hexagon:2611.00329v1

Genuinely multipartite bound entanglement of three qubits

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Abstract

If a three-party state is positive under partial transposition (PPT) across each of its three bipartitions, no local protocol can distil any pure entanglement from it: neither singlets between any two parties nor GHZ states. All of its entanglement is bound. We show that such a state can nevertheless be genuinely multipartite entangled (GME), meaning that it cannot be prepared by mixing states in each of which some party is unentangled from the rest. This already happens for three qubits, the smallest system in which the question arises and a common experimental setting for multipartite entanglement; to our knowledge no three-qubit example was known. An explicit mixture of five orthogonal states (one GHZ-type, three W-type and one product state) is PPT across every cut and GME throughout an exact parameter interval, and its rank, five, is the minimum possible. Genuine entanglement is proved by a witness acting only on the five-dimensional support, whose nonnegativity on the biseparable states in that support reduces to a sum of squares on its curves of product vectors. Such states are invisible to the PPT-mixture test, the standard semidefinite relaxation used to certify GME, yet they form a set with nonempty interior among density matrices: an explicit full-rank example remains GME under 2.33% white noise, while 2.336% makes it biseparable. We also construct extreme examples with the smallest possible partial-transpose ranks $(5, 5, 5, 5)$. All our explicit states, including the full-rank one, lie in the mixed W class: they are mixtures of pure states with vanishing three-tangle, so their genuine entanglement needs no GHZ-type component. Full PPT therefore does not imply biseparability, even for three qubits.

1 Introduction

Entanglement among three or more parties comes in two kinds. A state is *biseparable* if it is a mixture of states each separable across one of the bipartitions $A|BC$, $B|AC$, $C|AB$; the bipartition may differ from term to term. Otherwise it is *genuinely multipartite entangled* (GME). Each term of a biseparable mixture is entangled at most within one pair of parties, so GME is the multipartite property that experiments with several qubits usually aim to certify [17, 18]. Showing that a mixed state is GME is harder than showing it entangled across each cut: a state can be entangled across every bipartition and still be a mixture of states separable across different ones. Biseparability is a property of the convex hull of three sets of separable states, not of each set separately.

Entanglement can also be distillable or bound. Entanglement across a cut is *distillable* if many copies can be converted to singlets across that cut by local operations and classical communication (LOCC), and *bound* otherwise. A state that is positive under partial transposition (PPT) across a cut is undistillable across it [16], and PPT is also necessary for separability [14, 15]. For three qubits A, B, C we call a state *fully PPT* when its partial transposes across $A|BC$, $B|AC$ and $C|AB$ are all positive. Every LOCC protocol among the three parties is in particular LOCC across each cut, and such operations preserve PPT across that cut [16, 17]. Consequently no pure state entangled across any cut, neither singlets between two parties nor GHZ states, can be distilled from a fully-PPT state, even asymptotically. Whatever entanglement a fully-PPT state has is bound.

This paper asks whether bound entanglement of this extreme kind can still be genuinely multipartite, and answers yes for three qubits. A fully-PPT GME state carries entanglement that no

single bipartition accounts for and from which no pure entanglement can be distilled: it is present only collectively. The question also has a practical side. The standard way to certify GME in experiments and numerics relaxes biseparability to *PPT mixtures*, mixtures of states each PPT across one cut, which gives a semidefinite program [2, 3]. A fully-PPT state is already a one-term PPT mixture, so its genuine entanglement is invisible to this test and to any test built from PPT conditions across cuts and their mixtures. Novo, Moroder and Gühne conjectured that for three qubits PPT mixtures and biseparable states coincide [3]; Ha and Kye refuted this with a state PPT across one chosen cut [4]. A fully-PPT GME state is the strongest form of the counterexample.

Three qubits are where such a state is least expected. Each cut is 2×4, just beyond the 2×2 and 2×3 systems where PPT and separability coincide [15], and low rank forces separability: a fully-PPT three-qubit state of rank at most four is separable across a cut, hence biseparable [22, 10, 11, 7]. A fully-PPT GME state therefore has rank at least five. Examples are known for more qubits and for higher local dimension (Section 7.4). For three qubits, earlier constructions and numerical states were entangled but not shown to be GME, or were PPT across one cut only [4, 19, 33], and Singh and Banik noted that no explicit example is known and derived necessary rank conditions for one [1].

Our method follows the physics of the convex hull. Every pure state in a decomposition of ρ lies in its support, so a biseparable decomposition of a low-rank state can use only those vectors of the support that are products across some cut. We choose a five-dimensional support R, spanned by five orthogonal vectors (one GHZ-type, three W-type and $|111\rangle$), in which the cut-product vectors form one explicit curve per cut, parametrised by a single complex number and related by cyclic permutation of the qubits. A Hermitian form that is nonnegative on these three curves is nonnegative on every biseparable state supported on R; its negative expectation in our state then proves GME. For the form we use, nonnegativity on the curves is an explicit factorisation into squares. Full PPT is a separate and elementary step, since cyclic and parity symmetry reduce each partial transpose to two 4×4 blocks. Robustness to noise needs a test stronger than PPT; we use exact rational certificates from the second level of the symmetric-extension hierarchy, together with explicit biseparable decompositions that bound the noise thresholds from above.

Our main results are the following.

(1) *Minimal-rank family* (Theorem 1). A mixture ρ_t of five orthogonal vectors (GHZ-type, W-type and $|111\rangle$) has rank five and is fully PPT and GME for $3.257 \lesssim t \lesssim 3.389$ (exact endpoints in the theorem).

(2) *A witness that sees only the support* (Proposition 2). One diagonal Hermitian form, nonnegative on the product curves by a square factorisation, excludes biseparable mixtures across cuts, with no semidefinite-program solver.

(3) *Extreme states* (Sections 2 and 3). Extreme fully-PPT GME states have partial-transpose rank profiles $(5,6,6,6)$ and, the smallest possible, $(5,5,5,5)$; the latter lie in an analytic family that is fully PPT and GME on the whole window $1/\sqrt{3} < \tau < 1$ (one parameter modulo local filters). Every state of the family is extreme in the fully-PPT set, and near every member the extreme states of this profile form a 27-dimensional manifold with nine local-filter moduli.

(4) *One support, both phases* (Section 5). One rank-five support carries both biseparable and GME fully-PPT states, separated by an exact transition inside the strictly PPT region, so the support alone does not decide genuine entanglement.

(5) *Robustness* (Section 6). Fully-PPT GME states have positive volume. An explicit full-rank fully-PPT state remains GME under 2.33% white noise, 1% independent depolarisation per qubit, or a trace-distance perturbation of 7.3×10^{-3}; its white-noise threshold is known to within half a per cent in relative terms. A surface state of the $(5,5,5,5)$ family tolerates about 0.43%.

(6) *Structure* (Sections 2.2 and 7). No fully-PPT GME state is invariant under the exchange of two qubits, so none is permutation invariant [3], and our family is not locally equivalent to

its complex conjugate, not even by invertible local filters. Every explicit state we construct, including both extreme states and the full-rank state $\tilde{\rho}$, belongs to the mixed W class (Theorem 12), as conjectured for PPT entangled three-qubit states by Acín et al. [7]. For every rank-five fully-PPT GME state the product vectors of its support form three rational quartic curves, so supported witnesses of the kind used here detect all of them (Theorem 26).

Section 7 places the results relative to PPT mixtures and earlier work. The main text gives the proof strategy; the computations are in Appendices A, B, C and D, in the order of the sections they serve. Appendices E and F prove the product-kernel lemma and the moment cone of Section 7 and the kernel-penalty lift of the supported witness, and Appendix G lists the proof dependencies. All theorems, propositions, lemmas and corollaries are proved, analytically or with the exact computations named next; numerical observations appear only in remarks and are labelled as such. Exact rational or algebraic computations are used only for the extremality of σ_6 and the independent extremality check on the $(5,5,5,5)$ surface, the moduli count, the phase-coexistence minors and the noise certificates. The verification scripts with their captured output are in a companion repository [34], together with a Lean 4 formalisation of the main analytic results and of the extremality of one state of the surface; the noise certificates, the moduli count, the extremality of σ_6 and of the whole surface, and the minimality of rank five are checked by the exact scripts or cited, not formalised.

2 The rank-five family

We first separate two tasks: enforce PPT by symmetry, then exclude biseparability using only product vectors in the state's support. Appendix A supplies the coordinate proofs.

Let $\omega = e^{2\pi i/3}$. In the computational basis of $(\mathbb{C}^2)^{\otimes 3}$ define the five *unnormalised* vectors

$$\begin{aligned} u &= |000\rangle - |011\rangle - |101\rangle - |110\rangle, \quad v = |111\rangle, \qquad w = |001\rangle + |010\rangle + |100\rangle, \\ e &= |011\rangle + \omega|101\rangle + \omega^2|110\rangle, \qquad o = |100\rangle + \omega|010\rangle + \omega^2|001\rangle. \end{aligned} \tag{1}$$

Here $u/2$ is a local-Clifford rotation of the GHZ state and w, e, o are locally related W states. Global parity is $Z^{\otimes 3}$, with $Z = \mathrm{diag}(1,-1)$; it has eigenvalue $+1$ on u, e (an even number of ones) and -1 on v, w, o. The cyclic permutation Π of the qubits, $\Pi|abc\rangle = |cab\rangle$, fixes u, v, w and multiplies e and o by ω^2. Together, $Z^{\otimes 3}$ and Π split $(\mathbb{C}^2)^{\otimes 3}$ into two two-dimensional Π-invariant sectors, $\mathrm{span}\{|000\rangle, |011\rangle + |101\rangle + |110\rangle\}$ and $\mathrm{span}\{v, w\}$, and four one-dimensional sectors spanned by $e, \bar{e}$ (even) and $o, \bar{o}$ (odd). Exchanging two qubits swaps $e \leftrightarrow \bar{e}$ and $o \leftrightarrow \bar{o}$, and a PPT state invariant under such an exchange is separable across a cut (Corollary 3). The support below therefore keeps u from the first sector, all of the second, and only one orientation e, o of the one-dimensional sectors. Put $B = [u, v, w, e, o]$, an 8×5 matrix with $B^\dagger B = \mathrm{diag}(4, 1, 3, 3, 3)$, and $R = \mathrm{ran}\, B$. For $t > 0$ set

$$\rho_t = \frac{t\, uu^\dagger + 3t\, vv^\dagger + 3\, ww^\dagger + 3\, ee^\dagger + oo^\dagger}{7(t+3)}. \tag{2}$$

The weights $(t, 3t, 3, 3, 1)$ are a one-parameter choice for which the PPT conditions of the two parity blocks, derived below, leave a nonempty interval in t. For a three-qubit operator X write X^{T_c} for its partial transpose on qubit $c \in \{A, B, C\}$; the *rank profile* of X is $(\mathrm{rank}\, X, \mathrm{rank}\, X^{T_A}, \mathrm{rank}\, X^{T_B}, \mathrm{rank}\, X^{T_C})$. For a vector x, $\|x\|$ is the Euclidean norm; for an operator X, $\|X\|_1$ is the trace norm, $\|X\|_2 = (\mathrm{tr}\, X^\dagger X)^{1/2}$ the Hilbert–Schmidt norm and $\|X\|_\infty$ the operator norm (largest singular value).

Theorem 1 (the rank-five fully-PPT GME family)**.** *Every ρ_t has rank five. It is PPT at all three single-qubit cuts exactly when*

$$t_- \le t \le t_+, \qquad t_- = \frac{15 + 9\sqrt{17}}{16} \approx 3.257, \qquad t_+ = \frac{2(\sqrt{37} - 1)}{3} \approx 3.389, \tag{3}$$

and every state in this interval is genuinely multipartite entangled (GME). In the interior the rank profile is $(5,8,8,8)$*, and at the endpoints it is* $(5,7,7,7)$*. Rank five is minimal among fully-PPT GME three-qubit states.*

For minimality, any fully-PPT three-qubit state of rank at most four is separable across a 2×4 cut by the low-rank PPT theorem [22, 10], and is therefore biseparable.

Idea of the PPT proof. Cyclic invariance makes the three partial transposes unitarily equivalent, and global parity splits $\rho_t^{T_A}$ into two 4×4 blocks (the partial transpose commutes with the real diagonal $Z^{\otimes 3}$, whose eigenspaces are spanned by $|000\rangle, |011\rangle, |101\rangle, |110\rangle$ and by $|001\rangle, |010\rangle, |100\rangle, |111\rangle$). The even-block determinant forces $t \geq t_-$. In that range the odd-block determinant requires $t \leq t_+$, and the first three leading minors of both blocks are positive on the resulting interval. Sylvester's criterion then gives PPT. At an endpoint one block drops to rank three, which gives the profiles. The minors are in Appendix A; genuine entanglement is proved below by a witness on the range.

2.1 Cut-product rays and a supported witness

A biseparable decomposition of a state supported on R uses pure vectors in R that are products across some cut. A Hermitian form nonnegative on all these vectors can therefore exclude biseparability on R. For $s>0$ put $u_s = |000\rangle - s(|011\rangle + |101\rangle + |110\rangle)$, $B_s = [u_s, v, w, e, o]$ and $R_s = \operatorname{ran} B_s$, so that $u_1 = u$, $B_1 = B$, $R_1 = R$. An operator supported on R_s is written $B_s X B_s^\dagger$ with a 5×5 coefficient matrix X; subscripts identify the corresponding state. A pure cut-product vector $B_s q$ has projector $B_s q q^\dagger B_s^\dagger$, so its coefficient matrix is $qq^\dagger$ and $\operatorname{tr}(H\, qq^\dagger) = q^\dagger H q$ with no metric factor $B_s^\dagger B_s$.

Proposition 2 (supported witnesses)**.** *Let* $s>0$.

(a) The $A|BC$*-product rays in* R_s *are the rays of* $B_s q_s(z)$*,* $z\in\mathbb{C}$*, with*

$$q_s(z) = \big(2z,\ -z^2(z^2+3s),\ z^2-s,\ -z(z^2+s),\ z^2+s\big)^T, \tag{4}$$

together with the ray of v *(the point* $z=\infty$*). The other two cuts are given by the cyclic images.*

(b) Let $H_s = \operatorname{diag}(-12s^3,\ 1,\ 11s^2,\ -6s,\ 14s^2)$ *in the basis* (u_s, v, w, e, o)*. For* $z = x+iy$ *and* $r = |z|^2$*,*

$$q_s(z)^\dagger H_s q_s(z) = (r-s)^2\big[(r-5s)^2 + 12s\,x^2\big] \ \geq 0. \tag{5}$$

At $s=1$ *this is the witness* $H_5 := H_1$ *used for the first family,*

$$H_5 = \operatorname{diag}(-12, 1, 11, -6, 14), \qquad q_1(z)^\dagger H_5 q_1(z) = (r-1)^2\big[(r-5)^2 + 12x^2\big]. \tag{6}$$

Since H_s *is diagonal, it is invariant under the cyclic phases acting on* (u_s, v, w, e, o)*. Consequently* $\operatorname{tr}(H_s X) \geq 0$ *whenever* $B_s X B_s^\dagger$ *is biseparable, including mixtures across different cuts; no symmetry hypothesis on* X *is needed.*

Idea of the proof. (a) A vector of R_s is a product across $A|BC$ with first-qubit ray $(1,z)^T$ exactly when its two BC rows are proportional with ratio z: four linear equations in the five coordinates, with a one-dimensional solution $q_s(z)$ for each finite z. (b) Since H_s is diagonal, only the squared moduli of the entries of $q_s(z)$ enter, and the sum collapses to a perfect-square factorisation; a rescaling removes s. Diagonal forms commute with the cyclic phases, so the other cuts give the same values, and every positive decomposition of an operator supported on R_s uses vectors in R_s, so mixtures across cuts are controlled. The computations are in Appendix A.

Proof of the genuine entanglement in Theorem 1. Write $\rho_t = B X_t B^\dagger$. Then

$$\operatorname{tr}(H_5 X_t) = \frac{29-9t}{7(t+3)} < 0 \qquad \text{whenever } t > 29/9, \tag{7}$$

and $t_- > 29/9$. Proposition 2(b) therefore proves GME throughout (3). □

For a concrete member, take $t = 10/3$. Then the normalised versions of the orthogonal vectors (u, v, w, e, o) occur with probabilities $(40, 30, 27, 27, 9)/133$, all leading minors in (27) are positive, and $\mathrm{tr}(H_5 X_{10/3}) = -3/133$. Thus this fully-PPT GME state is certified by hand. Appendix F lifts the supported witness to a global witness with an explicit noise radius around this state (Remark 32), again without a semidefinite-program solver.

2.2 Reflection symmetry and local chirality

The states select one cyclic phase orientation of the W vectors e, o; exchanging two qubits reverses it. The following obstruction explains why this asymmetry matters.

Corollary 3 (reflection symmetry destroys the construction)**.** *(a) A three-qubit state that is PPT at $A|BC$ and commutes with the swap of B and C is separable across $A|BC$. Hence no fully-PPT GME state is invariant under the exchange of two qubits.*

(b) For every fully-PPT three-qubit state ρ, the permutation average $\mathcal{S}(\rho) = \frac{1}{6}\sum_{\pi\in S_3} U_\pi \rho U_\pi^\dagger$ is separable across each of the three cuts, so

$$D(\rho, \mathrm{BS}) \le \tfrac{1}{2}\|\rho - \mathcal{S}(\rho)\|_1, \tag{8}$$

where $D(\rho,\sigma) = \frac{1}{2}\|\rho-\sigma\|_1$ is the trace distance, $D(\rho,\mathrm{BS}) = \inf_{\sigma\in\mathrm{BS}} D(\rho,\sigma)$ and BS is the set of biseparable states.

(c) For any three-qubit ρ that is PPT at $A|BC$, pinching with the triplet and singlet projectors P_3, P_1 of BC gives an explicit $A|BC$-separable state, and

$$D(\rho, \mathrm{BS}) \le \|P_3\rho P_1\|_1 = \tfrac{1}{4}\|\rho - S_{BC}\rho S_{BC}\|_1, \tag{9}$$

where S_{BC} is the swap of B and C. For a cyclically invariant fully-PPT ρ, (8) gives the same value $\frac{1}{4}\|\rho - S\rho S^\dagger\|_1$ for every transposition S; (9) needs only PPT at $A|BC$.

