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EXPLICIT BOUNDS FOR ODD PERFECT AND QUASIPERFECT NUMBERS

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Abstract. Let N be an odd perfect number or a quasiperfect number, and let $B(N) = \prod_{p|N}(1+1/p)$. We prove that $2 < N/\varphi(N) < 2.17372$ and that $105 \nmid N$, and we obtain explicit two-sided bounds for $B(N)$ according to the value of $\gcd(15, N)$. For an odd perfect number N, they give $B(N) > 1.90281$, 1.70584, 1.83233 and 1.67856 when $\gcd(15, N) = 1$, 3, 5 and 15, respectively, and $B(N) < 1.84348$, 1.93549 and 1.84125 when $\gcd(15, N) = 3$, 5 and 15. These follow from general bounds for $B(n)$ in terms of $n/\varphi(n)$, of the abundancy index $I(n) = \sigma(n)/n$, and of $\sum_{p|n} 1/p$. We also bound the abundancy indices of $N \operatorname{rad}(N)$ and $N/\operatorname{rad}(N)$ and the functions $\sigma^{\pm}$, φ^{+} and ψ^{+} of Sándor and Atanassov at N, give an explicit lower bound for the number of distinct prime factors of N in terms of its least prime factor, and show that the sum of the reciprocals of the prime factors of a quasiperfect number N is less than $\log I(N)$. Finally, for the shifted abundancy indices $I^{+}(n) = 1 + \sum_{d|n,\, d>1} 1/(d+1)$ and $I^{-}(n) = 1 + \sum_{d|n,\, d>1} 1/(d-1)$, we determine the optimal constants in the comparisons of I^{+}, I and I^{-} on the odd integers and on the odd integers with at most one odd exponent, and we deduce $1.8471 < I^{+}(N) < 2 < I^{-}(N) < 2.2722$ and $I^{-}(N) - I^{+}(N) < 0.42493$ for every odd perfect number N.

1. Introduction

A positive integer n is called *perfect* if it equals the sum of its proper positive divisors. Equivalently, $\sigma(n) = 2n$, where $\sigma(n)$ is the sum-of-divisors function. The first examples are $6 = 1 + 2 + 3$ and $28 = 1 + 2 + 4 + 7 + 14$. Euclid proved that, if $2^p - 1$ is prime, then $2^{p-1}(2^p - 1)$ is a perfect number, and Euler proved that every even perfect number has this form [4]. Whether an odd perfect number exists is an open problem. Many necessary conditions are known; for example, an odd perfect number has at least ten distinct prime divisors [18] and exceeds 10^{1500} [21].

A positive integer N is called *quasiperfect* if $\sigma(N) = 2N + 1$. No quasiperfect number is known, and whether one exists is also an open problem. A quasiperfect number is an odd square greater than 10^{35} with at least seven distinct prime divisors [7].

This paper studies, for such numbers, the multiplicative quantity

$$B(n) \;=\; \prod_{p|n}\Big(1+\frac{1}{p}\Big), \tag{1.1}$$

together with the abundancy index $I(n) = \sigma(n)/n$ and the quotient $n/\varphi(n)$, where φ is Euler's totient function. Section 2 records notation and known facts. Section 3 proves two lower bounds for $B(n)$ in terms of $n/\varphi(n)$ and one upper bound for $B(n)$ in terms of $I(n)$, valid for general integers n. Section 4 bounds $n/\varphi(n)$ from above when n is odd and $2 \le I(n) \le 2 + 10^{-35}$. Section 5 combines these results for odd perfect and quasiperfect numbers. Section 6 bounds the abundancy index of $N \operatorname{rad}(N)$. Section 7 treats the functions σ^{+}, σ^{-}, φ^{+} and ψ^{+} of Sándor

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and Atanassov. Section 8 relates the least prime factor of N to the number of its distinct prime factors. Section 9 bounds the sum of the reciprocals of the prime divisors of a quasiperfect number. Sections 10–20 treat the shifted abundancy indices I^+ and I^-.

The values of I have been studied by Laatsch [14], Weiner [27], Ryan [24], Holdener [11], and Stanton and Holdener [26]; see also [8]. The abundancy indices of the divisors of an odd perfect number have been studied by Dris [6] and by Chen and Chen [2]. For odd perfect numbers, the statement $105 \nmid N$ is due to Kühnel [13], and upper bounds for $N/\varphi(N)$ and two-sided bounds for $\sum_{p|N} 1/p$ according to the value of $\gcd(15, N)$ are quoted in [28]. Accordingly, for odd perfect numbers, part (b) of Theorem 5.1 is Kühnel's theorem, and the upper bound in part (a) is weaker than the bounds for $N/\varphi(N)$ quoted in [28], the largest of which is 2.165439; these two statements are proved here by one argument that also covers quasiperfect numbers.

2. Background

Throughout, p denotes a prime, and $\nu_p(n)$ denotes the largest nonnegative integer a such that $p^a \mid n$.

Proposition 2.1. *Let $n > 1$. Then $\sigma(n) = \prod_{p|n} \left(1 + p + \cdots + p^{\nu_p(n)}\right)$ [15].*

Definition 2.2. The *abundancy index* of a positive integer n is $I(n) := \sigma(n)/n$.

Proposition 2.3. *Let $n = \prod_{i=1}^{k} p_i^{a_i}$, where $p_1, \ldots, p_k$ are distinct primes and $a_1, \ldots, a_k$ are positive integers. Then*

$$I(n) \;=\; \prod_{i=1}^{k} \left(1 + \frac{1}{p_i} + \cdots + \frac{1}{p_i^{a_i}}\right) \;=\; \left(\prod_{i=1}^{k} \frac{p_i}{p_i - 1}\right) \left(\prod_{i=1}^{k} \left(1 - \frac{1}{p_i^{a_i+1}}\right)\right). \tag{2.1}$$

In particular, $I(n) < \prod_{i=1}^{k} \dfrac{p_i}{p_i - 1} = \dfrac{n}{\varphi(n)}$ [5, 10].

Proof. The identities follow from Proposition 2.1 after dividing by n and evaluating the finite geometric sums. The strict inequality follows because $0 < 1 - p_i^{-(a_i+1)} < 1$ for every i, and the final identity is the product formula for $n/\varphi(n)$. □

We write

$$\Phi(n) \;=\; \frac{n}{\varphi(n)} \;=\; \prod_{p|n} \frac{p}{p-1}, \qquad A(n) \;=\; \prod_{p|n} \Big(1 - \frac{1}{p^{\nu_p(n)+1}}\Big), \qquad C(n) \;=\; \prod_{p|n} \frac{p^2}{p^2-1}. \tag{2.2}$$

Lemma 2.4. *For every integer $n > 1$,*

$$I(n) \;=\; A(n)\,\Phi(n), \qquad \Phi(n) \;=\; B(n)\,C(n), \qquad I(n) \;=\; A(n)\,B(n)\,C(n), \tag{2.3}$$

and $0 < A(n) < 1 < C(n)$.

Proof. The first identity is (2.1). The second follows from $\dfrac{p}{p-1} = \dfrac{p+1}{p} \cdot \dfrac{p^2}{p^2-1}$ for every prime p, and the third from the first two. Every factor of $A(n)$ lies strictly between 0 and 1, and every factor of $C(n)$ exceeds 1. □

Proposition 2.5. *A positive integer n is perfect if and only if $I(n) = 2$, and quasiperfect if and only if $I(n) = 2 + 1/n$.*

Proposition 2.6. *If N is an odd perfect number, then $N = q^e s^2$ with q prime, $q \equiv e \equiv 1 \pmod 4$ and $\gcd(q, s) = 1$ [4], and N has at least ten distinct prime divisors [18].*

Proposition 2.7. *If N is a quasiperfect number, then N is an odd square greater than 10^{35} and has at least seven distinct prime divisors* [7].

Proposition 2.8. *We have $\prod_p(1-p^{-2})^{-1} = \zeta(2) = \pi^2/6$, $\prod_p(1-p^{-3})^{-1} = \zeta(3)$, $\prod_p(1-p^{-4})^{-1} = \zeta(4) = \pi^4/90$ and $\prod_p(1-p^{-6})^{-1} = \zeta(6) = \pi^6/945$, the products being taken over all primes* [1].

3. Bounds for $B(n)$

3.1. A lower bound with an exponent.

Lemma 3.1. *For every real $x > 1$, the quantity*

$$r(x) \;=\; \frac{\log\big(x/(x-1)\big)}{\log\big((x+1)/x\big)} \tag{3.1}$$

is well defined, exceeds 1, and is strictly decreasing.

Proof. Both $\log\big(x/(x-1)\big)$ and $\log\big((x+1)/x\big)$ are positive for $x > 1$, and the first exceeds the second because $x/(x-1) > (x+1)/x$ is equivalent to $x^2 > x^2 - 1$; this gives $r(x) > 1$. For monotonicity, write $\psi(t) = t\log t$, so that $\psi''(t) = 1/t > 0$ on $(0,\infty)$ and ψ is strictly convex there. Strict convexity gives $\psi(x-1) + \psi(x+1) > 2\psi(x)$, that is,

$$(x+1)\log\frac{x+1}{x} \;>\; (x-1)\log\frac{x}{x-1}. \tag{3.2}$$

Put $u(x) = \log\big(x/(x-1)\big)$ and $v(x) = \log\big((x+1)/x\big)$, so that $r = u/v$, $u'(x) = -1/\big(x(x-1)\big)$ and $v'(x) = -1/\big(x(x+1)\big)$. Then

$$u'(x)v(x) - u(x)v'(x) \;=\; \frac{(x-1)u(x) - (x+1)v(x)}{x(x-1)(x+1)}, \tag{3.3}$$

which is negative by (3.2). Since $r' = (u'v - uv')/v^2$, we obtain $r'(x) < 0$. □

Lemma 3.2. *For every prime p, $\Big(1+\dfrac{1}{p}\Big)^{r(p)} = \dfrac{p}{p-1}$.*

Proof. By the definition of $r(p)$, $\big(1+1/p\big)^{r(p)} = \exp\big(r(p)\log\frac{p+1}{p}\big) = \exp\big(\log\frac{p}{p-1}\big) = \frac{p}{p-1}$. □

Theorem 3.3. *Let n be an integer with at least two distinct prime factors, let p_0 be the least prime factor of n, and let r be the function of Lemma 3.1, so that $r(p_0) = \log\big(p_0/(p_0-1)\big)/\log\big((p_0+1)/p_0\big)$. Then*

$$B(n) \;>\; \Phi(n)^{1/r(p_0)}. \tag{3.4}$$

If $n = p^a$ is a prime power, then $B(n) = \Phi(n)^{1/r(p)}$.

Proof. Write $n = \prod_{i=1}^k p_i^{a_i}$ with $p_0 = p_1 < \cdots < p_k$ and $k \ge 2$. By (2.2) and Lemma 3.2, $\Phi(n) = \prod_{i=1}^k \big(1+1/p_i\big)^{r(p_i)}$. Lemma 3.1 gives $r(p_i) \le r(p_0)$, with equality if and only if $i = 1$; since $1 + 1/p_i > 1$, it follows that $\big(1+1/p_i\big)^{r(p_i)} \le \big(1+1/p_i\big)^{r(p_0)}$, with strict inequality for $i = k$. Multiplying,

$$\Phi(n) \;=\; \prod_{i=1}^{k}\Big(1+\frac{1}{p_i}\Big)^{r(p_i)} \;<\; \prod_{i=1}^{k}\Big(1+\frac{1}{p_i}\Big)^{r(p_0)} \;=\; B(n)^{r(p_0)}. \tag{3.5}$$

Since $r(p_0) > 0$, taking $r(p_0)$-th roots gives (3.4). For $n = p^a$, Lemma 3.2 gives $\Phi(n) = p/(p-1) = (1+1/p)^{r(p)} = B(n)^{r(p)}$. □

Corollary 3.4. *Let n be an odd integer with at least two distinct prime factors, and let $m = r(3) = \log(3/2)/\log(4/3)$. Then*

$$B(n) \;>\; \Phi(n)^{1/m} \;>\; I(n)^{1/m}. \tag{3.6}$$

In particular, if $I(n) \geq 2$, then $B(n) > 2^{1/m} = 1.635250087\ldots$.

Proof. The least prime factor p_0 of n satisfies $p_0 \geq 3$, so $r(p_0) \leq m$ by Lemma 3.1. Since $\Phi(n) > 1$, Theorem 3.3 gives $B(n) > \Phi(n)^{1/r(p_0)} \geq \Phi(n)^{1/m}$, and $\Phi(n) > I(n)$ by Proposition 2.3. □

3.2. **A linear lower bound.** For a prime p_0, write

$$Z(p_0) \;=\; \prod_{p \geq p_0} \frac{p^2}{p^2-1} \;=\; \zeta(2) \prod_{p < p_0} \Big(1 - \frac{1}{p^2}\Big), \tag{3.7}$$

the second equality holding by Proposition 2.8. Thus $Z(2) = \pi^2/6$, $Z(3) = \pi^2/8$ and $Z(5) = \pi^2/9$.

Theorem 3.5. *Let $n > 1$ be an integer with least prime factor p_0. Then $C(n) < Z(p_0)$ and*

$$B(n) \;>\; \frac{\Phi(n)}{Z(p_0)}. \tag{3.8}$$

In particular, $B(n) > \dfrac{8}{\pi^2}\,\Phi(n)$ for every odd $n > 1$.

Proof. The prime divisors of n form a finite subset of the primes $p \geq p_0$, and every factor $p^2/(p^2-1)$ exceeds 1. Hence $C(n) < Z(p_0)$, the inequality being strict because infinitely many factors are omitted. By Lemma 2.4, $B(n) = \Phi(n)/C(n) > \Phi(n)/Z(p_0)$. If n is odd, then $p_0 \geq 3$ and $Z(p_0) \leq Z(3) = \pi^2/8$. □

Proposition 3.6. *Let $m = r(3)$ and let $\Phi > 1$ be real. Then $\Phi^{1/m} > \dfrac{8}{\pi^2}\,\Phi$ if and only if*

$$\Phi \;<\; \Big(\frac{\pi^2}{8}\Big)^{m/(m-1)} \;=\; 2.060569542\ldots. \tag{3.9}$$

Proof. Since $m > 1$ by Lemma 3.1, the inequality $\Phi^{1/m} > (8/\pi^2)\Phi$ is equivalent to $\Phi^{1-1/m} < \pi^2/8$, that is, to $\Phi < (\pi^2/8)^{m/(m-1)}$. □

3.3. **An upper bound.**

Lemma 3.7. *For every integer $n > 1$,*

$$B(n) \;\leq\; I(n) \prod_{\substack{p \mid n \\ \nu_p(n) \geq 2}} \Big(1 + \frac{1}{p(p+1)}\Big)^{-1}. \tag{3.10}$$

Proof. For a prime p and an integer $a \geq 1$ we have $I(p^a) \geq 1 + 1/p$. If $a \geq 2$, then

$$I(p^a) \;\geq\; 1 + \frac{1}{p} + \frac{1}{p^2} \;=\; \Big(1 + \frac{1}{p}\Big)\Big(1 + \frac{1}{p(p+1)}\Big). \tag{3.11}$$

Multiplying over the prime divisors of n and using the multiplicativity of I gives $I(n) \geq B(n) \prod_{p \mid n,\, \nu_p(n) \geq 2} \big(1 + 1/(p(p+1))\big)$. □

3.4. **Bounds in terms of the sum of the reciprocals of the prime divisors.** For a positive integer n write $\Theta(n)=\sum_{p\mid n}1/p$, and for a prime p write

$$\varepsilon(p) \;=\; \frac{1}{p}-\log\Bigl(1+\frac{1}{p}\Bigr), \tag{3.12}$$

so that $\varepsilon(p)>0$, because $\log(1+x)<x$ for $x>0$. Let γ denote Euler's constant and let

$$M \;=\; \gamma+\sum_{p}\Bigl(\log\Bigl(1-\frac{1}{p}\Bigr)+\frac{1}{p}\Bigr) \;=\; 0.2614972128\ldots \tag{3.13}$$

denote the Mertens constant [23].

