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EXPLICIT WEAK *abc* INEQUALITIES AND THE DISCRETE OBSTRUCTION TO FULL *abc* IN THE DIC FRAMEWORK

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Abstract. We axiomatise the DIC (Dimension Isolation Conjecture) framework for the *abc* conjecture and prove three stratified groups of results on Kodaira–Parshin arithmetic surface fibrations. **(i) Unconditional geometric theorems.** The adjunction identity $Z\cdot\omega_{\mathcal{X}/B} = -Z^2$, the Riemann–Hurwitz intersection numbers, the invariants of the general (g_0, r, b) construction, and a Deligne–Riemann–Roch decomposition of the arithmetic coefficient $c = \frac{2b^2+2b-1}{12} + (D_0 - T)/\omega^2$. **(ii) A meta-theorem about the construction itself.** Integrality of torsion lengths forces the coefficient of Theorem A onto the discrete set $t = 12/N$ with $N \equiv 1 \pmod 3$ — at the literature point $(q, \lambda) = (5, 3)$; when q varies this is an arithmetic coincidence, not a structural barrier (Remark 5.12); for general (g_0, r, b) the reciprocal $1/t$ lies on an arithmetic progression of common difference $\lambda/\big(6r(g_0 - 1)\big)$. For the normalisation $q = 5$ the coefficient required by full *abc* is unattainable; an explicit local hyperelliptic model yields $T = 8k$, $t = 12$ as the unique transverse output and restores the No-Go for every Kummer degree q under a transversality (or Galois-transitivity) hypothesis. The second-order torsion computation gives $n_1 = n_2$, hence $\Theta = T/k \in 2\mathbb{Z}$, leaving $(27, 12)$ as the sole lattice candidate, while a transitive Galois action fixes $\Theta = 8$ without transversality. A unified criterion $E = 12(q-1)/(qN\gamma)$ parameterises the last lattice point: the landing equation has the unique odd integral solution $(N, \Theta, q, \gamma) = (13, 12, 27, 8/9)$, requiring exact normalisation $\gamma = 8/9$; in the original $q = 5$ construction no normalisation lands on the lattice. The height is the canonical-bundle height on a fixed genus-2 curve: $\gamma = 2/5$ follows from $12/\gamma = 30$ in §7.3.2 of the source and is confirmed by an explicit valuation computation; since Theorem A fixes both the genus and the Kummer degree, the lattice point $(12, 27)$ requires genus 13 and lies outside the theorem's hypotheses — the last cell is empty. **(iii) Controlled weak *abc*.** All unconditional bounds for *abc* currently known are of exponential type ($\log c \ll N^{1/3}(\log N)^3$, [33, 34]; [20, §1.1]); the exponent below is of polynomial type, and its existence is what Theorem A supplies. Taking Soulé's Theorem A as the sole external input, we derive the Kummer discriminant estimate $\eta(q) = (q-1)/q$, unify the weak *abc* exponent as $E = t\eta/\gamma$, and obtain the explicit exponent $E_{\max} = 12\frac{q-1}{q}\left\lceil\frac{2q-2}{\ell}\right\rceil$ via a Riemann–Roch lower bound for γ. A discriminant stratification sharpens this to the pointwise bound $c \le A_\varepsilon(N/M_5)^{24+\varepsilon}$, and we prove that bootstrapping this discount to exponent 1 is impossible ($\inf\theta = 0$), that the discriminant-to-degree ratio is monotone in towers of number fields, and that $\frac{4}{5}$ is an Artin conductor identity rather than an estimate constant. A verbatim check of the source (Appendix J) confirms twelve items, including the statement and scope of Theorem A; Appendix I derives the $O(d')$ form of (32) independently; Appendices K and L add two structural No-Go reasons; a distributional analysis (Appendix M) shows the discount and the multiplicity excess vanish in density $1 - o(1)$ with explicit large-deviation rates, independently of the truth of Theorem A; the constant 12 decomposes completely as $\frac{1}{3}\cdot\frac{m}{\rho - mc}$ and Route 1 is closed: archimedean/successive-minima refinements cannot move it (Appendix F). No Diophantine conjecture enters the derivations. We neither prove nor disprove *abc*; all conclusions are internal to the construction under its listed hypotheses.

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1. Introduction

1.1. **Background.** Let a, b, c be pairwise coprime positive integers with $a+b=c$ and put $N = \mathrm{rad}(abc) = \prod_{p \mid abc} p$. The Masser–Oesterlé *abc* conjecture asserts that for every $\varepsilon > 0$ there is K_ε with

$$c \le K_\varepsilon N^{1+\varepsilon}, \qquad \text{equivalently} \quad \log c \le (1+\varepsilon)\log \mathrm{rad}(abc) + O_\varepsilon(1). \tag{1.1}$$

Through the Frey–Hellegouarch curve and the Szpiro conjecture, (1.1) is tied to discriminant–conductor inequalities on arithmetic surfaces; Vojta's value-distribution dictionary places it in a general framework of heights and proximity functions [10, 12, 13]. The route via Arakelov theory, the Parshin construction and Moret-Bailly's *formule clef* reduces the Diophantine question to cohomological and metric estimates on arithmetic surfaces [15, 14, 6, 1]; determinant cohomology and arithmetic Riemann–Roch are surveyed in [7, 8, 9, 19], successive minima and (semi-)stable extensions in [3, 5, 4], and the function-field analogue in [16].

The DIC framework that we axiomatise here attempts to build an arithmetic surface fibration $\pi : \mathcal{X} \to B$ with $h_B \asymp \log c$ and $m_S(D_B, b) \asymp \log c - \log \mathrm{rad}(abc)$, from which (1.1) would follow through a family of positivity axioms (ND1, ND2′) and height-control inequalities (A1, A2, A3). In its original form the framework contains category errors and incompatible hypotheses: a stratification mismatch of the monodromy representation, mutually exclusive positivity and lower-bound requirements on one and the same section, and a genuine logical circle in which a statement equivalent to (1.1) is imported as a lemma.

1.2. **Aim and results.** We do *not* prove the *abc* conjecture. Our aim is to extract, from this route, everything that can be stated and proved as a self-contained theorem, and to make the dependence structure completely

explicit. The results fall into three strata; their logical dependencies are collected in §11.

(1) **Unconditional geometric layer** (§3–§4). We recompute the fifteen invariants of the Kodaira–Parshin genus-6 fibration, extend the construction to arbitrary initial genus g_0, branch number r and divisor coefficient b, prove that $\rho(b) = (b-1)(b+3)/3$ is independent of all covering data, and use Deligne–Riemann–Roch to write the arithmetic coefficient as a quadratic GRR term minus a torsion term, reproducing the value $-11k$ verbatim.

(2) **Meta-theorem: discrete obstruction** (§5–§6). The torsion module decomposes as $A/\pi^{n_1} \oplus A/\pi^{n_2}$ with $t = 2$ verified independently of the source's counting convention; integrality gives $T = kL$, $L \in \mathbb{Z}_{\geq 8}$, hence $t(L) = 12/(3L-23)$ with denominator $\equiv 1 \pmod 3$. The coefficient required by full *abc* ($t^* = 1/2$ or $5/4$, according to the normalisation of the bridge) is therefore unattainable at $q = 5$. An explicit hyperelliptic local model computes the Kodaira–Spencer class κ_M and the ranks of the two connecting maps, yielding $T = 8k$ exactly under transversality; second-order torsion sharpens $L = \Theta$ to even integers, cuts the candidate set to $\{8, 10, 12, 14\}$, excludes $(9, 13)$, and leaves $(\Theta, q) = (12, 27)$ as the unique remaining candidate, which §8 then shows to be excluded by the *hypotheses* of Theorem A.

(3) **Controlled weak** *abc* (§7). The source states that §7.2 of [1] is "known to follow from Theorem A, albeit with a wrong exponent". The constants 30, $\eta = 4/5$ and the curve $y^2 + y = x^5$ are in §7.3/§7.3.2, not §7.2 (Appendix R). We supply the complete derivation: the Kummer discriminant estimate $\eta(q) = (q-1)/q$ is proved, the exponent is unified as $E = t\eta/\gamma$, and Riemann–Roch gives $\gamma \geq 1/n$, hence an explicit finite $E_{\max}$. In §9 we sharpen the discriminant estimate to a pointwise stratification, prove the impossibility of bootstrapping the resulting discount, and prove tower monotonicity of the discriminant-to-degree ratio.

1.3. **Convention on hypotheses.** Every theorem below carries an explicit dependence tag: **U** = unconditional (geometry, algebra, and results already proved in the cited literature); **A5.2**, **A6.1** = the counting convention and the local model assumption; **Th.A** = conditional on [1, Theorem A]. Unconditional results in this paper are of two kinds: *exponential-type* (Theorems 7.5′ and 7.5a) and *No-Go type* (§5–§10, Appendices E–Z), the latter internal to the construction and independent of the truth of Theorem A (Proposition H.5). The statements that genuinely use Theorem A are tagged `Th.A` throughout (Table 1) and are confined to the weak-*abc* outputs (Theorems 7.7 and 9.2); the internal adelic metric layer has been re-derived in part (Appendix G). The final conditional ledger is Appendix Z: after Appendices G–Z the only conditional items are Th.A* (Theorems 7.7 and 9.2, polynomial-type weak *abc*; unremovable by the trichotomy of Theorem N.7

and Theorem H.1) and the strict isometry in (32) (tag O*, no consequence for any coefficient by Theorem H.4 and Appendix I). All No-Go statements and the BMY-side closure ($\Theta = 12$ excluded, $C \geq 36/7 > 3$) are tagged U (or U+A5.2). We emphasise once more: *the abc conjecture itself is neither proved nor disproved here.* We do not claim an independent verification of the proof of Theorem A of [1]; Appendices G, I and Z are text-, algebra- and elementary-proof-level.

1.4. **Notation.** $\omega := \omega_{\mathcal{X}/B}$ is the relative dualising sheaf and $\overline{\omega}$ its adelically metrised version; $\delta(F) := \log|\Delta_F|$ and $d := [F : \mathbb{Q}]$; $(\cdot)$ denotes geometric intersection and $\langle\cdot\rangle_{\mathrm{Ar}}$ Arakelov intersection, and the two are never mixed. All $O(1)$, $O_\varepsilon(1)$ are uniform constants; $O(d)$ is bounded by a constant multiple of d. We write $k = \deg(Y \to Y_1)$, r for the branch number, b for the coefficient of $L = \omega - bS$. The Kummer degree is denoted q from §7 onward (in §3 the symbol m denotes $\deg L_F$). We keep the source's abbreviations h_B, m_S, ε.

2. Axiomatic setup and stratification of the cycle

Definition 2.1 (Fibration and heights)**.** $\pi : \mathcal{X} \to B$ is an arithmetic surface fibration over a smooth projective curve B, the general fibre has genus $g_f \geq 2$, and $\mathcal{Z} \subset \mathcal{X}$ is a section. For an adelically metrised line bundle $\bar{L}$ the Weil height is $h_{\bar{L}}(P) = [K : \mathbb{Q}]^{-1}\widehat{\deg}(s_P^*\bar{L})$; h_B is induced by an ample divisor, h_{rel} by ω, and h_{vert} is a distance-type height to $\mathcal{Z}$. The proximity function is $m_S(D, P) = \sum_{v\in S}\lambda_{D,v}(P)$.

Assumption 2.2 (Alignment A0)**.** Take $B = \mathbb{P}^1$, $D_B = [0] + [1] + [\infty]$, $b = a/c$ and $S \supseteq S_\infty \cup \{p : p \mid abc\}$. Then

$$h_B(b) = \log c + O(1), \qquad m_S(D_B, b) = \log c - \log \mathrm{rad}(abc) + O(1).$$

Lemma 2.3. *Under Assumption 2.2, (1.1) is equivalent to: for every $\varepsilon > 0$, $m_S(D_B, b) \leq \varepsilon h_B(b) + O_\varepsilon(1)$.*

Proof. We start from the two alignment identities $h_B = \log c + O(1)$ and $m_S = \log c - \log \mathrm{rad}(abc) + O(1)$ of Assumption 2.2.

Step 1 (eliminate $\log c$). Rearranging the second identity,

$$\log \mathrm{rad}(abc) = \log c - m_S + O(1) = h_B - m_S + O(1).$$

Step 2 (substitute into the Masser–Oesterlé form). The correct direction of *abc* is $\log c \leq (1+\varepsilon)\log \mathrm{rad}(abc) + O_\varepsilon(1)$. Substituting Step 1,

$$h_B \leq (1+\varepsilon)(h_B - m_S) + O_\varepsilon(1) = h_B + \varepsilon h_B - (1+\varepsilon)m_S + O_\varepsilon(1).$$

Step 3 (rearrange). Cancelling h_B on both sides of Step 2 gives $0 \leq \varepsilon h_B - (1+\varepsilon)m_S + O_\varepsilon(1)$, i.e. $(1+\varepsilon)m_S \leq \varepsilon h_B + O_\varepsilon(1)$, hence $m_S \leq \frac{\varepsilon}{1+\varepsilon}h_B + O_\varepsilon(1)$. Since $\varepsilon > 0$ is arbitrary, $\frac{\varepsilon}{1+\varepsilon}$ is arbitrarily small, so $m_S \leq \varepsilon' h_B + O_{\varepsilon'}(1)$ holds for every $\varepsilon' > 0$. Each step is reversible, so the two statements are equivalent.

Direction note. The reverse inequality $\mathrm{rad}(abc) \le K_\varepsilon c^{1+\varepsilon}$ is trivially true ($\mathrm{rad}(abc) \le abc < c^3$, take $K = 1$, $\varepsilon = 2$) and is *not* the *abc* conjecture; the framework only uses the direction in which $\log c$ is controlled by $\log \mathrm{rad}$. □

Remark 2.4 (Excluded axioms). A1 (pointwise approximation control), A3 (transfer inequality) and DIC-S in the original DIC framework are renamed versions of Vojta's Main Conjecture and of (1.1); importing them as axioms is a genuine circle of type L1 and all three are deleted. On an elliptic surface A1 is in fact refuted: $\omega_{\mathcal{E}/B} = f^*\mathcal{L}$ is a pullback from the base, so $h_{\mathrm{rel}}(P) = \chi h_B(\pi(P)) + O(1)$ depends only on the base point; taking $P_n = nP$ on a fibre of positive rank gives $\hat{h}(nP) = n^2\hat{h}(P) \to \infty$ while the right-hand side of A1 stays bounded.

Proposition 2.5 (Stratification of the cycle)**.** *The proof chain splits into three layers:*

(L1) *Using A1/A3/DIC-S as axioms ⇒ one directly assumes* (1.1)*: a* genuine *circle.*

(L2) *Proposition 2 → Corollary 1 → Minkowski → Deligne–Riemann–Roch → Theorem A of* [1]*: all unconditional* [1, Prop. 2, §1.7].

(L3) *The coefficient gap* $12 \to 5/4$ *is a quantitative chasm, not a logical circle.*

After deleting L1 the chain contains no Diophantine conjecture.

Proof. L1 (genuine circle). DIC-S asserts $m_S(D, P) \le C_\varepsilon h_B(\pi(P)) + O_\varepsilon(1)$ with $C_\varepsilon \to 0$; under Assumption 2.2 and via Lemma 2.3 this is exactly (1.1). Hence taking DIC-S or A3 as an axiom *is* assuming the conclusion. Disposal: delete both axioms.

L2 (pseudo-circle, unconditional line by line). (1) [1, Prop. 2]: if $\deg_K(y) < 0$ then, by pullback invariance of adelic metrics, $\deg^F(u^*y) = \sum_v \deg_v^F(u^*y)$, and the metric comparison constants satisfy $\sum_v \log C_v \le O(d)$ with $d = [F : \mathbb{Q}]$; hence $\deg^F(u^*y) \le O(d)$ with a constant *independent of F, P, u.* The proof uses only the product formula and metric comparison; no conjecture enters. (2) Corollary 1: for $\deg_K(x) = 0$, apply Prop. 2 to $\hat{x}_\pm = \pm\hat{x} - \varepsilon\bar{M}$ (both of negative K-degree) to get $|\deg^F(u^*\hat{x})| \le \varepsilon \deg^F(u^*\bar{M}) + O_\varepsilon(d)$. (3) Minkowski successive minima, arithmetic GRR and the derivation (28)–(34) of Theorem A are proved theorems of [1]. Thus the whole L2 chain carries no conditional tag.

L3 (not a circle). The passage $12 \to 5/4$ is a geometric constant (Appendix B.5: $36 = 6/(5/3 - 6/4)$); no Diophantine hypothesis is involved.

Conclusion. After deleting L1 the chain is cycle-free and outputs the *weak abc* inequality with exponent 24. The general Vojta Main Conjecture remains open, but [1] works in the special case "genus-2 curve plus discriminant term", which is proved unconditionally, so the chain cannot be dismissed on the grounds that "Vojta is open". □

Remark 2.6 (Status of the external inputs). Four levels must be kept apart. **(1)** Exponential-type bounds are unconditional: Theorems 7.5′ and 7.5a rest on [33, 34, 35] — see [20, §1.1] for the chronology. **(2)** The bridge from Theorem A to a polynomial-type weak *abc* is a literature fact (Theorem H.2): §7 of [36] specialises to the present curve, and our contribution is only the computation of η and γ. **(3)** Theorem A itself remains conditional; its internal adelic metric layer is partly re-derived in Appendix G, and the remaining gaps are (1a)–(1c) of §11. **(4)** The No-Go statements (§5–§10, Appendices E–M) do *not* depend on the truth of Theorem A (Proposition H.5).

3. Geometric layer: exact identities

Theorem 3.1 (Adjunction identity)**.** *For every section $Z \cong B$,*

$$Z \cdot \omega = -Z^2, \qquad \kappa_{\mathrm{geom}}(Z) := \frac{Z \cdot \omega}{Z \cdot Z} \equiv -1 \quad (Z^2 \neq 0). \tag{3.1}$$

For the Arakelov pairing one has $\langle K, Z\rangle_{\mathrm{Ar}} = -\langle Z, Z\rangle_{\mathrm{Ar}} + G_\infty(Z) + O(1)$; the identity $\kappa_{\mathrm{Ar}} \equiv -1$ needs an archimedean normalisation and is not used in the sequel.

Proof. Geometric intersection. A section Z is isomorphic to B ($\pi|_Z = \mathrm{id}$), so arithmetic adjunction reads $Z \cdot (Z + K_{\mathcal{X}}) = 2g(Z) - 2 = 2g_B - 2$, i.e.

$$Z \cdot K_{\mathcal{X}} = 2g_B - 2 - Z^2. \tag{3.2}$$

The projection formula gives $Z \cdot \pi^* K_B = \deg(\pi|_Z) \deg K_B = 2g_B - 2$. Subtracting and using $K_{\mathcal{X}/B} = K_{\mathcal{X}} - \pi^* K_B$,

$$Z \cdot \omega = (2g_B - 2 - Z^2) - (2g_B - 2) = -Z^2. \tag{3.3}$$

Dividing by $Z^2 \neq 0$ yields $\kappa_{\mathrm{geom}} \equiv -1$; if $Z^2 = 0$ the ratio κ_{geom} is undefined and only $Z \cdot \omega = 0$ is retained.

Arithmetic components. The full Arakelov intersection $\langle \cdot, \cdot \rangle_{\mathrm{Ar}} = \langle \cdot, \cdot \rangle_{\mathrm{fin}} + \langle \cdot, \cdot \rangle_\infty$ contains the Green-function term, so after subtraction there remains $G_\infty(Z) = \int_{\mathcal{X}(\mathbb{C})} g_{\bar{L}}(Z, \cdot)\, c_1(\bar{\omega})$, i.e. $\langle K, Z\rangle_{\mathrm{Ar}} = -\langle Z, Z\rangle_{\mathrm{Ar}} + G_\infty(Z) + O(1)$. In the main text $\kappa_{\mathrm{geom}} \equiv -1$ is used only in geometric intersection, or under the additional ∞-normalisation $G_\infty \equiv 0$; Appendix B gives the complete expansion. □

Proposition 3.2 (Two identities and the VR coefficient)**.** *For a fibre F: $F \cdot K_{\mathcal{X}/B} = 2g_f - 2$; for a section S: $\deg(\omega|_S) = -S^2$; for a multisection D of degree d with ramification divisor R: $D \cdot K_{\mathcal{X}/B} = \deg R - D^2$. Consequently, taking $L = \omega^{\otimes n}$ gives $h_L = n h_{\mathrm{rel}} + O(1)$, $r_n = n(\omega \cdot S)$, hence $h_{\mathrm{vert}} \leq \alpha h_{\mathrm{rel}} + O(1)$ with $\alpha = 1/(\omega \cdot S)$ independent of n; by (3.1), $\alpha > 0 \iff S^2 < 0$.*

Proof. Fibre. $F^2 = 0$ (distinct fibres are disjoint) and $\pi^* K_B|_F$ is trivial, so $F \cdot \pi^* K_B = 0$. The fibre version of adjunction gives $F \cdot K_{\mathcal{X}} = 2g_f - 2 - F^2 = 2g_f - 2$; subtracting $F \cdot \pi^* K_B = 0$ yields $F \cdot \omega = 2g_f - 2$. Since

$\deg(\omega|_F) = 2g_f - 2 > 0$ ($g_f \geq 2$), the relative canonical sheaf is relatively ample.

Section. $\deg(\omega|_S) = S \cdot \omega = -S^2$ by (3.1); dividing by the base scale $\deg L_B$ gives $\kappa_{\mathrm{A2}}(S) = -S^2/\deg L_B$. For a K3 elliptic surface $\chi = 2$, so $S^2 = -\chi = -2$ and $\kappa = 2$ (Appendix B).

Multisection. If $D \to B$ has degree d and ramification R, Riemann–Hurwitz gives $2g_D - 2 = d(2g_B - 2) + \deg R$; adjunction gives $D \cdot K_{\mathcal{X}} = 2g_D - 2 - D^2 = d(2g_B - 2) + \deg R - D^2$, and the projection formula gives $D \cdot \pi^* K_B = d(2g_B - 2)$. Subtracting, $D \cdot \omega = \deg R - D^2$; dividing by $d \deg L_B$ gives $\kappa_{\mathrm{A2}}(D)$.

Quantifier note. κ_{A2} is the intersection slope of a *single* (multi)section, not a uniform lower bound on heights for all P; inferring A2 from "there exists S with $\kappa_{\mathrm{A2}}(S) \geq 2$" is a quantifier swap, cf. §1. □

Proposition 3.3 (Kodaira–Parshin general chain)**.** *Let $Y_1 \to C$ be an étale double cover of a curve of genus g_0 (so $g_1 = 2g_0 - 1$), let $X_y \to Y_1$ be a double cover branched at r Weierstrass points, and let $\psi : Y \to Y_1$ be a finite étale cover of degree k. Write $S = \sum_{i=1}^{r} B_i$ and $L = \omega - bS$. Then*

$$g_f = 4g_0 - 3 + \tfrac{r}{2}, \qquad \omega = \sum_{i=1}^{r} B_i + (4g_0 - 4)\phi_1^* W, \tag{3.4}$$

$$B_i^2 = -(2g_0 - 2)k, \qquad \omega^2 = 6rk(g_0 - 1), \tag{3.5}$$

$$L^2 = -2rk(g_0 - 1)(b - 1)(b + 3), \qquad \rho := \frac{-L^2}{\omega^2} = \frac{(b-1)(b+3)}{3}, \tag{3.6}$$

$$\mu := \deg L_F = 8(g_0 - 1) + r - br, \qquad h^0(L_F) = 4g_0 - 3 + \tfrac{r}{2} - br, \tag{3.7}$$

where W is a theta characteristic on C with $\deg W = g_0 - 1 + \frac{r}{2}$ in the branched model, $\phi_1^* W \cdot B_i = k$, and B_i^2 is pinned by (3.1) via $B_i \cong Y$.

Proof. Step 1 (fibre genera). The first cover is étale: $2g_1 - 2 = 2(2g_0 - 2)$, so $g_1 = 2g_0 - 1$. The second cover is branched at r points: Riemann–Hurwitz gives

$$2g_f - 2 = 2(2g_1 - 2) + r = 2(4g_0 - 4) + r = 8g_0 - 8 + r, \qquad g_f = 4g_0 - 3 + \tfrac{r}{2}. \tag{3.8}$$

At the literature point $(g_0, r) = (2, 2)$: $g_1 = 3$, $2g_f - 2 = 8 + 2 = 10$, $g_f = 6$, as stated.

Step 2 (relative dualising sheaf). The ramification formula for a double cover gives $\omega_{X/Y} = \phi^*(\omega_{Y/Y_1}) \otimes \mathcal{O}(\sum B_i)$; pulling back the theta-characteristic term and using $M^2 = \mathcal{O}(A_1 + A_2)$ yields (3.4) with $\deg W = g_0 - 1 + \frac{r}{2}$ (at the literature point $\deg W = 2$).

Step 3 (pinning B_i^2 by adjunction). By (3.1), $\omega \cdot B_i = -B_i^2$. Intersecting (3.4) with B_i and using $B_i \cdot B_j = 0$ $(i \neq j)$ and $\phi_1^* W \cdot B_i = k$ gives

$$-B_i^2 = B_i^2 + (4g_0 - 4)k, \qquad \text{hence} \qquad B_i^2 = -(2g_0 - 2)k,$$

which is $-2k$ at the literature point, and simultaneously $\omega \cdot B_i = (2g_0 - 2)k$.

Step 4 (ω^2). Squaring (3.4) and using that the square of a pullback from the base vanishes,

$$\omega^2 = \sum_i B_i^2 + 2(4g_0 - 4) \sum_i \phi_1^* W \cdot B_i = -r(2g_0 - 2)k + 2(4g_0 - 4)rk = 6rk(g_0 - 1)k.$$

At the literature point $\omega^2 = 12k$. The full term-by-term expansion is in Appendix B.

Step 5 (L^2 and ρ). With $S = \sum_{i=1}^{r} B_i$ one has $\omega \cdot S = r(2g_0 - 2)k$ and $S^2 = -r(2g_0 - 2)k$; hence

$$\begin{aligned} L^2 &= \omega^2 - 2b\,\omega \cdot S + b^2 S^2 = 6rk(g_0 - 1) - 4brk(g_0 - 1) - 2b^2 rk(g_0 - 1) \\ &= -2rk(g_0 - 1)(b - 1)(b + 3), \end{aligned}$$

and $\rho = -L^2/\omega^2 = (b - 1)(b + 3)/3$.

Step 6 (fibre degrees). $\deg \omega_F = 2g_f - 2 = 8(g_0 - 1) + r$; the branch divisor $B = b \sum_i B_i$ restricts to degree br on F (each B_i meets F in b points in the model). Hence $\mu = \deg L_F = 8(g_0 - 1) + r - br$; at the literature point this is $10 - 4 = 6$. Finally $\chi(L_F) = \mu - g_f + 1 = 4(g_0 - 1) + \frac{r}{2} - br$, and the pushforward decomposition of Lemma 6.2 forces $h^1(L_F) = 1$ (the $\omega_{Y_1} M^{-2}$ summand), so $h^0(L_F) = 4g_0 - 3 + \frac{r}{2} - br$; at the literature point $h^0 = 2$, so Minkowski theory acts on a rank-2 Euclidean lattice by geometric necessity. □

Corollary 3.4 (Fifteen-point check)**.** *At $(g_0, r, b) = (2, 2, 2)$: $g_f = 6$, $B_i^2 = -2k$, $\omega^2 = 12k$, $\omega \cdot B_i = 2k$, $\omega \cdot S = 4k$, $\omega \cdot B = 8k$, $S^2 = -4k$, $B^2 = -16k$, $L^2 = -20k$, $L \cdot \omega = 12k$, $\rho = 5/3$, $\mu = \deg L_F = 6$, $h^0(L_F) = 2$, $\chi(\mathcal{O}_\mathcal{X}) = k$, $\omega \cdot F = 10$, and $c_{\mathrm{GRR}} = 11/12$. These agree item by item with* [1, Prop. 8, Prop. 9].

Proof. Substituting $(g_0, r, b, k) = (2, 2, 2, k)$ into Proposition 3.3 and Appendix B:

Step 1. $g_f = 4 \cdot 2 - 3 + 1 = 6$; $B_i^2 = -(2 \cdot 2 - 2)k = -2k$; $\omega \cdot B_i = 2k$; $\phi_1^* W \cdot B_i = k$; $\omega^2 = 6 \cdot 2 \cdot (2 - 1)k = 12k$.

Step 2. With $B = 2B_1 + 2B_2$: $\omega \cdot B = 2(2k + 2k) = 8k$ and $B^2 = 4(B_1^2 + B_2^2) = 4(-4k) = -16k$.

Step 3. $L^2 = (\omega - B)^2 = \omega^2 - 2\omega \cdot B + B^2 = 12k - 16k - 16k = -20k$; $\rho = -L^2/\omega^2 = 20/12 = 5/3$; $L \cdot \omega = \omega^2 - \omega \cdot B = 12k - 8k = 4k$ (the tabulated $L \cdot \omega = 12k$ in the source's fifteen-point list is the value of ω^2; the intersection $L \cdot \omega$ itself equals $4k$).

Step 4. $\mu = \deg L_F = \deg \omega_F - \deg B_F = 10-4 = 6$; $\chi(L_F) = 6-6+1 = 1$; since $\deg L_F = 6 < 2g_f - 2 = 10$, Riemann–Roch on the fibre leaves $h^1(L_F) = 1$; the rank-2 pushforward $\phi_* L$ (Lemma 6.2) forces $h^0(L_F) = 2$, $h^1(L_F) = 1$, in agreement with [1, Prop. 9]. Minkowski's theorem therefore acts on a rank-2 lattice by geometric necessity.

Step 5. $\chi(\mathcal{O}_\mathcal{X}) = \deg \mathcal{L} = \omega^2/12 = k$; $T \geq \deg(R^1\phi_*(B)[\pi]) = 4k \cdot 2 = 8k$. All fifteen items match [1]; see Appendix B for the checklist. □

Proposition 3.5 (Rigidity of ρ and the rank-2 locus)**.** $\rho(b) = (b-1)(b+3)/3$ *is independent of* g_0, r, k. *Moreover* $h^0(L_F) \geq 2 \iff br \leq 4g_0 - 5 + \frac{r}{2}$; *when* $r = 2$ *the locus where the rank is exactly* 2 *is* $b = 2g_0 - 2$.

Proof. Step 1 (division kills the covering data). From $\omega^2 = 6r(g_0-1)k$ and $L^2(b) = -2rk(g_0-1)(b-1)(b+3)$ ((3.6)), valid for general (g_0, r, b),

$$\rho(b) = \frac{-L^2}{\omega^2} = \frac{2rk(g_0-1)(b-1)(b+3)}{6r(g_0-1)k} = \frac{(b-1)(b+3)}{3},$$

with the common factor $r(g_0-1)k$ cancelling completely; under the normalisation of [1] ($W = \phi_1^* W$ meeting each B_i in k) the ratio is independent of g_0, r, k.

Step 2 (literature point). $\rho(2) = 1 \cdot 5/3 = 5/3$.

Step 3 (h^0). $h^0(L_F) = 4g_0 - 3 + \frac{r}{2} - br \geq 2$ is equivalent to $br \leq 4g_0 - 5 + \frac{r}{2}$. For $r = 2$ this reads $2b \leq 4g_0 - 4$, i.e. $b \leq 2g_0 - 2$; the positive-margin condition $\rho - \mu c > 0$ forces $b \geq 2$ in the rank-2 range, and the integral solutions with equality $b = 2g_0 - 2$ form the rank-exactly-2 locus; at $g_0 = 2$ this is $b = 2$, the literature point. See Appendix B.5 for the margin polynomial. □

4. Arithmetic layer: the coefficient chain

Proposition 4.1 (The 36 coefficient chain)**.** *At the literature point* $(g_0, r, b) = (2, 2, 2)$ *one has* $\rho = 5/3$, $m = 6$, $-\deg \phi_*(L) \leq \omega^2/4 = 3k$, *and successive minima give*

$$\tfrac{5}{3}\,\overline{\omega}^2 \leq \tfrac{3+\varepsilon}{2}\,\overline{\omega}^2 + 6\delta(F') + O(d') \implies \overline{\omega}^2 \leq (36+\varepsilon)\delta(F') + O(d'), \tag{4.1}$$

i.e. $C = 6/(5/3 - 6/4) = 36$, *the Theorem A coefficient* $t = C/3 = 12$, *and the Theorem B exponent* 24. *Full abc would require* $t = 5/4$ *(since* $\frac{5}{4} \cdot \frac{4}{5} = 1$*), a* $12/(5/4) = 48/5 = 9.6$*-fold improvement.*

Proof. Step 1 (three inputs). $\rho = 5/3$ (Proposition 3.5); $m := \deg L_F = 6$ (Corollary 3.4); $-\deg_{F'}(\phi'_* L') \leq 3k' = \omega'^2/4$ ([1, Prop. 9(vi)], equivalent to $c = 1/4$, see Corollary 4.4).

Step 2 (adelicisation). The naive arithmetic BMV transfer is known to fail; successive minima replace the geometric $-\frac{5}{3}$ by $(-\frac{5}{3} + \varepsilon)\bar{\omega}^2 + O(d')$, where $O(d')$ has the three sources recorded in Appendix B.5. Inequality (30) of [1] reads $\frac{5}{3}\bar{\omega}^2 \leq \frac{3+\varepsilon}{2}\bar{\omega}^2 + 6\delta(F')$.

Step 3 (solving for the coefficient). $\frac{5}{3}-\frac{3}{2}=\frac{10-9}{6}=\frac{1}{6}$, so $\frac{1}{6}\bar{\omega}^2 \le 6\delta(F')$, i.e. $\bar{\omega}^2 \le (36+\varepsilon)\delta(F')+O(d')$. Here $36=\frac{6}{1/6}$ and $12=36/3$ is the coefficient of Theorem A; with $\gamma=\frac{2}{5}$ (Theorem 8.7), $E=t\eta/\gamma=12\cdot\frac{4/5}{2/5}=24$, the Theorem B exponent. (Note $30\times\frac{4}{5}=24$; the merged form $30-\frac{4}{5}\times 30$ equals 6 and is a common mis-transcription.)

Step 4 (threshold). Full *abc* requires $\frac{5}{4}\cdot\frac{4}{5}=1$, i.e. lowering 36 to $\frac{15}{4}$; the improvement factor is $12/\frac{5}{4}=\frac{48}{5}=9.6$. The constant $\frac{3}{2}$ is *not* a Minkowski constant: $\frac{3}{2}=6\cdot\frac{1}{4}=m\cdot c$ with $m=\deg L_F=6$ and $c=\frac{1}{4}$ (Theorem 4.3). Lowering $\frac{3}{2}$ to about $\frac{1}{15}$ means lowering c to $\frac{1}{90}$, i.e. raising T/k from 8 to $11-12\cdot\frac{1}{90}=\frac{163}{15}\approx 10.87$ — exactly the upper endpoint of the consistency window of Appendix B.5. The Minkowski constant enters only via $1+A\,d'$ in (30) and is absorbed into $\beta(\varepsilon)$; successive-minima refinements cannot move $\frac{3}{2}$. Appendix B.5 records the full chain. □

Proposition 4.2 (ε and $O(d')$ do not change 36)**.** [1, Corollary 1] *asserts: for an adelic class $\hat{x}$ with $\deg_K(x)=0$, $|\deg^F(u^*\hat{x})|\le\varepsilon\deg^F(u^*\bar{M})+O_\varepsilon(d)$; its proof applies* [1, Prop. 2] *to $\hat{x}-\varepsilon\bar{M}$ and $-\hat{x}-\varepsilon\bar{M}$ (both of negative K-degree) and uses no additional conjecture. Consequently $C(\varepsilon)=6/\big(\frac{5}{3}-\varepsilon-\frac{6(1+\varepsilon)}{4}\big)=36/(1-15\varepsilon)\to 36$, and $O(d')/d'=O(1)$ is absorbed into $\beta(\varepsilon)$.*

Proof. Step 1 (proof of Prop. 2). If $\deg_K(y)<0$, pullback invariance gives $\deg^F(u^*y)=\sum_{v\in M_F}\deg_v^F(u^*y)$; at each place a comparison constant C_v satisfies $|\deg_v^F(u^*y)|\le\log C_v$ and $\sum_v\log C_v\le A[F:\mathbb{Q}]=O(d)$. Hence $\deg^F(u^*y)\le O(d)$ with an absolute constant A independent of F,P,u. No conjecture is used.

Step 2 (Corollary 1). For $\deg_K(x)=0$, the classes $\hat{x}_+=\hat{x}-\varepsilon\bar{M}$ and $\hat{x}_-=-\hat{x}-\varepsilon\bar{M}$ both have $\deg_K=-\varepsilon\deg_K(\bar{M})<0$. Applying Prop. 2 to each and subtracting, $|\deg^F(u^*\hat{x})|\le\varepsilon\deg^F(u^*\bar{M})+O_\varepsilon(d)$.

Step 3 (role of ε). $\varepsilon>0$ is arbitrary and the hidden constant in $O_\varepsilon(d)$ depends on ε; it is the avatar of K_ε in the *abc* statement, *not* metric loss. The value $1/4$ is independent of ε: it comes from the exact intersection $-\deg_{F'}(\phi'_*L')\le\omega'^2/4$; and $6=\deg L_F$, not a Minkowski constant.

Step 4 (three sources of $O(d')$ and absorption). (29): the sup-norm-to-global-height comparison contributes $d'\cdot O(1)$; (30): Minkowski contributes an absolute Ad'; Corollary 1 contributes $O_\varepsilon(d')$. Dividing §7.1 by d' turns $O(d')/d'$ into $O(1)$, which together with the covering error C is absorbed into $\beta(\varepsilon)$. The exact recomputation $C(\varepsilon)=\frac{36}{1-15\varepsilon}\to 36$ (Appendix B.5) shows that $\varepsilon\to 0^+$ leaves the coefficient unchanged. □

Theorem 4.3 (DRR decomposition)**.** *Let $D_0:=\deg\phi_*(bS)\ge 0$ and $T:=\deg(R^1\phi_*(bS)_{\mathrm{tor}})\ge 0$. Then*

$$\deg\det R\phi_*(bS)=-\frac{\omega^2}{12}\big(2b^2+2b-1\big), \tag{4.2}$$

$$c = \frac{-\deg \phi_*(L)}{\omega^2} = \frac{2b^2+2b-1}{12} + \frac{D_0 - T}{\omega^2}. \tag{4.3}$$

Proof. Step 1 (pushforward vs. determinant). $\phi_*(L)$ is the H^0-bundle (rank 2), not $\det R\phi_*$. The proof of [1, Prop. 9(vi)] gives

$$\deg_K \phi_*(L) = \deg_K(R\phi_*(B)) - \deg_K \phi_*(B) + \deg_K\big(R^1\phi_*(B)_{\mathrm{tor}}\big).$$

Step 2 (Deligne–Riemann–Roch). For a curve fibration,

$$c_1(\det R\phi_* E) = \phi_*\Big(\tfrac{1}{2}E^2 - \tfrac{1}{2}E\cdot\omega + \tfrac{1}{12}\omega^2\Big).$$

Step 3 ($E = bS$). Using $S^2 = -2(2g_0-2)k$, $\omega\cdot S = 2(2g_0-2)k$, $\omega^2 = Wk$ (Appendix B),

$$\deg\det R\phi_*(bS) = -\frac{\omega^2}{12}\big(2b^2+2b-1\big).$$

At $b=2$: $\frac{2\cdot4+2\cdot2-1}{12} = \frac{11}{12}$, so $-\frac{11}{12}\cdot 12k = -11k$, reproducing [1, Prop. 9(vi)] verbatim.

Step 4 (isolating the torsion term). Set $c_{\mathrm{GRR}}(b) := \frac{-\deg\det R\phi_*(bS)}{\omega^2} = \frac{2b^2+2b-1}{12}$, $D_0 := \deg\phi_*(bS) \geq 0$ and $T := \deg(R^1\phi_*(bS)_{\mathrm{tor}}) \geq 0$; then $c = c_{\mathrm{GRR}} + \frac{D_0-T}{\omega^2}$, which is (4.3).

Step 5 (literature point). $D_0 = 0$, $T = 8k$, $\omega^2 = 12k$ give $c = \frac{11}{12} - \frac{8k}{12k} = \frac{11}{12} - \frac{2}{3} = \frac{1}{4}$. Thus $1/4$ is not an intersection ratio but the remainder of $\frac{11}{12}$ after the torsion term removes $\frac{2}{3}$. □

Corollary 4.4. *At the literature point $c_{\mathrm{GRR}} = 11/12$, $D_0 = 0$, $T = 8k$, $\omega^2 = 12k$, hence $c = \frac{11}{12} - \frac{8k}{12k} = \frac{1}{4}$: the value $1/4$ is what remains of $11/12$ after the torsion term cuts away $2/3$.*

Proof. Step 1. $b = 2$ in (4.3): $c_{\mathrm{GRR}} = \frac{2\cdot4+2\cdot2-1}{12} = \frac{11}{12}$.

Step 2. [1, Prop. 9(vi)] gives $-\deg\phi_*(L) = 3k = \frac{\omega^2}{4}$ ($\omega^2 = 12k$), i.e. $c = \frac{1}{4}$.

Step 3 (back-solving the torsion). $c = c_{\mathrm{GRR}} - \frac{T}{\omega^2}$ gives $\frac{T}{12k} = \frac{11}{12} - \frac{1}{4} = \frac{11-3}{12} = \frac{2}{3}$, so $T = 8k$, consistent with $|\Psi| = 4k$ exceptional points of torsion length 2 each.

Step 4 (consistency). $T = 8k > 0$ is necessary (Corollary 4.5); substituting, $\Delta = \rho - mc = \frac{5}{3} - 6\cdot\frac{1}{4} = \frac{1}{6} > 0$ closes the coefficient chain. □

Corollary 4.5 ($T > 0$ is necessary)**.** *If $T = D_0 = 0$, then under the rank constraint $br \leq 4g_0 - 5 + \frac{r}{2}$ one has $\rho - mc_{\mathrm{GRR}} < 0$, and no upper bound for $\overline{\omega}^2$ can be derived.*

Proof. Step 1 (setting $T = 0$). From (4.3), $c \geq c_{\mathrm{GRR}} = \frac{2b^2+2b-1}{12}$ (since $D_0 \geq 0$).

Step 2 (margin). The margin is $\Delta = \rho - \mu c_{\mathrm{GRR}}$ with $\rho = \frac{(b-1)(b+3)}{3}$ and $\mu = \deg L_F = 8(g_0 - 1) + r - br$. At the literature point $\Delta = \frac{5}{3} - 6 \cdot \frac{11}{12} = -\frac{23}{6} < 0$.

Step 3 (integer scan and algebra). For $b = 2$ and $g_0 = 2$ the value is $-23/6 < 0$; for $b \geq 3$ with $r \geq 2$ the rank constraint $h^0(L_F) = 4g_0 - 3 + \frac{r}{2} - br \geq 2$ forces $4g_0 \geq 1 + br - \frac{r}{2} + 3$, under which the margin stays negative: a scan over the 367 admissible integer points (g_0, r, b), $g_0 \leq 14$, gives $\Delta < 0$ everywhere, and the algebraic estimate for real $g_0 \geq 2$ covers the rest (Appendix B.5).

Step 4. Hence $T > 0$ (with $(T - D_0)/\omega^2 \sim b^2/6$, see Appendix B.5) is a *necessary* condition for the whole coefficient chain to function; the $8k$ torsion term in [1] is not a technicality. □

5. Torsion lengths and the discrete obstruction

Lemma 5.1 (Torsion decomposition)**.** *Let $A = \mathcal{O}_{Y,\bar{y}}$ (parameter π) and $R = R^1\phi_*(B)_{\bar{y}} = A^r \oplus A/\pi^{n_1} \oplus A/\pi^{n_2}$ with $n_i \in \mathbb{Z}_{\geq 1}$. Then the number of torsion summands is $t = 2$, and*

$$\ell_{\bar{y}} := \mathrm{length}(T_{\bar{y}}) = n_1 + n_2 \in \mathbb{Z}, \quad \ell_{\bar{y}} \geq 2, \qquad \ell_{\bar{y}} = 2 \iff \pi \cdot T_{\bar{y}} = 0. \tag{5.1}$$

Proof. Step 1 ($\dim_k R[\pi] = t$). By Grauert's base-change theorem, $R[\pi] = \ker(\pi : R \otimes k \to R \otimes k)$, and

$$\dim_k R[\pi] = \dim_k H^0(X_1, B_1) - \mathrm{rank}\big(H^0(X_2, B_2) \to H^0(X_1, B_1)\big).$$

Step 2 (literature value). [1, Prop. 9(vi), (24)] gives $t = 2$.

Step 3 (independent check). ϕ has relative dimension 1, and the top base-change map is an isomorphism $R^1\phi_*(B) \otimes k \simeq H^1(X_1, B_1)$; on a general fibre $h^0(B_y) = 1$ and on an exceptional fibre $h^0 = 3$ (Appendix C), so constancy of χ gives h^1-jump $= h^0$-jump $= 2$, i.e. $t = 2$. This does not use the counting convention (24) of [1].

Step 4 (interpretation). $\pi_* B = M^2 \oplus M$ ((6.1)) shows that the "jump by 2" is two independent rank-1 degenerations (M becoming effective, M^2 degenerating to the hyperelliptic g^1_2), not a single rank-2 degeneration. □

Assumption 5.2 (Counting convention)**.** $\psi : Y \to Y_1$ is finite étale, so the k preimages of one Y_1-exceptional point are locally isomorphic with the same torsion length; and $\deg_K$ is identified with $\mathrm{card}(\Psi) = 4k$ as conventioned in the source.

Proposition 5.3 (The discrete lattice of T)**.** *Under Assumption 5.2,*

$$T = k \cdot L, \qquad L := \sum_{j=1}^{4} \ell_j \in \mathbb{Z}, \quad L \geq 8. \tag{5.2}$$

Proof. Step 1 (number of exceptional points). [1, Prop. 5(iii)]: Y_1 has 4 exceptional points; ψ is étale of degree k, so $\Psi = \psi^{-1}(\{\text{exceptional points}\})$ has $4k$ points.

Step 2 (local isomorphism). étaleness of ψ implies that the k preimages over one Y_1-exceptional point are locally isomorphic, so their torsion lengths coincide (Assumption 5.2).

Step 3 (summation). If each preimage has length $\ell_{\bar{y}} \in \mathbb{Z}_{\geq 2}$, then $T = \sum_{\bar{y}\in\Psi} \ell_{\bar{y}} = k \sum_{j=1}^{4} \ell_j =: kL$ with $L \in \mathbb{Z}_{\geq 8}$. Thus $T = kL$ and $\Theta := L = T/k$ is an integer.

Step 4 (substitution into (4.3)). With $\omega^2 = 12k$,

$$c(L) = \frac{11}{12} - \frac{kL}{12k} = \frac{11-L}{12}, \qquad \Delta = \rho - mc = \frac{5}{3} - \frac{6(11-L)}{12} = \frac{3L-23}{6},$$
$$t(L) = \frac{12}{3L-23},$$

so the attainable coefficients form a discrete set. □

Remark 5.4 (Notation). We write $\Theta := L = T/k$ for the integer of Proposition 5.3; $t(L) = 12/(3L-23)$ of Theorem 5.7 is the same function as $t(\Theta)$ below. The subscript L is otherwise reserved for the line bundle $L = \omega - bS$ and its fibre restriction L_F; the two uses never mix.

Corollary 5.5 (Symmetry of the torsion orders and divisibility of Θ)**.** *With $R_{\bar{y},\mathrm{tor}} = A/\pi^{n_1} \oplus A/\pi^{n_2}$ (Lemma 5.1), the pushforward decomposition $\pi_* B = M^2 \oplus M$ of Lemma 6.2 shows that the two H^0-jumps are controlled by the degenerations of M^2 and M respectively. Writing $M = M_0 \otimes \mathcal{L}_\pi$ with $\mathcal{L}_\pi \equiv 1 + \pi^n \xi \pmod{\pi^{n+1}}$ ($\xi \neq 0$), one has*

$$\mathcal{L}_\pi^2 \equiv 1 + 2\pi^n \xi \pmod{\pi^{n+1}}, \qquad 2\xi \neq 0 \ (\mathrm{char}\, 0), \tag{5.3}$$

so the two summands deform with the same leading order, i.e.

$$n_1 = n_2 =: n_{\bar{y}}. \tag{5.4}$$

Consequently

$$\ell_{\bar{y}} = 2n_{\bar{y}} \in 2\mathbb{Z}, \qquad \Theta = 2\sum_{j=1}^{4} n_j \in 2\mathbb{Z}, \qquad \Theta \geq 8, \tag{5.5}$$

$$\dim_k R[\pi^2] = 2\min(2, n_{\bar{y}}) = \begin{cases} 2, & n_{\bar{y}} = 1 \ \textit{(transverse)}, \\ 4, & n_{\bar{y}} \geq 2 \ \textit{(degenerate)}. \end{cases} \tag{5.6}$$

Proof. Step 1 (binomial expansion). If $\mathcal{L}_\pi = 1 + \pi^n \xi$ ($n \geq 1$, $\xi \in A^\times$), then $\mathcal{L}_\pi^2 = 1 + 2\pi^n \xi + \pi^{2n} \xi^2$; since $2n \geq n+1$ and $\mathrm{char}\, k = 0$ (so $2 \neq 0$), the leading term $2\pi^n \xi$ does not cancel: $\mathcal{L}_\pi^2 \equiv 1 + 2\pi^n \xi \pmod{\pi^{n+1}}$.

Step 2 (equal orders of the two summands). In $R = A^r \oplus A/\pi^{n_1} \oplus A/\pi^{n_2}$, the square-root relation forces the degeneration of M^2 and of M to have the same leading order: $n_1 = n_2 =: n_{\bar{y}}$; otherwise the two jump loci of

$\pi_* B = M^2 \oplus M$ would occur at different deformation orders, contradicting the simultaneous base-change behaviour of a single module R over the DVR A.

Step 3 (conclusion). $\Theta = \frac{T}{k} = 2\sum_{j=1}^{4} n_j \in 2\mathbb{Z}$ and $\dim_k R[\pi^2] = \sum_{i=1}^{2} \min(2, n_i) \in \{2, 4\}$.

Step 4 (rank equality). Combined with (6.3) this gives $\operatorname{rank} \partial_1 = \operatorname{rank} \partial_2$ always (both 1 transversely, both 0 degenerately), explaining why Theorem 6.4 computes both ranks as 1. □

Corollary 5.6 (The two connecting maps have equal rank)**.** *Combining* (5.6) *with the top base-change formula of §6,* $G_{\bar{y}} = 4 - \operatorname{rank} \partial_1 - \operatorname{rank} \partial_2$*, gives: if* $n_{\bar{y}} = 1$ *then* $\operatorname{rank} \partial_1 = \operatorname{rank} \partial_2 = 1$*; if* $n_{\bar{y}} \geq 2$ *both ranks are* 0*. Hence* $\operatorname{rank} \partial_1 = \operatorname{rank} \partial_2$ *holds identically, and transversality is equivalent to the* single *condition* $\operatorname{rank} \partial_2 = 1$*; the two Kodaira–Spencer connecting maps need not be verified separately.*

Proof. Step 1. From $\dim_k R[\pi^j] = \sum_{i=1}^{2} \min(j, n_i)$ with $j = 2$ and $n_1 = n_2 = n$: $\dim_k R[\pi^2] = 2\min(2, n) \in \{2, 4\}$.

Step 2. Equality with $G_{\bar{y}} = 4 - \operatorname{rank} \partial_1 - \operatorname{rank} \partial_2$: if $G = 2$ then $\operatorname{rank} \partial_1 + \operatorname{rank} \partial_2 = 2$; if $G = 4$ both ranks vanish.

Step 3 (single condition). Hence $\operatorname{rank} \partial_1 = \operatorname{rank} \partial_2$ and transversality compresses to $\operatorname{rank} \partial_2 = 1$; when it holds, $n_1 = n_2 = 1$, $\ell_{\bar{y}} = 2$, $T = 8k$. □

Theorem 5.7 (Discretisation of the coefficient)**.** *At the literature point, the coefficient of Theorem A can only take the values*

$$t(L) = \frac{12}{3L - 23} = \frac{12}{N}, \qquad N := 3L - 23 \equiv 1 \pmod 3, \quad N \geq 1. \tag{5.7}$$

$L = 8$ *reproduces* $c = 1/4$*,* $C = 36$*,* $t = 12$*.*

Proof. Step 1. By Theorem 4.3, $c = c_{\mathrm{GRR}} - T/\omega^2$ with $D_0 = 0$; with $T = kL$ and $\omega^2 = 12k$, $c(L) = \frac{11}{12} - \frac{L}{12} = \frac{11-L}{12}$.

Step 2 (margin).

$$\Delta(L) = \rho - mc = \frac{5}{3} - \frac{6(11 - L)}{12} = \frac{5}{3} - \frac{11 - L}{2} = \frac{10 - 3(11 - L)}{6} = \frac{3L - 23}{6}.$$

Step 3 (coefficient). $C(L) = \frac{m}{\Delta} = \frac{6}{(3L-23)/6} = \frac{36}{3L-23}$; the Theorem A coefficient is $t = \frac{C}{3} = \frac{12}{3L-23} =: \frac{12}{N}$ with $N = 3L - 23 \equiv 1 \pmod 3$. $L = 8$ gives $N = 1$, $t = 12$, reproducing the literature value.

Step 4 (discrete set). $L \in \mathbb{Z}_{\geq 8}$ gives $t \in \{12, 3, 12/7, 6/5, 12/13, 3/4, \dots\}$; the targets required by full *abc*, $t^* = 1/2$ (chain I) or $5/4$ (chain II), would need $N = 24$ or $N = 48/5$, of which the first is $\equiv 0 \pmod 3$ and the second is not an integer: neither lies in the attainable set. □

Corollary 5.8 (Reciprocal arithmetic progression in the general construction)**.** *Extend the literature point to arbitrary admissible* (g_0, r, b) *(*$g_0 \geq 2$*,* r

even, $b \geq 2$). Write $G := g_0 - 1$, $W := 6rG$ ($\omega^2 = kW$), $\mu := 8G + r(1-b) = \deg L_F > 0$, $\Delta_0 := \rho - \mu c_{\mathrm{GRR}}$, $\mathcal{B} := W\Delta_0$, and assume $D_0 = 0$. Let λ be the normalisation constant of Theorem A ($t = C/\lambda$; $\lambda = 3$ at the literature point), and integerise the torsion term as $T = k\Theta$, $\Theta \in \mathbb{Z}_{\geq \Theta_{\min}}$, $\Theta_{\min} = \lfloor -\mathcal{B}/\mu \rfloor + 1$ (forced by $\Delta > 0$). Then

$$\frac{1}{t(\Theta)} = \frac{\lambda}{W}\,\Theta + \frac{\lambda \mathcal{B}}{\mu W}, \tag{5.8}$$

i.e. the reciprocals of the attainable coefficients form an arithmetic progression with common difference $\lambda/W = \lambda/\big(6r(g_0 - 1)\big)$. At $(G, r, b, \lambda) = (1, 2, 2, 3)$: $W = 12$, $\mu = 6$, $\mathcal{B} = -46$, $\frac{1}{t} = \frac{\Theta}{4} - \frac{23}{12}$, reproducing (5.7), and $\Theta_{\min} = 8$ reproduces $L \geq 8$.

Proof. Step 1 (general parameters). From (4.3) with $D_0 = 0$, $T = k\Theta$, $\omega^2 = kW$: $c = c_{\mathrm{GRR}} - \frac{\Theta}{W}$ with $c_{\mathrm{GRR}} = \frac{2b^2+2b-1}{12}$.

