LAURENT PHENOMENON ALGEBRAS ON MINUSCULE GRASSMANNIANS AND THEIR MIRRORS

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Abstract. We construct Laurent phenomenon (LP) seeds whose upper algebras are the homogeneous coordinate rings of the half-spin Grassmannians and the Freudenthal variety. The frozen coordinate lines form a Coxeter orbit and define an anticanonical divisor. For these varieties and for the Cayley plane with the seed of Daisey and Ducat, we identify the Keel–Yu mirror of the affine boundary complement with an explicit upper LP algebra after specializing the curve-class parameters to one. We determine the theta function of each boundary component and hence the superpotential. The half-spin formulas are uniform in the rank. In the exceptional cases we compute the theta functions from contact orders and Laurent divisibility. The mirror identifications hold over the integers, and the intrinsic theta basis is valuatively independent in every reachable seed.

hexagon:2611.00328v2

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2020 *Mathematics Subject Classification.* Primary 13F60, 14J33; Secondary 14M15, 14M17, 14M25.

Key words and phrases. Laurent phenomenon algebras, minuscule Grassmannians, mirror symmetry, theta functions, Newton polytopes, Coxeter boundaries.

The author was supported in part by the National Natural Science Foundation of China (Nos. 12131015 and 12571038).

Introduction

Laurent phenomenon algebras (LP algebras), introduced by Lam and Pylyavskyy, generalize cluster algebras by allowing irreducible exchange polynomials with more than two terms [14]. A seed assigns an exchange polynomial to each *mutable variable*. Mutation replaces one such variable and leaves the coefficient variables, called *frozen variables*, fixed. Daisey and Ducat constructed a finite-type LP algebra on the homogeneous coordinate ring of the Cayley plane [5]. We construct LP seeds for the half-spin Grassmannians and the Freudenthal variety. In all three families we identify the intrinsic mirror of the open variety obtained by inverting the frozen variables.

For a simple group G, a *Coxeter element* in its Weyl group is a product of the simple reflections, each taken once. Its order is the Coxeter number $h(G)$. In the Cayley-plane seed the frozen coordinate lines form a full orbit of a Coxeter element. For a parabolic subgroup $P \subsetneq G$, let the *Fano index* of G/P be the largest positive integer dividing its anticanonical class. This index equals $h(G)$ precisely when P is maximal and its fundamental weight is *minuscule*, meaning that the weights of the corresponding irreducible representation form one Weyl-group orbit (Proposition 1.1). In that case a full Coxeter orbit of linear sections defines an anticanonical divisor. We ask when these sections can serve as frozen variables of an LP seed.

For example, Scott's cluster algebra on $\mathrm{Gr}(k, N)$ has the N cyclic interval Plücker coordinates as frozen variables [19]. Projective space admits a rank-zero seed, and the relation $uv = \sum_{i=1}^{r-1} a_i b_i$ defines a rank-one seed on the even-dimensional quadric D_r/P_1, with the a_i, b_i frozen. We consider the half-spin Grassmannians $X_n = \mathrm{OG}^+(n, 2n)$ for $n \geq 4$, the Cayley plane E_6/P_6, and the Freudenthal variety E_7/P_7. Here P_i is the maximal parabolic at node i in Bourbaki numbering. The polarized isomorphism $B_n/P_n \simeq D_{n+1}/P_{n+1}$ allows us to treat the type-B case together with type D.

Coordinate rings and anticanonical boundaries. Let $\mathcal{O}_X(1)$ denote the primitive ample generator of $\mathrm{Pic}(X)$. Write

$$R_X = \bigoplus_{d \geq 0} H^0(X, \mathcal{O}_X(d)), \qquad D = \sum_{i=1}^{h} \{a_i = 0\},$$

where $h = h(G)$ and $a_1, \ldots, a_h$ span the coordinate lines in the chosen Coxeter orbit. Write $\widehat{X} = \operatorname{Spec} R_X$ for the affine cone. We call D the *Coxeter boundary*. The seeds are defined over $S_X = \mathbb{C}[a_1, \ldots, a_h]$, with these variables kept polynomial. Write Σ_X for the chosen seed. Its *mutable rank* is the number of mutable variables. Following [14], we write $\mathcal{A}(\Sigma_X)$ for the S_X-algebra generated by the mutable variables in all reachable seeds and $\mathcal{U}(\Sigma_X)$ for the *upper LP algebra*, the intersection of all seed Laurent rings over S_X. Daisey and Ducat use the upper algebra convention [5, Definition 2.6].

Theorem A. *For each seed Σ_X described in Part I, the upper LP algebra $\mathcal{U}(\Sigma_X)$ equals the homogeneous coordinate ring R_X. For the half-spin Grassmannians and the Cayley plane we also have $\mathcal{A}(\Sigma_X) = R_X$. The half-spin seed has mutable rank $\binom{n-2}{2}$, with interval Pfaffians as mutable variables. The Cayley-plane seed has rank five and is due to Daisey and Ducat. The Freudenthal seed has mutable rank ten and eighteen frozen variables. The half-spin seeds have infinitely many cluster variables for $n \geq 7$, as does the Freudenthal seed.*

For the half-spin Grassmannians, we combine the multidimensional cube recurrence of Henriques–Speyer [10] with the wiring-diagram construction of Lam and Pylyavskyy [14]. Signed Pfaffians satisfy the exchange relations, and two complementary wiring charts identify the upper LP algebra with R_{X_n}. Every spinor coordinate is a cluster variable, yielding the asserted equality with $\mathcal{A}(\Sigma_{X_n})$. For the Freudenthal variety, five mutations connect our seed to an auxiliary chart. Laurent expansions and prime factorization in these charts give the coordinate-ring equality. In both families, alternating two mutations produces infinitely many

variables in the stated ranks. The finite-type conjecture of Daisey and Ducat concerns the existence of a positive seed [5, Conjecture 6.1].

Intrinsic mirrors and boundary functions. For each X above, consider the affine boundary complement $U_X = \operatorname{Spec} R_X[(a_1 \cdots a_h)^{-1}]$. It is a smooth affine *log Calabi–Yau variety*. The logarithmic volume form of the initial torus extends over U_X without zeros or poles and has at most simple poles along the compactifying boundary. Choose a smooth projective compactification Y as in [12, §2], and write $\mathrm{NE}(Y)$ for the monoid of effective numerical curve classes. The *intrinsic mirror algebra* A_Y of Keel and Yu has a theta basis $\{\vartheta_m\}$ over the monoid ring $\mathbb{Z}[\mathrm{NE}(Y)]$ [12, 11]. Here m ranges over the integral points of the *essential skeleton* of U_X. In the initial torus these are the monomial valuations of integral weight, hence are indexed by the cocharacter lattice. We study the algebra

$$A_{\mathrm{int}}(U_X)_1 = A_Y \otimes_{\mathbb{Z}[\mathrm{NE}(Y)],\, q^\beta \mapsto 1} \mathbb{C}.$$

The specialization is independent of the compactification [12, Proposition 17.3(1) and Remark 17.8]. The subscript 1 records the specialization of the curve-class parameters. The Laurent coefficient characters remain independent. We write Θ_m for the Laurent expansion of ϑ_m in a specified chart. When we fix the torus coordinates and their identification with the theta labels, we call the chart *marked*.

A *Newton polytope* is the convex hull of the exponents with nonzero coefficient in a Laurent polynomial. The order of vanishing along each boundary component b determines a theta label. We identify the corresponding *boundary theta function* W_b on the mirror. Their sum $W = \sum_b W_b$ is the *superpotential*.

Theorem B. *For each X, we construct a mirror seed $\Sigma_X^\vee$ and regular functions W_b for which the intrinsic Laurent expansion induces an isomorphism*

$$\Phi_X : A_{\mathrm{int}}(U_X)_1 \xrightarrow{\sim} \mathcal{U}(\Sigma_X^\vee).$$

It preserves the full lattice of coefficient characters and identifies each boundary theta function with W_b. These upper LP algebras are finitely generated, normal, and factorial.

The table gives the mutable ranks of the coordinate-ring and mirror seeds. The unit rank is the rank of the mirror's unit group modulo constants.

X	$\dim \widehat{X}$	Source rank	Mirror rank	Unit rank	Mirror seed
D_n/P_n	$\binom{n}{2}+1$	$\binom{n-2}{2}$	$n(n-3)/2$	$n+1$	§6.2
E_6/P_6	17	5	10	7	§8.1.1
E_7/P_7	28	10	20	8	§8.2.1

For the Cayley-plane mirror, Proposition 8.1 also proves that the generated LP algebra equals the upper LP algebra.

We use the intrinsic Laurent expansion of [8, Proposition A.5]. Regular functions on each minuscule variety bound the possible exponents. Contact counts determine the wall functions, and Laurent divisibility identifies the boundary theta functions within these bounds. For the Cayley plane the resulting functions generate the proposed mirror algebra. For the half-spin and Freudenthal varieties we show that the initial and adjacent charts account for all prime divisors, then apply [8, Proposition B.8].

For the half-spin family, let $m = n-1$. In the triangular coordinates y_{ab} $(1 \le a \le b \le m)$, with $\mathfrak{q}$ the character dual to scalar multiplication on the cone, the superpotential is

$$W = y_{11} + \sum_{1 \le a < b \le m} \frac{y_{a+1,b}}{y_{ab}} + \sum_{1 \le a \le b < m} \frac{y_{a,b+1}}{y_{ab}} + \frac{\mathfrak{q}}{y_{mm}}.$$

The Newton polytopes of the boundary sums control the possible missing divisors. The Cayley-plane mirror has a smooth complete-intersection presentation. For the Freudenthal mirror, twenty binomial exchanges give eighteen boundary functions and a forty-term superpotential. Its global comparison uses an exact finite calculation of divisorial valuations.

Integral forms and theta bases. In a Laurent chart, the *Gauss valuation* of weight w assigns to a nonzero Laurent polynomial the minimum of $\langle a, w\rangle$ over its exponents a, with the convention $\nu(0) = +\infty$. A family is *valuatively independent* if, whenever finitely many scalars are chosen and at least one is nonzero, the valuation of the resulting sum is the minimum valuation among its nonzero summands. Here the scalars carry the trivial valuation. An *extended cluster monomial* is a monomial of coefficient one in a single seed, with nonnegative exponents on mutable variables and arbitrary exponents on invertible frozen variables.

Theorem C. *The comparisons descend to the specified integral counting algebras and upper LP algebras over their Laurent coefficient rings. In every reachable seed, the intrinsic theta functions are integral Laurent polynomials and are valuatively independent for every real Gauss valuation. Thus the Newton polytope of a finite signed sum is the convex hull of the Newton polytopes of its nonzero theta terms. Every regular extended cluster monomial agrees up to sign with an intrinsic theta function. The theta function is the signed representative whose least term in the marked chart has positive coefficient, for the order fixed in §5.*

The integral comparison follows from primitive exchange polynomials and the coefficient-one leading terms of the theta functions. For valuative independence we use [8, Theorem 4.10], which extends the methods of [3] to these Laurent charts. In the D_4 mirror, every theta function is an extended cluster monomial. The basis is *atomic* in the LP seed charts. A function has nonnegative integral Laurent coefficients in every chart exactly when its theta coefficients are nonnegative integers.

The projective complement. A frozen linear section trivializes the tautological line bundle on $X^\circ = X \setminus D$, giving $U_X \simeq X^\circ \times \mathbb{G}_m$. The split torus factor contributes the character $\mathfrak{q}$ to the mirror coefficient ring. By the torus product formula [8, Lemma 3.7],

$$A_{\mathrm{int}}(U_X)_1 \simeq A_{\mathrm{int}}(X^\circ)_1[\mathfrak{q}^{\pm 1}].$$

The fibre $\mathfrak{q} = 1$ is therefore the marked intrinsic mirror of the projective complement. This specialization of a torus character is distinct from the earlier specialization $q^\beta = 1$.

We work over $\mathbb{C}$ except in the integral comparisons of Part II. We write $\mathbb{N} = \mathbb{Z}_{\geq 0}$.

Part 1. Laurent phenomenon structures

1. The index condition and Coxeter boundaries

Let G be a connected simple complex algebraic group, with Coxeter number $h(G)$. We may replace G by its simply connected cover. For a Fano variety X, let $\iota(X)$ be the largest positive integer dividing $-K_X$ in $\mathrm{Pic}(X)$.

Proposition 1.1. *For a proper parabolic subgroup $P \subset G$, the index satisfies $\iota(G/P) \leq h(G)$. Equality holds if and only if P is maximal at a minuscule node.*

Proof. Let I be the set of simple roots of the Levi subgroup and let J be its complement. Write ρ and ρ_I for the half-sums of the positive roots of G and of the Levi, and let w_I be the longest Levi Weyl-group element. The anticanonical weight is $2\rho - 2\rho_I = \rho + w_I\rho$ [13, §2.7]. Here $\alpha_j^\vee$ denotes a simple coroot and ht denotes height in the simple coroot basis. For $j \in J$, its coefficient at the fundamental weight ω_j is

$$r_j = \langle \rho + w_I\rho, \alpha_j^\vee\rangle = 1 + \mathrm{ht}(w_I\alpha_j^\vee), \qquad \iota(G/P) = \gcd_{j\in J} r_j.$$

The coefficient of $\alpha_j^\vee$ in $w_I\alpha_j^\vee$ is one, so this is a positive coroot. Let $\Theta^\vee$ be the highest root of the dual root system. It is the unique positive coroot of maximal height, namely $h(G) - 1$. Thus $r_j \leq h(G)$.

If the gcd is $h(G)$, then every $r_j = h(G)$ and $w_I\alpha_j^\vee = \Theta^\vee$. The left side has coefficient zero at each node of $J \setminus \{j\}$, whereas the highest coroot has full support. Thus $J = \{j\}$. Its coefficient at j is one, which is precisely the minuscule criterion $\langle \omega_j, \Theta^\vee\rangle = 1$.

Conversely, suppose $J = \{j\}$ and ω_j is minuscule. Pairing the anticanonical weight $r_j\omega_j$ with $\Theta^\vee$ gives $r_j = h(G) - 1 + \mathrm{ht}(w_I\Theta^\vee)$. The last coroot still has coefficient one at j, so it is positive. Therefore $r_j \geq h(G)$, and equality follows from the upper bound. □

The use of the dual root system matters in non-simply-laced type. For example, the cominuscule variety C_r/P_r ($r \geq 2$) has index $r+1$, whereas $h(C_r) = 2r$ [2, §2.1]. With Bourbaki numbering, the complete list is

$$A_n/P_i\ (1 \leq i \leq n), \quad B_n/P_n, \quad C_n/P_1, \quad D_n/P_1, D_n/P_{n-1}, D_n/P_n, \quad E_6/P_1, E_6/P_6, E_7/P_7.$$

Type A has the Grassmannian cluster structures of [19]; C_n/P_1 is projective space, and D_n/P_1 is an even quadric. We concentrate on the half-spin and exceptional varieties. The polarized isomorphism $B_n/P_n \simeq D_{n+1}/P_{n+1}$ accounts for type B.

Suppose H is the primitive ample generator of $\mathrm{Pic}(G/P)$ and $-K_{G/P} = hH$. A product of h nonzero linear sections defines an anticanonical divisor. A full Coxeter orbit specifies such sections compatibly with the Weyl-group action. The existence of an LP chart with these sections frozen is an additional assertion, established by the constructions below. The affine cone has dimension $\dim(G/P)+1$, so such a seed has mutable rank $\dim(G/P)+1-h$.

Proposition 1.2. *Fix a simple-root system and a fundamental weight ω_k of a finite irreducible root system. As c ranges over products of all the simple reflections, each used once, the sets $\langle c\rangle\omega_k$ are Weyl-equivalent.*

Proof. Since the finite Dynkin diagram is a tree, any two Coxeter words are related by cyclic rotations and commutations of commuting reflections [6, Propositions 2.3 and 4.1]. Commutations leave the element unchanged. Consider a rotation from $c = s_i v$ to $c' = v s_i = s_i c s_i$. The reflection s_i takes $\langle c\rangle\omega_k$ to $\langle c'\rangle(s_i\omega_k)$. For $i \neq k$ we have $s_i\omega_k = \omega_k$. For $i = k$ all factors of v fix ω_k, so $s_i\omega_k = (c')^{-1}\omega_k$. In either case the image is $\langle c'\rangle\omega_k$. □

2. Upper Laurent phenomenon algebras

We use Lam–Pylyavskyy's convention for LP mutation [14, Section 2], including the normalized exchange Laurent polynomials. Let S be a unique factorization domain (UFD), let $r \geq 1$, and write $\mathcal{F} = \mathrm{Frac}(S)(x_1, \ldots, x_r)$. An *LP seed* consists of the algebraically independent variables x_i and irreducible polynomials $F_i \in S[x_1, \ldots, x_r]$ such that F_i is independent of x_i and is not divisible by any x_j. Following Lam–Pylyavskyy, we associate to F_i an *exchange Laurent polynomial* $\widehat{F}_i = F_i/M_i$, where M_i is a monomial in the other mutable variables. Mutation replaces x_i by $x_i' = \widehat{F}_i/x_i$ and updates the exchange polynomials by the substitution and cancellation rule of [14, §2]. The x_i are the mutable variables. Together with the frozen variables they form the *extended cluster*.

2.1. The Laurent intersection.

Write $\mu_i t$ for mutation of a seed t in direction i and $t' \sim t$ when t' is obtained from t by a sequence of mutations. The Laurent ring of t, its *upper bound*, and the upper LP algebra are

$$L(t) = S[x_1^{\pm 1}, \ldots, x_r^{\pm 1}], \qquad \mathrm{UB}(t) = L(t) \cap \bigcap_{i=1}^{r} L(\mu_i t), \qquad \mathcal{U}(t) = \bigcap_{t' \sim t} L(t').$$

The LP algebra $\mathcal{A}(t)$ is the S-algebra generated by all cluster variables of seeds mutation-equivalent to t. The Laurent phenomenon gives $\mathcal{A}(t) \subseteq \mathcal{U}(t)$ [14].

We use the following LP starfish theorem of Fei [7, Theorem 1.3].

Theorem 2.1. *The finite upper bound is invariant under LP mutation over S. Consequently,*

$$\mathcal{U}(t) = L(t) \cap \bigcap_{i=1}^{r} L(\mu_i t).$$

We apply the theorem over polynomial coefficient rings in Part I and Laurent coefficient rings in Part II. It allows repeated exchange polynomials and nontrivial normalizing monomials. In rank zero, the finite upper bound and the upper LP algebra both equal S.

2.2. **Laurent regularity and boundary valuations.** Set $L_0 = L(t)$ and $L_i = L(\mu_i t)$. Expand an element of L_0 as $f = \sum_k x_i^k f_{i,k}$, with $f_{i,k} \in S[x_j^{\pm 1} : j \neq i]$. Substitution of $x_i = F_i/(M_i x_i')$ gives

$$f \in L_i \quad \Longleftrightarrow \quad F_i^{-k} \mid f_{i,k} \quad \text{for every } k < 0. \tag{2.1}$$

Indeed, M_i is a unit in this coefficient ring, and distinct powers of x_i' cannot cancel. On a fixed finite support, these conditions are linear equations for the coefficients of f.

No mutable variable divides F_i, so it remains a nonunit prime in $S[x_j^{\pm 1} : j \neq i]$ and hence in L_i. We call $\nu_i = \mathrm{ord}_{F_i}$ in the adjacent Laurent ring the corresponding *boundary valuation*. It satisfies

$$\nu_i(x_j) = \delta_{ij}, \qquad \nu_i(f) = \min_k \{k + \mathrm{ord}_{F_i}(f_{i,k})\}.$$

The independent coordinate x_i' again prevents cancellation. The starfish theorem therefore gives

$$\mathcal{U}(t) = \{f \in L_0 : \nu_i(f) \geq 0 \text{ for all } i\}. \tag{2.2}$$

For later use, suppose that $F_i = F$ for $i \in I$, that F is independent of the variables indexed by I, and that no other normalization factors occur. Then

$$\widehat{F}_i = \frac{F}{\prod_{j \in I \setminus \{i\}} x_j}, \qquad x_i' = \frac{F}{\prod_{j \in I} x_j}.$$

The coefficient of $\prod_{i \in I} x_i^{k_i}$ must then be divisible by $F^{\max(\{0\} \cup \{-k_i : i \in I\})}$. This condition for every exponent vector suffices for regularity in the adjacent charts indexed by I. This follows by imposing (2.1) for each i.

Pompili determines the units of an upper LP algebra and proves factoriality when it equals its upper bound [16, Proposition 5.14(1) and Theorem 5.19]. Daisey proves primality of the cluster variables over a coefficient UFD by computing their boundary orders [4, Lemma 3.4]. We give the valuation argument here to identify the other prime elements explicitly.

Proposition 2.2. *The upper LP algebra $\mathcal{U}(t)$ is a UFD. If P is irreducible in L_0, then $P^\circ = P \prod_i x_i^{-\nu_i(P)}$ is prime in $\mathcal{U}(t)$. These elements and the initial variables represent all prime elements up to associates, and $\mathcal{U}(t)^\times = S^\times$.*

Proof. All boundary orders of P° are zero, so $P^\circ \in \mathcal{U}(t)$ by (2.2). If P° divides an element of $\mathcal{U}(t)$ in L_0, the quotient has nonnegative boundary orders and belongs to $\mathcal{U}(t)$. Primality in L_0 therefore implies primality in $\mathcal{U}(t)$. Likewise, for $f \in \mathcal{U}(t)$ we have $f \in x_i \mathcal{U}(t)$ exactly when $\nu_i(f) > 0$, so each x_i is prime. Factor a nonzero $f \in \mathcal{U}(t)$ in L_0 and normalize its irreducible factors. The result is

$$f = c \prod_i x_i^{\nu_i(f)} \prod_j (P_j^\circ)^{e_j}, \qquad c \in S^\times,$$

with nonnegative exponents. This is a finite factorization into primes; no Noetherian hypothesis is required. Finally, a unit and its inverse have boundary order zero, so the mutable exponents in a Laurent unit must vanish. □

2.3. **Rank-one charts and extension across codimension two.** Let $s_i = \prod_{j \neq i} x_j$ and $B_0 = S[x_1, \ldots, x_r, x_1', \ldots, x_r']$. Localizing the finite intersection at s_i gives

$$\mathcal{U}(t)[s_i^{-1}] = B_0[s_i^{-1}] = L_0 \cap L_i = \frac{S[x_j^{\pm 1} : j \neq i][x_i, x_i']}{(x_i x_i' - F_i/M_i)}. \tag{2.3}$$

All other adjacent rings contain L_0 after this localization, and the last equality follows from (2.1). If S is a finitely generated $\mathbb{C}$-algebra, the open set $X^\circ = \mathrm{Spec}\, B_0 \setminus V(s_1, \ldots, s_r)$ is quasi-affine of finite type and has ring of global functions $\mathcal{U}(t)$. If S is smooth over $\mathbb{C}$, these *rank-one charts* are smooth whenever $F_i = 0$ is smooth in $\mathrm{Spec}\, S[x_j^{\pm 1} : j \neq i]$. For polynomial or Laurent polynomial coefficient rings, the rational logarithmic forms in the mutable and coefficient coordinates agree up to sign under mutation.

Proposition 2.3. *Let $B_0 \subseteq B \subseteq \mathcal{U}(t)$ be an intermediate ring and let $J = (s_1, \dots, s_r)B$. Here $(B :_{\operatorname{Frac} B} J^m)$ consists of the fractions u such that $uJ^m \subseteq B$. Then*

$$\mathcal{U}(t) = \bigcap_i B[s_i^{-1}] = \bigcup_{m \geq 0} (B :_{\operatorname{Frac} B} J^m).$$

If B is Noetherian, then $\operatorname{depth}_J B \geq 2$ suffices for $B = \mathcal{U}(t)$. In particular, this equality holds if B is Noetherian and normal and $V(J)$ has codimension at least two.

Proof. Write $D(s_i)$ for the principal open set where s_i is invertible. By (2.3), the three rings agree there. A fraction regular on every $D(s_i)$ has denominator ideal containing a power of each s_i, hence a power of J. This proves the saturation formula. For normal B, the open set contains every height-one point when $V(J)$ has codimension at least two. Intersecting the corresponding local rings gives B. The depth condition likewise gives extension across $V(J)$. □

Lemma 2.4. *Let $X = G/P$ be a positive-dimensional homogeneous variety with Picard group generated by the primitive ample line bundle $\mathcal{O}_X(1)$. Then its homogeneous coordinate ring R_X is factorial, and every nonzero element of degree one is prime.*

Proof. The homogeneous embedding is projectively normal, and R_X is generated in degree one; see [1]. The punctured cone is $\mathcal{O}_X(-1)^\times$. Since the vertex has codimension at least two, the class-group formula for the affine cone [9, II, Exercise 6.3] gives

$$\operatorname{Cl}(R_X) = \operatorname{Pic}(X)/\mathbb{Z}[\mathcal{O}_X(1)] = 0.$$

Thus R_X is factorial. A nonzero degree-one element is irreducible in a positively graded domain with degree-zero part $\mathbb{C}$, and is therefore prime. □

3. Half-spin Grassmannians

Let $n \geq 4$ and $m = n-1$. Write $X = \mathrm{OG}^+(n, 2n)$ for one component of the Grassmannian of maximal isotropic subspaces of $\mathbb{C}^{2n}$, and $R_X = \mathbb{C}[\widehat{X}]$ for its cone ring. The half-spin polarization H is the primitive ample generator. Thus $\dim X = m(m+1)/2$ and $\dim R_X = m(m+1)/2 + 1$. We index the coordinates by subsets of $[m] = \{1, \dots, m\}$; the resulting boundary also gives the type-B_m construction in Remark 3.3.

3.1. **Signed principal Pfaffians and the interval seed.** Let p_I, for even subsets $I \subseteq [m+1]$, denote the degree-one spinor coordinates. Our convention on the standard big cell is $p_I = t \operatorname{Pf}(B_I), \qquad p_\varnothing = t$, where t is the cone coordinate, B is a skew-symmetric $(m+1) \times (m+1)$ matrix, and B_I is its principal submatrix with indices in increasing order. These coordinates generate R_X and satisfy the quadratic *Wick relations*; see [18]. For every $S \subseteq [m]$, define

$$S^+ = \begin{cases} S, & |S| \text{ even}, \\ S \cup \{m+1\}, & |S| \text{ odd}, \end{cases} \qquad \epsilon(k) = \begin{cases} -1, & k \equiv 0 \pmod 4, \\ 1, & \text{otherwise}, \end{cases} \qquad q_S = \epsilon(|S|) p_{S^+}.$$

This is a signed relabeling of all spinor coordinates. In particular, $q_\varnothing = -p_\varnothing$. Write

$$z = q_\varnothing, \qquad X_{ij} = q_{[i,j]} \quad (1 \leq i \leq j \leq m), \qquad X_{i,i-1} = z.$$

The frozen and mutable variables are, respectively,

$$\begin{aligned} \mathcal{B}_m &= \{z\} \cup \{X_{1j} : 1 \leq j \leq m\} \cup \{X_{im} : 2 \leq i \leq m\}, \\ \mathcal{M}_m &= \{X_{ij} : 2 \leq i \leq j \leq m-1\}. \end{aligned} \tag{3.1}$$

We use the polynomial coefficient ring $S_m = \mathbb{C}[\mathcal{B}_m]$; the frozen variables are not inverted. Define

$$F_{ii} = X_{i-1,i-1}X_{i,i+1} + zX_{i-1,i+1} + X_{i+1,i+1}X_{i-1,i}, \qquad 2 \leq i \leq m-1, \tag{3.2}$$

and, for $2 \le i < j \le m-1$, define

$$\begin{aligned} F_{ij} = {} & X_{i-1,j-1}X_{i+1,j}X_{i,j+1} + X_{i+1,j-1}X_{i-1,j}X_{i,j+1} \\ & + X_{i-1,j+1}X_{i,j-1}X_{i+1,j} + X_{i+1,j+1}X_{i,j-1}X_{i-1,j}. \end{aligned} \tag{3.3}$$

All factors on the right are members of the extended cluster.

