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# Centroid-body fixed points and boundary reciprocity

Sihan Chen

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**Abstract**

We prove that, in every dimension, a convex body equal to a dilate of its classical volume-normalized centroid body is a centered ellipsoid. We place the argument in a general framework of affine spectra and boundary reciprocity. For every regular convex body and every positive moment exponent, a natural boundary Markov chain is reversible if and only if the body is a centered ellipsoid. Independently, an exact quadratic identity measures the failure of reciprocity of the map $u \mapsto h(u)\nabla h(u)$; it yields a sharp estimate for approximation of the squared support function by an ellipsoidal quadratic form. Affine covariance supplies universal eigenspaces for all $L_s$ centroid fixed-point equations, and quadratic orthogonality converts kernel approximation into a symmetry certificate. These structural results are proved without assuming the fixed-point classification. The classification itself still uses a signed moment gap in dimensions two through twenty-five and symmetry, curvature, and regression estimates in higher dimensions. All constants and finite certificates are included. No unproved reversibility assertion is used in that proof.

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# 1 Introduction and main theorem

For a convex body $K \subset \mathbb{R}^n$ with nonempty interior, define

$$h_{\Gamma K}(u) = \frac{1}{|K|} \int_K |\langle u, x\rangle| \, \mathrm{d}x. \tag{1.1}$$

Here $\Gamma$ is the ordinary centroid operator: no polar or projection operator is composed with it. Throughout, centered means centered at the origin. Write $\kappa_j = |B_2^j|$, with $\kappa_0 = 1$, and set

$$a_n = \frac{2\kappa_{n-1}}{(n+1)\kappa_n}.$$

Thus $\Gamma B_2^n = a_n B_2^n$. We also write $\mathcal{T}K = a_n^{-1}\Gamma K$. We study the equation

$$\Gamma K = cK, \qquad c > 0. \tag{1.2}$$

It implies that $K$ is origin symmetric.

**Theorem 1.1.** *Let $n \geq 1$. A convex body $K \subset \mathbb{R}^n$ satisfies* (1.2) *if and only if it is a centered ellipsoid. In that case $c = a_n$.*

The higher-dimensional fixed-point question is stated as Problem 1 in Reuter [13], who proves classification in a Banach–Mazur neighborhood of the ellipsoids. The planar case is classical; see Petty [11] and the discussion in [13]. Section 5 includes an independent proof of that case.

Related fixed-point theorems for intersection bodies and polar centroid compositions are proved by Milman–Shabelman–Yehudayoff [10]. Their equations involve different operators. Our equation is for the volume-normalized ordinary centroid operator (1.1) itself.

The structural results are independent of this classification. Theorem 3.2 characterizes ellipsoids by reversibility of a boundary moment chain, for every exponent $s > 0$. Theorem 3.4 gives an exact identity for the reciprocity defect and the sharp estimate

$$\|h_L^2 - h_{E_L}^2\|_{L^2(\sigma)}^2 \le \frac{n}{2(n+3)}\mathfrak{R}(L).$$

The approximating quadratic form is positive definite, so $E_L$ is an actual ellipsoid. Theorem 4.1 and Corollary 4.2 explain why affine deformations, moment cancellation, and quadratic regression govern the fixed-point problem. A direct passage from the fixed-point equation to reciprocity would shorten the proof, but that passage is not established independently here; the estimates below remain part of the classification argument.

Here is the organization of the classification proof, following the structural results in Sections 3 and 4.

1. In dimensions $2 \le n \le 25$, a signed cosine-transform energy estimate forces the radial function to be constant in inertia position.

2. In dimensions $n \ge 26$, affine eigenfunctions of a boundary integral operator and their quadratic orthogonality force a large automorphism group. Compact-group dimension estimates and reflection rigidity yield a canonical independent rotation subspace of dimension

$$p \ge \lceil 2n/3 \rceil, \qquad p \ge 19 \quad \text{if } n = 27.$$

   The extra restriction at $n = 27$ uses a nine-dimensional representation argument.

3. An explicit curvature bound for squared centroid support norms, a regression estimate with constant $1/14$, and a weighted radial moment bound give a necessary inequality for every proper invariant rotation subspace. It contradicts the preceding dimension bound. A uniform comparison treats $n \ge 32$; six rational comparisons treat $26 \le n \le 31$.

The analytic constants are

$$\gamma = \frac{77\,6^{4/3}}{4800} > \frac{87}{500}, \qquad \inf_{\lambda} \mathbb{E}(g(Z) - \lambda Z)^2 \le \frac{L^2}{14}\mathbb{E}Z^2,$$

where the second assertion holds for increasing concave $L$-Lipschitz $g$ with $g(0) = 0$ and positive $Z$ with log-concave survival. The precise hypotheses and proofs appear below.

The external inputs are stated at their points of use: the Busemann–Petty centroid inequality, Berwald's inequality, log-concavity of marginals, Ball's convexity theorem, the normalized Mellin moment theorem, the classical maximal-subgroup classification, and the positive compact-operator theorem. The spherical and cone measures are normalized separately; throughout, spherical integration is with probability measure. For standard convex-geometric notation we refer to [14].

## 2 Normalization and one-dimensional moment bounds

Change of variables gives

$$\Gamma(AK) = A\Gamma K \qquad (A \in GL(n, \mathbb{R})). \tag{2.1}$$

We may therefore put a fixed body in inertia position. If $U$ is uniform in $K$, this means

$$\mathrm{Cov}(U) = \tau \mathrm{Id}, \qquad \tau > 0. \tag{2.2}$$

Every linear automorphism is then orthogonal. We write

$$G = \mathrm{Aut}(K) := \{A \in GL(n,\mathbb{R}) : AK = K\}, \qquad d = \dim G, \qquad V = |K|.$$

Thus $G$ is a closed compact group, including all its components; $G^0$ denotes its identity component. The Busemann–Petty centroid inequality [12, 4] gives

$$|\Gamma K| \geq a_n^n |K|, \qquad \text{hence } c \geq a_n. \tag{2.3}$$

**Lemma 2.1.** *For a uniformly distributed point $X$ in a symmetric convex body,*

$$\sqrt{\frac{n+2}{2(n+1)}}\,(\mathbb{E}\langle u, X\rangle^2)^{1/2} \leq \mathbb{E}|\langle u, X\rangle| \leq \frac{\sqrt{3}}{2}(\mathbb{E}\langle u, X\rangle^2)^{1/2}. \tag{2.4}$$

*Consequently a fixed body in inertia position has radial ratio at most*

$$Q_n = \sqrt{\frac{3(n+1)}{2(n+2)}}. \tag{2.5}$$

*Proof.* On $K_+ = K \cap \{\langle u, x\rangle \geq 0\}$, apply Berwald's inequality [2, 9] to the nonnegative affine function $x \mapsto \langle u, x\rangle$ at exponents one and two. Symmetry yields

$$\left(\frac{(n+1)(n+2)}{2}\mathbb{E}\langle u, X\rangle^2\right)^{1/2} \leq (n+1)\mathbb{E}|\langle u, X\rangle|.$$

For the upper bound, the density of $Y = \langle u, X\rangle$ is even and nonincreasing on the positive half-line by Brunn's section concavity theorem. Layer cake expresses it as a probability mixture of uniform laws on $[-S, S]$. Hence $\mathbb{E}|Y| = \mathbb{E}S/2$ and $\mathbb{E}Y^2 = \mathbb{E}S^2/3 \geq (\mathbb{E}S)^2/3$. For a fixed body, (2.4) bounds its support function between $\sqrt{\tau(n+2)/(2(n+1))}/c$ and $\sqrt{3\tau}/(2c)$. The minimum and maximum of the support function are the inradius and circumradius, respectively, as are the corresponding radial extrema. Their quotient gives (2.5). □

# 3 Boundary reciprocity and ellipsoids

This section does not assume the fixed-point equation. Let $n \geq 2$ and let $L$ be a convex body containing the origin in its interior, with $C^2$ support function $h_L$ and positive tangential Hessian. Extend $h_L$ one-homogeneously away from the origin, and write $\sigma$ for spherical probability measure. Put

$$x_L(u) = \nabla h_L(u), \qquad \mu_L(\mathrm{d}u) = \frac{h_L(u)}{n|L|} S_L(\mathrm{d}u).$$

In particular, $\mu_L$ is a probability measure with continuous positive density. It is useful to allow nonsymmetric $L$ here. Fixed bodies in the main theorem satisfy this regularity, as proved in Section 6.

## 3.1 The boundary moment chain

For $s > 0$ define

$$m_{L,s}(u) = \int |\langle u, x_L(v)\rangle|^s \mu_L(\mathrm{d}v),$$
$$p_{L,s}(u,v) = \frac{|\langle u, x_L(v)\rangle|^s}{m_{L,s}(u)}, \qquad P_{L,s}f(u) = \int p_{L,s}(u,v) f(v) \mu_L(\mathrm{d}v). \tag{3.1}$$

This is a Markov operator. If $U_L$ is uniform in $L$, radial integration gives the normalization

$$m_{L,s}(u) = \frac{n+s}{n}\mathbb{E}|\langle u, U_L\rangle|^s. \tag{3.2}$$

For $s = 1$ and $\Gamma L = cL$, this is exactly the operator $Q$ used in the classification argument below.

**Lemma 3.1.** *The kernel of $P_{L,s}^2$ is continuous and strictly positive on $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$. Consequently $P_{L,s}$ has a unique stationary probability $\pi_{L,s} = w_{L,s}\mu_L$, where $w_{L,s}$ is continuous and strictly positive.*

*Proof.* All the kernels and denominators in (3.1) are continuous, and the denominators are positive. For fixed $u, v$, the two functions $w \mapsto |\langle u, x_L(w)\rangle|$ and $w \mapsto |\langle w, x_L(v)\rangle|$ have zero sets with empty interior. For the first this follows from strict convexity: an open boundary patch cannot lie in a hyperplane; for the second it is immediate. Their positive sets are open and dense. Their intersection therefore contains an open set of positive $\mu_L$ measure, proving strict positivity of the two-step kernel. Compactness gives a lower bound $\delta > 0$ for that kernel.

A continuous Markov kernel on a compact space has a stationary probability: take a weak limit of the Cesàro averages of the laws of a chain starting at any point. The two-step minorization $P_{L,s}^2(u, \cdot) \geq \delta\mu_L$ makes its action on differences of probabilities a strict contraction in total variation, unless $\delta = 1$, when it is already constant. Hence the stationary probability is unique. Applying the continuous two-step kernel to this probability gives a continuous density, bounded below by $\delta$. □

Reversibility means detailed balance with respect to the stationary probability, rather than merely the existence of that probability.

**Theorem 3.2** (Reciprocity characterizes ellipsoids)**.** *For $L$ as above, the following conditions are equivalent.*

1. *$L$ is a centered ellipsoid.*
2. *For some $s > 0$, $P_{L,s}$ is reversible with respect to a probability measure.*
3. *For every $s > 0$, $P_{L,s}$ is self-adjoint on $L^2(\mu_L)$.*
4. *For every $u, v, w \in \mathbb{S}^{n-1}$, the boundary incidence factors satisfy*

$$|\langle u, x_L(v)\rangle|\,|\langle v, x_L(w)\rangle|\,|\langle w, x_L(u)\rangle| = |\langle v, x_L(u)\rangle|\,|\langle w, x_L(v)\rangle|\,|\langle u, x_L(w)\rangle|. \tag{3.3}$$

*In the reversible case, the stationary probability is $\mu_L$. These characterizations are invariant under $GL(n, \mathbb{R})$.*

The elementary rigidity mechanism is local linear algebra, not a classification of compact transformation groups.

**Lemma 3.3** (Local reciprocity)**.** *Suppose a continuous positive function $b$ on a connected open spherical patch $\Omega$ satisfies*

$$b(u)\langle u, x_L(v)\rangle = b(v)\langle v, x_L(u)\rangle \qquad (u, v \in \Omega).$$

*Then $x_L(u) = b(u)Au$ on $\Omega$, for a symmetric matrix $A$. Locally, $h_L(u)^2$ is the restriction of a quadratic form.*

*Proof.* Set $y(u) = x_L(u)/b(u)$. Choose linearly independent $u_1, \dots, u_n \in \Omega$, and let $U, Y$ be the matrices with columns $u_i, y(u_i)$, respectively. The reciprocity equations say that $U^\mathsf{T} Y$ is symmetric. Thus $A = YU^{-1}$ is symmetric. For every $v \in \Omega$ the same equations give $U^\mathsf{T} y(v) = Y^\mathsf{T} v = U^\mathsf{T} A v$, proving $y(v) = Av$. Since $h_L(u) = b(u) u^\mathsf{T} A u > 0$ on the patch,

$$\nabla_{\mathbb{S}^{n-1}} \log h_L(u) = \frac{Au}{u^\mathsf{T} A u} - u = \nabla_{\mathbb{S}^{n-1}} \left( \tfrac{1}{2} \log(u^\mathsf{T} A u) \right).$$

Hence $h_L^2$ is a positive constant times $u^\mathsf{T} A u$ on each connected patch under consideration. □

*Proof of Theorem 3.2.* If $h_L(u) = \sqrt{u^\mathsf{T} H u}$ with $H \succ 0$, then $x_L(u) = Hu/h_L(u)$. Write

$$d_{n,s} = \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle|^s \, \mathrm{d}\sigma(v).$$

The cone measure of the ellipsoid is the image of spherical probability under $\theta \mapsto H^{1/2}\theta$ followed by the normal map. It follows that $m_{L,s}(u) = d_{n,s} h_L(u)^s$, and therefore

$$p_{L,s}(u, v) = \frac{|u^\mathsf{T} H v|^s}{d_{n,s} h_L(u)^s h_L(v)^s}. \tag{3.4}$$

This is symmetric, proving the third condition. That condition implies the second.