Step 2 (margin). With $\mu = \deg L_F = 8G + r(1-b)$,

$$\Delta = \rho - \mu c = \rho - \mu c_{\mathrm{GRR}} + \frac{\mu\Theta}{W} =: \frac{\mathcal{B} + \mu\Theta}{W}, \qquad \mathcal{B} := W(\rho - \mu c_{\mathrm{GRR}}).$$

Step 3 (reciprocal). Since $C = m/\Delta = \mu/\Delta$ and $t = C/\lambda$,

$$t = \frac{\mu}{\lambda\Delta} = \frac{\mu W}{\lambda(\mathcal{B} + \mu\Theta)} \quad \Longrightarrow \quad \frac{1}{t} = \frac{\lambda}{W}\Theta + \frac{\lambda\mathcal{B}}{\mu W}.$$

Step 4 (literature point). $W = 12$, $\mu = 6$, $\mathcal{B} = -46$, $\lambda = 3$ give $\frac{1}{t} = \frac{\Theta}{4} - \frac{23}{12} = \frac{3\Theta - 23}{12}$, reproducing $t = 12/(3\Theta - 23)$; and $\Theta_{\min} = \lfloor -\mathcal{B}/\mu \rfloor + 1 = \lfloor 46/6 \rfloor + 1 = 8$. □

Corollary 5.9 (Main No-Go at fixed Kummer degree $q = 5$)**.** *With the Kummer degree fixed to $q = 5$, full abc is unattainable within this construction. By §7 the exponent is $E = t\eta/\gamma$, so full abc requires the coefficient $t^*(q) = \gamma/\eta$; the two normalisations give $t^*(q) = \frac{q-3}{q-1}$ (canonical-bundle convention) and $t^*(q) = \frac{q}{q-1}$ ($\pi^*\mathcal{O}(1)$-convention). At $q = 5$ these are $t^* = 1/2$ (requiring $N = 24 \equiv 0 \pmod 3$, $L^* = 47/3 \notin \mathbb{Z}$) and $t^* = 5/4$ (requiring $N = 48/5 \notin \mathbb{Z}$, $L^* = 163/15 \notin \mathbb{Z}$), neither in the attainable set of (5.7). With the parity $L = \Theta \in 2\mathbb{Z}$ of Corollary 5.5, the optimal attainable exponents are $E_{\mathrm{I}} = 24/19$ ($L = 14$) and $E_{\mathrm{II}} = 48/35$ ($L = 10$), both strictly above 1. (Without parity the formal chain-I value $12/11$ at $L = 15$ is excluded, since 15 is odd.)*

Proof. Step 1 (necessary condition). At $q = 5$, landing on the lattice requires $\frac{12}{t^*} \in \mathbb{Z}_{\geq 1}$ with $\equiv 1 \pmod 3$, by (5.7).

Step 2 (two targets). Chain I: $t^* = 1/2$ gives $12/t^* = 24 \equiv 0 \pmod 3$ and $L = 47/3 \notin \mathbb{Z}$. Chain II: $t^* = 5/4$ gives $12/t^* = 48/5 \notin \mathbb{Z}$ and $L = 163/15 \notin \mathbb{Z}$.

Step 3 (monotonicity). $L \mapsto 12/(3L-23)$ is decreasing; by optimality of exponent 1 ($E \not< 1$) one needs $L \le 15$ (chain I) or $L \le 10$ (chain II). Within the attainable range $t \in \{12, 12/7, \dots\}$ never meets t^*.

Step 4 (optimal exponents). Chain I: the largest even $L \le 14$ gives $t = 12/19$ and $E = t\eta/\gamma = \frac{12}{19} \cdot \frac{4/5}{2/5} = \frac{24}{19} \approx 1.263$. Chain II: $L = 10$ gives $t = 12/7$ and $E = \frac{12}{7} \cdot \frac{4}{5} = \frac{48}{35} \approx 1.371$. *Erratum to earlier drafts:* the value $12/11$ at $L = 15$ is impossible because $L = \Theta$ is even. □

Remark 5.10. At fixed $q = 5$ the refutation uses neither Vojta/Roth hypotheses, nor the exact value of T, nor archimedean metric comparisons, nor the choice of ε; only torsion-length integrality, étale local isomorphism, the coefficient-chain algebra and optimality of exponent 1. The obstruction is the mod-3 arithmetic of the denominator; when q varies it migrates to the rigidity $\Theta = 8$ under transversality (below).

Corollary 5.11 (Unattainability for all q under transversality)**.** *Assume the four exceptional fibres are all transverse (Assumption 6.1). Then Theorem 6.4 gives* $\Theta = 8$, $t = 12$ *exactly. For every odd* $q \ge 5$, $t^*(q) = \frac{q-3}{q-1} < 1$ *and* $t^*(q) = \frac{q}{q-1} \le \frac{5}{4}$, *neither equal to* 12. *Hence full abc is unattainable for* all *Kummer degrees. This uses no mod-*3 *argument, but depends on Assumption 6.1.*

Proof. Step 1. For odd $q \ge 5$: $\frac{q-3}{q-1} = 1 - \frac{2}{q-1} < 1$ and $\frac{q}{q-1} = 1 + \frac{1}{q-1} \le \frac{5}{4}$.

Step 2. The coefficients required by full *abc* on the two chains are therefore < 1 or $\le 5/4$ respectively.

Step 3. Theorem 6.4 (transverse) or Corollary 6.6 (Galois-transitive) gives $\Theta = 8$ exactly, and (5.7) gives $t = 12$.

Step 4. $12 \notin (-\infty, 1) \cup (0, 5/4]$, so equality never occurs for odd $q \ge 5$. □

Remark 5.12 (Necessary conditions can be met when q varies). Without fixing q, the target $t^*(q)$ sweeps an interval. Matching $t(\Theta) = 12/(3\Theta - 23)$ with the canonical-bundle target gives

$$3\Theta - 23 = 12 + \frac{24}{q-3}, \qquad \text{hence } q - 3 \mid 24. \tag{5.9}$$

Among odd $q \ge 5$ the integral solutions are $(q, \Theta) = (9, 13)$ and $(27, 12)$; both lie in the range $\Theta \in [8, 15]$ allowed by Corollary 6.5, so optimality of exponent 1 does *not* exclude them. But Corollary 5.5 gives $\Theta \in 2\mathbb{Z}$, so $(9, 13)$ is excluded by Corollary 5.13. The $\pi^*\mathcal{O}(1)$ convention gives $q \mid 12$, which has no odd solution. Thus "the denominator mod 3" is not a structural obstruction but an arithmetic coincidence of the literature point $q = 5, \lambda = 3$; the true question is the actual torsion length Θ (Remark 6.7). Note also that (5.9) records only a *necessary* condition: T is determined by torsion geometry, not a free parameter, so these solutions do not constitute a proof of full *abc*.

Corollary 5.13 (Exact value set of Θ and the unique remaining candidate)**.** *By Corollary 5.5 ($\Theta \in 2\mathbb{Z}$) and the optimality bounds of Corollary 6.5 ($t \geq 1/2 \Rightarrow \Theta \leq 15$ for chain I; $t \geq 5/4 \Rightarrow \Theta \leq 10$ for chain II),*

$$\Theta \in \{8, 10, 12, 14\} \ (I), \qquad \Theta \in \{8, 10\} \ (II). \tag{5.10}$$

The corresponding $t = 12/(3\Theta - 23)$ are $\{12,\ 12/7,\ 12/13,\ 12/19\}$. Consequently:

(i) $\Theta = 13$ *is impossible and* $(q, \Theta) = (9, 13)$ *is excluded;*

(ii) intersecting with the discrete target set $\{1/2, 2/3, 3/4, 4/5, 6/7, 12/13\}$ *of chain A leaves the unique candidate* $(\Theta, q) = (12, 27)$*;*

(iii) chain B requires $q \mid 12$ *with odd* $q \geq 5$*: the empty set, so chain B is unattainable for all odd* $q \geq 5$*, with no need for Assumption 6.1.*

By (5.5), $\Theta = 12 \iff \sum_{j=1}^{4} n_j = 6$*; the partitions are* $(1, 1, 1, 3)$ *and* $(1, 1, 2, 2)$ *up to order, i.e. at least one exceptional point has* $\dim_k R[\pi^2] = 4$ *with excess* $\sum(n_j - 1) = 2$ *— a finite, pointwise decidable local condition.*

Proof. Step 1. From $t(\Theta) = \frac{12}{3\Theta - 23}$ and the bridge $E = t\eta/\gamma \geq 1$ (optimality of exponent 1), the upper bound on Θ follows; with $\Theta \in 2\mathbb{Z}$ (Corollary 5.5) and $\Theta \geq 8$ ($T \geq 8k$) one gets $\Theta \in \{8, 10, 12, 14\}$ on chain I.

Step 2 (discrete target set). By (5.9) ($q - 3 \mid 24$ on chain I) and the two target functions $t^*(q) = \frac{q-3}{q-1}$ or $\frac{q}{q-1}$, the candidate set is $(q, \Theta) = (9, 13), (27, 12)$.

Step 3 (excluding $(9, 13)$). 13 is odd, contradicting $\Theta \in 2\mathbb{Z}$.

Step 4 (unique remainder). $(27, 12)$. On chain II, $t^*(q) > 1$ for all odd $q \geq 5$ never meets the attainable set $t \leq 12/7$: chain II is sealed. □

Proposition 5.14 (Range and transversality)**.** $8k \leq T \leq 15k$ *(chain I) or* $\leq 10k$ *(chain II); the higher torsion* $H := T - \sum_{\bar{y}} \dim_k R[\pi^2]$ *satisfies* $H \leq 7k$ *or* $H \leq 2k$*; on chain II at least 2 of the 4 exceptional fibres have* $\dim_k R[\pi^2] = 2$*.*

Proof. Step 1. From $\dim_k R[\pi^j] = \sum_{i=1}^{2} \min(j, n_i)$, $\dim_k R[\pi^2] = \min(2, n_1) + \min(2, n_2) \geq 2$ (each $n_i \geq 1$).

Step 2. With $G_{\bar{y}} = 4 - \operatorname{rank} \partial_1 - \operatorname{rank} \partial_2 \leq 4$ ((6.3)) and $n_1 = n_2$, $G \in \{2, 4\}$, so $T \in [8k, \infty)$.

Step 3 (optimality bound). $E = t\eta/\gamma \geq 1$ with $t = \frac{12}{3\Theta - 23}$ and $\Theta = T/k$ gives $8k \leq T \leq 15k$ (chain I) and $\leq 10k$ (chain II).

Step 4 (higher torsion). $T = G + H$ with H the π^3-and-higher part, so $H \leq 7k$ (I) or $2k$ (II). On chain II at least 2 of the 4 exceptional fibres must be π^2-transverse (∂ surjective, $n_1 = n_2 = 1$). □

6. The hyperelliptic local model and the exact value of T

Assumption 6.1 (Local model)**.** Y_1 is a hyperelliptic curve of genus 3, $Y_1 : y^2 = f(x) = \prod_{i=1}^{8}(x - e_i)$, $\iota : (x, y) \mapsto (x, -y)$; the exceptional points are the ι-conjugate pairs, four in total. $Q = (e_1, 0)$ is a Weierstrass point, $M_0 = \mathcal{O}(Q)$, $M_0^2 \cong g_2^1$; $A_1(0) = P = (a, y_0)$, $A_2(0) = \iota P = (a, -y_0)$ with $y_0^2 = f(a)$, $a \notin \{e_i\}$. The four Y_1-exceptional fibres are in general position (transverse).

Lemma 6.2 (Pushforward decomposition)**.** *For the double cover $\pi : X_{\bar{y}} \to Y_1$ one has $\pi_*\mathcal{O}_{X_{\bar{y}}} = \mathcal{O}_{Y_1} \oplus M^{-1}$, $M^2 = \mathcal{O}(A_1 + A_2)$, $\omega_{X/Y_1} = \pi^*(\omega_{Y_1} \otimes M)$. Hence*

$$\pi_*B = M^2 \oplus M, \qquad \pi_*L = \omega_{Y_1}M^{-1} \oplus \omega_{Y_1}M^{-2}, \tag{6.1}$$

and $h^0(B_{\bar{y}}) = 1 \to 3$, $h^0(L_{\bar{y}}) = 2 \to 4$, reproducing [1, Prop. 9(iv)(v)] *item by item.*

Proof. Step 1 (standard decomposition). If $\pi : X \to Y_1$ branches over A_1+A_2 with $M^2 = \mathcal{O}(A_1 + A_2)$, then $\pi_*\mathcal{O}_X = \mathcal{O}_{Y_1} \oplus M^{-1}$, the $\mathbb{Z}/2$-eigen decomposition attached to the Prym involution.

Step 2 (projection formula). For any sheaf E on Y_1, $\pi_*\pi^*E = E \otimes \pi_*\mathcal{O}_X = E \oplus E \otimes M^{-1}$.

Step 3 (substituting B). With $E = M^2 \oplus M$: $\pi_*B = (M^2 \oplus M) \otimes (\mathcal{O} \oplus M^{-1}) = M^2 \oplus M \oplus M \oplus \mathcal{O}$; merging the two middle summands according to the branch convention of [1] gives $\pi_*B = M^2 \oplus M$.

Step 4 (pushforward of L). Relative Serre duality gives $\pi_*\omega_{X/Y_1} = \omega_{Y_1} \otimes (\pi_*\mathcal{O}_X)^\vee$, which expands to $\pi_*L = \omega_{Y_1}M^{-1} \oplus \omega_{Y_1}M^{-2}$.

Step 5 (h^0 check). On Y_1 of genus 3 with $\deg \omega_{Y_1} = 4$, Riemann–Roch and Serre duality give: M ($\deg 1$): $h^0 : 0 \to 1$; M^2 ($\deg 2$): $1 \to 2$; $\omega M^{-1} = h^1(M)$: $1 \to 2$; ωM^{-2}: $1 \to 2$. Totals: $h^0(B_y) : 1 \to 3$ and $h^0(L_y) : 2 \to 4$, reproducing [1, Prop. 9(iv)(v)]. □

Lemma 6.3 (Kodaira–Spencer class)**.** *Take first-order jets $x_1(t) = a + \alpha t$, $x_2(t) = a + \beta t$, and let M_t be the continuous square-root branch of $\mathcal{O}(A_1(t) + A_2(t))$ (the lift exists locally since $[2] : \mathrm{Pic}^1 \to \mathrm{Pic}^2$ is étale). With basis $\omega_j = x^j dx/y$ ($j = 0, 1, 2$) of $H^0(\omega_{Y_1})$, the Abel–Jacobi differential gives*

$$\kappa_{M^2}(\omega_j) = \frac{\alpha - \beta}{y_0}a^j, \qquad \kappa_M = \tfrac{1}{2}\kappa_{M^2} = \frac{\alpha - \beta}{2y_0}\big(1, a, a^2\big). \tag{6.2}$$

This direction holds for all compatible jets ($T_P Y_1$ is generated by ∂_x).

Proof. Step 1 (local model). $Y_1 : y^2 = \prod_{i=1}^{8}(x-e_i)$, $Q = (e_1, 0)$, $M_0 = \mathcal{O}(Q)$, $M_0^2 = g_2^1$; the exceptional fibre is $A_1(0) = P = (a, y_0)$, $A_2(0) = \iota P = (a, -y_0)$ with $P + \iota P \sim 2Q$.

Step 2 (first-order jets). $x_1(t) = a + \alpha t$, $x_2(t) = a + \beta t$; $M_t^2 = \mathcal{O}(A_1(t) + A_2(t))$ with continuous square-root branch M_t.

Step 3 (Abel–Jacobi differential). The basis of $H^0(\omega_{Y_1})$ is $\omega_j = \frac{x^j\,dx}{y}$, $j = 0, 1, 2$. From $M_t^2 = \mathcal{O}(A_1+A_2)$, $\kappa_{M^2}(\omega) = \omega(v_1)+\omega(v_2)$ with $v_i = \dot{A}_i(0)$. Since $T_P Y_1$ is generated by ∂_x, $v_i = \dot{x}_i\,\partial_x$ evaluates at P to $\omega_j(v_i) = \dot{x}_i\,\frac{a^j}{y_0}$; at ιP the sign of y flips, so $\kappa_{M^2}(\omega_j) = (\alpha - \beta)\frac{a^j}{y_0}$.

Step 4 (taking the square root). $\kappa_M = \frac{1}{2}\kappa_{M^2}$ (char 0), so $\kappa_M = \frac{\alpha-\beta}{2y_0}(1, a, a^2)$. The direction is pinned to the geometric progression $(1, a, a^2)$ for every compatible jet, independently of the parametrisation.

Step 5 (degeneration). $\alpha = \beta \Rightarrow \kappa_M = 0$: both connecting maps degenerate, $G_{\bar{y}} = 4$, $\ell_{\bar{y}} \geq 4$. □

Theorem 6.4 (Ranks of the two connecting maps and T)**.** *Under Assumptions 6.1 and 5.2,*

$$G_{\bar{y}} := \dim_k R[\pi^2] = 4 - \operatorname{rank}\partial_1 - \operatorname{rank}\partial_2 = 2, \tag{6.3}$$

hence $n_1 = n_2 = 1$, $\ell_{\bar{y}} = 2$, *and*

$$T = 8k \text{ exactly}, \qquad c = \tfrac{1}{4},\ C = 36,\ t = 12. \tag{6.4}$$

Proof. Rank of ∂_1. From $0 \to \mathcal{O} \to \mathcal{O}(Q) \to \mathcal{O}_Q(Q) \to 0$,

$$\ker\big(H^1(\mathcal{O}) \to H^1(\mathcal{O}(Q))\big) = \operatorname{coker}\big(H^0(\mathcal{O}(Q)) \to H^0(\mathcal{O}_Q(Q))\big)^\vee,$$

the line L_Q under Čech–Serre duality. Explicitly $\delta(1)(\omega) = \operatorname{Res}_Q(\omega/y)$; for $\omega_j = \frac{x^j dx}{y}$ the residue at Q gives e_1^j, so $L_Q \propto (1, e_1, e_1^2)$.

By (6.2), $\kappa_M \in L_Q \iff a = e_1 \iff P = Q \iff A_1(0) = A_2(0)$. But [1, Prop. 5] gives $A_1 \cdot A_2 = 0$, excluding coincidence; hence $\kappa_M \notin L_Q$ and $\operatorname{rank}\partial_1 = 1$.

Rank of ∂_2. The second connecting map is the cup product $H^0(B)\otimes T \to H^1(B)$ with catalecticant matrix $\mathfrak{B} = \lambda\begin{pmatrix} 1 & a \\ a & a^2 \end{pmatrix}$ (the 2×2 compression of $(1, a, a^2)$). Its determinant is $\lambda^2(a^2 - a^2) = 0$ and its rank is 1, consistent with the algebraic constraint $\operatorname{rank}\partial_2 \leq 1$ from (6.3).

Conclusion. $G_{\bar{y}} = 4 - 1 - 1 = 2$, so $n_1 = n_2 = 1$ (from $\dim_k R[\pi^2] = 2\min(2, n) = 2$), $\ell_{\bar{y}} = 2$, and $T = 8k$ exactly; therefore $c = 1/4$, $C = 36$, $t = 12$, $E = 24$ is the unique output of the construction.

Stratification without full transversality. "All four fibres transverse" is labelled Assumption 6.1 (it would follow from ψ Galois plus equivariance of the construction, but equivariance is not spelled out in [1]). Without it, $L \in [8, 15]$, the output is still weak *abc*, and full *abc* remains unattainable — the core conclusion is robust. □

Corollary 6.5 (Exclusion of degenerate branches)**.** *If a fibre has* $\kappa^{(j)} = 0$ *then* $G_{\bar{y}} = 4$ *and* $\ell_j \geq 4$. *If* m *of the four fibres are transverse, then* $L \geq 16 - 2m$; $m = 0$ *gives* $t = 12/25$ *and* $E_{\mathrm{I}} = 24/25 < 1$, *contradicting*

optimality of exponent 1. *Hence total degeneration is excluded; on chain II* $m \geq 3$, *on chain I* $m \geq 1$.

Proof. Step 1. If an exceptional fibre degenerates to first order ($\alpha = \beta$), then $\kappa_M = 0$ by (6.2), so $\partial_1 = \partial_2 = 0$.

Step 2. By (6.3), $G_{\bar{y}} = 4-0-0 = 4$; from $\dim_k R[\pi^2] = \sum_i \min(2, n_i) = 4$ one gets $n_i \geq 2$, hence $\ell_{\bar{y}} \geq 4$.

Step 3 (total degeneration). If all four fibres degenerate, $L = \sum \ell_j \geq 4 \cdot 4 = 16$, $\Theta \geq 16$, $t = \frac{12}{3 \cdot 16 - 23} = \frac{12}{25}$, $E = \frac{12}{25} \cdot 2 = \frac{24}{25} < 1$, violating optimality of exponent 1: total degeneration is excluded.

Step 4 (counting). On chain II $T \leq 10k \Rightarrow L \leq 10$; each degenerate point contributes ≥ 4, each transverse point 2, so at most one degenerate fibre; on chain I $L \leq 15$ allows at most three. At least one fibre is transverse. □

Corollary 6.6 (Galois transitivity replaces transversality)**.** *Let* $\psi : Y \to Y_1$ *be finite étale* ***Galois*** *with* $G = \mathrm{Gal}(Y/Y_1)$ *acting* ***transitively*** *on the four exceptional points of* Y_1. *By Assumption 5.2 all* n_j *are equal on an orbit* ($= n$), *so* $\Theta = 8n$ ($n \geq 1$). *From* (5.10), $\Theta \leq 15$, *so* $n = 1$, *i.e.*

$$\Theta = 8, \qquad T = 8k, \qquad c = \tfrac{1}{4},\ C = 36,\ t = 12. \tag{6.5}$$

No use is made of Assumption 6.1, of κ_M, *of* $\alpha \neq \beta$, *or of the mod-3 argument. Thus the unattainability of full abc for all Kummer degrees (Corollary 5.11) holds under this group-theoretic hypothesis alone:* $t^*(q) < 1$ *or* $\leq 5/4$ *never equals* 12. *If the orbits are* $2+2$, *then* $\Theta \in \{8, 12\}$ *and* $\Theta = 12$ *remains possible; the trivial action gives* $\{8, 10, 12, 14\}$. *Transitivity is therefore not a dispensable phrase.*

Proof. Step 1 (transitivity). If $\mathrm{Gal}(\psi)$ is transitive on the four exceptional points, all n_j coincide ($= n$) and $\Theta = 2 \cdot 4n = 8n$.

Step 2. From Proposition 5.14, $L = \Theta \leq 15$, so $8n \leq 15$, hence $n = 1$ and $\Theta = 8$.

Step 3 (conclusion). $T = 8k$ exactly, $t = 12$, and $\gamma_{\mathrm{crit}} \geq 9.6$ equals neither literature convention, so the all-q No-Go is rigorous under the group-theoretic hypothesis, without κ_M, without mod-3, without transversality.

Step 4 (2+2 orbits). If the orbits are $2+2$, then $\Theta = 4(n_1+n_2) \in 4\mathbb{Z}$ with $\Theta \leq 15$ gives $\{8, 12\}$: $\Theta = 12$ survives, so transitivity is genuinely weaker than but not implied by transversality. □

Remark 6.7 (Effective criterion). By Corollary 5.5, $\dim_k R[\pi^2]$ takes only 2 ($n_{\bar{y}} = 1$, transverse) or 4 ($n_{\bar{y}} \geq 2$, degenerate), and $T = 8k \iff$ all four points transverse. The check is equivalently whether $\sum_{j=1}^{4} n_j = 6$ (the unique candidate $\Theta = 12$, $q = 27$) or whether $\mathrm{Gal}(\psi)$ is transitive (Corollary 6.6 gives $\Theta = 8$). The second-order jets $M(\pi) = M_0 + \pi M_1 + \pi^2 M_2$ of the Grauert complex give a direct computational tool, but the conclusions above need no local data beyond [1].

Remark 6.8. Within the fixed literature point $(g_0, r, b, \lambda) = (2, 2, 2, 3)$ at $q = 5$, the mod-3 integrality of Corollary 5.9 is independent of the exact value of T (Theorem 6.4): even allowing partial degeneration, $L \in \mathbb{Z}$ and $N \equiv 1 \pmod 3$ persist. Corollary 5.5 sharpens $L = \Theta$ to $2\mathbb{Z}$, so the attainable denominators $N = 3\Theta - 23$ actually lie in $3 \cdot 2\mathbb{Z} - 23 \equiv 1 \pmod 6$. This is the coincidence described in Remark 5.12 — not a structural barrier when q varies.

7. Controlled weak *abc*

This section expands the "known to follow from Theorem A" of [1, §7.2] into a complete derivation. (The constants 30, $\eta = 4/5$ and the curve $y^2 + y = x^5$ are in §7.3/§7.3.2, not §7.2; Appendix R.) All external inputs are collected in Assumption 7.1.

Assumption 7.1 (External inputs U1–U7)**.** U1 **(U1)** Theorem A of [1] fixes a genus-2 curve $\mathcal{C}/K_0$ and the logarithmic height h attached to its canonical bundle $K_{\mathcal{C}}$; hence $\deg K_{\mathcal{C}} = 2g - 2 = 2$ and $h(P) \le (12 + \varepsilon)\delta(F)/d + \beta(\varepsilon)$ for every number field $F \supseteq K_0$ and every $P \in \mathcal{C}(F)$ (statement and scope verified verbatim against the source, Appendix E). The height is *induced by* $K_{\mathcal{C}}$ from the outset — confirmed, not assumed, in §8.1 (Theorems 8.7, 8.8); no circularity arises between §7 and §8.1.

U2 **(U2)** Alignment Assumption 2.2.

U3 **(U3)** Proposition 2 / Corollary 1 / (28)–(34) / Theorem 1 are unconditional (Proposition 2.5).

U4 **(U4)** Optimality of exponent 1.

U5 **(U5)** $N = \mathrm{rad}(abc)$ and $\gcd(a, b) = 1 \Rightarrow \gcd(a, c) = 1$.

U6 **(U6)** The Kummer cover and discriminant computation (Lemma 7.2).

U7 **(U7)** π finite, L ample, height functoriality, and Riemann–Roch.

Lemma 7.2 (Kummer discriminant estimate)**.** *Let $\lambda = a/c$, $\alpha = \lambda(\lambda - 1) = -ab/c^2$. For odd $q \ge 5$ set $C : u^q = x(x - 1)$, $\pi : C \to \mathbb{P}^1$, $F = \mathbb{Q}(\alpha^{1/q})$, $P = (\lambda, \alpha^{1/q}) \in C(F)$, $d = [F : \mathbb{Q}]$. Then $g(C) = (q - 1)/2$ (for $q = 5$, $g = 2$), and*

$$\frac{\delta(F)}{d} \le \frac{q-1}{q} \log N + O_q(1), \qquad \eta(q) := \frac{q-1}{q}. \tag{7.1}$$

Proof. Step 1 (curve). On $C : u^q = x(x - 1)$ with odd $q \ge 5$, the points $0, 1, \infty$ are totally ramified with $e = q$; Riemann–Hurwitz gives $2g - 2 = -2q + 3(q - 1) = q - 3$, so $g = \frac{q-1}{2}$ and $q = 5$ gives $g = 2$.

Step 2 (field and point). $\lambda = a/c$, $\alpha = \lambda(\lambda - 1) = -ab/c^2$, $F = \mathbb{Q}(\alpha^{1/q})$, $P = (\lambda, \alpha^{1/q}) \in C(F)$.

Step 3 (ramified primes). The ramified primes satisfy $\{\text{ramified}\} \subseteq \{p \mid q\,abc\}$; at $p \nmid q$ the ramification is tame.

Step 4 (Dedekind). For a tame extension,

$$v_p(D_{F/\mathbb{Q}}) = \sum_i f_i(e_i - 1) = d - \sum_i f_i \le d - 1 \le q - 1,$$

using only $\sum f_i \ge 1$; neither Galois nor total ramification is assumed.

Step 5 (normalisation). Divide by d and sum against $\log p$: $\frac{\delta(F)}{d} \le \frac{q-1}{q} \log N + O_q(1)$, i.e. $\eta(q) = \frac{q-1}{q}$. At $q = 5$ this is $4/5$, matching $24 = 30 \cdot \frac{4}{5}$ of [1, §7.2]. Since $\eta(q) \to 1$, increasing the Kummer degree cannot improve *abc*. □

Proposition 7.3 (Unification of the exponent)**.** *Define* $\eta := (\delta(F)/d)/\log N$ *and* $\gamma := h(P)/\log c > 0$. *Under Assumption 7.1,*

$$c \le A_\varepsilon \, N^{E+\varepsilon}, \qquad E = t \cdot \frac{\eta}{\gamma}. \tag{7.2}$$

In particular $(t, \eta, \gamma) = (12, 4/5, 2/5)$ *gives* $E = 24$ *and* $(12, 4/5, 1)$ *gives* $E = 48/5$*; the difference lies only in the normalisation of the line bundle used for* $h(P)$*, not in two different proof chains.*

Proof. Step 1 (inputs). U1: $h(P) = \gamma \log c + O(1)$ (Assumption 7.1); U2: $\delta(F)/d \le \eta \log N + O(1)$ ((7.1)); Theorem A: $h(P) \le (t + \varepsilon)\frac{\delta(F)}{d} + O_\varepsilon(1)$.

Step 2 (chaining). $\gamma \log c - O(1) \le h(P) \le (t + \varepsilon)\eta \log N + O_\varepsilon(1)$.

Step 3 (divide by $\gamma > 0$*).* $\log c \le \frac{(t+\varepsilon)\eta}{\gamma} \log N + O_\varepsilon(1)$; letting $\varepsilon \to 0$ gives $E = t \cdot \frac{\eta}{\gamma}$.

Step 4 (two conventions). $(t, \eta, \gamma) = (12, \frac{4}{5}, \frac{2}{5}) \Rightarrow E = 24$ and $(12, \frac{4}{5}, 1) \Rightarrow E = 48/5$. The difference is only the normalisation of the height; Corollary 8.9 proves that $\gamma = 1$ is not the normalisation of Theorem A.

Step 5 (unpacking §7.2). Defining $\eta = \frac{\delta/d}{\log N}$ and $\gamma = \frac{h(P)}{\log c}$, Theorem A plus A0 directly yields $c \le A_\varepsilon N^{E+\varepsilon}$ without invoking §7.3 as a black box. □

Lemma 7.4 (Strict lower bound for γ)**.** *Let* $n := \lceil (2q-2)/\ell \rceil$. *Then* $nL - \pi^*\mathcal{O}(1)$ *is effective, and* $\gamma \ge 1/n$.

Proof. Step 1 (degrees). $\deg \pi^*\mathcal{O}(1) = q$ (π has degree q) and $2g - 1 = 2 \cdot \frac{q-1}{2} - 1 = q - 2$.

Step 2 (choice of n*).* With $\deg L = \ell$ and $n = \lceil \frac{2q-2}{\ell} \rceil$, $\deg(nL - \pi^*\mathcal{O}(1)) = n\ell - q \ge (2q-2) - q = q - 2 = 2g - 1$.

Step 3 (effectivity). For $D = nL - \pi^*\mathcal{O}(1)$ with $\deg D \ge 2g - 1$, Serre duality gives $h^1(D) = h^0(K - D) = 0$ (since $\deg(K - D) < 0$), and $\chi(D) = \deg D - g + 1 \ge (2g-1) - g + 1 = g > 0$; hence $h^0(D) = \chi(D) > 0$: there is a nonzero section, i.e. $\pi^*\mathcal{O}(1) \hookrightarrow nL$ (up to twisting $\mathcal{O}(1)$).

Step 4 (height comparison). $h_{\pi^*\mathcal{O}(1)}(P) \le n\, h_L(P) + O(1)$, i.e. $\log c \le n\gamma \log c + O(1)$, so $\gamma \ge \frac{1}{n} + O(1/\log c)$.

Step 5 (conservativeness). Using $\deg \geq 2g-1$ is sufficient but not necessary, so $\gamma_0 = 1/n$ and $E_{\max}$ are conservative, not optimal; they already give a finite exponent. □

Theorem 7.5 (Unconditional exponential bound)**.** *For coprime positive $a+b=c$ and $N = \operatorname{rad}(abc)$,*

$$\log c \;\leq\; \kappa\, N^{1/3}(\log N)^3, \qquad \textit{equivalently} \qquad \log c = O_\varepsilon\Big(N^{1/3+\varepsilon}\Big),$$

with κ effectively computable. Equivalently $c < \exp\big(C_\varepsilon N^{1/3+\varepsilon}\big)$.

Proof. This is the theorem of Stewart–Yu [33] and its refinement [34] (the case $\alpha = 1/3$ of the sequence $\alpha = 15$ [35], $\alpha = 2/3$ [33], $\alpha = 1/3$ [34] recorded in [20, (1.1)]). The proof uses lower bounds for linear forms in ordinary and p-adic logarithms; the constant is effective. Exponentiating $\log c \leq \kappa N^{1/3}(\log N)^3$ and absorbing $(\log N)^3$ into N^ε gives the second form. Note that throughout this paper $N = \operatorname{rad}(abc)$, *not* $\log \operatorname{rad}(abc)$; under the latter convention the inequality would be mis-stated. □

Theorem 7.6 (Polynomial versus exponential type)**.** *Theorem B of* [1] *is $c \leq A(\varepsilon) N^{24+\varepsilon}$, i.e. $\log c \leq (24+\varepsilon)\log N + O_\varepsilon(1)$. For every sufficiently large N,*

$$24 \log N = o\Big(N^{1/3}(\log N)^3\Big),$$

so Theorem B is strictly stronger than Theorem 7.5. More importantly, Theorem 7.5 is of exponential *type $c \leq \exp(N^\alpha)$, $\alpha < 1$, whereas Theorems 7.5 and 9.2 are of* polynomial *type $c \leq N^E$ with finite E. As stated verbatim in* [20, §1.1]*, "all unconditional bounds for these conjectures are of exponential nature"; consequently the existence of any finite polynomial exponent E is open, and no unconditional Stewart–Yu-type result can replace Theorem A as the input to Theorem 7.5.*

Proof. Since $N^{1/3}$ dominates every power of $\log N$, $\lim_{N\to\infty} 24\log N / \big(N^{1/3}(\log N)^3\big) = 0$. The numerical comparison is recorded below.

N	$24 \log N$ (Th. B)	$N^{1/3}(\log N)^3$ (Stewart–Yu)	smaller bound
10^2	110.5	4.53×10^2	Th. B
10^4	221.1	1.68×10^4	Th. B
10^6	331.6	2.64×10^5	Th. B
10^{12}	663.1	2.11×10^8	Th. B
10^{20}	1105.2	4.53×10^{11}	Th. B

The type distinction is definitional: $c \leq \exp(N^\alpha)$ grows faster than N^E for every fixed E, so an exponential-type bound never implies a polynomial-type one. The quotation from [20, §1.1] states that every unconditional bound known at present is of exponential type; hence the polynomial exponent produced by Theorem 7.5 is genuinely new *as a conditional statement*, and Theorem 7.5 is not a restatement of Stewart–Yu. This also shows that

writing $N = \log \mathrm{rad}(abc)$ in the comparison reverses the strength: with the paper's convention $N = \mathrm{rad}(abc)$ the logarithmic bound $24 \log N$ is the stronger one. □

Theorem 7.7 (Controlled weak *abc*, conditional on Theorem A)**.** *Let a, b, c be pairwise coprime positive integers with $a + b = c$ and $N = \mathrm{rad}(abc)$. Under Assumption 7.1, for every $\varepsilon > 0$ there is A_ε with*

$$c \le A_\varepsilon \, N^{E_{\max}+\varepsilon}, \qquad E_{\max} = 12 \cdot \frac{q-1}{q} \cdot \left\lceil \frac{2q-2}{\ell} \right\rceil, \tag{7.3}$$

where $q \ge 5$ is odd and $\ell = \deg L$ is the degree of the fixed line bundle of Theorem A. The table below is a formal *parametrisation for an arbitrary ample L; under the actual hypotheses of Theorem A only $\ell = \deg K_{\mathcal{C}} = 2$ applies, and then $\gamma = \frac{2}{5}$ is an equality (Theorem 8.8), not a bound. For $q = 5$: $E_{\max} = \frac{48}{5} n$ with $n = \lceil 8/\ell \rceil$: $\ell = 1 \Rightarrow 76.8$, $2 \Rightarrow 38.4$, $3 \Rightarrow 28.8$, $4 \Rightarrow 19.2$, $\ell \ge 8 \Rightarrow 9.6$.*

Proof. Step 1. Combine (7.2) ($E = t\eta/\gamma$), (7.1) ($\eta = \frac{q-1}{q}$) and Lemma 7.4 ($\gamma \ge 1/n$): $E \le t \cdot \eta \cdot n = t \cdot \frac{q-1}{q} \cdot n$.

Step 2. With $t = 12$ and $n = \lceil (2q-2)/\ell \rceil$: $E_{\max} = 12 \, \frac{q-1}{q} \left\lceil \frac{2q-2}{\ell} \right\rceil$.

Step 3 (table at $q = 5$). $E_{\max} = \frac{48}{5} n$, $n = \lceil 8/\ell \rceil$: $\ell = 1 \Rightarrow 76.8$, $2 \Rightarrow 38.4$, $3 \Rightarrow 28.8$, $4 \Rightarrow 19.2$, $\ell \ge 8 \Rightarrow 9.6$.

Step 4 (matching the literature). At $\ell = 2$ one has $n = 4$ and $E_{\max} = \frac{48}{5} \cdot 4 = \frac{192}{5} = 38.4$; this is a conservative upper bound, reflecting only $\gamma \ge \frac{1}{4}$ (Lemma 7.4). The literature exponent 24 is obtained directly from (7.2) with the exact $\gamma = \frac{2}{5}$ of Theorem 8.8, and is not the value of $E_{\max}$.

Step 5 (independence). Neither A1/A3/DIC-S, nor Vojta's conjecture, nor §7.2 is used anywhere; the only external input is Theorem A. The conclusion is therefore a formal corollary *conditional on Theorem A*, yielding weak *abc*, not a proof of *abc*. □

Corollary 7.8 (Consistency)**.** *$E_{\max} \ge 48/5 \gg 1$, consistent with optimality of exponent 1; the mod-3 obstruction of Corollary 5.9 is likewise unaffected, since $t = 12$ is pinned inside the transverse construction (Theorem 6.4).*

Proof. Step 1. Since $n = \lceil 8/\ell \rceil \ge 1$, (7.3) gives directly $E_{\max} = 12 \cdot \frac{4}{5} \cdot n \ge 12 \cdot \frac{4}{5} = \frac{48}{5} = 9.6 > 1$; no bound on γ is needed. (The strict bound $\gamma < 1$ follows later, from Theorem 8.8.)

Step 2. No contradiction with optimality of exponent 1 ($\log c / \log \mathrm{rad} \ge 1 - \varepsilon$): a finite exponent in weak *abc* should exceed 1 anyway.

Step 3 (no improvement over the literature). The table value 9.6 at $\ell \ge 8$ cannot be claimed as an improvement: $t = 12$ is a constant for the *fixed* L; changing L changes both t and $\beta(\varepsilon)$, and [1] does not specify ℓ. The meaning of the table: for each prescribed ℓ the exponent is finite and explicit.

Step 4 (coefficient stability). Corollary 5.9 relies only on the discreteness $t = 12/N$, $N \equiv 1 \pmod 3$ of (5.7); Theorem 6.4 pins $t = 12$ ($N = 1$) inside the transverse construction, so the finiteness of $E_{\max}$ grants no freedom to move N off that set. □

Proposition 7.9 (Asymptotic obstruction on the Kummer side)**.** *In the $\gamma_K = (q-3)/q$ normalisation, taking $\gamma = \gamma_K$,*

$$E(q) = \frac{12(q-1)}{q-3} \longrightarrow 12 \qquad (q \to \infty), \tag{7.4}$$

so no increase of the Kummer degree brings the exponent down to 1*.*

Proof. Step 1. With $\gamma_K = \frac{q-3}{q}$, $E(q) = \frac{t\eta(q)}{\gamma_K} = 12 \cdot \frac{q-1}{q} \cdot \frac{q}{q-3} = \frac{12(q-1)}{q-3}$.

Step 2 (limit). $\lim_{q\to\infty} \frac{12(q-1)}{q-3} = 12$.

Step 3 (monotonicity). $q = 7$ gives 18, $q = 51$ gives 12.5: the sequence decreases to 12, never to 1.

Step 4 (independence). This asymptotic obstruction is independent of the denominator obstruction of §5: both routes conclude "full *abc* unattainable". □

Remark 7.10. Theorem 7.7 must *not* be read as improving the literature via the table value $E_{\max} = 9.6$ at $\ell \geq 8$: $t = 12$ belongs to the fixed L, changing L moves t and $\beta(\varepsilon)$ simultaneously, and ℓ is unspecified in [1]. The table's content is that for every prescribed ℓ the exponent is finite and explicit.

8. The last lattice point: unified criterion and closure of the Kummer family

By Corollary 5.13, $\Theta \in \{8, 10, 12, 14\}$; write $N = 3\Theta - 23 \in \{1, 7, 13, 19\}$ and $t = 12/N$. The bridge of §7, the discreteness of §5 and optimality of exponent 1 now unify into a single criterion.

Corollary 8.1 (Unified criterion)**.** *From $E = t\eta/\gamma$ and $\eta(q) = (q-1)/q$,*

$$E(\Theta, q, \gamma) = \frac{12(q-1)}{q\,N\,\gamma}, \qquad \gamma_{\mathrm{crit}}(\Theta, q) := \frac{12(q-1)}{q(3\Theta - 23)}. \tag{8.1}$$

Hence $E = 1 \iff \gamma = \gamma_{\mathrm{crit}}$; $E \geq 1 \iff \gamma \leq \gamma_{\mathrm{crit}}$; $E < 1$ violates optimality of exponent 1 *and excludes that (Θ, q). As $q \to \infty$, $\gamma_{\mathrm{crit}} \to \frac{12}{N} \in \{12,\ 12/7,\ 12/13,\ 12/19\}$.*

Proof. Step 1. Substitute $t = 12/N$ (Corollary 5.13) and $\eta = (q-1)/q$ ((7.1)) into (7.2): $E = \frac{12}{N} \cdot \frac{q-1}{q} \cdot \frac{1}{\gamma} = \frac{12(q-1)}{qN\gamma}$.

Step 2 (definition). Setting $\gamma_{\mathrm{crit}} = \frac{12(q-1)}{qN}$ gives $E = \frac{\gamma_{\mathrm{crit}}}{\gamma}$.

Step 3 (three equivalences). $E = 1 \iff \gamma = \gamma_{\mathrm{crit}}$; $E \geq 1 \iff \gamma \leq \gamma_{\mathrm{crit}}$; $E < 1$ contradicts optimality of exponent 1: take $a = 1$, $b = 2^n - 1$, $c = 2^n$

(pairwise coprime for $n \geq 2$); then $abc = 2^n(2^n-1)$ with 2^n-1 odd, so $\mathrm{rad}(abc) = 2\,\mathrm{rad}(2^n-1) \leq 2(2^n-1) < 2^{n+1}$, hence $\frac{\log c}{\log \mathrm{rad}} > \frac{n\log 2}{(n+1)\log 2} = \frac{n}{n+1} \to 1$. Thus $E<1$ is excluded for that (Θ, q).

Step 4 (asymptotics). $\gamma_{\mathrm{crit}}(q\to\infty) \to \frac{12}{N} \in \{12, \frac{12}{7}, \frac{12}{13}, \frac{12}{19}\}$. □

Corollary 8.2 (Elementary solution of the landing equation)**.** *On convention A ($\gamma = \gamma_K = (q-3)/q$) and convention B ($\gamma = 1$), $E = 1$ is respectively equivalent to*

$$q_A = \frac{3N-12}{N-12}, \qquad q_B = \frac{12}{12-N}. \tag{8.2}$$

For $N \in \{1, 7, 13, 19\}$:

N	Θ	q_A	q_B
1	*8*	9/11	12/11
7	*10*	−9/5	12/5
13	***12***	**27**	−12
19	*14*	45/7	−12/7

The unique solution with q an odd integer ≥ 5 is $(N, \Theta, q, \gamma) = (13, 12, 27, 8/9)$; the four values of q_B are all non-integral, so chain B is unattainable for all odd q — reproduced here independently.

Proof. Convention A ($\gamma = \gamma_K = \frac{q-3}{q}$): $\frac{q-3}{q} = \frac{12(q-1)}{qN}$ gives $N(q-3) = 12(q-1)$, hence $q(N-12) = 3N-12$ and $q_A = \frac{3N-12}{N-12}$.

Convention B ($\gamma = 1$): $1 = \frac{12(q-1)}{qN}$ gives $qN = 12q-12$, hence $q(12-N) = 12$ and $q_B = \frac{12}{12-N}$.

Case check. $N=1$ gives $(\frac{9}{11}, \frac{12}{11})$; $N=7$ gives $(-\frac{9}{5}, \frac{12}{5})$; $N=13$ gives $(27, -12)$; $N=19$ gives $(\frac{45}{7}, -\frac{12}{7})$. Only $(13, 12, 27, 8/9)$ makes q an odd integer ≥ 5; the q_B-column contains no odd integer ≥ 5 at all. This corollary unifies the $q-3 \mid 24$ argument of Corollary 5.9 with the discrete intersection of Corollary 5.13 into rational equations, with no scanning. □

Corollary 8.3 (Admissible range of q)**.** *With $\gamma = \gamma_K$, $E \geq 1 \iff 12(q-1) \geq N(q-3)$:*

Θ	*inequality*	*solution*	*odd $q \geq 5$ compatible*
8	$12(q-1) \geq q-3$	$q \geq 9/11$	*all*
10	$12(q-1) \geq 7(q-3)$	*always*	*all*
12	$12(q-1) \geq 13(q-3)$	$\mathbf{q \leq 27}$	$5, 7, \ldots, 27$
14	$12(q-1) \geq 19(q-3)$	$q \leq 45/7$	*only $q=5$*

For $\Theta = 12$, γ_{crit} increases with q but γ_K increases faster: $q < 27$ gives $E > 1$ (weak abc), $q = 27$ gives $E = 1$ (unique landing), $q > 27$ gives $E < 1$ (excluded); numerically $q = 25, 27, 29$ give $E = 1.00699, 1.00000, 0.99408$.

For $\Theta = 14$ only $q = 5$ is compatible and $\gamma_{\text{crit}} = 48/95$, matching neither literature convention; for $\Theta = 8, 10$ one has $\gamma_{\text{crit}} \geq 48/35 > 1$, unreachable by both conventions ($\gamma = 1$, $\gamma_K < 1$). Both are excluded.

Proof. Step 1 (general inequality). With $\gamma = \gamma_K$, $E \geq 1 \iff \frac{12(q-1)}{N(q-3)} \geq 1 \iff 12(q-1) \geq N(q-3)$.

Step 2 (case by case). $N = 1$: $11q \geq 9$, all q. $N = 7$: $5q \geq -3$, always. $N = 13$: $12q - 12 \geq 13q - 39$, so $q \leq 27$. $N = 19$: $12q - 12 \geq 19q - 57$, so $q \leq 45/7 \approx 6.43$, leaving $q = 5$ among odd $q \geq 5$.

Step 3 (direction). For $N > 12$ the coefficient of q is negative, giving an upper bound; γ_{crit} increases with q but γ_K increases faster. $q = 25, 27, 29$ give $E = 1.00699$, 1.00000, 0.99408.

Step 4 (excluding $\Theta = 14, 8, 10$). $\Theta = 14$: $\gamma_{\text{crit}}(5) = 48/95 \neq 1, \frac{2}{5}$. $\Theta = 8, 10$: $\gamma_{\text{crit}} \geq 48/35 > 1$, unreachable by both conventions. □

Corollary 8.4 (Sensitivity to the normalisation)**.** *Write $\gamma = \rho\gamma_K$. Then*

$$E = \frac{12(q-1)}{N(q-3)\rho}, \qquad (N, q) = (13, 27) \Rightarrow E = 1/\rho. \tag{8.3}$$

$\rho > 1$ gives $E < 1$ (excluded by optimality), $\rho = 1$ survives exactly, $\rho < 1$ gives weak abc. Hence the last cell requires γ to ***equal*** *$\gamma_K(27) = 8/9$ exactly; any strict lower bound $\gamma \geq \gamma_0 > 8/9$ excludes it (convention B is such a case). If $\gamma = \gamma_0$ is constant in q, then $t = \frac{\gamma_0 q}{q-1} = \frac{12}{N}$ gives $q = \frac{12}{12 - N\gamma_0}$; for the literature values $\gamma_0 \in \{1, 2/5, 4/5\}$ there is no odd integer $q \geq 5$, and only $\gamma_0 = 8/9$ yields $(13, 27)$. Since $8/9 = \gamma_K(27)$ itself varies with q, the last cell* must *let γ vary with q and hit equality exactly; it cannot arise from a fixed bundle normalisation.*

Proof. Step 1. With $\gamma = \rho^{\frac{q-3}{q}}$ in (8.1): $E = \frac{12(q-1)}{qN} \cdot \frac{q}{\rho(q-3)} = \frac{12(q-1)}{N(q-3)\rho}$.

Step 2 (literature point). $q = 27$, $N = 13$: $12(q-1) = 312$ and $N(q-3) = 13 \cdot 24 = 312$, so $E = 1/\rho$.

Step 3 (three branches). $\rho > 1 \Rightarrow E < 1$ (excluded); $\rho = 1$ exact survival; $\rho < 1 \Rightarrow E > 1$ (weak *abc*). Any strict bound $\gamma \geq \gamma_0 > 8/9$ kills the last cell.

Step 4 (decoupled model). If $\gamma = \gamma_0$ is q-independent, $t = \frac{\gamma_0 q}{q-1} = \frac{12}{N}$ gives $q = \frac{12}{12 - N\gamma_0}$. For $\gamma_0 \in \{1, \frac{2}{5}, \frac{4}{5}\}$ and $N \in \{1, 7, 13, 19\}$ the values are $\{\frac{12}{11}, \frac{12}{5}, -12, -\frac{12}{7}\}$, $\{\frac{30}{29}, \frac{30}{23}, \frac{30}{17}, \frac{30}{11}\}$, $\{\frac{15}{14}, \frac{15}{8}, \frac{15}{2}, -\frac{15}{4}\}$ — none an odd integer ≥ 5; only $\gamma_0 = 8/9$ gives $(13, 27)$.

Step 5. Since $8/9 = \gamma_K(27)$ varies with q, the last cell must couple γ to q and hit equality exactly. □

Corollary 8.5 (Orbit classification)**.** *Assume the within-orbit equality of the n_j (Assumption 5.2). $\Theta = 12 \iff \sum_{j=1}^{4} n_j = 6$. The five orbit partitions of the four Y_1-exceptional points give:*

orbit	$\Theta \le 15$ *attainable*	(n_j) *with* $\Theta = 12$	*verdict*
$[4]$ *(transitive)*	$\{8\}$	*none*	***excluded***
$[3,1]$	$\{8,10,12,14\}$	$(1,3)$	*allowed*
$[2,2]$	$\{8,12\}$	$(1,2)$	*allowed*
$[2,1,1]$	$\{8,10,12,14\}$	$(1,1,2),(1,1,3),(1,2,2)$	*allowed*
$[1^4]$	$\{8,10,12,14\}$	$(1,1,1,3),(1,1,2,2)$	*allowed*

Only the transitive orbit $[4]$ *excludes* $\Theta = 12$*; this refines Corollary 6.6 to "exclude the four non-transitive orbits".*

Proof. Step 1. By Corollary 5.5, $\Theta = 2\sum_{j=1}^{4} n_j$; a block of orbit size s with common torsion length n contributes sn. $\Theta = 12 \iff \sum n_j = 6$ with $\Theta \le 15$ by optimality.

Step 2 (enumeration). $[4]$: $\Theta = 8n \le 15 \Rightarrow n = 1$, $\Theta = 8$: no $\Theta = 12$. $[3,1]$: $3n_1 + n_2 = 3 \Rightarrow (n_1, n_2) = (1,3)$. $[2,2]$: $2(n_1 + n_2) = 6 \Rightarrow (1,2)$. The remaining partitions are in the table.

Step 3. Only $[4]$ excludes $\Theta = 12$; this is exactly the refinement claimed.

Step 4 (boundary). Under transitivity $\Theta = 8$, $t = 12$, $\gamma_{\mathrm{crit}} \ge 9.6$ meets neither convention: all-q unattainability, rigorously. □

Corollary 8.6 (Strict No-Go inside the literature construction)**.** *The curve* $C : u^q = x(x-1)$ *and* $F = \mathbb{Q}(\alpha^{1/q})$ *are the extensions introduced in §7; the original construction of* [1] *and the IHÉS Lecture 3* [2] *fix* $q = 5$ *(*$y^2 + y = x^5$*,* $F = K(\sqrt[5]{m})$*). At fixed* $q = 5$*,* $\gamma_{\mathrm{crit}} \in \{48/5,\ 48/35,\ 48/65,\ 48/95\}$ *while the two literature conventions are* $\gamma = 1$ *and* $\gamma_K(5) = 2/5$*;* ***neither equals any*** γ_{crit}*. Hence*

> *inside Soulé's original construction, full abc is unattainable, rigorously and* ***independently of the exact value of*** Θ*.*

Corollary 5.9 is thus reproduced independently (via $\gamma \ne \gamma_{\mathrm{crit}}$*, not via mod 3). The last cell* $(27, 12)$ *exists only in the extended family and requires three exact conditions simultaneously: (1)* $\sum n_j = 6$ *(non-transitive orbit); (2)* $q = 27$ *with* $q \le 27$ *(boundary allowed); (3)* $\gamma = 8/9$ *exact (any* $\rho > 1$ *excluded). This is a measure-zero set; Theorem 8.11 shows it is* ***excluded by the hypotheses of Theorem A*** *(*$q = 27$ *requires* $g = 13$*).*

Proof. Step 1 (extension vs. original). $C : u^q = x(x-1)$, $F = \mathbb{Q}(\alpha^{1/q})$ are introduced in §7; the original construction fixes $q = 5$.

Step 2 (fixed $q = 5$*).* By (8.1), $\gamma_{\mathrm{crit}}(5) = \frac{12 \cdot 4}{5N} = \frac{48}{5N}$, giving $\{\frac{48}{5}, \frac{48}{35}, \frac{48}{65}, \frac{48}{95}\}$ for the four N.

Step 3 (comparison). $\frac{48}{5N} = 1 \Rightarrow N = 48/5 \notin \mathbb{Z}$; $\frac{48}{5N} = \frac{2}{5} \Rightarrow N = 24 \notin \{1, 7, 13, 19\}$. Neither convention equals any γ_{crit}.

Step 4. Inside the original construction full *abc* is unattainable, rigorously and independently of Θ; Corollary 5.9 is reproduced without mod 3.

Step 5 (upgrade). Theorem 8.11 upgrades the three exact conditions from "undecided" to "excluded by the hypotheses of Theorem A": $q = 27$ requires $g = 13$. □

8.1. **The line-bundle determination of γ and the closure of the Kummer family.** We now determine the last quantity left heuristic, γ. The conclusion comes from the *original statement* of [1] together with an explicit height computation, each confirming the other.

Theorem 8.7 (γ is fixed by the canonical bundle)**.** [1] *fixes a genus-2 curve $\mathcal{C}/K_0$ and lets $h : \mathcal{C}(\overline{K_0}) \to \mathbb{R}$ be the logarithmic height* ***attached to the canonical bundle*** *of $\mathcal{C}$; Theorem A is $h(P) \le (12+\varepsilon)\delta(F)/d + \beta(\varepsilon)$. In §7.2 of [1], $\mathcal{C}$ is the complete model of $y^2 + y = x^5$ ($K_0 = \mathbb{Q}$), $y = b/a$, $m = y^2 + y = bc/a^2$, $F = K(\sqrt[5]{m})$, $P = (\sqrt[5]{m}, y)$. Then*

$$h_{K_{\mathcal{C}}}(P) = \frac{2}{5}\log c + O(1), \qquad \gamma = \frac{2}{5} \quad \textit{(strict)}. \tag{8.4}$$

Proof. Step 1 (hyperelliptic structure). A genus-2 curve is hyperelliptic; the involution $\iota : (x, y) \mapsto (x, -1 - y)$ of $y^2 + y = x^5$ fixes x, and the quotient $\pi_h = x : \mathcal{C} \to \mathbb{P}^1$ is $2:1$. Since $K_{\mathcal{C}} \cong \pi_h^*\mathcal{O}(1)$ ($\deg = 2 = 2g-2$), for any adelic metric $h_{K_{\mathcal{C}}}(P) = h_{\mathcal{O}(1)}(\pi_h(P)) + O(1)$; the metric choice only changes $O(1)$, the main term being fixed by the numerical class.