Theorem 3.1. *For every $m \ge 3$, (3.1)–(3.3) define a Laurent phenomenon seed Σ_m in* $\mathrm{Frac}(R_X)$. *Its $2m$ frozen coordinate lines form a full Coxeter orbit. Its mutable rank is $\binom{m-1}{2}$, and its initial exchange Laurent polynomials satisfy $\widehat{F}_{ij} = F_{ij}$. With polynomial frozen coefficients,*

$$\mathcal{A}_{S_m}(\Sigma_m) = \mathcal{U}_{S_m}(\Sigma_m) = R_X.$$

Every nonfrozen q_S is a cluster variable in a reachable wiring seed.

There are $m-2$ quadratic three-term exchanges and $\binom{m-2}{2}$ cubic four-term exchanges. Their first mutants therefore have degrees one and two, respectively. The cube recurrence below gives the degree-one mutations.

3.2. **The Coxeter orbit and the anticanonical boundary.** For type D_{m+1} take

$$\alpha_i = e_i - e_{i+1} \quad (1 \le i \le m), \qquad \alpha_{m+1} = e_m + e_{m+1}, \qquad c_D = s_1 \cdots s_{m+1},$$

where the rightmost reflection acts first. Then

$$e_1 \longmapsto e_2 \longmapsto \cdots \longmapsto e_m \longmapsto -e_1, \qquad e_{m+1} \longmapsto -e_{m+1}.$$

Encode the half-spin weight $\frac{1}{2}\sum_{i=1}^{m+1} e_i - \sum_{i\in I} e_i$, with $|I|$ even, by $S = I \cap [m]$. The induced operation on labels is

$$c_D(S) = \{i+1 : i \in S,\ i < m\} \cup \begin{cases} \{1\}, & m \notin S, \\ \varnothing, & m \in S. \end{cases}$$

Its orbit through $\varnothing$ is

$$\varnothing, [1,1], [1,2], \ldots, [1,m], [2,m], \ldots, [m,m]. \tag{3.4}$$

The $2m$ entries are distinct and are exactly the frozen labels. Replacing weight vectors by dual coordinate functions gives the same permutation of their one-dimensional weight spaces.

If $\mathcal{S}$ is the tautological bundle, then

$$T_X \simeq \Lambda^2 \mathcal{S}^*, \qquad \det \mathcal{S}^* \simeq \mathcal{O}(2H), \qquad -K_X = 2mH.$$

Thus the sum of the frozen hyperplane divisors is anticanonical. The frozen coordinates are pairwise nonassociate degree-one primes by Lemma 2.4, so the components are reduced and distinct.

3.3. **The signed cube recurrence.** For distinct $a < b < c$ outside $S \subseteq [m]$, use Sa, Sab, and similar expressions for unions with the indicated indices. The Wick relations give

$$q_{Sac}q_{Sb} = q_S q_{Sabc} + q_{Sab}q_{Sc} + q_{Sbc}q_{Sa}. \tag{3.5}$$

This identity holds for every common subset S. To check the signs, write $P_T = p_{T^+}$. The unsigned identity is

$$P_{Sac}P_{Sb} + P_S P_{Sabc} = P_{Sab}P_{Sc} + P_{Sbc}P_{Sa}.$$

For even $|S|$, this is the four-index principal-Pfaffian identity with additional indices $a, b, c, m+1$. For odd $|S|$, it is the odd-base Wick identity on the same four additional indices. Ordering common indices before the additional indices introduces in every product the same sign: each element of $\{a, b, c, m+1\}$ occurs exactly once outside the two copies of S. The identities therefore hold also when elements of S are interleaved with a, b, c. Finally, the identity $\epsilon(k)\epsilon(k+3) = -\epsilon(k+1)\epsilon(k+2)$ converts the unsigned identity into (3.5).

Equation (3.5) is the multidimensional cube recurrence of Henriques–Speyer [10]. Their bigraded projective quotient for m cube directions is $\mathrm{IG}(m-1, 2m)$. The grading here is

the ordinary homogeneous grading of the half-spin cone, in which each principal Pfaffian coordinate has degree one.

3.4. **Wiring diagrams and LP mutation.** We use a *reduced wiring diagram* for the longest permutation on m wires, labeled $1, \dots, m$ from top to bottom at the left boundary. Label each chamber by the set of wires above it. The unbounded chambers have the labels (3.4). The reduced word $(s_1)(s_2 s_1)\cdots(s_{m-1}\cdots s_1)$ has chamber collection $\{\varnothing\} \cup \{[i,j] : 1 \le i \le j \le m\}$. Its bounded singleton chambers are triangular, and its other bounded chambers are quadrilateral.

For an arbitrary bounded chamber r, let $s_1, \dots, s_d$ be its neighboring chambers across its sides, in cyclic order. Let c_i denote the chamber opposite r at the vertex between sides s_i and s_{i+1}, with cyclic subscripts. Lam–Pylyavskyy's wiring rule [14, Section 7.5] is

$$F_r = \sum_{i=1}^{d} q_{c_i} \prod_{j \notin \{i, i+1\}} q_{s_j}. \tag{3.6}$$

The side and corner labels are all distinct. Reading the interval diagram gives precisely (3.2) and (3.3).

We first check that the chamber variables form a transcendence basis. Indeed, a triangular braid move replaces Sb by Sac and has coordinate transformation (3.5); commutations of crossings on disjoint pairs of wires leave the chamber set unchanged. Reduced diagrams are connected by these moves, equivalently by rhombus-tiling flips [10, Theorem 2.2]. Every subset $S \subseteq [m]$ occurs as a chamber: choose a permutation with the elements of S in its first $|S|$ positions and extend a reduced word for that permutation to one for the longest permutation. Hence every q_S is rational in the chamber variables of any fixed diagram. Those variables number $m(m+1)/2 + 1 = \operatorname{trdeg}_{\mathbb{C}} R_X$, and therefore form a transcendence basis. In particular S_m is a polynomial ring.

Lemma 3.2. *The polynomials* (3.6) *satisfy the LP seed axioms, are pairwise nonassociate, and satisfy* $\widehat{F}_r = F_r$. *A braid move at a triangular chamber induces an LP mutation of seeds.*

Proof. For a fixed chamber, regard F_r as a linear polynomial in its corner variables over the polynomial ring in the remaining variables. Its coefficients are monomials in the side variables and have greatest common divisor one: every side variable is absent from two summands. It is irreducible over the fraction field of the coefficient ring, and Gauss's lemma proves irreducibility in the full polynomial ring. No variable divides F_r, and q_r does not occur in it.

The variables occurring in F_r are the $2d$ labels of its side and corner neighbors. For each wire bounding r, exactly three neighboring labels differ from r in that wire: the label across the side and the two labels across its corners. Membership of every other wire is constant. For $d \ge 4$ the majority of the $2d$ labels therefore recovers r one wire at a time. For $d = 3$, all six neighbors agree with r off the three boundary wires. In the given wiring diagram these three wires cut the plane into seven regions. The neighbors realize six of their sign patterns. The seventh gives the label of the triangle. Thus the variables occurring in F_r determine r, so distinct exchanges are nonassociate.

To verify normalization, observe that q_t occurs in F_r if and only if q_r occurs in F_t: the two chambers then share a side or a crossing. If they do not meet, F_r and F_t remain coprime after localization at variables, because they are distinct irreducibles and neither is divisible by a variable. If they meet, write F_r as a polynomial in q_t. Its nonzero constant coefficient is independent of q_r. The irreducible polynomial F_t has positive q_r-degree and nonzero q_r-constant coefficient, and therefore cannot divide that constant coefficient even in the Laurent ring. Modulo F_t, substituting $q_t = F_t/q_t'$ into F_r leaves precisely this coefficient. These are the LP normalization tests, so $\widehat{F}_r = F_r$.

Now let x be a triangular chamber, with sides a, b, c and opposite corners α, β, γ, labeled so that $xx' = \alpha a + \beta b + \gamma c$. For a mutable side chamber a, its exchange has the form $F_a = M(\beta b + \gamma c) + xP$. The LP update sets $a = 0$ in F_x, substitutes $x = (\beta b + \gamma c)/x'$, clears

the denominator, and cancels the common factor $\beta b+\gamma c$. The result is $F'_a = Mx' + P$, which is exactly the new wiring polynomial. For a mutable corner chamber α, write $F_\alpha = xN + Q$. Its updated exchange is $F'_\alpha = (\beta b + \gamma c)N + x'Q$, again the new wiring polynomial. Cyclic permutation accounts for the other side and corner chambers. All more distant exchanges are unchanged. The new wiring polynomials are irreducible and divisible by no variable, so the update requires no further cancellation. □

It follows that every wiring seed is reachable from Σ_m, and every nonfrozen spinor coordinate occurs in one of them. Since the spinor coordinates generate R_X, this proves

$$R_X \subseteq \mathcal{A}_{S_m}(\Sigma_m). \tag{3.7}$$

3.5. **The coordinate-ring equality.** For any wiring seed, let f be the product of its mutable coordinates. Its Laurent ring is $(R_X)_f$. Indeed, every spinor coordinate is a cluster or frozen variable, hence Laurent over S_m by [14, Theorem 5.1]. Conversely, the Laurent generators belong to $(R_X)_f$.

Take the interval diagram and its 180-degree rotation, and let f, g be their respective products. The rotation replaces a label by its complement. Each bounded interval has both endpoints in $[2, m-1]$, so its complement is the union of two disjoint nonempty end intervals. The two diagrams therefore have disjoint sets of mutable coordinates. These coordinates are pairwise nonassociate primes by Lemma 2.4 and their distinct weights. Thus a fraction regular in both localizations belongs to R_X. Since both seeds are reachable, (3.7) gives

$$R_X \subseteq \mathcal{A}_{S_m}(\Sigma_m) \subseteq \mathcal{U}_{S_m}(\Sigma_m) \subseteq (R_X)_f \cap (R_X)_g = R_X.$$

This proves Theorem 3.1.

Remark 3.3 (Transfer to type B). Intersection with a nondegenerate hyperplane gives the polarized isomorphism $D_{m+1}/P_{m+1} \simeq B_m/P_m$: a maximal isotropic $(m+1)$-plane meets the hyperplane in an isotropic m-plane, and each such m-plane has one extension in either even-dimensional component. The half-spin representation restricts to the spin representation, with the same subset coordinates. The type-B Coxeter element $s_1 \cdots s_m$ acts by the negative cycle $e_1 \mapsto \cdots \mapsto e_m \mapsto -e_1$, giving exactly (3.4). Thus the coordinate-ring theorem and the mirror theorem, including its boundary functions, transfer by replacing the type-D index by $m+1$. The two smaller cases $B_1/P_1 \simeq \mathbb{P}^1$ and $B_2/P_2 \simeq \mathbb{P}^3$ have polynomial coordinate rings and no mutable variables.

3.6. **The examples D_4 and D_5.** For $m = 3$ (type D_4), the seed has rank one, with exchange polynomial $F_{22} = X_{11}X_{23} + zX_{13} + X_{33}X_{12}$. Its other cluster variable is $q_{\{1,3\}}$, giving the quadric. For $m = 4$ (D_5), write

$$(x, y, w) = (X_{22}, X_{23}, X_{33}), \qquad (a, b, c, d, e, f, g) = (X_{11}, X_{12}, X_{13}, X_{14}, X_{24}, X_{34}, X_{44}).$$

Then

$$F_x = ay + bw + zc, \qquad F_y = dxw + cfx + bew + zce, \qquad F_w = fx + gy + ze.$$

This is the rank-three wiring example with ten mutable cluster variables and sixteen seeds, up to the coefficient relabeling in [14, Example 7.7]; see also [5, Section 4.2]. Its eight wiring seeds are a proper part of its LP mutation class.

3.7. **Infinite type for $n \geq 7$.**

Proposition 3.4. *The Coxeter-boundary seeds of Theorem 3.1 have infinitely many cluster variables for D_n with $n \geq 7$.*

Proof. We first give a rank-two reduction in D_7, so $m = 6$. Index mutation directions by the intervals in the initial seed. In order from left to right, perform the six mutations

$$\mu_{X_{33}}, \quad \mu_{X_{44}}, \quad \mu_{X_{23}}, \quad \mu_{X_{33}}, \quad \mu_{X_{45}}, \quad \mu_{X_{24}}. \tag{3.8}$$

After these mutations, let Y_{ij} denote the variable in the direction originally occupied by X_{ij}. Write

symbol	a	u	b	v	d	e	f	g	k
variable	Y_{22}	Y_{33}	Y_{44}	Y_{55}	Y_{34}	Y_{45}	Y_{24}	Y_{35}	Y_{25}

and the unchanged frozen coordinates

$$z = q_{\varnothing}, \quad A = q_{12}, \quad B = q_{1234}, \quad C = q_{12345}, \quad D = q_{3456}, \quad E = q_{56}, \quad H = q_6.$$

Here subscripts such as 1234 denote subsets, whereas Y_{ij} records the original mutation direction. With $T = vD + gE$ and $S = ud + gzB$, the mutation rule gives

$$F_u = T\big[adfB + bA(dC + gB) + kAB(vd + gz)\big] + bdef,$$
$$F_v = E\big[S\{adf + kz(u + gA)\} + bz(u + gA)(dC + gB)\big] + bdf(uH + ez).$$

Their exchange Laurent polynomials are

$$\widehat{F}_u = F_u/d, \qquad \widehat{F}_v = F_v/d. \tag{3.9}$$

The denominator d is independent of both u and v. Reading off the leading coefficients gives

$$\deg_v F_u = \deg_u F_v = 2, \qquad [v^2]F_u = dkABD, \qquad [u^2]F_v = dkzE.$$

Here $[v^2]$ and $[u^2]$ denote coefficient extraction. Freeze every current mutable variable except u and v using the LP freezing construction [14, Proposition 3.7]. The new coefficient ring inverts those variables; in particular d is a unit, so (3.9) changes the rank-two exchange polynomials only by coefficient units. They have degrees $(2, 2)$. The rank-two classification [14, Theorem 6.4 and Corollary 6.5] therefore supplies infinitely many distinct cluster variables, even up to units of the enlarged coefficient ring. The freezing construction realizes their mutation path in the original D_7 algebra.

For $n > 7$, the same rank-two reduction is obtained by freezing. In the initial D_n interval seed, $n \geq 7$, keep the ten mutable variables $\{X_{ij} : 2 \leq i \leq j \leq 5\}$ and freeze all other mutable variables, whose right index is at least six. Each exchange at a variable we keep involves only intervals in $[6]$ and is exactly the corresponding D_7 exchange. The resulting coefficient ring is obtained from the D_7 polynomial coefficient ring by inverting some of its frozen coordinates and adjoining independent coefficient variables, some invertible. The initial normalizers are trivial, so no additional factor enters this identification. The six mutations (3.8) therefore give the same pair of quadratic exchanges over this coefficient extension. Multiplication by coefficient units preserves their degrees. A second application of freezing lifts the infinite rank-two path to the original D_n seed. □

4. The exceptional varieties

The Cayley-plane and Freudenthal cones have dimensions 17 and 28. Their Coxeter boundaries contain 12 and 18 linear coordinates, respectively, so the mutable ranks are 5 and 10. We work over the polynomial rings generated by these frozen coordinates.

4.1. **The Cayley plane.** We use the labels of Daisey and Ducat [5, Section 5]. The 27 degree-one coordinates split into Coxeter orbits

$$(a_i)_{i\in\mathbb{Z}/12}, \qquad (x_i)_{i\in\mathbb{Z}/12}, \qquad (z_i)_{i\in\mathbb{Z}/3}.$$

Take $a_1, \ldots, a_{12}$ as frozen variables. When a subscript on z is an element of $\mathbb{Z}/12$, it is reduced modulo 3. Set

$$y_i = x_i z_i - a_i x_{i-1} - a_{i+7} x_{i+1} \qquad (i \in \mathbb{Z}/12).$$

In particular $y_3 = x_3 z_3 - a_3 x_2 - a_{10} x_4$ is a degree-two coordinate function, and we take $\mathbf{x}_6 = (x_1, x_2, x_3, x_4, y_3)$ as our initial cluster. Over $\mathbb{C}[a_1, \ldots, a_{12}]$ its ordinary exchange

polynomials are

$$\begin{aligned}
F_{x_1} &= y_3 + a_{12}a_1,\\
F_{x_2} &= a_2x_1(y_3 + a_{10}x_4) + a_9x_3(y_3 + a_1a_{12}) + x_1x_3(a_7x_4 + a_4a_{12}),\\
F_{x_3} &= y_3 + a_3x_2 + a_{10}x_4,\\
F_{x_4} &= a_{11}(y_3 + a_3x_2) + x_3(a_4x_1 + a_6x_2 + a_1a_9),\\
F_{y_3} &= x_1x_4(a_5x_2 + a_7x_3 + a_2a_{10}) + a_{12}x_3(a_4x_1 + a_1a_9) + a_{12}x_2(a_6x_3 + a_8x_4 + a_3a_{11}).
\end{aligned} \tag{4.1}$$

These are the exchanges of [5, Proposition 5.3]. Their associated Laurent polynomials satisfy

$$\widehat{F}_{x_2} = F_{x_2}/x_1, \qquad \widehat{F}_v = F_v \quad (v \neq x_2).$$

Indeed, substituting $x_1 = (y_3 + a_1a_{12})/x_1'$ into F_{x_2} introduces one common factor $y_3 + a_1a_{12}$; the other normalization orders are zero.

Daisey and Ducat prove that the LP algebra of (4.1) has 264 seeds up to equivalence and 32 mutable cluster variables. Its exchange polynomials and the Laurent expansions of its cluster variables have positive coefficients [5, Theorem 5.4 and Corollary 5.5].

The 32 mutable variables consist of the fifteen linear coordinates x_i, z_j, twelve quadratics y_i, two cubics, and three quartics. We use one of these quartics, t_1, in the mirror chart of Section 8.1.

4.2. **The Freudenthal seed.** Let R_7 be the homogeneous coordinate ring of E_7/P_7, in the coordinate convention of [5, Section 6.1], reproduced in Appendix A.1, with the upper signs in their equations (e)–(g). Subscripts of a_i, x_i, y_i are read modulo 18. The Coxeter orbit $a_0, \dots, a_{17}$ gives the frozen variables, with polynomial coefficient ring $S_{E_7} = \mathbb{C}[a_0, \dots, a_{17}]$.

We take the seven linear functions

$$(h, b, c, d, e, f, g) = (x_1, x_{11}, x_9, x_{16}, x_6, x_{14}, x_4)$$

and set

$$\begin{aligned}
r &= x_4y_{15} - a_4x_{14}, & q &= x_{11}y_0 - a_{11}x_1,\\
u &= x_1x_{11}y_8 - a_1x_{11}x_9 - a_{11}x_1y_5 - a_{12}x_1x_6.
\end{aligned}$$

The cluster $\mathbf{x} = (h, b, c, d, e, f, g, r, q, u)$ has degrees $(1, 1, 1, 1, 1, 1, 1, 2, 2, 3)$. This choice gives the binomial mirror of Subsection 8.2. For its exchange polynomials, write

$$\begin{aligned}
A_0 &= g\{b(a_0f + a_1a_{14} + a_{15}h) + a_{11}a_{16}h\} + a_{17}b(a_4f + r),\\
A_1 &= g\{b(a_0f + a_1a_{14} + a_{15}h) + h(a_{11}a_{16} + a_{12}d)\} + a_{17}a_4bf,\\
A_2 &= a_3f + dg,\\
A_3 &= a_1a_6c + a_2ce + a_4a_9h + a_5ch + a_8gh.
\end{aligned}$$

Define

$$F_h = a_1bc + u, \qquad F_b = a_{11}h + q, \qquad F_g = a_4f + r$$

and

$$\begin{aligned}
F_c &= a_9hF_g + a_8fgh + gu,\\
F_d &= A_0 + a_{16}gq,\\
F_e &= a_6qF_h + h(a_6a_{11} + a_7b)(br + u) + bhq(a_4a_9 + a_5c + a_8g),\\
F_f &= a_{14}gF_h + h\{cg(a_{11}a_{16} + a_{12}d + a_{15}b) + r(a_{13}c + a_{14}a_9)\},\\
F_r &= cA_2(eA_1 + dgu) + dgq(fA_3 + gu) + a_{16}a_3cefgq,\\
F_q &= eA_1 + br(a_{17}e + dg) + dgu,\\
F_u &= F_r|_{u=0} + r\{dgh[f(a_{10}e + a_{11}a_6 + a_7b) + e(a_{13}c + a_{14}a_9)] + a_{17}a_3bcef\}.
\end{aligned}$$

The exchange Laurent polynomial is $\widehat{F}_\xi = F_\xi/M_\xi$, where

ξ	h	b	c	d	e	f	g	r	q	u
M_ξ	1	1	g	bg	hb	h	1	1	g	dg.

Theorem 4.1. *These functions and exchanges define a rank-ten LP seed Σ_{E_7} over the polynomial coefficient ring S_{E_7}, with*

$$\mathcal{A}(\Sigma_{E_7}) \subseteq \mathcal{U}(\Sigma_{E_7}) = R_7.$$

The LP algebra has infinitely many cluster variables.

Proof. Proposition A.3 obtains this seed from the auxiliary Freudenthal seed by five LP mutations over S_{E_7}. Propositions A.1 and A.2 identify the upper algebra of that mutation class with R_7 and exhibit infinitely many cluster variables. The Laurent phenomenon gives $\mathcal{A}(\Sigma_{E_7}) \subseteq \mathcal{U}(\Sigma_{E_7})$. □

Part 2. Algebraic mirrors and intrinsic comparison

5. Newton reconstruction and intrinsic comparison

For a boundary complement U from Part I, let T be its initial torus, with character lattice M_T and cocharacter lattice $N_T = \mathrm{Hom}(M_T, \mathbb{Z})$. We write $\mathscr{M} = A_{\mathrm{int}}(U)_1$.

For $w \in N_T \otimes_{\mathbb{Z}} \mathbb{R}$ and a nonzero Laurent polynomial $f = \sum_{a \in M_T} c_a z^a$, its support, Newton polytope, and Gauss valuation are

$$\mathrm{supp}(f) = \{a : c_a \neq 0\}, \qquad \mathrm{Newt}(f) = \mathrm{conv}\,\mathrm{supp}(f), \qquad \nu_w(f) = \min_{c_a \neq 0} \langle a, w \rangle.$$

We use the term of least weight as the leading term whenever the minimum is attained at a unique exponent.

The ring $\mathcal{O}(U)$ is a localization of the factorial ring R_X of Lemma 2.4. Away from the cone vertex the cone is a $\mathbb{G}_m$-bundle over X. The inversion removes the vertex, so U is smooth. Thus $\mathrm{Pic}(U) = 0$, and $\mathscr{M}$ is a finitely generated normal domain by [8, Corollary 10.4]. On every dense open torus of $\mathrm{Spec}\,\mathscr{M}$ the standard logarithmic form extends to a generator of the dualizing sheaf [8, Proposition 9.1]. Its order at every prime divisor is zero. Here and below, the order of a logarithmic volume form at a divisor means its order as a rational top-degree form.

We use the *intrinsic Laurent expansion* of [8, Lemma A.4 and Proposition A.5]. In a marked torus whose standard logarithmic volume form agrees with that of U up to a nonzero scalar, suppose the mutable coordinates are regular on U and the frozen coordinates are units. These conditions refer to the original seed characters, before the change to Newton coordinates. Counting disks with one input gives an injective algebra map after setting $q^\beta = 1$, under the finiteness condition below. The trace of a product is its coefficient at ϑ_0. By multiplicativity of the Laurent expansion and the constant-term formula of [8, Lemma A.4 and Proposition A.5], it equals the constant term of the product of the expansions. On the prepared compactification of [8, §A.1], the expansions are Laurent polynomials in a chamber where a regular function has a minimizing Newton vertex negative in every mutable coordinate. In each family we exhibit the chamber and use regular source functions to bound the support before computing the coefficients.

All character changes include the frozen directions. We take the differences of exponents in the Laurent supports and choose a basis of the saturation of their span in M_T. We extend this basis to a basis of M_T. The dual coordinates are $z_1, \dots, z_r, \kappa_1, \dots, \kappa_c$; the coefficient ring is $\mathscr{S} = \mathbb{C}[\kappa_1^{\pm 1}, \dots, \kappa_c^{\pm 1}]$. If $U = V \times T'$ with T' a split torus of cocharacter lattice $N_{T'}$, the product formula [8, Lemma 3.7] gives

$$A_{\mathrm{int}}(V \times T')_1 \simeq A_{\mathrm{int}}(V)_1 \otimes_{\mathbb{C}} \mathbb{C}[N_{T'}].$$

It sends the theta function of a pair (m, n) to $\vartheta_m \otimes z^n$. This formula restores the torus factors after the comparison in the remaining coordinates.

5.1. **Support bounds and coefficient recognition.** Suppose there are s coordinate divisors on U outside its torus. The *contact map* $\mathsf{N} : \mathbb{Z}^s \to N_T$ sends the basis vector for each divisor to the cocharacter given by the orders of vanishing of torus characters along it. In our applications N is injective. Thus the possible corrections form the pointed monoid $\mathsf{N}\mathbb{Z}^s_{\geq 0}$. To compare monomials in a marked theta expansion, fix a linear functional equal to one on the columns of N and break ties by a lexicographic order in a lattice basis. We transport this order under each change of characters. Every nonzero correction increases it, so the straight term is the unique least term [8, Propositions A.2–A.3]. For a label $m \in N_T$ and regular functions ψ_j with minimizing exponents h_j in the chosen chamber, set

$$\delta_j(m) = \langle h_j, m\rangle - \min_{a \in \operatorname{supp} \psi_j} \langle a, m\rangle.$$

Let H have rows h_j, and let $\delta(m)$ have entries $\delta_j(m)$. For a vector d of contact multiplicities, the slope inequalities for the regular functions ψ_j [8, Lemma A.1] give

$$d \geq 0, \qquad H\mathsf{N}d \geq -\delta(m).$$

If a nonnegative row vector λ satisfies $\lambda H\mathsf{N} \leq -\epsilon\mathbf{1}$ for some $\epsilon > 0$, then $\sum_i d_i \leq \lambda\delta(m)/\epsilon$. Conversely, if $d \geq 0$ and $H\mathsf{N}d \geq 0$ force $d = 0$, Farkas' lemma gives such a vector λ over $\mathbb{Q}$.

