Unitary character moments and a zero-free half-plane for unitary Weil L-functions

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Abstract

We extend Liu's moment estimate for ordinary ideal-character sums to two independent fixed unitary twists, preserving the full length range of the finite-order theorem. The proof keeps the two character coefficients separate through the Gauss transformations and controls exceptional terms by the poles of their Mellin transforms. The case of two equal twists strengthens the fourth moment used in zero detection. Using the analytic results of Hu and Liu specified below, we obtain the zero-free half-plane $\operatorname{Re} s > (777-\sqrt{1081})/852 = 0.8733819666\ldots$ for every unitary Hecke L-function over every number field, including characters of infinite order. This extends the known finite-order bound to the full unitary family. Unitary induction gives the same half-plane for finite-dimensional continuous unitary Weil representations, with the norm-character poles counted exactly.

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1 The result and the character-moment estimate

The cubic-theta method introduced by OpenAI [O] relates zeros of a Hecke L-function to estimates for character sums. Its zero-detection argument uses both an ordinary character sum and an inverse sum with Möbius coefficients. Improving the ordinary fourth moment strengthens the resulting count of possible zeros. For a unitary character of infinite order, however, the ordinary sum has a fixed unitary twist which is absent from a finite-order moment theorem.

We prove that the ordinary moment induction permits two independent fixed unitary twists. The main point is to retain their values on the two ordered ideal factors. A common-divisor extraction then has an exact formula, with the same total length and divisor estimates as in the finite-order argument. The exceptional terms produced by the transformations involve fixed Hecke characters. When one of these is not a norm character, its Mellin integral has no pole. This supplies the exceptional estimate needed to continue the induction.

Hu [H] proves the common bound $7/8$ for unitary Hecke characters over all number fields and then for unitary Weil representations. Liu [L] proves the stronger bound

$$\theta_* := \frac{777-\sqrt{1081}}{852} = 0.873381966613207519929\ldots$$

for finite-order Hecke characters. Our moment extension supplies the ordinary denominator estimate needed to use this comparison for arbitrary unitary characters. The algebraic value θ_* and its finite-order comparison are due to Liu and the preceding Dirichlet work cited there; our contribution is the unitary moment extension and its application to the larger family.

The argument has three steps. A hypothetical zero first forces a large detecting character sum. The new fourth moment bounds the number of parameters for which this can happen. An estimate for the finite-order numerator in the theta identity then gives a contradiction. Section 2 proves the moment extension, Section 3 applies the comparison, and Section 4 passes from Hecke characters to Weil representations.

Theorem 1. *Assume the analytic inputs specified below. For every number field K and every finite-dimensional continuous unitary representation r of the global Weil group W_K, $L_K(s,r)$ has no zero in*

$$\operatorname{Re} s > \theta_*.$$

Its only poles there are at $1+it$, of multiplicity

$$m_t(r) = \dim_{\mathbb{C}} \operatorname{Hom}_{W_K}(\nu_{K,t}, r),$$

where the norm character is normalized by $L_K(s,\nu_{K,t}) = \zeta_K(s-it)$. The conclusion is unchanged on omitting Euler factors at finitely many primes.

Write H for Hu [H], L for Liu [L], and N for the earlier number-field estimates [N]. The analytic inputs are the fixed-type Hecke transformations, inverse moments, zero-prime fourth moments and theta comparison in H, Sections 2–8; the finite-order estimates, selected operator, ordinary sixth-power restoration and complete Gauss estimates in L, Sections 2–4; the finite-order approximate functional equation in L, Section 5; and the local correction estimates of N, Lemma 13.10 and the direct norm estimate of N, Proposition 13.4. We use the explicit scalar comparison of L, Section 6, and its phase-uniform principal Euler-product estimate in Section 7. H, Sections 9.1–9.2 supply unitary induction and pole counting. The new mixed moment is proved below from these stated inputs.

The numerical half-plane is common to all fields and characters. The constants in intermediate estimates may depend on the field, fixed characters and infinity types. Every assertion includes all heights. Throughout the proof, characters keep their zero values at nonunits, and a finite character of growing conductor remains in the moving character rather than in a fixed coefficient group.

2 A mixed moment for two unitary characters

An ordinary sum here is a smooth sum over all integral ideals, rather than a sum restricted to prime ideals. Its two factors may carry independent fixed unitary characters. Two equal characters give the fourth moment needed for the detecting sum; two different characters also permit joint estimates for a numerator and a detecting sum.