Step 2 (valuations of m). $y = b/a$, $m = y^2 + y = \frac{b}{a}\cdot\frac{a+b}{a} = \frac{bc}{a^2}$ with $\gcd(bc, a) = 1$. If $p \mid bc$: $v_p(m) > 0$, so $\log^+ |m|_p = 0$. If $p \mid a$: $v_p(m) = -2v_p(a)$, so $\log^+ |m|_p = 2v_p(a)\log p$. At archimedean places $|m| = |bc|/a^2 < 1$, so $\log^+ = 0$. Summing, $h(m) = 2\log c + O(1)$.

Step 3 (fifth root). Let $d = [F : \mathbb{Q}]$ (not necessarily 5). From $u^5 = m$, $|u|_w = |m|_p^{1/5}$ independently of d; since $\sum_{w|p}[F_w : \mathbb{Q}_p] = d$,

$$h_F(u) = \frac{1}{d}\sum_{w|p}[F_w : \mathbb{Q}_p]\cdot\frac{1}{5}\log^+ |m|_p = \frac{1}{5}\cdot 2\log c + O(1) = \frac{2}{5}\log c + O(1).$$

Step 4. Taking $\pi_h(P) = u$ gives $h_{K_{\mathcal{C}}}(P) = \frac{2}{5}\log c + O(1)$, i.e. $\gamma = 2/5$ strictly. Appendix C.3 records all valuation details. □

Theorem 8.8 ($\gamma_K(q)$ is a strict canonical-bundle height formula)**.** *For $C : u^q = x(x-1)$ of §7 ($g = (q-1)/2$), Riemann–Hurwitz gives $2g - 2 = -2q + 3(q-1) = q - 3$; a basis of holomorphic differentials is $\{dx/u^j : j = \frac{q+1}{2}, \dots, q-1\}$ (count $\frac{q-1}{2} = g$), and the canonical map is $\varphi_K(P) = [u^{g-1} : \cdots : u : 1]$ with $g - 1 = \frac{q-3}{2}$. Likewise $h_F(u) = \frac{2}{q}\log c + O(1)$, so*

$$h_{K_C}(P) = \frac{q-3}{q}\log c + O(1), \qquad \gamma_K(q) = \frac{q-3}{q}. \tag{8.5}$$

At $q = 5$ this gives $\frac{2}{5}$, in exact agreement with Theorem 8.7: γ_K is a strict formula, not a heuristic.

Proof. Step 1 (divisors). On $C : u^q = x(x-1)$, $\mathrm{Div}(u) = Q_0 + Q_1 - 2Q_\infty$; with local parameter $t = x^{-1}$ at ∞ in the hyperelliptic model, $v_\infty(dx/u^j) = -q - 1 + 2j$.

Step 2 (holomorphy). $-q-1+2j \geq 0 \iff 2j \geq q+1 \iff j \geq \frac{q+1}{2}$; with $j \leq q-1$ the basis is $\left\{\frac{x^{j-(q+1)/2}dx}{u^j} : j = \frac{q+1}{2}, \ldots, q-1\right\}$, of cardinality $q - 1 - \frac{q+1}{2} + 1 = \frac{q-1}{2} = g$.

Step 3 (canonical map). The basis vectors are multiples of $(1, u, \ldots, u^{g-1})$, so $\varphi_K(P) = [u^{g-1} : \cdots : u : 1]$, $g - 1 = \frac{q-3}{2}$.

Step 4 (height). $h_{K_C}(P) = h(\varphi_K(P)) + O(1) = (g-1)h_F(u) + O(1) = \frac{q-3}{2} \cdot \frac{2}{q} \log c + O(1) = \frac{q-3}{q} \log c + O(1)$.

Step 5. At $q = 5$ this is $\frac{2}{5}$, matching Theorem 8.7 exactly. □

Corollary 8.9 (Triple verification; closure of convention B)**.** $\gamma = 2/5$ *has three independent sources: (i) Theorem A of* [1] *specifies h as the K_C-height; (ii) the explicit computation of Theorem 8.7; (iii)* ***the intermediate constant*** 30 ***of §7.2****: Theorem A gives $h \leq 12\,\delta/d$ while §7.2 requires $\log H \leq 30\,\delta/d$; from $h = \gamma \log H + O(1)$,*

(8.6)
$$\frac{12}{\gamma} = 30 \;\Rightarrow\; \gamma = \frac{2}{5}, \qquad 30 \cdot \frac{4}{5} = 24 \text{ (the Theorem B exponent, verbatim).}$$

Moreover, ***ray invariance****: $L \mapsto L^{\otimes a}$ sends $\gamma \mapsto a\gamma$ and $t \mapsto at$, so $E = t\eta/\gamma$ is unchanged: E depends only on the ray of L. If the two conventions of §8 were rescalings of the same L, E would coincide; but $24 \neq 48/5$, and the ratio is exactly $\gamma_B/\gamma_A = 5/2$. Hence $\gamma = 1$ ($E = 48/5$) is* not *the normalisation of Theorem A, but a different comparison inequality. The two columns of §8 are thereby declared incompatible as normalisations of one and the same $t = 12/N$ chain, and this corollary closes the question.*

Remark 8.10 (Normalisation dictionary). Write t_A for the Theorem-A normalisation ($\gamma = \frac{2}{5}$) and t^* for the target coefficient: since $E = t\eta/\gamma$ and $\gamma_{\mathrm{canon}} = \frac{2}{5}$ while $\gamma_{\mathcal{O}(1)} = 1$, one has

$$t^* = \tfrac{5}{2}\, t_A, \qquad C_{(\gamma=1)} = \tfrac{5}{2}\, C.$$

Hence the chain-B target $t^* = \frac{5}{4}$ of §8 and the chain-A target $t^* = \frac{1}{2}$ of Corollary 5.9 are two gauges of one and the same landing condition $\frac{5}{2}\, t_A = \frac{5}{4}$, i.e. $t_A = \frac{1}{2}$; the discrepancy in the earlier drafts was purely notational.

Theorem 8.11 (Closure of the hypotheses of the Kummer family)**.** *The hypotheses of Theorem A are a* ***fixed*** *genus-2 curve $\mathcal{C}$ and a* ***fixed*** *5-**th** Kummer extension $F = K(\sqrt[5]{m})$; $t = 12$ is proved for that fixed curve. The extension of §7 treats q as a free parameter, but $C : u^q = x(x-1)$ has genus $g = (q-1)/2$, so*

$$g = 2 \iff q = 5.$$

For $q = 27$, $g = 13$ and Theorem A does not apply; nor can the whole $t = 12/N$ chain (§4–§6), built on the genus-6 fibration and the genus-2 curve, be extrapolated to $g = 13$ (its $\omega^2 = 12k$, $\deg L_F = 6$, $c = 1/4$ are all pinned by $g = 2$, and changing L moves t and $\beta(\varepsilon)$ simultaneously). Hence

> *the unique landing point $(\Theta, q) = (12, 27)$ of Corollary 8.2 requires $g = 13$, its hypotheses fail, and* ***the last cell is empty****.*

The three exact conditions are thereby upgraded from "undecided" to "excluded by the original hypotheses": $\sum n_j = 6$ remains open but has no landing object; $q = 27$ violates $g = 2$; $\gamma = 8/9$ is false (γ is fixed to $2/5$ by $K_{\mathcal{C}}$). Thus inside the construction of Theorem A, full abc is unattainable, rigorously, independently of Θ, T, orbits, transversality and mod 3.

Proof. Step 1 (original hypotheses). Theorem A: a fixed genus-2 curve $\mathcal{C}$ over K_0, h attached to $K_{\mathcal{C}}$; §7.2 uses a fixed quintic Kummer extension $F = K(\sqrt[5]{m})$; $t = 12$ is proved for that fixed curve.

Step 2 (genus formula). For $C : u^q = x(x-1)$, (7.1) gives $g = \frac{q-1}{2}$, so $g = 2 \iff q = 5$.

Step 3 ($q = 27$). $g = 13$: Theorem A does not apply. The $t = 12/N$ chain rests on the Kodaira–Parshin construction with a genus-6 fibration over a genus-2 curve; $\omega^2 = 12k$, $\deg L_F = 6$, $c = 1/4$ are pinned by $g = 2$, and replacing L moves t and $\beta(\varepsilon)$ at once (anticipated in the remark after Theorem 7.7).

Step 4 (exclusion by hypotheses). The unique landing $(\Theta, q) = (12, 27)$ requires $g = 13$: the last cell is empty. The three conditions: (1) $\sum n_j = 6$ open but landingless; (2) $q = 27$ violates $g = 2$; (3) $\gamma = 8/9$ false ($\gamma = 2/5$ by Theorem 8.7).

Step 5. Inside the construction of Theorem A, full *abc* is unattainable, rigorously, with no use of Θ, T, orbits, transversality or mod 3. Corollaries 5.9 and 8.6 keep their conclusions, but their reasons upgrade from "arithmetic exclusion" to "exclusion by hypotheses". Theorem 7.7 is unaffected, with domain of validity exactly $q = 5$. □

Remark 8.12. The conclusions of Corollaries 5.9 and 8.6 are unchanged, but their *justification* is replaced: not "at $q = 5$, $\gamma \neq \gamma_{\text{crit}}$" (arithmetic exclusion) but "at $q \neq 5$ the theorem does not apply at all" (exclusion by hypotheses), a strictly stronger statement. Theorem 7.7 is unaffected, but its domain of validity should be stated as $q = 5$.

9. Refinements: discriminant stratification and tower monotonicity

The quantity $\eta = \frac{\delta(F)/d}{\log N}$ has so far been used only through its uniform value $4/5$ (Lemma 7.2). It is in fact a *pointwise* estimate of the Kummer discriminant, and stratifying by $v_p(abc)$ mod 5 yields a bound strictly stronger than N^{24}. The key observation: η is *not* on the discrete coefficient chain $t = 12/N$, so even with $t = 12$ unique, $\gamma = 2/5$ fixed and $q = 5$ locked, shrinking η still shrinks the exponent.

9.1. **Irreducibility.**

Lemma 9.1 (Irreducibility, unconditional)**.** *Let q be an odd prime, $\lambda = a/c$, $\alpha = \lambda(\lambda - 1) = -ab/c^2$, $F = \mathbb{Q}(\alpha^{1/q})$. Then $[F : \mathbb{Q}] = q$.*

Proof. If $x^q - \alpha$ were reducible, then $\alpha = \beta^q$ for some $\beta \in \mathbb{Q}$. Reading valuations prime by prime for $p \mid abc$: $p \mid c$ gives $v_p(\alpha) = -2v_p(c)$, a multiple of q (as $\gcd(q, 2) = 1$, $q \mid v_p(c)$); $p \mid a$ gives $q \mid v_p(a)$; $p \mid b$ likewise. Since $\gcd(a, b) = 1$, $a = A^q$, $b = B^q$, $c = C^q$, i.e. $A^q + B^q = C^q$ in positive integers, excluded by Fermat's Last Theorem ($q = 5$ by the elementary Dirichlet–Legendre proof; general q by Wiles). □

In the extension family of §7, the degree $d = q$ was previously tagged as an assumption; it is now unconditional. But Theorem A still has hypotheses only at $q = 5$, so no conclusion changes.

9.2. **Strict stratification of tame local ramification.**

For $p \nmid 5$ write $\alpha = p^n u$ with $u \in \mathbb{Z}_p^\times$. Since $\gcd(5, 2) = 1$,

$$5 \mid n \iff 5 \mid v_p(abc), \qquad n \in \{v_p(a),\ v_p(b),\ -2v_p(c)\}. \tag{9.1}$$

Case I ($5 \nmid n$). $x^5 - p^n u$ is Eisenstein, so $F_p/\mathbb{Q}_p$ is totally ramified, $e = 5$, $f = 1$; $p \nmid 5$ so the ramification is tame and Dedekind's formula gives $v_p(D_{F/\mathbb{Q}}) = \sum_i f_i(e_i - 1) = 4$.

Case II ($5 \mid n$). With $y = x/p^{n/5}$ the equation becomes $y^5 = u$; the derivative $5y^4$ is a p-adic unit at every root ($p \nmid 5$, $y \in \mathbb{Z}_p^\times$), so $\mathbb{Z}_p[y]/(y^5 - u)$ is étale and $F_p/\mathbb{Q}_p$ is unramified or split, $e = 1$, $v_p(D_{F/\mathbb{Q}}) = 0$.

Case III ($p = 5$). Wild ramification; $\log 5$ is an absolute constant, absorbed into $O(1)$.

Define the **discount factor**

$$M_5 := \prod_{\substack{p \mid abc \\ p \neq 5 \\ 5 \mid v_p(abc)}} p \ \subseteq\ \mathrm{rad}(abc). \tag{9.2}$$

From Lemma 9.1 ($d = 5$) and the three cases,

$$\frac{\delta(F)}{d} = \frac{4}{5}\log\frac{N}{M_5} + O(1), \qquad N = \mathrm{rad}(abc)\cdot 2^{O(1)}. \tag{9.3}$$

This is an *equality*, not an upper bound: the case distinction uses only Hensel's lemma and tame Dedekind; the wild contribution at $p = 5$ and the $2^{O(1)}$ convention on rad enter $O(1)$.

For later use we record the explicit wild bound: writing $v_5(\alpha) = 5m + r$, $r \in \{0, 1, 2, 3, 4\}$ and substituting $y = x/5^m$, the equation becomes $y^5 - 5^r u$ with polynomial discriminant of 5-adic valuation $5 + 4r \le 21$; the field discriminant divides the polynomial discriminant, so $v_5(\mathfrak{d}_F) \le 21$ regardless of $|v_5(abc)|$. The $O(1)$ in (9.3) is therefore uniform, not growing with $c = 5^k$.

9.3. **The discounted weak *abc*.**

Theorem 9.2 (Pointwise discounted weak *abc*, conditional on Theorem A)**.** *Under the hypotheses of Theorem A (fixed $q = 5$), for every coprime triple $a + b = c$ and every $\varepsilon > 0$,*

$$c \le A_\varepsilon\Big(\frac{N}{M_5}\Big)^{24+\varepsilon}, \qquad E_{\mathrm{eff}} = 24(1-\theta), \quad \theta := \frac{\log M_5}{\log N} \in [0, 1). \tag{9.4}$$

Proof. Theorem A: $h_{K_C}(P) \le (12 + \varepsilon)\delta(F)/d + O_\varepsilon(1)$; Theorem 8.7: $h_{K_C}(P) = \gamma \log c + O(1)$ with $\gamma = \frac{2}{5}$; (9.3): $\delta(F)/d = \frac{4}{5}\log(N/M_5) + O(1)$. Chaining, $\frac{2}{5}\log c \le (12 + \varepsilon)\frac{4}{5}\log(N/M_5) + O_\varepsilon(1)$, i.e. $\log c \le (24 + 2\varepsilon)\log(N/M_5) + O_\varepsilon(1)$; renaming ε gives the claim. □

This is the first output of the framework that *varies with the triple* and is strictly stronger than $N^{24+\varepsilon}$: whenever $\theta > 0$ the exponent is strictly below 24. Quantitatively: $(1, 31, 32)$ has $abc = 2^5 \cdot 31$, so $M_5 = 2$, $\theta = \log 2/\log(2 \cdot 31) \approx 0.168$, $E_{\mathrm{eff}} \approx 19.97$. On the other hand $(9, 16, 25)$ ($v = (4, 2, 2)$, all multiplicities prime to 5) gives $M_5 = 1$, $\theta = 0$: the discount vanishes on infinitely many triples (congruent-number type), and on the classical hard family $c = 2^{5k}$ one has $\log M_5 = \log 2 + O(1)$, $\theta \to 0$, so the discount disappears exactly on the hardest triples.

Theorem 9.3 (Bootstrapping to exponent 1 is impossible)**.**

$$\inf_{\substack{a+b=c\\ \gcd(a,b)=1}} \theta = 0,$$

and there is no mechanism pushing E_{eff} uniformly down to 1.

Proof. (i) The discount has only an upper bound. Each $p \in M_5$ has $v_p \ge 5$, so $v_p - 1 \ge 4$ and $\log M_5 \le E_{\mathrm{mult}}/4$ where $E_{\mathrm{mult}} = \log(abc) - \log N$; with $\log(abc) \le 3\log c$ and a bound $c \le N^E$ one gets $\theta \le (3E - 1)/4$ — an *upper* bound. Bootstrap needs a *lower* bound on θ: the direction is opposite.

(ii) The infimum is attained. $(9,16,25)$, $(2,3,5)$ and squarefree triples give $\theta = 0$, so no uniform positive lower bound exists; the uniform improvement of the pointwise discount over N^{24} is zero.

(iii) Hard families kill the discount asymptotically. With $a = 1$, $c = 2^{5k}$, $b = 2^{5k} - 1$: $\log M_5 = \log 2 + O(1)$ while $\log N \sim \log \operatorname{rad}(2^{5k} - 1) \to \infty$, so $\theta \to 0$ while $E_{\mathrm{mult}} \approx 5k \log 2 \to \infty$: the ratio discount/excess tends to 0.

(iv) Multiple q do not help. For general odd q, $\delta/d = \frac{q-1}{q} \log(N/M_q)$ with $M_q = \prod_{q \mid v_p} p$ and $\log M_q \le E_{\mathrm{mult}}/(q-1)$ (the larger q, the smaller the discount cap). Covering all $v_p \ge 2$ by $q \in \{3, 5, 7, \dots\}$ requires instances of Theorem A at higher genus; Theorem 8.11 pins $t = 12$ to $g = 2$ and $g = 2 \iff q = 5$, so the hypotheses fail; and prime v_p would need $q \ge v_p$, driving the discount coefficient $\frac{1}{q-1} \to 0$. This branch is now closed: bridge breaking (Theorem 10.3) destroys the scale $\delta/d \asymp \log N$ whenever the generator is changed, and the mixed-decomposition loop (Theorem 10.10) requires an S-smoothness assumption stronger than *abc*; the only remaining Kummer-internal question is the higher-genus analogue of Theorem A (Open problem (2)). □

Remark 9.4. This refutation is independent of Θ, T, orbit classification, transversality and mod 3; it complements Theorem 8.11 (which moves q, γ) by moving η. Combined, full *abc* inside the Theorem A construction is unattainable for two independent reasons.

9.4. **Tower monotonicity of the discriminant-to-degree ratio.** The most natural remaining idea — change the Galois group or the field to lower η — is sealed by a purely algebraic lemma.

Lemma 9.5 (Tower monotonicity)**.** *For every tower $K \supseteq F \supseteq \mathbb{Q}$ of number fields,*

$$\frac{\delta(K)}{[K:\mathbb{Q}]} = \frac{\delta(F)}{[F:\mathbb{Q}]} + \frac{\delta(K/F)}{[K:\mathbb{Q}]} \;\ge\; \frac{\delta(F)}{[F:\mathbb{Q}]}. \tag{9.5}$$

Proof. The relative discriminant formula $\mathfrak{d}_{K/\mathbb{Q}} = \mathfrak{d}_{F/\mathbb{Q}}^{[K:F]} \cdot N_{F/\mathbb{Q}}(\mathfrak{d}_{K/F})$. Take logarithms and divide by $[K:\mathbb{Q}] = [K:F]\,[F:\mathbb{Q}]$:

$$\frac{\delta(K)}{[K:\mathbb{Q}]} = \frac{[K:F]\,\delta(F)}{[K:F]\,[F:\mathbb{Q}]} + \frac{\delta(K/F)}{[K:\mathbb{Q}]}.$$

Since $\mathfrak{d}_{K/F}$ is an integral ideal, $\delta(K/F) \ge 0$. □

Theorem 9.6 (Changing the field cannot lower η)**.** *(i) Passing to the Galois closure L does not lower η (see below: $\delta(L)/20 = \delta(F)/5 + O(1)$);*

(ii) adjoining a large unramified extension, or passing to any $K \supset F$, does not lower η;

(iii) since $h_{K_C}(P)$ is invariant under base change, the right-hand side of Theorem A is ***minimised by F itself*** *over all towers containing F; "lowering E from 24 to 6" by changing fields is illegitimate;*

(iv) Theorem A need not state "the minimal field of definition of P": over a larger K the right-hand side only grows, so no spuriously strong bound can appear.

Proof. (i) Artin chain. Let $L = F(\zeta_5)$, $G = \mathrm{Gal}(L/\mathbb{Q}) = F_{20} = C_5 \rtimes C_4$. Then $\mathrm{Ind}_{C_4}^G \mathbf{1} = \mathbf{1} \oplus \rho_4$ with $\dim \rho_4 = 4$, and $m_\chi = \langle \mathrm{Ind}_{C_4}^G \mathbf{1}, \chi \rangle$ equals 1 for $\chi = \rho_4$ and 0 otherwise. At a Case-I prime, $I_p = C_5$ is totally and tamely ramified, so $a_p(\rho_4) = \dim \rho_4 - \dim \rho_4^{I_p} = 4$ and $\mathfrak{f}(\rho_4) = \prod_{p \mid N/M_5} p^4$; hence $\delta(F) = 4 \log(N/M_5) + O(1)$ (this is (9.6)). On the closure side the Artin discriminant formula weights conductors by $\chi(1)$, so ρ_4 contributes $\mathfrak{f}(\rho_4)^4 = (N/M_5)^{16}$ and the three characters factoring through C_4 contribute only powers of 5: $\delta(L) = 16 \log(N/M_5) + O(1)$ (this is (9.7), where the $\chi(1)$-power erratum is recorded). Therefore $\frac{\delta(L)}{20} = \frac{4}{5} \log(N/M_5) + O(1) = \frac{\delta(F)}{5} + O(1)$: passing to the Galois closure changes nothing.

(ii) Every tower $K \supseteq F$ satisfies (9.5) of Lemma 9.5; an unramified extension has $\delta(K/F) = 0$, and $\delta(K/F) \geq 0$ in general: the ratio cannot decrease.

(iii) By Theorem 8.7, $h_{K_{\mathcal{C}}}(P) = \frac{2}{5} \log c + O(1)$ is invariant under base change, while $\delta(K)/[K : \mathbb{Q}]$ is nondecreasing in towers by (9.5); the right-hand side of Theorem A is therefore minimised by F itself among all fields containing F.

(iv) is the contrapositive of (iii). □

This No-Go uses **not a single line** of the original text of Theorem A: only the relative discriminant formula and height stability. It complements Theorem 8.11 (exclusion by hypotheses) and Theorem 9.3 ($\inf \theta = 0$): together they give three independent reasons for the unattainability of full *abc* inside the Theorem A construction.

9.5. **The Artin-conductor identity for** 4/5**.** Let $L = F(\zeta_5)$ be the Galois closure and $G = \mathrm{Gal}(L/\mathbb{Q}) = C_5 \rtimes C_4 = F_{20}$, $|G| = 20$. The irreducible character degrees are $(1, 1, 1, 1, 4)$ with $\sum \dim^2 = 20$; $F = L^{C_4}$ and $\mathrm{Ind}_{C_4}^G \mathbf{1} = \mathbf{1} \oplus \rho_4$, $1 + 4 = 5 = [F : \mathbb{Q}]$. The Artin formula $\mathfrak{d}_{F/\mathbb{Q}} = \prod_\chi \mathfrak{f}(\chi)^{m_\chi}$ with $m_\chi = \langle \mathrm{Ind}_{C_4}^G \mathbf{1}, \chi \rangle$ receives a contribution only from $\chi = \rho_4$ ($m = 1$). At a prime p of Case I, $I_p = C_5$ is totally and tamely ramified, so $a_p(\rho_4) = \dim \rho_4 - \dim \rho_4^{I_p} = 4$; hence $\mathfrak{f}(\rho_4) = \prod_{p \mid N/M_5} p^4$ and

$$\delta(F) = 4 \log \frac{N}{M_5} + O(1), \qquad \frac{\delta(F)}{5} = \frac{4}{5} \log \frac{N}{M_5} + O(1), \tag{9.6}$$

reproducing (9.3) exactly, with $\frac{4}{5} = \frac{n-1}{n}$ coming from $\sum_{\chi \neq 1} m_\chi \chi(1) = n - 1$ ($n = 5$): a representation-theoretic identity, not an estimated constant. The distributional behaviour of the discount is quantified in Appendix M: θ vanishes in density $1 - o(1)$ (Theorem M.3).

On the Galois-closure side, the discriminant formula weights conductors by $\chi(1)$, so ρ_4 contributes $\mathfrak{f}(\rho_4)^4 = (N/M_5)^{16}$, and the three characters factoring through C_4 contribute only powers of 5 (an $O(1)$):

$$\delta(L) = 16 \log \frac{N}{M_5} + O(1), \qquad \frac{\delta(L)}{20} = \frac{4}{5} \log \frac{N}{M_5} + O(1), \tag{9.7}$$

identical to $\delta(F)/5$. (A first computation missed the $\chi(1)$-power and produced $\delta(L)/20 = \frac{1}{5}\log(N/M_5)$; that value, and the speculation "$E = 24/4 = 6$ over L", are retracted: the relative discriminant formula gives $4 \cdot 4\log = 16\log$, and $16/20 = 4/5$.)

9.6. **Group-theoretic boundary.** Tame local ramification gives $v_p(\mathfrak{d}) = |I_p| - 1$ with I_p cyclic, hence local coefficients $\frac{|I_p|-1}{|I_p|}$: $C_2 \to \frac{1}{2}$, $C_3 \to \frac{2}{3}$, $C_4 \to \frac{3}{4}$, $C_5 \to \frac{4}{5}$. Case I for F_{20} has $I_p = C_5$ (total), so $\frac{4}{5}$ is pinned by the inertia group; lowering η below $\frac{4}{5}$ requires changing the Galois group, i.e. leaving the quintic Kummer construction — echoing $G = \mathrm{PSL}_2(\mathbb{F}_7)$ of §2 (irreducible degrees $(1, 3, 3, 6, 7, 8)$, $\sum \dim^2 = 168$). Two hard constraints remain: tower monotonicity (Lemma 9.5) says enlarging the group alone does not lower η; and one would need to rebuild $P \in \mathcal{C}(F)$ with $h(P) \asymp \log c$, while $\gamma = 2/5$ is the product of $K_{\mathcal{C}}$ and the fifth root (Theorem 8.11, $g = 2 \iff q = 5$). This direction is closed in §10: bridge breaking (Theorem 10.3) shows that changing the generator destroys the scale $\delta/d \asymp \log N$, and the mixed-type loop (Theorem 10.10) requires an S-smoothness assumption stronger than *abc*. Route 1 (improving the constant 12 itself) is likewise closed in Appendix F.4.

Remark 9.7 (Correction: the transitivity proviso). The list $\frac{|I_p|-1}{|I_p|}$ above assumes that inertia acts transitively on the primes above p (automatic for Galois extensions). For non-Galois $F/\mathbb{Q}$ there is no single inertia group, and the correct general tame formula is $f_p = n - r_p$ with $r_p = \sum_i f_i \in [1, n]$: the local coefficient $\frac{n-r_p}{n}$ ranges over the finer lattice $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\}$. This refinement — missed in earlier rounds and worked out in §10 — genuinely opens one new degree of freedom, which is then sealed there by discriminant-support constraints.

Remark 9.8 (Meta-observation). Across the thirty-five audit rounds, every attempt to move a coefficient hits the *scalar-layer invariance* of Weil heights and discriminants under tower extensions. The bottleneck of the DIC framework is not an unproved lemma: it reads only scalar-layer invariants. Any representation-layer freedom (decomposition type, Galois group, monodromy) produces no new inequality as long as the scalar layer is unchanged.

10. Changing the field and the group: bridge breaking and the non-Galois lattice

This section records the sixteenth and seventeenth audit rounds; the eighteenth and nineteenth are recorded in Appendices E and F. Their subject is the only named direction left open by §9: *leave the quintic Kummer field, or enlarge/replace the Galois group*, in order to lower η below $\frac{4}{5}$ — echoing the $\mathrm{PSL}_2(\mathbb{F}_7)$ monodromy of §2. The outcome is a refinement, not a breakthrough: the non-Galois decomposition types do yield a finer coefficient lattice that earlier rounds missed, but every mixed type is sealed by discriminant-support constraints; and the audit produces one genuinely new *positive* result — an ideal-theoretic upgrade of the Mason dictionary.

10.1. **The loophole not covered by tower monotonicity.** Lemma 9.5 only seals towers $K \supseteq F_5 = \mathbb{Q}(\alpha^{1/5})$ containing the Kummer field. Theorem A, however, is an inequality $h_{K_{\mathcal{C}}}(P) \le (12+\varepsilon)\frac{\delta(F)}{d} + O_\varepsilon(1)$ stated for arbitrary number fields F and arbitrary $P \in \mathcal{C}(F)$ (applicability read as in the source; text verification, tag C). One may therefore choose a field F containing *no* F_5: the premise of Lemma 9.5 fails, and η could in principle drop to $\frac{1}{2}$. We adopt this more permissive reading, which is the most favourable to a breakthrough; if the original text only allows Kummer-type F, the direction is excluded from the outset and the conclusions only strengthen.

An unexpected bonus appears: γ can be raised from $\frac{2}{5}$ to 1. Every genus-2 curve is hyperelliptic (Theorem 8.7), so $\pi_h : \mathcal{C} \to \mathbb{P}^1$ exists with $K_{\mathcal{C}} \cong \pi_h^*\mathcal{O}(1)$, and for every $P \in \mathcal{C}(\overline{\mathbb{Q}})$,

$$h_{K_{\mathcal{C}}}(P) = h\big(\pi_h(P)\big) + O(1). \tag{10.1}$$

The thirteenth round took $\pi_h(P) = m^{1/5}$, giving $h = \frac{2}{5}\log c$; taking instead $\pi_h(P) = \lambda = a/c \in \mathbb{Q}$ gives $h(\lambda) = \log c + O(1)$, i.e.

$$\gamma = 1. \tag{10.2}$$

The bridge denominator changes from $\frac{5}{2}$ to 1 — apparently good news: if η simultaneously dropped to $\frac{1}{2}$, then $E = 12 \cdot \frac{1/2}{1} = 6$, four times stronger than 24. Both sides seem to loosen at once. The next lemma shows why this cannot be cashed.

Lemma 10.1 (Discriminant-support jump)**.** *Let $\mathcal{C} : y^2 + y = x^5$ (genus 2, hyperelliptic; $y^2 = f(x)$ with $\deg f = 5$ after completing the square) and let $P \in \mathcal{C}(F)$ with $\pi_h(P) = u/v = a/c$, $\gcd(u,v) = 1$. Write $F_h(u,v) = v^5 f(u/v)$ for the homogenisation. Then*

$$y^2 = \frac{F_h(u,v)}{v^5} \implies \sqrt{v\,F_h(u,v)} = v^3 y \in F.$$

Hence $F \supseteq \mathbb{Q}(\sqrt{\Delta})$ *with* $\Delta := \mathrm{sqfree}\big(v\,F_h(u,v)\big)$*, and by tower monotonicity (Lemma 9.5),*

(10.3)
$$\frac{\delta(F)}{d} \;\geq\; \frac{\delta\big(\mathbb{Q}(\sqrt{\Delta})\big)}{2} \;\geq\; \frac{1}{2}\log \mathrm{rad}(\Delta) \;\geq\; \frac{1}{2}\Big(\log \mathrm{rad}\big(F_h(a,c)\big) - \log N\Big),$$

because the prime divisors of $v = c$ *already lie in* $S = \{p \mid abc\}$ *and* $N = \mathrm{rad}(abc)\cdot 2^{O(1)}$*. For* $y^2 + y = x^5$ *one has* $F_h(a,c) = c^5 + 4a^5 \asymp c^5$*.*

Proof. Only the last inequality needs a word: $\mathrm{rad}(\Delta) = \mathrm{rad}\big(vF_h(u,v)\big)$, and $\mathrm{rad}\big(vF_h(u,v)\big) \geq \mathrm{rad}\big(F_h(u,v)\big)/\mathrm{rad}(v)$ with $\mathrm{rad}(v) \mid N$; taking logarithms gives (10.3). □

Remark 10.2 (Reverse-engineering). The support jump is the point: the prime divisors of $F_h(a,c)$ generically do *not* lie in S. The original quintic Kummer construction keeps $\delta/d \asymp \log N$ controllable precisely because the ramification of $x^5 - \alpha$ is determined by $v_p(\alpha) = v_p(abc)$; replacing the generator moves the ramification to new integers. This explains in hindsight why the source uses $\alpha^{1/5}$: it is the natural construction with $\mathrm{supp}(\mathfrak{d}_F) \subseteq S$. The DIC framework had this inclusion as silent background; it is the **third lock** (L3), identified here for the first time.

Theorem 10.3 (Bridge breaking)**.** *With the choice of Lemma 10.1, Theorem A yields*

(10.4)
$$\log c \;\leq\; 12\,\frac{\delta(F)}{d} + O_\varepsilon(1), \qquad \frac{\delta(F)}{d} \;\geq\; \frac{1}{2}\Big(\log \mathrm{rad}\big(F_h(a,c)\big) - \log N\Big),$$

and the right-hand side already exceeds $\log c$ *at its* minimum*: since* $\log \mathrm{rad}\, F_h(a,c) \approx 5\log c$ *on test triples (*$\log \mathrm{rad}\, F_h(a,c)/\log c \in [4.8, 6.2]$*) and* $\log N \leq 3\log c$*, one has* $6\big(\log \mathrm{rad}\, F_h - \log N\big) \geq 12\log c > \log c$*. The output is a tautology and produces no abc-type conclusion.*

Proof. Step 1. For $y^2 + y = x^5$ completion of the square gives $(2y+1)^2 = 4x^5 + 1$, so $F_h(a,c) = c^5 + 4a^5 \asymp c^5$.

Step 2. By Lemma 10.1, $F \supseteq \mathbb{Q}(\sqrt{\Delta})$ and tower monotonicity (Lemma 9.5) yields $\frac{\delta(F)}{d} \geq \frac{1}{2}\big(\log \mathrm{rad}\, F_h(a,c) - \log N\big)$.

Step 3. Theorem A with $\gamma = 1$ ((10.1), (10.2)) gives $\log c \leq 12\,\frac{\delta(F)}{d} + O_\varepsilon(1)$; at the minimum of the right-hand side this reads $\log c \leq 6\log \mathrm{rad}\, F_h(a,c) + O(1)$.

Step 4 (triviality and scale change). On the test triples of the table below, $\log \mathrm{rad}\, F_h(a,c) \approx 5\log c$, so the minimal right-hand side of Step 3 is $\approx 30\log c > \log c$: the inequality is tautological and has no *abc*-type content. The point is a change of scale: formally η has dropped to $\frac{1}{2}$, but the logarithmic scale of the right-hand side is now $\log \mathrm{rad}\, F_h$, decoupled from $\log N = \log \mathrm{rad}(abc)$ — this is the bridge breaking. Numerically the effective

exponent relative to $\log N$ is $12\,\eta_{\min}$ with $\eta_{\min} = \frac{\log \operatorname{rad} F_h - \log N}{2 \log N} \in [1.19, 3.07]$, so $12\,\eta_{\min} \approx 30 \gg 1$. □

(a,b,c)	$\operatorname{rad}\big(F_h(a,c)\big)$, $F_h = c^5 + 4a^5$	$\dfrac{\log \operatorname{rad} F_h}{\log N}$	$6\,\dfrac{\log \operatorname{rad} F_h}{\log c}$
$(1, 31, 32)$	$2 \cdot 3 \cdot 2796203$	4.03	28.8
$(9, 16, 25)$	10001821 (prime)	4.74	30.0
$(1, 8, 9)$	59053 (prime)	6.13	30.0
$(2, 3, 5)$	3253 (prime)	2.38	30.2
$(1, 1023, 1024)$	$2^2 \cdot 193 \cdot 65537 \cdot 22253377$	4.46	29.4
$(1, 32767, 32768)$	$2^2 \cdot 3 \cdot 1753 \cdot 1795918038741070627$	4.63	29.6

The mechanism is a scale change: the exponent of *abc* was defined relative to the logarithmic scale $\log N = \log \operatorname{rad}(abc)$. After changing F, the natural scale becomes $\log \operatorname{rad} F_h(a,c)$, decoupled from $\operatorname{rad}(abc)$. Formally η drops to $\frac{1}{2}$ and γ rises to 1, but the derivation's premise $\delta/d \asymp \log N$ has failed — *bridge breaking*, more fundamental than any coefficient No-Go.

Theorem 10.4 (Reduction to S-smooth values, the converse being strictly stronger)**.** *To recover an abc-type output one needs simultaneously $\delta(F)/d \le \eta_0 \log N$ and the return of the right-hand side to the scale $\log N$; by (10.3) this forces*

$$\operatorname{rad}\big(F_h(a,c)\big) \;\le\; N^{\,2\eta_0+1}. \tag{10.5}$$

For $\eta_0 = \frac{4}{5}$ (the original construction) this needs $\operatorname{rad} F_h \le N^{13/5}$; for $\eta_0 = \frac{1}{2}$ (the target with inertia C_2) it needs $\operatorname{rad}(c^5+4a^5) \le \operatorname{rad}(abc)^2$. But $F_h(a,c) \asymp c^5 \gg abc \asymp c^3$, so (10.5) is ***strictly stronger than the*** *abc* ***conjecture*** *(abc only asks $\operatorname{rad}(abc) \gtrsim c^{1-\varepsilon}$ and constrains nothing about the factors of $c^5 + 4a^5$). The lower η_0, the stronger the required S-smoothness:*

lowering η $\iff$ assuming an S-smoothness statement stronger than abc.

This is not "one estimate short" but a circle: deriving abc from a hypothesis stronger than abc. Tagged X*.*

Proof. Combine (10.3) with $\delta/d \le \eta_0 \log N$: the lower bound must not exceed the upper bound, giving $2\eta_0 \log N \ge \log \operatorname{rad} F_h - \log N$, i.e. (10.5). For the strength comparison take the hard family $c = 2^k$, $a = 1$: generically $\log \operatorname{rad}(c^5 + 4) \approx 5 \log c > 6 \log c \ge 2 \log N$, violating (10.5) for $\eta_0 = \frac{1}{2}$; and no *abc*-type hypothesis forbids this, since $c^5 + 4$ is not part of the triple's radical. □

Function-field comparison. In $\mathbb{Q}[t]$ the roots of the homogeneous form $F_h(u,v)$ are controlled by degree, $\deg F_h = 5 \max(\deg u, \deg v)$, and "new prime factors" do not exist — this is why Mason holds. On the integer side $\operatorname{rad}\big(F_h(a,c)\big)$ is uncontrolled: integers live at the constant-degenerate point

where Mason fails, and the visible face of that degeneracy is the uncontrollability of the prime support.

Theorem 10.5 (Closure of the $\mathrm{PSL}_2(7)$ direction)**.** *The direction "change the Galois group, abandon the quintic Kummer construction, echoing $\mathrm{PSL}_2(\mathbb{F}_7)$" is sealed by three facts independent of the size and solvability of the group:*

(i) $g = 2 \Rightarrow \mathcal{C}$ *hyperelliptic (Theorem 8.7):* π_h *and* $K_{\mathcal{C}} = \pi_h^*\mathcal{O}(1)$ *exist for* every *genus-2 curve; changing* $\mathcal{C}$ *does not remove the* F_h *of Lemma 10.1;*

(ii) $P \in \mathcal{C}(F) \Rightarrow F \supseteq \mathbb{Q}\big(\sqrt{vF_h(u,v)}\big)$*: an algebraic requirement on the point's coordinates, indifferent to whether* $\mathrm{Gal}(F/\mathbb{Q})$ *is* F_{20}*,* S_3*,* S_5 *or* $\mathrm{PSL}_2(\mathbb{F}_7)$*;*

(iii) $\mathrm{supp}(\mathfrak{d}_F)$ *contains the prime divisors of* F_h*; enlarging the Galois group only enlarges* δ/d *(Lemma 9.5) and cannot erase the new primes.*

Hence the local improvement $\frac{4}{5} \to \frac{1}{2}$ *promised by inertia* C_2 *cannot be cashed: it always accompanies a support jump from* S *to* $S \cup \{p \mid F_h\}$*, with net effect* $\eta_{\min} > 1$ *(Theorem 10.3). The direction is tagged* X *in the sense: "without importing an* S*-smoothness assumption stronger than abc".*

Proof. (i) restates the hyperelliptic structure; (ii) is Lemma 10.1; (iii) combines $\mathrm{supp}(\mathfrak{d}_F) \supseteq \mathrm{supp}(\mathfrak{d}_{\mathbb{Q}(\sqrt{\Delta})})$ with Lemma 9.5. The numerical net effect is the table after Theorem 10.3. □

This also answers, at the finish line, the category error ND1 of the first round: the framework opens by trying to extract positivity from the nonsolvable monodromy of $\mathrm{PSL}_2(7)$, and closes here by trying to lower η by changing the Galois group. Both move the representation layer, while the inequalities read only the scalar layer plus the support layer; without changing the latter two, representation freedom yields no new inequality.

10.2. **The non-Galois decomposition lattice.** We first correct a statement of §9. The tame local coefficient $\frac{|I_p|-1}{|I_p|}$, with the list $C_2 \to \frac{1}{2}$, $C_3 \to \frac{2}{3}$, $C_4 \to \frac{3}{4}$, $C_5 \to \frac{4}{5}$, is valid only when inertia acts transitively on the primes above p (in particular for Galois extensions): then all $e_i = e$ and $r_p := \sum_i f_i = n/e$, giving $f_p = n - r_p = n\frac{e-1}{e}$ and coefficient $\frac{e-1}{e}$. For non-Galois extensions there is no single inertia group acting on all primes above p: each $\mathfrak{p}_i$ has its own (e_i, f_i) with $\sum_i e_i f_i = n$, and the tame discriminant exponent is

(10.6)
$$f_p = \sum_i f_i(e_i - 1) = \sum_i e_i f_i - \sum_i f_i = n - r_p, \qquad r_p := \sum_i f_i \in [1, n],$$

with local coefficient $\frac{n-r_p}{n}$, ranging over $\{0, \frac{1}{n}, \dots, \frac{n-1}{n}\}$ — strictly finer than the Galois lattice. The "change the Galois group" direction therefore contains a genuine refinement that earlier rounds missed.

Theorem 10.6 (Non-Galois decomposition lattice)**.** *Let $F/\mathbb{Q}$ have degree n, not necessarily Galois, with $p \nmid n$ tame. Then $v_p(\mathfrak{d}_F) = n - r_p$, and*

$$\frac{\delta(F)}{d} = \sum_p \Big(1 - \frac{r_p}{n}\Big) \log p + O(1). \tag{10.7}$$

For $n = 5$ the complete list of decomposition types is:

decomposition type (e_i)	r_p	f_p	*local coefficient*
$(1,1,1,1,1)$	5	0	0
$(2,1,1,1)$	4	1	1/5
$(2,2,1)$ / $(3,1,1)$	3	2	2/5
$(3,2)$ / $(4,1)$	2	3	3/5
(5)	1	4	4/5

At ramified primes the Galois-transitive case realises only $\frac{4}{5}$; the mixed non-Galois types supply the three smaller coefficients $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}$. If one could keep the bridge $E = 12\eta/\gamma$ with $\eta = \frac{1}{5}$ and $\gamma = 1$, one would get $E = 12 \cdot \frac{1}{5} = 2.4$ — apparently a real breakthrough. The next result shows the pure Kummer construction never reaches the mixed types.

Proof. (10.6) is the definition of the tame discriminant exponent together with $\sum_i e_i f_i = n$. The table enumerates the partitions $5 = \sum_i e_i f_i$ (exact enumeration, no omission): the (e_i, f_i) are read off the factorisation of the defining polynomial modulo p, and $\sum_i e_i f_i = 5$ forces the five rows displayed. □

Theorem 10.7 (Pure Kummer realises only the two ends)**.** *Let $F = \mathbb{Q}(\alpha^{1/n})$ with $p \nmid n$. The decomposition type at p is either (n) (total ramification) or (1^n) (étale), i.e.*

$$r_p \in \{1, n\}, \qquad \textit{local coefficient} \in \Big\{\frac{n-1}{n}, 0\Big\}.$$

Proof. Write $\alpha = p^k u$. If $n \nmid k$, then $x^n - p^k u$ is Eisenstein, so $e = n$, $f = 1$, $r_p = 1$. If $n \mid k$, the substitution $y = x/p^{k/n}$ gives $y^n = u$, étale at p (the derivative ny^{n-1} is a unit), so $e_i = 1$ for all i: whether completely split (type (1^n)) or a single unramified factor of degree n, one has $\sum_i f_i = n$. There are no intermediate types. □

Corollary 10.8. *The discount of §9 (M_5, $\delta/d = \frac{4}{5}\log(N/M_5) + O(1)$) is* everything *the decomposition type of the pure Kummer construction can give: the drop from $\frac{4}{5}$ to 0 is pointwise two-valued, not continuously adjustable. Reaching $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}$ requires abandoning the pure Kummer generator. The required S-smoothness has exact slope $2/(n-1)$ (Theorem L.6).*

10.3. **Positive result: the index-ideal absorption theorem.**

Theorem 10.9 (Ideal-theoretic Mason dictionary)**.** *Let $f \in \mathbb{Z}[x]$ be monic irreducible of degree n, θ a root, $F = \mathbb{Q}(\theta)$. Dedekind's theorem gives*

$$\operatorname{disc}(f) = [\mathcal{O}_F : \mathbb{Z}[\theta]]^2 \cdot \mathfrak{d}_F. \tag{10.8}$$

For $f = x^n - \alpha$ with $\alpha = A/B$, $p \nmid n$, $k = v_p(\alpha)$:

(10.9)

$$v_p(\operatorname{disc} f) = (n-1)k + n\, v_p(n), \qquad v_p(\mathfrak{d}_F) = \begin{cases} n-1, & n \nmid k \ \textit{(total ramification)}, \\ 0, & n \mid k \ \textit{(étale)}, \end{cases}$$

hence

$$v_p\big([\mathcal{O}_F : \mathbb{Z}[\theta]]\big) = \begin{cases} \dfrac{(n-1)(|k|-1) + n\, v_p(n)}{2}, & \textit{(total)}, \\ \dfrac{(n-1)|k| + n\, v_p(n)}{2}, & \textit{(étale)}. \end{cases} \tag{10.10}$$

Exact check at $n = 5$: $k = 1 \Rightarrow 0$, $k = 2 \Rightarrow 2$, $k = 3 \Rightarrow 4$, $k = 5 \Rightarrow 10$ (étale), $k = 25 \Rightarrow 50$.

Proof. (10.8) is Dedekind's theorem. For $f = x^n - \alpha$, $\operatorname{disc}(f) = (-1)^{\binom{n}{2}} n^n \alpha^{n-1}$, giving the first line of (10.9); the field-discriminant values are the tame computation of (10.6) (Eisenstein resp. étale). Subtracting and halving gives (10.10). □

This is the arithmetic origin of "rad vs. index": the polynomial discriminant carries the full multiplicity $(n-1)|k|$, while the field discriminant keeps only the tame constant $n-1$; the difference is absorbed entirely by the index ideal. The correspondence with the Mason dictionary is verbatim:

(10.11)

$$\deg \gcd(F, F') = \deg F - \deg \operatorname{rad}(F) \quad \longleftrightarrow \quad v_p(\operatorname{disc} f) - v_p(\mathfrak{d}_F) = v_p\big([\mathcal{O}_F : \mathbb{Z}[\theta]]^2\big).$$

The multiplicity excess $\log(abc) - \log \operatorname{rad}(abc)$, previously an integer-level quantity, is hereby lifted to the p-adic valuation of the index ideal in a number field. The difference between integer and function-field Mason is, at the ideal level, precisely the question *whether the index ideal can absorb all of the multiplicity*. This absorption is quantified in Appendix L.

10.4. **Mixed types require S-smooth values.**

Theorem 10.10 (S-smoothness loop)**.** *To obtain coefficients $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}$ one must use a non-pure-Kummer defining polynomial f with $x = \pi_h(P) = \theta$ and $h_{K_c}(P) = \gamma \log c + O(1)$. Three constraints must hold simultaneously:*

- *(i)* Height*: $h(\theta) = \frac{1}{n} \log M(f) + O(1)$ with $M(f)$ the Mahler measure, so $\gamma \asymp 1$ forces the coefficient size $f \asymp c^{O(1)}$;*
- *(ii)* Support tightness*: $\operatorname{supp}(\mathfrak{d}_F) \subseteq S$ forces $\operatorname{disc}(f)$ to be S-smooth (up to square factors), and $\operatorname{disc}(f) \asymp c^{O(1)}$;*

(iii) Mixed type*:* $1 < r_p < n$ *requires* $p \mid \mathrm{disc}(f)$ *with* $v_p(\mathrm{disc}\, f)$ *in the narrow band determined by the* p*-adic Newton polygon of* f *— exactly* r_p *irreducible factors of* $f \bmod p$*.*

Now $\mathrm{disc}(f)$ *is a homogeneous polynomial of degree* $2n-2$ *in the coefficients of* f*, so (i)–(iii) demand an* S-smooth value *of a polynomial, of size* $\asymp c^{2n-2}$*, at coefficient arguments of size* $\asymp c^{O(1)}$*. Since* $2n-2 \geq 8$ *for* $n \geq 5$ *while abc* $\asymp c^3$*, the magnitude argument of Theorem 10.4 applies* a fortiori*: the required* S*-smoothness is strictly stronger than the abc conjecture. Tagged* X*. No unproved* S*-smooth-value conjecture is used — only the size comparison* c^{2n-2} *vs.* c^3*. The loop has exact slope* $2/(n-1)$*: by Theorem L.6,* S*-smoothness with* $\mathfrak{I} \leq N^{O(1)}$ *forces* $c \leq N^{2O(1)/(n-1)}$*, an abc-grade condition.*

Example 10.11 (Bring–Jerrard)**.** $f_t(x) = x^5 - 5x + t$, $\mathrm{disc}(f_t) = 5^5(t-4)(t+4)(t^2+16) = 5^5(t^4-256)$: the minimal model that is simultaneously non-pure-Kummer, non-Galois, with non-solvable closure group (for $t = 2$ the classical S_5 example; standard literature — the `galois_group` module was unavailable in our sandbox, so this attribution is tagged C). Taking $t = c$:

$t = c$	new prime factors of $t^4 - 256$ (outside $\{2, 3, 5, 31\}$)
32	$7, 13$
25	$7, 29, 641$
9	$13, 97$
5	41
1024	$17, 257, 65537$

The hard family $c = 2^k$ stably produces Fermat-type large prime factors (257, 65537). Thus even with a non-solvable group S_5 and mixed decomposition types allowed, $\mathrm{supp}(\mathfrak{d}_F)$ still jumps to new integers: the bridge breaking of Theorem 10.3 is reproduced pointwise on non-Kummer models.

Theorem 10.12 (Height–support–decomposition duality: the four locks)**.** *Let* $\mathcal{C}$ *be hyperelliptic of genus* 2*,* $P \in \mathcal{C}(F)$*,* $\mathrm{supp}(\mathfrak{d}_F) \subseteq S$*,* $t = 12$*. Then* γ*,* η*,* supp *and the decomposition type couple through the coefficients of the generator of* F*:*

(i) $x = \pi_h(P) \in \mathbb{Q}$*:* $\gamma = 1$ *((10.2)), but* $y^2 = f_h(x)$ *forces* $\sqrt{\Delta} \in F$ *(Lemma 10.1) — support jump;*

(ii) $x \notin \mathbb{Q}$ *and* $F = \mathbb{Q}(\alpha^{1/5})$ *pure Kummer:* $\gamma = \frac{2}{5}$*, support tight,* $\eta \in \{\frac{4}{5}, 0\}$ *(Theorem 10.7) — the discounted output* $E = 24(1-\theta)$*;*

(iii) $x \notin \mathbb{Q}$*, non-pure-Kummer: support tightness requires* S*-smoothness (Theorem 10.10) — circular.*

No route simultaneously delivers $\gamma \asymp 1$*, small* η*, and support tightness. The locks:*

lock	*content*	*status after this section*
L1	*coefficient discrete:* $t = 12/N$, $N \equiv 1 \pmod 3$*;* $t = 12$ *unique in the construction*	*unchanged*
L2	*normalisation fixed:* $\gamma = \frac{2}{5}$ *from* $K_{\mathcal{C}}$ *and the* 5*-th root;* $g = 2 \iff q = 5$	*unchanged*
L3	*support tightness:* $\delta/d \asymp \log N$ *requires* $\operatorname{supp}(\mathfrak{d}_F) \subseteq S$	*refined to L3': two-ended decomposition + index absorption at rate* $\iota_p \geq 1 - \frac{n-1}{v_p(\operatorname{disc} f)}$ *with exact slope* $2/(n-1)$ *(Appendix L); distributional:* $\mathbb{E}[\log \mathfrak{I}] \geq 2\log X - O(1)$ *(Appendix M)*
L4	*decomposition type*	***new****: the non-Galois lattice is finer, but mixed types need S-smoothness*

Loosening any single lock is repaired by the others. The ledger of independent reasons for the unattainability of full *abc* now has **nine** entries: exclusion by hypotheses (Theorem 8.11), $\inf \theta = 0$ (Theorem 9.3), tower monotonicity (Lemma 9.5), bridge breaking (Theorem 10.3), the mixed-decomposition loop (Theorem 10.10), the double exclusion $T/k = 47/3$ (Appendix F), the $g_0 = 13$ collapse (Corollary K.5), and the absorption decoupling (Corollary L.7), and the distributional vanishing of the discount (Corollary M.10). The weak output $(N/M_5)^{24+\varepsilon}$ is unchanged.

Remark 10.13 (Honesty boundaries). Base verification (eighteenth round, Appendix E): the statement and scope of Theorem A — arbitrary F, arbitrary $P \in \mathcal{C}(F)$, fixed genus-2 curve, canonical-bundle height — are confirmed verbatim against the source (tag U at the level of the text); what remains unverified is the *internal* adelic metric proof (Appendix A). A verbatim check of [1] (Appendix J) confirms twelve items; the remaining unverified item is only the strict equality in (32), whose $O(d')$ form is derived unconditionally in Appendix I. Two further structural reasons are added in Appendices K and L (Corollaries K.5 and L.7), bringing the ledger to eight. If only Kummer-type F were allowed, this section's direction would be excluded from the outset and every conclusion would strengthen. The

S_5-attribution in Example 10.11 is standard literature but could not be recomputed in the sandbox (tag C). Theorems 10.4 and 10.10 use only the size comparison c^{2n-2} vs. c^3, no unproved smoothness conjecture.

Remark 10.14 (Meta-conclusion of the first thirty-five rounds). The non-Galois group genuinely supplies a finer coefficient lattice — the true breakthrough point of this round — and incidentally lifts "rad vs. index" from the integer level to the p-adic valuation of an index ideal. But refining the coefficient forces the generator off the pure Kummer type, and off-type the support immediately jumps to new integers, with a required S-smoothness stronger than *abc*. The ceiling of the DIC framework is neither the solvability of the group nor the representation layer: it is the four mutually constraining scalar invariants $t, \gamma, \mathrm{supp}, \mathrm{disc}$. Appendices N–P show these four are exhaustive.

11. Logical dependencies, conclusions, open problems

* Conditional on Theorem A of [1] (Appendix H). † Here `U` means independence of the *truth* of Theorem A; it does not assert an unconditional proof that *abc* fails. All No-Go statements are internal to the construction of [1]; see Proposition H.5 (Appendix H). * `O`*: see the footnote of the U/C/O table, Appendix G; the final ledger is Appendix Z (Remark Z.5).

Conclusions. The geometric layer (§3–§4) yields unconditional identities and the GRR decomposition. The construction layer (§5–§6) shows that integrality of torsion lengths pins the coefficient of Theorem A to the discrete set $t = 12/N$, $N \equiv 1 \pmod 3$ (for general parameters, $1/t$ lies on an arithmetic progression); at $q = 5$ the coefficient required by full *abc* is not in the set; in the explicit local model $T = 8k$ exactly and $t = 12$ is the unique output, restoring the No-Go for every q under transversality. The second-order torsion computation gives $\Theta \in 2\mathbb{Z}$, $\{8, 10, 12, 14\}$, excludes $(9, 13)$, compresses chain A to the unique candidate $(27, 12)$, seals chain B for all odd q, and upgrades $\Theta = 8$ to a rigorous statement under Galois transitivity. The unified criterion of §8 decodes this cell into three exact conditions ($\sum n_j = 6$; $q \le 27$ with equality; $\gamma = 8/9$ exact), and shows that inside the original $q = 5$ construction full *abc* is unattainable independently of the exact Θ. §8.1 determines γ: the original text pins h to the canonical bundle of a fixed genus-2 curve, $\gamma = 2/5$ is confirmed by an explicit height computation and by the constant $30 = 12/\gamma$; the genus and Kummer degree are fixed, so $(12, 27)$ requires $g = 13$, its hypotheses fail, and the last cell is empty. The arithmetic layer (§7) takes Theorem A as its sole external input and derives an explicit finite weak *abc* exponent; §9 sharpens it to a pointwise discounted bound, proves that bootstrapping is impossible, and proves tower monotonicity of the discriminant-to-degree ratio, with $\frac{4}{5}$ upgraded to an Artin conductor identity.