Lemma 5.1. *Suppose the contact map satisfies* [12, Assumption 20.27] *for* $\mathsf{N}\mathbb{Z}^s_{\geq 0}$. *Fix the incoming direction of an infinitesimal cylinder, or fix the label and generic evaluation point of a local theta expansion. Only finitely many numerical curve classes contribute with contact vector in a given finite set.*

Proof. Choose k larger than the total degree of every allowed contact vector. Since N is injective, the corresponding monomials survive modulo the kth power of the ideal generated by nonzero contact monomials. Keel–Yu prove finiteness in this truncation for infinitesimal cylinders and for local theta expansions in [12, Lemmas 20.29 and 20.31], respectively. □

Given a finite set of possible exponents, we impose the divisibility conditions (2.1) on the unknown coefficients [8, Lemma B.1 and §B.5.1]. Once we know the theta function is Laurent in the adjacent charts, its coefficients satisfy these linear equations. If the solution with leading coefficient one is unique, it gives the theta function. The ancillary files give the matrices and nonzero maximal minors for the exceptional generators and boundary functions.

5.2. **Contact equations in toric blowup charts.** Let U be smooth and contain a torus T with character lattice M_T and cocharacter lattice N_T. Suppose an open part of U is obtained from toric ray charts by blowing up centres $F_a = 0$ in their boundary divisors and removing the strict transforms of those divisors. We index the resulting coordinate divisors by $a \in I$ and write $n_a \in N_T$ for their primitive cocharacters, so $\langle \chi, n_a\rangle$ is the order of the character χ along divisor a. Near a generic contact the chart has the form $x_a w_a = F_a$ with the other torus characters invertible and F_a independent of x_a. Distinct divisors may have the same centre equation. These coordinate divisors lie inside U.

Use the open set on which the chosen compactification and the toric blowup charts agree. We describe the maps whose images lie in this set away from the two marked ends. We use *spine* and *toric tail* in the sense of [12, §§4,9]: the spine records the marked tropical curve, and the entire punctured tail beyond an end cut must map to the torus. A curve is *skeletal* if it satisfies the equivalent conditions of [12, Theorem 8.18 and Definition 8.19]; in particular, the convex hull of its marked points maps into the essential skeleton.

Proposition 5.2. *Fix two nonzero end directions, a contact vector* $(b_a)_{a\in I}$, *and a geometric evaluation point* c *over a generic monomial point of* T. *Suppose the relevant counted maps have stable domains with three distinguished points, meet the coordinate divisors transversely at distinct points, and avoid the complement of the toric blowup charts away from their marked ends. Normalize the ends and evaluation to* $0, \infty, 1$ *and write* $m \in N_T$ *for the incoming*

direction. Write $t_{a,1},\dots,t_{a,b_a}$ for the contact points on the normalized domain. Along each such map f, the characters satisfy

$$\chi(f(z)) = \chi(c)z^{\langle \chi,m\rangle}\prod_{a\in I}\prod_{\alpha=1}^{b_a}\left(\frac{z-t_{a,\alpha}}{1-t_{a,\alpha}}\right)^{\langle \chi,n_a\rangle}, \qquad \chi\in M_T. \tag{5.1}$$

The contact parameters satisfy the centre equations $F_a=0$ after substitution, and the outgoing direction is $-m-\sum_a b_a n_a$. Suppose some linear functional is strictly positive on every contact direction that occurs. For a two-ended expansion with the outer cut at the Gauss point of the unit disk, all contacts satisfy $0<|t_{a,\alpha}|<1$. If there is also a finite inner cut, they retract strictly between the cuts.

Conversely, consider solutions of these equations with distinct contact parameters in the prescribed annuli. Assume that the interpolated maps lie in the common toric blowup model, have the prescribed spine, and satisfy the open-stratum and toric-tail conditions at both ends. On this locus, the incidence scheme and the geometric evaluation fibre agree. An isolated solution at which evaluation is étale contributes one. Parameters of the same divisor type are unordered.

Proof. Work over a completed algebraic closure of the completed residue field of the chosen monomial point, the analytic point defined by the Gauss valuation. The evaluation fixes the value $\chi(c)$ of every character. The smooth, stable-domain assertion of [12, Lemmas 8.20–8.21] allows the normalization to a three-pointed $\mathbb{P}^1$ in the stated locus. The divisor of each character is determined by the two ends and the contacts. A rational function with that divisor is unique up to a constant, which evaluation at 1 fixes. This proves (5.1) and the balancing formula at infinity. Characters invertible on U have no unmarked zeros or poles.

At a simple contact of type a, the map lifts to $x_a w_a = F_a$ exactly when F_a vanishes there. This is divisibility by a local parameter on the domain. The quotient is the value of w_a; a further condition $w_a\neq 0$, when required, must be checked and is not implied by the centre equation alone. The prescribed leading coefficients at $0,\infty$ give the two marked boundary contacts. The toric-tail condition of [12, Construction 9.3] requires each punctured tail to map to the torus. The leg leading to the evaluation marking at 1 is contracted [12, Lemma 8.22]. Near its attachment point, (5.1) gives the slope along this leg as the sum of the contact vectors with $|1-t_{a,\alpha}|<1$. Applying the positive linear functional shows that this sum can vanish only when it is empty. The outer toric tail then excludes all remaining parameters with $|t_{a,\alpha}|\geq 1$. The inner tail, when present, imposes the strict bound at the other cut.

Conversely, the formulas construct a map to the compactification under the stated hypotheses. Recovering the contacts and interpolating the characters are inverse constructions in families on this open locus. They identify the incidence scheme with the evaluation fibre. An isolated étale point has length one. The three marks fix the parametrization, contacts of the same type are unordered, and each map determines its own curve class. □

To apply the proposition to the full counting fibre, we must also exclude curves outside its stated locus. Once the maps are in the appropriate moduli space, [12, Lemmas 3.9 and 8.21] allow generic evaluation to avoid fixed codimension-two loci and ensure generic transversality. Lemma 5.1 permits one generic choice for the finitely many relevant classes. We check the end-stratum and toric-tail conditions in each family.

5.3. **From local expansions to the full mirror.** Once the primitive walls are identified, consistency gives the adjacent expansions. Their finiteness follows from the next lemma whenever the denominator has nonnegative coefficients and a positive constant term.

Lemma 5.3. *Let f be a nonnegative formal series in finitely many translates of a pointed monoid, and let D be a nonnegative polynomial with positive constant term. If Df is a finite Laurent polynomial, so is f.*

Proof. The positive constant term gives $\operatorname{supp} f \subseteq \operatorname{supp}(Df)$. □

Let $L_0, \dots, L_r$ be the initial and adjacent Laurent rings of the proposed mirror seed. In each family the Laurent expansion map Φ has image $\mathscr{R}$ satisfying

$$\mathscr{S}[x_i, x_i' : 1 \le i \le r] \subseteq \mathscr{R} = \Phi(\mathscr{M}) \subseteq \bigcap_{i=0}^{r} L_i.$$

The intersection is the full upper LP algebra by Theorem 2.1. For the Cayley plane we exhibit its generators in $\mathscr{R}$. For the half-spin and Freudenthal varieties we use [8, Proposition B.8]. Since $\mathscr{R}$ is noetherian and normal, it suffices to show that no height-one prime contains two distinct initial mutable coordinates.

For the latter method, let w be the character-value vector of a normalized divisorial valuation of volume order zero, with $w \neq 0$, and write $m_f(w) = \min_{a \in \operatorname{supp} f} \langle a, w \rangle$. The divisorial valuation formula of [8, Proposition B.9] gives an irreducible Laurent polynomial q in the characters of value zero. More precisely, write $w = dw_0$, with w_0 primitive, and choose a character t with $\langle t, w_0 \rangle = 1$. Write a Laurent polynomial as $\sum_j t^j f_j$ with f_j in the group ring of $\ker w_0$. Then

$$\nu\left(\sum_j t^j f_j\right) = \min_j \{dj + \operatorname{ord}_q(f_j)\}.$$

Consequently, every regular f with $m_f(w) < 0$ satisfies $q^{-m_f(w)} \mid \operatorname{in}_w(f)$. Here $\operatorname{in}_w(f)$ is the sum of the terms of least w-weight. If every irreducible factor of this initial form has multiplicity at most b_f, then $m_f(w) \ge -b_f$.

Corollary 5.4. *Let f be regular with $m_f(w) < 0$. If its initial form is a character times an irreducible Laurent polynomial of multiplicity one, then w is primitive, $m_f(w) = -1$, and $\nu(f) = 0$. For this valuation, all such negative faces define the same hypersurface in the torus with character lattice* $\ker w_0$.

Proof. The divisibility forces $-m_f(w) = 1$. Since $m_f(w)$ is divisible by d, we have $d = 1$. The preceding formula for ν gives value zero on the lowest power of t and nonnegative value on every other term. The irreducible factor at the lowest power is associated to q. □

6. Algebraic mirrors of the half-spin varieties

Fix $n \ge 4$ and set

$$m = n - 1, \qquad N = \binom{n}{2} + 1, \qquad r_0 = \binom{n-2}{2}, \qquad r = \frac{n(n-3)}{2}.$$

We use the interval seed of Section 3, writing $h = q_\varnothing$, $X_{ij} = q_{[i,j]}$, and $X_{i,i-1} = h$. Let

$$\mathcal{M} = \{(i,j) : 2 \le i \le j \le m-1\}, \qquad \mathcal{B}_n = \{h, X_{11}, \dots, X_{1m}, X_{2m}, \dots, X_{mm}\}.$$

The interval seed has N coordinates, of which the r_0 indexed by $\mathcal{M}$ are mutable. Its mirror has r mutable coordinates, $n+1$ coefficient units, and $2n-2$ boundary functions indexed by $\mathcal{B}_n$.

6.1. **An integral system of Newton coordinates.** On the interval torus, define the following characters:

$$\chi_{B_i} = \frac{X_{i+2,i+2}X_{i,i+1}}{X_{ii}X_{i+1,i+2}} \qquad (1 \le i \le m-3), \tag{6.1}$$

$$\chi_{E_i} = \frac{X_{i,m-2}X_{i-1,m}}{X_{i-1,m-2}X_{im}} \qquad (2 \le i \le m-2),$$

$$\chi_{R_{ij}} = \frac{X_{ij}X_{i-1,j+1}X_{i+1,j+2}}{X_{i+1,j+1}X_{i-1,j}X_{i,j+2}} \qquad (2 \le i \le j \le m-2),$$

$$\chi_P = \frac{hX_{m-2,m}}{X_{m-2,m-2}X_{m-1,m}}, \qquad \chi_Q = \frac{X_{m-2,m-1}X_{mm}}{X_{m-2,m-2}X_{m-1,m}}. \tag{6.2}$$

To these r characters we adjoin

$$g_0 = \begin{cases} X_{1,m-2}, & n \text{ odd}, \\ X_{2,m-1}, & n \text{ even}, \end{cases} \qquad (g_1, \dots, g_n) = (X_{1,m-1}, X_{1m}, X_{2m}, \dots, X_{mm}). \tag{6.3}$$

Here ρ ranges over the labels B_i, E_i, R_{ij}, P, Q occurring above. Let C be the matrix whose rows are the exponents of (χ_ρ, g_j) in the interval coordinates, with the mutable coordinates first. For an interval coordinate a, the column Ce_a records its exponent in each of the new characters.

Proposition 6.1. *The characters in (6.1)–(6.3) form an integral basis. The χ_ρ generate the saturated support-difference lattice of the spinor-coordinate expansions. Its rank is $r = N - (n+1)$.*

Proof. Work in the quotient of the character lattice by the subgroup generated by the χ_ρ. Write x_{ij} for the additive class of X_{ij}, and let $P_j = x_{1j}$ and $S_i = x_{im}$, so $P_m = S_1$. For $m \geq 4$, set

$$\Delta_{ij} = x_{ij} + x_{i-1,j+1} - x_{i-1,j} - x_{i,j+1}.$$

The R relations say $\Delta_{ij} = \Delta_{i+1,j+1}$, and the E relations say $\Delta_{i,m-2} + \Delta_{i,m-1} = 0$. Hence, with $s = \Delta_{22}$, $\Delta_{ij} = (-1)^{j-i}s$. Let $\mathbf{1}_{\mathcal{P}}$ be 1 if the condition $\mathcal{P}$ holds and 0 otherwise. Set $A_i = \mathbf{1}_{i\,\text{even}}$ and $D_j = \sum_{t=j}^{m-1}(-1)^t$. Summing the differences gives

$$x_{ij} = P_j + S_i - S_1 + sA_iD_j \qquad (2 \leq i \leq j \leq m).$$

The remaining B relations become

$$P_{i+1} - P_i = S_{i+1} - S_{i+2} + \mathbf{1}_{m\,\text{even}}(-1)^i s.$$

Writing $[h]$ for the class of h, the Q and P relations give

$$P_{m-1} - P_{m-2} + S_m - S_{m-1} - \mathbf{1}_{m\,\text{even}}s = 0, \qquad [h] = P_{m-2} + S_{m-1} - S_1.$$

For m even, s is determined integrally by P_{m-2}, P_{m-1} and the S_i. For m odd, $P_{m-2} = P_{m-1} + S_m - S_{m-1}$, and $s = x_{2,m-1} - P_{m-1} - S_2 + S_1$. In either parity the classes of (6.3) therefore generate the quotient over $\mathbb{Z}$. Together with the χ_ρ they give N generators of a lattice of rank N, hence a basis. For $m = 3$, (6.3) already contains all interval coordinates except X_{11} and h. The characters χ_Q and χ_P in (6.2) recover these in turn.

Each χ_ρ has homogeneous half-spin weight zero. The interval coordinates have weights $(1, \mathbf{1}_{1\in I}, \dots, \mathbf{1}_{m\in I}, |I| \bmod 2)$, whose rank is $m + 2 = n + 1$. Since the lattice generated by the χ_ρ is saturated and has complementary rank, it is the kernel of this grading. By Lemma B.1, every χ_ρ occurs alongside the constant term in a normalized complementary-interval coordinate. Its exponent is therefore a support difference. These differences generate the kernel. □

Write

$$z = (B_i, E_i, R_{ij}, P, Q; \kappa_0, \dots, \kappa_n)$$

for the dual torus coordinates and let $\mathscr{S}_n = \mathbb{C}[\kappa_0^{\pm 1}, \dots, \kappa_n^{\pm 1}]$. We use Z_ρ for any of the mutable mirror coordinates B_i, E_i, R_{ij}, P, Q. The corresponding source character is χ_ρ. An original cocharacter v gives the mirror monomial z^{Cv}.

6.2. **The mirror seed.** For $a=(i,j)\in\mathcal{M}$, set $c_a=Ce_a$ and define

$$\mathcal{E}_a = z^{[c_a]_+} + z^{[-c_a]_+}. \tag{6.4}$$

Here $[c]_+$ is the coordinatewise positive part of c. The exchange polynomials are

$$F^\vee_{B_i}=\mathcal{E}_{i+1,i+1}, \qquad F^\vee_{R_{ij}}=\mathcal{E}_{i,j+1},$$
$$F^\vee_{E_i}=\mathcal{E}_{i,m-1}, \qquad F^\vee_P=F^\vee_Q=\mathcal{E}_{m-1,m-1}.$$

For each a, let $\mathcal{I}_a$ be the set of positions assigned to $\mathcal{E}_a$. The sets with more than one element are

$$\{P,Q\}, \qquad \{E_i,R_{i,m-2}\} \quad (2\le i\le m-2). \tag{6.5}$$

For $\rho\in\mathcal{I}_a$, the exchange Laurent polynomial and the new variable are

$$\widehat{F}^\vee_\rho=\frac{\mathcal{E}_a}{\prod_{\sigma\in\mathcal{I}_a\setminus\{\rho\}}Z_\sigma}, \qquad Z'_\rho=Y_a:=\frac{\mathcal{E}_a}{\prod_{\sigma\in\mathcal{I}_a}Z_\sigma}. \tag{6.6}$$

Thus the r mutations of the initial seed give r_0 distinct new variables.

Proposition 6.2. *Equations (6.4)–(6.6) define an LP seed $\Sigma^\vee_n$ over $\mathscr{S}_n$. This seed is obtained by Newton reconstruction from the interval-Pfaffian seed. Its mutable rank is $r=n(n-3)/2$.*

Proof. Each c_a is primitive because C is unimodular. Hence $1+z^{c_a}$ is irreducible in the full Laurent ring, and clearing its coprime monomial denominator gives the irreducible polynomial $\mathcal{E}_a$. This polynomial is independent of its assigned variable since $C_{\rho,a}=0$ for $\rho\in\mathcal{I}_a$. Distinct columns give nonassociate binomials. Setting a variable to zero in another binomial leaves either a monomial or that binomial itself. Neither is divisible by $\mathcal{E}_a$. For $\rho\in\mathcal{I}_a$, normalization removes one factor Z_σ for each $\sigma\in\mathcal{I}_a\setminus\{\rho\}$. This proves (6.6).

For the Newton reconstruction, the complementary-interval functions have mutable leading gradients $-e_a$. At the adjacent label

$$\mu_a=-\sum_{\rho\in\mathcal{I}_a}e_\rho+[-c_a]_+, \tag{6.7}$$

Lemma B.4 shows that the regular-function inequalities allow exactly the two exponents μ_a and μ_a+c_a. Their normalized binomial is $z^{\mu_a}(1+z^{c_a})=\mathcal{E}_a/\prod_{\rho\in\mathcal{I}_a}Z_\rho$, as in (6.6). Proposition B.5 identifies this binomial with the intrinsic theta function. □

For odd n, the exchanges contain no coefficient variables. For even n, only κ_0 occurs, in the exchange assigned to $(2,m-1)$. For $n=4$ the two mutable variables have the same exchange and the same mutation:

$$F^\vee_P=F^\vee_Q=1+\kappa_0, \qquad P'=Q'=\frac{1+\kappa_0}{PQ}.$$

The polynomial $1+\kappa_0$ is irreducible and noninvertible in $\mathscr{S}_4$. For D_5, the five mutable variables are B_1,E_2,R_{22},P,Q, with

$$F^\vee_{B_1}=E_2R_{22}+PQ,$$
$$F^\vee_{E_2}=F^\vee_{R_{22}}=B_1+Q,$$
$$F^\vee_P=F^\vee_Q=B_1+R_{22}.$$

6.3. **Boundary functions.** Let e_h,e_{ij} denote the standard basis indexed by the interval coordinates. Define $W_h=z^{Ce_h}$ and $W_{1m}=z^{Ce_{1m}}$. For the other frozen coordinates, set

$$W_{1k}=\sum_{t=0}^{m-k-1}z^{C(e_{1k}+\sum_{j=1}^{t}e_{j+1,j+k})} \qquad (1\le k<m), \tag{6.8}$$

$$W_{im}=\sum_{t=0}^{m-i}z^{C(e_{im}+\sum_{j=1}^{t}e_{i,m-j})} \qquad (2\le i\le m). \tag{6.9}$$

For each of the $2m$ frozen coordinates b, the initial label of W_b is Ce_b.

Proposition 6.3. *Every W_b belongs to $\mathcal{U}(\Sigma_n^\vee)$. Their sum has $m(m-1)+2$ distinct Laurent monomials, each with coefficient one.*

Proof. Appendix B.6 shows that whenever a mutable coordinate has negative exponent along either chain, its exponent is -1 in two consecutive terms. Their ratio is z^{c_a} for the corresponding exchange, so the coefficient of degree -1 is divisible by $\mathcal{E}_a$. By (2.1) and the starfish theorem, every W_b lies in the upper algebra. In (6.11) the terms are indexed by the distinct horizontal and vertical edges of the triangular array and its two ends, giving $2\binom{m}{2}+2$ monomials. □

6.4. **The triangular potential and its Newton polytope.** Let $\mathbf{1} = e_h + \sum_{i\le j} e_{ij}$ and set $\mathfrak{q} = z^{C\mathbf{1}} = \prod_{j=0}^{n} \kappa_j$. For $1 \le a \le b \le m$, let $i = m+1-b$ and define

$$v_{ab} = e_h + \sum_{r=i+1}^{m}\sum_{s=r}^{m} e_{rs} + \sum_{r=1}^{i}\sum_{\ell=0}^{a-2} e_{r,r+\ell}, \qquad y_{ab} = z^{Cv_{ab}}. \tag{6.10}$$

Empty sums are zero. The characters $\mathfrak{q}, y_{ab}$ form an integral basis by Lemma B.6. In this basis,

$$\begin{aligned} W_h &= y_{11}, & W_{1m} &= \mathfrak{q}/y_{mm}, \\ H_k := W_{1k} &= \sum_{b=k+1}^{m} \frac{y_{k+1,b}}{y_{kb}} & &(1 \le k < m), \\ V_i := W_{im} &= \sum_{a=1}^{m+1-i} \frac{y_{a,m+2-i}}{y_{a,m+1-i}} & &(2 \le i \le m). \end{aligned} \tag{6.11}$$

Consequently the potential is

$$W = y_{11} + \sum_{1\le a<b\le m} \frac{y_{a+1,b}}{y_{ab}} + \sum_{1\le a\le b<m} \frac{y_{a,b+1}}{y_{ab}} + \frac{\mathfrak{q}}{y_{mm}}. \tag{6.12}$$

Fix $\mathfrak{q} \neq 0$ and let Δ_m be the Newton polytope of W in the y_{ab}. Give the triangular set $P_m = \{(a,b) : 1 \le a \le b \le m\}$ the componentwise order. Its *order polytope* $\mathcal{O}(P_m)$ consists of the order-preserving maps $P_m \to [0,1]$. With $\mathbf{h} = (a+b-1)_{(a,b)\in P_m}$, we have

$$\Delta_m^\circ = 2m\,\mathcal{O}(P_m) - \mathbf{h},$$

where Δ_m° denotes the polar with convention $\langle v, u\rangle \ge -1$. Indeed, the vertices defining Δ_m are

$$e_{11}, \quad -e_{mm}, \quad e_{a+1,b} - e_{ab}\ (a<b), \quad e_{a,b+1} - e_{ab}\ (b<m).$$

Substituting $u = 2m\lambda - \mathbf{h}$ turns the polar inequalities into $0 \le \lambda_{ab} \le 1$ and the order inequalities. The order polytope is integral [25, Corollary 1.3], and $\mathbf{h}/(2m)$ lies in its interior. Thus Δ_m is reflexive. The scale $2m = 2n-2$ is the Fano index of the spinor variety.

7. The intrinsic mirror theorem in type D

Let

$$U_n = \operatorname{Spec} \mathbb{C}[\widehat{\mathrm{OG}^+(n,2n)}][(\prod_{b\in\mathcal{B}_n} b)^{-1}], \qquad \mathcal{U}_n^\vee = \mathcal{U}(\Sigma_n^\vee).$$

We identify $\mathcal{U}_n^\vee$ with the intrinsic mirror algebra of U_n, matching the boundary functions constructed in the preceding section.

Theorem 7.1. *For every $n \ge 4$, local intrinsic expansions induce an isomorphism*

$$\Phi_n : A_{\mathrm{int}}(U_n)_1 \xrightarrow{\sim} \mathcal{U}_n^\vee.$$

It identifies the lattice of coefficient characters and satisfies $\Phi_n(\vartheta_{Ce_b}) = W_b$ for all $b \in \mathcal{B}_n$. The algebra $\mathcal{U}_n^\vee$ is finitely generated, normal, and factorial, with unit group $\mathscr{S}_n^\times$. Its superpotential is (6.12).

We first identify the torus charts and their boundary functions. The Newton polytopes of the boundary functions then show that these charts meet every prime divisor.

7.1. **The local comparison.** Let $\mathscr{M}_n = A_{\mathrm{int}}(U_n)_1$ and let $L = \mathscr{S}_n[Z_\rho^{\pm 1}]$ be the mirror Laurent ring. Recall $Y_a = z^{\mu_a}(1 + z^{c_a})$ from (6.7) and (6.6).

Proposition 7.2. *There is an injective algebra homomorphism*

$$\Phi_n : \mathscr{M}_n \hookrightarrow L, \qquad M_n = \Phi_n(\mathscr{M}_n),$$

with

$$\mathscr{S}_n[Z_\rho, Y_a, W_b] \subseteq M_n \subseteq \mathcal{U}_n^\vee.$$

The adjacent and boundary functions are the following theta functions:

$$\Phi_n(\vartheta_{\mu_a}) = Y_a, \qquad \Phi_n(\vartheta_{Ce_b}) = W_b.$$

The initial torus and each adjacent torus are open charts of $\operatorname{Spec} \mathscr{M}_n$.

Proof. The complementary interval coordinates have mutable leading gradients $-e_t$ for $t \in \mathcal{M}$. They bound each contact multiplicity by (B.7) and exclude nonconstant attached branches in the chosen chamber. The integral Laurent realization of [8, Proposition A.5] therefore gives Φ_n, as verified in Appendix B. The bounds vanish on the initial coordinate labels, so the Z_ρ and the coefficient characters and their inverses belong to its image.

Proposition B.5 identifies each primitive wall crossing with the corresponding LP mutation, proves Laurentness in the adjacent chart, and gives $\Phi_n(\vartheta_{\mu_a}) = Y_a$. The intersection of the initial and adjacent Laurent rings is $\mathcal{U}_n^\vee$ by the starfish theorem, so $M_n \subseteq \mathcal{U}_n^\vee$. For a boundary label Ce_b, adjacent divisibility equates the coefficients on successive prefixes of its interval chain. The term with exponent Ce_b has coefficient one, giving $\Phi_n(\vartheta_{Ce_b}) = W_b$ (Appendix B.5). In each of these charts the image contains all its coordinates. Inverting their product gives the full Laurent ring, as claimed. □

7.2. **Triangular valuations.** Let M be a normal finitely generated subalgebra of L satisfying $\mathscr{S}_n[Z_\rho, Y_a, W_b] \subseteq M \subseteq \mathcal{U}_n^\vee$, and suppose the initial logarithmic form has order zero along every prime divisor of $\operatorname{Spec} M$. For such a divisor E, write

$$w_\rho = \operatorname{ord}_E(Z_\rho) \geq 0, \qquad \lambda_{ab} = \operatorname{ord}_E(y_{ab}).$$

The coefficient units have order zero. The map $w \mapsto \lambda$ is injective. For each diagonal $0 \leq d < m$, its entries satisfy

$$\sum_{a=1}^{m-d} \lambda_{a,a+d} = 0, \qquad \sum_{j=1}^{a} \lambda_{j,j+d} \geq 0 \quad (a < m-d). \tag{7.1}$$

Let ϵ_{ab} be the array with a single 1 at (a, b), and write $T_{a,d} = \sum_{j=1}^{a} \lambda_{j,j+d}$. For $w = e_\rho$ the array λ depends on Z_ρ as follows:

B_i	$\epsilon_{1,m-i} - \epsilon_{2,m-i+1}$
R_{ij}	$\epsilon_{j-i+2,m+1-i} - \epsilon_{j-i+3,m+2-i}$
$E_i,\ a = m-i$	$\epsilon_{a,a} + \epsilon_{a,a+1} - \epsilon_{a+1,a+2} - \epsilon_{a+2,a+2}$
P	$\epsilon_{11} + \epsilon_{12} - \epsilon_{23} - \epsilon_{33}$
Q	$\epsilon_{12} - \epsilon_{23}$.

The diagonal partial sums of each array are nonnegative, and its diagonal totals are zero. Lemma B.7 recovers w from the partial sums.

Diagonal rigidity. In the triangular array, a horizontal arrow increases a by one and a vertical arrow increases b by one. Its value is the entry at its head minus the entry at its tail. The next two lemmas concern real arrays satisfying (7.1).

Lemma 7.3. *If $\lambda \neq 0$ satisfies (7.1), then some horizontal arrow and some vertical arrow have negative value.*

Proof. The outermost diagonal has a single entry and is zero. Suppose first that all horizontal arrows are nonnegative. If the next outer diagonal is zero, the horizontal arrows into the current diagonal force every entry except its first to be nonnegative. The first is nonnegative because $T_{1,d} \geq 0$. Since the total on the diagonal is zero, every entry vanishes. Descending induction makes the whole array zero.

