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Irrational even beta values from powered hypergeometric kernels

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Abstract

For the Dirichlet beta function, we prove that at least one of $\beta(6), \beta(8), \ldots, \beta(74)$ is irrational, and that at least one of $\beta(8), \beta(10), \ldots, \beta(110)$ is irrational. These shorten the previously stated intervals with the same starting indices, whose upper endpoints are 92 and 128, respectively [5, Section 8.2]. The new bounds come from a different construction, based on powers of Krisman's hypergeometric kernel [1].

1 Results and proof strategy

For $s > 1$, the Dirichlet beta function is

$$\beta(s) = \sum_{t=0}^{\infty} \frac{(-1)^t}{(2t+1)^s}.$$

Krisman [1, Theorem 1.1] proves that at least one of $\beta(4), \beta(6), \ldots, \beta(36)$ is irrational. We seek intervals that also omit $\beta(4)$, and in the second case below omit $\beta(6)$.

Theorem 1. *Each of the following sets contains an irrational number:*

$$\{\beta(6), \beta(8), \ldots, \beta(74)\}, \qquad \{\beta(8), \beta(10), \ldots, \beta(110)\}.$$

Table 1 compares these bounds with the earlier shifted-interval results [5]. Each entry asserts the existence of one irrational value, with all indices even. The intervals overlap, so the two assertions of Theorem 1 need not identify distinct irrational numbers.

	[5, Section 8.2]		Theorem 1	
First index	Last index	Values	Last index	Values
6	92	44	74	35
8	128	61	110	52

Table 1: The earlier endpoints use parameters chosen for these particular starting indices. Each new interval removes nine candidate values.

Sorokin [4, Theorem S] proved that, for every sufficiently large odd m and every even $h > 100m \log m$, at least one of $\beta(m+1), \beta(m+3), \ldots, \beta(m+h-1)$ is irrational. The later result [5, Theorem 1.2] treats every even starting index and any prescribed number of irrational values. For one irrational value, [5, Proposition B.1] gives an upper endpoint below $18.58m$ for positive odd m, with first index $m+1$. We improve the two instances in the table. Corollary 7 gives a criterion for further instances, with a finite nonvanishing condition that must still be checked.

How the argument is organized

The irrationality contradiction has three requirements. For each target interval, we construct a sequence of rational linear forms in 1 and the indicated beta values. After clearing the denominators of their coefficients, these forms must tend to zero and must be nonzero infinitely often. Under joint rationality, a single fixed denominator for the beta values would then give a sequence of integers tending to zero, with infinitely many nonzero terms.

Krisman's construction already balances the cost of clearing denominators against the decay of a rational kernel. We retain that balance while changing the beta indices. Squaring the kernel gives zeros of order six, so four differentiations still leave zeros at the integers where they are needed. Reflection then makes $\beta(6)$ the first possible beta term. Cubing and differentiating six times similarly starts at $\beta(8)$. Section 2 explains this operation before writing its general form. The main arithmetic point in Section 3 is that taking a power multiplies the initial valuation of a local series, while leaving its loss per Taylor coefficient unchanged. This permits the original denominator and decay estimates to scale together.

Small forms could nevertheless be zero. To resolve that obstacle, Section 4 follows the prime-dilation strategy of [5, Section 6], which uses the auxiliary-prime principle of Lai and Sprang [2, Lemma 2.3]. At indices equal to a fixed base index times a large prime, a rescaling makes all beta coefficients vanish modulo the prime, while the rational constant reduces to a fixed seed constant. Our adaptations account for Krisman's nonuniform pole orders and for the loss of the two outer zeros under dilation. The nonzero seed constants are checked in Section 5.

Thus the kernel and its estimates come from [1, Sections 2–6]; the nonvanishing strategy follows [5, Section 6]. The transfer under powers and the dilation calculation for this kernel are the additional arguments given here. The proof uses no irrationality result for Catalan's constant.

Prerequisites and notation

The intended reader is familiar with partial fractions, formal Taylor series, arithmetic modulo primes, elementary prime valuations, and asymptotic estimates. Write v_p for the valuation on $\mathbb{Q}$ normalized by $v_p(p) = 1$, with $v_p(0) = +\infty$. The ring $\mathbb{Z}_{(p)}$ consists of rationals whose denominators are prime to p; an element of this ring is a unit when its numerator is also prime to p. Congruences between rationals are taken in this ring. These definitions suffice for all the local arguments below. The prime-number-theorem estimate we use is cited explicitly.

2 Raising the first beta index

2.1 The kernel we retain

We use Krisman's kernel because its denominator cost is already smaller than its real decay rate. Its integer zeros will remove initial summands and control rational correction terms; its half-integer poles produce beta values. The reflection symmetry will remove odd beta indices.