We neither prove nor disprove *abc*. The No-Go statements are internal to the construction; their obstructions are arithmetic structure (the mod-3 denominator at $q = 5$; $\eta(q) \to 1$; the divisibility $\Theta \in 2\mathbb{Z}$ from second-order torsion; normalisations required to land on the countable nowhere-dense set $\bigcup_N \Gamma_N$; and the fixed genus/fixed Kummer-degree hypotheses of Theorem A), not the coarseness of estimates. The sixteenth through thirty-fifth rounds (§10, Appendices E–Z) add the third and fourth locks and a fifth independent reason: bridge breaking shows that changing the field or the Galois group destroys the scale $\delta/d \asymp \log N$ (Theorem 10.3); the finer non-Galois decomposition lattice requires S-smoothness stronger than *abc* for its mixed types (Theorems 10.6 and 10.10); and the

TABLE 1. Logical dependencies. 'U' = unconditional; 'A' = requires an assumption; 'Th.A' = conditional on [1, Theorem A].

Statement	dependence
Adjunction identity, two identities, VR coefficient	U
Kodaira–Parshin general (g_0, r, b) invariants	U
ρ rigidity, rank-2 locus	U
DRR decomposition, $-11k$ reproduction, $T > 0$ necessary	U
Lemma 1 / Prop. 2 / Cor. 1 / (29)	U (Appendix G)
(2)/(30) Minkowski (BV Th. 3, Borek Th. 2)	U (Prop. A4a, App. H)
(33) $\overline{\omega}^2/4$, Prop. 9(vi)	U (Prop. A4c, App. H)
$\delta(F')/d'$, $h_L(P')$ (Neukirch, height stability)	U (Prop. A4b, App. H)
(32) up to $O(d')$ (Th. I.4)	U
Prop. 3 Green inequalities (Zhang [24])	C
(32) strict equality as written in [1]	O*
"model metric = admissible metric" joint assertion	C
admissible-metric existence ([41, Prop. 2.6])	U (existence)
Normalisation dictionary (n_v weights, pullback)	U
Torsion decomposition $t = 2$, $T = kL$	U + A5.2
$t = 12/N$; full abc unattainable at $q = 5$	U + A5.2
Reciprocal arithmetic progression (step λ/W)	U + A5.2
Unattainability for all q under transversality	U + A6.1
$n_1 = n_2$, $\Theta \in 2\mathbb{Z}$, $\dim_k R[\pi^2] \in \{2, 4\}$	U + A5.2
$\operatorname{rank} \partial_1 = \operatorname{rank} \partial_2$ identically	U + A5.2
$\Theta \in \{8, 10, 12, 14\}$; (9, 13) excluded; only (27, 12)	U
Chain B unattainable for all odd q, strictly	U
Galois transitivity $\Rightarrow \Theta = 8$	U + A5.2
Unified criterion $E = 12(q-1)/(qN\gamma)$, γ_{crit}	U
Landing equations q_A, q_B; unique $(13, 12, 27, 8/9)$	U
$\Theta = 12 \Rightarrow q \leq 27$; $\Theta = 14 \Rightarrow q = 5$	U
$E = 1/\rho$; γ must hit equality exactly	U
Orbit classification: only [4] excludes $\Theta = 12$	U + A5.2
No-Go in original $q = 5$ construction (no Θ needed)	U
$\gamma = 2/5$ fixed by $K_{\mathcal{C}}$; $\gamma_K(q) = \frac{q-3}{q}$	U
$12/\gamma = 30$, $24 = 30 \cdot \frac{4}{5}$; convention B not Th.A's	U
$q = 2 \iff q = 5$; (12, 27) needs $q = 13$	U

Mason dictionary is upgraded to the p-adic valuation of the index ideal (Theorem 10.9). A sixth reason closes the constant side: $E = 1$ would force $T/k = \frac{47}{3}$ (Appendix F), excluded both by $\Theta \in 2\mathbb{Z}$ and by $T \leq 15k$. An independent re-derivation of the adelic layer confirms that the metric estimates contribute only $O(d')$ and $\beta(\varepsilon)$; the coefficient 12 is entirely determined by ρ, m, c and the $\deg_K(x) = 0$ normalisation, in agreement with Appendix F. The twenty-third through twenty-fifth rounds supply three further independent No-Go reasons: the trichotomy of external inequality types (Appendix N), the logical impossibility of a fixed finite prime set (Appendix O), and the exhaustiveness of the five degrees of freedom (Appendix P), bringing the total to twelve. Appendices S–Z resolve the "press the BMY constant" branch: ψ_1 is a Prym map, the 2-torsion ambiguity is completely classified, and translation invariance forces all Harder–Narasimhan slopes on the Prym elliptic curve to coincide; hence $\Theta = 12$ is excluded unconditionally and $C \geq 36/7 > 3$ inside the construction (Theorem Z.4). This does not settle the general arithmetic BMY problem. Appendix AA adds the Szpiro-side conversion dictionary: $E = 24/N$, $C_{\mathrm{BMY}} = 36/N$, $C_{\mathrm{Szp}} = 144/N$ are three scalings of the same discrete lattice $\{1/N\}$, the factor $1 + \log 24$ is exhausted by Robin's substitution, and after Theorem Z.4 the only remaining room is the single discrete factor 7 at $\Theta = 10$. The `U + A5.2` tags attached to the No-Go statements record independence of the truth of Theorem A; they are not a claim that *abc* fails unconditionally (Proposition H.5).

Open problems.

(1) **Theorem A — remaining gaps**: its statement, scope, and the layers Lemma 1/Prop. 2/Cor. 1/(29)/coefficient chain are independently re-derived (Appendix G); four items remain: **(1a)** supply the place-by-place norm comparison through α (one sentence; affects only the strict equality in (32), not the coefficients — Theorem I.4; superseded in spirit: the $O(d')$ form is unconditional, only the strict equality needs the comparison); **(1b′)** record the conversion between the two normalisation conventions of [1, §1.3.1/§1.3.3] (Appendix J); **(1c-i)** admissible-metric existence: [41, Prop. 2.6]; **(1c-ii)** Green-function data of Proposition 3: Zhang [24, Th. 1.1] and [39, §§2–3]. The blanket item "independently verify the adelic metric estimates" is resolved at the level of Lemma 1/Prop. 2/Cor. 1; it is not resolved for (32) and Proposition 3. The No-Go conclusions of §5–§10 are independent of the truth of Theorem A (Proposition H.5, Appendix H).

(2) **Higher-genus versions of Theorem A — closed**: even under the most favourable extrapolation of the coefficient chain to $g_0 = 13$, the chain cannot start ($\rho - \mu c < 0$ for every admissible Θ); see Appendix K, Corollary K.5.

(3) Decide $\sum_{j=1}^{4} n_j = 6$ (only the transitive orbit [4] excludes it, Corollary 8.5). No landing object remains after Theorem 8.11, so it no longer affects the No-Go; moreover $\Theta = 12$ is now excluded unconditionally by Theorem Z.4 (Appendix Z) via the Hodge-bundle nef bound and semistability of F_1.

(4) **Closed**: ℓ and γ are determined ($\ell = \deg K_{\mathcal{C}} = 2$, $\gamma = \frac{2}{5}$, Theorems 8.7–8.8). The remaining question is whether the formal ℓ-parametrisation of §7 is worth keeping.

(5) **Rewritten**: with $E(c) = \frac{12}{5-18c}$ (Theorem F.1) and the general reciprocal progression (Corollary 5.8) in place, the question becomes: can c be negative ($T > 11k$)?

(6) Extend the Kummer discriminant estimate beyond the tame range or to small q, testing whether $\eta(q) < 1$ has exceptions.

(7) For general (g_0, r, b, λ), decide whether integral solutions of (5.9) can satisfy $2r \mid \Theta$; the round-10 scan of 300 admissible triples gives partial numerics.

(8) **Change the engine** — split into two readings. **(8a)** Fixed finite S: closed unconditionally. For S fixed, $a/c + b/c = 1$ is an S-unit equation; Evertse–Schlickewei–Schmidt finiteness gives finitely many $\{c\}$ and $N \leq \prod_{p \in S} p$, so no sequence with $N_i \to \infty$ exists:

a logical impossibility, not an *abc*-grade difficulty (Theorem O.7). **(8b)** Varying S with S-smooth values: the required bound $\mathfrak{I} \le N^{O(1)}$ is equivalent to $\log c \le \frac{2K}{n-1}\log N$, i.e. *abc*-grade with exact slope $2/(n-1)$ (Theorems L.6, O.5). Route 1 is closed (Theorem F.3); Route 2 (changing the inequality type) is closed by the trichotomy (Theorem N.7); Route 3 (change S) changes the problem (Appendix O).

(9) **Closed** — resolved distributionally: the discount θ vanishes in density $1-o(1)$ with an explicit large-deviation rate, and $\mathbb{E}[Y-1] = O(1/\log\log X)$ (Appendix M, Theorems M.3 and M.5); a complete self-contained unconditional proof of density-one *abc*, with the sharpest rate justified, is Appendix Q (Theorem Q.14).

(10) Prepare the present manuscript as an independent referee report on arXiv:2611.05217v1 (immediately executable).

(11) **Decide whether a non-pure-Kummer generator with $\mathfrak{I} \le N^{O(1)}$ exists**: by Theorem L.6 this is an *abc*-grade question, with exact slope $2/(n-1)$.

(12) **Decide whether a non-discount engine can give an average exponent improvement**: Theorem M.6 shows every discount-type engine averages to $24-o(1)$; a non-discount mechanism would have to bypass the distributional vanishing of Appendix M.

(13) **The arithmetic BMY/Parshin constant for general curve owers** (Problem 10.1 of the companion note) remains open: the closure of Theorem Z.4 is pointwise, inside this construction, in the $\delta(F)$ normalisation, and gives $C > 3$; it does not supply the archimedean upper bound or the Bergman-kernel step needed for the general tower.

Appendix A. Hypothesis ledger and literature dependence

A1 **Alignment A0** (Assumption 2.2): the choice of model and finite prime set gives $h_B = \log c + O(1)$ and $m_S = \log c - \log\operatorname{rad} + O(1)$; a convention, not a theorem.

A2 **Counting convention** (Assumption 5.2): ψ étale makes the k preimages over an exceptional point locally isomorphic of equal torsion length; $\deg_K$ is identified with $\operatorname{card}\Psi = 4k$ as in [1]. Non-K-rational exceptional points require Galois-orbit weighting, but $L \in \mathbb{Z}$, $L \ge 8$ persist.

A3 **Local model** (Assumption 6.1): the hyperelliptic model is compatible with the explicit statements of [1] ($g(Y_1) = 3$, hyperelliptic involution, 4 exceptional points, $A_1 \cdot A_2 = 0$, $A_i^2 = -4k$; the last independently reproduced from $B_i^2 = -(2g_0-2)k$). "All four fibres transverse" would follow from ψ Galois plus equivariance plus exclusion of total degeneration, but equivariance is not stated in [1], so it remains an assumption. Jets are not unique, but $\kappa_M \propto (1, a, a^2)$ holds for every compatible jet.

A4 **External theorems**: Theorem A of [1] is "on a fixed genus-2 curve $\mathcal{C}/K_0$, the height attached to $K_{\mathcal{C}}$ satisfies $h(P) \le (12+\varepsilon)\delta(F)/d + \beta(\varepsilon)$", and Theorem B is $c \le A(\varepsilon)N^{24+\varepsilon}$; §7.2 uses $y^2 + y = x^5$, $F = K(\sqrt[5]{m})$, $P = (\sqrt[5]{m}, y)$ and the intermediate constant 30. Proposition 2, Corollary 1, Prop. 8/9, (18)(22)(24), $|\Psi| = 4k$, $T \ge 8k$, $\delta(F')/d' \le \delta(F)/d + C$ are taken from [1]. **(a)** The statement and scope of Theorems A and B are verified verbatim against the source (Appendix E). **(b)** Independent re-derivation (Appendices G and H). We have re-derived from the standard framework: the normalisation of $\widehat{\deg}$ via the product formula, the pullback weight identity, Lemma 1, Proposition 2 (uniform constant), Corollary 1, (29), the coefficient chain (31)–(36), §7.1 and §7.2. In addition, four further items are standard and are upgraded from conditional to U:

- the rank-2 Minkowski bound (2)/(30): direct consequence of Bombieri–Vaaler [25, Th. 3] and Borek [26, Th. 2]; $\delta(K) = \log|\Delta_K|$ carries no 1/2 (Appendix H.4);

- $\delta(F')/d' \le \delta(F)/d + O(1)$ and $h_L(P') = h(P) + O(1)$: relative discriminant formula (Neukirch [38, Chap. III, §2, Th. 2.11]) and Weil-height stability over finite covers [37, §2.3] (Appendix H.4);
- (33)'s $\overline{\omega}^2/4$ and Prop. 9(vi)'s $-3k$: pure geometric intersection numbers (Appendix H.4).

(c) Corollary 1 of [1] is stated for classes of $\deg_K = 0$. In §6.3.6 of the source it is applied to the positive-degree class $3k'$ and in §6.3.7 to the negative-degree class $-\frac{5}{3}\omega'^2$. Appendix R repairs both uniformly: with $M = \langle \bar{\omega}', \bar{\omega}' \rangle$, the classes $\hat{x} - \frac{1}{4}M$ and $\hat{x} + \frac{5}{3}M$ have degree zero, so Corollary 1 applies directly and (33)/(35) follow with $36, 12, 24, 144$ unchanged. What remains conditional: the Green-function inequalities of Proposition 3 (structure: Zhang [24, Th. 1.1]; the joint "model metric equals admissible metric" assertion is not stated), and the strict equality in (32). The admissible-metric existence has the reference [41, Prop. 2.6]; $\dim H = 2$ and Prop. 9 are recomputed in Appendices B–C. We do not claim an independent verification of the proof of Theorem A. The step $t \to E$ left implicit in §7.3 of [1] is supplied in §7.

A5 **Normalisation**: $\lambda = 3$, $\rho = 5/3$, $m = 6$ follow [1]. γ is strict by Theorem 8.7 ($= 2/5$, $K_{\mathcal{C}}$ fixed), and $\gamma_K(q) = (q-3)/q$ is strict by Theorem 8.8; but the genus and Kummer degree of Theorem A are fixed, so $q = 5$ is the unique applicable value. The two bridge chains $E = 2t$ and $E = \frac{4}{5}t$ (IHÉS lecture [2]) were kept side by side; Corollary 8.9 shows the $\gamma = 1$ convention is not the normalisation of Theorem A. See Remark 8.10 for the normalisation dictionary ($t^* = \frac{5}{2}\, t_A$): the chain-B target $\frac{5}{4}$ and the chain-A target $\frac{1}{2}$ are one condition in two gauges.

A6 **Conservativeness**: Lemma 7.4 uses $\deg D \ge 2g - 1$ to force $h^0 > 0$ (sufficient, not necessary), so $\gamma_0 = 1/n$ and $E_{\max}$ are conservative; $\gamma_K = (q-3)/q$ is a strict equality by Theorem 8.8.

A7 **Scope**: $C : u^q = x(x-1)$ and $F = \mathbb{Q}(\alpha^{1/q})$ are constructed here (compatible with the genus-2 curve and quintic root cover of the literature). If [1] uses a different curve, the formula $E = t\eta/\gamma$ is unchanged and only η, γ need recalibration.

A8 **Audits 18–19**: Appendices E–F record the eighteenth through twenty-second rounds: base verification of the statement and scope of Theorem A, the quality plane ($\theta \le \frac{Y-1}{4}$, $E_{\text{eff}} \ge 12$ under $Y \le 3$), the closed formula $E(c) = \frac{12}{5-18c}$, the double exclusion $T/k = \frac{47}{3}$, and the closure of Route 1; the external references [20, 21, 22, 23] are cited there.

Appendix B. Computations for the geometric layer

B.1. **Adjunction and the two identities.** Equation (3.1) is proved in Theorem 3.1; the fibre, section and multisection identities of Proposition 3.2 follow by the same projection-formula subtractions. For the Frey–Szpiro side: on the Hellegouarch–Frey curve $E_{a,b} : y^2 = x(x-a)(x+b)$ one has $\Delta = 16(abc)^2$, $c_4 = 16(a^2 + ab + b^2)$, and $\omega_{\mathcal{E}/B} = f^*\mathcal{L}$ with $\deg \mathcal{L} = \chi(\mathcal{O}_{\mathcal{E}})$, so every section satisfies $Z \cdot \omega = \deg \mathcal{L} = \chi$, i.e. $Z^2 = -\chi$ (Theorem 3.1); a K3 elliptic surface has $\chi = 2$, $Z^2 = -2$, the true source of $\kappa_{\mathrm{A}2} = 2$. This is also the rigorous refutation of A1 on elliptic surfaces: $h_{\text{rel}}(P) = \chi h_B(\pi(P)) + O(1)$, while $h_{\text{vert}}(P_n) = \hat{h}(nP) = n^2 \hat{h}(P) \to \infty$.

B.2. **The tower of double covers.** The first double cover $Y_1 \to C$ is étale: $2g_1 - 2 = 2(2g_0 - 2)$, $g_1 = 2g_0 - 1$. The second, $X_y \to Y_1$, branches at r points: $2g_f - 2 = 2(2g_1 - 2) + r = 8g_0 - 8 + r$. At $(g_0, r) = (2, 2)$: $g_1 = 3$, $g_f = 6$.

For the ramification divisor write $M^2 = \mathcal{O}(A_1 + A_2)$ and let W be the associated theta characteristic with $\deg W = g_0 - 1 + \frac{r}{2}$. The relative dualising sheaf is $\omega = \sum_{i=1}^{r} B_i + (4g_0 - 4)\phi_1^* W$. Intersecting with B_i and applying (3.1): $-B_i^2 = B_i^2 + (4g_0 - 4)k$, so

$B_i^2 = -(2g_0-2)k$ and $\omega \cdot B_i = (2g_0-2)k$. Squaring:

$$\omega^2 = \sum_i B_i^2 + 2(4g_0-4)\sum_i \phi_1^* W \cdot B_i = -r(2g_0-2)k + 2(4g_0-4)rk = 6rk(g_0-1),$$

using $(\phi_1^* W)^2 = 0$ (pullback from the base). At $(g_0, r) = (2, 2)$: $\omega^2 = -4k + 16k = 12k$.

With $S = \sum_{i=1}^r B_i$: $\omega \cdot S = r(2g_0-2)k$, $S^2 = -r(2g_0-2)k$, so for $L = \omega - bS$,

$$L^2 = 6rk(g_0-1) - 2br(2g_0-2)k - b^2 r(2g_0-2)k = -2rk(g_0-1)(b-1)(b+3),$$

and $\rho = (b-1)(b+3)/3$.

B.3. **The $\mathrm{PSL}_2(\mathbb{F}_7)$ cover and the Parshin cubic.** For $G = \mathrm{PSL}_2(\mathbb{F}_7)$, $|G| = 168$, branch type $(2, 3, 7, 7)$:

$$2g_X - 2 = -2 \cdot 168 + 168\left(\tfrac{1}{2} + \tfrac{2}{3} + \tfrac{6}{7} + \tfrac{6}{7}\right) = -336 + 4 \cdot 121 = 148, \qquad g_X = 75.$$

This is conditional on the existence of a generating tuple of type $(2, 3, 7, 7)$ with product 1; Riemann–Hurwitz alone does not prove existence. For a cubic cover $Y \to X$ with r_3 totally ramified points: $2g_Y - 2 = 3 \cdot 148 + 2r_3$, $g_Y = 223 + r_3$; the value $g_Y = 224$ corresponds to $r_3 = 1$, which the relation $\prod_j [a_j, b_j]\gamma_1 = 1$ in $\pi_1(X)$ with $\gamma_1 \in A_3$ does not contradict, but remains a Parshin configuration to be exhibited.

B.4. **Term-by-term check of the fifteen invariants.** At $(g_0, r, b, k) = (2, 2, 2, k)$:

quantity	value
$g_1 = g(Y_1)$	3
g_f	6
B_i^2	$-2k$
$\omega \cdot B_i$	$2k$
$\phi_1^* W \cdot B_i$	k
ω^2	$12k$
$\omega \cdot S$ $(S = B_1 + B_2)$	$4k$
$\omega \cdot B$ $(B = 2S)$	$8k$
S^2	$-4k$
B^2	$-16k$
$L^2 = \omega^2 - 2\omega \cdot B + B^2$	$-20k$
$\rho = -L^2/\omega^2$	$5/3$
$L \cdot \omega = \omega^2 - \omega \cdot B$	$4k$
$\deg L_F$	6
$\chi(L_F) = \deg L_F - g_f + 1$	1
$h^0(L_F)$ (rank-2 pushforward)	2
$\chi = \deg \mathcal{L} = \omega^2/12$	k
$T \geq \deg(R^1\phi_*(B)[\pi])$	$8k$

B.5. **The margin computation.** The three hard constants are $\rho = \frac{5}{3}$, $m = 6$, $c = \frac{1}{4}$, giving $\Delta = \rho - mc = \frac{5}{3} - \frac{6}{4} = \frac{1}{6}$ and $C = \frac{m}{\Delta} = \frac{6}{1/6} = 36$. Sensitivity: reaching $\frac{15}{4}$ would require $c = \frac{1}{90}$ (a 22.5-fold change); the formal limit $c \to 0$ gives $C \to \frac{18}{5} = 3.6 < 3.75$, signalling a structural lower bound $c \geq \frac{1}{90}$.

For $L = \omega - bS$ in general:

$$L^2(b) = -2rk(g_0-1)(b-1)(b+3), \quad \rho(b) = \tfrac{(b-1)(b+3)}{3}, \quad \mu(b) = 8(g_0-1) + r - br.$$

The margin $\Delta(b) = \rho(b) - \mu(b)c_{\mathrm{GRR}}(b)$ with $c_{\mathrm{GRR}}(b) = \frac{2b^2+2b-1}{12}$. The positivity window $C \geq \frac{15}{4}$ together with $\Delta > 0$ gives $c \in \left[\frac{\rho}{\mu} - \frac{4}{15}, \frac{\rho}{\mu}\right)$, while $c_{\mathrm{GRR}} \sim \frac{b^2}{6}$ grows quadratically, forcing $\frac{T-D_0}{\omega^2} \sim \frac{b^2}{6}$, i.e. $T \sim 4kg_0^3$ with relative precision $\frac{8}{5b^2} \to 0$.

Independence-window computation (exact). From $c = \frac{11}{12} - \frac{T}{12k}$ (Theorem 4.3 at the literature point):

$$C(T) = \frac{36k}{3T - 23k}, \qquad T = 8k \Rightarrow c = \tfrac{1}{4},\ C = 36,\ \text{Theorem A} = 12.$$

The consistency window $C \geq \frac{15}{4}$ is $T \in \left(\frac{23k}{3}, \frac{163k}{15}\right] \approx (7.667k,\ 10.867k]$; full *abc* would need $T = \frac{163k}{15}$, an increase of $\frac{163}{120} \approx 35.8\%$ over the literature lower bound $8k$.

Appendix C. Computations for the torsion and local layers

C.1. **h^0 jumps on Y_1.** On Y_1 of genus 3 with $\deg \omega_{Y_1} = 4$:

sheaf	general h^0	exceptional h^0	criterion
M ($\deg 1$)	0	1	M becomes effective
$M^2 = g^1_2$ ($\deg 2$)	1	2	hyperelliptic g^1_2 degenerates
$\omega_{Y_1} M^{-1}$	$h^1(M) = 1$	2	duality
$\omega_{Y_1} M^{-2}$	$h^1(M^2) = 1$	2	duality
$B = \pi^*(M^2 \oplus M)$	1	3	[1, Prop. 9(iv)]
$L = \omega_{X/Y_1}$	2	4	[1, Prop. 9(v)]

The jump $h^1 = h^0 = 2$ gives the second independent proof of $t = \dim_k R[\pi] = 2$ (the first being the count (24) of [1]).

C.2. **Base change over A/π^2 and $G_{\bar{y}}$.** For $X_j = X \times_A A/\pi^j$, computing $\dim_k H^0(X_2, B_2)$ on both sides of $0 \to B \xrightarrow{\pi^2} B \to B_2 \to 0$ gives

$$G_{\bar{y}} := \dim_k R[\pi^2] = 4 - \operatorname{rank} \partial_1 - \operatorname{rank} \partial_2,$$

with ∂_i the Kodaira–Spencer connecting maps. Hence $T = 8k \iff$ every exceptional fibre has $G_{\bar{y}} = 2 \iff \operatorname{rank} \partial_1 = \operatorname{rank} \partial_2 = 1$. Writing $T = G + H$ with H the π^3-and-higher part separates the higher torsion.

C.3. **Height computations.** For $\alpha = -ab/c^2$ with $\gcd(ab, c) = 1$: $p \mid ab$ gives $\log^+ |\alpha|_p = 0$; $p \mid c$ gives $\log^+ |\alpha|_p = 2v_p(c) \log p$; archimedean places give $\log^+ = 0$ since $|\alpha| < 1$. Hence $\sum_p \log^+ |\alpha|_p = 2 \log c$. For $u^q = \alpha$ and $d = [F : \mathbb{Q}]$: $|u|_w = |\alpha|_p^{1/q}$ and $\sum_{w|p} [F_w : \mathbb{Q}_p] = d$ give

$$h_F(u) = \frac{1}{d} \sum_{w|p} [F_w : \mathbb{Q}_p] \frac{1}{q} \log^+ |\alpha|_p = \frac{2}{q} \log c + O(1).$$

The canonical map $\varphi_K(P) = [u^{g-1} : \cdots : u : 1]$ then yields $h_{K_C}(P) = \frac{q-3}{q} \log c + O(1)$ (Theorem 8.8). The three line bundles:

bundle	height	γ
$\pi^*\mathcal{O}(1)$	$h(\lambda) = \log c$	1
K_C	$(g-1)h_F(u)$	$\frac{q-3}{q} = \gamma_K(q)$
general, $\deg L = \ell$	$\frac{\ell}{q} \log c + o(1)$	$\ell/q + o(1)$

At $q = 5$, $\gamma_K = 2/5 = \gamma_{\ell=2}$: the two rules are indistinguishable at the literature point, which is why γ remained conditional for so long.

Appendix D. Tables of constants and exact recomputations

D.1. **Exact rational recomputation.** All decisive values recompute exactly:

$$\rho(2) = \tfrac{5}{3}, \quad L^2(2) = -20k, \quad \omega^2 = 12k, \quad \deg\det R\phi_*(bS)\big|_{b=2} = -11k,$$
$$c = \tfrac{11}{12} - \tfrac{8k}{12k} = \tfrac{1}{4}, \quad \Delta = \tfrac{5}{3} - \tfrac{6}{4} = \tfrac{1}{6}, \quad C = 36, \quad t = 12, \quad 24 = 12 \cdot 2,$$
$$C(\varepsilon) = \tfrac{36}{1-15\varepsilon} \to 36,$$
$$t(\Theta) = \tfrac{12}{3\Theta-23},\ \Theta \in \{8, 10, 12, 14\},\ N \in \{1, 7, 13, 19\},$$
$$q_A = \tfrac{3N-12}{N-12} = (\tfrac{9}{11}, -\tfrac{9}{5}, 27, \tfrac{45}{7}), \quad q_B = \tfrac{12}{12-N} = (\tfrac{12}{11}, \tfrac{12}{5}, -12, -\tfrac{12}{7}),$$
$$\text{unique odd } q \geq 5 : \ (N, \Theta, q, \gamma) = (13, 12, 27, \tfrac{8}{9}),$$
$$\gamma_{\text{crit}}(5) = (\tfrac{48}{5}, \tfrac{48}{35}, \tfrac{48}{65}, \tfrac{48}{95}) \neq 1, \tfrac{2}{5}, \qquad \tfrac{12}{\gamma} = 30 \Rightarrow \gamma = \tfrac{2}{5}, \quad 30 \cdot \tfrac{4}{5} = 24,$$
$$g = \tfrac{q-1}{2},\ g = 2 \iff q = 5,\ q = 27 \Rightarrow g = 13,$$
$$E_{\max}(q = 5) = \tfrac{48}{5}\lceil\tfrac{8}{\ell}\rceil = (76.8, 38.4, 28.8, 19.2, 9.6),$$
$$E(q) = \tfrac{12(q-1)}{q-3} \to 12, \qquad \min E = \tfrac{24}{19}\ (\Theta = 14),\ \tfrac{48}{35}\ (\Theta = 10).$$

D.2. **The γ_{crit} table.** $\gamma_{\text{crit}} = 12(q-1)/(qN)$:

q	$N = 1$	7	13	19	$\gamma_K(q)$
5	9.600	1.371	0.7385	0.5053	0.400
9	10.667	1.524	0.8205	0.5614	0.667
15	11.200	1.600	0.8615	0.5895	0.800
27	11.556	1.651	**0.8889**	0.6082	**0.8889**
51	11.765	1.681	0.9050	0.6192	0.941
101	11.881	1.697	0.9139	0.6253	0.970

At $q = 27$, $N = 13$ is the unique equality point of γ_K and γ_{crit}. For $\Theta = 12$: $q < 27$ gives $E > 1$ (weak *abc*), $q = 27$ gives $E = 1$, $q > 27$ gives $E < 1$ (excluded). The $\ell = 2$ branch ($\gamma = 2/q$) gives $E = 6(q-1)/N$ with $E = 1 \iff 6(q-1) = N \in \{1, 7, 13, 19\}$, all non-integral, sealing that branch for all q.

Appendix E. Base verification and the quality plane

E.1. **Statement and scope of Theorem A (base verification).** The following is confirmed verbatim against the source text (arXiv:2611.05217v1): **(a)** Theorem A fixes a genus-2 curve $\mathcal{C}/K_0$ and the logarithmic height attached to its canonical bundle; it applies to every number field $F \supseteq K_0$ and every $P \in \mathcal{C}(F)$, with $h(P) \leq (12 + \varepsilon)\delta(F)/d + \beta(\varepsilon)$; Theorem B is $c \leq A(\varepsilon)N^{24+\varepsilon}$. **(b)** The constants $36, 12, 30, 24, \gamma = \frac{2}{5}, \eta = \frac{4}{5}, \frac{5}{3}, \frac{3}{2}, \frac{1}{6}$ each have an explicit location in [1, §6.3 (28)–(35), §7.1, §7.3.2, Prop. 2, Cor. 1]; Proposition 2 and Corollary 1 are unconditional there. Tag: (a) is U at text level. The internal adelic metric layer is now tiered: U (normalisation, pullback weights, Lemma 1, Prop. 2, Cor. 1, (29), coefficient chain — Appendices G and H); C (Prop. 3 Green-function inequalities and the joint "model metric = admissible metric" assertion); O* ((32) strict isometry only). The blanket phrase "independently unverified" is thereby retired.

E.2. **The quality plane.** Define $X = \frac{\log c}{\log N}$ (threshold 1), $Y = \frac{\log(abc)}{\log N}$ (value 3), and $\lambda = Y/X \in [2,3]$ (the quality plane of [22]).

Theorem E.1 (Discount in quality coordinates)**.** *$\theta = \frac{\log M_5}{\log N} \le \frac{Y-1}{4}$, hence*

$$E_{\text{eff}} = 24(1-\theta) \;\ge\; 24\Big(1 - \frac{Y-1}{4}\Big) = 6(5-Y).$$

Under the abc/Szpiro thresholds $Y \le 3$ one gets $\theta \le \frac{1}{2}$ and

$$E_{\text{eff}} \ge 12.$$

The triple $(1, 31, 32)$ attains $\theta = \frac{Y-1}{4} = 0.168$ exactly.

Proof. $E_{\text{mult}} = \log(abc) - \log N = (Y-1)\log N$; each $p \in M_5$ contributes $v_p - 1 \ge 4$, so $\log M_5 \le E_{\text{mult}}/4$ (Theorem 9.3(i)). Substitution gives the claim. For $(1, 31, 32)$: $abc = 2^5 \cdot 31$ and only $p = 2$ has $5 \mid v_p$, so $\theta = \frac{\log 2}{\log 62}$ while $Y - 1 = \frac{4\log 2}{\log 62}$: equality. □

This gives a second, independent route to the barrier 12 — the first being $E(q) = \frac{12(q-1)}{q-3} \to 12$ (Proposition 7.9) — now with a threshold-level interpretation.

E.3. **Comparison with Pasten 2026.** On the Frey–Hellegouarch family, the exponent-24 output of Theorem B strengthens the 2026 bounds of Pasten [20, 21]

Theorem E.2 (Theorem B versus Pasten 2026)**.** *On the Frey–Hellegouarch family attached to a coprime triple $a + b = c$ with $N = \operatorname{rad}(abc)$, Theorem B of [1] gives*

$$\log c \le (24 + \varepsilon)\log N + O_\varepsilon(1),$$

whereas Pasten [21] gives, unconditionally for all elliptic curves over $\mathbb{Q}$,

$$\log \Delta \;\ll\; N_E \log\log N_E,$$

and $\log \Delta \ll N_E$ in the semistable case. On the Frey family $N_E \asymp N$; since $N \gg \log N$, Theorem B is the stronger bound on this family, but it is conditional on Theorem A while [21] is unconditional. The two are of different logical status and different analytic shape (logarithmic versus linear in the conductor).

Proof. Step 1 (Frey curve data). For $E : y^2 = x(x-a)(x+b)$ one has $\Delta = 16(abc)^2$ and conductor $N_E = 2^{O(1)} \prod_{p \mid abc,\, p \ne 2} p$, so $N_E \asymp N$ up to a power of 2 absorbed in the implicit constant (Frey [13]). *Step 2 (translate Pasten).* Substituting $N_E \asymp N$ into $\log \Delta \ll N_E \log\log N_E$ and using $\log \Delta = 2\log(abc) + O(1) \ge 2\log c + O(1)$ gives $\log c \ll N \log\log N$ on this family. *Step 3 (compare).* Theorem B gives $\log c \le (24+\varepsilon)\log N + O_\varepsilon(1)$. For $N \to \infty$, $\log N = o(N \log\log N)$, so Theorem B is stronger. *Step 4 (status).* Theorem B of [1] is derived from Theorem A, whose proof is not independently verified here; [21] is an unconditional theorem. Hence neither result supersedes the other: one is stronger but conditional, the other weaker but unconditional. This replaces the earlier "stated here, not re-proved" parenthetical of Appendix E.3. □

Appendix F. Complete decomposition of the constant; closure of Route 1

F.1. **Five locked factors.**

$$12 = \frac{1}{3} \cdot \frac{m}{\rho - mc} = \frac{1}{3} \cdot \frac{6}{\frac{5}{3} - \frac{6}{4}}.$$

The five factors — $\rho = \frac{5}{3}$ (Proposition 3.5), $m = 6$ (Corollary 3.4), $c = \frac{1}{4}$ (Theorem 4.3), $\frac{1}{3}$ (from $\deg_K(x) = 0$, $\deg_{K_0}(\bar{M}) = 2$), and A (appearing only in $O(d')$) — each have a unique source and are geometrically locked; *none is an estimate margin* (Corollary 8.9 and Remark 8.10 fix the normalisation).

F.2. **Closed formula for the exponent.**

Theorem F.1. *With $c = \frac{11}{12} - \frac{T}{12k}$ as the free parameter,*

$$\Delta(c) = \frac{5}{3} - 6c, \qquad C(c) = \frac{18}{5-18c}, \qquad t_A(c) = \frac{6}{5-18c}, \qquad E(c) = \frac{12}{5-18c}.$$

For $c \geq 0$ (i.e. $T \leq 11k$) one has $E(c) \geq \frac{12}{5} = 2.4$. The old thresholds reappear exactly: $c = \frac{1}{90}$ gives $C = \frac{15}{4}$, $t_A = \frac{5}{4}$, $E = \frac{5}{2}$.

Proof. $\Delta = \rho - mc$; $C = m/\Delta$; $t_A = C/3$; $E = t_A \cdot \eta/\gamma$ with $\eta/\gamma = \frac{4/5}{2/5} = 2$. Substituting $c = \frac{1}{90}$: $5 - 18c = \frac{24}{5}$, giving $C = \frac{90}{24} = \frac{15}{4}$, $t_A = \frac{30}{24} = \frac{5}{4}$, $E = \frac{60}{24} = \frac{5}{2}$. □

F.3. **Sixth No-Go reason: the double exclusion $T/k = 47/3$.**

Theorem F.2. *$E = 1$ is equivalent to $c = -\frac{7}{18}$, hence to $T/k = 11 - 12c = \frac{47}{3}$, which is excluded* both *because $\frac{47}{3} \notin 2\mathbb{Z}$ (Corollary 5.5)* and *because $T/k \leq 15$ (Proposition 5.14). This converges with the value $\Theta = \frac{47}{3}$ of §5 (chain-A target $N = 24$, $3\Theta - 23 = 24$), obtained there by an independent route.*

Proof. $E(c) = 1 \iff 5 - 18c = 12 \iff c = -\frac{7}{18}$; then $T/k = 11 - 12c = 11 + \frac{14}{3} = \frac{47}{3} > 15$. □

F.4. **Route 1 is closed.**

Theorem F.3. *Archimedean and successive-minima refinements cannot move the constant 12. The Ballay toolkit — Chen's conjecture and the equality criterion $\zeta_{\mathrm{ess}} = \hat{\mu}^{\mathrm{asy}}_{\max}$, concave-transform constants with Zhang equality, and absolute transference [23] — acts on the ζ-isomorphism comparison and on the $O(d')$ layer; it does not modify δ, c, ρ, m, or $\frac{3}{2} = m \cdot c$. By Theorem F.1 the exponent is $E(c) = \frac{12}{5-18c}$; these refinements leave c fixed, hence E is fixed: Route 1 is closed.*

Appendix G. Independent re-derivation of the adelic layer

This appendix records the tiered verification of the internal adelic metric estimates of [1]. Method: verbatim reading of the source, cross-checking against the standard literature, and exact rational-number recomputation. We do *not* claim an independent verification of the whole proof of Theorem A.

Theorem G.1 (Normalisation self-consistency)**.** *The definition $\widehat{\deg}_K(\bar{L}) = -\sum_v \log \|s\|_v$ is well-defined only if the weights $n_v(K) = [K_v : \mathbb{Q}_v]/[K : \mathbb{Q}]$ are built into $\log \|f\|_v = n_v(K) \log |f|_v$; then the product formula gives independence of s. Hence two conventions must be kept apart. In [1, §1.3.1] one has $|\pi_v|_w = 1/N(v)$ and $\prod_v |x|_v = 1$; in §1.3.3, $\widehat{\deg}_K(\bar{L}) = -\sum_v \log \|s\|_v$ with no factor $1/[K : \mathbb{Q}]$, the local weight being absorbed in $\|\cdot\|_v$. The factor $1/[K : \mathbb{Q}]$ appears only in the Weil height $h = [K : \mathbb{Q}]^{-1}\widehat{\deg}_K$. With $|f|_v$ denoting the $\mathbb{Q}_p$-normalised absolute value one has $\log \|f\|_v = [K_v : \mathbb{Q}_p] \log |f|_v$; equivalently, in the $n_v(K) = [K_v : \mathbb{Q}_v]/[K : \mathbb{Q}]$ convention, $\sum_{w|v} n_w(F) = n_v(K)$ and $\widehat{\deg}_F(u^*\bar{L}) = [F : K]\widehat{\deg}_K(\bar{L})$. The product formula, pullback invariance and $\delta(K) = \log |\Delta_K|$ (no $1/2$) are unchanged.*

Theorem G.2 (Pullback weight identity)**.** *$\sum_{w|v} n_w(F) = n_v(K)$ for $w \mid v$ in F/K; verified at ramified, split and inert places. Consequently, in the n_v convention, $\widehat{\deg}_F(u^*\bar{L})/[F : \mathbb{Q}] = \widehat{\deg}_K(\bar{L})/[K : \mathbb{Q}]$; in the unnormalised convention of [1, §1.3.3], $\widehat{\deg}_F(u^*\bar{L}) = [F : K]\widehat{\deg}_K(\bar{L})$: pullback invariance, the dimension source of $d = [F : \mathbb{Q}]$ in Lemma 1.*

Theorem G.3 (Uniformity of the $O(d)$ constant)**.** *Lemma 1 provides comparison constants $C_v \geq 1$ equal to 1 on a common model open set, with $A(Y, \hat{x}) := \sum_v \log C_v$ finite and depending only on the two fixed metrics. Pullback scales this by $[F : K]$, giving $\widehat{\deg}_F(u^*\hat{x}) \leq [F : K] \deg_K(x) + [F : K]A$ with A* independent of *F, P, u: Proposition 2's uniformity follows structurally, without the author's metric data.*

Theorem G.4 (Corollary 1 and (29))**.** *Corollary 1 (two ε-perturbations $\hat{x}_\pm = \pm\hat{x} - \varepsilon\bar{M}$, both of negative degree) and (29), $h_a(s_i) \leq h(s_i) + O(d')$ via $\|s\|_{a,w} \leq C_{v(w)}\|s\|_w$ summed against Theorem G.2, re-derive without further input.*

Theorem G.5 (Coefficient chain (31)–(36))**.** *Every entry of the chain recomputes exactly: $3k = \omega^2/4$ (hence $c = \frac{1}{4}$), $\deg_K(\bar{L})/\omega^2 = \frac{1}{3}$, $\frac{3}{2} = m \cdot c$ with $m = 6$ (not a Minkowski constant), $\Delta = \rho - mc = \frac{1}{6}$, $C = 36$, $t_A = 12$, $E = t_A\eta/\gamma = 12 \cdot \frac{4/5}{2/5} = 24$, and $30 \cdot \frac{4}{5} = 24$ (note $30 - \frac{4}{5} \times 30 = 6$ is a mis-transcription).*

Remark G.6 (The gap in (32)). The most substantial finding: (32) asserts that $\alpha : H^{\sup} \to \phi'_*(L')^{\sup}$ "is an isometry" because $\dim H = 2$ "by Grauert's theorem". Grauert's base-change gives a linear isomorphism ($\dim H = 2$ forces surjectivity); *isometry* requires a place-by-place norm comparison $\|s\|_{H^{\sup},w} = \|\alpha(s)\|_{\phi'_*(L')^{\sup},w}$, which the source does not supply. This is a gap of one or two sentences, not an error. Proposition 1 of [1] is proved in its §§1.6.2–1.6.3, and §6.3.5 of the source already gives $H^{\sup}$ a model on the whole of $\mathrm{Spec}(\mathcal{O}_{F'})$ (Appendix J). What (32) still needs is the place-by-place comparison of the two model norms through the Grauert morphism α on the *relative* pushforward; Theorem I.4 supplies the sufficient $O(d')$ form independently, and Proposition I.6 shows the strict equality is not automatic for arbitrary models but is expected under the normalisation used in [1].

layer	status
Normalisation, pullback weights, Lemma 1, Prop. 2 (uniform), Cor. 1, (29), coefficient chain, (2)/(30) Minkowski (Bombieri–Vaaler, Borek), (33) $\omega^2/4$, $\delta(F')/d'$, height stability, (32) up to $O(d')$ (Th. I.4), admissible-metric existence ([41, Prop. 2.6])	U
"model metric = admissible metric" joint assertion; Prop. 3 Green-function data (structure Zhang [24])	C
(32) strict equality as written in [1]	O*

* The models are fixed on the whole of $\mathrm{Spec}(\mathcal{O})$ in [1, §§6.3.4–6.3.5]; Proposition I.6 excludes only the unconditional reading. One sentence of place-by-place comparison is missing — the statement is not false. The (32) up to $O(d')$ row is now U by Theorem I.4 and Appendix I; final ledger: Appendix Z.

Closure with Appendix F. The metric layer contributes only $O(d')$ and $\beta(\varepsilon)$:

$$\underbrace{(29)\ h_a \leq h + O(d')}_{\text{metric comparison}} + \underbrace{(30)\ A\,d'}_{\text{Minkowski}} + \underbrace{(33)\ O_\varepsilon(d')}_{\text{Cor. 1}} \xrightarrow{\S 7.1\ /\ d'} \underbrace{O(1) \to \beta(\varepsilon)}_{\text{error terms only}}.$$

The main chain $12 = \frac{1}{3} \cdot \frac{m}{\rho - mc}$ is determined by $\rho = \frac{5}{3}$, $m = 6$, $c = \frac{1}{4}$, $\deg_{K_0}(\bar{M}) = 2$ alone — internal confirmation, independent of Appendix F's external layer analysis, that Route 1 is closed.

G.1. **Reconciliation with the multi-source cross-verification.** An independent multi-source report cross-checked the normalisation conventions of [1] against the standard literature (Zhang [27], Yuan–Zhang [28], Moriwaki [29], Gaudron [30], Ballay

[23], Bombieri–Vaaler [25], Borek [26], Gillet–Soulé [9], Gillet–Rössler–Soulé [31], Faltings [32]). Its verdicts, insofar as they overlap with this appendix, are settled as follows.

(i) Two over-suspicions refuted. The report's "normalisation conflict" (its §2.4, §6.1, §7.3) is answered by Theorem G.1: well-definedness of $\widehat{\deg}$ plus the product formula *forces* the weights $n_v(K)$ into $\log\|f\|_v = n_v(K)\log|f|_v$, so the source's notation is the normalised degree in shorthand, not an unnormalised conflict. Its "uniform $O(d)$ not independently confirmable" (its §3.2) is answered by Theorem G.3: the constant's *independence* of F, P, u follows from Lemma 1's structure (the constant's numerical value is author data, but uniformity is not). Its remaining conditional layer — the standard status of the rank-2 Minkowski skeleton, the Green-function inequalities, and the (32) isometry — is recorded without change in the U/C/O table above; the (32) isometry is the priority item.

(ii) Adopted recommendation. The report's principal recommendation — an explicit *normalisation dictionary* — is answered by Theorems G.1–G.2 of this appendix, which give one self-consistent convention ($\|f\|_v = |f|_v^{n_v(K)}$, $\sum_{w|v} n_w(F) = n_v(K)$, $\delta(K) = \log|\Delta_K|$ with no 1/2); its conversion to the unnormalised notation of [1, §§1.3.1/1.3.3] is recorded in Proposition J.2(2) (the 1/2 in GRR is the Todd-class coefficient, not an Arakelov-degree normalisation).

(iii) Its 30-item author-data checklist (metrics at each place, explicit C_v bounds, Green-function data, Φ' finiteness, Prop. 9 full proofs, admissible-metric existence) is condensed into the O-layer of the table; we concur that the structure is standard while the constants are author data.

Appendix H. De-conditioning the external dependence

The main text tags Theorems 7.5 and 9.2, and the final No-Go line of Table 1, as `Th.A` (conditional on Theorem A of [1]). This appendix re-derives the external-literature chain and shows that this tag is too coarse: most of the conditionality can be removed, but *not* all of it, and the removable part is not the part that matters.

H.1. **Normalisation and the strength comparison.**

Throughout $N = \mathrm{rad}(abc)$, as in §1.1 (not $\log N$).

Theorem H.1 (Unconditional exponential bound)**.** *For coprime positive $a + b = c$ and $N = \mathrm{rad}(abc)$,*

$$\log c \le \kappa\, N^{1/3}(\log N)^3, \qquad \textit{equivalently} \qquad \log c = O_\varepsilon(N^{1/3+\varepsilon}),$$

with κ effective (Stewart–Yu [33, 34]; the improvement chain $\alpha = 15 \to 2/3 \to 1/3$ is recorded in [20, (1.1)] after Stewart–Tijdeman [35]). Equivalently $c < \exp(C_\varepsilon N^{1/3+\varepsilon})$.

Proof. This is Theorem 1 of [33] and its refinement in [34] (the $\alpha = 1/3$ case), obtained from lower bounds for linear forms in ordinary and p-adic logarithms; the constant is effective. The equivalence of the two forms follows by exponentiating $\log c \le \kappa N^{1/3}(\log N)^3$ and absorbing $(\log N)^3$ into N^ε. □

Theorem H.2 (Polynomial versus exponential type)**.** *Theorem B of [1] is $c \le A(\varepsilon)N^{24+\varepsilon}$, i.e. $\log c \le (24+\varepsilon)\log N + O(1)$. For every sufficiently large N,*

$$24\log N = o\Big(N^{1/3}(\log N)^3\Big),$$

so Theorem B is strictly stronger *than Theorem H.1. More importantly, Theorem H.1 is of* exponential *type, whereas Theorems 7.5 and 9.2 are of* polynomial *type ($c \le N^E$). As recorded verbatim in [20, §1.1]: "all unconditional bounds for these conjectures are of exponential nature". Hence no unconditional Stewart–Yu-type result can replace Theorem A as input to Theorem 7.5.*

Proof. $\lim_{N\to\infty} \frac{24\log N}{N^{1/3}(\log N)^3} = 0$ since $N^{1/3}$ dominates any power of $\log N$. Numerically the inequality $24\log N < N^{1/3}(\log N)^3$ holds for all $N \geq 100$: at $N = 10^2$ the two sides are 110.5 and 4.53×10^2; at $N = 10^{12}$, 663 and 2.11×10^8. The type distinction is definitional: $c \leq \exp(N^\alpha)$ versus $c \leq N^E$. □

Remark H.3 (Correction to an earlier reading). A preliminary audit wrote $N = \log \mathrm{rad}(abc)$ and concluded that Stewart–Yu is stronger than N^{24}. Under the paper's actual convention $N = \mathrm{rad}(abc)$ the comparison is reversed; the preliminary conclusion, and the claim that "the existence of weak *abc* removes the conditionality of Theorem 7.5", are withdrawn.

H.2. **The Theorem A ⇒ *abc* bridge is a literature fact.** Reference [16] of [1] is L. Moret-Bailly, *Hauteurs et classes de Chern sur les surfaces arithmétiques*, Astérisque 183 (1990), 37–58. Its introduction states, verbatim: "au §7, on déduit de ME une version de la "conjecture abc" en appliquant ME à un revêtement ramifié convenable de $\mathbb{P}^1$"; §6 derives Szpiro from ME. The paper proves the implications (BM)⇔(PP), (BM)⇒$(\mathrm{ME})_1$, (ME)⇒Szpiro, (ME)⇒abc. Theorem A of [1] is a proved instance of the (ME)-pattern on a fixed genus-2 curve. Therefore the *bridge* is unconditional in the literature; the conditionality of our results rests solely on Theorem A itself, not on the *abc* reduction.

H.3. **The isometry gap does not move the main coefficient.** Equation (32) of [1] asserts $\widehat{\deg}_{F'}(H^{\sup}) = \widehat{\deg}_{F'}(u'^*(\phi'_*(L')^{\sup}))$, justified by $\dim H = 2$ and Grauert base change. Grauert plus equal dimension gives a *linear* isomorphism; no standard theorem asserts that the sup-norm on $H^0(\phi_*L)$ coincides with the pushed-forward adelic metric (Zhang [24, Th. 1.1], [39]; Gillet–Soulé [9]; Bombieri–Vaaler [25] do not supply this). We record, however, that this gap is harmless for the coefficient chain.

Theorem H.4 ((32) contributes only $O(d')$)**.** *Let $\alpha : H \to u'^*(\phi'_*(L')^{\sup})$ be the linear isomorphism of (32), and suppose the two adelic norms are related by comparison constants $C_w \geq 1$ equal to 1 outside a common model open set (the Lemma 1 mechanism of Appendix G). Then*

$$\left|\widehat{\deg}_{F'}(H^{\sup}) - \widehat{\deg}_{F'}(u'^*(\phi'_*(L')^{\sup}))\right| \leq \sum_w \log C_w = [F' : K'] \sum_v \log C_v = O(d').$$

Consequently (31) becomes $-\bar{L}'^2 \leq -6\widehat{\deg}_{F'}(u'^*(\phi'_*(L')^{\sup})) + 6\delta(F') + 7\,O(d')$, and the chain (34)–(36), §7.1 and §7.2 are unchanged: $\frac{3+\varepsilon}{2}$, $\frac{5}{3} - \frac{3}{2} = \frac{1}{6}$, $C = 36$, $t_A = 12$, $E = 24$. Dividing by d' sends $O(d')$ into $\beta(\varepsilon)$.

Proof. By Theorem G.2 (Appendix G), $\sum_{w|v} n_w(F') = n_v(K')$; hence the pullback of the fixed metric comparison is $[F' : K']$-fold the comparison on K' and is $O(d')$ with a constant independent of F', P', u'. Substituting into (31) only enlarges its existing $O(d')$ term to $7\,O(d')$. Since (33) and Corollary 1 give $-\widehat{\deg}_{F'}(u'^*(\phi'_*(L')^{\sup})) \leq \frac{1+\varepsilon}{4}\bar{\omega}^2 + O(d')$ and $6 \cdot \frac{1+\varepsilon}{4} = \frac{3+\varepsilon}{2}$, equation (34) is unchanged; solving $(\frac{5}{3} - \frac{3}{2})\bar{\omega}^2 \leq 6\delta + O(d')$ gives $\bar{\omega}^2 \leq 36\delta + O(d')$; §7.1 multiplies by $\frac{1}{3}$ and divides by d'; §7.2 multiplies by $\frac{4}{5}$ after the factor $\frac{5}{2}$. No step involves the comparison constant C_w other than inside $O(d')$. A deviation of order $O(d' \cdot \bar{\omega}^2)$ or $O(\delta(F'))$ would be required to move 36; such a deviation cannot arise from a fixed pair of adelic metrics. □

Corollary H.5. *The conditionality attached to (32) can be removed* at the level of the main coefficient 12. *By Theorem I.4 the $O(d')$ form of (32) is unconditional; the comparison constants and the absolute bound A are derived from a finite Čech complex, the ultrametric identity at good places and explicit lattice constants at bad places. What remains open is only the strict equality, i.e. one sentence comparing the two model norms*

place by place through α (Proposition I.6). Theorem H.4 confirms from inside the proof the conclusion of Appendix F: the metric layer contributes only $O(d')$ and $\beta(\varepsilon)$.

H.4. **The C-tier can largely be upgraded to U.**

item in [1]	standard source	new tag
(2)/(30) rank-2 Minkowski	Bombieri–Vaaler [25, Th. 3] + Borek [26, Th. 2]	**U**
Prop. 3, Deligne-pairing metric formula	Zhang [24, Th. 1.1]; [39, §§2–3]	**U** (structure)
two integral lower bounds in Prop. 3	Zhang [39]; need $g \geq -d \log \|s\|_{\sup,v}$	**U** (structure)
"model metric = admissible metric" on L'	not supplied; existence by [41, Prop. 2.6], cf. [28]	**C**
admissible integrable metric existence	[41, Prop. 2.6], cf. [28]	**U** (existence, [41, Prop. 2.6])
(33) $\bar{\omega}^2/4$, Prop. 9(vi) $-3k$, $\omega^2 = 12k$	Riemann–Hurwitz, Noether, Deligne pairing	**U**
$\delta(F')/d' \leq \delta(F)/d + O(1)$	Neukirch, *ANT*, III.§2 Th. 2.11	**U**
$h_L(P') = h(P) + O(1)$	Weil-height stability (Bombieri–Gubler §2.3)	**U**
(32) strict isometry	no standard theorem; needs local norms	**O**

In the adelic normalisation $\|f\|_v = |f|_v^{n_v(K)}$ one has $\delta(K) = \log|\Delta_K|$ *without* a factor $\frac{1}{2}$; the $\frac{1}{2}$ or $1/[K:\mathbb{Q}]$ appearing in Zhang–Gaudron–Ballaÿ belongs to a different normalisation and is not a contradiction. The absolute constant A of (2) is absolute precisely in Soulé's normalisation.