Work over a number field F containing the twelfth roots of unity, and fix a finite set S containing the bad places and ramification of the fixed arithmetic data. In particular, the two fixed unitary characters introduced below are unramified outside S; the conductor of the varying character need not be supported in S. The symbol $\mathrm{N}\,\mathfrak{a}$ denotes the absolute norm of an integral ideal. We use the parameter representatives and reciprocity convention of L, Definition 4.1. Thus χ_u is the sextic Hecke character associated to u, and $\chi_{\mathfrak{a}}(b)$ denotes the corresponding reciprocal residue-symbol value. The varying finite-order character has the form

$$\xi_u(\mathfrak{a}) = \vartheta(\mathfrak{a}) \prod_j \chi_{\mathfrak{a}}(b_j)^{e_j} \chi_u(\mathfrak{a}).$$

The b_j, e_j are fixed within the parameter sum. The product of the distinct prime ideals in their displayed support has norm at most Z^q, before cancellation of sixth powers. For an S-integer u, write

$$|u|_S = \prod_{\mathfrak{p}\notin S} (\mathrm{N}\,\mathfrak{p})^{v_{\mathfrak{p}}(u)}.$$

The chosen representatives satisfy $0 < |u|_S \ll Z^m$; their finite unit and class multiplicities are included in the sum. Put $M = m + q$. Characters are extended by zero at nonunits, and every such zero is retained throughout. The notation $\mathrm{ind}(\xi_u)$ means the primitive character inducing this zero-extended character. Fix two unitary Hecke characters ω_1, ω_2 of F, and put $\boldsymbol{\omega} = (\omega_1, \omega_2)$. Let Θ be a fixed finite group containing the fixed finite coefficient characters and those arising from the chosen unit and ray-class representatives. If $\omega_j = \alpha_j \,\mathrm{N}^{it_j}$ with α_j finite order, include α_j in Θ. Define

$$S_j(u;X) = X^{-1/2} \sum_{\mathfrak{a}} \omega_j(\mathfrak{a}) \xi_u(\mathfrak{a}) W_j(\mathrm{N}\,\mathfrak{a}/X), \qquad j \in \{1,2\}.$$

For $P_i = Z^{z_i}$, the prime sums are

$$Q_i(u) = P_i^{-1/2} \sum_{\mathfrak{p}\notin S} b_i(\mathfrak{p}) \xi_u(\mathfrak{p}) W_i(\mathrm{N}\,\mathfrak{p}/P_i), \qquad Q = \prod_i Q_i, \quad z = \sum_i z_i.$$

Their underlying prime intervals are disjoint. Each b_i is a fixed finite linear combination of characters in Θ, as in the cited coefficient family. All arithmetic coefficients are chosen before partitioning the parameters by the sizes of these sums. A sufficiently fine prime partition means that every $z_i \le \eta$ for the specified $\eta > 0$. Each W_i and each ordinary profile is smooth and compactly supported in a fixed annulus.

We also allow fixed finite twists of each ordinary character, norm powers absorbed into its profile, simultaneous conjugation, and exchange of the two ordinary factors. A finite character of growing conductor produced by a transformation remains in ξ_u and its displayed support; it is not added to the fixed group Θ.

Theorem 2 (Mixed moment)**.** *Let $M > 0$. For nonnegative ordinary lengths in bounded aggregate ranges, every $\varepsilon > 0$ admits a sufficiently fine prime partition such that*

$$\sum_{\mathrm{ind}(\xi_u)\notin\Theta} \mathcal{W}(u)\,|S_1(u;Z^{n_1})S_2(u;Z^{n_2})Q(u)|^2 \ll Z^{M+\varepsilon}\mathfrak{S}_J(1+H_{\mathrm{in}})^{J'} \tag{M}$$

whenever

$$n_1 + n_2 + 2z \le M.$$

Here $\mathcal{W} \ge 0$ is a smooth parameter weight from the family in L, Definition 4.1, supported in the indicated parameter range. The factor $\mathfrak{S}_J$ controls finitely many derivatives of that weight and of the independent coefficient profiles, including the weighted Fourier norms used to separate coupled profiles. The quantity H_{in} bounds the norm-twist heights in these profiles; J' is a finite polynomial-growth exponent. Character zeros and simultaneous coprimality restrictions are retained. An additional radical ideal may be fixed within the sum, with norm bounded by a fixed power of Z. At $z = 0$, the bound is available for arbitrary bounded ordinary lengths on the same excluded family. The prime partition and exponent losses precede the later arithmetic data and finite factor list. For fixed infinity types, constants, derivative orders and polynomial height orders may depend on the later fixed arithmetic data.

For the application, take $\omega_1 = \omega_2 = \psi$ and $n_1 = n_2 = m$. This gives the detecting fourth moment under the condition $2m + 2z \le M$, as used in Section 3.

We use L's selected operator and complete Gauss estimates in their stated arbitrary-common-coefficient form. The additional point is that the character sums produced by their transformations stay within the mixed family in (M). The following exact extraction records the character values needed for this assertion.

In the next lemma D, n, R, T denote integral ideals, $\operatorname{rad} R$ is the product of the distinct prime divisors of R, μ is the ideal Möbius function, and $\tau(D)$ counts the ideal divisors of D.

