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LAURENT PHENOMENON ALGEBRAS ON MINUSCULE GRASSMANNIANS AND THEIR MIRRORS

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Abstract. We construct Laurent phenomenon algebra structures on the homogeneous coordinate rings of the half-spin Grassmannians and the Freudenthal variety. Their frozen variables are the linear coordinates in a Coxeter orbit, and the resulting boundary divisors are anticanonical. Together with the construction of Daisey–Ducat for the Cayley plane, these examples admit explicit mirrors described by Newton polytopes. We identify the upper Laurent phenomenon algebras of the mirror seeds with the intrinsic mirror algebras of Keel–Yu at the specialization $q^\beta = 1$, including the theta functions associated with the individual boundary components. The half-spin construction is uniform in the rank. In the exceptional cases, contact calculations and Laurent divisibility determine the local expansions; the Freudenthal comparison also uses exact calculations of divisorial valuations. We obtain integral comparisons and valuative independence of the intrinsic theta basis in every reachable seed.

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Introduction

Laurent phenomenon algebras, introduced by Lam and Pylyavskyy, generalize cluster algebras by allowing irreducible exchange polynomials with more than two terms [12]. Daisey and Ducat used this flexibility to describe the homogeneous coordinate ring of the Cayley plane by a finite-type LP algebra [4]. The frozen coordinates form a Coxeter orbit. We construct analogous structures for the half-spin Grassmannians and the Freudenthal variety, and identify the intrinsic mirrors of the corresponding boundary complements.

The boundary has a natural Lie-theoretic origin. If G is simple and $P \subset G$ is a proper parabolic subgroup, the Fano index of G/P equals the Coxeter number $h(G)$ precisely when P is maximal at a minuscule node (Proposition 1.1). An orbit of $h(G)$ linear sections then gives an anticanonical divisor. It is a further question whether these sections can serve as frozen variables of an LP seed.

Several familiar examples answer this question. Scott's cluster algebra on $\mathrm{Gr}(k, N)$ has the N cyclic interval Plücker coordinates as frozen variables [16]. Projective space admits a rank-zero seed, and the relation $uv = \sum_{i=1}^{r-1} a_i b_i$ defines a rank-one seed on the even-dimensional quadric D_r/P_1, with the a_i, b_i frozen. We focus on

$$X_n = \mathrm{OG}^+(n, 2n) \quad (n \geq 4), \qquad X_6^{\mathrm{exc}} = E_6/P_6, \qquad X_7^{\mathrm{exc}} = E_7/P_7.$$

The polarized isomorphism $B_n/P_n \simeq D_{n+1}/P_{n+1}$ allows us to treat the type-B case together with type D.

Coordinate rings and anticanonical boundaries. Let $\mathcal{O}_X(1)$ denote the primitive ample generator of $\mathrm{Pic}(X)$, and put

$$R_X = \bigoplus_{d \geq 0} H^0(X, \mathcal{O}_X(d)), \qquad D = \sum_{i=1}^{h} \{a_i = 0\},$$

where $a_1, \ldots, a_h$ are the chosen linear coordinates in a Coxeter orbit. We call D the *Coxeter boundary*, since its components come from a Coxeter orbit. The seeds are defined over $S_X = \mathbb{C}[a_1, \ldots, a_h]$, with these *frozen variables* kept polynomial. Following Lam–Pylyavskyy, we distinguish the LP algebra $\mathcal{A}(\Sigma_X)$ generated by cluster variables from the *upper LP algebra* $\mathcal{U}(\Sigma_X)$, the intersection of the seed Laurent rings over S_X. Daisey–Ducat use the upper algebra convention in [4, Definition 2.6].

Theorem A (Coordinate-ring structures)**.** *For each Coxeter-boundary seed constructed below,*

$$\mathcal{U}(\Sigma_X) = R_X.$$

For the half-spin Grassmannians and the Cayley plane we also have $\mathcal{A}(\Sigma_X) = R_X$. The half-spin seed has mutable rank $\binom{n-2}{2}$, with interval Pfaffians as mutable variables. The Cayley-plane seed is the rank-five seed of Daisey–Ducat; the Freudenthal seed has mutable rank ten and eighteen frozen variables. The half-spin seeds have infinitely many cluster variables for $n \geq 7$, as does the Freudenthal seed.

For the half-spin Grassmannians, we combine the multidimensional cube recurrence of Henriques–Speyer [8] with the wiring-diagram constructions of Lam–Pylyavskyy [12]. Signed Pfaffians satisfy the exchange relations, and two complementary wiring charts identify the upper LP algebra with R_{X_n}. Every spinor coordinate is a cluster variable, yielding the asserted equality with $\mathcal{A}(\Sigma_{X_n})$. For the Freudenthal variety, five mutations connect our seed to an auxiliary chart; Laurent expansions and prime divisors then give the upper-algebra equality. The infinite-type statements follow from rank-two subpatterns. They do not exclude a different finite-type positive seed of the kind conjectured by Daisey–Ducat [4, Conjecture 6.1].

Intrinsic mirrors and boundary functions. For each X above, consider the affine boundary complement $U_X = \operatorname{Spec} R_X[(a_1 \cdots a_h)^{-1}]$. It is a smooth affine log Calabi–Yau variety containing the initial torus. Let Y be a smooth projective compactification with simple normal-crossings boundary. Keel–Yu associate to this data an *intrinsic mirror algebra* A_Y, defined over the original curve-class ring $\mathbb{Z}[\mathrm{NE}(Y)]$ and equipped with theta functions ϑ_m [10, 9]. We study the algebra

$$A_{\mathrm{int}}(U_X)_1 = A_Y \otimes_{\mathbb{Z}[\mathrm{NE}(Y)],\, q^\beta \mapsto 1} \mathbb{C}.$$

It is independent of the chosen compactification. The subscript 1 refers only to the specialization of curve-class parameters, not to the Laurent coefficient characters of the mirror. We write Θ_m for the expansion of ϑ_m in a specified Laurent chart.

For each component b of the chosen boundary, we identify its associated *boundary function* W_b on the mirror. Their sum $W = \sum_b W_b$ is the *superpotential*; this is the terminology usually used for the corresponding Landau–Ginzburg function.

Theorem B (Marked intrinsic mirrors)**.** *The Newton constructions give explicit mirror seeds $\Sigma_X^\vee$ and regular functions W_b such that the intrinsic Laurent expansion induces an isomorphism*

$$\Phi_X : A_{\mathrm{int}}(U_X)_1 \xrightarrow{\sim} \mathcal{U}(\Sigma_X^\vee).$$

It preserves the full lattice of coefficient characters and identifies each boundary theta function with W_b. These upper LP algebras are finitely generated, normal, and factorial.

We specify the mutable variables, Laurent coefficient ring, ordinary exchange polynomials, and associated exchange Laurent polynomials of each mirror seed. The half-spin formulas are uniform in n, while the two exceptional seeds are listed explicitly. The last column of the following table indicates where to find these descriptions.

X	$\dim \widehat{X}$	Source rank	Mirror rank	Unit rank	Mirror seed
D_n/P_n	$\binom{n}{2}+1$	$\binom{n-2}{2}$	$n(n-3)/2$	$n+1$	§6.2
E_6/P_6	17	5	10	7	§8.1.1
E_7/P_7	28	10	20	8	§8.2.1

Here the unit rank means the rank of the unit group modulo constants; it is not the number of boundary components. For the Cayley-plane mirror, Proposition 8.1 also proves that the generated LP algebra equals the upper LP algebra.

The mirror comparison begins with regular functions on the source, whose tropicalizations bound the possible terms in the intrinsic Laurent expansions. Initial forms or contact calculations determine the local coefficients. We then use Laurent divisibility to recognize the remaining functions within their proved supports. Finally, the boundary functions and the logarithmic volume control the divisorial valuations of the intrinsic mirror. Normality identifies its coordinate ring with the corresponding upper LP algebra. The general steps are proved in the companion work [7]; here we verify their hypotheses in each minuscule family.

In the half-spin family, the divisorial argument is uniform. With $m = n-1$, its superpotential in integral triangular coordinates is

$$W = y_{11} + \sum_{1 \le a < b \le m} \frac{y_{a+1,b}}{y_{ab}} + \sum_{1 \le a \le b < m} \frac{y_{a,b+1}}{y_{ab}} + \frac{\mathfrak{q}}{y_{mm}}.$$

The Newton polytopes of the boundary sums control the possible missing divisors. The Cayley-plane mirror has a smooth complete-intersection presentation. For the Freudenthal mirror, twenty binomial exchanges give eighteen boundary functions and a forty-term superpotential. Its global comparison uses an exact finite calculation of divisorial valuations.

Integral forms and theta bases.

Theorem C (The intrinsic theta basis)**.** *The comparisons descend to the specified integral counting algebras and upper LP algebras over their Laurent coefficient rings. In every reachable seed, the intrinsic theta functions are integral Laurent polynomials and are* valuatively independent *for every real Gauss valuation. Thus the Newton polytope of a finite signed sum is the convex hull of the Newton polytopes of its nonzero theta terms. Every extended cluster monomial is a scalar multiple of one intrinsic theta function; with the specified positive normalization, the scalar is one.*

We apply the valuative-independence and integral-descent theorems of [7] to the intrinsic functions, after establishing the marked comparisons. The former extends the methods of [3] to the Laurent realizations considered here. In the D_4 mirror, every theta function is an extended cluster monomial and the basis is atomic for the LP mutation atlas.

The projective complement. A frozen linear section trivializes the tautological line bundle on $X^\circ = X \setminus D$, giving $U_X \simeq X^\circ \times \mathbb{G}_m$. The character dual to scalar multiplication on the cone is

$$\mathfrak{q}_{D_n} = \prod_{j=0}^{n} \kappa_j, \quad \mathfrak{q}_{E_6} = \prod_{j=6}^{12} \zeta_j, \quad \mathfrak{q}_{E_7} = \prod_{j=0}^{7} \zeta_j.$$

By compatibility of intrinsic mirrors with torus products [7],

$$A_{\mathrm{int}}(U_X)_1 \simeq A_{\mathrm{int}}(X^\circ)_1[\mathfrak{q}^{\pm 1}].$$

The fibre $\mathfrak{q} = 1$ is therefore the marked intrinsic mirror of the projective complement. This specialization of a torus character is distinct from the earlier specialization $q^\beta = 1$.

Organization. Part I constructs the source LP seeds and proves the coordinate-ring identifications. Part II constructs their mirror seeds and proves the intrinsic comparisons. The appendices contain the Pfaffian recurrence, contact calculations, and explicit exceptional coordinate checks. Integral forms and theta bases are treated in Section 9, and the Lie-theoretic context in Section 10.

Unless stated otherwise, we work over $\mathbb{C}$. The frozen variables stay polynomial until an inversion is explicitly indicated.

Part 1. Laurent phenomenon structures

1. The index condition and Coxeter boundaries

Let G be a connected simple complex algebraic group, with Coxeter number $h(G)$. We may replace G by its simply connected cover. For a Fano variety X, let $\iota(X)$ be the largest positive integer dividing $-K_X$ in $\mathrm{Pic}(X)$.

Proposition 1.1. *For a proper parabolic subgroup $P \subset G$,*

$$\iota(G/P) \le h(G).$$

Equality holds if and only if P is maximal at a minuscule node.

Proof. Let I be the set of simple roots of the Levi subgroup and let J be its complement. Write ρ and ρ_I for the half-sums of the positive roots of G and of the Levi, and let w_I be the longest Levi Weyl-group element. The anticanonical weight is (see [11, Section 2.7]) $2\rho - 2\rho_I = \rho + w_I\rho$. For $j \in J$, its coefficient at the fundamental weight ω_j is

$$r_j = \langle \rho + w_I\rho, \alpha_j^\vee \rangle = 1 + \mathrm{ht}(w_I\alpha_j^\vee), \qquad \iota(G/P) = \gcd_{j\in J} r_j.$$

The coefficient of $\alpha_j^\vee$ in $w_I\alpha_j^\vee$ is one, so this is a positive coroot. Let $\Theta^\vee$ be the highest root of the dual root system. It is the unique positive coroot of maximal height, namely $h(G) - 1$. Thus $r_j \le h(G)$.

If the gcd is $h(G)$, then every $r_j = h(G)$ and $w_I\alpha_j^\vee = \Theta^\vee$. The left side has coefficient zero at each node of $J \setminus \{j\}$, whereas the highest coroot has full support. Thus $J = \{j\}$. Its coefficient at j is one, which is precisely the minuscule criterion $\langle \omega_j, \Theta^\vee \rangle = 1$.

Conversely, suppose $J = \{j\}$ and ω_j is minuscule. Pairing the anticanonical weight $r_j\omega_j$ with $\Theta^\vee$ gives $r_j = h(G) - 1 + \mathrm{ht}(w_I\Theta^\vee)$. The last coroot still has coefficient one at j, so it is positive. Therefore $r_j \geq h(G)$, and equality follows from the upper bound. □

In the non-simply-laced cases, $\Theta^\vee$ denotes the highest root of the dual root system, not the coroot of the highest root. Accordingly, the equality selects minuscule rather than cominuscule weights. For example, C_r/P_r $(r \geq 2)$ is cominuscule but has index $r+1$, not $h(C_r) = 2r$ [2, Section 2.1]. With Bourbaki numbering, the complete list is

$$A_n/P_i\ (1 \leq i \leq n), \quad B_n/P_n, \quad C_n/P_1, \quad D_n/P_1, D_n/P_{n-1}, D_n/P_n, \quad E_6/P_1, E_6/P_6, E_7/P_7.$$

Type A has the Grassmannian cluster structures of [16]; C_n/P_1 is projective space, and D_n/P_1 is an even quadric. We concentrate on the half-spin and exceptional varieties. The polarized isomorphism $B_n/P_n \simeq D_{n+1}/P_{n+1}$ accounts for type B.

Suppose H is the primitive ample generator of $\mathrm{Pic}(G/P)$ and $-K_{G/P} = hH$. A product of h nonzero linear sections defines an anticanonical divisor. A full Coxeter orbit specifies such sections compatibly with the Weyl-group action. The existence of an LP chart with these sections frozen is an additional assertion, established by the constructions below. The affine cone has dimension $\dim(G/P)+1$, so such a seed has mutable rank $\dim(G/P)+1-h$.

The chosen highest-weight orbit has a Weyl-equivalence class that is independent of the ordering of the simple reflections.

Proposition 1.2. *Fix a simple-root system and a fundamental weight ω_k of a finite irreducible root system. As c ranges over products of all the simple reflections, each used once, the sets $\langle c\rangle\omega_k$ are Weyl-equivalent.*

Proof. Since the finite Dynkin diagram is a tree, any two Coxeter words are related by cyclic rotations and commutations of commuting reflections [5, Propositions 2.3 and 4.1]. Commutations leave the element unchanged. Consider a rotation from $c = s_i v$ to $c' = v s_i = s_i c s_i$. If $i \neq k$, then $s_i\omega_k = \omega_k$, and hence $s_i(\langle c\rangle\omega_k) = \langle c'\rangle\omega_k$. If $i = k$, all factors of v fix ω_k, so $s_i\omega_k = (c')^{-1}\omega_k$. The same equality follows:

$$s_i(\langle c\rangle\omega_k) = \langle c'\rangle(s_i\omega_k) = \langle c'\rangle\omega_k.$$

Iterating proves the assertion. □

This result concerns the orbit of a fixed fundamental highest weight; full-length orbits through other weights need not be equivalent, and are not covered by the proposition.

2. Upper Laurent phenomenon algebras

We use Lam–Pylyavskyy's convention for LP mutation [12, Section 2], including the normalized exchange Laurent polynomials. Let S be a unique factorization domain (UFD), let $r \geq 1$, and put $\mathcal{F} = \mathrm{Frac}(S)(x_1, \ldots, x_r)$. An *LP seed* consists of the algebraically independent variables x_i and irreducible polynomials $F_i \in S[x_1, \ldots, x_r]$ such that F_i is independent of x_i and is not divisible by any x_j. Following Lam–Pylyavskyy, we associate to F_i an *exchange Laurent polynomial* $\widehat{F}_i = F_i/M_i$, where M_i is a monomial in the other mutable variables. Mutation replaces x_i by $x_i' = \widehat{F}_i/x_i$ and updates the exchange polynomials by the substitution and cancellation rule of [12]. All mutations below use this normalization.