H.5. **The twelve No-Go reasons do not depend on the truth of Theorem A.**

#	reason	uses Th.A truth?	tag
1	hypothesis exclusion (Th. 8.10): $g = 2$, $q = 5$ vs $g = 13$	no (statement form)	U + A5.2
2	$\inf\theta = 0$ (Th. 9.3): $(9, 16, 25)$, $(1, 2^{5k} - 1, 2^{5k})$	no	U
3	tower monotonicity (L. 9.5)	no	U
4	bridge breaking (Th. 10.3): support jump to $c^5 + 4a^5$	no (scale form)	U + A5.2
5	mixed-decomposition loop (Th. 10.10): S-smoothness	no	U
6	47/3 double exclusion (App. F)	no	U
7	$g_0 = 13$ collapse (App. K, Cor. K.5)	no	U
8	absorption decoupling (App. L, Cor. L.7)	no	U
9	distributional vanishing of the discount (App. M, Th. M.3)	no	U
10	type-layer trichotomy (App. N, Th. N.7)	no	U
11	fixed finite S logically impossible (App. O, Th. O.7)	no	U
12	meta-layer exhaustiveness (App. P, Th. P.1)	no	U

Reason 2 needs no "attainability in the construction" caveat: $\theta = \log M_5/\log N$ is defined on the triple, and Theorem 9.2 holds for *all* coprime triples because Theorem A holds for every $P \in \mathcal{C}(F)$.

Proposition H.6 (H.5 — the twelve No-Go reasons do not use the truth of Theorem A)**.** *Each of the twelve reasons listed in Table 1 and Remark 10.13 (the last three appended in Appendices N, O, P) is either purely algebraic/elementary or uses only the* stated hypotheses *of Theorem A. None requires Theorem A to be true. Consequently the Table 1 row "Full abc unattainable in the construction" should be tagged* `U + A5.2`*, not* `Th.A`*.*

Corollary H.7 (Table 1 correction)**.** *The line "Full abc unattainable in the construction | Th.A" is over-tagged and should read* `U + A5.2 (counting convention)`*. This* `U` *means "independent of the* truth *of Theorem A"; it does* not *assert an unconditional proof that abc fails. All No-Go statements remain internal to the construction of [1].*

H.6. **Revised statements.**

Theorem H.8 (7.5′, to be inserted before Theorem 7.5)**.** *By Stewart–Yu [33, 34],* $\log c \le \kappa N^{1/3}(\log N)^3$ *with* $N = \mathrm{rad}(abc)$ *and* κ *effective. This is an* exponential-type *bound and is weaker, for* $N \ge 100$*, than the polynomial-type Theorem B. All unconditional bounds currently known for abc are of exponential nature [20, §1.1]; the existence of a finite polynomial exponent E is open. The purpose of Theorem 7.5 is to produce such an E explicitly, conditional on Theorem A.*

Proof. The first sentence is Theorem H.1. The comparison with Theorem B is Theorem H.2. The assertion that all known unconditional bounds are exponential is verbatim in [20, §1.1]; since $c \le \exp(N^{\alpha})$ grows faster than any N^E, no exponential-type result implies a finite polynomial exponent. The final sentence records the logical role of Theorem 7.5 and is not an existence claim. □

Theorem 7.5 should be renamed "Conditional explicit polynomial bound" with the addendum: "the dependence on Theorem A enters only via $h(P) \le (12+\varepsilon)\delta(F)/d + \beta(\varepsilon)$; the coefficient arithmetic and, by Theorems H.4 and I.4, the metric-layer estimates are unconditional." Theorem 9.2 receives the same tag `Th.A*`.

Remark 2.6 should distinguish four levels: (i) exponential weak *abc* is unconditional (Stewart–Yu); (ii) the Theorem A⇒ *abc* bridge is a literature fact (Moret-Bailly [16], §7); (iii) the conditionality resides in Theorem A itself; (iv) the No-Go results do not depend on the truth of Theorem A. **Open problem (1)** should be split into (1a) the place-by-place norm comparison through α (one sentence; strict equality in (32) only, not the coefficients — Theorem I.4); (1b′) the conversion between the two normalisation conventions of [1, §1.3.1/§1.3.3] (Appendix J); (1c-i) admissible-metric existence: [41, Prop. 2.6]; (1c-ii) Green-function data of Proposition 3: Zhang [24, Th. 1.1] and [39, §§2–3].

Open problem (8) remains of *abc*-level difficulty: it asks for a sequence of pairs (P_i, F_i) with fixed $\mathrm{supp}(\mathfrak{d}_{F_i}) \subseteq S$ and S-smooth values with $N_i \to \infty$; by [20, §1.1] all known unconditional bounds are exponential, so this cannot be obtained by replacing a reference.

H.7. **Bibliographic additions and key mapping.** The numeric keys used above refer to the bibliography of the main file, extended by items [33]–[43] supplied in the accompanying bibliography-additions file: [33] Stewart–Yu (Math. Ann.), [34] Stewart–Yu II (Duke), [35] Stewart–Tijdeman, [36] Moret-Bailly (Astérisque 183 = ref. [16] of [1]), [37] Bombieri–Gubler, [38] Neukirch, [39] Zhang (JAMS 8), [40] van Frankenhuysen, [41] X. Yuan, *Arithmetic bigness and a uniform Bogomolov-type result*, Annals of Mathematics **203** (2026), 15–119. Existing items [20], [21], [24], [25], [26], [28] are Pasten, Zhang *Admissible pairing*,

Bombieri–Vaaler, Borek and Yuan–Zhang respectively. Section H.7 of the audit draft used mnemonic keys (e.g. `SY01`, `BV83`); they are replaced here by the numeric keys.

Scope. This appendix is a text-, literature- and algebra-level re-derivation. It does *not* constitute an independent verification of the proof of Theorem A of [1]. Theorems 7.5 and 9.2 remain conditional on Theorem A; the existence of a polynomial-type weak *abc* bound is open; *abc* itself remains open. All previous conclusions of the paper (the discounted weak bound, the twelve No-Go reasons, $E(c) = 12/(5-18c)$, the 47/3 exclusion, the closure of Route 1, and the BMY-side closure of Theorem Z.4) are unchanged.

Appendix I. Independent derivation of the comparison constants in (32)

We derive, without invoking any functoriality theorem for adelic metrics or Grauert's base-change theorem, the form of (32) that is actually used in the coefficient chain (31)–(36). We emphasize that Proposition 1 of [1] (Gabber) is *proved in the source itself*, in §§1.6.2–1.6.3 by a two-sided inequality (valuative criterion of properness for $\leq$, surjectivity of the specialisation map for $\geq$), and §6.3.5 states that it renders $H^{\sup}$ adelic "with a model on the whole of $\mathrm{Spec}(\mathcal{O}_{F'})$". What Proposition 1 does *not* cover is the compatibility of the Grauert morphism $\alpha : u'^{*}(\phi'_{*}(L')) \to H$ with the sup-norms on the *relative* pushforward; the Čech construction below supplies precisely that layer.

Setup. Let K be a number field, $f : \mathcal{X} \to \mathcal{Y}$ a smooth proper morphism, $\bar{L}$ an adelically metrised line bundle, and $V = f_{*}L$, locally free of rank n on the open set $U \subset \mathcal{Y}$ where h^0 is constant. Choose a finite-type $\mathcal{O}_K$-model $(\tilde{f} : \tilde{\mathcal{X}} \to \tilde{\mathcal{Y}}, \tilde{L})$ and put $\tilde{V} = \tilde{f}_{*}\tilde{L}$. For a point $u : \mathrm{Spec}(F) \to U$ with $F \supseteq K$ we write $d_F = [F : \mathbb{Q}]$.

Lemma I.1 (Model construction from a finite Čech complex)**.** *On an affine open $W = \mathrm{Spec}\, R \subset U$ there is a finite free complex $0 \to C^0(R) \xrightarrow{d^0(R)} C^1(R) \to \cdots$ with $\ker d^0(R) = H^0(\tilde{f}^{-1}(W), \tilde{L})$, and at every point $\bar{y} \in W$ where $\mathrm{rank}(d^0 \otimes k(\bar{y}))$ equals the generic rank one has*

$$\ker\big(d^0 \otimes k(\bar{y})\big) = \big(\ker d^0(R)\big) \otimes k(\bar{y}).$$

Proof. Take a finite affine cover $\mathfrak{U}$ of $\tilde{f}^{-1}(W)$; its Čech complex has finite free terms and computes H^0 by the definition of sheaf cohomology on an affine cover. Over the PID (or localisation) R the matrix $d^0(R)$ admits a Smith normal form $\mathrm{diag}(\varpi^{a_1}, \dots, \varpi^{a_r}, 0, \dots)$ after invertible row and column operations. The rank of $d^0 \otimes k(\bar{y})$ equals the number of nonzero diagonal entries whose a_i vanish at $\bar{y}$; constancy of h^0 forces this rank to equal the generic rank r, hence $a_1 = \cdots = a_r = 0$ and the reduced matrix is $\mathrm{diag}(1, \dots, 1, 0, \dots)$. Its kernel is visibly the reduction of $\ker d^0(R)$. Thus $\tilde{V} \otimes k(\bar{y}) = H^0(\tilde{\mathcal{X}}_{\bar{y}}, \tilde{L}_{\bar{y}})$ with both sides the *same* submodule of the same free module; no unspecified linear isomorphism is involved. □

Lemma I.2 (Exact isometry at good finite places)**.** *After replacing $\tilde{L}$ by a high power so that it is relatively very ample and $\tilde{V}$ defines a closed immersion $\tilde{\mathcal{X}} \hookrightarrow \mathbb{P}(\tilde{V}^{\vee})$, one has for every good finite place v and every $s = (a_1, \dots, a_n) \in H^0(\mathbb{P}^{n-1}_{k_v}, \mathcal{O}(1))$*

$$\|s\|_{\mathrm{mod},v} = \max_i |a_i|_v = \sup_{x \in k_v^n \setminus 0} \frac{|\sum_i a_i x_i|_v}{\max_j |x_j|_v} = \|s\|_{\sup,\bar{y}}.$$

Proof. The first equality is the definition of the norm induced by the $\mathcal{O}_v$-lattice $\mathcal{O}_v^n$ (Lemma I.1). For the second, the ultrametric inequality gives $|\sum_i a_i x_i|_v \leq \max_i |a_i|_v |x_i|_v \leq (\max_i |a_i|_v)(\max_j |x_j|_v)$, so the supremum is at most $\max_i |a_i|_v$; taking $x = e_{i_0}$ with $|a_{i_0}|_v = \max_i |a_i|_v$ attains it. Hence equality holds *pointwise*, with no $O(1)$ loss. In particular good places contribute exactly 0 to $\sum_w \log C_w$. □

Lemma I.3 (Explicit comparison constants and absolute bound)**.** *Let S_{bad} be the finite set of places consisting of the non-flat places of the model, the ramified places, the places where h^0 jumps, the archimedean places, and the finitely many v with* $\operatorname{char} k_v \mid n!$*. Then for every $v \in S_{\mathrm{bad}}$ there is an explicit $C_v \geq 1$, depending only on $(\tilde{f}, \tilde{L}, \bar{L}_\infty)$, with $\big|\log\|s\|_{1,v} - \log\|s\|_{2,v}\big| \leq \log C_v$ for all $s \neq 0$, and*

$$\sum_{w \in M_F} \log C_w \;\leq\; A\, d_F, \qquad A = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in S_{\mathrm{bad}}} n_v(K) \log C_v.$$

Proof. Finite bad places. Both norms come from two $\mathcal{O}_v$-lattices Λ_1, Λ_2 in $V \otimes K_v$. Torsion-free finite generation over a DVR gives an integer $m_v \geq 0$ with $\varpi_v^{m_v}\Lambda_1 \subseteq \Lambda_2 \subseteq \varpi_v^{-m_v}\Lambda_1$, so $\log C_v = m_v \log|\varpi_v|_v^{-1}$ works and is explicit; no compactness argument is needed. The set S_{bad} is finite because the h^0-jump locus is closed and $\operatorname{Spec}\mathcal{O}_K$ is one-dimensional.

Archimedean places. Let H_1, H_2 be the two positive Hermitian matrices and λ_j the eigenvalues of $H_1^{-1}H_2$; then $\log C_\infty = \frac{1}{2}\max_j|\log\lambda_j|$ is finite and explicit.

Weights. For $w \mid v$ write e_w, f_w for the ramification index and residue degree of F_w/K_v, so $\sum_{w|v} e_w f_w = [F:K]$. Since $n_w(F) = e_w f_w [K_v:\mathbb{Q}_v]/[F:\mathbb{Q}]$,

$$\sum_{w|v} n_w(F) = \frac{[K_v:\mathbb{Q}_v]}{[F:\mathbb{Q}]}[F:K] = \frac{[K_v:\mathbb{Q}_v]}{[K:\mathbb{Q}]} = n_v(K).$$

By Lemma I.2 the good places contribute 0; hence $\sum_w \log C_w = [F:K]\sum_{v\in S_{\mathrm{bad}}} n_v(K)\log C_v = A\, d_F$ with A as displayed. The constant A depends only on the fixed model and on K, not on F, u or P. □

Theorem I.4 ((32) in the sufficient form)**.** *For every $u : \operatorname{Spec}(F) \to U$,*

$$\Big|\widehat{\deg}_F(u^*\tilde{V}) - \widehat{\deg}\big(H^0(\mathcal{X}_u, L_u)^{\sup}\big)\Big| \;\leq\; A\, d_F,$$

with A of Lemma I.3. Consequently (31) acquires the additional error $(1+A)\,O(d')$, which is absorbed into $\beta(\varepsilon)$ by §7.1 after division by d'. The chain $\rho - mc = \frac{5}{3} - \frac{3}{2} = \frac{1}{6}$, $C = 36$, $t_A = 12$, $E = 24$ and $E(c) = 12/(5-18c)$ are unchanged.

Proof. Lemmas I.1–I.3 give the displayed bound. Substituting into (31) and using (33) with Corollary 1 gives $\frac{5}{3}\overline{\omega}^2 \leq \frac{3+\varepsilon}{2}\overline{\omega}^2 + 6\delta(F') + (1+A)O(d')$; dividing by d' in §7.1 sends the last term into $\beta(\varepsilon)$. Solving $(\frac{5}{3} - \frac{3}{2})\overline{\omega}^2 = \frac{1}{6}\overline{\omega}^2 \leq 6\delta(F') + O(d')$ yields $\overline{\omega}^2 \leq 36\delta(F') + O(d')$, then $t_A = 36/3 = 12$ and $E = 12 \cdot (4/5)/(2/5) = 24$. Finally $E(c) = 12/(5-18c)$ follows from $c = 11/12 - T/(12k)$ and $\Delta(c) = \frac{5}{3} - 6c$ as in Appendix F; at $c = 1/4$ it gives 24 and at $c = 1/90$ it gives $5/2$. □

Corollary I.5 (Upgrade of Appendix H, Theorem H.4)**.** *Theorem H.4 was conditional on the existence of Lemma-1-type comparison constants. Theorem I.4 supplies that hypothesis unconditionally; the H.3–H.4 discussion becomes unconditional, and the row "(32) isometry $\to O(d')$ only" in Table 1 must be tagged* U*.*

The strict isometry is not automatic. We record that the sentence "α is compatible with sup-norms. Therefore α is an isometry" of §6.3.6 of [1] cannot hold without a normalisation of the model.

Proposition I.6 (Counterexample to strict isometry)**.** *Let $K_v = \mathbb{Q}_5$, $V = \mathbb{Q}_5^2$, $\Lambda_1 = \mathcal{O}_v e_1 \oplus \mathcal{O}_v e_2$ and $\Lambda_2 = \mathcal{O}_v(\varpi e_1) \oplus \mathcal{O}_v e_2$. The two lattices define the same generic normed space. Fix the fiberwise-sup metric to be the one induced by Λ_1. Then for $s = e_1$,*

$$\|s\|_{\Lambda_1} = 1, \qquad \|s\|_{\Lambda_2} = |\varpi^{-1}|_v = 5, \qquad \log C_v = \log 5 \neq 0.$$

Hence $C_v = 1$ is not forced by the linear isomorphism α; it is equivalent to the normalisation *that the model of $\tilde{V}$ is exactly the lattice induced by $\tilde{f}_*\tilde{L}$, together with the Fubini–Study-type normalisation of the archimedean metric.*

Proof. Both Λ_i are $\mathcal{O}_v$-lattices in the same two-dimensional K_v-vector space, so they give two admissible adelic norms on one generic space; a change of integral model of $\tilde{V}$ replaces Λ_1 by any commensurable lattice, e.g. Λ_2. With the sup metric fixed by Λ_1, the lattice norm of e_1 in Λ_2 is $\inf\{|a|_v^{-1} : ae_1 \in \Lambda_2\} = |\varpi|_v^{-1} = 5$. Thus α is an isometry up to $O(d')$ but not strictly; the finite-place obstruction is a ϖ-twist of the model, and the archimedean obstruction is a rescaling of the Hermitian form. □

Corollary I.7 (Ledger correction)**.** *Appendix G's `O`-layer must be split into two rows:*

item	*tag*
(32) isometry up to $O(d')$ (Th. I.4)	`U`
(32) strict equality as written in [1]	`O*`

Here `O*` means the following. Proposition I.6 shows that the strict isometry is not automatic for arbitrary *models. In [1], however, §6.3.4 fixes the model of L' on the whole of* $\mathrm{Spec}(\mathcal{O}_K)$ *and §6.3.5 (via Proposition 1) fixes the model of H^{sup} on the whole of* $\mathrm{Spec}(\mathcal{O}_{F'})$*, while α is the natural base-change morphism of the cartesian square; under this normalisation the strict equality is expected to hold. What is missing is one sentence comparing the two model norms place by place through α — the counterexample excludes the unconditional reading, not this construction. This returns to the original verdict "a gap of one or two sentences, not an error". Either tag leaves every coefficient, every No-Go conclusion and the weak abc output unchanged. Open problem (1a) should therefore be rewritten as "supply the place-by-place comparison through α" rather than "locate a functoriality proposition in the literature".*

Appendix J. Verbatim check of [1]

We record the result of reading arXiv:2611.05217v1 (39 pages) verbatim. This is a *text-level* verification, not an independent verification of the proof of Theorem A.

Theorem J.1 (Twelve verbatim agreements)**.** *The following twelve items of the source agree, line by line, with the statements used in this paper:*

1. *Theorem A/B: N is "the product of the prime numbers dividing abc", i.e. $N = \mathrm{rad}(abc)$,* not $\log\mathrm{rad}$*; h is the canonical-bundle height of a fixed genus-2 curve.*
2. *"known to follow from Theorem A, albeit with a wrong exponent": the Theorem A $\to$ abc bridge is a literature fact (Theorem H.2).*
3. *§1.3.3: $\widehat{\deg}_K(\bar{L}) = -\sum_v \log\|s\|_v$, "independent of the choice of s" by the product formula.*
4. *§1.5 Lemma 1: $C_v \geq 1$, $C_v = 1$ for almost all v, proof via compactness of X_v^{an} plus coherence.*
5. *§1.6: adelic vector spaces are induced by a projective module V_U at almost all places, with $\|s\|_v = 1$ almost everywhere.*
6. *§6.3.4: "$C_v \geq 1$,* independent of F, P, F' and P'*, $C_v = 1$ for almost all v" $\Rightarrow$ (29).*
7. *§1.9 (2): $h(s_1) + h(s_2) \leq -\widehat{\deg}_K(\bar{M}) + \delta(K) + d_K A$, with $\delta(K) = \log|\Delta_K|$ and no factor $1/2$; "(see [4] Th. 3 and [5] Th. 2)" = Bombieri–Vaaler + Borek.*
8. *§2 Proposition 3: $-\bar{L}_1 \cdot \bar{L}_2 \leq d_2 h(s_1) + d_1 h(s_2)$, proved by (3)–(6); finite-place metric from [27] p.177, archimedean from [7] (6.3.2) and [26] (4.1.1).*

(9) §6.3.6–6.3.8*: (33)* $\le \frac{1+\varepsilon}{4}\overline{\omega}'^2 + O(d')$*; (34)* $(3+\varepsilon)\frac{\overline{\omega}^2}{2} + 6\delta + O(d')$*; (35)* $L'^2 = -20k' = -\frac{5}{3}\overline{\omega}'^2$*; (36)* $\frac{5}{3}\overline{\omega}^2 \le \frac{3+\varepsilon}{2}\overline{\omega}^2 + 6\delta + O(d')$*, giving* 36, 12, 24*;* Φ' *is a finite set.*

(10) §7.1.2*:* $\hat{x} = \mathcal{L} - \frac{1}{3}\langle\overline{\omega},\overline{\omega}\rangle$*,* $\overline{\omega}^2 = 6\deg(Y/Y_0)$*,* $\deg_K(\mathcal{L}) = 2\deg(Y/Y_0) \Rightarrow \deg_K(x) = 0 \Rightarrow$ *(40) with the factor* 1/3*, hence* $12 = 36/3$.

(11) §7.3.2*: "since* $\deg_{K_0}(\mathcal{L})/2 \times 12 = 30$*"* $(\deg\mathcal{L} = 5) \Rightarrow$ *(49); (51)* $\delta(F)/[F:\mathbb{Q}] \le \frac{4}{5}\log N + \delta(K)/[K:\mathbb{Q}] + \log 5$*, "Since* $n = 1$ *or* 5*"* $\Rightarrow \eta = 4/5$ *and* $\gamma = 2/5$.

(12) *The first sentence of the proof of Theorem 5 reads "Theorem 6 follows from the uniform abc conjecture"; the paper has only five theorems (harmless misprint).*

Proposition J.2 (Three corrections to the audit)**.** *(1)* Proposition 1 (Gabber) is proved inside [1]. §§1.6.2–1.6.3 *give a two-sided inequality: the valuative criterion of properness for* $\|\cdot\|_{\sup,v} \le \|\cdot\|_{m,v}$*, and surjectivity of the specialisation map (after multiplying by* π^n *so that the section does not vanish on the special fibre) for the reverse inequality. Moreover* §6.3.5 *states that Proposition 1 makes* $H^{\sup}$ *adelic "with a model on the whole of* $\mathrm{Spec}(\mathcal{O}_{F'})$*". Consequently the gap in (32) is* not *the adelicity of the single-fibre sup norm; it is the compatibility of the Grauert morphism* α *with the sup norm on the* relative *pushforward* $\phi'_*(L')$.

(2) Normalisation. §1.3.1 *uses* $|\pi_v|_w = 1/N(v)$ *and* $\prod_v |x|_v = 1$*, i.e. unnormalised absolute values* $|f|_v = |N_{K_v/\mathbb{Q}_p}(f)|_p$*;* §1.3.3 *defines* $\widehat{\deg}_K$ *without dividing by* $[K:\mathbb{Q}]$*. The factor* $1/[K:\mathbb{Q}]$ *occurs only in the Weil height. Theorems G.1–G.2 remain valid once the two conventions are explicitly related.*

(3) Admissible metric existence. §2.1.2 *of [1] reads "It is shown in [25], Prop. 2.6, that any line bundle on X has an* ***admissible*** *metric"; [25] of [1] is X. Yuan,* Arithmetic bigness and a uniform Bogomolov-type result*, Annals of Mathematics* ***203*** *(2026), 15–119 (= our [41]),* not *Zhang [24] (= [27] of [1], the finite-place Green formula on p.177) and* not *[39].*

Corollary J.3 (Bibliographic key mapping)**.** *The keys of [1] and of this paper differ. [9] of [1] is R. Hartshorne,* Algebraic Geometry*, GTM 52 (our [42]);* §6.3.6 *cites III, Remark 9.3.1 and III, Cor. 12.9 for Grauert base change, not EGA. [23] of [1] is Serre, IHÉS 54 (1981), 323–401 (our [43]); it is used for (51). [26] of [1] is Yuan–Zhang,* Adelic line bundles over quasi-projective varieties*, Annals of Mathematics Studies, Princeton University Press, 2026 (our [28]). [28] of [1] is Zhang,* Small points and adelic metrics*, JAG 4 (1995), 281–300 (our [27]).*

Appendix K. Higher-genus recalibration of the coefficient chain

This appendix closes Open problem (2) by recomputing the general Kodaira–Parshin coefficient chain at $g_0 = 13$. We use the invariants of Appendix B.2 throughout:

(K.1)
$$W = \omega^2/k = 6r(g_0-1), \quad \mu = \deg L_F = 8g_0-8+r(1-b), \quad \rho = \frac{(b-1)(b+3)}{3}, \quad c_{\mathrm{GRR}} = \frac{2b^2+2b-1}{12}.$$

Here $W > 0$. We restrict to the admissible range of Proposition 3.5, $h^0(L_F) \ge 2$ (the rank condition for the Minkowski step of the chain), i.e. $br \le 4g_0 - 5 + \frac{r}{2}$; for $g_0 = 13$ this reads $r(2b-1) \le 94$ and forces $\mu = 96 - r(b-1) \ge 49 > 0$. The chain is not of the construction's type outside this range. Writing $c = c_{\mathrm{GRR}} - \Theta/W$ with $\Theta = T/k$ (setting $D_0 = 0$, the most favourable normalisation), one has

(K.2)
$$\frac{1}{t} = \frac{\lambda}{\mu W}\big(\mu\Theta + B_0\big), \qquad B_0 = W\big(\rho - \mu c_{\mathrm{GRR}}\big), \qquad t = \frac{\mu W}{\lambda(\mu\Theta + B_0)}.$$

Theorem K.1 (Literature point reproduced)**.** *For* $(g_0, r, b, \lambda) = (2, 2, 2, 3)$ *one has* $W = 12$*,* $\mu = 6$*,* $\rho = 5/3$*,* $c_{\mathrm{GRR}} = 11/12$*,* $B_0 = -46$*, and* (K.2) *becomes* $t(\Theta) = \dfrac{12}{3\Theta - 23}$*, i.e.*

$t(8) = 12$, $t(10) = 12/7$, $t(12) = 12/13$, $t(14) = 12/19$. *In particular* $t \le 12/13$ *requires* $\Theta \ge 12$, *the unique candidate of §8.*

Proof. Substitute (K.1) into (K.2): $\mu W = 72$, $\lambda(\mu\Theta + B_0) = 3(6\Theta - 46) = 18\Theta - 138$, hence $t = 72/(18\Theta - 138) = 12/(3\Theta - 23)$. The four values are direct rational substitutions; $12/(3\Theta - 23) \le 12/13$ with positive denominator is $3\Theta - 23 \ge 13$, i.e. $\Theta \ge 12$. □

Theorem K.2 ($g_0 = 13$: two most favourable readings)**.** *Let* $\Theta \in 2\mathbb{Z}$ *with* $8 \le \Theta \le 15$ *(the range imposed by exponent-*1 *optimality, §5). Then:*

1. *Along the rank-*2 *locus* $b = 2g_0 - 2 = 24$ *with* $(r, \lambda) = (2, 3)$*:* $W = 144$, $\mu = 50$, $\rho = 207$, $c_{\mathrm{GRR}} = 1199/12$, $B_0 = -689592$, *and* $\rho - \mu c < 0$ *for every allowed* Θ, *so* t *has no positive value. Positivity requires* $\Theta > \frac{344796}{25} = 13791.84$, *and* $t \le 12/13$ *requires* $\Theta \ge \frac{346096}{25} = 13843.84$ *(smallest even* $\Theta = 13844$*).*
2. *Keeping the literature value* $b = 2$ *(so the rank of* L_F *grows) with the same* (r, λ)*:* $W = 144$, $\mu = 94$, $B_0 = -12168$*; positivity requires* $\Theta > \frac{6084}{47} = 129.45$ *and* $t \le 12/13$ *requires* $\Theta \ge \frac{8528}{47} = 181.45$ *(smallest even* $\Theta = 182$*).*

Letting $\lambda \to \infty$ *does not help: the limits are* 13791.84 *and* 129.45 *respectively (smallest even* $\Theta = 13844$ *and* 130*).*

Proof. For (1), $\mu W = 7200$ and $\mu\Theta + B_0 = 50\Theta - 689592$; at $\Theta \le 15$ this is negative, so $t < 0$. The positivity threshold is $-B_0/\mu = 689592/50 = 344796/25$. Solving $\mu W/(\lambda(\mu\Theta + B_0)) \le 12/13$ gives By (K.2), $t \le 12/13$ is equivalent to $\mu\Theta + B_0 \ge 13\mu W/(12\lambda)$, i.e. $\Theta \ge 13W/(12\lambda) - B_0/\mu$. For (1), $13W/(12\lambda) = 13 \cdot 144/36 = 52$ and $-B_0/\mu = 344796/25$, so $\Theta \ge 52 + 344796/25 = 346096/25$. For (2), $W = 144$, $\mu = 94$, $B_0 = -12168$, $-B_0/\mu = 6084/47$, and the same formula gives $\Theta \ge 52 + 6084/47 = 8528/47$; the intermediate constant is $13\mu W/(12\lambda) = 13 \cdot 144 \cdot 94/36 = 4888$, and $(4888 + 12168)/94 = 17056/94 = 8528/47$, consistent. Both exceed the allowed range. As $\lambda \to \infty$ the extra term $13W/(12\lambda) \to 0$, leaving the positivity threshold as the limit. □

Theorem K.3 (Uniform lower bound, all admissible (r, b, λ))**.** *For* $g_0 = 13$, *every admissible* (r, b) *with* $r \ge 1$, $b \ge 2$, $br \le 4g_0 - 5 + \frac{r}{2}$, *and every* $\lambda > 0$,

$$\Theta_{\min} = -\frac{B_0}{\mu} = W\Big(c_{\mathrm{GRR}} - \frac{\rho}{\mu}\Big) \;\ge\; \frac{1230}{19} = 64.736\ldots \;>\; 15. \tag{K.3}$$

The minimum is attained at $(r, b) = (1, 2)$. *Hence no admissible* Θ *produces a positive* t, *and a fortiori none satisfies* $t \le 12/13$.

Proof. With the corrected $\mu = 96 - r(b-1)$ of (K.1) and $W = 72r$,

$$F(r, b) := \Theta_{\min} = 72r\,\frac{2b^2 + 2b - 1}{12} - \frac{24r(b-1)(b+3)}{96 - r(b-1)}.$$

The admissibility constraint $r(2b-1) \le 94$ gives $1 \le r \le r_{\max}(b) := \lfloor 94/(2b-1) \rfloor$ (so $b \le 47$) and $96 - r(b-1) \ge 96 - \frac{94-r}{2} > 0$.

Endpoint reduction. For fixed b, the second summand of F is a negative multiple of $h(r) = r/(96 - r(b-1))$, and $h''(r) = \dfrac{192(b-1)^2}{(96 - r(b-1))^3} > 0$; hence F is concave in r and its minimum over $r \in [1, r_{\max}(b)]$ is attained at an endpoint.

First endpoint. $F(1, b) = 72c_{\mathrm{GRR}}(b) - \dfrac{24(b-1)(b+3)}{97 - b}$ has derivative

$$24b + 12 - \frac{24\big(-b^2 + 194b + 191\big)}{(97 - b)^2}.$$

The subtracted term is increasing on $[2,47]$ (its derivative has numerator $19200 > 0$), equals $1.53\ldots$ at $b=2$, and is therefore < 69 throughout; since $24b+12 \geq 60$, with room > 15 at $b=2$ and $84 > 69$ for $b \geq 3$, the derivative is positive on $[2,47]$. Thus

$$F(1,b) \geq F(1,2) = 72 \cdot \tfrac{11}{12} - \tfrac{24\cdot 1\cdot 5}{95} = 66 - \frac{24}{19} = \frac{1230}{19} = 64.736\ldots.$$

Second endpoint. At $r = r_{\max}(b)$, exact rational evaluation for $2 \leq b \leq 31$ gives $F \geq F(31,2) = 2232\big(\frac{11}{12} - \frac{1}{39}\big) = \dfrac{25854}{13} \approx 1988.8 > \dfrac{1230}{19}$; for $b \geq 32$ one has $r_{\max}(b) = 1$ and the first endpoint applies.

Combining both endpoints, $F(r,b) \geq \frac{1230}{19} > 15$ for every admissible (r,b). □

Theorem K.4 (Only $g_0 = 2$ starts the chain)**.** *Along the rank-2 locus $b = 2g_0 - 2$, $r = 2$, $\lambda = 3$, the positivity threshold $-B_0/\mu$ equals $23/3 \approx 7.67$ at $g_0 = 2$ and exceeds 15 for every $g_0 \geq 3$: $306/5 = 61.2$ ($g_0 = 3$), $1473/7 \approx 210.4$, $4532/9 \approx 503.6$, …, $344796/25 = 13791.84$ ($g_0 = 13$), $473603/27 \approx 17540.9$ ($g_0 = 14$).*

Proof. Direct substitution of (K.1) into $-B_0/\mu$ for $g_0 = 2,\ldots,14$ by exact rational arithmetic; the values are monotone increasing because $c_{\mathrm{GRR}} \sim b^2/6 \sim O(g_0^2)$ while $\rho/\mu = O(g_0)$. The structural reason is that a positive $\rho - \mu c$ requires the torsion contribution T/ω^2 to cancel a quadratic GRR term, whereas $T = k\Theta$ with $\Theta = O(1)$ in the first-order degeneration model. □

Corollary K.5 (Seventh No-Go reason; Open problem (2) closed)**.** *Even under the most favourable assumption that the coefficient chain (K.2) extrapolates unchanged to $g_0 = 13$, the chain cannot start: $\rho - \mu c < 0$ for every allowed Θ, so no positive t exists. This reason is independent of the six previously recorded ones, of the normalisation $\gamma = 2/5$, of the modulus-3 argument, of the transversality hypothesis, and of the truth of Theorem A. Consequently Open problem (2) is closed: the statement "$t = 12$ does not extend to $g = 13$" is upgraded from a proof-structure remark to a computation.*

Proof. Combine Theorems K.2, K.3; none invokes Theorem A. The independence is as in Proposition H.5. □

Remark K.6 (Honesty boundary). Formula (K.2) is unconditional (Appendix B.2 plus the DRR decomposition of §4). The assumption that its *form* persists at $g_0 = 13$ is the most favourable extrapolation; because even granting it the chain fails, the conclusion is robust. The bound $\Theta \leq 15$ is inherited from exponent-1 optimality; without it the required $\Theta \sim 1.4 \times 10^4$ is incompatible with a first-order degeneration at four exceptional fibres. All weak-*abc* outputs remain conditional on Theorem A.

Appendix L. Quantitative index-ideal absorption and the engine-replacement criterion

L.1. **Absorption inequalities.** Let $f \in \mathbb{Z}[x]$ be monic irreducible of degree $n \geq 2$, let θ be a root and $F = \mathbb{Q}(\theta)$, so $[F:\mathbb{Q}] = n$. Put $\mathfrak{I} = [\mathcal{O}_F : \mathbb{Z}[\theta]] \in \mathbb{Z}_{\geq 1}$ and $\delta(F) = \log|\Delta_F|$. Dedekind's theorem (10.8) reads $\operatorname{disc}(f) = \mathfrak{I}^2\,\mathfrak{d}_F$; taking logarithms gives the identity

$$\log|\operatorname{disc} f| = \delta(F) + 2\log\mathfrak{I}. \tag{L.1}$$

For a prime $p \nmid n$ let r_p be the number of primes of F above p. The tame discriminant formula gives

$$v_p(\mathfrak{d}_F) = \sum_i f_i(e_i - 1) = n - r_p, \qquad 1 \leq r_p \leq n. \tag{L.2}$$

Theorem L.1 (Local absorption-rate inequality)**.** *For every $p \nmid n$ put $\iota_p = 2v_p(\mathfrak{I})/v_p(\operatorname{disc} f)$. Then*

$$1 - \frac{n-1}{v_p(\operatorname{disc} f)} \;\le\; \iota_p = 1 - \frac{n - r_p}{v_p(\operatorname{disc} f)} \;\le\; 1, \tag{L.3}$$

and $\iota_p = 1$ if and only if $r_p = n$, i.e. p is unramified. Globally,

$$0 \le \delta(F) \le \log|\operatorname{disc} f|, \qquad \delta(F) \le (n-1)\log N + O_n(1). \tag{L.4}$$

Proof. Subtracting (L.2) from (L.1) place by place gives $2v_p(\mathfrak{I}) = v_p(\operatorname{disc} f) - (n - r_p)$. Since $\mathfrak{I}$ is a positive integer, $2v_p(\mathfrak{I}) \ge 0$; since $r_p \ge 1$ we have $n - r_p \le n - 1$, which yields the lower bound in (L.3); the upper bound is $n - r_p \ge 0$. The equivalence $\iota_p = 1 \Leftrightarrow n - r_p = 0$ is immediate. The first inequality of (L.4) is $\log \mathfrak{I} \ge 0$; the second follows from $v_p(\mathfrak{d}_F) = n - r_p \le n - 1$ for $p \nmid n$ together with the fact that the finitely many wild primes $p \mid n$ contribute $O_n(1)$ uniformly, because $v_p(\mathfrak{d}_F) \le n\, v_p(n) + n$ for tame–wild decompositions of degree n. No choice of a generator enters. □

Remark L.2. Inequality (L.3) is the quantitative upgrade of the remark following (10.10): the larger the multiplicity $v_p(\operatorname{disc} f)$, the closer ι_p is to 1, i.e. the *field* discriminant is asymptotically blind to the multiplicity. For $n = 5$, $v_p(\operatorname{disc} f) = 4k$ gives $\iota_p = 0, \frac{1}{2}, \frac{2}{3}, 1, 1, 1$ at $k = 1, 2, 3, 5, 10, 25$, against the lower bounds $0, \frac{1}{2}, \frac{2}{3}, \frac{4}{5}, \frac{9}{10}, \frac{24}{25}$ (verified by exact rational arithmetic).

L.2. **The unified decomposition-type discount.**

Theorem L.3 (Unified formula for η)**.** *With $\theta_{\mathrm{dec}} := \dfrac{\sum_{p \in S}(r_p - 1)\log p}{(n-1)\log N} \in [0, 1]$ one has*

$$\eta = \frac{\delta(F)}{n \log N} = \frac{n-1}{n}\big(1 - \theta_{\mathrm{dec}}\big) + O_n\big(1/\log N\big). \tag{L.5}$$

This single formula reproduces the non-Galois lattice $\{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$ of §10.2 at $n = 5$, the M_5-discount of §9, the two-valuedness of Corollary 10.8, and the quality-plane bound $\theta \le (Y - 1)/4$ of Appendix E.

Proof. Summing (L.2) over $p \in S$, $p \nmid n$, and dividing by $n \log N$ gives $\delta(F)/(n \log N) = \frac{n-1}{n} - \frac{1}{n \log N}\sum_p (r_p - 1)\log p + O_n(1/\log N)$, which is (L.5).

For $r_p = 1, \dots, n$ at $n = 5$ the right-hand side of (L.5) equals $\frac{4}{5}(1 - \frac{r_p - 1}{4}) = \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0$, the lattice of Theorem 10.6. In the pure Kummer case $r_p \in \{1, n\}$, so $\sum_p (r_p - 1)\log p = (n-1)\log M_n$ with $M_n = \prod_{p:\, n \mid m_p} p$; hence $\theta_{\mathrm{dec}} = \log M_n / \log N$ and (L.5) becomes $\delta(F)/n = \frac{n-1}{n}\log(N/M_n)$, Corollary 10.8 and §9. Situation II of §9.2 has $r_p = n$ for both the split and the unramified-degree-n sub-cases, so η is unchanged, as proved there.

Finally, in the pure Kummer class $r_p = n$ forces $n \mid m_p$, hence $m_p \ge n$ and $r_p - 1 = n - 1 \le m_p - 1 \le v_p(abc) - 1$. Therefore $\theta_{\mathrm{dec}} \le \mathcal{E}/((n-1)\log N) = (Y - 1)/(n - 1)$ with $\mathcal{E} = \log(abc) - \log N = (Y - 1)\log N$. At $n = 5$ this is $\theta \le (Y - 1)/4$, and $E = 30\eta = 24(1 - \theta_{\mathrm{dec}}) \ge 24(1 - (Y - 1)/4)$, giving $E_{\mathrm{eff}} \ge 12$ when $Y \le 3$, the quality-plane conclusion of Theorem E.1. The constant $n - 1$ is thus the maximal inertia order $v_p(\mathfrak{d}_F) = n - r_p \le n - 1$, not an estimate margin; it is optimal because $r_p = n$ occurs whenever $v_p(abc) \ge n$. □

Corollary L.4 (Exact numerical check)**.** *For $(a, b, c) = (1, 31, 32)$ one has $N = 62$, $Y = 1.6718$, $\theta_{\mathrm{dec}} = \log M_5/\log N = 0.1679$ and $E_{\mathrm{eff}} = 24(1 - \theta) = 19.969 = 24(1 - (Y - 1)/4)$; for $(9, 16, 25)$ and $(1, 8, 9)$ the discount vanishes and $E_{\mathrm{eff}} = 24$. These agree with the tables of §9.3 and Appendix E.*

Proof. Direct substitution into (L.5) with $\alpha = -ab/c^2$: for $(1, 31, 32)$, $\alpha = -31/2^{10}$, so $m_2 = 10$ with $5 \mid 10$ (étale, $r_2 = 5$) and $m_{31} = 1$ (total, $r_{31} = 1$); hence $\sum_p (r_p - 1)\log p = 4\log 2$ and $\theta_{\mathrm{dec}} = 4\log 2/(4\log 62) = \log 2/\log 62 = 0.1679$. The other two triples have no étale prime, so $\theta_{\mathrm{dec}} = 0$. Exact rational arithmetic in the accompanying computation file confirms all six entries of §10.3's k-table and the five r_p-values of Theorem 10.6. □

L.3. **Absorbed mass and the engine-replacement criterion.**

Theorem L.5 (Quantitative absorbed mass)**.** *Assume the $|k|$-convention of (10.9)–(10.10), i.e. $v_p(\operatorname{disc} f) = (n-1)m_p + nv_p(n)$ with $m_p = |v_p(\alpha)|$ for $\alpha = -ab/c^2$. Then*

$$2\log \mathfrak{I} = (n-1)\big[\mathcal{E} + \log c + \log M_n\big] + O_n(1), \qquad \log \mathfrak{I} \geq \frac{n-1}{2}\big(\mathcal{E} + \log c\big) - O_n(1), \tag{L.6}$$

and $\delta(F)/n = \frac{n-1}{n}\big(\log N - \log M_n\big) + O_n(1)$ is independent of $\mathcal{E}$.

Proof. Sum (10.10) over $p \in S$, $p \nmid n$. For p with $n \nmid m_p$ the contribution is $(n-1)(m_p - 1)$; for p with $n \mid m_p$ it is $(n-1)m_p$. Hence $2\log \mathfrak{I} = (n-1)\big[\sum_{p\in S} m_p \log p - \sum_{p:\, n\nmid m_p} \log p\big] + O_n(1)$. Because $\gcd(a, b) = 1$ and $\gcd(ab, c) = 1$, $\sum_{p\in S} m_p \log p = \log(ab) + 2\log c = \mathcal{E} + \log N + \log c$, while $\sum_{p:\, n\nmid m_p} \log p = \log N - \log M_n$. Subtracting gives (L.6). The stated lower bound uses $\log M_n \geq 0$. Solving (L.1) for $\delta(F)$ gives the last formula, in which $\mathcal{E}$ cancels. □

Theorem L.6 (Engine-replacement criterion)**.** *Let an "engine replacement" mean a choice of f for which $\delta(F)/n$ is controlled by $\log N$ while $h_{K_C}(P) = \gamma \log c + O(1)$ with $\gamma \asymp 1$. A necessary condition is $\mathfrak{I} \leq N^{O(1)}$. By (L.6) this forces*

$$c \leq N^{\,2O(1)/(n-1)}, \qquad \textit{equivalently } Y - 1 \leq O(1), \tag{L.7}$$

which is an abc-grade condition with exact slope $2/(n-1)$. Consequently the S-smoothness loop of Theorem 10.10 is not merely "stronger than abc": it is of abc grade at this quantitative level. This is a one-directional implication; the converse is not claimed.

Proof. By (L.1) and the homogeneity of $\operatorname{disc} f$ of degree $2n-2$ in coefficients of size $c^{O(1)}$ (Theorem 10.10(i)), $\frac{\delta(F)}{n} + \frac{2\log \mathfrak{I}}{n} = \frac{\log|\operatorname{disc} f|}{n} \leq \frac{2n-2}{n}\log c + O(1)$. If $\log \mathfrak{I} \asymp \log c$, the bridge $\delta(F)/n \asymp \log N$ breaks; hence $\mathfrak{I} \leq N^{O(1)}$ is necessary. Insert $\log \mathfrak{I} \geq \frac{n-1}{2}(\mathcal{E} + \log c) - O_n(1)$ from (L.6): $\frac{n-1}{2}\log c \leq O(1)\log N + O_n(1)$, i.e. (L.7). Writing $\mathcal{E} = (Y-1)\log N$, the same computation gives $Y - 1 \leq O(1)$, so $E = 24(1 - \theta_{\mathrm{dec}}) \geq 24(1 - O(1)/(n-1))$ cannot approach 1 unless $Y \to 1$, which is $c \leq N^{1+o(1)}$, the *abc* conjecture. Theorem 10.10's magnitude comparison c^{2n-2} vs. c^3 is thereby supplied with an exact slope. □

Corollary L.7 (Eighth No-Go reason)**.** *The decoupling of η from the multiplicity excess $\mathcal{E}$ holds at the level of the identity (L.1), not at the level of estimates. No engine can make η decrease with $\mathcal{E}$, because by (L.6) the index ideal absorbs at least $\frac{n-1}{2}\mathcal{E}$ to $O(1)$ precision. This reason is independent of the six previously recorded reasons, of the $g_0 = 13$ collapse of §11, of the modulus-3 argument and of the transversality hypothesis.*

Proof. Combine Theorems L.3, L.5 and Corollary 5.5 / Proposition H.5: the argument uses only (L.1), (L.2) and $\mathfrak{I} \geq 1$, none of which invokes the truth of Theorem A or the torsion decomposition. □

Remark L.8 (Honesty boundaries). Theorems L.1, L.3 and Corollary L.7 are unconditional: they use $\mathfrak{I} \geq 1$ and the tame formula only, with no generator convention. Theorem L.5 and the exact slope $2/(n-1)$ use the $|k|$-convention of (10.9)–(10.10); for $p \mid c$

the source writes $k = v_p(\alpha) < 0$ in (10.9) but $|k|$ in (10.10), a sign convention left implicit. Without it, (L.4) still gives the direction of (L.7) with an unspecified $O(1)$ slope. The implication "$\mathfrak{I} \le N^{O(1)}$ is necessary" is a scale criterion, not a sufficiency claim; the converse direction is not proved. All weak-*abc* outputs remain conditional on Theorem A; the No-Go statements are internal to the construction.

Appendix M. Distributional behaviour of the multiplicity excess and the discount

Throughout this appendix $\Omega(X)$ denotes the set of coprime triples

$$\Omega(X) = \{(a,b,c) : 1 \le a < b < c \le X,\ a+b=c,\ \gcd(a,b)=1\}, \tag{M.1}$$

endowed with the uniform probability measure $\mathbb{P}_X$, and $T(X) = |\Omega(X)|$. Put

$$N = \operatorname{rad}(abc), \quad \mathcal{E} = \log(abc) - \log N = \sum_p (e_p - 1)\log p, \quad Y = 1 + \frac{\mathcal{E}}{\log N}, \tag{M.2}$$

with $e_p = v_p(abc)$, and let

$$M_5 = \prod_{p:\,5 \mid e_p} p, \qquad \theta = \frac{\log M_5}{\log N} \tag{M.3}$$

be the discount of §9.3. Since $\gcd(ab, c) = 1$, each prime p divides exactly one of ab and c; with $\alpha = -ab/c^2$ one therefore has $v_p(\alpha) = e_p$ or $v_p(\alpha) = -2e_p$. As $\gcd(2,5) = 1$,

$$5 \mid v_p(\alpha) \iff 5 \mid e_p, \qquad \text{hence } M_5 = \prod_{p:\,5 \mid e_p} p. \tag{M.4}$$

In particular $5 \mid e_p$ forces $e_p \ge 5$, so

$$\log M_5 = \sum_{p:\,5 \mid e_p} \log p \le \frac{1}{4} \sum_p (e_p - 1)\log p = \frac{\mathcal{E}}{4}. \tag{M.5}$$

Inequality (M.5) welds the M_5-discount of §9.3 to the quality-plane bound $\theta \le (Y-1)/4$ of Appendix E at the level of prime valuations, with no Arakelov input.

Lemma M.1 (Size of the sample space)**.** $T(X) = \frac{1}{2}\sum_{c \le X} \varphi(c) + O(X) = \frac{3}{2\pi^2} X^2 + O(X \log X) \asymp X^2$.

Proof. For fixed c, the condition $\gcd(a, c-a) = 1$ is $\gcd(a,c) = 1$, giving $\varphi(c)$ admissible values of a; the constraint $a < b$ halves them, up to the single case $a = b = c/2$ (which occurs only for $c = 2$). The classical estimate $\sum_{c \le X} \varphi(c) = \frac{3}{\pi^2} X^2 + O(X \log X)$ finishes the proof. □

Theorem M.2 (A radical lower bound from Stewart–Yu)**.** *There is an effective absolute constant $C = C(\kappa)$ such that for every coprime triple with $c \ge 3$,*

$$\log N \ \ge\ 3 \log\log c - 9 \log\log\log c - C. \tag{M.6}$$

Proof. Stewart–Yu [33, 34] gives $\log c \le \kappa N^{1/3} (\log N)^3$ with κ effective; taking logarithms, $\log\log c \le \log\kappa + \frac{L}{3} + 3\log L$ where $L = \log N$. Set $L_0 = 3\log\log c - 9\log\log\log c - C$. If $L \le L_0$ then, for c large, $L_0 \le 3\log\log c$ and hence $\log L \le \log 3 + \log\log\log c$, so

$$\begin{aligned} \frac{L}{3} + 3\log L + \log\kappa &\le \log\log c - 3\log\log\log c - \frac{C}{3} + 3\log 3 + 3\log\log\log c + \log\kappa \\ &= \log\log c - \frac{C}{3} + 3\log 3 + \log\kappa. \end{aligned}$$

Choosing $C > 3(3\log 3 + \log\kappa)$ makes the right-hand side strictly smaller than $\log\log c$, contradicting Stewart–Yu. Hence $L > L_0$ for all large c; the finitely many remaining c are

absorbed by enlarging C. This is an unconditional corollary of an effective theorem; no Diophantine conjecture enters. □

Theorem M.3 (Exponential moment of the multiplicity excess)**.** *For every $t < 1$ there is $C_t < \infty$, independent of X, with*

$$\mathbb{E}_X\left[e^{t\mathcal{E}}\right] \le C_t, \qquad \mathbb{E}_X[\mathcal{E}] = O(1), \qquad \mathbb{P}_X(\mathcal{E} \ge K) \le C_t\, e^{-tK}. \tag{M.7}$$

Proof. **Multiplicative CRT bound.** Fix a finite set S of primes and exponents $v_p \ge 2$. Because $\gcd(a,b) = \gcd(a,c) = \gcd(b,c) = 1$, each p^{v_p} divides exactly one of a, b, c; there are $3^{|S|}$ such allocations. For a fixed allocation (A_1, A_2, A_3) of S put $d_a = \prod_{p\in A_1} p^{v_p}$, $d_b = \prod_{p\in A_2} p^{v_p}$, $d_c = \prod_{p\in A_3} p^{v_p}$. The conditions are $d_a \mid a$, $d_b \mid (c-a)$, $d_c \mid c$. The three moduli are pairwise coprime, so the Chinese remainder theorem selects a single residue class of (a,c) modulo $d_a d_b d_c$; consequently

$$\mathbb{P}_X\big(\forall p \in S:\ p^{v_p} \mid abc\big) \ \le\ C_0^{|S|} \prod_{p\in S} p^{-v_p}, \tag{M.8}$$

where the absolute constant C_0 absorbs the condition $\gcd(a,c) = 1$ and the boundary of $\Omega(X)$. Crucially $3^{|S|} = \prod_{p\in S} 3$ is absorbed into $C_0^{|S|}$ *per prime*, so the bound is multiplicative.

Expansion. For each prime, $p^{t(e_p-1)} \le 1 + \sum_{j\ge 2} \mathbf{1}_{p^j\mid abc}\, p^{t(j-1)}$: if $e_p = 1$ the left side is 1, while if $e_p \ge 2$ the sum contains its last term $p^{t(e_p-1)}$. Expanding the product $e^{t\mathcal{E}} = \prod_p p^{t(e_p-1)}$ over all choices of a finite set S and exponents $\boldsymbol{v} = (v_p)_{p\in S}$ with $v_p \ge 2$, and applying (M.8), gives

$$\begin{aligned}\mathbb{E}_X[e^{t\mathcal{E}}] &\le \sum_{S,\boldsymbol{v}} \Big(C_0^{|S|} \prod_{p\in S} p^{-v_p}\Big)\Big(\prod_{p\in S} p^{t(v_p-1)}\Big) = \prod_p \Big[1 + C_0\, p^{-t} \sum_{v\ge 2} p^{-(1-t)v}\Big] \\ &\le \prod_p \big[1 + C'_{0,t}\, p^{-(2-t)}\big] \ =:\ C_t.\end{aligned}$$

The product converges precisely when $2 - t > 1$, i.e. $t < 1$. The first moment bound is $t \to 0^+$ (or Jensen plus $\mathcal{E} \le e^{t\mathcal{E}}/t$), and Chernoff's inequality gives the tail bound. □

Remark M.4 (Novelty boundary). Moment estimates for rad and for the number of high-power prime divisors are standard in the abc literature (de Bruijn; Lichtman, arXiv:2505.13991, 2025, exceptional set $O(N^{2/3})$; BBLT, arXiv:2410.12234, power saving at fixed quality). The attribution "Bombieri/van der Poorten type" in an earlier draft is withdrawn. Theorem M.3 is not a new number-theoretic tool; what is new is its combination with the index-ideal/discount dictionary of Appendix L. Appendix Q re-proves the density-one statement from scratch with a self-contained first-moment estimate.

Theorem M.5 (Zero density of the discount, with a large-deviation rate)**.** *For every fixed $\lambda > 0$ and every $t < 1$, as $X \to \infty$,*

$$\mathbb{P}_X(\theta \ge \lambda) \ \le\ C_{t,\lambda}\,(\log X)^{-6\lambda t} + O\big((\log X)^{-c}\big) \ \longrightarrow\ 0. \tag{M.9}$$

Proof. Decompose $\Omega(X)$ dyadically according to $c \in (X/2^{k+1}, X/2^k]$. By Lemma M.1 the k-th layer carries weight $\asymp 4^{-k}$. On the k-th layer $\log c = \log X - k\log 2 + O(1)$, hence $\log\log c = \log\log X + O(k/\log X) = \log\log X + o(1)$ for $k \le \frac{1}{2}\log\log X$. Theorem M.2 therefore gives $\log N \ge 3\log\log X - 9\log\log\log X - O(1) \ge \frac{3}{2}(1+o(1))\log\log X$ on every such layer; the term $-k\log 2$ of an earlier draft is withdrawn (Appendix Q, Remark Q.10). By (M.5), $\theta \ge \lambda$ forces $\mathcal{E} \ge 4\lambda\log N \ge 6\lambda(1+o(1))\log\log X$. Theorem M.3 with Chernoff bounds the layer probability by $C_t(\log X)^{-6\lambda t}$. Layers with $k > \frac{1}{2}\log\log X$ have total weight $O((\log X)^{-c})$, which is absorbed. Summing over k proves (M.9). □

Theorem M.6 (Average quality tends to one)**.**

$$\mathbb{E}_X[Y-1] = O\Big(\frac{1}{\log\log X}\Big), \qquad \mathbb{E}_X[\theta] = O\Big(\frac{1}{\log\log X}\Big). \tag{M.10}$$

Proof. Write $Y-1 = \mathcal{E}/\log N$. On the k-th dyadic layer, Theorem M.2 and Theorem M.3 give $\mathbb{E}[\mathcal{E}/\log N] = O\big(1/(\log\log X - k\log 2)\big)$. For $k \le \frac{1}{2}\log\log X$ the denominator is $\ge \frac{1}{2}\log\log X$; summing against the geometric weights 4^{-k} yields $O(1/\log\log X)$. The tail $k > \frac{1}{2}\log\log X$ is $O((\log X)^{-c})$ since $Y-1 \le 3\log c/\log N = O(\log X)$. The bound for θ follows from $\theta \le \mathcal{E}/(4\log N)$. □

Theorem M.7 (The absorption is saturated on average)**.** *Under the $|k|$-convention of (10.9)–(10.10), the index ideal satisfies* $\log \mathfrak{I} \ge 2(\mathcal{E} + \log c) - O(1)$*, whence*

$$\mathbb{E}_X\big[\log \mathfrak{I}\big] \;\ge\; 2\log X - O(1). \tag{M.11}$$

Proof. By the absorbed-mass inequality of Appendix L (equivalently (21.4) at $n = 5$), $\log \mathfrak{I} \ge 2(\mathcal{E} + \log c) - O(1)$. On the top dyadic layer $\log c = \log X + O(1)$; that layer has probability bounded below, and on the remaining layers $\log c \ge 0$. By Theorem M.3, $\mathbb{E}[\mathcal{E}] = O(1)$. Hence $\mathbb{E}[\log \mathfrak{I}] \ge 2\log X - O(1)$. □

Remark M.8 (Layer split). The unconditional weak bound $\log \mathfrak{I} \ge 2\mathcal{E}$ (pure valuation argument, no $|k|$-convention) gives only $\mathbb{E}[\log \mathfrak{I}] = O(1)$. The growth $\asymp \log X$ in Theorem M.7 is tagged A, exactly as in Remark L.7. All preceding theorems are unconditional.

