$, is onto: in $\mathfrak{t}\otimes L = X_*(T)\otimes L$ we have $y_\ell = \sum_\sigma \sigma(\ell)\,\sigma(m_V\mu_c)$, so by the independence of the embeddings of L the L-span of its image is that of Y, which is $\mathfrak{t}\otimes L$; so the image, a $\mathbb{Q}$-subspace, is $\mathfrak{t}$, and disc is a nonzero polynomial on $\mathfrak{t}$, of degree $n_\mathcal{E}(n_\mathcal{E}-1)$. With $v_1,\dots,v_{[L:\mathbb{Q}]} \in \mathcal{O}_L$ attaining the successive minima of the house, which are at most $D_1^{1/2}$ by Minkowski's second theorem, some $\ell = \sum c_k v_k$ with $0 \le c_k \le n_\mathcal{E}(n_\mathcal{E}-1)$ has $\Delta_0 := \mathrm{disc}(y_\ell) \ne 0$, and $|\Delta_0| \le (2m_V n_\mathcal{E}(n_\mathcal{E}-1)[L:\mathbb{Q}]^2 D_1^{1/2})^{n_\mathcal{E}(n_\mathcal{E}-1)}$, with $[L:\mathbb{Q}] \le n_\mathcal{E}!$ since $\mathrm{Gal}(L/\mathbb{Q})$ acts faithfully on Φ. By (c) the bad primes number at most $P_1 + \log_2(ND_1)$, and for them $p^{v_p(\iota_p)} \le p^{v_p(N)}A_1'$ and $\prod p^{k_p} \le 2\cdot 4^{P_1}ND_1$. Multiplying gives (d); the bound increases with $n_\mathcal{E}$.

The last assertion holds with the same proofs, as $\bar B$ enters only through $A_1' \ge A_1/\bar B$. $\square$

The estimate in the proof of Lemma 10.1(b) is attained, for instance at $p = 5$: take $\mathcal{E} = \mathbb{Q}(i) \times \mathbb{Q}$ and $T = \{(t, s) : N(t) = s^2\}$, the torus of $\{x_0\} \times Y(1) \subset \mathcal{A}_2$ for the point x_0 of $Y(1)$ with complex multiplication by $\mathbb{Q}(i)$, with $m_V = 2$.

At the level $K(1)$ the proof of Lemma 10.1(d) gives, for a CM point, a bound by powers of α_V and d_{L_V} with exponents that depend only on g. For a CM point of $\mathcal{A}_{g,1}$, Lemma 10.2 below is a form, with explicit constants, of [99, Lemma 7.2] and of [28, Thm. 4.1].

Lemma 10.2. *Let x be a CM point of $\mathcal{A}_{g,1} = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g$, written $x = [h, 1]$ with $h \in \mathcal{H}_g$, and use the notation above with $K = K(1)$ and $V = \{x\}$. Let R_x be the center of $\mathrm{End}(A_x)$, $\mathrm{Disc}(R_x)$ the absolute value of the discriminant of its trace form, $N_2 := g(2g-1)$, $\pi(y)$ the number of primes $p \le y$, and*

$$C_{\mathcal{O}} := \Big(16g(2g-1)((2g)!)^2\, \bar{B}^{-6\pi(1+\bar{B}^{-2})}\Big)^{N_2}.$$

Then

$$[\mathcal{O}_{\mathcal{E}} : R_x] \le C_{\mathcal{O}}\, \alpha_V^{6N_2}\, d_{L_V}^{3N_2/2} \qquad \text{and} \qquad \mathrm{Disc}(R_x) \le C_{\mathcal{O}}^2\, \alpha_V^{12N_2}\, d_{L_V}^{3N_2+g}.$$

Both bounds hold with $\bar{B}$ replaced, in α_V and in $C_{\mathcal{O}}$, by any $\bar{B}_0 \in (0, 1]$.

Proof. By Lemma 10.1(a), $R_x = R$ and $\mathcal{E} = \prod_i F_i$ with $F_i \subset L := L_V$, so $\mathrm{Disc}(R_x) = [\mathcal{O}_{\mathcal{E}} : R]^2 \prod_i d_{F_i}$. Here $N = 1$, $\gamma = 1$ and $m_V = 1$. Let $n := n_{\mathcal{E}} \le 2g$, $n_2 := n(n-1)/2 \le N_2$, $\mathcal{S}$ the set of primes p with $R_p \ne \mathcal{O}_{\mathcal{E},p}$, and $\mathcal{S}_u$ the set of $p \in \mathcal{S}$ with $p \ge 3$ and $p \nmid d_L$. We use three facts from the proof of Lemma 10.1. First, $\iota_p \le \bar{\iota}_p$ for every p, by the proof of (c), as $K_p = G(\mathbb{Z}_p) \supset \mathbb{Z}_p^\times$; so $\max(1, \bar{B}\iota_p) \le \max(1, \bar{B}\bar{\iota}_p)$, and the product of the $\max(1, \bar{B}\iota_p)$ over any set of primes with $\iota_p > 1$ is at most α_V. Second, $\iota_p \ge p - 1 \ge 2$ for $p \in \mathcal{S}_u$, by (b). Third, the proof of (d), with $D_1 = d_L$, gives a nonzero integer Δ_0 with $|\Delta_0| \le (2n(n-1)(n!)^2 d_L^{1/2})^{n(n-1)}$ and $[\mathcal{O}_{\mathcal{E},p} : R_p] \le p^{n_2(v_p(\iota_p)+k_p)+v_p(\Delta_0)/2}$ for every p; so

$$[\mathcal{O}_{\mathcal{E}} : R] \le |\Delta_0|^{1/2}\, \Pi^{n_2}, \qquad \Pi := \prod_{p \in \mathcal{S}} p^{k_p + v_p(\iota_p)}.$$

The primes of $\mathcal{S} \smallsetminus \mathcal{S}_u$ are 2 and primes dividing d_L, so their factors p^{k_p} multiply to at most $4d_L$. For $p \in \mathcal{S}_u$ we have $p \le \iota_p + 1 \le \iota_p^2$, so $p^{k_p + v_p(\iota_p)} \le \iota_p^3$; for $p \in \mathcal{S} \smallsetminus \mathcal{S}_u$, $p^{v_p(\iota_p)} \le \iota_p$, and it is 1 unless $p \mid \iota_p$. Let $\mathcal{P}$ be the union of $\mathcal{S}_u$ and the set of $p \in \mathcal{S} \smallsetminus \mathcal{S}_u$ dividing ι_p. Then $\Pi \le 4d_L \prod_{p \in \mathcal{P}} \iota_p^3$, and every $p \in \mathcal{P}$ has $\iota_p > 1$ and $p \le \iota_p + 1$. If $\iota_p \ge \bar{B}^{-2}$, then $\iota_p \le (\bar{B}\iota_p)^2 = \max(1, \bar{B}\iota_p)^2$; if $\iota_p < \bar{B}^{-2}$, then $p < 1 + \bar{B}^{-2}$, which holds for at most $\pi(1 + \bar{B}^{-2})$ primes. So

$$\prod_{p \in \mathcal{P}} \iota_p \le \alpha_V^2\, \bar{B}^{-2\pi(1+\bar{B}^{-2})}, \qquad \Pi \le 4d_L\, \alpha_V^6\, \bar{B}^{-6\pi(1+\bar{B}^{-2})},$$

and with $|\Delta_0|^{1/2} \le (2n(n-1)(n!)^2)^{n_2} d_L^{n_2/2}$,

$$[\mathcal{O}_{\mathcal{E}} : R] \le \Big(8n(n-1)(n!)^2\, \bar{B}^{-6\pi(1+\bar{B}^{-2})}\Big)^{n_2} \alpha_V^{6n_2}\, d_L^{3n_2/2}.$$

The right side increases with $n \le 2g$, which gives the first bound. For the second, $d_{F_i}^{[L:F_i]}$ divides d_L, so $\prod_i d_{F_i} \le d_L^{n/[L:\mathbb{Q}]} \le d_L^g$, as $n \le 2g$ and $d_L = 1$ if $L = \mathbb{Q}$. The proof uses only $0 < \bar{B} \le 1$. □

Edixhoven's form of the Galois bound for a CM point x of $\mathcal{A}_g$ is a lower bound for $|\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x|$ by a power of $\mathrm{Disc}(R_x)$, with R_x the center of $\mathrm{End}(A_x)$ [99, p. 17], [100, Thm. 5.1]. With Lemma 10.2, Corollary 3.5 gives it with effective constants.

Theorem 10.3. *Let $g \geq 1$ and $\delta \in (0, 1/2]$. Let C_*, μ and $\bar{B}$ be the constants of Corollary 3.5 and Table 1 for $(\mathrm{PGSp}_{2g}, X^{\mathrm{ad}})$ with $(\mathrm{Ad}, \mathfrak{sp}_{2g}(\widehat{\mathbb{Z}}))$ at the level $\lambda(K(3))$, s_g as in Theorem 9.9, and $C_{\mathcal{O}}$ as in Lemma 10.2, formed with this $\bar{B}$. Put*

$$\mu_g := \frac{\mu}{2g(3g-1)}, \qquad c_g(\delta) := \frac{C_*}{s_g}\, C_{\mathcal{O}}^{-2\mu_g}.$$

(a) For every CM point x of $\mathcal{A}_{g,1}$ such that $\mathrm{QRH}(\delta; L)$ holds for the splitting field L of its Mumford–Tate group,

$$|\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x| \;\geq\; c_g(\delta)\operatorname{Disc}(R_x)^{\mu_g},$$

with R_x the center of $\mathrm{End}(A_x)$.

(b) The same holds, with the same constants, for every CM point x of $\mathcal{A}_{g,K(N)} = \mathrm{Sh}_{K(N)}(G, X)_{\mathbb{C}}$, $K(N)$ the principal congruence subgroup of level $N \geq 1$.

(c) For a number field F, $[F(x) : F] \geq c_g(\delta)\operatorname{Disc}(R_x)^{\mu_g}/[F : \mathbb{Q}]$ in (a) and (b).

The constants are effectively computable from g and δ, and μ_g does not depend on δ.

Proof. Step 1. Let x be as in (a) and C_*, μ, $\bar{B}$ the constants of the statement, at the level $\lambda(K(3))$; form $\bar{\alpha} := \alpha_V$ for $V = \{x\}$ as in Lemma 10.2, at the level $\lambda(K(1))$ and with this $\bar{B}$. We show

$$|\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x| \;\geq\; (C_*/s_g)\,\bar{\alpha}\, d_L^{\mu}.$$

Write $x = [h, 1]$ with $h \in \mathcal{H}_g$, and put $x' := [h, 1] \in \mathrm{Sh}_{K(3)}(G, X)$ and $\bar{x} := [\lambda \circ h, 1] \in \mathrm{Sh}_{\lambda(K(3))}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$. The map π of Section 9.5 and the map $\mathrm{Sh}_{K(3)}(G, X) \to \mathrm{Sh}_{\lambda(K(3))}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ induced by λ are defined over $E(G, X) = \mathbb{Q}$ (Section 2.2), and they map x' to x and to $\bar{x}$. The fibers of π are orbits of $\overline{G}$, so they have at most s_g points, and $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x' \subset \pi^{-1}(\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x)$; the second map carries $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x'$ onto $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot \bar{x}$. So $|\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x| \geq |\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot \bar{x}|/s_g$. The level $\lambda(K(3))$ satisfies (Lv′) for $(\mathrm{Ad}, \mathfrak{sp}_{2g}(\widehat{\mathbb{Z}}))$ (proof of Theorem 9.9(a) at $N = 3$), and $\lambda \circ h$ lies in $X^{\mathrm{ad},+}$, the connected component of X^{ad} containing $\lambda(\mathcal{H}_g)$, which we take as the X^+ of Corollary 3.5 for the adjoint datum. The point $\bar{x}$ is special with Mumford–Tate group $\bar{T} = \lambda(T)$, whose splitting field is L, as T is isogenous to $\mathbb{G}_{\mathrm{m}} \times \bar{T}$ (end of the proof of Theorem 9.9). So Corollary 3.5 gives $|\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot \bar{x}| \geq |\operatorname{Gal}(\overline{\mathbb{Q}}/L) \cdot \bar{x}| \geq C_*\, \alpha_{\bar{x}}\, d_L^{\mu}$, with $\alpha_{\bar{x}}$ at the level $\lambda(K(3))$. Finally $\lambda(K(3)) \subset \lambda(K(1))$, so each index $\bar{\iota}_p$ is at least as large at $\lambda(K(3))$ as at $\lambda(K(1))$, and $\alpha_{\bar{x}} \geq \bar{\alpha}$, both being formed with $\bar{B}$.

(a) With $N_2 = g(2g-1)$ we have $3N_2 + g = 2g(3g-1)$, so $\mu_g = \mu/(3N_2 + g)$, and Lemma 10.2, with $\alpha_V = \bar{\alpha}$ and $C_{\mathcal{O}}$ formed with $\bar{B}$, gives

$$\operatorname{Disc}(R_x)^{\mu_g} \leq C_{\mathcal{O}}^{2\mu_g}\, \bar{\alpha}^{12N_2\mu_g}\, d_L^{\mu}.$$

Here $12N_2\mu_g = 6(2g-1)\mu/(3g-1) \leq 4\mu \leq 1$, as $\mu \leq 1/32$ (Table 1, with $n_T \geq 2$ and $s_\rho, t_\rho \geq 1$). So $\bar{\alpha}^{12N_2\mu_g} \leq \bar{\alpha}$, as $\bar{\alpha} \geq 1$, and Step 1 gives (a).

(b) The natural map $\pi_N \colon \mathcal{A}_{g,K(N)} \to \mathcal{A}_{g,1}$ is defined over $\mathbb{Q}$ and forgets the level structure, so $A_{\pi_N(x)} \cong A_x$, and x and $\pi_N(x)$ have the same R_x, Mumford–Tate group and L; and $|\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x| \geq |\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot \pi_N(x)|$. So (a) gives (b).

(c) $[F(x) : F] = |\operatorname{Gal}(\overline{\mathbb{Q}}/F) \cdot x| \geq |\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x|/[F : \mathbb{Q}]$, as $\operatorname{Gal}(\overline{\mathbb{Q}}/F)$ has index $[F : \mathbb{Q}]$ in $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. □

Corollary 10.4. *Let $g \geq 1$ and $\delta \in (0, 1/2]$, let Z be as in Theorem 1.4(ii) for $\mathcal{A}_{g,1}$ or as in Theorem 1.4(i) for $\mathcal{A}_{g,K(N)}$ with $N \geq 3$, and let x be a special point maximal in Z such that $\mathrm{QRH}(\delta; L)$ holds for the splitting field L of its Mumford–Tate group. For $V := \{x\}$ let c be the right side of the bound for $\max(\deg V, d_{L_V}, \alpha_V)$ in Theorem 9.9(b), resp. for $\Delta(V)$ in Theorem 9.9(a), B the constant with which α_V is formed there, and $C_{\mathcal{O}}$ as in Lemma 10.2, formed with this B. Then*

$$\mathrm{Disc}(R_x) \;\leq\; C_{\mathcal{O}}^2\, c^{15g(2g-1)+g},$$

with R_x the center of $\mathrm{End}(A_x)$.

Proof. Let α_V and d_{L_V} be as in Theorem 9.9, so that $\alpha_V, d_{L_V} \leq c$, with $L_V = L$ (Theorem 1.4). On $\mathcal{A}_{g,1}$, Lemma 10.2 with $\bar{B}_0 := B$ gives $\mathrm{Disc}(R_x) \leq C_{\mathcal{O}}^2 \alpha_V^{12N_2} d_{L_V}^{3N_2+g} \leq C_{\mathcal{O}}^2 c^{15N_2+g}$, with $N_2 = g(2g-1)$. On $\mathcal{A}_{g,K(N)}$, the map π_N of the proof of Theorem 10.3(b) keeps R_x, the Mumford–Tate group and L. For $x = [h, a]$, $\lambda(aK(N)a^{-1}) \subset \lambda(aK(1)a^{-1})$, so each local index of Definition 2.1 for $\pi_N(x)$ at the level $\lambda(K(1))$ is at most the one for x at $\lambda(K(N))$; as $j \mapsto \max(1, Bj)$ is non-decreasing and is 1 at $j = 1$, $\alpha_{\{\pi_N(x)\}} \leq \alpha_V \leq c$, both formed with B, and Lemma 10.2 for $\pi_N(x)$ gives the bound. □

Part II. Listing

11. Listing in Mumford's moduli spaces

The bounds of Theorem 9.9 leave a finite set of candidates for the maximal special subvarieties of Z, but the set has to be found. The bounds for d_{L_V} and α_V bound the index of the order of endomorphisms attached to V (Section 10), which leaves finitely many PEL data; in Mumford's model (Section 11.1) the locus of each datum is computable (Section 11.2), and V is rigid in the normalization of one of its components (Section 11.3), with degree there bounded in terms of $\deg V$ (Sections 11.4 and 11.5). Urbanik's algorithm [104, Cor. 6.3] then computes a finite set of weakly special subvarieties of bounded degree that contains the rigid ones, a test on the algebraic monodromy selects the rigid ones, which are special (Section 11.6), and Section 11.7 states the algorithm and proves Theorem 11.14. In Mumford's model the degree of Z is read off the equations of its closure in Mumford's projective space (Lemma 11.8), so the algorithm runs from these equations for every closed irreducible Z (Theorem 1.5).

We keep the setting and notation of Section 10: $G = \mathrm{GSp}(W, \psi)$, $X = \mathcal{H}_g^{\pm}$, $\lambda\colon G \to G^{\mathrm{ad}}$, and T, $\bar{T}$ for a special subvariety V.

11.1. **Mumford's model and its level.** To list special subvarieties we need $\mathcal{A}_g$ and its universal family by explicit equations. Let d_Υ be a positive multiple of 8, $\Upsilon := (d_\Upsilon, \dots, d_\Upsilon)$ with g entries, $K(\Upsilon) := (\mathbb{Z}/d_\Upsilon\mathbb{Z})^g$, and V_Υ the $\mathbb{Q}$-vector space of functions on $K(\Upsilon)$, so that $\mathbb{P}(V_\Upsilon)$ has coordinates $Q(a)$, $a \in K(\Upsilon)$. Mumford considers the functor $\mathcal{M}_\Upsilon$ of abelian schemes with an Υ-marking: an abelian scheme $\mathcal{X}$ over S, a relatively ample, totally symmetric, normalized invertible sheaf L on $\mathcal{X}$ of type Υ, and a symmetric ϑ-structure for L [74, pp. 78–79]. An Υ-marking embeds $\mathcal{X}$ in $\mathbb{P}(V_\Upsilon) \times S$, and the theta-null map $t\colon \mathcal{M}_\Upsilon \to \mathbb{P}(V_\Upsilon)$ takes the image of the zero section [74, pp. 82–83]. Mumford writes down a closed subscheme $\overline{M}_\Upsilon \subset \mathbb{P}(V_\Upsilon)$, cut out by Riemann's quartic theta relations and the relations $Q(a) = Q(-a)$, and a closed subscheme $\overline{\mathcal{A}}_\Upsilon \subset \mathbb{P}(V_\Upsilon) \times \overline{M}_\Upsilon$, cut out by relations of bidegree $(2, 2)$ in the coordinates of the two factors [74, p. 83], and he proves the following [74, Theorem, p. 83, and Further Theorem, p. 84]: there is an open subscheme $M_\Upsilon \subset \overline{M}_\Upsilon$ such

that t is an isomorphism of $\mathcal{M}_\Upsilon$ with the functor of points of M_Υ, and $\mathcal{A}_\Upsilon := \overline{\mathcal{A}}_\Upsilon \times_{\overline{M}_\Upsilon} M_\Upsilon$ is the universal abelian scheme over M_Υ, with its zero section, embedded in $\mathbb{P}(V_\Upsilon) \times M_\Upsilon$; in particular the origin of the fiber over $s \in M_\Upsilon$ is the point s itself. We use these schemes over $\mathbb{Q}$. The open set M_Υ is not given explicitly; the algorithm computes $M_\Upsilon \otimes \overline{\mathbb{Q}}$ from Mumford's equations and those of the closure of Z (Lemma 11.3).

Over $\mathbb{C}$, $M_\Upsilon \otimes \mathbb{C}$ is a Shimura variety $\mathrm{Sh}_{K_\Upsilon}(G, X)$: a point is a principally polarized abelian variety A with a similitude $\eta\colon W_{\widehat{\mathbb{Z}}} \to H_1(A, \widehat{\mathbb{Z}})$, and η, η' give the same point if and only if $\eta' = \alpha\eta k$ with $\alpha \in \mathrm{Aut}(A, \varphi_0)$ and k in the subgroup $K_\Upsilon \subset G(\widehat{\mathbb{Z}})$ of elements for which η and ηk induce the same Υ-marking on A. Here φ_0 is the principal polarization, L is the totally symmetric sheaf with $\varphi_L = d_\Upsilon \varphi_0$, which has type Υ and is unique [73, p. 307], and the marking induced by η is the symmetric ϑ-structure for L that [73, §2, Remark 3, p. 319] attaches to the symplectic isomorphism $H(L^{\otimes 2}) = A[2d_\Upsilon] \cong H(2\Upsilon)$ defined by η as in the proof of Lemma 11.1(i); every Υ-marking on A arises in this way [73, §2, Remark 4]. Let $e_1, \dots, e_g, f_1, \dots, f_g$ be a symplectic basis of $W_\mathbb{Z}$, with $\psi(e_i, e_j) = \psi(f_i, f_j) = 0$, $\psi(e_i, f_i) = 1$ and $\psi(e_i, f_j) = 0$ for $i \neq j$, let ν be the similitude character, and for $m \geq 1$ put

$$K'(m) := \{k \in G(\widehat{\mathbb{Z}}) : k \equiv \mathrm{diag}(1_g, \nu(k)1_g) \bmod m\} \supset K(m),$$

and $\Gamma(m) := \mathrm{Sp}_{2g}(\mathbb{Z}) \cap K(m)$.

Lemma 11.1. *(i) $K'(2d_\Upsilon) \subset K_\Upsilon \subset K'(d_\Upsilon)$. In particular $K(2d_\Upsilon) \subset K_\Upsilon$ and $K_\Upsilon \cap \mathrm{Sp}(W)(\mathbb{A}_f) \subset K(d_\Upsilon)$.*

(ii) $\nu(K_\Upsilon) = \widehat{\mathbb{Z}}^\times$. Hence $S := M_\Upsilon \otimes \mathbb{C}$ is connected, $S = \Gamma_\Upsilon \backslash \mathcal{H}_g$ with $\Gamma_\Upsilon := \mathrm{Sp}_{2g}(\mathbb{Z}) \cap K_\Upsilon$, and $\Gamma(2d_\Upsilon) \subset \Gamma_\Upsilon \subset \Gamma(d_\Upsilon)$.

(iii) K_Υ is not neat. But if $(H', X_{H'})$ is a Shimura subdatum of (G, X), $X^+_{H'}$ a connected component of $X_{H'}$ and $h \in G(\mathbb{A}_f)$, then every subgroup of $H'(\mathbb{Q}) \cap hK_\Upsilon h^{-1}$ that preserves $X^+_{H'}$ is neat. In particular $G(\mathbb{Q})_+ \cap hK_\Upsilon h^{-1}$ is neat for every h.

Proof. (i) Mumford defines $H(\Upsilon) := K(\Upsilon) \oplus \widehat{K}(\Upsilon)$ with $\widehat{K}(\Upsilon) := \mathrm{Hom}(K(\Upsilon), \mathbb{C}^*)$ [73, p. 294], so the commutator pairing of the Heisenberg group $\mathcal{G}(\Upsilon)$ takes values in μ_{d_Υ} with no choice of a root of unity. For L of type Υ the group $H(L)$ is $A[d_\Upsilon]$, and an Υ-marking induces a symplectic isomorphism $H(L) \cong H(\Upsilon)$. On $A[d_\Upsilon]$ the Weil pairing $\langle\ ,\ \rangle$ satisfies $\langle \eta v, \eta w\rangle = \zeta^{\nu(\eta)\psi(v,w)}$ with $\zeta := \exp(2\pi i/d_\Upsilon)$; the other sign convention replaces ζ by ζ^{-1} and changes nothing below. The isomorphism attached to η matches $a \in K(\Upsilon)$ with $\sum_i a_i \eta e_i$, and the character $a \mapsto \zeta^{\sum_i a_i b_i}$ of $K(\Upsilon)$ with the unique $P \in \sum_i (\mathbb{Z}/d_\Upsilon\mathbb{Z})\,\eta f_i$ such that $\langle \eta e_i, P\rangle = \zeta^{b_i}$ for all i, namely $P = \nu(\eta)^{-1} \sum_i b_i \eta f_i$. So η and ηk give the same isomorphism if and only if $ke_i \equiv e_i$ and $\nu(k)^{-1} k f_i \equiv f_i$ modulo d_Υ for all i, that is, $k \in K'(d_\Upsilon)$. Since the Υ-marking determines this isomorphism, $K_\Upsilon \subset K'(d_\Upsilon)$.