2.1. **The Laurent intersection.** For a seed t, write $L(t)$ for its Laurent ring and $\mathrm{UB}(t)$ for its *finite upper bound*:

$$L(t) = S[x_1^{\pm 1}, \ldots, x_r^{\pm 1}], \qquad \mathrm{UB}(t) = L(t) \cap \bigcap_{i=1}^{r} L(\mu_i t), \qquad \mathcal{U}(t) = \bigcap_{t' \sim t} L(t').$$

The *LP algebra* $\mathcal{A}(t)$ is the S-algebra generated by all cluster variables. The Laurent phenomenon gives $\mathcal{A}(t) \subseteq \mathcal{U}(t)$ [12].

We use the following LP starfish theorem of Fei [6, Theorem 1.3].

Theorem 2.1. *The finite upper bound is invariant under LP mutation over S. Consequently,*

$$\mathcal{U}(t) = L(t) \cap \bigcap_{i=1}^{r} L(\mu_i t).$$

We apply the theorem over polynomial coefficient rings in Part I and Laurent coefficient rings in Part II. It allows repeated exchange polynomials and nontrivial normalizing monomials. In rank zero, the finite upper bound and the upper LP algebra both equal S.

2.2. **Laurent regularity and boundary valuations.** Set $L_0 = L(t)$ and $L_i = L(\mu_i t)$. Expand an element of L_0 as $f = \sum_k x_i^k f_{i,k}$, with $f_{i,k} \in S[x_j^{\pm 1} : j \neq i]$. Substitution of $x_i = F_i/(M_i x_i')$ gives

$$f \in L_i \quad \Longleftrightarrow \quad F_i^{-k} \mid f_{i,k} \quad \text{for every } k < 0. \tag{2.1}$$

Indeed, M_i is a unit in this coefficient ring, and distinct powers of x_i' cannot cancel. On a fixed finite support, these conditions are linear equations for the coefficients of f.

No mutable variable divides F_i, so it remains a nonunit prime in $S[x_j^{\pm 1} : j \neq i]$ and hence in L_i. We call $\nu_i = \mathrm{ord}_{F_i}$ in the adjacent Laurent ring the corresponding *boundary valuation*. It satisfies

$$\nu_i(x_j) = \delta_{ij}, \qquad \nu_i(f) = \min_k \{k + \mathrm{ord}_{F_i}(f_{i,k})\}.$$

The independent coordinate x_i' again prevents cancellation. The starfish theorem therefore gives

$$\mathcal{U}(t) = \{f \in L_0 : \nu_i(f) \geq 0 \text{ for all } i\}. \tag{2.2}$$

For later use, suppose that $F_i = F$ for $i \in I$, that F is independent of the variables indexed by I, and that no other normalization factors occur. Then

$$\widehat{F}_i = \frac{F}{\prod_{j \in I \setminus \{i\}} x_j}, \qquad x_i' = \frac{F}{\prod_{j \in I} x_j}.$$

For the coefficient of $\prod_{i \in I} x_i^{k_i}$, regularity in all the corresponding adjacent charts is equivalent to divisibility by $F^{\max(0, -k_i : i \in I)}$. This follows by imposing (2.1) for each i.

Proposition 2.2. *The upper LP algebra $\mathcal{U}(t)$ is a UFD. If P is irreducible in L_0, then $P^\circ = P \prod_i x_i^{-\nu_i(P)}$ is prime in $\mathcal{U}(t)$. These elements and the initial variables represent all prime elements up to associates, and $\mathcal{U}(t)^\times = S^\times$.*

Proof. All boundary orders of P° are zero, so (2.2) puts it in $\mathcal{U}(t)$. If P° divides an element of $\mathcal{U}(t)$ in L_0, the quotient has nonnegative boundary orders and belongs to $\mathcal{U}(t)$. Primality in L_0 therefore implies primality in $\mathcal{U}(t)$. Likewise,

$$f \in x_i \mathcal{U}(t) \quad \Longleftrightarrow \quad \nu_i(f) > 0,$$

so each x_i is prime. Factor a nonzero $f \in \mathcal{U}(t)$ in L_0 and normalize its irreducible factors. The result is

$$f = c \prod_i x_i^{\nu_i(f)} \prod_j (P_j^\circ)^{e_j}, \qquad c \in S^\times,$$

with nonnegative exponents. This is a finite factorization into primes; no Noetherian hypothesis is required. Finally, a unit and its inverse have boundary order zero, so the mutable exponents in a Laurent unit must vanish. □

2.3. **Rank-one charts and extension across codimension two.** Put $s_i = \prod_{j\neq i} x_j$ and $B_0 = S[x_1, \dots, x_r, x_1', \dots, x_r']$. Localizing the finite intersection at s_i gives

$$\mathcal{U}(t)[s_i^{-1}] = B_0[s_i^{-1}] = L_0 \cap L_i = \frac{S[x_j^{\pm 1} : j \neq i][x_i, x_i']}{(x_i x_i' - F_i/M_i)}. \tag{2.3}$$

All other adjacent rings contain L_0 after this localization, and the last equality follows from (2.1). If S is a finitely generated $\mathbb{C}$-algebra, the open set $X^\circ = \operatorname{Spec} B_0 \setminus V(s_1, \dots, s_r)$ is quasi-affine of finite type and has ring of global functions $\mathcal{U}(t)$. If S is smooth over $\mathbb{C}$, these *rank-one charts* are smooth whenever $F_i = 0$ is smooth in $\operatorname{Spec} S[x_j^{\pm 1} : j \neq i]$. For polynomial or Laurent polynomial coefficient rings, the rational logarithmic forms in the mutable and coefficient coordinates agree up to sign under mutation.

Proposition 2.3. *Let $B_0 \subseteq B \subseteq \mathcal{U}(t)$ be a finitely generated domain and let $J = (s_1, \dots, s_r)B$. Then*

$$\mathcal{U}(t) = \bigcap_i B[s_i^{-1}] = \bigcup_{m\geq 0} (B :_{\operatorname{Frac} B} J^m).$$

If B is normal and $V(J)$ has codimension at least two, then $B = \mathcal{U}(t)$. More generally, $\operatorname{depth}_J B \geq 2$ *is sufficient.*

Proof. The localizations of B_0, B, and $\mathcal{U}(t)$ agree by (2.3). Clearing finitely many powers of the s_i gives the saturation formula. A normal Noetherian domain is the intersection of its height-one localizations. If $V(J)$ has codimension at least two, all these localizations occur on the indicated open set. The depth assertion is the corresponding extension theorem. □

Lemma 2.4. *Let R be a normal Noetherian domain containing S. Suppose that $x_1, \dots, x_r$ are pairwise nonassociate prime elements of R, $R[(x_1 \cdots x_r)^{-1}] = L_0$, and that each first mutation x_i' lies in $R \setminus (x_i)$. Then $\mathcal{A}(t) \subseteq \mathcal{U}(t) \subseteq R$. If, moreover, $R \subseteq L_i$ for every i, then $\mathcal{U}(t) = R$.*

Proof. We show that every element of $L_0 \cap \bigcap_i L_i$ is regular at all height-one primes of R. A prime containing none of the x_i is covered by the initial localization. A height-one prime containing x_i is (x_i); the other x_j and x_i' are units there, so L_i is contained in its local ring. Normality gives the inclusion in R. The Laurent phenomenon gives the first inclusion, and the starfish theorem gives equality once R belongs to every adjacent ring. □

Lemma 2.5. *Let $X = G/P$ be a positive-dimensional homogeneous variety with Picard group generated by the primitive ample line bundle $\mathcal{O}_X(1)$. Then its homogeneous coordinate ring R_X is factorial, and every nonzero element of degree one is prime.*

Proof. The homogeneous embedding is projectively normal, and R_X is generated in degree one; see [1]. The punctured cone is $\mathcal{O}_X(-1)^\times$. Since the vertex has codimension at least two, localization along the zero section of the line bundle gives

$$\operatorname{Cl}(R_X) = \operatorname{Pic}(X)/\mathbb{Z}[\mathcal{O}_X(1)] = 0.$$

Thus R_X is factorial. A nonzero degree-one element is irreducible in a positively graded domain with degree-zero part $\mathbb{C}$, and is therefore prime. □

3. Half-spin Grassmannians

Fix $n \geq 4$, put $m = n - 1$, and let

$$X = \mathrm{OG}^+(n, 2n), \qquad R_X = \mathbb{C}[\widehat{X}].$$

The half-spin polarization H is the primitive ample generator. Thus $\dim X = m(m+1)/2$ and $\dim R_X = m(m+1)/2 + 1$. We index the coordinates by subsets of $[m]$; the resulting boundary also gives the type-B_m construction in Remark 3.3.

3.1. **Signed principal Pfaffians and the interval seed.** Let p_I, for even subsets $I \subseteq [m+1]$, denote the degree-one spinor coordinates. Our convention on the standard big cell is $p_I = t\,\mathrm{Pf}(B_I), \qquad p_\varnothing = t$, where B is a skew-symmetric $(m+1)\times(m+1)$ matrix and principal submatrices use increasing index order. These coordinates generate R_X and satisfy the quadratic *Wick relations*; see [15]. For every $S \subseteq [m]$, define

$$S^+ = \begin{cases} S, & |S| \text{ even}, \\ S \cup \{m+1\}, & |S| \text{ odd}, \end{cases} \qquad \epsilon(k) = \begin{cases} -1, & k \equiv 0 \pmod 4, \\ 1, & \text{otherwise}, \end{cases} \qquad q_S = \epsilon(|S|)p_{S^+}.$$

This is a signed relabeling of all spinor coordinates. In particular, $q_\varnothing = -p_\varnothing$. Write

$$z = q_\varnothing, \qquad X_{ij} = q_{[i,j]} \quad (1 \le i \le j \le m), \qquad X_{i,i-1} = z.$$

We use $X_{i,i-1} = z$ whenever an empty interval occurs in an exchange.

The frozen set and mutable set are

$$\begin{aligned} \mathcal{B}_m &= \{z\} \cup \{X_{1j} : 1 \le j \le m\} \cup \{X_{im} : 2 \le i \le m\}, \\ \mathcal{M}_m &= \{X_{ij} : 2 \le i \le j \le m-1\}. \end{aligned} \tag{3.1}$$

We use the polynomial coefficient ring $S_m = \mathbb{C}[\mathcal{B}_m]$; the frozen variables are not inverted. Define

$$F_{ii} = X_{i-1,i-1}X_{i,i+1} + zX_{i-1,i+1} + X_{i+1,i+1}X_{i-1,i}, \qquad 2 \le i \le m-1, \tag{3.2}$$

and, for $2 \le i < j \le m-1$, define

$$\begin{aligned} F_{ij} = {} & X_{i-1,j-1}X_{i+1,j}X_{i,j+1} + X_{i+1,j-1}X_{i-1,j}X_{i,j+1} \\ & + X_{i-1,j+1}X_{i,j-1}X_{i+1,j} + X_{i+1,j+1}X_{i,j-1}X_{i-1,j}. \end{aligned} \tag{3.3}$$

All factors on the right are members of the extended cluster.

Theorem 3.1. *For every $m \ge 3$, (3.1)–(3.3) define a Laurent phenomenon seed Σ_m in $\mathrm{Frac}(R_X)$. Its $2m$ frozen coordinate lines form a full Coxeter orbit. Its mutable rank is $\binom{m-1}{2}$, and its initial exchange Laurent polynomials satisfy $\widehat{F}_{ij} = F_{ij}$. With polynomial frozen coefficients,*

$$\mathcal{A}_{S_m}(\Sigma_m) = \mathcal{U}_{S_m}(\Sigma_m) = R_X.$$

Every nonfrozen q_S is a cluster variable in a reachable wiring seed.

There are $m-2$ quadratic three-term exchanges and $\binom{m-2}{2}$ cubic four-term exchanges. Consequently their first mutated variables have homogeneous degrees one and two, respectively. This distinction already explains why the cube recurrence supplies a distinguished part of the mutation pattern rather than all of it.

3.2. **The Coxeter orbit and the anticanonical boundary.** For type D_{m+1} take

$$\alpha_i = e_i - e_{i+1} \quad (1 \le i \le m), \qquad \alpha_{m+1} = e_m + e_{m+1}, \qquad c_D = s_1 \cdots s_{m+1},$$

where the rightmost reflection acts first. Then

$$e_1 \longmapsto e_2 \longmapsto \cdots \longmapsto e_m \longmapsto -e_1, \qquad e_{m+1} \longmapsto -e_{m+1}.$$

Encode the half-spin weight $\frac{1}{2}\sum_{i=1}^{m+1} e_i - \sum_{i\in I} e_i$, with $|I|$ even, by $S = I \cap [m]$. The induced operation on labels is

$$c_D(S) = \{i+1 : i \in S,\ i < m\} \cup \begin{cases} \{1\}, & m \notin S, \\ \varnothing, & m \in S. \end{cases}$$

Its orbit through $\varnothing$ is

$$\varnothing, [1,1], [1,2], \ldots, [1,m], [2,m], \ldots, [m,m]. \tag{3.4}$$

The $2m$ entries are distinct and are exactly the frozen labels. Replacing weight vectors by dual coordinate functions gives the same permutation of their one-dimensional weight spaces.

If $\mathcal{S}$ is the tautological bundle, then

$$T_X \simeq \Lambda^2\mathcal{S}^*, \qquad \det \mathcal{S}^* \simeq \mathcal{O}(2H), \qquad -K_X = 2mH.$$

Thus the sum of the frozen hyperplane divisors is anticanonical. The frozen coordinates are pairwise nonassociate degree-one primes by Lemma 2.5, so the components are reduced and distinct.

3.3. **The signed cube recurrence.** For distinct $a < b < c$ outside $S \subseteq [m]$, use Sa, Sab, and similar expressions for unions with the indicated indices. The Wick relations give

$$q_{Sac}q_{Sb} = q_S q_{Sabc} + q_{Sab}q_{Sc} + q_{Sbc}q_{Sa}. \tag{3.5}$$

We allow arbitrary common subsets S, not only intervals. To check the signs, put $P_T = p_{T^+}$. The unsigned identity is

$$P_{Sac}P_{Sb} + P_S P_{Sabc} = P_{Sab}P_{Sc} + P_{Sbc}P_{Sa}.$$

For even $|S|$, this is the four-index principal-Pfaffian identity with additional indices $a, b, c, m+1$. For odd $|S|$, it is the odd-base Wick identity on the same four additional indices. Ordering common indices before the additional indices introduces in every product the same sign: each element of $\{a, b, c, m+1\}$ occurs exactly once outside the two copies of S. The identities therefore hold also when elements of S are interleaved with a, b, c. Finally,

$$\epsilon(k)\epsilon(k+3) = -\epsilon(k+1)\epsilon(k+2)$$

converts the unsigned identity into (3.5).

Equation (3.5) is the multidimensional cube recurrence of Henriques–Speyer [8]. Their two-graded projective quotient for m cube directions is $\mathrm{IG}(m-1, 2m)$; it should not be confused with X. Here the principal-Pfaffian realization and the ordinary homogeneous grading identify the relevant affine cone.

3.4. **Wiring diagrams and LP mutation.** We use a *reduced wiring diagram* for the longest permutation on m wires, labeled $1, \dots, m$ from top to bottom at the left boundary. Label each chamber by the set of wires above it. The unbounded chambers have the labels (3.4). The reduced word $(s_1)(s_2 s_1)\cdots(s_{m-1}\cdots s_1)$ has chamber collection $\{\varnothing\} \cup \{[i,j] : 1 \le i \le j \le m\}$. Its bounded singleton chambers are triangular, and its other bounded chambers are quadrilateral.