These are upper bounds on the distance to BS*, not entanglement tests. They show, however, that a fully-PPT GME state must have coherence between the singlet and triplet sectors of every pair of qubits (otherwise (9) vanishes at that cut), and in particular must break every pair-exchange symmetry.*

Proof. (a) The swap-invariant state is block diagonal in the triplet ($\dim 3$) and singlet ($\dim 1$) subspaces of BC, and its partial transpose on A preserves these blocks. Each block is PPT on 2×3 or 2×1, where PPT implies separability; this is the mechanism of Novo, Moroder and Gühne [3, Sec. III.2]. (b) $\mathcal{S}(\rho)$ is fully PPT and invariant under every transposition, so (a) applies at each cut; $\mathcal{S}(\rho) \in \mathrm{BS}$ gives (8). (c) The pinched state $P_3\rho P_3 + P_1\rho P_1$ is PPT at $A|BC$ (the partial transpose on A commutes with the pinching on BC) and block diagonal as in (a), hence separable; its trace distance from ρ is at most $\|P_3\rho P_1\|_1$. Since $S_{BC} = P_3 - P_1$, $\rho - S_{BC}\rho S_{BC} = 2(P_3\rho P_1 + P_1\rho P_3)$, whose trace norm is $4\|P_3\rho P_1\|_1$. For cyclic ρ, $\mathcal{S}(\rho) = (\rho + S\rho S^\dagger)/2$ for each transposition S, so (8) becomes $\frac{1}{4}\|\rho - S\rho S^\dagger\|_1$, and cyclic invariance makes this independent of S. □

The obstruction concerns reflection alone. The real vectors u_s, v, w and the pair e, o of (1) are invariant under the antiunitary map $K\circ S_{BC}$, the swap of B and C followed by complex conjugation in the computational basis (for example $S_{BC}e = |011\rangle + \omega^2|101\rangle + \omega|110\rangle = \bar{e}$). Hence every state with real coefficients in these vectors, including ρ_t and all states of Section 3, is invariant under this combined reflection and conjugation. The GME states among them are invariant under neither separately: not under the reflection by part (a), and not under conjugation, which exchanges e and o with $\bar{e}$ and $\bar{o}$, outside their supports.

Corollary 4 (intrinsic local chirality)**.** *(a) For every $t > 0$, ρ_t and its complex conjugate are not related by local unitaries. The same holds for physical spin time reversal, which differs from complex conjugation by a local unitary.*

(b) *For every $s > 0$, no invertible local filter $F_A \otimes F_B \otimes F_C$ maps R_s onto its complex conjugate. Hence no state with range R_s, in particular no ρ_t, σ_6 (Corollary 5), $\rho_{\tau,a}$ with $1/\sqrt{3} < \tau < 1$ (Theorem 7) or state of the phase-coexistence line (Proposition 15), is related to its complex conjugate or its spin time reversal by an invertible local filter. The stabiliser of R_s in $\mathrm{PGL}(2,\mathbb{C})^{\times 3}$ is isomorphic to S_3, acting by the same Möbius map on the three qubits.*

Idea of the proof. (a) is the unitary case of (b), since $\operatorname{ran}\rho_t = R_1$ for every $t > 0$. (b) R_s contains exactly three fully product rays, $[|1\rangle^{\otimes 3}]$ and $[(|0\rangle \pm i\sqrt{s}|1\rangle)^{\otimes 3}]$, and a filter must permute them; a Möbius map is fixed by its action on three points, so the filter is $F_0^{\otimes 3}$ up to a scalar, which commutes with the cyclic shift. The shift has eigenvalue ω^2 on $e, o \in R_s$ but not on $\overline{R_s}$. See Appendix A.

2.3 An extreme point of profile $(5,6,6,6)$

Adjusting the weight of v makes both partial-transpose parity blocks singular and yields an extreme fully-PPT state.

Corollary 5 (an extreme fully-PPT GME state of profile $(5,6,6,6)$)**.** *Set $t = t_-$ and $b = 8(9-t)/5$ ($t \approx 3.257$, $b \approx 9.189$). Then*

$$\sigma_6 = \frac{t\,uu^\dagger + b\,vv^\dagger + 3\,ww^\dagger + 3\,ee^\dagger + oo^\dagger}{4t + b + 21} \tag{10}$$

is an extreme point of the set of fully-PPT three-qubit states and is GME. Its coefficient matrix is $X_6 = \operatorname{diag}(t, b, 3, 3, 1)/(4t + b + 21)$.

Idea of the proof. The choice $t = t_-$ makes the even partial-transpose block singular, and that of b makes the odd block singular, each with a positive 3×3 leading minor, so every partial transpose has rank six. GME follows from the witness H_5. By the supported-face criterion of [33], extremality means that the only Hermitian D supported on $\operatorname{ran}\sigma_6$ whose partial transposes annihilate the kernels of those of σ_6 is a multiple of σ_6; an exact rank computation on the 25-dimensional space of coefficient matrices shows this (Appendix A).

3 Profile $(5,5,5,5)$: local-filter orbits

How little partial-transpose rank can a fully-PPT GME state have? We deform the GHZ-type vector and tune the weights to make both partial-transpose parity blocks singular. One of the two remaining parameters is an invertible local filter, so a single parameter controls PPT and GME. Appendix B gives the proofs.

The first family has partial-transpose ranks seven or eight. A fully-PPT state of rank at most four is separable [22, 10, 11, 7], and the same theorem bounds each partial-transpose rank from below: if $\operatorname{rank}\rho^{T_c} \le 4$, then ρ^{T_c}, and hence ρ, is separable across that cut [22, 10, 11]. Thus $(5,5,5,5)$ is the smallest possible profile for fully-PPT GME. The construction below attains it, including at extreme points of the fully-PPT set.

For real $\alpha > 0$, $0 < \beta < \alpha$ and $\gamma > 0$ put $h = \sqrt{\alpha^2 + \beta^2}$, $\kappa = \alpha^2 - \beta^2$, $s = \alpha/\gamma$ and

$$O = \frac{\kappa}{\beta}, \qquad E = \frac{\kappa(h+\beta)^2}{\alpha^2\gamma}, \qquad G = \frac{\alpha(E - \alpha s)}{\beta}, \qquad V = \frac{\alpha^2E^2 - 2\alpha s\kappa E + (\alpha s)^2\kappa}{\beta\kappa}, \tag{11}$$

$$\rho(\alpha,\beta,\gamma) = \gamma\,u_s u_s^\dagger + \beta\,ww^\dagger + V\,vv^\dagger + G\,(wv^\dagger + vw^\dagger) + E\,ee^\dagger + O\,oo^\dagger, \tag{12}$$

with u_s, v, w, e, o as in Section 2. The coefficients are chosen to make both partial-transpose blocks singular. Where they come from (at $\alpha = \gamma = 1$, so $\beta = \tau$): the odd block of ρ^{T_A} has positive 2×2 leading minor $3\tau O$, so it has rank two only if its 3×3 leading minor $3O\,(\tau(\tau + O) - 1)$ vanishes, which gives $O = \kappa/\tau$; the last row and column must then lie in the span of the others,

which gives $G = (E-1)/\tau$ and $V = E^2/O + \tau G^2 = (E^2 - 2\kappa E + \kappa)/(\tau\kappa)$ for every E, and the remaining parameter E is fixed by making the even block singular. The operator is homogeneous of degree one in (α, β, γ); its essential parameters are $\tau = \beta/\alpha \in (0,1)$ and $a = \gamma/\alpha > 0$. Write $\rho_{\tau,a} = \rho(1, \tau, a)$ and $\rho_\tau = \rho_{\tau,1}$, with coefficient matrix X_τ. These symbols denote *unnormalised* operators, not the normalised state ρ_t and matrix X_t of Section 2; division by their trace gives physical density matrices. Positive scaling does not change PPT, rank or separability.

Lemma 6 (a τ-independent supported witness)**.** *Let H_0 be the Hermitian form on $\mathbb{C}^5$ in the basis (u, v, w, e, o) with diagonal $(-24 - 8\sqrt{3},\ 1,\ 9,\ 12 + 4\sqrt{3},\ 12 + 12\sqrt{3})$ and $(H_0)_{vw} = (H_0)_{wv} = -\nu$, $\nu = 7 + 4\sqrt{3}$. Put $y_1 = 3 + 2\sqrt{3}$, $x_0 = 2\sqrt{3} - 3$, $p = 26 + 16\sqrt{3}$, $q = \nu^2$ (so $x_0^2 q = y_1^2$), and for $z = x + iy$ let $\xi = x^2$, $\zeta = y^2$, $r = \xi + \zeta$. Then, with q_1 of (4),*

$$r\, q_1^\dagger H_0 q_1 = \zeta(r-1)^2(r-y_1)^2 + \xi(r-x_0)^2(r^2 + pr + q) + 16\nu\, \xi\zeta r \ \geq 0, \tag{13}$$

so H_0 is nonnegative on all cut-product rays of R, and $\operatorname{tr}(H_0 X) \geq 0$ whenever $BXB^\dagger$ is biseparable.

Idea of the proof. At fixed $r = |z|^2$ the form $q_1^\dagger H_0 q_1$ is a quadratic in $\xi = x^2$, so it is fixed by its two axis values (z imaginary and z real) and its leading coefficient. Each axis value is a perfect square times a manifestly nonnegative factor. The computation is in Appendix B.

Theorem 7 (fully-PPT GME states of profile $(5,5,5,5)$)**.** *For $1/\sqrt{3} < \tau < 1$ and every $a > 0$, the normalised $\rho_{\tau,a}$ is fully PPT and GME, with rank profile $(5,5,5,5)$. The family has the following structure.*

(a) Each operator is cyclically and parity invariant, with range contained in $R_{1/a}$, and its partial transposes have rank at most five. In the stated interval the range equals $R_{1/a}$.

(b) For $F_\lambda = \operatorname{diag}(1, \lambda)^{\otimes 3}$,

$$\rho(\alpha, \beta, a\alpha) = a F_\lambda\, \rho(\alpha, \beta, \alpha) F_\lambda^\dagger, \qquad \lambda^2 = a^{-1}.$$

Thus varying a is an invertible local filter; PPT and GME depend only on τ.

(c) The window is sharp within this ansatz: $\rho_{\tau,a}$ is not positive for $0 < \tau < 1/\sqrt{3}$. At $\tau = 1/\sqrt{3}$ it has rank four and is separable across every cut. As $\tau \uparrow 1$ its normalisation tends to a fully separable state (Remark 10).

The GME proof uses the fixed supported witness H_0 of Lemma 6 when $1/\sqrt{3} < \tau \leq 3/4$, and H_5 from Proposition 2 when $3/4 \leq \tau < 1$; the case $a \neq 1$ follows because the local filter of (b) preserves biseparability, full PPT and ranks.

Idea of the proof. Homogeneity and the local filter reduce the problem to $\alpha = \gamma = 1$. With $Q = \tau + \sqrt{1 + \tau^2}$, the odd parity block of $\rho_\tau^{T_A}$ has an explicit Gram factorisation of rank two, and the even block has vanishing determinant and a third leading minor that changes sign at $\tau = 1/\sqrt{3}$. Together with positivity of ρ_τ, these give the sharp window and ranks; cyclic symmetry covers the other cuts. The two witnesses cover the window by one-variable sign checks.

Corollary 8 (an explicit extreme state with rational weights and profile $(5,5,5,5)$)**.** *Let $\tilde{u} = 2|000\rangle - |011\rangle - |101\rangle - |110\rangle$, let v, w, e, o be as in (1), and*

$$N = 825000\, \tilde{u}\tilde{u}^\dagger + 1440000\, ww^\dagger + 2021184\, vv^\dagger + 145200\, (wv^\dagger + vw^\dagger) + 951720\, ee^\dagger + 450625\, oo^\dagger, \qquad \sigma = \frac{N}{16323219}. \tag{14}$$

Then σ has rank five and is invariant under cyclic permutations and global parity; $N = 30000\, \rho(55, 48, 110)$ is the member $(\alpha, \beta, \gamma) = (55, 48, 110)$ of Theorem 7, the Pythagorean choice $(m, n, a) = (8, 3, 2)$ with $\alpha = m^2 - n^2$, $\beta = 2mn$ and $\gamma = a\alpha$. Here $\tau = 48/55$ and $s = 1/2$; all six

coefficients in (12) are rational. The projector coefficients are rational, but the matrix entries of σ in the computational basis belong to $\mathbb{Q}(\omega)$, since e and o contain cube-root phases. All three partial transposes of σ are positive semidefinite of rank five (at $A|BC$ the even and odd blocks have ranks 3 and 2). The state σ is an extreme point of the set of fully-PPT three-qubit states and is GME, and $(1-p)\sigma + pI/8$ is GME for every $p < 4.05 \times 10^{-3}$ (the certificate is in Appendix B, using the witness of Corollary 19). More generally, every *normalised $\rho_{\tau,a}$ of Theorem 7 ($1/\sqrt{3} < \tau < 1$, $a > 0$) is an extreme point of the set of fully-PPT three-qubit states, and an edge state across every cut: no product vector across a cut lies in its range with its partial conjugate in the range of the corresponding partial transpose.*

Idea of the proof. Theorem 7 gives positivity and ranks, and $H_{1/2}$ detects GME directly. The noise bound uses the level-two witness Y_* of Corollary 19. For extremality, a 2×4 PPT state of birank $(5,5)$ that is entangled across a cut is extreme in that cut's PPT set, because a state on the boundary of its face has rank or partial-transpose rank at most four and is then separable [22, 10]; independently, the cyclic, parity and reflection symmetries split the rank-24 face-direction calculation into blocks whose minors are explicit polynomials in $Q = \tau + \sqrt{1+\tau^2}$ with no root in the window (Appendix B).

Proposition 9 (the profile-$(5,5,5,5)$ states are not isolated)**.** *Near every state ρ of Theorem 7 (in particular near σ of Corollary 8), the fully-PPT three-qubit states of rank profile $(5,5,5,5)$ form a smooth real manifold $\mathcal{M}$ of dimension 27, and every state of $\mathcal{M}$ near ρ is GME and an extreme point of the set of fully-PPT states. Invertible local filters ($\rho \mapsto F\rho F^\dagger/\operatorname{tr}(F\rho F^\dagger)$, $F = F_1 \otimes F_2 \otimes F_3$) act on $\mathcal{M}$ with orbits of dimension 18, so $\mathcal{M}$ has nine local moduli. The construction of Theorem 7 is only a two-parameter cyclic-symmetric family (one parameter modulo filtering), not a classification.*

Idea of the proof. Rank-five states have 54 real parameters. Each rank-five partial transpose imposes $(8-5)^2 = 9$ smooth equations; the symmetry blocks of their differentials have explicit minors in Q that do not vanish on the window, so they are independent at every state of the surface. The implicit function theorem gives dimension 27, and local filters remove 18 parameters (the stabiliser of the support is finite, Corollary 4). GME and extremality persist nearby (Appendix B).

Remark 10 (the two endpoints; two witnesses are needed). At $\tau = 1/\sqrt{3}$ ($Q = \sqrt{3}$) the operator ρ_τ is a limit of fully-PPT states, hence positive semidefinite with all three partial transposes positive semidefinite, and it has rank four, so it is separable across each cut by the 2×4 low-rank theorem [22, 10]. Explicitly, at $a = 1$ its coefficient matrix $X_{1/\sqrt{3}} = \operatorname{diag}(1, 3\sqrt{3}, \sqrt{3}/3, 2, 2\sqrt{3}/3) + \sqrt{3}\,(E_{vw} + E_{wv})$ is a positive combination of four $A|BC$-product rays, at the zeros $z = \pm i\sqrt{y_1}, \pm\sqrt{x_0}$ of $q_1(z)^\dagger H_0 q_1(z)$ in (13):

$$X_{1/\sqrt{3}} = \frac{3\sqrt{3}-5}{48} \sum_{\pm} q_1(\pm i\sqrt{y_1}) q_1(\pm i\sqrt{y_1})^\dagger + \frac{3\sqrt{3}+5}{48} \sum_{\pm} q_1(\pm\sqrt{x_0}) q_1(\pm\sqrt{x_0})^\dagger, \tag{15}$$

with weights $(3\sqrt{3} \mp 5)/48 > 0$. Cyclic permutation gives the decompositions across $B|CA$ and $C|AB$, and the local filter in Theorem 7(b) carries them to every $a > 0$. It is not fully separable. Writing the product vectors of Proposition 2(a) as $B_s q_s(z) = (|0\rangle + z|1\rangle) \otimes \phi_s(z)$, the BC coefficient matrix of $\phi_s(z)$ has determinant $-3(z^2+s)^2$, so the only fully product rays in R_s are $|111\rangle$ and $(|0\rangle \pm i\sqrt{s}|1\rangle)^{\otimes 3}$. A fully separable state with range in $R_{1/a}$ therefore has rank at most three, while $\rho_{1/\sqrt{3},a}$ has rank four.

At the upper endpoint, for every fixed $a > 0$ the limit is fully separable and has rank profile $(2,2,2,2)$. Indeed the weights satisfy $(O, E, G, V) \to (0, 0, -1/a, 1/a^2)$ as $\tau \uparrow 1$, giving

$$\rho_{1,a} := \lim_{\tau\uparrow 1} \rho_{\tau,a} = a\, u_{1/a} u_{1/a}^\dagger + (w - a^{-1}v)(w - a^{-1}v)^\dagger.$$

Put $|\eta_\pm(a)\rangle = (\sqrt{a}|0\rangle \pm i|1\rangle)/\sqrt{a+1}$. Expanding the product vectors gives

$$\frac{\rho_{1,a}}{\operatorname{tr}\rho_{1,a}} = \frac{1}{2}\left[(|\eta_+(a)\rangle\langle\eta_+(a)|)^{\otimes 3} + (|\eta_-(a)\rangle\langle\eta_-(a)|)^{\otimes 3}\right], \qquad \operatorname{tr}\rho_{1,a} = \frac{(a+1)^3}{a^2}. \tag{16}$$

The two local rays are distinct for every finite $a > 0$, and remain distinct after conjugating any subset of factors. Thus both the state and its three partial transposes have rank exactly two. The GME interval therefore terminates at states separable across every cut at both ends; only the upper endpoint is fully separable.

Together, (15) and (16) explain the origin of H_0 and show that no single witness detects ρ_τ on the whole window $1/\sqrt{3} < \tau < 1$, so two witnesses are needed. Only the compression of a witness to R matters, and it is a form H on $\mathbb{C}^5$ nonnegative on the cut-product rays of R. If $\mathrm{tr}(HX_\tau) < 0$ on the whole window, continuity gives $\mathrm{tr}(HX_{1/\sqrt{3}}) \le 0$ and $\mathrm{tr}(HX_1) \le 0$, where X_1 is the coefficient matrix of $\rho_{1,1}$. Both endpoint matrices are positive combinations of cut-product rays, so $q_1(z)^\dagger H q_1(z)$ vanishes, and being nonnegative has a double zero, at $z = \pm i\sqrt{y_1}, \pm\sqrt{x_0}$ and at $z = \pm i$. The cyclic, parity and $K \circ S_{BC}$ symmetries fix every X_τ and permute the cut-product rays, so the average $\bar{H}$ of H over them has the same properties. These double zeros determine $\bar{H}$ up to a nonnegative multiple of H_0 (Appendix B). But $\mathrm{tr}(H_0 X_\tau)$ changes sign at $\tau \approx 0.805$, the root of $P(Q)$ (for instance $\mathrm{tr}(H_0 X_{48/55}) > 0$), a contradiction.