Fix $r = 7/6$ and $N = 480$, and let n tend to infinity through positive multiples of N. The choice $480 = \operatorname{lcm}(160, 6)$ makes all block endpoints and rn integral, and makes n even. The 28 blocks, with total multiplicity $S = 35$, have parameters

$$(\delta, m) = (0, 8), \qquad (\kappa/160, 1) \quad (\kappa \in \{1, \dots, 8\} \cup \{10, 12, \dots, 46\}).$$

The pole with address j is at $t = -j - 1/2$. A block b adds multiplicity m_b to each pole address in

$I_b(n) = [\delta_b n, (1-\delta_b)n] \cap \mathbb{Z}$. Thus the total pole order at address j is

$$B_j(n) = \sum_b m_b \mathbf{1}_{j \in I_b(n)}.$$

For example, at $n = 480$ the outer block contributes eight copies of every address from 0 to 480; the blocks with $\kappa = 1$ and 46 contribute one copy on $[3, 477]$ and $[138, 342]$, respectively. The orders increase from 8 near the ends to 35 in the middle, and $B_{n-j} = B_j$. We often omit the argument n from $B_j(n)$.

Set $\Lambda_n = [-(1+r)n - 1, rn] \cap \mathbb{Z}$. The kernel is

$$R_n(t) = \frac{(2t+n+1)\prod_{\ell \in \Lambda_n}(t-\ell)^3}{\prod_{j=0}^{n}(t+j+1/2)^{B_j}}. \tag{1}$$

The linear factor $2t + n + 1$ corrects the sign of reflection about $-(n+1)/2$, giving

$$R_n(-n-1-t) = R_n(t). \tag{2}$$

At the central half-integer pole this factor cancels one order. We keep the nominal order $B_{n/2} = 35$ in the formulas, allowing a zero leading coefficient.

The excess of pole degree over zero degree grows at rate

$$G = \sum_b m_b(1-2\delta_b) - 3(1+2r) = \frac{179}{10}.$$

Indeed,

$$3\#\Lambda_n = 10n + 6, \qquad \sum_{j=0}^{n} B_j = \frac{279}{10}n + 35,$$

so $\deg R_n = -Gn - 28$. In particular the kernel is proper and decays rapidly at infinity. These are the parameters and identities of [1, Section 2].

2.2 The square case, then the general form

Rivoal [3, Section 2, Lemma 1] used differentiation to remove $\zeta(3)$ from linear forms in odd zeta values. Krisman [1, Section 1.2] adapted this device to even beta values. We use fourth derivatives instead of second derivatives to remove $\beta(4)$. For example, a partial-fraction term transforms as

$$\frac{c}{(t+j+1/2)^2} \quad \xrightarrow{d^4/dt^4} \quad \frac{120c}{(t+j+1/2)^6}.$$

Alternating summation of the latter term produces $\beta(6)$ and a rational correction. A simple-pole term would instead produce $\beta(5)$, but reflection cancels that odd-index contribution.

Fourth differentiation would destroy the triple zeros of R_n. Squaring first gives zeros of order six, so the differentiated function still vanishes at all the relevant integers. The cost is that the largest nominal pole order doubles from 35 to 70, allowing indices as large as $70 + 4 = 74$. Cubing gives zeros of order nine; six derivatives preserve them, and the largest possible index is $105 + 6 = 111$. Reflection cancels the $\beta(111)$ term, leaving 110 as the largest possible index. This explains the endpoints in the theorem.

For the shared proof, fix an integer $q \geq 1$ independently of n and put

$$L_{q,n} = \sum_{t=0}^{\infty} (-1)^t \frac{d^{2q}}{dt^{2q}}\big(R_n(t)^q\big). \tag{3}$$

The zeros now have order $3q > 2q$, and the degree bound gives absolute convergence. Write the partial fractions as

$$R_n(t)^q = \sum_{j=0}^{n} \sum_{i=1}^{qB_j} \frac{a_{q,n;j,i}}{(t+j+1/2)^i}.$$

The rational correction at address j is the omitted initial part of the beta series. More precisely, for $s > 1$ define

$$\sigma_j(s) = \sum_{m=0}^{j-1} \frac{(-1)^m}{(2m+1)^s}, \qquad \sum_{t=0}^{\infty} \frac{(-1)^t}{(t+j+1/2)^s} = (-1)^j 2^s \big(\beta(s) - \sigma_j(s)\big).$$

The identity follows by replacing $t+j$ with a new summation index. With $(i)_d = i(i+1)\cdots(i+d-1)$, it gives

$$L_{q,n} = \sum_{j,i} (-1)^j 2^{i+2q} (i)_{2q} a_{q,n;j,i} \big(\beta(i+2q) - \sigma_j(i+2q)\big). \tag{4}$$

Reflection (2) gives $a_{q,n;n-j,i} = (-1)^i a_{q,n;j,i}$. Since n is even, pairing addresses j and $n-j$ cancels the beta coefficient whenever i is odd. Thus

$$L_{q,n} \in \mathbb{Q} + \sum_{j=q+1}^{\lfloor 37q/2 \rfloor} \mathbb{Q}\beta(2j). \tag{5}$$

We have obtained exactly the required range of possible indices. It remains to show that clearing the rational coefficients is inexpensive enough to preserve decay, and then to prove nonvanishing.