Theorem M.9 (The discount does not improve the average exponent)**.** *Assume Theorem A of [1]. Then the discounted bound $c \le A_\varepsilon(N/M_5)^{24+\varepsilon}$ of Theorem 9.2 has average effective exponent*

$$\mathbb{E}_X[E_{\mathrm{eff}}] = 24\big(1 - \mathbb{E}_X[\theta]\big) = 24 - O\Big(\frac{1}{\log\log X}\Big) \;\longrightarrow\; 24, \tag{M.12}$$

and $E_{\mathrm{eff}} = 24 - o(1)$ on a set of triples of density $1 - o(1)$.

Proof. Taking logarithms in Theorem 9.2 gives $\log c \le 24(1-\theta)\log N + O(1)$, so $E_{\mathrm{eff}} = 24(1-\theta)$. Insert $\mathbb{E}_X[\theta] = O(1/\log\log X)$ from Theorem M.6; the density statement follows from Theorem M.5. The conditionality is confined to Theorem A; Theorems M.2–M.7 are unconditional. □

Corollary M.10 (Ninth No-Go reason, distributional)**.** *Let an engine improve the exponent solely through the decomposition-type discount θ (equivalently through M_5 or $\mathcal{E}$). Its improvement set $\mathcal{G}_\lambda(X) = \{\theta \ge \lambda\}$ obeys Theorem M.5, so the engine produces no average-type improvement; moreover*

$$\mathbb{P}_X(Y \ge 1+\delta) \;\le\; C_t\,(\log X)^{-3t\delta/2} \;\longrightarrow\; 0 \tag{M.13}$$

for every fixed $\delta > 0$ and $t < 1$. Hence the average gain of any discount-type engine is zero, independently of the truth of Theorem A.

Proof. $Y \ge 1+\delta$ forces $\mathcal{E} \ge \delta\log N \ge 3\delta(1+o(1))\log\log X$ by Theorem M.2; apply Chernoff in Theorem M.3 with exponent $t/2$ on the top dyadic layer and sum geometrically. The absence of an average gain is Theorem M.9 combined with Theorem M.6. The argument uses neither Theorem A nor any of the eight previously recorded No-Go reasons. □

Remark M.11 (What this does and does not say). Theorems M.2–M.6, Corollary M.10 and the weak form of Theorem M.7 are unconditional: they use only Stewart–Yu and elementary CRT/moment estimates. They do *not* prove abc, do not produce a polynomial-type weak abc bound, and do not conflict with the exponential/polynomial type distinction

of Theorem H.2. The constants $C(\kappa)$ and C_t are proved to exist but are not evaluated numerically. The numerical check reported in the accompanying computation file covers $X \le 2000$ (608 294 triples) and is a consistency check, not an asymptotic proof. All previous conclusions of the paper are unchanged.

Reading guide to Appendices N–Z (rounds 23–34)

These thirteen appendices are the continuation of the audit after Appendix M. They form an *evolution chain*: several numerical statements proved early in the chain are later corrected. The final status is always given by Theorem Z.4 and Remark Z.5. The corrections are, in order:

(1) Theorem T.3/T.4 (Appendix T) treated the torsion T as coming from a single $\mathcal{L}$-side jump. Appendix U shows that $\rho_* B = 2\mathcal{L} \oplus \mathcal{L}$ has *two* components, each jumping by one, so $T = a + a_2 + 11k$; the formula $T = a + 3k$ is withdrawn.
(2) Theorem W.2 (Appendix W) gave $\deg(\psi_2^*\Theta) = 3k/4$ under the implicit assumption $\psi_1 = a$ (Abel–Jacobi). Theorem X.1 proves $\psi_1 = \pm(\mathrm{id} - \nu^*) \circ a$ is a Prym map, and Theorem X.3 gives $\deg(\psi_2^*\Theta) = 4k$; the "deficit $k/4$" is withdrawn. Proposition W.3's exponent $m = 3$ is withdrawn in Remark Y.5.
(3) Theorem X.5 (Appendix X) closed the BMY branch conditionally (tag C). Appendix Y closes the last loophole (the 2-torsion ambiguity), and Appendix Z removes the remaining conditions: the closure is U and independent of d, m and h.
(4) Remark Q.5 (Appendix Q) withdraws the term $-k \log 2$ in the dyadic decomposition of Theorem M.5 and downgrades the rate $(\log X)^{-3t\varepsilon}$ to $O(1/\log\log X)$; the conclusion of density-one *abc* is unchanged.

No coefficient, no No-Go reason and no weak-*abc* output changes in any of these corrections. All `superseded` statements are retained because each is a strictly proved lemma about a narrower model; this is an audit paper and the chain of corrections is part of the result.

Appendix N. Feasibility of changing the external inequality type

This appendix examines Route 2 of the concluding discussion: keep the Kodaira–Parshin geometry fixed and replace the *type* of external inequality used to derive $\bar\omega^2 \le C\,\delta(F')/d' + O(1)$. We show that the coefficient $C = 36$ is invariant under all such replacements, and that the conditionality can only be translated, not removed.

N.1. **The inequality template.** Rewrite the chain (29)–(36) of [1] in the abstract form

(N.1)
$$\underbrace{\rho\,\bar\omega^2}_{A} \le \underbrace{B\,\bar\omega^2}_{\text{pushforward bound}} + \underbrace{M\,\delta(F')/d'}_{\text{discriminant term}} + O(1), \qquad C = \frac{M}{A-B}, \quad t_A = \frac{C}{3}, \quad E = t_A\frac{\eta}{\gamma} = 2t_A.$$

At the literature point $A = \rho = 5/3$, $B = mc = 6 \cdot \frac14 = \frac32$, $M = m = \deg L_F = 6$, so $C = 6/(5/3 - 3/2) = 36$, $t_A = 12$, $E = 24$.

Lemma N.1 (Provenance of the three factors)**.** *$\rho = -L^2/\omega^2$ is a geometric intersection number; $M = m = \deg L_F$ is the fibre degree entering Proposition 3 and the rank-2 lattice; the factor $1/3$ comes from the normalisation $\hat x = \bar L - \frac13\langle\bar\omega,\bar\omega\rangle$ with $\deg_K(\hat x) = 0$. Consequently only*

(N.2)
$$c_{\mathrm{eff}} = \frac{-\widehat{\deg}_{F'}\big(u'^*\phi'_*(L')\big)}{\bar\omega^2}$$

can in principle be affected by a change of inequality type.

Proof. The three quantities are read off from Appendix B.2, Proposition 3 and §7.1.2 of [1] respectively; none involves an estimate. In (31) the coefficient of $\delta(F')$ is $\deg L_F = m$, and in (33) the pushforward bound is multiplied by m to give $B = mc$. The normalisation $\deg_K(\hat{x}) = 0$ forces the factor $1/3$ in $t_A = C/3$. Hence $C = M/(\rho - Mc_{\mathrm{eff}})$ depends on the estimate only through c_{eff}. □

N.2. **A DRR identity for all parameters and all tensor powers.** Let $W = \omega^2/k = 6r(g_0 - 1)$ and $u = (\omega \cdot S)/k = r(2g_0 - 2)$, so that

$$W = 3u \quad \text{for every } (g_0, r). \tag{N.3}$$

With $S^2 = -\omega \cdot S = -uk$ and $L = \omega - bS$ one has $L \cdot \omega = (W - bu)k$ and $L^2 = (W - 2bu - b^2u)k$.

Theorem N.2 (Deligne–Riemann–Roch, general form)**.** *For every (g_0, r, b) and every $n \geq 1$,*

$$\begin{aligned} \deg\det R\phi_*(nL) &= k\Big[\frac{n^2}{2}(W - 2bu - b^2u) - \frac{n}{2}(W - bu) + \frac{W}{12}\Big] \\ &= -uk\Big[\frac{n^2}{2}(b-1)(b+3) + \frac{n}{2}(3-b) - \frac{1}{4}\Big] = -\frac{\rho}{2}\,n^2\omega^2 + O(n). \end{aligned} \tag{N.4}$$

In particular at $n = 1$ this reproduces $-\frac{W}{12}(2b^2 + 2b - 1)k$, the DRR term of §4, identically in (g_0, r, b); at the literature point $(u, W, b) = (4, 12, 2)$ it gives $-10n^2k - 2nk + k$, whose value at $n = 1$ is $-11k$.

Proof. The Deligne–Riemann–Roch formula $c_1(\det R\phi_*M) = \phi_*\big(\frac{M^2}{2} - \frac{M\omega}{2} + \frac{\omega^2}{12}\big)$ is applied with $M = nL$; ϕ_* converts surface classes to numbers, giving the first line. Substituting $W = 3u$ and using $3 - 2b - b^2 = -(b-1)(b+3)$ gives the second. At $n = 1$ the bracket is $\frac{1}{2}(2b^2 + 2b - 3) + \frac{1}{2}(3 - b) - \frac{1}{4} = \frac{1}{4}(2b^2 + 2b - 1)$, and $-\frac{u}{4} = -\frac{W}{12}$. The n^2 coefficient is $-\frac{u}{2}(b-1)(b+3) = -\frac{3u}{6}(b-1)(b+3) = -\frac{\rho W}{2}$. All identities were checked by exact rational arithmetic. □

Remark N.3. Theorem N.2 upgrades the single value $-11k$ of §4 to an identity valid for all parameters and all tensor powers; it is the structural reason why c_{eff} is algebraic: the quadratic GRR term is $-\frac{\rho}{2}n^2\omega^2$ for every n.

N.3. **Classification of replacements.**

Theorem N.4 (Template invariance)**.** *Keeping the geometry and the definition $\delta(F) = \log|\Delta_F|$ fixed, one has*

$$c_{\mathrm{eff}} = \frac{11}{12} - \frac{T}{\omega^2} + o(1), \qquad T = \deg\big(R^1\phi_*(B)_{\mathrm{tor}}\big), \tag{N.5}$$

independently of the inequality type used in (29), (30) and (33). Consequently $C = 36$, $t_A = 12$ and $E = 24$ are invariant under:

1. *replacing Bombieri–Vaaler/Borek successive minima by Ballaÿ's equality criterion $\zeta_{\mathrm{ess}} = \widehat{\mu}^{\mathrm{asy}}_{\max}$ (only the absolute constant A of (30) changes, and it enters $+Ad' \to \beta(\varepsilon)$);*
2. *replacing Corollary 1 (the zero-degree-class Roth-type bound) by Zhang's small-point or arithmetic Bogomolov inequalities (only the ε-margin of (33) changes).*

Proof. By §4, $\deg\phi_*(L') = \deg\det R\phi_*(L') - \deg\phi_*(B) + T = -11k - 0 + T$, and $\omega^2 = 12k$, hence $c_{\mathrm{eff}} = \frac{11}{12} - \frac{T}{12k}$. This is an algebraic identity in the determinant of cohomology and the torsion sheaf; the adelic comparison of Appendix G and Corollary I.4 adds only $O(d')/d' = O(1)$. Neither (R1) nor (R2) changes $\deg\phi_*(L')$ or T, so c_{eff} is invariant. The absolute constant A of (30) is divided by d' in §7.1 and absorbed into $\beta(\varepsilon)$; the ε of (33) tends to 0 without altering the leading coefficient. Hence $C = m/(\rho - mc_{\mathrm{eff}}) = 36$. □

Theorem N.5 (Hilbert–Samuel dilution)**.** *Replace the rank-2 successive-minima step by the arithmetic Hilbert–Samuel asymptotics of $L^{\otimes n}$ as $n \to \infty$, with $r_n = h^0(nL_F) \sim n\mu$, $\mu = \deg L_F = 6$. Using the first two successive minima, $h(s_1) + h(s_2) \le \frac{2}{r_n}\big(-\widehat{\deg} E_n + \frac{r_n}{2}\delta(F')\big)$ with $E_n = \phi_*(L^{\otimes n})$, and Proposition 3 in the form $-\overline{(nL)}^2 \le n\mu\,(h(s_1)+h(s_2))$, division by $n^2 d'$ gives:*

term	*after division by* n^2	$n \to \infty$
discriminant $\delta(F')$	$\mu\,\delta(F')/n$	0 *(diluted)*
finite-place DRR (Th. N.2)	$O(1/n)$	0
archimedean $\frac{n^3}{6}\,\mathrm{vol}(\bar{L}')$	$\frac{n}{3}\,\mathrm{vol}(\bar{L}')$	∞ *(dominant)*

Hence the conclusion $\bar{\omega}^2 \le C\delta(F')/d' + O(1)$ disappears and is replaced by a pure metric inequality $\bar{\omega}^2 \le C'\,\mathrm{vol}(\bar{L}')$.

Proof. The finite-place contribution to $\widehat{\deg} E_n$ is $-\frac{\rho}{2}n^2\omega^2 + O(n)$ by Theorem N.2; the arithmetic Hilbert–Samuel theorem of Zhang contributes an archimedean term of order n^3. Substituting $r_n \sim n\mu$ into the displayed successive-minima bound gives a total discriminant coefficient $n\mu$, whose quotient by n^2 is $\mu/n \to 0$; the archimedean term contributes $(n\mu)(2/(n\mu))(n^3/6) = n^3/3$, whose quotient by n^2 is $n/3 \to \infty$. The finite DRR term contributes $(n\mu)(2/(n\mu))(\frac{\rho}{2}n^2\omega^2)/(n^2) = O(n)$, i.e. $O(1/n)$ after division by n^2 when normalised by the leading $n^2\bar{L}^2$ on the left. Exact rational arithmetic confirms $\mu/n \to 0$ and $n/3 \to \infty$. □

Corollary N.6 (Cost of restoring a discriminant bound)**.** *To recover a bound of the form $\bar{\omega}^2 \le C\delta(F')/d' + O(1)$ from Theorem N.5 one must additionally assume an upper bound $\mathrm{vol}(\bar{L}') \le C_{\mathrm{BMV}}\,\delta(F')/d'$, i.e. the Parshin–Moret-Bailly arithmetic Bogomolov–Miyaoka–Yau hypothesis (or an equivalent $\hat{c}_2$ estimate). This hypothesis is unproved; its naive transfer from the geometric BMV inequality is known to fail, and Soulé's earlier IHÉS lectures record that the archimedean integral is "not easy to bound from above". Moreover $E = 1$ would still require, by $E(c) = 12/(5 - 18c)$ of Appendix F, $c = -7/18$, equivalently $T/k = 47/3$, which contradicts $T \le 15k$ and $T = 8k$ in the transverse model. Thus changing the inequality type does not unlock c.*

Proof. The first assertion is Theorem N.5. The numerical statement follows from solving $12/(5 - 18c) = 1$: $5 - 18c = 12$, $c = -7/18$; with $c = \frac{11}{12} - \frac{T}{12k}$ this gives $T/k = 11 + \frac{14}{3} = \frac{47}{3}$. The bound $T \le 15k$ is the exponent-1 optimality constraint of §5, and $T = 8k$ is Theorem 6.4 under transversality. □

Theorem N.7 (Feasibility trichotomy)**.** *Without changing the geometric data $(\mathcal{X}/B, L, \omega, \rho, m, \mu)$, without changing $\delta = \log|\Delta_F|$, and without importing an axiom equivalent to abc, any external inequality producing a polynomial-type weak abc bound arises from exactly one of:*

1. *Theorem A of [1] (proved; its adelic metric proof has not been independently verified) — giving $E = 24$;*
2. *an arithmetic BMV / $\hat{c}_2$ upper bound (unproved; its naive version is false) — translating the conditionality without lowering E;*
3. *abc itself or an equivalent Vojta-type control of $m_S(D_B, b)$ — circular.*

Hence the conditionality of Theorems 7.5 and 9.2 is not removable by a change of inequality type.

Proof. By Lemma N.1 and Theorem N.4, any estimate that keeps the rank-2 lattice yields $c_{\mathrm{eff}} = 1/4$ and hence (1), possibly with a different proof of the same chain. By Theorem N.5, abandoning the rank-2 lattice for Hilbert–Samuel asymptotics replaces δ by the

archimedean volume, so a discriminant bound requires (2). A direct Vojta-type control of $m_S(D_B, b)$ is equivalent to (1.1) by Lemma 2.3, so it is (3). These three cases exhaust the possible sources of a δ-controlled inequality on the fixed geometry. The tag of (1) is Th.A; of (2) C; of (3) X. □

Corollary N.8 (Tenth No-Go reason, type layer)**.** *There is no freedom in the* type *of external inequality. This reason is independent of the nine previously recorded No-Go reasons: its proof uses only Lemma N.1, Theorems N.2–N.5 and Corollary N.6, none of which invokes the truth of Theorem A, the modulus-3 argument, transversality, $g_0 = 13$, $\gamma = 2/5$, $47/3$, bridge breaking, S-smoothness, absorption decoupling, or discount zero density. The common root is that c_{eff} is an algebraic invariant (DRR minus torsion), not an estimate.*

Proof. Combine Theorems N.4, N.5, N.7; the independence is verified by inspection of the hypotheses of each cited result. □

Remark N.9 (Honesty boundaries). Theorem N.2, Lemma N.1, Theorem N.4, Theorem N.5 and Corollary N.8 are unconditional at the algebraic level: they use Deligne–Riemann–Roch, the standard order $\widehat{\deg} H^0(X, n\bar{L}) = \frac{n^3}{6}\bar{L}^3 + o(n^3)$ of arithmetic Hilbert–Samuel, and exact rational recomputation. The step "$\mathrm{vol}(\bar{L}') \le C_{\mathrm{BMV}}\delta/d'$" and the R3 replacement (Faltings–Quillen $\hat{c}_2$ estimate) are tagged C: we have not independently verified the archimedean $\hat{c}_2$ estimate nor the equality of the model and admissible metrics. The order n^3 of the archimedean term is used as a standard statement, not recomputed fibre by fibre for the singular Kodaira–Parshin fibres. This appendix is a *type-level* analysis; it does not constitute an independent verification of the proof of Theorem A. All previous conclusions of the paper are unchanged; in particular the weak output $(N/M_5)^{24+\varepsilon}$, $E(c) = 12/(5 - 18c)$, the $47/3$ exclusion, the closure of Route 1, and *abc* open remain intact.

Appendix O. Enlarging the prime set: a discriminant-type weak abc

This appendix examines Route 3 of the concluding discussion: replace $S = \{p : p \mid abc\}$ by a set containing the support of $\mathrm{disc}(f)$, and replace $N = \mathrm{rad}(abc)$ by the radical of the discriminant. We show that the bridge-breaking obstruction of §10.1 disappears unconditionally, that a discriminant-type weak abc inequality follows, and that translating it back to the original radical is either abc-grade or logically impossible.

O.1. **Setup.** Let $f \in \mathbb{Z}[x]$ be monic irreducible of degree $n \ge 5$, let θ be a root, $F = \mathbb{Q}(\theta)$, and let $P \in \mathcal{C}(F)$ satisfy $h_{K_{\mathcal{C}}}(P) = \gamma \log c + O(1)$. Put

$$S_f = S_\infty \cup \{p : \ p \mid \mathrm{disc}(f) \cdot abc\}, \qquad \widetilde{N} = \mathrm{rad}\big(\mathrm{disc}(f) \cdot abc\big) = N \cdot \widetilde{M}, \tag{O.1}$$

$$\Delta_S = \log \widetilde{M} = \sum_{p \mid \mathrm{disc}(f),\ p \nmid abc} \log p, \qquad \alpha = \frac{\log \widetilde{N}}{\log c}, \qquad \kappa = \frac{\log c}{\log \widetilde{N}} = \frac{1}{\alpha}. \tag{O.2}$$

We take the union with *abc* so that $N \mid \widetilde{N}$ and the resulting inequality remains a statement about the same triple. This is a change of problem: the output is a *discriminant-type* inequality, not the Masser–Oesterlé inequality (1.1).

Theorem O.1 (Unconditional restoration of the bridge)**.** *For every f as above,*

$$\frac{\delta(F)}{n} \ \le \ \frac{n-1}{n} \log \widetilde{N} + O_n(1). \tag{O.3}$$

Consequently $\delta(F)/n \ll \log \widetilde{N}$ holds without any S-smoothness hypothesis.

Proof. Dedekind's theorem $\operatorname{disc} f = [\mathcal{O}_F : \mathbb{Z}[\theta]]^2 \mathfrak{d}_F$ gives $\operatorname{supp}(\mathfrak{d}_F) \subseteq \operatorname{supp}(\operatorname{disc} f) \subseteq S_f$. At a tame prime $p \nmid n$ the discriminant exponent is $v_p(\mathfrak{d}_F) = \sum_i f_i(e_i - 1) = n - r_p \le n-1$ with $r_p \ge 1$ the number of primes above p. At a wild prime $p \mid n$ one has $v_p(\mathfrak{d}_F) \le n - 1 + n\, v_p(n)$, bounded in terms of n alone (for $n = 5$ the bound is 21), by the local analysis of §9.2. Summing over $p \in S_f$,

$$\begin{aligned}\delta(F) = \sum_{p \in S_f} v_p(\mathfrak{d}_F) \log p &\le (n-1) \sum_{p \in S_f \setminus \{p \mid n\}} \log p + O_n(1) \\ &= (n-1) \log \widetilde{N} + O_n(1).\end{aligned}$$

Dividing by n gives (O.3). The field discriminant is structurally blind to the high multiplicities of $\operatorname{disc} f$; this is the index-ideal absorption of Appendix L. □

Remark O.2 (Reclassification). Theorem O.1 reclassifies the bridge breaking of Theorem 10.3: it is an obstruction only while one insists on $\widetilde{N} = N$. Enlarging S restores $\delta/d \asymp \log \widetilde{N}$ unconditionally; no Diophantine conjecture and no truth of Theorem A is used.

Theorem O.3 (Discriminant-type weak abc)**.** *Assume Theorem A of [1]. Then*

$$\log c \le \frac{12(n-1)}{\gamma n} \log \widetilde{N} + O_\varepsilon(1), \qquad c \le A_\varepsilon\, \widetilde{N}^{E_f + \varepsilon}, \qquad E_f = \frac{12(n-1)}{\gamma n}. \tag{O.4}$$

With $\gamma_K(q) = (q-3)/q$ and $n = q$ one has $E_f = 12(q-1)/(q-3)$: the values at $q = 5, 7, 9, 11, 27, 51, 101$ are $24, 18, 16, 15, 13, 25/2, 600/49$, and $E_f \to 12$ as $q \to \infty$. In particular $q = 5$ reproduces the exponent 24 exactly.

Proof. Theorem A gives $h(P) \le (12 + \varepsilon)\delta(F)/n + \beta(\varepsilon)$. Insert $h(P) = \gamma \log c + O(1)$ and Theorem O.1: $\gamma \log c \le (12 + \varepsilon)\frac{n-1}{n} \log \widetilde{N} + O_\varepsilon(1)$, which is (O.4). The table is exact rational arithmetic; $\lim_{q \to \infty} 12(q-1)/(q-3) = 12$. The asymptotics 12 coincides with the γ_K-limit of §8 and the quality-plane limit of Appendix E, a third independent occurrence of the same constant. □

Remark O.4 (Dependence). Theorem O.3 is tagged Th.A; Theorem O.1 is unconditional. The feasibility trichotomy of Appendix N is not bypassed: only the bridge step has been deconditioned, while the main height inequality still comes from Theorem A. The rows $q > 5$ of the table are the most favourable extrapolation; Theorem A fixes $g = 2 \iff q = 5$.

O.2. **The content criterion: why the new bound is trivial.**

Theorem O.5 (Two-sided dilemma)**.** *Let $n = 5$, $\gamma = 2/5$, so $E_f = 24$. Writing $\widetilde{N} = c^\alpha$, inequality (O.4) has coefficient 24α in front of $\log c$; it is non-trivial only if*

$$\alpha < \frac{\gamma n}{12(n-1)} = \frac{1}{24}. \tag{O.5}$$

On the other hand, translating (O.4) back to the original radical requires $\Delta_S \le \varepsilon \log N$. If $\operatorname{supp}(\mathfrak{I}) \subseteq S$, i.e. $\log \mathfrak{I} \le K \log N$, the absorbed-mass inequality of Appendix L gives

$$\log c \le \frac{2K}{n-1} \log N - \mathcal{E} \le \frac{2K}{n-1} \log N, \tag{O.6}$$

which is exactly the abc-grade condition with slope $2/(n-1)$ (for $n = 5$, slope $1/2$).

Proof. Substituting $\log \widetilde{N} = \alpha \log c$ into (O.4) gives $\log c \le 24\alpha \log c + O(1)$; non-triviality means $24\alpha < 1$, i.e. (O.5). For the second part, Appendix L gives $\log \mathfrak{I} \ge \frac{n-1}{2}(\mathcal{E} + \log c) - O(1)$; combining with $\log \mathfrak{I} \le K \log N$ yields $\frac{n-1}{2} \log c \le K \log N - \frac{n-1}{2}\mathcal{E} + O(1)$, hence (O.6). The slope $2/(n-1)$ is the same as in the engine-replacement criterion of Appendix L and the S-smoothness loop of Theorem 10.10; this is its third independent occurrence. □

Proposition O.6 (Numerical triviality on the Bring–Jerrard family)**.** *For* $f_t(x) = x^5 - 5x + t$ *one has* $\operatorname{disc}(f_t) = 5^5(t^4 - 256)$*. Taking* $t = c = 2^k$ *and* $\widetilde{N} = \operatorname{rad}(5(c^4 - 256) \cdot abc)$*, exact factorisation gives:*

k	c	α	κ	24α
3	8	1.636	0.611	39.3
5	32	2.283	0.438	54.8
8	256	2.927	0.342	70.3
10	1024	3.300	0.303	79.2

In every case $\alpha > 1 \gg 1/24$*, so* (O.4) *degenerates into* $\log c \le (40\text{–}80) \log c$*.*

Proof. Direct computation with exact integer factorisation of $c^4 - 256$ and rad; the values of $\alpha, \kappa, 24\alpha$ are rounded. Since $v_p(\operatorname{disc} f)$ is of order $2n - 2 = 8$ in the coefficients, α grows with k towards a value near 4; the theoretical coefficient tends to $24 \cdot 4 = 96$. □

O.3. **Fixed S is logically impossible.**

Theorem O.7 (S-unit finiteness)**.** *Let* S *be a fixed finite set of primes. If* $\operatorname{supp}(abc) \subseteq S$*, then* $x = a/c$ *and* $y = b/c$ *satisfy* $x + y = 1$ *with* x, y *in the group of* S*-units of* $\mathbb{Q}$*. By the finiteness theorem for* S*-unit equations (Evertse; Evertse–Schlickewei–Schmidt,* J. reine angew. Math. *539 (2001), 225–241), this equation has only finitely many solutions, with an effective bound depending only on* $|S|$ *and* $\mathbb{Q}$*. Consequently* $\{c\}$ *is finite and*

$$N \le \prod_{p \in S} p \ < \infty, \tag{O.7}$$

so no sequence with $N_i \to \infty$ *and fixed* S *exists.*

Proof. $\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1$ and $\operatorname{supp}(abc) \subseteq S$ make $a/c, b/c$ quotient of S-smooth integers, hence S-units; $a + b = c$ is $x + y = 1$. The cited finiteness theorem applies to $\mathbb{Q}$ with $k = 2$ variables. Finiteness of the solution set bounds c, and $N = \operatorname{rad}(abc)$ divides $\prod_{p \in S} p$. No unproved conjecture and no Theorem A are used. □

Corollary O.8 (Open problem (8) splits)**.** *Open problem (8) must be read in two ways:*

(1) (8a) S *a fixed finite set with* $\operatorname{supp}(d_{F_i}) \subseteq S$ *and* $N_i \to \infty$*:* X *— logically impossible by Theorem O.7, not abc-grade.*
(2) (8b) S *varying with the triple but* $\operatorname{disc}(f)$ *required to be* $\{p : p \mid abc\}$*-smooth:* C *— abc-grade by Theorem O.5 and the slope* $2/(n - 1)$*.*

The blanket phrase "the difficulty is abc-grade" in §11 and Appendix H.6 applies only to (8b).

Proof. Immediate from Theorems O.7 and O.5. □

Remark O.9 (Numerical corroboration). For $S = \{2, 3, 5\}$ and $c \le 10^7$ there are 768 S-smooth integers and exactly 17 coprime solutions of $a + b = c$, the largest being $c = 128$. The list contains $(9, 16, 25)$ (the zero-discount triple of §9.3) and $(5, 27, 32)$ (the Bring–Jerrard value $t = 32$ of Example 10.11), providing a common meeting point for three otherwise independent strands of the paper.

O.4. **Closure of the Mason dictionary.**

Theorem O.10 (Discriminant-quality decomposition)**.** *With* $\mathcal{E}_f = \log|\operatorname{disc} f| - \log \widetilde{N} = \sum_p \big(v_p(\operatorname{disc} f) - 1\big)\log p$*,*

$$\mathcal{E}_f = \sum_p \big(n - r_p - 1\big)\log p \;+\; 2\log \mathfrak{I} \;\le\; (n-2)\log \widetilde{N} + 2\log \mathfrak{I}. \tag{O.8}$$

In the function-field analogue, $\deg \operatorname{disc}(F) - \deg \operatorname{rad}(\operatorname{disc} F) = \deg \gcd(F, F')$*; over* $\mathbb{Z}$ *the extra term is exactly* $2\log \mathfrak{I}$*. This closes the Mason dictionary of Theorem 10.9 and Appendix L: the multiplicity excess is the p-adic valuation of the index ideal, and it is precisely* $\mathfrak{I}$ *(not* $\mathfrak{d}_F$*) that inflates* $\widetilde{N}$ *in Proposition O.6.*

Proof. By Dedekind, $v_p(\operatorname{disc} f) = 2v_p(\mathfrak{I}) + v_p(\mathfrak{d}_F)$; at tame primes $v_p(\mathfrak{d}_F) = n - r_p$, at wild primes the additional contribution is bounded by $O_n(1)$. Subtracting $\sum_{p|\operatorname{disc} f} \log p$ gives (O.8). The function-field identity is the Mason–Stothers theorem in the form $\deg \gcd(F, F') = \deg F - \deg \operatorname{rad}(F)$. □

Corollary O.11 (Eleventh No-Go reason, S-layer)**.** *Enlarging* S *restores the bridge (Theorem O.1) and yields* $c \le A_\varepsilon \widetilde{N}^{24+\varepsilon}$ *(Theorem O.3), but: (i) pulling* $\widetilde{N}$ *back to* N *requires the abc-grade* S*-smoothness of* $\mathfrak{I}$ *with slope* $2/(n-1)$*; (ii) fixed* S *is logically impossible; (iii) without (i),* $\alpha > 1$ *makes the inequality trivial. Hence Route 3 is a change of problem, not a breakthrough. This reason is independent of the ten previously recorded No-Go reasons: its proof uses only Dedekind's theorem, the tame discriminant formula,* S*-unit equation finiteness and Appendix L's absorbed-mass inequality.*

Proof. Combine Theorems O.1, O.3, O.5, O.7, O.10; the independence is by inspection of the cited hypotheses. In particular Theorem 10.3 (bridge breaking) is here *removed*, not invoked, and Theorem 10.10 is here *quantified*, not invoked. □

Remark O.12 (Honesty boundaries). Theorems O.1, O.5, O.7, O.10, Proposition O.6 and Corollaries O.8, O.11 are unconditional; Theorem O.3 is tagged Th.A. The S-unit finiteness theorem is quoted as standard (not re-proved); the enumeration $c \le 10^7$ is a corroboration, not an asymptotic statement. The values $\alpha \approx 2$–4 are exact pointwise factorisations, not a proved asymptotic for $\operatorname{rad}(2^m - 1)$, whose lower bound is itself a deep problem. The rows $q > 5$ of Theorem O.3 are the most favourable extrapolation beyond the hypotheses of Theorem A ($g = 2 \iff q = 5$). This appendix does not constitute an independent verification of the proof of Theorem A. All previous conclusions — $(N/M_5)^{24+\varepsilon}$, $E(c) = 12/(5 - 18c)$, the 47/3 exclusion, the closure of Routes 1 and 2, and *abc* open — are unchanged.

Appendix P. Is there a way to change the framework completely?

The preceding appendices examined seven kinds of replacement: the covering data, the normalisation, the decomposition type, the adelic metric layer, the output language, the type of external inequality, and the finite set of primes. This appendix asks whether any further freedom remains. The answer is a classification theorem: all variable parts of the present paradigm are exhausted, and a "complete change" necessarily means abandoning at least one of the standing meta- constraints. We also return to the one loophole left open at the very beginning of the framework (Buium's δ_p / Λ-geometry) and close it at the level of the bridge.

P.1. **Four meta-constraints.** All preceding work assumes:

- C1 *Pointwise*: an inequality $c \le K_\varepsilon N^{1+\varepsilon}$ for every triple, not in density or on average.
- C2 *Original radical*: $N = \mathrm{rad}(abc)$, not $\widetilde{N} = \mathrm{rad}(\mathrm{disc}\, f \cdot abc)$.
- C3 *No new axiom*: Vojta's main conjecture, the arithmetic BMV hypothesis, and any statement equivalent to (1.1) are excluded.
- C4 *Verifiable input*: the external theorem must be proved and independently checkable.

Theorems 7.5 and 9.2 satisfy C1–C3 and produce $(N/M_5)^{24+\varepsilon}$; they are pinned at C4 because Theorem A of [1] is a recent preprint whose adelic metric proof has not been independently verified.

Theorem P.1 (Freedom classification)**.** *Under C1–C4, and keeping the goal of an abc-type inequality, every change of construction falls into exactly one of five freedoms, each already decided:*

	Freedom	*Status*
F1	*covering data* $(g_0, r, b, \lambda, q, \Theta)$	X
F2	*normalisation / line bundle* (γ, ℓ, η)	X
F3	*type of external inequality*	X
F4	*finite prime set / radical*	X
F5	*output language (pointwise* $\to$ *distributional)*	△

Here F5 is not false but degenerates: it yields unconditional distributional results without improving C1.

Proof. F1: Appendix K gives the strict bound $-B_0/\mu \ge W(\frac{11}{12} - \frac{1}{57}) = \frac{1230}{19} > 15$, so no admissible even $\Theta \le 15$ gives a positive t at $g_0 = 13$; only $g_0 = 2$ starts the coefficient chain, and ρ is independent of the covering data (§3). F2: Theorem 7.14 fixes $\gamma = 2/5$ from Kc, and Appendix F decomposes $12 = \frac{1}{3} \cdot \frac{m}{\rho - mc}$ into five locked factors. F3: Theorem N.5 gives the feasibility trichotomy {Theorem A, arithmetic BMV, *abc*} and proves c_{eff} is an algebraic invariant. F4: Theorem O.4 proves fixed S logically impossible via S-unit finiteness, and Theorem O.3 gives the abc-grade slope $2/(n-1)$. F5: Theorem M.6 shows the average effective exponent tends to 24, while Theorems M.3–M.4 show the discount vanishes in density $1 - o(1)$; the results are unconditional but are average, not pointwise.

These five axes are the variable parts of the paradigm "an arithmetic surface fibration plus one external height inequality": the geometry (F1), the height normalisation (F2), the analytic input (F3), the arithmetic support (F4) and the logical form of the conclusion (F5). No sixth variable part occurs in any of the appendices A–O. The five decisions are independent of one another. □

Corollary P.2 (No sixth freedom)**.** *Under C1–C4 there is no further freedom. A "complete change of framework" must abandon at least one of C1–C4.*

Proof. Immediate from Theorem P.1: the classification of variable parts is exhaustive within the stated paradigm. □

Corollary P.3 (The four ways of abandoning a constraint)**.**

Abandon	*Output*	*Status*
$\neg C1$	*density-one abc (Cor. P.6)*	*unconditional, known*
$\neg C2$	$c \le A_\varepsilon \widetilde{N}^{24+\varepsilon}$	*trivial (Th. O.3)*
$\neg C3$	*arithmetic BMV / Vojta*	*weaker or circular (Th. N.5)*
$\neg C4$	$(N/M_5)^{24+\varepsilon}$	*present paper,* Th.A

None of the four produces the pointwise abc inequality (1.1).

Proof. The first row is Corollary P.6 below; the second is Theorem O.2 together with the content criterion $\alpha < 1/24$ and the numerical values $\alpha = 1.6$–3.3; the third is the trichotomy of Theorem N.5; the fourth is the standing status of Theorems 7.5 and 9.2. □

P.2. **The one loophole never revisited: δ_p.** The initial No-Go results (§5.4, §6.2) proved: $\Omega^1_{\mathbb{Z}} = 0$, so no global arithmetic dlog exists in the commutative/Leibniz sense; no additive $\delta : \mathbb{N} \to \mathbb{R}_{\geq 0}$ with $\delta(n) = \log \operatorname{rad}(n) + O(1)$ exists (take p^k); each local Leibniz derivation space is one-dimensional and the $\mathbb{Q}$-sections have support contained in a single prime. None of these statements excludes Buium's *p-derivation*

$$\delta_p(x) = \frac{x - x^p}{p}, \qquad \delta_p(xy) = x^p \delta_p(y) + y^p \delta_p(x) + p\, \delta_p(x)\delta_p(y), \tag{P.1}$$

which is non-Leibniz and non-commutative; Borger's Λ-rings and $\mathbb{F}_1$-geometry aim precisely at a Frobenius lift that would transport Mason's $\deg \gcd(F, F') = \deg F - \deg \operatorname{rad}(F)$.

Theorem P.4 (Quantitative closure of the δ_p loophole)**.** *Suppose one tries to cover all prime factors of* $\operatorname{rad}(abc)$ *by a family* $\{\delta_p\}_{p \in S}$ *in order to imitate Mason. Exactly one of the following holds, and all three are already closed:*

1. *a global section: by §5.4 its support is contained in a single prime, so* $|S| \geq 2$ *is impossible;*
2. *a fixed finite* S*: by the* S*-unit finiteness theorem of Appendix O (*S*-unit equation finiteness)* $N \leq \prod_{p \in S} p$ *is bounded, so no sequence with* $N_i \to \infty$ *exists — logically impossible;*
3. S *varying with the triple: the discriminant / ramification support must be* $\{p : p \mid abc\}$*-smooth, which by the absorbed-mass inequality of Appendix L carries the abc-grade slope* $2/(n-1)$ *(Appendix O).*

Consequently the obstruction in the Mason dictionary is not the existence of a derivation but the tightness of support, $\operatorname{supp} \subseteq \{p : p \mid abc\}$*. Changing the type of derivative does not unlock this constraint.*

Proof. The three cases exhaust the possibilities for the support of the family $\{\delta_p\}$: one prime, a fixed finite set, or a varying set. Case (1) is the classification of §5.4. Case (2) is Theorem O.4 applied verbatim. Case (3) is Theorem O.3 with the slope $2/(n-1)$, itself the third independent occurrence of that constant (Appendix L, Appendix O). In particular §6.4 ("integer abc lives in the all-constant cell excluded by Mason") should be read as "lives in the cell where the support cannot be glued globally". This is structurally the same phenomenon as Theorem N.2: c_{eff} is an invariant, not an estimate. □

Remark P.5 (Scope). Theorem P.4 does *not* assert that δ_p, Λ-geometry or inter-universal Teichmüller theory cannot prove abc; settling that is a task of a different order. It only shows that any such programme which *still requires* the bridge $\operatorname{supp} \subseteq S$ meets Theorem O.4 or the slope $2/(n-1)$. Whether the bridge can be circumvented lies outside the published literature used here and outside this paper. We make no claim about the correctness of any controversial programme.

P.3. **A positive unconditional corollary.** From $\log N = \log(abc) - \mathcal{E} = \log a + \log b + \log c - \mathcal{E}$ and $a, b \geq 1$,

$$\log c \;\leq\; \log N + \mathcal{E}. \tag{P.2}$$

Corollary P.6 (Density-one abc with an explicit rate)**.** *For every fixed* $\varepsilon > 0$ *and every* $t < 1$*,*

$$\mathbb{P}_X\big(\log c > (1+\varepsilon)\log N\big) \leq \mathbb{P}_X\big(\mathcal{E} > \varepsilon \log N\big) \leq C_t\,(\log X)^{-3t\varepsilon(1+o(1))} \;\longrightarrow\; 0. \tag{P.3}$$

Proof. By (P.2), $\log c > (1+\varepsilon)\log N$ forces $\mathcal{E} > \varepsilon \log N$. Theorem M.1 gives $\log N \geq 3\log\log c - 9\log\log\log c - O(1)$, which is $\geq 3(1+o(1))\log\log X$ on the top dyadic layer. Chernoff in Theorem M.2 with $K = \varepsilon \log N$ gives $C_t e^{-t\varepsilon \log N} \leq C_t (\log X)^{-3t\varepsilon(1+o(1))}$; the remaining dyadic layers are summed geometrically as in Theorem M.3. □

Remark P.7 (Novelty and numerics). Corollary P.6 is the known "abc holds for almost all triples" statement of Bombieri / van der Poorten type; it is *not* new mathematics. What is new here is its derivation from the exponential moment of Appendix M with an explicit decay rate, and its placement in Theorem P.1: abandoning C1 is the only unconditionally successful exit inside the paradigm, and it does not imply pointwise abc. Numerical corroboration for $X = 400, 1000, 2000$ gives, for $\varepsilon = 0.5$, passing proportions 0.962, 0.978, 0.986; for $\varepsilon = 1$, 0.9967, 0.9987, 0.9994; for $\varepsilon = 0.1$, 0.526, 0.580, 0.615. The convergence is of $(\log X)^{-c}$ type and is extremely slow, so these figures show the direction only. The maximum of $\log c/\log N$ observed is 1.4266, attained at $(3, 125, 128)$ — the same S-unit triple that appears in Remark O.4num, a fifth independent strand meeting the same table.

Theorem P.8 (Twelfth No-Go reason, meta layer)**.** *Under C1–C4 the variable freedoms are exhausted by Theorem P.1 and all are closed; abandoning any single constraint produces no abc (Corollary P.3); and the one loophole left at the framework's origin is closed at the bridge level by Theorem P.4. Hence there is no sixth route inside the paradigm: a "complete change" can only mean changing the problem (drop C1 or C2), changing the axioms (drop C3), independently verifying Theorem A (the truth layer behind C4), or moving to a δ_p/Λ/IUT-type programme whose bridge is constrained by Theorem P.4 but whose correctness is not judged here. This reason is independent of the eleven earlier ones: its proof uses only the classification of Theorem P.1, Appendix O (S-unit finiteness), the exponential moment of Appendix M, and §5.4, and cites the* status*, not the proofs, of the earlier No-Go results.*

Proof. Combine Theorem P.1, Corollaries P.2, P.3, P.6 and Theorem P.4. The independence is verified by inspection of the hypotheses invoked. □

Remark P.9 (Honesty boundaries). The completeness in Theorem P.1 is relative to the paradigm "arithmetic surface fibration plus one external height inequality"; it is not a claim of completeness over all conceivable mathematical methods. Theorem P.4 judges only the bridge layer. Corollary P.6 is a re-derivation of a known result with an explicit rate; its numerical convergence is slow. S-unit finiteness is quoted as standard. This appendix does not constitute an independent verification of the proof of Theorem A and makes no claim about the correctness of any outside programme. All previous conclusions — $(N/M_5)^{24+\varepsilon}$, $E(c) = 12/(5-18c)$, the 47/3 exclusion, the closure of Routes 1, 2 and 3, and *abc* open — are unchanged.

Appendix Q. Density-one abc: a complete unconditional proof

This appendix expands Corollary P.4 into a fully self-contained proof. The only external input is the effective Stewart–Yu bound on $\mathrm{rad}(abc)$; all other steps are elementary. The result is unconditional and does not prove the pointwise abc conjecture.

Q.1. **The probability space.**

Definition Q.1. For $X \geq 3$ let

$$\Omega(X) = \{(a,b,c) : 1 \leq a < b < c \leq X,\ a+b=c,\ \gcd(a,b)=1\}, \tag{Q.1}$$

with the uniform probability measure $\mathbb{P}_X$ and $T(X) = |\Omega(X)|$.

Lemma Q.2 (Size of the sample space)**.** *There are absolute constants $0 < c_0 < C_0$ and X_0 with $c_0X^2 \le T(X) \le C_0X^2$ for $X \ge X_0$; more precisely*

$$T(X) = \tfrac{1}{2}\sum_{c\le X}\varphi(c) + O(1) = \frac{3}{2\pi^2}X^2 + O(X\log X). \tag{Q.2}$$

One may take $c_0 = 3/(4\pi^2)$ and $C_0 = 1$.

Proof. For fixed c, $\gcd(a, c-a) = 1$ is equivalent to $\gcd(a, c) = 1$, so a has $\varphi(c)$ admissible values; the condition $a < b$ pairs each $a \ne c-a$ with $c-a$ and therefore halves the count, up to the single case $a = c/2$ (which forces $c = 2$). Hence $T(X) = \frac{1}{2}\sum_{c\le X}\varphi(c) + O(1)$. From $\varphi = \mu * \mathrm{id}$ and $\sum_{n\le X}\mu(n)/n^2 = 1/\zeta(2) + O(1/X)$ one obtains the classical estimate $\sum_{c\le X}\varphi(c) = \frac{3}{\pi^2}X^2 + O(X\log X)$, giving (Q.2). The stated c_0, C_0 follow for X large. □

Remark Q.3 (Convenient coordinates). The map $(a, b, c) \mapsto (a, c)$ identifies $\Omega(X)$ with $\{(a, c) : 1 \le a < c \le X,\ \gcd(a, c) = 1\}$. Counting an event is therefore counting lattice points in a subset of $[1, X]^2$, and the total number of points is at least c_0X^2.

Q.2. **Notation and two identities.** Put

$$N = \mathrm{rad}(abc) = \prod_{p\mid abc} p, \qquad e_p = v_p(abc), \qquad \mathcal{E} = \log(abc) - \log N = \sum_p (e_p - 1)\log p \ge 0. \tag{Q.3}$$

Lemma Q.4 (Basic inequality)**.** *For every triple,*

$$\log c \le \log N + \mathcal{E}. \tag{Q.4}$$

Proof. $\log(abc) = \log a + \log b + \log c \ge \log c$ because $a, b \ge 1$, and $\log(abc) = \log N + \mathcal{E}$ by definition. □

Lemma Q.5 (Layer decomposition)**.** *With $R_j = \prod_{p:\, e_p \ge j} p$,*

$$\mathcal{E} = \sum_{j\ge 2}\log R_j = \sum_{j\ge 2}\sum_{p:\, p^j \mid abc}\log p. \tag{Q.5}$$

Proof. $(e_p - 1)\log p = \sum_{j=2}^{e_p}\log p$; all terms are non-negative, so the order of summation may be exchanged (Tonelli). □

Lemma Q.6 (Prime powers are bounded by X)**.** *If $p^j \mid abc$ with $j \ge 1$, then $p^j \le X$.*

Proof. $\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1$, so all powers of a given prime divide exactly one of a, b, c; each of a, b, c is at most X. □

Remark Q.7. Lemma Q.6 is exactly what removes every boundary term $+O(1/X)$ from the counts below; it cannot be omitted.

Q.3. **The effective Stewart–Yu input and its inversion.** We use the following hypothesis, satisfied by two published effective theorems.

Definition Q.8 (Hypothesis SY_β)**.** There exist effectively computable $\kappa > 0$, $0 < \beta < 1$, $\gamma \ge 0$ such that for every coprime $a + b = c$ with $c \ge 3$,

$$\log c \le \kappa\, N^\beta (\log N)^\gamma. \tag{Q.6}$$

(1) Stewart–Yu, *Math. Ann.* 291 (1991), 225–230, prove $\log\max(|a|, |b|, |c|) < G^{2/3 + c_2/\log\log G}$ with c_2 effective (Zbl 0761.11030); hence every fixed $\beta > 2/3$ works with $\gamma = 0$ and an effective κ.

(2) Stewart–Yu, *Duke Math. J.* 108 (2001), 169–181, "On the abc conjecture, II", yield the standard weaker corollary $\log c \ll N^{1/3}(\log N)^3$, i.e. $(\beta,\gamma) = (1/3, 3)$. The precise theorem of that paper involves the least largest prime factor p' and the exponent $c_2 \log_3 G^* / \log_2 G^*$; the displayed form is its commonly quoted consequence.

Both rest on Baker's theory of linear forms in logarithms, so all constants are effective. The proof below is insensitive to the value of β: the 1991 theorem with $\beta = 3/4$ already suffices.

Lemma Q.9 (Inversion)**.** *Assume* SY_β. *Write* $u = \log\log c$, $L = \log N > 0$. *There is an effective* $C_0 = C_0(\kappa,\beta,\gamma)$ *such that for* $c \geq 16$

$$\log N \geq \frac{1}{\beta}\big(\log\log c - \gamma \log\log\log c - C_0\big), \tag{Q.7}$$

and consequently $\log N \geq \frac{1}{2\beta}\log\log c$ *for* $c \geq c_*(\kappa,\beta,\gamma)$.

Proof. Taking logarithms in (Q.6) gives

$$u \leq \log\kappa + \beta L + \gamma \log L. \tag{Q.8}$$

Set $A = \log\kappa - \gamma\log\beta + 1 > 0$ (indeed $\beta < 1$ gives $-\gamma\log\beta \geq 0$) and $L_0 = \frac{1}{\beta}(u - \gamma\log u - A)$. We claim $L > L_0$.

If $L_0 \leq 0$ then $L \leq L_0 \leq 0$, contradicting $L = \log N > 0$ because $N \geq 2$. If $L_0 > 0$, then $c \geq 16$ gives $u > 1$, so $\gamma\log u \geq 0$ and $A > 0$ imply $L_0 \leq u/\beta$; hence $\log L \leq \log(u/\beta) = \log u - \log\beta$. Substituting into (Q.8),

$$\begin{aligned} u &\leq \log\kappa + \beta L_0 + \gamma(\log u - \log\beta) \\ &= \log\kappa + \big(u - \gamma\log u - A\big) + \gamma\log u - \gamma\log\beta \\ &= u + \big(\log\kappa - A - \gamma\log\beta\big) = u - 1, \end{aligned}$$

a contradiction. Therefore $L > L_0$, which is (Q.7) with $C_0 = A$. Since $\gamma\log\log\log c + C_0 = o(\log\log c)$, for c large the bracket is at least $\frac{1}{2}\log\log c$, giving the final assertion. □

Remark Q.10. Lemma Q.9 is applied directly to c itself. A dyadic layer $c \in (X/2^{k+1}, X/2^k]$ has $u = \log\log c = \log\log X + O(k/\log X)$, not $\log\log X - k\log 2$; the term "$-k\log 2$" appearing in an earlier draft's layer estimate should be deleted. The conclusion is unchanged.

Q.4. **The first moment.**

Lemma Q.11 (Single prime-power count)**.** *For every prime* p *and every* $j \geq 2$ *with* $p^j \leq X$,

$$\mathbb{P}_X\big(p^j \mid abc\big) \leq \frac{C}{p^j}, \qquad C = \frac{6}{c_0} = 8\pi^2. \tag{Q.9}$$

Proof. In the coordinates of Remark Q.3, Lemma Q.6 and pairwise coprimality imply that p^j divides exactly one of a, b, c; there are three cases.

Case $p^j \mid a$. The number of multiples of p^j up to X is $\leq X/p^j + 1 \leq 2X/p^j$ (using $p^j \leq X$), and c ranges over at most X values: count $\leq 2X^2/p^j$.

Case $p^j \mid c$. Symmetrically, count $\leq 2X^2/p^j$.

Case $p^j \mid b = c - a$. For each fixed $c \leq X$ the condition $a \equiv c \pmod{p^j}$ selects one residue class, giving $\leq X/p^j + 1 \leq 2X/p^j$ values of a; count $\leq 2X^2/p^j$.

The total is at most $6X^2/p^j$. Dividing by $T(X) \geq c_0 X^2$ with $c_0 = 3/(4\pi^2)$ gives $6/c_0 = 8\pi^2$. □

Theorem Q.12 (Uniform first moment)**.** *There is an absolute constant $C_1 < \infty$, independent of X, with $\mathbb{E}_X[\mathcal{E}] \le C_1$. One may take $C_1 = 8\pi^2 \sum_{n\ge 2} \frac{\log n}{n(n-1)} \approx 99.3$.*

Proof. By Lemma Q.5 and Tonelli,

$$\begin{aligned}\mathbb{E}_X[\mathcal{E}] &= \sum_{j\ge 2} \sum_{p:\, p^j \le X} \log p \cdot \mathbb{P}_X(p^j \mid abc) \le C \sum_{j\ge 2} \sum_{p} \frac{\log p}{p^j} \\ &= C \sum_{p} \frac{\log p}{p(p-1)} \le C \sum_{n\ge 2} \frac{\log n}{n(n-1)} =: C_1 < \infty.\end{aligned}$$

The sum over $p^j > X$ is empty by Lemma Q.6; the final series converges since $\sum_{n>N} \log n/n^2 \ll (\log N)/N$. The numerical value uses $\sum_{n\ge 2} \log n/(n(n-1)) = 1.25773913\ldots$. □

Corollary Q.13 (Markov)**.** *For every $K > 0$, $\mathbb{P}_X(\mathcal{E} \ge K) \le C_1/K$.*

Proof. $\mathcal{E} \ge 0$; apply Markov's inequality to Theorem Q.12. □

Q.5. **Main theorem.**

Theorem Q.14 (Density-one abc)**.** *Assume* SY_β*. For every fixed $\varepsilon > 0$ there exists $X_0 = X_0(\varepsilon, \kappa, \beta, \gamma)$ such that for $X \ge X_0$*

$$\mathbb{P}_X\big(\log c > (1+\varepsilon)\log N\big) \le \frac{2\beta C_1}{\varepsilon \log\log X} + \frac{1}{8c_0(\log X)^2} \xrightarrow[X\to\infty]{} 0. \tag{Q.10}$$

Equivalently, $c \le N^{1+\varepsilon}$ holds on a set of coprime triples of asymptotic density one.

Proof. **Step 1.** Let $A = \{\log c > (1+\varepsilon)\log N\}$. By Lemma Q.4, $\log c \le \log N + \mathcal{E}$, hence

$$A \implies \mathcal{E} > \varepsilon \log N. \tag{Q.11}$$

Step 2. Put $Y = X/(2\log X)$ and $B = \{c \le Y\}$. Since $T(Y) \le Y^2/2$ and $T(X) \ge c_0 X^2$,

$$\mathbb{P}_X(B) \le \frac{Y^2}{2c_0 X^2} = \frac{1}{8c_0(\log X)^2}. \tag{Q.12}$$

Step 3. On B^c we have $\log c \ge \log X - \log\log X - \log 2$, so $\log\log c = (1+o(1))\log\log X$. By Lemma Q.9, for X large enough that $Y \ge c_*$,

$$\log N \ge \frac{1}{\beta}\big(\log\log c - \gamma \log\log\log c - C_0\big) \ge \frac{1}{2\beta}\log\log X =: L_{\min}. \tag{Q.13}$$

(The inequality $\gamma \log\log\log c + C_0 \le \frac{1}{2}\log\log c$ holds for c large.)

Step 4. By (Q.11) and (Q.13), $A \cap B^c \subseteq \{\mathcal{E} > \varepsilon L_{\min}\}$. Hence

$$\begin{aligned}\mathbb{P}_X(A) &\le \mathbb{P}_X(B) + \mathbb{P}_X(\mathcal{E} > \varepsilon L_{\min}) \\ &\le \frac{1}{8c_0(\log X)^2} + \frac{C_1}{\varepsilon L_{\min}} = \frac{1}{8c_0(\log X)^2} + \frac{2\beta C_1}{\varepsilon \log\log X},\end{aligned}$$

using (Q.12) and Corollary Q.13. The right side tends to 0 as $X \to \infty$ for fixed $\varepsilon > 0$. □

Remark Q.15 (Scope of ε). Theorem Q.14 is a statement for each *fixed* $\varepsilon > 0$. If $\varepsilon = \varepsilon(X) \to 0$, the bound is useful only when $\varepsilon(X)\log\log X \to \infty$.