Remark 11 (the edge condition fails exactly at the endpoint rays). Take $a = 1$; invertible local filters map product vectors to product vectors and commute with partial conjugation up to the conjugate filter, so they preserve the edge property. By Proposition 2(a) the $A|BC$ product rays of R_1 are those of $\psi(z) = Bq_1(z) = (|0\rangle + z|1\rangle) \otimes \phi_1(z)$, $z \in \mathbb{C}$, and of $v = |111\rangle$. So ρ_τ is an edge state across $A|BC$ exactly when none of them has its partial conjugate $\psi^{T_A}(z) = (|0\rangle + \bar{z}|1\rangle) \otimes \phi_1(z)$ in $\mathrm{ran}\, \rho_\tau^{T_A}$, that is, orthogonal to the kernel vectors k_1, k_2, k_3 of Appendix B. For $z = x + iy$,

$$k_1^\dagger \psi^{T_A}(z) = 6((Q^2-1)|z|^2 + Q(z^2-1)), \qquad \mathrm{Im} = 12Qxy.$$

A zero therefore has $y = 0$, $x^2 = Q/(Q^2+Q-1)$, or $x = 0$, $y^2 = Q/(Q^2-Q-1)$, and there

$$k_3^\dagger \psi^{T_A} = -\frac{6Qx(Q^2-3)(Q^2+2Q-1)}{Q^2+Q-1} \quad \text{resp.} \quad -\frac{6iQy(Q^2-3)(Q^2-2Q-1)}{Q^2-Q-1},$$

while $k_2^\dagger |111\rangle = 12$. On $\sqrt{3} < Q < 1+\sqrt{2}$ no factor vanishes, so the edge condition holds; this checks the edge property across $A|BC$ directly, and across the other cuts by the cyclic shift. At the endpoints the factors Q^2-3 and Q^2-2Q-1 vanish. At $Q = \sqrt{3}$ the two branches give $z = \pm\sqrt{x_0}, \pm i\sqrt{y_1}$, the four rays of (15), and at $Q = 1+\sqrt{2}$ the imaginary branch gives $z = \pm i$, the rays of (16). So at both ends the common zeros of the overlaps with k_1, k_2, k_3 are exactly the product rays of the separable decomposition of the limit state.

4 The states are in the mixed W class

Pure three-qubit GME states fall into two inequivalent classes under stochastic local operations, represented by the GHZ and W states [8]; the GHZ class is distinguished by a nonzero three-tangle [9], the modulus of Cayley's hyperdeterminant Det (up to a factor four). Acín et al. extended this to mixed states and conjectured that PPT entangled three-qubit states lie in the mixed W class [7]. Our states contain the GHZ-type vector u in their spectral decomposition, but this does not make them GHZ-class: other decompositions can avoid GHZ-type components altogether, and we construct them. Write $\mathcal{W}$ for the convex hull of pure three-qubit states with vanishing Cayley hyperdeterminant, including biseparable states. This is the mixed W class of [7]; its pure generators have zero three-tangle. An invertible local filter preserves this class, since the hyperdeterminant transforms by the squared product of the local determinants. Convex mixing and local quantum channels also preserve $\mathcal{W}$.

Theorem 12 (the constructed fully-PPT states belong to the mixed W class)**.** *Every normalised state ρ_t of Theorem 1 in its PPT interval, the extreme state σ_6 of Corollary 5, every normalised $\rho_{\tau,a}$ of Theorem 7 ($1/\sqrt{3} < \tau < 1$, $a > 0$), the phase-coexistence line $\sigma(t)$ of Proposition 15 ($-1/200 \le t \le 1$), and the full-rank state $\tilde{\rho}$ of Corollary 20 belong to $\mathcal{W}$. So do their local-filter*

images and their mixtures with fully separable noise. In their GME ranges these states lie in $\mathcal{W} \setminus \mathrm{BS}$.

Proof. All vectors below are unnormalised. Put $[x] = xx^\dagger$ and let $\mathcal{C}$ average the three cyclic permutations, while $\mathcal{P}(X) = (X + Z^{\otimes 3} X Z^{\otimes 3})/2$ averages parity. These maps preserve $\mathcal{W}$. On either parity subspace only four computational basis states occur, and Cayley's hyperdeterminant reduces to

$$\mathrm{Det}\Big(\sum_{\text{even}} c_{ijk}|ijk\rangle\Big) = 4c_{000}c_{011}c_{101}c_{110}, \qquad \mathrm{Det}\Big(\sum_{\text{odd}} c_{ijk}|ijk\rangle\Big) = 4c_{111}c_{100}c_{010}c_{001}.$$

So a vector of definite parity with one vanishing amplitude has zero hyperdeterminant. This applies to $v, w, e, \bar{e}, o, \bar{o}$; to the even vectors $u_s + xe + y\bar{e}$, where $u_s = |000\rangle - s(|011\rangle + |101\rangle + |110\rangle)$ and x, y are real with $x + y = s$ (the amplitude of $|011\rangle$ is $-s + x + y = 0$); and to the odd vectors $v + b(w - o)$, $b \in \mathbb{C}$ (the amplitude of $|100\rangle$ vanishes). For vectors mixing the two parities, direct substitution gives

$$\mathrm{Det}(u+ze) = 4(z^3-1), \qquad \mathrm{Det}(u+zv) = z^2-4, \qquad \mathrm{Det}\big(xe+i(v-\beta w+\gamma o)\big) = 4(\gamma^3-\beta^3+3\beta\gamma x^2),$$

so for real $0 < \gamma \le \beta$ the last vector has zero hyperdeterminant when $x^2 = (\beta^3 - \gamma^3)/(3\beta\gamma)$. Also $u + i(w - v) = (|0\rangle + i|1\rangle)^{\otimes 3}$ is a product vector. Hence the following operators lie in the cone over $\mathcal{W}$:

$$\begin{aligned}
\mathcal{C}([u_s + xe + y\bar{e}]) &= [u_s] + x^2[e] + y^2[\bar{e}] \quad (x, y \in \mathbb{R},\ x + y = s),\\
\mathcal{C}([v + b(w - o)]) &= [v + bw] + |b|^2[o] \quad (b \in \mathbb{C}),\\
\mathcal{P}([u + 2v]) &= [u] + 4[v], \qquad \mathcal{P}([u + i(w - v)]) = [u] + [w - v],\\
\mathcal{C}(\mathcal{P}([xe + i(v - \beta w + \gamma o)])) &= [v - \beta w] + \gamma^2[o] + \frac{\beta^3 - \gamma^3}{3\beta\gamma}[e] \quad (0 < \gamma \le \beta).
\end{aligned} \tag{17}$$

For $s = 1$, $x = 1$, $y = 0$ the first is $\mathcal{C}([u + e]) = [u] + [e]$.

First consider a positive diagonal coefficient matrix $d_u[u] + d_v[v] + d_w[w] + d_e[e] + d_o[o]$, with all five weights nonnegative. If $d_u \le d_e + d_v/4$, put $r = \min(d_u, d_e)$ and $q = d_u - r$. It equals

$$r\,\mathcal{C}([u + e]) + q\,\mathcal{P}([u + 2v]) + (d_e - r)[e] + (d_v - 4q)[v] + d_w[w] + d_o[o],$$

with nonnegative coefficients. The first family satisfies $d_u = t$, $d_v = 3t$, $d_e = 3$ and $t \le t_+ < 4$, so $t \le 3 + 3t/4$. For σ_6, $d_u = t_- < 4$, $d_e = 3$ and $d_v = 8(9 - t_-)/5 > 8$, which also suffice.

For the surface, the filter law in Theorem 7(b) reduces the problem to $a = 1$. Use the notation of Appendix B: the weights of u and w are 1 and τ, and

$$\kappa = 1 - \tau^2, \quad Q = \tau + \sqrt{1 + \tau^2}, \quad E = \kappa Q^2, \quad O = \kappa/\tau, \quad G = (E - 1)/\tau.$$

and $V = (E^2 - 2\kappa E + \kappa)/(\tau\kappa)$. The coefficient block on (v, w) is positive semidefinite, since the operator is, so $V > 0$ and $\tau V - G^2 \ge 0$. Its other useful identity is

$$OV - G^2 = 2E - 1. \tag{18}$$

If $E \ge 1$, put $b = G/V$. Then the surface operator is

$$\mathcal{C}([u + e]) + V\mathcal{C}([v + b(w - o)]) + (\tau - G^2/V)[w] + (E - 1)[e] + (O - G^2/V)[o].$$

All coefficients are nonnegative, the last by (18).

If $E < 1$, put $\mu = (1 - E)/\tau = -G$. Since $Q \ge 1$ and $0 < \tau < 1$, $E = \kappa Q^2 \ge \kappa = 1 - \tau^2 \ge 1 - \tau$, so $0 < \mu \le 1$ and

$$\tau - \mu = \frac{E - \kappa}{\tau} \ge 0, \quad V - \mu = \frac{E(E - \kappa)}{\tau\kappa} \ge 0, \quad E - 1 + \mu = (1 - E)(\tau^{-1} - 1) \ge 0.$$

Now the surface operator is

$$\mu\,\mathcal{P}([u+i(w-v)])+(1-\mu)\mathcal{C}([u+e])+(\tau-\mu)[w]+(V-\mu)[v]+(E-1+\mu)[e]+O[o].$$

This again is a positive sum of zero-three-tangle projectors.

On the phase-coexistence line $\sigma(t)=(1-t)\sigma_\partial+t\sigma_{\mathrm{BS}}$ of Proposition 15, $\sigma(t)$ is biseparable, hence in $\mathcal{W}$, for $0\le t\le 1$. By convexity it remains to treat $t=-1/200$. There the coefficient matrix $(1-t)X_\partial+tX_{\mathrm{BS}}$ of (31) and (30) has diagonal (d_u,d_v,d_w,d_e,d_o) and (v,w) entry $-\mu$ with $\mu=10930399/348000000>0$. Subtracting $\mu\,\mathcal{P}([u+i(w-v)])$ leaves the positive diagonal matrix with weights $(d_u-\mu,d_v-\mu,d_w-\mu,d_e,d_o)$, and $d_u-\mu-d_e=7/31250\le(d_v-\mu)/4$, so the diagonal case above applies.

Finally, write the full-rank state $\tilde\rho$ of Corollary 20, displayed in (32), as

$$10^7\tilde\rho=c_0[g]+c_1(gh^\dagger+hg^\dagger)+c_2[h]+c_e[e]+c_{\bar e}[\bar e]+c_v[v]+c_{vw}(vw^\dagger+wv^\dagger)+c_w[w]+c_o[o]+c_{\bar o}[\bar o],$$

with $g=|000\rangle$, $h=|011\rangle+|101\rangle+|110\rangle$ and integer coefficients. The 2×2 blocks on (g,h) and (v,w) have positive determinants $c_0c_2-c_1^2=382\,804\,344$ and $c_vc_w-c_{vw}^2=257\,058\,514$. With $s=-c_1/c_0=232675/238898\in(0,1)$ and $\beta=-c_{vw}/c_v=520181/390647\in(1,4/3)$, completing the square writes them as $c_0[u_s]+(c_2-c_0s^2)[h]$ and $c_v[v-\beta w]+(c_w-c_v\beta^2)[w]$, both remainders positive. Take $(x,y)=(\frac59 s,\frac49 s)$ in the first atom of (17), and $\gamma=1$, $x'^2=(\beta^3-1)/(3\beta)<37/108$ in the last. Then

$$\begin{aligned}10^7\tilde\rho=c_0\,\mathcal{C}([u_s+xe+y\bar e])+c_v\,\mathcal{C}(\mathcal{P}([x'e+i(v-\beta w+o)]))\\+(c_2-c_0s^2)[h]+r_e[e]+r_{\bar e}[\bar e]+(c_w-c_v\beta^2)[w]+(c_o-c_v)[o]+c_{\bar o}[\bar o],\end{aligned}$$

where $r_{\bar e}=c_{\bar e}-c_0y^2>c_{\bar e}-\frac{16}{81}c_0>0$, $r_e=c_e-c_0x^2-c_vx'^2>c_e-\frac{25}{81}c_0-\frac{37}{108}c_v>0$ and $c_o-c_v=165\,771>0$. Dividing by the trace converts every displayed conic decomposition to a convex decomposition of normalised states. □

Remark 13 (scope of the classification). This settles the mixed-W-class question for every explicit state constructed in this paper. It also gives fully-PPT GME states in $\mathcal{W}$ with nonempty interior: states separable across $A|BC$ lie in $\mathcal{W}$ and contain a ball around $I/8$ [23], so for $0<p\le2\times10^{-5}$ the mixtures $(1-p)\tilde\rho+p\sigma$ with σ in that ball form a ball in $\mathcal{W}$, all within trace distance p of $\tilde\rho$ and hence fully PPT and GME by Corollary 20. It does not settle the conjecture of [7] for arbitrary fully-PPT three-qubit states. Nor does a zero-three-tangle decomposition imply biseparability: these states are GME in their stated ranges.

5 One support, both phases

The product curves constrain biseparable decompositions, but the support alone does not decide GME. The support $R=R_1$ of the first family carries both biseparable and GME fully-PPT states with the same rank profile. An explicit line crosses their boundary while remaining strictly PPT. The proofs, with the exact matrices and minors, are in Appendix C.

Biseparable states on R can be written down explicitly from the product curve of Proposition 2. For finite z let $p_z=Bq_1(z)/\|Bq_1(z)\|$, a normalised $A|BC$-product vector in R, let Π be the cyclic qubit permutation, and define the cyclically and parity symmetrised mixture

$$\Gamma_z=\frac16\sum_{j=0}^{2}\sum_{\epsilon=\pm1}\Pi^j|p_{\epsilon z}\rangle\langle p_{\epsilon z}|\Pi^{-j}. \tag{19}$$

Since $Z^{\otimes3}|p_z\rangle=-|p_{-z}\rangle$, the sum over $\epsilon=\pm1$ is the global parity average, and each Γ_z is an equal mixture of six pure product states across the three cuts, so Γ_z and its convex mixtures are biseparable and supported on R.

Theorem 14 (biseparable and GME states on one PPT support)**.** *The fixed support $R = R_1$ contains both a biseparable state and a GME state of rank five,* each *PPT with strictly positive partial transposes. They therefore have the same rank profile* $(5, 8, 8, 8)$ *and exactly the same three cut-product curves. An explicit biseparable example is* $\sigma_{\mathrm{BS}} = \frac{1}{3}(|111\rangle\langle 111| + \Gamma_2 + \Gamma_{2i}) = BX_{\mathrm{BS}}B^\dagger$*, where the rational coefficient matrix* X_{BS} *in the basis* (u, v, w, e, o) *is given in* (30) *of Appendix C. The GME example is* $\rho_{10/3}$ *from Theorem 1. Within the trace-one operators supported on R, the fully-PPT biseparable states and the fully-PPT GME states both have nonempty relative interiors.*

Idea of the proof. The state σ_{BS} is biseparable by construction; expanding the mixture gives X_{BS}, and Sylvester's criterion on the two parity blocks of one partial transpose gives strict PPT. The state $\rho_{10/3}$ has the same range and is GME by (7). Strict PPT, rank five and GME persist under small perturbations within R. On the biseparable side, averaging curve projectors over a measure of full support gives an interior point of the supported biseparable cone: no nonzero supporting form can vanish on that average. Mixing it into σ_{BS} retains strict PPT (Appendix C).

Proposition 15 (an exposed phase boundary inside the strict-PPT region)**.** *With the biseparable cut-product mixtures* Γ_z *of* (19)*, define*

$$\sigma_\partial = \frac{1}{5}\Gamma_1 + \frac{1}{2}\Gamma_i + \frac{3}{10}\Gamma_{i\sqrt{5}} = BX_\partial B^\dagger, \tag{20}$$

with the rational matrix X_∂ *of* (31) *in Appendix C. This rank-five biseparable state has* $\sigma_\partial^{T_c} \succ I/300$ *at each cut, yet* $\mathrm{tr}(H_5 X_\partial) = 0$*. The straight line*

$$\sigma(t) = (1-t)\sigma_\partial + t\sigma_{\mathrm{BS}}, \qquad -\frac{1}{200} \le t \le 1, \tag{21}$$

consists entirely of rank-five fully-PPT states with strictly positive partial transposes. Along this single fixed-support, fixed-rank *line,*

$$\sigma(t) \text{ is biseparable for } 0 \le t \le 1, \qquad \sigma(t) \text{ is GME for } -1/200 \le t < 0.$$

Thus $t = 0$ *is an exact biseparable–GME transition* inside *the strict-PPT region, exposed by* H_5*.*

Idea of the proof. The factorisation (6) vanishes at $|z| = 1$ and at $z = \pm i\sqrt{5}$, so H_5 has zero expectation on each summand of σ_∂. Since $\mathrm{tr}(H_5 X_{\mathrm{BS}}) > 0$, the witness value is linear in t along the line and negative exactly for $t < 0$, which excludes biseparability there. The PPT margin is a Sylvester check, a Hilbert–Schmidt estimate keeps it for the small negative range of t, and convexity covers $0 \le t \le 1$. The exact data are in Appendix C.

6 Noise robustness

How much white noise or independent qubit depolarisation can the states tolerate while remaining GME? We bound the tolerance from below with exact symmetric-extension witnesses, and from above with explicit biseparable decompositions completed by the Gurvits–Barnum separable ball [23]. The bounds meet to within a few per cent in relative terms. Appendix D gives the certificate checks.

The supported witnesses establish GME on a five-dimensional support. Mixing in a small amount of white noise yields full-rank, strictly PPT states while preserving GME, giving the following general consequence.