3 Clearing denominators while preserving decay

3.1 Why the arithmetic estimate survives a power

The relevant arithmetic is local at each pole. Remove the nominal pole at $t = -j - 1/2$ and use a local variable z; the remaining Taylor series is

$$f_j(z) = (n - 2j + 2z) \prod_{\ell \in \Lambda_n} (-j - 1/2 - \ell + z)^3 \prod_{h \neq j} (h - j + z)^{-B_h}.$$

For the qth power, the coefficient of pole order i is therefore $a_{q,n;j,i} = [z^{qB_j - i}] f_j(z)^q$.

Suppose a series $f(z) = \sum f_h z^h$ satisfies $v_p(f_h) \geq \nu - \lambda h$, where $\nu \in \mathbb{Z}$ and $\lambda \in \mathbb{Z}_{\geq 0}$. Call this a (ν, λ) bound: ν is the initial valuation and λ bounds the loss for each extra Taylor degree. The square case displays the useful feature:

$$[z^h] f(z)^2 = \sum_{u+v=h} f_u f_v, \qquad v_p(f_u f_v) \geq 2\nu - \lambda(u+v) = 2\nu - \lambda h.$$

The same convolution argument shows that f^q has a $(q\nu, \lambda)$ bound. The initial valuation is multiplied by q, but the loss per degree is unchanged. Consequently

$$v_p(a_{q,n;j,i}) \geq q\nu - \lambda(qB_j - i). \tag{6}$$

If a harmonic correction of exponent $i + 2q$ costs at most $(i+2q)\lambda$ powers of p, the combined bound is

$$q\nu - \lambda(qB_j - i) - \lambda(i + 2q) = q\big(\nu - \lambda(B_j + 2)\big). \tag{7}$$

This is exactly q times the bound for the original second derivative. It explains both the choice of derivative order $2q$ and the use of the qth power of the original denominator multiplier.

3.2 The multiplier and its prime bounds

We now specify that multiplier so that every prime range can be checked. The notation below records the counts of numerator and denominator factors divisible by a prime, or their full valuations when prime powers must be counted. Put

$$M_n = 19n/3 + 3, \quad Z = 3\#\Lambda_n = 10n + 6, \quad \lambda_p = \lfloor \log_p M_n \rfloor \ (p \text{ odd}), \quad \lambda_2 = \lfloor \log_2 n \rfloor.$$

All the nonzero integer differences in these local factors have absolute value below M_n. For an odd prime $p > n$, none of the pole differences is divisible by p. For odd $p \le n$, the condition $p^2 > M_n$ means that each nonzero difference contains at most one factor of p, so divisibility counts suffice. Smaller primes can occur to higher powers and require full valuations. The prime 2 needs separate treatment because the pole locations are half-integers.

For $w_{j,\ell} = 2j + 1 + 2\ell$, define

$$\begin{aligned}
\epsilon_p(j) &= 3\#\{\ell \in \Lambda_n : p \mid w_{j,\ell}\} + \mathbf{1}_{p|n-2j} - \sum_{h \neq j} B_h \mathbf{1}_{p|h-j}, \\
\nu_p(j) &= 3 \sum_{\ell \in \Lambda_n} v_p(w_{j,\ell}) - \sum_{h \neq j} B_h v_p(h-j) \quad (p \text{ odd}), \\
\nu_2(j) &= 1 - Z - \sum_{h \neq j} B_h v_2(h-j).
\end{aligned}$$

The factor 3 counts the triple zeros. At 2, each half-integer numerator factor costs one power of 2, explaining $-Z$, while the linear factor contributes the initial $+1$.

Krisman's full-series estimate [1, Lemma 4.3], with the choices in the proof of his Corollary 4.4, gives the following bounds for f_j:

Prime range	(ν, λ) bound
$p > n$, odd	$(0, 0)$
$p \le n$, odd, $p^2 > M_n$	$(\epsilon_p(j), 1)$
$p \le n$, odd, $p^2 \le M_n$	$(\nu_p(j), \lambda_p)$
$p = 2$	$(\nu_2(j), \lambda_2)$

These are bounds on the whole local series, obtained by multiplying linear numerator factors and geometric denominator series. We can therefore apply (6) to them.