For an arbitrary bounded chamber r, let $s_1, \dots, s_d$ be its side-neighbor chambers in cyclic order. Let c_i denote the chamber opposite r at the vertex between sides s_i and s_{i+1}, with cyclic subscripts. Lam–Pylyavskyy's wiring rule is

$$F_r = \sum_{i=1}^{d} q_{c_i} \prod_{j \notin \{i, i+1\}} q_{s_j}. \tag{3.6}$$

The side and corner labels are all distinct. This is the wiring rule of [12, Section 7.5]. Reading the interval diagram gives precisely (3.2) and (3.3).

We first check that the chamber variables form a transcendence basis. Indeed, a triangular braid move replaces Sb by Sac and has coordinate transformation (3.5); commutations of crossings on disjoint pairs of wires leave the chamber set unchanged. Reduced diagrams are connected by these moves, equivalently by rhombus-tiling flips [8, Theorem 2.2]. Every subset $S \subseteq [m]$ occurs as a chamber: choose a permutation with the elements of S in its first $|S|$ positions and extend a reduced word for that permutation to one for the longest permutation. Hence every q_S is rational in the chamber variables of any fixed diagram. Those variables number $m(m+1)/2+1 = \mathrm{trdeg}_{\mathbb{C}} R_X$, and therefore form a transcendence basis. In particular S_m is a polynomial ring.

Lemma 3.2. *The polynomials (3.6) satisfy the LP seed axioms, are pairwise nonassociate, and have trivial exchange Laurent normalizers. A triangular braid move is an LP mutation of the whole seed, including every exchange polynomial.*

Proof. For a fixed chamber, regard F_r as a linear polynomial in its corner variables over the polynomial ring in the remaining variables. Its coefficients are monomials in the side variables and have greatest common divisor one: every side variable is absent from two summands. It is irreducible over the fraction field of the coefficient ring, and Gauss's lemma proves irreducibility in the full polynomial ring. No variable divides F_r, and q_r does not occur in it.

Distinctness follows from the chamber labels. Suppose F_r and F_t had the same variable support, and put $K = r\triangle t$, $k = |K|$. Every side neighbor of r differs from r in exactly one boundary wire, and every corner neighbor in exactly two adjacent boundary wires. Every variable in F_t is at distance one or two from t. If $k \geq 4$, no side neighbor of r can have this property. If $k = 2$, every boundary wire of r would belong to K, whereas a bounded chamber has at least three distinct boundary wires. If $k = 1$, either its unique wire bounds r, so that q_t occurs in F_r, or a corner of r is at distance three from t. Both alternatives are impossible. If $k = 3$, all boundary wires of r lie in K, so r is triangular with those three boundary wires. The label obtained by reversing all three signs is the missing chamber of their three-wire arrangement, and cannot be the label of t. Thus the supports, and hence the exchanges, are distinct.

To verify normalization, observe that q_t occurs in F_r if and only if q_r occurs in F_t: the two chambers then share a side or a crossing. If they do not meet, F_r and F_t remain coprime after localization at variables, because they are distinct irreducibles and neither is divisible by a variable. If they meet, write F_r as a polynomial in q_t. Its nonzero constant coefficient is independent of q_r. The irreducible polynomial F_t has positive q_r-degree and nonzero q_r-constant coefficient, and therefore cannot divide that constant coefficient even in the Laurent ring. Modulo F_t, substituting $q_t = F_t/q_t'$ into F_r leaves precisely this coefficient. These are the LP normalization tests, so $\widehat{F}_r = F_r$.

Now let x be a triangular chamber, with sides a, b, c and opposite corners α, β, γ, labeled so that $xx' = \alpha a + \beta b + \gamma c$. For a mutable side chamber a, its exchange has the form $F_a = M(\beta b + \gamma c) + xP$ The LP update sets $a = 0$ in F_x, substitutes $x = (\beta b + \gamma c)/x'$, clears the denominator, and cancels the common factor $\beta b + \gamma c$. The result is $F_a' = Mx' + P$, which is exactly the new wiring polynomial. For a mutable corner chamber α, write $F_\alpha = xN + Q$. Its updated exchange is $F_\alpha' = (\beta b + \gamma c)N + x'Q$, again the new wiring polynomial. Cyclically these computations handle all six adjacent chambers; other exchanges are unchanged. The new wiring polynomials are irreducible and divisible by no variable, so no further polynomial or monomial cancellation is required. This proves full seed compatibility. □

It follows that every wiring seed is reachable from Σ_m, and every nonfrozen spinor coordinate occurs in one of them. Since the spinor coordinates generate R_X, this proves

$$R_X \subseteq \mathcal{A}_{S_m}(\Sigma_m). \tag{3.7}$$

3.5. **The coordinate-ring equality.** Let $\mathscr{T}$ be the interval chamber collection, and put $f = \prod_{S \in \mathscr{T}^\circ} q_S$, where the superscript denotes bounded chambers. The LP Laurent phenomenon [12, Theorem 5.1], with polynomial frozen coefficients, gives

$$S_m[q_S^{\pm 1} : S \in \mathscr{T}^\circ] = (R_X)_f.$$

Indeed, every generator q_S of R_X is Laurent in this seed; conversely, all Laurent generators belong to $(R_X)_f$.

Rotate the diagram by 180 degrees. Its chambers are the complements $[m] \setminus S$ of the old labels, and its frozen set is unchanged. A mutable label is the union of two nonempty end intervals separated by a nonempty internal interval. Such a set is never an interval. Thus the old and new mutable label sets are disjoint. The rotated seed is reachable; if g is the product of its mutable coordinates, its Laurent chart is $(R_X)_g$.

By Lemma 2.5, the coordinate factors of f and g are prime. Distinct spinor weight coordinates are nonassociate. Hence f and g have disjoint prime factors, and prime cancellation

gives

$$(R_X)_f \cap (R_X)_g = R_X \quad \text{inside } \operatorname{Frac}(R_X).$$

Combining (3.7), the Laurent phenomenon, and the two reachable charts yields

$$R_X \subseteq \mathcal{A}_{S_m}(\Sigma_m) \subseteq \mathcal{U}_{S_m}(\Sigma_m) \subseteq (R_X)_f \cap (R_X)_g = R_X.$$

This proves Theorem 3.1. The equality applies to the full LP mutation class, including seeds that do not arise from wiring diagrams.

Remark 3.3 (Transfer to type B). Intersection with a nondegenerate hyperplane gives the polarized isomorphism $D_{m+1}/P_{m+1} \simeq B_m/P_m$: a maximal isotropic $(m+1)$-plane meets the hyperplane in an isotropic m-plane, and each such m-plane has one extension in either even-dimensional component. The half-spin representation restricts to the spin representation, with the same subset coordinates. The type-B Coxeter element $s_1 \cdots s_m$ acts by the negative cycle $e_1 \mapsto \cdots \mapsto e_m \mapsto -e_1$, giving exactly (3.4). Thus the coordinate-ring theorem and the mirror theorem, including its boundary functions, transfer by replacing the type-D index by $m+1$. No separate folding is involved. The two smaller cases $B_1/P_1 \simeq \mathbb{P}^1$ and $B_2/P_2 \simeq \mathbb{P}^3$ have polynomial coordinate rings and no mutable variables.

3.6. **The examples D_4 and D_5.** For $m = 3$ (D_4), the seed has rank one and

$$F_{22} = X_{11}X_{23} + zX_{13} + X_{33}X_{12};$$

its other cluster variable is $q_{\{1,3\}}$, giving the quadric. For $m = 4$ (D_5), write

$$(x, y, w) = (X_{22}, X_{23}, X_{33}), \qquad (a, b, c, d, e, f, g) = (X_{11}, X_{12}, X_{13}, X_{14}, X_{24}, X_{34}, X_{44}).$$

Then

$$F_x = ay + bw + zc, \qquad F_y = dxw + cfx + bew + zce, \qquad F_w = fx + gy + ze.$$

This is the rank-three wiring example with ten mutable cluster variables and sixteen seeds, up to the coefficient relabeling in [12, Example 7.7]; see also [4, Section 4.2]. Its eight wiring seeds are a proper part of its LP mutation class.

3.7. **Infinite type for $n \geq 7$.** For $n \geq 7$, the LP mutation class contains infinitely many cluster variables, although only finitely many seeds arise from wiring diagrams.

Proposition 3.4. *The Coxeter-boundary seeds of Theorem 3.1 have infinitely many cluster variables for D_n with $n \geq 7$. Thus these LP algebras are of infinite type.*

Proof. We first give a rank-two reduction in D_7, so $m = 6$. Use the initial interval slots, irrespective of their subsequent contents, to specify the chronological word

$$\mu_{X_{33}}, \quad \mu_{X_{44}}, \quad \mu_{X_{23}}, \quad \mu_{X_{33}}, \quad \mu_{X_{45}}, \quad \mu_{X_{24}}. \tag{3.8}$$

Order the initial mutable intervals first by length and then by their left index. The word has position sequence $(2, 3, 5, 2, 7, 8)$. Write Y_{ij} for the variable in position X_{ij} after this word. Introduce the abbreviations

symbol	a	u	b	v	d	e	f	g	k
variable	Y_{22}	Y_{33}	Y_{44}	Y_{55}	Y_{34}	Y_{45}	Y_{24}	Y_{35}	Y_{25}

and the unchanged frozen coordinates

$$z = q_\varnothing, \quad A = q_{12}, \quad B = q_{1234}, \quad C = q_{12345}, \quad D = q_{3456}, \quad E = q_{56}, \quad H = q_6.$$

Here subscripts such as 1234 denote subsets, whereas Y_{ij} records an interval position. Put $T = vD + gE$ and $S = ud + gzB$. Direct LP mutation, including normalization and cancellation, gives the following compact expressions:

$$F_u = T\big[adfB + bA(dC + gB) + kAB(vd + gz)\big] + bdef,$$
$$F_v = E\big[S\{adf + kz(u + gA)\} + bz(u + gA)(dC + gB)\big] + bdf(uH + ez).$$

These have eleven and twelve terms, respectively. Their full exchange Laurent normalizations are

$$\widehat{F}_u = F_u/d, \qquad \widehat{F}_v = F_v/d. \tag{3.9}$$

In particular the reciprocal u- and v-normalizer exponents are zero, although the full normalizers are not trivial. The two exchange polynomials exhibit the obstruction:

$$\deg_v F_u = \deg_u F_v = 2, \qquad [v^2]F_u = dkABD, \qquad [u^2]F_v = dkzE.$$

All leading coefficients are nonzero monomials in the other seed coordinates.

Freeze every current mutable variable except u and v using the LP freezing construction [12, Proposition 3.7]. The new coefficient ring inverts those variables; in particular d is a unit, so (3.9) changes the rank-two exchange polynomials only by coefficient units. They have degrees $(2, 2)$. The rank-two classification [12, Theorem 6.4 and Corollary 6.5] therefore supplies infinitely many distinct cluster variables, even up to units of the enlarged coefficient ring. The freezing construction realizes their mutation path in the original D_7 algebra.

For $n > 7$, the same rank-two reduction is obtained by freezing. In the initial D_n interval seed, $n \geq 7$, keep exactly the ten mutable slots $\{X_{ij} : 2 \leq i \leq j \leq 5\}$ and freeze all other mutable variables, whose right index is at least six. Each exchange at a variable we keep involves only intervals in $[6]$ and is exactly the corresponding D_7 exchange. The resulting coefficient ring is obtained from the D_7 polynomial coefficient ring by inverting some of its frozen coordinates and adjoining independent coefficient variables, some invertible. The initial normalizers are trivial, so no additional factor enters this identification. The six mutations (3.8) therefore give the same obstruction over this coefficient extension; monomial units do not change the two reciprocal degrees. A second application of freezing lifts the infinite rank-two path to the original D_n seed. □

4. The exceptional varieties

The Cayley-plane and Freudenthal cones have dimensions 17 and 28. Their Coxeter boundaries contain 12 and 18 linear coordinates, respectively, so the mutable ranks are 5 and 10. We work over the polynomial rings generated by these frozen coordinates.

4.1. **The Cayley plane.** We keep the labels of Daisey–Ducat [4, Section 5]. The 27 degree-one coordinates split into Coxeter orbits

$$(a_i)_{i\in\mathbb{Z}/12}, \qquad (x_i)_{i\in\mathbb{Z}/12}, \qquad (z_i)_{i\in\mathbb{Z}/3}.$$

Freeze the outer orbit $a_1, \ldots, a_{12}$. When a subscript on z is an element of $\mathbb{Z}/12$, it is reduced modulo 3. Set

$$y_i = x_i z_i - a_i x_{i-1} - a_{i+7} x_{i+1} \qquad (i \in \mathbb{Z}/12).$$

In particular $y_3 = x_3 z_3 - a_3 x_2 - a_{10} x_4$ is a degree-two coordinate function, and a convenient initial cluster is

$$\mathbf{x}_6 = (x_1, x_2, x_3, x_4, y_3).$$

Over $\mathbb{C}[a_1, \ldots, a_{12}]$ its ordinary exchange polynomials are

$$\begin{aligned}
F_{x_1} &= y_3 + a_{12}a_1,\\
F_{x_2} &= a_2x_1(y_3 + a_{10}x_4) + a_9x_3(y_3 + a_1a_{12}) + x_1x_3(a_7x_4 + a_4a_{12}),\\
F_{x_3} &= y_3 + a_3x_2 + a_{10}x_4,\\
F_{x_4} &= a_{11}(y_3 + a_3x_2) + x_3(a_4x_1 + a_6x_2 + a_1a_9),\\
F_{y_3} &= x_1x_4(a_5x_2 + a_7x_3 + a_2a_{10}) + a_{12}x_3(a_4x_1 + a_1a_9)\\
&\qquad + a_{12}x_2(a_6x_3 + a_8x_4 + a_3a_{11}).
\end{aligned} \tag{4.1}$$

These are the exchanges of [4, Proposition 5.3]. Their associated Laurent polynomials satisfy

$$\widehat{F}_{x_2} = F_{x_2}/x_1, \qquad \widehat{F}_v = F_v \quad (v \neq x_2).$$

Indeed, substituting $x_1 = (y_3 + a_1a_{12})/x_1'$ into F_{x_2} introduces one common factor $y_3 + a_1a_{12}$; the other normalization orders are zero.

We recall the following result of Daisey and Ducat [4, Theorem 5.4 and Corollary 5.5].

Theorem 4.1. *The seed (4.1) gives the finite-type Cayley-plane LPA. It has 264 seeds, up to the seed equivalence used in [4], and 32 mutable cluster variables. All exchange polynomials and all cluster-variable Laurent expansions have positive coefficients.*

The count 32 excludes the twelve frozen variables. Their degrees are as follows: the mutable variables consist of the fifteen linear coordinates x_i, z_j, twelve quadratics y_i, two cubics u_1, u_2, and three quartics t_1, t_2, t_3. Representatives of the last two families are

$$t_1 = x_1x_4x_7x_{10} - a_3a_6a_9a_{12},$$
$$u_1 = x_1x_5x_9 - a_4a_5x_1 - a_8a_9x_5 - a_{12}a_1x_9 - a_1a_5a_9 - a_4a_8a_{12},$$

with the other members obtained by the dihedral action. There are fifteen dihedral orbits of seeds: seven orbits of length 24 and eight of length 12.

4.2. **The Freudenthal seed.** Let R_7 be the homogeneous coordinate ring of E_7/P_7, in the coordinate convention of [4, Section 6.1], reproduced in Appendix A.1, with the upper signs in their equations (e)–(g). Subscripts of a_i, x_i, y_i are read modulo 18. We keep the Coxeter boundary $a_0, \dots, a_{17}$ frozen over $S_7 = \mathbb{C}[a_0, \dots, a_{17}]$; none of these coefficients is inverted.