If all vertical arrows are nonnegative, a zero outer diagonal forces every entry of the current diagonal except its last to be nonpositive. The last is also nonpositive, being the negative of a nonnegative proper partial sum. Again the zero total and descending induction make the whole array zero. Both conclusions contradict $\lambda \neq 0$. □

Lemma 7.4. *Let λ satisfy (7.1). Suppose every arrow has nonnegative value except the four directed sides of the square*

$$(k,\ell), \quad (k,\ell+1), \quad (k+1,\ell), \quad (k+1,\ell+1), \qquad 1 \leq k < \ell \leq m-1,$$

and each of these four arrows has value -1. Let $d = \ell - k$. If $d \geq 2$, then $\lambda = \epsilon_{k,\ell} - \epsilon_{k+1,\ell+1}$. If $d = 1$, then for some $t \in [0,1]$,

$$\lambda = \epsilon_{k,k+1} - \epsilon_{k+1,k+2} + t(\epsilon_{k,k} - \epsilon_{k+2,k+2}).$$

If λ is the image of $w \in \mathbb{Z}^r_{\geq 0}$ under the map above, then $w = e_\rho$ for some ρ.

Proof. Write $A^{(s)}_a = \lambda_{a,a+s}$. Whenever the relevant arrows are nonnegative, their inequalities are

$$A^{(s+1)}_{a-1} \leq A^{(s)}_a \leq A^{(s+1)}_a. \tag{7.2}$$

At the first and last positions the missing bounds are respectively $A^{(s)}_1 \geq 0$ and $A^{(s)}_{m-s} \leq 0$, by the partial-sum conditions.

Every diagonal of gap greater than d is zero, by downward induction using (7.2). On diagonal d, the vertical exception from (k,ℓ) to $(k,\ell+1)$ forces the value 1 at (k,ℓ). The horizontal exception from $(k,\ell+1)$ to $(k+1,\ell+1)$ forces the value -1 at $(k+1,\ell+1)$. The bounds on interior and extreme entries force all other entries to vanish.

On diagonal $d-1$, the other two exceptional arrows both force the entry at $(k+1,\ell)$ to be zero. The bounds (7.2) leave only two other possible nonzero entries: the entry at position k lies in $[0,1]$, and that at position $k+2$ lies in $[-1,0]$. The zero total makes these t and $-t$. If $d \geq 2$, the entry at position $k+1$ of diagonal $d-2$ satisfies $t \leq A^{(d-2)}_{k+1} \leq 0$. Thus $t = 0$, and the remaining inner diagonals vanish by downward induction. If $d = 1$ there is no further inner diagonal, giving the stated family.

For an array coming from the seed, equations (B.11)–(B.12) finish the argument. When $d \geq 2$, the only nonzero proper diagonal partial sum is $T_{k,d} = 1$, so exactly one B_i or R_{ij} has order one. When $d = 1$, the only nonzero diagonal-one partial sum is $T_{k,1} = 1$, and the diagonal-zero partial sums are $T_{k,0} = T_{k+1,0} = t$. The recovery formulas give orders t and $1-t$ to the two variables in the corresponding repeated-exchange pair, and order zero to every other initial variable. Since t is integral, exactly one member of the pair has order one. □

Boundary Newton simplices.

Lemma 7.5. *For any positive-dimensional face of the Newton polytope of a boundary function in (6.11), divide its face polynomial by one of its monomials. The result defines a smooth irreducible hypersurface in the torus whose character lattice is generated by the exponent differences on the face. This lattice is saturated in the full character lattice and hence in any intermediate lattice.*

Distinct horizontal boundary functions have disjoint sets of column directions, and so do distinct vertical boundary functions. The sets for H_k and V_i intersect in the single column c_{ij}, where $j = i + k - 1$, when (i, j) is mutable, and are disjoint otherwise. If positive-dimensional faces of two distinct boundary Newton polytopes are translates, each is an edge joining consecutive exponents of its chain. The two chains belong to some H_k and V_i and share the column of that edge.

Proof. In the initial indexing, the successive support differences are

$$H_k: \quad c_{2,k+1}, c_{3,k+2}, \dots, c_{m-k,m-1},$$
$$V_i: \quad c_{i,m-1}, c_{i,m-2}, \dots, c_{i,i}.$$

All are distinct columns of the unimodular matrix C. A cumulative chain built from part of an integral basis is a unimodular simplex: its differences from the first vertex are related to those columns by a triangular integral matrix with diagonal entries one. Every face of this simplex is unimodular in its affine lattice. A nonmonomial face polynomial is therefore, after a Laurent monomial change and a unit factor, $1 + t_1 + \cdots + t_s$ with $s \geq 1$. This is smooth and irreducible on the corresponding torus. Saturation passes to every intermediate lattice because the corresponding quotient embeds in the ambient torsion-free quotient.

The index difference $j - i$ distinguishes the horizontal chains, and the first index i distinguishes the vertical chains. This proves the disjointness assertions and the formula for their unique possible intersection. Because all c_a are part of a basis, the linear spans of two sets of columns intersect in the span of their set-theoretic intersection. Equal nonzero face-direction spaces therefore require one horizontal and one vertical chain and must be the line $\mathbb{R}c_{ij}$. Each face is consequently an edge. A difference between two vertices of a cumulative chain is a sum of consecutive distinct basis columns, so it is parallel to a single column only when the vertices are consecutive at that column. □

Proposition 7.6. *For every prime divisor of* $\operatorname{Spec} M$*, the vector w is zero or a coordinate vector e_ρ.*

Proof. Suppose $w \neq 0$. Then $\lambda \neq 0$, so Lemma 7.3 gives a horizontal boundary function H_k and a vertical one V_i with negative minimum value. The corresponding face polynomial has at least two terms, since the boundary function is regular at the divisor. Each such polynomial defines a reduced irreducible hypersurface by Lemma 7.5. Corollary 5.4 then shows that all these faces have minimum -1 and define the same hypersurface. Their Newton polytopes are translates.

Lemma 7.5 now identifies the faces as consecutive edges of exactly one H_k and one V_i, at their shared column. The four corresponding arrows bound one elementary square and have value -1. All other arrow values are nonnegative by integrality. Lemma 7.4 gives $w = e_\rho$. □

7.3. **Global comparison.**

Proof of Theorem 7.1. Since U_n is smooth and factorial, $\operatorname{Pic}(U_n) = 0$. Its intrinsic mirror algebra is finitely generated and normal by [8, Corollary 10.4]. Proposition 7.2 identifies the initial torus as an open chart of $\operatorname{Spec} M_n$. Its logarithmic volume form has order zero along every prime divisor by [8, Proposition 9.1]. Proposition 7.6 thus shows that two distinct initial mutable coordinates cannot both vanish along a prime divisor. Since M_n contains every first mutation and is Laurent in the initial and adjacent charts, [8, Proposition B.8] gives $M_n = \mathcal{U}_n^\vee$. The local comparison identifies the boundary functions, and Proposition 2.2 gives factoriality and the unit group. □

Remark 7.7. Under $B_n/P_n \simeq D_{n+1}/P_{n+1}$, the signed subset coordinates and prefix–suffix Coxeter boundary agree (Remark 3.3). The mirror theorem, including its boundary functions, therefore transfers by replacing the type-D index with $n+1$. For $B_1/P_1 = \mathbb{P}^1$ and $B_2/P_2 = \mathbb{P}^3$, the complements of the frozen coordinate divisors in the affine cones are tori. Their mirrors are the dual tori with the sums of the boundary characters as potentials.

7.4. **The quadric case D_4.** For $n = 4$ the mirror has two mutable variables P, Q, with common exchange polynomial $1 + \kappa_0$. Set

$$\mathscr{S}_4 = \mathbb{C}[\kappa_0^{\pm 1}, \dots, \kappa_4^{\pm 1}], \qquad R = \frac{1+\kappa_0}{PQ}.$$

The first mutation in either position gives R. Subsequent mutations recover P or Q. Thus P, Q, R generate the LP algebra $\mathcal{A}_4^\vee = \mathcal{A}(\Sigma_4^\vee)$ over $\mathscr{S}_4$.

Proposition 7.8. *There are isomorphisms preserving the boundary functions*

$$A_{\mathrm{int}}(U_4)_1 \simeq \mathcal{A}_4^\vee = \mathcal{U}_4^\vee = \mathscr{S}_4[P, Q, R]/(PQR - 1 - \kappa_0).$$

For $a, b \in \mathbb{Z}$, $k \in \mathbb{Z}^5$ and $d = \max(0, -a, -b)$, the theta function with label $(a, b; k)$ is

$$\Theta_{(a,b;k)} = \kappa^k P^a Q^b (1+\kappa_0)^d = \kappa^k P^{a+d} Q^{b+d} R^d.$$

Proof. The complementary spinor coordinate bounds the contact degree by d. Since the unique contact direction is the κ_0 direction, write the intrinsic expansion as $\kappa^k P^a Q^b f(\kappa_0)$, where $\deg f \le d$ and $f(0) = 1$. Laurentness in the adjacent P chart forces $(1+\kappa_0)^{\max(0,-a)}$ to divide f, and the Q chart forces exponent $\max(0, -b)$. The combined exponent is their maximum, namely d. Hence $f = (1+\kappa_0)^d$.

The monomials $P^i Q^j R^\ell$ with $i, j, \ell \ge 0$ and $\min(i, j, \ell) = 0$ form a free $\mathscr{S}_4$-basis of this quotient: use the monic relation to reduce every factor PQR. Their images are linearly independent in $\mathscr{S}_4[P^{\pm 1}, Q^{\pm 1}]$, since the exponent pair $(i - \ell, j - \ell)$ determines the triple in this normal form. The same divisibility calculation describes the upper LP algebra. This proves the presentation and the asserted equalities. □

The six boundary functions, in Coxeter order, are

$$P, \quad R, \quad \kappa_1 Q, \quad \kappa_2 P, \quad \kappa_3 R, \quad \kappa_4 Q.$$

In particular $W = (1+\kappa_2)P + (\kappa_1 + \kappa_4)Q + (1+\kappa_3)R$.

8. Exceptional mirrors

8.1. **The Cayley-plane mirror.** Let R_6 be the Cayley-plane coordinate ring, with the twelve frozen coordinates of Section 4. Set

$$U_{\mathrm{C}} = \operatorname{Spec} R_6[(a_1 \cdots a_{12})^{-1}], \qquad \mathscr{S}_{E_6} = \mathbb{C}[\zeta_6^{\pm 1}, \dots, \zeta_{12}^{\pm 1}].$$

For the Newton construction we choose the chart

$$(x_1, x_7, z_3, x_4, t_1), \qquad t_1 = x_1 x_4 x_7 x_{10} - a_3 a_6 a_9 a_{12}.$$

The chart belongs to family J of Daisey and Ducat [5, Theorem 5.4]. Appendix C.1 gives the three mutations relating it to their representative. The integral Newton characters in Appendix C determine the following rank-ten mirror seed $\Sigma_{E_6}^\vee$.

8.1.1. *The seed and affine model.* The mutable variables are $(s, u, v, p_1, \dots, p_7)$. Define

$$P = \prod_{i=1}^{7} p_i, \qquad \Pi = p_1 p_2 p_3, \qquad T = p_5 p_6 p_7, \qquad Q = p_4 T,$$
$$R = p_2 p_3 p_6 p_7, \qquad L = p_3 T, \qquad H = p_1 p_2 p_4 p_7,$$
$$E = (1+R)(1+Q) + uvL(1+H).$$

The exchange polynomials F_x and exchange Laurent polynomials $\widehat{F}_x$ are

x	F_x	$\widehat{F}_x$
s	E	E/u
u	$1+Q$	$1+Q$
v	$1+s+Q$	$(1+s+Q)/Q$
p_i	$1+s$	$(1+s)/(P/p_i)$.

(8.1)

Mutating once gives four distinct variables:

$$f = \frac{E}{su}, \qquad \bar{u} = \frac{1+Q}{u}, \qquad \bar{v} = \frac{1+s+Q}{vQ}, \qquad k = \frac{1+s}{P}. \tag{8.2}$$

The mutation word (u, v, s) gives one further function,

$$g = \frac{E + s(1+R)}{svT} = \frac{p_4(1+R)\bar{v} + p_3 u(1+H)}{s}. \tag{8.3}$$

Define

$$\mathscr{B}_6 = \frac{\mathbb{C}[p_1, \dots, p_7, k, u, \bar{u}, v, \bar{v}, f, g]}{\left(\begin{array}{l} u\bar{u} - 1 - Q, \\ v\bar{v} - 1 - \Pi k, \\ Tvg - uf - 1 - R, \\ p_4 \bar{v} f - \bar{u} g - p_3(1+H) \end{array} \right)}. \tag{8.4}$$

In this ring, $s = Pk - 1 = Qv\bar{v} - u\bar{u}$.

Proposition 8.1. *The ring $\mathscr{B}_6$ is a smooth integral ten-dimensional complete intersection. The formulas above identify it with the ordinary and upper LP algebras of the coefficient-free seed (8.1). Restoring the seven coefficient characters gives*

$$\mathcal{A}(\Sigma_{E_6}^{\vee}) = \mathcal{U}(\Sigma_{E_6}^{\vee}) = \mathscr{S}_{E_6} \otimes_{\mathbb{C}} \mathscr{B}_6.$$

The initial and adjacent tori meet every prime divisor.

Proof. The affine calculations in Appendix C.3 cover the model by $D(s)$ and $D(Pk)$, since $Pk - s = 1$. On $D(s)$ the last two equations eliminate f, g with determinant s. On $D(Pk)$ the unimodular change of characters eliminates $Q, \Pi k, R, H$. These are smooth integral charts of dimension ten. The same calculation proves that $s, u, v, p_1, \dots, p_7$ are distinct prime elements and that along the zero divisor of each, its mutation partner is generically nonzero. By normality, the ring is the intersection of the Laurent rings of these eleven charts. The starfish theorem identifies this intersection with the upper LP algebra. All fourteen generators occur under mutation, so the generated and upper LP algebras coincide as well. □

8.1.2. *The twelve boundary functions.* The boundary functions have the following polynomial representatives:

$$\begin{aligned} W_1 &= k, & W_2 &= p_4, \\ W_3 &= p_1 p_2 p_4 p_5 k f - p_2 \bar{u} - p_5 v(1+H), & W_4 &= p_1, \\ W_5 &= p_7, & W_6 &= \zeta_6 g, \\ W_7 &= \zeta_7 p_2, & W_8 &= \zeta_8 p_6, \\ W_9 &= \zeta_9(kf - p_7 v), & W_{10} &= \zeta_{10} p_3, \\ W_{11} &= \zeta_{11} p_5, & W_{12} &= \zeta_{12}(p_1 p_5 p_6 k g - p_6 \bar{v} - p_1 u). \end{aligned}$$

Although these representatives contain minus signs, their initial Laurent expansions have positive coefficients. For the three representatives with subtractions, they are

$$\begin{aligned} W_3 &= \frac{E + s(1+Q)}{su\, p_3 p_6 p_7}, \\ W_9 &= \zeta_9 \frac{E + s((1+R)(1+Q) + uvL)}{suP}, \\ W_{12} &= \zeta_{12} \frac{E + s(2+Q+R+uvL) + s^2}{sv\, p_2 p_3 p_4 p_5 p_6 p_7^2}. \end{aligned}$$

They follow from (8.4) and $Pk = 1 + s$. The superpotential is $W = \sum_{i=1}^{12} W_i$.

8.1.3. *Intrinsic identification.*

Theorem 8.2. *Local theta expansions induce an isomorphism*

$$A_{\mathrm{int}}(U_{\mathrm{C}})_1 \xrightarrow{\sim} \mathscr{S}_{E_6} \otimes_{\mathbb{C}} \mathscr{B}_6.$$

The theta function indexed by the divisor $a_i = 0$ maps to W_i for each $1 \le i \le 12$.

Proof. The integral change of characters splits off a seven-dimensional torus, giving $U_{\mathrm{C}} = X \times \mathbb{G}_m^7$. For X, the four wall calculations in Appendix C.4 identify the adjacent charts. Equation (C.7) embeds the intrinsic algebra in the intersection of their Laurent rings with the initial Laurent ring, which is $\mathscr{B}_6$ by Proposition 8.1. By Proposition C.1, its image contains $f, \bar{u}, \bar{v}, k, g$. Together with the initial coordinates these generate $\mathscr{B}_6$, proving equality. The same proposition identifies the three remaining boundary functions W_3, W_9, W_{12} after removing their coefficient units. Restoring the torus factor by [8, Lemma 3.7] gives the asserted isomorphism with all twelve boundary labels. □

8.2. **The Freudenthal mirror.** Let

$$U_{\mathrm{F}} = \operatorname{Spec} R_7[(a_0 \cdots a_{17})^{-1}], \qquad \mathscr{S}_{E_7} = \mathbb{C}[\zeta_0^{\pm 1}, \dots, \zeta_7^{\pm 1}].$$

For the seed of Theorem 4.1, the twenty characters χ_i in Appendix D define the chamber $\mathrm{val}(\chi_i) > 0$. The dual mutable coordinates are $v_1, \dots, v_{20}$. The remaining eight dual characters are the coefficient variables $\zeta_0, \dots, \zeta_7$.

8.2.1. *The seed and boundary functions.* Let $I_1 = \{1, 5, 15, 18\}$, $I_2 = \{6, 7, 12, 13, 20\}$, $I_3 = \{11, 14\}$, and $I_4 = \{16, 17, 19\}$. The ordinary exchanges are

i	E_i
I_1	$v_{11} + v_3$
I_2	$v_2 + v_8 v_9$
I_3	$v_1 v_4 + v_{19} v_3$
I_4	$v_{14} v_2 + v_{20} v_4$
2	$v_{10} v_{16} v_{17} v_9 + v_{12} v_{13} v_{20} v_4 v_6$
3	$v_1 v_{15} v_{17} v_{18} v_5 + v_{11} v_{14}$
4	$v_{10} v_{15} v_{16} v_{17} v_{19} + v_{11} v_{14} v_2$
8	$v_{13} v_{20} + v_{17}$
9	$v_{12} v_{13} v_{19} v_6 v_7 + v_2$
10	$v_{12} v_4 + v_2 v_5$.

For $i \in I_\ell$ the exchange Laurent polynomial is $\widehat{E}_i = E_i / \prod_{j \in I_\ell \setminus \{i\}} v_j$. For $i = 2, 3, 4, 8, 9, 10$, it equals E_i. These formulas define the mirror LP seed $\Sigma_{E_7}^\vee$.

For the boundary functions we use the following mutation partners:

$$\eta_1 = \frac{v_{11} + v_3}{v_1 v_5 v_{15} v_{18}}, \quad \eta_2 = \frac{v_2 + v_8 v_9}{v_6 v_7 v_{12} v_{13} v_{20}},$$
$$\eta_3 = \frac{v_1 v_4 + v_{19} v_3}{v_{11} v_{14}}, \quad \eta_4 = \frac{v_{14} v_2 + v_{20} v_4}{v_{16} v_{17} v_{19}},$$
$$\tau_8 = \frac{v_{13} v_{20} + v_{17}}{v_8}, \quad \tau_9 = \frac{v_{12} v_{13} v_{19} v_6 v_7 + v_2}{v_9}, \quad \tau_{10} = \frac{v_{12} v_4 + v_2 v_5}{v_{10}}.$$

Write $W_j = \epsilon_j \overline{W}_j$, where $\epsilon_j = \zeta_j$ for $0 \le j < 8$ and $\epsilon_j = 1$ otherwise. The eighteen functions are

$$\begin{aligned}
&\overline{W}_0 = v_{15}, && \overline{W}_1 = v_5, && \overline{W}_2 = v_{18},\\
&\overline{W}_3 = \frac{v_{17} + v_{14}\eta_1}{v_3}, && \overline{W}_4 = \eta_3, && \overline{W}_5 = \frac{v_{10}v_{15} + v_{11}\eta_4}{v_4},\\
&\overline{W}_7 = v_{13}, && \overline{W}_9 = v_7, && \overline{W}_{10} = v_{12},\\
&\overline{W}_{11} = v_6, && \overline{W}_{12} = \tau_8, && \overline{W}_{13} = \eta_2,\\
&\overline{W}_{14} = \frac{v_{10}v_7 + \eta_4\tau_9}{v_2}, && \overline{W}_{16} = \eta_1, && \overline{W}_{17} = v_1,
\end{aligned}$$

with

$$\overline{W}_6 = \frac{v_{16}\tau_8}{v_{13}v_{20}} + \frac{v_{16}v_{17}v_9}{v_{13}v_2v_{20}} + \frac{v_6\tau_{10}}{v_2}, \qquad \overline{W}_8 = \frac{v_{19} + v_{20}\eta_2}{v_9},$$
$$\overline{W}_{15} = \frac{v_{16}}{v_3} + \frac{v_{18}\tau_{10}}{v_4} + \frac{v_{15}v_{16}v_{17}v_{18}v_5\eta_3}{v_3v_4}.$$

The superpotential $W = \sum_{j=0}^{17} W_j$ has forty distinct Laurent monomials in the full character lattice, all with coefficient one. The degree character is $\mathfrak{q}_{E_7} = \prod_{j=0}^{7} \zeta_j$.

8.2.2. *Newton reconstruction.*

Proposition 8.3. *The seed $\Sigma_{E_7}^{\vee}$ and the functions W_j are the Newton reconstructions from Σ_{E_7} in the chamber $\mathrm{val}(\chi_i) > 0$. All W_j belong to $\mathcal{U}(\Sigma_{E_7}^{\vee})$. The character matrix is unimodular, and the twenty χ_i span the saturated support-difference lattice.*

Proof. The linear coordinates and the cubic mutation $f^\sharp = F_f/(hf)$ form a family of 57 regular functions. The least monomial of $G = x_{15}^2 x_2 y_{14} x_5 y_{11} x_{10} x_{13} x_0^2 y_1^3$ has exponent $(-1, \dots, -1)$ in the ten mutable coordinates. The regular-function inequalities therefore bound a contact correction $\beta \in \mathbb{Z}_{\ge 0}^{10}$ by $\sum_i \beta_i \le \delta_G(m)$ at label m. Appendix D gives the finite support and adjacent Laurent-divisibility calculations. At each facet the bounds allow only the two exponents of the proposed exchange. Their difference is primitive and the two monomials are coprime, so the binomial is irreducible. Normalization divides by every other variable with the same exchange, once each. No other factor occurs, since setting a variable to zero in a different binomial leaves either a monomial or a nonassociate binomial. Proposition 8.4 determines the wall coefficients by counting the corresponding contacts. For each boundary label, the linear equations for Laurentness in the adjacent charts have a one-dimensional solution space. Taking the least coefficient to be one gives W_j. The starfish theorem [7, Theorem 1.3] then gives $W_j \in \mathcal{U}(\Sigma_{E_7}^{\vee})$. The determinant and support-lattice calculations give the last assertion. □

8.2.3. *Intrinsic comparison.*

Proposition 8.4. *Let $\beta^{(i)}$ be the nonzero contact vector at facet i in §D.1, and let b_i be the sum of its contact directions, counted with multiplicity. In the χ_j-coordinates, b_i is the oriented exponent difference of E_i. At that facet of the chamber $\mathrm{val}(\chi_j) > 0$, the primitive wall function is $1 + v^{b_i}$, where $v^{b_i} = \prod_{j=1}^{20} v_j^{(b_i)_j}$.*

At twelve facets the curve meets just one coordinate divisor, and the centre equation has a simple root after reduction. The other eight facets require five counts with two contacts and three with three contacts. Appendix D.2.2 computes these counts.

Proposition 8.5. *Let ν be a normalized divisorial valuation for which the logarithmic volume form has order zero and $\nu(\zeta_j) = 0$ for all coefficient characters. Suppose $w = (\nu(v_1), \dots, \nu(v_{20}))$ is nonnegative and that every first mutation v_i' and every boundary function W_j has nonnegative valuation. Then $w = 0$ or $w = e_i$ for some i.*

The proof of Proposition 8.5 in Appendix D.2.4 considers all eighty possible irreducible factors of face polynomials of negative weight. For each factor, an exact computation with rational inequalities leaves only zero and the coordinate vectors as possible values of w.

Theorem 8.6. *There is an isomorphism of $\mathscr{S}_{E_7}$-algebras*

$$A_{\mathrm{int}}(U_{\mathrm{F}})_1 \xrightarrow{\sim} \mathcal{U}(\Sigma^{\vee}_{E_7})$$

carrying the theta function indexed by $a_j = 0$ to W_j for every $0 \le j \le 17$. The upper LP algebra is finitely generated, normal, and factorial, and its unit group is $\mathscr{S}^{\times}_{E_7}$. Its superpotential is $W = \sum_{j=0}^{17} W_j$.

Proof. The regular function in Proposition 8.3 has a Newton vertex negative in every mutable direction. It gives the marked Laurent realization of [8, Proposition A.5] in the primary chamber. Proposition 8.4 and §D.2.3 identify the twenty first mutations and all eighteen boundary theta functions. The intrinsic image $\mathscr{R}_{E_7}$ therefore satisfies

$$\mathscr{S}_{E_7}[v_i, v_i', W_j] \subseteq \mathscr{R}_{E_7} \subseteq \mathcal{U}(\Sigma^{\vee}_{E_7}).$$

It is finitely generated and normal, and the logarithmic volume on its initial torus has order zero along every prime divisor [8, Corollary 10.4 and Proposition 9.1]. By Proposition 8.5, two distinct initial mutable coordinates cannot both vanish along such a divisor. The codimension-one criterion [8, Proposition B.8] gives $\mathscr{R}_{E_7} = \mathcal{U}(\Sigma^{\vee}_{E_7})$, and Proposition 2.2 gives factoriality and the unit group. □

9. The integral theta basis

9.1. **Integral descent.** At each mutation choose the primitive integral representative of the exchange polynomial, up to sign, and use the same normalizing monomial as over $\mathbb{C}$. Let $\mathcal{U}^{\vee}_{X,\mathbb{Z}}$ be the upper LP algebra over $\mathbb{Z}[\kappa_j^{\pm 1}]$, with ζ in place of κ in the exceptional cases. Write $A_{\mathrm{int},\mathbb{Z}}(U_X)_1$ for the integral counting algebra at the specialization $q^{\beta} = 1$.

Theorem 9.1. *The isomorphisms of Theorem B restrict to*

$$A_{\mathrm{int},\mathbb{Z}}(U_X)_1 \xrightarrow{\sim} \mathcal{U}^{\vee}_{X,\mathbb{Z}}.$$

The intrinsic basis has nonnegative integral structure constants and nonnegative integral expansions in the marked initial chart. Every theta function is a Laurent polynomial with integer coefficients in every LP seed.

Proof. The structure constants are counts, and the local expansions in the marked chart have nonnegative integer coefficients and distinct monic least terms for the order of §5 [8, Propositions A.3 and A.5]. By Theorem B, each expansion is Laurent over $\mathbb{C}$ in every seed. Suppose it is integral in one seed. For a mutation in direction i, each coefficient of x_i^{-d}, $d > 0$, is divisible over $\mathbb{C}$ by F_i^d in the other Laurent coordinates, by (2.1). The polynomial F_i^d is primitive. Gauss's lemma gives the same divisibility over $\mathbb{Z}$, and the normalizing monomial is a unit. Induction along a mutation path proves integral Laurentness.