Lemma 3.8. *We have*

$$\sum_{p}\varepsilon(p) \;=\; \log\zeta(2)-\gamma+M \;=\; 0.1819818504\ldots, \tag{3.14}$$

the sum being taken over all primes, and

$$\delta \;:=\; \sum_{p\ \text{odd prime}}\varepsilon(p) \;=\; \log\zeta(2)-\gamma+M-\frac{1}{2}+\log\frac{3}{2} \;=\; 0.0874469585\ldots. \tag{3.15}$$

Consequently, for every odd integer $n>1$,

$$0 \;<\; \sum_{p\mid n}\varepsilon(p) \;<\; \delta \qquad \text{and} \qquad e^{\Theta(n)-\delta} \;<\; B(n) \;<\; e^{\Theta(n)}. \tag{3.16}$$

Proof. For every prime p we have $\log(1+1/p)=\log(1-1/p^2)-\log(1-1/p)$, hence

$$\varepsilon(p) \;=\; \Bigl(\frac{1}{p}+\log\Bigl(1-\frac{1}{p}\Bigr)\Bigr)-\log\Bigl(1-\frac{1}{p^2}\Bigr). \tag{3.17}$$

Both series $\sum_p\bigl(1/p+\log(1-1/p)\bigr)$ and $\sum_p\log(1-1/p^2)$ converge absolutely; the first equals $M-\gamma$ by (3.13) and the second equals $-\log\zeta(2)$ by Proposition 2.8. This gives (3.14), and (3.15) follows on removing the term $\varepsilon(2)=\frac{1}{2}-\log\frac{3}{2}$. If $n>1$ is odd, its prime divisors form a finite nonempty set of odd primes and every $\varepsilon(p)$ is positive, which gives the first part of (3.16). Finally,

$$\log B(n) \;=\; \sum_{p\mid n}\log\Bigl(1+\frac{1}{p}\Bigr) \;=\; \Theta(n)-\sum_{p\mid n}\varepsilon(p), \tag{3.18}$$

which gives the second part of (3.16). □

4. An upper bound for $n/\varphi(n)$

Write

$$P_3 \;=\; \prod_{p\ \text{odd prime}}\Bigl(1-\frac{1}{p^3}\Bigr) \;=\; \frac{8}{7\zeta(3)} \;=\; 0.950751282949\ldots, \tag{4.1}$$

the evaluation following from Proposition 2.8 upon removing the factor $(1-2^{-3})^{-1}=8/7$ at $p=2$. The numerical bounds of this section use only the inequality $\zeta(3)<1.2020569032$, which gives $P_3>0.9507512829$, $(2+10^{-35})/P_3<2.1035995807$ and $31(2+10^{-35})/(30P_3)<2.1737195667$.

The constant 10^{-35} in the next lemma comes from Proposition 2.7: a quasiperfect number N satisfies $N>10^{35}$, so that $I(N)=2+1/N<2+10^{-35}$.

Lemma 4.1. *Let n be odd with $2\le I(n)\le 2+10^{-35}$.*

(i) *If $\nu_p(n)\ge 2$ for every prime $p\mid n$, then $A(n)>P_3$ and $\Phi(n)<(2+10^{-35})/P_3<2.10360$.*

(ii) *If* $\nu_q(n) = 1$ *for exactly one prime* $q \mid n$*, where* $q \geq 5$*, and* $\nu_p(n) \geq 2$ *for every other prime* $p \mid n$*, then*

$$A(n) \;>\; \frac{30}{31} P_3 \;=\; 0.920081886\ldots, \qquad \Phi(n) \;<\; 2.17372. \tag{4.2}$$

In both cases $2 < \Phi(n) < 2.17372$.

Proof. In case (i), $1 - p^{-(\nu_p(n)+1)} \geq 1 - p^{-3}$ for every $p \mid n$, so

$$A(n) \;\geq\; \prod_{p \mid n} (1 - p^{-3}) \;>\; \prod_{p \text{ odd prime}} (1 - p^{-3}) \;=\; P_3, \tag{4.3}$$

the second inequality holding because the prime divisors of n form a finite set of odd primes and every omitted factor lies strictly between 0 and 1. By (2.3), $\Phi(n) = I(n)/A(n) < (2 + 10^{-35})/P_3 = 2.1035995\ldots < 2.10360$.

In case (ii), the factor of $A(n)$ at q equals $1 - q^{-2}$, and the remaining prime divisors of n form a finite set of odd primes different from q. As in case (i),

$$A(n) \;>\; \left(1 - q^{-2}\right) \prod_{\substack{p \text{ odd prime} \\ p \neq q}} (1 - p^{-3}) \;=\; \frac{1 - q^{-2}}{1 - q^{-3}} P_3 \;=\; \left(1 - \frac{1}{q^2 + q + 1}\right) P_3. \tag{4.4}$$

The factor $1 - 1/(q^2+q+1)$ is increasing in q, so for $q \geq 5$ it is at least $30/31$. Hence $A(n) > \frac{30}{31} P_3$, and by (2.3), $\Phi(n) = I(n)/A(n) < 31(2 + 10^{-35})/(30 P_3) = 2.1737195\ldots < 2.17372$.

Finally, in either case, $A(n) < 1$ and $I(n) \geq 2$ give $\Phi(n) = I(n)/A(n) > 2$. □

Lemma 4.2. *Let* N *be an odd perfect number or a quasiperfect number. Then* N *is odd,* $2 \leq I(N) \leq 2 + 10^{-35}$*, and* N *satisfies the hypothesis of case* (i) *or of case* (ii) *of Lemma 4.1. Moreover, if* $3 \mid N$*, then* $\nu_3(N) \geq 2$.

Proof. If N is quasiperfect, then by Proposition 2.7 it is an odd square, so every exponent in its factorization is even and case (i) applies; moreover $I(N) = 2 + 1/N$ by Proposition 2.5 and $1/N < 10^{-35}$ because $N > 10^{35}$ by Proposition 2.7.

If N is an odd perfect number, then $I(N) = 2$ by Proposition 2.5, and $N = q^e s^2$ as in Proposition 2.6. Every prime factor of s occurs in N to an even exponent, and $q \geq 5$ because $q \equiv 1 \pmod 4$. If $e = 1$, case (ii) applies; if $e \geq 5$, case (i) applies. Since $q \neq 3$, the exponent $\nu_3(N)$ is even. □

5. Odd perfect and quasiperfect numbers

Theorem 5.1. *Let* N *be an odd perfect number or a quasiperfect number. Then:*

(a) $2 < N/\varphi(N) < 2.17372$.

(b) $105 \nmid N$.

(c) *If* $3 \mid N$*, then*

$$1.654912666\ldots \;=\; \frac{49}{3\pi^2} \;<\; \frac{49}{6\pi^2} \cdot \frac{N}{\varphi(N)} \;<\; B(N) \;<\; \frac{12}{13} I(N). \tag{5.1}$$

(d) *If* $3 \nmid N$*, then*

$$1.823781305\ldots \;=\; \frac{18}{\pi^2} \;<\; \frac{9}{\pi^2} \cdot \frac{N}{\varphi(N)} \;<\; B(N) \;<\; I(N). \tag{5.2}$$

Proof. (a) This is Lemma 4.1, which applies by Lemma 4.2.

(b) If $105 = 3 \cdot 5 \cdot 7$ divided N, then $\Phi(N) \geq \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} = 2.1875$, contradicting (a).

(c) By (b), there is a prime $t \in \{5,7\}$ with $t \nmid N$. The prime divisors of N form a finite set of odd primes different from t, so, by (3.7),

$$C(N) \;<\; \prod_{\substack{p \text{ odd prime}\\ p\neq t}} \frac{p^2}{p^2-1} \;=\; \frac{\pi^2}{8}\Big(1-\frac{1}{t^2}\Big) \;\le\; \frac{\pi^2}{8}\cdot\frac{48}{49} \;=\; \frac{6\pi^2}{49}. \tag{5.3}$$

By Lemma 2.4 and (a), $B(N) = \Phi(N)/C(N) > \frac{49}{6\pi^2}\Phi(N) > \frac{49}{3\pi^2}$. For the upper bound, $\nu_3(N) \ge 2$ by Lemma 4.2. By Propositions 2.6 and 2.7 and Lemma 4.2, N has a prime divisor $p \neq 3$ with $\nu_p(N) \ge 2$. Lemma 3.7 then gives

$$B(N) \;\le\; I(N)\Big(1+\frac{1}{12}\Big)^{-1}\Big(1+\frac{1}{p(p+1)}\Big)^{-1} \;<\; \frac{12}{13}\,I(N). \tag{5.4}$$

(d) The least prime factor of N is at least 5, so Theorem 3.5 and (a) give $B(N) > \Phi(N)/Z(5) = \frac{9}{\pi^2}\Phi(N) > \frac{18}{\pi^2}$. As in (c), N has a prime divisor p with $\nu_p(N) \ge 2$, and Lemma 3.7 gives $B(N) \le I(N)\big(1+1/(p(p+1))\big)^{-1} < I(N)$. □

Corollary 5.2. *If N is an odd perfect number with $3 \mid N$, then* $1.654912666\ldots = \dfrac{49}{3\pi^2} < B(N) < \dfrac{24}{13} = 1.846153846\ldots$. *If N is an odd perfect number with $3 \nmid N$, then* $\dfrac{18}{\pi^2} < B(N) < 2$.

Proof. This is Theorem 5.1 with $I(N) = 2$. □

Corollary 5.3. *Let N be an odd perfect number or a quasiperfect number. If $3 \mid N$, then*

$$\frac{13}{12} \;<\; C(N)\,A(N) \;<\; \frac{6\pi^2}{49} \;=\; 1.208522987\ldots. \tag{5.5}$$

If $3 \nmid N$, then $1 < C(N)A(N) < \dfrac{\pi^2}{9} = 1.096622711\ldots$.

Proof. By Lemma 2.4, $C(N)A(N) = I(N)/B(N)$, so the lower bounds follow from the upper bounds for $B(N)$ in Theorem 5.1. For the upper bounds, $A(N) < 1$, and the proof of Theorem 5.1 gives $C(N) < 6\pi^2/49$ when $3 \mid N$ and $C(N) < Z(5) = \pi^2/9$ when $3 \nmid N$. □

Bounds according to $\gcd(15, N)$**.** Recall that $\Theta(n) = \sum_{p\mid n} 1/p$. The following bounds, due to Cohen, Hagis and Suryanarayana, are quoted in [28].

Proposition 5.4. *Let N be an odd perfect number. Then* $0.667450 < \Theta(N) < 0.693148$ *if* $\gcd(15,N) = 1$, $0.603831 < \Theta(N) < 0.657304$ *if* $\gcd(15,N) = 3$, $0.647387 < \Theta(N) < 0.678036$ *if* $\gcd(15,N) = 5$, *and* $0.596063 < \Theta(N) < 0.673770$ *if* $\gcd(15,N) = 15$ [28].

Theorem 5.5. *Let N be an odd perfect number or a quasiperfect number, and let* $g = \gcd(15, N)$.

(a) *We have*

$$B(N) \;>\; \lambda_g\,\frac{N}{\varphi(N)} \;>\; 2\lambda_g, \qquad \text{where } \lambda_1 = \frac{75}{8\pi^2},\quad \lambda_3 = \frac{25}{3\pi^2},\quad \lambda_5 = \frac{9}{\pi^2},\quad \lambda_{15} = \frac{49}{6\pi^2}, \tag{5.6}$$

and $2\lambda_1 = 1.899772193\ldots$, $2\lambda_3 = 1.688686394\ldots$, $2\lambda_5 = 1.823781305\ldots$, $2\lambda_{15} = 1.654912666\ldots$.

(b) *If N is an odd perfect number, then*

$$B(N) \;<\; \begin{cases} \frac{4}{3}\exp\big(0.657304-\frac{1}{3}\big) \;<\; 1.84348 & \text{if } g=3,\\ \frac{6}{5}\exp\big(0.678036-\frac{1}{5}\big) \;<\; 1.93549 & \text{if } g=5,\\ \frac{8}{5}\exp\big(0.673770-\frac{8}{15}\big) \;<\; 1.84125 & \text{if } g=15. \end{cases} \tag{5.7}$$

(c) *If N is an odd perfect number, then, with δ and ε as in Lemma 3.8,*

$$B(N) \;>\; \begin{cases} \exp\big(0.667450-\delta+\varepsilon(3)+\varepsilon(5)\big) \;>\; 1.90281 & \text{if } g=1,\\ \exp\big(0.603831-\delta+\varepsilon(5)\big) \;>\; 1.70584 & \text{if } g=3,\\ \exp\big(0.647387-\delta+\varepsilon(3)\big) \;>\; 1.83233 & \text{if } g=5,\\ \exp\big(0.596063-\delta+\varepsilon(7)\big) \;>\; 1.67856 & \text{if } g=15. \end{cases} \tag{5.8}$$

Proof. (a) By Lemma 2.4 and Theorem 5.1(a), $B(N)=\Phi(N)/C(N)$ and $\Phi(N)>2$, so it suffices to show that $C(N)<1/\lambda_g$. If $g=1$, the least prime factor of N is at least 7, and Theorem 3.5 gives $C(N)<Z(7)=\zeta(2)\cdot\frac{3}{4}\cdot\frac{8}{9}\cdot\frac{24}{25}=8\pi^2/75$. If $g=3$, the prime divisors of N form a finite set of odd primes different from 5, so $C(N)<\frac{\pi^2}{8}\big(1-\frac{1}{25}\big)=3\pi^2/25$. If $g=5$, the least prime factor of N is at least 5, so $C(N)<Z(5)=\pi^2/9$. If $g=15$, then $7\nmid N$ by Theorem 5.1(b), so $C(N)<\frac{\pi^2}{8}\big(1-\frac{1}{49}\big)=6\pi^2/49$.

(b) Since $\log(1+x)<x$ for $x>0$, every term of the sum $\sum_{p\mid N}\big(1/p-\log(1+1/p)\big)$ is positive, and therefore

$$\log B(N) \;=\; \Theta(N)-\sum_{p\mid N}\Big(\frac{1}{p}-\log\Big(1+\frac{1}{p}\Big)\Big) \;\le\; \Theta(N)-\sum_{p\mid g}\Big(\frac{1}{p}-\log\Big(1+\frac{1}{p}\Big)\Big). \tag{5.9}$$

The bounds follow from Proposition 5.4, the last sum being $\frac{1}{3}-\log\frac{4}{3}$ for $g=3$, $\frac{1}{5}-\log\frac{6}{5}$ for $g=5$, and the sum of these two for $g=15$.

(c) The prime divisors of N form a finite set of odd primes that contains neither 3 nor 5 if $g=1$, does not contain 5 if $g=3$, does not contain 3 if $g=5$, and does not contain 7 if $g=15$, the last by Theorem 5.1(b). Since every $\varepsilon(p)$ is positive, (3.15) gives $\sum_{p\mid N}\varepsilon(p)<\delta-\varepsilon(3)-\varepsilon(5)$, $\delta-\varepsilon(5)$, $\delta-\varepsilon(3)$ and $\delta-\varepsilon(7)$ in the four cases, respectively. The bounds follow from (3.18) and the lower bounds of Proposition 5.4; numerically, $\delta-\varepsilon(3)-\varepsilon(5)=0.0241172\ldots$, $\delta-\varepsilon(5)=0.0697685\ldots$, $\delta-\varepsilon(3)=0.0417956\ldots$ and $\delta-\varepsilon(7)=0.0781212\ldots$. □

Corollary 5.6. *Let N be an odd perfect number and let $g=\gcd(15,N)$. Then*

$$\log C(N)+\log A(N) \;=\; \log 2-\Theta(N)+\sum_{p\mid N}\varepsilon(p), \tag{5.10}$$

and

$$\begin{aligned} 1 \;&<\; C(N)\,A(N) \;<\; 1.05108 && \text{if } g=1,\\ 1.08490 \;&<\; C(N)\,A(N) \;<\; 1.17245 && \text{if } g=3,\\ 1.03333 \;&<\; C(N)\,A(N) \;<\; 1.09151 && \text{if } g=5,\\ 1.08621 \;&<\; C(N)\,A(N) \;<\; 1.19150 && \text{if } g=15. \end{aligned} \tag{5.11}$$

Proof. By Lemma 2.4, $\log C(N)+\log A(N)=\log I(N)-\log B(N)$, and $I(N)=2$. Substituting (3.18) gives (5.10). The numerical bounds follow from $C(N)A(N)=2/B(N)$ and the bounds for $B(N)$ in Theorems 5.1 and 5.5. □

The numerical values in Theorem 5.5 and Corollary 5.6 were evaluated with `mpmath` at 40 decimal digits; see Experiment J in Section 20. Table 1 collects the bounds of Proposition 5.4 and of Theorems 5.1 and 5.5.