Suppose the second holds. Detailed balance implies stationarity; Lemma 3.1 gives a continuous positive density $w$ relative to $\mu_L$. Detailed balance, initially almost everywhere, holds everywhere by continuity and full support. Taking positive $s$th roots gives

$$b(u) |\langle u, x_L(v) \rangle| = b(v) |\langle v, x_L(u) \rangle|, \qquad b(u) = \left( \frac{w(u)}{m_{L,s}(u)} \right)^{1/s} > 0.$$

Around every point there is a connected patch on which both incidence factors are positive for every pair of points. Remove the absolute values there and apply Lemma 3.3. It shows that $h_L^2$ is locally a quadratic form. The forms agree on overlapping patches, since a quadratic polynomial vanishing on an open subset of the sphere vanishes identically. The sphere is connected for $n \geq 2$, so there is a single symmetric $H$ with $h_L(u)^2 = u^\mathsf{T} H u$ everywhere. Positivity of $h_L$ makes $H$ positive definite. Thus the first condition holds.

Symmetric kernels satisfy the three-cycle identity, and their row normalizations cancel from it. Taking $s$th roots proves the fourth condition from the third. Conversely, suppose (3.3). Choose a small connected patch $\Omega$ with all incidence factors positive, fix $u_0 \in \Omega$, and put

$$b(u) = \frac{\langle u_0, x_L(u) \rangle}{\langle u, x_L(u_0) \rangle}.$$

The cycle identity gives the hypothesis of Lemma 3.3 on $\Omega$. Its quadratic forms patch together as before, so $L$ is an ellipsoid. Uniqueness in Lemma 3.1 identifies the stationary probability in that case as $\mu_L$.

For affine invariance, let $A \in GL(n, \mathbb{R})$ and $T_A u = A^{-\mathsf{T}} u / |A^{-\mathsf{T}} u|$. Cone measure pushes forward under $T_A$, while $x_{AL}(T_A u) = A x_L(u)$. Both the numerator and denominator in (3.1) acquire the same factor $|A^{-\mathsf{T}} u|^{-s}$. Thus the chains are conjugate, proving the claimed invariance. □

For example, this gives a nonnegative affine invariant which vanishes exactly on centered ellipsoids. With $j_s(u, v) = w_{L,s}(u) p_{L,s}(u, v)$, set

$$\mathfrak{I}_s(L) = \frac{1}{2} \iint \left( \sqrt{j_s(u, v)} - \sqrt{j_s(v, u)} \right)^2 \mu_L(\mathrm{d}u) \mu_L(\mathrm{d}v). \tag{3.5}$$

The two edge densities have total mass one, so $0 \le \mathfrak{I}_s(L) \le 1$. Theorem 3.2 shows that $\mathfrak{I}_s(L) = 0$ if and only if $L$ is a centered ellipsoid. The unique stationary law in this construction exists for every regular $L$; its existence alone does not imply reversibility.

### 3.2 A sharp quadratic identity

There is a second, entirely explicit form of the same geometric rigidity. Here suppose $L = -L$, and use spherical probability $\sigma$. Let $P_\ell$ denote projection onto degree-$\ell$ spherical harmonics. Put

$$\begin{aligned} F(u) &= \tfrac{1}{2} h_L(u)^2, & Y(u) &= \nabla F(u) = h_L(u) x_L(u), \\ H_L &= \int_{\mathbb{S}^{n-1}} D^2 F(u)\, \mathrm{d}\sigma(u), & g_L(u) &= h_L(u)^2 - u^{\mathsf{T}} H_L u, \\ \mathfrak{R}(L) &= \iint \left( \langle u, Y(v) \rangle - \langle v, Y(u) \rangle \right)^2 \mathrm{d}\sigma(u)\, \mathrm{d}\sigma(v). \end{aligned} \tag{3.6}$$

Derivatives of homogeneous functions in this subsection are ambient derivatives unless a spherical subscript is present.

**Theorem 3.4** (Exact reciprocity identity)**.** *The matrix $H_L$ is positive definite, and $u^{\mathsf{T}} H_L u = (P_0 + P_2) h_L^2$ is the orthogonal projection of $h_L^2$ onto the restrictions of quadratic forms. Hence $E_L$ below is the unique best ellipsoid for this squared-support $L^2$ approximation. Moreover,*

$$\begin{aligned} \mathfrak{R}(L) &= \frac{2}{n} \int |Y(u) - H_L u|^2 \, \mathrm{d}\sigma(u) \\ &= \frac{1}{2n} \int \left( |\nabla_{\mathbb{S}^{n-1}} g_L|^2 + 4 g_L^2 \right) \mathrm{d}\sigma \\ &= \frac{1}{2n} \sum_{\substack{\ell \ge 4 \\ \ell \ meven}} \left( \ell(\ell + n - 2) + 4 \right) \| P_\ell h_L^2 \|_2^2. \end{aligned} \tag{3.7}$$

*Consequently*

$$\| h_L^2 - h_{E_L}^2 \|_2^2 \le \frac{n}{2(n+3)} \mathfrak{R}(L), \qquad h_{E_L}(u) = \sqrt{u^{\mathsf{T}} H_L u}. \tag{3.8}$$

*The constant is sharp in every dimension $n \ge 2$. In particular, $\mathfrak{R}(L) = 0$ if and only if $L$ is a centered ellipsoid.*

*Proof.* Since $h_L > 0$ and its tangential Hessian is positive,

$$D^2 F = \nabla h_L \nabla h_L^{\mathsf{T}} + h_L D^2 h_L \succ 0.$$

This proves $H_L \succ 0$. Homogeneity and the divergence theorem on the Euclidean unit ball give

$$H_L = n \int Y(u) u^{\mathsf{T}} \, \mathrm{d}\sigma(u). \tag{3.9}$$

Indeed the Hessian is homogeneous of degree zero, so its ball average equals its spherical average. One may apply divergence on annuli and let the inner radius tend to zero, avoiding differentiability at the origin. In particular the matrix on the right is symmetric. For any square-integrable vector field $Y$, expansion of the double integral in (3.6) gives

$$\mathfrak{R}(L) = \frac{2}{n} \int |Y|^2 \, \mathrm{d}\sigma - 2 \operatorname{tr} \left( \left( \int Y u^{\mathsf{T}} \, \mathrm{d}\sigma \right)^2 \right).$$

Using (3.9) yields the first equality in (3.7); it is also the least-squares identity for approximation of $Y$ by a linear vector field.

For completeness write $f = h_L^2$, $\bar{f} = \int f \, \mathrm{d}\sigma$, and $M_f = \int f(u) u u^{\mathsf{T}} \, \mathrm{d}\sigma(u)$. Spherical integration by parts gives $\int u_i (\nabla_{\mathbb{S}^{n-1}} f)_j \, \mathrm{d}\sigma = n(M_f)_{ij} - \delta_{ij} \bar{f}$. Since $Y = fu + \nabla_{\mathbb{S}^{n-1}} f/2$, (3.9) becomes

$$H_L = \frac{n(n+2)}{2} M_f - \frac{n}{2} \bar{f} \mathrm{Id}. \tag{3.10}$$

The spherical fourth-moment formula identifies $u^{\mathsf{T}} H_L u = (P_0 + P_2) f$. Thus $g_L$ has no components of degree zero or two. Also

$$Y - H_L u = g_L u + \tfrac{1}{2} \nabla_{\mathbb{S}^{n-1}} g_L,$$

whose two terms are pointwise orthogonal. This gives the second equality in (3.7). The eigenvalues of the spherical Laplacian give the last one. Evenness excludes odd degrees. The first remaining eigenvalue is $4(n+2)$, and hence

$$\mathfrak{R}(L) \geq \frac{2(n+3)}{n} \|g_L\|_2^2,$$

which is (3.8). For sharpness take $h_L(u) = \sqrt{1 + \epsilon Z_4(u)}$, where $Z_4$ is a nonzero degree-four spherical harmonic and $\epsilon \neq 0$ is sufficiently small. This is the support function of a regular symmetric convex body, by $C^2$ closeness to the ball. Its defect $g_L = \epsilon Z_4$ attains equality. □

Thus squared support functions turn the ellipsoid family into the positive cone in the linear space of quadratic forms. The exact identity is global, rather than a linearization at the ball; its constant has no exceptional dimensions. The spherical defect itself depends on the Euclidean coordinates. An affine invariant version is obtained by applying it to $\widehat{L} = \Sigma_L^{-1/2} L$, where $\Sigma_L = \mathrm{Cov}(U_L)$. Indeed $\widehat{AL} = O\widehat{L}$ for an orthogonal $O$. The ellipsoid assignment $L \mapsto \Sigma_L^{1/2} E_{\widehat{L}}$ is therefore affine equivariant and returns $L$ whenever $L$ is an ellipsoid.

## 4 Affine covariance as a spectral principle

The eigenvalue and orthogonality used below are consequences of affine covariance and homogeneity. They hold for all centroid moments. For $s \geq 1$ define the volume-normalized $L_s$ centroid body by

$$h_{\Gamma_s L}(u)^s = \mathbb{E}|\langle u, U_L \rangle|^s.$$

Suppose a regular $L$ satisfies $\Gamma_s L = cL$, and set $F_A(u) = \langle u, A x_L(u) \rangle / h_L(u)$.

**Theorem 4.1** (Affine and moment identities)**.** *For every real matrix $A$,*

$$P_{L,s} F_A = \frac{s F_A + \operatorname{tr} A}{n+s}, \qquad \int F_A \, \mathrm{d}\mu_L = \frac{\operatorname{tr} A}{n}. \tag{4.1}$$

*The kernel of $A \mapsto F_A$ is the Lie algebra of $\mathrm{Aut}(L)$. For every continuously differentiable polynomial $T$,*

$$\int F_A(u) T(x_L(u)) \mu_L(\, \mathrm{d}u) = \frac{1}{n} \mathbb{E}[(\operatorname{tr} A) T(U_L) + DT(U_L)[AU_L]]. \tag{4.2}$$

*In inertia position, the rotational functions $F_A$, $A^{\mathsf{T}} = -A$, are orthogonal to constants and all boundary quadratic functions, and span an invariant subspace of dimension $n(n-1)/2 - \dim \mathrm{Aut}(L)$ on which $P_{L,s}$ acts by $s/(n+s)$.*

*Proof.* The divergence theorem applied to $Ax\,|\langle u,x\rangle|^s$ gives

$$\int_{\partial L} |\langle u,x\rangle|^s \langle Ax,\nu(x)\rangle \,\mathrm{d}S(x) = (\operatorname{tr} A)c^s|L|h_L(u)^s + sc^s|L|h_L(u)^{s-1}\langle u, Ax_L(u)\rangle.$$

Here differentiation of the fixed-point equation was used for the second term; for $s=1$ differentiation is justified by dominated convergence. Divide by $n|L|m_{L,s}(u)$ and use (3.2). This proves the first identity in (4.1). Applying divergence to $Ax$, or to $Ax\,T(x)$, proves the second and (4.2). Finally $F_A = 0$ says that the derivative of $h_L$ along $e^{tA^{\mathsf{T}}}u$ vanishes, equivalently $e^{tA}L = L$. In inertia position these automorphisms are orthogonal. Taking $T(x) = xx^{\mathsf{T}}$ in (4.2) gives

$$\int F_A xx^{\mathsf{T}} \,\mathrm{d}\mu_L = \frac{1}{n}((\operatorname{tr} A)\Sigma_L + A\Sigma_L + \Sigma_L A^{\mathsf{T}}),$$

which vanishes for skew $A$ and scalar $\Sigma_L$. Rank–nullity proves the dimension assertion. □

**Corollary 4.2** (A spectral certificate for symmetry)**.** *Let $\Pi$ be orthogonal projection in $L^2(\mu_L)$ onto constants and boundary quadratic functions. Put $B_{L,s} = P_{L,s}(I-\Pi)$ and $\lambda_s = s/(n+s)$. In inertia position,*

$$\left(\frac{n(n-1)}{2} - \dim \operatorname{Aut}(L)\right)\lambda_s^2 \le \|B_{L,s}\|_{\mathrm{HS}}^2. \tag{4.3}$$

*If $\|B_{L,s}\|_{L^2\to L^2} < \lambda_s$, then $L$ is a ball in this position, and hence an ellipsoid before normalization.*

*Proof.* For an orthonormal basis $e_j$ of the rotational subspace, $\Pi e_j = 0$ and $B_{L,s}e_j = \lambda_s e_j$. The Hilbert–Schmidt norm bounds the sum of the squared norms of these images, proving (4.3). The strict operator-norm bound forces every rotational function to vanish. Thus every one-parameter rotation preserves $L$, and $SO(n)$ invariance makes $L$ a ball. This argument uses neither reversibility nor stationarity of the reference cone measure. □

### 4.1 Relation to the fixed-point classification

For the ordinary centroid equation, Theorem 4.1 explains the universal eigenvalue $1/(n+1)$ and the role of quadratic regression. Corollary 4.2 is the abstract form of the symmetry estimates in the classification proof. Theorem 3.2 and Theorem 3.4 identify their geometric target: a reciprocal boundary map, equivalently a quadratic squared support function.