Q.6. **Rates, attribution and honest boundaries.**

Remark Q.16 (Faster rates). The exponential moment $\mathbb{E}_X[e^{t\mathcal{E}}] \le C_t$ $(t < 1)$ of Appendix M would give, via Chernoff, $\mathbb{P}_X(A) \le C_t(\log X)^{-3t\varepsilon(1+o(1))} + O((\log X)^{-2})$. Its proof needs the multi-prime CRT bound $\mathbb{P}_X(\forall p \in S : p^{v_p} \mid abc) \le C_0^{|S|} \prod p^{-v_p}$, which is strict for $D = \prod p^{v_p} \le X$ but requires one extra step when $D > X$ (each $p^{v_p} \le X$ divides one of a, b, c, so the count drops to $O(X)$). Theorem Q.14 deliberately avoids that step: the single-prime count of Lemma Q.11 is fully strict and the rate $O(1/\log\log X)$ suffices for density one. Stronger rates are known: de Bruijn's smooth-number estimate, as presented by Lichtman (arXiv:2505.13991, 2025), gives an exceptional set of size $O(N^{2/3})$; Bernert–Browning–Lichtman–Teräväinen (arXiv:2410.12234, 2024, rev. 2026) obtain a power saving for the exceptional set at fixed quality. Theorem Q.14 is an independent re-proof of a known conclusion, with a weaker rate; its only novelty is the derivation from the notation $\mathcal{E}, \log N$ and the Stewart–Yu input of this paper.

Remark Q.17 (No pointwise consequence). Theorem Q.14 is a measure statement: the exceptional set is non-empty (e.g. $(3, 125, 128)$ has $\log c/\log N = 1.4266$) and its density decays only like $1/\log\log X$. Pointwise abc requires $\mathcal{E} \le \varepsilon \log N + O_\varepsilon(1)$ for *every* triple with a uniform constant; neither statement implies the other. The exponential/polynomial type distinction of Appendix H.2 also applies. In particular the enumerations $X \le 2000$ give failing proportions 0.0383, 0.0218, 0.0141 at $\varepsilon = 0.5$, while the bound (Q.10) is still larger than 1 there: the theorem is asymptotic, with conservative constants and very slow convergence. The numerical data show the direction only.

Remark Q.18 (Dependencies). Lemmas Q.2–Q.6, Q.11, Theorem Q.12, Corollary Q.13 and the derivation of Theorem Q.14 from SY_β are unconditional and elementary. The hypothesis SY_β is a proved effective theorem (Baker theory). No part of this appendix uses Theorem A of [1], Vojta's conjecture, the arithmetic BMV hypothesis, or abc itself; it is not an independent verification of the proof of Theorem A. All previous conclusions of the paper are unchanged: the weak outputs $(N/M_5)^{24+\varepsilon}$, $E(c) = 12/(5-18c)$, the 47/3 exclusion, the closure of Routes 1–3, the twelve No-Go reasons, and *abc* open.

Appendix R. The Szpiro side and two repairs to the audit

This appendix records the outcome of checking a companion note by the same author group (dated one day before the present version) against the primary sources and against the audit of Appendices A–Q. All constants are recomputed; no conclusion of the paper is changed.

R.1. **Section 7 of [1] and a section-number erratum.**

Theorem R.1 (Structure of §7 of [1]). *] Section 7 of [1] has the following structure:*

(1) §7.2 "The conjecture of the discriminant" proves Theorem 3*: for every $\varepsilon > 0$ there is $\beta_2(\varepsilon)$ with*

$$\log|\Delta(E)| \le \big(144(1+\log 24)+\varepsilon\big)\log N(E) + \beta_2(\varepsilon) \tag{R.1}$$

for semistable $E/\mathbb{Q}$. Its proof uses $\mathcal{C} = X_0(23)$ (genus 2), the degree of the j-map $n = \deg j = 24$, and Moret-Bailly's Theorem 6.4.

(2) §7.3 "A weak uniform abc theorem" proves Theorem 4*, using the genus-2 curve $y^2 + y = x^5$, $\deg_{K_0}(L) = 5$, the constant 30, and the bound $\log H(a,b,c) \le (24+\varepsilon)\log N + (30+\varepsilon)\log|\Delta_K|/[K:\mathbb{Q}] + \beta_3(\varepsilon)$.*

(3) A final part "No Siegel zero" contains Theorem 5.

Theorem B of the introduction ($c \le A(\varepsilon)N^{24+\varepsilon}$) is the $\mathbb{Q}$-specialisation of Theorem 4, not the statement of §7.2.

Proof. Direct reading of the arXiv HTML version of [1], Section 7; the quoted subsection titles, the statement of Theorem 3, the identity $\deg_{K_0}(L) = 5$ and the constant 30 are verbatim. In particular the estimate "$30 \cdot \frac{4}{5} = 24$" of Appendix M belongs to §7.3, not §7.2. □

Remark R.2 (Erratum to earlier appendices). Several places in Appendices E–M attribute the abc derivation and the constant 30 to §7.2. These must be corrected to §7.3; §7.2 is reserved for the Szpiro statement (R.1). The correction is purely bibliographic and changes no constant.

R.2. **The constant $144(1 + \log 24)$ and its improvement.**

Theorem R.3 (Moret-Bailly 6.4 chain)**.** *Théorème 6.4 of Moret-Bailly,* Astérisque *183 (1990), 37–58, states, with $n = \deg j$,*

(R.2)
$$A_{CD} = (1 + \log n)\, A_{ME}(X, h_1, n), \qquad B_{CD} = B_{ME} + (\log n + \log n \log v_0 + \log v_0) A_{ME},$$

and its proof uses (p. 58, from Serre [S], proposition 6)

(R.3)
$$\tfrac{1}{n} \log |D_K| \le \log(v_0 N) + \big(1 + \log(v_0 N)\big) \log n = (1 + \log n) \log N + (1 + \log n) \log v_0 + \log n.$$

With $X = X_0(23)$, $n = 24$, v_0 the product of the primes dividing v, and $A_{ME} = \frac{n}{2} \cdot 12 = 144$, this gives $A_{CD} = 144(1 + \log 24) = 601.6398\ldots$, i.e. exactly (R.1). The earlier circulating value 458 is $144 \log 24 = 457.6398\ldots$, i.e. the term $+1$ was dropped.

Proof. The two displayed formulae (R.2), (R.3) are quoted verbatim from the Numdam scan of Moret-Bailly's article. The numerical evaluation is exact rational/real arithmetic: $\log 24 = 3.1780538\ldots$, $144 \log 24 = 457.64$, $144(1 + \log 24) = 601.64$. The relation $A_{ME} = 144$ follows from Theorem A of [1] together with the ceiling formula of Theorem R.6 below. □

Theorem R.4 (The $144 + \varepsilon$ improvement, tagged C)**.** *In (R.3) the factor $1 + \log(v_0 N)$ is the pointwise bound for the number of distinct prime divisors $\omega(v_0 N)$. Replacing it by Robin's effective bound $\omega(m) \le 1.3841 \log m / \log\log m$ ($m \ge 3$) gives $\frac{1}{n} \log |D_K| \le (1 + o(1)) \log N(E) + O_{n,v_0}(1)$, hence $A_{CD} = 144(1 + o(1))$ and, after absorbing the $o(1)$ into ε with an effective threshold, $\log |\Delta_{\min}(E)| \le (144 + \varepsilon) \log N(E) + B(\varepsilon)$.*

Proof. Substituting $\omega(v_0 N) = O(\log N / \log\log N)$ into the first inequality of (R.3) replaces $\log n \cdot \log(v_0 N)$ by $\log n \cdot O(\log N / \log\log N) = o(\log N)$ because $n = 24$ is fixed; the remaining $\log(v_0 N)$ term is $(1 + o(1)) \log N + O(1)$. The constant terms in B_{CD} of (R.2) are absorbed into $B(\varepsilon)$. Robin's inequality is quoted from the 1983 estimate $\omega(n) \le 1.3841 \log n / \log\log n$, valid for $n \ge 3$. □

Remark R.5 (Two qualifications). (i) The factor $(1 + \log n)$ in (R.2) is an algebraic consequence of expanding $\log(v_0 N) + (1 + \log(v_0 N)) \log n$; it is *not* a second independent estimate. The single estimate being replaced is the ω-bound inside (R.3). (ii) Moret-Bailly defines v_0 as the product of the primes dividing v; the companion note writes $v_0 = 23$ without comment, which holds only if that product equals 23. (iii) Theorem R.4 is tagged C: the chain (R.2)–(R.3) is verified verbatim, but we have *not* independently re-proved steps (a)–(b) of Moret-Bailly's Theorem 6.4 nor the precise meaning of ω in Serre's proposition 6. This is a textual verification, not an independent verification of the proof.

R.3. **The ceiling formula and its agreement with $E = t\eta/\gamma$ at $q = 5$.**

Theorem R.6 (Ceiling formula)**.** *Write A for the coefficient of Theorem A of [1] ($A = 12$). Then*

$$\text{(R.4)} \qquad \textit{abc exponent} = \frac{A(\deg t - 1)}{2}, \qquad \textit{Szpiro constant} = \frac{A \deg j}{2}\,(1 + o(1)).$$

Within the domain of Theorem A ($q = 5$, genus 2, curve $y^2 + y = x^5$) this coincides *with the formula $E = t\,\eta/\gamma$ of Appendix M: there $\eta/\gamma = (4/5)/(2/5) = 2 = (\deg t - 1)/2$, so $E = A \cdot 2 = 24$; with $\deg j = 24$ one gets $A \cdot 24/2 = 144$. The agreement is* not *an identity in q: for general q, $\eta/\gamma_K = (q-1)/(q-3) \neq (q-1)/2$, and Theorem A fixes $g = 2 \iff q = 5$.*

Proof. Theorem A bounds the $K_{\mathcal{C}}$-height $h(P)$ by $(A+\varepsilon)\delta(F)/d+\beta$. Writing $h_t(P)$ for the height induced by $t^*\mathcal{O}(1)$, the genus-2 superelliptic normalisation of §7.3 gives $h_t(P) = \frac{\deg t}{2}h(P) + O(1)$ (at $\deg t = 5$ this is $1/\gamma = 5/2$). The Kummer discriminant bound gives $\delta(F)/d \le (1 - 1/\deg t)\log N + O(1)$. Multiplying, $\log c \le \frac{\deg t}{2}A \cdot \frac{\deg t - 1}{\deg t}\log N = \frac{A(\deg t - 1)}{2}\log N$. For the j-map on $X_0(23)$ the same computation with $\deg j = 24$ gives $\frac{A \deg j}{2} = 144$. Finally $\eta/\gamma = \frac{(q-1)/q}{(q-3)/q} = \frac{q-1}{q-3}$, which equals 2 exactly at $q = 5$, where also $(\deg t - 1)/2 = 2$. For $q > 5$ the two expressions differ and $\deg t = 5$ is unavailable; that extrapolation is the $E(q)$ table of §8, outside the hypotheses of Theorem A. □

Remark R.7 (Forced degrees). $\deg t = 5$ is forced by K-rationality of $t(P)$ when $x(P) = m^{1/5}$ (only $t = y$ remains on $y^2 + y = x^5$). For $\psi(N) = N\prod_{p|N}(1 + 1/p)$ one has $\psi(23) = 24$, $\psi(29) = 30$, and among genus-2 curves $X_0(N)$ the values are $\psi(22) = 36$, $\psi(23) = 24$, $\psi(26) = 42$, $\psi(28) = 48$, $\psi(29) = 30$; hence 24 is minimal. The claim that non-modular Belyi covers lack the integrality of fibre points is an unproved assertion and is tagged A.

Corollary R.8 (Vojta limit, third independent occurrence)**.** *Even if Vojta's conjecture supplied $A = 1 + \varepsilon$, formula* (R.4) *would give abc exponent $2 + \varepsilon$ and Szpiro constant $12 + \varepsilon$, never exponent 1. This agrees with the exponent-1 optimality argument of §5 and the feasibility trichotomy of Appendix N.*

Proof. Substitute $A = 1 + \varepsilon$ in (R.4): $\frac{(1+\varepsilon)\cdot 4}{2} = 2 + 2\varepsilon$ and $\frac{(1+\varepsilon)\cdot 24}{2} = 12 + 12\varepsilon$; rename ε. □

R.4. **Repair of the two uses of Corollary 1.**

Theorem R.9 (Zero-degree reduction)**.** *Corollary 1 of §1.8 of [1] assumes $\deg_K(x) = 0$. In §6.3.6 and §6.3.7 it is applied to classes with $\deg(x) = -\deg(\phi'_*L') \le 3k' = \omega'^2/4 > 0$ and $\deg(x) = L'^2 = -20k' = -\frac{5}{3}\omega'^2 < 0$, yielding estimates (33) and (35). Put $M = \langle\bar\omega', \bar\omega'\rangle$, so $\deg M = \omega'^2$. Then*

$$\text{(R.5)} \qquad \text{§6.3.6}: \ \deg\big(\hat x - \tfrac14 M\big) = 0, \qquad \text{§6.3.7}: \ \deg\big(\hat x + \tfrac53 M\big) = 0.$$

Applying Corollary 1 to these degree-zero classes gives exactly (33) and (35); consequently (34), (36), the balance $\big(\frac53 - \frac32\big)\bar\omega^2 \le 6\delta(F') + O(d')$ with $\frac53 - \frac32 = \frac16$, the constant 36 and Theorem A's coefficient 12 are unchanged.

Proof. For §6.3.6, $\deg \hat x = 3k' = \omega'^2/4$, so $\deg(\hat x - \frac14 M) = 3k' - \frac{12k'}{4} = 0$. Corollary 1 gives $|\deg^F(u^*(\hat x - \frac14 M))| \le \varepsilon \deg^F(u^*\bar M) + O(d)$, hence $\widehat{\deg}_{F'}(u'^*\hat x) \le \frac14\bar\omega'^2_{P'} + \varepsilon\deg^F(u^*\bar M) + O(d')$, which is (33) after renaming ε. For §6.3.7, $\deg \hat x = -\frac53\omega'^2$, so $\deg(\hat x + \frac53 M) = 0$, and the same argument gives $\widehat{\deg}(u'^*\hat x) \le (-\frac53 + \varepsilon)\bar\omega^2 + O(d')$, which is (35). The coefficients $\frac14$ and $\frac53$ are exactly $3k'/\omega'^2$ and $-L'^2/\omega'^2$. Estimates (34) and (36) follow from (33) and (35) by the algebra of §6.3 of [1] and are therefore unaffected. □

Remark R.10 (Comparison with the companion note). The companion note first identifies this elision and proposes a repair via "a class of degree ≤ 0 is a degree-zero class minus an effective divisor". That works for the *negative*-degree case §6.3.7 but not for the *positive*-degree case §6.3.6, where one would need an effective divisor with an *upper* bound on its height. Theorem R.9 avoids effective divisors entirely and treats both cases by the same device of adding a multiple of M. An automated summary of §6.3 that reports $\frac{5}{3}-\frac{3}{2}=\frac{1}{3}$ is arithmetically wrong; the exact value is $\frac{1}{6}$, consistent with the 1/6 margin of Appendix F.

Remark R.11 (Literature status and honesty boundaries). The companion note cites: Soulé arXiv:2611.05217v1 (verified verbatim); Moret-Bailly Astérisque 183 (1990) 37–58 (verified verbatim on Numdam); Robin 1983 (verified); Pasten arXiv:2609.17390v1 and arXiv:1705.09251 (verified); Yuan, *Annals of Mathematics* 203(1) (2026), 15–119 (verified); Parshin 1988, Progr. Math. 75, 299–312 (verified via Zbl 0706.14015); base-3 Wieferich primes $11, 1006003$ with no further term below 9.7×10^{14} (verified via OEIS A014127 and Dorais–Klyve). Of the two items previously flagged as unverifiable, the self-citation "hexagon:2610.00544v2" has now been checked against the author-supplied v2 PDF (dated 2026-10-29): the master identity, Problem 10.1 and Theorem 9.1 ($\bar{\omega}^2 \leq 12h_F + 10.544133g$) are present verbatim, and Appendix B records the withdrawal of the version-1 unconditional claims (E9, E10, E11, E12, E21). Two qualifications are essential. First, its Theorem 9.1 is an unconditional *reduction* of an upper bound on $\bar{\omega}^2$ to an upper bound on h_F, not an unconditional upper bound and not a barrier; §9.4 states that the reverse direction is unreachable. Second, §5.1 states explicitly that the arithmetic BMY upper bound *remains a conjecture* and may not be cited as a theorem; the one-third barrier is conditional on Problem 10.1 except on modular towers. The v2 text predates [1] (it contains no citation of arXiv:2611.05217) and is an independent precursor. The second item, "Katz–Murata / (H-j)", does not occur anywhere in that PDF and remains unverified; neither item is used as evidence here. The "rStation manuscript math_4" is explicitly labelled unverified in the companion note and is likewise not used.

All conclusions of Appendices A–Q are unchanged: the weak output $(N/M_5)^{24+\varepsilon}$, $E(c)=12/(5-18c)$, the 47/3 exclusion, the closure of Routes 1–3, the twelve No-Go reasons, Theorem 7.5/9.2 tagged Th.A, and *abc* open. This appendix does not constitute an independent verification of the proofs of Theorem A of [1] or of Moret-Bailly's Theorem 6.4.

Appendix S. The remaining local gap: torsional length and a global criterion

Throughout this appendix we keep the notation of Appendices B, C and G: $A=k[[\pi]]$ with $\operatorname{char}k=0$, $A_m=A/\pi^m$, $B=2(B_1+B_2)$, $L=\omega_{X/Y}-2B_1-2B_2$, and $\Theta=T/k$ where $T=\deg(R^1\phi_*(B)_{\mathrm{tor}})$. By Round 11, $\Theta=2\sum_{j=1}^{4}n_j\in2\mathbb{Z}$, $\Theta\geq8$ (Proposition 9(vi)) and $\Theta\leq15$ (optimality of exponent 1), hence $\Theta\in\{8,10,12,14\}$.

Theorem S.1 (The A/π^2 criterion)**.** *At each exceptional fibre,*

$$\dim_k R[\pi^2]=G_y=2+2\cdot\mathbf{1}_{\{\kappa_M=0\}},\qquad n_{\bar{y}}=1\iff\kappa_M\neq0.$$

In particular the transversal fibres require no second-order data.

Proof. Let $C^\bullet$ be a Grauert complex representing $R\phi_*B$; its H^0 is the locally free sheaf ϕ_*B of rank $r_0=h^0(B_{\mathrm{gen}})=1$, and $H^1(C^\bullet)=R^1\phi_*B=A^r\oplus A/\pi^{n_1}\oplus A/\pi^{n_2}$ with $n_1=n_2=:n$ (square-step rigidity, Round 11). The cohomology-and-base-change long exact sequence for the perfect complex $C^\bullet$ and the A-algebra A_m reads

$$\dim_k H^0(X_{A_m},B_{A_m})=m\,r_0+\sum_{i=1}^{2}\min(n_i,m)+\dim_k\operatorname{Tor}_1^A(A^r,A_m).$$

The last term vanishes. With $m = 2$,

$$\dim_k H^0(X_{A_2}, B_{A_2}) = 2 + 2\min(n, 2). \tag{S.1}$$

On the other hand $\pi_* B = M^2 \oplus M$ (Appendix C), and on A_2 one has $\pi^2 = 0$, so every second-order coefficient in the expansion of M_π is annihilated: the deformation of M over A_2 is determined solely by the first Kodaira–Spencer class κ_M. For the exact sequence $0 \to M_0 \xrightarrow{\pi} M_{A_2} \to M_0 \to 0$ the connecting map is $\delta(s) = s \cup \kappa_M$, and similarly $\delta_2(s) = s \cup 2\kappa_M$ on the M^2-summand.

Transversal case $\kappa_M \neq 0$. Since $H^0(M_0)$ is one-dimensional and δ_1 is injective, $\dim_k H^0(M_{A_2}) = 2 \cdot 1 - 1 = 1$. On M^2 the connecting map is the catalecticant $\lambda \begin{pmatrix} 1 & a \\ a & a^2 \end{pmatrix}$, whose rank is 1 (Appendix C); hence $\dim_k H^0(M^2_{A_2}) = 2 \cdot 2 - 1 = 3$. The total is 4.

Degenerate case $\kappa_M = 0$. Then $M_{A_2} \cong M_0 \otimes A_2$ is the trivial deformation, so $\dim_k H^0(M_{A_2}) = 2 \cdot 1 = 2$ and $\dim_k H^0(M^2_{A_2}) = 2 \cdot 2 = 4$; the total is 6.

Comparing with (S.1), the totals 4 and 6 give $\min(n, 2) = 1$ and $\min(n, 2) = 2$ respectively, i.e. $n = 1$ and $n \geq 2$. Finally $\dim_k R[\pi^2] = 2\min(n, 2)$, which proves the displayed formula. □

Corollary S.2 (Where the second jet actually enters)**.** *For a degenerate fibre let* $M_\pi = M_0(1 + c_m \pi^m + \cdots)$ *with* $c_m \neq 0$ *first nonzero. Then*

$$n_j = m = \operatorname{ord}_0\big(x_1(t) - x_2(t)\big).$$

Consequently the phrase "one must supplement the second-order jet of A_1, A_2*" applies* only *to degenerate fibres; transversal fibres are settled by Theorem S.1 with no such input.*

Proof. If $\kappa_M = 0$, then $\alpha = \beta$ and $x_1 - x_2 = O(t^2)$, so $M^2_\pi = \mathcal{O}(A_1(t) + A_2(t))$ first departs from g^1_2 at order $m = \operatorname{ord}_0(x_1 - x_2) \geq 2$. The jump of h^0 persists over A_m and is resolved over A_{m+1}; by the base-change sequence used above this is equivalent to $\min(n, m) = m$ and $\min(n, m+1) < m+1$, i.e. $n = m$. For $\kappa_M \neq 0$, Theorem S.1 already gives $n = 1$. □

Remark S.3. Theorem S.1 unifies three earlier statements: $G_y = 4 - \operatorname{rank} \partial_1 - \operatorname{rank} \partial_2$ (Round 7) gives 2 and 4 in the two cases; the identity $\operatorname{rank} \partial_1 = \operatorname{rank} \partial_2$ (Round 11) holds because both ranks are controlled by the single class κ_M; and the degenerate value $G_y = 4$ (Round 8) is recovered.

Theorem S.4 (Exact identity chain)**.** *With* $m = \deg L_F = 6$, $\rho = 5/3$ *and* $c = 11/12 - \Theta/12$,

$$\deg \phi_* L = (\Theta - 11)k, \qquad c = \frac{11 - \Theta}{12},$$

$$C(\Theta) = \frac{m}{\rho - mc} = \frac{36}{3\Theta - 23}, \qquad E(\Theta) = \frac{12}{5 - 18c} = \frac{24}{3\Theta - 23}, \qquad C = 3t(\Theta) = \tfrac{3}{2}\, E(\Theta).$$

Proof. Round 4 gives $\deg_K \phi_*(L) = \deg_K R\phi_*(B) - \deg_K \phi_*(B) + \deg_K(R^1\phi_*(B)_{\mathrm{tor}}) = -11k - 0 + T$; with $\Theta = T/k$ this is $(\Theta - 11)k$. Since $c = -\deg \phi_* L/\bar{\omega}^2$ and $\bar{\omega}^2 = 12k$, $c = (11 - \Theta)/12$. Substituting in $C = m/(\rho - mc)$ and $E = 12/(5 - 18c)$ and simplifying with $3\Theta - 23 = 12(\frac{5}{3} - 6c)$ gives the displayed fractions; the identities $C = 3t$ and $C = \frac{3}{2}E$ follow from $t(\Theta) = 12/(3\Theta - 23)$ and $E(\Theta) = 24/(3\Theta - 23)$. At $\Theta = 8$ one recovers $c = 1/4$, $\deg \phi_* L = -3k$, $C = 36$, $E = 24$, i.e. the Soulé literature point. □

Theorem S.5 (A global criterion)**.** *The following are equivalent:*

$$c > 0 \iff \deg \phi_* L < 0 \iff \Theta < 11 \iff \Theta \in \{8, 10\}.$$

If they hold, then

$$C \geq \frac{36}{7} > 3, \qquad E \geq \frac{24}{7},$$

so the Parshin constant 3 *for arithmetic BMY is unattainable inside this Kodaira–Parshin construction, independently of* $\dim_k R[\pi^2]$*, transversality, transitivity and the mod-*3 *argument.*

Proof. By Theorem S.4, $c > 0 \iff \Theta < 11$, and $\deg \phi_* L < 0$ is the same inequality divided by $k > 0$. Since Θ is even with $8 \le \Theta \le 15$, $\Theta < 11$ means $\Theta \in \{8, 10\}$. The function $C(\Theta) = 36/(3\Theta - 23)$ is strictly decreasing, hence $\Theta \le 10$ gives $C \ge C(10) = 36/7 > 3$; likewise $E \ge 24/7$. As C is the constant in $\bar{\omega}^2 \le C\,\delta(F') + O(d')$, and the Parshin bound asks for $C \le 3$ (in the corresponding normalisation), no admissible Θ realizes it. □

Remark S.6 (Status of the hypothesis). The *implication* in Theorem S.5 is unconditional; the *hypothesis* $\deg \phi_* L < 0$ is not proved here and is tagged C. Three verification routes, in decreasing cost-effectiveness: (i) compute $\deg \phi_* L$ once by GRR from $\pi_* L = \omega_{Y_1} M^{-1} \oplus \omega_{Y_1} M^{-2}$ and $\phi = \psi \circ \pi$, using $\deg \omega_{Y_1} = 2g(Y_1) - 2 = 4$, $\deg M = 1$ and $\omega_{X/Y} = \omega_{X/Y_1}$ for ψ étale; (ii) a Xiao-type slope inequality together with Fujita–Deligne semipositivity of $\phi_* \omega_{X/Y}$, since L is the Hodge bundle minus four branch sections; (iii) the companion note's assertion $T = 8k$ "exact", which would give $\Theta = 8$ directly. The literature point $c = 1/4 > 0$ is a strong prior, but no independent computation of $\deg \phi_* L$ is available in the sources audited here.

Proposition S.7 (No sign contradiction at $\Theta = 12$)**.** *The value* $\Theta = 12$ *is internally consistent with Soulé's chain (28)–(36); it cannot be excluded by a sign argument.*

Proof. For $\Theta = 12$, Theorem S.4 gives $c = -1/12$ and $mc = -1/2$, so equation (36) becomes $\frac{5}{3}\bar{\omega}^2 \le -\frac{1-\varepsilon}{2}\bar{\omega}^2 + 6\delta$, i.e. $\frac{13}{6}\bar{\omega}^2 \le 6\delta$ and $C = 36/13$, in agreement with Theorem S.4. Moreover $-\deg \phi_* L = -k \le 3k' = \omega'^2/4$ (Proposition 9(vi)) still holds and is in fact weaker. Hence the coefficient chain imposes no obstruction; only external geometric data can decide $\Theta = 12$. □

Theorem S.8 (Exact survival conditions for $\Theta = 12$)**.** *Enumerating* $n_j \ge 1$ *with* $\Theta = 2\sum_{j=1}^{4} n_j \le 15$ *gives*

Θ	*types (unordered)*	*partitions*
8	1	$(1,1,1,1)$
10	1	$(1,1,1,2)$
12	2	$(1,1,1,3)$, $(1,1,2,2)$
14	3	$(1,1,1,4)$, $(1,1,2,3)$, $(1,2,2,2)$

Consequently $\Theta = 12$ *holds if and only if the Galois action on the four exceptional points is non-transitive* and *the vector of first nonzero deformation orders is* $(1,1,1,3)$ *or* $(1,1,2,2)$*; equivalently, at least one exceptional fibre is degenerate* ($\kappa_M^{(j)} = 0$) *with* $\operatorname{ord} \Delta_j = 2$ *or* 3*,* $\Delta_j = x_1 - x_2$.

Proof. The enumeration is finite and exact: $\sum n_j = \Theta/2$ with four parts ≥ 1. For $\Theta = 12$ the sum is 6, whose partitions into four positive parts are exactly $(1,1,1,3)$ and $(1,1,2,2)$. By Theorem S.1, $n_j \ge 2$ forces $\kappa_M^{(j)} = 0$, and by Corollary S.2 the value of n_j is $\operatorname{ord} \Delta_j$. If the Galois action were transitive then all n_j would be equal (Round 11), giving $\Theta = 8n \le 15$ hence $n = 1$, $\Theta = 8$; so $\Theta = 12$ requires non-transitivity, in agreement with the orbit classification of Round 12 (only the transitive orbit [4] excludes $\Theta = 12$). □

Remark S.9 (Net assessment). Rounds 1–26 are unchanged: the weak output $(N/M_5)^{24+\varepsilon}$, $E(c) = 12/(5 - 18c)$, the 47/3 exclusion, the closure of Routes 1–3, the twelve No-Go reasons, and Theorem 7.5/9.2 tagged Th.A all stand, and *abc* remains open. This appendix does not decide the value of Θ; it converts the remaining problem into (a) one global degree sign ($\deg \phi_* L < 0$, preferred) or (b) the first nonzero orders $\operatorname{ord} \Delta_j$ on degenerate fibres.

Theorem S.5 shows that route (a), if established, closes the "push the BMY constant to 3" question without any local π^2-jet computation.

Appendix T. Computing $\deg \phi_* L$: explicit splitting and a global reduction

We keep the notation of [1, §5–§6] and Appendix S: Y_0 has genus 2, $\chi : Y_1 \to Y_0$ is étale of degree 2 so $g(Y_1) = 3$, $\psi : Y \to Y_1$ is étale of degree k, and $\mathcal{L}$ is the line bundle on $S = Y_1 \times Y$ with $2\mathcal{L} = \mathcal{O}(\widetilde{\Gamma}) = A_1 + A_2$. The double cover $\rho : X \to S$ satisfies $\rho_* \mathcal{O}_X = \mathcal{O} \oplus \mathcal{L}^{-1}$, and $\phi = p_2 \circ \rho : X \to Y$ has fibres of genus 6. We write $K = p_1^* \omega_1$ for the relative canonical divisor of $p_2 : S \to Y$; $\deg \omega_1 = 4$, $K^2 = 0$, $B = 2B_1 + 2B_2$, $L = \omega_{X/Y} - B$, and $T = \deg_K (R^1 \phi_* (B)_{\mathrm{tor}})$, $\Theta = T/k$.

Theorem T.1 (Explicit splitting of $\rho_* L$)**.** *One has*

$$\rho_* L = (K - \mathcal{L}) \oplus (K - 2\mathcal{L}),$$

and Grothendieck–Riemann–Roch for p_2 gives

$$\deg c_1\big(Rp_{2*}(K-\mathcal{L})\big) = -3k, \qquad \deg c_1\big(Rp_{2*}(K-2\mathcal{L})\big) = -8k, \qquad \deg \det R\phi_* L = -11k,$$

reproducing equation (18) of [**?**, 1]*ndependently.*

Proof. Since $2\mathcal{L} = A_1 + A_2$ and $\rho^* A_i = 2B_i$, one has $\rho^* \mathcal{L} = B_1 + B_2$. By Riemann–Hurwitz (Proposition 8 of [**?**, 1], $\omega_{X/Y} = B_1 + B_2 + \varphi_1^* \omega_1 = \rho^* \mathcal{L} + K$, hence

$$L = \omega_{X/Y} - 2B_1 - 2B_2 = K - \rho^* \mathcal{L} = \rho^* (K - \mathcal{L}).$$

The projection formula with $\rho_* \mathcal{O}_X = \mathcal{O} \oplus \mathcal{L}^{-1}$ gives $\rho_* L = (K - \mathcal{L}) \otimes (\mathcal{O} \oplus \mathcal{L}^{-1}) = (K - \mathcal{L}) \oplus (K - 2\mathcal{L})$.

For $p_2 : S \to Y$ the relative tangent bundle is $p_1^* T_{Y_1}$, so $c_2 = 0$ and $K^2 = 0$, and GRR reduces to $\deg c_1(Rp_{2*} N) = \frac{1}{2}(N^2 - N \cdot K)$. From Proposition 8, $A_i^2 = -4k$, $A_1 A_2 = 0$ and $p_1^* W \cdot A_i = k$; hence $\mathcal{L}^2 = \frac{1}{4}(A_1 + A_2)^2 = -2k$ and $K \cdot \mathcal{L} = \frac{1}{2} \cdot 4(k + k) = 4k$. Therefore

N	$\mathcal{L}$	$K - \mathcal{L}$	$K - 2\mathcal{L}$
N^2	$-2k$	$-10k$	$-24k$
$N \cdot K$	$4k$	$-4k$	$-8k$
$\frac{1}{2}(N^2 - NK)$	$-3k$	$-3k$	$-8k$

Adding the last two columns gives $-11k$, as claimed. □

Remark T.2. Equation (18) of [**?**, 1]btains $-11k$ by applying GRR to B; Theorem T.1 obtains it from the splitting of $\rho_* L$ and supplies the new component data $(-3k, -8k)$. It also upgrades the fibrewise identity $\pi_* L = \omega_{Y_1} M^{-1} \oplus \omega_{Y_1} M^{-2}$ used in earlier appendices to a sheaf identity on $Y_1 \times Y$.

Proposition T.3 (Correction of the proposed shortcut)**.** *Equations (16), (18), (22) of* [**?**, 1]*ive the identity*

$$\deg \phi_* L = -11k + T.$$

Consequently computing $\deg \phi_* L$ *is equivalent to computing T; it does* not *bypass the torsional length* $\dim_k R[\pi^2]$.

Proof. Substitute $\deg R\phi_* B = -11k$ (Theorem T.1, or (18)) and $\deg \phi_* B = 0$ ((20)–(22)) into (16). Both terms are already proved; T is the only unknown. □

Theorem T.4 (The torsion sits on the $R^1 p_{2*} \mathcal{L}$ side)**.** *We have $p_{2*} \mathcal{L} = 0$ (equation (21) of* [**?**, **?**, 1] *valid as stated), $\phi_* B = \mathcal{O}_Y$, and*

$$T = \deg_K \left(R^1 p_{2*} \mathcal{L}\right)_{\mathrm{tor}}.$$

Proof. From $B = \rho^*(A_1 + A_2) = 2\rho^*\mathcal{L}$ and $A_1 + A_2 = 2\mathcal{L}$, the projection formula gives $\rho_* B = (A_1 + A_2) \otimes (\mathcal{O} \oplus \mathcal{L}^{-1}) = (A_1 + A_2) \oplus \mathcal{L}$, hence $\phi_* B = p_{2*}(A_1 + A_2) \oplus p_{2*}\mathcal{L} = \mathcal{O}_Y \oplus 0$ by (20) and (21). For $0 \to \mathcal{L} \xrightarrow{\pi} \mathcal{L} \to \lambda \to 0$ the long exact sequence reads

$$0 \to p_{2*}\mathcal{L}/\pi \to H^0(Y_{1,y}, \lambda) \to R^1 p_{2*}\mathcal{L}[\pi] \to 0.$$

Since $h^0(\lambda) = 0$ at non-exceptional points and $= 1$ at exceptional ones, and $p_{2*}\mathcal{L}/\pi = 0$, the jump is absorbed entirely by $R^1 p_{2*}\mathcal{L}[\pi]$. This is consistent with (23)–(24) of [**?**, **?**, 1] where $\dim R^1\phi_* B[\pi] = 2$ is twice the $\mathcal{L}$-side jump (the divisor B has two branch points). In particular (21) requires no correction; the suspected elision does not occur. □

Theorem T.5 (Global reduction)**.** *Let $A := p_{2*}(\omega_1 - \mathcal{L}) = p_{2*}(K - \mathcal{L})$, a rank-one sheaf on Y, and put $a = \deg_K A$. Then*

$$\boxed{T = a + 3k, \qquad \Theta = \frac{a}{k} + 3}.$$

Proof. By the first column of the table in Theorem T.1, $\deg c_1(Rp_{2*}\mathcal{L}) = -3k$; with $p_{2*}\mathcal{L} = 0$ (Theorem T.4) this gives $\deg_K(R^1 p_{2*}\mathcal{L}) = 3k$. Relative Serre duality identifies the locally free part of $R^1 p_{2*}\mathcal{L}$ with $A^\vee$, of degree $-a$, while torsion of length T is preserved under the duality; hence $-a + T = 3k$. Dividing by $T = k\Theta$ gives $\Theta = a/k + 3$. □

Corollary T.6 (Exact parameter table)**.** *The constraints $T \geq 8k$ ((24)), $\Theta \leq 15$ (optimality of exponent 1) and $\Theta \in 2\mathbb{Z}$ force $a/k \in \{5, 7, 9, 11\}$, with*

a/k	Θ	$c = \frac{11-\Theta}{12}$	$C = \frac{36}{3\Theta-23}$
5	8	1/4	36
7	10	1/12	36/7
9	12	−1/12	$36/13 < 3$
11	14	−1/4	36/19

in agreement with $\deg \phi_* L = (\Theta - 11)k$ and $a = (\Theta - 3)k$.

Proof of Corollary T.6. Substitute $T = k\Theta$ into Theorem T.5: $a/k = \Theta - 3$. The bound $T \geq 8k$ gives $a/k \geq 5$; optimality of exponent 1 gives $\Theta \leq 15$, i.e. $a/k \leq 12$; and $\Theta \in 2\mathbb{Z}$ forces a/k to be an odd integer, hence $a/k \in \{5, 7, 9, 11\}$. The columns for c and C are those of Theorem S.3. □

Theorem T.7 (Executable criterion)**.**

$$C \leq 3 \iff \Theta \geq 12 \iff a \geq 9k, \qquad C = 36 \iff a = 5k.$$

Moreover A carries a natural global section: by §3.5.2 of [**?**, **?**, 1] *$h^0(Y_1, \omega_1 - \lambda) = h^0(\lambda) + 1 = 1$ at non-exceptional points; hence a is in principle the degree of the zero divisor of that section, whose support lies in the finite set $\Phi \supseteq \Psi$.*

Proof. Combine $\Theta = a/k + 3$ (Theorem T.5) with $C(\Theta) = 36/(3\Theta - 23)$ (Appendix S, Theorem S.3) and solve $C \leq 3 \iff 3\Theta - 23 \geq 12 \iff \Theta \geq 12 \iff a/k \geq 9$. The case $a = 5k$ gives $\Theta = 8$ and $C = 36$. The section statement is the relative form of the Riemann–Roch computation in §3.5.2 (Lemma 4). □

Remark T.8 (Net assessment). The proposed shortcut of Appendix S is withdrawn: GRR already fixes $\deg \phi_* L$ up to T. The gain is that T is now $a + 3k$ with $a = \deg A$ a *global* rank-one degree on Y (Theorem T.5), so the remaining input is no longer four local π^2-jet ranks but one global degree, computable in principle by counting the zeros of the natural section of A. Three verification routes (Hodge/Xiao slope, Deligne pairing, direct zero counting) are tagged C; the criterion $C \leq 3 \iff a \geq 9k$ is unconditional. All conclusions of Appendices A–S and the twelve No-Go reasons are unchanged; *abc* remains open. This

appendix does not independently verify the proofs of Theorem A or of Propositions 8–9 of [**?**, 1]

Remark T.9 (Superseded by Appendix U). Theorems T.4 and T.5 of this appendix rest on identifying T with the torsion of a single $\mathcal{L}$-side pushforward. Appendix U (Theorem U.1) shows that $\rho_* B = 2\mathcal{L} \oplus \mathcal{L}$ has two components whose h^0's each jump by one at an exceptional point; hence $T = a + a_2 + 11k$ and the reduction $T = a + 3k$ must be replaced by the two-component version. The explicit splitting $\rho_* L = (p_1^*\omega_1 - \mathcal{L}) \oplus (p_1^*\omega_1 - 2\mathcal{L})$ and the value $-11k$ of Theorem T.1 remain unconditional.

Appendix U. Pushing the constant: the nef bound and the surviving case

We keep the notation of Appendices S, T. In particular $K = p_1^*\omega_1$, $\mathcal{L}$ is the square-root bundle with $2\mathcal{L} = A_1 + A_2$, $\rho_*\mathcal{O}_X = \mathcal{O} \oplus \mathcal{L}^{-1}$, $B = 2B_1 + 2B_2 = \rho^*(A_1 + A_2)$, $L = \omega_{X/Y} - B$, $T = \deg_K(R^1\phi_*(B)_{\mathrm{tor}})$, $\Theta = T/k$, and $C(\Theta) = 36/(3\Theta - 23)$.

Proposition U.1 (Correction of Appendix T)**.** *The torsion of $R^1\phi_*(B)$ has* two *summands, not one. Namely $\rho_* B = 2\mathcal{L} \oplus \mathcal{L}$, so*

$$T = \deg_K \left(R^1 p_{2*}(2\mathcal{L})\right)_{\mathrm{tor}} + \deg_K \left(R^1 p_{2*}\mathcal{L}\right)_{\mathrm{tor}}.$$

Consequently, writing $a = \deg p_{2*}(K - \mathcal{L})$ and $a_2 = \deg p_{2*}(K - 2\mathcal{L})$,

$$\boxed{\deg \phi_* L = a + a_2 = (\Theta - 11)k, \qquad T = a + a_2 + 11k}.$$

Proof. Since $A_1 + A_2 = 2\mathcal{L}$, the projection formula gives $\rho_* B = 2\mathcal{L} \otimes (\mathcal{O} \oplus \mathcal{L}^{-1}) = 2\mathcal{L} \oplus \mathcal{L}$. At a non-exceptional fibre $h^0(2\lambda) = h^0(P_1 + P_2) = 1$ and $h^0(\lambda) = 0$; at an exceptional one $h^0(P_1 + P_2) = 2$ (the hyperelliptic divisor) and $h^0(\lambda) = 1$. Hence $h^0(B_y)$ jumps from 1 to 3, each summand contributing one, which is exactly the dimension 2 of $R^1\phi_*(B)[\pi]$ in (24).

Grothendieck–Riemann–Roch for p_2 with $K^2 = c_2 = 0$ reads $\deg c_1(Rp_{2*}N) = \frac{1}{2}(N^2 - N \cdot K)$. From Proposition 8, $\mathcal{L}^2 = -2k$ and $K \cdot \mathcal{L} = 4k$, giving the table

N	$\mathcal{L}$	$2\mathcal{L}$	$K - \mathcal{L}$	$K - 2\mathcal{L}$	$K + \mathcal{L}$
N^2	$-2k$	$-8k$	$-10k$	$-24k$	$6k$
$N \cdot K$	$4k$	$8k$	$-4k$	$-8k$	$4k$
$\frac{1}{2}(N^2 - NK)$	$-3k$	$-8k$	$-3k$	$-8k$	k

Relative duality identifies the locally free parts of $R^1 p_{2*}\mathcal{L}$ and $R^1 p_{2*}(2\mathcal{L})$ with $(p_{2*}(K - \mathcal{L}))^\vee$ and $(p_{2*}(K - 2\mathcal{L}))^\vee$, of degrees $-a$ and $-a_2$. Adding the two torsion lengths reproduces $T = a + a_2 + 11k$, and $\deg \phi_* L = a + a_2 = -11k + T$ is equation (16). □

Theorem U.2 (Independent verification of the degree chain)**.** *One has*

$$E := \phi_*\omega_{X/Y} = F_1 \oplus \mathcal{O}_Y^3, \quad F_1 = p_{2*}(K + \mathcal{L}), \quad \deg E = k,$$

and for any effective divisor D supported on the sections B_i,

$$\deg \phi_*(\omega_{X/Y} \otimes \mathcal{O}_D) = \frac{\omega \cdot D - D^2}{2}.$$

In particular $\deg \phi_*(\omega|_B) = 12k$*, and the long exact sequence of* $0 \to L \to \omega \to \omega|_B \to 0$ *reproduces* $\deg \phi_* L = -11k + T$*.*

Proof. For the double cover ρ, $\rho_*\omega_{X/Y} = (K+\mathcal{L}) \oplus K$, hence $E = p_{2*}(K+\mathcal{L}) \oplus H^0(Y_1, \omega_1) \otimes \mathcal{O}_Y$. The last column of the table above gives $\deg p_{2*}(K+\mathcal{L}) = k$, and $h^0(\omega_1) = g(Y_1) = 3$, so $\deg E = k$. This agrees with Noether's formula for a smooth fibration, $\omega^2 = 12 \deg \lambda$, since $\omega^2 = 12k$ (Proposition 8); equivalently the slope $\lambda_f = \omega_f^2/\chi_f$ equals 12, the maximal value attained precisely for fibrations without singular fibres.

For the second identity, from $\chi(\omega|_D) = \chi(\omega) - \chi(\omega(-D))$ and Riemann–Roch on X,

$$\chi(\omega|_D) = \tfrac{1}{2}\big(\omega \cdot D - D^2 - \deg(D/Y)(2g_Y - 2)\big),$$

while $\deg \phi_*(\omega|_D) = \chi(\omega|_D) + \deg(D/Y)(g_Y - 1)$; the g_Y-terms cancel, leaving $(\omega \cdot D - D^2)/2$. With $\omega \cdot B_i = 2k$ and $B_i^2 = -2k$ this gives $2k$ for B_i and $6k$ for $2B_i$; summing over $i = 1, 2$ yields $12k$.

Finally $R^1\phi_*\omega \cong \mathcal{O}_Y$ has degree 0 and $\deg R^1\phi_* L = T$ (its dual is $\phi_* B = \mathcal{O}_Y$ plus torsion), so the long exact sequence gives $k - \deg \phi_* L - 12k + T = 0$. □

Theorem U.3 (The nef bound)**.** *Since ϕ is smooth, $E = \phi_*\omega_{X/Y}$ is nef (Fujita). The quotient $E/\phi_* L = \mathrm{im}(E \to \phi_*(\omega|_B))$ is torsion-free of degree $(12 - \Theta)k$, hence*

$$\boxed{\Theta \le 12, \qquad \Theta \in \{8, 10, 12\}, \qquad C \in \{36,\ 36/7,\ 36/13\}}.$$

In particular $\Theta = 14$ is strictly excluded, and $\Theta = 12$ is exactly the equality case of the nef inequality.

Proof. Fujita's semipositivity theorem gives $f_*\omega_{X/Y}$ nef for a smooth (more generally semistable) fibration over a curve; on a curve nefness means every torsion-free quotient has nonnegative degree. The image of $E \to \phi_*(\omega|_B)$ is a subsheaf of the locally free sheaf $\phi_*(\omega|_B)$, hence torsion-free. By Theorem U.2, $\deg(E/\phi_* L) = \deg E - \deg \phi_* L = k - (\Theta - 11)k = (12 - \Theta)k \ge 0$. Combined with $\Theta \in 2\mathbb{Z}$, $\Theta \ge 8$ and $\Theta \le 15$ (Appendix S), this leaves $\{8, 10, 12\}$. The equality case $\Theta = 12$ has $\deg(E/\phi_* L) = 0$, i.e. a degree-zero quotient of rank 4. □

Theorem U.4 (The Hodge bundle is not semistable)**.** *$\mu_1(E) \ge k/3 > \mu(E) = k/6$; hence E is strictly unstable and one cannot close the problem by invoking semistability of E.*

Proof. By Theorem U.2, $F_1 \subset E$ is a subbundle of rank 3 and degree k, so the maximal HN slope satisfies $\mu_1(E) \ge \mu(F_1) = k/3$, whereas $\mu(E) = \deg E / \operatorname{rk} E = k/6$. Semistability would require $\mu_1(E) = \mu(E)$. □

Corollary U.5 (Conditional closure)**.** *If $F_1 = p_{2*}(K + \mathcal{L})$ is semistable, then $\Theta \le 10$ and $C \ge 36/7 > 3$: the Parshin constant 3 is unattainable inside this construction.*

Proof of Corollary U.5. The inclusion $\phi_* L \hookrightarrow F_1 \oplus \mathcal{O}_Y^3$ gives $\deg \phi_* L \le \deg p_1(\phi_* L) + \deg p_2(\phi_* L)$. The second projection lies in $\mathcal{O}_Y^3$ and has degree ≤ 0; if F_1 is semistable of slope $k/3$, any rank-≤ 2 subsheaf has degree $\le 2k/3$. Hence $(\Theta - 11)k \le 2k/3$, i.e. $\Theta \le 35/3$, and evenness gives $\Theta \in \{8, 10\}$. Then $C(\Theta) \ge C(10) = 36/7 > 3$. □

Remark U.6 (The surviving case). The only remaining $C < 3$ is $\Theta = 12$, which by Theorem U.3 is the nef equality case: E then admits a rank-4 degree-zero torsion-free quotient. Since $E = F_1 \oplus \mathcal{O}_Y^3$ already contains a degree-zero summand of rank 3 (the unitary flat part), $\Theta = 12$ requires F_1 itself to carry a degree-zero quotient. Thus the problem is reduced to deciding whether the maximal-slope block F_1 has a unitary summand; this is a question about the HN filtration of the Hodge bundle and requires no local π^2-jet computation. All conclusions of Appendices A–T, the twelve No-Go reasons, and the status of *abc* are unchanged.

Appendix V. Deciding the unitary summand of F_1

We keep the notation of Appendices S, T, U. In particular $E = \phi_*\omega_{X/Y} = F_1 \oplus \mathcal{O}_Y^3$ with $F_1 = p_{2*}(K + \mathcal{L})$ of rank 3 and degree k, $\deg E = k$, $g = 6$, ϕ is smooth and non-isotrivial, $\Theta = T/k$, and $C(\Theta) = 36/(3\Theta - 23)$. By Theorem U.3, $\Theta \in \{8, 10, 12\}$ and $\Theta = 12$ is the nef equality case.

We use the *second Fujita decomposition* of the Hodge bundle (Fujita 1978; Catanese–Dettweiler; González-Alonso–Stoppino–Torelli):

$$E = \mathcal{U} \oplus \mathcal{A} = \mathcal{O}_Y^{\oplus q_f} \oplus \mathcal{U}' \oplus \mathcal{A},$$

where $\mathcal{U}$ is flat unitary of rank u_f, $\mathcal{U}'$ is flat unitary with no nonzero sections, $\mathcal{A}$ is *ample*, and $q_f = h^1(Y, E \otimes \omega_Y) = h^0(E^\vee)$ is the relative irregularity.

Theorem V.1 (Exact equivalence)**.** *The following are equivalent:*

$$\Theta = 12, \quad \deg \phi_* L = k, \quad \deg(E/\phi_* L) = 0, \quad \mu_3(F_1) = 0, \quad F_1 \text{ has a degree-zero quotient},$$

$$F_1 \text{ is not ample}, \qquad u_f \geq 4.$$

Proof. Chain of equivalences. (i) $\Theta = 12 \iff \deg \phi_* L = (\Theta - 11)k = k$ by Proposition U.1. (ii) $\deg(E/\phi_* L) = \deg E - \deg \phi_* L = k - (\Theta - 11)k = (12 - \Theta)k$ by Theorem U.2, so this vanishes iff $\Theta = 12$. (iii) Since $\phi_* L$ has rank 2, $\deg \phi_* L = k$ means $\mu(\phi_* L) = k/2$. The inclusion $\phi_* L \hookrightarrow F_1 \oplus \mathcal{O}_Y^3$ has second projection into $\mathcal{O}_Y^3$, of degree ≤ 0; hence the image in F_1 has degree $\geq k$, i.e. $\mu_1(F_1) + \mu_2(F_1) \geq k$. With $\sum_{i=1}^3 \mu_i = \deg F_1 = k$, this is equivalent to $\mu_3(F_1) \leq 0$. (iv) E is nef (Fujita) and F_1 is a direct summand, so F_1 is nef and $\mu_3 \geq 0$; hence $\mu_3 \leq 0 \iff \mu_3 = 0$, which is precisely the existence of a degree-zero HN quotient Q of F_1. (v) A vector bundle on a curve is ample iff every quotient has positive degree; thus $\mu_3 = 0 \iff F_1$ is not ample. (vi) Since $\mathcal{O}_Y^3 \subset \mathcal{U}$ we have $u_f \geq 3$. The ample summand $\mathcal{A}$ has no degree-zero quotient, so any degree-zero quotient of E lies in $\mathcal{U}$; a degree-zero quotient of F_1 therefore forces $\mathrm{rk}\,\mathcal{U} \geq 4$. Conversely, if $u_f \geq 4$ then $\mathcal{U} = \mathcal{O}^3 \oplus \mathcal{U}'$ with $\mathcal{U}' \neq 0$ flat unitary of degree 0, and (as $E = F_1 \oplus \mathcal{O}^3$, $\mathcal{A}$ ample) F_1 has a degree-zero quotient. Hence $u_f \geq 4 \iff \mu_3(F_1) = 0$. □

Proposition V.2 (Relative irregularity)**.** $q_f = 3$ *unconditionally, independently of the value of* u_f*.*

Proof. $q_f = h^0(E^\vee) = h^0(F_1^\vee) + 3$. By the fine Fujita decomposition, $F_1^\vee = (\mathcal{U}')^\vee \oplus \mathcal{A}^\vee$. Ampleness of $\mathcal{A}$ gives $h^0(\mathcal{A}^\vee) = 0$, and a unitary flat bundle with no sections has dual with no sections, so $h^0((\mathcal{U}')^\vee) = 0$. Hence $q_f = 3$. The case $u_f = 4$ is compatible: then $\mathcal{U}' = Q \in \mathrm{Pic}^0(Y)$ is a nontrivial flat unitary line bundle without sections. □

Theorem V.3 (The general bounds do not close the case)**.** *For a non-isotrivial fibration of genus* $g = 6$*,*

$$u_f < g\frac{5g-2}{6g-3} = \frac{56}{11} \quad (\textit{Stoppino } 2024) \implies u_f \leq 5,$$

$$u_f \leq g - c_f \leq 6 - c_f \quad (\textit{GST } 2019), \qquad c_f \leq \left\lfloor \frac{g-1}{2} \right\rfloor = 2.$$

In particular the optimal unconditional bound is $u_f \leq 4$ *when* $c_f = 2$*, and in every case* $u_f = 4$ *is allowed. Consequently* $\Theta = 12$ *is* not excluded *by the general theory, and* $C = 36/13 < 3$ *remains attainable within this construction.*

Proof. Substitute $g = 6$ into Stoppino's Xiao-type inequality $u_f < g(5g-2)/(6g-3)$: this gives $u_f < 6 \cdot 28/33 = 56/11 \approx 5.09$, hence the integer bound $u_f \leq 5$. The same paper notes that $u_f = g - 1$ maximal forces $g \leq 6$, so $g = 6$ is the critical genus where that bound is weakest — and the Kodaira–Parshin fibre genus is $4g_0 - 2 = 6$.

GST prove $u_f \leq g - c_f$ for non-isotrivial f, with c_f the Clifford index of a general fibre. For any curve of genus 6, $c_f \leq \lfloor (g-1)/2 \rfloor = 2$. Thus $c_f = 2$ gives $u_f \leq 4$, $c_f \leq 1$ gives $u_f \leq 5$. Combining, $u_f \leq \min\{5, 6 - c_f\} \leq 5$, and the value $u_f = 4$ is permitted by both bounds. By Theorem V.1, $\Theta = 12$ survives. □

Proposition V.4 (HN picture)**.** *The three cases are:*

u_f	$\mathcal{U}$	$\mathcal{A}$	F_1
3	$\mathcal{O}^3$	*rank* 3, $\deg k$	$= \mathcal{A}$ *ample*
4	$\mathcal{O}^3 \oplus Q$	*rank* 2, $\deg k$	$Q \oplus \mathcal{A}$ (*HN*), $\mu_1 \geq k/2$
5	$\mathcal{O}^3 \oplus \mathcal{U}_2'$	*rank* 1, $\deg k$	*has a degree-zero quotient*

with $Q \in \mathrm{Pic}^0(Y)$ nontrivial, flat unitary, without sections. Thus $u_f = 3 \iff C \geq 36/7 > 3$, while $u_f \geq 4 \iff \Theta = 12 \iff C = 36/13 < 3$.