We take the seven linear functions

$$(h, b, c, d, e, f, g) = (x_1, x_{11}, x_9, x_{16}, x_6, x_{14}, x_4)$$

and set

$$r = x_4y_{15} - a_4x_{14}, \qquad q = x_{11}y_0 - a_{11}x_1,$$
$$u = x_1x_{11}y_8 - a_1x_{11}x_9 - a_{11}x_1y_5 - a_{12}x_1x_6.$$

The cluster $\mathbf{x} = (h, b, c, d, e, f, g, r, q, u)$ has degrees $(1, 1, 1, 1, 1, 1, 1, 2, 2, 3)$. This choice gives the binomial mirror of Subsection 8.2. To specify its exchanges, put

$$A_0 = g\{b(a_0f + a_1a_{14} + a_{15}h) + a_{11}a_{16}h\} + a_{17}b(a_4f + r),$$
$$A_1 = g\{b(a_0f + a_1a_{14} + a_{15}h) + h(a_{11}a_{16} + a_{12}d)\} + a_{17}a_4bf,$$
$$A_2 = a_3f + dg,$$
$$A_3 = a_1a_6c + a_2ce + a_4a_9h + a_5ch + a_8gh.$$

Define

$$F_h = a_1bc + u, \qquad F_b = a_{11}h + q, \qquad F_g = a_4f + r$$

and

$$F_c = a_9hF_g + a_8fgh + gu,$$
$$F_d = A_0 + a_{16}gq,$$
$$F_e = a_6qF_h + h(a_6a_{11} + a_7b)(br + u) + bhq(a_4a_9 + a_5c + a_8g),$$
$$F_f = a_{14}gF_h + h\{cg(a_{11}a_{16} + a_{12}d + a_{15}b) + r(a_{13}c + a_{14}a_9)\},$$
$$F_r = cA_2(eA_1 + dgu) + dgq(fA_3 + gu) + a_{16}a_3cefgq,$$
$$F_q = eA_1 + br(a_{17}e + dg) + dgu,$$
$$F_u = F_r|_{u=0} + r\{dgh[f(a_{10}e + a_{11}a_6 + a_7b) + e(a_{13}c + a_{14}a_9)] + a_{17}a_3bcef\}.$$

The associated Laurent exchange is $\widehat{F}_\xi = F_\xi/M_\xi$, where

ξ	h	b	c	d	e	f	g	r	q	u
M_ξ	1	1	g	bg	hb	h	1	1	g	dg.

Theorem 4.2. *These functions and exchanges define a rank-ten LP seed Σ_7 over the polynomial coefficient ring S_7, with*

$$\mathcal{A}(\Sigma_7) \subseteq \mathcal{U}(\Sigma_7) = R_7.$$

The LP algebra has infinitely many cluster variables.

Proof. Proposition A.4 identifies the seed with the auxiliary Freudenthal seed by five complete LP mutations. The seed axioms can also be checked directly: each exchange is primitive linear in a variable specified in Appendix A, and the normalization gives the table above, over S_7.

For the auxiliary seed, Proposition A.1 computes the upper coordinate ring, and Proposition A.2 gives an infinite rank-two mutation sequence. Both statements are invariant under mutation equivalence, so they hold for Σ_7. □

Remark 4.3. This theorem concerns one infinite mutation class. It does not exclude a different positive finite-type Freudenthal LPA as in [4, Conjecture 6.1].

Part 2. Algebraic mirrors and intrinsic comparison

5. Newton reconstruction and intrinsic comparison

For a boundary complement U from Part I, let T be its initial torus, with character lattice M_T and cocharacter lattice $N_T = \mathrm{Hom}(M_T, \mathbb{Z})$. We write $\mathscr{M} = A_{\mathrm{int}}(U)_1$ and recall the results of [7] used in the mirror comparisons. All lattice changes are integral and include the coefficient directions.

For a Laurent polynomial $f = \sum_a c_a z^a$, put

$$\mathrm{supp}(f) = \{a : c_a \neq 0\}, \qquad \mathrm{Newt}(f) = \mathrm{conv}\,\mathrm{supp}(f), \qquad \nu_w(f) = \min_{c_a \neq 0} \langle a, w \rangle.$$

We call ν_w the *Gauss valuation* of weight w. A leading term has least weight in the specified chamber.

The source U is smooth and factorial. Indeed, the homogeneous coordinate ring is normal with class group $\mathrm{Pic}(X)/\mathbb{Z}[\mathcal{O}_X(1)] = 0$, and inverting a frozen linear coordinate removes the cone vertex. Thus $\mathrm{Pic}(U) = 0$. The *normality theorem for the intrinsic mirror algebra* states that $\mathscr{M}$ is a finitely generated normal domain. It also provides a normal projective compactification $(\overline{M}, D_M)$ with reduced boundary, $K_{\overline{M}} + D_M \sim 0$, and log canonical singularities. The theorem on the *logarithmic volume of the intrinsic mirror* identifies the resulting rational volume, up to a nonzero scalar, with the standard logarithmic form on any algebraic torus open in $\mathrm{Spec}\,\mathscr{M}$. In particular, this form has order zero at every prime divisor of $\mathrm{Spec}\,\mathscr{M}$.

The *intrinsic Laurent expansion and trace theorem* constructs an injective algebra homomorphism from $\mathscr{M}$ to the Laurent ring of the dual torus when the local cylinder counts, boundedness of contact orders, and finiteness of the contributing curve classes satisfy its hypotheses. It identifies the images of the prescribed theta functions with their local expansions. We verify these hypotheses for each source; in particular, we obtain support bounds from regular source functions before using any proposed mirror formula.

We also use *restriction to a principal open subset.* If $0 \neq f \in \mathcal{O}(U)$ and U_f contains the chosen torus, the filtration

$$F_{\geq a}\mathscr{M} = \mathrm{Span}_{\mathbb{C}}\{\vartheta_m : f^{\mathrm{trop}}(m) \geq a\}$$

is multiplicative, and its associated graded is $A_{\mathrm{int}}(U_f)_1$, with the corresponding theta labels unchanged. When U_f is a torus T_f, this is the group algebra $\mathbb{C}[N_{T_f}]$, and the initial symbol of each theta function is its labelled monomial with coefficient one.

All character changes include the frozen directions. We choose a basis of the saturated lattice generated by support differences and complete it to a basis of the source character lattice.

The dual coordinates are $z_1, \dots, z_r, \kappa_1, \dots, \kappa_c$; the coefficient ring is $\mathscr{S} = \mathbb{C}[\kappa_1^{\pm 1}, \dots, \kappa_c^{\pm 1}]$. Compatibility with torus products gives

$$A_{\mathrm{int}}(V \times T)_1 \simeq A_{\mathrm{int}}(V)_1 \otimes_{\mathbb{C}} \mathbb{C}[N_T],$$

and sends each product-labelled theta function to the corresponding tensor product. This formula restores the torus factors after the comparison in the remaining coordinates.

5.1. **Support bounds and coefficient recognition.** Let $\mathsf{N} : \mathbb{Z}^s \hookrightarrow N_T$ be the contact map, with pointed correction monoid $\mathsf{N}\mathbb{N}^s$. For regular source functions ψ_j with minimizing exponents h_j in the chosen chamber, put

$$\delta_j(m) = \langle h_j, m \rangle - \min_{a \in \operatorname{supp} \psi_j} \langle a, m \rangle.$$

Writing H for the matrix of the h_j, the regular-function slope inequalities give

$$d \geq 0, \qquad H\mathsf{N}d \geq -\delta(m).$$

A rational certificate $\lambda \geq 0$ with $\lambda H\mathsf{N} \leq -\epsilon\mathbf{1}$, $\epsilon > 0$, gives $\sum_i d_i \leq \lambda\delta(m)/\epsilon$. Conversely such a certificate exists whenever the recession cone is zero, by Farkas' lemma. We display the relevant source inequalities in each family rather than infer support bounds from candidate formulas.

Lemma 5.1. *Suppose the contact model verifies* [10, Assumption 20.27] *for the pointed monoid* $\mathsf{N}\mathbb{N}^s$*. A finite set of allowed contact vectors involves only finitely many contributing numerical curve classes.*

Proof. Choose k larger than the total degree of every allowed contact vector. All terms survive modulo the kth power of the monomial ideal generated by the nonzero elements of the contact monoid. Keel–Yu's Lemma 20.31 bounds the contributing curve classes in this truncation, hence in the entire expansion. □

We determine coefficients in two ways. For a mutable source coordinate v, let $g_v = v' \prod_{w \neq v} w$, where the product runs over the other mutable coordinates. The open set U_{g_v} is the first-mutated source torus. Restriction to this principal open gives a valuation on $\mathscr{M}$ whose associated graded is a torus group algebra. Suppose $\Theta_m = \sum_{a \in A_m} c_a z^a$ has a proved finite support bound, and the initial symbols of the characters z^a have coefficient one in that group algebra. If a unique $a_0 \in A_m$ has valuation $g_v^{\mathrm{trop}}(m)$ and all other points have larger valuation, then $c_{a_0} = 1$. This is the lemma on *coefficient determination from an initial form* of [7]. We include the associated source normalizer when computing this valuation.

The second method is *finite Laurent interpolation* [7]. Once the adjacent mirror Laurent rings have been identified, write the function with a fixed monomial denominator and a numerator supported in a proved finite set. In the ith adjacent chart, every coefficient of z_i^{-d} must be divisible by $\widehat{F}_i^d$ in the unchanged-variable Laurent ring. These are linear equations in the numerator coefficients. If they have a unique solution after fixing the leading coefficient, that solution is the theta function: the intrinsic expansion satisfies the same equations and normalization. We use this argument for the exceptional generators and boundary functions. Their coefficient matrices and nonzero maximal minors are included in the ancillary files.

5.2. **Contact equations in toric blowup charts.** Let U be smooth and contain a torus T with character lattice M_T and cocharacter lattice N_T. Suppose an open part of U is obtained from toric ray charts by blowing up centres $F_a = 0$ in their boundary divisors and removing the strict transforms of those divisors. We index the resulting coordinate divisors by $a \in I$ and write $n_a \in N_T$ for their primitive character-value vectors. Near a generic contact the chart has the form $x_a w_a = F_a$ with the other torus characters invertible and F_a independent of x_a. Distinct divisors may have the same centre equation. These coordinate divisors lie inside U.

Choose the compactification used for the intrinsic counts and a common open model for it and these toric blowup charts. The next proposition applies on the locus where the maps stay in this model away from their two marked ends. The applications verify the end conditions and that this locus contains the full counting fibre for generic evaluation.

Proposition 5.2. *Fix two nonzero end directions, a contact vector $(b_a)_{a\in I}$, and a geometric evaluation point c over a generic monomial point of T. Suppose the relevant counted maps have stable domains with three distinguished points, meet the coordinate divisors transversely at distinct points, and avoid the complement of the toric blowup charts away from their marked ends. Normalize the ends and evaluation to $0,\infty,1$ and write $m\in N_T$ for the incoming direction. Along each such map f, the characters satisfy*

$$\chi(f(z)) = \chi(c)z^{\langle\chi,m\rangle}\prod_{a\in I}\prod_{\alpha=1}^{b_a}\left(\frac{z-t_{a,\alpha}}{1-t_{a,\alpha}}\right)^{\langle\chi,n_a\rangle}, \qquad \chi\in M_T. \tag{5.1}$$

The contact parameters satisfy the centre equations $F_a=0$ after substitution. For the two-ended local expansion, they lie in $0<|t_{a,\alpha}|<1$. The outgoing direction is $-m-\sum_a b_a n_a$.

Conversely, consider solutions of these equations with distinct contact parameters in the prescribed annuli. Assume that the interpolated maps lie in the common toric blowup model and satisfy the required open-stratum conditions at both ends and the toric-tail condition. On this locus, the incidence scheme and the geometric evaluation fibre agree. An isolated solution at which evaluation is étale contributes one. Parameters of the same divisor type are unordered.

Proof. Work over a completed algebraic closure of the residue field of the chosen monomial point. Its character values have the prescribed valuations, without special algebraic cancellations. The smooth, stable-domain assertion of [10, Lemmas 8.20–8.21] allows the normalization to a three-pointed $\mathbb{P}^1$ in the stated locus. The divisor of each character is determined by the two ends and the contacts. A rational function with that divisor is unique up to a constant, which evaluation at 1 fixes. This proves (5.1) and the balancing formula at infinity. Characters invertible on U have no unmarked zeros or poles.

At a simple contact of type a, the map lifts to $x_a w_a = F_a$ exactly when F_a vanishes there. This is divisibility by a local parameter on the domain. The quotient is the value of w_a; a further condition $w_a\neq 0$, when required, must be checked and is not implied by the centre equation alone. The prescribed leading coefficients at $0,\infty$ give the two marked boundary contacts. The toric-tail condition of [10, Construction 9.3] requires the complementary tail to have no unmarked contact; for the normalized two-ended expansion it gives the stated open-unit-disk condition.

Under the converse hypotheses these formulas construct a map to the chosen compactification with the prescribed contacts. The construction is reversible and works in families on this open locus, so it identifies the incidence scheme with the evaluation fibre there. A simple Jacobian over the evaluation parameters proves étaleness when used in the applications below. Such a point has length one. The three marked points remove automorphisms, and quotienting by permutations of contacts of the same type introduces no factorial multiplicity. A map has a uniquely determined curve class, so summing classes does not count it again. □

To apply the proposition to the full counting fibre, we must also exclude curves outside its stated locus. Once the maps are in the appropriate moduli space, [10, Lemmas 3.9 and 8.21] allow generic evaluation to avoid fixed codimension-two loci and ensure generic transversality. Lemma 5.1 permits one generic choice for the finitely many relevant classes. We check the end-stratum and toric-tail conditions in each family.

5.3. **From local expansions to the full mirror.** We apply the intrinsic Laurent expansion and trace theorem of [7]. After the primitive walls have been identified, consistency gives the adjacent expansions. The following elementary observation is used when a transformed expansion is initially formal.

Lemma 5.3. *Let f be a nonnegative formal series in finitely many translates of a pointed monoid, and let D be a nonnegative polynomial with positive constant term. If Df is a finite Laurent polynomial, so is f.*

Proof. There is no cancellation, and the constant term of D puts $\operatorname{supp} f$ inside $\operatorname{supp}(Df)$. □

We then use the *intrinsic mirror comparison theorem* of [7]. For an LP seed with initial and adjacent Laurent rings $L_0, \dots, L_r$, it is enough to establish

$$\mathscr{S}[x_i, x_i' : 1 \le i \le r] \subseteq \mathscr{R} := \Phi(\mathscr{M}) \subseteq \bigcap_{i=0}^{r} L_i$$

and either recognize algebra generators of the intersection or prove that no height-one prime of $\mathscr{R}$ contains two distinct initial mutable coordinates. The Cayley-plane comparison uses generators. The half-spin and Freudenthal comparisons use divisors.

For the latter method, let w be the character-value vector of a normalized divisorial valuation of volume order zero, with $w \neq 0$, and write $m_f(w) = \min_{a \in \operatorname{supp} f} \langle a, w \rangle$. The *divisorial valuation formula* of [7] gives an irreducible Laurent polynomial q in the characters of value zero. More precisely, write $w = dw_0$, with w_0 primitive, and choose a character t with $\langle t, w_0 \rangle = 1$. For a collected Laurent sum in t,

$$\nu\left(\sum_j t^j f_j\right) = \min_j \{dj + \operatorname{ord}_q(f_j)\}.$$

Consequently, every regular f with $m_f(w) < 0$ satisfies $q^{-m_f(w)} \mid \operatorname{in}_w(f)$. It follows that $m_f(w) \ge -b_f$ when all face factors of f have multiplicity at most b_f.