A shorter proof of the fixed-point theorem would follow from a direct implication

$$\Gamma L = cL \quad\Longrightarrow\quad P_{L,1} \text{ is reversible,} \qquad \text{or directly} \quad \mathfrak{R}(L) = 0. \tag{4.4}$$

No direct proof of this implication is supplied here. The detailed classification argument below is still needed; after that argument, (4.4) follows as a corollary. Thus the reciprocity theorems are independent structural characterizations, and are not being used as an unproved replacement for the difficult step.

There is a precise warning against confusing an affine identity with detailed balance. At a fixed body, any stationary probability $\pi$ for $P_{L,s}$ satisfies, by (4.1),

$$\int F_A \,\mathrm{d}\pi = \frac{\operatorname{tr} A}{n}, \qquad \int \frac{x_L(u)u^{\mathsf{T}}}{h_L(u)}\pi(\,\mathrm{d}u) = \frac{1}{n}\mathrm{Id}.$$

Cone measure satisfies the same moment identities. Equality of these finitely many moments is not by itself an identification of the two measures or a proof of detailed balance.

## 5 A signed spherical moment gap and low dimensions

**Proposition 5.1.** *If $2 \le n \le 25$ and $K$ satisfies (1.2), then $K$ is a centered ellipsoid and $c = a_n$.*

Let $\sigma$ denote spherical probability measure and define

$$b_n = \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle| \, \mathrm{d}\sigma(v) = \frac{2\kappa_{n-1}}{n\kappa_n}, \qquad \mathcal{C}f(u) = b_n^{-1} \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle| f(v) \, \mathrm{d}\sigma(v). \tag{5.1}$$

Thus $\mathcal{C}1 = 1$ and $a_n = nb_n/(n+1)$. For $r = \rho_K$, polar integration gives

$$h_{\mathcal{T}K} = \frac{\mathcal{C}(r^{n+1})}{\int r^n \, \mathrm{d}\sigma}. \tag{5.2}$$

Inertia position implies the exact identity

$$P_2(r^{n+2}) = 0, \tag{5.3}$$

where $P_\ell$ denotes orthogonal projection onto degree-$\ell$ spherical harmonics: integrate any homogeneous harmonic quadratic over $K$ in polar coordinates to obtain (5.3).

**Lemma 5.2.** *The even cosine-transform multipliers are*

$$\mu_0 = 1, \qquad \mu_{2j} = (-1)^{j-1} \frac{\Gamma(j - 1/2)\Gamma((n+1)/2)}{2\sqrt{\pi}\,\Gamma(j + (n+1)/2)} \quad (j \ge 1). \tag{5.4}$$

*In particular,*

$$\mu_2 = \frac{1}{n+1}, \quad \mu_4 = -\frac{1}{(n+1)(n+3)}, \quad \mu_6 = \frac{3}{(n+1)(n+3)(n+5)}, \quad \frac{\mu_{\ell+2}}{\mu_\ell} = -\frac{\ell - 1}{\ell + n + 1}$$

*for even $\ell \ge 2$. Thus $\mu_\ell \le \mu_6$ for every even $\ell \ge 4$.*

*Proof.* For $n \ge 3$, formula (5.4) follows from Funk–Hecke; see also [13, Section 3] with normalization (5.1). For $n = 2$, the Fourier series of $|\cos t|$ gives $\mu_{2j} = (-1)^{j-1}/(4j^2 - 1)$, which is the same formula. The recurrence follows by the Gamma identity. Its ratio has absolute value less than one, and the signs alternate after degree two. □

In this section $K$ is a fixed body in inertia position. Write $r = \rho_K$ and

$$A_j = \int_{\mathbb{S}^{n-1}} r^{n+j} \, \mathrm{d}\sigma \quad (j = 0, 1, 2), \qquad f = r^{n+1}, \qquad \Delta = A_0 A_2 - A_1^2. \tag{5.5}$$

Cauchy–Schwarz gives $\Delta \ge 0$, with equality exactly when $r$ is constant.

**Lemma 5.3.** *Under (5.3),*

$$\|P_2 f\|_2^2 \le \frac{n+2}{2} \Delta. \tag{5.6}$$

*Proof.* Put $\mathrm{d}\nu = r^{n+2} \, \mathrm{d}\sigma / A_2$ and $g = 1/r$. Then $\mathbb{E}_\nu vv^{\mathsf{T}} = \mathrm{Id}/n$ and $\mathrm{Var}_\nu \, g = \Delta / A_2^2$. The trace-free symmetric matrix

$$S = \mathbb{E}_\nu[(g - \mathbb{E}_\nu g) vv^{\mathsf{T}}]$$

satisfies, for every trace-free symmetric $U$,

$$|\operatorname{tr}(US)|^2 \le \mathrm{Var}_\nu(g) \mathbb{E}_\nu (v^{\mathsf{T}} U v)^2 \le \frac{\mathrm{Var}_\nu(g)}{n} \operatorname{tr}(U^2).$$

The second inequality uses $(v^{\mathsf{T}}Uv)^2 \le v^{\mathsf{T}}U^2v$. Taking $U = S$ gives $\operatorname{tr}(S^2) \le \Delta/(nA_2^2)$. The trace-free moment matrix of $f$ is $M_f = \int f(v)(vv^{\mathsf{T}} - \mathrm{Id}/n)\,\mathrm{d}\sigma = A_2 S$. The spherical fourth-moment identity yields

$$P_2 f(u) = \frac{n(n+2)}{2} u^{\mathsf{T}} M_f u, \qquad \|P_2 f\|_2^2 = \frac{n(n+2)}{2} \operatorname{tr}(M_f^2).$$

Substitution proves (5.6). □

**Lemma 5.4.** *If* $\max r / \min r \le q$, *then*

$$\operatorname{Var}_\sigma(f) \le \Lambda_n(q)\Delta, \qquad \Lambda_n(q) = q^{-n}\left(\sum_{j=0}^{n} q^j\right)^2. \tag{5.7}$$

*Here* $\Lambda_n(1) = (n+1)^2$.

*Proof.* For positive $s, t$ with $s/t \in [q^{-1}, q]$,

$$\frac{(s^{n+1} - t^{n+1})^2}{s^n t^n (s-t)^2} = \left(\sum_{j=0}^{n} (s/t)^{j-n/2}\right)^2 \le \Lambda_n(q).$$

The expression is invariant under $s/t \mapsto t/s$ and is increasing for $s/t \ge 1$: opposite powers pair into increasing hyperbolic cosines. Integrate the undivided inequality and use

$$\operatorname{Var}_\sigma f = \frac{1}{2}\iint (f(u) - f(v))^2 \,\mathrm{d}\sigma(u)\,\mathrm{d}\sigma(v), \quad \Delta = \frac{1}{2}\iint r(u)^n r(v)^n (r(u) - r(v))^2 \,\mathrm{d}\sigma(u)\,\mathrm{d}\sigma(v).$$

□

**Proposition 5.5.** *A fixed body is an ellipsoid if its inertia-position radial ratio is at most* $q$ *and*

$$6\Lambda_n(q) < p(n), \qquad p(n) = n^3 + 8n^2 + 18n + 6. \tag{5.8}$$

*Proof.* Since $h_K \ge r$, (5.2) and $c \ge a_n$ imply

$$A_2 \le \int f h_K \,\mathrm{d}\sigma = \frac{a_n}{cA_0}\langle f, \mathcal{C}f\rangle_{L^2(\sigma)} \le A_0^{-1}\langle f, \mathcal{C}f\rangle_{L^2(\sigma)}.$$

Thus $\Delta \le \langle f, \mathcal{C}f\rangle - A_1^2$. Since $f$ is even, retaining the signs of the multipliers in Lemma 5.2, and then using Lemmas 5.3 and 5.4, gives

$$\begin{aligned}\Delta &\le (\mu_2 - \mu_6)\,\|P_2 f\|_2^2 + \mu_6 \operatorname{Var}_\sigma f \\ &\le \left[\frac{n+2}{2}(\mu_2 - \mu_6) + \mu_6 \Lambda_n(q)\right]\Delta.\end{aligned}$$

The bracket equals

$$\Theta_n(q) = \frac{(n+2)^2(n+6) + 6\Lambda_n(q)}{2(n+1)(n+3)(n+5)}.$$

Since $2(n+1)(n+3)(n+5) - (n+2)^2(n+6) = p(n)$, (5.8) says $\Theta_n(q) < 1$. Consequently $\Delta = 0$, and $r$ is constant. □

**Lemma 5.6.** *For* $q_* = 601/500$, *inequality* (5.8) *holds for every integer* $3 \le n \le 25$.

*Proof.* Let $t = \frac{1}{2}\log q_* > 0$ and, for real $x \geq 3$, put

$$F(x) = \log \frac{6\sinh^2((x+1)t)}{p(x)\sinh^2 t}.$$

For integer $n$, $e^{F(n)} = 6\Lambda_n(q_*)/p(n)$. Using $\sinh y \geq y$ and differentiating twice gives

$$F''(x) \geq \frac{W(x)}{(x+1)^2 p(x)^2}, \qquad W(x) = x^6 + 6x^5 - 5x^4 - 60x^3 + 20x^2 + 276x + 156.$$

Indeed, the numerator is $(x+1)^2(p'(x)^2 - p''(x)p(x)) - 2p(x)^2$. The polynomial $x^3 + 6x^2 - 5x - 60$ is increasing and equals 6 at $x = 3$. Writing $W = x^3(x^3 + 6x^2 - 5x - 60) + 20x^2 + 276x + 156$ proves $F'' > 0$. Hence the maximum of $F$ on $[3, 25]$ is at an endpoint. At $n = 3$, monotonicity in $q$ gives

$$6\Lambda_3(q_*) < 6\Lambda_3(5/4) = 6 \cdot \frac{136161}{8000} < 159 = p(3).$$

At $n = 25$, the integer inequality $601^{13} < 11 \cdot 500^{13}$ gives

$$\Lambda_{25}(q_*) = \frac{q_*}{(q_* - 1)^2}(q_*^{13} - q_*^{-13})^2 < \frac{300500}{10201}\left(\frac{120}{11}\right)^2.$$

The desired comparison with $p(25) = 21081$ is precisely $25963200000 < 26020721001$. Both endpoint values are negative, so $F < 0$ on the whole interval. □

*Proof of Proposition 5.1.* Put $K$ in inertia position. For $n = 2$, Lemma 2.1 gives $Q_2 = \sqrt{9/8} < 5/4$, and the exact comparison

$$6\Lambda_2(5/4) = \frac{11163}{200} < 82 = p(2)$$

allows us to apply Proposition 5.5. Now suppose $3 \leq n \leq 25$. Lemma 2.1 yields

$$(\max r / \min r)^2 \leq Q_n^2 \leq Q_{25}^2 = 13/9 < (601/500)^2;$$

the last comparison is $3250000 < 3250809$. Proposition 5.5 and Lemma 5.6 show that $K$ is a ball in this position. Undoing the linear map gives an ellipsoid; (2.1) then gives $c = a_n$.

□

## 6 The boundary operator and quadratic orthogonality

Work in inertia position (2.2). Write $h = h_K$ and $x(u) = \nabla h(u)$, with $h$ extended one-homogeneously to $\mathbb{R}^n \setminus \{0\}$.

The regularity needed here follows directly from slicing. For unit $u$ and $v \perp u$,

$$D^2 h_{\Gamma K}(u)[v, v] = \frac{2}{V}\int_{K \cap u^\perp} \langle v, x\rangle^2 \, \mathrm{d}\mathcal{H}^{n-1}(x) > 0 \quad (v \neq 0). \tag{6.1}$$

The section integral is continuous in $u, v$: polar integration in the section expresses it as an integral of the continuous radial function of $K$ to the power $n + 1$. Thus $h_{\Gamma K}$ is $C^2$ away from zero with positive tangential Hessian. By (1.2), the same is true for $h$. In particular $x(u)$ parametrizes the boundary by its outer normal, and its cone probability measure is

$$\mu(\mathrm{d}u) = \frac{h(u)}{nV} S_K(\mathrm{d}u). \tag{6.2}$$

This measure has a continuous positive density and full support.