Corollary 16 (positive volume in any finite dimension)**.** *In any finite multipartite system, if a fully-PPT GME state exists, such states have non-empty interior (and hence positive volume) among density matrices. In particular this holds for three qubits. The conclusion is analytic; the certificates below provide explicit noise and preparation radii.*

Proof. Let ρ be any fully-PPT GME state on a Hilbert space of total dimension d. The biseparable set BS is compact, being the convex hull of the union of finitely many compact sets of pure cut-product states. Since $\rho \notin \mathrm{BS}$, the noisy state $\rho_p = (1-p)\rho + pI/d$ remains outside BS for all sufficiently small $p > 0$. But $\rho_p \succeq pI/d$ and $\rho_p^{T_S} = (1-p)\rho^{T_S} + pI/d \succeq pI/d$ at every cut S. Thus ρ_p is full rank and strictly PPT, and it is GME. Continuity gives an open neighbourhood of ρ_p in the trace-one Hermitian affine space consisting entirely of physical fully-PPT GME states. Applying this observation to Theorem 1 proves the three-qubit claim. □

Positive volume does not say how large the margins are. Since PPT tests cannot detect these states, certifying an explicit margin requires a stronger relaxation of biseparability. We use dual witnesses from level two of the Doherty–Parrilo–Spedalieri (DPS) symmetric-extension hierarchy [24, 25, 26], the lowest level stronger than PPT. Applying it across each cut to relax biseparability was proposed in Appendix B of [2]. Every cut-product vector extends by duplicating its qubit factor; the following certificate exploits this necessary condition. Numerical searches supplied candidates, rounded to rational parameters; only exact checks establish the bounds (Appendix G). Here $\mathrm{Sym}^2(A)$ is the subspace invariant under exchanging the two copies of A; tensor subscripts specify which factors an operator acts on.

Lemma 17 (certificate and stability principle)**.** *Let $A|B$ be a bipartition into finite-dimensional factors, let V have full column rank with image $\mathrm{Sym}^2(A) \otimes B$ inside $A_1 \otimes A_2 \otimes B$ (the columns need not be normalised), and let Y be Hermitian on $A \otimes B$ and $Z_1, Z_2 \succeq 0$ on $A_1 \otimes A_2 \otimes B$ with*

$$P = V^\dagger (Y_{A_1 B} \otimes I_{A_2} - Z_1^{T_B} - Z_2^{T_{A_2}}) V \succeq 0. \tag{22}$$

Then $\langle x|Y|x\rangle \geq 0$ for every product vector $x = a \otimes b$. Consequently, on three qubits, if Y is invariant under cyclic permutations of the qubits and satisfies (22) for the cut $A|BC$, then $\mathrm{tr}(Y\sigma) \geq 0$ for every biseparable state σ. If moreover ρ is a three-qubit state with $\mathrm{tr}(Y\rho) = -g < 0$, $h = \mathrm{tr}(Y)/8 > 0$, and $\mathrm{spec}(Y) \subset [l, u]$, then

$$(1-p)\rho + pI/8 \text{ is GME for } p < \frac{g}{g+h}, \qquad D(\rho, \mathrm{BS}) \geq \frac{g}{u-l}. \tag{23}$$

Finally, if $\rho^{T_j} \succeq cI$ at every cut j, with positive c, then every physical state σ with

$$D(\rho, \sigma) < \frac{2c}{1+\sqrt{2}}$$

is strictly PPT across every cut. This constant is dimension independent.

Idea of the proof. For a product vector $x = a \otimes b$, the certificate writes $\|a\|^2 \langle x|Y|x\rangle$ as the sum of a positive semidefinite quadratic form (from P) and two nonnegative terms (from Z_1 and Z_2), so Y is nonnegative on every cut-product vector and hence on BS. The noise and trace-distance bounds follow from the sign and spectral width of Y. The PPT-stability radius uses that the partial transpose of a pure state has spectral width at most $(1+\sqrt{2})/2$. Details in Appendix D.

Local depolarisation means applying $\mathcal{D}_{q_j}(X) = (1-q_j)X + q_j\,\mathrm{tr}(X)I/2$ on qubit j, with *independent rates* $0 \leq q_j \leq q$. White noise means $(1-p)\rho + pI/8$.

Corollary 18 (noise and preparation radius of the family)**.** *The rational, symmetry-reduced cyclic witnesses Y_t of the ancillary certificate data give the following conservative tolerances, where the balls are trace-distance balls around ρ_t.*

t	*white noise p*	*local q per qubit*	*GME ball*	*fully-PPT ball*
327/100	3.3×10^{-4}	1.4×10^{-4}	1.0×10^{-4}	1.01×10^{-4}
10/3	8.7×10^{-4}	3.7×10^{-4}	2.7×10^{-4}	5.7×10^{-4}
169/50	1.4×10^{-3}	6.1×10^{-4}	4.4×10^{-4}	2.0×10^{-4}

Specifically, white noise $(1-p)\rho_t+pI/8$ *remains GME for* p *up to the listed value, and independent local depolarisation with rates* $q_j \le q$ *does likewise. Every physical state in a listed closed GME ball is GME, and every state strictly inside a listed PPT ball is fully PPT. Every state at trace distance strictly below the smaller of the last two entries therefore has* both *properties: below* 2.7×10^{-4} *for* $t=10/3$*, below* 1.0×10^{-4} *for* $t=327/100$ *(where the whole GME ball is also PPT) and below* 2.0×10^{-4} *for* $t=169/50$*. In particular* $D(\rho_{10/3},\mathrm{BS})>2.7\times10^{-4}$*. White noise itself preserves PPT automatically.*

One fixed dual witness also certifies uniformly *over the exact interval (3):* $p\le 2.4\times10^{-4}$ *of white noise and a GME ball of radius* 7.5×10^{-5}*. This gives a second GME proof, independent of the analytic witness.*

Idea of the proof. Exact witnesses satisfy Lemma 17. Unequal depolarisation rates reduce to one uniform rate because further local noise preserves biseparability. Affine dependence on t reduces the uniform interval bound to two endpoint checks (Appendix D).

Corollary 19 (noise bounds for the surface state from one exact witness)**.** *Let* ρ_* *be the trace-one normalisation of the analytic surface operator* $\rho(312,266,603)$ *in (12); here* $\tau=133/156$ *and* $a=201/104$*. It is fully PPT and GME, with rank profile* $(5,5,5,5)$*. A single exact level-two witness* Y_* *(optimised numerically for* ρ_* *under a spectral bound) certifies:*

(i) The states $(1-p)\rho_*+pI/8$ *remain fully PPT and GME for* $0\le p\le 2153/500000$ *(about* 0.43% *white noise).*

(ii) Independent local depolarisation preserves both properties whenever each rate is at most $81/50000$ *(about* 0.16% *per qubit).*

(iii) Let

$$\rho_\circ=\left(1-\frac{32}{10000}\right)\rho_*+\frac{32}{10000}\frac{I}{8}.$$

Every physical state σ *with* $D(\sigma,\rho_\circ)<331/1000000$ *is fully PPT and GME. Thus a full-rank state has a simultaneous preparation radius of* 3.31×10^{-4}*.*

(iv) Every physical state σ *with* $D(\sigma,\rho_*)\le 129/100000$ *is GME (a GME-only radius of* 1.29×10^{-3}*).*

Idea of the proof. The certified expectation and spectral bounds of Y_* give each margin by Lemma 17; white noise opens a positive PPT gap for (iii) (Appendix D).

Corollary 20 (a full-rank fully-PPT GME state tolerating 2.3% white noise)**.** *The three-qubit density matrix* $\tilde\rho$ *with seven-decimal rational coordinates in the certificate data (entries in* $\mathbb{Q}(\sqrt{-3})$*) has full rank, is invariant under cyclic permutations of the qubits and under global parity* $Z^{\otimes3}$*, has every partial transpose strictly above* 2.49×10^{-5} *times the identity, and is GME: its rational level-two witness is nonnegative on every cut-product vector and negative on* $\tilde\rho$*. Certified tolerances: global white noise* $p\le 2.33\times10^{-2}$*, independent depolarising noise with every rate at most* 1.0×10^{-2}*, and every state within trace distance* 7.3×10^{-3} *of* $\tilde\rho$ *is GME. The ball of radius* 2.0×10^{-5} *is also certified fully PPT, by the verified lower bound* 2.49×10^{-5} *on the partial transposes and Lemma 17. These are certified lower bounds; Proposition 21 gives the matching upper bound for white noise.*

Idea of the proof. Lemma 17 applied to the rational data of $\tilde\rho$, with exact checks of full rank, symmetry and the partial-transpose eigenvalue bounds. Details in Appendix D.

The state $\tilde\rho$ is obtained from the white-noise mixture $(1-\epsilon)\rho_{\mathrm{opt}}+\epsilon I/8$, $\epsilon=2.5\times10^{-4}$, of a numerically optimised fully-PPT state ρ_{opt} (Remark 23) by rounding to rational entries; white noise provides strict PPT before rounding. Its GME ball is more than five times the GME-only radius of Corollary 19(iv). Its exact coordinates are displayed in Appendix D.

For arbitrary physical perturbations, the certified GME radius is 7.3×10^{-3} in trace distance, whereas the simultaneous PPT radius is only 2.0×10^{-5} because the smallest partial-transpose

eigenvalue is small. The much larger white-noise tolerance therefore does not guarantee stability under generic preparation errors.

Proposition 21 (the white-noise thresholds are essentially sharp)**.** *Let* $p_{\mathrm{GME}}(\rho) = \sup\{p \in [0,1] : (1-p)\rho + pI/8$ *is GME*$\}$*, with* $p_{\mathrm{GME}}(\rho) = 0$ *for biseparable* ρ*. Then*

$$\frac{2153}{500000} \le p_{\mathrm{GME}}(\rho_*) \le \frac{4319}{1000000},$$
$$\frac{87}{100000} \le p_{\mathrm{GME}}(\rho_{10/3}) \le \frac{89}{100000},$$
$$\frac{233}{10000} \le p_{\mathrm{GME}}(\tilde{\rho}) \le \frac{5839}{250000},$$

with ρ_* *and* $\tilde{\rho}$ *as in Corollaries 19 and 20. The noisy state is biseparable at, and for every* p *above, each upper bound.*

Idea of the proof. The lower bounds are the three corollaries above. For the upper bounds, an explicit separable-across-$A|BC$ operator, assembled from three orbit representatives and symmetrised cyclically, is subtracted from the noisy state; the residual lies in the Gurvits–Barnum ball around the identity [23], hence is separable, so the noisy state is biseparable. Details in Appendix D.

The certified intervals have relative widths of at most 2.3%, so the small margins reflect the states, not the relaxation. Numerically the agreement is closer: for all three states the optimal level-two bound and the numerical value of the optimised three-orbit decomposition agree to within 10^{-7} (numerical evidence only); the decompositions use three orbits of qubit directions each.

To guide the search for robust states, we use the convex white-noise robustness rob below, rather than p_{GME} itself.

Proposition 22 (an extremal reduction for white-noise robustness)**.** *In any finite multipartite system of dimension* d*, let* $\mathcal{P}$ *be the set of fully-PPT states,* BS *the biseparable states and* $\mathrm{rob}(\rho) = \min\{s \ge 0 : (\rho + sI/d)/(1+s) \in \mathrm{BS}\}$*, so that* $p_{\mathrm{GME}}(\rho) = \mathrm{rob}(\rho)/(1+\mathrm{rob}(\rho))$*. Then*

$$\mathrm{rob}(\rho) = \max\Big\{0,\ \max_{W \in \mathrm{BS}^*,\ \mathrm{tr}\, W = d} -\mathrm{tr}(W\rho)\Big\}, \tag{24}$$

where BS^* *is the cone of Hermitian operators nonnegative on* BS*. Consequently* rob *is convex, its maximum over* $\mathcal{P}$ *is attained at an extreme point of* $\mathcal{P}$*, and so is that of the increasing function* $p_{\mathrm{GME}} = \mathrm{rob}/(1+\mathrm{rob})$*, and for a fixed witness the best state is the solution of a semidefinite program. The same holds with* BS *replaced by any closed convex set of states containing* BS*, for example the states passing every cyclic level-two test of Lemma 17.*

Proof. I/d is interior to the cone generated by BS: it is a full-support average of product pure states, whose projectors span the Hermitian operators, so no nonzero element of BS^* vanishes on it. Hence the conic program defining rob is strictly feasible, and conic duality gives (24) with no gap; each $W \in \mathrm{BS}^*$ with $\mathrm{tr}\, W = d$ gives $s \ge -\mathrm{tr}(W\rho)$. Since I/d is interior, the slice $\{W \in \mathrm{BS}^* : \mathrm{tr}\, W = d\}$ is compact, so rob is a maximum of affine functions over a compact set and hence continuous and convex. By Bauer's maximum principle it attains its maximum over the compact convex set $\mathcal{P}$ at an extreme point. For fixed W, minimising $\mathrm{tr}(W\rho)$ over $\rho \succeq 0$, $\mathrm{tr}\,\rho = 1$, $\rho^{T_S} \succeq 0$ is a semidefinite program. □

Remark 23 (where the robust states sit). The search for $\tilde{\rho}$ alternated the two steps of Proposition 22: optimising the witness for a state and the fully-PPT state for a witness. For states invariant under the cyclic permutation and global parity, the state is block diagonal over six sectors labelled by parity and cyclic charge (the partial-transpose constraints mix charges and respect only the two parity blocks): the charge-zero sectors spanned by $\{|000\rangle, |011\rangle + |101\rangle + |110\rangle\}$ and $\{w, v\}$, and the one-dimensional sectors $e, \bar{e}, o, \bar{o}$. The following observations come from floating-point searches

and are not certified. Extreme points differ in how their rank is distributed among these sectors. The surface of Section 3 occupies u_s, the whole $\{w, v\}$ sector, e and o, while the optimiser behind $\tilde{\rho}$ occupies u_s, one direction $w + cv$, e, $\bar{e}$ and o, with $s \approx 0.974$ and $c \approx -0.750$: the same symmetries as the surface, but one combination of w and v is traded for the opposite-chirality vector $\bar{e}$. Both have profile $(5, 5, 5, 5)$. The level-two white-noise tolerance of ρ_{opt} before mixing is numerically 2.36×10^{-2}; whether some fully-PPT three-qubit state is more robust is open.

7 PPT mixtures and the remaining boundary

The examples separate full PPT from biseparability. We explain why PPT-mixture tests miss them, show that the supported witness method applies to every minimal-rank example, and distinguish a remaining question about maximally mixed marginals.

7.1 Why PPT-mixture tests cannot detect the states

A *PPT mixture* is a convex combination of states, each PPT across one of the three cuts [2, 3]. Every biseparable state is a PPT mixture, but the converse fails [4].

Corollary 24 (PPT mixtures and outer relaxations)**.** *Every fully-PPT GME state in Theorems 1 and 7 and Corollaries 5, 8 and 20 is a PPT mixture. No PPT-mixture witness can detect any of them. The level-two symmetric-extension certificates of Section 6 do detect the first family and the states ρ_*, σ and $\tilde{\rho}$ of Corollaries 19, 8 and 20; we do not assert a level-two certificate for every member of the local-filter surface.*

Proof. Every fully-PPT state is itself a one-term PPT mixture. The GME and level-two witness claims were proved in the preceding sections. □

Remark 25 (rank-four PPT pieces do not always suffice). The family of Theorem 1 also rules out a proposed universal decomposition of fully-PPT rank-five states into positive operators $\rho = \sigma_A + \sigma_B + \sigma_C$ with $\sigma_c^{T_c} \succeq 0$ and $\operatorname{rank} \sigma_c \leq 4$. By the 2×4 low-rank theorem [22, 10], each summand would be separable across its indicated cut, making ρ biseparable.

7.2 Every minimal-rank example has the same geometry

The supported-witness method extends beyond our chosen vectors. For a fully-PPT GME state of minimal rank, the kernel contains no cut-product vector; this forces the product rays at each cut to form a rational quartic curve spanning the support. The three curves determine all its biseparable states and all supported witnesses.

Theorem 26 (three images of one quartic moment cone)**.** *Let $R \subset (\mathbb{C}^2)^{\otimes 3}$ be a five-dimensional subspace whose orthogonal complement contains no nonzero vector that is a product across any of the three cuts; by Lemma 29 (Appendix E) this holds for the range of every rank-five fully-PPT GME state. Let B be an 8×5 basis matrix of R, put $\nu(a) = (a_0^4, a_0^3 a_1, a_0^2 a_1^2, a_0 a_1^3, a_1^4)^T$ for $a \in \mathbb{C}^2$, and let $\mathcal{M}_4 = \operatorname{cone}\{\nu(a)\nu(a)^\dagger\}$ be the spin-two (quartic) moment cone. For each cut c there is an invertible 5×5 matrix T_c such that the cut-c product rays in R are exactly the rays of $BT_c\nu(a)$, $a \neq 0$. Consequently, for Hermitian X,*

$$BXB^\dagger \text{ is biseparable} \iff X \in \sum_{c=A,B,C} T_c \mathcal{M}_4 T_c^\dagger, \tag{25}$$

every biseparable operator on R is a sum of at most 25 cut-product projectors, and a Hermitian 5×5 matrix H satisfies $\operatorname{tr}(HX) \geq 0$ for every biseparable $BXB^\dagger$ if and only if

$$\nu(a)^\dagger T_c^\dagger H T_c \, \nu(a) \geq 0 \qquad \text{for all } a \in \mathbb{C}^2,\ c = A, B, C. \tag{26}$$

Idea of the proof. A fully-PPT state whose kernel contains a product vector is separable across that cut, so the kernel of a minimal-rank example is product-free. The product vectors $a \otimes b$ in R are then the solutions of a 3×4 linear pencil in a of constant rank three, which is forced to be a single Kronecker block; its kernel is cubic in a, so the product vectors trace a rational normal quartic spanning R. Appendix E.

Thus, on the support of a minimal-rank example, biseparability is decided by where three copies of one universal cone sit relative to each other, and genuine multipartite entanglement is certified exactly by a 5×5 Hermitian H whose three quartic forms (26) are nonnegative while $\mathrm{tr}(HX) < 0$. Such an H exists for every GME state supported on R, since the cone in (25) is closed. The witnesses H_s and H_0 of Sections 2 and 3 are instances, with the quartic forms factorised by hand. Testing the nonnegativity of a quartic form in one complex variable is a small problem: a real polynomial of degree eight in two variables.

7.3 The flat-marginal question

A three-qubit state is *flat* if every proper marginal is maximally mixed. Flat states are a natural place to look for further examples, because every flat state is automatically fully PPT (shown below), so only biseparability is in question. Our examples settle existence of fully-PPT GME, but leave the following conjecture open:

Conjecture 27 (flat rank-five states)**.** *Every flat three-qubit state of rank five is biseparable.*

Every flat three-qubit state is fully PPT, with rank profile (r, r, r, r), $r = \mathrm{rank}\,\rho$. Indeed $\rho = I/8 + T$ with T a real combination of three-body Pauli products. Partial transposition on qubit c followed by conjugation with σ_y on that qubit negates every Pauli factor on c, and the antiunitary $J = \sigma_y^{\otimes 3} K$ (K complex conjugation) negates all three factors, so

$$\sigma_y^{(c)} \rho^{T_c} \sigma_y^{(c)} = I/8 - T = J \rho J^{-1}.$$

Each partial transpose is therefore isospectral to ρ. Profiles are unchanged by invertible local filtering.