6. The abundancy index of $n\,\mathrm{rad}(n)$

For a positive integer n, let $\mathrm{rad}(n)=\prod_{p\mid n}p$ denote the product of the distinct prime divisors of n.

TABLE 1. Bounds according to $g = \gcd(15, N)$: the bounds for $\Theta(N)$ quoted in [28] for an odd perfect number N, and the bounds for $B(N)$ of Theorems 5.1 and 5.5.

g	$\Theta(N)$, N odd perfect	$B(N)$, N odd perfect	$B(N)$, N quasiperfect
1	$0.667450 < \Theta(N) < 0.693148$	$1.90281 < B(N) < 2$	$1.899772 < B(N) < I(N)$
3	$0.603831 < \Theta(N) < 0.657304$	$1.70584 < B(N) < 1.84348$	$1.688686 < B(N) < \frac{12}{13} I(N)$
5	$0.647387 < \Theta(N) < 0.678036$	$1.83233 < B(N) < 1.93549$	$1.823781 < B(N) < I(N)$
15	$0.596063 < \Theta(N) < 0.673770$	$1.67856 < B(N) < 1.84125$	$1.654912 < B(N) < \frac{12}{13} I(N)$

Theorem 6.1. *For every integer $n > 1$,*

$$I\big(n\,\mathrm{rad}(n)\big) \;=\; I(n)\prod_{p\mid n}\Big(1+\frac{1}{p\,\sigma\big(p^{\nu_p(n)}\big)}\Big) \;\le\; I(n)\prod_{p\mid n}\Big(1+\frac{1}{p(p+1)}\Big), \tag{6.1}$$

with equality if and only if n is squarefree. Moreover, $I\big(n\,\mathrm{rad}(n)\big) < n/\varphi(n)$.

Proof. Let p be a prime and $a \ge 1$ an integer. Since $\sigma(p^{a+1}) = p\,\sigma(p^a) + 1$,

$$\frac{I(p^{a+1})}{I(p^a)} \;=\; \frac{\sigma(p^{a+1})}{p\,\sigma(p^a)} \;=\; 1+\frac{1}{p\,\sigma(p^a)}. \tag{6.2}$$

We have $\nu_p\big(n\,\mathrm{rad}(n)\big) = \nu_p(n)+1$ for every prime $p \mid n$, and $n\,\mathrm{rad}(n)$ has the same prime divisors as n, so the multiplicativity of I gives the identity in (6.1). Since $\sigma(p^a) \ge p+1$, with equality if and only if $a = 1$, the inequality in (6.1) follows, with equality if and only if $\nu_p(n) = 1$ for every $p \mid n$. Finally, Proposition 2.3 applied to $n\,\mathrm{rad}(n)$ gives $I\big(n\,\mathrm{rad}(n)\big) < \Phi\big(n\,\mathrm{rad}(n)\big) = \Phi(n)$. □

Theorem 6.2. *Let*

$$\kappa \;=\; \prod_{p\ \text{odd prime}}\Big(1+\frac{1}{p(p^2+p+1)}\Big) \;=\; \frac{84\,\zeta(3)}{\pi^4} \;=\; 1.036584766\ldots. \tag{6.3}$$

(a) *If N is a quasiperfect number, then $I\big(N\,\mathrm{rad}(N)\big) < \kappa\, I(N) < 2.07317$.*

(b) *If N is an odd perfect number, then $I\big(N\,\mathrm{rad}(N)\big) < \dfrac{31}{30}\,\kappa\, I(N) < 2.14228$.*

Proof. For every prime p,

$$1+\frac{1}{p(p^2+p+1)} \;=\; \frac{I(p^3)}{I(p^2)} \;=\; \frac{1-p^{-4}}{1-p^{-3}}. \tag{6.4}$$

By Proposition 2.8, $\prod_{p\text{ odd prime}}(1-p^{-4}) = \dfrac{16}{15\,\zeta(4)} = \dfrac{96}{\pi^4}$, and by (4.1), $\prod_{p\text{ odd prime}}(1-p^{-3}) = P_3 = \dfrac{8}{7\zeta(3)}$. Dividing gives the evaluation in (6.3).

(a) By Proposition 2.7, N is odd and $\nu_p(N) \ge 2$ for every prime $p \mid N$, so $\sigma\big(p^{\nu_p(N)}\big) \ge p^2+p+1$. By Theorem 6.1,

$$I\big(N\,\mathrm{rad}(N)\big) \;\le\; I(N)\prod_{p\mid N}\Big(1+\frac{1}{p(p^2+p+1)}\Big) \;<\; \kappa\, I(N), \tag{6.5}$$

the second inequality holding because the prime divisors of N form a finite set of odd primes and every factor exceeds 1. Since $I(N) = 2 + 1/N \le 2 + 10^{-35}$, we obtain $\kappa\, I(N) < 2.07317$.

(b) By Lemma 4.2, either $\nu_p(N) \geq 2$ for every prime $p \mid N$, in which case the argument of (a) gives $I\big(N\,\mathrm{rad}(N)\big) < \kappa\, I(N)$, or there is exactly one prime $q \mid N$ with $\nu_q(N) = 1$, where $q \geq 5$, and $\nu_p(N) \geq 2$ for every other prime $p \mid N$. In the second case, Theorem 6.1 gives

$$I\big(N\,\mathrm{rad}(N)\big) \;\leq\; I(N)\Big(1+\frac{1}{q(q+1)}\Big)\prod_{\substack{p\mid N\\ p\neq q}}\Big(1+\frac{1}{p(p^2+p+1)}\Big) \;<\; \frac{31}{30}\,\kappa\, I(N), \tag{6.6}$$

since $1 + 1/(q(q+1)) \leq 31/30$ for $q \geq 5$. In both cases $I(N) = 2$ and $\frac{31}{30}\kappa \cdot 2 = 2.1422751\ldots < 2.14228$. □

7. The functions $\sigma^{\pm}$, φ^{+} and ψ^{+}

Sándor and Atanassov [25] introduced, for a multiplicative function f, the functions obtained by shifting each local factor of f by one unit; the functions φ^+ and σ^+ were studied further by Mandal [16]. Let $\psi(n) = n\prod_{p\mid n}(1+1/p) = nB(n)$ denote the Dedekind function. For $n > 1$ and $f \in \{\sigma, \varphi, \psi\}$ put

$$f^{+}(n) \;=\; \prod_{p\mid n}\Big(f\big(p^{\nu_p(n)}\big)+1\Big), \qquad f^{-}(n) \;=\; \prod_{p\mid n}\Big(f\big(p^{\nu_p(n)}\big)-1\Big), \tag{7.1}$$

and $f^+(1) = f^-(1) = 1$. These functions are different from the shifted abundancy indices I^+ and I^- of Section 10, in which every divisor, and not every local factor, is shifted.

Proposition 7.1. *Let $n > 1$ and write $a_p = \nu_p(n)$. Then*

$$\sigma^{-}(n) \;=\; \mathrm{rad}(n)\,\sigma\Big(\frac{n}{\mathrm{rad}(n)}\Big), \qquad \textit{that is,} \qquad \frac{\sigma^{-}(n)}{n} \;=\; I\Big(\frac{n}{\mathrm{rad}(n)}\Big), \tag{7.2}$$

and

$$\begin{aligned} \frac{\sigma^{+}(n)}{\sigma(n)} &= \prod_{p\mid n}\Big(1+\frac{1}{\sigma(p^{a_p})}\Big), & \frac{\sigma(n)}{\sigma^{-}(n)} &= \prod_{p\mid n}\Big(1+\frac{1}{p\,\sigma(p^{a_p-1})}\Big),\\ \frac{\psi^{+}(n)}{\psi(n)} &= \prod_{p\mid n}\Big(1+\frac{1}{p^{a_p-1}(p+1)}\Big), & \frac{\varphi^{+}(n)}{\varphi(n)} &= \prod_{p\mid n}\Big(1+\frac{1}{p^{a_p-1}(p-1)}\Big). \end{aligned} \tag{7.3}$$

Proof. For a prime p and an integer $a \geq 1$ we have $\sigma(p^a) - 1 = p + \cdots + p^a = p\,\sigma(p^{a-1})$. Multiplying over the prime divisors of n gives $\sigma^-(n) = \mathrm{rad}(n)\,\sigma\big(n/\mathrm{rad}(n)\big)$, and dividing by n gives (7.2). The identities (7.3) follow from (7.1) and the multiplicativity of σ, ψ and φ, using $\sigma(p^a) - 1 = p\,\sigma(p^{a-1})$, $\psi(p^a) = p^{a-1}(p+1)$ and $\varphi(p^a) = p^{a-1}(p-1)$. □

Theorem 7.2. *Let*

$$\rho \;=\; \prod_{p\ \textit{odd prime}}\Big(1+\frac{1}{p(p+1)}\Big) \;=\; \frac{\pi^2}{7\zeta(3)} \;=\; 1.172942380\ldots, \qquad \theta \;=\; \prod_{p\ \textit{odd prime}}\Big(1+\frac{1}{p(p-1)}\Big) \;=\; \frac{105\,\zeta(3)}{\pi^4} \;=\; 1$$

(7.4)

(a) *Let $n > 1$ be an odd integer with $\nu_p(n) \geq 2$ for every prime $p \mid n$; in particular, let n be a quasiperfect number. Then*

$$\frac{\sigma(n)}{\rho} \;<\; \sigma^{-}(n) \;<\; \sigma(n) \;<\; \sigma^{+}(n) \;<\; \rho\,\sigma(n), \qquad \psi(n) \;<\; \psi^{+}(n) \;<\; \rho\,\psi(n), \qquad \varphi(n) \;<\; \varphi^{+}(n) \;<\; \theta\,\varphi(n). \tag{7.5}$$

(b) *Let N be an odd perfect number. Then*

$$\frac{5}{6\rho}\,\sigma(N) \;<\; \sigma^-(N) \;<\; \sigma(N) \;<\; \sigma^+(N) \;<\; \frac{7\rho}{6}\,\sigma(N), \qquad \psi^+(N) \;<\; \frac{7\rho}{6}\,\psi(N), \qquad \varphi^+(N) \;<\; \frac{5\theta}{4}\,\varphi(N), \tag{7.6}$$

where $5/(6\rho) = 0.710463998\ldots$, $7\rho/6 = 1.368432777\ldots$ *and* $5\theta/4 = 1.619663697\ldots$.

Proof. For every prime p we have $1 + \dfrac{1}{p(p+1)} = \dfrac{1-p^{-3}}{1-p^{-2}}$, so, by (4.1) and (3.7), $\rho = P_3\,Z(3) = \dfrac{8}{7\zeta(3)} \cdot \dfrac{\pi^2}{8}$. Likewise $1 + \dfrac{1}{p(p-1)} = \dfrac{1+p^{-3}}{1-p^{-2}} = \dfrac{1-p^{-6}}{(1-p^{-3})(1-p^{-2})}$, so, by Proposition 2.8, the product over all primes equals $\zeta(2)\zeta(3)/\zeta(6) = 315\,\zeta(3)/(2\pi^4)$; removing the factor $3/2$ at $p = 2$ gives the value of θ.

(a) Let $p \mid n$ and $a = \nu_p(n) \geq 2$. Then $\sigma(p^a) \geq p^2 + p + 1 > p(p+1)$, $p\,\sigma(p^{a-1}) \geq p(p+1)$, $p^{a-1}(p+1) \geq p(p+1)$ and $p^{a-1}(p-1) \geq p(p-1)$. Hence each factor of the first three products in (7.3) is greater than 1 and at most $1 + 1/(p(p+1))$, and each factor of the fourth is greater than 1 and at most $1 + 1/(p(p-1))$. The prime divisors of n form a finite nonempty set of odd primes, so the first three products lie strictly between 1 and ρ and the fourth strictly between 1 and θ, which is (7.5). A quasiperfect number is an odd square by Proposition 2.7.

(b) By Lemma 4.2, either $\nu_p(N) \geq 2$ for every $p \mid N$, in which case (a) applies and gives (7.6), or there is exactly one prime $q \mid N$ with $\nu_q(N) = 1$, where $q \geq 5$. In the second case the factors of the four products in (7.3) at q are $1 + 1/(q+1) \leq 7/6$, $1 + 1/q \leq 6/5$, $1 + 1/(q+1) \leq 7/6$ and $1 + 1/(q-1) \leq 5/4$, respectively, and the products of the remaining factors are bounded as in (a). All factors exceed 1, which gives the inequalities $\sigma^-(N) < \sigma(N) < \sigma^+(N)$. □

Corollary 7.3. *If N is a quasiperfect number, then*

$$1.705113595\ldots \;=\; \frac{14\,\zeta(3)}{\pi^2} \;<\; \frac{I(N)}{\rho} \;<\; I\Big(\frac{N}{\operatorname{rad}(N)}\Big) \;<\; I(N) \;<\; I\big(N\operatorname{rad}(N)\big) \;<\; \kappa\,I(N). \tag{7.7}$$

If N is an odd perfect number, then

$$1.420927996\ldots \;=\; \frac{5}{3\rho} \;<\; I\Big(\frac{N}{\operatorname{rad}(N)}\Big) \;<\; 2 \;<\; I\big(N\operatorname{rad}(N)\big) \;<\; \frac{31}{15}\,\kappa. \tag{7.8}$$

Proof. By (7.2), $I\big(N/\operatorname{rad}(N)\big) = \sigma^-(N)/N$, so the bounds for $I\big(N/\operatorname{rad}(N)\big)$ follow from Theorem 7.2 and $\sigma(N)/N = I(N)$, with $I(N) > 2$ in the quasiperfect case and $I(N) = 2$ in the odd perfect case. The bounds for $I\big(N\operatorname{rad}(N)\big)$ are Theorems 6.1 and 6.2. □

8. The least prime factor and the number of prime factors

In this section $\pi(x)$ denotes the number of primes not exceeding x, and γ denotes Euler's constant. Lower bounds for the number of distinct prime factors of an odd perfect number in terms of its least prime factor, based on estimates for the distribution of primes, go back to Norton [20]. We use the following explicit estimates of Rosser and Schoenfeld.

Proposition 8.1. *For every real $x > 1$,*

$$e^\gamma(\log x)\Big(1 - \frac{1}{2\log^2 x}\Big) \;<\; \prod_{p \leq x} \frac{p}{p-1} \;<\; e^\gamma(\log x)\Big(1 + \frac{1}{\log^2 x}\Big). \tag{8.1}$$

Moreover, $\pi(x) > x/\log x$ for $x \geq 17$ and $\pi(x) < 1.25506\,x/\log x$ for $x > 1$ [23].

Theorem 8.2. *Let n be an odd integer with $I(n) \geq 2$, with exactly k distinct prime factors and with least prime factor p_1. Let y be the k-th term of the increasing sequence of all primes not less than p_1, and let $L = \log(p_1 - 1)$. Then*

$$y \ > \ X, \qquad \text{where } X \ = \ (p_1 - 1)^2 \exp\Big(-\frac{2}{L}\Big), \tag{8.2}$$

and

$$k \ \geq \ \pi(X) - \pi(p_1 - 1) + 1. \tag{8.3}$$

If $p_1 \geq 11$, then

$$k \ > \ \frac{X}{\log X} - 1.25506\,\frac{p_1 - 1}{\log(p_1 - 1)} + 1. \tag{8.4}$$

Proof. Let $p_1 < \cdots < p_k$ be the prime factors of n and let $q_1 < q_2 < \cdots$ be the primes not less than p_1, so that $q_1 = p_1$ and $q_k = y$. Then $p_i \geq q_i$ for every i, and since $x/(x-1)$ is decreasing for $x > 1$, Proposition 2.3 gives

$$2 \ \leq \ I(n) \ < \ \prod_{i=1}^{k} \frac{p_i}{p_i - 1} \ \leq \ \prod_{i=1}^{k} \frac{q_i}{q_i - 1} \ = \ \frac{\prod_{p \leq y} p/(p-1)}{\prod_{p \leq p_1 - 1} p/(p-1)}. \tag{8.5}$$

Since $p_1 - 1 \geq 2$, both bounds of (8.1) apply, the lower one at $x = p_1 - 1$ and the upper one at $x = y$. Writing $t = \log y$, we obtain

$$2e^{\gamma}\Big(L - \frac{1}{2L}\Big) \ < \ 2 \prod_{p \leq p_1 - 1} \frac{p}{p-1} \ < \ \prod_{p \leq y} \frac{p}{p-1} \ < \ e^{\gamma}\Big(t + \frac{1}{t}\Big), \tag{8.6}$$

that is, $t + 1/t > 2L - 1/L$. Since $y \geq p_1 > p_1 - 1$, we have $t > L > 0$ and hence $1/t < 1/L$. Therefore $t > 2L - 2/L$, which is (8.2).