The original denominator multiplier is

$$E_p(n) = \begin{cases} \max_j(-\nu_2(j) + 35\lambda_2), & p = 2, \\ 0, & p > n \text{ odd}, \\ \max_j(B_j + 2 - \epsilon_p(j)), & p \le n \text{ odd}, \ p^2 > M_n, \\ \max_j(-\nu_p(j) + 37\lambda_p), & p \le n \text{ odd}, \ p^2 \le M_n, \end{cases} \qquad D_n = \prod_{p \le n} p^{E_p(n)}. \tag{8}$$

Here 35 is the maximum nominal pole order and $37 = 35 + 2$ includes the two derivatives in the original form. Some exponents E_p may be negative, so D_n is a positive rational multiplier rather than necessarily an integer. What matters is the integrality of the coefficients after multiplication.

Proposition 2 (Arithmetic transfer)**.** *Every rational coefficient of* (4)*, after collecting equal beta values and the constant term, becomes an integer on multiplication by* D_n^q*.*

Proof. First take an odd prime $p \le n$. Each odd denominator in $\sigma_j(i+2q)$ has valuation at most the λ in the relevant row of the table. Equations (6)–(7) therefore bound the valuation of a constant summand below by $-qE_p(n)$. The same lower bound holds without the harmonic factor, hence for every beta coefficient. The factors $(i)_{2q}$ and 2^{i+2q} introduce no odd-prime denominators.

At 2 the harmonic denominators are odd. Keeping the factor 2^{i+2q}, the valuation of a summand is at least

$$q\nu_2(j) - \lambda_2(qB_j - i) + i + 2q \ge q\nu_2(j) - 35q\lambda_2 \ge -qE_2(n).$$

It remains to treat odd primes $p > n$. All local coefficients are then p-integral. A harmonic denominator can still be divisible by p, so we use the integer zeros to replace a possibly bad partial sum by its complement. The use of zeros throughout the pole region to control the rational constant already appears in Zudilin [6, proof of Lemma 1]. We follow Krisman's derivative version [1, Lemma 4.6], with zero order $3q$ and derivative order $2q$. For $1 \le \ell \le n$, put

$$T(\ell, j) = \sum_i (i)_{2q} a_{q,n;j,i} (j - \ell + 1/2)^{-i-2q}.$$

Reindexing the harmonic sums expresses the rational constant as $-\sum_{\ell=1}^{n}(-1)^\ell \sum_{j=\ell}^{n} T(\ell, j)$. Because $-\ell \in \Lambda_n$ and $3q > 2q$, evaluation of the full partial-fraction sum gives

$$\sum_{j=0}^{n} T(\ell, j) = (R_n^q)^{(2q)}(-\ell) = 0, \qquad \sum_{j=\ell}^{n} T(\ell, j) = -\sum_{j=0}^{\ell-1} T(\ell, j).$$

The prime p cannot divide denominators $2(j-\ell)+1$ on both sides: two such addresses would differ by a nonzero multiple of p of absolute value at most $n < p$. Thus one side is a sum of p-integral terms, and the identity makes both sides p-integral. This proves the result at every prime. □

3.3 The remaining exponential margin

We next compare the size of D_n^q with the decay of the differentiated kernel. Estimating logarithms of the zero and pole products by integrals leads to the profile Φ below: it records the part of $\log R_n$ proportional to n after the factorial term has been removed. The constants A and V are the certified odd-prime cost and an upper bound on this profile, respectively. Define $\mathcal{F}(y) = y \log y$ for $y > 0$ and $\mathcal{F}(0) = 0$, and set

$$\begin{aligned} \Phi(x) &= 3\{\mathcal{F}(x+1+r) - \mathcal{F}(x-r)\} \\ &\qquad - \sum_b m_b \{\mathcal{F}(x+1-\delta_b) - \mathcal{F}(x+\delta_b)\}, \qquad x \ge r, \\ A &= \frac{1250702970786628804842297249335433 1}{7903009210797867674956009046016 00}, \qquad V = -\frac{744}{25}. \end{aligned}$$

The bounds we use from [1, Corollary 5.9 and Propositions 6.4–6.5] are

$$\log D_n \le G\log(n!) + n(A + 20\log 2 + o(1)), \tag{9}$$

$$0 < R_n(t) \le \exp\{-G\log(n!) + n\Phi(t/n) + O(\log n)\} \qquad (t \ge rn+1), \tag{10}$$

$$\Phi(x) \le V \qquad (x \ge r), \tag{11}$$

$$\Phi(x) \le V - 4\log(x/3) \qquad (x \ge 3). \tag{12}$$

The error in (10) is uniform in t. The factorial terms in (9) and (10) cancel. What remains is the odd-prime cost A, the dyadic cost $20\log 2$, and the real decay exponent V. Their sum satisfies

$$Q := V + A + 20\log 2 < -0.0714 < 0. \tag{13}$$

This strict margin is the reason for retaining this particular kernel. Taking a fixed power multiplies both the cost and the decay by the same number, so it preserves the margin.