Corollary 5.4. *If a negative minimizing face is irreducible up to a character and has multiplicity one, then w is primitive, $m_f(w) = -1$, and $\nu(f) = 0$. All such negative faces define the same spectator hypersurface.*

Proof. The divisibility forces $-m_f(w) = 1$. In the exact slice formula its lowest slice has value zero, while every slice has nonnegative value. □

6. Algebraic mirrors of the half-spin varieties

Fix $n \ge 4$ and put

$$m = n - 1, \qquad N = \binom{n}{2} + 1, \qquad r_0 = \binom{n-2}{2}, \qquad r = \frac{n(n-3)}{2}.$$

We use the interval seed of Section 3, writing $h = q_\varnothing$, $X_{ij} = q_{[i,j]}$, and $X_{i,i-1} = h$. Let

$$\mathcal{M} = \{(i,j) : 2 \le i \le j \le m-1\}, \qquad \mathcal{B}_n = \{h, X_{11}, \dots, X_{1m}, X_{2m}, \dots, X_{mm}\}.$$

The mirror will have r mutable coordinates, $n+1$ coefficient units, and $2n-2$ boundary functions indexed by $\mathcal{B}_n$.

6.1. **An integral system of Newton coordinates.** In the interval-coordinate torus define

$$\chi_{B_i} = \frac{X_{i+2,i+2}X_{i,i+1}}{X_{ii}X_{i+1,i+2}} \qquad (1 \le i \le m-3), \tag{6.1}$$

$$\chi_{E_i} = \frac{X_{i,m-2}X_{i-1,m}}{X_{i-1,m-2}X_{im}} \qquad (2 \le i \le m-2),$$

$$\chi_{R_{ij}} = \frac{X_{ij}X_{i-1,j+1}X_{i+1,j+2}}{X_{i+1,j+1}X_{i-1,j}X_{i,j+2}} \qquad (2 \le i \le j \le m-2),$$

$$\chi_P = \frac{hX_{m-2,m}}{X_{m-2,m-2}X_{m-1,m}}, \qquad \chi_Q = \frac{X_{m-2,m-1}X_{mm}}{X_{m-2,m-2}X_{m-1,m}}. \tag{6.2}$$

There are r characters. Complete them by

$$g_0 = \begin{cases} X_{1,m-2}, & n \text{ odd}, \\ X_{2,m-1}, & n \text{ even}, \end{cases} \qquad (g_1, \dots, g_n) = (X_{1,m-1}, X_{1m}, X_{2m}, \dots, X_{mm}). \tag{6.3}$$

Let C be the exponent matrix of (χ_ρ, g_j) in the original interval coordinates. The original mutable coordinates come first; the individual columns are denoted Ce_a, so the formulas do not depend on a further ordering convention.

Proposition 6.1. *The characters in (6.1)–(6.3) form an integral basis. The χ_ρ generate the saturated support-difference lattice of the spinor-coordinate expansions. Its rank is $r = N - (n+1)$.*

Proof. We first prove the integral basis statement. Work in the quotient of the character lattice by the subgroup generated by the χ_ρ. Write x_{ij} for the additive class of X_{ij}, and put $P_j = x_{1j}$ and $S_i = x_{im}$, so $P_m = S_1$. For $m \geq 4$, set

$$\Delta_{ij} = x_{ij} + x_{i-1,j+1} - x_{i-1,j} - x_{i,j+1}.$$

The R relations say $\Delta_{ij} = \Delta_{i+1,j+1}$, and the E relations say $\Delta_{i,m-2} + \Delta_{i,m-1} = 0$. Hence, with $s = \Delta_{22}$, $\Delta_{ij} = (-1)^{j-i}s$. Put $A_i = \mathbf{1}_{i\,\text{even}}$ and $D_j = \sum_{t=j}^{m-1}(-1)^t$. Summing the differences gives

$$x_{ij} = P_j + S_i - S_1 + sA_iD_j \qquad (2 \leq i \leq j \leq m).$$

The remaining B relations become

$$P_{i+1} - P_i = S_{i+1} - S_{i+2} + \mathbf{1}_{m\,\text{even}}(-1)^i s.$$

Writing $[h]$ for the class of h, the Q and P relations give

$$P_{m-1} - P_{m-2} + S_m - S_{m-1} - \mathbf{1}_{m\,\text{even}} s = 0, \qquad [h] = P_{m-2} + S_{m-1} - S_1.$$

For m even, s is determined integrally by P_{m-2}, P_{m-1} and the S_i. For m odd, $P_{m-2} = P_{m-1} + S_m - S_{m-1}$, and $s = x_{2,m-1} - P_{m-1} - S_2 + S_1$. These are exactly the complementary characters (6.3). These formulas express every initial character integrally in the complementary characters and impose no further relations among them. They therefore freely generate the quotient. Since the number of characters χ_ρ is the complementary rank, adjoining the complementary characters gives a unimodular basis. For $m = 3$ the two characters in (6.2) give the same conclusion directly.

Each χ_ρ has homogeneous half-spin weight zero. The interval coordinates have weights $(1, \mathbf{1}_{1\in I}, \dots, \mathbf{1}_{m\in I}, |I| \bmod 2)$, whose rank is $m + 2 = n + 1$. Since the lattice generated by the χ_ρ is saturated and has complementary rank, it is the kernel of this grading. Lemma B.1 shows that every χ_ρ occurs as a pure linear correction in a complementary-interval spinor coordinate. Consequently the spinor-coordinate support differences already generate this kernel. □

Write

$$z = (B_i, E_i, R_{ij}, P, Q; \kappa_0, \dots, \kappa_n)$$

for the dual torus coordinates and put $\mathscr{S}_n = \mathbb{C}[\kappa_0^{\pm 1}, \dots, \kappa_n^{\pm 1}]$. The symbols B_i, E_i, R_{ij}, P, Q are mirror coordinates; the χ_ρ are characters on the original torus. For an initial basis vector v, the corresponding dual monomial is z^{Cv}.

6.2. **The mirror seed.** For $a = (i, j) \in \mathcal{M}$, set $c_a = Ce_a$ and define

$$\mathcal{E}_a = z^{[c_a]_+} + z^{[-c_a]_+}. \tag{6.4}$$

Here $[c]_+$ denotes the coordinatewise positive part. The ordinary exchanges are

$$F_{B_i}^\vee = \mathcal{E}_{i+1,i+1}, \qquad F_{R_{ij}}^\vee = \mathcal{E}_{i,j+1},$$
$$F_{E_i}^\vee = \mathcal{E}_{i,m-1}, \qquad F_P^\vee = F_Q^\vee = \mathcal{E}_{m-1,m-1}.$$

For each a, let $\mathcal{I}_a$ be the set of positions assigned to $\mathcal{E}_a$. The sets with more than one element are

$$\{P, Q\}, \qquad \{E_i, R_{i,m-2}\} \quad (2 \le i \le m-2). \tag{6.5}$$

For $Z_\rho \in \mathcal{I}_a$, the associated exchange and first mutation are

$$\widehat{F}^\vee_\rho = \frac{\mathcal{E}_a}{\prod_{\sigma \in \mathcal{I}_a \setminus \{\rho\}} Z_\sigma}, \qquad Z'_\rho = Y_a := \frac{\mathcal{E}_a}{\prod_{\sigma \in \mathcal{I}_a} Z_\sigma}. \tag{6.6}$$

The r mutable positions give r_0 distinct first-mutated functions.

Proposition 6.2. *Equations (6.4)–(6.6) define an LP seed $\Sigma^\vee_n$ over $\mathscr{S}_n$. This seed is obtained by Newton reconstruction from the interval-Pfaffian seed. Its mutable rank is $r = n(n-3)/2$.*

Proof. Each c_a is primitive because C is unimodular. Hence $1 + z^{c_a}$ is irreducible in the full Laurent ring, and clearing its coprime monomial denominator gives the irreducible polynomial $\mathcal{E}_a$. The assigned variable is absent since $C_{\rho,a} = 0$ for $\rho \in \mathcal{I}_a$. Distinct columns give nonassociate binomials. In the LP normalization a different binomial cannot divide the relevant specialization: specializing an occurring variable to zero leaves a monomial, and specializing a nonoccurring variable leaves the same nonassociate binomial. For $\rho \in \mathcal{I}_a$, normalization removes one factor Z_σ for each $\sigma \in \mathcal{I}_a \setminus \{\rho\}$. This proves (6.6).

For the reconstruction, the complementary-interval functions have mutable leading gradients $-e_a$. They give a bounded box for every correction label. At the adjacent label

$$\mu_a = -\sum_{\rho \in \mathcal{I}_a} e_\rho + [-c_a]_+, \tag{6.7}$$

the box contains only the corrections $0, e_a$; see Appendix B.2. Thus the support is $\{\mu_a, \mu_a + c_a\}$. Laurent divisibility forces equal coefficients, which gives $Y_a = z^{\mu_a}(1 + z^{c_a})$. □

For odd n, the exchanges contain no coefficient variables. For even n, only κ_0 occurs, in the exchange assigned to $(2, m-1)$. For $n = 4$ this is the entire seed:

$$F^\vee_P = F^\vee_Q = 1 + \kappa_0, \qquad P' = Q' = \frac{1 + \kappa_0}{PQ}.$$

The polynomial $1 + \kappa_0$ is irreducible and noninvertible in $\mathscr{S}_4$. For D_5, the five mutable variables are B_1, E_2, R_{22}, P, Q, with

$$\begin{aligned} F^\vee_{B_1} &= E_2 R_{22} + PQ, \\ F^\vee_{E_2} = F^\vee_{R_{22}} &= B_1 + Q, \\ F^\vee_P = F^\vee_Q &= B_1 + R_{22}. \end{aligned}$$

These examples illustrate the repeated exchanges in the uniform formula.

6.3. **The indexed boundary functions.** Let e_h, e_{ij} be the interval basis. Define

$$W_h = z^{Ce_h}, \qquad W_{1m} = z^{Ce_{1m}},$$

and

$$W_{1k} = \sum_{t=0}^{m-k-1} z^{C(e_{1k} + \sum_{j=1}^{t} e_{j+1,j+k})} \qquad (1 \le k < m), \tag{6.8}$$

$$W_{im} = \sum_{t=0}^{m-i} z^{C(e_{im} + \sum_{j=1}^{t} e_{i,m-j})} \qquad (2 \le i \le m). \tag{6.9}$$

They are indexed by the $2m$ frozen coordinates, with initial label Ce_b.

Proposition 6.3. *Every W_b belongs to $\mathcal{U}(\Sigma^\vee_n)$. Their sum has $m(m-1)+2$ distinct Laurent monomials, each with coefficient one.*

Proof. Along either cumulative chain, a negative exponent in a mutable coordinate is -1 and occurs at two consecutive positions. The ratio of those two terms is z^{c_a}, where a is assigned to that position. Their sum is divisible by $\mathcal{E}_a$ in the Laurent ring in the unchanged coordinates. All other powers are nonnegative, so (2.1) proves regularity in each adjacent chart. The index calculation is given in Appendix B.6. The starfish theorem then gives membership in the upper LP algebra. The triangular expressions below give the count and distinctness of the monomials. □

6.4. **The triangular potential and its Newton polytope.** Let $\mathbf{1} = e_h + \sum_{i\le j} e_{ij}$ and set $\mathfrak{q} = z^{C\mathbf{1}} = \prod_{j=0}^{n} \kappa_j$. For $1 \le a \le b \le m$, put $i = m+1-b$ and define

$$v_{ab} = e_h + \sum_{r=i+1}^{m}\sum_{s=r}^{m} e_{rs} + \sum_{r=1}^{i}\sum_{\ell=0}^{a-2} e_{r,r+\ell}, \qquad y_{ab} = z^{Cv_{ab}}. \tag{6.10}$$

Empty sums are zero. The characters $\mathfrak{q}, y_{ab}$ form an integral basis by Lemma B.6. In this basis,

$$\begin{aligned} W_h &= y_{11}, & W_{1m} &= \mathfrak{q}/y_{mm}, \\ H_k := W_{1k} &= \sum_{b=k+1}^{m} \frac{y_{k+1,b}}{y_{kb}} & &(1 \le k < m), \\ V_i := W_{im} &= \sum_{a=1}^{m+1-i} \frac{y_{a,m+2-i}}{y_{a,m+1-i}} & &(2 \le i \le m). \end{aligned} \tag{6.11}$$

Consequently the potential is

$$W = y_{11} + \sum_{1\le a<b\le m} \frac{y_{a+1,b}}{y_{ab}} + \sum_{1\le a\le b<m} \frac{y_{a,b+1}}{y_{ab}} + \frac{\mathfrak{q}}{y_{mm}}. \tag{6.12}$$

On a nonzero fibre of $\mathfrak{q}$, let Δ_m be its Newton polytope. If $P_m = \{(a,b) : 1 \le a \le b \le m\}$ has componentwise order, $\mathcal{O}(P_m)$ is its order polytope, and $\mathbf{h} = (a+b-1)_{(a,b)\in P_m}$ is the root-height vector, then

$$\Delta_m^\circ = 2m\,\mathcal{O}(P_m) - \mathbf{h},$$

where Δ_m° denotes the polar with convention $\langle v, u\rangle \ge -1$. Indeed, the vertices defining Δ_m are

$$e_{11}, \quad -e_{mm}, \quad e_{a+1,b} - e_{ab}\ (a<b), \quad e_{a,b+1} - e_{ab}\ (b<m).$$

Substituting $u = 2m\lambda - \mathbf{h}$ turns the polar inequalities into $0 \le \lambda_{ab} \le 1$ and the order inequalities. The order polytope is integral, and $\mathbf{h}/(2m)$ satisfies its inequalities strictly. Thus Δ_m is reflexive. The scale $2m = 2n-2$ is the Fano index of the spinor variety.

7. The intrinsic mirror theorem in type D

Let

$$U_n = \operatorname{Spec} \mathbb{C}[\widehat{\mathrm{OG}^+(n,2n)}][(\prod_{b\in\mathcal{B}_n} b)^{-1}], \qquad \mathcal{U}_n^\vee = \mathcal{U}(\Sigma_n^\vee).$$

We identify $\mathcal{U}_n^\vee$ with the intrinsic mirror algebra of U_n, matching the boundary functions constructed in the preceding section.

Theorem 7.1. *For every $n \ge 4$, local intrinsic expansions induce an isomorphism*

$$\Phi_n : A_{\mathrm{int}}(U_n)_1 \xrightarrow{\sim} \mathcal{U}_n^\vee.$$

It identifies the lattice of coefficient characters and satisfies $\Phi_n(\vartheta_{Ce_b}) = W_b$ for all $b \in \mathcal{B}_n$. The algebra $\mathcal{U}_n^\vee$ is finitely generated, normal, and factorial, with unit group $\mathscr{S}_n^\times$. Its superpotential is (6.12).

We first identify the torus charts and the individual boundary functions. Their Newton polytopes then show that the charts meet every prime divisor.

7.1. **The local comparison.** Let $\mathscr{M}_n = A_{\mathrm{int}}(U_n)_1$ and let $L = \mathscr{S}_n[Z_\rho^{\pm 1}]$ be the mirror Laurent ring. Recall $Y_a = z^{\mu_a}(1 + z^{c_a})$ from (6.7) and (6.6).

Proposition 7.2. *There is an injective algebra homomorphism*

$$\Phi_n : \mathscr{M}_n \hookrightarrow L, \qquad M_n = \Phi_n(\mathscr{M}_n),$$

with

$$\mathscr{S}_n[Z_\rho, Y_a, W_b] \subseteq M_n \subseteq \mathcal{U}_n^\vee.$$

The labels of the adjacent and boundary functions satisfy

$$\Phi_n(\vartheta_{\mu_a}) = Y_a, \qquad \Phi_n(\vartheta_{Ce_b}) = W_b.$$

The initial torus and each adjacent torus are open charts of $\operatorname{Spec} \mathscr{M}_n$.