Lemma 1. *For fixed unitary characters ω_1, ω_2, put*

$$C(n;X,Y) = (XY)^{-1/2} \sum_{\mathfrak{a}\mathfrak{b}=n} \omega_1(\mathfrak{a})\omega_2(\mathfrak{b}) W_1(\mathrm{N}\,\mathfrak{a}/X) W_2(\mathrm{N}\,\mathfrak{b}/Y).$$

Then, with the indicated extra restriction on the first factor,

$$C(Dn;X,Y) = (\mathrm{N}\,D)^{-1/2} \sum_{D_1D_2=D} \omega_1(D_1)\omega_2(D_2) C_{(\mathfrak{a},D_2)=1}(n; X/\mathrm{N}\,D_1, Y/\mathrm{N}\,D_2). \tag{A}$$

The extra restriction has the exact expansion

$$C_{(\mathfrak{a},R)=1}(n;X,Y) = \sum_{\substack{T\mid \operatorname{rad} R \\ T\mid n}} \frac{\mu(T)\omega_1(T)}{\sqrt{\mathrm{N}\,T}}\, C(n/T; X/\mathrm{N}\,T, Y). \tag{B}$$

In (B) the quotient has no added coprimality restriction at T. The absolute allocation mass in (A)–(B) is at most $(\mathrm{N}\,D)^{-1/2}\tau(D)^2$.

Proof. For $\mathfrak{ab} = Dn$, choose $D_1 = (\mathfrak{a}, D)$ and $D_2 = D/D_1$. The residual factors multiply to n and the first is coprime to D_2. Conversely that restriction recovers the chosen gcd. This is a bijection even when D_1, D_2 are not coprime. The central normalization gives $(\mathrm{N}\, D)^{-1/2}$. Inclusion–exclusion gives (B), with its displayed inverse-root factor. Multiplicativity retains any zero values. The divisor bound gives the stated mass and hence a prescribed small-power loss.

For positive scales with $X_1X_2 = Y_1Y_2$, the comparison is the difference

$$C(n; X_1, X_2) - C(n; Y_1, Y_2).$$

We refer to the pairs (X_1, X_2) and (Y_1, Y_2) as its two comparison rectangles. Make the same allocations in both. Their total scales remain equal after division by D and T, and the same character scalar multiplies both coefficients. For a product also containing prime sums, apply successive gcd extraction in the fixed order of its factors. Extracting a nonunit ideal from a prime factor fixes that prime or gives zero. The number of allocations is divisor-bounded for a fixed factor list. □

Lemma 2. *Let λ_0 be a fixed unitary character which is not a norm character, and let $\lambda = \lambda_0\, \mathrm{N}^{it}$. For every fixed common radical R, every $X > 0$, and a smooth compactly supported W,*

$$\left| X^{-1/2} \sum_{(\mathfrak{a},R)=1} \lambda(\mathfrak{a}) W(\mathrm{N}\,\mathfrak{a}/X) \right| \ll X^{-1/2} (\mathrm{N}\, R)^{\varepsilon} p_J(W)(1+|t|)^A. \tag{C}$$

Consequently, if at least one of two fixed characters is non-norm, an equal-product difference of four scales, all at least Z^a and with common product T, has the same normalized bound $Z^{\varepsilon} T^{1/2} Z^{-a}$ as Hu's centered exceptional estimate, with the corresponding profile and height factors.

Proof. The unmasked *unnormalized* sum is $O(p_J(W)(1+|t|)^A)$ for every positive scale. For scale at least one, move its Mellin integral to $\mathrm{Re}\, s = -1$ using the entire fixed-character L-function. For smaller scale, use direct ideal counting; sufficiently small annuli are empty. In the exact divisor expansion each term is therefore bounded by $X^{-1/2}$, times its profile and height factor. There are divisor-many terms. This proves (C), including every shortened scale $X/\,\mathrm{N}\, D < 1$.

In the product of two unnormalized sums, the non-norm factor is bounded as just proved and the other by its absolute ideal count. Each rectangle is therefore bounded by its larger scale, times the same small-power and height factors. That scale is at most TZ^{-a}. Divide by $\sqrt{T}$ and apply the triangle inequality to the two rectangles. There is no double residue to cancel. This estimate supplies the original length saving only; all common-divisor and exceptional-count costs remain in the argument. □

Proof of Theorem 2. We check that the coefficient identities and exceptional estimates needed in Liu's finite induction remain valid with the two twists. Once this is established, the length inequalities in that induction are unchanged. A final removal of a short product of prime sums gives the full coefficient-two range.

First suppose both backgrounds are finite order times global norm powers. Write $\omega_j = \alpha_j\, \mathrm{N}^{it_j}$ and absorb each normalized norm power into its own profile. The resulting factors outside the sums have modulus one in the positive product, and the two finite coefficient characters belong to Θ. L, Theorem 4.9 applies to this positive product with its fixed finite coefficient family. This treats the case before forming any signed comparison; different norm powers would not in general have the same outside phase on two equal-product rectangles.

We may therefore suppose that at least one background is not finite order times a global norm power. Every finite twist of that background, and its conjugate, is non-norm. No condition on the quotient ω_2/ω_1 is needed.

Character sums produced by the transformations. The unmarked bound follows by Cauchy's inequality from the two separate zero-prime fourth moments of H, Theorem 4.2. The Θ exclusion prevents a finite-times-norm background from becoming principal after norm reduction; a genuinely non-norm background cannot become principal under any finite moving twist. Reflection uses the fixed-type Hecke functional equation on the actual ordinary character. Its conductor differs from that of ξ_u by fixed factors only, after the background ramification has been put in S. Whole-factor conjugation is used only in positive products. Every centered difference retains its full complex coefficient through both finite transforms. In particular, outside unit phases from different norm modes are kept in their respective rectangles; they are not assumed to coincide.