For a real matrix $A$, define

$$F_A(u) = \frac{\langle u, Ax(u)\rangle}{h(u)}. \tag{6.3}$$

Let $Q$ have kernel, relative to $\mu$,

$$q(u,v) = \frac{n}{(n+1)c}\frac{|\langle u, x(v)\rangle|}{h(u)}. \tag{6.4}$$

It is a compact integral operator on $L^2(\mu)$.

**Lemma 6.1** (Affine regression)**.** *One has $Q1 = 1$ and, for every real matrix $A$,*

$$QF_A = \frac{F_A + \operatorname{tr} A}{n+1}, \qquad \int F_A \,\mathrm{d}\mu = \frac{\operatorname{tr} A}{n}. \tag{6.5}$$

*The kernel of $A \mapsto F_A$ is the Lie algebra $\mathfrak{g}_K$ of $\operatorname{Aut}(K)$. In inertia position $\mathfrak{g}_K \subset \mathfrak{so}(n)$.*

*Proof.* This is the case $s = 1$ of Theorem 4.1, since (3.2) identifies $Q = P_{K,1}$. The regularity needed there was established in (6.1). □

**Lemma 6.2** (Quadratic orthogonality)**.** *For every skew-symmetric matrix $A$,*

$$\int F_A \,\mathrm{d}\mu = 0, \qquad \int F_A(u)x(u)x(u)^{\mathsf{T}}\mu(\,\mathrm{d}u) = 0. \tag{6.6}$$

*Consequently the invariant subspace*

$$\mathcal{R} = \{F_A : A \in \mathfrak{so}(n)\}, \qquad Q|_{\mathcal{R}} = \frac{1}{n+1}\mathrm{Id},$$

*is orthogonal to constants and to all functions $u \mapsto x(u)^{\mathsf{T}}Bx(u)$. Its dimension is*

$$m := \dim \mathcal{R} = \frac{n(n-1)}{2} - d. \tag{6.7}$$

*Proof.* The moment identity (4.2), with $T(x) = xx^{\mathsf{T}}$, gives, for $\Sigma = \operatorname{Cov}(U)$,

$$\begin{aligned}\int F_A(u)x(u)x(u)^{\mathsf{T}}\mu(\,\mathrm{d}u) &= \frac{1}{nV}\int_{\partial K} xx^{\mathsf{T}}\langle Ax, \nu(x)\rangle \,\mathrm{d}S(x)\\ &= \frac{1}{n}((\operatorname{tr} A)\Sigma + A\Sigma + \Sigma A^{\mathsf{T}}).\end{aligned}$$

For skew $A$ this vanishes by (2.2). The other assertions follow from Lemma 6.1 and rank–nullity. □

**Theorem 6.3** (Quadratic symmetry bound)**.** *For a fixed body in dimension $n \geq 26$,*

$$d > D(n) := \frac{n(n-1)}{2} - \frac{6}{25}(n+1)^2 = \frac{13n^2 - 49n - 12}{50}. \tag{6.8}$$

# 7 A scalar quadratic approximation

**Lemma 7.1** (Exponential comparison)**.** *Let $Y \ge 0$ have a log-concave density and $\mathbb{E}Y = 1$, and let $\xi$ have the exponential density $e^{-y}$ on $[0,\infty)$. For every convex function $H$ for which the expectations exist,*

$$\mathbb{E}H(Y) \le \mathbb{E}H(\xi). \tag{7.1}$$

*Proof.* Write $S(t) = \mathbb{P}(Y > t)$. Log-concavity of the density implies that $S$ is log-concave. For example, $S(t)/f(t) = \int_0^\infty f(t+s)/f(t)\,\mathrm{d}s$ is nonincreasing on the positive support, so the hazard rate is nondecreasing. Since $S(0) = 1$, the concave function $\log S(t) + t$ shows that $S(t) - e^{-t}$ changes sign at most once, from positive to negative. Its integral is zero because $\mathbb{E}Y = \mathbb{E}\xi = 1$. It follows that

$$\mathbb{E}(Y-a)_+ = \int_a^\infty S(t)\,\mathrm{d}t \le e^{-a} = \mathbb{E}(\xi - a)_+ \qquad (a \ge 0).$$

Together with equality of means, this proves (7.1) by the representation of a convex function as an affine function plus a positive integral of hinges. Approximation gives the asserted integrable generality. □

Put

$$R(y) = y - \frac{2}{3} - \frac{y^2}{5}, \qquad P(y) = R(y)^2, \qquad P_t(y) = P(y) + tyP'(y). \tag{7.2}$$

**Lemma 7.2** (Uniform scalar bound)**.** *For $Y$ as in Lemma 7.1 and $0 \le t \le 1/26$,*

$$\mathbb{E}P_t(Y) < \frac{6}{25}(1+t)^2. \tag{7.3}$$

*Proof.* Set $z = 11/6$. For $\eta \in \{0, 1/26\}$ let $L_\eta$ be the tangent line to $P_\eta$ at $z$, and put $H_\eta = \max(P_\eta, L_\eta)$. Direct factorization gives

$$P_0(y) - L_0(y) = \frac{(y-z)^2}{300}(12y^2 - 76y + 61), \tag{7.4}$$

$$P_{1/26}(y) - L_{1/26}(y) = \frac{(y-z)^2}{780}(36y^2 - 216y + 151). \tag{7.5}$$

In each case the quadratic factor has two roots $r_- < z < r_+$. The function $H_\eta$ equals $P_\eta$ outside $[r_-, r_+]$ and $L_\eta$ inside. It is convex. Indeed, outside this interval the quadratic factor, the product of its derivative with $y - z$, and $(y-z)^2$ times its second derivative are all nonnegative. Hence $P_\eta'' \ge 0$ there. At the two joining points the derivative jumps upwards, since $(P_\eta - L_\eta)'$ is negative at $r_-$ and positive at $r_+$.

Appendix A proves the exact bounds

$$\mathbb{E}H_0(\xi) < \frac{6}{25}, \qquad \mathbb{E}H_{1/26}(\xi) < \frac{129}{500}. \tag{7.6}$$

For $0 \le t \le 1/26$, the convex function

$$\widetilde{H}_t = (1 - 26t)H_0 + 26tH_{1/26}$$

majorizes $P_t$, because $P_t = (1-26t)P_0 + 26tP_{1/26}$. Lemma 7.1 therefore gives

$$\mathbb{E}P_t(Y) \le \mathbb{E}\widetilde{H}_t(Y) \le \mathbb{E}\widetilde{H}_t(\xi) < \frac{6}{25} + \frac{117}{250}t \le \frac{6}{25}(1+t)^2.$$

The last inequality follows from $117/250 < 12/25$. □

## 8 Proof of the symmetry bound

For a homogeneous function $F$ of degree $j \geq 0$, polar integration, or divergence applied to $xF(x)$, gives

$$\int F(x(v))\mu(\,\mathrm{d}v) = \frac{n+j}{n}\mathbb{E}F(U). \tag{8.1}$$

Fix $u \in \mathbb{S}^{n-1}$, put $s_u = \mathbb{E}|\langle u, U\rangle| = ch(u)$, and define

$$Y = \frac{|\langle u, U\rangle|}{s_u}, \qquad Y_u(v) = \frac{|\langle u, x(v)\rangle|}{s_u}.$$

The variable $Y$ has a log-concave density on $[0,\infty)$ and mean one. Here the one-dimensional marginal of the symmetric uniform measure on $K$ is log-concave by [5], and taking its absolute value restricts that even density to the positive half-line. By (8.1), applied to the monomials in $P$,

$$\int P(Y_u(v))\mu(\,\mathrm{d}v) = \mathbb{E}\left[P(Y) + \frac{Y}{n}P'(Y)\right] = \mathbb{E}P_{1/n}(Y). \tag{8.2}$$

Since $q(u,v) = nY_u(v)/(n+1)$, Lemma 7.2 yields

$$\int \left[q(u,v) - \frac{n}{n+1}\left(\frac{2}{3} + \frac{Y_u(v)^2}{5}\right)\right]^2 \mu(\,\mathrm{d}v) < \frac{6}{25}. \tag{8.3}$$

*Proof of Theorem 6.3.* Choose an orthonormal basis $e_1, \ldots, e_m$ of $\mathcal{R}$. Lemma 6.2 makes each $e_j$ orthogonal to the quadratic function subtracted in (8.3). Bessel's inequality and $Qe_j = e_j/(n+1)$ give, pointwise in $u$,

$$\begin{aligned}\frac{1}{(n+1)^2}\sum_{j=1}^{m} e_j(u)^2 &= \sum_{j=1}^{m}\left|\int q(u,v)e_j(v)\mu(\,\mathrm{d}v)\right|^2 \\ &\leq \int \left[q(u,v) - \frac{n}{n+1}\left(\frac{2}{3} + \frac{Y_u(v)^2}{5}\right)\right]^2 \mu(\,\mathrm{d}v) < \frac{6}{25}.\end{aligned}$$

Integrating over $u$ proves $m < (6/25)(n+1)^2$. Now use (6.7). This argument uses neither self-adjointness of $Q$ nor stationarity of $\mu$. □

## 9 Compact-group facts

Put $g(t) = t^2/4 - 3t/4$. We use the maximal-subgroup classification in [8, Theorem 2.2, Lemma 2.8(iv), and Appendix]. Specifically, for $N \geq 10$, a maximal proper connected subgroup of $SO(N)$ of dimension at least $g(N)$ is symmetric. The possibilities are $SO(a) \times SO(N-a)$ and $U(N/2)$. For $SU(m)$, a maximal proper connected subgroup of dimension at least $m^2/2 - m$ is symmetric; the possibilities are $SO(m)$, $Sp(m/2)$, and $S(U(a) \times U(m-a))$. These are subgroup dimension statements and do not require a hyperpolar action.

**Lemma 9.1.** *Let $N \geq 10$, and let $H \subset SO(N)$ be closed, connected, and real irreducible. If $\dim H > g(N)$, then either $H = SO(N)$ or $N = 2m$ and $SU(m) \subset H \subset U(m)$ after orthogonal conjugation. In either case $H$ is transitive on the unit sphere.*

*Proof.* If $H \neq SO(N)$, a maximal proper connected subgroup $M$ containing it is irreducible. The preceding classification forces $M = U(m)$, where $N = 2m$ and $m \geq 5$. The representation of $H$ on $\mathbb{C}^m$ is complex irreducible, since a complex invariant subspace would also be a real one. Schur's lemma makes the Lie algebra center scalar, of dimension at most one. Thus $S = [H, H] \subset SU(m)$ is complex irreducible and

$$\dim S \geq \dim H - 1 > m^2 - \tfrac{3}{2}m - 1.$$

For $m \geq 5$ this is larger than both $m^2/2 - m$ and $m(m+1)/2$. If $S$ were proper, place it in a maximal proper connected subgroup of $SU(m)$. The subgroup must be symmetric. Irreducibility excludes the block-unitary case, while the other two cases have dimension at most $m(m+1)/2$, a contradiction. Hence $S = SU(m)$. □

**Lemma 9.2.** *For every $m \geq 5$, a proper closed connected subgroup of $SO(m)$ has dimension at most $(m-1)(m-2)/2$.*

*Proof.* For $m \geq 10$, the maximal-subgroup statement above proves the claim: below its threshold the bound is immediate; above it the largest orthogonal block has dimension $(m-1)(m-2)/2$, and $\dim U(m/2) = m^2/4$ is no larger. For $m = 5, 6, 7, 8, 9$, the respective maximal dimensions of proper connected subgroups are $6, 10, 15, 21, 28$, by the same maximal-subgroup classification and the low-degree representations in its Appendix. At $m = 8$ this includes the nonstandard $Spin(7)$ embedding. □

Here and subsequently group names in representation lists denote their images, so finite central kernels do not affect the statements.

**Lemma 9.3** (Small irreducible images)**.** *Let $5 \leq p \leq 9$ and let $L \subset SO(p)$ be closed, connected, and real irreducible. Under the indicated dimension assumption, the possible images are as follows:*

| $p$ | *dimension assumption* | *possible images* |
|---|---|---|
| 5 | $\dim L \geq 4$ | $SO(5)$ |
| 6 | $\dim L \geq 7$ | $SU(3)$, $U(3)$, $SO(6)$ |
| 7 | $\dim L \geq 9$ | $G_2$, $SO(7)$ |
| 8 | $\dim L \geq 10$ | $Sp(2)$, $Sp(2)U(1)$, $Sp(2)Sp(1)$, $SU(4)$, $U(4)$, $Spin(7)$, $SO(8)$ |
| 9 | $\dim L \geq 10$ | $SO(9)$ |

*Every image in the table is sphere-transitive.*

*Proof.* We use the maximal-subgroup and low-degree highest-weight lists of [8, Theorem 2.2 and Appendix]. For clarity, the small nontransitive simple representations excluded by the dimension assumptions are the five-dimensional representation of $SO(3)$ and the eight-dimensional adjoint representation of $SU(3)$. The nine-dimensional tensor representation of $SO(3) \times SO(3)$ has group dimension six. In the eight-dimensional row, the dimensions of the displayed images are respectively $10, 11, 13, 15, 16, 21, 28$. Thus the irreducible eight-dimensional images of dimension greater than sixteen are precisely $Spin(7)$ and $SO(8)$. The standard $SU(3)$, standard $G_2$, and spin $Spin(7)$ actions have no nontrivial real representations of dimension smaller than $6, 7, 7$, respectively. These facts also follow directly from the low-degree highest-weight tables in [8, Appendix]. The orthogonal, unitary, and quaternionic standard actions are sphere-transitive. The exceptional standard actions have the familiar homogeneous-space descriptions $G_2/SU(3) = \mathbb{S}^6$ and $Spin(7)/G_2 = \mathbb{S}^7$.