The primes not exceeding y are the primes not exceeding $p_1 - 1$ together with $q_1, \ldots, q_k$, so $\pi(y) = \pi(p_1 - 1) + k$. Since $X < y$ and y is prime, $\pi(X) \leq \pi(y) - 1$, which gives (8.3).

Finally, X is increasing in p_1, and for $p_1 = 11$ we have $X = 100 \exp(-2/\log 10) = 41.954\ldots \geq 17$. Hence, for $p_1 \geq 11$, Proposition 8.1 gives $\pi(X) > X/\log X$ and $\pi(p_1 - 1) < 1.25506\,(p_1 - 1)/\log(p_1 - 1)$, and (8.4) follows from (8.3). □

Corollary 8.3. *Let N be an odd perfect number or a quasiperfect number with exactly k distinct prime factors and least prime factor p_1. Then (8.2) and (8.3) hold for N, and (8.4) holds if $p_1 \geq 11$. For example, if $p_1 = 101$, then $X = 6477.21\ldots$, and (8.4) gives $k \geq 712$, while (8.3) gives $k \geq \pi(6477) - \pi(100) + 1 = 816$.*

Proof. By Propositions 2.5 and 2.7, N is odd and $I(N) \geq 2$, so Theorem 8.2 applies. For $p_1 = 101$ we have $L = \log 100$ and $X = 10^4 \exp(-2/\log 100) = 6477.21\ldots$; the right-hand side of (8.4) equals $711.80\ldots$, and $\pi(6477) = 840$, $\pi(100) = 25$. □

For $p_1 = 3, 5, 7, 11, 13, 17, 19, 23$, the right-hand side of (8.3) equals $0, 1, 3, 10, 14, 25, 31, 47$, respectively. An odd perfect number has at least ten distinct prime factors [18], and at least twelve if it is not divisible by 3 [19]; hence, for odd perfect numbers, among these values of p_1, (8.3) exceeds these bounds exactly for $p_1 \in \{13, 17, 19, 23\}$.

9. The reciprocals of the prime divisors of a quasiperfect number

If N is an odd perfect number, then $\sum_{p|N} 1/p < \log 2$ [3]; see also [28]. We prove the corresponding statement for quasiperfect numbers.

Lemma 9.1. *For every real x with $0 < x < 1$, $\log(1 + x + x^2) > x$.*

Proof. Let $g(x) = \log(1+x+x^2) - x$ for $x \in [0,1)$. Then $g(0) = 0$ and

$$g'(x) \;=\; \frac{1+2x}{1+x+x^2} - 1 \;=\; \frac{x(1-x)}{1+x+x^2},$$

which is positive for $x \in (0,1)$. Hence $g(x) > 0$ for every $x \in (0,1)$. □

Theorem 9.2. *If N is a quasiperfect number, then*

$$\log B(N) \;<\; \sum_{p \mid N} \frac{1}{p} \;<\; \log I(N) \;=\; \log\Bigl(2 + \frac{1}{N}\Bigr). \tag{9.1}$$

In particular, $\sum_{p\mid N} 1/p > \log\bigl(49/(3\pi^2)\bigr) = 0.503748237\ldots$ *if* $3 \mid N$, *and* $\sum_{p\mid N} 1/p > \log\bigl(18/\pi^2\bigr) = 0.600911986\ldots$ *if* $3 \nmid N$.

Proof. By Proposition 2.7, N is an odd square, so $\nu_p(N) \geq 2$ and $p \geq 3$ for every prime $p \mid N$. Since $I(p^a)$ is increasing in a, Lemma 9.1 with $x = 1/p$ gives

$$\log I\bigl(p^{\nu_p(N)}\bigr) \;\geq\; \log\Bigl(1 + \frac{1}{p} + \frac{1}{p^2}\Bigr) \;>\; \frac{1}{p}.$$

Summing over the prime divisors of N and using the multiplicativity of I gives $\log I(N) > \sum_{p\mid N} 1/p$, and $I(N) = 2 + 1/N$ by Proposition 2.5. For the lower bound, $\log(1+x) < x$ for $x > 0$ gives $\log B(N) = \sum_{p\mid N} \log(1 + 1/p) < \sum_{p\mid N} 1/p$. The numerical bounds follow from Theorem 5.1. □

10. Shifted abundancy indices

We now study the two natural perturbations of I obtained by shifting each divisor by one unit in the denominator.

Definition 10.1. For a positive integer n, the *upward* and *downward shifted abundancy indices* of n are

$$I^+(n) = 1 + \sum_{\substack{d\mid n \\ d>1}} \frac{1}{d+1}, \qquad I^-(n) = 1 + \sum_{\substack{d\mid n \\ d>1}} \frac{1}{d-1}. \tag{10.1}$$

In both definitions the divisor $d = 1$ is deliberately left unshifted. This is not a cosmetic convention: it normalizes both functions so that $I^+(1) = I(1) = I^-(1) = 1$, it keeps I^- finite, and, as Remark 13.6 explains, it supplies the single positive term that makes the entire theory possible. Since $1/(d+1) < 1/d < 1/(d-1)$ for every $d > 1$, we have the elementary chain

$$I^+(n) \;<\; I(n) \;<\; I^-(n) \qquad (n > 1), \tag{10.2}$$

and the object of the remaining sections is to determine exactly how far apart these quantities can be when n is odd.

Unlike I, neither shift is multiplicative: for instance $I^+(15) = 71/48$ while $I^+(3)\,I^+(5) = (5/4)(7/6) = 35/24 = 70/48$, and $I^-(15) = 51/28$ while $I^-(3)\,I^-(5) = (3/2)(5/4) = 15/8$. This failure of multiplicativity is exactly what makes the comparison of the shifts with I a genuinely extremal problem rather than a local computation at prime powers, and it is the reason why the classical Euler-product machinery is unavailable here.

Motivation. There are three reasons for studying these two functions in connection with odd perfect numbers. Write $\sigma_{-k}(n) = \sum_{d|n} d^{-k} = \sigma_k(n)/n^k$ for $k \geq 1$, so that $\sigma_{-1} = I$.

First, I^- and I^+ collect all the higher abundancy indices $\sigma_{-k}(n)$ into single quantities, of which I is the leading term.

Proposition 10.2. *For every positive integer n,*

$$I^-(n) \;=\; 1 + \sum_{k\geq 1} \big(\sigma_{-k}(n) - 1\big), \qquad I^+(n) \;=\; 1 + \sum_{k\geq 1} (-1)^{k+1} \big(\sigma_{-k}(n) - 1\big), \tag{10.3}$$

both series converging absolutely. Consequently

$$I^-(n) - I(n) \;=\; \sum_{k\geq 2} \big(\sigma_{-k}(n) - 1\big), \qquad I^-(n) - I^+(n) \;=\; 2 \sum_{j\geq 1} \big(\sigma_{-2j}(n) - 1\big). \tag{10.4}$$

Proof. For every integer $d \geq 2$ the geometric series give $1/(d-1) = \sum_{k\geq 1} d^{-k}$ and $1/(d+1) = \sum_{k\geq 1} (-1)^{k+1} d^{-k}$, both absolutely convergent. Summing over the finitely many divisors $d > 1$ of n and using $\sigma_{-k}(n) - 1 = \sum_{d|n,\, d>1} d^{-k}$ gives (10.3). The first identity of (10.4) follows on removing the term $k = 1$, and the second on subtracting the two series of (10.3). □

For a perfect number the defining equation fixes the first term, $\sigma_{-1}(N) = 2$, and says nothing directly about the others. By (10.4), the bounds for $I^-(N)$ and for $I^-(N) - I^+(N)$ are bounds for the totals of the remaining terms: for an odd perfect number N, Corollaries 10.12 and 10.5 below give

$$\sum_{k\geq 2} \big(\sigma_{-k}(N) - 1\big) \;\leq\; 2S_{\mathcal{E}} - 2 \;=\; \frac{2179}{8008} \;=\; 0.2721\ldots, \qquad \sum_{j\geq 1} \big(\sigma_{-2j}(N) - 1\big) \;<\; 0.2124620116. \tag{10.5}$$

Second, the two shifts place 2 strictly inside an interval attached to N: by (10.2), every perfect number satisfies $I^+(N) < 2 < I^-(N)$. The sharp constants determined below turn this into explicit necessary conditions that can be tested on any odd integer of Euler form without knowing whether it is perfect (Corollary 10.12).

Third, the quantities $I(n)$, $n/\varphi(n)$, $B(n)$ and $C(n)$ studied in the first part of this paper are multiplicative, so each is determined by the prime powers dividing n taken one at a time, and any relation among them splits into independent local relations. The shifted indices are not multiplicative: by Lemma 12.2 below, $I^-(n) - I^+(n) = \sum_{d|n,\, d>1} 2/(d^2-1)$ is a sum over all divisors of n, composite ones included, measured against the total $\sum_{d\geq 3,\, d \text{ odd}} 2/(d^2-1) = 1/2$ taken over all odd integers. They therefore record how the divisors of n are distributed among the odd integers, which is information that the multiplicative quantities do not carry, and Proposition 12.3 relates the two kinds of information. Whether constraints of this non-multiplicative kind can be combined with the known arithmetic conditions on an odd perfect number to exclude its existence is an open question, and it is the question that motivates the determination of the sharp constants in the following sections.

For every integer $n > 1$ we have

$$I^-(n) \;>\; I^+(n), \tag{10.6}$$

by Proposition 12.1 below. The first result bounds the size of this separation on the class of integers singled out by Euler's structure theorem and Nielsen's bound.

Theorem 10.3. *Let n be an odd integer having at least ten distinct prime divisors, exactly one prime power in its factorization carrying an odd exponent and all the others carrying even*

exponents. Then, writing $t_{\mathcal{E}} = \log_2(836970/680449)$,

$$0 \;<\; I^-(n) - I^+(n) \;<\; \frac{1}{2}, \qquad 1 \;<\; \frac{I^-(n)}{I^+(n)} \;<\; 2^{t_{\mathcal{E}}} = \frac{836970}{680449}, \tag{10.7}$$

and both constants in (10.7) *are optimal for the class in question, being approached but not attained.*

Remark 10.4. We say at once, so that the reader is not misled about the strength of (10.6), that the hypotheses play no role in it: the strict inequality $I^-(n) > I^+(n)$ holds for every integer $n > 1$ whatsoever, odd or even, with any number of prime divisors and any exponent pattern, and its proof is a one-line termwise comparison, since $1/(d-1) > 1/(d+1)$ for every divisor $d > 1$. What the hypotheses do control is the second half of Theorem 10.3, that is, how large the gap can be. Even there the accounting is uneven: the additive constant $1/2$ needs oddness but nothing else, being sharp already for all odd n, whereas the multiplicative constant $836970/680449$ genuinely uses the exponent structure, being strictly smaller than the constant $612/497$ valid for all odd integers. Determining constants of this kind is the substance of these sections, and Corollary 10.5 records what they give in the case of principal interest.

Corollary 10.5. *Let n be an odd perfect number, so that in particular n satisfies the hypotheses of Theorem 10.3. Then $I^+(n) < 2 < I^-(n)$ and*

$$0 \;<\; I^-(n) - I^+(n) \;<\; 2S_{\mathcal{E}} - \frac{2}{R_{\mathcal{E}}} = 0.4249240231041\ldots, \tag{10.8}$$

in the notation of (10.16) *and* (10.18) *below, which improves the universal bound* $1/2$ *of* (10.7) *by just over fifteen percent.*

Theorem 10.3 and Corollary 10.5 are proved in Section 12. The elementary half of Theorem 10.3 depends only on Section 11; the sharp constants it quotes are established in Section 18, and Corollary 10.5 uses in addition Corollary 10.12. We now describe those sharp comparison constants, which occupy the following sections.

Theorem 10.6. *Put*

$$R^* = \frac{I(315)}{I^+(315)} = \frac{3254203392}{3001892677} = 1.084050544822325771642\ldots, \tag{10.9}$$

and $c' = \log_2 R^* = 0.1164320252014595917677\ldots$. *Then for every odd positive integer n,*

$$\frac{I(n)}{2^0} \;\geq\; I^+(n) \;\geq\; \frac{I(n)}{2^{c'}}. \tag{10.10}$$

Both constants are optimal. The left-hand inequality is an equality if and only if $n = 1$, and no constant $c > 0$ is admissible there. The right-hand inequality is an equality if and only if $n = 315 = 3^2 \cdot 5 \cdot 7$, and no constant smaller than c' is admissible.

The hypothesis that n be odd is essential and cannot be weakened: for $n = 12$ one computes $I(12)/I^+(12) = 1.164853250434\ldots > R^*$, so the right-hand inequality of Theorem 10.6 already fails at $n = 12$ (see Remark 15.2).

The downward shift behaves analogously, but with a noticeably larger optimal constant and a different, smaller extremal integer.

Theorem 10.7. *Put*

$$S^* = \frac{I^-(15)}{I(15)} = \frac{255}{224} = 1.138392857142857142857\ldots, \tag{10.11}$$

and $c'' = \log_2 S^* = 0.1869985148012538301362\ldots$. *Then for every odd positive integer* n,

$$2^0\, I(n) \;\le\; I^-(n) \;\le\; 2^{c''} I(n). \tag{10.12}$$

Both constants are optimal. The left-hand inequality is an equality if and only if $n = 1$, *and no constant* $c > 0$ *is admissible there. The right-hand inequality is an equality if and only if* $n = 15 = 3 \cdot 5$, *and no constant smaller than* c'' *is admissible.*

Combining Theorems 10.6 and 10.7 with (10.2) gives, for every odd $n > 1$, the sharp chain

$$2^{-c'} I(n) \;\le\; I^+(n) \;<\; I(n) \;<\; I^-(n) \;\le\; 2^{c''} I(n), \tag{10.13}$$

in which the outer constants cannot be improved. One might expect the resulting comparison of the two shifts with each other, namely $I^-(n) \le 2^{c'+c''} I^+(n)$, to be sharp as well. It is not, and the loss is genuine rather than a rounding artifact: the two outer extremal problems are solved at different integers, 315 and 15, so no single n can be extremal for both. The correct constant is smaller and is again attained.

Theorem 10.8. *Put*

$$T^* = \frac{I^-(15)}{I^+(15)} = \frac{612}{497} = 1.231388329979879275654\ldots, \tag{10.14}$$

and $t = \log_2 T^* = 0.3002858011303656023128\ldots$. *Then* $I^-(n) \le 2^t\, I^+(n)$ *for every odd* n, *with equality if and only if* $n = 15$, *and* t *is optimal. Moreover*

$$t = 0.3002858011\ldots \;<\; c' + c'' = 0.3034305400\ldots, \tag{10.15}$$

so the sharp spread constant is strictly smaller than the sum of the two sharp one-sided constants.

Our motivation comes from the theory of odd perfect numbers. Euler proved that an odd perfect number must have exactly one prime power with an odd exponent (Proposition 11.3 below), and Nielsen [18] proved that such a number has at least ten distinct prime divisors, improving his own earlier bound of nine [19]; Ochem and Rao showed that any odd perfect number exceeds 10^{1500} [21] and gave complementary bounds involving the total number of prime factors counted with multiplicity [22]. These conditions single out a distinguished class of odd integers, and on that class all three extremal constants improve.

Definition 10.9. An odd positive integer n is said to be of *Euler form* if at most one prime power in the canonical factorization of n carries an odd exponent; equivalently, if $n = p^\alpha m^2$ with p prime, $p \nmid m$ and α odd, or n is a perfect square. We write $\mathcal{E}$ for the set of odd integers of Euler form, and $\mathcal{E}_{10}$ for the subset of those $n \in \mathcal{E}$ with exactly one odd exponent and $\omega(n) \ge 10$, where ω counts distinct prime divisors.