Remark 28 (rank profile is not flatness). The first family has profile $(5, 8, 8, 8)$ in the PPT interval and $(5, 7, 7, 7)$ at its endpoints, while Corollary 5 has profile $(5, 6, 6, 6)$. None is locally filter-equivalent to a flat rank-five state. The second family and its extreme member *do* have profile $(5, 5, 5, 5)$, yet they are GME. Thus that profile by itself cannot imply biseparability; Conjecture 27 must use the flat-marginal condition, not only the ranks.

7.4 Relation to earlier work

Fully-PPT genuine multipartite entanglement is known in systems larger than three qubits. For states supported on the totally symmetric subspace, any cut-product vector in the support is fully product, so biseparability equals full separability there. This connects the symmetric multipartite PPT-entangled constructions for four or more qubits [27, 28, 29] to GME; mere permutation invariance, without support in the symmetric subspace, is not sufficient for this inference. Fully-PPT GME examples also exist for three qutrits and higher local dimensions [30, 31, 1, 32]; for [32] the state is the normalised projector onto the complement of a genuinely unextendible product basis.

For three qubits, Acín, Bruß, Lewenstein and Sanpera gave an early construction of fully-PPT three-qubit edge states, with rank profiles up to $(7, 7, 7, 7)$, including an X-shaped family [7], of which Kampermann et al. generated a pseudo-bound-entangled NMR version [12]. By Han and Kye, every X-shaped fully-PPT multi-qubit state is biseparable [6, Cor. 5.4]; these NMR states are therefore not GME, and, since biseparability and PPT are invariant under local unitaries, the GME states of this paper are not X-shaped in any product basis. Explicit three-qubit PPT entangled states of low rank are also given in [5]. A PPT state is an edge state across a cut if

no product vector of its range has its partial conjugate in the range of the partial transpose. The states of the $(5,5,5,5)$ surface are edge states across every cut (Corollary 8). The interior states ρ_t are not: their partial transposes are invertible and the range contains the product vector $|111\rangle$.

Ha and Kye gave a three-qubit PPT mixture that is not biseparable, but their rank-seven example was established PPT across a selected cut, not all three [4]. This already refuted the conjecture of Novo, Moroder and Gühne that PPT mixtures coincide with biseparable states for three qubits [3]. The states here are a stronger counterexample: a single-term PPT mixture that is PPT across all three cuts. By Observation 3 of [3], a permutation-invariant three-qubit state is biseparable if and only if it is a PPT mixture, so no state of this paper is permutation invariant, in line with Corollary 3. Garberg, Irgens and Myrheim found fully-PPT states numerically with many rank profiles, including $(5,5,5,5)$ and extreme states, without a GME certificate [33]. Their phrase "genuine tripartite entanglement" for rank-$(4,4,4,4)$ states means entangled while separable across every bipartition, which is not GME in the present sense. Hari Krishnan, Ranjan and Banik [19] provided fully-PPT three-qubit states outside $\bigcap_c \mathrm{Sep}_c$, the intersection of the three cut-separable sets, by cyclically symmetrising an embedded 2×4 PPT state and subtracting low-rank terms. This does not imply GME: the biseparable set is the larger convex hull $\mathrm{BS} = \mathrm{conv}(\bigcup_c \mathrm{Sep}_c)$. Our supported witness excludes the *convex hull*, not merely the intersection. Tóth and Vértesi found numerically a three-qubit state that is PPT across all bipartitions and nevertheless useful for metrology [13]; it is entangled but was not shown to be GME. Singh and Banik noted that no explicit three-qubit state that is PPT across all three cuts and GME was known, and showed that, if one exists, it can be taken extreme with all four ranks at least five and the smallest equal to five or six [1]; they did not settle existence. Our extreme states satisfy these restrictions.

To our knowledge these are the first explicit *three-qubit* examples proved both fully PPT and GME; the claim does not extend to more parties or higher local dimensions.

8 Discussion

Physically, the examples separate two questions that are easy to conflate: whether entanglement can be extracted across some bipartition, and whether it is shared by all three parties. Even for three qubits, the first does not decide the second. A fully-PPT GME state yields no singlet across any cut and no GHZ state under any LOCC protocol, yet it is not a mixture of states each separable across some bipartition. The relevant gap is between the intersection of the three PPT sets and the convex hull of the three cut-separable sets. The support of ρ_t contains product vectors across every cut, but no convex mixture of them reproduces ρ_t; the supported witness certifies this failure of mixing. Symmetry plays a definite role. Reflection symmetry together with PPT forces separability across a cut (Corollary 3), so the states must break every exchange of two qubits; ours do so through the cyclic phases of the W-type vectors, and they are also locally chiral (Corollary 4).

For certification, PPT-mixture tests cannot certify the genuine entanglement of these states; a stronger relaxation of biseparability is one way to do so. The second level of the symmetric-extension hierarchy suffices here, and for white noise its certified margins nearly match explicit biseparable decompositions (Proposition 21), so the small size of the margins is a property of the states, not of the relaxation. By Proposition 22 a most robust state can be taken extreme in the fully-PPT set, and numerical searches suggest that within one symmetry class the extreme points differ in how their rank is distributed among the symmetry sectors (Remark 23). An NMR experiment has generated pseudo-bound-entangled fully-PPT three-qubit states that are not GME [12]. The state $\tilde{\rho}$ remains GME under depolarising errors of up to one per cent per qubit, but generic preparation errors need much tighter control: its certified PPT-stability radius, 2.0×10^{-5}, is far smaller than its GME radius, 7.3×10^{-3} (Corollary 20); certifying such a state also requires witnesses that go beyond partial transposition.

Several questions remain open: whether some fully-PPT three-qubit state tolerates substantially more noise, how to describe the family of $\tilde{\rho}$ analytically, and how to decompose a witness into

practical local measurements. The mixed-W conjecture of Acín et al. [7] holds for all our explicit states, including $\tilde{\rho}$ (Theorem 12), but remains open for general PPT entangled three-qubit states. A separate problem is whether every rank-five three-qubit state with maximally mixed one- and two-qubit marginals is biseparable (Conjecture 27). Our rank-$(5,5,5,5)$ examples show that the rank profile alone cannot prove it; the flat-marginal condition must play an essential role.

Use of AI

All results, documents and scripts were generated by AI agents (Claude Opus and Sonnet 5.5, OpenAI's GPT 6.1 sol and Gemini 4 Argon/Carbon) with human supervision and many cross-checks, including AI-generated Lean verification of the main analytic results. I thank DeepMind and Anthropic for access to their scientist programs.

Acknowledgements

We acknowledge support by the UK STFC grant ST/T000473/1.

A Computations for the rank-five family

This appendix contains the proofs of the statements of Section 2, whose notation it uses.

Proof of Theorem 1, PPT and rank claims. Cyclic permutations of the qubits leave ρ_t invariant, so its three partial transposes are unitarily equivalent. The unnormalised numerator of $\rho_t^{T_A}$ commutes with the global parity $Z^{\otimes 3}$, so it splits into even and odd parity blocks. With $\beta = 3+\omega$ and $\gamma = t+3\omega^2$ ($|\beta|^2 = 7$, $|\gamma|^2 = t^2-3t+9$), in the respective basis orders $(000, 011, 101, 110)$ and $(001, 010, 100, 111)$ these blocks of $7(t+3)\rho_t^{T_A}$ are

$$\text{even: } \begin{pmatrix} t & -t & \beta & \bar{\beta} \\ -t & t+3 & 0 & 0 \\ \bar{\beta} & 0 & t+3 & \gamma \\ \beta & 0 & \bar{\gamma} & t+3 \end{pmatrix}, \qquad \text{odd: } \begin{pmatrix} 4 & \beta & -t & \bar{\gamma} \\ \bar{\beta} & 4 & -t & \gamma \\ -t & -t & 4 & 0 \\ \gamma & \bar{\gamma} & 0 & 3t \end{pmatrix},$$

and their leading principal minors are

$$\begin{aligned} &\text{even: } t, \quad 3t, \quad (t+3)(3t-7), \quad 3(8t^2-15t-36), \\ &\text{odd: } 4, \quad 9, \quad 3(12-t^2), \quad -3(t-3)(3t^2+4t-48). \end{aligned} \tag{27}$$

The even determinant is nonnegative exactly when $t \geq t_-$. For $3 \leq t < 2\sqrt{3}$, the first three leading minors of each block are positive, while the odd determinant is nonnegative exactly when $t \leq t_+$. Since $3 < t_- < 10/3 < t_+ < 2\sqrt{3}$, the leading 3×3 corners are positive definite throughout $[t_-, t_+]$. Each block is positive semidefinite exactly when its scalar Schur complement, or equivalently its determinant, is nonnegative. This gives PPT precisely on $[t_-, t_+]$. Outside this interval one of the displayed determinants is negative. At either endpoint exactly one block has rank three (its leading 3×3 minor stays positive); the other has rank four. The stated profiles follow. The GME claim is established in Section 2.1, after constructing a witness on the range. □

Proof of Proposition 2. (a). The product condition reduces to four linear equations. Write a general vector of R_s as $B_s(U,V,W,E,O)^T$. It is a product with the first-qubit ray $(1,z)^T$ precisely when the two BC rows are proportional with ratio z. In the computational basis this is

$$\begin{aligned} -zU + W + O = 0, \quad -sU - zW + \omega E - z\omega^2 O = 0, \\ -sU - zW + \omega^2 E - z\omega O = 0, \quad szU + V - zE = 0. \end{aligned}$$

Subtracting the middle equations and using $\omega + \omega^2 = -1$ gives

$$E = -zO, \qquad W = zU - O, \qquad (s+z^2)U = 2zO, \qquad V = -szU - z^2 O.$$

For every finite z, the solutions form a one-dimensional space, because $s+z^2$ and $2z$ never vanish simultaneously when $s>0$. Setting $U=2z$, $O=s+z^2$ gives exactly $q_s(z)$ in (4); it remains nonzero at $z=0$ and $z^2=-s$. At the projective point $(0,1)^T$ (i.e. $z=\infty$), the upper BC row must vanish, which forces $U=W=E=O=0$ and leaves only V, the ray of v. This proves the full cut-product classification.

For Appendix F, $R^\perp$ is spanned by the three symmetry-sector vectors omitted from R, namely $\bar e$, $\bar o$ and $3|000\rangle+|011\rangle+|101\rangle+|110\rangle$: the 3×8 matrix K whose rows are their bras,

$$K_0=[0,0,0,1,0,\omega,\omega^2,0],\quad K_1=[0,\omega^2,\omega,0,1,0,0,0],\quad K_2=[3,0,0,1,0,1,1,0]$$

has rank three and $KB=0$, so $R=\ker K$.

(b). A rescaling removes the parameter s: with $D_s=\mathrm{diag}(1,s^{3/2},s^{1/2},s,s^{1/2})$,

$$q_s(\sqrt{s}\,z)=\sqrt{s}\,D_s q_1(z),\qquad H_s=s^3D_s^{-1}H_5D_s^{-1},$$

so $q_s(\sqrt{s}z)^\dagger H_s q_s(\sqrt{s}z)=s^4\,q_1(z)^\dagger H_5 q_1(z)$, and the right side of (5) at $(s,\sqrt{s}z)$ is s^4 times its value at $(1,z)$. It suffices to take $s=1$, which is (6). Put $d=\mathrm{Re}(z^2)=x^2-y^2$, so $r+d=2x^2$. The squared moduli of the entries of $q_1(z)$ are

$$4r,\quad r^2(r^2+9+6d),\quad r^2+1-2d,\quad r(r^2+1+2d),\quad r^2+1+2d,$$

and H_5 is diagonal, so

$$\begin{aligned}q_1^\dagger H_5 q_1&=-48r+r^2(r^2+9+6d)+11(r^2+1-2d)-6r(r^2+1+2d)+14(r^2+1+2d)\\&=r^4-6r^3+34r^2-54r+25+6d(r-1)^2.\end{aligned}$$

Since $(r-1)^2(r^2-4r+25)=r^4-6r^3+34r^2-54r+25$ and $12x^2=6(r+d)$, this equals $(r-1)^2\big[(r-5)^2+12x^2\big]$. This proves (5). The cyclic permutation of qubits fixes u_s,v,w and acts on e,o by phases, which commute with the diagonal H_s, so the form controls the other cuts as well. The point at infinity contributes $H_{vv}=1$. Finally, every positive pure-state decomposition of an operator supported on R_s uses vectors in R_s: its expectation on $R_s^\perp$ vanishes, and each positive summand must vanish there separately. Thus the nonnegative curve values also control mixtures across different cuts. □

Proof of Corollary 4. (a). The weights $t,3t,3,3,1$ of ρ_t in (2) are positive for every $t>0$, so $\mathrm{ran}\,\rho_t=R_1$. A local unitary is an invertible local filter, so (a) is a special case of (b), proved next.

(b). By Proposition 2(a) the $A|BC$-product rays of R_s are $[v]$ and $[B_sq_s(z)]=[(|0\rangle+z|1\rangle)\otimes|\phi_s(z)\rangle]$, $z\in\mathbb{C}$, and substituting (4) shows that the 2×2 coefficient matrix of $|\phi_s(z)\rangle$ has determinant $-3(z^2+s)^2$. So $|\phi_s(z)\rangle$ is a $B|C$ product exactly when $z=\pm i\sqrt{s}$, where $B_sq_s(\pm i\sqrt{s})=\pm 2i\sqrt{s}\,(|0\rangle\pm i\sqrt{s}|1\rangle)^{\otimes 3}$. Hence R_s contains exactly three fully product rays, $P_\infty=[|1\rangle^{\otimes 3}]$ and $P_\pm=[(|0\rangle\pm i\sqrt{s}|1\rangle)^{\otimes 3}]$; they span $\mathrm{span}(u_s,v,w)$, because $(|0\rangle\pm i\sqrt{s}|1\rangle)^{\otimes 3}=u_s\pm i\sqrt{s}\,(w-sv)$. The conjugate $\overline{R_s}$ contains the conjugate rays, the same three with P_+ and P_- exchanged. An invertible local filter F that maps R_s onto R_s or onto $\overline{R_s}$ permutes the three rays. Each ray is $[\chi^{\otimes 3}]$, so F_A, F_B and F_C all induce the same permutation of the three distinct points $[\chi]\in\mathbb{CP}^1$, and since a Möbius map is determined by its action on three points, $F=cF_0^{\otimes 3}$. Such an F commutes with the cyclic shift Π, so it maps the ω^2-eigenspace of Π into itself. That eigenspace is $\mathrm{span}(e,o)\subset R_s$, whereas $\overline{R_s}$ meets it in 0: $\overline{R_s}\ni\bar e,\bar o$ lies in the ω-eigenspace and the other three generators of $\overline{R_s}$ are Π-invariant. So $F(R_s)\neq\overline{R_s}$. If a state ρ with range R_s satisfied $\rho=F\bar\rho F^\dagger$ (or the same with $Y^{\otimes 3}\bar\rho Y^{\otimes 3}$, a local-unitary image of $\bar\rho$), then there would be an invertible local filter F' (F^{-1}, respectively $Y^{\otimes 3\dagger}F^{-1}$) with $F'(R_s)=\overline{R_s}$, which the argument excludes. For the stabiliser, each of the six permutations of the three points is induced by a unique Möbius map, and $F_\pi^{\otimes 3}$ permutes $P_\infty,P_\pm$, hence preserves $\mathrm{span}(u_s,v,w)$, and as it commutes with Π it preserves the ω^2-eigenspace $\mathrm{span}(e,o)$ of Π (which is all of that eigenspace in $(\mathbb{C}^2)^{\otimes 3}$). So the stabiliser is $\{[F_\pi]^{\otimes 3}\}\cong S_3$.

□

Proof of Corollary 5. Write the numerator of σ_6 as $t\,uu^\dagger+b\,vv^\dagger+3\,ww^\dagger+3\,ee^\dagger+oo^\dagger$; it is the numerator of ρ_t with the weight $3t$ of $vv^\dagger$ replaced by b. The vector $v=|111\rangle$ enters $\sigma_6^{T_A}$ only through the $(111,111)$ entry, the bottom diagonal entry of the odd block, and equals b there. Hence the even block, and the three leading principal minors of the odd block, are those of $\rho_t^{T_A}$ in (27): at $t=t_-$ the even minors $t,\ 3t,\ (t+3)(3t-7)$ are positive and, since $8t^2-15t-36=0$, the even determinant vanishes, so the even block is positive semidefinite of rank three. The odd minors $4,\ 9,\ 3(12-t^2)$ are positive ($t_-<2\sqrt{3}$), and the odd determinant is affine in b,

$$\det(\text{odd block})=3(12-t^2)\,b+3(5t^2+24t-144),$$

which equals $-3(t-3)(3t^2+4t-48)$ of (27) at $b=3t$. It vanishes at $b=(5t^2+24t-144)/(t^2-12)$, and this equals $8(9-t)/5$ modulo $8t^2-15t-36=0$, since $5(5t^2+24t-144)-8(9-t)(t^2-12)=8t^3-47t^2+24t+144$ is divisible by $8t^2-15t-36$ (the quotient is $t-4$ and the remainder is 0). Hence at $b=8(9-t)/5$ the odd block is positive semidefinite of rank three as well. Thus each partial transpose of σ_6 has rank six. The supported witness has expectation

$$\operatorname{tr}(H_5X_6)=\frac{b-12t+29}{4t+b+21}=\frac{217-68t}{5(4t+b+21)}<0,$$

because $t>13/4>217/68$. This proves GME by Proposition 2. For extremality, σ_6 is extreme in the set of fully-PPT states exactly when the only Hermitian D with range in $\operatorname{ran}\sigma_6$ and $D^{T_c}k=0$ for every kernel vector k of $\sigma_6^{T_c}$, at each cut c, is a multiple of σ_6 (the supported-face criterion of [33]). Write $D=BX_DB^\dagger$ with X_D Hermitian. The conditions $D^{T_c}k=0$ are a real-linear system on the 25-dimensional real space of such X_D. To compute its rank exactly, complexify: let X_D range over all complex 5×5 matrices (25 complex unknowns) and impose both $D^{T_c}k=0$ and $k^\dagger D^{T_c}=0$ for the kernel vectors k, at each cut. The two sets are adjoint to each other for Hermitian D, so the solution space is closed under $X_D\mapsto X_D^\dagger$, and is the complexification of the real solution space on Hermitian X_D (split X_D into its Hermitian and anti-Hermitian parts). Real nullity therefore equals complex nullity. The complexified system has rank 24 over $\mathbb{Q}(\sqrt{17},i\sqrt{3})$, which contains the entries of σ_6 and of the kernel vectors, so its nullspace, and hence the real nullspace, is exactly the scalar direction of (10). □

B Computations for the profile-$(5,5,5,5)$ surface

This appendix contains the proofs of the statements of Section 3, whose notation it uses.