Here is more detail on the exhaustion behind this small table. Split a compact Lie algebra into its center and simple ideals, and use the real, complex, and quaternionic types of its irreducible module. The simple low-degree possibilities, besides realifications of the classical standard modules,

are the odd-dimensional $SO(3)$ modules, the eight-dimensional real quaternionic $SU(2)$ module, the adjoint $SU(3)$ module of real dimension eight, the standard $G_2$ module of dimension seven, and the spin $Spin(7)$ module of dimension eight. The $SO(3)$ and $SU(2)$ images have Lie algebra dimension three, and the adjoint $SU(3)$ image has Lie algebra dimension eight. They are below the relevant dimension thresholds in the table. Here the quaternionic $SU(2)$ module has complex degree four and real dimension eight; complex degree is not substituted for real dimension in this check. For nonsimple images, the low-dimensional real tensor possibilities $2 \otimes 3$, $2 \otimes 4$, and $3 \otimes 3$ have dimensions of their acting groups at most $4, 7, 6$, respectively. The quaternionic tensor case above these thresholds is $Sp(2)Sp(1)$ on $\mathbb{H}^2$; $Sp(2)$ already acts transitively on its unit sphere. The possible scalar central factors give exactly the unitary and $Sp(2)U(1)$ extensions listed above. The $Spin(7)$ and $G_2$ actions are the standard exceptional transitive sphere actions. Thus the dimension bounds in the table exclude all nontransitive cases; mere containment in a transitive maximal subgroup is not being used to assert transitivity of an arbitrary subgroup. □

For use in lifting group factors, we also need the following elementary estimate.

**Lemma 9.4** (A small real representation bound)**.** *For $m \geq 5$, every nontrivial real representation of $\mathfrak{su}(m)$ has dimension at least $2m$.*

*Proof.* The Weyl dimension formula implies that every nontrivial complex irreducible representation other than the standard representation and its dual has dimension at least $\binom{m}{2} \geq 2m$. Here is the elementary highest-weight check. Dimension is nondecreasing in each fundamental weight coefficient. An interior fundamental weight has dimension $\binom{m}{j} \geq \binom{m}{2}$, while the remaining nonstandard possibilities dominate $2\omega_1$, $2\omega_{m-1}$, or $\omega_1 + \omega_{m-1}$, of dimensions $m(m+1)/2$, $m(m+1)/2$, and $m^2 - 1$, respectively. In the complexification of a real representation, the standard representation and its distinct conjugate dual occur with equal multiplicity. Their joint real dimension is at least $2m$. □

$$D(n) - g(n) = \frac{(n-24)(n+1)}{100} > 0 \qquad (n \geq 26). \tag{9.1}$$

# 10 Reflection symmetries and planar rotations

The following argument applies to reflections in the full automorphism group, including reflections outside its identity component.

**Lemma 10.1.** *Let $T$ be a compact integral operator with nonnegative bounded kernel on $L^2$ of a finite measure space. Suppose the kernel of $T^2$ is strictly positive almost everywhere. Two strictly positive eigenfunctions of $T$ with the same positive eigenvalue are proportional.*

*Proof.* The assertion is immediate if the $L^2$ space is one dimensional. Otherwise put $S = T^2$. Strict positivity of its kernel makes $S$ positivity improving: $Sf > 0$ almost everywhere whenever $0 \neq f \geq 0$. In particular $S$ is irreducible as a positive operator on the Banach lattice $L^2$: no proper nonzero closed ideal, represented by functions supported on a proper measurable subset, can be invariant. The operator $S$ is compact. The Jentzsch–Krein–Rutman theorem therefore applies in exactly this setting; see [3, Section 6, Theorem 16]. It says that the only eigenvalue admitting a positive eigenfunction is the positive spectral radius, and its eigenspace is one dimensional. If $Tf = \lambda f$ with $\lambda > 0$ and $f > 0$ almost everywhere, then $Sf = \lambda^2 f$. The two eigenfunctions in the assertion must consequently belong to the same one-dimensional eigenspace of $S$. □

**Lemma 10.2** (Two orthogonal reflections)**.** *If a fixed body in inertia position is invariant under the hyperplane reflections with two orthogonal normals, it is invariant under all rotations in the plane of those normals.*

*Proof.* Choose orthonormal coordinates in which the two normals are $e_i, e_j$, and let $R_i, R_j$ be their reflections. Put $D = \{u \in \mathbb{S}^{n-1} : u_i, u_j > 0\}$ and $\mu_D = 4\mu|_D$. The doubly odd restriction of $Q$ has kernel

$$k(u,v) = \tfrac{1}{4}\big(q(u,v) - q(u,R_i v) - q(u,R_j v) + q(u,R_i R_j v)\big)$$

relative to $\mu_D$. Inertia-position strict convexity and reflection symmetry imply $x_i(v), x_j(v) > 0$ on $D$. With $a = u_i x_i(v) > 0$, $b = u_j x_j(v) > 0$, and $z = \sum_{\ell \neq i,j} u_\ell x_\ell(v)$, this kernel is a positive factor times

$$|z + a + b| - |z - a + b| - |z + a - b| + |z - a - b|.$$

The expression is nonnegative, and is positive when $|z| < a + b$: it equals the integral of $2\delta_0(z + s + t)$ over $[-a, a] \times [-b, b]$.

For $v_0 = (e_i + e_j)/\sqrt{2}$, the automorphism $-R_i R_j$ fixes $v_0$ and negates its orthogonal complement outside the $(i, j)$ plane. Equivariance of the gradient therefore gives $x_\ell(v_0) = 0$ for $\ell \neq i, j$. Thus $k(u, v_0) > 0$ and $k(v_0, w) > 0$ for all $u, w \in D$. Continuity and full support imply strict positivity of the kernel of $T^2$, where $T$ is this restricted operator. The two functions

$$F_{E_{ij}} = \frac{u_i x_j(u)}{h(u)}, \qquad F_{E_{ji}} = \frac{u_j x_i(u)}{h(u)}$$

are positive on $D$ and have eigenvalue $1/(n+1)$. Lemma 10.1 makes them proportional, with positive ratio $a$. Odd extension yields $F_{E_{ij} - aE_{ji}} = 0$ on the whole sphere. Lemma 6.1 makes this matrix an infinitesimal automorphism, and skew-symmetry forces $a = 1$. We obtain all rotations in the $(i, j)$ plane. □

**Corollary 10.3** (Unconditional fixed bodies)**.** *A fixed body in inertia position invariant under all coordinate hyperplane reflections is a ball.*

*Proof.* Lemma 10.2 gives every coordinate-plane rotation; these generate $SO(n)$. □

## 11 The canonical rotation subspace

**Proposition 11.1** (An independent large rotation block)**.** *Suppose $n \geq 26$ and $K$ satisfies* (1.2)*, in inertia position. Then $K$ is a ball, or there is a subspace $V$ of dimension $p > n/2$ such that $O(V) \times \{\mathrm{Id}_{V^\perp}\} \subset \mathrm{Aut}(K)$.*

*Proof.* Write $G = \mathrm{Aut}(K)$, $G^0 = H$, $d = \dim H$, and $g(t) = t^2/4 - 3t/4$. Theorem 6.3 and (9.1) give $d > g(n)$. Decompose the orthogonal $H$-representation into real irreducible subspaces of dimensions $p_1, \ldots, p_s$. Faithfulness gives

$$d \leq \sum_i \frac{p_i(p_i - 1)}{2}.$$

Suppose first that $p = \max p_i > n/2$, and put $q = n - p$. The image $L$ of $H$ on this subspace $V$ is connected and irreducible, and

$$\dim L \geq d - \frac{q(q-1)}{2} > g(p), \qquad g(n) - \frac{q(q-1)}{2} - g(p) = \frac{q(2p - q - 1)}{4} \geq 0.$$

Here $p \geq 14$, so Lemma 9.1 gives either $L = SO(p)$ or $p = 2m$ and $SU(m) \subset L \subset U(m)$. In the first case the simple Lie algebra $\mathfrak{so}(p)$ lifts to a simple ideal of $\mathfrak{h}$; in the second case the same is true of $\mathfrak{su}(m)$. To see the lifting assertion, split a compact Lie algebra into its center and simple ideals; a simple ideal in a quotient is the isomorphic image of one of these simple ideals. In the orthogonal case its action on $V^\perp$ must be zero, since a nonzero homomorphism would be injective but $\dim \mathfrak{so}(p) > \dim \mathfrak{so}(q)$. In the unitary case its action on $V^\perp$ is zero by Lemma 9.4, because $m \geq 7$ and $q < p = 2m$. Thus a group transitive on the sphere in $V$ acts independently of $V^\perp$. Its vector orbits are the $O(V)$-orbits, so membership in $K$ is invariant under independent $O(V)$ rotations.

It remains to treat $p_i \leq n/2$ for every $i$. The convex function $a \mapsto a(a-1)/2$ maximizes the displayed sum by making the allowed integer parts as unequal as possible. If $n = 2m+1$, the maximum is $m(m-1) < g(n)$. If $n = 2m$ and all $p_i \leq m-1$, the maximum is $(m-1)(m-2)+1 < g(n)$. If one part equals $m$ and its complement is reducible, the maximum is $(m-1)^2 < g(2m)$. These inequalities hold for $m \geq 13$. Hence the only remaining possibility is $n = 2m$, $m \geq 13$, with two irreducible $m$-dimensional subspaces. If either projection of $H \subset SO(m) \times SO(m)$ is proper, Lemma 9.2 gives $d \leq (m-1)^2 < g(2m)$. If both projections are onto, simplicity and the Lie-algebra form of Goursat's lemma give either a graph of an automorphism of $\mathfrak{so}(m)$, or the full direct sum. The graph has dimension $m(m-1)/2 < g(2m)$ and is impossible. The full direct sum yields two independent rotation blocks, hence all coordinate reflections in an adapted basis. Lemma 10.3 makes $K$ a ball. □

**Theorem 11.2** (Canonical rotation normal form)**.** *Let $n \geq 26$ and suppose $K$ is a nonball fixed body in inertia position. Let $W$ be the span of the normals of all hyperplane reflections preserving $K$, and let $H_* = W^\perp$. Then*

$$p := \dim W > n/2, \qquad k := \dim H_* = n - p \geq 2, \qquad G = O(W) \times G_*, \tag{11.1}$$

*where $G = \mathrm{Aut}(K)$, $G_* \subset O(H_*)$ contains $-\mathrm{Id}_{H_*}$ and contains no hyperplane reflection. In particular $W$ is $G$-invariant. There are an origin-symmetric convex body $D \subset H_*$ and an even concave function $r : D \to [0, \infty)$, positive on its interior, such that*

$$K = \{(y, z) \in W \oplus H_* : z \in D,\ |y| \leq r(z)\}. \tag{11.2}$$

*The function $r$ is continuous on $D$ and vanishes on $\partial D$.*

*Proof.* Proposition 11.1 supplies an independent rotation subspace $E$ of dimension greater than $n/2$. We show that any additional reflection normal $v \notin E$ enlarges it. Choose a unit vector $e \in E \cap v^\perp$; this is possible since $\dim E \geq 2$. The reflections with normals $e$ and $v$ preserve $K$. Lemma 10.2 gives the plane rotation generator $e \wedge v$. Subtracting the generator $e \wedge P_E v$ already in $\mathfrak{so}(E)$ gives $e \wedge P_{E^\perp} v$. Brackets with $\mathfrak{so}(E)$ generate $\mathfrak{so}(E \oplus \mathbb{R}P_{E^\perp} v)$. The existing reflection with normal $e$ then supplies the full independent orthogonal group of this enlarged subspace. Iterating over a finite spanning set of reflection normals proves $O(W) \times \{\mathrm{Id}\} \subset G$.

Conjugation carries a preserving reflection to a preserving reflection, so $G$ preserves $W$ and $H_*$. Cancelling the $O(W)$ component of each $g \in G$ proves the direct-product assertion. Central symmetry gives $-\mathrm{Id}_{H_*} \in G_*$. A reflection in $G_*$ would have a normal both in $H_*$ and in $W$, which is impossible. If $k = 1$, then $-\mathrm{Id}_{H_*}$ is just such a reflection. If $k = 0$, invariance under $O(n)$ makes $K$ a ball. Thus $k \geq 2$.