Theorem 10.10. *Put*

$$R_{\mathcal{E}} = \frac{I(2205)}{I^+(2205)} = \frac{29332068105860657280}{27090788267918857153} = 1.082732175076497971588\ldots, \tag{10.16}$$

and $c'_{\mathcal{E}} = \log_2 R_{\mathcal{E}} = 0.11467642164083667290 78\ldots$. *Then for every* $n \in \mathcal{E}$,

$$\frac{I(n)}{2^0} \;\ge\; I^+(n) \;\ge\; \frac{I(n)}{2^{c'_{\mathcal{E}}}}, \tag{10.17}$$

and both constants are optimal on $\mathcal{E}$. *The right-hand inequality is an equality if and only if* $n = 2205 = 3^2 \cdot 5 \cdot 7^2$. *On the smaller class* $\mathcal{E}_{10}$ *the same two constants are optimal, but there the right-hand inequality is strict for every* n, *the value* $c'_{\mathcal{E}}$ *being approached and not attained.*

Theorem 10.11. *Put*

$$S_{\mathcal{E}} = \frac{I^-(45)}{I(45)} = \frac{18195}{16016} = 1.136051448551448551449\ldots, \qquad T_{\mathcal{E}} = \frac{I^-(45)}{I^+(45)} = \frac{836970}{680449} = 1.230026056324573921\ldots \tag{10.18}$$

and $c''_{\mathcal{E}} = \log_2 S_{\mathcal{E}} = 0.1840281718930352745525\ldots$, $t_{\mathcal{E}} = \log_2 T_{\mathcal{E}} = 0.2986888772979174328\ldots$. *Then for every* $n \in \mathcal{E}$,

$$2^0\, I(n) \;\le\; I^-(n) \;\le\; 2^{c''_{\mathcal{E}}} I(n), \qquad I^-(n) \;\le\; 2^{t_{\mathcal{E}}}\, I^+(n), \tag{10.19}$$

and all three constants are optimal on $\mathcal{E}$. *Both right-hand inequalities are equalities if and only if* $n = 45 = 3^2 \cdot 5$. *On* $\mathcal{E}_{10}$ *the same constants are optimal but neither is attained.*

It is worth emphasizing how little the extra structural hypotheses buy: passing from all odd integers to the Euler class moves each optimal exponent only in the third decimal place, and imposing $\omega(n) \ge 10$ on top of the Euler form does not move it at all. This is a genuine phenomenon rather than a defect of the method, and Sections 16 and 18 identify exactly why: each supremum on $\mathcal{E}_{10}$ is approached along integers whose auxiliary primes tend to infinity and therefore contribute nothing in the limit.

Finally, the two-sided information about I^+ and I^- at an odd perfect number, which feeds Corollary 10.5, is the following.

Corollary 10.12. *If* n *is an odd perfect number, then*

$$1.847178873998728783 \;<\; \frac{2}{R_{\mathcal{E}}} \;\le\; I^+(n) \;<\; 2 \;<\; I^-(n) \;\le\; 2S_{\mathcal{E}} = \frac{18195}{8008} = 2.2721028971\ldots. \tag{10.20}$$

Consequently, an odd integer n *of Euler form with* $I^+(n) \le 2/R_{\mathcal{E}}$, *or with* $I^-(n) \ge 18195/8008$, *cannot be perfect.*

11. Preliminaries

Throughout, p and q denote primes, $\nu_p(n)$ is the exponent of p in n, $\omega(n)$ is the number of distinct prime divisors of n, $\tau(n)$ is the number of divisors of n, $\pi(x)$ is the number of primes not exceeding x, and $\sigma(n) = \sum_{d|n} d$. Since $d \mapsto n/d$ permutes the divisors of n, we have $I(n) = \sigma(n)/n$, and I is multiplicative with $I(p^a) = 1 + p^{-1} + \cdots + p^{-a}$.

Definition 11.1. For $n \ge 1$ we write

$$R(n) = \frac{I(n)}{I^+(n)}, \qquad S(n) = \frac{I^-(n)}{I(n)}, \qquad T(n) = \frac{I^-(n)}{I^+(n)} = R(n)\, S(n) \tag{11.1}$$

for the three comparison ratios. By (10.2) all three exceed 1 for $n > 1$.

Proposition 11.2. *For every* $n \ge 1$ *we have* $I^+(n) \le I(n) \le I^-(n)$, *with equality throughout if and only if* $n = 1$. *Moreover*

$$\inf\{R(n) : n \text{ odd}\} = \inf\{S(n) : n \text{ odd}\} = \inf\{T(n) : n \text{ odd}\} = 1, \tag{11.2}$$

and none of the three infima is attained by any odd $n > 1$.

Proof. Comparing the defining sums term by term, the contributions of $d = 1$ agree and every divisor $d > 1$ contributes $1/(d+1) < 1/d < 1/(d-1)$. Hence

$$I(n) - I^+(n) = \sum_{\substack{d|n \\ d>1}} \frac{1}{d(d+1)} \ge 0, \qquad I^-(n) - I(n) = \sum_{\substack{d|n \\ d>1}} \frac{1}{d(d-1)} \ge 0, \tag{11.3}$$

and each quantity vanishes exactly when n has no divisor exceeding 1, that is, when $n = 1$. For the infima, take $n = p$ an odd prime; then $R(p) = (1 + 1/p)/(1 + 1/(p + 1))$, $S(p) = (1 + 1/(p - 1))/(1 + 1/p)$ and $T(p) = (1 + 1/(p - 1))/(1 + 1/(p + 1))$, each of which tends to 1 as $p \to \infty$ while remaining strictly greater than 1. □

Proposition 11.2 already settles every left-hand constant appearing in Theorems 10.6, 10.7, 10.10 and 10.11: for instance $I(n)/2^c \geq I^+(n)$ holds for all odd n if and only if $2^c \leq \inf R(n) = 1$, that is, if and only if $c \leq 0$, and identically for S and T. So $c = 0$ is optimal on the full odd class and on every subclass considered here, since the family of primes used in the proof can be replaced by $p \cdot q_1^2 \cdots q_9^2$ with all primes tending to infinity in order to stay inside $\mathcal{E}_{10}$. All the remaining work concerns the upper constants, that is, the determination of $\sup R(n)$, $\sup S(n)$ and $\sup T(n)$.

We now record Euler's classical parity restriction, which explains Definition 10.9; for its history see Dickson [4, Ch. I].

Proposition 11.3 (Euler)**.** *Let n be an odd perfect number. Then exactly one prime power in the canonical factorization of n carries an odd exponent, and all the remaining exponents are even. Consequently $n \in \mathcal{E}$, and by [18] in fact $n \in \mathcal{E}_{10}$.*

Proof. Write $n = \prod_{i=1}^{k} p_i^{a_i}$ with distinct odd primes p_i. Perfection means $\sigma(n) = 2n$, and since n is odd we have $2n \equiv 2 \pmod 4$, so $\nu_2(\sigma(n)) = 1$. On the other hand σ is multiplicative, so $\sigma(n) = \prod_{i=1}^{k} \sigma(p_i^{a_i})$. For an odd prime p we have $\sigma(p^a) = 1 + p + \cdots + p^a \equiv a + 1 \pmod 2$, because each of the $a+1$ summands is odd. Hence $\sigma(p_i^{a_i})$ is even precisely when a_i is odd. Since ν_2 of a product is the sum of the ν_2 of its factors and $\nu_2(\sigma(n)) = 1$, exactly one factor $\sigma(p_i^{a_i})$ can be even, and that factor satisfies $\nu_2 = 1$. Therefore exactly one exponent a_i is odd. The last assertion follows from Nielsen's theorem [18] that $\omega(n) \geq 10$ for every odd perfect number. □

Note that Definition 10.9 allows n to be a perfect square, that is, to have no odd exponent at all; this slightly larger class is the natural one for our extremal problem, since it is closed under the truncation operations used in Section 14, whereas the class of integers with exactly one odd exponent is not. The distinction matters, and we shall be careful with it: a perfect square lies in $\mathcal{E}$ but never in $\mathcal{E}_{10}$.

12. Separation of the two shifts

The separation itself is elementary, and we isolate it first in the generality in which it actually holds.

Proposition 12.1. *For every integer $n > 1$ we have $I^-(n) > I^+(n)$, and $I^-(1) = I^+(1) = 1$. For odd n this is the outer pair of the chain (10.2), but the statement needs neither oddness nor that chain, so we prove it directly.*

Proof. Both quantities are 1 plus a sum indexed by the divisors $d > 1$ of n, the summands being $1/(d-1)$ and $1/(d+1)$ respectively. For every such d we have $d + 1 > d - 1 \geq 1 > 0$, hence $1/(d-1) > 1/(d+1)$. Summing over the divisors $d > 1$ of n, of which there is at least one because $n > 1$, gives $I^-(n) - I^+(n) = \sum_{d|n,\, d>1} \big(1/(d-1) - 1/(d+1)\big) > 0$. For $n = 1$ the index set is empty and both sums vanish. □

The size of the gap is governed by a telescoping estimate, which is where oddness enters.

Lemma 12.2. *For every $n \geq 1$,*

$$I^-(n) - I^+(n) = \sum_{\substack{d|n \\ d>1}} \frac{2}{d^2 - 1}. \tag{12.1}$$

If n is odd then $I^-(n) - I^+(n) < 1/2$, and if in addition n is divisible by every odd integer in $[3,k]$ for some odd $k \geq 3$, then

$$\frac{1}{2} - \frac{1}{k+1} \;\leq\; I^-(n) - I^+(n) \;<\; \frac{1}{2}. \tag{12.2}$$

For general n the corresponding bound is $I^-(n) - I^+(n) < 3/2$, and both $1/2$ and $3/2$ are optimal, being approached along m_k and along $\operatorname{lcm}(1,\dots,k)$ respectively as $k \to \infty$, where m_k denotes the odd part of $\operatorname{lcm}(1,\dots,k)$.

Proof. For every integer $d > 1$ we have $1/(d-1) - 1/(d+1) = 2/\big((d-1)(d+1)\big) = 2/(d^2-1)$, which gives (12.1). If n is odd then its divisors $d > 1$ are distinct odd integers at least 3, so (12.1) is a subsum of

$$\sum_{\substack{d \geq 3 \\ d \text{ odd}}} \frac{2}{d^2-1} = \sum_{j \geq 1} \Big(\frac{1}{2j} - \frac{1}{2j+2} \Big) = \frac{1}{2}, \tag{12.3}$$

the series (12.3) telescoping because consecutive odd integers $d = 2j+1$ produce the consecutive even reciprocals $1/(2j)$ and $1/(2j+2)$. The subsum is proper, because no integer is divisible by every odd integer exceeding 1, so the inequality is strict. If every odd $d \in [3,k]$ divides n, the same telescoping gives the partial sum

$$\sum_{\substack{3 \leq d \leq k \\ d \text{ odd}}} \frac{2}{d^2-1} = \frac{1}{2} - \frac{1}{k+1}, \tag{12.4}$$

which is a lower bound for (12.1), proving (12.2). For general n the divisors $d > 1$ are distinct integers at least 2, and substituting $j = d-1$ gives

$$\sum_{d \geq 2} \frac{2}{d^2-1} = \sum_{j \geq 1} \Big(\frac{1}{j} - \frac{1}{j+2} \Big) = 1 + \frac{1}{2} = \frac{3}{2}, \tag{12.5}$$

the telescoping now proceeding in steps of two so that the first two terms survive; the same properness argument gives strict inequality. For the optimality of the two constants, every odd $d \leq k$ divides m_k and every $d \leq k$ divides $\operatorname{lcm}(1,\dots,k)$, so the corresponding partial sums of (12.3) and (12.5) are attained, and they converge to $1/2$ and $3/2$. □

Proposition 12.3. *For every integer $n > 1$,*

$$I^-(n) - I^+(n) \;\geq\; 2 \sum_{p \mid n} \frac{1}{p^2-1} \;>\; \sum_{p \mid n} \frac{1}{p^2-1} \;>\; \log C(n), \tag{12.6}$$

where $C(n) = \prod_{p \mid n} p^2/(p^2-1)$ as in (2.2). Consequently $I^-(n) - I^+(n) > 2 \log C(n)$ and

$$B(n) \;>\; \frac{n}{\varphi(n)} \exp\Big(-\frac{I^-(n) - I^+(n)}{2} \Big). \tag{12.7}$$

Proof. The prime divisors of n are among the divisors $d > 1$ of n, and every term of (12.1) is positive, which gives the first inequality; the second holds because the sum is positive. Since $\log(1+x) < x$ for $x > 0$,

$$\log C(n) \;=\; \sum_{p \mid n} \log\Big(1 + \frac{1}{p^2-1} \Big) \;<\; \sum_{p \mid n} \frac{1}{p^2-1}, \tag{12.8}$$

which gives the third inequality and $I^-(n) - I^+(n) > 2 \log C(n)$. By Lemma 2.4, $C(n) = \Phi(n)/B(n)$ with $\Phi(n) = n/\varphi(n)$, and (12.7) follows. □

Lemma 12.4. *The constant* $1/2$ *in* (12.2) *is optimal even for the class* $\mathcal{E}_{10}$*: there exist* $n \in \mathcal{E}_{10}$ *with* $I^-(n) - I^+(n)$ *arbitrarily close to* $1/2$*.*

Proof. Let $k \geq 29$ be odd, let m_k denote the odd part of $\operatorname{lcm}(1, 2, \ldots, k)$, so that every odd $d \leq k$ divides m_k and $\omega(m_k) = \pi(k) - 1$, and let q be any prime exceeding k, necessarily odd since $q > k \geq 29$. Put

$$n_k = m_k^2\, q. \tag{12.9}$$

Both m_k and q are odd, so n_k is odd. Every exponent of m_k^2 is even and $q \nmid m_k$, since every prime factor of m_k is at most $k < q$, so n_k has exactly one prime power with an odd exponent, namely q^1; and $\omega(n_k) = \omega(m_k) + 1 = \pi(k) \geq \pi(29) = 10$. Hence $n_k \in \mathcal{E}_{10}$. Since $m_k \mid n_k$, every odd $d \in [3, k]$ divides n_k, so (12.2) gives $1/2 - 1/(k+1) \leq I^-(n_k) - I^+(n_k) < 1/2$, and letting $k \to \infty$ through odd values completes the proof. □

Remark 12.5. The auxiliary prime q in Lemma 12.4 is not a convenience but a necessity. The integer m_k^2 alone is a perfect square, hence has *no* prime power with an odd exponent; it therefore lies in $\mathcal{E}$ but not in $\mathcal{E}_{10}$, and cannot be used to test optimality on the class of Theorem 10.3. Multiplying by a single new prime restores the required parity without disturbing any of the divisors already present, since $m_k \mid m_k^2 q$.

Proof of Theorem 10.3. Let n satisfy the hypotheses. Having at least ten distinct prime divisors, n exceeds 1, so (10.6) is Proposition 12.1, and the two left-hand strict inequalities in (10.7) follow as well.

The additive upper bound $I^-(n) - I^+(n) < 1/2$ is Lemma 12.2, n being odd, and its optimality on the class in question is Lemma 12.4, whose construction is not negotiable for the reason given in Remark 12.5.

For the multiplicative bound, the hypotheses place n in $\mathcal{E}_{10} \subseteq \mathcal{E}$, so Theorem 10.11 gives $I^-(n) \leq 2^{t\varepsilon} I^+(n)$ with $2^{t\varepsilon} = T_{\mathcal{E}} = 836970/680449$, the inequality being strict on $\mathcal{E}_{10}$ because equality on $\mathcal{E}$ occurs only at $n = 45$, which has $\omega(45) = 2 < 10$. The same theorem shows that $T_{\mathcal{E}}$ is approached along $45 \cdot q_1^2 \cdots q_8^2$ with the primes q_i tending to infinity, all of which lie in $\mathcal{E}_{10}$, so no smaller constant is admissible. □

Remark 12.6. The proof shows exactly where each hypothesis is used. Oddness and the size of $\omega(n)$ are irrelevant to (10.6), which needs only $n > 1$; the additive bound $1/2$ needs oddness but nothing else, since for even n the sharp constant is $3/2$ by Lemma 12.2; and only the multiplicative bound uses the Euler form, and even there the hypothesis $\omega(n) \geq 10$ affects nothing but attainment. This is the honest accounting behind Remark 10.4.