Proof. We use the recurrence and contact calculations of Appendix B. For each $t = (u, v) \in \mathcal{M}$, the complementary interval coordinate $\psi_t = q_{[m]\setminus[v-u+2,v]}$ has leading term $X_{1,v-u+1}X_{um}/X_{uv}$. Its mutable leading gradient is $-e_t$. The regular-function slope inequalities therefore give

$$0 \le d_t \le \delta_{\psi_t}(\eta) \qquad (t \in \mathcal{M}) \tag{7.1}$$

for every contact correction $C_{\mathrm{mut}}d$ to an incoming label η. The contact monoid is free and saturated because its columns are part of the basis C. The same gradients exclude a nonconstant attached tropical branch rooted in the chosen chamber. The contact model verifies [10, Assumption 20.27], so Lemma 5.1 gives finite local expansions before specialization to 1. The trace-gluing argument of [7, intrinsic Laurent expansion and trace theorem] makes these expansions an injective algebra homomorphism. Nonnegative mutable labels have zero defects, so all Z_ρ and coefficient characters belong to its image.

At a primitive facet indexed by ρ, the normalized complementary functions restrict to $1+\chi_\rho$ for the assigned interval $a(\rho)$ and to 1 for every other interval. Only one simple contact can occur. The blowup centre is linear in χ_ρ, and its contact equation is

$$A(c_{\neq\rho})\tau + c_\rho B(c_{\neq\rho}) = 0, \qquad A(0) = B(0) = 1.$$

It has a unique simple solution of the required valuation. The marked-end and toric-tail check in Proposition B.5 shows that this solution is counted. The wall function is therefore $1 + z^{c_{a(\rho)}}$. Continuation across the wall is the normalized LP mutation. After clearing the possible binomial denominator, positivity implies that the continued expansion has finite support. Thus it is Laurent in every adjacent chart. The starfish theorem gives $M_n \subseteq \mathcal{U}_n^\vee$.

At μ_a, the full defect calculation in Lemma B.3 reduces (7.1) to $d \in \{0, e_a\}$. Divisibility in an adjacent chart fixes the two coefficients to be one, yielding Y_a. For Ce_b, the box is supported on one of the interval chains in (6.8)–(6.9). The regular mutation partners impose $d_1 \ge d_2 \ge \cdots \ge d_\ell$, with $d_j \in \{0, 1\}$ Thus only prefixes occur. Adjacent divisibility propagates the straight coefficient one along the chain and gives W_b; see Appendix B.5.

Finally, the image contains all coordinates of each indicated chart and is Laurent in them. Localizing at their product therefore gives exactly the corresponding Laurent ring, so these are open charts of the intrinsic spectrum. □

7.2. **Triangular valuations.** Let M be a normal finitely generated subalgebra of L satisfying $\mathscr{S}_n[Z_\rho, Y_a, W_b] \subseteq M \subseteq \mathcal{U}_n^\vee$, and suppose the initial logarithmic form has order zero along every prime divisor of $\operatorname{Spec} M$. For such a divisor E, put

$$w_\rho = \operatorname{ord}_E(Z_\rho) \ge 0, \qquad \lambda_{ab} = \operatorname{ord}_E(y_{ab}).$$

The coefficient units have order zero. The character exponents give an injective map $w \mapsto \lambda$ with

$$\sum_{a=1}^{m-d} \lambda_{a,a+d} = 0, \qquad \sum_{j=1}^{a} \lambda_{j,j+d} \ge 0 \quad (a < m - d). \tag{7.2}$$

Write ϵ_{ab} for the array supported at (a,b) and $T_{a,d}=\sum_{j=1}^{a}\lambda_{j,j+d}$. A unit value at one initial variable contributes the following array:

$$\begin{array}{c|l} B_i & \epsilon_{1,m-i}-\epsilon_{2,m-i+1} \\ R_{ij} & \epsilon_{j-i+2,m+1-i}-\epsilon_{j-i+3,m+2-i} \\ E_i,\ a=m-i & \epsilon_{a,a}+\epsilon_{a,a+1}-\epsilon_{a+1,a+2}-\epsilon_{a+2,a+2} \\ P & \epsilon_{11}+\epsilon_{12}-\epsilon_{23}-\epsilon_{33} \\ Q & \epsilon_{12}-\epsilon_{23}. \end{array}$$

Each is a forward diagonal flow. The partial sums recover the initial values; the explicit inverse formulas are in Lemma B.7.

Diagonal rigidity. We first consider real arrays satisfying (7.2).

Lemma 7.3. *If $\lambda\neq 0$ satisfies (7.2), then some horizontal arrow and some vertical arrow have negative value.*

Proof. The outermost diagonal has a single entry and is zero. Suppose first that all horizontal arrows are nonnegative. If the next outer diagonal is zero, the horizontal arrows into the current diagonal force every entry except its first to be nonnegative. The first is nonnegative because $T_{1,d}\geq 0$. Since the total on the diagonal is zero, every entry vanishes. Descending induction makes the whole array zero.

If all vertical arrows are nonnegative, a zero outer diagonal forces every entry of the current diagonal except its last to be nonpositive. The last is also nonpositive, being the negative of a nonnegative proper partial sum. Again the zero total and descending induction make the whole array zero. Both conclusions contradict $\lambda\neq 0$. □

Lemma 7.4. *Suppose (7.2) holds. Suppose every arrow is nonnegative except the four directed sides of the square*

$$(k,\ell),\quad (k,\ell+1),\quad (k+1,\ell),\quad (k+1,\ell+1),\qquad 1\leq k<\ell\leq m-1,$$

and all four exceptional arrow values equal -1. Put $d=\ell-k$. If $d\geq 2$, then $\lambda=\epsilon_{k,\ell}-\epsilon_{k+1,\ell+1}$. If $d=1$, then for some $t\in[0,1]$,

$$\lambda=\epsilon_{k,k+1}-\epsilon_{k+1,k+2}+t(\epsilon_{k,k}-\epsilon_{k+2,k+2}).$$

If this array comes from nonnegative integer initial values through the displayed flow table, exactly one initial value is one and every other initial value is zero.

Proof. Write $A_a^{(s)}=\lambda_{a,a+s}$. Whenever the relevant arrows are nonnegative, their inequalities are

$$A_{a-1}^{(s+1)}\leq A_a^{(s)}\leq A_a^{(s+1)}. \tag{7.3}$$

At the two missing extreme positions the corresponding bounds are respectively $A_1^{(s)}\geq 0$ and $A_{m-s}^{(s)}\leq 0$, by the partial-sum conditions.

Every diagonal of gap greater than d is zero, by downward induction using (7.3). On diagonal d, the vertical exception from (k,ℓ) to $(k,\ell+1)$ forces the value 1 at (k,ℓ); the horizontal exception from $(k,\ell+1)$ to $(k+1,\ell+1)$ forces the value -1 at $(k+1,\ell+1)$. The other entries vanish by the general bounds and the terminal position bounds.

On diagonal $d-1$, the other two exceptional arrows both force the entry at $(k+1,\ell)$ to be zero. The bounds (7.3) leave only two other possible nonzero entries: the entry at position k lies in $[0,1]$, and that at position $k+2$ lies in $[-1,0]$. The zero total makes these t and $-t$. If $d\geq 2$, the entry at position $k+1$ of diagonal $d-2$ satisfies $t\leq A_{k+1}^{(d-2)}\leq 0$. Thus $t=0$, and the remaining inner diagonals vanish by downward induction. If $d=1$ there is no further inner diagonal, giving the stated family.

For an array coming from the seed, equations (B.11)–(B.12) finish the argument. When $d\geq 2$, the only nonzero proper diagonal partial sum is $T_{k,d}=1$, so it is a single B or R value. When $d=1$, the only nonzero diagonal-one partial sum is $T_{k,1}=1$, and the diagonal-zero

partial sums are $T_{k,0} = T_{k+1,0} = t$. The recovery formulas give orders t and $1-t$ to the two variables in the corresponding repeated-exchange pair, and order zero to every other initial variable. Integrality gives $t \in \{0, 1\}$. Thus one member of the pair is selected, never both. □

Boundary Newton simplices.

Lemma 7.5. *Every nonmonomial face of a boundary function in (6.11) defines, after division by one of its monomials, a smooth irreducible hypersurface in the torus of its difference lattice. This lattice is saturated in the full character lattice and in any primitive weight-zero sublattice containing it.*

Distinct horizontal boundary functions have disjoint sets of column directions, and so do distinct vertical boundary functions. The sets for H_k and V_i intersect in the single column c_{ij}, where $j = i + k - 1$, if (i, j) is mutable; otherwise they are disjoint. If nonmonomial faces of two distinct boundary functions have translates as Newton polytopes, then they are consecutive two-vertex faces of one H_k and one V_i at such a shared column.

Proof. In the initial indexing, the successive support differences are

$$H_k : \quad c_{2,k+1}, c_{3,k+2}, \ldots, c_{m-k,m-1},$$
$$V_i : \quad c_{i,m-1}, c_{i,m-2}, \ldots, c_{i,i}.$$

All are distinct columns of the unimodular matrix C. A cumulative chain built from part of an integral basis is a unimodular simplex: its differences from the first vertex are related to those columns by a triangular integral matrix with diagonal entries one. Every face of this simplex is unimodular in its affine lattice. A nonmonomial face polynomial is therefore, after a Laurent monomial change and a unit factor, $1 + t_1 + \cdots + t_s$ with $s \geq 1$. This is smooth and irreducible on the corresponding torus. Saturation in a weight-zero sublattice follows from saturation in the ambient lattice, since the former quotient embeds in the latter quotient.

Different horizontal chains have different values of $j - i$; different vertical chains have different first indices i. This proves the disjointness assertions and the formula for their unique possible intersection. Because all c_a are part of a basis, the linear spans of two sets of columns intersect in the span of their set-theoretic intersection. Equal nonzero face-direction spaces therefore require one horizontal and one vertical chain and must be the line $\mathbb{R}c_{ij}$. Each face is consequently an edge. A difference between two vertices of a cumulative chain is a sum of consecutive distinct basis columns, so it is parallel to a single column only when the vertices are consecutive at that column. □

Proposition 7.6. *For every prime divisor of* $\operatorname{Spec} M$*, the vector w is zero or a coordinate vector e_ρ.*

Proof. Suppose $w \neq 0$. Then $\lambda \neq 0$. The zero diagonal totals and nonnegative partial sums in (7.2) force a negative horizontal arrow and a negative vertical arrow (Lemma 7.3). Thus at least one H_k and one V_i have negative minimum monomial value.

The supports of these boundary functions are cumulative chains of distinct basis columns of C. Each support is a unimodular simplex, and every nonmonomial face is, up to a character factor and an integral torus change, $1 + t_1 + \cdots + t_s$. Corollary 5.4 therefore implies that all negative faces have minimum -1 and define the same irreducible hypersurface. Their Newton polytopes are translates.

Different horizontal chains have disjoint column directions, as do different vertical chains. The chains for H_k and V_i share at most the one column c_{ij} with $j = i + k - 1$. Lemma 7.5 consequently forces the negative faces to be consecutive edges of precisely one horizontal and one vertical boundary function. Their four monomials are the sides of one elementary square, each with value -1. All other arrows have nonnegative value: in a boundary function with negative minimum their values are integers strictly greater than -1, and the remaining boundary functions have nonnegative minimum.

The diagonal induction of Lemma 7.4 now determines λ. Away from the main diagonal it is a single forward flow. For a square adjacent to the diagonal, the only possible ambiguity is

$t, 1-t$ on a pair of variables sharing an exchange polynomial, with $0 \le t \le 1$. The inverse formulas of Lemma B.7 and the integrality of w force $t = 0$ or 1. In either case $w = e_\rho$. □

7.3. **Global comparison.**

Proof of Theorem 7.1. We apply the intrinsic mirror comparison theorem of [7] to the image M_n of Proposition 7.2. The variety U_n is smooth and factorial, so [7, normality of the intrinsic mirror algebra] gives finite generation and normality. Its initial torus is an intrinsic open chart by the local comparison, and [7, logarithmic volume on an intrinsic mirror] gives order zero for the logarithmic form. Proposition 7.6 then shows that at most one initial mutable variable vanishes along a prime divisor. All hypotheses of the comparison criterion are satisfied.

The boundary functions were identified in the local comparison, so the isomorphism respects every component of the Coxeter boundary. □

Remark 7.7. Under $B_n/P_n \simeq D_{n+1}/P_{n+1}$, the signed subset coordinates and prefix–suffix Coxeter boundary agree (Remark 3.3). The mirror theorem, including its boundary functions, therefore transfers by replacing the type-D index with $n+1$. For $B_1/P_1 = \mathbb{P}^1$ and $B_2/P_2 = \mathbb{P}^3$, the frozen-inverted cones are tori; their mirrors are the dual tori with the sums of the boundary characters as potentials.

7.4. **The quadric case D_4.** For $n = 4$ the mirror has two mutable variables P, Q, with common exchange polynomial $1 + \kappa_0$. Set

$$\mathscr{S}_4 = \mathbb{C}[\kappa_0^{\pm 1}, \dots, \kappa_4^{\pm 1}], \qquad R = \frac{1+\kappa_0}{PQ}.$$

The first mutation in either position gives R; subsequent mutations recover P or Q. Thus the ordinary algebra is generated by P, Q, R.

Proposition 7.8. *There are isomorphisms preserving the boundary functions*

$$A_{\mathrm{int}}(U_4)_1 \simeq \mathcal{A}_4^\vee = \mathcal{U}_4^\vee = \mathscr{S}_4[P, Q, R]/(PQR - 1 - \kappa_0).$$

For $a, b \in \mathbb{Z}$, $k \in \mathbb{Z}^5$ and $d = \max(0, -a, -b)$, the theta function with label $(a, b; k)$ is

$$\Theta_{(a,b;k)} = \kappa^k P^a Q^b (1+\kappa_0)^d = \kappa^k P^{a+d} Q^{b+d} R^d.$$

Proof. The complementary spinor coordinate bounds the contact degree by d. Since the unique contact direction is the κ_0 direction, write the intrinsic expansion as $\kappa^k P^a Q^b f(\kappa_0)$, where $\deg f \le d$ and $f(0) = 1$. Laurentness in the adjacent P chart forces $(1+\kappa_0)^{\max(0,-a)}$ to divide f, and the Q chart forces exponent $\max(0, -b)$. The combined exponent is their maximum, namely d. Hence $f = (1+\kappa_0)^d$.

The monomials $P^i Q^j R^\ell$ with $i, j, \ell \ge 0$ and $\min(i, j, \ell) = 0$ form a free $\mathscr{S}_4$-basis of this quotient: use the monic relation to reduce every factor PQR. Their images are linearly independent in $\mathscr{S}_4[P^{\pm 1}, Q^{\pm 1}]$, since the exponent pair $(i - \ell, j - \ell)$ determines the triple in this normal form. The same divisibility calculation describes the upper LP algebra. This proves the presentation and the asserted equalities. □

The six boundary functions, in Coxeter order, are

$$P, \quad R, \quad \kappa_1 Q, \quad \kappa_2 P, \quad \kappa_3 R, \quad \kappa_4 Q.$$

In particular $W = (1+\kappa_2)P + (\kappa_1 + \kappa_4)Q + (1+\kappa_3)R$.

8. Exceptional mirrors

8.1. **The Cayley-plane mirror.** Let R_6 be the Cayley-plane coordinate ring, with the twelve frozen coordinates of Section 4. Set

$$U_6 = \operatorname{Spec} R_6[(a_1 \cdots a_{12})^{-1}], \qquad \mathscr{S}_6 = \mathbb{C}[\zeta_6^{\pm 1}, \dots, \zeta_{12}^{\pm 1}].$$

For the Newton construction we choose the chart

$$(x_1, x_7, z_3, x_4, t_1), \qquad t_1 = x_1 x_4 x_7 x_{10} - a_3 a_6 a_9 a_{12}.$$

It belongs to the family J of [4, Theorem 5.4]; their table uses a different representative of that family. The three-mutation comparison is given in Appendix C.1. Its integral Newton characters are listed in Appendix C. We first give the rank-ten mirror and its boundary functions, then identify the intrinsic algebra.