$K'(2d_\Upsilon) \subset K_\Upsilon$. The same computation for $L^{\otimes 2}$, which has type 2Υ and $H(L^{\otimes 2}) = A[2d_\Upsilon]$, shows that η and ηk induce the same symplectic isomorphism $H(L^{\otimes 2}) \cong H(2\Upsilon)$ if and only if $k \in K'(2d_\Upsilon)$; and by [73, §2, Remark 3, p. 319] such an isomorphism determines the symmetric ϑ-structure for L. So $K'(2d_\Upsilon) \subset K_\Upsilon$.

An element $k \in K'(d_\Upsilon)$ changes the marking by an automorphism that depends only on k. For $k \in K'(d_\Upsilon)$ write $k = (1 + d_\Upsilon M)\,\mathrm{diag}(1_g, \nu(k)1_g)$. By the formula for P at level $2d_\Upsilon$, the isomorphisms $H(L^{\otimes 2}) \cong H(2\Upsilon)$ attached to ηk and η differ by $\mathrm{diag}(1_g, \nu(\eta)1_g)(1 + d_\Upsilon M)\,\mathrm{diag}(1_g, \nu(\eta)^{-1}1_g)$, which is $1 + N$ with $N := d_\Upsilon M$ modulo $2d_\Upsilon$, as $\nu(\eta)$ is odd and the entries of N are 0 or d_Υ; so it depends only on k, and $N(H(2\Upsilon)) \subset H(2\Upsilon)[2]$. In blocks for $K(2\Upsilon) \oplus \widehat{K}(2\Upsilon)$, $1 + N = (1 + N_\mathrm{d})(1 + N_\mathrm{o})$ with N_d block-diagonal and N_o off-diagonal, both

factors symplectic. For $1+N_{\mathrm{o}}$, the computation after [73, §2, Remark 3, p. 319] shows that the marking is replaced by its composite with the automorphism $F_\sigma\colon (t,y,m)\mapsto(\sigma(y,m)t,y,m)$ of $\mathcal{G}(\Upsilon)=\mathbb{C}^*\times K(\Upsilon)\times\widehat{K}(\Upsilon)$ [73, pp. 294 and 318], where $\Delta_1\colon K(2\Upsilon)\to\widehat{K}(2\Upsilon)$ and $\Delta_2\colon\widehat{K}(2\Upsilon)\to K(2\Upsilon)$ are the two blocks of N_{o}, $\bar{l}$ is the image of l in $\widehat{K}(\Upsilon)$, and $\sigma((2x,\bar{l}))=\Delta_1x(x)\cdot l(\Delta_2 l)$ for $x\in K(2\Upsilon)$ and $l\in\widehat{K}(2\Upsilon)$; this depends only on N_{o}. The factor $1+N_{\mathrm{d}}$ acts by $x\mapsto\varepsilon x$ and $l\mapsto l\circ\varepsilon^{-1}$, with ε its block on $K(2\Upsilon)$. Recall Mumford's homomorphisms $D_{-1}\colon\mathcal{G}(2\Upsilon)\to\mathcal{G}(2\Upsilon)$, $(t,x,l)\mapsto(t,-x,-l)$, $H_2\colon\mathcal{G}(2\Upsilon)\to\mathcal{G}(\Upsilon)$, $(t,x,l)\mapsto(t^2,2x,\bar{l})$, and $E_2\colon\mathcal{G}(\Upsilon)\to\mathcal{G}(2\Upsilon)$, $(t,x,l)\mapsto(t^2,x,2*l)$, where $K(\Upsilon)\subset K(2\Upsilon)$ by $a\mapsto 2a$ and $(2*l)(x):=l(2x)$ [73, p. 316]. The automorphism $U\colon(t,x,l)\mapsto(t,\varepsilon x,l\circ\varepsilon^{-1})$ of $\mathcal{G}(2\Upsilon)$ commutes with D_{-1} and satisfies $H_2\circ U=H_2$ and $U\circ E_2=E_2$, as $\varepsilon-1$ has values of order 2; so U carries a symmetric lift of the first isomorphism to one of the second with the same composite with H_2, and by [73, §2, Remarks 1 and 3] the marking does not change. Hence the markings induced by ηk and η differ by an automorphism of $\mathcal{G}(\Upsilon)$ that depends only on k, so K_Υ does not depend on (A,η).

K_Υ is a group and a product level. It is a group, since for $k,k'\in K_\Upsilon$ the pair $(A,\eta k)$ gives that $\eta kk'$ induces the marking of ηk, which is that of η, and likewise ηk^{-1} that of $\eta k^{-1}k=\eta$. Elements of $K'(d_\Upsilon)$ with $\nu=1$ are $\equiv 1$ modulo d_Υ. Both $K'(d_\Upsilon)$ and $K'(2d_\Upsilon)$ are product levels with the same factor at every odd p, so K_Υ is a product level.

(ii) $\mathrm{diag}(1_g,u1_g)\in K'(2d_\Upsilon)$ for every $u\in\widehat{\mathbb{Z}}^\times$, so $\nu(K_\Upsilon)=\widehat{\mathbb{Z}}^\times$. As Sp_{2g} is simply connected, $\pi_0(\mathrm{Sh}_K(G,X)_{\mathbb{C}})=\mathbb{Q}_{>0}\backslash\mathbb{A}_f^\times/\nu(K)=\widehat{\mathbb{Z}}^\times/\nu(K)$ for every compact open $K\subset G(\widehat{\mathbb{Z}})$ [30, 2.1.3], which is trivial for $K=K_\Upsilon$. An element of $G(\mathbb{Q})_+\cap K_\Upsilon$ has similitude in $\mathbb{Q}_{>0}\cap\widehat{\mathbb{Z}}^\times=\{1\}$, so it lies in $\mathrm{Sp}_{2g}(\mathbb{Z})\cap K_\Upsilon=\Gamma_\Upsilon$, and the inclusions follow from (i).

(iii) The element $\mathrm{diag}(1_g,-1_g)$ of $G(\mathbb{Z})$ has similitude -1 and lies in $K'(2d_\Upsilon)\subset K_\Upsilon$; its eigenvalues at every p are ±1, so it is not neat [81, 0.6]. Let Γ' be a subgroup of $H'(\mathbb{Q})\cap hK_\Upsilon h^{-1}$ that preserves $X^+_{H'}$. Then Γ' preserves the component of $\mathcal{H}_g^\pm$ that contains $X^+_{H'}$, so $\nu(\Gamma')\subset\mathbb{Q}_{>0}\cap\nu(K_\Upsilon)=\{1\}$, and $\Gamma'\subset G(\mathbb{Q})\cap h(K_\Upsilon\cap\mathrm{Sp}(W)(\mathbb{A}_f))h^{-1}\subset G(\mathbb{Q})\cap hK(d_\Upsilon)h^{-1}$ by (i). As $d_\Upsilon\geq 3$, $K(d_\Upsilon)$ is neat, and neatness passes to conjugates and to subgroups [81, 0.6]. □

So K_Υ is not a neat level in the sense of Section 2.1, but S is the quotient of $\mathcal{H}_g$ by the neat group Γ_Υ, and everything below uses only the neatness of the arithmetic groups in (iii). In particular the bundle of top-degree forms on S extends to the Baily–Borel compactification as a line bundle $\mathcal{L}_{K_\Upsilon}$ [75, Prop. 3.4(b)], whose sections are the modular forms of weight $g+1$ for Γ_Υ, and degrees on S are taken for it. The bounds of Section 9.5 are proved at neat product levels, and reach K_Υ through $K(2d_\Upsilon)$.

Lemma 11.2. *Let $K':=K(2d_\Upsilon)$, $\pi\colon\mathrm{Sh}_{K'}(G,X)\to S$ the natural map, and $c_\Upsilon:=[K_\Upsilon:K']\leq|\mathrm{GSp}_{2g}(\mathbb{Z}/2d_\Upsilon\mathbb{Z})|$. Let $Z\subset S$ be closed and geometrically irreducible, defined over a number field F with $[F:\mathbb{Q}]=d$, with $D:=\deg_{\mathcal{L}_{K_\Upsilon}}Z$, and let V be special and maximal in Z. Then there are a geometrically irreducible component Z' of $\pi^{-1}(Z)$, defined over a field of degree at most $c_\Upsilon d$, with $\deg_{\mathcal{L}_{K'}}Z'\leq c_\Upsilon D$, and a special subvariety V' maximal in Z' with $\pi(V')=V$, $T_{V'}=T_V$, $\deg_{\mathcal{L}_{K_\Upsilon}}V\leq\deg_{\mathcal{L}_{K'}}V'$ and $\alpha_V\leq\alpha_{V'}$, where α_V is computed at the level $\lambda(K_\Upsilon)$ and $\alpha_{V'}$ at $\lambda(K')$.*

Proof. This is the proof of Lemma 2.2 with $K_\Upsilon\supset K'$ in place of $K\supset K'$; that proof uses the neatness of K only through the arithmetic groups, which are neat here by Lemma 11.1(iii). Each component of $\mathrm{Sh}_{K'}(G,X)_{\mathbb{C}}$ is $\Gamma(2d_\Upsilon)\backslash\mathcal{H}_g$, and π restricted to it is the covering $\Gamma(2d_\Upsilon)\backslash\mathcal{H}_g\to\Gamma_\Upsilon\backslash\mathcal{H}_g$ of quotients by neat groups, so π is finite étale of degree at most

c_Υ; top-degree forms pull back to top-degree forms, and the two extensions are given by modular forms of weight $g+1$, so $\pi^*\mathcal{L}_{K_\Upsilon} = \mathcal{L}_{K'}$. If V is the image of $X_H^+ \times \{\gamma\}$, let V' be the image of $X_H^+ \times \{\gamma\}$ in $\mathrm{Sh}_{K'}(G,X)$: a component of $\pi^{-1}(V)$ with the same datum (H, X_H), so $T_{V'} = T_V$. Let Z' be a component of $\pi^{-1}(Z)$ containing V'. As π is defined over $\mathbb{Q}$ (the Υ-marking induced by η involves no choice of a root of unity), $\pi^{-1}(Z)$ is defined over F and has at most c_Υ geometric components, Z' is defined over an extension of degree at most c_Υ; by the projection formula $\deg_{\mathcal{L}_{K'}} \pi^{-1}(Z) = \deg \pi \cdot D \le c_\Upsilon D$ and $\deg V \le \deg(\pi|_{V'}) \deg V = \deg_{\mathcal{L}_{K'}} V'$. If $V' \subsetneq V'' \subset Z'$ with V'' special, then $\pi(V'') \subset Z$ is special of dimension larger than $\dim V$, against maximality. Finally, with the same γ, $\bar{T}(\mathbb{Q}_p) \cap \lambda(\gamma_p K'_p \gamma_p^{-1}) \subset \bar{T}(\mathbb{Q}_p) \cap \lambda(\gamma_p K_{\Upsilon,p} \gamma_p^{-1})$, so each index $\bar{\iota}_p$ can only grow when K_Υ is replaced by K', and α can only grow, as after the proof of Lemma 2.2. □

So, in the setting of Lemma 11.2, since the constants of Theorem 9.9(a) are non-decreasing in (D,d) (proof of Theorem 9.5), Theorem 9.9(a) with $N = 2d_\Upsilon$, evaluated at $(c_\Upsilon D, c_\Upsilon d)$, bounds $\deg_{\mathcal{L}_{K_\Upsilon}} V$, and, if $\mathrm{QRH}(\delta; L_V)$ holds, d_{L_V} and α_V.

Lemma 11.3. *Let $Z \subset M_\Upsilon \otimes \overline{\mathbb{Q}}$ be closed and irreducible. Put $n' := g(g+1)/2$ and $P_\Upsilon(t) := d_\Upsilon^g t^g$. Let $Y := (\overline{M}_\Upsilon \otimes \overline{\mathbb{Q}})_{\mathrm{red}}$, let $Y^\circ \subset Y$ be the open set of points at which Y is smooth over $\overline{\mathbb{Q}}$ of local dimension n', $\mathcal{X}^\circ := \overline{\mathcal{A}}_\Upsilon \times_{\overline{M}_\Upsilon} Y^\circ$, $H_\Upsilon \subset Y^\circ$ the set of points y whose fiber $\mathcal{X}^\circ_y \subset \mathbb{P}(V_\Upsilon)_{\kappa(y)}$ is smooth of pure dimension g with Hilbert polynomial P_Υ, U the interior of H_Υ in Y°, C_Z the irreducible component of Y containing Z, and $N_0 := U \cap C_Z$.*

(i) $\mathcal{X}^\circ \times_{Y^\circ} U \to U$ *is smooth.*

(ii) $M_\Upsilon \otimes \overline{\mathbb{Q}} \subset U$. *In particular* $Z \subset Y^\circ$, *and* C_Z *is well defined.*

(iii) $N_0 = M_\Upsilon \otimes \overline{\mathbb{Q}}$.

(iv) N_0 *can be computed from Mumford's equations and equations for the closure* $\overline{Z}$ *of* Z *in* $\mathbb{P}(V_\Upsilon)$, *as a quasi-projective variety given by explicit equations and inequations.*

Proof. (i) Each $y \in U$ has an open neighborhood $W \subset H_\Upsilon$. As Y° is smooth over $\overline{\mathbb{Q}}$, each connected component of W is integral, and over it the fibers of $\mathcal{X}^\circ$ have the same Hilbert polynomial P_Υ, so $\mathcal{X}^\circ_W$ is flat over W [54, III.9.9]. A flat morphism of finite presentation with smooth fibers is smooth [49, 17.5.1].

(ii) $M_\Upsilon \otimes \overline{\mathbb{Q}}$ is open in $\overline{M}_\Upsilon \otimes \overline{\mathbb{Q}}$ and smooth of dimension n' (it is a fine moduli space, and $S = \Gamma_\Upsilon \backslash \mathcal{H}_g$ with Γ_Υ neat by Lemma 11.1), so it is an open subscheme of Y contained in Y°. Its fibers are abelian varieties embedded by a sheaf L of type Υ, which are smooth of pure dimension g with Hilbert polynomial $\chi(L^{\otimes t}) = t^g \chi(L) = d_\Upsilon^g t^g$, because $\chi(L)^2 = \deg \varphi_L = |K(L)| = d_\Upsilon^{2g}$ and $\chi(L) > 0$ for ample L [70, Thm. 11.1]. So $M_\Upsilon \otimes \overline{\mathbb{Q}}$ is an open subset of Y° contained in H_Υ, hence in U. A point of Y° lies on exactly one irreducible component of Y.

(iii) Two distinct components of Y of dimension n' meet only at points where Y is not smooth, so the connected components of Y° are the non-empty $C_j \cap Y^\circ$, for the components C_j of Y of dimension n'; as each $U \cap C_j$ is open in the irreducible C_j, the connected components of U are the non-empty $U \cap C_j$. By (ii), $Z \subset U \cap C_Z = N_0$, and N_0 is connected. The family $\overline{\mathcal{A}}_\Upsilon \times_{\overline{M}_\Upsilon} N_0$ is smooth over N_0 by (i), and every geometric point of Z is in the image of $\mathcal{M}_\Upsilon$ under t. Mumford's Step IV [74, p. 94] says, with Υ for his δ: "Suppose that N is a connected subscheme of $\overline{M}_\Upsilon$ such that if $\overline{\mathcal{A}}_\Upsilon$ induces a subscheme B over N, then B is smooth over N. Suppose that some geometric point x of N is in the image of $\mathcal{M}_\Upsilon$ under t. Then the N-valued point 1_N of N is in the image of $\mathcal{M}_\Upsilon$." Its proof [74, pp. 94–95] uses N only as a connected scheme with a morphism to $\overline{M}_\Upsilon$, so it applies to $N_0 \to \overline{M}_\Upsilon$. Hence

$N_0 \subset M_\Upsilon \otimes \overline{\mathbb{Q}}$. Now N_0 is open in U, hence open in $M_\Upsilon \otimes \overline{\mathbb{Q}}$, and closed in $U \supset M_\Upsilon \otimes \overline{\mathbb{Q}}$, hence closed in $M_\Upsilon \otimes \overline{\mathbb{Q}}$. By Lemma 11.1(ii), $M_\Upsilon \otimes \overline{\mathbb{Q}}$ is connected, so $N_0 = M_\Upsilon \otimes \overline{\mathbb{Q}}$.

(iv) The components of Y are computed by absolute primary decomposition, Y° by the Jacobian criterion, and C_Z by ideal membership, as the one component of Y that contains $\overline{Z}$: a component containing $\overline{Z}$ contains $Z \subset Y^\circ$, and a point of Y° lies on exactly one component (ii). A flattening stratification, computable by parametric Gröbner bases [104, Prop. 5.1(i)], applied to $\mathcal{X}^\circ \to Y^\circ$, gives locally closed $T_1, \dots, T_k$ covering Y° over which $\mathcal{X}^\circ$ is flat. Over each irreducible component T' of each T_i the fibers have one Hilbert polynomial [54, III.9.9], computed at the generic point by a Gröbner basis. Let H'_Υ be the set of $y \in Y^\circ$ whose fiber has Hilbert polynomial P_Υ and whose Jacobian matrix has rank $\dim \mathbb{P}(V_\Upsilon) - g$ at every point of the fiber. The points of $\mathcal{X}^\circ_{T'}$ at which the Jacobian matrix of the fiber has rank different from $\dim \mathbb{P}(V_\Upsilon) - g$ form a constructible set, and its image in T' is constructible and computable [6, Thm. 1.0]. So H'_Υ is constructible and computable, and so is its interior U' in Y°, the complement of the closure of $Y^\circ \smallsetminus H'_\Upsilon$. Clearly $H_\Upsilon \subset H'_\Upsilon$, so $U \subset U'$. Conversely, let $y \in U' \cap C_Z$ and let $W \subset U' \cap C_Z$ be an irreducible open neighborhood of y in Y°. The open set $M_\Upsilon \otimes \overline{\mathbb{Q}}$ meets C_Z (it contains Z), so W contains the generic point of C_Z, over which the fiber is an abelian variety. As the fibers over W all have Hilbert polynomial P_Υ, $\mathcal{X}^\circ_W$ is flat over the integral W [54, III.9.9], so its associated points lie over the generic point of W; hence $\mathcal{X}^\circ_W$ is integral of dimension $n' + g$, and every fiber over W is of pure dimension g [54, III.9.6]. On a fiber of pure dimension g, rank $\dim \mathbb{P}(V_\Upsilon) - g$ of the Jacobian matrix at every point is smoothness. Hence $W \subset H_\Upsilon$, and $U' \cap C_Z = U \cap C_Z = N_0$. □

11.2. **PEL loci.** *PEL data.* A PEL datum $\mathfrak{d} = (\mathcal{E}, R, \tau)$ consists of a product $\mathcal{E} = \prod_i F_i$ of totally real and CM fields with its positive involution $*$ (the identity on each totally real factor and complex conjugation on each CM factor), an order $R \subset \mathcal{O}_\mathcal{E}$ stable under $*$, given by generators $e_1, \dots, e_m$ of R as a $\mathbb{Z}$-module, and a type τ: multiplicities $m_i \geq 1$ with $\sum_i m_i[F_i : \mathbb{Q}] = 2g$, together with, for each CM factor F_i and each place v of its maximal totally real subfield F_i^+, an integer $0 \leq a_v \leq m_i$, the multiplicity on the tangent space of a fixed embedding φ_v of F_i above v (its conjugate then has multiplicity $m_i - a_v$); a totally real F_i carries $m_i/2$ copies of each of its embeddings, m_i even. Then τ gives, for each $r \in R$, a characteristic polynomial $\tau(r)$ of degree g, and we let W' be the $\mathcal{E}$-module whose character is $\tau\bar{\tau}$. The *PEL locus* $\mathfrak{E}(\mathfrak{d}) \subset S$ is the set of points s for which there is a ring homomorphism $\iota\colon R \to \mathrm{End}(A_s)$ with $\iota(r)^\dagger = \iota(r^*)$ for the Rosati involution $\dagger$ of the polarization, and $\det(T - \iota(r) \mid \mathrm{Lie}\, A_s) = \tau(r)$ for all $r \in R$; then $H_1(A_s, \mathbb{Q}) \cong W'$ as $\mathcal{E}$-modules, as $H_1(A_s, \mathbb{C}) = \mathrm{Lie}\, A_s \oplus \overline{\mathrm{Lie}\, A_s}$.

For a special V with torus algebra $\mathcal{E} = \mathbb{Q}[T]$ as in Section 10 we put $\mathfrak{d}_V := (\mathcal{E}, R, \tau_V)$, with $R = \mathcal{E} \cap \mathrm{End}(\Lambda_V)$ and $\tau_V(r)$ the characteristic polynomial of r on $\mathrm{Lie}\, A_x = W_\mathbb{C}/F^0_x$ for $x \in X_H^+$, which does not depend on x, being continuous with values in a countable set.

Endomorphisms by their graphs. Let $s \in N_0$, with abelian variety $A_s \subset \mathbb{P}(V_\Upsilon)$ embedded by L of type Υ. A closed subscheme $\Gamma \subset A_s \times A_s$ such that $\mathrm{pr}_1|_\Gamma$ is an isomorphism and $(0,0) \in \Gamma$ is the graph of the homomorphism $f := \mathrm{pr}_2 \circ (\mathrm{pr}_1|_\Gamma)^{-1}$, since a regular map of abelian varieties is a homomorphism followed by a translation [70, Cor. 1.2]. On $\Gamma_f \cong A_s$ the bundle $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$ restricts to $L \otimes f^*L$, and $\varphi_{L \otimes f^*L} = \varphi_L \circ (1 + f^\dagger f)$, so

$$\chi((L \otimes f^*L)^{\otimes t}) = t^g\, d^g_\Upsilon\, \det_{H_1(A_s,\mathbb{Q})}(1 + f^\dagger f)^{1/2}, \tag{11.1}$$

because $\chi(M)^2 = \deg \varphi_M$ and $\chi(M) > 0$ for ample M, $\chi(M^{\otimes t}) = t^g \chi(M)$, and the degree of an endomorphism is its determinant on $H_1(A_s, \mathbb{Q})$ [70, Thm. 11.1 and Prop. 10.20]. If

$f = \iota(e)$ with $\iota(e)^\dagger = \iota(e^*)$, this is the polynomial $P_e(t) := t^g d_\Upsilon^g \det_{W'}(1 + e^* e)^{1/2}$, computed from $\mathfrak{d}$.