Conversely, an element of $\mathcal{U}^{\vee}_{X,\mathbb{Z}}$ has a finite complex theta expansion by Theorem B. In the marked chart its least coefficient is the coefficient of the unique theta summand with that least exponent, hence is an integer. Subtract this integer multiple of the theta function. The remainder is integral and has one fewer summand. Induction gives integral coefficients throughout. □

9.2. **Valuative independence of the intrinsic basis.**

Theorem 9.2. *In each comparison of Theorem B, the intrinsic theta basis is valuatively independent for every real monomial valuation in the marked initial chart and in every*

reachable LP seed. In particular, in any such seed and for any finite family of complex coefficients c_m, not all zero,

$$\mathrm{Newt}\left(\sum_m c_m\Theta_m\right) = \mathrm{conv}\bigcup_{c_m\neq 0}\mathrm{Newt}(\Theta_m).$$

Every regular extended cluster monomial equals an intrinsic theta function up to sign. Of the two signed representatives, the theta function is the one whose least term in the marked initial chart has positive coefficient.

Proof. In each family the marked chart comes from a regular function with a Newton vertex negative in every mutable coordinate. Thus [8, Theorem 4.10] gives valuative independence there. The full lattice change is unimodular, and LP mutation preserves the logarithmic volume up to sign. Valuative independence therefore holds in every reachable seed [8, Proposition A.13]. A regular monomial M in a seed has a one-point Newton polytope. Every constituent of its theta expansion must therefore be a scalar multiple of the same monomial in that seed. Linear independence leaves a single term, say $M = c\Theta_m$. By Theorem 9.1, c is an integer. In the seed where M is a monomial of coefficient one, the coefficient of M in Θ_m is $1/c$, also an integer. Hence $c = \pm 1$. The least term of Θ_m in the marked chart has coefficient one, which determines the sign. □

In all three families, the integral theta basis is *atomic in the intrinsic local charts.* This means that an element of the integral algebra has nonnegative coefficients in the intrinsic local expansion at every generic point of the marked source skeleton precisely when it is a nonnegative integer combination of theta functions [8, Theorem A.11]. The theorem applies on the same compactification as the Laurent realization. For the quadric below we also prove atomicity using just its three LP charts.

Corollary 9.3. *Let P be a bounded rational polytope in $M_{\mathbb{R}}$, where M is the character lattice of the marked initial chart, and define*

$$A_P = \mathcal{U}(\Sigma_X^\vee) \cap \mathrm{Span}_{\mathbb{C}}\{z^a : a \in P \cap M\}.$$

Then

$$A_P = \bigoplus_{\mathrm{Newt}(\Theta_m)\subseteq P} \mathbb{C}\Theta_m.$$

In particular the theta labels occurring in A_P belong to the finite set $P \cap M$. On a known finite support, Laurentness in the adjacent charts imposes finitely many linear equations on the coefficients.

Proof. Apply the preceding theorem to a theta expansion of an element of A_P. Every nonzero constituent has Newton polytope contained in P. Conversely, each theta function on the right belongs to A_P. Its label occurs as an exponent in its marked expansion. □

We can now describe equality in the smallest-polytope property of [8, Theorem 2.1]. In the marked chart, write $\Delta(m) = \mathrm{Newt}(\Theta_m)$ and let $\Delta^-(m)$ be the convex hull of the lattice points of $\Delta(m)$ other than its vertices. We allow the empty hull and set $A_\varnothing = 0$. An element is *monic* if its least term has coefficient one.

Proposition 9.4. *Every monic element with least exponent m has Newton polytope containing $\Delta(m)$. Those with Newton polytope exactly $\Delta(m)$ form the affine space $\Theta_m + A_{\Delta^-(m)}$. In particular, Θ_m is the unique such element precisely when $A_{\Delta^-(m)} = 0$.*

Proof. Distinct theta functions have distinct monic least terms, so the coefficient of Θ_m in the expansion is one. The Newton polytope formula gives the first assertion. Suppose another constituent supported in $\Delta(m)$ contains a vertex of $\Delta(m)$. Choose a weight minimized uniquely at that vertex. Both theta functions then have a single least term at the same exponent. A linear combination cancelling these two terms contradicts valuative independence.

Thus every other constituent is supported in $\Delta^-(m)$. Conversely, adding an element of $A_{\Delta^-(m)}$ changes no vertex coefficient, so the least term and the polytope are preserved. □

9.3. **The quadric mirror.**

Corollary 9.5. *The basis of Proposition 7.8 is valuatively independent and Laurent-positive in every seed over the specified integral coefficient ring. Its elements are precisely the extended cluster monomials, and it is atomic for the LP atlas.*

Proof. By Proposition 7.8 the theta functions are exactly the monomials $\kappa^k P^i Q^j R^\ell$ with $i, j, \ell \geq 0$ and $\min(i, j, \ell) = 0$. Each is a monomial in one of the three clusters. In the P, Q chart, eliminating R by $R = (1 + \kappa_0)/(PQ)$ gives nonnegative coefficients. If $\ell > 0$, either $i = 0$ or $j = 0$, so every term then has a negative P- or Q-exponent. Hence the coefficient of $\kappa^k P^i Q^j$ for $i, j \geq 0$ in any theta expansion is exactly the basis coefficient of that monomial. The same argument in the other two charts detects all basis coefficients. Positivity in the three charts is therefore equivalent to being a nonnegative integer combination of theta functions. Valuative independence follows from Theorem 9.2. □

10. Lie-theoretic structure

The coefficient torus of each mirror is determined by the degrees and weights of the minuscule coordinates. In type D, the Newton polytope of the potential also recovers the minuscule root poset.

10.1. **The coefficient torus and the weight lattice.** Let G be simply connected, let $T_G \subset G$ be a maximal torus, and write $\ell = \operatorname{rank} G$. Cone scaling and T_G act on R_X. Let Λ be the character lattice of the initial torus of U_X. The homomorphism $\gamma_X : \Lambda \to \mathbb{Z} \oplus X^*(T_G)$ sends each initial coordinate to its homogeneous degree and its weight as a function. Its image Γ_X is the character lattice of the effective degree-and-weight torus acting on the cone.

Let $L_X \subset \Lambda$ be the saturated lattice generated by the support differences in the Newton construction.

Proposition 10.1. *For each of the three families,*

$$L_X = \ker(\gamma_X), \qquad \Lambda/L_X \simeq \Gamma_X, \qquad \operatorname{rank} \Gamma_X = \ell + 1.$$

The mirror coefficient lattice is canonically $L_X^\perp = \operatorname{Hom}(\Gamma_X, \mathbb{Z})$. Consequently the coefficient torus is dual to the effective degree-and-weight torus acting on the cone.

Proof. The initial coordinates and the regular functions used to construct L_X are homogeneous T_G-eigenvectors. Two monomials in the Laurent expansion of one such function have the same degree and weight. Hence $L_X \subseteq \ker(\gamma_X)$. The action of $\mathbb{G}_m \times T_G$ on the cone has finite kernel. Indeed, a torus element acting as a scalar on this representation is central, and the simple group G has finite centre. Since the initial coordinates generate the function field, their degree-and-weight characters have rank $\ell + 1$.

By Proposition 6.1 and Appendices C–D, the $\dim \widehat{X} - \ell - 1$ Newton characters form part of an integral basis of Λ. Thus L_X is saturated and has the same rank as $\ker(\gamma_X)$, so the two lattices agree. In the dual basis, the mirror coefficient characters span the annihilator of L_X. This annihilator is $\operatorname{Hom}(\Lambda/L_X, \mathbb{Z}) = \Gamma_X^*$. □

The coefficient torus is isogenous to $\mathbb{G}_m \times T_G^\vee$, where $T_G^\vee$ is the dual torus. Its rank is $n + 1$, 7, or 8 according as the type is D_n, E_6, or E_7. The inclusion of the coefficient ring gives a morphism

$$\operatorname{Spec} \mathcal{U}(\Sigma_X^\vee) \longrightarrow \operatorname{Spec} \mathbb{C}[\Gamma_X^*].$$

The character $\mathfrak{q}$ corresponds to cone scaling. Fixing its value leaves ℓ independent coefficient parameters on the dual torus.

10.2. **The minuscule root poset in type D.** For $X = \mathrm{OG}^+(n, 2n)$ and $m = n-1$, the triangular poset P_m is the minuscule poset of the half-spin representation. Write $\varepsilon_1, \dots, \varepsilon_n$ for the standard orthonormal basis in the Euclidean realization of D_n. The identification with the positive roots outside the Levi of P_n is

$$(a, b) \longmapsto \beta_{ab} = \varepsilon_{n-b} + \varepsilon_{n-a+1}.$$

The root order agrees with the poset order, and $\mathrm{ht}(\beta_{ab}) = a+b-1 = \mathbf{h}_{ab}$. Thus the horizontal and vertical ratios in the potential are indexed by covering relations in the minuscule root poset. The other two terms correspond to its minimum and maximum.

If Δ_m is the Newton polytope of the fixed-$\mathfrak{q}$ potential, Section 6 gives $\Delta_m^\circ = (2n-2)\mathcal{O}(P_m) - \mathbf{h}$. For the minuscule embedding, Spacek and Wang identify this order polytope with a Newton–Okounkov body and prove that the Plücker coordinates form a Khovanskii basis [23, Theorem 1.1]. In particular, for every integer $d \geq 0$,

$$\#(d\mathcal{O}(P_m) \cap \mathbb{Z}^{P_m}) = \dim H^0(X, \mathcal{O}_X(d)).$$

10.3. **Comparison with canonical homogeneous-space mirrors.** Rietsch constructs Landau–Ginzburg models for G/P using the Langlands dual group and an open Richardson variety [17]. For cominuscule G/P, Spacek computes the potential on a Lusztig torus [20]. Spacek and Wang give canonical Plücker-coordinate models for all cominuscule spaces and identify them with Rietsch's models [22]. Their exceptional and orthogonal constructions are given in [21, 24]. Our construction starts from the complement of the Coxeter boundary and identifies the theta function of each component. To compare the resulting Landau–Ginzburg models, one must identify the affine domains and their potentials. Already for the six-dimensional quadric, the two affine domains differ.

Example 10.2 (The six-dimensional quadric)**.** At $\mathfrak{q} = 1$, the D_4 mirror in Section 7 has coordinate ring

$$\frac{\mathbb{C}[\kappa_0^{\pm1}, \kappa_1^{\pm1}, \kappa_2^{\pm1}, \kappa_3^{\pm1}, P, Q, R]}{(PQR - 1 - \kappa_0)}.$$

Its units modulo constants are generated by the four κ_i. The morphism defined by these units has a nonsmooth fibre precisely over $\kappa_0 = -1$, an irreducible hypersurface in the unit torus.

The canonical mirror of Q^6 has coordinate ring

$$\frac{\mathbb{C}[A^{\pm1}, B^{\pm1}, C^{\pm1}, D^{\pm1}, x, x', y, y']}{(xx' - C - D,\ yy' - AB - D)};$$

this is the specialization $m = 4$, $p_0 = 1$ of [15, §5.2, equation (51)], with $(A, B, C, D) = (p_3, p_3', p_6, \delta_1)$ and $(x, x', y, y') = (p_1, p_5, p_2, p_4)$. Its units modulo constants are generated by A, B, C, D. Indeed, localizing at x, y makes any unit a Laurent monomial. Regularity of that unit and its inverse along $x = 0$ and $y = 0$ forces the exponents of x, y to vanish. Its unit morphism has nonsmooth fibres over the two distinct irreducible hypersurfaces $C + D = 0$ and $AB + D = 0$. An isomorphism of affine varieties identifies their groups of units and hence these morphisms up to an automorphism and translation of the target torus. The numbers of irreducible components of the nonsmooth fibre loci differ, so the two affine domains are not isomorphic.

A uniform construction of the Coxeter-boundary mirrors from root subgroups or generalized minors remains open.

Appendix A. Freudenthal coordinate ring and chart comparisons

We prove Theorem 4.1 by passing through the auxiliary seed $\Sigma^\circ_{E_7}$. Until the comparison of the two seeds in §A.7, we omit the superscript $\circ$ from its variables $h, b, c, d, e, f, g, r, q, u$. In particular, d denotes y_{10} in the auxiliary seed and x_{16} in the primary seed.

A.1. **Coordinate convention.** We follow the upper-sign convention of [5, Section 6.1]. The Coxeter element shifts the indices of a_i, x_i, y_i modulo 18 and interchanges z_1, z_2. The cyclic translates of the following eight quadrics generate the defining ideal.

$$\begin{aligned}
x_1x_2 &= a_3x_{17} + a_2y_0 + a_1y_3 + a_0x_4 + a_{17}a_4,\\
y_{14}y_1 &= x_{15}y_4 - a_0x_{10} - a_{15}x_5 + x_0y_{11} + x_{13}x_2,\\
y_2y_1 &= a_2y_{12} + y_5x_0 + y_{16}x_3 + a_1y_9 + a_5a_{16},\\
x_4y_1 &= x_2y_5 + a_5x_{17} + a_4y_{16} - a_2y_8 - a_1x_7,\\
x_2x_6 &= x_4y_4 - a_4z_2 - a_5y_0 - a_3y_8 + a_1a_7,\\
y_1y_3 &= x_{17}x_5 + x_2z_1 + a_0x_7 + a_4x_{15} - a_2y_{11},\\
y_4y_{16} &= y_1z_2 - y_8x_0 - y_{12}x_2 - a_1x_{10} + a_5a_{15},\\
z_1z_2 &= y_3y_{12} + y_7y_{16} + y_8y_{17} + x_1x_{10} + x_3x_{12} + x_5x_{14}\\
&\quad + a_0a_9 + a_2a_{11} - a_3a_{12} + a_4a_{13} + a_6a_{15}.
\end{aligned} \tag{A.1}$$

Gaussian elimination over $\mathbb{Q}$ gives rank 133 for the coefficient matrix of the 144 translates. The elimination data are in `anc/freudenthal_quadric_rank_certificate.json`.

A.2. **Coordinates and exchanges.** Let $R_7 = \mathbb{C}[\widehat{E_7/P_7}]$. Its 56 linear coordinates are a_i, x_i, y_i for $i \in \mathbb{Z}/18$ and z_1, z_2. The first three families are Coxeter orbits of length 18, while z_1, z_2 form an orbit of length two. The frozen variables a_i generate the coefficient ring $S_{E_7} = \mathbb{C}[a_0, \dots, a_{17}]$.

The three higher-degree functions in the auxiliary cluster are

$$\begin{aligned}
r &= x_{17}y_{10} - a_{17}x_9, \qquad q = x_6y_{13} - a_6x_{14},\\
u &= x_{14}x_4y_7 - a_{14}x_4x_6 - a_3x_{14}x_9 - a_4x_{14}y_{10}.
\end{aligned} \tag{A.2}$$

The initial cluster is

$$(h, b, c, d, e, f, g, r, q, u) = (x_1, x_{11}, x_9, y_{10}, x_6, x_{14}, x_4, r, q, u). \tag{A.3}$$

The cluster variables have degrees $(1, 1, 1, 1, 1, 1, 1, 2, 2, 3)$ in this order.

A.3. **Exchange polynomials.** Set

$$P = a_6a_{11}f + a_7bf + a_9a_{14}e + a_{10}ef + a_{12}de + a_{13}ce.$$

The ten ordinary exchange polynomials are

$$\begin{aligned}
F_h &= efq(a_0dg + a_3r + a_{17}(a_3c + a_4d)) + df(er + u)(a_1a_6 + a_2e) + a_1dq(u + a_{14}eg),\\
F_b &= a_{11}(a_6f + q) + e(a_9a_{14} + a_{10}f + a_{12}d + a_{13}c),\\
F_c &= fg(a_6a_{11} + a_7b + a_8d + a_{10}e) + a_9(a_4df + a_{14}eg + u),\\
F_d &= r + a_{17}c, \quad F_e = q + a_6f, \quad F_f = u + a_{14}eg,\\
F_g &= u + a_3cf + a_4df, \qquad F_q = hP + b(er + u),\\
F_r &= g(a_{16}d + a_{17}b)(hP + bu) + hdgq(a_{11}a_{16} + a_{15}b)\\
&\quad + bq\big(dg(a_0f + a_1a_{14}) + a_{17}f(a_3c + a_4d) + a_{17}u\big),\\
F_u &= c\,F_r|_{u=0} + br\left\{cf\big(d(a_1a_6 + a_2e) + a_3q\big) + h\big(F_c|_{u=0} + a_5cdf + a_{13}ceg\big)\right\}.
\end{aligned}$$

The associated Laurent exchange polynomials are $\widehat{F}_v = F_v/M_v$, where

v	h	b	c	d	e	f	g	r	q	u
M_v	def	e	f	1	1	1	1	bg	1	bd

Thus $v' = F_v/(vM_v)$.

Each F_v is independent of v and has no mutable monomial factor. In the order $h, b, c, d, e, f, g, r, q, u$, the polynomials are primitive and linear in $a_0, a_{10}, a_8, r, q, u, u, a_{15}, a_7, a_5$, respectively. The linear coefficients are monomials coprime to the constant

terms, so Gauss's lemma proves irreducibility over $\mathbb{C}$. The LP normalization gives the monomials M_v in the table.

A.4. **The first mutations.** Mutation in each of the ten directions gives the following polynomials in the linear coordinates.

$$\begin{aligned}
&b' = x_{12}, \quad d' = x_{17}, \quad e' = y_{13},\\
&c' = x_4x_8 - a_3a_9, \qquad q' = x_{11}y_0 - a_{11}x_1,\\
&f' = x_4y_7 - a_4y_{10} - a_3x_9, \qquad g' = x_{14}y_7 - a_{14}x_6,\\
&r' = x_{14}(y_2y_7 - x_3y_{10} - x_1x_8 + a_3x_{11}) - x_6(a_{13}x_1 + a_{14}y_2),\\
&h' = x_2q - a_7(a_2x_{14} + a_1y_{13}) - (a_1a_6 + a_2x_6)x_{12},\\
&u' = x_9(x_1y_{16} - a_2x_{14} - a_1y_{13}) - a_9x_1x_{17}.
\end{aligned}$$

Their exchange identities $vM_vv' = F_v$ hold by substitution in the injective Laurent chart constructed below.

A.5. **The upper coordinate ring.**

Proposition A.1. *The rank-ten seed* (A.3) *satisfies* $\mathcal{A}(\Sigma^\circ_{E_7}) \subseteq \mathcal{U}(\Sigma^\circ_{E_7}) = R_7$. *Every cluster variable is regular on the Freudenthal cone.*

Proof. To construct the initial chart, let

$$L_0 = S_{E_7}[h^{\pm1}, b^{\pm1}, c^{\pm1}, d^{\pm1}, e^{\pm1}, f^{\pm1}, g^{\pm1}, r^{\pm1}, q^{\pm1}, u^{\pm1}].$$

We compute Laurent expressions in L_0 for the 56 coordinates from the cyclic quadratic relations. The complete expressions are in `anc/freudenthal_source_coordinates.json`, and the adjacent-chart divisibilities are recorded in `anc/freudenthal_source_divisibilities.json`. All identities can be checked over $\mathbb{Q}$. For example,

$$x_{17} = \frac{r + a_{17}c}{d}, \qquad y_{13} = \frac{q + a_6f}{e}, \qquad y_7 = \frac{u + a_{14}eg + a_3cf + a_4df}{fg}.$$

The expressions satisfy the 144 cyclic translates of the defining quadrics in (A.1) and hence define a homomorphism $R_7 \to L_0$. The eighteen frozen coordinates, seven initial linear coordinates, and three functions in (A.2) map to the 28 independent indeterminates of L_0. The image has transcendence degree 28, so the homomorphism is injective because R_7 is a 28-dimensional domain. After inverting the ten mutable functions, we obtain $R_7[(hbcdefgrqu)^{-1}] = L_0$. Applying (2.1) to the negative powers of each mutable variable gives 204 divisibility conditions, including those at poles of order two. The identities in the second file verify all of them. Thus R_7 is contained in all ten adjacent Laurent rings.

By Lemma 2.4, R_7 is factorial and each initial degree-one variable is prime. For the higher-degree variables, we grade by degree and by the weight for the rank-seven maximal torus. All 56 degree-one weights are distinct. The connected torus acts trivially on the finite set of irreducible factors of a semi-invariant, up to associates. Each such factor is therefore itself a semi-invariant. The same argument applies to the grading by degree. Thus a reducible degree-two or degree-three variable has a degree-one coordinate factor.

Since r, q, u are units in the initial Laurent ring, such a degree-one factor must be a unit there as well. The Laurent expressions show that the only degree-one coordinates invertible in L_0 are h, b, c, d, e, f, g, since the frozen coordinates have not been inverted. For each of these seven coordinates v, none of $r/v, q/v, u/v$ belongs to the weight space it would have to occupy in R_7. In degree one, this amounts to comparison with a single coordinate when the weight occurs. In degree two, the coordinate products of the required weight span the weight space since R_7 is generated in degree one, and there are at most six such products in each case. It follows that r, q, u are prime. The ten initial variables have distinct degree-and-weight vectors, so no height-one prime contains two of them. The first mutants lie in R_7 by their

polynomial formulas. Proposition 2.3 now identifies R_7 with the upper LP algebra over S_{E_7}. The Laurent phenomenon gives $\mathcal{A}(\Sigma^\circ_{E_7}) \subseteq R_7$. □

A.6. **Infinite type.**

Proposition A.2. *The rank-ten Freudenthal seed* (A.3) *is of infinite type. More precisely, after mutating first at h and then at g, freezing the other eight mutable variables leaves a rank-two LP pattern of exchange degrees* $(1, 4)$.

Proof. Write $H = h'$ and $G = g'$ for the two new variables, and set

$$\begin{aligned} K &= u + a_3cf + a_4df, \\ \mathsf{A} &= (a_{16}d + a_{17}b)P + dq(a_{11}a_{16} + a_{15}b), \\ \mathsf{B} &= b\big((a_{16}d + a_{17}b)u + dq(a_0f + a_1a_{14})\big), \\ \mathsf{C} &= deq(a_0f + a_1a_{14}), \\ \mathsf{D} &= a_{17}efq(a_3c + a_4d) + du\big(f(a_1a_6 + a_2e) + a_1q\big). \end{aligned}$$

The initial exchanges satisfy

$$F_h|_{r=0} = g\mathsf{C} + \mathsf{D}, \qquad F_r = gh\mathsf{A} + g\mathsf{B} + a_{17}bqK.$$

Since $\widehat{F}_h = F_h/(def)$, the first mutation gives

$$F_r^{(h)} = g^2\mathsf{AC} + g(\mathsf{AD} + defH\mathsf{B}) + a_{17}bdefqHK.$$

In the resulting seed $\widehat{F}_g = F_g = K$. We substitute $g = K/G$, multiply by G^2, and divide out the common factor K. This gives the ordinary exchange polynomial

$$\mathcal{Q}(d) = K\mathsf{AC} + G(\mathsf{AD} + defH\mathsf{B}) + a_{17}bdefqHG^2. \tag{A.4}$$

There is no further factor of K in (A.4). Indeed, at $u = -2$ with all other variables equal to 1, we have $K = 0$ and $\mathcal{Q} = -57$.

Both mutations leave $F_d = r + a_{17}c$ unchanged. In the final seed the associated exchanges in these two directions are

$$\widehat{F}_d = r + a_{17}c, \qquad \widehat{F}_r = \frac{\mathcal{Q}(d)}{befGq}.$$

All the factors b, e, f, G, q become coefficient units when the eight variables H, b, c, e, f, G, q, u are frozen. In particular the active exchange degrees are unchanged. From (A.4) we obtain

$$\begin{aligned} [d^4]\mathcal{Q} &= a_4a_{12}a_{16}e^2fq(a_0f + a_1a_{14}) \neq 0, \\ [d^0]\mathcal{Q} &= a_3a_{17}^2bcefGq(a_6a_{11}f + a_7bf + a_9a_{14}e + a_{10}ef + a_{13}ce) \neq 0. \end{aligned}$$

Hence the two degrees are exactly $(1, 4)$.

The rank-two classification gives infinitely many denominator vectors for the degree pair $(1, 4)$ [14, Theorem 6.4 and Corollary 6.5]. Freezing commutes with mutation in the two remaining directions [14, Proposition 3.7], so these vectors already distinguish infinitely many cluster variables of the rank-ten pattern. □

A.7. **Comparison of the Freudenthal charts.** We number the auxiliary variables as in (A.3) and read mutation words from left to right, keeping the positions fixed.

Proposition A.3. *The word* $(4, 9, 8, 10, 4)$ *gives the primary seed of Theorem 4.1, including its stated global polynomial coordinates. It fixes each frozen coordinate and identifies both the generated and upper LP algebras.*

Proof. Use $r^\circ, q^\circ, u^\circ$ for the auxiliary functions in (A.2) and r, q, u for the primary ones in Section 4.2. Substitution in the five successive exchange relations gives the replacements

$$y_{10} \mapsto x_{17}, \quad q^\circ \mapsto q, \quad r^\circ \mapsto r, \quad u^\circ \mapsto u, \quad x_{17} \mapsto x_{16}.$$

The final exchanges and their normalizing monomials are those of Section 4.2.

For a direct irreducibility check, in the given order the ten primary exchange polynomials are linear in $a_1, a_{11}, a_4, a_0, a_1, a_1, a_4, a_2, a_0, a_2$, respectively. Each leading coefficient is a monomial coprime to the constant term. The LP normalizations are taken over S_{E_7}.

The five mutations fix S_{E_7}. The two seeds have the same mutation class, hence the same generated and upper LP algebras. □

Appendix B. Spinor mirror calculations

We prove the recurrence and triangular-coordinate identities used in Sections 6 and 7. Throughout, $m = n - 1$ and the characters and the sets $\mathcal{I}_a$ are those of (6.1)–(6.5).

B.1. **Normalized cube recurrence.** We normalize each interval exchange by its first summand. Extend the character notation by

$$\rho_{s,t} = \chi_{R_{st}} \quad (s \le t \le m-2), \qquad \rho_{s,s-1} = \chi_{B_{s-1}} \quad (s \le m-2), \qquad \rho_{m-1,m-2} = \chi_Q,$$

and set $\xi_s = \chi_{E_s}$ for $2 \le s \le m-2$ and $\xi_{m-1} = \chi_P$. Introduce the monomials

$$V_{i,m-1} = \xi_i, \qquad V_{ij} = \rho_{i,j-1}\rho_{ij}V_{i+1,j+1} \quad (j \le m-2). \tag{B.1}$$

An empty product equals one. Some of the index sets are empty when $n = 4$. If A_{ij} is the first summand of the interval exchange F_{ij}, then cancellation of monomials in the interval coordinates gives

$$\begin{aligned} N_{ii} &:= F_{ii}/A_{ii} = 1 + \rho_{i,i-1} + V_{ii}, \\ N_{ij} &:= F_{ij}/A_{ij} = 1 + \rho_{i,j-1} + V_{ij} + \rho_{i,j-1}V_{i+1,j} \quad (i < j). \end{aligned} \tag{B.2}$$

At the terminal positions, these specialize to

$$N_{m-1,m-1} = 1 + \chi_P + \chi_Q, \qquad N_{i,m-1} = 1 + \rho_{i,m-2} + \xi_i + \rho_{i,m-2}\xi_{i+1}.$$

To prove (B.1), set $D_{it} = \frac{X_{it}X_{i-1,t+1}}{X_{i-1,t}X_{i,t+1}}$. Then $\rho_{it} = D_{it}/D_{i+1,t+1}$ and $\xi_i = D_{i,m-2}D_{i,m-1}$, so the products telescope. The same cancellation, with $X_{i,i-1} = h$, applies when $i = j$ and gives the four ratios in (B.2).