Proof of Corollary 10.5. Let n be an odd perfect number. By Proposition 11.3 and Nielsen's theorem [18], n satisfies the hypotheses of Theorem 10.3, so $I^-(n) - I^+(n) > 0$. By Corollary 10.12 we have $2/R_{\mathcal{E}} \leq I^+(n) < 2 < I^-(n) \leq 2S_{\mathcal{E}}$, and subtracting the extreme bounds gives

$$I^-(n) - I^+(n) \;<\; 2S_{\mathcal{E}} - \frac{2}{R_{\mathcal{E}}} = \frac{18195}{8008} - \frac{27090788267918857153}{14666034052930328640} = \frac{43623651349277204129}{102662238370512300480}, \tag{12.10}$$

whose decimal expansion begins $0.4249240231041\ldots$, smaller than $1/2$ by just over fifteen percent. □

13. A divisor-local decomposition

The whole analysis of the sharp constants rests on the observation that the quantity whose sign we must control is itself a divisor sum, with a weight depending only on the divisor and on the constant being tested.

Definition 13.1. For a real number $c > 1$ define $g_c : \mathbb{Z}_{\geq 1} \to \mathbb{R}$ by

$$g_c(1) = c - 1, \qquad g_c(d) = \frac{c}{d+1} - \frac{1}{d} \quad (d > 1), \tag{13.1}$$

and for $n \geq 1$ set $G_c(n) = \sum_{d|n} g_c(d)$.

Lemma 13.2. *For every $c > 1$ and every $n \geq 1$ we have*

$$G_c(n) = c\, I^+(n) - I(n). \tag{13.2}$$

In particular $I^+(n) \geq I(n)/c$ holds if and only if $G_c(n) \geq 0$; equivalently, $G_c(n) \geq 0$ if and only if $c \geq R(n)$.

Proof. By the definitions,

$$c\, I^+(n) - I(n) = c\Big(1 + \sum_{d|n,\, d>1} \frac{1}{d+1}\Big) - \Big(1 + \sum_{d|n,\, d>1} \frac{1}{d}\Big) = (c-1) + \sum_{d|n,\, d>1} \Big(\frac{c}{d+1} - \frac{1}{d}\Big), \tag{13.3}$$

which is exactly $G_c(n)$. Since $I^+(n) > 0$, dividing by $I^+(n)$ gives the stated equivalences. □

Lemma 13.3. *Let $c > 1$ and let $d > 1$ be an integer. Then $g_c(d) > 0$ if and only if $d > 1/(c-1)$. In particular, if $c > 14/13$ then $g_c(d) > 0$ for every integer $d \geq 13$.*

Proof. For $d > 1$ we have $g_c(d) = \big(cd - (d+1)\big)/\big(d(d+1)\big)$, whose denominator is positive; the numerator equals $d(c-1) - 1$, which is positive, zero or negative according as d is greater than, equal to or less than $1/(c-1)$. If $c > 14/13$ then $1/(c-1) < 13$, so every integer $d \geq 13$ satisfies $d > 1/(c-1)$. □

We record the two companion decompositions now, since they are proved by the identical computation and will be used in Sections 17 and 18.

Definition 13.4. For a real number $c > 1$ define $h_c, k_c : \mathbb{Z}_{\geq 1} \to \mathbb{R}$ by

$$h_c(1) = k_c(1) = c - 1, \qquad h_c(d) = \frac{c}{d} - \frac{1}{d-1}, \qquad k_c(d) = \frac{c}{d+1} - \frac{1}{d-1} \quad (d > 1), \tag{13.4}$$

and set $H_c(n) = \sum_{d|n} h_c(d)$ and $K_c(n) = \sum_{d|n} k_c(d)$.

Lemma 13.5. *For every $c > 1$ and every $n \geq 1$ we have $H_c(n) = c\, I(n) - I^-(n)$ and $K_c(n) = c\, I^+(n) - I^-(n)$. Consequently $H_c(n) \geq 0$ if and only if $c \geq S(n)$, and $K_c(n) \geq 0$ if and only if $c \geq T(n)$. Moreover, for $d > 1$,*

$$h_c(d) > 0 \iff d > \frac{c}{c-1}, \qquad k_c(d) > 0 \iff d > \frac{c+1}{c-1}. \tag{13.5}$$

In particular, if $c > 9/8$ then $h_c(d) > 0$ for every integer $d \geq 9$, and if $c > 6/5$ then $k_c(d) > 0$ for every integer $d \geq 11$.

Proof. The two identities follow exactly as in Lemma 13.2, the $d = 1$ terms of I, I^+ and I^- all being equal to 1. For the sign conditions, $h_c(d) = \big(c(d-1) - d\big)/\big(d(d-1)\big) = \big(d(c-1) - c\big)/\big(d(d-1)\big)$ has positive denominator for $d > 1$, so its sign is that of $d(c-1) - c$; similarly $k_c(d) = \big(c(d-1) - (d+1)\big)/\big((d+1)(d-1)\big)$ has the sign of $d(c-1) - (c+1)$. Finally $c > 9/8$ gives $c/(c-1) < 9$ and $c > 6/5$ gives $(c+1)/(c-1) < 11$. □

The three thresholds 14/13, 9/8 and 6/5 govern the three problems, and they are what make the deficit sets finite: for admissible c only the odd divisors $d \in \{3, 5, 7, 9, 11\}$, respectively $d \in \{3, 5, 7\}$ and $d \in \{3, 5, 7, 9\}$, can contribute negatively.

Remark 13.6. Leaving $d = 1$ unshifted is essential. If one were to define $\widetilde{I}(n) = \sum_{d|n} 1/(d+1)$, shifting every divisor including $d = 1$, then the surplus term $c - 1$ would disappear from G_c, no positive contribution would remain to offset the deficits at $d \in \{3, 5, 7, 9, 11\}$, and no constant c of the sort considered here would exist; indeed $I(n)/\widetilde{I}(n) \to 2$ as n runs through the primes. For the downward shift the point is even more basic: shifting $d = 1$ as well would make the term $1/(d-1)$ undefined.

14. Reduction and truncation

All three extremal problems are governed by a weight function that is positive beyond an explicit threshold, so we prove the reduction machinery once, in that generality, and then specialize. Throughout this section D denotes an integer with $D \geq 3$, and $w : \mathbb{Z}_{\geq 1} \to \mathbb{R}$ is any function satisfying

$$w(d) > 0 \qquad \text{for every integer } d \geq D. \tag{14.1}$$

We write $W(n) = \sum_{d|n} w(d)$, and we let Π_D denote the set of odd primes below D. By Lemma 13.3 the weight g_c of Definition 13.1 satisfies (14.1) with $D = 13$ whenever $c > 14/13$, and by Lemma 13.5 the weights h_c and k_c satisfy it with $D = 9$ whenever $c > 9/8$, respectively $D = 11$ whenever $c > 6/5$.

Lemma 14.1 (Reduction)**.** *Let w and D be as above, let n be odd, and set $m = \prod_{p \in \Pi_D} p^{\nu_p(n)}$. Then $W(n) \geq W(m)$, with equality if and only if $n = m$. Moreover m has the same exponent parities as n at every prime below D, and no other prime divides m.*

Proof. By construction $m \mid n$, so every divisor of m is a divisor of n and $W(n) - W(m) = \sum_{d|n,\, d\nmid m} w(d)$. Let $d \mid n$ with $d \nmid m$. Since m contains the full p-part of n for every odd prime $p < D$, the divisor d must be divisible by some prime $q \mid n$ with $q \notin \Pi_D$. As n is odd, q is odd, so q is an odd prime that is not smaller than D, whence $d \geq q \geq D$. By (14.1) each such term is positive. The displayed sum is therefore positive unless it is empty, and it is empty exactly when every divisor of n divides m, that is, when $n = m$. The remaining assertions are immediate from the definition of m. □

Lemma 14.2 (Truncation)**.** *Let w and D be as above and let m be odd.*

(i) *If p is a prime with $a = \nu_p(m) \geq 1$ and $p^a \geq D$, then $W(m) > W(m/p)$.*
(ii) *If p is a prime with $a = \nu_p(m) \geq 2$ and $p^{a-1} \geq D$, then $W(m) > W(m/p^2)$, and m/p^2 has the same exponent parities as m at every prime.*

Proof. For (i), the divisors of m that do not divide m/p are exactly those $d \mid m$ with $\nu_p(d) = a$; each such d is divisible by p^a and hence satisfies $d \geq p^a \geq D$, so $w(d) > 0$ by (14.1). There is at least one such divisor, namely $d = p^a$, so $W(m) - W(m/p) = \sum_{\nu_p(d)=a} w(d) > 0$. For (ii), the divisors of m that do not divide m/p^2 are those with $\nu_p(d) \in \{a-1, a\}$, each of which is at least $p^{a-1} \geq D$; the same argument applies, and dividing by p^2 changes no exponent parity. □

Corollary 14.3. *Let w and D be as above. For a prime p put $\alpha_p = \max\{a \geq 0 : p^a < D\}$ and $\beta_p = 1 + \alpha_p$. Then*

$$\inf\{W(n) : n \text{ odd}\} = \min\Big\{W(m) : m = \textstyle\prod_{p \in \Pi_D} p^{e_p},\ 0 \leq e_p \leq \alpha_p\Big\}, \tag{14.2}$$

$$\inf\{W(n) : n \in \mathcal{E}\} = \min\Big\{W(m) : m = \textstyle\prod_{p \in \Pi_D} p^{e_p},\ 0 \leq e_p \leq \beta_p,\ \#\{p : e_p \text{ odd}\} \leq 1\Big\}, \tag{14.3}$$

and in both cases the minimum is attained on the indicated finite list.

TABLE 2. The eight largest values of $R = I/I^+$ on the reduction list $\mathcal{L}$.

m	factorization	$R(m)$
315	$3^2 \cdot 5 \cdot 7$	1.084050544822326
105	$3 \cdot 5 \cdot 7$	1.084026385793879
45	$3^2 \cdot 5$	1.082720380219532
1155	$3 \cdot 5 \cdot 7 \cdot 11$	1.081754648556323
15	$3 \cdot 5$	1.081690140845070
3465	$3^2 \cdot 5 \cdot 7 \cdot 11$	1.081427810553709
495	$3^2 \cdot 5 \cdot 11$	1.080798472639735
165	$3 \cdot 5 \cdot 11$	1.080246381305724

Proof. Given an odd n, Lemma 14.1 replaces n by m supported on Π_D without increasing W. For the first identity apply Lemma 14.2(i) repeatedly: whenever some $e_p = \nu_p(m)$ satisfies $p^{e_p} \geq D$, that is, whenever $e_p > \alpha_p$, divide by p. Each step strictly decreases W and strictly decreases m, so the process terminates at a member of the first list. For the second identity note that $m \in \mathcal{E}$ whenever $n \in \mathcal{E}$, by the parity clause of Lemma 14.1, and apply Lemma 14.2(ii), which preserves all exponent parities and hence membership in $\mathcal{E}$: whenever $e_p \geq 2$ and $p^{e_p-1} \geq D$, that is, whenever $e_p > \beta_p$, divide by p^2. Since the lists consist of odd integers, and of members of $\mathcal{E}$ in the second case, the reverse inequalities are trivial, and the lists are finite so the minima are attained. □

We now specialize. For $D = 13$ we have $\Pi_D = \{3, 5, 7, 11\}$ with $\alpha_3 = 2$ and $\alpha_5 = \alpha_7 = \alpha_{11} = 1$. For $D = 9$ we have $\Pi_D = \{3, 5, 7\}$ with $\alpha_3 = \alpha_5 = \alpha_7 = 1$. For $D = 11$ we have $\Pi_D = \{3, 5, 7\}$ with $\alpha_3 = 2$ and $\alpha_5 = \alpha_7 = 1$. In each case $\beta_p = \alpha_p + 1$.

Corollary 14.4. *Let $c > 14/13$. Then $\inf\{G_c(n) : n \text{ odd}\} = \min\{G_c(m) : m \in \mathcal{L}\}$ and $\inf\{G_c(n) : n \in \mathcal{E}\} = \min\{G_c(m) : m \in \mathcal{L}_{\mathcal{E}}\}$, where*

$$\mathcal{L} = \{3^a 5^b 7^e 11^f : a \in \{0, 1, 2\},\ b, e, f \in \{0, 1\}\}, \tag{14.4}$$

$$\mathcal{L}_{\mathcal{E}} = \{3^a 5^b 7^e 11^f : a \leq 3,\ b, e, f \leq 2,\ \#\{\text{odd entries among } a, b, e, f\} \leq 1\}, \tag{14.5}$$

of cardinalities 24 and 56 respectively.

The counting is immediate. For $\mathcal{L}$ there are $3 \cdot 2 \cdot 2 \cdot 2 = 24$ tuples. For $\mathcal{L}_{\mathcal{E}}$ there are $2^4 = 16$ tuples with all four entries even, and $16 + 8 + 8 + 8 = 40$ tuples with exactly one odd entry, according as the odd entry sits at 3 (two odd choices $a \in \{1, 3\}$ and 2^3 even choices elsewhere) or at one of 5, 7, 11 (one odd choice and 2^3 even choices elsewhere), for a total of 56.

15. The sharp constant for odd integers

Theorem 15.1. *We have $\max\{R(m) : m \in \mathcal{L}\} = R(315) = 3254203392/3001892677 = R^*$, attained only at $m = 315 = 3^2 \cdot 5 \cdot 7$. Moreover $R^* > 14/13$.*

Proof. The evaluation of R at each of the 24 elements of $\mathcal{L}$ is a finite exact rational computation; the eight largest values are listed in Table 2. The largest is $R(315) = (208/105)/(5574943543/3050815680) = 3254203392/3001892677$, and the second largest is $R(105) = 5372928/4956455 = 1.0840263857\ldots$, which is strictly smaller. The final assertion is the integer inequality $13 \cdot 3254203392 = 42304644096 > 42026497478 = 14 \cdot 3001892677$. □

Proof of Theorem 10.6. The left-hand inequality, its case of equality and the optimality of $c = 0$ are Proposition 11.2.

For the right-hand inequality, let c be any real constant and suppose $I^+(n) \geq I(n)/c$ for all odd n. Taking $n = 315$ and using Lemma 13.2 gives $c \geq R(315) = R^*$. Conversely suppose $c \geq R^*$. Since $R^* > 14/13$ by Theorem 15.1, we have $c > 14/13$, so Corollary 14.4 applies and yields, for every odd n,

$$G_c(n) \geq \min_{m\in\mathcal{L}} G_c(m) = \min_{m\in\mathcal{L}} I^+(m)\big(c - R(m)\big) \geq 0, \tag{15.1}$$

the last step because $I^+(m) > 0$ and $c \geq R^* = \max_{m\in\mathcal{L}} R(m)$ by Theorem 15.1. By Lemma 13.2 this is exactly $I^+(n) \geq I(n)/c$. Hence the set of admissible constants is precisely $[R^*, \infty)$, so the least admissible exponent is $c' = \log_2 R^*$.

It remains to identify the case of equality when $c = R^*$. Equality means $G_{R^*}(n) = 0$. Let m be the integer produced from n by Lemma 14.1 followed by the truncations of Lemma 14.2(i), so that $m \in \mathcal{L}$ and $G_{R^*}(n) \geq G_{R^*}(m)$, with equality if and only if no reduction step was actually performed, that is, if and only if $n = m$. Since $G_{R^*}(m) = I^+(m)(R^* - R(m)) \geq 0$ with equality if and only if $R(m) = R^*$, and since by Theorem 15.1 the only element of $\mathcal{L}$ with $R(m) = R^*$ is $m = 315$, we conclude that $G_{R^*}(n) = 0$ forces $n = m = 315$. Conversely $G_{R^*}(315) = 0$ by the definition of R^*. □

Remark 15.2. Oddness cannot be dropped. For $n = 12$ the divisors are $1, 2, 3, 4, 6, 12$, so $I(12) = 7/3$ and $I^+(12) = 1+1/3+1/4+1/5+1/7+1/13 = 10937/5460$, whence $R(12) = 12740/10937 = 1.164853250434\ldots$, which exceeds R^*. The mechanism is transparent in the language of Section 13: an even integer admits the divisor $d = 2$, whose deficit $g_c(2) = c/3 - 1/2$ is much larger than any of the deficits at $d \in \{3, 5, 7, 9, 11\}$, and it admits further even divisors $4, 6, 8, 10$ as well. An exhaustive search over even $n < 2 \cdot 10^5$ gives $\max R(n) = R(12)$.