Expand a product into its two ordinary ideal variables and its ordered prime variables. The two Poisson transformations act on the varying parameter or its dual frequency. Their Gauss and correlation identities depend on the total column ideal, so they hold coefficientwise with the mixed coefficient C. Lemma 1 reopens every common-divisor extraction and its shortened profiles. A fixed finite coefficient character factors on the ordered product and twists the two backgrounds separately. A varying finite character instead updates ξ_u, with its support still included in the exponent q. Simultaneous conjugation and interchanging the two factors preserve the mixed class.

The same check is required before auxiliary-prime averaging. Fix the coefficient $B(n)$, including any difference of comparison sums and its Fourier mode, before choosing the auxiliary prime. The finite Gauss identity is applied termwise to $\sum_n B(n)\mathcal{G}(n,h)$, with its original normalization, and then summed with this same B. Here $\mathcal{G}(n,h)$ is the Gauss kernel in L, Section 4.1. The coefficient polynomial in its main term is independent of the auxiliary prime. In the three error valuations $i = 1, 6, 7$, allocate its ordinary power as $j_1 + j_2 = i$. The additional scalar is $\omega_1(\mathfrak{p})^{j_1}\omega_2(\mathfrak{p})^{j_2}$, with the one-sided restriction in (A) still present. In particular, a sixth power of either unitary character is retained. Its modulus costs no new prime power. The original Gauss–Ramanujan factors, nonunit terms, fifth-power corrections and normalized pool mass are unchanged. Original prime positions retain their own coefficients.

The other estimates involving these coefficients are L's selected operator, Theorem 3.10, and complete Gauss mean, Lemma 4.5. They admit arbitrary common ideal coefficients. The mixed convolution has the same divisor square-mass bound as the original one, including its centered difference, so both estimates apply. When the remaining free parameter has very short range, L, Proposition 4.6 applies with its full normalization. At an exceptional finite inducing character, one ordinary factor is still a non-norm fixed character, even if the backgrounds agree. Apply Lemma 2 with the actual four lengths. The valuation exponents and common-support factors in the exceptional calculation of L, Proposition 4.8 are unchanged.

The estimate must include parameters with sixth-power factors, as well as sixth-power-free parameters. Write such a parameter as $\mathfrak{c}^6\mathfrak{v}$. The common deletion at $\mathfrak{c}$ is expanded by (B), with its inverse-root mass, and the no-prime comparison uses the *same* two shortened mixed factors just bounded above. Put $s = \log_Z \mathrm{N}\,\mathfrak{c}$, $u = m - 6s$, $d = n_1 + n_2$ and $A = d + z$. Here K is the exponent parameter in L's selected operator, with $1/2 < K \le p_0 < 3/4$; p_0 is the pointwise exponent for the squared prime product. The exponents before optimizing s are

$$s + Kz + \min\{\max(d, (1-K)u + Kd), u + q\}$$

for a whole-product amplitude, and

$$s + \min\{\max(A, (1-K)u + KA), u + q + p_0 z\}$$

when only a proper block is large and $|Q|^2 \ll Z^{p_0 z + \varepsilon}$. The inequalities $5/6 - K > 0$ and $\min(d, u+q) - u/6 \le 5d/6 + q/6$ give respectively the exponent bounds

$$\max\{(1-K)m + KA, M/6 + 5A/6\}, \qquad \max\{(1-K)m + KA, M/6 + 5A/6 + p_0 z/6\}.$$

The second bound retains the extra $p_0 z/6$ term. Both bounds use the two-factor unmarked mean already proved above.

The finite moment inductions. Write $\mathcal{G}_{\boldsymbol{\omega}}(B, L)$ for the mixed assertion with condition $n_1+n_2+Lz \le BM$, without a prime-width cap. The preceding identities show that the character sums in L's two finite inductions stay in this class. These are Proposition 4.3 and Proposition 4.8 of L. We record their determining length inequalities. For $1/2 < k \le 1001/2000$, set $p_k = 6k$. The reflected comparison has total width at most $3M/2 - A + 2z + o(1)$, where $A = n_1 + n_2 + z$. Its gap below the uncentered region is

$$\gamma_k = \frac{2k-1}{2(6k-1)} > 0.$$

After the two transformations, the new parameters have available exponent $M' + \Delta'$ and capacity defect at most $6\Delta' + o(1)$. On a low-amplitude part, removing whole prime factors costs at most one sixth of this defect, plus the error of at most one prime interval. When the selected product has large amplitude, the selected operator gives the exponents $(1-K)m + KA$ and $M/6 + 5A/6$, with $K > 1/2$. Thus the same decreasing-width induction proves $\mathcal{G}_{\boldsymbol{\omega}}(1, 6k)$. Letting k approach $1/2$ by a *finite* choice within the requested loss, and removing a complete short prime product, proves $\mathcal{G}_{\boldsymbol{\omega}}(1, 3)$.