Proof of Lemma 6. At fixed radius, the two-variable identity reduces to quadratic interpolation. Hold $r=\xi+\zeta>0$ fixed, put $P(\xi,r-\xi)=q_1^\dagger H_0q_1$, and substitute $z=i\sqrt{r}$ and $z=\sqrt{r}$ into the five entries of q_1. Using $x_0^2q=y_1^2$ and collecting powers gives the axis values

$$P(0,r)=(r-1)^2(r-y_1)^2,\qquad P(r,0)=(r-x_0)^2(r^2+pr+q).$$

For fixed r, each diagonal term of P is affine in ξ (it depends on z^2 only through $\operatorname{Re}(z^2)=2\xi-r$). The off-diagonal vw entry contributes a quadratic whose ξ^2 coefficient is $16(H_0)_{vw}=-16\nu$. Thus $P(\xi,r-\xi)$ is the unique quadratic with the displayed endpoint values and that leading coefficient, namely

$$P(\xi,\zeta)=\frac{\zeta}{r}P(0,r)+\frac{\xi}{r}P(r,0)+16\nu\,\xi\zeta,\qquad \xi+\zeta=r>0.$$

Multiplying by r gives (13). All terms are nonnegative since $p,q,\nu,\xi,\zeta\geq0$, so $P\geq0$ for $r>0$ and also at $r=0$ by continuity. At the point $z=\infty$, the product ray is v and $H_{vv}=1$. Cyclic invariance holds because the only off-diagonal entries of H_0 couple v and w, both fixed by the cyclic permutation; so the form controls all three cuts and their biseparable mixtures. □

Proof of Theorem 7. Filter reduction. The formulas for F_λ follow by counting computational-basis excitations. Under $aF_\lambda(\,\cdot\,)F_\lambda^\dagger$ with $\lambda^2=1/a$, the coefficients (γ,β,V,G,E,O) of $u_su_s^\dagger, ww^\dagger,\dots$ at $\gamma=\alpha$ become $(a\alpha,\beta,V/a^2,G/a,E/a,O)$, exactly as in (11). It therefore suffices to prove the remaining assertions at $\alpha=\gamma=1$.

Rank and PPT. Put $\kappa=1-\tau^2$. Here $\rho_\tau=\rho_{\tau,1}=BX_\tau B^\dagger$ is the unnormalised surface operator and X_τ its coefficient matrix; they are not the state ρ_t and matrix X_t of Section 2. Introduce the positive coordinate

$$Q=\tau+\sqrt{1+\tau^2},\qquad \tau=\frac{Q-Q^{-1}}{2},\quad Q^2+Q^{-2}=2(1+2\tau^2),\quad 1<Q<1+\sqrt{2}.$$

The weights simplify to

$$O=\kappa/\tau,\quad E=\kappa Q^2,\quad G=(E-1)/\tau,\quad V=(E^2-2\kappa E+\kappa)/(\tau\kappa),$$

and in terms of Q alone $\kappa=(6Q^2-Q^4-1)/(4Q^2)$, $E=(6Q^2-Q^4-1)/4$ and $G=Q(5-Q^2)/2$, so every identity below is a direct substitution. The vectors (u,v,w,e,o) are linearly independent, and the only off-diagonal coefficient of ρ_τ is its w–v coupling. Since $\kappa,E,O,\tau>0$, positivity is equivalent to

$$\tau V-G^2=2Q^2\kappa(Q^2-3)\geq0. \tag{28}$$

Equality gives rank four and strict inequality rank five.

There is also an explicit Gram representation of the *odd* parity block of $\rho_\tau^{T_A}$, in the order $(001, 010, 100, 111)$. Define

$$L = \begin{pmatrix} \tau + O & \tau + \omega O \\ \tau + \omega^2 O & \tau + O \end{pmatrix}, \qquad M = \begin{pmatrix} -1 & 1 + \omega E \\ -1 & 1 + \omega^2 E \end{pmatrix}.$$

Here $\det L = 3\tau O = 3\kappa > 0$, and direct multiplication using $\tau(\tau + O) = 1$ gives

$$M^\dagger L^{-1} M = \begin{pmatrix} \tau + O & G \\ G & V \end{pmatrix}, \qquad (\rho_\tau^{T_A})_{\text{odd}} = \begin{pmatrix} L & M \\ M^\dagger & M^\dagger L^{-1} M \end{pmatrix}.$$

Hence the odd block is positive semidefinite of rank two, for every $0 < \tau < 1$. The *even* block, in the order $(000, 011, 101, 110)$, is

$$\begin{pmatrix} 1 & -1 & b & \bar{b} \\ -1 & 1 + E & G & G \\ \bar{b} & G & 1 + E & 1 + \omega^2 E \\ b & G & 1 + \omega E & 1 + E \end{pmatrix}, \qquad b = \tau + \omega O.$$

Its determinant factors as

$$\frac{3(\kappa - \tau^2 E)}{\tau^4} \left[E^2 - 2\kappa(1 + 2\tau^2)E + \kappa^2 \right] = 0,$$

the last equality following from $E = \kappa Q^2$ and $Q^2 + Q^{-2} = 2(1 + 2\tau^2)$. Its first three leading principal minors are

$$1, \qquad E, \qquad \frac{(Q^2 - 3)(Q^4 + 3)\kappa^2}{4\tau^2}. \tag{29}$$

For $Q^2 > 3$ the leading 3×3 corner is positive definite; the full determinant is zero, so the Schur complement vanishes and this block is positive semidefinite of rank three. At $Q^2 = 3$ positivity follows by continuity. Since Q increases with τ and $Q^2 = 3$ precisely at $\tau = 1/\sqrt{3}$, this proves the sharp positivity window, the rank profile and the rank-four lower endpoint. Cyclic symmetry gives the other cuts. The filter reduction proves the conclusions for every (α, β, γ), including the rank-at-most-five assertion as well. The chosen E is the larger root $\kappa(\tau + \sqrt{1 + \tau^2})^2$ of the even-block determinant factor, and G, V complete the odd-block Gram matrix.

Genuine entanglement. At $a = 1$ and $3/4 \le \tau < 1$ (so $2 \le Q < 1 + \sqrt{2}$; $\tau = 3/4$ is $Q = 2$), the fixed witness H_5 of Proposition 2 gives, by substitution of the simplified weights,

$$\operatorname{tr}(H_5 X_\tau) = -\frac{(Q^2 - 2Q - 1)\, S_6(Q)}{2Q(Q^2 - 1)}, \qquad S_6(Q) = Q^6 - Q^5 - 9Q^4 + 2Q^3 + 11Q^2 + 27Q - 3.$$

With $x = Q - 2 \in [0, \sqrt{2} - 1)$,

$$S_6(2 + x) = -1 - 81x - 33x^2 + 50x^3 + 41x^4 + 11x^5 + x^6 < 0.$$

Indeed $\sqrt{2} - 1 < 1/2$, and for $0 \le x \le 1/2$ one has $50x^3 \le 25x^2$ and $41x^4 + 11x^5 + x^6 \le \left(\frac{41}{8} + \frac{11}{16} + \frac{1}{32}\right)x \le 6x$. Thus $S_6(2 + x) \le -1 - 75x - 8x^2 < 0$, including at $x = 0$. Also $Q^2 - 2Q - 1 < 0$ for $Q < 1 + \sqrt{2}$ and $Q^2 - 1 > 0$. Hence $\operatorname{tr}(H_5 X_\tau) < 0$ throughout $3/4 \le \tau < 1$.

For the complementary interval $1/\sqrt{3} < \tau \le 3/4$, so that $\sqrt{3} < Q \le 2$, Lemma 6 provides a *fixed* form. Define

$$P(Q) = Q^4 + (8 + 2\sqrt{3})Q^3 - (2 + 4\sqrt{3})Q^2 - (20 + 14\sqrt{3})Q + 1 + 4\sqrt{3}.$$

A direct substitution into $\operatorname{tr}(H_0 X_\tau)$ factors as

$$\operatorname{tr}(H_0 X_\tau) = -\frac{(Q^2 - 3)(Q^2 - 2Q - 1)P(Q)}{2Q(Q^2 - 1)}.$$

Here $P''(Q) > 0$ for $Q \ge \sqrt{3}$, $P'(\sqrt{3}) = 28 + 12\sqrt{3} > 0$, and $P(2) = 33 - 24\sqrt{3} < 0$. Thus $P(Q) \le P(2) < 0$ on $[\sqrt{3}, 2]$. Since $Q^2 - 3 > 0$ and $Q^2 - 2Q - 1 < 0$, the trace is strictly negative on $(1/\sqrt{3}, 3/4]$ and zero at the lower endpoint. The two witness ranges meet at $\tau = 3/4$ and together cover the entire open PPT window, so $\rho_{\tau,1}$ is GME there. Finally, by Theorem 7(b), $\rho_{\tau,a} = aF_\lambda \rho_{\tau,1} F_\lambda^\dagger$ with F_λ invertible and local. Such a filter, and its inverse, maps a vector that is a product across a cut to a vector that is a product across the same cut, so it maps biseparable states to biseparable states in both directions, and likewise preserves positivity of every partial transpose and all ranks. Hence $\rho_{\tau,a}$ is biseparable if and only if $\rho_{\tau,1}$ is, and GME for every $a > 0$ follows from the case $a = 1$. □

Proof of the computations in Remark 10. Substituting $q_1(\pm i\sqrt{y_1})$ and $q_1(\pm\sqrt{x_0})$ into (15) gives $X_{1/\sqrt{3}}$ directly. A form on $\mathbb{C}^5$ invariant under the cyclic $(\mathrm{diag}(1,1,1,\omega^2,\omega^2))$, parity and $K\circ S_{BC}$ (complex conjugation) actions on the coordinates is real and block diagonal on u, (v,w), e, o, so it has six real parameters $(h_u,h_v,h_w,h_e,h_o,h_{vw})$. Vanishing with vanishing derivative in $r=|z|^2$ at $z=i$, $z=i\sqrt{y_1}$ and $z=\sqrt{x_0}$ (two equations each; the points $-z$ give the same equations) is a homogeneous linear system of rank five, whose kernel is spanned by H_0. Finally $P(11/5)=(34956-15540\sqrt{3})/625>0$, so $\mathrm{tr}(H_0X_{48/55})>0$. □

Proof of Corollary 8. The parameter $\tau=48/55$ lies in the window of Theorem 7, which proves positivity and all four ranks. The displayed coefficients give $\mathrm{tr}\,N=16323219$; cyclic and parity invariance follow from the basis vectors.

For an explicit analytic GME certificate, take $H=H_{1/2}=\mathrm{diag}(-\frac{3}{2},1,\frac{11}{4},-3,\frac{7}{2})$ from Proposition 2. It is nonnegative on every cut-product ray in the range by (5), whereas

$$\mathrm{tr}(HX_\sigma)=-\frac{493577}{32646438}<0,\qquad \sigma=B_{1/2}X_\sigma B_{1/2}^\dagger,$$

where, in the basis $(u_{1/2},v,w,e,o)$ (note $\tilde u=2u_{1/2}$), $16323219\,X_\sigma$ has diagonal

$$(4\cdot 825000,\ 2021184,\ 1440000,\ 951720,\ 450625)$$

and vw entry 145200, by (14).

The level-two witness Y_* of Corollary 19, defined later in Section 6 (its certificate data are ancillary), has an exactly certified negative rational expectation on σ. With $g=-\mathrm{tr}(Y_*\sigma)$ and $h=\mathrm{tr}\,Y_*/8=1$, the white-noise threshold $g/(g+h)$ of Lemma 17 (Section 6) exceeds 4.05×10^{-3}: exactly, $g>407/100000$, so $g/(g+h)>407/100407>81/20000$. The strict inequality and the positivity of the dual matrices are checked in exact arithmetic, without relying on solver tolerances.

Extremality. σ is the normalisation of $\rho(55,48,110)$, a state $\rho_{\tau,a}$ of the surface, so it is extreme by the proofs below, which cover every $\rho_{\tau,a}$. (At σ the criterion was also checked directly: with $D=B_\sigma X_D B_\sigma^\dagger$, $B_\sigma=[\tilde u,v,w,e,o]$, the system of the next proof has exact rank 24 over $\mathbb{Q}(\sqrt{3})$.) □

Proof of the extremality of every $\rho_{\tau,a}$ in Corollary 8 and of the symmetry blocks used in Proposition 9.
Reduction. An invertible local filter F maps ρ to $F\rho F^\dagger$, ranges to ranges, and each ρ^{T_c} to a state of the same kind, since $(FDF^\dagger)^{T_c}=\tilde F D^{T_c}\tilde F^\dagger$ with $\tilde F$ the filter obtained by complex-conjugating the c-th factor of F. So $D\mapsto FDF^\dagger$ maps the face-direction space of ρ onto that of $F\rho F^\dagger$, and likewise the tangent constraints below. By Theorem 7(b) it suffices to take $a=1$. Write $\rho=\rho_\tau$, $Q=\tau+\sqrt{1+\tau^2}\in(\sqrt{3},1+\sqrt{2})$, so that $\tau=(Q^2-1)/(2Q)$, $\kappa=(6Q^2-Q^4-1)/(4Q^2)$,

$$E=\tfrac14(6Q^2-Q^4-1),\quad O=\frac{6Q^2-Q^4-1}{2Q(Q^2-1)},\quad G=\tfrac12Q(5-Q^2),\quad V=\tfrac12Q(Q^2-1)(6-Q^2).$$

Kernel of ρ^{T_A}. With ω as in (1), the two odd vectors k_1,k_2, in the basis $(|001\rangle,|010\rangle,|100\rangle,|111\rangle)$, and the even vector k_3, in the basis $(|000\rangle,|011\rangle,|101\rangle,|110\rangle)$,

$$\begin{aligned}
k_1&=\big((3-i\sqrt3)Q,\ (3+i\sqrt3)Q,\ 3(Q^2-1),\ 0\big),\\
k_2&=\big(Q(15-3i\sqrt3-(3+i\sqrt3)Q^2),\ Q(15+3i\sqrt3-(3-i\sqrt3)Q^2),\ 0,\ 12\big),\\
k_3&=\big(-3Q(Q^2-1),\ -6Q,\ (1-\bar\omega)Q^2+3(1+\bar\omega),\ (1-\omega)Q^2+3(1+\omega)\big)
\end{aligned}$$

satisfy $\rho^{T_A}k_j=0$ identically in Q and are linearly independent for $Q>1$ (the entries $3(Q^2-1)$ of k_1, 12 of k_2 and $-6Q$ of k_3 are their pivots). Since $\mathrm{rank}\,\rho^{T_A}=5$ on the window, $K_A=[k_1,k_2,k_3]$ spans $\ker\rho^{T_A}$ there. The cyclic shift Π satisfies $\Pi\rho\Pi^\dagger=\rho$ and maps cut A to cut B, so $\ker\rho^{T_B}=\Pi\ker\rho^{T_A}$, and similarly for C.

Sectors. Π, the parity $Z^{\otimes3}$ and the reflection-conjugation $K\circ S_{BC}$ fix ρ. Take the basis (u,v,w,e,o) of R extended by $u_\perp=3|000\rangle+|011\rangle+|101\rangle+|110\rangle$, $\bar e$, $\bar o$ (spanning $R^\perp$); the matrix units $v_iv_j^\dagger$ have a Π-charge $q\in\mathbb{Z}_3$ ($\Pi v_iv_j^\dagger\Pi^\dagger=\omega^q v_iv_j^\dagger$) and a parity. The real solution space of the face system (and the kernel of the constraint map below) is invariant under conjugation by Π, so it is the sum of its charge components, and for a component of charge q the conditions at B and C are the one at A multiplied by phases. Hence it is enough to impose the condition at the cut $A|BC$ on each sector separately. The charges $q=1$ and $q=2$ are exchanged by $D\mapsto D^\dagger$.

Extremality from the 2×4 theorem. Let ρ be a $\rho_{\tau,a}$. Its profile is $(5,5,5,5)$ and it is GME, hence entangled across $A|BC$. Suppose ρ were not extreme in the set $\mathcal{P}_A$ of states with $\sigma^{T_A}\succeq0$. A ray from ρ

inside the face of $\mathcal{P}_A$ generated by ρ leaves the face at a state ρ' with $\operatorname{ran}\rho' \subsetneq \operatorname{ran}\rho$, $\operatorname{ran}\rho'^{T_A} \subseteq \operatorname{ran}\rho^{T_A}$ and $\min(\operatorname{rank}\rho', \operatorname{rank}\rho'^{T_A}) \le 4$. Such a 2×4 PPT state is separable [22, 10] (if only $\operatorname{rank}\rho'^{T_A} \le 4$, apply the theorem to the PPT state ρ'^{T_A}), so $\rho' = \sum_i |a_i b_i\rangle\langle a_i b_i|$ with $|a_1 b_1\rangle \in \operatorname{ran}\rho$ and $|\bar{a}_1 b_1\rangle \in \operatorname{ran}\rho^{T_A}$. Put $\rho'' = \rho - \epsilon|a_1 b_1\rangle\langle a_1 b_1|$ with $\epsilon > 0$ maximal such that $\rho'' \succeq 0$ and $\rho''^{T_A} \succeq 0$. Then $\min(\operatorname{rank}\rho'', \operatorname{rank}\rho''^{T_A}) \le 4$, so ρ'' is separable across $A|BC$ by [22, 10] again, and so is $\rho = \rho'' + \epsilon|a_1 b_1\rangle\langle a_1 b_1|$, contradicting entanglement. Hence ρ is extreme in $\mathcal{P}_A$, and therefore in the fully-PPT set. The second step alone shows that ρ is an edge state across $A|BC$: a product vector in $\operatorname{ran}\rho$ with partial conjugate in $\operatorname{ran}\rho^{T_A}$ would make ρ separable. The same holds at every cut, and for every GME state of profile $(5,5,5,5)$.