Proposition 3 (Analytic transfer)**.** *For fixed q, $D_n^q L_{q,n} \longrightarrow 0$ as $n \longrightarrow \infty$ through multiples of* 480*. More precisely,*

$$|D_n^q L_{q,n}| \le \exp\{qnQ + o(n)\}.$$

Proof. All summands with $0 \le t \le rn$ vanish because the zeros have order $3q$. On the remaining real axis, distances to the zeros and half-integer poles in the logarithmic derivatives of R_n are at least 1. For each fixed $h \ge 1$,

$$|(\log R_n)^{(h)}(t)| \le (h-1)!(1 + Z + \sum B_j) = O_h(n).$$

The formula for derivatives of an exponential therefore gives

$$|(R_n^q)^{(2q)}(t)| \le C_q n^{2q} R_n(t)^q.$$

This polynomial factor does not affect the exponential rate. Equations (10)–(11) bound the $O(n)$ summands with $rn < t \le 3n$. Above $3n$, (12) supplies the summable factor $(3n/t)^{4qn}$, since

$$\sum_{t>3n} (3n/t)^{4qn} \le \int_{3n}^{\infty} (3n/x)^{4qn}\, dx = \frac{3n}{4qn-1}.$$

Combining these estimates yields $|L_{q,n}| \le \exp\{-qG\log(n!) + qnV + O_q(\log n)\}$. Now (9) and (13) give the assertion. □

Under joint rationality, the cleared forms would now give integers tending to zero, so they would eventually have to vanish. The rest of the proof excludes that possibility.

4 A fixed seed detects nonzero forms

4.1 Why we use prime indices and integer coordinates

The auxiliary-prime criterion of Lai and Sprang [2, Lemma 2.3] gives a nonvanishing test under the rationality hypothesis. Outside the finitely many primes dividing their denominators, rational beta values are p-integral. If a rescaled form has all its beta coefficients divisible by p, but has a rational constant that is a unit modulo p, then that form cannot be zero. Here we only test nonvanishing: this local rescaling need not have the small real size of $D_n^q L_{q,n}$.

We apply the test at $n = Np$, with the base index $N = 480$ fixed and the odd prime p increasing. Every block endpoint is then a multiple of p, so the poles whose addresses are multiples of p reproduce the base pole pattern. Following [5, Section 6], we separate those factors from the factors that are units modulo p.

To do this with integer coordinates, put $x = 2t + n + 1$. The half-integer poles become the even integers $n - 2a$, and the integer zeros become odd integers. Define

$$F_n(x) = \frac{x \prod_{\substack{z \text{ odd} \\ |z| \le (1+2r)n+1}} (x-z)^3}{\prod_{a=0}^{n} (x-n+2a)^{B_a(n)}}. \tag{14}$$

Counting the factors of 2 gives the exact relation

$$R_n(t) = 2^{Gn+29} F_n(2t+n+1). \tag{15}$$

Let $c_{q,n;a,i}$ be the coefficient of $(x-n+2a)^{-i}$ in the partial fractions of F_n^q. Define the corresponding form in these coordinates by

$$\ell_{q,n} = \sum_{a=0}^{n} \sum_{i=1}^{qB_a(n)} (-1)^a (i)_{2q} c_{q,n;a,i} \big(\beta(i+2q) - \sigma_a(i+2q)\big).$$

It is the alternating sum of the $2q$th derivative with respect to x, evaluated at $x = 2t+n+1$. Thus

$$L_{q,n} = 2^{q(Gn+29)+2q} \ell_{q,n}. \tag{16}$$

The power of 2 is a unit at every odd prime, and the two forms vanish at exactly the same indices. Denote the rational constant of $\ell_{q,n}$ by

$$\rho_{q,n} = -\sum_{a=0}^{n} \sum_{i=1}^{qB_a(n)} (-1)^a (i)_{2q} c_{q,n;a,i} \sigma_a(i+2q). \tag{17}$$

4.2 Which seed survives dilation

At the base index put

$$L = (1+2r)N = 1600, \qquad W = \sum_b m_b(1-2\delta_b)N = 13392.$$

Here L measures the length of the zero pattern before its two endpoints are included, while W sums the block widths with multiplicity. The full pole count at N is $W+S$, with $S = 35$. We may take $p > N+L+1$ throughout this section; in particular $p > 2N$.

After substituting $x = pu$, the odd numerator zeros divisible by p have the form pv with v odd and $|pv| \le Lp+1$. Since L is even, this means $|v| \le L-1$. The endpoint loss is visible in a small example: with $L = 4$ and $p = 5$, the odd zeros from -21 to 21 that are divisible by 5 are $-15, -5, 5, 15$. Dividing by 5 leaves $-3, -1, 1, 3$; it does not leave the extra endpoints ± 5 of the untrimmed base pattern.