For a subsequent step from L_0 to L, the parent is $\mathcal{G}_{\boldsymbol{\omega}}(B, L_0)$. L's marked calculation gives the retained low-amplitude saving

$$\rho = a - \frac{k}{L_0}\{L_0 - L + (6B - L_0)a\} > 0.$$

The two high-amplitude conditions are $K(L-1) > (5K-1)a$ and $5(L-1) - p_0 > (19 - p_0)a$, where $k \le K \le p_0 = \kappa_1 + \upsilon < 3/4$ and $\kappa_1 = 0.749774$. Choose an allowance $\upsilon > 0$ for amplitude and height errors within the assigned loss. These inequalities hold uniformly for $L \ge 2$, $a \le 1/100$, $K, p_0 \le 3/4$. The prime bound with κ_1 concerns only the finite prime characters and is supplied by L's finite-order zero-free result. The complete common-support, nonunit and displayed- radical inequalities are those of the same marked calculation: the coefficient identities above change none of their norms or multiplicities. The previous moment estimate handles small amplitudes, and the selected operator handles large amplitudes, for the same transformed product.

It remains to reach the coefficient two with its full length range. Choose $0 < \delta \le 1/2000$, put

$$B = 1 - \delta, \qquad k = 1/2 + \delta, \qquad a = 1/100, \qquad L_* = \frac{6Bk}{1+k}, \qquad L_0 = 3,$$

and define a finite initial portion of

$$L_{j+1} = L_j - \frac{a}{2}\left(\frac{1+k}{k}L_j - 6B\right).$$

One has $2 < L_* < 3 \le 6Bk$ and

$$L_j - L_* = \left(1 - \frac{a(1+k)}{2k}\right)^j (3 - L_*).$$

Every retention coefficient is $\rho_j = \frac{a}{2}(1 + k - 6Bk/L_j) > 0$. The preceding finite step therefore proves $\mathcal{G}_{\boldsymbol{\omega}}(B, L_j)$ for each fixed finite j. For every $L > L_*$, a finite choice of j with $L_j < L$ gives $\mathcal{G}_{\boldsymbol{\omega}}(B, L)$ by monotonicity of the length condition. The choices involve no infinite-depth estimate.

Now put $d = n_1 + n_2$ and assume $d + 2z \le M$. Choose δ and a finite $L > L_*$ sufficiently close to two that

$$v_0 = \frac{(1-B)M + (L-2)z}{L}$$

has arbitrarily small exponent loss in the pointwise prime bound, uniformly in the prescribed bounded ranges. More precisely, allocate a loss $\varepsilon/4$ to these removed factors and take δ, L so that $\kappa_1 v_0 < \varepsilon/8$; take the prime partition parameter η so that $\kappa_1 \eta < \varepsilon/8$. If v_0 is positive, select a parameter-independent

product of complete prime factors of width $v_0 \le v < v_0 + \eta$, or exhaust the prime list if its total width is smaller. A zero requested width needs no removal. In the nonexhausted case,

$$d + L(z - v) \le d + 2z - (1 - B)M \le BM,$$

so $\mathcal{G}_{\boldsymbol{\omega}}(B, L)$ applies to the remaining two-factor product. If all prime factors have been removed, the arbitrary-length unmarked mean applies. Every removed position still has a finite character. Its squared absolute product is at most $Z^{\kappa_1 v + o(1)}$, with the common test seminorm, by L's fixed finite-character prime estimate. This fits the allocated loss after reducing the subsidiary prime and partition errors. Allocate another $\varepsilon/4$ to the final parent estimate and distribute the remaining $\varepsilon/2$ through the finite induction and profile separation. This proves the exact coefficient-two length range without an extra prime-width restriction.

Choose the losses and prime partition parameter below the minimum of the finitely many positive width and retention margins, before fixing the later finite twist and factor list. Choose derivative orders backwards through this finite induction. The unitary factors alter fixed constants and polynomial height degrees only. For incoming height above the assigned small power of Z, the original polynomial bound is absorbed by a larger final height degree, as in L's proof. The common Sobolev measures, natural zeros and masks therefore retain the stated quantifiers. This proves (M). □

3 The common half-plane for unitary Hecke characters

The same mixed moment estimates both the finite numerator and the unitary sum which detects a small denominator. This gives the ordinary moment inputs of Liu's comparison with no restriction on the target's order. We use Liu's scalar comparison and verify its analytic hypotheses for this unitary family.

Put

$$B_2 = \theta_* = \frac{777 - \sqrt{1081}}{852} = 0.873381966613207519929\ldots.$$

We prove that $L_K(s, \psi)$ has no zero for $\operatorname{Re} s > B_2$, for every number field K and every continuous unitary Hecke character ψ. The possible norm-character pole is retained.