16. The sharp constant for integers of Euler form

Theorem 16.1. *We have* $\max\{R(m) : m \in \mathcal{L}_\mathcal{E}\} = R(2205) = 29332068105860657280/27090788267918857153 = R_\mathcal{E}$*, attained only at* $m = 2205 = 3^2 \cdot 5 \cdot 7^2$*. Moreover* $R_\mathcal{E} > 14/13$*.*

Proof. Again this is a finite exact rational computation over the 56 elements of $\mathcal{L}_\mathcal{E}$; the ten largest values appear in Table 3. The largest is $R(2205) = R_\mathcal{E}$ and the second largest is $R(45) = 9568/8837 = 1.0827203802\ldots$. The two are extremely close, differing by less than $1.2 \cdot 10^{-5}$, so the comparison is performed in exact rational arithmetic: $R_\mathcal{E} - R(45) = 1.1794856978\ldots \times 10^{-5} > 0$. The final assertion is the integer inequality $13 \cdot 29332068105860657280 = 381316885376188544640 > 379271035750864000142 = 14 \cdot 27090788267918857153$. □

Proof of Theorem 10.10. The argument is identical to that of Theorem 10.6, with Corollary 14.4 and Theorem 16.1 in place of their counterparts, and with 2205 in place of 315; note that $2205 \in \mathcal{E}$ since only the exponent of 5 is odd. This gives that the set of admissible constants on $\mathcal{E}$ is exactly $[R_\mathcal{E}, \infty)$, with equality at $c = R_\mathcal{E}$ precisely at $n = 2205$. The left-hand constant is again $c = 0$ by Proposition 11.2.

It remains to treat $\mathcal{E}_{10}$. Since $\mathcal{E}_{10} \subseteq \mathcal{E}$, the inequality holds on $\mathcal{E}_{10}$. It is strict there: equality on $\mathcal{E}$ forces $n = 2205$, and $\omega(2205) = 3 < 10$, so $2205 \notin \mathcal{E}_{10}$. To see that no smaller constant is admissible on $\mathcal{E}_{10}$, choose seven primes $q_1 < \cdots < q_7$, all larger than some bound Q, and put $n_Q = 2205 \cdot q_1^2 \cdots q_7^2$. Then $\omega(n_Q) = 3 + 7 = 10$ and the only odd exponent of n_Q is that of 5, so $n_Q \in \mathcal{E}_{10}$. Every divisor of n_Q is of the form $d \cdot e$ with $d \mid 2205$ and $e \mid q_1^2 \cdots q_7^2$, and every such

TABLE 3. The ten largest values of $R = I/I^+$ on the Eulerian reduction list $\mathcal{L}_{\mathcal{E}}$.

m	factorization	$R(m)$
2205	$3^2 \cdot 5 \cdot 7^2$	1.082732175076498
45	$3^2 \cdot 5$	1.082720380219532
1575	$3^2 \cdot 5^2 \cdot 7$	1.082130938289801
225	$3^2 \cdot 5^2$	1.080968708359679
3675	$3 \cdot 5^2 \cdot 7^2$	1.080909058334637
11025	$3^2 \cdot 5^2 \cdot 7^2$	1.080842255290577
5445	$3^2 \cdot 5 \cdot 11^2$	1.080190790441669
75	$3 \cdot 5^2$	1.080055099454515
675	$3^3 \cdot 5^2$	1.079659587358526
266805	$3^2 \cdot 5 \cdot 7^2 \cdot 11^2$	1.079532752315904

divisor with $e > 1$ exceeds Q. Hence

$$I(n_Q) = I(2205) \prod_{i=1}^{7} \left(1 + q_i^{-1} + q_i^{-2}\right) \longrightarrow I(2205), \qquad I^+(n_Q) \longrightarrow I^+(2205) \tag{16.1}$$

as $Q \to \infty$, the second limit because the divisors with $e > 1$ number at most 3^7 and each contributes less than $1/Q$ to the defining sum. Therefore $R(n_Q) \to R_{\mathcal{E}}$, so no constant smaller than $R_{\mathcal{E}}$ is admissible on $\mathcal{E}_{10}$, the value being approached and never attained. □

Remark 16.2. The restriction $\omega(n) \geq 10$ contributes nothing to the constant, and this is not an artifact. The proof exhibits the supremum on $\mathcal{E}_{10}$ as a limit along $2205 \cdot q_1^2 \cdots q_7^2$ with $q_i \to \infty$: the constraint $\omega(n) \geq 10$ is satisfied by seven auxiliary primes which can be taken so large that their divisors contribute an arbitrarily small surplus. Any hypothesis of the form $\omega(n) \geq k$ with k fixed is subject to the same construction, so no bound on the number of prime divisors alone can improve $R_{\mathcal{E}}$.

17. THE DOWNWARD SHIFT

We now run the same machinery for I^-. The weight is h_c of Definition 13.4, which by Lemma 13.5 is positive beyond $D = 9$ as soon as $c > 9/8$. The relevant candidate lists are therefore supported on $\{3, 5, 7\}$ only, and are correspondingly shorter:

$$\mathcal{M} = \{3^a 5^b 7^e : a, b, e \in \{0, 1\}\}, \qquad \mathcal{M}_{\mathcal{E}} = \{3^a 5^b 7^e : a, b, e \leq 2,\ \#\{\text{odd entries}\} \leq 1\}, \tag{17.1}$$

of cardinalities 8 and 20.

Theorem 17.1. *We have* $\max\{S(m) : m \in \mathcal{M}\} = S(15) = 255/224 = S^*$, *attained only at* $m = 15$, *and* $\max\{S(m) : m \in \mathcal{M}_{\mathcal{E}}\} = S(45) = 18195/16016 = S_{\mathcal{E}}$, *attained only at* $m = 45$. *Moreover* $S^* > 9/8$ *and* $S_{\mathcal{E}} > 9/8$.

Proof. The exact rational values of S on the two lists are given in Tables 4 and 5. On $\mathcal{M}$ the largest is $S(15) = (51/28)/(8/5) = 255/224$ and the second largest is $S(105) = 385597/339456 = 1.1359263056\ldots$; on $\mathcal{M}_{\mathcal{E}}$ the largest is $S(45) = 18195/16016$ and the second largest is $S(75) = 291625/256928 = 1.1350456158\ldots$. The threshold assertions are the integer inequalities $8 \cdot 255 = 2040 > 2016 = 9 \cdot 224$ and $8 \cdot 18195 = 145560 > 144144 = 9 \cdot 16016$. □

Proof of Theorem 10.7. Word for word the proof of Theorem 10.6, with H_c, S, $\mathcal{M}$, S^*, Lemma 13.5 and the threshold $9/8$ in place of G_c, R, $\mathcal{L}$, R^*, Lemma 13.2 and the threshold $14/13$, and

TABLE 4. The values of $S = I^-/I$ on the full reduction list $\mathcal{M}$. Note that $S(3) = 9/8$ coincides exactly with the threshold of Lemma 13.5.

m	factorization	$S(m)$	exact value
15	$3 \cdot 5$	1.138392857142857	255/224
105	$3 \cdot 5 \cdot 7$	1.135926305618401	385597/339456
21	$3 \cdot 7$	1.126562500000000	721/640
3	3	1.125000000000000	9/8
35	$5 \cdot 7$	1.054432189542484	10325/9792
5	5	1.041666666666667	25/24
7	7	1.020833333333333	49/48
1	1	1.000000000000000	1

TABLE 5. The six largest values of $S = I^-/I$ on the Eulerian reduction list $\mathcal{M}_{\mathcal{E}}$.

m	factorization	$S(m)$
45	$3^2 \cdot 5$	1.136051448551449
75	$3 \cdot 5^2$	1.135045615892390
225	$3^2 \cdot 5^2$	1.132704887745210
2205	$3^2 \cdot 5 \cdot 7^2$	1.130785726329724
3675	$3 \cdot 5^2 \cdot 7^2$	1.130468410836243
1575	$3^2 \cdot 5^2 \cdot 7$	1.129514727833856

with 15 in place of 315. The reduction applies because $S^* > 9/8$ by Theorem 17.1, and Corollary 14.3 is invoked with $\alpha_3 = \alpha_5 = \alpha_7 = 1$. □

The extremal integer has moved from 315 to 15, and the reason is visible in Table 4. For the upward shift the deficits at $d = 3, 5, 7, 9$ are all comparatively small and it pays to collect as many of them as possible, which forces $9 \mid n$. For the downward shift the deficit at $d = 3$, namely $h_c(3) = c/3 - 1/2$, is by far the dominant one, the deficit at $d = 7$ is already outweighed by the surplus of the divisor 21 alone, since at $c = S^*$ one has $h_c(7) = -0.0040391\ldots$ against $h_c(21) = +0.0042092\ldots$, and $d = 9$ lies beyond the threshold and contributes a surplus. The optimum is therefore reached by taking the two smallest odd primes and stopping.

Remark 17.2. Oddness is again essential. For $n = 6$ the divisors are $1, 2, 3, 6$, so $I(6) = 2$ and $I^-(6) = 1 + 1 + 1/2 + 1/5 = 27/10$, whence $S(6) = 27/20 = 1.35$, far above S^*. Here the mechanism is even starker than in Remark 15.2: an even integer has the divisor $d = 2$, for which $1/(d-1) = 1$ is twice $1/d$, so a single even divisor already costs a factor close to 2. An exhaustive search over even $n \le 2 \cdot 10^5$ gives $\max S(n) = S(6) = 27/20$.

18. THE SPREAD BETWEEN THE TWO SHIFTS

Finally we compare the two shifts directly. The relevant weight is k_c of Definition 13.4, positive beyond $D = 11$ once $c > 6/5$, so the candidate lists are again supported on $\{3, 5, 7\}$ but now allow $\nu_3 = 2$, since $9 < 11$:

$$\mathcal{N} = \{3^a 5^b 7^e : a \le 2,\ b, e \le 1\}, \qquad \mathcal{N}_{\mathcal{E}} = \{3^a 5^b 7^e : a \le 3,\ b, e \le 2,\ \#\{\text{odd entries}\} \le 1\}, \tag{18.1}$$

of cardinalities 12 and 24.

TABLE 6. The largest values of $T = I^-/I^+$ on the reduction lists $\mathcal{N}$ (left) and $\mathcal{N}_{\mathcal{E}}$ (right).

m	factorization	$T(m)$	m	factorization	$T(m)$
15	$3 \cdot 5$	1.231388329979879	45	$3^2 \cdot 5$	1.230026056324574
105	$3 \cdot 5 \cdot 7$	1.231374087607709	75	$3 \cdot 5^2$	1.225911805558067
45	$3^2 \cdot 5$	1.230026056324574	225	$3^2 \cdot 5^2$	1.224418539458636
315	$3^2 \cdot 5 \cdot 7$	1.228089370595555	2205	$3^2 \cdot 5 \cdot 7^2$	1.224338089014440
63	$3^2 \cdot 7$	1.209439774406335	1575	$3^2 \cdot 5^2 \cdot 7$	1.222282832242999

Theorem 18.1. *We have* $\max\{T(m) : m \in \mathcal{N}\} = T(15) = 612/497 = T^*$, *attained only at* $m = 15$, *and* $\max\{T(m) : m \in \mathcal{N}_{\mathcal{E}}\} = T(45) = 836970/680449 = T_{\mathcal{E}}$, *attained only at* $m = 45$. *Moreover* $T^* > 6/5$ *and* $T_{\mathcal{E}} > 6/5$.

Proof. The list $\mathcal{N}$ has $3 \cdot 2 \cdot 2 = 12$ elements and $\mathcal{N}_{\mathcal{E}}$ has $8 + 8 + 4 + 4 = 24$ elements, counted as in Section 14. The exact evaluations appear in Table 6. The two runners-up are $T(105) = 1348818306/1095376555 = 1.2313740876\ldots$ and $T(75) = 5762510/4700591 = 1.2259118055\ldots$; the first of these differs from T^* by only $1.42 \cdot 10^{-5}$, so the comparison is carried out in exact rational arithmetic. The threshold assertions are $5 \cdot 612 = 3060 > 2982 = 6 \cdot 497$ and $5 \cdot 836970 = 4184850 > 4082694 = 6 \cdot 680449$. □

Proofs of Theorems 10.8 and 10.11. Both are word for word the argument given for Theorem 10.6, with K_c, T, $\mathcal{N}$, T^* and the threshold $6/5$ in place of G_c, R, $\mathcal{L}$, R^* and $14/13$ for Theorem 10.8, and with the Eulerian lists $\mathcal{M}_{\mathcal{E}}$ and $\mathcal{N}_{\mathcal{E}}$ furnished by the second half of Corollary 14.3 for Theorem 10.11. Note that $45 \in \mathcal{E}$, only the exponent of 5 being odd.

For the statement about $\mathcal{E}_{10}$ in Theorem 10.11, argue as in the proof of Theorem 10.10: since $\omega(45) = 2 < 10$, equality on $\mathcal{E}$ cannot occur inside $\mathcal{E}_{10}$, so both inequalities are strict there; and taking eight primes $q_1 < \cdots < q_8$ all larger than Q and setting $n_Q = 45 \cdot q_1^2 \cdots q_8^2$ gives $\omega(n_Q) = 10$, exactly one odd exponent, and $I(n_Q) \to I(45)$, $I^{\pm}(n_Q) \to I^{\pm}(45)$ as $Q \to \infty$, whence $S(n_Q) \to S_{\mathcal{E}}$ and $T(n_Q) \to T_{\mathcal{E}}$. So no smaller constant is admissible on $\mathcal{E}_{10}$.

It remains to verify the strict inequality $t < c' + c''$ asserted in Theorem 10.8. Since $T = R \cdot S$ pointwise, we have $T^* = \sup T \le (\sup R)(\sup S) = R^* S^*$, and the inequality is strict because the two suprema are attained at different integers: $R(15) = 384/355 < R^* = R(315)$, so $T^* = R(15)S(15) < R^* S^*$. Numerically $t = 0.3002858011303656\ldots$ against $c' + c'' = 0.3034305400027134\ldots$. □

Proof of Corollary 10.12. Let n be an odd perfect number. By Proposition 11.3 we have $n \in \mathcal{E}_{10} \subseteq \mathcal{E}$, and $I(n) = \sigma(n)/n = 2$. Theorem 10.10 gives $I^+(n) \ge I(n)/R_{\mathcal{E}} = 2/R_{\mathcal{E}} = 1.847178873998728783486\ldots$, and Theorem 10.11 gives $I^-(n) \le S_{\mathcal{E}}\, I(n) = 2S_{\mathcal{E}} = 18195/8008$. The remaining two inequalities $I^+(n) < 2 < I^-(n)$ are Proposition 11.2 together with $I(n) = 2$. The final assertion is the contrapositive. □

Corollary 18.2. *If n is a quasiperfect number, then*

$$\frac{2}{R_{\mathcal{E}}} < \frac{I(n)}{R_{\mathcal{E}}} < I^+(n) < I(n) < I^-(n) < S_{\mathcal{E}}\, I(n), \qquad I^-(n) < T_{\mathcal{E}}\, I^+(n), \tag{18.2}$$

where $I(n) = 2 + 1/n$.

Proof. A quasiperfect number n is an odd perfect square [7], so $n \in \mathcal{E}$ by Definition 10.9, and $I(n) = 2 + 1/n > 2$. Theorems 10.10 and 10.11 give $I^+(n) \ge I(n)/R_{\mathcal{E}}$, $I^-(n) \le S_{\mathcal{E}} I(n)$ and $I^-(n) \le T_{\mathcal{E}} I^+(n)$, with equality only for $n = 2205$ and $n = 45$, neither of which is a perfect square. The inner inequalities are Proposition 11.2. □

Remark 18.3. The failure of $c' + c''$ to be sharp is not a small effect of the kind one could hope to remove by a more careful estimate. It reflects a genuine structural incompatibility: the upward shift is most costly on integers divisible by 9, since 9 lies below its threshold 13, whereas the downward shift is most costly on integers with as few small divisors as possible beyond 3 and 5, since 9 lies above its threshold 9 and already yields a surplus. No odd integer can be simultaneously extremal for both, and the discrepancy $c' + c'' - t = 0.0031447388723478\ldots$ measures exactly this tension.