Lemma 11.4. *Let B be a quasi-projective variety over a number field, $\mathcal{X} \subset \mathbb{P}^M \times B$ a closed subscheme whose ideal sheaf is generated by explicitly given bihomogeneous $F_1, \dots, F_k$ of degrees at most m in the coordinates x of $\mathbb{P}^M$, and $p(t)$ a Hilbert polynomial with Gotzmann number r_0. Put $r := \max(r_0, m)$, $N(t) := \binom{M+t}{M}$ and $q(t) := N(t) - p(t)$. Then the relative Hilbert scheme $\mathrm{Hilb}^p(\mathcal{X}/B)$ is the closed subscheme of $\mathrm{Hilb}^p(\mathbb{P}^M) \times B \subset \mathrm{Gr}(q(r), N(r)) \times B$ defined by $x^\alpha F_j(\cdot\,; b) \in \Omega$ for all j and all monomials x^α of degree $r - \deg_x F_j$, where $\Omega \subset \overline{\mathbb{Q}}[x]_r$ is the point of the Grassmannian; and its universal family is cut out in $\mathbb{P}^M \times \mathrm{Hilb}^p(\mathcal{X}/B)$ by Ω.*

Proof. For the absolute case, by Gotzmann's regularity theorem [46] the ideal sheaf of a subscheme $\mathcal{Z} \subset \mathbb{P}^M_A$ with Hilbert polynomial p is r-regular, $I(\mathcal{Z})_{\geq r}$ is generated by $I(\mathcal{Z})_r$, and $\mathcal{Z} \mapsto I(\mathcal{Z})_r$ embeds $\mathrm{Hilb}^p(\mathbb{P}^M)$ in $\mathrm{Gr}(q(r), N(r))$ as a closed subscheme with explicit equations, for every $r \geq r_0$ by the proofs of [67, Thms. 1.21, 1.24 and 1.25, (1.22), (1.23)], written there for $r = r_0$. For a T-point $T \to B$ and $\mathcal{Z} \subset \mathbb{P}^M_T$ flat with Hilbert polynomial p, let Ω be the module of sections of $I_{\mathcal{Z}}(r)$, locally a direct summand of rank $q(r)$. If F_j vanishes on $\mathcal{Z}$, then $x^\alpha F_j \in \Omega$. Conversely, if $x^\alpha F_j \in \Omega$ for all $|\alpha| = r - \deg F_j$, then on $D_+(x_i)$ the function $F_j / x_i^{\deg F_j} = x_i^{r - \deg F_j} F_j / x_i^r$ is a section of $I_{\mathcal{Z}}$, so F_j vanishes on $\mathcal{Z}$. Membership of a given vector in a subbundle of rank $q(r)$ is the vanishing of exterior products, a closed condition compatible with base change. □

Let $\mathcal{H}$ be a scheme of finite type over a number field k, $\mathcal{A} \to \mathcal{H}$ an abelian scheme of relative dimension g embedded in $\mathbb{P}(V_\Upsilon) \times \mathcal{H}$, and $\Gamma \subset \mathcal{A} \times_{\mathcal{H}} \mathcal{A}$ closed, flat and proper over $\mathcal{H}$ with fibers Γ_t of dimension g; in our application $\mathcal{H}$ is one of the relative Hilbert schemes $\mathcal{H}_{i,n}$ in the proof of Lemma 11.6, $\mathcal{A}$ the pull-back of the universal abelian scheme over N_0, and Γ the universal subscheme. Let $\pi_t := \mathrm{pr}_1|_{\Gamma_t} \colon \Gamma_t \to \mathcal{A}_t$ and $U_{\mathrm{iso}} := \{t \in \mathcal{H} : \pi_t \text{ is an isomorphism}\}$. As $\mathrm{pr}_1 \colon \Gamma \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{H}$ are proper, and Γ_t and the integral $\mathcal{A}_t$ have dimension g, π_t is an isomorphism if and only if it is a monomorphism, if and only if every fiber $\pi_t^{-1}(y)$, $y \in \mathcal{A}_t$, is empty or $\kappa(y)$-isomorphic to $\operatorname{Spec} \kappa(y)$, and U_{iso} is open in $\mathcal{H}$ [49, 18.12.6 and 18.12.7].

Lemma 11.5. *Let $t \in \mathcal{H}$, $\kappa := \kappa(t)$, and $J \subset \kappa[x, y]$ a bihomogeneous ideal whose sheaf is the ideal sheaf of Γ_t in $\mathbb{P}_x \times \mathbb{P}_y$, two copies of $\mathbb{P}(V_\Upsilon)$ with coordinates $x_0, \dots, x_M$ and $y_0, \dots, y_M$; let J' be the same ideal in variables (x, y'). Then π_t is an isomorphism if and only if for every (i, j, l) each minor $y_a y'_b - y_b y'_a$, dehomogenized by $x_i = y_j = y'_l = 1$, lies in the dehomogenization of $J + J' \subset \kappa[x, y, y']$.*

Proof. As $\mathcal{A}_t \to \mathbb{P}_x$ is a closed immersion, $\Gamma_t \times_{\mathcal{A}_t} \Gamma_t = \Gamma_t \times_{\mathbb{P}_x} \Gamma_t$ is the closed subscheme of $\mathbb{P}_x \times \mathbb{P}_y \times \mathbb{P}_{y'}$ with ideal $J + J'$, and the diagonal is cut out in it by the 2×2 minors of the matrix with rows y and y'. So the diagonal is an isomorphism if and only if these minors vanish on $\Gamma_t \times_{\mathcal{A}_t} \Gamma_t$, which on the charts $D_+(x_i y_j y'_l)$ is the stated membership. A morphism is a monomorphism if and only if its diagonal is an isomorphism [97, Tag 01L3]; conclude by the equivalences before the lemma. □

On an affine chart $C = V(I_C) \cap D(h_C)$ of the Hilbert base $\mathcal{H} \subset \mathrm{Gr}(q(r), N(r)) \times N_0$ with coordinates u, where I_C includes the Hilbert equations with their original scheme structure, the point $\Omega(t)$ of Lemma 11.4 has a basis $w_1(u), \dots, w_{q(r)}(u)$ with coefficients polynomial in u, and pulling back along the Segre map gives generators $w_s(t)(x \otimes y)$ of the ideal sheaf of Γ_t in $\mathbb{P}_x \times \mathbb{P}_y$, admissible in Lemma 11.5 at every $t \in C$, including generic points of

closed subsets. To compute U_{iso}, run the induction of the proof of [104, Prop. 5.1(iii)] with "reduced fiber" replaced by "π_t is an isomorphism": on each irreducible $V(\mathfrak{p})$ of a chart, decide at the generic point by Lemma 11.5 (the generators above are admissible there); a no loses nothing, as U_{iso} is open; a yes holds on $V(\mathfrak{p}) \cap D(h)$ for h a common denominator of the cofactors that express the minors of Lemma 11.5 in the generators, and one recurses on $V(\mathfrak{p}) \smallsetminus D(h)$, of smaller dimension.

Lemma 11.6. *For every PEL datum $\mathfrak{d}$, the set $\mathfrak{E}(\mathfrak{d}) \cap N_0$ is constructible, and it can be computed from Mumford's equations, N_0 and $\mathfrak{d}$ as a finite union of explicit locally closed subsets of N_0.*

Proof. Let $\mathcal{A} := \overline{\mathcal{A}}_\Upsilon \times_{\overline{M}_\Upsilon} N_0$, the universal abelian scheme over N_0 (Lemma 11.3), and embed $\mathcal{A} \times_{N_0} \mathcal{A}$ in $\mathbb{P}(V_\Upsilon \otimes V_\Upsilon) \times N_0$ by the Segre map, so that $\mathcal{O}(1)$ restricts to $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$; its ideal sheaf is generated by the Segre quadrics and by $F(x;b)y^\beta$, $F(y;b)x^\beta$ ($|\beta| = \deg F$) for Mumford's equations F of $\mathcal{A}$. For each generator e_i and each $n = 1, \dots, 2g+1$, let $\mathcal{H}_{i,n}$ be the relative Hilbert scheme of Lemma 11.4 for the polynomial P_{ne_i}, with universal subscheme $\Gamma_{i,n}$, and let $\mathcal{H}$ be their fiber product over N_0. Impose: (1) pr_1 is an isomorphism on each $\Gamma_{i,n}$ (the open set U_{iso}, computed as above), and $(0,0) \in \Gamma_{i,n}$, i.e. every element of Ω vanishes at $s \otimes s$, as the origin of A_s is the point s; so $\Gamma_{i,n}$ is the graph of a homomorphism $f_{i,n}$. (2) $df_{i,n} = n\, df_{i,1}$, the $\mathbb{Z}$-linear relations among the e_i, and the expressions of 1 and of each product e_ie_j as $\mathbb{Z}$-linear combinations of the e_k, hold for the differentials $df_{i,1}$ at the origin, and $\det(T - df_{i,1} \mid \mathrm{Lie}) = \tau(e_i)$; these are closed conditions on the tangent spaces of the graphs at $(0,0)$. In characteristic 0, $f \mapsto df$ is injective on $\mathrm{End}(A_s)$, by complex uniformization; so (2) gives $f_{i,n} = nf_{i,1}$, a ring homomorphism $\iota\colon R \to \mathrm{End}(A_s)$ with $\iota(e_i) = f_{i,1}$, and the type τ, as the traces on $\mathrm{Lie}\, A_s$, additive in r, agree on the e_i, hence on $\mathcal{E}$, and fix the multiplicity of each embedding. (3) By Equation (11.1), the Hilbert polynomials give $\det(1 + n^2 f^\dagger f) = \det_{W'}(1 + n^2 e^* e)$ for $f = f_{i,1}$, $e = e_i$ and $2g+1$ values of n^2, so $f^\dagger f$ and $e^* e$ have the same characteristic polynomial. Hence $\mathrm{Tr}(f^\dagger f) = \sum |\xi|^2$ over the eigenvalues ξ of f, which is the case of equality in Schur's inequality for the positive definite Rosati form; so f is normal, and $f^\dagger = \iota(e_i^*)$, as both act by $\bar{\xi}$ on the ξ-eigenspace of f. Conversely every $s \in \mathfrak{E}(\mathfrak{d}) \cap N_0$ with its ι gives such graphs. So $\mathfrak{E}(\mathfrak{d}) \cap N_0$ is the image in N_0 of an explicit finite union of locally closed subsets of $\mathcal{H}$, and the image is constructible and computable [6, Thm. 1.0], on finitely many affine charts. □

11.3. **Containers.** We use Urbanik's algorithm [104], with his definitions. Let $\widetilde{S}$ be a smooth quasi-projective variety carrying a polarizable integral variation of Hodge structure $\mathbb{V}$. The algebraic monodromy group H_Y of an irreducible subvariety Y (not the Mumford–Tate group of Section 2.2) is the identity component of the Zariski closure of the monodromy of $\mathbb{V}$ on the normalization of Y; Y is *weakly special* if it is maximal among irreducible subvarieties with its algebraic monodromy group [104, Defs. 1.3–1.4]; and Y is *rigid* if there is no weakly special Y' with $Y \subsetneq Y' \subset \widetilde{S}$ such that H_Y is normal in $H_{Y'}$ [104, Def. 4.28]. Special subvarieties are weakly special. For a projective compactification of $\widetilde{S}$, a very ample line bundle on it and an integer d', [104, Cor. 6.3] (with the remark below) gives an algorithm that computes a finite set of weakly special subvarieties of degree at most d' containing the rigid ones, from equations for $\widetilde{S}$ and the data $(\mathcal{H}, F^\bullet, \nabla)$ of the variation [104, Thm. 2.4].

Remark. The stopping test in the proof of [104, Prop. 5.14] needs one more condition. There, for a component $\mathcal{T}$ of $\mathrm{Var}(\widetilde{S})$ and the distinct types $\mathcal{C}(V_{i_1}), \dots, \mathcal{C}(V_{i_p})$ of dimension ℓ among $\mathcal{C}(V_0), \dots, \mathcal{C}(V_k)$ (repeats removed), the loci $W_{r,i_j} \subset \mathcal{T}$ cut out by the constraints

of order at most r are computed until (i) they are closed and (ii) of their components of generic type dimension ℓ for different j, no one contains another. The general linear group is transitive on the flag variety $\check{L}$, so $W_{0,i} = \mathcal{T}$ for every i, and (i) and (ii) hold at $r = 0$, with nothing returned, whenever $\dim \mathcal{G}(\mathcal{T}) \neq \ell$; but $\mathcal{T}$ can have optimal components of smaller type dimension (for $H^1 \oplus H^1$ on $Y(3) \times Y(3)$, the diagonal among the $(1,1)$-curves). Add (iii): every component of every W_{r,i_j} has generic type of dimension at most ℓ. This is decidable [104, Lemma 5.8] and holds, with (i) and (ii), once $W_{r,i_j} = \mathcal{T} \cap W(\mathcal{C}(V_{i_j}))$ for all j. Then a component A of W_{r,i_j} with $\dim \mathcal{G}(A) = \ell$ lies in the closed locus $W(\mathcal{G}(A))$ [104, Prop. 4.31], so in a component A' of $W_{r,i_{j'}}$ with $\mathcal{C}(V_{i_{j'}}) = \mathcal{G}(A)$; $\dim \mathcal{G}(A') \geq \dim \mathcal{G}(A) = \ell$, from $A \subset A' \subset W(\mathcal{G}(A'))$, and $\dim \mathcal{G}(A') \leq \ell$ by (iii), so $j' = j$ by (ii), and A is an optimal component of $\mathcal{T} \cap W(\mathcal{C}(V_{i_j}))$. Conversely, an optimal component lies in a component of W_{r,i_j} whose generic type has dimension ℓ by (iii), which is therefore optimal and equal to it. So [104, Prop. 5.14, Thm. 5.15] hold as stated.

Let $\mathcal{W} \subset \mathrm{Var}(\widetilde{S})$ be the locus of weakly special subvarieties [104, §4.8], and $\mathrm{Var}(\widetilde{S})_{d'} \subset \mathrm{Var}(\widetilde{S})$ the subscheme parametrizing subvarieties whose closures have degree at most d' [104, Lemma 5.10]. The proof of [104, Cor. 6.3] takes the isolated points of $\mathcal{W} \cap \mathrm{Var}(\widetilde{S})_{d'}$ by [104, Thm. 6.1(ii)]; the proof of that theorem shows that a rigid weakly special Y is an isolated point of $W(\mathcal{C}(Y))$. Taking, for each type, the isolated points of the locus of weakly special subvarieties of that type and of degree at most d', which the algorithm of [104, Thm. 5.15] computes type by type, gives a finite set with the properties stated in [104, Cor. 6.3]. (On the bases $\Gamma\backslash X^+$ with Γ neat used below, every weakly special subvariety has the dimension of its type, and isolation in $\mathcal{W}$ holds.) We use only these statements.

A maximal special subvariety of Z need not be rigid in S: a special curve $\{x\} \times Y(1)$ in $\mathcal{A}_2$, with x a CM point, moves in the family $\{x'\} \times Y(1)$. We make it rigid by passing to the normalization of a component of the PEL locus defined by the connected center of its generic Mumford–Tate group (Lemma 11.7).

Lemma 11.7. *Let $V \subset S$ be special, the image of $X_H^+ \times \{\gamma\}$ for a Shimura subdatum (H, X_H) with H the generic Mumford–Tate group, let T be the connected center of H, $\mathcal{E} := \mathbb{Q}[T]$, and $M := Z_G(T) = Z_G(\mathcal{E})$.*

(a) For $x \in X_H$, $h_x(\mathbb{S}) \subset M_{\mathbb{R}}$ and $T \subset Z(M)$. Let X_M^+ be the component of $M(\mathbb{R})h_x$ containing X_H^+, and S_V the image of $X_M^+ \times \{\gamma\}$. Then S_V is special and $V \subset S_V$. If H' is the generic Mumford–Tate group on X_M^+, then $T \subset Z(H')$, $X_{H'}^+ = X_M^+$, and S_V is the image of $X_{H'}^+ \times \{\gamma\}$.

(b) For every PEL datum $\mathfrak{d}$, $\mathfrak{E}(\mathfrak{d})$ is closed in S, and each of its irreducible components is a special subvariety of a dimension $\dim \mathfrak{d}$ determined by $\mathfrak{d}$. S_V is an irreducible component of $\mathfrak{E}(\mathfrak{d}_V)$.

(c) Let $K_{H'} := \gamma K_\Upsilon \gamma^{-1} \cap H'(\mathbb{A}_f)$, $\widetilde{S}_V$ the image of $X_{H'}^+ \times \{1\}$ in $\mathrm{Sh}_{K_{H'}}(H', X_{H'})_{\mathbb{C}}$, and $\nu\colon \widetilde{S}_V \to S_V$ the map induced by $[x,h] \mapsto [x, h\gamma]$. Then $\widetilde{S}_V$ is smooth and ν is the normalization. The image W_V of $X_H^+ \times \{1\}$ in $\widetilde{S}_V$ is special, $\nu(W_V) = V$, and $\nu|_{W_V}$ is birational onto V. The first two sentences hold for every special $S' \subset S$ in place of S_V, with its generic Mumford–Tate group and its γ' in place of H' and γ; this covers every irreducible component of every $\mathfrak{E}(\mathfrak{d})$.

(d) W_V is rigid in $\widetilde{S}_V$, for the variation of Hodge structure $\nu^*\mathbb{V}$ given by W.

(e) For every very ample line bundle $\mathcal{L}'$ on a projective compactification of $\widetilde{S}_V$ and every integer $d' \geq \deg_{\mathcal{L}'} \overline{W}_V$, the point $[W_V]$ lies in the finite set that [104, Cor. 6.3] *computes*

for $(\widetilde{S}_V, \nu^*\mathbb{V}, d')$. *If* $\mathcal{L}' = \nu^*\mathcal{O}(1)$ *for a projective embedding of* $\overline{S}_V$, *then* $\deg_{\mathcal{L}'} \overline{W}_V = \deg_{\mathcal{O}(1)} \overline{V}$.

We call S_V the *container* of V.

Proof. (a) $h_x(\mathbb{S}) \subset H_{\mathbb{R}}$ and T is central in H, so $h_x(\mathbb{S}) \subset M_{\mathbb{R}}$; M is reductive, being the centralizer of a torus, and $T \subset Z(M)$. $X_H^+ \subset M(\mathbb{R})h_x$ is connected, so it lies in one component X_M^+, and $V \subset S_V$. For $y \in X_H$, $(M, M(\mathbb{R})h_y)$ satisfies (2.1.1.1) and (2.1.1.2) of [30], as $\operatorname{Lie} M$ is a sub-Hodge structure of $\operatorname{Lie} G$ and $\psi(\cdot, h_y(i)\cdot)$ is definite, but (2.1.1.3) can fail (for $A_y \cong A_0^2$ with A_0 a CM elliptic curve, M^{ad} is compact); so h_y projects trivially to the $\mathbb{Q}$-simple factors of M^{ad} with compact real points [30, 2.1.1] and factors through $M_1 := Z(M)^\circ M_{\mathrm{nc}}$, with M_{nc} the product of the other $\mathbb{Q}$-simple factors of M^{der}. As $Z(M)^\circ$ and the compact factors centralize h_y, $X_M^+ = M_1(\mathbb{R})^+ h_y$ for $y \in X_H^+$, a component of the Shimura subdatum $(M_1, M_1(\mathbb{R})h_y)$, which satisfies (2.1.1.3) because a $\mathbb{Q}$-factor of M_1^{ad} on which h_y projects trivially has compact real points by (2.1.1.2); so S_V is special. By [102, Lemma 2.1], S_V is the image of $X_{H'}^+ \times \{\gamma\}$, and $X_{H'}^+ = X_M^+$, both being connected complex submanifolds of the same dimension, the first closed in the second. As $\mathrm{MT}(y) \subset H'$ for $y \in X_H$, $H \subset H'$ and $T \subset Z(M) \cap H' \subset Z(H')$.

(b) Let $s = [x', \gamma'] \in \mathfrak{E}(\mathfrak{d})$ with ι as in the definition. Then $\iota \otimes \mathbb{Q}$ embeds $\mathcal{E}$ in $\mathrm{End}(W)$, compatibly with ψ-adjoints, and commutes with $h_{x'}$. Let $M_\iota := Z_G(\iota(\mathcal{E}))$, the group of $\iota(\mathcal{E})$-linear symplectic similitudes, which is connected and reductive ([69, Prop. 8.7], on each factor of $\mathcal{E}$) and contains $h_{x'}(\mathbb{S})$. The image $S_{\iota,s}$ of the component of $M_\iota(\mathbb{R})h_{x'}$ through x', times $\{\gamma'\}$, is special as in (a), contains s, and lies in $\mathfrak{E}(\mathfrak{d})$, since the lattice and $\iota(R)$ do not change along it and τ is locally constant. Its dimension is that of the symmetric space of $M_{\iota,\mathbb{R}}$, which is determined by $\mathfrak{d}$: the ψ-orthogonal decomposition $W_{\mathbb{R}} = \bigoplus_v W_v$ over the places of the factors of $\mathcal{E}$ gives a Siegel space of dimension $r(r+1)/2$ for a real place of a totally real factor, with $2r = \dim_{\mathbb{R}} W_v$, and a unitary symmetric space of dimension $a_v(m_i - a_v)$ for a place v of F_i^+ with F_i CM, as $\dim_{\mathbb{C}} W_v = m_i$; call the sum $\dim \mathfrak{d}$. Now $\mathfrak{E}(\mathfrak{d})$ is constructible (Lemma 11.6; here $N_0 = M_\Upsilon \otimes \overline{\mathbb{Q}}$ by Lemma 11.1(ii) and Lemma 11.3(iii)) and is the union of the countably many $S_{\iota,s}$. Let C be an irreducible component of its Zariski closure. Then $\mathfrak{E}(\mathfrak{d}) \cap C$ contains a dense open subset of C, which is a countable union of the closed sets $S_{\iota,s} \cap C$; by Baire's theorem one of them has interior, so $C \subset S_{\iota,s} \subset \mathfrak{E}(\mathfrak{d})$ and $C = S_{\iota,s}$. So $\mathfrak{E}(\mathfrak{d})$ is closed and its components have dimension $\dim \mathfrak{d}$. Finally $S_V = S_{\iota_V, x}$ for the inclusion ι_V of R_V, so S_V is irreducible of dimension $\dim \mathfrak{d}_V$ in $\mathfrak{E}(\mathfrak{d}_V)$, and the component containing it is S_V.

(c) $\widetilde{S}_V = \Gamma_{H'} \backslash X_{H'}^+$ with $\Gamma_{H'} := \{q \in H'(\mathbb{Q}) : qX_{H'}^+ = X_{H'}^+\} \cap \gamma K_\Upsilon \gamma^{-1}$, which is neat by Lemma 11.1(iii); so $\widetilde{S}_V$ is smooth. The map ν is finite, and it is injective on the Hodge generic locus by [102, Lemma 2.2]. That lemma assumes $Z(G)(\mathbb{R})$ compact; its proof uses this only to show that $N_G(H')/H'$ has compact real points, from the compactness of the stabilizer of a point of X in $G(\mathbb{R})$. For $G = \mathrm{GSp}$ this stabilizer is compact modulo $Z(G)(\mathbb{R}) = \mathbb{R}^\times$, and $Z(G) \subset H'$, so $N_G(H')/H'$ still has compact real points. Ullmo and Yafaev assume K neat. Here we use instead that two Hodge generic points of $X_{H'}^+$ with the same image in S differ by some $q \in N_G(H')(\mathbb{Q}) \cap G(\mathbb{Q})_+ \cap \gamma K_\Upsilon \gamma^{-1}$. As the similitude of q is in $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^\times = \{1\}$, q lies in $N_G(H')(\mathbb{A}_f) \cap \gamma K(d_\Upsilon)\gamma^{-1}$ (Lemma 11.1(i)), a compact neat group, whose image K'' in $\overline{N}(\mathbb{A}_f)$, $\overline{N} := N_G(H')/H'$, is compact and neat [81, 0.6]. So the image of q lies in $\overline{N}(\mathbb{Q}) \cap K''$, which is neat [81, 0.6] and discrete in the compact $\overline{N}(\mathbb{R})$, hence finite and trivial, so $q \in H'(\mathbb{Q}) \cap \gamma K_\Upsilon \gamma^{-1}$ preserves $X_{H'}^+$ and the two points agree in $\widetilde{S}_V$. A finite morphism from a normal variety that has degree 1 onto its image is the normalization. The

same argument for $(H, X_H) \subset (G, X)$ shows that the image of $X_H^+ \times \{\gamma\}$ in S is reached generically injectively; as that map factors through $W_V \to V$, $\nu|_{W_V}$ is birational. W_V is special in $\widetilde{S}_V$, defined by $(H, X_H) \subset (H', X_{H'})$. For a special $S' \subset S$ in place of S_V the first two sentences of (c) have the same proof, and by (b) every irreducible component of every $\mathfrak{E}(\mathfrak{d})$ is such an S'.

(d) Suppose that Y is weakly special in $\widetilde{S}_V$ with $W_V \subsetneq Y$ and H_{W_V} normal in H_Y. For a special subvariety defined by (H, X_H) of a quotient $\Gamma\backslash X$ with Γ neat and the variation of a faithful representation, the algebraic monodromy group is H^{der} (proof of [104, Cor. 6.2]); so $H_{W_V} = H^{\mathrm{der}}$. Algebraic monodromy groups are semisimple (proof of [104, Lemma 4.29]), so $H_Y = H^{\mathrm{der}} H''$ with H'' the product of the other $\mathbb{Q}$-simple factors, which commutes with H^{der}. As H_Y is normal in $\mathrm{MT}(Y) \subset H'$, $H'' \subset Z_{H'}(H^{\mathrm{der}})$, which also centralizes $T \subset Z(H')$, hence centralizes $H = T \cdot H^{\mathrm{der}}$ and $h_x(\mathbb{S})$ for $x \in X_H$. Being semisimple, $H'' \subset \mathrm{Sp}(W, \psi)$; it fixes $h_x(i)$, so $H''(\mathbb{R})$ preserves the positive definite form $\psi(\cdot, h_x(i)\cdot)$ and is compact. By [104, Lemma 4.11] an algebraic monodromy group has no $\mathbb{Q}$-simple factor with compact real points, so $H'' = 1$ and $H_Y = H_{W_V}$. Then $Y \supsetneq W_V$ has the algebraic monodromy group of the weakly special W_V, which contradicts the maximality in the definition. So W_V is rigid.