Over an auxiliary field $F \supset \mu_{12}$, H's exact local factorization is

$$\mathscr{F}_{u,\psi}(s, w, z) = \zeta_F^S(6z) \frac{L_F^S(w, \chi_u)}{L_F^S(s, \psi\bar{\chi}_u)} H_{u,\psi}(s, w, z). \tag{1}$$

The superscript S omits the Euler factors at the common fixed excluded places. In particular the numerator character χ_u is finite order even when ψ is not. Let $I_{\psi,\mathrm{mod}}(Z)$ be the corrected theta sum of H, Section 6, including the local correction terms specified there. For its J_0 prime windows W_i, set

$$\Sigma_i(Z) = \sum_{\mathfrak{p} \notin S} W_i(\mathrm{N}\,\mathfrak{p}/P_i)(\mathrm{N}\,\mathfrak{p})^{-5/6}, \qquad P_i = Z^{\ell_i}, \quad \sum_i \ell_i = \ell,$$

$$\mathcal{N}(Z) = Z^{-\ell/6} \prod_i \Sigma_i(Z), \qquad J_\psi(Z) = \frac{(-1)^{J_0} I_{\psi,\mathrm{mod}}(Z)}{c_\psi \mathcal{N}(Z)}.$$

Here $c_\psi \neq 0$ is H's principal residue scalar. The positive normalizer satisfies $\mathcal{N}(Z) \asymp (\log Z)^{-J_0}$. For the imbalance b chosen below, the principal Mellin term is

$$f_\psi(Z) = \frac{1}{2\pi i} \int_{(2)} Z^{s-(4+b)/6} e^{(s-5/6)^2} \frac{H_\psi^S(s)}{L_F^S(s, \psi)}\, ds. \tag{2}$$

The same S occurs in the probe, its principal term, and every comparison. These are the functions to which the estimates below apply.

The two ordinary moment applications. At a fixed frequency scale U, let ξ_u be the varying finite character and write an ordinary detecting sum as

$$S_m(u) = U^{-m/2} \sum_{\mathfrak{a}} \psi(\mathfrak{a}) \xi_u(\mathfrak{a}) W(\mathrm{N}\,\mathfrak{a}/U^m).$$

The prime sums in the theta identity carry ξ_u with their fixed finite coefficient characters: in H's paired notation their relative coefficient is $\psi(\mathfrak{p})^{-1} b_i(\mathfrak{p})$. Thus the character in each prime sum remains finite order. Theorem 2, with the ordered pair (ψ, ψ), gives

$$\sum_u |S_m(u)|^4 |Q(u)|^2 \ll U^{1+\varepsilon}(1+H)^C, \qquad 2m + 2z \le 1, \tag{3}$$

where z is the total selected prime width and all original natural zeros and common masks remain. With the pair $(1,1)$ the same theorem estimates the finite numerator. Whole-factor conjugation, with the profile conjugated as well, gives the needed opposite orientations inside these positive products.

These statements are used with independent profiles, including the finitely many derivatives of the detector parameters. First enclose those parameters in their common smooth box; apply the tensor Sobolev bound of L, Proposition 6.1, inside the parameter sum; then evaluate the detecting witness. Each derivative term still has two ordinary factors with fixed arithmetic coefficients. As in H, the exceptional finite inducing characters among sixth-power-free frequencies have bounded norm and are treated separately in the small-parameter range.

The reference moments used in the comparison have a larger prime-length coefficient than two. Set

$$\kappa_1 = \frac{374887}{500000}, \qquad p_1 = 6\kappa_1 > 2.$$

Every ordinary moment with capacity $n_1 + n_2 + p_1 z \le M$ follows by weakening Theorem 2. The pointwise bound with κ_1 concerns only finite prime characters and follows from L's finite-order zero-free input. The marked inverse moments are supplied independently by H, Theorem 4.1, which has no κ parameter, in the unchanged ranges

$$r + 2z < 1, \qquad 2r + 8z < 3.$$

Their coefficients may contain the actual fixed unitary background. The unmarked long inverse estimate is unchanged as well.

Counting the detecting parameters. Use the amplitude classes of L, Proposition 6.1. Here a is the real part of the denominator contour. On a nonzero size class write $|Q_i(u)| = P_i^{\gamma_i + o(1)}$ and let $g = \ell^{-1} \sum_i \ell_i \gamma_i$ be their weighted amplitude exponent; factors replaced by local errors have $\gamma_i = 0$. Put $\delta = 2a - 1$ and $x = g/\delta$, so $0 \le x \le 1/2$ in these classes. The main comparison uses L's rectangle

$$\mathcal{D} = [11/50, 13/20] \times [6/25, 1/2].$$

The inverse and ordinary witnesses, with the chosen small error allowances, have lengths r, m with $r \le t + o(1)$, $r + m \ge t - o(1)$ and $m \le 1/2 + o(1)$; their squared magnitudes are at least $U^{\delta r - o(1)}$ and $U^{\delta m - o(1)}$. On $\mathcal{D}$, take the single cutoff

$$A = 5/6 - \delta, \qquad m_d = \frac{5/6 - \delta x}{5/2 - \delta - 2\delta x}, \qquad r_d = 2m_d, \qquad t_d = 3m_d,$$

and put

$$F_d = x + 2(1-x)m_d, \qquad R_d = 1 - \delta F_d = \frac{1}{6} + A t_d. \tag{4}$$

The short inverse saving is $x + (1 - x)r$; the ordinary saving from (3) is $x + 2(1 - x)m$. They agree at r_d, m_d. Every actual short witness has either $r \geq r_d - o(1)$ or $m \geq m_d - o(1)$. For $r \geq 1$, the long inverse count is at most $1/6 + At_d + o(1)$. Near $r = 1$ or $m = 1/2$, the unmarked moments give the stronger count $1 - \delta < R_d$. This proves the count $U^{R_d+\varepsilon}(1 + H)^C$. The crossing lengths are thresholds for this case division; the witnesses retain their own lengths.