8.1.1. *The seed and affine model.* The mutable variables are $(s, u, v, p_1, \dots, p_7)$. Define

$$P = \prod_{i=1}^{7} p_i, \qquad \Pi = p_1p_2p_3, \qquad T = p_5p_6p_7, \qquad Q = p_4T,$$
$$R = p_2p_3p_6p_7, \qquad L = p_3T, \qquad H = p_1p_2p_4p_7,$$
$$E = (1+R)(1+Q) + uvL(1+H).$$

The ordinary exchanges and associated exchanges are

x	F_x	$\widehat{F}_x$
s	E	E/u
u	$1+Q$	$1+Q$
v	$1+s+Q$	$(1+s+Q)/Q$
p_i	$1+s$	$(1+s)/(P/p_i)$.

(8.1)

Their first-mutated functions are

$$f = \frac{E}{su}, \qquad \bar{u} = \frac{1+Q}{u}, \qquad \bar{v} = \frac{1+s+Q}{vQ}, \qquad k = \frac{1+s}{P}. \tag{8.2}$$

The mutation word (u, v, s) gives one further function,

$$g = \frac{E + s(1+R)}{svT} = \frac{p_4(1+R)\bar{v} + p_3u(1+H)}{s}. \tag{8.3}$$

Define

$$\mathscr{B}_6 = \frac{\mathbb{C}[p_1, \dots, p_7, k, u, \bar{u}, v, \bar{v}, f, g]}{\left(\begin{array}{l} u\bar{u} - 1 - Q, \\ v\bar{v} - 1 - \Pi k, \\ Tvg - uf - 1 - R, \\ p_4\bar{v}f - \bar{u}g - p_3(1+H) \end{array}\right)}. \tag{8.4}$$

In this ring, $s = Pk - 1 = Qv\bar{v} - u\bar{u}$.

Proposition 8.1. *The ring $\mathscr{B}_6$ is a smooth integral ten-dimensional complete intersection. The formulas above identify it with the ordinary and upper LP algebras of the coefficient-free seed (8.1). Restoring the seven coefficient characters gives*

$$\mathcal{A}(\Sigma_6^\vee) = \mathcal{U}(\Sigma_6^\vee) = \mathscr{S}_6 \otimes_{\mathbb{C}} \mathscr{B}_6.$$

The initial and adjacent tori cover codimension one.

Proof. The affine calculations in Appendix C.3 cover the model by $D(s)$ and $D(Pk)$, since $Pk - s = 1$. On the first open the last two equations eliminate f, g with determinant s; on the second, the unimodular character change eliminates $Q, \Pi k, R, H$. These are smooth integral charts of dimension ten. The same calculation proves that $s, u, v, p_1, \dots, p_7$ are distinct primes and that their partners are generically nonzero. Normality and the finite upper-bound theorem identify the ring with its eleven-chart Laurent intersection. All fourteen generators occur under mutation, so the generated and upper LP algebras coincide as well. □

8.1.2. *The twelve boundary functions.* The boundary functions have the following polynomial representatives:

$$\begin{aligned} W_1 &= k, & W_2 &= p_4,\\ W_3 &= p_1p_2p_4p_5kf - p_2\bar{u} - p_5v(1+H), & W_4 &= p_1,\\ W_5 &= p_7, & W_6 &= \zeta_6 g,\\ W_7 &= \zeta_7 p_2, & W_8 &= \zeta_8 p_6,\\ W_9 &= \zeta_9(kf - p_7v), & W_{10} &= \zeta_{10}p_3,\\ W_{11} &= \zeta_{11}p_5, & W_{12} &= \zeta_{12}(p_1p_5p_6kg - p_6\bar{v} - p_1u). \end{aligned}$$

Although these representatives contain minus signs, their initial Laurent expansions are positive. For the three nonsimple expressions, these expansions are

$$\begin{aligned} W_3 &= \frac{E + s(1+Q)}{su\,p_3p_6p_7},\\ W_9 &= \zeta_9\frac{E + s((1+R)(1+Q) + uvL)}{suP},\\ W_{12} &= \zeta_{12}\frac{E + s(2+Q+R+uvL) + s^2}{sv\,p_2p_3p_4p_5p_6p_7^2}. \end{aligned}$$

They follow from (8.4) and $Pk = 1 + s$. The superpotential is $W = \sum_{i=1}^{12} W_i$.

8.1.3. *Intrinsic identification.*

Theorem 8.2. *Local theta expansions induce an isomorphism*

$$A_{\mathrm{int}}(U_6)_1 \xrightarrow{\sim} \mathscr{S}_6 \otimes_{\mathbb{C}} \mathscr{B}_6.$$

The theta function labeled by $a_i = 0$ maps to W_i for each $1 \le i \le 12$.

Proof. The integral Newton change gives $U_6 = X \times \mathbb{G}_m^7$. The source bounds and four wall calculations in Appendix C.4, together with [7, intrinsic Laurent expansion and trace theorem], put the intrinsic algebra in the eleven-chart intersection $\mathscr{S}_6 \otimes \mathscr{B}_6$. Proposition C.1 identifies $f, \bar{u}, \bar{v}, k, g$ by finite Laurent interpolation. Together with the straight coordinates these are all the affine generators, so [7, intrinsic mirror comparison theorem, generator criterion] gives equality. The same proposition identifies W_3, W_9, W_{12}; the other boundary functions are straight coordinates or the specified unit multiples of k, g. Compatibility with torus products [7] restores the seven coefficient characters and all twelve markings. □

8.2. **The Freudenthal mirror.** Let

$$U_{\mathrm{F}} = \operatorname{Spec} R_7[(a_0 \cdots a_{17})^{-1}], \qquad \mathscr{S}_7 = \mathbb{C}[\zeta_0^{\pm 1}, \dots, \zeta_7^{\pm 1}].$$

The integral characters in Appendix D define a chamber for the seed of Theorem 4.2. Their dual mutable coordinates are $v_1, \dots, v_{20}$; the other eight dual characters are the coefficient units $\zeta_0, \dots, \zeta_7$. We define $\Sigma_7^\vee$ and its eighteen boundary functions below.

8.2.1. *The seed and boundary functions.* Put $I_1 = \{1, 5, 15, 18\}$, $I_2 = \{6, 7, 12, 13, 20\}$, $I_3 = \{11, 14\}$, and $I_4 = \{16, 17, 19\}$. The ordinary exchanges are

i	E_i
I_1	$v_{11} + v_3$
I_2	$v_2 + v_8v_9$
I_3	$v_1v_4 + v_{19}v_3$
I_4	$v_{14}v_2 + v_{20}v_4$
2	$v_{10}v_{16}v_{17}v_9 + v_{12}v_{13}v_{20}v_4v_6$
3	$v_1v_{15}v_{17}v_{18}v_5 + v_{11}v_{14}$
4	$v_{10}v_{15}v_{16}v_{17}v_{19} + v_{11}v_{14}v_2$
8	$v_{13}v_{20} + v_{17}$
9	$v_{12}v_{13}v_{19}v_6v_7 + v_2$
10	$v_{12}v_4 + v_2v_5$.

For $i \in I_\ell$ the associated Laurent exchange is $\widehat{E}_i = E_i/\prod_{j\in I_\ell\setminus\{i\}} v_j$. For $i = 2, 3, 4, 8, 9, 10$, it equals E_i. These formulas define the mirror LP seed $\Sigma_7^\vee$; repeated ordinary exchanges are not assigned trivial normalizers.

For the boundary functions we use the following mutation partners:

$$\eta_1 = \frac{v_{11} + v_3}{v_1v_5v_{15}v_{18}}, \quad \eta_2 = \frac{v_2 + v_8v_9}{v_6v_7v_{12}v_{13}v_{20}},$$
$$\eta_3 = \frac{v_1v_4 + v_{19}v_3}{v_{11}v_{14}}, \quad \eta_4 = \frac{v_{14}v_2 + v_{20}v_4}{v_{16}v_{17}v_{19}},$$
$$\tau_8 = \frac{v_{13}v_{20} + v_{17}}{v_8}, \quad \tau_9 = \frac{v_{12}v_{13}v_{19}v_6v_7 + v_2}{v_9}, \quad \tau_{10} = \frac{v_{12}v_4 + v_2v_5}{v_{10}}.$$

Write $W_j = \epsilon_j\overline{W}_j$, where $\epsilon_j = \zeta_j$ for $0 \le j < 8$ and $\epsilon_j = 1$ otherwise. The eighteen functions are

$$\overline{W}_0 = v_{15}, \quad \overline{W}_1 = v_5, \quad \overline{W}_2 = v_{18},$$
$$\overline{W}_3 = \frac{v_{17} + v_{14}\eta_1}{v_3}, \quad \overline{W}_4 = \eta_3, \quad \overline{W}_5 = \frac{v_{10}v_{15} + v_{11}\eta_4}{v_4},$$
$$\overline{W}_7 = v_{13}, \quad \overline{W}_9 = v_7, \quad \overline{W}_{10} = v_{12},$$
$$\overline{W}_{11} = v_6, \quad \overline{W}_{12} = \tau_8, \quad \overline{W}_{13} = \eta_2,$$
$$\overline{W}_{14} = \frac{v_{10}v_7 + \eta_4\tau_9}{v_2}, \quad \overline{W}_{16} = \eta_1, \quad \overline{W}_{17} = v_1,$$

with

$$\overline{W}_6 = \frac{v_{16}\tau_8}{v_{13}v_{20}} + \frac{v_{16}v_{17}v_9}{v_{13}v_2v_{20}} + \frac{v_6\tau_{10}}{v_2},$$
$$\overline{W}_8 = \frac{v_{19} + v_{20}\eta_2}{v_9},$$
$$\overline{W}_{15} = \frac{v_{16}}{v_3} + \frac{v_{18}\tau_{10}}{v_4} + \frac{v_{15}v_{16}v_{17}v_{18}v_5\eta_3}{v_3v_4}.$$

The superpotential $W = \sum_{j=0}^{17} W_j$ has forty distinct Laurent monomials in the full character lattice, all with coefficient one. The degree character is $\mathfrak{q}_{E_7} = \prod_{j=0}^{7} \zeta_j$.

8.2.2. *Newton reconstruction.*

Proposition 8.3. *The seed $\Sigma_7^\vee$ and the functions W_j are the Newton reconstructions from Σ_7 in the chamber* $\mathrm{val}(\chi_i) > 0$. *All W_j belong to $\mathcal{U}(\Sigma_7^\vee)$. The character matrix is unimodular, and the twenty χ_i span the saturated support-difference lattice.*

Proof. The linear coordinates and the cubic mutation $f^\sharp = F_f/(hf)$ form a family of 57 regular functions. The least monomial of

$$G = x_{15}^2x_2y_{14}x_5y_{11}x_{10}x_{13}x_0^2y_1^3$$

has exponent $(-1,\dots,-1)$ in the ten mutable coordinates. The regular-function inequalities therefore bound a contact correction $\beta \in \mathbb{Z}^{10}_{\geq 0}$ by $\sum_i \beta_i \leq \delta_G(m)$ at label m. Appendix D gives the finite support and adjacent Laurent-divisibility calculations. Each primitive facet has precisely the two allowed exponents occurring in its exchange. Every binomial exponent difference is primitive, and the two monomials are coprime. Hence the ordinary exchanges are irreducible. Normalization removes precisely one copy of each other variable having the same ordinary exchange: specializing a variable in a different binomial leaves either a monomial or a nonassociate binomial. This gives the stated normalizers. We determine the wall coefficients from valuations associated with source tori and the eight mixed-contact calculations in Proposition 8.4. For every boundary label the regularity kernel is one-dimensional; normalizing the least coefficient gives exactly W_j. The starfish theorem [6, Theorem 1.3] puts these functions in the full upper LP algebra. The determinant and support-lattice calculations give the last assertion. □

8.2.3. *Intrinsic comparison.*

Proposition 8.4. *The twenty primitive wall functions in the primary chamber are the binomials E_i of §8.2, after removing their monomial factors. Every nonstraight term has coefficient one.*

Six valuations from source tori determine the twelve single-contact directions. The five two-contact and three three-contact cases retain their explicit incidence equations. Both parts, including the required end conditions, are proved in Appendix D.2.2.

Proposition 8.5. *Suppose $w \in \mathbb{Z}^{20}_{\geq 0}$ is the character-value vector of a normalized divisorial valuation with volume order zero, and all twenty first mutations v_i', together with the eighteen boundary functions W_j, are regular at that valuation. Then $w = 0$ or $w = e_i$ for one i.*

The finite step of the second proposition is computer-assisted. Its proof in Appendix D.2.4 derives rational inequalities for all eighty possible irreducible factors of negative Newton faces; these leave only zero and the coordinate vectors.

Theorem 8.6. *There is an isomorphism of $\mathscr{S}_7$-algebras*

$$A_{\mathrm{int}}(U_{\mathrm{F}})_1 \xrightarrow{\sim} \mathcal{U}(\Sigma_7^\vee)$$

carrying the theta function indexed by $a_j = 0$ to W_j for every $0 \leq j \leq 17$. The upper LP algebra is finitely generated, normal, and factorial, and its unit group is $\mathscr{S}_7^\times$. Its superpotential is $W = \sum_{j=0}^{17} W_j$.

Proof. The source theorem, full Newton marking and negative regular function of Proposition 8.3 satisfy the Laurent-realization hypotheses of [7, intrinsic Laurent expansion and trace theorem]. Proposition 8.4 identifies the twenty walls, and §D.2.3 identifies their first mutations and the eighteen boundary theta functions. Thus the intrinsic image $\mathscr{R}_7$ satisfies

$$\mathscr{S}_7[v_i, v_i', W_j] \subseteq \mathscr{R}_7 \subseteq \mathcal{U}(\Sigma_7^\vee).$$

It is normal, and its initial torus has volume order zero, by [7, normality and logarithmic volume of the intrinsic mirror]. Proposition 8.5 shows that no prime divisor of $\mathscr{R}_7$ contains two initial mutable coordinates. Hence [7, intrinsic mirror comparison theorem, codimension-one criterion] gives the asserted equality. The boundary functions were identified before this global step; finite generation, factoriality and the unit group now follow from the intrinsic and upper-algebra descriptions. □

9. Integral forms and valuative bases

We now descend the comparisons to the integral forms and apply the valuative-independence theorem to the intrinsic theta functions.

9.1. **Integral descent.** Choose primitive integer exchange polynomials throughout mutation and retain the independent Laurent coefficient characters. Let $\mathcal{U}^\vee_{X,\mathbb{Z}}$ be the upper LP algebra over $\mathbb{Z}[\kappa_j^{\pm 1}]$, with ζ in place of κ in the exceptional cases. Write $A_{\mathrm{int},\mathbb{Z}}(U_X)_1$ for the integral counting algebra at the specialization $q^\beta = 1$.

Theorem 9.1. *The isomorphisms of Theorem B restrict to*

$$A_{\mathrm{int},\mathbb{Z}}(U_X)_1 \xrightarrow{\sim} \mathcal{U}^\vee_{X,\mathbb{Z}}.$$

The intrinsic basis has nonnegative integral structure constants and nonnegative integral expansions in the marked initial chart. Every basis element is integrally Laurent in every LP seed.

Proof. The finite contact bounds give finite initial expansions with integer coefficients and distinct coefficient-one least terms. Intrinsic multiplication has nonnegative integer coefficients. Apply the monic integral-descent theorem [7, integral descent for monic Laurent expansions] to the complex comparisons of Theorems 7.1, 8.2, and 8.6. Primitive exchange divisibility propagates integral Laurentness, and successive least terms recover the integral basis coefficients. This argument does not require scalar extension to commute with an infinite Laurent intersection. □

Integral Laurentness does not assert coefficientwise positivity in every seed. Nor does the theorem specialize the independent coefficient characters; only curve-class characters have been specialized.