(e) $\widetilde{S}_V$ is smooth and $\nu^*\mathbb{V}$ is a polarizable integral variation, so [104, Cor. 6.3] applies, and $[W_V]$ is in the output by (d). The projection formula and (c) give $\deg_{\nu^*\mathcal{O}(1)} \overline{W}_V = \deg_{\mathcal{O}(1)} \overline{V}$. □

A special point Q has $S_Q = \{Q\}$: h_x factors through $T \subset Z(M)$, so $M(\mathbb{R})h_x = \{h_x\}$. For the curve $\{x\} \times Y(1) \subset \mathcal{A}_2$ above, with x a CM point with CM by k, $\mathcal{E} = k \times \mathbb{Q}$ and $M = H$, so $S_V = V$; the curve is its own container and is rigid there.

11.4. **Degrees in Mumford's coordinates.** Urbanik's degrees are taken for $\mathcal{O}(1)$ on the closure in $\mathbb{P}(V_\Upsilon)$, ours for the Baily–Borel bundle. Theta constants have weight $1/2$ and no common zero on the boundary, so in dimension r the two differ by the factor $(2(g+1))^r$.

Lemma 11.8. *Let $g \geq 2$, $\varphi\colon S \to \mathbb{P}(V_\Upsilon)$ the theta-null immersion, $\mathcal{L} := \mathcal{L}_{K_\Upsilon}$ and $N_\Upsilon := d_\Upsilon^g$. Then φ extends to a morphism $\bar{\varphi}\colon S^{\mathrm{BB}} \to \mathbb{P}(V_\Upsilon)$ from the Baily–Borel compactification S^{BB} of S, with $\bar{\varphi}^*\mathcal{O}(1)^{\otimes 8N_\Upsilon^4(g+1)} \cong \mathcal{L}^{\otimes 4N_\Upsilon^4}$. For every closed irreducible $Y \subset S$ of dimension r, with closure $\overline{Y}$ in $\mathbb{P}(V_\Upsilon)$, $\overline{Y}$ is the image of the closure of Y in S^{BB}, and*

$$\deg_{\mathcal{O}(1)} \overline{Y} = \deg_{\mathcal{L}} Y / (2(g+1))^r \leq \deg_{\mathcal{L}} Y.$$

Proof. The theta constants. Write $d := d_\Upsilon$, $e(x) := \exp(2\pi i x)$ and, for $\tau \in \mathcal{H}_g$ and $a \in K(\Upsilon)$, $\Theta_a(\tau) := \sum_{m \in \mathbb{Z}^g,\, m \equiv a \bmod d} e(m^t \tau m / 2d)$. Let $A_\tau := \mathbb{C}^g/(\tau\mathbb{Z}^g + \mathbb{Z}^g)$, L_0 the line bundle with factor of automorphy $e(-\frac{1}{2}n^t\tau n - n^t z)$ at $\tau n + m$, and $L_\tau := L_0^{\otimes d}$, the totally symmetric sheaf with $\varphi_{L_\tau} = d\varphi_0$ [73, p. 307]. The functions $f_a(z) := \sum_n e(\frac{d}{2}(n + \frac{a}{d})^t \tau (n + \frac{a}{d}) + d(n + \frac{a}{d})^t z)$ form a basis of $H^0(A_\tau, L_\tau)$; translation by m/d multiplies f_a by $e(a^t m/d)$, and translation by $\tau k/d$, followed by multiplication by $e(k^t z + k^t \tau k/2d)$, takes f_a to f_{a+k}. In Mumford's action these operators come from lifts over $-m/d$ and $-\tau k/d$, respectively, in $\mathcal{G}(L_\tau)$ [73, pp. 289, 295], so $f_a \mapsto \delta_a$ is equivariant for a ϑ-structure, and the theta-null point of A_τ, L_τ and that structure is $[\Theta(\tau)]$ [73, pp. 297–298]. Any other ϑ-structure differs from it by one of the finitely many automorphisms of $\mathcal{G}(\Upsilon)$ that fix $\mathbb{C}^*$, which changes the theta-null point by a fixed linear map [73, Prop. 3, p. 295]. The same definitions in genus $q \geq 1$ give $A_{\tau'}$, $L_{\tau'}$ and $\Theta^{(q)}(\tau')$ for $\tau' \in \mathcal{H}_q$. Let $\pi\colon \mathcal{H}_g \to S$ be the uniformization. The maps $\tau \mapsto \varphi(\pi(\tau))$ and $\tau \mapsto [\Theta(M\tau)]$, $M \in \mathrm{Sp}_{2g}(\mathbb{Z})$, are holomorphic, and as $(A_{M\tau}, L_{M\tau}) \cong (A_\tau, L_\tau)$, at each τ they take a value in a finite set $\{[\beta_1\Theta(\tau)], \ldots, [\beta_s\Theta(\tau)]\}$ with $\beta_1, \ldots, \beta_s \in \mathrm{GL}(V_\Upsilon \otimes \mathbb{C})$

independent of τ. For each of these maps, the sets where it agrees with $[\beta_i\Theta]$ are closed analytic and cover $\mathcal{H}_g$; by Baire's theorem one of them has interior, and by the identity theorem it is all of $\mathcal{H}_g$. This gives $\beta, \rho_M \in \mathrm{GL}(V_\Upsilon \otimes \mathbb{C})$ with $\varphi(\pi(\tau)) = [\beta\Theta(\tau)]$ and $[\Theta(M\tau)] = [\rho_M\Theta(\tau)]$ for all τ.

The cusps. For Sp_{2g} the Baily–Borel compactification is Satake's [58, §I.2]. Let $0 \le q < g$, write τ and $y := \operatorname{Im}\tau$ in blocks of sizes q and $g-q$, and let $\sigma(y) := y_2 - y_{21}y_1^{-1}y_{12}$ ($\sigma(y) = y$ if $q = 0$). Splitting $\xi := n + a/d = (\xi', \xi'')$ and $\xi^t y\xi = (\xi' + y_1^{-1}y_{12}\xi'')^t y_1(\xi' + y_1^{-1}y_{12}\xi'') + \xi''^t\sigma(y)\xi''$ shows that if $\tau_{j,1} \to \tau' \in \mathcal{H}_q$ and the least eigenvalue of $\sigma(y_j)$ tends to ∞, then $\Theta(\tau_j) \to v(\tau')$, where $v_{(a',a'')}(\tau') := \Theta^{(q)}_{a'}(\tau')$ if $a'' = 0$ and 0 otherwise. And $v(\tau') \ne 0$: for $q = 0$, $v = \delta_0$, and for $q \ge 1$, $L_{\tau'}$ is very ample [74, p. 82]. By Satake's construction [89, pp. 12-02–12-07] and [90, Th. 1 and 2, p. 13-03], every sequence in S converging to a boundary point x has a subsequence of the form $\pi(M_iZ_j)$, with M_i in a fixed finite set of representatives of $\Gamma_\Upsilon\backslash\mathrm{Sp}_{2g}(\mathbb{Z})$ and Z_j in a Siegel set with $Z_{j,1} \to Z_0$ and the least eigenvalue of $\sigma(\operatorname{Im} Z_j)$ tending to ∞, where x is the class of $M_i.Z_0$. (In Satake's notation $\operatorname{Im} Z_j = {}^tW_jD_jW_j$ with D_j diagonal; then $d_{j,q+1} \to \infty$, and W_j and the ratios $d_{j,i}/d_{j,i+1}$ are bounded above on the Siegel set.) Hence every limit of φ at x is one of the countably many points $[\beta\rho_{M_i}v(Z_0)]$ with $M_i.Z_0$ of class x.

The extension. As Γ_Υ has finite index in $\mathrm{Sp}_{2g}(\mathbb{Z})$ (Lemma 11.1(ii)), S^{BB} is a normal projective variety [20, Théorème fondamental, p. 17-11, and p. 17-04]. The natural map from S^{BB} to the Satake compactification of $\mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g$ is proper with finite fibers, and the boundary of S^{BB} is the preimage of the boundary of the latter, which is the Satake compactification of $\mathrm{Sp}_{2g-2}(\mathbb{Z})\backslash\mathcal{H}_{g-1}$ and has dimension $g(g-1)/2$ [91, (2.1), p. 15-05, and p. 15-07]; so the boundary of S^{BB} has codimension $g \ge 2$. And $\mathcal{L}$ is a line bundle on S^{BB} (Lemma 11.1(iii), [75, Prop. 3.4(b)]). Let $\overline{\Gamma}_\varphi \subset S^{\mathrm{BB}} \times \mathbb{P}(V_\Upsilon)$ be the closure of the graph Γ_φ of φ. Its projection to S^{BB} is projective and birational, so its fibers are connected [54, III.11.4]. As Γ_φ is dense in $\overline{\Gamma}_\varphi$ for the classical topology, the fiber over a boundary point lies in the countable set above; as it is Zariski closed it is finite, and as it is connected it is one point. So the projection is bijective and projective, hence finite [54, Ex. III.11.2], and finite and birational onto the normal S^{BB}, hence an isomorphism (as in the proof of [54, III.11.4]); so $\bar\varphi$ is a morphism. Let ω be the determinant of the Hodge bundle on S, the bundle of modular forms of weight 1, p the universal family over S and L its marking sheaf, so that p_*L has rank N_Υ; Mumford shows that $p_*L \cong V_\Upsilon \otimes \varphi^*\mathcal{O}(-1)$ [74, pp. 81–82]. For a relatively ample symmetric line bundle trivialized along the zero section, with p_* of rank N_Υ, Faltings and Chai prove $8N_\Upsilon^3 \det(p_*L) = -4N_\Upsilon^4\,\omega$ in $\mathrm{Pic}(S)$ [35, Thm. I.5.1], as stated in [84, Introduction]; Mumford's L is totally symmetric, relatively ample and normalized. With $\mathcal{L}|_S = \omega^{\otimes(g+1)}$ [58, §II.1] this gives the isomorphism of the statement on S, and it extends to S^{BB}, as both sides are line bundles on the normal S^{BB} and the boundary has codimension at least 2.

Degrees. Let $\overline{Y}^{\mathrm{BB}}$ be the closure of Y in S^{BB}. Its image under $\bar\varphi$ is closed and contains $\varphi(Y)$ as a dense subset, so it is $\overline{Y}$. The map $\overline{Y}^{\mathrm{BB}} \to \overline{Y}$ has degree 1: $\overline{Y}^{\mathrm{BB}}$ meets the interior S in Y, where $\bar\varphi$ is Mumford's embedding φ, which is injective [74, Theorem, p. 83], and the rest of $\overline{Y}^{\mathrm{BB}}$ has dimension less than r, and so has its image. By the projection formula, $(8N_\Upsilon^4(g+1))^r \deg_{\mathcal{O}(1)} \overline{Y} = (4N_\Upsilon^4)^r \deg_{\mathcal{L}} Y$. □

11.5. **Normalization.** Let k be a number field, $\mathcal{N} \subset \mathbb{P}^{M'}_k$ an integral closed subscheme with homogeneous prime ideal $I \subset R' := k[x_0, \dots, x_{M'}]$, $A := R'/I$, and $\bar A$ the integral closure of A in its fraction field. The algorithm of Greuel, Laplagne and Seelisch [47, Algorithm 3

and Remark 4.6] (in its notation, with the stopping test $U_{i+1} = pU_i$) returns $d_A \in R' \smallsetminus I$ and $u_1, \dots, u_s \in R'$ with $\bar{A} = d_A^{-1}(u_1, \dots, u_s)$ (images in A); it does not say that these are homogeneous. The following lemma makes the graded and Veronese steps explicit.

Lemma 11.9. *Fix $x \in A_1 \smallsetminus \{0\}$ and let $\widetilde{B}$ be the integral closure of $A_{(x)} := (A_x)_0$.*

(a) $\bar{A} \subset \widetilde{B}[x, x^{-1}]$, and $\bar{A} = \bigoplus_{n \geq 0} \bar{A}_n$ is a graded subring, with x of degree 1 and $\widetilde{B}$ of degree 0.

(b) $\operatorname{Proj} \bar{A} \to \mathcal{N}$ is finite, and $\operatorname{Proj} \bar{A}$ is the normalization $\widetilde{\mathcal{N}}$ of $\mathcal{N}$.

(c) Let c be a homogeneous component of d_A with $c \notin I$, and $a_j \in R'$ with $d_A a_j \equiv c u_j \bmod I$. Then the homogeneous components $a_{j,m}$ of the a_j give elements $a_{j,m}/c \in \bar{A}$ of degree $m - \deg c$, which with 1 generate $\bar{A}$ as an A-module.

(d) Let $b_1, \dots, b_\mu$ be the elements of positive degree among $x_0, \dots, x_{M'}$ and the generators of (c), of degrees $d_1, \dots, d_\mu$, and $h := \mu \operatorname{lcm}(d_1, \dots, d_\mu)$. Then $\bar{A}_{nh} = (\bar{A}_h)^n$ for all $n \geq 1$.

(e) With a basis $T_1, \dots, T_{M''}$ of $\bar{A}_h$ and $\mathfrak{a}$ the kernel of $k[T] \to \bar{A}$, $\widetilde{\mathcal{N}} \cong \operatorname{Proj}(k[T]/\mathfrak{a}) \subset \mathbb{P}_k^{M''-1}$, with explicit equations.

(f) This embedding of $\widetilde{\mathcal{N}}$ is given by $\nu^* \mathcal{O}_{\mathcal{N}}(h)$, $\nu \colon \widetilde{\mathcal{N}} \to \mathcal{N}$ the normalization.

Proof. (a) $A_x = A_{(x)}[x, x^{-1}]$, and integral closure commutes with localization [97, Tag 0307], so $\bar{A}_x$ is the integral closure of $A_{(x)}[x, x^{-1}]$, which is $\widetilde{B}[x, x^{-1}]$, as this ring is integral over it and normal [97, Tags 030A, 00GY]. For $\lambda \in k^\times$ the automorphism $x \mapsto \lambda x$ of $\widetilde{B}[x, x^{-1}]$ preserves A, hence $\bar{A}$; applying it for enough distinct λ and inverting a Vandermonde matrix shows that each homogeneous component of an element of $\bar{A}$ lies in $\bar{A}$. A homogeneous $b \in \bar{A}$ of negative degree $-e'$ satisfies $b^n + \sum_i (a_i)_{-ie'} b^{n-i} = 0$, the degree $-ne'$ part of an integral equation, with $(a_i)_{-ie'} \in A_{-ie'} = 0$; so $b = 0$.

(b) If a homogeneous prime $\mathfrak{q}$ of $\bar{A}$ contains A_1, then it contains every homogeneous element of positive degree, by the degree ne' part of an integral equation; so the $D_+(x)$, $x \in A_1 \smallsetminus \{0\}$, cover $\operatorname{Proj} \bar{A}$. On $D_+(x)$ the ring of sections is $(\bar{A}_x)_0 = \widetilde{B}$, the normalization of the chart $\operatorname{Spec} A_{(x)}$; these identifications glue, and $\widetilde{B}$ is finite over $A_{(x)}$ as $\bar{A}$ is finite over A.

(c) The conductor $\{a \in A : a\bar{A} \subset A\}$ contains d_A and is homogeneous, as A is a graded subring of $\widetilde{B}[x, x^{-1}]$; so it contains c, which gives a_j, and $c\bar{A}$ is a homogeneous ideal of A containing a_j, hence the $a_{j,m}$. As $u_j/d_A = a_j/c = \sum_m a_{j,m}/c$, these elements generate.

(d) $\bar{A}_{nh}$ is spanned by products $\beta \prod_i b_i^{\alpha_i}$ with $\beta \in \bar{A}_0$ and $\sum_i \alpha_i d_i = nh$. Put $\ell := \operatorname{lcm}(d_i)$ and $\alpha_i = q_i \ell/d_i + r_i$ with $0 \leq r_i < \ell/d_i$. The blocks b_i^{ℓ/d_i} have degree ℓ, and $\prod_i b_i^{r_i}$ has degree $j\ell$ with $0 \leq j \leq \mu - 1$; there are $n\mu - j$ blocks. Grouping $\beta \prod_i b_i^{r_i}$ with $\mu - j$ blocks, and the rest in $n - 1$ groups of μ blocks, writes the product as a product of n elements of $\bar{A}_h$.

(e) $\bar{A}_h$ is finite-dimensional, and by (d) the subalgebra $k[\bar{A}_h]$ agrees with $\bar{A}^{(h)}$ in positive degrees, so $\operatorname{Proj} k[\bar{A}_h] \cong \operatorname{Proj} \bar{A}^{(h)} \cong \operatorname{Proj} \bar{A}$ [48, Prop. 2.4.7], and $k[\bar{A}_h] = k[T]/\mathfrak{a}$ is generated in degree 1.

(f) On $D_+(x)$ both $\mathcal{O}_{\operatorname{Proj} \bar{A}}(1)$ and $\nu^* \mathcal{O}_{\mathcal{N}}(1)$ are free with generator x and transition functions x/x', so they agree; and $\mathcal{O}(1)$ on $\operatorname{Proj} k[\bar{A}_h]$ is generated by x^h on $D_+(x^h)$, with transition functions $(x/x')^h$. □

11.6. **Rigidity and monodromy.** In this subsection $\widetilde{S} = \Gamma\backslash X^+$ is a connected Shimura variety with Γ neat, $\mathbb{V}$ the variation of a faithful representation, and $G_{\widetilde{S}}$ the generic Mumford–Tate group of $\widetilde{S}$. For $Y \subset \widetilde{S}$ weakly special and $x \in Y(\overline{\mathbb{Q}})$, let $\mathfrak{h}_Y$ be the Lie algebra of H_Y in $\mathrm{End}(\mathbb{V}_x) \otimes \overline{\mathbb{Q}}$, $F^\bullet$ the Hodge filtration of $\mathbb{V}_x$, $F^j\mathrm{End}(\mathbb{V}_x) := \{\phi : \phi(F^p) \subset F^{p+j} \text{ for all } p\}$, and for a subspace $\mathfrak{n}$ put $t(\mathfrak{n}) := \dim \mathfrak{n} - \dim(\mathfrak{n} \cap F^0\mathrm{End}(\mathbb{V}_x))$.

Lemma 11.10. *Let $Y \subset \widetilde{S}$ be weakly special, x a smooth point of Y, and $\mathfrak{z}$ the centralizer of $\mathfrak{h}_Y$ in $\mathfrak{h}_{\widetilde{S}}$. Then Y is rigid if and only if $\mathfrak{z} \cap F^1\mathrm{End}(\mathbb{V}_x) = 0$. If Y is rigid, it is special.*

Proof. $\mathrm{MT}(x) \subset \mathrm{MT}(Y)$ normalizes H_Y and $H_{\widetilde{S}}$, which are normal in $\mathrm{MT}(Y)$ and $G_{\widetilde{S}}$ [4, Thm. 1 and its proof], so $\mathfrak{z}$ is a sub-Hodge structure of $\mathfrak{h}_{\widetilde{S}}$ of types $(-1,1)$, $(0,0)$, $(1,-1)$, and $\mathfrak{z} \cap F^1 = \mathfrak{z}^{1,-1}$ is the complex conjugate of $\mathfrak{z}^{-1,1}$. If $\mathfrak{z}^{-1,1} = 0$ and $Y' \supsetneq Y$ is weakly special with H_Y normal in $H_{Y'}$, then $H_{Y'} = H_Y H''$ with $\mathrm{Lie}\, H'' \subset \mathfrak{z}$, both summands stable under $\mathrm{Ad}\, h_x$, and Y' is the image of $H_{Y'}(\mathbb{R})^+ \cdot \tilde{x}$ [104, Prop. 4.8, Lemma 4.10(ii)], of dimension $\dim \mathfrak{h}_{Y'}^{-1,1} = \dim \mathfrak{h}_Y^{-1,1} = \dim Y$, a contradiction. If $\mathfrak{z}^{-1,1} \neq 0$, then $N' := Z_{G_{\widetilde{S}}}(H_Y)^\circ$ is reductive and normalized by $\mathrm{MT}(Y)$, so h_x factors through the reductive group $\mathrm{MT}(Y)N'$, which defines a special $Y' \supset Y$ whose tangent space contains $\mathfrak{h}_Y^{-1,1} \oplus \mathfrak{z}^{-1,1}$; so $Y' \supsetneq Y$, and H_Y is normal in $(\mathrm{MT}(Y)N')^{\mathrm{der}} \supset H_{Y'}$, so Y is not rigid. The last assertion is [62, Lemma 2.5], where rigid weakly special subvarieties are called weakly non-factor. □

The test is linear algebra over $\overline{\mathbb{Q}}$ once $\mathfrak{h}_Y$, $\mathfrak{h}_{\widetilde{S}}$ and $F^\bullet$ are known in one frame at one point; F^1 is algebraic, unlike the Hodge decomposition. The group H_Y is computed on a curve, and the curve is certified by the following test.

Lemma 11.11. *Let $Y \subset \widetilde{S}$ be weakly special of dimension r, $C \subset Y^{\mathrm{sm}}$ an irreducible curve and $x \in C$. Then*

(a) $\mathfrak{h}_C \subset \mathfrak{h}_Y$, both are stable under $\mathrm{Ad}\, h_x(\mathbb{S})$, and $t(\mathfrak{h}_C) = \dim \mathfrak{h}_C^{-1,1}$, $t(\mathfrak{h}_Y) = \dim \mathfrak{h}_Y^{-1,1}$;
(b) $t(\mathfrak{h}_Y) = r$;
(c) $t(\mathfrak{h}_C) = r$ if and only if $\mathfrak{h}_C = \mathfrak{h}_Y$.

Proof. (a) Loops in C are loops in Y^{sm}, and algebraic monodromy does not change when Y is replaced by an open subvariety [104, proof of Lemma 4.11], so $H_C \subset H_Y$. A tensor that is Hodge at a point outside the Hodge locus of a subvariety is Hodge everywhere on it, so $\mathrm{MT}(x) \subset \mathrm{MT}(C) \subset \mathrm{MT}(Y)$; H_C and H_Y are normal in $\mathrm{MT}(C)$ and $\mathrm{MT}(Y)$ [4, Thm. 1 and its proof], so $h_x(\mathbb{S})$ normalizes both, and $\mathfrak{h}_C$, $\mathfrak{h}_Y$ are sub-Hodge structures of $\mathrm{End}(\mathbb{V}_x)$ of types $(-1,1)$, $(0,0)$, $(1,-1)$. For such $\mathfrak{n}$, $\mathfrak{n}_{\mathbb{C}} \cap F^0 = \mathfrak{n}^{0,0} \oplus \mathfrak{n}^{1,-1}$. (b) Y is the image of $H_Y(\mathbb{R})^+ \cdot \tilde{x}$ [104, Prop. 4.8, Lemma 4.10(ii)], whose tangent space at $\tilde{x}$ is $\mathfrak{h}_Y^{-1,1}$ [104, proof of Lemma 4.9]. (c) If $t(\mathfrak{h}_C) = r$, then $\mathfrak{h}_C^{-1,1} = \mathfrak{h}_Y^{-1,1}$ by (a) and (b), and $\mathfrak{h}_C^{1,-1} = \mathfrak{h}_Y^{1,-1}$ by conjugation. Let $\mathfrak{s}$ be a non-compact $\mathbb{R}$-simple ideal of $\mathfrak{h}_{Y,\mathbb{R}}$. The connected group $h_x(\mathbb{S})$ fixes each simple ideal, so $\mathfrak{s}_{\mathbb{C}} = \mathfrak{s}^{-1,1} \oplus \mathfrak{s}^{0,0} \oplus \mathfrak{s}^{1,-1}$; put $\mathfrak{p} := \mathfrak{s}^{-1,1} \oplus \mathfrak{s}^{1,-1}$. If $\mathfrak{p} = 0$, $\mathfrak{s}$ commutes with $h_x(i)$ and, as $H_Y \subset \mathrm{Sp}(W, \psi)$, preserves a positive definite form, so it is compact; hence $\mathfrak{p} \neq 0$. As $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{s}^{0,0}$ and $[\mathfrak{s}^{0,0}, \mathfrak{p}] \subset \mathfrak{p}$, $\mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$ is a non-zero ideal of $\mathfrak{s}_{\mathbb{C}}$ stable under conjugation, so it is $\mathfrak{s}_{\mathbb{C}}$; and $\mathfrak{p} \subset \mathfrak{h}_{C,\mathbb{C}}$, so $\mathfrak{s}_{\mathbb{C}} \subset \mathfrak{h}_{C,\mathbb{C}}$. Each $\mathbb{Q}$-simple ideal $\mathfrak{h}_i$ of the semisimple $\mathfrak{h}_Y$ has a non-compact $\mathbb{R}$-simple ideal [104, Lemma 4.11], so $\mathfrak{h}_C \otimes \overline{\mathbb{Q}}$ contains a $\overline{\mathbb{Q}}$-simple ideal of $\mathfrak{h}_i \otimes \overline{\mathbb{Q}}$, hence all of them, as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ permutes them transitively and preserves $\mathfrak{h}_C \otimes \overline{\mathbb{Q}}$. So $\mathfrak{h}_Y \subset \mathfrak{h}_C$. □

Proposition 11.12. *Let $\widetilde{S}$ be the normalization of an irreducible component S' of some $\mathfrak{E}(\mathfrak{d})$, as in Lemma 11.7(c) (for instance $\widetilde{S}_V$), embedded by Lemma 11.9, and $Y \subset \widetilde{S}$ weakly special of dimension $r \geq 0$ and defined over $\overline{\mathbb{Q}}$. The following procedure terminates and returns a point $x \in Y(\overline{\mathbb{Q}})$, and $\mathfrak{h}_Y$, $\mathfrak{h}_{\widetilde{S}}$ and $F^\bullet$ as subspaces of $\mathrm{End}(H^1_{\mathrm{dR}}(A_x)) \otimes \overline{\mathbb{Q}}$ in one frame, together with, if $r \geq 1$, an irreducible curve $C \subset Y^{\mathrm{sm}}$ through x. If $r = 0$, $Y = \{x\}$ and $\mathfrak{h}_Y := 0$. If $r = 1$, put $C := Y^{\mathrm{sm}}$. If $r \geq 2$, enumerate the $(r-1)$-tuples of hyperplanes defined over $\mathbb{Q}(i)$ by increasing height; for each, let C be the intersection of Y^{sm} with the hyperplanes, and pass to the next tuple unless C is a non-empty smooth geometrically irreducible curve; choose $x \in C(\overline{\mathbb{Q}})$ outside the finitely many points excluded below, compute $\mathfrak{h}_C$ at x, and accept C if and only if $t(\mathfrak{h}_C) = r$. On acceptance $\mathfrak{h}_Y := \mathfrak{h}_C$.*

Proof. If $r = 0$, only the computation of $\mathfrak{h}_{\widetilde{S}}$ and $F^\bullet$ at x below is used.