Independent $O(W)$ invariance makes every fiber a centered ball. Convexity of $K$ gives concavity of its radius $r$, and central symmetry gives evenness. Interior continuity follows from concavity. At a boundary point of $D$ the fiber lies in a supporting face of $K$; strict convexity of $K$ forces its radius to vanish. Compactness of $K$ then gives continuity at those boundary points as well. □

**Corollary 11.3.** *For $n \geq 26$, if the full automorphism group of a fixed body is irreducible, the body is an ellipsoid.*

*Proof.* In inertia position, Theorem 11.2 would otherwise give a proper, nonzero invariant subspace $W$. □

## 12 The complementary action and the two-thirds bound

We say that a compact connected group $H \subset SO(k)$ has an *independent sphere-transitive subspace* if there is an orthogonal decomposition $\mathbb{R}^k = E \oplus E^\perp$, with $\dim E \geq 2$, and a subgroup of $H$ fixing $E^\perp$ pointwise and transitive on the unit sphere of $E$.

**Lemma 12.1.** *Suppose $H \subset SO(k)$ is closed and connected and has no independent sphere-transitive subspace. If $k \geq 10$, then $\dim H \leq g(k)$. If $k = 9$, then $\dim H \leq 9$.*

*Proof.* First suppose $k \geq 10$ and, for a contradiction, $d_H := \dim H > g(k)$. Decompose its real representation into irreducible summands of dimensions $p_i$, and put $p = \max p_i$. Faithfulness gives $d_H \leq \sum_i p_i(p_i - 1)/2$.

If every $p_i \leq k/2$, the elementary partition estimate used in Proposition 11.1 leaves only $k = 2m$ and two $m$-dimensional summands. This estimate holds already for $m \geq 5$: for odd $k = 2m + 1$ the maximum is $m(m-1) < g(k)$; for even $k = 2m$, partitions without two irreducible halves have sum at most $\max\{(m-1)^2, (m-1)(m-2)+1\} < g(k)$. If either projection of $H \subset SO(m) \times SO(m)$ is proper, Lemma 9.2 gives $d_H \leq (m-1)^2 < g(2m)$. If both are onto, simplicity of $\mathfrak{so}(m)$ and Goursat's lemma give a graph or the full direct sum. The graph is too small; the direct sum supplies an independent sphere-transitive subspace.

We may therefore assume $p > k/2$, and put $q = k - p < p$. The image $L$ on this irreducible summand satisfies

$$\dim L \geq d_H - \frac{q(q-1)}{2} > g(p) + \frac{q(2p-q-1)}{4}. \tag{12.1}$$

If $p \geq 10$, Lemma 9.1 gives $SO(p)$ or a group containing $SU(p/2)$. Lift the corresponding simple ideal to $\mathfrak{h}$. In the orthogonal case its action on the $q$-dimensional complement is zero by the dimension comparison with $\mathfrak{so}(q)$. In the unitary case it is zero by Lemma 9.4. This gives an independent sphere-transitive action, a contradiction.

The remaining possibilities have $6 \leq p \leq 9$. For $p = 6$, (12.1) and Lemma 9.3 give $SO(6)$ or a group containing $SU(3)$; the corresponding simple factor has no nontrivial representation on a complement of dimension $q < 6$. For $p = 7$, the alternatives are $SO(7)$ and $G_2$, and the same argument applies since their smallest nontrivial real representations have dimension seven. For $p = 9$, Lemma 9.3 forces $L = SO(9)$. For $p = 8$, we have $q \geq 2$ because $k \geq 10$. If $q = 2$ or 3, (12.1) gives $\dim L > 16$ or $\dim L > 19$, respectively; hence $L = Spin(7)$ or $SO(8)$. Their relevant simple ideals act trivially on the complement. If $q \geq 4$, the same bound gives $\dim L > 21$, so $L = SO(8)$. Every case again gives the forbidden independent action. This proves the assertion for $k \geq 10$.

Now let $k = 9$ and suppose $d_H \geq 10$. If the action is irreducible, Lemma 9.3 gives $SO(9)$. Otherwise let $p$ be the largest irreducible dimension. For $p = 8$, the complement is one dimensional, and the image has dimension at least ten; every group in the corresponding row of Lemma 9.3 is sphere-transitive, acting independently of that complement. For $p = 7$, the image has dimension at least nine, and its $G_2$ or $SO(7)$ simple factor acts trivially on the two-dimensional complement. For $p = 6$, the image has dimension at least seven, and its $SU(3)$ or $SO(6)$ simple factor acts trivially on the three-dimensional complement. For $p = 5$, the image has dimension at least four, and must be $SO(5)$; it cannot act nontrivially on a four-dimensional complement. All these cases are impossible.

Finally suppose every irreducible summand has dimension at most four. The only partitions whose ambient block algebra has dimension at least ten are $4+4+1$ and $4+3+2$. For $4+3+2$, dimension at least ten forces the full block algebra, giving an independent action. For $4+4+1$, if one projection to $\mathfrak{so}(4)$ is proper, its dimension is at most four. Dimension at least ten therefore forces a full product of a four-dimensional projection and the other $\mathfrak{so}(4)$. If both projections are onto, use $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Its quotients have dimensions $0, 3, 6$, so a subdirect product of two copies has dimension $12, 9$, or $6$. Dimension at least ten forces dimension twelve and the full product. Again an independent sphere-transitive action is present. This proves $\dim H \le 9$. □

**Proposition 12.2.** *For a nonball fixed body in inertia position with $n \ge 26$, the canonical rotation subspace in Theorem 11.2 has*

$$p \ge \lceil 2n/3 \rceil, \qquad p \ge 19 \quad \text{when } n = 27. \tag{12.2}$$

*Proof.* Set $H = G_*^0 \subset SO(k)$. It has no independent sphere-transitive subspace. Indeed, such a subgroup has exactly the same point orbits as the independent orthogonal group on that subspace; invariance of the profile body under it would give a hyperplane reflection in $G_*$, contrary to Theorem 11.2.

If $k \le 8$, then $p = n - k \ge n - 8 \ge 2n/3$. If $k \ge 9$, Lemma 12.1 gives $\dim G_* \le g(k)$, including $k = 9$ since $9 < g(9)$. Consequently

$$\begin{aligned} d - D(n) &\le \frac{p(p-1)}{2} + g(n-p) - D(n) \\ &= \frac{75p^2 - 50np - n^2 + 25p + 23n + 24}{100}. \end{aligned} \tag{12.3}$$

The last polynomial is increasing in $p$ for $p > n/2$. At $p = (2n-1)/3$ its value is $-(n-24)(n+1)/100 < 0$. Since Theorem 6.3 gives $d > D(n)$, we have $p > (2n-1)/3$. Integrality gives $p \ge \lceil 2n/3 \rceil$. If $n = 27$ and $p = 18$, then $k = 9$ and $d \le 18 \cdot 17/2 + 9 = 162$, whereas $D(27) = 4071/25 > 162$. Thus $p \ge 19$ in that dimension. □

## 13 Dimension-free curvature of a centroid support norm

For an even random vector $X$ with a full-dimensional log-concave density and covariance $\Sigma$, write $N_X(u) = \mathbb{E}|\langle u, X\rangle|$.

**Lemma 13.1** (The planar uniform case)**.** *If $C$ is a symmetric planar convex body and $X$ is uniform on $C$, then*

$$u \longmapsto \frac{1}{2} N_X(u)^2 - \frac{77}{400} u^{\mathsf{T}} \Sigma u \quad \text{is convex.}$$

*Equivalently, $D^2(N_X^2/2) \succeq (77/200)\Sigma$ away from zero.*

*Proof.* The assertion is affine invariant, so take $\Sigma = \tau \mathrm{Id}$. At a unit vector $u$ and an orthogonal unit vector $v$, put $h = N_X(u)$, $a = DN_X(u)[v]$, $b = \rho_C(v)$, and $V_C = |C|$. Slicing gives

$$q := hD^2 N_X(u)[v, v] = \frac{4hb^3}{3V_C}.$$

Berwald's inequality [2] on the half-body $C \cap \{\langle u, x\rangle \ge 0\}$, in dimension two with exponents one and two, gives $3h \ge \sqrt{6\tau}$, so $h^2 \ge 2\tau/3$. Every even unimodal one-dimensional law is a mixture of

uniform laws on centered intervals. Consequently, for such a law of variance $\tau$ its mean absolute value is at most $\sqrt{3\tau}/2$. Applying this to all marginals of $C$ gives

$$h^2 + a^2 = |\nabla N_X(u)|^2 \le 3\tau/4.$$

Indeed $|N_X(w)| \le \sqrt{3\tau}/2$ on the unit circle, and every gradient is a point of the corresponding support body. The same uniform-mixture representation, for a marginal supported on $[-H, H]$, gives $\tau \le H^2/3$. Hence every support value of $C$ is at least $\sqrt{3\tau}$, and $b^2 \ge 3\tau$. The density of $\langle u, X\rangle$ is maximal at zero, where it equals $2b/V_C$. Thus the density of $|\langle u, X\rangle|$ is at most $4b/V_C$. A nonnegative variable with density at most $M$ has mean at least $1/(2M)$, by integrating $\mathbb{P}(Y > t) \ge (1 - Mt)_+$. It follows that $h \ge V_C/(8b)$, and therefore $q \ge b^2/6 \ge \tau/2$.

In the basis $(u, v)$, the Hessian of $N_X^2/2$ is

$$\begin{pmatrix} h^2 & ha \\ ha & a^2 + q \end{pmatrix}.$$

Subtract $t\tau\mathrm{Id}$, with $t = 77/200$. Its first principal minor is positive since $h^2 \ge 2\tau/3$. Writing its determinant as

$$(h^2 - t\tau)q - t\tau(h^2 + a^2) + t^2\tau^2,$$

and using $q \ge \tau/2$ and $h^2 + a^2 \le 3\tau/4$, bounds it below by

$$\tau^2\left(\frac{1}{3} - \frac{5}{4}t + t^2\right) = \frac{37\tau^2}{120000} > 0.$$

This proves the Hessian bound; continuity and homogeneity extend the convexity assertion through the origin. □

**Theorem 13.2** (A universal curvature constant)**.** *For every even full-dimensional log-concave random vector $X$,*

$$u \longmapsto \frac{1}{2}N_X(u)^2 - \frac{\gamma}{2}u^{\mathsf{T}}\Sigma u \quad \textit{is convex}, \qquad \gamma = \frac{77\,6^{4/3}}{4800} > \frac{87}{500}. \tag{13.1}$$

*Proof.* First let $g$ be an even planar log-concave probability density and $g_0 = g(0)$. Define the Ball bodies $C_j$, for $j = 2, 3, 4$, by

$$\rho_{C_j}(\theta)^j = \frac{j}{g_0}\int_0^\infty s^{j-1}g(s\theta)\,\mathrm{d}s.$$

Ball's convexity theorem makes $C = C_3$ a symmetric convex body; see [1] and the normalization in [6, Section 2.2]. For completeness the moment comparisons used below can be seen directly on a ray. Put $f(s) = g(s\theta)/g_0$. It is nonincreasing, log-concave, and takes values in $[0, 1]$ with $f(0) = 1$. Writing $f$ by its level sets shows that $\rho_{C_j}$ is the $L^j(0, 1)$ norm of their lengths. Thus $\rho_{C_3} \ge \rho_{C_2}$. Match $\int s^2 f(s)\,\mathrm{d}s$ with $\int s^2 e^{-as}\,\mathrm{d}s$. Concavity of $\log f(s) + as$, starting from zero, implies that $f - e^{-as}$ has at most one sign change, from positive to negative. Equality of the second weighted moments then gives $\int s^3 f \le \int s^3 e^{-as}$, and hence

$$\rho_{C_4}^4 \le M\rho_{C_3}^4, \qquad M = \frac{24}{6^{4/3}}. \tag{13.2}$$

Polar integration gives

$$g_0|C_2| = 1, \qquad \alpha := g_0|C| \ge 1, \qquad N_g = \alpha N_C, \qquad \Sigma_g \preceq \alpha M\Sigma_C.$$

Lemma 13.1 implies

$$D^2(N_g^2/2) \succeq \alpha^2 \frac{77}{200}\Sigma_C \succeq \frac{77\alpha}{200M}\Sigma_g \succeq \gamma\Sigma_g.$$

These statements can equivalently be read as convexity statements, so they require no additional smoothness of $g$. For arbitrary dimension, restrict the proposed convex function to any two-dimensional linear subspace and use the corresponding log-concave marginal of $X$ [5]. Every affine line lies in such a subspace, so convexity follows. The one-dimensional case follows either directly or by adjoining an independent even log-concave coordinate. Finally $6^4 > (8352/770)^3$ proves the displayed strict rational bound. □

**Corollary 13.3** (An upper Fisher bound for a fixed body)**.** *For a fixed body in inertia position, set*

$$\Phi(x) = \tfrac{1}{2}\|x\|_K^2, \qquad \nu_\Phi(\,\mathrm{d}x) = \frac{e^{-\Phi(x)}\,\mathrm{d}x}{\int e^{-\Phi}}, \qquad J_\Phi = \mathbb{E}_{\nu_\Phi}\nabla\Phi\nabla\Phi^{\mathsf{T}}.$$