19. Why cruder approaches cannot reach these constants

It is instructive to see that several natural strategies provably fall short, which explains why the divisor-local decomposition of Section 13 is needed.

Proposition 19.1. *For every odd n,*

$$I(n) - I^+(n) = \sum_{\substack{d\mid n\\ d>1}} \frac{1}{d(d+1)} < \log 2 - \frac{1}{2} = 0.1931471805599453\ldots, \tag{19.1}$$

$$I^-(n) - I(n) = \sum_{\substack{d\mid n\\ d>1}} \frac{1}{d(d-1)} < 1 - \log 2 = 0.3068528194400547\ldots, \tag{19.2}$$

and both bounds are best possible in the sense that each right-hand side is the supremum of the corresponding left-hand side over all finite sets of odd integers exceeding 1. The two constants sum to $1/2$, in agreement with the telescoping constant of Lemma 12.2.

Proof. The two identities are (11.3). Every divisor $d > 1$ of an odd n is an odd integer at least 3, and these are distinct, so the first sum is a subsum of $\sum_{k\ge1}\big(1/(2k+1) - 1/(2k+2)\big)$ and the second is a subsum of $\sum_{k\ge1}\big(1/(2k) - 1/(2k+1)\big)$, all terms being positive. By the alternating harmonic series $\log 2 = 1 - 1/2 + 1/3 - 1/4 + \cdots$, the first of these series equals $\log 2 - (1 - 1/2) = \log 2 - 1/2$ and the second equals $(1/2 - 1/3) + (1/4 - 1/5) + \cdots = 1 - \log 2$. Both subsums are proper because no odd n is divisible by every odd integer, giving strict inequalities. □

Corollary 19.2. *The estimates of Proposition 19.1 are insufficient to prove either half of Corollary 10.12. For an odd perfect number $I(n) = 2$, so the conclusion $I^+(n) \ge 2^{1-c'}$ is equivalent to $I(n) - I^+(n) \le 2 - 2^{1-c'}$, and already for the weaker exponent $c' = 1/7$ the right-hand side is $2 - 2^{6/7} = 0.1885526714721866\ldots < \log 2 - 1/2$. Likewise the conclusion $I^-(n) \le 2^{1+c''}$ is equivalent to $I^-(n) - I(n) \le 2^{1+c''} - 2$, and for the sharp exponent $c'' = \log_2(255/224)$ the right-hand side is $2(S^* - 1) = 31/112 = 0.2767857142\ldots < 1 - \log 2$. In both cases the crude bound falls on the wrong side, so no argument resting only on the size of the difference can succeed.*

A second natural route is to bound $I(n)$ from below using the hypothesis $\omega(n) \ge 10$ and then to absorb the difference. This also fails, and for a structural reason.

Proposition 19.3. *The hypothesis $\omega(n) \ge 10$, even together with membership in $\mathcal{E}$, implies no lower bound on $I(n)$ beyond the trivial $I(n) > 1$. Explicitly, for $n = 101 \cdot (103 \cdot 107 \cdot 109 \cdot 113 \cdot 127 \cdot 131 \cdot 137 \cdot 139 \cdot 149)^2$ one has $n \in \mathcal{E}_{10}$ and $I(n) = 1.0875712439\ldots$, and letting the ten primes tend to infinity drives $I(n)$ to 1.*

Proof. Multiplicativity gives $I(n) = \prod_{p^a \| n}(1 + p^{-1} + \cdots + p^{-a})$, and each factor tends to 1 as $p \to \infty$ uniformly in a, since it is bounded above by $(1 - 1/p)^{-1}$. Choosing ten primes larger than any prescribed bound, with one carrying exponent 1 and the others exponent 2,

produces elements of $\mathcal{E}_{10}$ with $I(n)$ arbitrarily close to 1. The stated numerical value is a direct evaluation. □

Finally, the naive termwise comparison is far too lossy. Since every divisor $d > 1$ of an odd n satisfies $d \geq 3$ and hence $d + 1 \geq 4$, one gets $1/\big(d(d+1)\big) \leq 1/(4d)$ and therefore $I^+(n) \geq \big(3I(n) + 1\big)/4$. This yields $I^+(n) \geq I(n)/c$ only when $I(n) \leq c/(4 - 3c)$, an upper bound on $I(n)$ rather than a universal statement, and for $c = R^*$ it requires $I(n) \leq 1.4495\ldots$, which fails for most of the integers of interest. The reason is that the local ratio $4/3$ at $d = 3$ is charged to every divisor, whereas in reality it is charged only once.

20. Numerical experiments

All computations were carried out in Python with `numpy` [9] for the sieves, `sympy` [17] for factorization and divisor enumeration, exact `Fraction` arithmetic for all rational identities, and `mpmath` [12] at 60 decimal digits for the high-precision evaluations. The accompanying script `experiments.py` reproduces every numerical statement in this paper.

Experiment A: exhaustive divisor sieve. A harmonic-type sieve accumulates $I(n)$, $I^+(n)$ and $I^-(n)$ simultaneously for all $n \leq N$ by adding $1/d$, $1/(d+1)$ and $1/(d-1)$ to the arithmetic progressions $d, 2d, 3d, \ldots$ for each $d \leq N$, in double precision. With $N = 10^7$ this covers $5 \cdot 10^6$ odd integers. The results are: $\max R(n) = 1.08405054482233$ at $n = 315$, $\max S(n) = 1.13839285714286$ at $n = 15$, and $\max T(n) = 1.23138832997988$ at $n = 15$, each agreeing with R^*, S^* and T^* to fourteen decimals; no odd $n \leq 10^7$ violates any of the three sharp bounds; and $I^+(n) < I(n) < I^-(n)$ for every odd n with $1 < n \leq 10^7$, confirming Proposition 11.2. The runners-up in the first problem are the integers $315q$ with q a large prime, whose ratios accumulate at R^* from below, exactly as the theory predicts, and similarly $15q$ for the other two. The sieve also returns $\max\big(I^-(n) - I(n)\big) = 0.28790584906385$ and $\max\big(I(n) - I^+(n)\big) = 0.17517080090658$ over odd $n \leq 10^7$, both comfortably below the limiting values $1 - \log 2$ and $\log 2 - 1/2$ of Proposition 19.1.

Experiments B and C: the reduction lists. All six reduction lists are evaluated in exact rational arithmetic: the 24 elements of $\mathcal{L}$ and the 56 of $\mathcal{L}_{\mathcal{E}}$ for the upward shift, the 8 of $\mathcal{M}$ and the 20 of $\mathcal{M}_{\mathcal{E}}$ for the downward shift, and the 12 of $\mathcal{N}$ and the 24 of $\mathcal{N}_{\mathcal{E}}$ for the spread; see Tables 2, 3, 4, 5 and 6. The six maxima are the exact fractions of (10.9), (10.11), (10.14), (10.16) and (10.18). The gaps to the runners-up are small in every case, namely 2.4159×10^{-5}, 1.1795×10^{-5}, 2.4666×10^{-3}, 1.0058×10^{-3}, 1.4242×10^{-5} and 4.1143×10^{-3} in the order just listed; the smallest of these are far too tight for double precision to settle, which is why exact arithmetic is used throughout.

Experiment D: brute-force cross-check. Independently of the reduction lemmas, a direct search over all odd $n \leq 2 \cdot 10^6$, with exponent parities computed from a smallest-prime-factor sieve, returns the six extremal integers 315, 2205, 15, 45, 15, 45 for R, R on $\mathcal{E}$, S, S on $\mathcal{E}$, T, T on $\mathcal{E}$ respectively, with maximal values agreeing to fourteen decimals with Theorems 15.1, 16.1, 17.1 and 18.1. This is a genuine check of the reduction lemmas themselves, since it does not use them.

Experiment E: Eulerian integers with at least ten prime divisors. Table 7 evaluates I, I^+, I^- and the three ratios at explicit members of $\mathcal{E}_{10}$, using 60-digit arithmetic and full divisor enumeration, for divisor counts up to $\tau(n) = 3906250$. All values obey Theorems 10.10 and 10.11. The last two rows show the two extreme regimes side by side: ten large primes drive all three ratios to 1, while a small Eulerian core with large auxiliary primes drives them towards the sharp constants.

TABLE 7. High-precision evaluation on explicit members of $\mathcal{E}_{10}$, with $P = \{3,5,7,11,13,17,19,23,29,31\}$ the ten smallest odd primes.

n	$\tau(n)$	I	I^+	I^-	R	S	T
$3\cdot(5\cdot7\cdots31)^2$	39366	2.871082	2.699378	3.153366	1.063609	1.098320	1.168182
$(3\cdots29)^2\cdot31$	39366	3.107207	2.923302	3.404504	1.062910	1.095680	1.164609
$3^5\cdot(5\cdots31)^2$	118098	3.225537	3.040003	3.524570	1.061031	1.092708	1.159397
$3\cdot(5\cdots31)^4$	3906250	2.906542	2.734750	3.188915	1.062818	1.097151	1.166072
$3\cdot(5\cdots37)^2$	118098	2.950776	2.778207	3.233966	1.062115	1.095971	1.164048
$3\cdot(5\cdots41)^2$	354294	3.024502	2.851226	3.308429	1.060772	1.093876	1.160353
$101\cdot(103\cdots149)^2$	39366	1.087571	1.086865	1.088290	1.000650	1.000661	1.001312
$2205\cdot(101\cdots131)^2$	39366	2.146937	1.992203	2.411320	1.077670	1.123144	1.210379

Experiment F: certificates. The six threshold comparisons on which Lemmas 13.3 and 13.5 and hence the whole reduction depend, namely $R^*, R_{\mathcal{E}} > 14/13$, $S^*, S_{\mathcal{E}} > 9/8$ and $T^*, T_{\mathcal{E}} > 6/5$, are verified as exact integer inequalities. No floating-point comparison enters any step of any proof in Sections 10–19.

Experiment G: Theorem 10.3. The separation $I^-(n) > I^+(n)$ was confirmed in exact rational arithmetic for every n with $1 < n \le 2\cdot10^4$, odd and even, and by the sieve of Experiment A for every odd $n \le 10^7$. The identity (12.1) was verified exactly over the same range. For the additive half of Theorem 10.3, the largest gap $I^-(n) - I^+(n)$ over odd $n \le 10^7$ is 0.46307664997043, attained at $n = 3828825 = 3^2\cdot5^2\cdot7\cdot11\cdot13\cdot17$, still short of the limit $1/2$ and approaching it only along integers divisible by all small odd numbers, exactly as (12.2) predicts. The witnesses $n_k = m_k^2 q$ of Lemma 12.4 were checked to lie in $\mathcal{E}_{10}$ and to satisfy (12.2) with equality in the partial sum: for $k = 29$, 31 and 101 the guaranteed lower bounds are $7/15$, $15/32$ and $25/51$ respectively, while $\omega(n_k) = 10$, 11 and 26 and the number of odd exponents is 1 in each case. For the multiplicative half, the largest value of $I^-(n)/I^+(n)$ over the Eulerian integers up to $2\cdot10^4$ is $T_{\mathcal{E}} = 836970/680449$ at $n = 45$, in agreement with Theorem 18.1. Finally the odd perfect bound of Corollary 10.5 was evaluated exactly as $2S_{\mathcal{E}} - 2/R_{\mathcal{E}} = 43623651349277204129/102662238370512300480 = 0.4249240231041\ldots$, an improvement of 15.02 percent over $1/2$.

Experiment H: constants for $B(n)$, $n/\varphi(n)$ and $I(n\,\mathrm{rad}(n))$. The constants of Sections 3–6 and 9 were evaluated with `mpmath` at 40 decimal digits: $2^{1/m}$ and the threshold of Proposition 3.6, the values $Z(2)$, $Z(3)$, $Z(5)$ and $Z(7)$ of (3.7), the constant P_3 of (4.1) together with the three inequalities that follow from $\zeta(3) < 1.2020569032$, the bounds 2.10360 and 2.17372 of Lemma 4.1, the constants of Theorem 5.1 and of Corollaries 5.2 and 5.3, and the constant κ of (6.3) with the bounds 2.07317 and 2.14228 of Theorem 6.2. The partial product of $1 + 1/\big(p(p^2+p+1)\big)$ over the odd primes below 10^6 lies below κ and within 10^{-12} of it. The identity (6.1) was verified in exact rational arithmetic for every n with $2 \le n \le 3000$.

Experiment I: the least prime factor and the number of prime factors. The two bounds (8.1) were verified at sample points $x \le 2\cdot10^5$, and the two bounds for $\pi(x)$ of Proposition 8.1 at sample points $x \le 10^5$. The right-hand side of (8.3) was computed for $p_1 = 3,5,7,11,13,17,19,23$, giving $0,1,3,10,14,25,31,47$. For $p_1 = 101$ the computation gives $X = 6477.2149\ldots$, the value $711.8029\ldots$ for the right-hand side of (8.4), $\pi(6477) = 840$ and $\pi(100) = 25$. For every prime p_1 with $3 \le p_1 < 200$, the least integer k such that the product of

$q/(q-1)$ over the first k primes $q \geq p_1$ exceeds 2 was computed directly; in every case the k-th such prime exceeds X and k satisfies (8.3), and for $p_1 \geq 11$ it satisfies (8.4).

Experiment J: the constant δ and the bounds according to $\gcd(15, N)$. At 40 decimal digits, $\log\zeta(2) - \gamma + M = 0.18198185041685527062\ldots$ and $\delta = 0.087446958525019652601\ldots$, in agreement with (3.14) and (3.15). The partial sum of $\varepsilon(p)$ over the primes below $2 \cdot 10^6$ equals $0.181981834241561\ldots$, which lies below the value (3.14) and within 10^{-7} of it, and the corresponding partial sum in (3.13) is within 10^{-7} of M. With the bounds of Proposition 5.4 as input, the three upper bounds of (5.7) evaluate to $1.84347566675\ldots$, $1.93548425624\ldots$ and $1.84124191162\ldots$, and the four lower bounds of (5.8) to $1.90281191154\ldots$, $1.7058482335\ldots$, $1.83233535283\ldots$ and $1.67856924672\ldots$; each lower bound exceeds the corresponding value $2\lambda_g$ of (5.6). The bounds of Corollary 5.6 were checked against these values.

Experiment K: general inequalities over odd integers. In the loop of Experiment D, the following statements were tested for every odd n with $3 \leq n \leq 2 \cdot 10^6$, using the factorization of n and the sieved values of $I^+(n)$ and $I^-(n)$, and no violation was found: the inequality (3.4) of Theorem 3.3 when n has at least two distinct prime factors; the inequality $B(n) > (8/\pi^2)\, n/\varphi(n)$ of Theorem 3.5; the inequality (3.10) of Lemma 3.7; the identity and the two inequalities of Theorem 6.1; the inequalities (3.16) of Lemma 3.8; and the inequalities (12.6) and (12.7) of Proposition 12.3.

Experiment L: the functions $\sigma^{\pm}$, ψ^+, φ^+ and the series for $I^{\pm}$. At 40 decimal digits, $\rho = 1.1729423808173193221\ldots$ and $\theta = 1.2957309578805061367\ldots$, in agreement with (7.4); the partial products of $1 + 1/\big(p(p+1)\big)$ and of $1 + 1/\big(p(p-1)\big)$ over the odd primes below 10^6 lie below ρ and θ and within 10^{-6} of them. The identity (7.2) and the four product formulas (7.3) were verified in exact arithmetic for every n with $2 \leq n \leq 3000$. The inequalities (7.5) hold for each of the 336 odd integers $n \leq 2 \cdot 10^5$ with $n > 1$ all of whose exponents are at least 2, and the inequalities (7.6) hold for each of the 25968 odd integers $n \leq 2 \cdot 10^5$ having exactly one exponent equal to 1, at a prime $q \geq 5$, and all other exponents at least 2. The three series identities of Proposition 10.2 were verified to 30 decimal digits for every n with $2 \leq n < 400$ and for $n = 945, 2205, 3465, 45045$.

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