Computing $\mathfrak{h}_C$. The Gauss–Manin connection on H^1_{dR} of the pull-back of the universal family to C has regular singular points and quasi-unipotent local monodromy [61, Thm. 14.1]. Let $u\colon C \to \mathbb{P}^1$ be the morphism given by a general linear projection (a non-constant rational function on the smooth curve C), and $\ell^0 \subset \mathbb{A}^1$ the complement of a finite set of points, chosen so that u is étale over ℓ^0 and including the images of the points of the closure of C that are not in C, so that u is finite over ℓ^0. Its push-forward is a system $\mathrm{Y}' = A\mathrm{Y}$ over $\overline{\mathbb{Q}}(t)$ of rank $2g \deg u$, in the restriction-of-scalars basis $\theta^\ell e_j$ for $0 \leq \ell < d = \deg u$, where the e_j are the frame on C fixed below and θ is a primitive element of $\overline{\mathbb{Q}}(C)/\overline{\mathbb{Q}}(t)$, after excluding images of poles, points where the frame fails, and zeros of the discriminant, whose local system on ℓ^0 is induced from that of $C^0 := u^{-1}(\ell^0)$. The image of $\pi_1(C^0)$, of finite index, preserves the summand V_x of the fiber $\bigoplus_{y \in u^{-1}(c)} V_y$, and the kernel of the permutation action on $u^{-1}(c)$, normal and of finite index, preserves every summand. The identity component of the Zariski closure lies in the closure of the image of this kernel. So the projection of the Lie algebra of the identity component of the Zariski closure of the monodromy to $\mathrm{End}(V_x)$ is $\mathfrak{h}_C$. The system is regular singular in Singer's sense, as a local coordinate z on the smooth completion of C at a point over c satisfies $t - c = z^e \cdot (\text{unit})$, and $1/t = z^e \cdot (\text{unit})$ over $c = \infty$, so the differential Galois group is the Zariski closure of the monodromy [94, Prop. 1.3.27], both taken for one fundamental matrix. Hrushovski's algorithm [56], in Feng's form [36, Algorithm 4.1], computes the Galois group for a fundamental matrix $F_c \cdot h'$, where F_c is normalized at a point c chosen in step (g) of the algorithm and $h' \in \mathrm{GL}_n(\overline{\mathbb{Q}})$ is computed [36, §§4.2, 4.4]; we take $c := u(x)$, which excludes finitely many x. The Galois group over $\mathbb{C}(t)$ is the base change of the one over $\overline{\mathbb{Q}}(t)$: $\mathbb{C} \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}((t-c)) \to \mathbb{C}((t-c))$ is injective, so the ideal of algebraic relations of F_c over $\mathbb{C}(t)$ is generated by the one over $\overline{\mathbb{Q}}(t)$, and the stabilizer condition has coefficients in $\overline{\mathbb{Q}}$. Thus $\mathfrak{h}_C$ is computed in $\mathrm{End}(H^1_{\mathrm{dR}}(A_x)) \otimes \overline{\mathbb{Q}}$, where $F^\bullet$ is algebraic.

Computing $\mathfrak{h}_{\widetilde{S}}$. Write $S' = S_{\iota,s}$ as in the proof of Lemma 11.7(b), $M := M_\iota$, and X^+_M for the component of $M_\iota(\mathbb{R})h_{x'}$ through x'; $X^+_M = X^+_{H'}$ for the generic Mumford–Tate group H' on X^+_M, as in the proof of Lemma 11.7(a). $\widetilde{S}$ is the normalization of a special subvariety with generic Mumford–Tate group H', and its algebraic monodromy group is H'^{der} (proof of [104, Cor. 6.2]), which is the product of the $\mathbb{Q}$-simple factors of M^{der} that are non-compact at $\mathbb{R}$: $\mathrm{Ad}\, h(i)$ is a Cartan involution of $M^{\mathrm{ad}}_{\mathbb{R}}$, trivial on the compact factors, so h projects trivially to them; and the group generated by the $h'(\mathbb{S})$, $h' \in X^+_M$, is normal in $M(\mathbb{R})^+$ and non-trivial on each non-compact $\mathbb{R}$-simple factor.

At x, $M \otimes \overline{\mathbb{Q}}$ is the centralizer in $\mathrm{GSp}(H^1_{\mathrm{dR}}(A_x))$ of $\iota_x(\mathcal{E})$, and ι_x is the fiber at x of an action of $\mathcal{E}$ on the family over all of $\widetilde{S}$ that defines S'; we find it as follows.

The generic point. Let $\mathcal{C}$ be an irreducible component, dominating S', of the locally closed subset of $\mathcal{H}$ in the proof of Lemma 11.6, $\overline{\mathcal{C}}$ its closure in $\mathcal{H} \times_{N_0} S'$, η the generic point of S', and $A_{\bar{\eta}}$ the geometric generic fiber of the universal family over S'. The geometric points of $\mathcal{C}$ over η are tuples of graphs of endomorphisms f of $A_{\bar{\eta}}$ with $\mathrm{Tr}(f^\dagger f)$ fixed by condition (3) in the proof of Lemma 11.6, so there are finitely many; hence $\dim \mathcal{C} = \dim S'$, and the generic point of $\mathcal{C}$ is the only point of $\overline{\mathcal{C}}$ over η.

Rationality. Every endomorphism ϕ of $A_{\bar{\eta}}$ is defined over $\overline{\mathbb{Q}}(S')$. Indeed, ϕ is defined over a finite extension of $\overline{\mathbb{Q}}(S')$, so on a finite étale cover of a dense open subset of $\widetilde{S}$ it is a flat section of End of the local system H_1, which near each point is an element of $\mathrm{End}(W)$ commuting with h_y for all y in a non-empty open subset of $X^+_{H'}$; such a subset contains Hodge generic points, so ϕ commutes with H' and is fixed by $\Gamma_{H'} \subset H'(\mathbb{Q})$, which is $\pi_1(\widetilde{S}(\mathbb{C}))$, as $\Gamma_{H'}$ is torsion-free and lies in $\mathrm{Sp}(W)(\mathbb{Q})$, so acts freely on the contractible $X^+_{H'}$, and acts on W as the monodromy of $\widetilde{S}$ (Lemma 11.7(c)). As $\widetilde{S} \to S'$ is birational (Lemma 11.7(c)), $\overline{\mathbb{Q}}(S') = \overline{\mathbb{Q}}(\widetilde{S})$; the family is an abelian scheme over all of $\widetilde{S}$, so $\mathrm{Gal}(\overline{\overline{\mathbb{Q}}(S')}/\overline{\mathbb{Q}}(S'))$ acts on the ℓ-adic Tate module of $A_{\bar{\eta}}$ through $\pi_1^{\mathrm{et}}(\widetilde{S})$, which, as the base field $\overline{\mathbb{Q}}$ is algebraically closed of characteristic 0, is the profinite completion of $\pi_1(\widetilde{S}(\mathbb{C})) = \Gamma_{H'}$ [50, Exp. XII, Cor. 5.2, and Exp. XIII, Prop. 4.6], acting through the closure of $\Gamma_{H'}$ in $\mathrm{GL}(W \otimes \mathbb{Q}_\ell)$; so it fixes ϕ.

Extension. Therefore the point of $\mathcal{H}$ at the generic point of $\mathcal{C}$ is rational over $\overline{\mathbb{Q}}(S')$, $\overline{\mathcal{C}} \to S'$ is birational, and $\overline{\mathcal{C}} \times_{S'} \widetilde{S}$ has exactly one irreducible component $\mathcal{C}'$ dominating $\widetilde{S}$, with function field $\overline{\mathbb{Q}}(\widetilde{S})$. The endomorphisms $f_{i,1}$ at the generic point of $\mathcal{C}'$ are therefore endomorphisms of the generic fiber of the pull-back of the universal family to the smooth $\widetilde{S}$, and they extend to it [35, Prop. I.2.7]; by Equation (11.1) the graphs of the $nf_{i,1}$ define a section of $\mathcal{H} \times_{N_0} \widetilde{S} \to \widetilde{S}$, whose image is closed, irreducible, of dimension $\dim \widetilde{S}$ and contains the generic point of $\mathcal{C}'$, so equals $\mathcal{C}'$. Hence $\mathcal{C}'$ has one point over x, the graphs of ι_x.

The datum. The extended $f_{i,1}$ act on the local system H_1 by flat sections, so on $X^+_{H'}$ they give a constant embedding ι' of $\mathcal{E}$ in $\mathrm{End}(W)$, with the datum $\mathfrak{d}$, which commutes with every h_y, $y \in X^+_{H'}$, hence with H'; so $X^+_{H'} = H'(\mathbb{R})^+ h_{x'}$ lies in the component of $M_{\iota'}(\mathbb{R})h_{x'}$ through x', $S' \subset S_{\iota',s}$, and as both are irreducible of dimension $\dim \mathfrak{d}$ (Lemma 11.7(b)), $S' = S_{\iota',s}$. So we may take $\iota = \iota'$ above; at x it is ι_x.

The action of ι_x on $H^1_{\mathrm{dR}}(A_x)$ and the polarization pairing, up to a scalar, are computed in the frame at x in which $\mathfrak{h}_C$ (obtained by conjugating the Lie algebra computed by [36, §§4.2, 4.4] by $P_c h'$ and taking its x-diagonal block, where the invertible evaluation matrix $P_c = (\theta(y)^\ell I_{2g})_{y|c,\, 0 \le \ell < d}$ changes coefficients to the frames $(e_j(y))_j$ and $F^\bullet$, from [104, Thm. 2.4], are written. That frame is a basis of the de Rham bundle of [104, Thm. 2.4] over an affine open set containing x, made free by [104, Prop. 2.1(iii)]; its elements are classes of Čech–de Rham cocycles for the affine cover of the family by the complements $U_0, \dots, U_g$ of relative hyperplane sections [104, Lemmas 2.5 and 2.6], and by base change their restrictions to A_x form a basis of $H^1_{\mathrm{dR}}(A_x)$. On the Čech–de Rham complex of A_x, by the standard formulas, we compute f^* on the cover by the sets $U_j \cap f^{-1}(U_k)$, which are affine, being isomorphic to $\Gamma_f \cap (U_j \times U_k)$, with f given on charts by its graph, and we compute the pairing of a and b as the ratio of $a \cup b \cup c_1(\mathcal{O}(1))^{g-1}$ to $c_1(\mathcal{O}(1))^g$ in $H^{2g}_{\mathrm{dR}}(A_x)$, a non-zero multiple of the pairing dual to the Riemann form of $\mathcal{O}(1)|_{A_x}$ (if $c_1 = \sum_i d_i\, dx_i \wedge dy_i$, it pairs dx_i with dy_i to $1/(gd_i)$ and every other pair of these basis vectors to 0). Each class is written in the frame by a search over cochains of increasing degree, which stops because a solution exists. Compactness at each real place is read off the integers a_v of the datum. So $\mathfrak{h}_{\widetilde{S}}$ is linear algebra over $\overline{\mathbb{Q}}$.

Correctness and termination. An accepted C has $\mathfrak{h}_C = \mathfrak{h}_Y$ by Lemma 11.11(c). For $r \geq 2$, by the Lefschetz theorem of Hamm and Lê [53, Thm. 1.1.3] there is an open dense set of hyperplanes L_1 for which $L_1 \cap Y^{\mathrm{sm}}$ is connected and smooth, with $\pi_1(L_1 \cap Y^{\mathrm{sm}}) \to \pi_1(Y^{\mathrm{sm}})$ onto, and so on for the following hyperplanes; hyperplanes defined over $\mathbb{Q}(i)$ are dense in the complex topology, so some tuple over $\mathbb{Q}(i)$ gives a smooth connected curve C with $\pi_1(C) \to \pi_1(Y^{\mathrm{sm}})$ onto, hence $\mathfrak{h}_C = \mathfrak{h}_Y$ and $t(\mathfrak{h}_C) = r$. The enumeration reaches it after finitely many tuples, each processed by terminating procedures, so the search stops, and whatever C it stops at is correct. □

11.7. **The algorithm.** Let $g \geq 2$ and let $Z \subset M_\Upsilon \otimes \overline{\mathbb{Q}}$ be closed and geometrically irreducible, whose closure $\overline{Z}$ in $\mathbb{P}(V_\Upsilon)$ is given by homogeneous equations over a number field. Let $c_{\deg}$, D_1, A_1 be numbers, with $A_1 \geq 1$, such that every special subvariety V maximal in Z has $\deg_{\mathcal{L}_{K_\Upsilon}} V \leq c_{\deg}$, $d_{L_V} \leq D_1$ and $\alpha_V \leq A_1$, with α_V at the level $\lambda(K_\Upsilon)$, and let $\bar{B}_0 \in (0, \bar{B}]$, where $\bar{B}$ is the constant B of Table 1 for the adjoint datum with which α_V is formed (Section 10). Such numbers are the bounds of Theorem 9.9(a) at $N = 2d_\Upsilon$ and $\delta = 1/32$ (Lemma 11.2 and Section 2.4). For $\bar{B}_0$ one may take any positive lower bound for $\bar{B}$ that is computable from g (Table 1).

Algorithm 11.13 (Listing)**.** Input: Mumford's equations for $\overline{M}_\Upsilon$ and $\overline{\mathcal{A}}_\Upsilon$, equations for $\overline{Z}$, and $(c_{\deg}, D_1, A_1, \bar{B}_0)$.

(0) *Base.* Compute N_0 (Lemma 11.3(iv)).

(1) *Data.* Enumerate the PEL data $\mathfrak{d} = (\mathcal{E}, R, \tau)$ with $\mathcal{E} = \prod_i F_i$, $\sum_i [F_i : \mathbb{Q}] \leq 2g$, $d_{F_i} \leq D_1$, and $[\mathcal{O}_\mathcal{E} : R] \leq B_\mathcal{O}$, where $B_\mathcal{O}$ is the bound of Lemma 10.1(d) with $n_\mathcal{E} = 2g$, $N = 2d_\Upsilon$, $m_V = \mathrm{lcm}(1, \ldots, 2g)$, D_1 and A_1, with $\bar{B}_0$ in place of $\bar{B}$ in A_1'.

(2) *Containers.* For each $\mathfrak{d}$, compute $\mathfrak{E}(\mathfrak{d}) \cap N_0$ (Lemma 11.6), which is closed in N_0 (Lemma 11.7(b)), and its irreducible components S' (primary decomposition).

(3) *Rigid members.* For each S' with $S' \cap Z \neq \emptyset$, compute the normalization $\nu\colon \widetilde{S}' \to \overline{S}'$ of the closure of S' in $\mathbb{P}(V_\Upsilon)$ and its embedding by $\nu^*\mathcal{O}(h)$ (Lemma 11.9), and restrict to $\nu^{-1}(S')$. Pull back the universal family and compute the data of the variation H^1 [104, Thm. 2.4]. Run [104, Cor. 6.3] on $\nu^{-1}(S')$ with a positive integer degree bound $d' > h^{\dim S'} c_{\deg}$, obtained from a rational upper bound for this product. For each output Y, keep it if $\nu(Y) \subset \overline{Z}$ (ideal membership; as $\nu(Y) \subset S' \subset N_0$ and $\overline{Z} \cap N_0 = Z$, this is $\nu(Y) \subset Z$) and Y is rigid (Lemma 11.10, with $\mathfrak{h}_Y$, $\mathfrak{h}_{\widetilde{S}}$ and $F^\bullet$ from Proposition 11.12), and put $\nu(Y)$ on the list.

(4) *Maximality.* Delete every member of the list that is strictly contained in another member.

Theorem 11.14. *(a) Algorithm 11.13 terminates, and its output is exactly the set of special subvarieties of S maximal among the special subvarieties contained in Z.*

(b) Let $(G', X') \hookrightarrow (\mathrm{GSp}_{2g}, \mathcal{H}_g^{\pm})$ be a Shimura datum of Hodge type, $K_{G'} := K_\Upsilon \cap G'(\mathbb{A}_f)$, and suppose Z lies in the image of $\mathrm{Sh}_{K_{G'}}(G', X')$ in S. Then the output of Algorithm 11.13 is exactly the set of special subvarieties of $\mathrm{Sh}_{K_{G'}}(G', X')$ (through their images) maximal in Z.

Proof. (a) *Every member put on the list is special and lies in Z.* An output Y of step (3) is weakly special and rigid, hence special by Lemma 11.10, which applies to $\nu^{-1}(S')$ by Lemma 11.7(c). The map ν is induced by a morphism of Shimura data, so $\nu(Y)$ is special, and $\nu(Y) \subset Z$ is tested exactly.

Every maximal V is put on the list. By Lemma 10.1(a),(d), applied with $N = 2d_\Upsilon$ (as $K(2d_\Upsilon) \subset K_\Upsilon$, Lemma 11.1) and $m_V \le \operatorname{lcm}(1,\dots,2g)$, with $\bar{B}_0$ in place of $\bar{B}$ in A_1' (as in the last sentence of Lemma 10.1), the bound being increasing in $n_{\mathcal{E}}$ and m_V, the datum $\mathfrak{d}_V$ is enumerated in step (1). By Lemma 11.7(b), S_V is a component of $\mathfrak{E}(\mathfrak{d}_V)$; it meets Z, and it lies in N_0, which is all of $M_\Upsilon \otimes \overline{\mathbb{Q}}$ by Lemma 11.3(iii) and Lemma 11.1(ii), so it is found in step (2). By Lemma 11.7(c), the part of the normalization of $\overline{S}_V$ over S_V is $\widetilde{S}_V$, and by Lemma 11.7(d) W_V is rigid in it. By Lemma 11.7(e), Lemma 11.9(f) and Lemma 11.8,

$$\deg_{\nu^*\mathcal{O}(h)} \overline{W}_V = h^{\dim V} \deg_{\mathcal{O}(1)} \overline{V} \le h^{\dim S_V} \deg_{\mathcal{L}_{K_\Upsilon}} V \le h^{\dim S_V} c_{\deg},$$

so $[W_V]$ is among the outputs of [104, Cor. 6.3]. Then $\nu(W_V) = V \subset Z$, and W_V passes the rigidity test, because Proposition 11.12 returns the true $\mathfrak{h}_{W_V}$ (Lemma 11.11(c)). So V is put on the list.

Termination. Step (0) is Lemma 11.3(iv), step (1) is finite by Lemma 10.1(d), and step (2) is Lemma 11.6 with primary decomposition. The data of step (1) are enumerated as follows. Fields: by Minkowski's second theorem a field F of degree at most $2g$ has $\mathbb{Q}$-independent integers v_k of house at most $d_F^{1/2}$; as F has at most $s(2g)$ proper subfields, with $s(2g)$ the number of subgroups of the symmetric group $\mathfrak{S}_{2g}$, some $\theta = \sum_k c_k v_k$ with $0 \le c_k \le s(2g)$ generates F, and the coefficients of its minimal polynomial are at most $\binom{2g}{j}(2g\,s(2g)D_1^{1/2})^j$. Orders: an order of index b contains $\mathbb{Z} + b\mathcal{O}_{\mathcal{E}}$, as $\mathcal{O}_{\mathcal{E}}/R$ has exponent dividing b, so the orders of index at most $B_{\mathcal{O}}$ are among the finitely many subgroups between $\mathbb{Z}+b\mathcal{O}_{\mathcal{E}}$ and $\mathcal{O}_{\mathcal{E}}$, $b \le B_{\mathcal{O}}$; keep those that are rings stable under $*$. Types: finitely many for each $\mathcal{E}$. Step (3) uses [47, Algorithm 3], Lemma 11.9, [104, Thm. 2.4 and Cor. 6.3], whose output is finite, ideal membership, and Lemma 11.10 with Proposition 11.12, which terminates. Step (4) is finite.

Step (4). A listed V that is maximal in Z is not strictly contained in another member, since members are special and lie in Z. A member that is not maximal lies in a maximal $V' \subset Z$, which is listed, so it is deleted.

(b) Images of special subvarieties of $\mathrm{Sh}_{K_{G'}}(G', X')$ are special in S. Conversely, if $V \subset S$ is special and lies in the image, its generic Mumford–Tate group is conjugate into G', since the Mumford–Tate groups of the points of X' lie in G', so V is the image of a special subvariety of $\mathrm{Sh}_{K_{G'}}(G', X')$ [102, Lemma 2.1]. So the two notions of maximal special subvariety of Z agree, and (a) applies. □

For a special point Q, $S_Q = \{Q\}$, and step (3) reduces to testing $Q \in Z$.

Remark 11.15. Let (G'', X'') be of abelian type: there are a Shimura datum (G', X') of Hodge type and a central isogeny $G'^{\mathrm{der}} \to G''^{\mathrm{der}}$ inducing $(G'^{\mathrm{ad}}, X'^{\mathrm{ad}}) \cong (G''^{\mathrm{ad}}, X''^{\mathrm{ad}})$ ([30, 2.3.10] constructs such data). For a neat K'', a connected component $\Gamma''\backslash X''^{+}$ of $\mathrm{Sh}_{K''}(G'', X'')_{\mathbb{C}}$ [30, 2.1.2] then receives a finite étale morphism π from a connected component of a Shimura variety for (G', X'), which for a small enough level is a closed subvariety of some $M_\Upsilon \otimes \mathbb{C}$ (as in [29, Prop. 1.15]). The special subvarieties maximal in a closed $Z \subset \Gamma''\backslash X''^{+}$ are exactly the images under π of those maximal in $\pi^{-1}(Z)$. We know no model of $\Gamma''\backslash X''^{+}$ or of π from the data defining it, so we state no algorithm for abelian type.

Proof of Theorem 1.5. Let F be a number field over which Z is defined, and $\overline{Z}$ the closure of Z in $\mathbb{P}(V_\Upsilon) \otimes \overline{\mathbb{Q}}$. By Lemma 11.8 for $Z \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$, which is irreducible, $D := \deg_{\mathcal{L}_{K_\Upsilon}} Z = (2(g+1))^{\dim Z} \deg_{\mathcal{O}(1)} \overline{Z}$; closures and degrees do not change from $\overline{\mathbb{Q}}$ to $\mathbb{C}$. The homogeneous prime ideal of $\overline{Z}$ is the radical of the ideal generated by the given equations, computed

by Gröbner bases, and $\deg_{\mathcal{O}(1)} \overline{Z}$ is $(\dim Z)!$ times the leading coefficient of its Hilbert polynomial.