For $0 \le a \le b \le c \le m$, set $Z(a,b,c) = q_{[1,a]\cup[b+1,c]}$. For $1 \le a < b < c \le m$, the cube with common set $[1,a-1]\cup[b+1,c-1]$ replaces $Z(a-1,b-1,c-1)$ by $Z(a,b,c)$. Mutating in lexicographic order on the triples gives the complementary-interval chart after $\binom{m}{3}$ mutations.

Lemma B.1. *For $1 \le a < b < c \le m$, the least term of $Z(a,b,c)$ in the positive χ chamber is*

$$\operatorname{lead} Z(a,b,c) = L(a,b,c) := \frac{X_{1a}X_{b-a+1,c}}{X_{b-a+1,b}}. \tag{B.3}$$

The quotient $P_{abc} = Z(a,b,c)/L(a,b,c)$ is a polynomial in the χ characters, with constant term one and linear part $\sum_{\rho\in\mathcal{I}_{(b-a+1,b)}} \chi_\rho$.

Proof. The third product in the cube recurrence gives the leading term in (B.3). The ratios of the other two leading products are the nonconstant monomials M_1, M_2 in (B.5) below. Induction therefore gives P_{abc} as a formal power series with constant term one. The Laurent phenomenon gives an integral Laurent polynomial in the interval coordinates [14, Theorem 5.1]. By homogeneity and the integral change of characters, P_{abc} belongs to $\mathbb{Z}[\chi^{\pm 1}]$. Its power series expansion therefore belongs to $\mathbb{Z}[\chi]$. The axis calculation in Lemma B.2 gives the linear part. □

For the initial cases $a = 0$, $a = b$, or $b = c$, we have $P_{abc} = 1$. Below, $\mathbf{1}_{\mathcal{P}}$ equals one when $\mathcal{P}$ holds and zero otherwise.

Lemma B.2. *For every mirror position ρ and every interval t,*

$$N_t|_{\chi_\sigma=0\ (\sigma\neq\rho)} = 1 + \mathbf{1}_{t=a(\rho)}\chi_\rho.$$

Moreover N_t has total degree at most one in the variables of its set $\mathcal{I}_t$. For $1 \le a < b < c \le m$,

$$P_{abc}|_{\chi_\sigma=0\ (\sigma\neq\rho)} = 1 + \mathbf{1}_{a(\rho)=(b-a+1,b)}\chi_\rho. \tag{B.4}$$

Proof. The exchange assertions follow from (B.2). If $j < m-1$, every nonconstant summand is divisible by the one assigned character $\rho_{i,j-1}$, and contains it just once. The remaining factors in the two longer products have other indices. If $j = m-1$, the formulas for $N_{i,m-1}$ give both assertions, including the positions that share an exchange polynomial.

For the cube recurrence, let $d = b-a+1$ and $\ell = m-c$. Dividing by the leading monomials gives

$$P_{abc}P_{a-1,b-1,c-1} = M_1 P_{a-1,b,c-1}P_{a,b-1,c} + M_2 P_{a,b-1,c-1}P_{a-1,b,c} + P_{a-1,b-1,c}P_{a,b,c-1}, \tag{B.5}$$

where $M_2 = \prod_{t=b-1}^{c-2} \rho_{d,t}$ and

$$M_1 = \begin{cases} \xi_d \prod_{t=b-1}^{m-3} \rho_{d,t}, & \ell = 0, \\ \xi_{d+\ell} \left(\prod_{t=b-1}^{c-2} \rho_{d,t}\right) \left(\prod_{k=0}^{\ell-1} \rho_{d+k,c-1+k}\right) \left(\prod_{k=0}^{\ell-2} \rho_{d+k+1,c-1+k}\right), & \ell \geq 1. \end{cases}$$

Every product of length at least two in these formulas contains two distinct characters. The horizontal indices differ, and a terminal ξ differs from each ρ.

We prove (B.4) by the lexicographic cube induction. If $c > b+1$, both M_1 and M_2 vanish on each axis. The factors $P_{a-1,b-1,c-1}$ and $P_{a-1,b-1,c}$ are either both degenerate or both indexed by the interval $(d, b-1)$ in (B.4). Their restrictions cancel, leaving the restriction of $P_{a,b,c-1}$ for the interval (d,b). If $c = b+1 < m$, only $M_2 = \rho_{d,b-1}$ can survive on a single axis. On its own axis all four accompanying factors have restriction one, and the restriction is $1 + \rho_{d,b-1}$. On every other axis the same two factors cancel. If $c = b+1 = m$, the only monomials that can survive are the two distinct characters ξ_d and $\rho_{d,m-2}$. Each contributes one on its own assigned axis and zero on the other. This proves the formula also for the two positions at the last interval. □

For $t = (u,v) \in \mathcal{M}$, write $\psi_t = q_{[m]\setminus[v-u+2,v]} = X^{h_t} P_t(\chi)$. Its leading term is $X_{1,v-u+1}X_{u,m}/X_{uv}$; hence $h_t^{\mathrm{mut}} = -e_t$. Lemma B.2 applies to P_t.

The full adjacent support is determined by the following tropical identity.

Lemma B.3. *For $1 \leq a < b < c \leq m$ and every $s \in \mathcal{M}$,*

$$-\min_{\alpha \in \operatorname{supp} P_{abc}} \langle \alpha, \mu_s \rangle = \mathbf{1}_{s=(b-a+1,b)}.$$

In particular the defect of ψ_t at μ_s is $\mathbf{1}_{t=s}$.

Proof. We use induction in the normalized recurrence. A character $\rho_{u,v-1}$ has value

$$-\mathbf{1}_{s=(u,v)} + \mathbf{1}_{s=(u+1,v)} + \mathbf{1}_{s=(u-1,v-1)} + \mathbf{1}_{s=(u,v+1)}$$

at μ_s, with inadmissible intervals omitted. A terminal character ξ_u has value $-\mathbf{1}_{s=(u,m-1)} + \mathbf{1}_{s=(u-1,m-2)}$. As above, let $d = b-a+1$. Write $T = \mathbf{1}_{s=(d,b)}$, $O = \mathbf{1}_{s=(d,b-1)}$, $A = \mathbf{1}_{s=(d+1,b)}$, $B = \mathbf{1}_{s=(d-1,b-1)}$, and $\varepsilon = \mathbf{1}_{c>b+1}$. If λ_i is the value of M_i, the three terms on the right of (B.5) have values

$$\lambda_1 - \varepsilon A - B, \qquad \lambda_2 - A - B, \qquad -O - \varepsilon T.$$

Summing the values of the factors of M_2 gives

$$\lambda_2 - A - B = -T + \mathbf{1}_{s=(d,c)} + \sum_{v=b+1}^{c-1} \mathbf{1}_{s=(d+1,v)} + \sum_{v=b}^{c-2} \mathbf{1}_{s=(d-1,v)}.$$

In the product for M_1, the negative contribution of each factor after the first is cancelled by a positive contribution from the preceding factor. The first factor also contributes positively at $(d-1, b-1)$ and, when $\varepsilon = 1$, at $(d+1, b)$. If M_1 has only one factor, $\varepsilon = 0$. Consequently $\lambda_1 \geq -T + B + \varepsilon A$. If $\varepsilon = 1$, the third term has value $-O - T$ and the others are no smaller. If $\varepsilon = 0$, the minimum of the second and third values is $-O - T$, since O and T cannot both be one. Subtracting the value $-O$ of $P_{a-1,b-1,c-1}$ gives $-T$. The cube recurrence is subtraction-free, so its tropicalization computes the Newton minimum. □

B.2. **The adjacent supports.** The least monomial of ψ_t has mutable exponent $-e_t$. The slope inequality therefore gives the bound $0 \le d_t \le \delta_{\psi_t}(\eta)$ for a correction to η; see (B.6)–(B.7) below. Lemma B.3 gives $\delta_{\psi_t}(\mu_s) = \mathbf{1}_{t=s}$.

Lemma B.4. *For $\eta = \mu_a$, the corrections satisfying (B.6) for every $f \in \mathcal{O}(U_n)$ are 0 and e_a. The corresponding exponents are μ_a and $\mu_a + c_a$. Assigning coefficient one to both gives*

$$z^{\mu_a}(1 + z^{c_a}) = \frac{\mathcal{E}_a}{\prod_{\rho \in \mathcal{I}_a} Z_\rho}.$$

Proof. The complementary coordinates and Lemma B.3 restrict the bounds (B.7) to $\{0, e_a\}$. The zero correction always satisfies the inequalities.

To see that e_a satisfies the inequality for every f, consider the least exponent h_f. For $(h_f)_a \ge 0$, the assertion follows from $\delta_f \ge 0$. If $(h_f)_a = -k < 0$, Laurent regularity in the adjacent chart implies that F_a^k divides the coefficient of X_a^{-k} in f. Let A_a be the least monomial of F_a and let γ_ρ be the exponent vector of χ_ρ in the interval coordinates. For $\rho \in \mathcal{I}_a$, the monomial $A_a\chi_\rho$ occurs in F_a by (6.1)–(6.2). Newton polytopes add under multiplication. Thus the Newton polytope of the terms with X_a-exponent $-k$ contains $h_f + k\gamma_\rho$. In the dual coordinates $\langle \gamma_\rho, \mu_a \rangle = -1$, so $\delta_f(\mu_a) \ge k = -(h_f)_a$. Thus e_a satisfies every regular-function inequality. Finally (6.7) gives the stated Laurent polynomial. □

B.3. **Finite expansions and primitive contacts.** The initial and adjacent tori cover the codimension-one points of U_n: its initial coordinates are nonassociate primes, and the localized chart at a is $X_a w_a = F_a$. Each centre $F_a = 0$ is smooth on the torus of the other coordinates, since a corner coordinate occurs linearly with a monomial coefficient. The punctured spinor cone is smooth, and boundary inversion removes its vertex. The initial logarithmic volume form extends across every rank-one chart, since the exchange replaces its wedge by its negative. It is therefore a nowhere vanishing form on U_n. These facts verify the smooth affine log Calabi–Yau hypotheses for the intrinsic algebra.

Along the ray for X_a, the common chart is the blowup of the ideal (X_a, F_a) with the strict transform of the toric boundary removed. The charts contain the generic points of all divisors in U_n. Skeletal curves avoid their codimension-two complement [12, Lemma 8.21]. Thus every unmarked toric contact is over one of these centres and has direction c_a, with positive integral multiplicity. In particular all realized cylinder bends lie in

$$\Pi = C_{\mathrm{mut}}\mathbb{Z}^{r_0}_{\ge 0}, \qquad r_0 = \binom{n-2}{2}.$$

The columns are independent and span a saturated sublattice, so Π is a free pointed toric monoid. This verifies [12, Assumption 20.27] for the given blowup model, including its nonbinomial centres.

For a regular function f with unique minimizing exponent h_f in the positive χ chamber, the principal-divisor slope formula [12, Lemma 15.3 and Theorem 15.7] gives, for a correction d to an incoming label η,

$$h_f^{\mathrm{mut}} d \ge -\delta_f(\eta), \qquad \delta_f(\eta) = \langle h_f C^{-1}, \eta \rangle - \min_{u \in \operatorname{supp} f} \langle u C^{-1}, \eta \rangle. \tag{B.6}$$

The least mutable exponent of ψ_t is $-e_t$, so the product of the complementary functions has a minimizing vertex negative in every mutable coordinate. Applying the inequality to each ψ_t gives

$$0 \le d_t \le \delta_{\psi_t}(\eta) \qquad (t \in \mathcal{M}). \tag{B.7}$$

The same slope formula excludes a nonconstant attached tropical branch rooted in this chamber. Such a branch comes from an unmarked analytic disk with one boundary point and image in U_n. If its vector directed from the root into the disk is $C_{\mathrm{mut}}d$, a regular function has nonnegative slope there: the slope is its number of zeros in the disk. Applying the complementary functions gives $-d_t \ge 0$ for every t. Hence $d = 0$. A disk with no contacts maps to the torus and has constant tropicalization.

The free contact monoid and (B.7), together with Lemma 5.1, give finitely many terms and curve classes before specialization to $q^\beta = 1$. The constant-term trace and the Laurent realization [8, Lemma A.4 and Proposition A.5] therefore give the injective algebra map

$$\Phi_n : \mathscr{M}_n \hookrightarrow \mathscr{S}_n[Z_\rho^{\pm 1}], \qquad M_n = \Phi_n(\mathscr{M}_n).$$

The zero-contact term is monic. For nonnegative mutable labels the only possible correction is zero. Thus the initial variables, the coefficient characters, and their inverses in the coefficient torus are straight theta functions in M_n.

B.4. **All primitive facet walls and the adjacent variables.**

Proposition B.5. *At a generic point of the positive-chamber facet* $\mathrm{val}(\chi_\rho) = 0$*, the intrinsic wall function is* $1 + z^{c_a(\rho)}$*. Moreover* $\Theta_{\mu_a} = Y_a$ *for every* $a \in \mathcal{M}$*.*

Proof. Take a sufficiently small transverse primitive crossing with incoming direction $-e_\rho$. The axis restriction in Lemma B.2 is the entire local face polynomial of each normalized complementary function. Its gradient changes by one precisely for $t = a(\rho)$. The slope formula for the infinitesimal cylinder consequently gives $0 \le d_t \le \mathbf{1}_{t=a(\rho)}$. Thus only the zero-contact configuration and one simple contact at $a = a(\rho)$ can occur. Tangency is automatic because $(c_a)_\rho = 0$.

We count the single-contact cylinders directly. Normalize the smooth three-marked domain at $0, \infty, 1$ and write the torus evaluation as $c = (c_\sigma)$. With contact parameter τ, its character map is

$$\chi_\sigma(w) = c_\sigma w^{-\delta_{\sigma\rho}} \left(\frac{w-\tau}{1-\tau}\right)^{(c_a)_\sigma}. \tag{B.8}$$

In the full character basis the same formula also determines the coordinates of the split torus factor. Every exponent of N_a pairs to zero with c_a, since the interval exchange is independent of X_a. The contact factors therefore cancel in the centre equation. By (B.2), $N_a = A(\chi_{\neq\rho}) + \chi_\rho B(\chi_{\neq\rho})$ with $A(0) = B(0) = 1$, and the incidence equation is

$$A(c_{\neq\rho})\tau + c_\rho B(c_{\neq\rho}) = 0.$$

Give c_ρ valuation $\delta > 0$ sufficiently small relative to the generic positive valuations of the other χ coordinates. The unique solution has valuation δ, and the derivative is a unit. Applying C^{-1} to (B.8), any interval coordinate has the form

$$X_s(w) = X_s(c) w^{-(C^{-1}e_\rho)_s} \left(\frac{w-\tau}{1-\tau}\right)^{\delta_{sa}}.$$

Thus only X_a vanishes at the unmarked contact, and it has a simple zero. Moreover $N_a(w) = A + c_\rho B/w$ has a simple zero there. Consequently F_a/X_a is nonzero at the contact, and the map extends through the rank-one chart.

On the annulus $0 < \mathrm{val}(w) < 2\delta$, the principal spine has its only bend at $\mathrm{val}(w) = \delta$. We can choose the tangential coordinates of c so that this bend is at the prescribed generic point of the facet [12, Definition 20.2]. Both exterior tails map to the torus away from their marked ends. The valuations of the leading coefficients at the two essential ends are $\mathrm{val}(c)$ and $\mathrm{val}(c) + \delta c_a$. All their χ entries are positive after shrinking δ, because $(c_a)_\rho = 0$. If a terminal ray bears another blowup centre, its normalized equation has residue one there. Thus both marked ends lie in open essential toric strata. Start with the partial toric fan consisting of the mutable-coordinate and terminal rays, and perform the specified centre blowups. Choose a smooth compactification preserving this partial model, in particular the open strata over all these rays. The simple incidence solution makes evaluation generically étale. We may therefore pass to a common resolution by [12, Lemmas 3.6 and 3.9] while avoiding its fixed codimension-two loci. Proposition 5.2 therefore identifies the simple incidence solution with a cylinder of multiplicity one. The toric curve with no contacts also contributes one, giving the wall function $1 + z^{c_a}$.

Continuation across this wall, followed by the unimodular basis change replacing e_ρ by μ_a, is the first LP mutation. The rational transform of any finite initial theta polynomial has as its only nonmonomial denominator a power of $1 + z^{c_a}$. The continued theta series has nonnegative coefficients, so Lemma 5.3 gives a Laurent polynomial in the adjacent chart. Applying this at every position and using the LP starfish theorem [7, Theorem 1.3] gives $M_n \subseteq \mathcal{U}_n^\vee$.

For $\eta = \mu_a$, Lemma B.3 and (B.7) restrict the theta support to $\{\mu_a, \mu_a + c_a\}$. Hence $\Theta_{\mu_a} = z^{\mu_a}(1 + \lambda z^{c_a})$. In any adjacent position $\rho \in \mathcal{I}_a$ both terms have exponent -1 in Z_ρ. Laurent divisibility by $\mathcal{E}_a$ forces $\lambda = 1$. Thus $\Theta_{\mu_a} = Y_a$, including the cases where several variables share an exchange polynomial. □

B.5. **Boundary theta functions.** The functions W_b are defined in (6.8)–(6.9).

Proof of the boundary identities in Proposition 7.2. The complementary coordinate ψ_t and each frozen coordinate b are distinct degree-one primes in the spinor coordinate ring. Hence ψ_t has order zero along $b = 0$, and its least b-exponent in the initial Laurent chart is zero. The leading numerator of $\psi_{(u,v)}$ is $X_{1,v-u+1}X_{u,m}$, so (B.7) restricts the corrections for Ce_{1k} to zero-one choices on

$$a_1 = (2, k+1),\ a_2 = (3, k+2), \dots,\ a_\ell = (m-k, m-1).$$

For Ce_{im} the corresponding chain is

$$a_1 = (i, m-1),\ a_2 = (i, m-2), \dots,\ a_\ell = (i, i).$$

The boxes for h and X_{1m} are zero.

Let $d_1, \dots, d_\ell \in \{0, 1\}$ be the allowed correction coefficients along either chain. For $j \geq 2$ apply (B.6) to the regular mutation partner F_{a_j}/X_{a_j}. Its least monomial is the first summand of F_{a_j} divided by X_{a_j}. On the chain its mutable gradient is exactly $e_{a_{j-1}} - e_{a_j}$, and its frozen defect at Ce_b is zero. For a prefix chain this follows from the factor $X_{u-1,v-1}/X_{uv}$; for a suffix chain it follows from $X_{u,v+1}/X_{uv}$. The singleton exchange has the same restrictions at the marked end. The other factors are off the chain. Thus $d_1 \geq d_2 \geq \cdots \geq d_\ell$. The only possible supports are the successive prefixes in (6.8)–(6.9).

Write their unknown coefficients as $\lambda_0, \dots, \lambda_\ell$, with $\lambda_0 = 1$. For the edge assigned to $a_j = (u, v)$ choose the mirror position $\rho_{u,v-1}$: it is $R_{u,v-1}$ for $u < v$, B_{u-1} for a nonterminal singleton, and Q for the terminal singleton. Exactly two prefix monomials have negative exponent at this position, those indexed by $j-1$ and j. Both exponents are -1. Their monomial ratio is $z^{c_{a_j}}$. Adjacent Laurentness consequently requires $1 + z^{c_{a_j}}$ to divide $\lambda_{j-1} + \lambda_j z^{c_{a_j}}$ in the Laurent ring in the unchanged coordinates. Since c_{a_j} is primitive and nonzero, this is equivalent to $\lambda_j = \lambda_{j-1}$. Since $\lambda_0 = 1$, all coefficients equal one and $\Theta_{Ce_b} = W_b$ for every b.

Proposition B.5 gives the remaining inclusions. After inverting the initial coordinates, or Y_a and the unchanged coordinates in an adjacent seed, the image is the corresponding Laurent ring. Hence these tori are open subsets of the intrinsic mirror. □

B.6. **Adjacent regularity of the boundary sums.** Index the monomials in each boundary chain from zero. On a prefix chain W_{1k}, the pairs of monomials with negative exponent in B_a, E_a, R_{ab} occur respectively when $k = 1, m-a, b-a+2$, at positions $(a-1, a), (a-2, a-1), (a-2, a-1)$. The pair for P, Q occurs when $k = 1$, at $(m-3, m-2)$. On a suffix chain W_{im}, the pairs for B_a, E_a, R_{ab} occur respectively when $i = a+1, a, a$, at $(m-a-2, m-a-1), (0, 1), (m-b-2, m-b-1)$; the pair for P, Q occurs when $i = m-1$, at $(0, 1)$. Empty indexed families are omitted. All other monomials have nonnegative exponent in the variable in question. In each pair, the ratio of the two monomials is z^{c_a} for the corresponding interval a. Their sum is therefore divisible by its exchange binomial in the remaining variables. This gives Laurent regularity in each adjacent chart and hence membership in the upper LP algebra by [7, Theorem 1.3]. Distinctness follows from the chain exponents, or equivalently from the monomial coordinates below.

B.7. **Integral triangular coordinates and their flows.** The vectors v_{ab} and the characters $y_{ab}, \mathfrak{q}$ are those of (6.10).

Lemma B.6. *The characters $\mathfrak{q}$ and y_{ab} form a basis of the character lattice of the initial mirror torus. Their boundary expressions are those of (6.11).*

Proof. Since C is unimodular, we may work in the initial lattice basis. The inverse formulas are

$$e_h = v_{11}, \qquad e_{1m} = \mathbf{1} - v_{mm}, \qquad e_{1j} = v_{j+1,m} - v_{jm} \quad (j < m).$$

For $2 \le i \le j \le m$, define

$$D(i,j) = v_{j-i+1,m+2-i} - v_{j-i+1,m+1-i}.$$

Directly subtracting the two sums in (6.10) gives $D(i,j) = \sum_{s=j}^{m} e_{is}$. Consequently

$$e_{im} = D(i,m), \qquad e_{ij} = D(i,j) - D(i,j+1) \quad (j < m).$$

Every initial basis vector has therefore been expressed integrally in these N vectors. This proves unimodularity.

For a horizontal arrow, with $i = m+1-b$, the same subtraction gives

$$v_{k+1,b} - v_{kb} = \sum_{r=1}^{i} e_{r,r+k-1} = e_{1k} + \sum_{t=1}^{m-b} e_{t+1,t+k}.$$

As b decreases from m to $k+1$, these are exactly the cumulative exponents of the coordinates defining W_{1k}. The vertical arrow with index a has exponent

$$v_{a,m+2-i} - v_{a,m+1-i} = \sum_{s=i+a-1}^{m} e_{is}.$$

As a decreases from $m+1-i$ to 1, these are exactly the cumulative exponents defining W_{im}. The other two boundary functions follow from the first two inverse formulas. □

We call the arrow $y_{a+1,b}/y_{ab}$ *horizontal* and $y_{a,b+1}/y_{ab}$ *vertical*. Give the initial variables nonnegative real values w_ρ and give every κ_j value zero. Set

$$\lambda_{ab} = w(y_{ab}), \qquad T_{a,d} = \sum_{j=1}^{a} \lambda_{j,j+d} \quad (0 \le d \le m-1,\ 1 \le a \le m-d).$$

Let ϵ_{ab} denote the array supported at the single cell (a,b) with value one.

Lemma B.7. *When $w_\rho = 1$ and all other weights are zero, the array (λ_{ab}) is given by the formula for ρ below.*

$$\begin{array}{c|l} B_i & \epsilon_{1,m-i} - \epsilon_{2,m-i+1} \\ R_{ij} & \epsilon_{j-i+2,m+1-i} - \epsilon_{j-i+3,m+2-i} \\ E_i\ (a = m-i) & \epsilon_{a,a} + \epsilon_{a,a+1} - \epsilon_{a+1,a+2} - \epsilon_{a+2,a+2} \\ P & \epsilon_{11} + \epsilon_{12} - \epsilon_{23} - \epsilon_{33} \\ Q & \epsilon_{12} - \epsilon_{23}. \end{array} \tag{B.9}$$

In particular,

$$T_{m-d,d} = 0, \qquad T_{a,d} \ge 0 \quad (a < m-d). \tag{B.10}$$

More precisely, set

$$p_1 = w_P, \qquad q_1 = w_Q, \qquad p_0 = p_{m-1} = 0,$$
$$p_a = w_{E_{m-a}}, \qquad q_a = w_{R_{m-a,m-2}} \quad (2 \le a \le m-2).$$

Then

$$\begin{aligned} T_{a,0} &= p_{a-1} + p_a && (1 \le a < m), \\ T_{a,1} &= p_a + q_a && (1 \le a < m-1), \end{aligned} \tag{B.11}$$

$$T_{a,d} = \begin{cases} w_{B_{m-d-1}}, & a = 1, \\ w_{R_{m+1-a-d,m-d-1}}, & a \geq 2, \end{cases} \qquad (d \geq 2,\ 1 \leq a < m-d). \tag{B.12}$$

The map $w \mapsto \lambda$ is injective.

Proof. Fix a cell (a, b) and let $L = m + 1 - b$. Write $A_r = \mathbf{1}_{r \leq L}$ and $U_d = \mathbf{1}_{a \leq d+1}$. The coefficient of e_{rs} in v_{ab} is $[v_{ab} : e_{rs}] = 1 - A_r U_{s-r}$. This remains valid for an empty interval $X_{r,r-1} = h$, since $U_{-1} = 0$. Pairing the exponent vector of $\chi_{R_{ij}}$ with this formula gives

$$(A_{i+1} - A_i)U_{j-i} + (A_{i-1} - A_{i+1})U_{j-i+1} + (A_i - A_{i-1})U_{j-i+2}.$$

It equals 1 exactly at $(a, b) = (j - i + 2, m + 1 - i)$, equals -1 exactly at $(j - i + 3, m + 2 - i)$, and is zero elsewhere. The character χ_{B_i} gives $(A_i - A_{i+2})U_0 + (A_{i+1} - A_i)U_1$, giving its two indicated cells. With $a_0 = m - i$, the character χ_{E_i} gives

$$A_i \mathbf{1}_{a \in \{a_0, a_0+1\}} - A_{i-1} \mathbf{1}_{a \in \{a_0+1, a_0+2\}}.$$

The domain $a \leq b$ restricts its nonzero cells to precisely the four shown in (B.9). Substituting $i = m - 1$ in this last calculation gives the exponent vector of χ_P, using $X_{m-1,m-2} = h$. Finally the character χ_Q gives $-(\mathbf{1}_{L=m-2})U_1 + (\mathbf{1}_{L=m-2} + \mathbf{1}_{L=m-1})U_0$, which is $\epsilon_{12} - \epsilon_{23}$.

Each contribution has zero sum on every diagonal. On diagonal zero the contribution of E_i is the sum of two successive differences of consecutive cells, and on diagonal one it is one such difference. The same description holds for P at $a = 1$. The contribution of Q is supported on diagonal one. In every case the proper diagonal partial sums are nonnegative, proving (B.10).

Taking the partial sums of these individual columns gives (B.11)–(B.12). These equations also recover w from λ. The sums on diagonals $d \geq 2$ give the corresponding B and R weights. On the last two diagonals, the recursion $p_a = T_{a,0} - p_{a-1}$ starts at $p_0 = 0$, and then $q_a = T_{a,1} - p_a$. □

Appendix C. Calculations for the Cayley-plane mirror

We describe the change of characters on the Cayley-plane torus, the affine charts, and the contact counts used in Theorem 8.2.