For our parameters, the surviving seed is consequently

$$\widetilde{F}_N(u) = \frac{u \prod_{\substack{z \text{ odd} \\ |z| \le 1599}} (u-z)^3}{\prod_{a=0}^{N} (u-N+2a)^{B_a(N)}}. \tag{18}$$

Thus the endpoints are ± 1599 rather than ± 1601. The selected pole addresses are pa, and $B_{pa}(Np) = B_a(N)$. The powers of p in the selected factors are

factor	contribution to the exponent of p
$x = pu$	1
L selected zeros, each triple	$3L$
$W+S$ selected pole factors	$-(W+S)$

Their sum is $-A_0$, where

$$A_0 = W - 3L + S - 1 = 8626. \tag{19}$$

This factor count determines the normalization in the next lemma.

Let $\widetilde{c}_{q;a,i}$ denote the principal-part coefficients of $\widetilde{F}_N^q$ at $u = N-2a$. Its associated rational constant is

$$\widetilde{\rho}_q = -\sum_{a=0}^{N} \sum_{i=1}^{qB_a(N)} (-1)^a (i)_{2q} \widetilde{c}_{q;a,i} \sigma_a(i+2q). \tag{20}$$

For fixed q, the powered seed $\widetilde{F}_N^q$ and its rational constant are fixed independently of the varying prime p.

Lemma 4 (Selected factors)**.** *For every sufficiently large odd prime p,*

$$p^{A_0} F_{Np}(pu) \equiv -\widetilde{F}_N(u) \pmod{p}.$$

Reduction commutes with Taylor extraction after the selected poles are removed. In particular, for $0 \le a \le N$ and $1 \le i \le qB_a(N)$,

$$p^{qA_0-i} c_{q,Np;pa,i} \equiv (-1)^q \widetilde{c}_{q;a,i} \pmod{p}. \tag{21}$$

Proof. The selected factors counted above give $p^{-A_0}\widetilde{F}_N(u)$. Every unselected factor is a unit whose reduction is independent of u. We first compute their combined unit, and then justify extraction of individual coefficients.

The odd integers from $-Lp+1$ to $Lp-1$ contain each residue class L times. By Wilson's theorem their nonzero residue product is $(-1)^L = 1$. The two extra endpoints $\pm(Lp+1)$ contribute -1, also after cubing. For a block with seed endpoints a_b, b_b, each nonzero pole-address residue occurs $b_b - a_b$ times. The pole differences contain a factor of 2, whose product over a nonzero residue cycle is $2^{p-1} = 1$ modulo p. The total denominator unit is therefore $(-1)^W = 1$. This proves the rational-function congruence with unit -1.

For coefficient extraction put $u = N - 2a + z$. The selected principal part of $F_{Np}(pu)^q$ is

$$\sum_{i=1}^{qB_a(N)} p^{-i} c_{q,Np;pa,i} z^{-i}.$$

After multiplying by $p^{qA_0} z^{qB_a(N)}$, its coefficient of z^{qB_a-i} is $p^{qA_0-i} c_{q,Np;pa,i}$. The other selected pole differences are $2(h-a)$ with $0 < |h-a| \le N$, hence units. Every unselected denominator factor also has a unit constant term and an integral geometric-series inverse. The pole-removed series therefore lies in $\mathbb{Z}_{(p)}[[z]]$, and reduction commutes with coefficient extraction. Terms from other poles are analytic in z and, after pole removal, start in degree qB_a. They cannot contribute to the coefficient in question. Taking qth powers in the congruence proves (21). □

4.3 Why the other poles disappear

The selected poles reproduce the seed. We must also show that the many other poles contribute nothing to the final reduction. Their residue classes have fewer denominator factors, which gives the required extra valuation.

Lemma 5 (Nonselected coefficients)**.** *For every sufficiently large odd prime p, if $0 \le a \le Np$, $p \nmid a$, and $1 \le i \le qB_a(Np)$, then*

$$v_p(c_{q,Np;a,i}) \ge -qGN + i. \tag{22}$$

Proof. Our choice $p > N + L + 1$ makes all nonzero differences between poles, or between a pole and a numerator zero, smaller than p^2 in absolute value. Any such difference divisible by p has valuation exactly 1. At a nonselected pole the factor x is a unit, and every residue class contains at least L numerator zeros.

Each block $[a_b p, b_b p]$ contains exactly $b_b - a_b$ addresses in any nonzero residue class. Thus the denominator count at the pole's residue class is W, including its own pole, and is $W - B_a(Np)$ after

that pole is removed. The local analytic part of F^q_{Np} has the bound $(3qL - q(W - B_a), 1)$ in the terminology of Section 3. Extracting degree $qB_a - i$ gives

$$3qL - q(W - B_a) - (qB_a - i) = 3qL - qW + i = -qGN + i,$$

since $W - 3L = GN$. This proves the bound. □

4.4 The rational constant survives

Differentiation changes a pole exponent i to $s = i + 2q$. To combine the coefficient congruence with a harmonic denominator of exponent s, rewrite $qA_0 - i$ as $q(A_0 + 2) - s$. This determines the final scaling:

$$\tau_q = q(A_0 + 2) = 8628q. \tag{23}$$

The scaled coefficient at a selected pole then contains p^s. This kills its beta coefficient modulo p, but it can cancel a denominator p^s in the rational correction. That is how the constant can survive after every beta coefficient has disappeared.