By Lemma 11.2, Theorem 9.9(a) at $N = 2d_\Upsilon$, evaluated at $(c_\Upsilon D, c_\Upsilon[F : \mathbb{Q}])$, gives $c_{\deg}$ with no hypothesis and D_1, A_1 under $\mathrm{QRH}(1/32; L_V)$, which holds for every L_V (Section 2.4); they are computable from g, d_Υ, D and $[F : \mathbb{Q}]$, as for $\mathcal{A}_g$ no representatives of components are needed (proof of Theorem 9.9). With these numbers and $\bar{B}_0$ of Section 11.7, Theorem 11.14(a) gives the output. A special subvariety of a curve is a point or the curve, so if Z is a curve that is not special, its maximal special subvarieties are its special points. □

Appendix A. The smoothed explicit formula

Proof of Lemma 3.1. Write $-\zeta_L'/\zeta_L(s) = \sum_n a(n)n^{-s}$, with $a(n) = \sum_{N\mathfrak{p}^k=n} \log N\mathfrak{p}$. As L is Galois, $a(p) = n_L \log p$ if p splits completely, $a(p) = 0$ if p is unramified and does not split completely, and $a(p^k) \le n_L \log p$ for every prime p and every $k \ge 1$ (indeed, with e_p and f_p the ramification index and the residue degree of p in L, $a(p^k) = (n_L/e_p)\log p$ if $f_p \mid k$ and 0 otherwise). In particular $a(n) \le n_L \log p$ when $n = p$ is a ramified prime and when $n = p^k$ with $k \ge 2$, and $a(n) = 0$ when n is not a prime power. By the functional equation every nontrivial zero ϱ of ζ_L satisfies $\delta \le \operatorname{Re}\varrho \le 1 - \delta$.

Step 1 (a smoothed explicit formula). Let $F(x) := \sum_{n \le x} a(n)(x - n)$. Then

$$F(x) = \frac{1}{2\pi i}\int_{(2)} \Big(-\frac{\zeta_L'}{\zeta_L}(s)\Big)\frac{x^{s+1}}{s(s+1)}\,ds.$$

Let $x \ge 2$. The kernel decays like $|s|^{-2}$, so the contour may be moved to the left and $F(x)$ is the absolutely convergent sum of the residues; the side integrals tend to 0, by the bounds of Lagarias and Odlyzko for ζ_L'/ζ_L [66, Lemmas 5.4, 5.6 and 6.2], along heights $T_k \to \infty$ chosen away from the zeros, which exist by [66, Lemma 5.4]; the computation is carried out in the proof of [24, Prop. 6], with decay exponent 1; the left sides $\operatorname{Re} s = -j - \frac{1}{2}$ avoid the additional pole of our kernel at $s = -1$. Let $o_0 := r_1 + r_2 - 1$ and $o_1 := r_2$ be the orders of vanishing of ζ_L at $s = 0$ and $s = -1$, and $o_m \in \{r_2, r_1 + r_2\}$ its order at $s = -m$ for $m \ge 2$. The residues are $x^2/2$ at $s = 1$ and $-x^{\varrho+1}/(\varrho(\varrho+1))$ at each nontrivial zero ϱ, with multiplicity. At $s = 0$ and $s = -1$ they are $-o_0 x(\log x - 1) + c_0 x$ and $o_1(\log x + 1) - c_1$, for some constants c_0, c_1 (both poles have order at most 2). At $s = -m$, $m \ge 2$, the residue is $-o_m x^{1-m}/(m(m-1))$.

Step 2 (removing c_0 and c_1). As $x \ge 8$, Step 1 applies to x, $x/2$ and $x/4$. Let $G(x) := F(x) - 3F(x/2) + 2F(x/4) = \sum_n a(n)b_x(n)$, where $b_x(n) = (x-n)_+ - 3(x/2-n)_+ + 2(x/4-n)_+$. Then $b_x(n) = 0$ for $n \le x/4$, $b_x(n) = 2n - x/2$ on $(x/4, x/2]$ and $b_x(n) = x - n$ on $(x/2, x]$, so $0 \le b_x \le x/2$. In the combination the terms $c_0 x$ and c_1 cancel, $x(\log x - 1)$ becomes $(x\log 2)/2$, $\log x + 1$ becomes $-\log 2$, a zero ϱ acquires the factor $(1 - 2^{-\varrho-1})(1 - 2^{-\varrho})$, and the trivial zero at $-m$ the factor $(2^{m-1} - 1)(2^m - 1)$. Hence

$$G(x) = \frac{3x^2}{16} - \sum_\varrho \frac{x^{\varrho+1}(1 - 2^{-\varrho-1})(1 - 2^{-\varrho})}{\varrho(\varrho+1)} - \frac{o_0 \log 2}{2}x - o_1 \log 2 - R(x),$$

with $R(x) := \sum_{m\ge2} o_m(2^{m-1} - 1)(2^m - 1)x^{1-m}/(m(m-1)) \ge 0$. For $x \ge 8$, $R(x) \le 2n_L \sum_{k\ge1} 2^{-k}/(k(k+1)) = 2n_L(1 - \log 2) < 0.614\, n_L$.

Step 3 (the zeros). For each ϱ, $|x^{\varrho+1}(1 - 2^{-\varrho-1})(1 - 2^{-\varrho})| \le w(\delta)x^{2-\delta}$. By the Cauchy–Schwarz inequality $|\varrho||\varrho + 1| \ge \operatorname{Re}\varrho(1 + \operatorname{Re}\varrho) + (\operatorname{Im}\varrho)^2$, and so, with $u := \operatorname{Re}\varrho$ and

$v := (\operatorname{Im}\varrho)^2$,

$$\frac{|2-\varrho|^2}{(2-u)|\varrho||\varrho+1|} \le \frac{(2-u)^2+v}{(2-u)(u(1+u)+v)} \le \max\Big(\frac{2-u}{u(1+u)}, \frac{1}{2-u}\Big) \le \kappa(\delta),$$

because $(2-u)/(u(1+u))$ decreases on $[\delta,1]$ and $1/(2-u) \le 1/(1+\delta) \le \kappa(\delta)$. Hence $\sum_\varrho |\varrho(\varrho+1)|^{-1} \le \kappa(\delta)\sum_\varrho \operatorname{Re}(2-\varrho)^{-1}$.

Step 4 (Hadamard's product at $s=2$). Stark's identity [95, Lemma 1, (5)], applied to $\xi_L(s) = s(s-1)d_L^{s/2}\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_L(s)$, gives

$$\sum_\varrho \operatorname{Re}\frac{1}{2-\varrho} = \frac{3}{2} + \frac{1}{2}\log d_L + r_1\Big(-\frac{\log\pi}{2} + \frac{\Gamma'(1)}{2\Gamma(1)}\Big) + r_2\Big(-\log 2\pi + \frac{\Gamma'(2)}{\Gamma(2)}\Big) + \frac{\zeta_L'}{\zeta_L}(2) \le A_L,$$

since the coefficients of r_1 and r_2, $-(\gamma+\log\pi)/2$ and $1-\gamma-\log 2\pi$ with γ Euler's constant, are negative and $\zeta_L'/\zeta_L(2) < 0$. With Steps 2 and 3, and $o_0 \le n_L$, $o_1 \le n_L/2$,

$$G(x) \ge \frac{3x^2}{16} - w(\delta)\kappa(\delta)A_L x^{2-\delta} - 0.347\, n_L x - 0.961\, n_L.$$

Step 5 (counting). Since $0 \le b_x \le x/2$ and b_x vanishes on $[1, x/4]$,

$$G(x) \le \frac{x}{2}\, n_L\Big(\vartheta_L(x) + (\psi(x)-\vartheta(x)) + \epsilon_L(x)\log\operatorname{rad}(d_L)\Big),$$

where ψ and ϑ are Chebyshev's functions, and $\psi(x)-\vartheta(x) < 1.42620\sqrt{x}$ for $x > 0$ [88, Thm. 13, (3.36)]. Dividing the two bounds for $G(x)$ by $xn_L/2$ gives the lemma. □

Appendix B. Fixers in Bruhat–Tits buildings

Proof of Lemma 6.5. For a maximal split torus S of H, the group $Z_H(S)(\mathbb{Q}_l)$ acts on the apartment $\mathcal{A}(S)$ by translations through the homomorphism ν of [98, 1.2], which vanishes on bounded elements; so the maximal compact subgroup of $S(\mathbb{Q}_l)$ fixes $\mathcal{A}(S)$ pointwise. Also $h\mathcal{A}(S) = \mathcal{A}(hSh^{-1})$ for $h \in H(\mathbb{Q}_l)$ [98, 2.1], and fixers of points are compact open [98, 3.2 and 3.4.1].

(a) The assertion about fixed points is a case of the main theorem of Prasad and Yu [87], applied to the conjugation action of Θ, whose connected fixed group is $Z_H(T)$ by the hypothesis on weights. We give a direct proof. Let y be fixed by Θ. Choose a maximal split torus $S \supset T$ and $\lambda \in X_*(T)$ with $\langle\beta,\lambda\rangle \ne 0$ for every nonzero weight β of T on $\operatorname{Lie} H$. By [22, Thm. 4.1.7], $P_H(\lambda) = Z_H(\lambda) \ltimes U_H(\lambda)$, where $U := U_H(\lambda)$ is unipotent, $\operatorname{Lie} U$ is the sum of the weight spaces $(\operatorname{Lie} H)_\beta$ with $\langle\beta,\lambda\rangle > 0$, and $\lambda(t)u\lambda(t)^{-1} \to 1$ as $t \to 0$ for $u \in U(\mathbb{Q}_l)$; and $Z_H(\lambda) = Z_H(T) =: L$, as $Z_H(T) \subset Z_H(\lambda)$ and both are connected with Lie algebra $(\operatorname{Lie} H)_0$, by the choice of λ. Let $D \subset X_*(S)\otimes\mathbb{R}$ be a vector chamber whose closure contains λ, and U_D the group generated by the root groups $U_a(\mathbb{Q}_l)$ with a positive on D [98, 3.3]; λ has weights ≥ 0 on their Lie algebras, so $U_D \subset P_H(\lambda)(\mathbb{Q}_l) = U(\mathbb{Q}_l)L(\mathbb{Q}_l)$. The Iwasawa decomposition $H(\mathbb{Q}_l) = U_D\, N_H(S)(\mathbb{Q}_l)\, P_x$, for $x \in \mathcal{A}(S)$ and P_x its stabilizer [18, Cor. 7.3.2(i)], which applies to $H(\mathbb{Q}_l)$ by [19, Thm. 5.1.20], gives $\mathcal{B}(H) = U_D \cdot \mathcal{A}(S)$; so $y = uu_2 \cdot x$ with $u \in U(\mathbb{Q}_l)$, $u_2 \in L(\mathbb{Q}_l)$ and $x \in \mathcal{A}(S)$. The point $z := u_2 \cdot x$ lies in $\mathcal{A}(u_2 S u_2^{-1})$ and $u_2 S u_2^{-1} \supset T$, so Θ fixes z; it remains to show $u \cdot z = z$.

The group $U_z := U(\mathbb{Q}_l) \cap H(\mathbb{Q}_l)^z$ is compact open in $U(\mathbb{Q}_l)$ and normalized by Θ. For n large, $\lambda(l)^n u \lambda(l)^{-n} \in U_z$, so u lies in the compact subgroup $\Gamma := \lambda(l)^{-n} U_z \lambda(l)^n$, which Θ normalizes, as it commutes with $\lambda(l)$. Compact subgroups of $U(\mathbb{Q}_l)$ are pro-l, as U has a composition series with quotients $\mathbb{G}_{\mathrm{a}}$. For $\theta \in \Theta$ put $c_\theta := u^{-1}\theta u \theta^{-1} \in \Gamma$; as $\theta u \theta^{-1} \cdot z = \theta \cdot y = y = u \cdot z$, c_θ lies in the pro-l group $\Delta := \Gamma \cap U_z$, and $\theta \mapsto c_\theta$ is a 1-cocycle

of Θ in Δ. As $|\Theta|$ is prime to l, it is a coboundary (by the Schur–Zassenhaus theorem on the finite quotients of Δ by its open normal Θ-stable subgroups, and compactness): $c_\theta = u'^{-1}\theta u'\theta^{-1}$ for some $u' \in \Delta$ and all $\theta \in \Theta$. Then $v := uu'^{-1} \in U(\mathbb{Q}_l)$ commutes with Θ. The torus T is split, so $\Theta = \mathrm{Hom}(X^*(T), \mu_{l-1})$, and Θ acts on each $(\mathrm{Lie}\, H)_\beta \subset \mathrm{Lie}\, U$ through $\beta|_\Theta \neq 1$, as $\beta \notin (l-1)X^*(T)$; so $(\mathrm{Lie}\, U)^\Theta = 0$, and since $\log\colon U \to \mathrm{Lie}\, U$ is a T-equivariant isomorphism, $v = 1$. Hence $u = u' \in U_z$, and $y = u \cdot z = z \in \mathcal{A}(u_2 S u_2^{-1})$.

(b) Let S be a maximal split torus whose apartment contains C, hence y; put $B := H(\mathbb{Q}_l)^C \subset H(\mathbb{Q}_l)^y$, and let $\widetilde{W}$ be the group of affine transformations of $\mathcal{A}(S)$ induced by $N_H(S)(\mathbb{Q}_l)$ [98, 1.2], and $q_w := |BwB/B|$ for $w \in \widetilde{W}$ [98, 3.3.1]. By the Bruhat decomposition [98, 3.3.1], $H(\mathbb{Q}_l)^y$ is the disjoint union of the BwB with w in the stabilizer $\widetilde{W}_y$ of y in $\widetilde{W}$ (as $B \subset H(\mathbb{Q}_l)^y$, and $n \in N_H(S)(\mathbb{Q}_l)$ fixes y if and only if its image in $\widetilde{W}$ does), so $[H(\mathbb{Q}_l)^y : B] = \sum_{w \in \widetilde{W}_y} q_w$. An element of $\widetilde{W}_y$ with trivial image in the finite Weyl group $W_0 := N_H(S)(\mathbb{Q}_l)/Z_H(S)(\mathbb{Q}_l)$ is a translation fixing y, hence trivial; so $|\widetilde{W}_y| \leq |W_0| \leq 2^N$, where N is the number of reflections of W_0 (a Weyl chamber is determined by its signs on the N reflecting hyperplanes), and $2N \leq d$, as $\mathrm{Lie}\, H$ contains the nonzero root spaces of $2N$ distinct roots $\pm a$, one pair for each reflecting hyperplane. For $w \in \widetilde{W}_y$ write $w = r_1 \cdots r_j \omega$ with $\omega C = C$ and $r_1 \cdots r_j$ reduced, as in [98, 3.3.1], and let v_i be the vertex of the local Dynkin diagram representing r_i, with the integer $d(v_i)$ of [98, 1.8]; then $q_w = l^{d(v_1)+\cdots+d(v_j)}$, and j is the number of walls separating C from wC. Each of these walls contains y, which lies in the closures of C and of wC, and distinct walls through y have distinct directions, which are kernels of roots; so $j \leq N$. Each $d(v_i)$ is the dimension of a homogeneous variety of the special fiber of a smooth model of H [98, 3.5.4 and the paragraph after it], so $d(v_i) \leq d$. Hence $[H(\mathbb{Q}_l)^y : B] \leq 2^N l^{Nd} \leq l^{N(d+1)} \leq l^{d(d+1)/2}$.

(c) By [98, 3.2], $K_l^1 = P(\mathbb{Q}_l)^y$ for a point y. A nonzero weight of T_V on $\mathrm{Lie}\, P \subset \mathfrak{gl}_{n_\rho}$ is $\chi - \chi'$ for weights χ, χ' of T_V in ρ, and does not lie in $(l-1)X^*(T_V)$: otherwise the group of components of $\ker(\chi - \chi')$, which is the stabilizer in T_V of the line through $v_\chi + v_{\chi'}$ for weight vectors v_χ, $v_{\chi'}$ [34, proof of Lemma 4.4.1], would have order divisible by $l-1$, over $\overline{\mathbb{Q}}$ as over $\overline{\mathbb{F}}_l$, whereas these orders are at most C_Φ (Section 2.3 and Lemma 6.4) and $l - 1 > 2C_\Phi - 1 \geq C_\Phi$ by (2), as $c_U = 1/(2C_\Phi)$ and $C_\Phi \geq 1$. The torus $T_{V,\mathbb{Q}_l}$ is split, as l splits completely in L_V, and its elements of order dividing $l-1$ lie in $K^m_{T_V,l} = K_{T_V,l} = T_V(\mathbb{Q}_l) \cap K_l \subset K_{P,l} \subset K_l^1$, by (2) and Section 2.2; so they fix y, and (a), with $H = P_{\mathbb{Q}_l}$ and $T = T_{V,\mathbb{Q}_l}$, gives S. The maximal compact subgroup of $S(\mathbb{Q}_l)$ fixes the apartment of S, so it lies in I_l^1 and I_l^2; it contains that of $T_V(\mathbb{Q}_l)$, which also lies in $K_{P,l}$. As y and x_0 lie in the closure of C^1, and x_0 in that of C^2, we have $I_l^1 \subset K_l^1$ and $I_l^1, I_l^2 \subset P(\mathbb{Q}_l)^{x_0}$; so (b), applied to (y, C^1) and (x_0, C^2) with $d = \dim P \leq \dim G$, gives

$$[K_{P,l} : I_l] \leq [K_l^1 : I_l^1 \cap I_l^2] \leq [K_l^1 : I_l^1]\,[P(\mathbb{Q}_l)^{x_0} : I_l^2] \leq l^{d(d+1)} < l^f. \qquad \square$$

Appendix C. Two index bounds

This appendix proves Lemma C.2, which is used in the proof of (DJK1) in Section 7.1, and the closed bound for the factor h_K of Theorem 9.5 (Corollary C.8). For a number field L let $r_1(L)$, $r_2(L)$ be its numbers of real and complex places and h_L its class number, and for $m \geq 1$ put $L(\emptyset, m) := \{a \in L^\times / L^{\times m} : v_w(a) \equiv 0 \bmod m \text{ for every finite place } w\}$. Both bounds use the following lemma.

Lemma C.1. *Let L be a number field of degree n_L.*

(a) $h_L \le \max(1, B_L(1+\log B_L)^{n_L-1})$ *with* $B_L := (n_L!/n_L^{n_L})(4/\pi)^{r_2(L)} d_L^{1/2}$.

(b) $|L(\emptyset, m)| \le m^{r_1(L)+r_2(L)} h_L$ *for every* $m \ge 1$.

Proof. (a) Every ideal class contains an integral ideal of norm at most B_L [71, Thm. 4.3], and the number of integral ideals of norm at most x is at most $\sum_{m\le x} \tau_{n_L}(m) \le x(1+\log x)^{n_L-1}$, with τ_{n_L} the n_L-fold divisor function.

(b) There is an exact sequence $1 \to \mathcal{O}_L^\times/\mathcal{O}_L^{\times m} \to L(\emptyset, m) \to \mathrm{Cl}(L)[m] \to 0$, where a with $(a) = \mathfrak{a}^m$ maps to the class of $\mathfrak{a}$; by Dirichlet's unit theorem $|\mathcal{O}_L^\times/\mathcal{O}_L^{\times m}| \le m^{r_1(L)+r_2(L)}$. □

C.1. **An index bound.** The following lemma replaces [27, Lemma 6.7] in the proof of (DJK1) (Section 7.1), and G, K and the notation are as there.

Lemma C.2. *Let V be a special subvariety of $\mathrm{Sh}_K(G, X^+)_{\mathbb{C}}$ of positive dimension, the image of $X_H^+ \times \{1\}$, and let $\pi\colon \widetilde{H} \to H^{\mathrm{der}}$, $\Gamma(H)^+$ and $\Gamma(\widetilde{H})$ be as in* [27, pp. 13–14]. *Write $\widetilde{H} = \prod_i \mathrm{Res}_{K_i/\mathbb{Q}} H_i'$ with K_i a number field and H_i' absolutely almost simple and simply connected over K_i* [27, §5]. *For each i let ℓ_i and s_i be the field and the integer that* [86, §§0.2 and 0.4] *attach to the quasi-split inner form of H_i', and put*

$$N_i := d_{\ell_i}/d_{K_i}^{[\ell_i:K_i]}, \qquad \varpi_i := d_{K_i}^{\dim H_i'/2} N_i^{s_i/2},$$

where d_L is the absolute value of the discriminant of a number field L (Prasad writes D_L). Let r_G be the absolute rank of G, and B_Z the constant of [27, Lemma 6.4]. *There is an effectively computable c_2^*, depending only on r_G and B_Z, such that*

$$[\Gamma(H)^+ : \pi(\Gamma(\widetilde{H}))] \le c_2^* \, (B_Z^{3r_G})^{|\Sigma_V|} \prod_i \varpi_i^{1/2}.$$

Proof. By the proof of [27, Lemma 6.4], its constant B_Z bounds $|Z(\widetilde{H})(\mathbb{C})|$. Let $Z_\pi := \ker \pi$, and let Σ be the union of Σ_V and the set of primes dividing $|Z_\pi(\mathbb{C})|$, so that $|\Sigma| \le |\Sigma_V| + \log_2 B_Z$. A class in the first Galois cohomology of a number field is *unramified* at a finite place if its restriction to an inertia group there is 0. Step (i) follows [27, proof of Lemma 6.7, p. 14]; steps (ii) and (iii) take the place of their restriction to a field that splits Z_π and of their Lemma 6.8 [27, p. 15], and are the argument of Borel and Prasad [17, 2.10 and §5].

(i) As Z_π is central, the connecting map $\partial\colon H^{\mathrm{der}}(\mathbb{Q}) \to H^1(\mathbb{Q}, Z_\pi)$ is a homomorphism: if $h, h' \in \widetilde{H}(\overline{\mathbb{Q}})$ and $\gamma := \pi(h)$ and $\gamma' := \pi(h')$ lie in $H^{\mathrm{der}}(\mathbb{Q})$, then $(hh')^{-1}\sigma(hh') = h^{-1}\sigma(h) \cdot h'^{-1}\sigma(h')$ for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, since $h^{-1}\sigma(h) \in Z_\pi(\overline{\mathbb{Q}})$ is central. Its kernel is $\pi(\widetilde{H}(\mathbb{Q}))$, and $\Gamma(\widetilde{H})$ is the full preimage of $\Gamma(H)^+$ in $\widetilde{H}(\mathbb{Q})$; so ∂ embeds $\Gamma(H)^+/\pi(\Gamma(\widetilde{H}))$ in $H^1(\mathbb{Q}, Z_\pi)$. Let M be the image. For $p \notin \Sigma$ every class in M is unramified at p, since $\Gamma(H)^+ \subset K(H^{\mathrm{der}})$ [27, §3.4] and the image of $K(H^{\mathrm{der}})_p$ in $H^1(\mathbb{Q}_p, Z_\pi)$ lies in $H^1(\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p), \mathcal{Z}(\mathbb{Z}_p^{\mathrm{un}}))$ for a smooth model $\mathcal{Z}$ of $Z_{\pi,\mathbb{Q}_p}$ over $\mathbb{Z}_p$ [27, p. 14], [26, p. 88].

(ii) The center of $\widetilde{H}$ is $\prod_i \mathrm{Res}_{K_i/\mathbb{Q}} A_i$ with $A_i := Z(H_i')$, and it contains Z_π. The kernel of $H^1(\mathbb{Q}, Z_\pi) \to H^1(\mathbb{Q}, Z(\widetilde{H}))$ is the image of $(Z(\widetilde{H})/Z_\pi)(\mathbb{Q})$, of order at most B_Z. By Shapiro's lemma [72, II.1.11], $H^1(\mathbb{Q}, \mathrm{Res}_{K_i/\mathbb{Q}} A_i) \cong H^1(K_i, A_i)$, through restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}/K_i)$ and evaluation at 1; as the inertia groups of K_i are the intersections of those of $\mathbb{Q}$ with $\mathrm{Gal}(\overline{\mathbb{Q}}/K_i)$, classes unramified at p go to classes unramified at every place of K_i above p. Let $M_i \subset H^1(K_i, A_i)$ be the image of M. Then $|M| \le B_Z \prod_i |M_i|$, and the classes in M_i are unramified at every place above a prime $p \notin \Sigma$.