C.1. **The Cayley-plane transition.** Starting with the ordered cluster $(x_1, x_2, x_3, x_4, y_3)$ of [5, Proposition 5.3], the mutations $(3, 2, 5)$ give $(x_1, x_7, z_3, x_4, t_1)$. The first exchange gives $z_3 = (y_3 + a_3x_2 + a_{10}x_4)/x_3$, and the next two give x_7 and $t_1 = x_1x_4x_7x_{10} - a_3a_6a_9a_{12}$. The five exchange polynomials have $2, 2, 5, 2, 12$ terms, respectively, and the cluster variables have degrees $1, 1, 1, 1, 4$. This chart is in family J of Daisey and Ducat, whose representative uses z_2 [5, Table 3].

C.2. **The Cayley-plane chart and Newton coordinates.** Let $\kappa = a_3a_6a_9a_{12}$. In the chart just constructed, consider the following ten characters of the initial torus.

$$\begin{aligned} \sigma &= t_1/\kappa, & \mu &= x_1x_7/(a_3a_9), \\ \nu &= x_1x_7/(a_6a_{12}), & \rho_1 &= a_4x_1/(a_1a_9), \\ \rho_2 &= a_7x_1x_4/(a_1a_{12}a_9), & \rho_3 &= a_{10}x_1x_7x_4/(a_1a_{12}a_3a_9), \\ \rho_4 &= a_2x_1z_3/(a_1a_{12}a_9), & \rho_5 &= a_{11}x_1x_7z_3/(a_1a_{12}a_6a_9), \\ \rho_6 &= a_8x_1x_7x_4z_3/(a_1a_{12}a_3a_6a_9), & \rho_7 &= a_5x_1^2x_7x_4z_3/(a_1a_{12}^2a_3a_6a_9). \end{aligned}$$

Their exponent vectors, followed by those of $a_6, \ldots, a_{12}$, form a unimodular basis of the seventeen-dimensional character lattice of the Cayley-plane torus. The characters of the mirror torus corresponding to the dual basis are $(s, u, v, p_1, \ldots, p_7, \zeta_6, \ldots, \zeta_{12})$, in that order.

Choose the chamber in which all ten characters of the initial torus have positive valuation. The normalized numerators of the regular functions x_{10}, z_2, y_3, respectively, are

$$N_x = 1 + \sigma, \qquad N_z = (1 + \mu)(1 + \nu) + \sigma, \qquad N_t = (1 + \nu)(1 + \rho_1 + \rho_2 + \rho_3) + \rho_4 + \cdots + \rho_7.$$

Their leading mutable denominators are $x_1x_7x_4$, $x_1x_7z_3$, and t_1. Thus the product $x_{10}z_2y_3$ has a minimizing vertex with mutable exponents $(-2,-2,-1,-1,-1)$ in the order x_1, x_7, z_3, x_4, t_1. For a label $m = (m_\sigma, m_\mu, m_\nu, m_{\rho_1}, \dots, m_{\rho_7})$, we write a correction in the five mutable directions as $b = (b_1, b_7, b_z, b_4, b_t) \in \mathbb{Z}^5_{\geq 0}$. Regularity gives

$$b_1 + b_7 + b_4 \leq A(m), \qquad b_1 + b_7 + b_z \leq B(m), \qquad b_t \leq C(m). \tag{C.1}$$

Here

$$\begin{aligned} A(m) &= \max(0, -m_\sigma), \\ B(m) &= \max(0, -m_\mu, -m_\nu, -m_\mu - m_\nu, -m_\sigma), \\ C(m) &= \max\bigl(0, -m_\nu, \{-m_{\rho_i}\}_{i=1}^{7}, \{-m_\nu - m_{\rho_i}\}_{i=1}^{3}\bigr). \end{aligned}$$

These are the tropical defects of x_{10}, z_2, y_3. Pairing their least mutable exponents with b gives the left sides of (C.1) after a change of sign. The number of possible corrections is at most

$$(C+1) \sum_{j=0}^{\min(A,B)} (j+1)(A-j+1)(B-j+1)$$

since there are $j+1$ choices for (b_1, b_7) with $b_1 + b_7 = j$, followed by independent choices for b_4, b_z, b_t. The intrinsic support bound follows from the model for contacts with the mutable-coordinate divisors and the slope inequality proved below.

C.3. **Affine charts and prime divisors.** Let $Y_6 = \operatorname{Spec} \mathscr{B}_6$, with $\mathscr{B}_6$ as in (8.4). Write $D(a)$ for the open set where a is invertible.

C.3.1. *The open set where s is invertible.* Let

$$Z = \operatorname{Spec} \frac{\mathbb{C}[p_1, \dots, p_7, k, u, \bar{u}, v, \bar{v}]}{(u\bar{u} - 1 - Q,\ v\bar{v} - 1 - \Pi k)}.$$

The two equations involve disjoint sets of variables, so Z is a product of two smooth fivefolds of the form $ab = 1 + c_1c_2c_3c_4$. At a singular point one would have $a = b = 0$ and hence all $c_i \neq 0$. But then the partial derivatives with respect to the c_i are nonzero. The defining polynomial is primitive and linear in a, so the hypersurface is also integral.

The last two equations of (8.4) are the linear system

$$\begin{pmatrix} u & -Tv \\ p_4\bar{v} & -\bar{u} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} -(1+R) \\ p_3(1+H) \end{pmatrix}.$$

Its determinant is s. Therefore $Y_6|_{s\neq 0} \simeq Z|_{s\neq 0}$, and Cramer's rule yields

$$sf = (1+R)\bar{u} + vL(1+H), \qquad sg = p_4(1+R)\bar{v} + p_3u(1+H).$$

These imply (8.2)–(8.3) on the initial torus.

C.3.2. *The open set where Pk is invertible.* Here all p_i and k are units. Let $J = \Pi k$ and $\widetilde{g} = Tg$. The eight characters $(Q, J, R, H, L, p_5, p_6, k)$ are a unimodular basis of the torus with coordinates $(p_1, \dots, p_7, k)$. The exponent matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \det = 1.$$

The inverse is

$$p_1 = HLp_6/(QR), \qquad p_2 = Rp_5/L,$$

$$p_3 = QJ/(Hp_5p_6k), \qquad p_4 = Q^2J/(HLp_5p_6k), \qquad p_7 = HLk/(QJ),$$

with p_5, p_6, k unchanged.

The defining equations eliminate four unit characters:

$$\begin{aligned} Q &= u\bar{u} - 1, & J &= v\bar{v} - 1, \\ R &= v\widetilde{g} - uf - 1, & H &= \frac{(u\bar{u} - 1)\bar{v}f - \bar{u}\widetilde{g} - L}{L}. \end{aligned} \tag{C.2}$$

Thus this open is a localization of $\mathbb{C}[u, \bar{u}, v, \bar{v}, f, \widetilde{g}, L^{\pm 1}, p_5^{\pm 1}, p_6^{\pm 1}, k^{\pm 1}]$, obtained by inverting the four numerators in (C.2). It is smooth and integral of dimension ten.

Since $Pk - s = 1$, these two open sets cover Y_6. Their intersection meets the initial torus, so Y_6 is smooth and integral of dimension ten. Its four defining equations have codimension four in affine 14-space and hence define a complete intersection.

C.3.3. *The coordinate divisors.* On $p_i = 0$ one has $s = -1$. If $i \leq 3$, the equation $v\bar{v} = 1 + \Pi k$ reduces to $v\bar{v} = 1$. If $i \geq 4$, the equation $u\bar{u} = 1 + Q$ reduces to $u\bar{u} = 1$. In either case the product description of Z shows that the quotient is a domain. Thus every p_i is prime.

On $u = 0$, the chart $D(s)$ is an open subset of the integral hypersurface in Z defined by $u = 0$. Indeed, $Q = -1$ and the other fivefold factor remains integral. The chart $D(Pk)$ is a localization of the ring in the preceding calculation with $u = 0$. These two integral open sets intersect, so $\mathscr{B}_6/(u)$ is a domain. Interchanging the roles of the two factors gives the same conclusion for v.

Finally, $s = 0$ implies $Pk = 1$, so this entire divisor is in the second open. There its equation is $(u\bar{u} - 1)v\bar{v} - u\bar{u} = 0$. As a polynomial in $\bar{v}$, its leading coefficient $(u\bar{u} - 1)v$ is coprime to its constant term $-u\bar{u}$. Hence it is irreducible, and the localized quotient is a nonzero domain. Thus s is prime as well. The ten prime divisors just described are distinct. The torus charts. All fourteen generators of (8.4) are cluster variables for (8.1). The variables u, v, p_i are initial, $k, \bar{u}, \bar{v}, f$ occur in adjacent clusters, and the mutations (u, v, s) give g. Also $s = Pk - 1$.

By the Laurent phenomenon [14, Theorem 5.1], $\mathscr{B}_6$ embeds in the Laurent ring of every seed. Localizing at the product of the variables of any initial or adjacent cluster gives its Laurent ring. The corresponding torus is therefore open in Y_6.

At the prime divisor $s = 0$, f is nonzero at the generic point; at $u = 0$ and $v = 0$, the same holds for $\bar{u}$ and $\bar{v}$; and at each $p_i = 0$, it holds for k. All other initial coordinates are units at the corresponding generic point. Hence these eleven tori meet every codimension-one point. Normality gives

$$\mathscr{B}_6 = L_0 \cap L_s \cap L_u \cap L_v \cap \bigcap_{i=1}^{7} L_{p_i}. \tag{C.3}$$

Every generator of $\mathscr{B}_6$ is a cluster variable, and the upper LP algebra lies in the intersection (C.3). The Laurent phenomenon now identifies both LP algebras of the coefficient-free seed with $\mathscr{B}_6$, proving Proposition 8.1.

The volume form and units. The initial logarithmic form

$$\omega = \frac{ds}{s} \wedge \frac{du}{u} \wedge \frac{dv}{v} \wedge \frac{dp_1}{p_1} \wedge \cdots \wedge \frac{dp_7}{p_7}$$

agrees up to sign with the analogous form on every adjacent LP torus: an LP exchange replaces one logarithmic differential by its negative plus a linear combination of the other differentials. Therefore ω has no zero or pole at any codimension-one point. Smoothness and extension of sections of a line bundle across codimension two make it a global nowhere-vanishing volume form.

By Proposition 2.2, the upper algebra $\mathscr{B}_6$ is factorial with unit group $\mathbb{C}^\times$. After restoring the seven coefficient characters its unit group is $\mathscr{S}_{E_6}^\times$.

C.4. **The torus factor and the contact directions.** The initial torus of U_{C} has a factor $H = (\mathbb{G}_m)^7$ that scales $a_6, \dots, a_{12}$ independently and fixes $\sigma, \mu, \nu, \rho_1, \dots, \rho_7$. In the order $(x_1, x_7, z_3, x_4, t_1)$, the following five vectors give the orders of these ten invariant characters along the mutable coordinate divisors.

	σ	μ	ν	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7
n_1	0	1	1	1	1	1	1	1	1	2
n_7	0	1	1	0	0	1	0	1	1	1
n_z	0	0	0	0	0	0	1	1	1	1
n_4	0	0	0	0	1	1	0	0	1	1
n_t	1	0	0	0	0	0	0	0	0	0

(C.4)

They are independent and generate a saturated sublattice. All entries are nonnegative. The normalized centre equations in the corresponding toric blowup charts are

x_1, x_7, x_4	$1 + \sigma$
z_3	$(1+\mu)(1+\nu) + \sigma$
t_1	$(1+\nu)(1+\rho_1+\rho_2+\rho_3) + \rho_4 + \rho_5 + \rho_6 + \rho_7.$

(C.5)

Thus three distinct toric boundary divisors carry the same centre equation $N_x = 1 + \sigma$.

The normalized exchange numerators in (C.5) are H-invariant. Hence the variables in the initial and adjacent clusters are H-eigenfunctions. The corresponding charts cover U_{C} in codimension one. By normality, their Laurent rings intersect in $\mathcal{O}(U_{\mathrm{C}})$, which is therefore $\mathbb{Z}^7$-graded. The units $a_6, \dots, a_{12}$ have the seven standard basis degrees. Each homogeneous element is consequently an invariant times a Laurent monomial in these units. This gives

$$U_{\mathrm{C}} \simeq X \times H, \qquad X = \operatorname{Spec} \mathcal{O}(U_{\mathrm{C}})^H.$$

We regard each eigenfunction as a function on X after division by the unique Laurent monomial in $a_6, \dots, a_{12}$ of the same H-degree. We use this convention for the mutable functions and for x_{10}, z_2, y_3. The divisor directions and centre equations in (C.4)–(C.5) are unaffected by division by these units.

C.5. **Contact interpolation on the Cayley-plane complement.** The Cayley-plane coordinate ring is factorial, and its five initial mutable functions are pairwise nonassociate primes. Its initial and adjacent LP charts therefore cover codimension one. On the open where the other coordinates are invertible, an associated exchange $x_i x_i' = F_i/M_i$ becomes $x_i w_i = F_i, \qquad w_i = M_i x_i'$. This is the blowup of the ideal (x_i, F_i) in the toric chart for the ray n_i, with the strict transform of its toric boundary removed. The derivatives of the three centre polynomials with respect to σ, σ, ρ_4, respectively, are all one. These characters are invertible on the corresponding ray orbits, and their exponents annihilate the corresponding ray directions. Thus each centre is smooth. Write X° for the union of the rank-one charts and the initial torus in X. Its complement has codimension at least two.

The punctured Cayley-plane cone is smooth because it is the complement of the zero section in the tautological line bundle. Inverting the boundary coordinates removes its vertex. The product decomposition therefore implies that X is smooth. On its initial torus take the standard logarithmic volume form. In a smooth rank-one chart $x_i w_i = F_i$, choose F_i as a local coordinate. The identity $(dx_i/x_i) \wedge dF_i = dx_i \wedge dw_i$ shows that this form extends without a zero or pole across the generic point of each missing mutable-coordinate divisor. The boundary complement is smooth, and the rank-one charts cover codimension one, so the form extends to a nowhere-vanishing form on X. In the unimodular character basis, its wedge with $\bigwedge_{j=6}^{12} d\log a_j$ gives the volume form on U_{C} up to sign. Together with the open torus this gives the log Calabi–Yau and maximal-boundary hypotheses of [12, Section 2].

Choose a compactification satisfying the hypotheses of [12]. The common toric-blowup model can be enlarged by the toric rays of this compactification. Away from a fixed codimension-two subset it agrees with X° and with the generic points of the essential boundary divisors. The counted maps lie in the smooth evaluation locus by [12, Lemmas

8.20–8.21]; generic avoidance then follows from [12, Lemma 3.9]. We also exclude the intersections of distinct coordinate divisors. After the curve-class bound below, one generic evaluation point suffices for all cases being counted.

To apply Proposition 5.2, consider the end directions m and $-m-\sum_i b_i n_i$. Substitution from (C.4) shows that, among the cases of (C.6) and the ten positive coordinate labels, an end lies on a centre ray only for the two straight curves with labels $\pm e_\sigma$. Add these finitely many end rays to the partial toric fan. The open toric divisors of the new rays are disjoint from the blowup centres, so nonzero leading angular coefficients suffice there. In the two exceptions the relevant centre polynomial N_t has residue one. Hence these marked ends also lie in the open essential stratum.

One can first complete this partial toric blowup without altering its open ray divisors. The incidence solutions then give maps with the required boundary contacts in that compactification. Their simple incidence Jacobians make evaluation generically étale. The deformation calculation in [12, Lemma 3.6] gives the trivial logarithmic tangent pullback, and the argument of [12, Lemma 3.9] lets generic such curves avoid every fixed codimension-two locus. Apply this to the centres of a common resolution with a compactification of X. The resulting identification preserves the generic evaluation fibre and the multiplicity of each solution. For arbitrary incoming labels we use only the necessary slope inequalities.

C.6. **Finite expansions in the adjacent charts.** Let $\mathcal{C}_+$ be the chamber in which all ten character valuations are positive. For an incoming label m, the total correction is $b_1n_1+b_7n_7+b_zn_z+b_4n_4+b_tn_t$ with $b_i\ge 0$. The three functions x_{10}, z_2, y_3 give exactly the bounds (C.1), with A, B, C defined in the preceding Newton-coordinate calculation. These follow from the principal-divisor slope formula for regular source functions.

These functions also exclude a nonconstant tropical branch attached in $\mathcal{C}_+$. Indeed, such a branch would give $b\ge 0$, $b\neq 0$, and the outgoing slope of every regular function at its root is nonnegative. The three slopes in question are $-b_1-b_7-b_4$, $-b_1-b_7-b_z$, and $-b_t$, which force $b=0$.

For each of the ten character coordinates, let $\mathcal{C}_i$ be the chamber where that coordinate is negative and all others are positive. Each entire orthant lies in one chamber of the three centre tropicalizations. The unique minimizing terms of the last two centres, and their slopes on b, give the following recession tests:

negative coordinate	x_{10} slope	z_2 slope	y_3 slope
σ	$-b_1-b_7-b_4+b_t$	$-b_1-b_7-b_z+b_t$	$-b_t$
μ	$-b_1-b_7-b_4$	$-b_z$	$-b_t$
ν	$-b_1-b_7-b_4$	$-b_z$	$b_1+b_7-b_t$
ρ_i	$-b_1-b_7-b_4$	$-b_1-b_7-b_z$	$\langle \rho_i, \sum b_j n_j\rangle - b_t$.

Nonnegativity of each triple, together with $b\ge 0$, forces $b=0$. Hence no nonconstant tropical branch is attached in these chambers, and they have bounded correction polytopes for every incoming label. The actual contact directions generate a free pointed correction monoid, including the bends of infinitesimal cylinders. Lemma 5.1 therefore bounds the contributing classes. The specialized local expansions are Laurent polynomials in all eleven chambers. The straight coefficient is one.

In $\mathcal{C}_i$, the theta labels $-e_i$ and e_j for $j\neq i$ are straight. To see this, trace a contributing spine backwards from its terminal point. Every intermediate exponent is its incoming label plus a nonnegative combination of (C.4). For $j\neq i$ these exponents have nonnegative j-component. Consequently those coordinates stay at least as large as their strictly positive values at the marked ends. The entire spine lies in $\mathcal{C}_i$, $\mathcal{C}_+$, or their common facet. Neither open chamber permits a bend. Every possible facet bend has zero i-component: on the σ facet the y_3 slope forces $b_t=0$; on the μ facet the x_{10}, y_3 slopes leave only b_z; on the ν facet the x_{10} slope leaves only b_z, b_t; on a ρ_i facet the x_{10}, z_2 slopes leave only b_t. In each case the remaining directions have zero component normal to that facet. Thus the i-component of the

spine direction is unchanged. It is -1 for $-e_i$, and zero for each unchanged coordinate e_j, so the backwards spine cannot reach the facet from its incoming end. There is no bend anywhere. The corresponding local expansions are therefore precisely z^{-e_i} and z^{e_j}. At a generic point of the common facet there is only the one integral hyperplane in question. The wall-crossing automorphism is therefore determined by its value on this one transverse monomial, by [12, Theorem 20.9 and Proposition 20.13].

The expansions in the positive chamber are as follows.

incoming label	Θ_m in $\mathcal{C}_+$	allowed contact types
$-e_\sigma$	E/s	$\varnothing, 1, 7, z, 4, \{4, z\}$
$-e_\mu$	$(1+Q)/u$	$\varnothing, z$
$-e_\nu$	$(1+s+Q)/v$	$\varnothing, z, t$
$-e_{\rho_i}$	$(1+s)/p_i$	$\varnothing, t.$

(C.6)

Every listed incidence scheme has length one. The formula for $-e_\sigma$ follows by placing an x-contact at $-\sigma$ and a z-contact at $-\sigma/((1+\mu)(1+\nu))$; the mixed $\{4, z\}$ contacts are distinct for generic evaluation. For a single contact in these cases, the relevant centre polynomial is $A + By$ in the transverse character y, with A and B of residue one at the evaluation point. The contact factor cancels from the centre equation, which reduces to $Ar + By(c) = 0$ for the contact parameter r and evaluation c. This gives one simple root of valuation $\mathrm{val}(y(c))$. For $-e_\nu$ the only additional possibility is $\{z, t\}$. With contact parameters r, t, subtraction of the two centre equations factors as

$$(r-t)\left[(1+\mu)(1-t)\left(1+\frac{t(\rho_4+\cdots+\rho_7)}{(1+\rho_1+\rho_2+\rho_3)(1-r)}\right)+\sigma r\right]=0.$$

All evaluation coordinates have positive valuation. Thus $1+\rho_1+\rho_2+\rho_3$ and $1-r$ are units, while $t(\rho_4+\cdots+\rho_7)/((1+\rho_1+\rho_2+\rho_3)(1-r))$ and σr have positive valuation. The bracket consequently has residue 1. Hence $r = t$, and the remaining centre equation gives $r = t = -\nu$. This collision is excluded. This proves (C.6).

The resulting wall functions are

$$E, \qquad 1+Q, \qquad 1+s+Q, \qquad 1+s \quad \text{(the last repeated seven times)}.$$

The common compactification and bounds (C.1) give the marked Laurent realization by [8, Proposition A.5]. Its adjacent expansions are related by these wall transformations. The associated normalizers are monomials in the unchanged variables, so these transformations give exactly the ten adjacent LP Laurent rings. Their straight coordinates form unimodular monomial systems; localizing the intrinsic algebra at them identifies the corresponding open tori. By the intersection calculation (C.3),

$$\Phi : A_{\mathrm{int}}(X)_1 \hookrightarrow \mathscr{B}_6 \subset \mathbb{C}[s^{\pm1}, u^{\pm1}, v^{\pm1}, p_1^{\pm1}, \ldots, p_7^{\pm1}]. \tag{C.7}$$

C.7. **Recognition of generators and markings by Laurent regularity.** The map (C.7) gives Laurent expansions in the initial and ten adjacent charts. On a fixed finite support, the slice divisibilities of [8, Lemma B.1] are linear conditions on the coefficients. We apply them in the notation of §8.1.

A finite refinement of the source bounds. Write a contact vector in the order $\beta = (\beta_1, \beta_7, \beta_z, \beta_4, \beta_t)$. We refine (C.1) using four further regular coordinates x_3, x_5, x_9, x_{11}. For each coordinate the table gives the negative mutable exponent of its least term, in the order $(x_1, x_7, z_3, x_4, t_1)$, and its tropical defect at each of the four incoming labels. We name the labels by the proposed functions g, W_3, W_9, W_{12}, omitting their coefficient characters.

	$-h^{\mathrm{mut}}$	$\delta(g)$	$\delta(W_3)$	$\delta(W_9)$	$\delta(W_{12})$
x_3	$(1,1,1,0,1)$	1	2	2	2
x_5	$(1,0,0,0,1)$	1	1	1	2
x_9	$(2,1,1,1,1)$	2	2	3	2
x_{11}	$(1,1,0,1,1)$	2	1	2	2

Substitution in the Cayley-plane quadrics gives the entries. By homogeneity all exponents after this normalization lie in the ten-dimensional character lattice of X.

Each of the four coordinates gives the inequality $-h_{x_i}^{\mathrm{mut}}\beta \le \delta_{x_i}(m)$. Together with the three coarse bounds, these inequalities give the following possible numerator supports, with monomials identified with their exponent vectors. The last column is the rank of the linear system imposed by Laurent regularity in the adjacent charts.

function	possible numerator monomials	size	rank
f	$\operatorname{supp} E$	6	5
$\bar{u}$	$1, Q$	2	1
$\bar{v}$	$1, Q, s, sQ$	4	3
k	$1, s$	2	1
g	$\operatorname{supp} E \cup \{s, sR\}$	8	7
W_3	$\operatorname{supp} E \cup \{s, sQ\}$	8	7
W_9	$\operatorname{supp} E \cup \{s, sR, sQ, sRQ, suvL\}$	11	10
W_{12}	$\operatorname{supp} E \cup \{s, sQ, sR, suvL, s^2\}$	11	10

The three initial bounds suffice for $f, \bar{u}, \bar{v}, k$. Adjacent divisibility forces the coefficient of sQ in the numerator of $\bar{v}$ to vanish. The denominators of the eight functions, in order, are

$$su, \quad u, \quad vQ, \quad P, \quad svT, \quad sup_3p_6p_7, \quad suP, \quad svp_2p_3p_4p_5p_6p_7^2.$$

Proposition C.1. *Each of these eight finite support spaces has a unique Laurent-regular function with incoming coefficient one. These functions are $f, \bar{u}, \bar{v}, k, g, W_3, W_9/\zeta_9, W_{12}/\zeta_{12}$, and hence are the corresponding intrinsic theta expansions.*

Proof. The table gives the ranks of the Laurent-divisibility systems. In each case a minor of size one less than the support size has determinant ± 1, and the proposed numerator is a nonzero kernel vector with incoming coefficient one. It is therefore the unique normalized solution. The exact matrices, minors and kernel vectors are in `anc/e6_recognition_certificates.json`. For W_{12}, the coefficient of s^{-1} is divisible by E, fixing six coefficients. The numerator then has the form $E+s(a_0+a_QQ+a_RR+a_LuvL)+a_2s^2$. The p_7^{-2} slice requires $(1+s)^2 \mid 1+a_0s+a_2s^2$, hence $a_0 = 2$ and $a_2 = 1$. The v^{-1} and p_2^{-1} slices then give $a_Q = a_R = a_L = 1$.

Each intrinsic expansion belongs to the indicated support space and satisfies these divisibilities by (C.7). The incoming coefficient one therefore identifies it uniquely. □

Appendix D. Calculations for the Freudenthal mirror

D.1. **Newton characters and support bounds.** In the Freudenthal seed of Theorem 4.1, order the mutable variables as $(h, b, c, d, e, f, g, r, q, u)$ and the frozen variables as $(a_0, \dots, a_{17})$. In this order the twenty Newton characters are as follows.

i	χ_i	i	χ_i
1	$bda_4a_{17}/(qa_3a_{16})$	11	$hqa_5/(bdga_4)$
2	$hcra_{13}/(gua_{14})$	12	$fa_{10}/(ca_{13})$
3	$qa_{16}/(hba_{15})$	13	$bfa_7/(cea_{13})$
4	fa_4/r	14	$ra_3/(dga_4)$
5	$da_1a_6/(ea_3a_{16})$	15	$bdga_0/(qa_3a_{16})$
6	$fa_6a_{11}/(cea_{13})$	16	$bega_{15}/(fqa_5)$
7	$a_9a_{14}/(ca_{13})$	17	$dega_{12}/(fqa_5)$
8	$u/(hea_{12})$	18	$da_2/(a_3a_{16})$
9	$u/(hfa_8)$	19	$ga_8/(ca_5)$
10	$bega_{14}/(fqa_6)$	20	$fqa_5/(era_{13})$.

Let C be the 28×28 matrix whose rows are the exponents of $\chi_1, \dots, \chi_{20}, a_0, \dots, a_7$ in the ordered seed. Integer elimination gives $\det C = 1$. The 56 linear coordinates have positive Laurent expansions with 6050 terms in total. They satisfy all 144 cyclic translates of the defining quadrics.

Each of these coordinates and the mutation $f^\sharp = F_f/(hf)$ has a unique term whose exponent is coordinatewise least in the new basis. That term has coefficient one. After division by it, the expansion is a polynomial in $\chi_1, \dots, \chi_{20}$ with constant term one. The Hermite normal form of the matrix of exponent differences for the 56 coordinates has nonzero part I_{20} in the χ_i-coordinates. Thus these differences generate the full lattice spanned by the χ_i.