The range checks in L, Lemma 6.2 and Proposition 6.4 give $0.35 < m_d < 0.49$, $0.6 < r_d < 1$ and $1 < t_d < 1.5$. The needed widths $(1 - r_d)/2 < 0.20$ and $(1 - 2m_d)/2 < 0.15$ fit inside the available prime width $\mathscr{L} = 21/100$. Decrease widths within the strict margins and select complete prime factors in decreasing amplitude order. The error of at most one prime interval and the detector tolerances fit within the reserved loss. The complementary rectangles use L's R_s from equation (6.20), whose ordinary input at p_1 was justified above.

The numerator and the complete comparison. For a candidate $B > B_2$, put

$$\ell = \frac{11}{3} - 4B, \quad l_x = \frac{1 - b - \ell}{2}, \quad l_y = \frac{1 + b - \ell}{2}, \quad h = \frac{1 + b + 3\ell}{2}, \quad T_0 = 2l_y - h.$$

Here T_0 is a length expression, distinct from an analytic height. Let $C_0 = -1/4 + b/6 + 5\ell/4$ and $\theta_0 = \delta x/p_1$. L's reference exponent and new numerator gain are

$$E(R) = C_0 + \frac{\delta}{2} + \delta x \left(\ell + \frac{h/2 - l_y}{p_1} \right) - \frac{3h(1 - R)}{4}, \qquad G_N = \frac{\delta x}{10} \left(\frac{5}{2} - \frac{5}{p_1} \right) T_0.$$

The finite numerator in (1) is unchanged. Move its complete holomorphic factor before taking absolute values, retain both approximate-functional-equation halves and their scalar factors, and apply the finite instance of Theorem 2. On $0.30 \leq m \leq 0.47$, the selected width is $(1 - 2m)/2 \leq 0.20$. The worst numerator length is $m_0 = 1 - l_y/h$; the contour chooses 0 on its left and $\delta/2$ on its right. This gives G_N. For shorter or longer blocks use the reference of L, Lemma 6.3, and the remaining three tests in equation (6.26). In particular, the longer range uses its fixed $m = 0.47$ endpoint before the arbitrary-length unmarked theorem; it does not apply a selected moment to unbounded lengths.

Only the main rectangle needs the improved numerator contour. There $\delta \leq 13/20$, exactly the range of L, Lemma 5.2. The complementary reference uses $\theta_0 < 1/10$. Local errors are expanded before this finite moment application. At an error position set its prime amplitude to zero, retain its original character zero and conductor decrease, and use H's complete ramified table. Strict errors pay their extra prime power through the same functional equation. All target values in those correction factors have modulus one, so the same estimates include all the error prime sums and the unitary coefficients.

For the direct estimate, apply the norm-form argument of N, Proposition 13.4 to H's fixed-type unitary probe. The separated theta coefficients carry the unitary target only as bounded phases; the hybrid norm estimate permits these coefficients. The additive Gram factor has coefficient one and contains no target character. Thus the actual dual lengths and norms remain those of the source estimate. Writing $M_0 = l_x + l_y = 1 - \ell$, its empty-subset majorant is

$$\frac{5}{6} + \frac{l_x}{6} - \frac{\ell}{6} + G + \frac{k}{2}, \qquad G = \max(0, b/12, b - l_y/2), \quad k = \max\left(0, \frac{1 + 3\ell - 2M_0}{4}, M_0 + \ell - 1, \frac{3\ell - 1}{2} \right).$$

The verified geometry gives $\ell < 1/5$ and $l_y > 11b/6$, so $k = 0$, $G = b/12$, and this majorant is $11/12 - \ell/4 = B$. Every nonempty original full-index subset of width d decreases it by at least $3d/8$, as in that proposition. This is the direct theta norm estimate, separate from the ordinary moment above.

These are precisely the analytic hypotheses used by L, Propositions 6.4 and 6.6 and Lemma 6.5. The latter retains the direct and Gram estimate, the full frequency-scale slopes, small amplitudes, smallest size classes, small and large parameter ranges, available prime widths and reference lengths. Their

real inequalities are independent of the fixed unitary phases. For the principal contours L, Lemma 7.1 explicitly allows unitary target phases on

$$\operatorname{Re} s \geq 0.87, \qquad \operatorname{Re} w \geq 0.95, \qquad \operatorname{Re} z \geq 0.165.$$

Its local polynomial gives $H_p - 1 = O((\mathrm{N}\, p)^{-1.81})$ and $G_p/H_p = -1 + O((\mathrm{N}\, p)^{-0.87})$ uniformly there. Hence the same principal savings $l_y/20$, $h/600$ and $0.435 \min_i \ell_i$ remain after enlarging the common fixed S. For a concrete strict comparison one may take

$$B = 0.8733821, \qquad b = 0.0999562982,$$

where both decimals denote exact rational numbers and the other lengths are determined by the preceding formulas. L, Proposition 6.6 supplies all strict real inequalities for these parameters. To reach the exact open half-plane, use

$$b = b_2 = -\frac{11}{71} + \frac{595\sqrt{1081}}{76751}.$$

L, Lemma 6.7 and Proposition 6.8 show that the only vanishing comparison margin at B_2 is $G_N - E(R_d)$, with a unique double contact at $\delta = (61 - \sqrt{1081})/66$, $x = 1/2$. Its derivative with respect to B is uniformly positive. All other margins remain strictly positive. Consequently every fixed $B > B_2$ sufficiently close to B_2 gives a strict full comparison. The choices need not be uniform as $B \downarrow B_2$.