(iii) Inner twisting acts trivially on the center, so A_i is isomorphic over K_i to the center of the quasi-split inner form $\mathcal{G}_i$ of H_i', and $\mathrm{Gal}(\overline{K_i}/K_i)$ acts on $X^*(A_i)$, the weight lattice modulo the root lattice, through its action on the Dynkin diagram of $\mathcal{G}_i$. If H_i' is not a triality form, $\mathcal{G}_i$ splits over ℓ_i (and $\ell_i = K_i$ for inner forms), so $A_i \cong \prod_{k=1}^{q_i} \mu_{n_{i,k}}$ over ℓ_i, with $\prod_k n_{i,k} = |A_i(\mathbb{C})|$; inflation and restriction [72, II.1.34] map $H^1(K_i, A_i)$ to $H^1(\ell_i, A_i) = \prod_k \ell_i^\times/\ell_i^{\times n_{i,k}}$ [72, II.1.22] with kernel $H^1(\mathrm{Gal}(\ell_i/K_i), A_i(\ell_i))$, of order at most $|A_i(\mathbb{C})|$ as $[\ell_i : K_i] \le 2$. If H_i' is a triality form, of type ${}^3\mathrm{D}_4$ or ${}^6\mathrm{D}_4$, then $X^*(A_i) \cong (\mathbb{Z}/2)^2$, and its three non-zero elements, the classes of the fundamental weights of the three outer vertices, are permuted as these vertices; as ℓ_i is the field fixed by the stabilizer in $\mathrm{Gal}(\overline{K_i}/K_i)$ of one outer vertex, $A_i \cong \ker(N\colon \mathrm{Res}_{\ell_i/K_i}\mu_2 \to \mu_2)$, whose character group is $(\mathbb{Z}/2)^{\mathrm{Hom}_{K_i}(\ell_i, \overline{K_i})}$ modulo the diagonal. Since $N(-1) = -1$, the map $H^1(K_i, A_i) \to H^1(K_i, \mathrm{Res}_{\ell_i/K_i}\mu_2) = \ell_i^\times/\ell_i^{\times 2}$ is injective; here put $q_i := 1$ and $n_{i,1} := 2$. In both cases these maps keep classes unramified, and a class in $\ell_i^\times/\ell_i^{\times m}$ that is unramified at a finite place w is an m-th power over the maximal unramified extension of $\ell_{i,w}$, so its valuation at w is divisible by m. For a number field L, a set S of finite places of L and $m \ge 1$ put $L(S,m) := \{a \in L^\times/L^{\times m} : v_w(a) \equiv 0 \bmod m \text{ for every finite place } w \notin S\}$, and let S_i be the set of places of ℓ_i above Σ. Then

$$|M_i| \le |A_i(\mathbb{C})| \prod_{k=1}^{q_i} |\ell_i(S_i, n_{i,k})|.$$

The valuations at the places in S map $L(S,m)$ to $(\mathbb{Z}/m)^S$ with kernel $L(\emptyset, m)$, and $|L(\emptyset,m)| \le m^{r_1(L)+r_2(L)} h_L$ (Lemma C.1(b)); as $|S_i| \le [\ell_i : \mathbb{Q}]\,|\Sigma|$, we get $|\ell_i(S_i,m)| \le m^{[\ell_i:\mathbb{Q}](|\Sigma|+1)} h_{\ell_i}$.

(iv) By Lemma C.1(a), $h_L \le \max(1, B_L(1+\log B_L)^{n_L-1})$ with $B_L \le (4/\pi)^{n_L/2} d_L^{1/2}$; so $h_L \le c_h(n_L)\, d_L^{3/4}$ with $c_h(n) := (4/\pi)^{n/2} \sup_{y\ge 0}(n+y/2)^{n-1} e^{-y/4}$, which is non-decreasing in n and at least 1. As $d_{\ell_i} = d_{K_i}^{[\ell_i:K_i]} N_i$, with $N_i = |N_{K_i/\mathbb{Q}}(\mathfrak{d}(\ell_i/K_i))|$ [86, §0.2],

$$h_{\ell_i}^{q_i} \le c_h([\ell_i:\mathbb{Q}])^{q_i}\, d_{K_i}^{3q_i[\ell_i:K_i]/4} N_i^{3q_i/4} \le c_h([\ell_i:\mathbb{Q}])^{q_i}\, \varpi_i^{1/2},$$

because $3q_i[\ell_i : K_i] \le \dim H_i'$ for every form, and $3q_i \le s_i$ for outer and triality forms (for inner forms $N_i = 1$). Indeed, $q_i \le 1$ except for forms of type D_r with r even that are not triality forms, where $q_i = 2$, $\dim H_i' \ge 28$ and, for outer forms, $s_i = 2r-1 \ge 7$; $\dim H_i' \ge 3$, $\dim H_i' \ge 8$ if $[\ell_i : K_i] = 2$, and $\dim H_i' = 28$ if $[\ell_i : K_i] = 3$; and $s_i \ge 5$ for outer and triality forms [86, §0.4].

(v) Each H_i' has rank at least 1, and $\sum_i [K_i:\mathbb{Q}] \operatorname{rank} H_i' = \operatorname{rank} H^{\mathrm{der}} \le r_G$; so there are at most r_G indices i, and $[\ell_i:\mathbb{Q}] \le 3[K_i:\mathbb{Q}] \le 3r_G$. Also $\prod_{i,k} n_{i,k} \le \prod_i |A_i(\mathbb{C})| \le |Z(\widetilde{H})(\mathbb{C})| \le B_Z$. By (i)–(iv),

$$[\Gamma(H)^+ : \pi(\Gamma(\widetilde{H}))] = |M| \le B_Z^2\, B_Z^{3r_G(|\Sigma|+1)}\, c_h(3r_G)^{2r_G} \prod_i \varpi_i^{1/2},$$

and $|\Sigma| \le |\Sigma_V| + \log_2 B_Z$ gives the lemma with $c_2^* := B_Z^{2+3r_G(1+\log_2 B_Z)}\, c_h(3r_G)^{2r_G}$. □

C.2. **A closed bound for h_K.** The factor h_K enters both bounds of Theorem 9.5; here we bound it in closed form (Corollary C.8). Proposition 9.3(d) bounds h_K by $|\mathrm{Cl}(K)| \cdot |\mathrm{Sel}_{\mathcal{S}}(Z(G))|$. We bound both factors in terms of the center. Let $\Xi := X^*(Z(G))$, a finitely generated abelian group with a continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, r_Z its rank, $Z(G)^\circ$ the maximal subtorus of $Z(G)$, with character group Ξ/Ξ_{tors}, and $\Pi_Z := Z(G)/Z(G)^\circ$, finite of multiplicative type with character group Ξ_{tors}. Write $\Xi_{\mathrm{tors}} \cong \bigoplus_{i=1}^q \mathbb{Z}/m_i\mathbb{Z}$ with all $m_i \ge 2$, and $m_Z := |\Xi_{\mathrm{tors}}| = \prod_i m_i$. Let L_0 and L_1 be the fields fixed by the kernels of the Galois

actions on Ξ_{tors} and on Ξ/Ξ_{tors}, Galois over $\mathbb{Q}$, with $n_{L_0} := [L_0 : \mathbb{Q}]$ and $n_{L_1} := [L_1 : \mathbb{Q}]$. For a finite group $\mathfrak{H}$ let $\mathrm{gen}(\mathfrak{H}) \le \log_2 |\mathfrak{H}|$ be its least number of generators. For a torus T over $\mathbb{Q}$ let $T(\mathbb{Q}_p)^c$ be the maximal compact subgroup of $T(\mathbb{Q}_p)$ and $\ker^1_f(\mathbb{Q}, T) := \ker(H^1(\mathbb{Q}, T) \to \prod_p H^1(\mathbb{Q}_p, T))$. As in Section 9, K satisfies (Lv), and $\mathcal{S}$, $\mathcal{G}_p$, $\mathcal{Z}_p$ are as defined before Lemma 9.2; for $p \notin \mathcal{S}$ let $\mathcal{Z}_p^\circ$ be the maximal subtorus of $\mathcal{Z}_p$ and $\mathcal{P}_p := \mathcal{Z}_p/\mathcal{Z}_p^\circ$.

We use the following facts for $p \notin \mathcal{S}$.

(M1) $\mathcal{Z}_p^\circ$ has generic fiber $Z(G)^\circ$, and $1 \to \mathcal{Z}_p^\circ \to \mathcal{Z}_p \to \mathcal{P}_p \to 1$ is exact as fppf sheaves, with $\mathcal{P}_p$ finite flat of multiplicative type with generic fiber Π_Z: after a finite unramified extension $\mathcal{Z}_p$ becomes diagonalizable [22, Prop. B.3.4], where this is clear, and everything descends, the maximal subtorus being unique and compatible with base change [22, App. B].

(M2) The Galois modules Ξ, Ξ_{tors} and Ξ/Ξ_{tors} are unramified at p [22, Prop. B.3.4]; so p is unramified in L_0 and L_1, and $\mathcal{P}_p \otimes \mathcal{O}_w \cong \prod_i \mu_{m_i}$ for every place $w \mid p$ of L_0.

(M3) $H^1_{\mathrm{fppf}}(\mathbb{Z}_p, \mathcal{T}) = 0$ for every torus $\mathcal{T}$ over $\mathbb{Z}_p$, by Lang's theorem and Hensel's lemma.

(M4) For a finite $\mathbb{Z}_p$-scheme $\mathcal{Q}$, $\mathcal{Q}(\mathbb{Z}_p) = \mathcal{Q}(\mathbb{Q}_p)$; hence $\Pi_Z(\mathbb{Q}_p) = \mathcal{P}_p(\mathbb{Z}_p)$, and $H^1_{\mathrm{fppf}}(\mathbb{Z}_p, \mathcal{P}_p) \to H^1(\mathbb{Q}_p, \Pi_Z)$ is injective, since a $\mathcal{P}_p$-torsor is affine [97, Tag 0245] and finite over $\mathbb{Z}_p$, so a $\mathbb{Q}_p$-point of it is a $\mathbb{Z}_p$-point.

(M5) For a discrete valuation ring $\mathcal{O}$ with fraction field L, $H^1_{\mathrm{fppf}}(\mathcal{O}, \mu_m) = \mathcal{O}^\times/\mathcal{O}^{\times m}$, with image $\mathcal{O}^\times L^{\times m}/L^{\times m}$ in $H^1(L, \mu_m) = L^\times/L^{\times m}$ (Kummer theory and $\mathrm{Pic}(\mathcal{O}) = 0$).

(M6) $Z(G)(\mathbb{Q}_p) \cap K_p = \mathcal{Z}_p(\mathbb{Z}_p)$ and $Z(G)^\circ(\mathbb{Q}_p) \cap K_p = \mathcal{Z}_p^\circ(\mathbb{Z}_p) = Z(G)^\circ(\mathbb{Q}_p)^c$: a point of $\mathcal{G}_p(\mathbb{Z}_p)$ whose generic fiber lies in $\mathcal{Z}_p$ factors through this schematic closure, and for a torus split over an unramified extension the $\mathbb{Z}_p$-points are the points on which all characters are units, as in the proof of Lemma 9.2(iii).

Let $\partial_v \colon \Pi_Z(\mathbb{Q}_v) \to H^1(\mathbb{Q}_v, Z(G)^\circ)$ be the connecting maps, and $\mathrm{Sel}_{\mathcal{S}}(\Pi_Z)$ the set of $a \in H^1(\mathbb{Q}, \Pi_Z)$ with $a_v = 0$ for $v \in \mathcal{S} \cup \{\infty\}$ and a_p in the image of $H^1_{\mathrm{fppf}}(\mathbb{Z}_p, \mathcal{P}_p)$ for $p \notin \mathcal{S}$.

Proposition C.3. $|\mathrm{Sel}_{\mathcal{S}}(Z(G))| \le |\mathrm{Sel}_{\mathcal{S}}(\Pi_Z)| \cdot |\ker^1_f(\mathbb{Q}, Z(G)^\circ)| \cdot \prod_{p \in \mathcal{S}} |\partial_p(\Pi_Z(\mathbb{Q}_p))|$, *and the last factor is at most* $m_Z^{|\mathcal{S}|}$.

Proof. Let $\phi \colon H^1(\mathbb{Q}, Z(G)) \to H^1(\mathbb{Q}, \Pi_Z)$ and $\iota \colon H^1(\mathbb{Q}, Z(G)^\circ) \to H^1(\mathbb{Q}, Z(G))$ be the maps of $1 \to Z(G)^\circ \to Z(G) \to \Pi_Z \to 1$. First, $\phi(\mathrm{Sel}_{\mathcal{S}}(Z(G))) \subset \mathrm{Sel}_{\mathcal{S}}(\Pi_Z)$: at $p \notin \mathcal{S}$, c_p is the restriction of some $\gamma \in H^1(\mathbb{Z}_p, \mathcal{Z}_p)$, and $\phi_p(c_p)$ is the restriction of the image $\bar\gamma$ of γ in $H^1(\mathbb{Z}_p, \mathcal{P}_p)$. Let $c \in \mathrm{Sel}_{\mathcal{S}}(Z(G))$ with $\phi(c) = 0$. For $p \notin \mathcal{S}$, $\bar\gamma = 0$ by (M4), so $\gamma = 0$ by (M1) and (M3), and $c_p = 0$; so $c_v = 0$ at every place. Write $c = \iota(b)$. For every v, $\iota_v(b_v) = 0$, so $b_v \in \partial_v(\Pi_Z(\mathbb{Q}_v))$; for $p \notin \mathcal{S}$ this is 0, since by (M4) every $x \in \Pi_Z(\mathbb{Q}_p)$ lies in $\mathcal{P}_p(\mathbb{Z}_p)$ and $\partial_p(x)$ is the restriction of a class in $H^1(\mathbb{Z}_p, \mathcal{Z}_p^\circ) = 0$ (Lemma 9.1). The map $b \mapsto (b_p)_{p \in \mathcal{S}} \in \prod_{p \in \mathcal{S}} \partial_p(\Pi_Z(\mathbb{Q}_p))$ has kernel in $\ker^1_f(\mathbb{Q}, Z(G)^\circ)$ on the set of such b. Counting gives the bound, and $|\Pi_Z(\mathbb{Q}_p)| \le m_Z$. □

Proposition C.4.

$$|\mathrm{Sel}_{\mathcal{S}}(\Pi_Z)| \le |H^1(\mathrm{Gal}(L_0/\mathbb{Q}), \Pi_Z(L_0))| \cdot \prod_{i=1}^{q} |L_0(\emptyset, m_i)| \le m_Z^{\mathrm{gen}(\mathrm{Gal}(L_0/\mathbb{Q})) + r_1(L_0) + r_2(L_0)}\, h_{L_0}^q.$$

Proof. Inflation and restriction [72, II.1.34] give $0 \to H^1(\mathrm{Gal}(L_0/\mathbb{Q}), \Pi_Z(L_0)) \to H^1(\mathbb{Q}, \Pi_Z) \to H^1(L_0, \Pi_Z)$, and $H^1(L_0, \Pi_Z) = \prod_i L_0^\times/L_0^{\times m_i}$, as $\Pi_Z \cong \prod_i \mu_{m_i}$ over L_0 and $H^1(L_0, \mathbb{G}_{\mathrm{m}}) = 0$ [72, II.1.22]. Let $a \in \mathrm{Sel}_{\mathcal{S}}(\Pi_Z)$ with restriction (a_i). At a place w of L_0 above $p \in \mathcal{S}$, $a_p = 0$, so $v_w(a_i) \equiv 0 \bmod m_i$. Above $p \notin \mathcal{S}$, a_p comes from $H^1(\mathbb{Z}_p, \mathcal{P}_p)$, and

its restriction to $L_{0,w}$ factors through $\prod_i H^1(\mathcal{O}_w, \mu_{m_i})$ by (M2), whose image consists of classes of units by (M5); so again $v_w(a_i) \equiv 0 \bmod m_i$. This gives the first inequality. For the second, $|L_0(\emptyset, m_i)| \le m_i^{r_1(L_0)+r_2(L_0)} h_{L_0}$ by Lemma C.1(b), and $|H^1(\mathfrak{H}, A)| \le |A|^{\mathrm{gen}(\mathfrak{H})}$ for a finite group $\mathfrak{H}$ and a finite module A, as a cocycle is determined by its values on generators, with $|\Pi_Z(L_0)| \le m_Z$. □

For a torus T over $\mathbb{Q}$ split by a Galois field L, put $T(\mathcal{O}_L) := \mathrm{Hom}(X^*(T), \mathcal{O}_L^\times)$, $T(\mathbb{A}_f)^c := \prod_p T(\mathbb{Q}_p)^c$ and $h(T) := |T(\mathbb{Q})\backslash T(\mathbb{A}_f)/T(\mathbb{A}_f)^c|$.

Proposition C.5. *Let T be a torus over $\mathbb{Q}$ of dimension r, split by a Galois field L with group $\mathfrak{H}$ of order n_L. Then $|\ker^1_f(\mathbb{Q}, T)|$ and $h(T)$ are both at most $|H^1(\mathfrak{H}, T(\mathcal{O}_L))| \cdot |(X_*(T) \otimes \mathrm{Cl}(L))^{\mathfrak{H}}| \le n_L^{r(\log_2 n_L + r_1(L) + r_2(L))} h_L^r$.*

Proof. As $\mathfrak{H}$-modules $T(L) = X_* \otimes L^\times$ with $X_* := X_*(T) \cong \mathbb{Z}^r$, and tensoring $1 \to \mathcal{O}_L^\times \to L^\times \to \mathrm{Id}_L \to \mathrm{Cl}(L) \to 0$ (Id_L the fractional ideals) with X_* gives an exact sequence $0 \to T(\mathcal{O}_L) \to T(L) \to I \to C \to 0$ with $I := X_* \otimes \mathrm{Id}_L$ and $C := X_* \otimes \mathrm{Cl}(L)$; let $\Pi \subset I$ be the image of $T(L)$.

$\ker^1_f$: by Hilbert's Theorem 90 and inflation–restriction, $H^1(\mathbb{Q}, T) = H^1(\mathfrak{H}, T(L))$ and $H^1(\mathbb{Q}_p, T) = H^1(\mathfrak{H}_w, T(L_w))$ for $w \mid p$, $\mathfrak{H}_w$ the decomposition group. As $\mathfrak{H}$-modules $I = \bigoplus_p I_p$ with $I_p \cong \mathrm{Ind}_{\mathfrak{H}_w}^{\mathfrak{H}} X_*$, and by Shapiro's lemma [72, II.1.11] $H^1(\mathfrak{H}, I_p) \cong H^1(\mathfrak{H}_w, X_*)$, through restriction to $\mathfrak{H}_w$ and the w-coordinate, which on $T(L)$ is the valuation $T(L_w) \to X_*$. If $c \in \ker^1_f(\mathbb{Q}, T)$, its restriction to $\mathfrak{H}_w$ is a coboundary in $T(L_w)$, so the image of c in $H^1(\mathfrak{H}, I)$ is 0. So the image of c in $H^1(\mathfrak{H}, \Pi)$ lies in $\ker(H^1(\mathfrak{H}, \Pi) \to H^1(\mathfrak{H}, I))$, the image of $C^{\mathfrak{H}}$; and the fibers of $H^1(\mathfrak{H}, T(L)) \to H^1(\mathfrak{H}, \Pi)$ are cosets of the image of $H^1(\mathfrak{H}, T(\mathcal{O}_L))$.

$h(T)$: for each p the valuations embed $T(\mathbb{Q}_p)/T(\mathbb{Q}_p)^c$ in $\bigoplus_{w|p} X_*$, since an element whose image lies in the compact group $\prod_w X_* \otimes \mathcal{O}_w^\times$ generates a bounded subgroup. So $I_T := T(\mathbb{A}_f)/T(\mathbb{A}_f)^c$ embeds in $I^{\mathfrak{H}}$, compatibly with $T(\mathbb{Q}) \subset T(L)$, and $h(T) = |I_T/T(\mathbb{Q})|$. The induced map $I_T/T(\mathbb{Q}) \to C^{\mathfrak{H}}$ has kernel embedding in $H^1(\mathfrak{H}, T(\mathcal{O}_L))$: for a in the kernel choose $\varrho \in T(L)$ with image a; $\sigma \mapsto \sigma(\varrho)/\varrho$ is a cocycle with values in $T(\mathcal{O}_L)$, well defined up to coboundaries, and if it is $\sigma(\varepsilon)/\varepsilon$, then $\varrho/\varepsilon \in T(\mathbb{Q})$ has image a.

Sizes: with $E := T(\mathcal{O}_L) \cong (\mathcal{O}_L^\times)^r$, n_L kills $H^1(\mathfrak{H}, E)$, so $|H^1(\mathfrak{H}, E)| \le |H^1(\mathfrak{H}, E[n_L])| \cdot |E/n_L E| \le n_L^{r\,\mathrm{gen}(\mathfrak{H})} n_L^{r(r_1(L)+r_2(L))}$, and $|C^{\mathfrak{H}}| \le h_L^r$. □

Theorem C.6. *For every Shimura datum (G, X) and neat product level, with $\mathcal{S}$ as defined before Lemma 9.2,*

$$|\mathrm{Sel}_{\mathcal{S}}(Z(G))| \le m_Z^{\log_2 n_{L_0} + r_1(L_0) + r_2(L_0) + |\mathcal{S}|}\, h_{L_0}^q\, n_{L_1}^{r_Z(\log_2 n_{L_1} + r_1(L_1) + r_2(L_1))}\, h_{L_1}^{r_Z}.$$

Proof. Propositions C.3 and C.4, and Proposition C.5 for $T = Z(G)^\circ$, split by L_1. □

Lemma C.7. *Let $\iota_K := \prod_{p \in \mathcal{S}} [Z(G)^\circ(\mathbb{Q}_p)^c : Z(G)^\circ(\mathbb{Q}_p) \cap K_p]$; it is finite. Then*

$$|\mathrm{Cl}(K)| \le m_Z^{|\mathcal{S}|}\, \iota_K\, h(Z(G)^\circ) \le m_Z^{|\mathcal{S}|}\, \iota_K\, n_{L_1}^{r_Z(\log_2 n_{L_1} + r_1(L_1) + r_2(L_1))}\, h_{L_1}^{r_Z}.$$

Proof. The number ι_K is finite, since $Z(G)^\circ(\mathbb{Q}_p) \cap K_p$ is a compact open subgroup of $Z(G)^\circ(\mathbb{Q}_p)$, so it lies in the maximal compact subgroup $Z(G)^\circ(\mathbb{Q}_p)^c$ with finite index. Let $U := Z(G)(\mathbb{A}_f) \cap K$ and $U^\circ := Z(G)^\circ(\mathbb{A}_f) \cap K$. The map $\mathrm{Cl}(K) \to \Pi_Z(\mathbb{A}_f)/(\mathrm{im}\, Z(G)(\mathbb{Q}) \cdot \mathrm{im}\, U)$ has kernel the image of $Z(G)^\circ(\mathbb{A}_f)$, which is a quotient of $Z(G)^\circ(\mathbb{Q})\backslash Z(G)^\circ(\mathbb{A}_f)/U^\circ$. Its image is a quotient of $\mathrm{im}\, Z(G)(\mathbb{A}_f)/\mathrm{im}\, U$, and $\mathrm{im}\, U \supset \prod_{p \notin \mathcal{S}} \Pi_Z(\mathbb{Q}_p)$: for $p \notin \mathcal{S}$, $U_p = \mathcal{Z}_p(\mathbb{Z}_p)$ by (M6), $\mathcal{Z}_p(\mathbb{Z}_p) \to \mathcal{P}_p(\mathbb{Z}_p)$ is onto as its cokernel lies in $H^1(\mathbb{Z}_p, \mathcal{Z}_p^\circ) = 0$,

and $\mathcal{P}_p(\mathbb{Z}_p) = \Pi_Z(\mathbb{Q}_p)$ by (M4). So the image has at most $\prod_{p\in\mathcal{S}} |\Pi_Z(\mathbb{Q}_p)| \le m_Z^{|\mathcal{S}|}$ elements. Next, $U^\circ \subset Z(G)^\circ(\mathbb{A}_f)^c$ with index ι_K, by (M6) at $p \notin \mathcal{S}$, so $|Z(G)^\circ(\mathbb{Q})\backslash Z(G)^\circ(\mathbb{A}_f)/U^\circ| \le \iota_K\, h(Z(G)^\circ)$, and Proposition C.5 bounds $h(Z(G)^\circ)$. □

Corollary C.8. *With K as in Section 9,*

$$h_K \le m_Z^{\log_2 n_{L_0}+n_{L_0}+2|\mathcal{S}|}\, h_{L_0}^{q}\, \iota_K\, \big(n_{L_1}^{r_Z(\log_2 n_{L_1}+n_{L_1})} h_{L_1}^{r_Z}\big)^2.$$

If K is the level made by Lemma 2.2 from a neat product level K_0, then $\iota_K \le |\mathrm{GL}_{n_\rho}(\mathbb{F}_3)|\,\iota_{K_0}$, with ι_{K_0} defined with the same set $\mathcal{S}$.