For each of the 57 functions ψ, write $N_\psi \subset \mathbb{Z}^{20}_{\geq 0}$ for the support after this division. Let H_ψ be the vector of the first ten exponents of the least term in the original seed. For the first twenty coordinates m of a label in the new basis, set $\delta_\psi(m) = -\min_{a \in N_\psi} \langle a, m \rangle$. A contact vector $\beta \in \mathbb{Z}^{10}_{\geq 0}$ must satisfy $H_\psi \beta \geq -\delta_\psi(m)$ for every ψ. The product $G = x_{15}^2 x_2 y_{14} x_5 y_{11} x_{10} x_{13} x_0^2 y_1^3$ has least term with original mutable exponent $(-1, \dots, -1)$. Hence $\sum_{j=1}^{10} \beta_j \leq \delta_G(m)$. It suffices to test the lattice points of this simplex against the 57 inequalities.

At the ith facet of the positive chamber, define

$$d_{\psi i} = \max\{a_i : a \in N_\psi,\ a_j = 0\ (j \neq i)\}.$$

At a point in the relative interior of the facet, only the powers on the ith axis minimize the weight. Their least and greatest exponents are 0 and $d_{\psi i}$, respectively. These exponents give the minimizing terms on the two sides of a sufficiently short transverse crossing. Applying the slope inequality on the two arms as in the proof of [8, Lemma A.1] gives $H_\psi \beta \geq -d_{\psi i}$, and tangency gives $\sum_j C_{ij} \beta_j = 0$. Summing the inequalities for the factors of G, with their multiplicities, again bounds $\sum_j \beta_j$. The table gives the unique nonzero vector satisfying these conditions at each facet. Here $e_h, \dots, e_u$ are the standard basis vectors in the ten original mutable directions.

i	β	i	β
$1, 5, 15, 18$	$e_h + e_u$	2	e_f
$6, 7, 12, 13, 20$	e_u	3	e_d
$11, 14$	$e_g + e_r + e_q$	4	e_g
$16, 17, 19$	e_r	8	$e_b + e_q$
		9	e_c
		10	$e_e + e_f + e_u$.

Applying C to the two allowed exponents at each facet and clearing the common Laurent monomial gives the support of the ordinary mirror exchange. Proposition 8.4 gives both coefficients.

For the boundary label Ce_{a_j}, the first twenty entries give the exponent vector m_j and the last eight give the coefficient character ϵ_j. Using all of N_ψ in the inequalities for m_j gives the following bounds and numbers of allowed exponents.

j	0	1	2	3	4	5	6	7	8
$\delta_G(m_j)$	0	0	0	3	3	2	3	0	2
support size	1	1	1	3	2	3	5	1	3
j	9	10	11	12	13	14	15	16	17
$\delta_G(m_j)$	0	0	0	2	1	3	6	2	0
support size	1	1	1	2	2	5	5	2	1.

For a Laurent polynomial $P = \sum_k v_i^k P_k$ with one of these support bounds, regularity in the ith adjacent chart is equivalent to $E_i^{-k} \mid P_k$ for every $k < 0$, by (2.1). The divisibility is taken in the Laurent ring of the unchanged coordinates, where the normalizing monomial is a unit. We solve the resulting linear equations for the boundary coefficients in §D.2.3.

D.2. **Intrinsic comparison in the primary chart.** Write $t_i = \chi_i$ and let N be the submatrix of C in its first twenty rows and first ten columns. Its columns n_v, indexed by the ten mutable variables in seed order, give the contact directions of the corresponding coordinate divisors. The remaining eight rows of C vanish in these columns. Since C is unimodular, the n_v span a direct summand of $\mathbb{Z}^{20}$. The degree and weight gradings make the

normalized coordinate expansions independent of $a_0, \dots, a_7$. These eight invertible characters therefore give a torus factor $U_{\mathrm{F}} = V_{\mathrm{F}} \times \mathbb{G}_m^8$, where V_{F} is the fibre over $(1, \dots, 1)$ of $(a_0, \dots, a_7)$. We work on V_{F} and recover the full mirror by the product formula [8, Lemma 3.7].

D.2.1. *The initial intrinsic chart.* Dividing the ordinary exchange polynomial F_v by its coordinatewise least monomial in the characters $(t_1, \dots, t_{20}, a_0, \dots, a_7)$ gives a polynomial $P_v(t)$ with positive coefficients and constant term one. Every $\alpha \in \operatorname{supp} P_v$ satisfies $\langle \alpha, n_v \rangle = 0$, since F_v is independent of v. Thus $P_v = 0$ defines the blowup centre on the open orbit of the toric divisor in direction n_v. For instance,

$$P_h = 1 + t_2t_3t_4t_5t_{10}t_{14}t_{16}t_{20}, \qquad P_b = 1 + t_2t_4t_6t_8t_{10}t_{11}t_{17}, \qquad P_g = 1 + t_4.$$

The ten polynomials P_v are recorded in `anc/e7_calibration.json`.

Theorem 4.1 identifies the rank-one charts along the ten coordinate divisors with the corresponding toric blowup charts. The logarithmic Jacobian identity for LP mutation identifies their volume forms up to sign. Each centre is smooth on the open orbit because its equation is linear in an invertible coordinate with monomial leading coefficient. These charts contain the generic points of all ten divisors outside the initial torus. We use a smooth compactification as in [12, Assumption 2.4]. After compatible refinements, it agrees with the toric compactification near the generic points of the essential boundary strata [12, Lemmas 2.2 and 2.6(2)].

For a fixed incoming label, the slope inequality for G bounds the contact vectors by the simplex of §D.1. It also excludes a nonconstant branch rooted in the positive chamber. The contact vector of such a branch would satisfy $-\sum_v \beta_v \geq 0$, forcing $\beta = 0$. A branch without contacts is toric and constant. Every bend is a nonnegative integral combination of the independent n_v, so Lemma 5.1 gives finiteness of the contributing numerical curve classes before specialization.

The Laurent realization [8, Proposition A.5] now gives finite theta expansions with distinct monic least terms and an injective algebra map

$$\Phi : A_{\mathrm{int}}(V_{\mathrm{F}})_1 \hookrightarrow \mathbb{C}[v_1^{\pm 1}, \dots, v_{20}^{\pm 1}].$$

If a label has nonnegative coordinates, all the δ_ψ vanish. Its theta function is therefore the corresponding monomial. Inverting the v_i therefore gives the initial torus of the intrinsic spectrum.

D.2.2. *Contact counts at the twenty facets.* Write $\beta^{(i)}$ for the unique nonzero contact vector allowed at facet i by §D.1. Its entries are zero or one. Twelve facets require one contact, five require two, and three require three.

Fix i and write $\beta = \beta^{(i)}$ and $\Delta = N\beta$. Tangency gives $\Delta_i = 0$. Let e_i be the ith standard basis vector in the Newton coordinates. Choose a point x on the facet whose nineteen positive coordinates are linearly independent over $\mathbb{Q}$. Work over a complete algebraically closed valued extension of the trivially valued field $\mathbb{C}$ whose valuation ν takes the required real values. Choose $\delta > 0$ rationally independent of the positive coordinates of x, and set $\omega = x + \delta(e_i - \Delta)$. The unmarked ends of the tropical curve have directions n_v with multiplicities β_v. The *twigs* are the branches outside the subtree spanned by the marked points. Their total degree is bounded by the sum of the norms of these unmarked directions, counted with multiplicity [12, Definitions 4.11–4.12 and 4.22]. This bound is independent of δ. With the two marked directions fixed, only finitely many extended curve classes can occur [12, Proposition 4.24 and Lemma 4.30(2)]. The classes of the local cylinders are also finite for this displacement [12, Lemma 20.29]. Thus we may choose one δ small enough that $\omega_j > 100\delta$ for $j \neq i$ and the spine meets the relevant walls only at x [12, Definition 20.2]. Let $c = (c_j)$ be a geometric evaluation point with $\nu(c_j) = \omega_j$. These weights are rationally independent and $\omega_i = \delta$.

For a counted cylinder, generic evaluation gives a smooth three-marked domain and transverse contacts in the rank-one charts [12, Lemmas 8.20–8.21]. Place the ends at $0, \infty$

and the evaluation marking at 1, and write r_v for the parameter of the contact of type v. By Proposition 5.2,

$$t_j(z) = c_j z^{-\delta_{ij}} \prod_{v:\,\beta_v=1} \left(\frac{z-r_v}{1-r_v}\right)^{(n_v)_j}, \tag{D.1}$$

where δ_{ij} is the Kronecker delta. To evaluate the centre equation $P_v(t(r_v)) = 0$, first cancel the factor belonging to r_v. It has exponent zero in every character of P_v. On the main annulus write $\lambda = \nu(z)$ and take the cuts at 0 and 2δ. Since the columns n_v are independent, a linear functional takes value one on each of them. Proposition 5.2 therefore excludes contacts on the contracted leg towards 1 and on either toric tail, including its attachment point. Hence $0 < \nu(r_v) < 2\delta$ and $\nu(1-r_v) = 0$ for every v. At the Gauss point of parameter λ, (D.1) has tropicalization

$$\omega - \lambda e_i + \sum_{v:\,\beta_v=1} \min\{\lambda, \nu(r_v)\}\, n_v.$$

As λ passes a contact value, the slope changes by the negative sum of the columns at that value. Independence makes each such change nonzero. The prescribed spine bends only at $\lambda = \delta$, so every contact parameter of a counted cylinder has valuation δ.

Conversely, consider a solution of the centre equations with distinct contacts of valuation δ. The preceding formula gives the prescribed bend at $\omega - \delta e_i + \delta\Delta = x$ and the two prescribed slopes. All characters in (D.1) are invertible away from the contacts and the two ends, so both punctured tails lie in the torus. At each r_v the corresponding mutable coordinate has a simple zero and the other source coordinates are units. The centre equation makes the rank-one chart coordinate F_v/x_v regular there, where x_v denotes the mutable coordinate at that ray.

Choose compatible toric and toroidal refinements [12, Lemmas 2.2 and 2.6] containing the rays generated by $-e_i$ and $e_i - \Delta$. Both vectors are primitive since $\Delta_i = 0$. At zero and infinity the valuation vectors of the leading coefficients in (D.1) are $\omega + \delta\Delta$ and ω, respectively. The first is $(I + \Delta e_i^{\mathsf{T}})\omega$, and this integral matrix has determinant one. Thus distinct characters on either end orbit have distinct valuations at the marked point. Every nonzero Laurent polynomial on that orbit over $\mathbb{C}$ has a unique term of least value there. The marked point consequently lies in every dense open subset of the orbit defined over $\mathbb{C}$, including the open set where the toric and chosen compactifications agree [12, Lemma 2.6(2)].

The extensions at the contacts and the ends give a stable map to the chosen compactification. Its boundary pullback consists of 0 and ∞, each with multiplicity one by primitivity. Its evaluation lies over the monomial point of weight ω. Hence the map is skeletal, its spine is the one computed above, and evaluation is étale there [12, Lemmas 8.20–8.22]. Proposition 5.2 identifies the resulting incidence scheme with the geometric fibre for this spine and its toric tails. The three marked points fix the parametrization, and each contact type occurs only once. Each reduced solution therefore contributes one.

Lemma D.1. *At each of the twelve facets with $\beta^{(i)} = e_v$, the cylinder count for that contact vector is one.*

Proof. For each such (i, v), the only terms of P_v supported on the t_i-axis are 1 and t_i, by the normalized exchanges in `anc/e7_calibration.json`. Let r be the contact parameter. Since every exponent of P_v pairs to zero with n_v, the contact factor in (D.1) cancels from the centre equation, leaving

$$P_v(c_1, \ldots, c_{i-1}, c_i/r, c_{i+1}, \ldots, c_{20}) = 0.$$

The prescribed spine requires $\nu(r) = \delta$. Write $r = c_i R$, with R a unit in the valuation ring. Each term involving a c_j for $j \neq i$ has positive valuation. The equation therefore reduces to $1 + R^{-1} = 0$ in the residue field. Its only root is -1, which is simple. After multiplication by a power of R, Hensel's lemma gives exactly one unit solution. Simplicity also shows that the

centre function on the domain has a simple zero at $z = r$. This point of the cylinder fibre has multiplicity one. □

Pairs. The pairs are (h, u) at $i = 1, 5, 15, 18$, and (b, q) at $i = 8$. Write s, t for the respective contact parameters. In (D.1), collect the terms of each P_v by their powers of c_i and of the factor for the other contact. The centre equations become

$$1 + Ms^{-\epsilon}\frac{1-t}{s-t} = 0, \qquad U + \left(V + \frac{W}{t}\right)\frac{1-s}{t-s} = 0,$$

where $\epsilon = 1$ for $i = 5, 8$ and $\epsilon = 0$ otherwise. Write $V = LV_0$, $W = Lc_iW_0$, and $M = Lc_i^{\epsilon}m$. The coefficients U, V_0, W_0 have residue one. The following monomials L, m both have valuation greater than 100δ.

i	L	m
$1, 15, 18$	$c_{14}c_{16}c_{20}c_3c_4$	$c_{10}c_2c_5$
5	$c_{14}c_{16}c_{20}c_3c_4$	$c_{10}c_2$
8	$c_{11}c_{17}c_4$	$c_{10}c_2c_6$.

If $\epsilon = 0$, elimination gives

$$s = (1 + M)t - M, \qquad t = -\frac{(1 + M)W}{UM + (1 + M)V}.$$

Both parameters have valuation δ, and $s - t = M(t - 1) \neq 0$. The linear equations give a unique simple solution. If $\epsilon = 1$, the second equation gives $s = \frac{Ut^2+Vt+W}{Ut+Vt+W}$. The first reduces, after removing the inadmissible factor $t - 1$, to

$$UVt^3 + (M(U + V)^2 + UW + V^2)t^2 + 2W(M(U + V) + V)t + (1 + M)W^2 = 0.$$

Writing $\ell = \nu(L)$, the coefficient valuations, from constant to cubic, are $2\ell + 2\delta, 2\ell + \delta, \ell + \delta, \ell$. The Newton polygon gives one simple root of valuation δ and two roots of valuation $(\ell+\delta)/2 > 2\delta$. For the admissible root, $\nu(s) = \delta$ and $s - t = -M(1 - t)/s \neq 0$. Thus each of the five pairs contributes one.

Triples of types g, r, q. At $i = 11, 14$, let s, r, t be the parameters of the contacts of types g, r, q, respectively. With $a = c_4$, the g-equation gives $s = (1+a)r - a$. Set $V = (r - t)/(1 - t)$ and $\widetilde{V} = (t - r)/(1 - r)$. Substitution gives the other two centre equations in the table.

i	r-equation	q-equation
11	$A + K(B + Y/r)/V = 0$	$1 + c_{11}a(B + Y/t)/(t\widetilde{V}) = 0$
14	$A(r) + K(r)(B + L/r)/V = 0$	$1 + c_{11}a(B + L/t)/\widetilde{V} = 0$.

We use $\equiv$ for equality of residues. At $i = 11$ the coefficients satisfy

$$A \equiv 1, \qquad K = \frac{a(1 + c_{14}(1 + a))}{1 + a}, \qquad \nu(B) > 100\delta, \quad \nu(Y/B) > 100\delta.$$

For $i = 14$, they satisfy

$$A(r) = A_0 + T/r, \quad A_0 \equiv 1, \quad K(r) = \frac{a}{1 + a}\left(1 + \frac{(1 + a)c_{14}}{r}\right),$$

with $\nu(B), \nu(L/B), \nu(T) > 100\delta$. These coefficients are obtained by collecting terms of the normalized exchange polynomials in `anc/e7_calibration.json`, keeping all eighteen frozen variables independent. Their monomial factors give the indicated valuations.

Write $H = K(B + Y/r)/A$ for $i = 11$ and $H = K(r)(B + L/r)/A(r)$ for $i = 14$. The r-equation reads $V = -H$, so $t = (r + H)/(1 + H)$. Throughout $0 < \nu(r) < 2\delta$, we have $\nu(H) > 2\delta$. Thus t and s have the same valuation as r. At $i = 11$, the q-equation becomes

$$t = -c_{11}\frac{aA}{K}(1 + H)\frac{B + Y/t}{B + Y/r}.$$

The factor multiplying $-c_{11}$ has residue one, so $\nu(r) = \delta$. Write $r = c_{11}R$. The unit factors in the fractions have residue one, so clearing denominators gives the residue equation $R + 1 = 0$ up to a unit at $R = -1$. Its root is simple, and Hensel's lemma gives exactly one lift.

At $i = 14$, the equation instead becomes

$$1 + \frac{(1+a)c_{14}}{r} = -c_{11}(1+a)A(r)(1+H)\frac{B + L/t}{B + L/r}.$$

Its right side has positive valuation, so again $\nu(r) = \delta$. Writing $r = c_{14}R$ again gives the simple residue equation $R + 1 = 0$. Hensel's lemma gives a unique lift. In both cases $H \neq 0$ at the solution: for $i = 14$ this follows from the nonzero right-hand side of the last equation. Hence $t - r = H(1 - r)/(1 + H)$ is nonzero. Its valuation exceeds that of $s - r = -a(1 - r)$, so the three contacts are distinct. The eliminations are reversible there, and the simple root gives a reduced point of the incidence scheme.

The triple e, f, u. At $i = 10$, write $z = c_{10}$, $a = c_2$, and let s, t, w be the contact parameters of types e, f, u, respectively. Collecting terms in the three centre equations gives

$$\begin{aligned}
&(s + zaZ)(s - w) + zaA(1 - w) = 0,\\
&t(t - w) + a(B + zY/t)t(1 - w) = 0,\\
&B + zY/w + (C + zLY/w)V + AV/U + DV^2 = 0,
\end{aligned}$$

where $U = (w - s)/(1 - s)$ and $V = (w - t)/(1 - t)$. To specify the coefficients, let $V_e = (s-w)/(1-w)$ and $V_f = (t-w)/(1-w)$. In these variables $P_e = 1+zaZ/s+zaA/(sV_e)$ and $P_f = 1+a(B+zY/t)/V_f$. The last equation is the expansion of P_u in powers of U, V and w^{-1}. In the normalized exchanges, B has residue one and a, A, Y, Z, C, D, L have valuation greater than 100δ. Moreover, A contains the terms c_{13} and c_6, and $Z = c_{11}c_{17}c_4c_8(c_{13} + c_6)$, so $\nu(Z/A) > 100\delta$.

Fix w with $0 < \nu(w) < 2\delta$. The first two equations are quadratic in s and t, respectively. In each equation, division by w^2 and substitution of s/w or t/w as variable gives the reduction $X(X - 1)$. The lift of 1 satisfies $\nu(s - w) > 2\delta$ or $\nu(t - w) > 2\delta$, respectively, by the original equation. The product of the two roots shows that the other root has valuation greater than 2δ. For the roots in the annulus,

$$U = \frac{zaA}{s + za(Z + A)}, \qquad V = \frac{a(B + zY/t)}{1 + a(B + zY/t)}.$$

Substitution in the third equation gives $B(1 + w/z)$ up to terms of valuation greater than $\min(0, \nu(w) - \delta)$. Hence $\nu(w) = \delta$ at any solution. Write $w = zW$, $s = w + aAS$, and $t = w + aT$. Divide the first two equations by zaA and at, respectively, and multiply the third by S. The residue equations are

$$WS + 1 = 0, \qquad T + 1 = 0, \qquad S + T = 0.$$

Their unique solution is $(W, S, T) = (-1, 1, -1)$. The denominators after rescaling are units there, and the reduced Jacobian has determinant -1. The multivariate Hensel lemma gives one simple lift. Conversely, the first two equations force S and T to be units whenever $\nu(w) = \delta$ and s, t lie in the annulus. Thus every admissible solution is this lift. The contacts are distinct because $\nu(s - w) = \nu(a) + \nu(A) > \nu(a) = \nu(t - w)$.

Proof of Proposition 8.4. Lemma D.1 gives coefficient one at the twelve facets with a single contact. At each of the remaining eight facets the preceding calculations give one reduced point of the cylinder fibre. Since the support bounds allow no other nonzero contact vector, these are all the nonconstant wall terms. □

D.2.3. *The first mutations and boundary functions.* Consistency of wall crossing [12, Proposition 20.35] gives the intrinsic expansion in each adjacent chart by the binomial transformation at its facet. Lemma 5.3 shows that these expansions are Laurent polynomials. Thus the

intrinsic functions are Laurent in the initial chart and all twenty adjacent charts. Their intersection is the upper LP algebra by the starfish theorem [7, Theorem 1.3].

Let $\Delta^{(i)} = N\beta^{(i)}$ and let $\mathbf{m}_i$ be the exponent of the normalizing monomial of the ith mirror exchange. The label of the first mutation is $m_i' = -[\Delta^{(i)}]_- - \mathbf{m}_i - e_i$, where $[\,\cdot\,]_-$ takes the coordinatewise minimum with zero. At m_i' the support inequalities bound $\sum_v \beta_v$ by the number of contacts in $\beta^{(i)}$ and admit only 0 and $\beta^{(i)}$. On these two monomials, adjacent Laurent divisibility requires the coefficients to agree. The least coefficient is one, so the theta function with label m_i' is v_i'.

For each of the eighteen boundary labels, take the support bound of §D.1. Every negative coordinate exponent is -1. In a coefficient divisible by an exchange binomial, the terms occur in pairs whose exponents differ by an odd multiple of the difference of the two exchange exponents. Modulo the binomial, the paired monomials are negatives of one another, so their coefficients agree. Joining each such pair gives a connected graph on the allowed monomials for every label. All coefficients therefore equal the least coefficient, which is one. This gives exactly W_j in each case. The eighteen systems, their normalized solutions and maximal minors of determinant ± 1 are recorded in `anc/e7_boundary_certificates.json`.

After restoring the coefficient torus, the intrinsic image $\mathscr{R}_{E_7}$ satisfies

$$\mathscr{S}_{E_7}[v_i, v_i', W_j] \subseteq \mathscr{R}_{E_7} \subseteq \mathcal{U}(\Sigma_{E_7}^{\vee}).$$

Inverting all v_i gives the initial torus. If $d_i = \prod_{j\neq i} v_j$, the two outer algebras have the same localization at d_i. Indeed, only the initial and ith adjacent Laurent charts impose conditions there. Thus

$$\mathscr{R}_{E_7}[d_i^{-1}] = \frac{\mathscr{S}_{E_7}[v_j^{\pm 1} : j \neq i][v_i, V_i]}{(v_i V_i - E_i)},$$

where V_i is v_i' times its normalizing monomial. We next show that these open sets meet every prime divisor.

D.2.4. *Linear inequalities for the divisorial valuations.* Consider a valuation as in Proposition 8.5, and write $w = (\nu(v_1), \dots, \nu(v_{20}))$. The set $\mathscr{F} = \{v_1', \dots, v_{20}', \overline{W}_0, \dots, \overline{W}_{17}\}$ contains 24 distinct Laurent polynomials. Each depends only on the mutable coordinates and has at most five terms. We may use $\overline{W}_j = W_j/\epsilon_j$ because the eight coefficient characters are units. They have value zero and contribute their $d\log$ factors to the full logarithmic volume form.

Suppose $w \neq 0$. Consider a w-homogeneous irreducible factor q of an initial form which, up to a Laurent monomial, defines the centre in the torus of weight-zero characters [8, Proposition B.9]. Assume first that $q = Av_p + B$, where $A = cv^a$ is a Laurent monomial with $c \neq 0$, and A, B are independent of v_p. Below we check this condition for every possible q. Choose $\alpha \in \operatorname{supp} q$ and write $\gamma = \langle \alpha, w\rangle$ for its weight. Factor the coordinatewise least Laurent monomial from each $f \in \mathscr{F}$ to write $f = v^{\ell}P(v)$ with P a polynomial. Set $d = \deg_{v_p} P$. In the new variable $u = Av_p + B$, expand

$$A^d P(v_p = (u - B)/A) = \sum_{j\geq 0} u^j R_j(v_{\neq p}).$$

Regularity of f implies

$$\langle \ell + e - da + j\alpha, w\rangle \geq -j \qquad (e \in \operatorname{supp} R_j). \tag{D.2}$$

Indeed, assign weight γ to u and consider a nonzero homogeneous part of weight h in the sum. Let j_0 be its least power of u. After division by u^{j_0} its constant coefficient is a nonzero Laurent polynomial in the other variables. It remains nonzero at the generic point of $q = 0$, where $v_p = -B/A$ and the other variables are algebraically independent. This homogeneous part has order exactly j_0 along $q = 0$. Apply [8, Proposition B.9] with the weight-zero equation q/v^{α}. The valuation of f is the minimum of $\langle \ell - da, w\rangle + h + j_0$ over the homogeneous parts, so regularity makes each of these numbers nonnegative. For each term, $h = \langle e, w\rangle + j\gamma$ and $j \geq j_0$, giving (D.2).

Proof of Proposition 8.5. Use the exchanges, normalizing monomials and boundary functions of §8.2. If every monomial in every member of $\mathscr{F}$ has nonnegative value, a nonnegative rational combination of these inequalities gives $\sum_i w_i \le 0$. Since $w \ge 0$, this forces $w = 0$.

Otherwise some $f \in \mathscr{F}$ has negative minimum weight. By [8, Proposition B.9], an irreducible factor of its initial form defines the centre up to a Laurent monomial. To include every possible initial form, take the sums over all subsets of two or more terms in each member of $\mathscr{F}$. The 100 sums have 80 distinct nonmonomial irreducible factors up to associates. Each is linear in some v_p, with monomial leading coefficient and nonzero constant coefficient in v_p. In particular, these factors remain irreducible after adjoining the coefficient characters, and (D.2) applies.

The one-parameter torus determined by w fixes each prime factor of a w-homogeneous Laurent polynomial up to a scalar, since its connected action preserves the finite set of irreducible components and the support of each factor. Thus each factor is homogeneous. For each candidate q, we have $\langle \alpha - \alpha', w\rangle = 0$ for all $\alpha, \alpha' \in \operatorname{supp} q$. Write these equations as $H_q w = 0$ and write the inequalities (D.2), taken over all $f \in \mathscr{F}$, as $A_q w \ge b_q$. There are rational vectors $\lambda_q \ge 0$ and μ_q satisfying

$$\lambda_q^t A_q + \mu_q^t H_q \le -(1, \dots, 1), \qquad -\lambda_q^t b_q \in \{0, 1\}.$$

Multiplying by $w \ge 0$ gives $\sum_i w_i \le -\lambda_q^t b_q$. The bound is zero for seventy factors and one for the ten distinct mirror exchange polynomials. The factorizations, inequalities and rational multipliers, including those for the first case, are given in `anc/e7_divisor_certificates.json`. Since the w_i are nonnegative integers, w is zero or a standard basis vector. □

Ancillary calculations. The ancillary files contain the coefficient equations for the Cayley plane and the Freudenthal boundary functions, the normalized Freudenthal exchanges, the 56 coordinate expansions, and the inequalities for the divisorial valuations. The accompanying scripts verify the 144 quadrics, the first mutations, the 204 divisibilities in the adjacent charts, and the primality calculations. The conventions and instructions are in `anc/README.md`.

Declaration of generative AI and AI-assisted technologies

The author used ChatGPT to help explore and formulate most of the proofs. The author subsequently revised and independently verified all arguments and takes full responsibility for the mathematical content of the paper.

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