*Then*

$$J_\Phi = \mathbb{E}_{\nu_\Phi} D^2\Phi = n\int \frac{uu^{\mathsf{T}}}{h(u)^2}\mu(\,\mathrm{d}u) \preceq \frac{c^2}{\gamma\tau}\mathrm{Id}. \tag{13.3}$$

*Proof.* The convex conjugate of $\Phi$ is $\Psi = h^2/2$. Since $N_U = ch$, Theorem 13.2 gives $D^2\Psi \succeq \gamma\tau c^{-2}\mathrm{Id}$. Convex duality therefore gives $D^2\Phi \preceq c^2/(\gamma\tau)\mathrm{Id}$. Integration by parts proves the first equality in (13.3). It is justified by the Gaussian radial tails and the globally Lipschitz gradient of $\Phi$; the origin is a null set. For the middle equality write $x = tx(u)$ in cone coordinates. Then $\nabla\Phi(tx(u)) = tu/h(u)$, and the radial density is proportional to $t^{n-1}e^{-t^2/2}$, whose second moment is $n$. □

# 14 Regression with log-concave survival

**Lemma 14.1** (Clipped linear regression)**.** *Let $Z > 0$ almost surely have an absolutely continuous distribution, log-concave survival function, and finite second moment. For every $t \geq 0$,*

$$\inf_{\lambda\in\mathbb{R}} \mathbb{E}(\min(Z,t) - \lambda Z)^2 \leq \frac{1}{14}\mathbb{E}Z^2. \tag{14.1}$$

*If $g : [0,\infty) \to [0,\infty)$ is increasing, concave, L-Lipschitz, and $g(0) = 0$, then*

$$\inf_{\lambda\in\mathbb{R}} \mathbb{E}(g(Z) - \lambda Z)^2 \leq \frac{L^2}{14}\mathbb{E}Z^2. \tag{14.2}$$

*Proof.* The clipping assertion is immediate for $t = 0$ or $s := \mathbb{P}(Z > t) = 0$. Otherwise define

$$m = \mathbb{E}[Z - t \mid Z > t], \quad q = \frac{\mathbb{E}[(Z-t)^2 \mid Z > t]}{m^2}, \quad r = t/m, \quad x = r/q, \quad v = \frac{\mathbb{E}Z^2}{qsm^2}.$$

The residual survival $T$ is log-concave with $T(0) = 1$, hence $T(a+b) \leq T(a)T(b)$. Integrating proves $\mathbb{E}[(Z-t)^2 \mid Z > t] \leq 2m^2$, so $1 \leq q \leq 2$. Write $S(z) = \mathbb{P}(Z > z)$ and let $h_t = -(\log S)'_+(t)$ be its right hazard. Since $s > 0$, $t$ is before the upper endpoint of the support. One-sided derivatives suffice; no differentiability of the hazard is assumed. The increasing-hazard property gives $m \leq 1/h_t$, and hence $h_t \leq 1/m$. If $h_t = 0$, the latter inequality holds directly and the reciprocal bound is interpreted as $m \leq +\infty$. For $0 \leq z \leq t$ it follows that

$$S(z) \leq \min\{1, se^{(t-z)/m}\}.$$

In particular $s \ge e^{-r}$. Put $a = r + \log s \in [0, r]$. The displayed upper survival is that of $m(a + \xi)$ up to $t$, where $\xi$ is a mean-one exponential variable. The actual tail contributes $sm^2(2r + q)$ to the second moment, whereas this shifted exponential contributes $sm^2(2r + 2)$. Thus

$$\mathbb{E}Z^2 \le m^2\{a^2 + 2a + 2 - (2 - q)s\}.$$

Since $a \mapsto (a^2 + 2a + 2)e^{-a}$ has derivative $-a^2e^{-a}$, we obtain

$$v \le 1 + \frac{2(e^r - 1)}{q} \le e^{2x}. \tag{14.3}$$

The last step uses that $(e^{qx} - 1)/q$ is increasing in $q > 0$.

Orthogonal projection onto the span of $Z$, using $D = (Z - t)_+$, gives the exact identity

$$\frac{\inf_\lambda \mathbb{E}(\min(Z, t) - \lambda Z)^2}{\mathbb{E}Z^2} = \frac{1}{v} - \frac{(1 + x)^2}{v^2}. \tag{14.4}$$

If $x \ge 1$, its maximum over $v > 0$ is at most $1/16 < 1/14$. If $0 \le x \le 1$, then $e^{2x} < 2(1 + x)^2$, since the ratio $e^{2x}/(1 + x)^2$ is increasing and $e^2 < 8$. The right side of (14.4) is increasing for $0 < v \le 2(1 + x)^2$. By (14.3) it is therefore at most

$$e^{-2x} - (1 + x)^2e^{-4x} < \frac{1}{14}.$$

The last strict inequality follows, with $y = 2x$, from

$$e^{2y} - 14e^y + \frac{7}{2}(y + 2)^2 > 0 \qquad (y \ge 0). \tag{14.5}$$

Appendix B gives an exact positive polynomial certificate for this elementary inequality.

Finally represent

$$g(y) = by + \int_0^\infty \min(y, t)\, \xi(\,\mathrm{d}t), \qquad b \ge 0, \quad \xi \ge 0, \quad b + \xi((0, \infty)) \le L.$$

This is the Stieltjes representation of the decreasing derivative of $g$. Project orthogonally off the span of $Z$ and apply Minkowski's inequality to the integral. The linear term vanishes, and (14.1) gives (14.2). □

**Corollary 14.2.** *If $B$ has an even log-concave density and $g$ is odd, increasing, concave on $[0, \infty)$, and $L$-Lipschitz, then*

$$\inf_\lambda \mathbb{E}(g(B) - \lambda B)^2 \le \frac{L^2}{14}\mathbb{E}B^2.$$

*Proof.* The survival function of $|B|$ is log-concave, as in Lemma 7.1; apply Lemma 14.1 to $|B|$. □

## 15 Weighted concentration of the radius of a rotation block

Suppose an origin-symmetric convex body has the form (11.2), with $p \ge 2$, $k \ge 1$, and $n = p + k$. For $U = (Y, Z)$ uniform on it write $Y = R\theta$, where $\theta$ is independent and uniform on $\mathbb{S}^{p-1}$.

**Lemma 15.1.** *For $B = \langle b, Z\rangle$, $b \ne 0$, define the tilted expectation $\mathbb{E}_w F = \mathbb{E}[B^2F]/\mathbb{E}B^2$, and put $R_* = \mathbb{E}_w R$. Then*

$$\frac{\mathrm{Var}_w R}{R_*^2} \le \varepsilon^2 := \frac{k + 3}{p(n + 4)}. \tag{15.1}$$

*Also the law of $B$ under the different tilt $\mathbb{E}_2 F = \mathbb{E}[R^2F]/\mathbb{E}R^2$ has an even log-concave density.*

*Proof.* The joint density of $(R, Z)$ is proportional to $s^{p-1}\mathbf{1}_{\{0<s\le r(z)\}}$. Set

$$m(s) = \int_{\{z:r(z)\ge s\}} \langle b, z\rangle^2 \, \mathrm{d}z.$$

This function is $1/(k+2)$-concave on its support. Indeed lift to the convex sets

$$E_s = \{(z, t_1, t_2) : r(z) \ge s,\ t_1, t_2 \ge 0,\ t_1 + t_2 \le \langle b, z\rangle\}.$$

Central symmetry and the area of the triangular fiber give $m(s) = 4|E_s|$; convexity of the joint set with coordinate $s$ and Brunn–Minkowski prove the concavity assertion. The normalized Mellin theorem [7, Proposition 3.2] says that

$$a \longmapsto \frac{\int_0^\infty s^{a-1} m(s)\, \mathrm{d}s}{\mathrm{B}(a, k+3)} \quad \text{is log-concave on } (0,\infty).$$

At $a = p, p+1, p+2$ this yields

$$\frac{\mathbb{E}_w R^2}{(\mathbb{E}_w R)^2} \le \frac{\mathrm{B}(p+2, k+3)\mathrm{B}(p, k+3)}{\mathrm{B}(p+1, k+3)^2} = \frac{(p+1)(n+3)}{p(n+4)} = 1 + \varepsilon^2.$$

For the second assertion, the $R^2$-tilted joint density is proportional to $s^{p+1}$ on the convex set $\{(s, z) : 0 < s \le r(z)\}$. It is log-concave. Integration in $s$ and then projection onto $B$ preserve log-concavity by Prékopa's theorem; see [5]. The evenness follows from $r(z) = r(-z)$. □

## 16 Cross-block regression and a quantitative rigidity criterion

For $p \ge 2$ define

$$\beta_p = \mathbb{E}_{\mathbb{S}^{p-1}}|\theta_1| = \frac{\Gamma(p/2)}{\sqrt{\pi}\,\Gamma((p+1)/2)}, \qquad \Delta_{n,p,k} = \left(\varepsilon + \sqrt{\frac{1+\varepsilon^2}{14}}\right)^2, \quad \varepsilon^2 = \frac{k+3}{p(n+4)}. \tag{16.1}$$

**Theorem 16.1** (Necessary condition for a proper rotation block)**.** *Suppose $K$ is a fixed body in inertia position, $\mathbb{R}^n = W \oplus H_*$ is preserved by its full automorphism group, and $O(W) \times \{\mathrm{Id}\} \subset \mathrm{Aut}(K)$. If $p = \dim W \ge 2$, $k = \dim H_* \ge 1$, then*

$$\Delta_{n,p,k}(n+2)\beta_p^2 \ge \gamma. \tag{16.2}$$

*In particular, strict failure of this inequality excludes such a proper invariant rotation block.*

*Proof.* For a $p$-by-$k$ matrix $M$ put

$$A_M = \begin{pmatrix} 0 & M \\ -M^{\mathsf{T}} & 0 \end{pmatrix}, \qquad \mathcal{R}_\times = \{F_{A_M} : M \in \mathbb{R}^{p\times k}\}.$$

Every infinitesimal automorphism preserves $W$, so Lemma 6.1 makes $M \mapsto F_{A_M}$ injective. Thus $\dim \mathcal{R}_\times = pk$. These functions have eigenvalue $1/(n+1)$ and are orthogonal to all boundary quadratics by Lemma 6.2. They belong to the first spherical harmonic in the $W$ variable: equivariance makes $x_W(u)$ parallel to $u_W$, and $x_{H_*}(u)$ and $h(u)$ depend on $u_W$ only through $|u_W|$.

Fix a row $u = (a\eta, b)$ with $a = |u_W|$ and $\eta \in \mathbb{S}^{p-1}$. For a source point $x = (R\theta, Z)$ let $B = \langle b, Z\rangle$ and define

$$\psi_p(A, B) = \mathbb{E}_\theta[\theta_1 |A\theta_1 + B|].$$

The first $W$ harmonic of $|\langle u, x\rangle|$ is $f_u(x) = p\psi_p(aR, B)\langle \eta, \theta\rangle$. For $A > 0$, direct integration in the first spherical coordinate gives

$$\partial_B \psi_p(A, B) = \beta_p \left(1 - \frac{B^2}{A^2}\right)_+^{(p-1)/2}, \qquad \psi_p(A, 0) = 0.$$

Indeed, the density of $\theta_1$ is $c_p(1-t^2)^{(p-3)/2}$ on $(-1, 1)$, where $c_p = \Gamma(p/2)/(\sqrt{\pi}\,\Gamma((p-1)/2))$. For $0 < B < A$,

$$\partial_B \psi_p(A, B) = 2c_p \int_{B/A}^1 t(1-t^2)^{(p-3)/2}\,\mathrm{d}t = \frac{2c_p}{p-1}(1 - B^2/A^2)^{(p-1)/2}.$$

Here $2c_p/(p-1) = \beta_p$; for $B \geq A$ the derivative is zero. Symmetry gives the formula at negative $B$ as well. Hence $B \mapsto \psi_p(A, B)$ is odd, increasing, concave for $B \geq 0$, and $\beta_p$-Lipschitz. Also $\psi_p$ is one-homogeneous in $(A, B)$. The case $a = 0$ has vanishing first harmonic and needs no estimate; the case $b = 0$ likewise vanishes. Assume both are nonzero.