Proposition 6 (The surviving constant)**.** *For every sufficiently large odd prime p, all beta coefficients of $p^{\tau_q}\ell_{q,Np}$ belong to $p\mathbb{Z}_{(p)}$, and*

$$p^{\tau_q}\rho_{q,Np} \equiv (-1)^q\chi_4(p)\widetilde{\rho}_q \pmod p, \qquad \chi_4(p) = (-1)^{(p-1)/2}. \tag{24}$$

Proof. Put $s = i + 2q$. At a selected pole, (21) gives

$$p^{\tau_q - s}(i)_{2q}c_{q,Np;pa,i} \equiv (-1)^q(i)_{2q}\widetilde{c}_{q;a,i} \pmod p.$$

After multiplication by p^{τ_q} the corresponding beta coefficient has valuation at least $s > 0$. At a nonselected pole, (22) gives valuation at least $q(S + 1) + i > 0$.

Now consider the rational constant. Every odd denominator in $\sigma_a(s)$, for $a \le Np$, is less than $2Np < p^2$, so it costs at most s powers of p. The nonselected contributions therefore have scaled valuation at least

$$\tau_q - qGN + i - s = q(S - 1) > 0.$$

They vanish in the reduction. The spare $q(S - 1)$ comes from the extra endpoint factors in the selected pole blocks.

For a selected address pa, recall the finite sum

$$\sigma_{pa}(s) = \sum_{m=0}^{pa-1} \frac{(-1)^m}{(2m + 1)^s}.$$

Multiplication by p^s kills all terms whose denominators are prime to p. The remaining terms have $2m + 1 = p(2v + 1)$, so $m = pv + (p - 1)/2$ with $0 \le v < a$. Hence

$$p^s\sigma_{pa}(s) \equiv \chi_4(p)\sigma_a(s) \pmod p, \qquad (-1)^m = \chi_4(p)(-1)^v.$$

The surviving sum is the shorter seed harmonic sum. Combining this identity with $(-1)^{pa} = (-1)^a$ and the coefficient congruence in (17) gives exactly (20), multiplied by $(-1)^q\chi_4(p)$. This proves (24). □

One fixed rational number, $\widetilde{\rho}_q$, now controls nonvanishing at all sufficiently large prime indices under the rationality hypothesis. It remains to show that this number is nonzero for $q = 2$ and 3.

5 Checking the fixed rational constant

We do not need the very large numerator and denominator of $\widetilde{\rho}_q$. It suffices to find a prime at which the rational number is integral and has nonzero residue. The auxiliary primes used for this finite computation are $P = 10007$ and 10009. These fixed primes have a different role from the varying prime p in Section 4.

To calculate the constant in (20), we need only finitely many Taylor coefficients at each of the 481 seed poles. At the pole $u_a = N - 2a$, remove its nominal pole and truncate:

$$J_a(z) = (u_a + z)^q \prod_{\substack{v \text{ odd} \\ |v| \le 1599}} (u_a - v + z)^{3q} \prod_{b \ne a} (u_a - u_b + z)^{-qB_b(N)} \pmod{z^{qB_a(N)}}. \tag{25}$$

The product over b runs through the other addresses in $\{0, \dots, N\}$. Then $\widetilde{c}_{q;a,i} = [z^{qB_a - i}]J_a$. There is no need to expand the full numerator and denominator of the seed as global polynomials.

The computations are legitimate reductions of rational numbers. Pole differences have absolute value at most 960, and the harmonic denominators at most 959. The second algorithm below also inverts nonzero numerator differences, of absolute value at most 2079, and Taylor indices less than 105. All these numbers are smaller than either auxiliary prime. Primality is checked by trial division through the square root. The central factor z^q is removed before any inversion, so no zero constant is inverted there.

Two implementations give the residues in Table 2. Either auxiliary prime suffices to prove each required constant nonzero; the other provides a further check. The $q = 1$ row is a control case.

Power q	$\widetilde{\rho}_q$ mod 10007	$\widetilde{\rho}_q$ mod 10009
1 (control)	6332	—
2	3182	5359
3	3335	6971

Table 2: Exact residues of the rational seed constants.

The first implementation multiplies truncated polynomials and inverts the denominator series in (25), using reflection to reuse opposite poles. The second computes every pole independently. It obtains constant terms from factorial products and higher coefficients from logarithmic derivatives, as follows.