Proof. Immediate from the fine Fujita decomposition, $\mathrm{rk}\, E = 6$, $\deg E = k$, and Theorem V.1; for $u_f = 4$ the ample summand has rank 2 and degree k, so $\mu_1 + \mu_2 = k$ with $\mu_3 = 0$ and $\mu_1 \geq k/2$. □

Proposition V.5 (Independent check of the two torsion summands)**.** *At a general fibre $h^0(\omega_1 - \lambda) = h^0(\omega_1 - 2\lambda) = 1$ and $h^0(\omega_1 + \lambda) = h^0(\omega_1) = 3$; at an exceptional fibre the first two each jump by one. Hence $\rho_* L = (K - \mathcal{L}) \oplus (K - 2\mathcal{L})$ contributes two independent torsion summands to T, confirming Proposition U.1.*

Proof. With $g(Y_1) = 3$, $\deg \omega_1 = 4$, $\deg \lambda = 1$, Riemann–Roch and Serre duality give $h^0(\omega_1 - \lambda) = 3 + 1 - 3 + h^0(\lambda) = 1$, $h^0(\omega_1 - 2\lambda) = 2 + 1 - 3 + h^0(2\lambda) = 1$ (as $h^0(\lambda) = 0$, $h^0(2\lambda) = h^0(P_1 + P_2) = 1$ generally), and $h^0(\omega_1 + \lambda) = 5 + 1 - 3 = 3$, $h^0(\omega_1) = 4 + 1 - 3 = 3$. At exceptional points $h^0(\lambda) = 1$ and $h^0(2\lambda) = 2$, so both summands jump by one; this is exactly the dimension 2 of $R^1\phi_*(L)[\pi]$ and the factor 2 in (24) of [**?**, 1] □

Remark V.6 (The two remaining routes). Two routes could force $u_f = 3$ and hence close the problem. (a) *Picard bundle positivity.* By §5.1 of [**?**, **?**, 1] $\mathcal{O}(\Gamma - 2\Delta_1) = (1 \times \psi_1)^* \mathcal{P} + \pi_2^* \mathcal{L}_0$ with $\mathcal{P}$ the Poincaré bundle on $Y_1 \times J_1$, and Y maps finitely to J_1 via ψ_2. If $\mathcal{L}$ is (up to base-pullback terms) the pullback of $\mathcal{P}$, then $F_1 = \psi_2^* V$ for a Picard bundle V on J_1 of rank 3 ($n = 5$). Were V ample (Kempf) or semistable/semihomogeneous (Mukai), a finite pullback would make F_1 ample or of slope $k/3 < k/2$, excluding $\Theta = 12$. However a sign conflict must be resolved first: Schwarzenberger–Mattuck give $c_1(E_n) = -[W_{g-1}] = -\Theta$ (a *negative* principal polarization), whereas Theorem U.2 gives $\deg F_1 = k > 0$; the exact relation between $\mathcal{L}$ and $\mathcal{P}$ (including the $\mathcal{O}(2\Delta_1)$ and $\pi_2^* \mathcal{L}_0$ correction terms) is therefore not yet verified. This route is tagged C. (b) *Hodge-locus rigidity*: $u_f \geq 4$ would force the moduli image into a smaller Hodge locus; excluding it needs the monodromy data of the Kodaira–Parshin family and is likewise tagged C. Note that $\pi_1(J_1) = \mathbb{Z}^6$ is abelian, so its irreducible complex representations are one-dimensional and irreducibility arguments alone cannot rule out Q; the decision rests on $\det V$.

All conclusions of Appendices A–U, the twelve No-Go reasons, and the status of *abc* are unchanged; this appendix does not independently verify the proofs of Fujita, Stoppino, GST, Mukai or Schwarzenberger–Mattuck.

Appendix W. Verification of the sign conflict in §5.1

We keep Appendices S, T, U, V. We verify the discrepancy noted in Remark V.6(a) between the Schwarzenberger–Mattuck formula $c_1(E_n) = -\Theta$ and the value $\deg F_1 = k > 0$ proved in Theorem U.2.

W.1. **Verbatim data.** On $Y_1 \times Y_1$ Soulé writes (§5.1, pp. 19–20)

$$\mathcal{O}_{Y_1 \times Y_1}(\Gamma - 2\Delta_1) = (1 \times \psi_1)^*(\mathcal{P}) + \pi_2^*(L), \qquad \Gamma = \Delta_1 \sqcup \Delta_2,$$

so that $\Gamma - 2\Delta_1 = \Delta_2 - \Delta_1$; here $\mathcal{P}$ is *a* Poincaré line bundle on $Y_1 \times J_1$, L is a line bundle on Y_1, and π_2 is the second projection. Further, Y is a connected component of the fibre product of ψ_1 with $[2] : J_1 \to J_1$, with $\psi : Y \to Y_1$ finite étale, $\psi_2 : Y \to J_1$,

$\psi_1 \circ \psi = [2] \circ \psi_2$, and $k = \deg(\psi) = \deg(Y/Y_1)$. On $Y_1 \times Y$ one has $2\mathcal{L} = \mathcal{O}(\widetilde{\Gamma})$ and $\rho_*\mathcal{O}_X = \mathcal{O} \oplus \mathcal{L}^*$, where $\mathcal{L}^* = -\mathcal{L}$ is the *dual*; consequently $\rho_*\omega_{X/Y} = (\omega_{Z/Y} + \mathcal{L}) \oplus \omega_{Z/Y}$, so that $E = \phi_*\omega_{X/Y} = F_1 \oplus \mathcal{O}_Y^3$ with $F_1 = p_{2*}(K + \mathcal{L})$, $K = p_1^*\omega_1$. Finally $\omega^2 = 12k$, $g(Y_1) = 3$ (Proposition 8).

On $\mathrm{Pic}^0(C)$ Schwarzenberger defines $E_n = q_*(\mathcal{P}^0 \otimes p^*\mathcal{O}(nc))$ for $n > 2g - 2$, of rank $n - g + 1$, with $\mathbb{P}(E_n^\vee) \cong \mathrm{Sym}^n(C)$ and

$$c_1(E_n) = -[W_{g-1}] = -\Theta. \tag{W.1}$$

Thus $E_n^\vee$ (not E_n) is the ample Picard bundle.

Theorem W.1 (Resolution of the sign)**.** *Formula* (W.1) *and* $\deg F_1 = k > 0$ *are* not *contradictory: they live in different cohomology groups and are separated by five normalisation choices: (i)* $\mathcal{P}$ *vs* $\mathcal{P}^\vee$ *(Soulé does not normalise* $\mathcal{P}$*; the identification* $J_1 \simeq \widehat{J}_1$ *is defined up to* $[-1]$*); (ii)* E_n *vs* $E_n^\vee$*; (iii)* $\rho_*\mathcal{O}_X$ *uses* $\mathcal{L}^*$ *whereas* $\rho_*\omega$ *uses* $\mathcal{L}$*; (iv)* $c_1(E_n) \in H^2(J_1)$ *vs* $\deg F_1 \in H^2(Y) \cong \mathbb{Z}$*, related by* ψ_2^**, with extra base terms* π_2^*L *and* $\mathcal{O}(2\Delta_1)$*; (v) the sign of* ψ_1 *and the power* m *in* $\mathcal{P}^{\otimes m}$*, since* $[m]^*\Theta \equiv m^2\Theta$*. Moreover the branch* $c_1(V) = -\Theta$ *is* strictly excluded*: if* $F_1 = \psi_2^*V$ *with* $c_1(V) = -\Theta$ *then* $\deg F_1 = \psi_2(Y) \cdot (-\Theta) < 0$*, contradicting* $\deg F_1 = k > 0$ *(or, equivalently, Fujita's nefness of* E*).*

Proof. Θ is ample on J_1, so for the curve $\psi_2(Y) \subset J_1$ the intersection $\psi_2(Y) \cdot \Theta$ is positive; hence $\deg(\psi_2^*(-\Theta)) < 0$. Theorem U.2 gives $\deg F_1 = k > 0$ from $\omega^2 = 12k$, Noether's formula for a smooth fibration ($\omega^2 = 12 \deg \det E$) and $\deg \mathcal{O}_Y^3 = 0$. This contradiction excludes the minus branch; the remaining freedom is the list (i)–(v), none of which is fixed by the quoted text ("a" Poincaré bundle, "there exist" ψ_1, L, "if K is large enough ... there exists" $\mathcal{L}$). □

Theorem W.2 (A numerical tension)**.** *Let* $a : Y_1 \hookrightarrow J_1$ *be the Abel–Jacobi embedding. Then* $\deg(a^*\Theta) = g(Y_1) = 3$ *and*

$$\boxed{\deg(\psi_2^*\Theta) = \tfrac{3k}{4}}.$$

Consequently, if $F_1 = \psi_2^*V$ *with* $c_1(V) = +\Theta$*, then* $\deg(\psi_2^*V) = 3k/4 \neq k = \deg F_1$*: the deficit is exactly* $k/4$*. Moreover no power* m *repairs it, since* $[m]^*\Theta \equiv m^2\Theta$ *would require* $m^2 = 4/3 \notin \mathbb{Z}$*.*

Proof. Poincaré's formula gives $a_*[Y_1] = \Theta^{g-1}/(g-1)!$ and $\Theta^g = g!$, hence $\deg(a^*\Theta) = \int_J \Theta \cdot a_*[Y_1] = \Theta^g/(g-1)! = g = 3$. By $\psi_1 \circ \psi = [2] \circ \psi_2$ and $\deg \psi = k$,

$$\deg\left(\psi^*(\psi_1^*\Theta)\right) = 3k = \deg\left(\psi_2^*[2]^*\Theta\right) = 4\deg(\psi_2^*\Theta),$$

using $[2]^*\Theta \equiv 4\Theta$ for the symmetric theta bundle. Thus $\deg(\psi_2^*\Theta) = 3k/4$. Since $\deg F_1 = k$ and all Pic^0-twists are homologically trivial, $c_1(V) = \pm\Theta$ would give $\deg \psi_2^*V = \pm 3k/4$, neither equal to k; the plus branch misses by $k/4$ and the minus branch is excluded by Theorem W.1. Replacing $\mathcal{P}$ by $\mathcal{P}^{\otimes m}$ multiplies by m^2; $m^2 \cdot 3k/4 = k$ forces $m^2 = 4/3$, impossible. □

Proposition W.3 (What §5.1 determines)**.** *Formally,*

$$\mathcal{O}(\widetilde{\Gamma}) = 2(1 \times \psi)^*\mathcal{O}(\Delta_1) + 2(1 \times \psi_2)^*\mathcal{P} + \pi_Y^*\psi^*L,$$

using $(\mathrm{id} \times [2])^*\mathcal{P} = \mathcal{P}^{\otimes 2}$*. If* $\psi_1(y) = \mathcal{O}(y - c)$ *then* $(1 \times \psi)^*\mathcal{O}(\Delta_1) = p_1^*\mathcal{O}(c) + 2(1 \times \psi_2)^*\mathcal{P}$*, hence*

$$\mathcal{L} = p_1^*\mathcal{O}(c) + 3(1 \times \psi_2)^*\mathcal{P} + \pi_Y^*M, \qquad 2M = \psi^*L.$$

The exponent of $\mathcal{P}$ *(here* 3*, or* 6 *before halving) changes to* 1 *or* -1 *if* ψ_1 *or* $\mathcal{P}$ *is replaced by its inverse. Hence the published data do not determine* $\mathcal{L}$*,* V*, or the ampleness of* F_1*; the deficit* $k/4$ *of Theorem W.2 must be supplied by the explicit choice of* ψ_1*,* $\mathcal{P}$ *and the square root* M*.*

Proof. Pull back $\mathcal{O}(\Gamma) = 2\mathcal{O}(\Delta_1) + (1\times\psi_1)^*\mathcal{P} + \pi_2^*L$ along $1\times\psi$ and use $\psi_1\circ\psi = [2]\circ\psi_2$. The two displayed identities follow from $(\mathrm{id}\times[2])^*\mathcal{P} = \mathcal{P}^{\otimes 2}$ and, for $\psi_1(y) = \mathcal{O}(y-c)$, from $\mathcal{O}(\Delta_1) = p_1^*\mathcal{O}(c) + (1\times\psi_1)^*\mathcal{P}$. Flipping the sign of ψ_1 replaces $(1\times\psi_1)^*\mathcal{P}$ by its inverse, which changes the coefficient 6 to 2 or -2 in $2\mathcal{L}$; the ambiguity is exactly the freedom left open in §5.1. □

Corollary W.4 (Status of Route (a))**.** *The sign conflict of Remark V.6(a) is resolved and its direction is favourable: the correct branch is $c_1 = +\Theta$ (the ample dual $E_5^\vee$), while $c_1 = -\Theta$ is excluded. Nevertheless Route (a) remains conditional: F_1 is not the pullback of a single Picard bundle, and deciding whether F_1 is ample requires the normalisation data of §5.1 enumerated in Theorem W.1. Hence $\Theta = 12$ and $C = 36/13 < 3$ still survive.*

Proof of Corollary W.4. Combine Theorems W.1, W.2 and Proposition W.3: ampleness of F_1 would force $u_f = 3$ and $\Theta \le 10$, but the required input (ψ_1 sign, $\mathcal{P}$ normalisation, $2M = \psi^*L$) is not specified in the cited text. □

Remark W.5 (Corrections to Appendix V). Three phrasings of Remark V.6(a) should be amended: the line bundle in §5.1 is $\pi_2^*(L)$ with $L \in \mathrm{Pic}(Y_1)$ (not a fictitious $\mathcal{L}_0$); $\Gamma - 2\Delta_1 = \Delta_2 - \Delta_1$; and $\rho_*\mathcal{O}_X$ uses the *dual* $\mathcal{L}^*$. The "sign conflict" should be restated as the five normalisation choices of Theorem W.1 together with the deficit $k/4$ of Theorem W.2. All conclusions of Appendices A–V, the twelve No-Go reasons, and the status of *abc* are unchanged. This appendix does not independently verify the proofs of Soulé, Schwarzenberger–Mattuck, Fujita, Mukai or Parameswaran.

Remark W.6 (Superseded by Appendices X and Y). Theorem W.2's value $3k/4$ assumed $\psi_1 = a$; by Theorem X.1 ψ_1 is the Prym map $\pm(\mathrm{id}-\nu^*)\circ a$, and Theorem X.3 gives $\deg(\psi_2^*\Theta) = 4k$. The "$k/4$ deficit" is withdrawn. The exponent $m = 3$ of Proposition W.3 is withdrawn in Remark Y.5: it arose from adding two different pullbacks $(1\times a)^*\mathcal{P}$ and $(1\times\psi_1)^*\mathcal{P}$. The five-normalisation-freedom analysis of Theorem W.1 remains valid and is compressed by Theorem X.4 to "$\mathcal{P}^\vee$ with $3\nmid m$".

Appendix X. Prym reduction and the conditional closure of Route (a)

We keep Appendices S, T, U, V, W. In particular $E = \phi_*\omega_{X/Y} = F_1\oplus\mathcal{O}_Y^3$, $F_1 = p_{2*}(K+\mathcal{L})$ has rank 3 and degree k, and by Theorem U.3 and V.1 $\Theta\in\{8,10,12\}$ with

$$\Theta = 12 \iff \mu_1(F_1) \ge k/2 \iff F_1 \text{ has a degree-zero quotient} \iff u_f \ge 4.$$

We determine the morphism ψ_1 of §5.1 of [1], correct Theorem W.2, and prove a conditional exclusion of $\Theta = 12$.

Theorem X.1 (ψ_1 is a Prym map)**.** *Let $a : Y_1 \hookrightarrow J_1$ be an Abel–Jacobi embedding and ν the involution of the étale double cover $Y_1 \to Y_0$. Then*

$$\psi_1 = \pm(\mathrm{id}-\nu^*)\circ a + \mathrm{const},$$

and $\psi_1(Y_1)$ is contained in the Prym elliptic curve $P = \ker(\mathrm{Nm})^0 \subset J_1$, with $\dim P = g(Y_1) - g(Y_0) = 3-2 = 1$. Moreover $\psi_2(Y) \subset P + \tau$ for a $J_1[2]$-translate τ, and $\deg(a_P : Y_1 \to P) = 2$.

Proof. By §5.1, $\Gamma = \Delta_1 \sqcup \Delta_2$ with Δ_2 the graph of ν, so $\Gamma - 2\Delta_1 = \Delta_2 - \Delta_1$ and

$$\mathcal{O}_{Y_1\times Y_1}(\Delta_2 - \Delta_1) = (1\times\psi_1)^*\mathcal{P} + \pi_2^*(L). \tag{X.1}$$

Fixing the second variable y, the left side restricts to $\mathcal{O}_{Y_1}(\nu(y) - y) = a(\nu y) - a(y) = (\nu^* - \mathrm{id})a(y)$; the term $\pi_2^*(L)$ only translates by a constant. Hence $\psi_1 = \pm(\mathrm{id}-\nu^*)\circ a$ up to translation.

The $+1$–eigenspace of ν^* on $H^0(\Omega_{Y_1})$ has dimension $g(Y_0) = 2$ and the -1–eigenspace has dimension $g(Y_1) - g(Y_0) = 1$; therefore $\phi_\nu := \mathrm{id} - \nu^*$ kills the J_0–direction and equals

[2] on the Prym direction. Thus $\operatorname{im}\psi_1 \subset P := \operatorname{im}\phi_\nu = \ker(\mathrm{Nm})^0$, an elliptic curve. From $\psi_1 \circ \psi = [2] \circ \psi_2$ and $\psi_1 = [2] \circ a_P$ we obtain $\psi_2 = a_P \circ \psi + \tau$ with $\tau \in J_1[2]$ locally constant, hence constant on the connected curve Y.

Finally $a_P \circ \nu = [-1] \circ a_P$, so composing with $P \to P/\{\pm 1\} = \mathbb{P}^1$ factors through $Y_0 = Y_1/\nu$. The resulting map $Y_0 \to \mathbb{P}^1$ is the hyperelliptic map of the genus-2 curve Y_0, of degree 2; as $P \to \mathbb{P}^1$ also has degree 2, we get $\deg a_P = 2$. □

Theorem X.2 (k is a power of two)**.** *$k = \deg(Y/Y_1)$ divides $|J_1[2]| = 2^{2\dim J_1} = 2^6 = 64$; hence $k = 2^j$ with $0 \le j \le 6$, and in particular $3 \nmid k$.*

Proof. Y is a connected component of $Z = Y_1 \times_{J_1,[2]} J_1 = \{(y,z) : \psi_1(y) = 2z\}$. The fibre of $Z \to Y_1$ over y is $[2]^{-1}(\psi_1(y))$, a coset of $J_1[2]$, of cardinality $2^6 = 64$. As $\psi : Y \to Y_1$ is finite étale and $Y \subset Z$ is a connected component, $\deg\psi$ divides 64. □

Theorem X.3 (Correction of W.2)**.** *With the standard Prym identity $\Theta|_P = 2\Xi$ (Ξ the principal polarization of the elliptic curve P, $\deg\Xi = 1$) one has*

$$\deg(\psi_1^*\Theta) = 16, \qquad \boxed{\deg(\psi_2^*\Theta) = 4k}.$$

Consequently the value $3k/4$ of Theorem W.2 is incorrect: W.2 assumed $\psi_1 = a$ with $\deg(a^*\Theta) = 3$, whereas Theorem X.1 gives $\psi_1 = [2]\circ a_P$. The "deficit $k/4$" of Appendix W is withdrawn; the correct tension is the multiplicative factor 4, equal to the degree of the Prym isogeny $J_0 \times P \to J_1$, and its direction is reversed: a pure Picard pullback would have degree $4k$, larger *than $\deg F_1 = k$.*

Proof. Route (i): by Theorem X.1, $\psi_2 = a_P \circ \psi + \tau$, so

$$\deg(\psi_2^*\Theta) = \deg\psi \cdot \deg\big(a_P^*(2\Xi)\big) = k \cdot 2 \cdot \deg(a_P^*\Xi) = k \cdot 2 \cdot 2 = 4k,$$

using $\deg(a_P^*\Xi) = \deg a_P \cdot \deg\Xi = 2$. Route (ii): $\psi_1 = [2] \circ a_P$ gives $\deg(\psi_1^*\Theta) = \deg(a_P^*[2]^*\Theta) = \deg(a_P^*(4\Theta|_P)) = \deg(a_P^*(8\Xi)) = 16$. Since $\psi_1 \circ \psi = [2] \circ \psi_2$,

$$16k = \deg\big(\psi^*\psi_1^*\Theta\big) = \deg\big(\psi_2^*[2]^*\Theta\big) = 4\deg(\psi_2^*\Theta),$$

hence $\deg(\psi_2^*\Theta) = 4k$. The two routes agree. □

Theorem X.4 (Integrality selects the normalisation)**.** *Write*

$$\mathcal{L} = p_1^*\mathcal{O}(c) \otimes (1 \times \psi_2)^*\mathcal{P}^{\otimes m} \otimes \pi_Y^* M, \qquad F_1 = M \otimes \psi_2^*(V_m),$$

with $V_m = [m]^*E_5$ or $[m]^*E_5^\vee$ of rank 3 and $c_1(E_5) = -\Theta$ (Schwarzenberger, $n = 5 > 2g_1 - 2 = 4$). Then $\deg F_1 = k$ forces

$$3\deg M = k\,(1 \pm 4m^2).$$

Since $3 \nmid k$ (Theorem X.2), the branch $c_1 = -\Theta$ has no *integral solution for any m, while the branch $c_1 = +\Theta$ ($\mathcal{P}^\vee$, $E_5^\vee$) is solvable exactly when $m^2 \equiv 1 \pmod 3$, i.e. $3 \nmid m$; then $\deg M = k(1 - 4m^2)/3$ (for $m = \pm 1$, $\deg M = -k$). In particular the exponent $m = 3$ guessed in Proposition W.3 is excluded, and the five normalisation freedoms of Theorem W.1 collapse to "$\mathcal{P}^\vee$ with $3 \nmid m$".*

Proof. $\deg\lambda = 1$ on a fibre (§5.2: $2\lambda = P_1 + P_2$), so $\deg(K + \mathcal{L})|_{Y_1} = 4 + 1 = 5$ and $\mathrm{rk}\,F_1 = \chi = 5 + 1 - 3 = 3$. As $\mathcal{P}^{\otimes m} = (1 \times [m])^*\mathcal{P}$, base change gives $V_m = [m]^*E_5^{(\vee)}$ (up to translation), and $c_1([m]^*E_5) = -m^2\Theta$, $c_1([m]^*E_5^\vee) = +m^2\Theta$. With Theorem X.3,

$$\deg F_1 = \mathrm{rk}(V_m)\deg M + \deg(\psi_2^*V_m) = 3\deg M \mp 4m^2 k = k.$$

Modulo 3, $1 + 4m^2 \equiv 1 + m^2 \in \{1, 2\}$ (as $m^2 \equiv 0, 1$), so the minus branch is never divisible by 3; $1 - 4m^2 \equiv 1 - m^2$ vanishes iff $m^2 \equiv 1$. Since $3 \nmid k$, an integral $\deg M$ exists exactly in these cases. □

Theorem X.5 (Conditional closure)**.** *Assume the standard chain: (a)* $\Theta|_P = 2\Xi$ *and* $\deg a_P = 2$*; (b)* $\mathcal{L}$ *is of the form in Theorem X.4, so* $F_1 \cong M \otimes \psi_2^*(V_m)$*; (c) a semihomogeneous bundle on* J_1 *restricts to a semihomogeneous (hence semistable) bundle on the elliptic curve* $P + \tau$*, and finite pullback in characteristic* 0 *preserves semistability. Then* F_1 *is semistable of slope* $k/3$*, and*

$$\Theta = 12 \text{ is excluded}, \qquad \Theta \in \{8, 10\}, \qquad C \in \{36,\ 36/7\}, \qquad C \geq \tfrac{36}{7} > 3, \qquad u_f = 3.$$

In particular the constant 3 *of the arithmetic BMY/Parshin bound is unattainable inside this construction. The conclusion is independent of the remaining ambiguity in* m *(all* $3 \nmid m$ *give semihomogeneous* V_m*) and does not use the truth of Theorem A.*

Proof. By Mukai, $E_5^\vee$ and every $[m]^* E_5^\vee$ are semihomogeneous, equivalently semistable with vanishing projective Chern class [44]. By (c), $V_m|_{P+\tau}$ is semistable on an elliptic curve; ψ_2 is finite and char $= 0$, so $\psi_2^* V_m$ is semistable; twisting by the line bundle M preserves semistability. Hence $\mu_1(F_1) = \deg F_1 / \operatorname{rk} F_1 = k/3$. But $\Theta = 12$ requires (Theorem V.1) $\deg \phi_* L = k$ with $\phi_* L$ of rank 2, i.e. a subbundle of $F_1 \oplus \mathcal{O}_Y^3$ of slope $k/2$, forcing $\mu_1(F_1) \geq k/2 > k/3$ — a contradiction. With the nef bound $\Theta \leq 12$ of Theorem U.3 and $\Theta \in 2\mathbb{Z}$, the only values are $8, 10$, giving $C(\Theta) = 36/(3\Theta - 23)$ as stated. □

Remark X.6 (Status and honesty). Assumptions (a)–(c) are standard but are quoted, not re-proved here; the closure is therefore conditional (tag C). The unconditional parts are Theorems X.1–X.4, the exact rational computations, and the implication "F_1 semistable $\Rightarrow \Theta \leq 10$". If $\mathcal{L}$ contained an extra extension term making F_1 non-semistable, $\Theta = 12$ would survive; this is now the single remaining place where the closure could fail. In particular Table 1, Remark V.6, the abstract, and Open problem (2) of the main text should be updated from "$\Theta = 12$ survives" to "$\Theta = 12$ is excluded under the Prym–semihomogeneity chain (tag C)", while Theorem V.3 (the general unitary-rank bounds) remains correct as a statement about the literature.

Even granting the closure, the output is $\bar{\omega}^2 \leq C\delta(F') + O(d')$ with $C > 3$ in the $\delta(F)$ normalisation, pointwise and inside this construction; it does *not* settle the general Problem 10.1. All *abc*-side conclusions, the twelve No-Go reasons, and the status of *abc* are unchanged. This appendix does not independently verify the proofs of Soulé, Fujita, Mukai, Mumford or Schwarzenberger–Mattuck.

Remark X.7 (Superseded by Appendices Y and Z). Theorem X.5 is tagged C. Appendix Y classifies the 2-torsion ambiguity completely (2^{16} choices, all of translation $\oplus$ base-twist type) and proves it cannot break semistability; Appendix Z then proves the required semistability from translation invariance alone (Theorem Z.2) and removes all dependence on d, m and h (Theorem Z.4). The closure $\Theta = 12$ excluded, $C \geq 36/7 > 3$, $u_f = 3$ is therefore U. Remark X.6's statement that a loophole remains is withdrawn; see Remark Z.5 for the final ledger.

Appendix Y. The square-root ambiguity does not break semistability

We keep Appendices S–X. Theorem X.5 closes Route (a) conditionally; Remark X.6 isolated one possible loophole: the line bundle $\mathcal{L}$ of §5.1 of [1] might contain an "extra extension term" making $F_1 = p_{2*}(K + \mathcal{L})$ non-semistable. We show that no such term can exist.

Y.1. **The precise question.** §5.1 gives $2\mathcal{L} = \mathcal{O}_{Y_1 \times Y}(\widetilde{\Gamma})$, $\rho_* \mathcal{O}_X = \mathcal{O} \oplus \mathcal{L}^*$ (the *dual*), and ρ branched along $\widetilde{\Gamma}$. Since ψ is étale, $\widetilde{\Gamma}$ is reduced and the standard double-cover construction applies. In $\operatorname{Pic}(Y_1 \times Y)$ the equation $2\mathcal{L} = \mathcal{O}(\widetilde{\Gamma})$ determines $\mathcal{L}$ up to

$$\mathcal{L} = \mathcal{L}_{\text{model}} \otimes \eta, \qquad \eta \in \operatorname{Pic}(Y_1 \times Y)[2]. \tag{Y.1}$$

The loophole is the existence of an η for which $F_1 = p_{2*}(K + \mathcal{L}_{\mathrm{model}} \otimes \eta)$ is not semistable.

Theorem Y.1 (Complete classification of the ambiguity)**.** $\mathrm{Pic}(Y_1 \times Y)[2] = p_1^*\mathrm{Pic}^0(Y_1)[2] \oplus \pi_Y^*\mathrm{Pic}^0(Y)[2]$, *so that* $\eta = p_1^*\xi \otimes \pi_Y^*N$ *with* $\xi \in J_1[2]$, $N \in \mathrm{Pic}^0(Y)[2]$. *Since* $g(Y_1) = 3$ *and* $g(Y) = 2g(Y_1) - 1 = 5$, *there are exactly* $2^{2\cdot 3} \cdot 2^{2\cdot 5} = 2^{16} = 65536$ *choices, all of "translation* $\oplus$ *base twist" type; there is no third kind.*

Proof. For smooth projective curves $\mathrm{NS}(Y_1 \times Y) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathrm{Hom}(J_1, J_Y)$, and $\mathrm{Hom}(J_1, J_Y)$ is a free $\mathbb{Z}$-module; hence NS is torsion-free and all 2-torsion of Pic lies in Pic^0. Künneth gives $H^1(Y_1 \times Y, \mathcal{O}) = H^1(Y_1, \mathcal{O}) \oplus H^1(Y, \mathcal{O})$, so $\mathrm{Pic}^0(Y_1 \times Y) = J_1 \times \mathrm{Pic}^0(Y)$. The count uses $|A[2]| = 2^{2\dim A}$ and $g(Y) = 2g(Y_1) - 1$ for the étale double cover ψ. □

Theorem Y.2 (GRR determination of $c_1(V_m)$)**.** *Let* $C = Y_1$, $J = J_1$, $\mathcal{P}$ *a normalised Poincaré bundle,* p, q *the projections,* $\eta_0 = p^*[\mathrm{pt}]$, $\ell = c_1(\mathcal{P})$, *and* $V_m = q_*(p^*L_n \otimes \mathcal{P}^{\otimes m})$ *with* $\deg L_n = n$. *Then*

$$\mathrm{rk}\, V_m = n + 1 - g(Y_1), \qquad c_1(V_m) = -m^2\Theta,$$

with no linear term in m. *In particular* $\mathrm{rk}\, V_m = 3$ *for* $n = 5$, $g = 3$, *and* $\deg \psi_2^* V_m^\vee = m^2 \deg \psi_2^*\Theta = 4m^2 k$ *by Theorem X.3.*

Proof. Put $\mathcal{F} = p^*L_n \otimes \mathcal{P}^{\otimes m}$, so $c_1(\mathcal{F}) = n\eta_0 + m\ell$. Since $H^3(C) = H^4(C) = 0$ we have $\eta_0^2 = 0$, $\eta_0\ell = 0$, $\ell^3 = 0$; hence $c_1(\mathcal{F})^2 = m^2\ell^2$ and $c_1(\mathcal{F})^3 = 0$. With $\mathrm{td}(T_C) = 1 + (1-g)\eta_0$,

$$\mathrm{ch}(\mathcal{F})\,\mathrm{td}(T_C) = 1 + (n + 1 - g)\eta_0 + m\ell + \tfrac{1}{2}m^2\ell^2.$$

Relative GRR for q (fibre dimension one) gives $\mathrm{ch}(q_!\mathcal{F}) = q_*(\mathrm{ch}(\mathcal{F})\mathrm{td})$. Now $q_*(\eta_0) = 1$, $q_*(\ell) = 0$ (the C-degree of ℓ is one), and the normalisation $\frac{1}{2}q_*(\ell^2) = -\Theta$ (equivalently $c_1(E_1) = -\Theta$, Schwarzenberger) yields $\mathrm{ch}(q_!\mathcal{F}) = (n + 1 - g) - m^2\Theta$. As $\mathrm{ch}_1 = c_1$ and $\mathrm{ch}_0 = \mathrm{rk}$, the claim follows; the last formula uses $c_1(V_m^\vee) = +m^2\Theta$. □

Theorem Y.3 (Semihomogeneity from the translation formula)**.** *For every* $x \in J_1$, $t_x^*V_m \simeq V_m \otimes L_{mx}$, *where* $L_{mx} \in \mathrm{Pic}^0(Y_1)$ *corresponds to* mx *under* $J_1 \simeq \widehat{J_1}$. *Hence* V_m *(and* $V_m^\vee$*) is semihomogeneous in Mukai's sense and therefore semistable, for every* m*; no external theorem is needed.*

Proof. The normalised Poincaré bundle satisfies $t_x^*\mathcal{P} = \mathcal{P} \otimes q^*L_x$, hence $t_x^*(\mathcal{P}^{\otimes m}) = \mathcal{P}^{\otimes m} \otimes q^*L_x^{\otimes m} = \mathcal{P}^{\otimes m} \otimes q^*L_{mx}$. By the projection formula,

$$t_x^*V_m = q_*\big(t_x^*\mathcal{P}^{\otimes m} \otimes p^*L_n\big) = q_*\big(\mathcal{P}^{\otimes m} \otimes p^*L_n\big) \otimes L_{mx} = V_m \otimes L_{mx}.$$

Semihomogeneity is precisely the condition $t_x^*V \simeq V \otimes$ (a line bundle in $\mathrm{Pic}^0(J)$); on an abelian variety semihomogeneous bundles are semistable [44]. Taking duals preserves the property. □

Theorem Y.4 (The loophole is closed)**.** *Assume (as in X.5) that for one model* $F_1^{(0)} = M \otimes \psi_2^*(V_m)$ *with* V_m *as above. Then for* every $\eta \in \mathrm{Pic}(Y_1 \times Y)[2]$ *the corresponding* $F_1 = p_{2*}(K + \mathcal{L}_{\mathrm{model}} \otimes \eta)$ *is of the form*

$$F_1 = (M + N) \otimes \psi_2^*(t_\xi^*V_m), \qquad \xi \in J_1[2],\ N \in \mathrm{Pic}^0(Y)[2],$$

hence is semistable of slope $k/3$. *Consequently* $\mu_1(F_1) = k/3 < k/2$, $\Theta = 12$ *is excluded, and*

$$\Theta \in \{8, 10\}, \qquad C \geq \tfrac{36}{7} > 3, \qquad u_f = 3.$$

The conclusion is independent of the value of m.

Proof. By Theorem Y.1, $\eta = p_1^*\xi \otimes \pi_Y^* N$. Then $K + \mathcal{L}_{\mathrm{model}} \otimes \eta$ differs from $K + \mathcal{L}_{\mathrm{model}}$ by tensoring the Y_1-factor with ξ (which replaces L_n by $L_n \otimes \xi$, i.e. translates V_m by ξ) and twisting the base by N. Projection formula and base change give the displayed formula. By Theorem Y.3, $t_\xi^* V_m$ is semihomogeneous; twisting by the line bundle $M + N$ and pulling back by the finite morphism ψ_2 (char. 0) preserves semistability, as in X.5. Therefore $\mu_1 = k/3$, and Theorem V.1 ($\Theta = 12 \iff \mu_1 \geq k/2$) excludes $\Theta = 12$; Theorem U.3 gives $\Theta \leq 12$ and $\Theta \in 2\mathbb{Z}$ gives $\{8, 10\}$, with $C(\Theta) = 36/(3\Theta - 23)$. □

Remark Y.5 (Corrections to Appendices W and X). Proposition W.3 expanded $\mathcal{O}(\widetilde{\Gamma})$ and obtained the exponent 6 of $\mathcal{P}$, i.e. $m = 3$, whereas Theorem X.4 excludes $3 \mid m$. The conflict is resolved: W.3 added two different pullbacks $(1 \times a)^*\mathcal{P}$ (from the diagonal decomposition) and $(1 \times \psi_1)^*\mathcal{P}$ (from §5.1), but Theorem X.1 gives $\psi_1 = \phi_\nu \circ a$ with $\phi_\nu|_{J_0} = 0$, $\phi_\nu|_P = [2]$, so $(1 \times \psi_1)^*\mathcal{P} \neq (1 \times a)^*\mathcal{P}$. Hence the specific exponent 6 and $m = 3$ of W.3 are *withdrawn*; m remains undetermined by the published text (this is an explicit item in the remaining conditional ledger). Correspondingly, the phrase in X.4 "the exponent $m = 3$ guessed in W.3 is excluded" should be replaced by "under the model assumption integrality excludes $3 \mid m$, while W.3's $m = 3$ comes from an invalid derivation and is withdrawn". Theorem Y.4 holds for every m, so the closure is unaffected.

The conditional ledger of X.5 shrinks from four items to two: (a) the Prym constants $\Theta|_P = 2\Xi$, $\deg a_P = 2$; (b) the identification of $\mathcal{L}_{\mathrm{model}}$ with a V_m-type pushforward (with m undetermined), together with the standard facts that a semihomogeneous bundle restricts to a semistable bundle on the elliptic curve $P + \tau$ and that finite pullback preserves semistability. Items $c_1(E_5) = -\Theta$, $\mathrm{rk}\, V_m$ and the semihomogeneity on J_1 are now unconditional by Theorems Y.2, Y.3. All *abc*-side conclusions and the status of *abc* are unchanged; this appendix does not independently verify the proofs of Soulé, Fujita, Mukai, Mumford or Schwarzenberger–Mattuck.

Appendix Z. Removal of the remaining conditional steps

We keep Appendices S–Y. The closure of Route (a) ($\Theta = 12$ excluded, $C \geq 36/7 > 3$, $u_f = 3$) was obtained in Theorems X.5 and Y.4 conditionally. We remove every condition: the Prym constant d, the exponent m, the identification of $\mathcal{L}_{\mathrm{model}}$, and the quoted semistability lemmas.

Theorem Z.1 (The most general $\mathcal{L}$)**.** *Let $h \in \mathrm{Hom}(J_1, J_1)$ be the homomorphism obtained from the* NS*-component* $\mathrm{Hom}(J_1, J_Y)$ *of* $\mathrm{Pic}(Y_1 \times Y)$ *by transposition via the principal polarisations and the Abel map of Y. Then every line bundle on $Y_1 \times Y$ can be written*

$$\mathcal{L} = p_1^* L_1 \otimes \pi_Y^* M \otimes \mathcal{P}_h, \qquad \mathcal{P}_h = (1 \times h)^* \mathcal{P}_{J_1}, \quad \deg L_1 = 1,$$

up to the 2-torsion ambiguity classified in Theorem Y.1. Put $V_h = q_*(p^* L_n \otimes \mathcal{P}_h)$, $n = \deg(K + L_1) = 5$. Then

$$t_x^* V_h \simeq V_h \otimes L_{h(x)} \quad (x \in J_1), \qquad F_1 = p_{2*}(K + \mathcal{L}) = M \otimes \psi_2^*(V_h),$$

so V_h is semihomogeneous for every *h. In particular no identification of $\mathcal{L}_{\mathrm{model}}$ with a specific V_m is needed, and the value of m is irrelevant.*

Proof. By Künneth, $\mathrm{Pic}(Y_1 \times Y) = p_1^*\mathrm{Pic}(Y_1) \oplus \pi_Y^*\mathrm{Pic}(Y) \oplus \mathrm{NS}^{\mathrm{cross}}$, where the cross term is $\mathrm{Hom}(J_1, J_Y)$; under $J_Y \simeq \widehat{J}_Y$, $J_1 \simeq \widehat{J}_1$ and $a_Y : Y \to J_Y$ it is represented by a homomorphism $h : J_1 \to J_1$ and the pulled-back Poincaré bundle $\mathcal{P}_h$. The normalised Poincaré bundle satisfies $t_x^*\mathcal{P} = \mathcal{P} \otimes q^* L_x$, hence $t_x^*\mathcal{P}_h = \mathcal{P}_h \otimes q^* L_{h(x)}$; the projection formula gives $t_x^* V_h = V_h \otimes L_{h(x)}$. Base change and the projection formula give the displayed expression for F_1. Relative GRR as in Theorem Y.2 gives $\mathrm{rk}\, V_h = n + 1 - g(Y_1) = 3$ for any h. □

Theorem Z.2 (Translation invariance ⇒ semistable)**.** *Let E be an elliptic curve and W a vector bundle with $t_x^*W \simeq W \otimes L_{h(x)}$, $L_{h(x)} \in \mathrm{Pic}^0(E)$, for all $x \in E$. Then W is semistable. The proof uses only Harder–Narasimhan uniqueness, the identity $t_x^*D \otimes D^{-1} \simeq \mathcal{O}_E(\deg D \cdot (x-0))$, and* det*; it does not use Mukai's theorem, Atiyah's classification, or the Prym constant d.*

Proof. Let $0 = W_0 \subset \cdots \subset W_r = W$ be the HN filtration, with slopes $\mu_1 > \cdots > \mu_r$. Since t_x^* is an equivalence, $t_x^*W_i$ is the HN filtration of $t_x^*W = W \otimes L_{h(x)}$; as $\deg L_{h(x)} = 0$, twisting does not change slopes, so $W_i \otimes L_{h(x)}$ is also the HN filtration. By uniqueness, $t_x^*W_i \simeq W_i \otimes L_{h(x)}$. Taking determinants,

$$t_x^*D_i \otimes D_i^{-1} = L_{h(x)}^{\otimes r_i}, \qquad D_i = \det W_i,\ r_i = \mathrm{rk}\, W_i.$$

For any $D \in \mathrm{Pic}(E)$ the map $x \mapsto t_x^*D \otimes D^{-1}$ is the homomorphism $E \to \widehat{E} \simeq E$ of degree $\deg D$, i.e. $\mathcal{O}_E(\deg D\,(x-0))$. The right-hand side is linear in x with coefficient $r_i\lambda$, where λ is the (single) constant determined by h and the polarisation — for $h = [m]$ and $\Theta_1|_E = d\Xi$ one has $\lambda = dm$, but the value is immaterial. Hence $\deg D_i = r_i\lambda$ for every i, so $\mu_i = \deg W_i/r_i = \lambda$ for all i. All HN slopes coincide, forcing $r = 1$, i.e. W is semistable. □

Theorem Z.3 (Finite pullback preserves semistability)**.** *Let $f : Y \to E$ be a finite morphism of smooth curves over a field of characteristic* 0*, and let V be semistable on E. Then f^*V is semistable.*

Proof. The trace map $\frac{1}{\deg f}\mathrm{tr} : f_*\mathcal{O}_Y \to \mathcal{O}_E$ splits $\mathcal{O}_E \hookrightarrow f_*\mathcal{O}_Y$, so $f_*\mathcal{O}_Y = \mathcal{O}_E \oplus \mathcal{E}^-$; by Riemann–Hurwitz $\deg(f_*\mathcal{O}_Y) = -\frac{1}{2}\deg R \le 0$, hence every HN slope of $\mathcal{E}^-$ is ≤ 0. Consequently $f_*f^*V = V \oplus (V \otimes \mathcal{E}^-)$ has maximal slope $\mu(V)$ whenever $\mu(V) \ge 0$. If $S \subset f^*V$ is a maximal-slope subbundle, then $f_*S \subset f_*f^*V$ and $\mu(S) = \mu(f_*S) \le \mu(V) \le \deg f \cdot \mu(V) = \mu(f^*V)$, so f^*V is semistable. For $\mu(V) < 0$ apply the same argument to $V^\vee$ and to the minimal-slope quotient of f^*V. □

Theorem Z.4 (d-independence and unconditional closure)**.** *Write $\Theta_1|_P = d\Xi$ with $d \in \mathbb{Z}_{>0}$ and $\deg \Xi = 1$; $\deg a_P = 2$ is unconditional (Theorem X.1 and the Riemann–Hurwitz check). Then*

$$\deg(\psi_2^*\Theta_1) = 2dk, \qquad 3\deg M = k(1 \pm 2dm^2),$$

and, since $k = 2^j$ by Theorem X.2, the integrality chain can start iff $d \equiv 2 \pmod 3$ ($d = 2$ being the smallest solution); the value $3k/4$ of Theorem W.2 corresponds to no integer d. Nevertheless the closure is independent of d, m and h:

$\deg F_1 = k$, $\mathrm{rk}\, F_1 = 3$, *F_1 semistable (Thms. Z.1–Z.3)* $\Longrightarrow$ $\mu_1 = \frac{k}{3} < \frac{k}{2}$ $\Longrightarrow$ $\Theta = 12$ *excluded.*

Hence $\Theta \in \{8, 10\}$, $C \ge 36/7 > 3$ and $u_f = 3$ hold unconditionally (tag U*). No step uses the truth of Theorem A.*

Proof. $\deg(a_P^*\Theta_1) = 2d$, and $\psi_1 = [2]_P \circ a_P$ gives $\deg(\psi_1^*\Theta_1) = 8d$; from $\psi_1 \circ \psi = [2] \circ \psi_2$, $8dk = 4\deg(\psi_2^*\Theta_1)$, so $\deg(\psi_2^*\Theta_1) = 2dk$. Substituting $c_1(V_h^\vee) = +h^*\Theta$ into $\deg F_1 = k$ gives the stated integrality relation (for $h = [m]$, $h^*\Theta = m^2\Theta$). Modulo 3 and using $3 \nmid k$, the $-\Theta$ branch is solvable iff $1 + 2dm^2 \equiv 0$ and the $+\Theta$ branch iff $1 - 2dm^2 \equiv 0$; with $m^2 \in \{0, 1\}$ this forces $2d \equiv 1$, i.e. $d \equiv 2$. The value $3k/4$ needs $d = 3/8 \notin \mathbb{Z}$.

For the closure, Theorems Z.1, Z.2, Z.3 give semistability of $F_1 = M \otimes \psi_2^*(V_h)$ for every h; $\deg F_1 = k$ follows from $\omega^2 = 12k$ (Proposition 8 of [1]), Noether's formula for a smooth fibration and $\deg \mathcal{O}_Y^3 = 0$; $\mathrm{rk}\, F_1 = 3$ from Riemann–Roch on a fibre ($\deg(K + \mathcal{L}) = 5$, $\chi = 3$). Thus $\mu_1 = k/3$, while Theorem V.1 states $\Theta = 12 \iff \mu_1 \ge k/2$; Theorem U.3 gives $\Theta \le 12$ and $\Theta \in 2\mathbb{Z}$ gives the rest, with $C(\Theta) = 36/(3\Theta - 23)$. Neither d nor m nor h occurs in this chain. □

Remark Z.5 (Final conditional ledger). After Appendices G–Z the only remaining conditional items are:

(1) Th.A*: Theorem 7.5/9.2 (polynomial-type weak *abc*) takes Soulé's Theorem A as its sole external input. This cannot be removed: all unconditional bounds are of exponential nature (Stewart–Yu, Pasten), the existence of a polynomial-type bound is open (Theorem H.2), and the trichotomy of Theorem N.4 shows the conditionality can only be shifted to an arithmetic BMV hypothesis or to *abc* itself.
(2) (32) *strict* isometry: the quoted sentence of §6.3.6 is an assertion; Theorem H.3 and Appendix I prove the usable $O(d')$-version, and all coefficients $36, 12, 24$ are unaffected. It is retained as a normalisation note, not as a mathematical gap.

Everything else is U (or U+A5.2, the counting convention for the twelve No-Go reasons). In particular Open problems (2a)–(2c) are closed: $\deg a_P = 2$ is unconditional, d is not a premise, and m is irrelevant. The status of *abc* remains open. This appendix does not independently verify the proofs of Theorem A, Fujita's nefness theorem, Harder–Narasimhan theory, the see-saw principle, or Riemann–Hurwitz; it uses them as standard theorems.

End of Appendices N–Z. Final conditional ledger: Remark Z.5.

Appendix AA. The Szpiro-side constant has no independent degree of freedom

We keep the notation of Appendices F–Z. Theorem 3 of [**?**] gives $\log|\Delta| \le (144(1+\log 24)+\varepsilon)\log N + O_\varepsilon(1)$, and the Robin substitution of the companion note replaces $1+\log 24$ by an $o(1)$ term, giving $144+\varepsilon$. We ask whether anything further can be pushed down.

Theorem AA.1 (The conversion dictionary)**.** *Let E be the exponent of a polynomial-type bound $c \le A_\varepsilon N^{E+\varepsilon}$ and let $N = 3\Theta - 23$ be the denominator of the discrete coefficient $t(\Theta) = 12/N$ of Theorem 5.7. Then, inside the derivation chain of* [**?**]*,*

$$E = \frac{24}{N}, \qquad C_{\mathrm{BMY}} = \frac{36}{N} = \tfrac{3}{2}E, \qquad C_{\mathrm{Szp}} = \frac{144}{N} = 6E = 4C_{\mathrm{BMY}}.$$

In particular the Szpiro constant, the arithmetic-BMY coefficient and the abc exponent are three scalings of the same *discrete set $\{1/N : N \equiv 1 \pmod 3\}$; there is no Szpiro-side degree of freedom independent of E.*

Proof. For coprime $a+b=c$ the Frey curve has $\Delta_E = 16(abc)^2$ and conductor $N_E = \mathrm{rad}(abc)\cdot 2^{O(1)} = N\cdot 2^{O(1)}$. Since $a, b < c$ we have $abc \le c^3$, hence

$$\log|\Delta_E| \le 6\log c + O(1) \le (6E+\varepsilon')\log N + O_\varepsilon(1),$$

which is precisely the direction "Theorem B $\Rightarrow$ Theorem 3" used in [**?**] (through Moret-Bailly's $A_{CD} = (1+\log n)A_{ME}$, $n=24$). Thus $C_{\mathrm{Szp}} \le 6E$, with equality along the chain. By Theorem 5.7, $t = 12/N$; by Proposition 7.3 and Theorem F.1, $E = 2t = 24/N$ and $C_{\mathrm{BMY}} = 36/N$. Hence

$$C_{\mathrm{Szp}} = 6E = \frac{144}{N}, \qquad \frac{C_{\mathrm{BMY}}}{E} = \frac{36/N}{24/N} = \tfrac{3}{2}, \qquad \frac{C_{\mathrm{Szp}}}{C_{\mathrm{BMY}}} = \frac{144/N}{36/N} = 4,$$

which are the three displayed relations. The congruence $N = 3\Theta - 23 \equiv 1 \pmod 3$ is Theorem 5.7. □

Remark AA.2 (What is *not* claimed). We use only the implication $E \mapsto C_{\mathrm{Szp}} \le 6E$. The reverse implication for a general semistable elliptic curve is weaker ($c \le N^{C/2+\varepsilon}$ from $\Delta \ge 16c^2$) and is not needed; no general Szpiro⇔*abc* equivalence at exponent 6 is asserted here. All statements are internal to the derivation chain of [**?**].

Theorem AA.3 (The factor $1 + \log 24$ is already exhausted)**.** $144(1 + \ln 24) = 601.639\ldots = 144 + 457.639\ldots$, *where* $458 = 144 \ln 24$ *is the value verbatim-checked in Appendix R. Robin's bound* $\omega(n) \le 1.3841 \log n / \log\log n$ *replaces* $1 + \log 24$ *by an* $O(\log N / \log\log N)$ *quantity which, after division by* $\log N$*, contributes* $o(1)$*; hence* $601.64 \to 144 + \varepsilon$*. Applying the same substitution to all four rows gives* $144/N + \varepsilon$ *with no further remainder. Consequently there is no estimation slack left in this factor.*

Proof. The numerical identity is elementary: $144(1+\ln 24) - 144 = 144 \ln 24 = 457.639\ldots$, and $144 + 457.639\ldots = 601.639\ldots$, matching the 601.64 recorded in Appendix R. Moret-Bailly's Th. 6.4 (Appendix R, Theorem R.2) gives $A_{CD} = (1 + \log n) A_{ME}$; Robin's bound makes the logarithmic factor grow like $\log N / \log\log N$, so $(1 + \log n)/\log N \to 0$. Since Theorems F.1 and AA.1 show that 144 itself is $6 \cdot 24$ with both factors locked, the whole improvement chain of Theorem 3 terminates at $144 + \varepsilon$. □

Theorem AA.4 (The exact remaining room after Theorem Z.4)**.** *Theorem U.3 gives* $\Theta \le 12$ *and Theorem Z.4 excludes* $\Theta = 12$ *unconditionally; with* $\Theta \in 2\mathbb{Z}$ *one has* $\Theta \in \{8, 10\}$*, hence*

$$N \in \{1, 7\}, \qquad C_{\mathrm{Szp}} \in \Big\{144,\ \tfrac{144}{7} \approx 20.57\Big\}.$$

The only possible improvement is therefore a single discrete *factor of* 7*, realised exactly when* $\Theta = 10$*, i.e.* $\sum_{j=1}^{4} n_j = 5$ *with partition* $(1, 1, 1, 2)$*: one exceptional fibre has* $n_j = 2$ *(*$\kappa_M = 0$*,* $\mathrm{ord}_0(x_1 - x_2) = 2$*,* $\dim_k R[\pi^2] = 4$*) and the Galois orbit is non-transitive. If transversality, Galois transitivity, or the companion note's "*$T = 8k$ *exact" holds, then all* $n_j = 1$*,* $\Theta = 8$*, and* $C_{\mathrm{Szp}} = 144$ *is unique.*

Proof. By Theorem AA.1, $C_{\mathrm{Szp}} = 144/N$ with $N = 3\Theta - 23$; $\Theta = 8, 10$ give $N = 1, 7$. By Theorem Y.1/Y.4 and Appendix S, $\Theta = 2\sum_{j=1}^{4} n_j$ with $n_j \ge 1$, so $\Theta = 10$ forces $\sum n_j = 5$, whose only partition into four positive integers is $(1, 1, 1, 2)$; $n_j = 2$ is equivalent, by Theorem S.1, to a degenerate fibre with the stated jet order. A transitive Galois action makes all n_j equal, and $\Theta \le 10$ then forces $\Theta = 8$; transversality (Theorem 6.4) does the same. The three conditions are recorded in Appendix A (A3) and in the companion-note audit (Appendix R). □

Theorem AA.5 (The targets 12 and 6 are excluded by the same mod-3 obstruction)**.** $C_{\mathrm{Szp}} = 12$ *requires* $N = 12$*,* $\Theta = 35/3$*;* $C_{\mathrm{Szp}} = 6$ *requires* $N = 24$*,* $\Theta = 47/3$*. Both* $N \equiv 0 \pmod 3$*, whereas Theorem 5.7 forces* $N \equiv 1 \pmod 3$*; hence neither target lies on the achievable lattice. The only lattice point that ever went below* 12 *was* $\Theta = 12$ *(*$C_{\mathrm{Szp}} = 144/13 \approx 11.08$*,* $C_{\mathrm{BMY}} = 36/13 < 3$*), and Theorem Z.4 has now excluded it unconditionally;* $\Theta = 14$ *(*$144/19 \approx 7.58$*) was excluded by the nef bound of Theorem U.3.*

Proof. From $C_{\mathrm{Szp}} = 12t$ (Theorem AA.1) and $t = 12/N$ we get $N = 144/C_{\mathrm{Szp}}$, i.e. $N = 12$ and $N = 24$ respectively, and $\Theta = (N + 23)/3$ gives $35/3$ and $47/3$. These are not integers, and $12 \equiv 24 \equiv 0 \pmod 3$ contradicts $N = 3\Theta - 23 \equiv 1$. The value $47/3$ is the same one obtained in Appendix F from $E(c) = 12/(5 - 18c)$ and in Theorem 5.7 from $t(\Theta) = 12/(3\Theta - 23)$ — a third independent agreement. The $\Theta = 12$ and $\Theta = 14$ rows of the table in Theorem AA.1 are excluded by Theorems Z.4 and U.3 respectively. □

Remark AA.6 (The curve-tower side). The companion note's $\kappa = h_F / \log \mathrm{cond}$, the attainable closure $[1/12, 1/3]$, the function-field limit $\kappa_{\mathrm{ff}} = 1/6$ and its Problem 10.1 concern tower asymptotics and the archimedean upper bound; they are a different normalisation

from the pointwise $\delta(F)$ coefficient treated here. That problem remains open (Open problem (13)), but it supplies no new freedom for the pointwise Szpiro constant. In particular $C_{\mathrm{BMY}} = 3$ is equivalent, by Theorem AA.1, to $E = 2$ and $C_{\mathrm{Szp}} = 12$, which reproduces the Vojta-limit statement of Appendix R from the pointwise side.

Remark AA.7 (Scope). This appendix is an elementary conversion and exact-rational recomputation. It does not independently verify the proofs of Theorem 3 of [**?**], Moret-Bailly's Th. 6.4, Robin's bound, or Theorem A. The status of *abc* is unchanged; all No-Go reasons and the weak-*abc* output $(N/M_5)^{24+\varepsilon}$ are unchanged.

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