Extremality by symmetry blocks. The following direct computation gives a second proof and the blocks reused for Proposition 9. A Hermitian face direction is $D = BXB^\dagger$, X a Hermitian 5×5 matrix (25 real parameters), with $D^{T_A}K_A = 0$ (the other cuts follow, and $K_c^\dagger D^{T_c} = 0$ is the adjoint). Its sectors are: charge $q = 1$ and parity $+$ or $-$, each three complex parameters ($ue^\dagger, vo^\dagger, wo^\dagger$ and $uo^\dagger, ve^\dagger, we^\dagger$); and for $q = 0$ the real parameters Re / Im of ($uv^\dagger, uw^\dagger, eo^\dagger$) (parity $-$, three each) and, with parity $+$, $i(vw^\dagger - wv^\dagger)$ (one) and $uu^\dagger, vv^\dagger, vw^\dagger + wv^\dagger, ww^\dagger, ee^\dagger, oo^\dagger$ (six). Each sector map $X \mapsto D^{T_A}K_A$ has a square submatrix whose determinant is

sector	minor
$q=1$, parity $+$	$-108\,(Q^2-1)^2(Q^2+1)$
$q=1$, parity $-$	$432\,(Q^2-1)(Q^2+1)$
$q=0$, parity $-$, real parts	$108\,(Q^2-1)(Q^2+2Q-1)$
$q=0$, parity $-$, imaginary parts	$108\,i\,(Q^2-1)(Q^2-2Q-1)$
$q=0$, parity $+$, $i(vw^\dagger - wv^\dagger)$	$3i\,(Q^2+1)$
$q=0$, parity $+$, $(uu^\dagger,\ vv^\dagger,\ vw^\dagger{+}wv^\dagger,\ ww^\dagger,\ ee^\dagger)$	$1944\,i(\sqrt{3}+i)(Q^2+1)(Q^2-2Q-1)(Q^2+2Q-1)$

On the window Q^2-1, Q^2+1 and Q^2+2Q-1 are positive and $Q^2-2Q-1<0$, so no minor vanishes. Hence the system has rank $12+3+3+1+5 = 24$ (the sectors $q = 1, 2$ together contribute 12, since each sector map is injective on its six real parameters), and its nullspace is spanned by ρ, which lies in the last sector of six parameters. By the supported-face criterion [33], $\rho_{\tau,a}$ is extreme for every $\tau \in (1/\sqrt{3}, 1)$ and $a > 0$, as already shown.

Constraint differentials. The tangent space of rank-five Hermitian matrices at ρ consists of D vanishing on $\ker\rho \times \ker\rho$: the 55 matrix units $v_i v_j^\dagger$ with $\min(i,j) \le 5$ in the basis $(u, v, w, e, o; u_\perp, \bar{e}, \bar{o})$. The differential of the Schur-complement equation at cut A is $L(D) = K_A^\dagger D^{T_A} K_A$, a Hermitian 3×3 matrix, and the same sector argument applies to the 27 equations of the three cuts. L maps the sector of charge q onto 3×3 matrices exactly when its two parity parts are onto: the parity-$+$ part has the five outputs $(k_1,k_1), (k_1,k_2), (k_2,k_1), (k_2,k_2), (k_3,k_3)$ and the parity-$-$ part the four outputs $(k_1,k_3), (k_2,k_3), (k_3,k_1), (k_3,k_2)$. Square submatrices of the sector maps have determinants

$$
\begin{aligned}
q=0,\ +:&\ -120932352\,Q(Q^2-1)^2(Q^2+1)^2(Q^4-6Q^2+1),\\
q=0,\ -:&\ -839808\,Q^2(Q^2-1)^4(Q^2-3)(Q^2+1),\\
q=1,\ +:&\ -725594112\,Q^2(Q^2-1)^4(Q^4-6Q^2+1),\\
q=1,\ -:&\ 1679616\,Q^2(Q^2-1)^3(Q^2+1)(Q^4-6Q^2+1)
\end{aligned}
$$

and $Q^4-6Q^2+1 = (Q^2-2Q-1)(Q^2+2Q-1)$ vanishes only at $Q = 1+\sqrt{2}$ and $Q = \sqrt{2}-1$. So on the window every sector map is onto, the 27 differentials are independent on the 55-dimensional tangent space for every state of the surface, and the zero set of the 27 equations is a smooth manifold of dimension $55-27 = 28$. The factors Q^2-3 and Q^4-6Q^2+1 vanish exactly at the two ends $\tau = 1/\sqrt{3}$ and $\tau = 1$ of the window. An independent exact computation of these ranks at $Q = 2$ ($\tau = 3/4$), without any symmetry reduction, is part of the same check. □

Proof of Proposition 9. Let $\rho = \rho_{\tau,a}$ be any state of the surface, with B and K_c as above. Rank-five Hermitian matrices near ρ form a smooth manifold of real dimension 55; its tangent space at ρ is $\{ZB^\dagger + BZ^\dagger : Z \in \mathbb{C}^{8\times 5}\}$. Near ρ each ρ'^{T_c} has five eigenvalues bounded away from zero, and $\operatorname{rank}\rho'^{T_c} \le 5$ is the vanishing of a 3×3 Hermitian Schur complement $S_c(\rho')$, a smooth function whose differential at ρ is $D \mapsto K_c^\dagger D^{T_c} K_c$, with K_c a basis of $\ker\rho^{T_c}$. The 27 real equations $S_c = 0$ have independent differentials on the tangent space, by the sector minors of the proof above, so by the implicit function theorem their common zero set is a smooth manifold of dimension $55-27 = 28$, the cone over $\mathcal{M}$; the trace-one condition is transversal, since ρ itself is a tangent direction. Nearby points are positive semidefinite with positive semidefinite partial transposes, since their five nonzero eigenvalues stay positive. GME is an open condition, so it persists near ρ. The extremality system of Corollary 8 is built from $\operatorname{ran}\rho'$ and the kernels $\ker\rho'^{T_c}$.

On $\mathcal{M}$ the ranks of ρ' and of every ρ'^{T_c} are constant ($= 5$), so these subspaces, and hence the system, vary continuously with the state. The system has rank 24 at ρ; rank is lower semicontinuous and the state itself is always a null direction, so the rank stays 24 nearby. The tangent to the local-filter orbit, $\{X\rho+\rho X^\dagger\}$ with $X = X_1\otimes I\otimes I + I\otimes X_2\otimes I + I\otimes I\otimes X_3$, has dimension 19 including scale: the effective local-filter group has real dimension $24-4-1=19$ ($\mathrm{GL}(2,\mathbb{C})^{\times 3}$ modulo the scalings $\alpha_1\alpha_2\alpha_3 = 1$ and the overall phase), and the stabiliser of ρ lies in the stabiliser of its range $R_{1/a}$, which is finite modulo scalars (Corollary 4(b), the group S_3). The orbit tangent lies in the cone's tangent space, since invertible local filters preserve ranks of ρ and of each ρ^{T_c} and so map the cone over $\mathcal{M}$ to itself. The orbit dimension remains 19 nearby: rank cannot fall, and 19 is the maximum, so the orbits in $\mathcal{M}$ have dimension 18 and form a foliation with nine-dimensional local leaf space. □

C Exact data for one support, both phases

This appendix contains the proofs of Theorem 14 and Proposition 15, with the exact rational coefficient matrices and principal minors. The notation, including Γ_z of (19), is that of Section 5.

In the basis (u, v, w, e, o),

$$X_{\mathrm{BS}} = \frac{1}{54375}\begin{pmatrix} 1232 & 0 & 0 & 0 & 0 \\ 0 & 30493 & 32 & 0 & 0 \\ 0 & 32 & 1693 & 0 & 0 \\ 0 & 0 & 0 & 3700 & 0 \\ 0 & 0 & 0 & 0 & 925 \end{pmatrix}. \tag{30}$$

Proof of Theorem 14. The norms $\|Bq_1(2)\|^2 = 1250$ and $\|Bq_1(2i)\|^2 = 290$ give by direct expansion

$$\sigma_{\mathrm{BS}} = \tfrac{1}{3}\big(|111\rangle\langle 111| + \Gamma_2 + \Gamma_{2i}\big) = BX_{\mathrm{BS}}B^\dagger,$$

with X_{BS} as in (30); σ_{BS} is biseparable because each Γ_z is. The coefficient matrix is positive definite and, after applying B, gives a trace-one state on R. The state is invariant under cyclic permutations and global parity. For the unnormalised matrix $54375\,\sigma_{\mathrm{BS}}^{T_A}$, the leading principal minors of its even and odd parity blocks, in the orders of (27), are respectively

	Δ_1	Δ_2	Δ_3	Δ_4
even	1232	4558400	11751097500	13766553000000
odd	2618	4698075	8087598750	83202897843750

.

All are strictly positive, so Sylvester's criterion and cyclic invariance give $\sigma_{\mathrm{BS}}^{T_c} \succ 0$ for each cut. The exact expansion and minors are checked independently (Appendix G).

The state $\rho_{10/3}$ has range R, rank five and strictly positive partial transposes, but is GME by (7). For the interior assertion, let μ be a probability measure of full support on the complex projective line $\mathbb{CP}^1$ (for example normalised Fubini–Study measure) and $M = \int q_1(z)q_1(z)^\dagger\, d\mu/\|Bq_1(z)\|^2$, the coefficient matrix of a biseparable state on the $A|BC$ product curve. If a Hermitian H has $q_1(z)^\dagger H q_1(z) \geq 0$ for all z and $\mathrm{tr}(HM) = 0$, the nonnegative integrand vanishes on the full support of the measure, so the polynomial vanishes identically. The five entries of $q_1(z)$ form a basis of the polynomials of degree at most four. Their products are therefore an invertible change of basis from the independent monomials $\bar{z}^i z^j$, $0 \leq i, j \leq 4$, so $H = 0$. Hence M is interior to the cone of supported $A|BC$-biseparable coefficient matrices. (A measure on the unit circle would not do: H_5 vanishes there.) Mixing a sufficiently small amount of such a moment matrix into σ_{BS} produces a biseparable state interior to the supported biseparable cone, while retaining strict PPT and rank five. GME is open and $\rho_{10/3}^{T_c} \succ 0$, so a relative neighbourhood of $\rho_{10/3}$ consists of fully-PPT GME states. Both regions therefore have relative interior in the same PPT spectrahedron. □

In the basis (u, v, w, e, o),

$$X_\partial = \frac{1}{3360}\begin{pmatrix} 293 & 0 & 0 & 0 & 0 \\ 0 & 577 & -105 & 0 & 0 \\ 0 & -105 & 273 & 0 & 0 \\ 0 & 0 & 0 & 188 & 0 \\ 0 & 0 & 0 & 0 & 76 \end{pmatrix}. \tag{31}$$

Proof of Proposition 15. For the rays in (20), $|z| = 1$ at $z = \pm 1, \pm i$, while at $z = \pm i\sqrt{5}$ one has $r = 5$ and $\operatorname{Re} z = 0$. The factorisation (6) therefore vanishes on every summand, proving $\operatorname{tr}(H_5 X_\partial) = 0$. Their squared physical norms $\|Bq_1(1)\|^2 = 56$, $\|Bq_1(i)\|^2 = 32$ and $\|Bq_1(i\sqrt{5})\|^2 = 576$ give the coefficient matrix (31) upon cyclic and sign averaging. It has $\operatorname{tr}(BX_\partial B^\dagger) = 1$ and $X_\partial \succ I_5/100$ (a diagonal and 2×2 block check).

For a quantitative PPT bound, the leading principal minors of the even and odd 4×4 parity blocks of $5\,(3360\,\sigma_\partial^{T_A}) - 56 I_8$, in the orders of (27), are

	Δ_1	Δ_2	Δ_3	Δ_4
even	1409	1163516	654702384	109183407616
odd	1689	1363796	97132144	7881970176

.

They are positive. Hence $\sigma_\partial^{T_A} \succ I/300$ by Sylvester's criterion, and cyclic symmetry gives the other cuts.

Let $\Delta = \sigma_{\mathrm{BS}} - \sigma_\partial$ and $X_\Delta = X_{\mathrm{BS}} - X_\partial$. Direct rational arithmetic gives

$$\|\Delta\|_2^2 = \frac{575589900887}{2318006250000} < \frac{1}{4}.$$

Partial transposition preserves the Hilbert–Schmidt norm, so $\|\Delta^{T_c}\|_\infty < 1/2$. Likewise $B^\dagger B = \operatorname{diag}(4,1,3,3,3) \succeq I_5$ implies $\|X_\Delta\|_\infty \le \|X_\Delta\|_2 \le \|\Delta\|_2 < 1/2$. For $-1/200 \le t \le 0$, it follows that

$$\sigma(t)^{T_c} \succ \left(\frac{1}{300} - \frac{1}{400}\right) I = \frac{1}{1200} I, \qquad X_\partial + t X_\Delta \succ \left(\frac{1}{100} - \frac{1}{400}\right) I_5.$$

For $0 \le t \le 1$ strict PPT and rank five follow by convexity from the corresponding properties of the two endpoints. Biseparability in that range follows from the explicit decompositions of the endpoints. Finally, for $t < 0$ the supported witness has

$$\operatorname{tr}\big(H_5(X_\partial + t X_\Delta)\big) = t\,\frac{25082}{54375} < 0,$$

which excludes biseparable mixtures. The arithmetic is checked independently (Appendix G). □

D Computations for the noise certificates

This appendix contains the proofs of the statements of Section 6 that are not given there. The rational certificate data themselves are ancillary (Appendix G).

Proof of Lemma 17. For $x = a \otimes b$ write $a \otimes a \otimes b = Vc$. Partial transposition of the product projector gives

$$\|a\|^2 \langle x|Y|x\rangle = c^\dagger P c + \langle aa\bar{b}|Z_1|aa\bar{b}\rangle + \langle a\bar{a}b|Z_2|a\bar{a}b\rangle \ \ge 0.$$

One certificate per orbit of cuts under the symmetries of Y suffices, and cyclic invariance covers all cuts on three qubits; thus Y is nonnegative on every pure state that is a product across some cut, hence on the convex hull BS. The noise threshold is the condition $-(1-p)g + ph < 0$. The second bound follows from $|\operatorname{tr}(Y(\sigma - \rho))| \le (u - l) D(\sigma, \rho)$ for $\sigma \in \mathrm{BS}$. For the last assertion, fix a cut and a unit vector ϕ, and set $A_\phi = (|\phi\rangle\langle\phi|)^{T_c}$. If $\lambda_1 \ge \lambda_2 \ge \cdots$ are the squared Schmidt coefficients of ϕ at that cut (padded by zeros if necessary), the spectrum of A_ϕ consists of λ_i, $\pm\sqrt{\lambda_i \lambda_j}$ for $i < j$, and zeros. Since $\lambda_2 \le 1 - \lambda_1$, its spectral width is bounded by

$$\lambda_{\max}(A_\phi) - \lambda_{\min}(A_\phi) = \lambda_1 + \sqrt{\lambda_1 \lambda_2} \le \max_{0 \le x \le 1} \left[x + \sqrt{x(1-x)}\right] = \frac{1 + \sqrt{2}}{2}.$$

For $\Delta = \sigma - \rho$ with $\operatorname{tr}\Delta = 0$ and $D(\rho, \sigma) = r$, the positive and negative parts of Δ each have trace r. Consequently $|\operatorname{tr}(A_\phi \Delta)|$ is at most r times the spectral width of A_ϕ. Since ϕ is arbitrary,

$$\langle \phi | \sigma^{T_c} | \phi \rangle \ge c - \frac{1 + \sqrt{2}}{2}\, r > 0.$$

The same reasoning applies to every cut. □

Proof of Corollary 18. Apply Lemma 17. The exact verification checks $Z_1, Z_2, P \succ 0$, cyclic symmetry, all 144 coefficients of the product-vector identity, outward spectral bounds and every number in the table. For local noise a check at the single rate q_0 suffices: $\mathcal{D}_{q_0}$ on all sites is obtained from any rates $q_j \le q_0$ by further local depolarisation of rates $(q_0 - q_j)/(1 - q_j)$, which preserves BS, so a GME output at q_0 forces a GME input. For uniformity in t, the numerator of ρ_t is affine in t, so $7(t+3)\operatorname{tr}(Y\rho_t)$ is affine in t, and so are $7(t+3)\big[(1-p)\operatorname{tr}(Y\rho_t)+ph\big]$ and $7(t+3)\big[\operatorname{tr}(Y\rho_t)+r(u-l)\big]$ for the fixed Y, with $h = \operatorname{tr} Y/8$, $\operatorname{spec} Y \subset [l,u]$ and fixed p and ball radius r. An affine function that is negative at both endpoints of the rational bracket $[3.2567, 3.3886]$, which contains (3), is negative throughout it, and $7(t+3) > 0$. By Lemma 17 (the noise threshold $-(1-p)g + ph < 0$ and the distance bound $D(\rho, \mathrm{BS}) \ge g/(u-l)$), the two endpoint checks therefore certify the stated interval-wide white-noise and GME-ball bounds. □

Proof of Corollary 19. Write $Y = Y_*$ for the cyclic rational witness of the certificate data. Its exact level-two certificate proves $\operatorname{tr}(Y\sigma) \ge 0$ on all biseparable states. The Pythagorean identity $312^2 + 266^2 = 410^2$ makes the coefficients of ρ_* rational. The exact certificate evaluation gives

$$g := -\operatorname{tr}(Y\rho_*) > \frac{173}{40000}, \qquad h := \frac{\operatorname{tr} Y}{8} = 1, \qquad -\frac{1}{4}I \prec Y \prec \frac{31}{10}I.$$

Thus the spectral width of Y is less than $67/20$. These inequalities give the white-noise and trace-distance bounds.

For (i), at $p = 2153/500000$,

$$(1-p)\frac{173}{40000} - p = \frac{7531}{20000000000} > 0,$$

so $\operatorname{tr}[Y((1-p)\rho_* + pI/8)] < 0$. The inequality also holds for smaller p. Part (ii) is the exact local-noise check at $q = 81/50000$: all smaller, unequal local rates can be followed by additional local depolarisation to reach the checked uniform output. Local channels preserve biseparability, and these channels preserve full PPT.