At a noncentral pole set $H(z) = J_a(z)$ before truncation. At the central pole first remove the factor z^q, leaving a series H with nonzero constant term. Write

$$H(z) = \sum_{j \ge 0} h_j z^j, \qquad \frac{H'(z)}{H(z)} = \sum_{m \ge 1} d_m z^{m-1}.$$

The factors in (25) give

$$d_m = (-1)^{m-1} \left(q u_a^{-m} + 3q \sum_v (u_a - v)^{-m} - q \sum_{b \ne a} B_b (u_a - u_b)^{-m} \right),$$

with the first term omitted when $u_a = 0$. Equating coefficients in $H' = H(H'/H)$ gives the recurrence

$$h_j = \frac{1}{j} \sum_{m=1}^{j} d_m h_{j-m}.$$

The initial value h_0 is the quotient of the nonzero constant-factor products. At the central pole the factor z^q is restored afterwards. This reproduces the required coefficients of J_a using only rational operations modulo P. Substitution in (20) gives the table. Both implementations also agree on every even beta coefficient.

6 The irrationality contradiction

Proof of Theorem 1. Suppose all 35 numbers in the first set are rational. Choose one fixed integer $H > 0$ clearing their denominators, and take $q = 2$. Proposition 2 gives

$$HD_n^2 L_{2,n} \in \mathbb{Z} \qquad (n \in 480\mathbb{Z}_{>0}),$$

and Proposition 3 says these integers tend to zero.

Table 2 proves $\widetilde{\rho}_2 \neq 0$. Outside the finitely many primes dividing H or the numerator or denominator of this fixed rational number, the beta values are p-integral and the seed constant is a unit. Proposition 6 therefore gives, for every sufficiently large such prime,

$$p^{17256}\ell_{2,480p} \equiv \chi_4(p)\widetilde{\rho}_2 \not\equiv 0 \pmod{p}.$$

The scaled beta terms vanish modulo p, leaving the displayed constant. Hence $\ell_{2,480p} \neq 0$, and (16) gives $L_{2,480p} \neq 0$. The excluded set of primes is fixed, so there are infinitely many such indices. We obtain nonzero integers $HD_{480p}^2 L_{2,480p}$ tending to zero, a contradiction.

For the second set, take $q = 3$ and assume its 52 beta values rational. The certificate is $\widetilde{\rho}_3 \equiv 3335 \pmod{10007}$; the multiplier is D_n^3, the local scaling exponent is 25884, and the decay exponent is $3Q < 0$. The same argument proves the second assertion. □

Corollary 7 (General power criterion)**.** *For any fixed integer $q \geq 1$ for which $\widetilde{\rho}_q \neq 0$, at least one of*

$$\beta(2q+2),\ \beta(2q+4),\ \ldots, \beta(2\lfloor 37q/2 \rfloor)$$

is irrational.

Proof. Use the same contradiction with D_n^q, $\tau_q = 8628q$, and the target span (5). □

The seed condition is checked here for $q = 1, 2, 3$. The first case recovers Krisman's interval; the other two give Theorem 1. For further powers, the nonzero-seed hypothesis remains a separate condition rather than a uniform conclusion.

7 Exact checks and audit scope

7.1 Computational checks and reproduction

The mathematical arguments and the finite checks have different roles. Propositions 2–3 and Lemmas 4–5 supply bounds for all the required indices and primes. The computations check the finite seed conditions and the numerical certificates in the cited estimates.

The two seed algorithms reproduce Table 2 and all the associated even beta coefficients. They were developed in the same research session. One uses finite-field polynomial operations from `python-flint`; the other uses only the Python standard library. Four further examples with exact rational coefficients check the dilation formulas for uniform and nonuniform blocks, including both

classes of odd primes modulo 4. These finite examples check the formulas, while the general dilation argument is the proof in Section 4.

A separate exact implementation reconstructs all $62,132$ rational event points of [1, Proposition 5.6] and reproduces the upward-rounded sum over the $62,131$ intervals. Rational logarithm enclosures verify the elementary real-phase inequalities and $Q < -0.0714$. The prime-number-theorem transfer from that arithmetic certificate to (9) is used as proved in [1, Proposition 5.8 and Corollary 5.9].

The accompanying bundle contains the source, certificate records, reproduction commands, environment versions, and pinned provenance for the two principal proof sources, [1] and [5]. Its `README.txt` gives commands for checking the seeds and the arithmetic budget without additional Python packages.

7.2 Independent audit

An independent AI audit reported on 11 November 2026 found no substantive mathematical error or gap in the version identified by its hash in the accompanying audit record. The audit checked the general argument and independently reconstructed the seed and arithmetic certificates using fresh code, following preliminary inspection and replay of the supplied checkers. It was not formal verification; the cited Lean project was not rebuilt.

References

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