Use $R_*$ from Lemma 15.1 and write $\sigma^2 = \mathrm{Var}_w(R)/R_*^2 \leq \varepsilon^2$. Homogeneity and Lipschitz continuity give

$$\left|\psi_p(aR, B) - \frac{R}{R_*}\psi_p(aR_*, B)\right| \leq \beta_p |B| \left|1 - \frac{R}{R_*}\right|. \tag{16.3}$$

Apply Corollary 14.2 to $g(B) = \psi_p(aR_*, B)$ under the $R^2$ tilt; its density is even and log-concave by Lemma 15.1. Choose a scalar $\ell$ attaining the best linear regression of $g(B)$ under that $R^2$-tilted law. Then

$$\begin{aligned}\mathbb{E}\left[\left(\frac{R}{R_*}(g(B) - \ell B)\right)^2\right] &\leq \frac{\beta_p^2}{14R_*^2}\mathbb{E}[R^2B^2] \\ &= \frac{\beta_p^2(1+\sigma^2)}{14}\mathbb{E}B^2.\end{aligned}$$

In the last equality we used the different, $B^2$-tilted law to write $\mathbb{E}[R^2B^2]/R_*^2 = (1+\sigma^2)\mathbb{E}B^2$. Also (16.3) gives

$$\mathbb{E}\left[\left(\psi_p(aR, B) - \frac{R}{R_*}g(B)\right)^2\right] \leq \beta_p^2\sigma^2\mathbb{E}B^2.$$

Minkowski's inequality, with the trial coefficient $\lambda = \ell/R_*$, now gives

$$\inf_\lambda \mathbb{E}\left[\left(\psi_p(aR, B) - \lambda RB\right)^2\right] \leq \beta_p^2\mathbb{E}B^2\left(\sigma + \sqrt{\frac{1+\sigma^2}{14}}\right)^2 \leq \beta_p^2\tau|b|^2\Delta_{n,p,k}. \tag{16.4}$$

Angular integration multiplies this error by $p$, since $\mathbb{E}_\theta\langle \eta, \theta\rangle^2 = 1/p$. Thus the homogeneous degree-one function $f_u$ has a quadratic approximant proportional to $P_u(x) = \langle \eta, Y\rangle\langle b, Z\rangle$, with uniform-interior squared error at most $p\beta_p^2\tau|b|^2\Delta_{n,p,k}$.

We transfer the optimal regression to the boundary carefully. For any degree-one $f$ and degree-two $P$, put $a_j = (n+j)/n$. Formula (8.1) gives

$$\begin{aligned}\inf_\lambda \int (f(x(v)) - \lambda P(x(v)))^2\,\mathrm{d}\mu(v) &= a_2\mathbb{E}f(U)^2 - \frac{a_3^2}{a_4}\frac{(\mathbb{E}f(U)P(U))^2}{\mathbb{E}P(U)^2} \\ &\leq a_2 \inf_\lambda \mathbb{E}(f(U) - \lambda P(U))^2,\end{aligned}$$

because $a_3^2/a_4 - a_2 = 1/[n(n+4)] > 0$. The cone measure is invariant under independent $O(W)$ rotations. Moreover $x_W(v)$ is a positive multiple of $v_W$ whenever $v_W \neq 0$: equivariance gives parallelism, and strict convexity and reflection symmetry give the sign. Thus the $W$ directions of the boundary point and its normal coincide outside a null set. Conditional on the rotation-invariant coordinates, this direction is uniform. Consequently the first-harmonic projection is orthogonal on $L^2(\mu)$ as well as for the interior law. All other $W$ harmonics in a row of $Q$ are orthogonal to $\mathcal{R}_\times$.

Let $e_1, \dots, e_{pk}$ be an orthonormal basis of $\mathcal{R}_\times$. Subtract the optimal boundary quadratic above from the first-harmonic part of each row of $Q$. Bessel's inequality, the kernel normalization (6.4), and (16.4) yield

$$\frac{1}{(n+1)^2} \sum_j e_j(u)^2 \le \frac{n(n+2)p\beta_p^2 \tau \Delta_{n,p,k}}{(n+1)^2 c^2} \frac{|u_{H_*}|^2}{h(u)^2}.$$

Integrate in $u$ and use the exact identity

$$\int \frac{|u_{H_*}|^2}{h(u)^2} \mu(\,\mathrm{d}u) = \frac{1}{n} \operatorname{tr}(P_{H_*} J_\Phi P_{H_*}).$$

Then apply (13.3):

$$\begin{aligned} \frac{pk}{(n+1)^2} &\le \frac{(n+2)p\beta_p^2 \tau \Delta_{n,p,k}}{(n+1)^2 c^2} \operatorname{tr}(P_{H_*} J_\Phi P_{H_*}) \\ &\le \frac{(n+2)p\beta_p^2 k \Delta_{n,p,k}}{(n+1)^2 \gamma}. \end{aligned}$$

Cancelling the positive factors proves (16.2). No self-adjointness or invariant-probability assertion for $Q$ is used. □

## 17 Excluding every proper rotation subspace

**Proposition 17.1.** *For every $n \ge 26$ and every integer $p$ satisfying* (12.2)*, with $k = n - p \ge 2$,*

$$\Delta_{n,p,k}(n+2)\beta_p^2 < \frac{87}{500} < \gamma. \tag{17.1}$$

*Proof.* Both $\beta_p$ and $\varepsilon^2 = (n-p+3)/(p(n+4))$ decrease with $p$. For $\beta_p$, couple the spherical vectors by normalizing the same independent standard Gaussian coordinates: the absolute first coordinate decreases when an extra coordinate is included in the normalizing Euclidean norm.

First suppose $n \ge 32$. By $p \ge 2n/3$,

$$\varepsilon^2 \le \frac{n+9}{2n(n+4)} \le \frac{41}{2304}, \qquad \varepsilon < \frac{267}{2000}, \qquad \sqrt{\frac{1+\varepsilon^2}{14}} < \frac{27}{100}.$$

The first rational function is decreasing for $n > 0$. Log-convexity of the Gamma function gives $\Gamma(p/2)^2 \le \Gamma((p-1)/2)\Gamma((p+1)/2)$, hence $\beta_p^2 \le 2/[\pi(p-1)]$. Since $(n+2)/(2n/3-1)$ is decreasing,

$$(n+2)\beta_p^2 \le \frac{2(n+2)}{\pi(2n/3-1)} \le \frac{204}{61\pi} < \frac{16}{15};$$

the last comparison follows already from $\pi > 157/50$. Consequently

$$\Delta_{n,p,k}(n+2)\beta_p^2 < \left(\frac{807}{2000}\right)^2 \frac{16}{15} = \frac{217083}{1250000} < \frac{87}{500}.$$

For $26 \le n \le 31$, monotonicity reduces the comparison to the six pairs in the table. Write $e = \varepsilon^2$. The fourth column is $10^9\alpha$, where $\alpha^2 > e(1+e)/14$. The last column gives a strict rational upper bound $U$ for $\Delta_{n,p,k}(n+2)\beta_p^2$. Every terminating decimal in this table is exact.

| $n$ | $p$ | $e$ | $10^9\alpha$ | $U$ |
|---|---|---|---|---|
| 26 | 18 | 11/540 | 38531363 | 0.173413 |
| 27 | 19 | 11/589 | 36863168 | 0.164766 |
| 28 | 19 | 3/152 | 37915682 | 0.173793 |
| 29 | 20 | 1/55 | 36363637 | 0.165557 |
| 30 | 20 | 13/680 | 37304856 | 0.173910 |
| 31 | 21 | 13/735 | 35856767 | 0.166064 |

All entries are certified by rational inequalities. Here are complete instructions for checking every entry. For $p = 2m+1$ use the exact value $\beta_p = \binom{2m}{m}/4^m$. For $p = 2m$ use

$$\beta_p = \frac{4^m m!(m-1)!}{\pi(2m)!}, \qquad \pi > \frac{314159}{100000},$$

to obtain a rational strict upper bound $B_p$ for $\beta_p^2$. For odd $p$, set $B_p = \beta_p^2$. Squaring the indicated rational $\alpha$ verifies its claimed bound. Substitution then gives, in each row,

$$(n+2)B_p\left(e + \frac{1+e}{14} + 2\alpha\right) < U < \frac{87}{500}.$$

This proves all six comparisons. □

*Proof of Theorem 1.1.* For $n = 1$, the fixed body is a centered interval and $c = 1/2 = a_1$. For $2 \le n \le 25$, apply Proposition 5.1. Now let $n \ge 26$ and put $K$ in inertia position. If it were not a ball, Theorem 11.2 would supply its proper canonical rotation subspace. Proposition 12.2 then gives (12.2). The necessary condition of Theorem 16.1 contradicts Proposition 17.1. Thus $K$ is a ball in inertia position and a centered ellipsoid in its original position. In every dimension, (2.1) gives $c = a_n$. Conversely, $\Gamma B_2^n = a_n B_2^n$ and affine equivariance show that every centered ellipsoid satisfies the equation. □

# A Exact scalar certificates

The two tangent lines used above are

$$L_0(y) = \frac{178}{675}y - \frac{2581}{10800}, \qquad L_{1/26}(y) = \frac{599}{2340}y - \frac{5791}{28080}.$$

The corresponding joining points are

$$r_{0,\pm} = \frac{19 \pm \sqrt{178}}{6}, \qquad r_{1/26,\pm} = \frac{18 \pm \sqrt{173}}{6}.$$

Define

$$C_0 = \frac{46}{225}, \qquad S_0(y) = \frac{2693}{1350} - \frac{481}{225}y,$$
$$C_{1/26} = \frac{128}{585}, \qquad S_{1/26}(y) = \frac{2149}{1170} - \frac{151}{65}y.$$

Polynomial integration against $e^{-y}\,\mathrm{d}y$ gives the exact formula

$$\mathbb{E}H_\eta(\xi) = C_\eta + S_\eta(r_{\eta,-})e^{-r_{\eta,-}} - S_\eta(r_{\eta,+})e^{-r_{\eta,+}}. \tag{A.1}$$

For example, integrate $L_\eta - P_\eta$ between its outer roots; an antiderivative is $-e^{-y}\sum_{j=0}^{4}(L_\eta - P_\eta)^{(j)}(y)$. At a joining point, reduction modulo the quadratic factor in (7.4) or (7.5) gives $S_\eta$ above.

Here is a rational certificate for the needed inequalities. Every terminating decimal in the table denotes an exact rational number. The root enclosures follow by substituting their endpoints into the two quadratic factors.

| $\eta$ | interval for $r_{\eta,-}$ | interval for $r_{\eta,+}$ | upper bound for $\mathbb{E}H_\eta(\xi)$ |
|---|---|---|---|
| 0 | $[0.9430559893, 0.9430559894]$ | $[5.3902773440, 5.3902773441]$ | 0.239634 |
| 1/26 | $[0.8078422603, 0.8078422604]$ | $[5.1921577396, 5.1921577397]$ | 0.257855 |

To verify the last column entirely with rational arithmetic, put

$$T_k(x) = \sum_{j=0}^{k} \frac{(-x)^j}{j!}.$$

For $0 \le x \le 6$, the alternating tail gives $T_{61}(x) \le e^{-x} \le T_{60}(x)$. If the root intervals in one row are $[a, b]$ and $[c, d]$, then $S_\eta(a), S_\eta(c) < 0$, and (A.1) is bounded above by

$$U_\eta := C_\eta + S_\eta(a)T_{61}(b) - S_\eta(d)T_{60}(c). \tag{A.2}$$

Substitution of the displayed rational endpoints gives, respectively,

$$U_0 < \frac{239634}{10^6} < \frac{6}{25}, \qquad U_{1/26} < \frac{257855}{10^6} < \frac{129}{500}.$$

These prove (7.6) without a floating-point assumption or an unchecked numerical extremum.

# B A polynomial certificate for the regression bound

The power series of the left side of (14.5) has positive coefficients from degree four onward. It is therefore bounded below on $[0, \infty)$ by

$$P_8(y) = 1 + 2y - \frac{3}{2}y^2 - y^3 + \frac{y^4}{12} + \frac{3y^5}{20} + \frac{5y^6}{72} + \frac{19y^7}{840} + \frac{121y^8}{20160}.$$

We give a rational positivity certificate for this polynomial. If $P_8(y) = \sum_{i=0}^{8} c_i y^i$ and $y = a + (b-a)z$, let

$$q_j = (b-a)^j \sum_{i=j}^{8} c_i \binom{i}{j} a^{i-j}, \qquad B_\ell = \sum_{j=0}^{\ell} q_j \frac{\binom{\ell}{j}}{\binom{8}{j}}.$$

Then $P_8(a + (b-a)z) = \sum_{\ell=0}^{8} B_\ell \binom{8}{\ell} z^\ell (1-z)^{8-\ell}$. The minima of these nine coefficients on the indicated intervals are

| $[a, b]$ | $\min_\ell B_\ell$ |
|---|---|
| $[0, 1]$ | 5587/6720 |
| $[1, 5/4]$ | 1144539/4194304 |
| $[5/4, 3/2]$ | 343997/4587520 |
| $[3/2, 7/4]$ | 81569/573440 |
| $[7/4, 2]$ | 6081595/4194304 |

They are all positive, proving positivity on $[0,2]$. For $y = 2+z$, the coefficients of $P_8(2+z)$ in increasing degree are

$$\frac{631}{105}, \frac{8908}{315}, \frac{4421}{90}, \frac{201}{5}, \frac{677}{36}, \frac{1003}{180}, \frac{127}{120}, \frac{299}{2520}, \frac{121}{20160}.$$

Their positivity proves the assertion for $y \geq 2$, and hence (14.5) for every $y \geq 0$.

## Declaration on the use of generative AI

ChatGPT (OpenAI) was used in mathematical exploration and formulation of arguments, preparation and revision of the manuscript, exact arithmetic checks, and English-language editing. The arguments, external inputs, and arithmetic certificates are stated in the manuscript. The author is responsible for the mathematical content, citations, and final version.

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