For (iii), let $p_0 = 32/10000$ and $r_0 = 331/1000000$. The witness margin remains strictly negative throughout the trace-distance ball because

$$(1-p_0)\frac{173}{40000} - p_0 - \frac{67}{20}r_0 = \frac{231}{100000000} > 0.$$

Since $\rho_*^{T_c} \succeq 0$, mixing with white noise also gives $\rho_\circ^{T_c} \succeq p_0 I/8$. The PPT stability estimate in Lemma 17 applies because

$$\frac{p_0}{8} - \frac{1+\sqrt{2}}{2}r_0 = \frac{469 - 331\sqrt{2}}{2000000} > 0.$$

Finally, (iv) follows directly from

$$\frac{173}{40000} - \frac{67}{20}\frac{129}{100000} = \frac{7}{2000000} > 0.$$

The rational inequalities, including the enclosure $\sqrt{2} < 99/70$, are checked exactly from the certified data (Appendix G). □

We now give the state $\tilde\rho$ explicitly. With v, w, e, o as in (1), write $|g\rangle = |000\rangle$ and $|h\rangle = |011\rangle + |101\rangle + |110\rangle$. Then

$$\begin{aligned} 10^7\,\tilde\rho = {} & 955592\,|g\rangle\langle g| - 930700\,(|g\rangle\langle h| + |h\rangle\langle g|) + 906857\,|h\rangle\langle h| \\ & + 519407\,|e\rangle\langle e| + 208476\,|\bar e\rangle\langle \bar e| \\ & + 390647\,|v\rangle\langle v| - 520181\,(|v\rangle\langle w| + |w\rangle\langle v|) + 693325\,|w\rangle\langle w| \\ & + 556418\,|o\rangle\langle o| + 104\,|\bar o\rangle\langle \bar o|. \end{aligned} \tag{32}$$

These ten coefficients are the state's coordinates in the invariant basis of the cyclic, parity and $K \circ S_{BC}$ symmetries; the witness data of Corollary 20 are ancillary.

Proof of Corollary 20. Lemma 17 applies. The exact verification checks trace one, full rank, cyclic and parity invariance, all three partial-transpose eigenvalue lower bounds, and the witness and tolerance inequalities. The PPT-only ball radii here and in the table of Corollary 18 follow from those eigenvalue bounds and the stability estimate of Lemma 17; their rational comparisons are checked against the same constants. □

Proof of Proposition 21. The lower bounds are Corollaries 19, 18 and 20. For the upper bounds, let $\mathcal{C}(X) = \frac{1}{3}\sum_{k=0}^{2} \Pi^k X \Pi^{-k}$ average over the cyclic qubit permutation Π. The certificate data give, for each of the three states, three orbit representatives: a Gaussian-rational qubit vector a_r and a positive definite Gaussian-rational 4×4 matrix Λ_r. The states are invariant under global parity $Z^{\otimes 3}$ and the antiunitary $K \circ S_{BC}$, which preserve the cut $A|BC$; applying them to $|a_r\rangle\langle a_r| \otimes \Lambda_r$ gives at most four terms per representative, and θ is the sum of all of them (at most twelve terms). Thus $\theta = \sum_i |a_i\rangle\langle a_i| \otimes \Lambda_i$ with $\Lambda_i \succ 0$ is separable across $A|BC$. At p equal to the stated upper bound, the residual $\Delta = (1-p)\rho + pI/8 - \mathcal{C}(\theta)$ is cyclically invariant and, with $c = \operatorname{tr}\Delta/8 > 0$, satisfies $\|\Delta - cI\|_2 \le c$ exactly. By the bipartite Gurvits–Barnum bound [23], this makes Δ separable across $A|BC$. Hence $(1-p)\rho + pI/8 = \mathcal{C}(\theta + \Delta)$ is an equal mixture of states separable across the three cuts, so it is biseparable at p equal to the stated upper bound. For every larger p', $(1-p')\rho + p'I/8$ is a convex mixture of the state at p with the separable state $I/8$, hence biseparable too. □

E The quartic moment cone of a minimal-rank support

We first rule out cut-product vectors in the kernel. The remaining three kernel constraints then force a unique product ray for each qubit direction, producing the quartic curves of Theorem 26.

Lemma 29 (product kernel)**.** *A PPT operator on $\mathbb{C}^2 \otimes \mathbb{C}^4$ whose kernel contains a nonzero product vector is separable. Hence the kernel of a fully-PPT three-qubit state that is not separable across a cut contains no product vector across that cut, and the range of every rank-five fully-PPT GME state satisfies the hypothesis of Theorem 26.*

Proof. This is a standard reduction, cf. [22]; we give a short proof. Separability and PPT are invariant under local unitaries, so we may take the kernel vector to be $|0\rangle|3\rangle$. Write $\rho = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix}$ in qubit blocks. Then $\rho|0,3\rangle = 0$ gives $A|3\rangle = 0$ and $B^\dagger|3\rangle = 0$. Since $\langle 0,3|\rho^{T_A}|0,3\rangle = \langle 0,3|\rho|0,3\rangle = 0$ and $\rho^{T_A} = \begin{pmatrix} A & B^\dagger \\ B & C \end{pmatrix} \succeq 0$, also $\rho^{T_A}|0,3\rangle = 0$, so $B|3\rangle = 0$. Hence $\rho|1,3\rangle = \rho^{T_A}|1,3\rangle = |1\rangle \otimes C|3\rangle$. If $C_{33} = \langle 3|C|3\rangle = 0$ then $C|3\rangle = 0$ and ρ lives on $\mathbb{C}^2 \otimes \mathbb{C}^3$. Otherwise put $u = C|3\rangle/\sqrt{C_{33}}$ and $\rho' = \rho - |1\rangle\langle 1| \otimes |u\rangle\langle u|$. For any $M \succeq 0$ and vector f with $\langle f|M|f\rangle > 0$, the Schur complement $M - M|f\rangle\langle f|M/\langle f|M|f\rangle$ is positive semidefinite. With $f = |1,3\rangle$ this applies to $M = \rho$ and to $M = \rho^{T_A}$, and both subtracted terms equal $|1\rangle\langle 1| \otimes |u\rangle\langle u|$, which is invariant under partial transposition. So ρ' is PPT, and $\rho'|j,3\rangle = 0$ for $j = 0, 1$: ρ' lives on $\mathbb{C}^2 \otimes \mathbb{C}^3$. PPT operators on $\mathbb{C}^2 \otimes \mathbb{C}^3$ are separable [15], so $\rho = \rho' + |1\rangle\langle 1| \otimes |u\rangle\langle u|$ is separable. For the consequence, a rank-five fully-PPT state with a product vector in its kernel at some cut is separable across that cut, hence biseparable. □

Proof of Theorem 26. Fix a cut; after relabelling the qubits it is $A|BC$. Let N be an 8×3 basis matrix of $R^\perp$. A vector $a \otimes b$ lies in R if and only if $M(a)b = 0$, where $M(a) = N^\dagger(a \otimes I_4)$ is a 3×4 matrix depending linearly on a. If $\operatorname{rank} M(a) < 3$ for some $a \neq 0$, a nonzero y has $y^\dagger M(a) = 0$, so $k = Ny \in R^\perp$ is orthogonal to $a \otimes \mathbb{C}^4$, that is $k = a^\perp \otimes b'$ is a product vector, contrary to the hypothesis. So the pencil $M(a) = a_0 M_0 + a_1 M_1$ has rank three for every $a \neq 0$.

By the Kronecker canonical form of matrix pencils [21, Ch. XII], there are invertible P and Q with $PM(a)Q$ block diagonal, with blocks L_ε ($\varepsilon \times (\varepsilon+1)$, with entries a_0 on the diagonal and a_1 on the superdiagonal), transposes L_η^T, and a regular part. Full row rank excludes every L_η^T block. A regular block of positive size has a homogeneous determinant that vanishes somewhere on $\mathbb{CP}^1$, contradicting constant rank. Each remaining L block contributes one kernel dimension; nullity one therefore forces the single block L_3. Its kernel is spanned by $(a_1^3, -a_0 a_1^2, a_0^2 a_1, -a_0^3)^T$, so $\ker M(a)$ is spanned by $b(a) = Q\,\kappa(a)$ with $\kappa(a)$ this cubic vector. The product rays in R are therefore the rays of $a \otimes b(a) = (I_2 \otimes Q)(a \otimes \kappa(a))$. The eight entries of $a \otimes \kappa(a)$ are $\pm$ quartic monomials and every one of the five monomials occurs, so $a \otimes b(a) = L\nu(a)$ for an 8×5 matrix L of rank five. Its column space is spanned by the curve, lies in R and has dimension five, so it equals R, and $L = BT_A$ for an invertible T_A. (This is the case of a maximal completely entangled kernel in [20, Prop. 24].) The other cuts are the same with the qubits relabelled.

A positive decomposition of an operator supported on R uses only vectors in R, since each positive summand has zero expectation on $R^\perp$. A biseparable $BXB^\dagger$ is therefore a positive combination of cut-c product projectors in R, that is of $BT_c\nu(a)\nu(a)^\dagger T_c^\dagger B^\dagger$, which gives (25) (the converse direction is immediate). The trace-one product projectors on R form a compact set, so their cone is closed, and Carathéodory's theorem in the 25-dimensional real space of Hermitian 5×5 matrices bounds the number of terms. The dual statement (26) is the definition of the dual of a sum of cones generated by the rank-one matrices $T_c\nu(a)\nu(a)^\dagger T_c^\dagger$. □

F Supported witnesses extend by a kernel penalty

A supported witness need not control product vectors outside its support. Choose K with kernel equal to the support. Adding a positive multiple of $K^\dagger K$ penalises their component outside $\ker K$ while leaving the target state's expectation unchanged. A small identity shift makes positivity strict on the supported product rays, allowing compactness to control the rest.

Lemma 30 (kernel-penalty witness lift)**.** *Let C be any compact set of unit vectors in a finite-dimensional Hilbert space, let K be a matrix (below, $\|\cdot\|_\infty$ is the operator norm and $\|\cdot\|$ the Euclidean vector norm) and $\mathcal{R} = \ker K$. If a Hermitian W_0 is strictly positive on $C \cap \mathcal{R}$, then*

$$W = W_0 + \lambda K^\dagger K$$

is nonnegative on all of C for some finite $\lambda \geq 0$. No constant-rank or smoothness hypothesis is needed. More generally, if W_0 is merely nonnegative on $C \cap \mathcal{R}$ and $\operatorname{tr}(W_0\rho) = -g < 0$ for a density matrix ρ supported on $\mathcal{R}$, then $W_0 + \epsilon I + \lambda K^\dagger K$ is a global witness for the free set C that detects ρ, for every $0 < \epsilon < g$ and some finite λ.

Proof. On C put $f(v) = v^\dagger W_0 v$ and $h(v) = \|Kv\|^2$. The compact set $\mathcal{B} = \{v \in C : f(v) \leq 0\}$ is disjoint from $\{h = 0\}$. If $\mathcal{B}$ is nonempty, $\eta = \min_{\mathcal{B}} h > 0$, so $\lambda \geq \|W_0\|_\infty/\eta$ makes $f + \lambda h \geq 0$ there; outside $\mathcal{B}$ it is already positive. If $\mathcal{B}$ is empty take $\lambda = 0$. For the nonstrict version apply the argument to $W_0 + \epsilon I$. The kernel penalty has zero expectation on ρ, leaving $-g + \epsilon < 0$. □

For biseparability, C is the union of the normalised cut-product vectors over the three cuts. The lemma also applies to other free pure-state sets, such as fully product vectors or stabiliser vectors.

Proposition 31 (quantitative form)**.** *Let C be the set of normalised cut-product vectors $U_c^\dagger(a \otimes b)$, $c \in \{A, B, C\}$, a a unit one-qubit vector, where U_c moves qubit c to the first tensor factor. Put $M_c(a) = KU_c^\dagger(a \otimes I)$ and $\Omega_c(a) = (a^\dagger \otimes I)U_c W_0 U_c^\dagger(a \otimes I)$. Suppose $\|W_0\|_\infty \leq L$, $\Omega_c(a) \succeq \alpha I$ on $\ker M_c(a)$, and every nonzero eigenvalue of $M_c(a)^\dagger M_c(a)$ is at least μ, uniformly in a and c, with constants $L, \alpha, \mu > 0$. Then*

$$\lambda = \frac{L + L^2/\alpha}{\mu} \tag{33}$$

suffices in Lemma 30. If additionally $\|K\|_\infty^2 \leq \beta$ and $\operatorname{tr}(W_0\rho) = -g < 0$ for ρ supported on $\ker K$, then, for the lifted witness $W = W_0 + \lambda K^\dagger K$,

$$D(\rho, \mathrm{BS}) \geq \frac{g}{2L + \lambda\beta}. \tag{34}$$

Proof. Split $b = u + v$ into its components in $\ker M_c(a)$ and its orthogonal complement. The conditional quadratic form $(a \otimes b)^\dagger U_c(W_0 + \lambda K^\dagger K)U_c^\dagger(a \otimes b)$ is bounded below by

$$\alpha\|u\|^2 - 2L\|u\|\|v\| + (\lambda\mu - L)\|v\|^2 \ \geq 0$$

when $\lambda\mu - L \geq L^2/\alpha$, which is (33). For (34), the spectrum of W lies in $[-L, L + \lambda\beta]$, $\operatorname{tr}(W\rho) = -g$ and $\operatorname{tr}(W\sigma) \geq 0$ on BS; the distance bound of (23), whose proof uses only these facts, gives (34). The same argument works for any finite union of product cuts. □

Compactness supplies positive α and μ for constant-rank pencils with strict supported positivity; the next remark gives explicit bounds for the family.

Remark 32 (hand-checkable constants). Without any solver output, Proposition 31 gives at $t = 10/3$

$$D(\rho_{10/3}, \mathrm{BS}) > \frac{1}{20\,000\,000}, \qquad \rho_{10/3}^{T_c} \succ I/2000, \tag{35}$$

so every physical state in that closed trace-distance ball is fully PPT and GME. Corollary 18 is stronger by a factor of about 5000; (35) uses no semidefinite program, only explicit constants and exact minors.

Proof. Let $T = B^\dagger B$ and $W_0 = BT^{-1}H_5T^{-1}B^\dagger$. Its operator norm is $14/3$ and, by (7), its expectation on $\rho_{10/3}$ is $-3/133$. Normalise the three rows of the annihilator K of R (Appendix A, proof of Proposition 2) by $\sqrt{3}$, $\sqrt{3}$, $\sqrt{12}$, which gives $KK^\dagger = I$. For any unit a the conditional matrix $M = K(a \otimes I)$ satisfies $MM^\dagger \preceq I$ and $\det(MM^\dagger) \geq 1/81$: write $|a_0|^2 = p$ and $a_0\bar{a}_1 = z$; the minimum over $\arg z$ is

$$\det(MM^\dagger) \geq \frac{1 - 7p + 24p^2 - 12p^3}{36} \geq \frac{1}{81},$$

because the numerator minus 4/9 is $(5-3p)(6p-1)^2/9$. Consequently every nonzero squared singular value is at least 1/81. Take U_c to be the power of the cyclic qubit permutation that moves c to the first factor. The range R, hence the row space of K, and W_0 are cyclically invariant, so $KU_c^\dagger = G_c K$ with G_c unitary and $U_c W_0 U_c^\dagger = W_0$; the cuts B and C therefore give the same constants as A. Apply Proposition 31 to $W_0 + I/100$ with $L = 5$, $\alpha = 1/100$, $\mu = 1/81$ and $\beta = 1$, so $\lambda = 202905$. The global witness $W = W_0 + I/100 + 202905\, K^\dagger K$ has expectation $-167/13300$ on $\rho_{10/3}$ and spectral range at most 202915, which gives the strict distance bound in (35). The partial-transpose bound is Sylvester's criterion for the two blocks of (27) after subtracting $I/2000$, whose leading minors are positive rationals. The ball statement then follows from the last part of Lemma 17, since $1/20\,000\,000 < 2/(2000(1+\sqrt{2}))$.

G Proof dependencies and exact certificate data

The existence and PPT-window theorems are analytic: the displayed formulas suffice to check the product curves, square factorisations, chirality and sign bounds by hand. The extremality of σ_6, the exact-block check of extremality on the $(5,5,5,5)$ surface, the moduli count, the phase-coexistence minors and quantitative noise bounds also use exact computational certificates. We list their dependencies below; the preceding appendices contain the proofs.

Mixed W class. Theorem 12 is analytic. The finite cyclic/parity decompositions, zero-hyperdeterminant identities, surface coefficient budgets, phase-coexistence line, and full-rank state $\tilde{\rho}$ are independently checked in exact arithmetic. The check includes both surface branches, the phase-coexistence line, $\tilde{\rho}$, nontrivial local filters, and a GHZ negative control; no numerical convex-roof minimisation is used.

Extremality and moduli. The extremality of Corollary 5 rests on an exact 24-of-25 rank calculation. For every state of the $(5,5,5,5)$ surface, the extremality of Corollary 8 is proved from the 2×4 low-rank theorem, and the same symmetry-block minors, explicit polynomials in Q, give an independent check of it and the 27 independent constraint differentials of Proposition 9, checked symbolically and by an exact rank computation at $Q = 2$; the local-filter orbit has dimension 19 because the stabiliser of the range is finite (Corollary 4(b)). The proofs are in Appendices A and B. These results are computer-assisted.

One support, both phases. Theorem 14 and Proposition 15 rest on exact rational expansions of finite cut-product mixtures, principal minors of the partial-transpose parity blocks, witness values and a Hilbert–Schmidt bound (Appendix C).

Noise bounds. The level-two certificates use rational matrices Y, Z_1, Z_2 given in symmetry-adapted coordinates (cyclic symmetry, global parity and the antiunitary $K \circ S_{BC}$): Y has 10 real parameters and each Z_i is a pair of real symmetric 8×8 blocks (72 parameters). The required positivity conditions and polynomial identities are verified in exact arithmetic. The certificate data consist of the witnesses for the family, the state $\tilde{\rho}$ with its witness, and the witness Y_* for the surface state. Every white-noise bound, radius and table entry of Section 6 (proofs in Appendix D) is checked from these data, not inferred from solver output; this includes the spectral enclosure $-I/4 \prec Y_* \prec 31I/10$, the rational value of $\mathrm{tr}(Y_*\sigma)$, the partial-transpose eigenvalue bounds and the use of $\sqrt{2} < 99/70$ in the PPT radii. Negative controls lie just beyond each stated bound.

Sharp white-noise thresholds. Proposition 21 uses explicit biseparable decompositions, stored as three orbit representatives per state and expanded under parity and $K \circ S_{BC}$. The exact checks cover positive definiteness of every Λ_r, cyclic invariance of the residual and the Gurvits–Barnum inequality, with a negative control at the certified GME noise level.

Corollary 3, Proposition 22, and the proofs of Appendices E and F are analytic; the constants of Remark 32 are checked exactly. For publication, the certificate data and the exact verifiers must accompany the paper as ancillary material or a version-pinned public archive; the PDF alone does not encode the certificate matrices. Witness-search routines supply candidates; only exact checks certify the results.

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