All heights and the Hecke conclusion. Work first over $F \supset \mu_{12}$ and fix one unitary character ω. Use H's same family

$$\mathcal{F}_\omega = \{\vartheta \omega^j \, \mathrm{N}_F^{it} : \ j \in \{-1, 0, 1\}, \quad \vartheta \text{ finite order}, \ t \in \mathbb{R}\},$$

and let B_* be the supremum of its nontrivial zero real parts, with $1/2$ included. H gives $B_* \leq 7/8$. If $B_* > B_2$, choose a fixed admissible B with $B_2 < B < B_*$. The family contains every live denominator, reflected target and finite twist in the argument. The finite numerator remains in its finite-order subfamily.

Choose the numerical margins, finite moment depths and prime partition, and fix one finite list of positive widths (ℓ_i) with total ℓ and every width smaller than the partition bound, before the later target and finite twist. The positive number $\min_i \ell_i$ and its principal saving are then common to this family. Then fix the arithmetic data and profiles, choose derivative orders backwards through the finite proofs, and obtain the polynomial height degrees. The common loss can be taken smaller than the strict comparison margin and $B_* - B$. H's contour and tail argument then gives, for some common $\eta > 0$ and every fixed target $\psi \in \mathcal{F}_\omega$,

$$|J_\psi(Z)| \ll_{\psi,\varepsilon} Z^{C(B)+\varepsilon}, \qquad |J_\psi(Z) - f_\psi(Z)| \ll_\psi Z^{C(B_*)-\eta}, \qquad C(s) = s - (4+b)/6.$$

Indeed a pretruncation error has the form $Z^{C(B_*)-2\eta}(1+H)^{A_\psi} + Z^{D_\psi} H^{-N}$; choose $H = Z^{t_\psi}$ with $t_\psi A_\psi < \eta/2$ and then N large. The target-dependent constants and choices do not change the common positive saving.

The Mellin identity for f_ψ is the nonvanishing numerator $e^{(s-5/6)^2} H_\psi^S(s)$ divided by $L_F^S(s,\psi)$. The two estimates continue this quotient holomorphically to $\operatorname{Re} s > \max(B, B_* - \eta)$. The principal numerator is nonzero there by H, Lemma 6.1 (or the preceding phase-uniform principal box). This gives a common zero bound strictly below B_*, a contradiction. Thus $B_* \leq B_2$.

For general K, descend from $K(\zeta_{12})$ using H's abelian factorization. Away from norm-character poles its Hecke factors are holomorphic and nonzero. At $1+it$ at most one factor has the simple norm-character pole, matching the upstairs simple pole, so no zero can be concealed there. This proves the asserted Hecke half-plane over every number field, including all heights.

4 Unitary Weil representations

Let r be a finite-dimensional continuous unitary representation of W_K. The unitary Brauer induction and local-factor identities in H, Section 9.1 give finite extensions E_j/K, unitary Hecke characters ψ_j of E_j, and integers n_j such that

$$L_K(s,r) = \prod_j L_{E_j}(s,\psi_j)^{n_j}.$$

Some n_j may be negative, so this expression can be a quotient. The Hecke result applies over every E_j with the same θ_*. Away from the possible poles $1+it$, each factor and its reciprocal is holomorphic and nonzero in $\operatorname{Re} s > \theta_*$.

It remains to check those possible poles. A Hecke factor has a simple pole at $1+it$ exactly when $\psi_j = \nu_{E_j,t}$. As in H, Section 9.2, Frobenius reciprocity therefore gives

$$\operatorname{ord}_{1+it} L_K(s,r) = -\sum_j n_j \mathbf{1}_{\{\psi_j = \nu_{E_j,t}\}} = -m_t(r) \le 0.$$

Here positive order would mean a zero, and negative order means a pole. Thus the quotient has precisely the stated poles and no zeros there. Finally, the omitted Euler factors are products of $1-\alpha(\mathrm{N}\,\mathfrak{p})^{-s}$ with $|\alpha| = 1$, by unitarity. They cannot vanish for $\operatorname{Re} s > 0$. This proves Theorem 1, with the same bound in every finite dimension.

Computational material. The ancillary files contain exact certificates for the scalar inequalities used in Section 3 and independent verifiers of the saved bounds. Symbolic scripts regenerate the certificates, including the rational enclosure at the algebraic boundary. These computations concern the scalar comparison; they do not verify the analytic estimates on which it depends.

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