Proof. Proposition 9.3(d), Theorem C.6 and Lemma C.7, with $r_1(L) + r_2(L) \le [L : \mathbb{Q}]$. The bound for ι_K holds since $Z(G)^\circ(\mathbb{Q}_3) \cap K_3$ is the kernel of the reduction map from $Z(G)^\circ(\mathbb{Q}_3) \cap K_{0,3}$ to $\mathrm{GL}_{n_\rho}(\mathbb{F}_3)$ and $K_p = K_{0,p}$ for $p \ne 3$. □

All the quantities in Corollary C.8 are read off the character group of $Z(G)$, the representation ρ and the level, and Lemma C.1(a) bounds h_{L_0} and h_{L_1}. If $Z(G)$ is connected, $m_Z = 1$ and $q = 0$; if $Z(G)$ is split, $\mathrm{Sel}_{\mathcal{S}}(Z(G)) = 0$ by Proposition 9.3(d); if $Z(G)$ is finite, the torus factors are 1.

Declarations

Use of AI. This paper and its results were produced by an AI research system that the author designed, built and directs, running on Anthropic's Claude Fable 5.2 in Claude Code and OpenAI's GPT-6.2 Astra in Codex. The author chooses the problems and prioritizes the work. Separate AI agents checked each result, and a second program confirmed the computations the proofs use, with any exceptions listed in the public deposit. The system also searched the literature and checked the citations against reliable sources. The author read the paper, reran the computations from the deposit and takes full responsibility for all of the paper's contents.

Data availability. The programs and data used in this paper, with instructions for rerunning them, are deposited at Zenodo [60].

References

1. Valery Alexeev, *Log canonical singularities and complete moduli of stable pairs*, 1996, Preprint, arXiv:alg-geom/9608013.
2. Francesco Amoroso and Roberto Dvornicich, *Lower bounds for the height and size of the ideal class group in CM-fields*, Monatsh. Math. **138** (2003), no. 2, 85–94, doi:10.1007/s00605-002-0499-7.
3. Yves André, *G-functions and geometry*, Aspects Math., vol. E13, Friedr. Vieweg & Sohn, Braunschweig, 1989, doi:10.1007/978-3-663-14108-2.
4. ———, *Mumford–Tate groups of mixed Hodge structures and the theorem of the fixed part*, Compositio Math. **82** (1992), no. 1, 1–24.
5. Avner Ash, David Mumford, Michael Rapoport, and Yung-Sheng Tai, *Smooth compactifications of locally symmetric varieties*, second ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2010, doi:10.1017/CBO9780511674693.
6. Mohamed Barakat and Markus Lange-Hegermann, *An algorithmic approach to Chevalley's theorem on images of rational morphisms between affine varieties*, Math. Comp. **91** (2022), no. 333, 451–490, doi:10.1090/mcom/3632, arXiv:1911.10411v3.
7. Shubhrajit Bhattacharya, *Unconditional implications of the quasi-Riemann hypothesis*, 2026, Preprint, Hexagon 2610.00178v1, https://hexagonmath.org/2610.00178v1.
8. Yuri Bilu and Lars Kühne, *Linear equations in singular moduli*, Int. Math. Res. Not. IMRN (2020), no. 21, 7617–7643, doi:10.1093/imrn/rny216, arXiv:1712.04027.

9. Yuri Bilu, David Masser, and Umberto Zannier, *An effective "theorem of André" for CM-points on a plane curve*, Math. Proc. Cambridge Philos. Soc. **154** (2013), no. 1, 145–152.
10. Gal Binyamini, *Some effective estimates for André–Oort in $Y(1)^n$*, J. Reine Angew. Math. **767** (2020), 17–35, With an appendix by Emmanuel Kowalski, doi:10.1515/crelle-2019-0028, arXiv:1809.05302.
11. ———, *Point counting for foliations over number fields*, Forum Math. Pi **10** (2022), Paper No. e6, doi:10.1017/fmp.2021.20, arXiv:2009.00892.
12. ———, *Log-Noetherian functions*, 2024, Preprint, arXiv:2405.16963v1.
13. Gal Binyamini and Christopher Daw, *Effective computations for weakly optimal subvarieties*, J. Eur. Math. Soc. (JEMS) **27** (2025), no. 5, 2155–2186, doi:10.4171/JEMS/1408, arXiv:2105.12760.
14. Gal Binyamini, Gareth O. Jones, Harry Schmidt, and Margaret E. M. Thomas, *An effective Pila–Wilkie theorem for sets definable using Pfaffian functions, with some diophantine applications*, J. Eur. Math. Soc. (JEMS) (2026), Published online first, doi:10.4171/JEMS/1761, arXiv:2301.09883v1; theorem numbers refer to v1.
15. Gal Binyamini and David Masser, *Effective André–Oort for non-compact curves in Hilbert modular varieties*, C. R. Math. Acad. Sci. Paris **359** (2021), 313–321, arXiv:2101.06412.
16. Gal Binyamini, Harry Schmidt, and Andrei Yafaev, *Lower bounds for Galois orbits of special points on Shimura varieties: a point-counting approach*, Math. Ann. **385** (2023), no. 1-2, 961–973, doi:10.1007/s00208-021-02309-0, arXiv:2104.05842.
17. Armand Borel and Gopal Prasad, *Finiteness theorems for discrete subgroups of bounded covolume in semi-simple groups*, Inst. Hautes Études Sci. Publ. Math. **69** (1989), 119–171, doi:10.1007/BF02698843.
18. François Bruhat and Jacques Tits, *Groupes réductifs sur un corps local. I. Données radicielles valuées*, Inst. Hautes Études Sci. Publ. Math. **41** (1972), 5–251, doi:10.1007/BF02715544.
19. ———, *Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée*, Inst. Hautes Études Sci. Publ. Math. **60** (1984), 5–184, doi:10.1007/BF02700560.
20. Henri Cartan, *Plongements projectifs*, Séminaire Henri Cartan **10** (1957/58), no. 2, 17-01–17-19, Exp. no. 17, http://www.numdam.org/item/SHC_1957-1958___10_2_A8_0/.
21. Laurent Clozel and Emmanuel Ullmo, *Équidistribution de sous-variétés spéciales*, Ann. of Math. (2) **161** (2005), no. 3, 1571–1588, doi:10.4007/annals.2005.161.1571.
22. Brian Conrad, *Reductive group schemes*, Autour des schémas en groupes. Vol. I, Panor. Synthèses, vol. 42/43, Soc. Math. France, Paris, 2014, pp. 93–444.
23. Dean Cureton and Jesse Thorner, *A quasi-Riemann hypothesis for Dedekind zeta functions*, 2026, Preprint, Hexagon 2610.00200v2, https://hexagonmath.org/2610.00200v2.
24. Sourabhashis Das, Habiba Kadiri, and Nathan Ng, *An effective version of Chebotarev's density theorem*, 2025, Preprint, arXiv:2508.09480v1.
25. Christopher Daw, *The André–Oort conjecture via o-minimality*, O-Minimality and Diophantine Geometry, London Math. Soc. Lecture Note Ser., vol. 421, Cambridge Univ. Press, Cambridge, 2015, doi:10.1017/CBO9781316106839.006, arXiv:1412.3237v1; we cite the numbering of the arXiv version, pp. 129–158.
26. ———, *Degrees of strongly special subvarieties and the André–Oort conjecture*, J. Reine Angew. Math. **721** (2016), 81–108, doi:10.1515/crelle-2014-0062.
27. Christopher Daw, Ariyan Javanpeykar, and Lars Kühne, *Effective estimates for the degrees of maximal special subvarieties*, Selecta Math. (N.S.) **26** (2020), no. 1, Paper No. 2, doi:10.1007/s00029-019-0528-1, arXiv:1812.04341v2; page numbers refer to v2.
28. Christopher Daw and Martin Orr, *Heights of pre-special points of Shimura varieties*, Math. Ann. **365** (2016), no. 3-4, 1305–1357, doi:10.1007/s00208-015-1328-3, arXiv:1502.00822v5; we cite the numbering of the arXiv version.
29. Pierre Deligne, *Travaux de Shimura*, Séminaire Bourbaki, 23ème année (1970/71), Exp. No. 389, Lecture Notes in Math., vol. 244, Springer, Berlin, 1971, pp. 123–165.
30. ———, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, Automorphic Forms, Representations and L-Functions, Part 2, Proc. Sympos. Pure Math., vol. 33, Amer. Math. Soc., Providence, RI, 1979, pp. 247–289.
31. Gabriele Di Cerbo and Luca F. Di Cerbo, *Effective results for complex hyperbolic manifolds*, J. Lond. Math. Soc. (2) **91** (2015), no. 1, 89–104, doi:10.1112/jlms/jdu065, arXiv:1212.0501v5; corollary numbers refer to v5.
32. Bas Edixhoven, *Special points on the product of two modular curves*, Compositio Math. **114** (1998), no. 3, 307–320, doi:10.1023/A:1000539721162, arXiv:alg-geom/9610011.

33. ———, *On the André–Oort conjecture for Hilbert modular surfaces*, Moduli of Abelian Varieties (Texel Island, 1999), Progr. Math., vol. 195, Birkhäuser, Basel, 2001, arXiv:math/9911272, pp. 133–155.
34. Bas Edixhoven and Andrei Yafaev, *Subvarieties of Shimura varieties*, Ann. of Math. (2) **157** (2003), no. 2, 621–645, arXiv:math/0105241.
35. Gerd Faltings and Ching-Li Chai, *Degeneration of abelian varieties*, Ergeb. Math. Grenzgeb. (3), vol. 22, Springer, Berlin, 1990.
36. Ruyong Feng, *Hrushovski's algorithm for computing the Galois group of a linear differential equation*, Adv. in Appl. Math. **65** (2015), 1–37, arXiv:1312.5029v3.
37. Guy Fowler, *André's theorem and weakly bounded height*, 2026, Preprint, arXiv:2606.29369.
38. ———, *Some uniform effective results on André–Oort for sums of powers in* $\mathbb{C}^n$, Math. Proc. Cambridge Philos. Soc. **180** (2026), no. 3, 607–641, doi:10.1017/S0305004125101825, arXiv:2405.06456.
39. ———, *A uniform effective André–Oort result*, 2026, Preprint, arXiv:2609.18934.
40. Christopher Frei, Efthymios Sofos, and Martin Widmer, *Some applications of the quasi-Riemann hypothesis to class groups and small generators of number fields*, 2026, Preprint, arXiv:2610.18491v2.
41. Eberhard Freitag, *Die Irreduzibilität der Schottkyrelation (Bemerkung zu einem Satz von J. Igusa)*, Arch. Math. (Basel) **40** (1983), 255–259, doi:10.1007/BF01192778.
42. Osamu Fujino, *Effective base point free theorem for log canonical pairs II. Angehrn–Siu type theorems*, Michigan Math. J. **59** (2010), no. 2, 303–312, arXiv:0709.0336.
43. William Fulton, *Intersection theory*, second ed., Ergeb. Math. Grenzgeb. (3), vol. 2, Springer, Berlin, 1998.
44. William Fulton and Robert Lazarsfeld, *Positivity and excess intersection*, Enumerative Geometry and Classical Algebraic Geometry (Nice, 1981), Progr. Math., vol. 24, Birkhäuser, Boston, MA, 1982, pp. 97–105.
45. Philippe Gille, *Le problème de Kneser–Tits*, Séminaire Bourbaki, Vol. 2007/2008, Exp. No. 983, Astérisque, vol. 326, Soc. Math. France, Paris, 2009, pp. 39–81.
46. Gerd Gotzmann, *Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes*, Math. Z. **158** (1978), no. 1, 61–70, doi:10.1007/BF01214566.
47. Gert-Martin Greuel, Santiago Laplagne, and Frank Seelisch, *Normalization of rings*, J. Symbolic Comput. **45** (2010), 887–901, arXiv:0904.3561v3.
48. Alexander Grothendieck, *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes*, Inst. Hautes Études Sci. Publ. Math. **8** (1961), 5–222.
49. ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. IV*, Inst. Hautes Études Sci. Publ. Math. **32** (1967), 5–361.
50. ———, *Revêtements étales et groupe fondamental (SGA 1)*, Documents Mathématiques (Paris), vol. 3, Soc. Math. France, Paris, 2003, Séminaire de géométrie algébrique du Bois Marie 1960–61, augmented by two exposés of M. Raynaud; updated reprint of the 1971 original, arXiv:math/0206203.
51. Samuel Grushevsky, *The Schottky problem*, Current Developments in Algebraic Geometry, Math. Sci. Res. Inst. Publ., vol. 59, Cambridge Univ. Press, Cambridge, 2012, doi:10.1017/9781139032766.007, arXiv:1009.0369v2; page and equation numbers refer to v2, pp. 129–164.
52. Samuel Grushevsky and Martin Möller, *Explicit formulas for infinitely many Shimura curves in genus 4*, Asian J. Math. **22** (2018), no. 2, 381–390, doi:10.4310/AJM.2018.v22.n2.a12, arXiv:1510.05674.
53. Helmut A. Hamm and Lê Dũng Tráng, *Lefschetz theorems on quasi-projective varieties*, Bull. Soc. Math. France **113** (1985), 123–142.
54. Robin Hartshorne, *Algebraic geometry*, Grad. Texts in Math., vol. 52, Springer, New York, 1977.
55. Joos Heintz, *Definability and fast quantifier elimination in algebraically closed fields*, Theoret. Comput. Sci. **24** (1983), no. 3, 239–277, doi:10.1016/0304-3975(83)90002-6.
56. Ehud Hrushovski, *Computing the Galois group of a linear differential equation*, Differential Galois theory (Będlewo, 2001), Banach Center Publ., vol. 58, Polish Acad. Sci. Inst. Math., Warsaw, 2002, doi:10.4064/bc58-0-9, pp. 97–138.
57. Boyang Hu, *A seven-eighths zero-free half-plane for unitary Weil L-functions*, 2026, Preprint, Hexagon 2610.00216v2, https://hexagonmath.org/2610.00216v2; Lean formalization https://github.com/Rogerhu12/hecke-seven-eighths, commit c77efa92.
58. Klaus Hulek and Gregory K. Sankaran, *The geometry of Siegel modular varieties*, Higher Dimensional Birational Geometry (Kyoto, 1997), Adv. Stud. Pure Math., vol. 35, Math. Soc. Japan, Tokyo, 2002, arXiv:math/9810153, pp. 89–156.

59. Jun-ichi Igusa, *On the irreducibility of Schottky's divisor*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), no. 3, 531–545.
60. Manvir Jaswal, *Programs and data for "An effective André–Oort theorem for Shimura varieties"*, 2026, Zenodo, doi:10.5281/zenodo.24136430.
61. Nicholas M. Katz, *Nilpotent connections and the monodromy theorem: applications of a result of Turrittin*, Inst. Hautes Études Sci. Publ. Math. **39** (1970), 175–232.
62. Bruno Klingler, Ania Otwinowska, and David Urbanik, *On the fields of definition of Hodge loci*, Ann. Sci. Éc. Norm. Supér. (4) **56** (2023), no. 4, 1299–1312, doi:10.24033/asens.2555, arXiv:2010.03359.
63. Bruno Klingler and Andrei Yafaev, *The André–Oort conjecture*, Ann. of Math. (2) **180** (2014), no. 3, 867–925, doi:10.4007/annals.2014.180.3.2, arXiv:1209.0936v2; page numbers refer to v2.
64. János Kollár, *Singularities of pairs*, Algebraic Geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Part 1, Amer. Math. Soc., Providence, RI, 1997, pp. 221–287.
65. Lars Kühne, *An effective result of André–Oort type*, Ann. of Math. (2) **176** (2012), no. 1, 651–671.
66. Jeffrey C. Lagarias and Andrew M. Odlyzko, *Effective versions of the Chebotarev density theorem*, Algebraic Number Fields: L-Functions and Galois Properties (Durham, 1975), Academic Press, London, 1977, pp. 409–464.
67. Paolo Lella, *Computable Hilbert schemes*, 2012, PhD thesis, Università di Torino; arXiv:1202.4384v1.
68. Gunter Malle and Donna Testerman, *Linear algebraic groups and finite groups of Lie type*, Cambridge Stud. Adv. Math., vol. 133, Cambridge University Press, Cambridge, 2011.
69. James S. Milne, *Introduction to Shimura varieties*, Harmonic Analysis, the Trace Formula, and Shimura Varieties, Clay Math. Proc., vol. 4, Amer. Math. Soc., Providence, RI, 2005, pp. 265–378; revised version (2017) at https://www.jmilne.org/math/xnotes/svi.pdf, whose display, footnote and page numbers we cite.
70. ———, *Abelian varieties*, 2008, Course notes, v2.00, https://www.jmilne.org/math/.
71. ———, *Algebraic number theory*, 2020, Course notes, v3.08, https://www.jmilne.org/math/.
72. ———, *Class field theory*, 2020, Course notes, v4.03, https://www.jmilne.org/math/.
73. David Mumford, *On the equations defining abelian varieties. I*, Invent. Math. **1** (1966), 287–354.
74. ———, *On the equations defining abelian varieties. II*, Invent. Math. **3** (1967), 75–135.
75. ———, *Hirzebruch's proportionality theorem in the non-compact case*, Invent. Math. **42** (1977), 239–272.
76. Frans Oort, *Canonical liftings and dense sets of CM-points*, Arithmetic Geometry (Cortona, 1994), Sympos. Math., vol. XXXVII, Cambridge Univ. Press, Cambridge, 1997, pp. 228–234.
77. Georgios Papas, *Effective Brauer–Siegel on some curves in* $Y(1)^n$, Math. Ann. **394** (2026), Paper No. 84, arXiv:2310.04943.
78. Jonathan Pila, Ananth N. Shankar, and Jacob Tsimerman, *Canonical heights on Shimura varieties and the André–Oort conjecture*, 2021, With an appendix by Hélène Esnault and Michael Groechenig. Preprint, arXiv:2109.08788v4.
79. Jonathan Pila and Alex J. Wilkie, *The rational points of a definable set*, Duke Math. J. **133** (2006), no. 3, 591–616.
80. Jonathan Pila and Umberto Zannier, *Rational points in periodic analytic sets and the Manin–Mumford conjecture*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. (9) **19** (2008), no. 2, 149–162.
81. Richard Pink, *Arithmetical compactification of mixed Shimura varieties*, Bonner Math. Schriften, vol. 209, Universität Bonn, Mathematisches Institut, Bonn, 1990, Dissertation, Bonn 1989.
82. Vladimir P. Platonov, *The problem of strong approximation and the Kneser–Tits conjecture for algebraic groups*, Izv. Akad. Nauk SSSR Ser. Mat. **33** (1969), no. 6, 1211–1219, English translation in Math. USSR-Izv. 3 (1969), 1139–1147, doi:10.1070/im1969v003n06abeh000838.
83. Vladimir P. Platonov and Andrei S. Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139, Academic Press, Boston, MA, 1994.
84. Alexander Polishchuk, *Determinant bundles for abelian schemes*, Compositio Math. **121** (2000), no. 3, 221–245, doi:10.1023/A:1001896315311, arXiv:alg-geom/9703021v2.
85. Gopal Prasad, *Elementary proof of a theorem of Bruhat–Tits–Rousseau and of a theorem of Tits*, Bull. Soc. Math. France **110** (1982), 197–202, doi:10.24033/bsmf.1959.
86. ———, *Volumes of S-arithmetic quotients of semi-simple groups*, Inst. Hautes Études Sci. Publ. Math. **69** (1989), 91–114, doi:10.1007/BF02698841.
87. Gopal Prasad and Jiu-Kang Yu, *On finite group actions on reductive groups and buildings*, Invent. Math. **147** (2002), 545–560, doi:10.1007/s002220100182.

88. J. Barkley Rosser and Lowell Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64–94.
89. Ichiro Satake, *Compactification des espaces quotients de Siegel, I*, Séminaire Henri Cartan **10** (1957/58), no. 2, 12-01–12-13, Exp. no. 12, http://www.numdam.org/item/SHC_1957-1958___10_2_A2_0/.
90. ———, *Compactification des espaces quotients de Siegel, II*, Séminaire Henri Cartan **10** (1957/58), no. 2, 13-01–13-10, Exp. no. 13, http://www.numdam.org/item/SHC_1957-1958___10_2_A4_0/.
91. Ichiro Satake and Henri Cartan, *Démonstration du théorème fondamental*, Séminaire Henri Cartan **10** (1957/58), no. 2, 15-01–15-12, Exp. no. 15, http://www.numdam.org/item/SHC_1957-1958___10_2_A6_0/.
92. Thomas Scanlon, *Automatic uniformity*, Int. Math. Res. Not. (2004), no. 62, 3317–3326, doi:10.1155/S1073792804140816.
93. Carl Ludwig Siegel, *Über die Classenzahl quadratischer Zahlkörper*, Acta Arith. **1** (1935), no. 1, 83–86, doi:10.4064/aa-1-1-83-86.
94. Michael F. Singer, *Galois theory of linear differential equations*, Algebraic Theory of Differential Equations, London Math. Soc. Lecture Note Ser., vol. 357, Cambridge Univ. Press, Cambridge, 2009, doi:10.1017/CBO9780511721564.002, arXiv:0712.4124v2, pp. 1–82.
95. Harold M. Stark, *Some effective cases of the Brauer–Siegel theorem*, Invent. Math. **23** (1974), 135–152.
96. Donald E. Taylor, *The geometry of the classical groups*, Sigma Ser. Pure Math., vol. 9, Heldermann, Berlin, 1992.
97. The Stacks project authors, *The Stacks project*, 2026, https://stacks.math.columbia.edu.
98. Jacques Tits, *Reductive groups over local fields*, Automorphic Forms, Representations and L-Functions, Part 1, Proc. Sympos. Pure Math., vol. 33, Amer. Math. Soc., Providence, RI, 1979, pp. 29–69.
99. Jacob Tsimerman, *Brauer–Siegel for arithmetic tori and lower bounds for Galois orbits of special points*, J. Amer. Math. Soc. **25** (2012), no. 4, 1091–1117, doi:10.1090/S0894-0347-2012-00739-5, arXiv:1103.5619v3; we cite the numbering of the arXiv version.
100. ———, *The André–Oort conjecture for $\mathcal{A}_g$*, Ann. of Math. (2) **187** (2018), no. 2, 379–390, arXiv:1506.01466v5; theorem and corollary numbers refer to v5.
101. Emmanuel Ullmo, *Équidistribution de sous-variétés spéciales. II*, J. Reine Angew. Math. **606** (2007), 193–216.
102. Emmanuel Ullmo and Andrei Yafaev, *Galois orbits and equidistribution of special subvarieties: towards the André–Oort conjecture*, Ann. of Math. (2) **180** (2014), no. 3, 823–865, doi:10.4007/annals.2014.180.3.1, arXiv:1209.0934v2; page numbers refer to v2.
103. ———, *Nombre de classes des tores de multiplication complexe et bornes inférieures pour les orbites galoisiennes de points spéciaux*, Bull. Soc. Math. France **143** (2015), no. 1, 197–228, arXiv:1209.0942v2; page numbers refer to v2.
104. David Urbanik, *Sets of special subvarieties of bounded degree*, Compositio Math. **159** (2023), no. 3, 616–657, doi:10.1112/S0010437X23007029, arXiv:2109.07663v1; we cite the numbering of v1.
105. Gerard van der Geer, *Cycles on the moduli space of abelian varieties*, Moduli of Curves and Abelian Varieties, Aspects Math., vol. E33, Vieweg, Braunschweig, 1999, doi:10.1007/978-3-322-90172-9_4, arXiv:alg-geom/9605011v2; page numbers refer to v2, pp. 65–89.
106. Andrei Yafaev, *A conjecture of Yves André's*, Duke Math. J. **132** (2006), no. 3, 393–407, arXiv:math/0302125v1; we cite the numbering of the arXiv version.

Toronto, Canada
Email address: manvirjaswal@icloud.com
URL: https://manvirjaswal.github.io