9.2. **Valuative independence of the intrinsic basis.**

Theorem 9.2. *In each comparison of Theorem B, the intrinsic theta basis is valuatively independent for every real monomial valuation in the marked initial chart and in every reachable LP seed. In particular, for every finite signed sum,*

$$\mathrm{Newt}\left(\sum_m c_m \Theta_m\right) = \mathrm{conv} \bigcup_{c_m \neq 0} \mathrm{Newt}(\Theta_m).$$

Every regular extended cluster monomial spans an intrinsic theta line. Under the established integral and positive normalization, it equals the corresponding theta function.

Proof. The source theorems and the regular negative-vertex functions verify the hypotheses of the valuative-independence theorem in [7]. The local realization in each comparison identifies its functions with the intrinsic theta functions; the full integral Newton marking only changes the character basis. The theorem therefore applies to these intrinsic functions. Exact Gauss transport through the exchange Laurent polynomials gives the assertion in every reachable seed. A regular seed monomial has singleton Newton support there, so valuative independence allows only one theta constituent. Integral descent and the specified positive normalization fix its scalar. □

The theorem applies to the entire mutation pattern. Positivity in every seed, and atomicity for the LP atlas, require additional arguments.

Corollary 9.3. *Let P be a bounded rational polytope in $M_\mathbb{R}$, where M is the full character lattice, and put*

$$A_P = \mathcal{U}(\Sigma^\vee_X) \cap \mathrm{Span}_\mathbb{C}\{z^a : a \in P \cap M\}.$$

Then

$$A_P = \bigoplus_{\mathrm{Newt}(\Theta_m) \subseteq P} \mathbb{C}\Theta_m.$$

In particular the theta labels occurring in A_P belong to the finite set $P \cap M$. Laurent-divisibility equations therefore give a finite recognition calculation once the corresponding support bounds are established.

Proof. Apply the preceding theorem to a theta expansion of an element of A_P. Every nonzero constituent has Newton polytope contained in P. The converse inclusion is immediate; the label is a monomial in its theta function. □

For completeness, the familiar vertex ambiguity can be described without introducing a second basis. Fix the pointing order and put $P_m = \mathrm{Newt}(\Theta_m)$ and $P_m^- = \mathrm{conv}((P_m \cap M) \setminus \mathrm{Vert}(P_m))$. We set $A_\varnothing = 0$.

Proposition 9.4. *Every monic element with least label m has Newton polytope containing P_m. The monic elements with least label m and Newton polytope exactly P_m form $\Theta_m + A_{P_m^-}$. Thus this representative is unique precisely when $A_{P_m^-} = 0$.*

Proof. Pointing makes Θ_m the least constituent of any such expansion, with coefficient one. Newton containment proves the first assertion. If another constituent in A_{P_m} had a vertex of P_m in its support, a generic weight exposing that vertex would give it and Θ_m the same least exponent. Cancelling the two coefficients would contradict valuative independence. All other constituents are therefore supported in P_m^-. The converse follows because adding such an element changes neither the vertex coefficients nor the least label. □

9.3. **The quadric mirror.**

Corollary 9.5. *The basis of Proposition 7.8 is valuatively independent and Laurent-positive in every seed over the specified integral coefficient ring. Its elements are precisely the extended cluster monomials, and it is atomic for the LP atlas. If intrinsic scattering expansions are coefficientwise positive at every generic endpoint, it is also atomic for the intrinsic scattering atlas.*

Proof. For $m = (a, b; k)$, put $d = \max(0, -a, -b)$. The theta polytope is the segment from m to $m + d\epsilon_0$, where ϵ_0 is the κ_0 direction. A function in this Newton window is $\kappa^k P^a Q^b f(\kappa_0)$ with $\deg f \le d$. Adjacent Laurentness requires $(1+\kappa_0)^d \mid f$, so the window is one-dimensional. Theorem 9.2 applies to every label.

The clusters are $\{P, Q\}$, $\{P, R\}$, and $\{Q, R\}$. At least one exponent in $\kappa^k P^{a+d} Q^{b+d} R^d$ is zero, so every theta function is a cluster monomial. In a chart where it is monomial, that exponent cannot occur in another theta function: Newton containment transports to a chart where the latter is monomial and would force proportionality. Its coefficient therefore detects its basis coefficient. This proves LP atomicity. The same detectors work in the intrinsic charts; under the stated scattering-positivity hypothesis they also prove scattering atomicity. □

10. Lie-theoretic structure

The mirror construction reflects the degree-and-weight lattice of the minuscule representation. In type D, its Newton polytope also encodes the minuscule root poset. These give two Lie-theoretic interpretations of the formulas. Neither identifies the entire mirror with a homogeneous space or an open Richardson variety.

10.1. **The coefficient torus and the weight lattice.**

Let G be simply connected, let $T_G \subset G$ be a maximal torus, and write $\ell = \operatorname{rank} G$. The degree grading and the T_G-action act together on R_X. Let Λ be the character lattice of the initial torus of U_X, and let

$$\gamma_X : \Lambda \longrightarrow \mathbb{Z} \oplus X^*(T_G)$$

send each initial coordinate to its homogeneous degree and its weight as a function. Put $\Gamma_X = \mathrm{im}(\gamma_X)$. Thus Γ_X is the character lattice of the effective degree-and-weight torus acting on the cone. We use this image lattice, rather than the whole lattice $\mathbb{Z} \oplus X^*(T_G)$, to account for the finite central kernel of the action.

Let $L_X \subset \Lambda$ be the saturated lattice generated by the support differences in the Newton construction.

Proposition 10.1. *For the three LP structures on the homogeneous coordinate rings in this paper,*

$$L_X = \ker(\gamma_X), \qquad \Lambda/L_X \simeq \Gamma_X, \qquad \operatorname{rank}\Gamma_X = \ell + 1.$$

The mirror coefficient lattice is canonically $L_X^\perp = \operatorname{Hom}(\Gamma_X, \mathbb{Z})$. *Consequently the coefficient torus is dual to the effective degree-and-weight torus acting on the cone.*

Proof. The initial coordinates and the regular functions used to construct L_X are homogeneous T_G-eigenvectors. Two monomials in the Laurent expansion of one such function have the same degree and weight. Hence $L_X \subseteq \ker(\gamma_X)$. The action of $\mathbb{G}_m \times T_G$ on the cone has finite kernel: a torus element acting as a scalar on the irreducible representation is central, and G is simple. Since the initial coordinates generate the function field, their degree-and-weight characters have rank $\ell + 1$.

The Newton characters form part of an integral basis of Λ, and there are $\dim \widehat{X} - \ell - 1$ of them; see Proposition 6.1 and Appendices C–D. Hence L_X is saturated and has the same rank as $\ker(\gamma_X)$. Their equality follows. Dualizing the resulting exact sequence gives

$$0 \longrightarrow \Gamma_X^* \longrightarrow \Lambda^* \longrightarrow L_X^* \longrightarrow 0.$$

The duals of the complementary characters of the initial torus are exactly the mirror coefficient characters, so they span $L_X^\perp$. □

The coefficient torus is isogenous to $\mathbb{G}_m \times T_G^\vee$, where $T_G^\vee$ is the dual torus. This explains the coefficient ranks $n + 1$, 7, and 8 for D_n, E_6, and E_7. It also explains why the number of mirror units is not the number h of chosen boundary components. The inclusion of the coefficient ring gives a morphism

$$\operatorname{Spec}\mathcal{U}(\Sigma_X^\vee) \longrightarrow \operatorname{Spec}\mathbb{C}[\Gamma_X^*].$$

The chosen lattice basis provides coordinates on its target. The character $\mathfrak{q}$ corresponds to cone scaling, and fixing it removes the one degree direction. These multiplicative parameters are not identified here with either the curve-class parameters of the intrinsic family or additive equivariant parameters in quantum cohomology.

10.2. **The minuscule root poset in type D.** For $X = \mathrm{OG}^+(n, 2n)$, put $m = n - 1$. The triangular poset

$$P_m = \{(a, b) : 1 \le a \le b \le m\}$$

with componentwise order is the minuscule poset for the half-spin representation. In the usual realization of the D_n roots, its identification with the positive roots outside the Levi of P_n is

$$(a, b) \longmapsto \beta_{ab} = \varepsilon_{n-b} + \varepsilon_{n-a+1}.$$

The root order agrees with the poset order, and $\mathrm{ht}(\beta_{ab}) = a + b - 1 = \mathbf{h}_{ab}$. Thus the horizontal and vertical ratios in the potential are indexed by covering relations in the minuscule root poset; the other two terms are indexed by its minimum and maximum.

If Δ_m is the Newton polytope of the fixed-$\mathfrak{q}$ potential, Section 6 gives $\Delta_m^\circ = (2n - 2)\mathcal{O}(P_m) - \mathbf{h}$. For the minuscule embedding, Spacek–Wang identify the order polytope with a Newton–Okounkov body and establish the corresponding Khovanskii-basis result [19, Theorem 1.1]. In particular,

$$\#\big(d\mathcal{O}(P_m) \cap \mathbb{Z}^{P_m}\big) = \dim H^0(X, \mathcal{O}_X(d)).$$

The polar relation therefore connects the mirror potential directly to the representation-theoretic polytope of the half-spin variety. This is a statement about the Newton polytope and a toric degeneration, not a presentation of the whole mirror as a flag variety.

10.3. **Comparison with canonical homogeneous-space mirrors.** Rietsch constructs Landau–Ginzburg models for G/P using the Langlands dual group and an open Richardson variety [14]. Spacek and Wang give canonical Plücker-coordinate models for all cominuscule spaces and identify them with Rietsch's models [18]; their exceptional and orthogonal constructions are given in [17, 20]. These results provide a natural comparison with our formulas. Here we study the complement of a specified Coxeter orbit, and the mirror theorem preserves its individual boundary labels. A comparison with a canonical homogeneous-space mirror must therefore identify both the affine domain and the potential. Agreement on a torus, equality of dimensions, or equality of coefficient ranks does not suffice.

Example 10.2 (The six-dimensional quadric)**.** Even the underlying affine domains can differ. At $\mathfrak{q} = 1$, the D_4 mirror in Section 7 has coordinate ring

$$\frac{\mathbb{C}[\kappa_0^{\pm1}, \kappa_1^{\pm1}, \kappa_2^{\pm1}, \kappa_3^{\pm1}, P, Q, R]}{(PQR - 1 - \kappa_0)}.$$

Its units modulo constants are generated by the four κ_i. The morphism defined by these units has a nonsmooth fibre precisely over $\kappa_0 = -1$, an irreducible hypersurface in the unit torus.

The canonical mirror of Q^6 has coordinate ring

$$\frac{\mathbb{C}[A^{\pm1}, B^{\pm1}, C^{\pm1}, D^{\pm1}, x, x', y, y']}{(xx' - C - D,\ yy' - AB - D)};$$

this is the specialization $m = 4$, $p_0 = 1$ of the exchange relations in [13, Section 5.2, equation (51)]. Its units modulo constants are generated by A, B, C, D. Indeed, localizing at x, y makes any unit a Laurent monomial; regularity of that unit and its inverse along $x = 0$ and $y = 0$ forces the exponents of x, y to vanish. Its unit morphism has nonsmooth fibres over the two distinct irreducible hypersurfaces $C + D = 0$ and $AB + D = 0$. An isomorphism of affine varieties identifies their groups of units and hence these morphisms up to an automorphism and translation of the target torus. The two domains are therefore not isomorphic. This example does not exclude a birational comparison of potentials, but it rules out an automatic identification of their affine completions.

A uniform construction of our Coxeter-boundary mirrors from root subgroups, generalized minors, or another Lie-theoretic model remains a separate question. Proposition 10.1 and the root-poset description give intrinsic constraints that such a construction would have to respect.

Appendix A. Freudenthal coordinate ring and chart comparisons

We prove the auxiliary Freudenthal coordinate-ring statements used in Theorem 4.2. In this appendix we suppress $\circ$ on the auxiliary variables $h, b, c, d, e, f, g, r, q, u$; in particular, $d = y_{10}$, whereas the primary seed uses $d = x_{16}$. The auxiliary seed is Σ_7°.

A.1. **Coordinate convention.** We follow the upper-sign convention of [4, Section 6.1]. Indices on a_i, x_i, y_i are read modulo 18; a Coxeter step shifts these indices and interchanges

z_1, z_2. The defining ideal is generated by the cyclic translates of the eight quadrics:

$$\begin{aligned}
x_1x_2 &= a_3x_{17} + a_2y_0 + a_1y_3 + a_0x_4 + a_{17}a_4,\\
y_{14}y_1 &= x_{15}y_4 - a_0x_{10} - a_{15}x_5 + x_0y_{11} + x_{13}x_2,\\
y_2y_1 &= a_2y_{12} + y_5x_0 + y_{16}x_3 + a_1y_9 + a_5a_{16},\\
x_4y_1 &= x_2y_5 + a_5x_{17} + a_4y_{16} - a_2y_8 - a_1x_7,\\
x_2x_6 &= x_4y_4 - a_4z_2 - a_5y_0 - a_3y_8 + a_1a_7,\\
y_1y_3 &= x_{17}x_5 + x_2z_1 + a_0x_7 + a_4x_{15} - a_2y_{11},\\
y_4y_{16} &= y_1z_2 - y_8x_0 - y_{12}x_2 - a_1x_{10} + a_5a_{15},\\
z_1z_2 &= y_3y_{12} + y_7y_{16} + y_8y_{17} + x_1x_{10} + x_3x_{12} + x_5x_{14}\\
&\quad + a_0a_9 + a_2a_{11} - a_3a_{12} + a_4a_{13} + a_6a_{15}.
\end{aligned} \tag{A.1}$$

The 144 cyclic translates span a 133-dimensional space of quadratic relations. Row reduction over $\mathbb{Q}$ gives this rank, which is unchanged over $\mathbb{C}$. The elimination data are in `verification/freudenthal_quadric_rank_certificate.json`.

A.2. **Coordinates and exchanges.** Let $R_7 = \mathbb{C}[\widehat{E_7/P_7}]$. Its 56 linear coordinates are a_i, x_i, y_i for $i \in \mathbb{Z}/18$ and z_1, z_2. The first three families are Coxeter orbits of length 18, while z_1, z_2 form an orbit of length two. We freeze the a_i and put $S_7 = \mathbb{C}[a_0, \dots, a_{17}]$.

Define global homogeneous functions

$$\begin{aligned}
r &= x_{17}y_{10} - a_{17}x_9,\\
q &= x_6y_{13} - a_6x_{14},\\
u &= x_{14}x_4y_7 - a_{14}x_4x_6 - a_3x_{14}x_9 - a_4x_{14}y_{10}.
\end{aligned} \tag{A.2}$$

The initial cluster is

$$(h, b, c, d, e, f, g, r, q, u) = (x_1, x_{11}, x_9, y_{10}, x_6, x_{14}, x_4, r, q, u). \tag{A.3}$$

Its degrees are $(1, 1, 1, 1, 1, 1, 1, 2, 2, 3)$.

A.3. **Exchange polynomials.** Set

$$P = a_6a_{11}f + a_7bf + a_9a_{14}e + a_{10}ef + a_{12}de + a_{13}ce.$$

The ten ordinary exchange polynomials are

$$\begin{aligned}
F_h &= efq(a_0dg + a_3r + a_{17}(a_3c + a_4d))\\
&\quad + df(er + u)(a_1a_6 + a_2e) + a_1dq(u + a_{14}eg),\\
F_b &= a_{11}(a_6f + q) + e(a_9a_{14} + a_{10}f + a_{12}d + a_{13}c),\\
F_c &= fg(a_6a_{11} + a_7b + a_8d + a_{10}e) + a_9(a_4df + a_{14}eg + u),\\
F_d &= r + a_{17}c,\\
F_e &= q + a_6f,\\
F_f &= u + a_{14}eg,\\
F_g &= u + a_3cf + a_4df,\\
F_r &= g(a_{16}d + a_{17}b)(hP + bu) + hdgq(a_{11}a_{16} + a_{15}b)\\
&\quad + bq\big(dg(a_0f + a_1a_{14}) + a_{17}f(a_3c + a_4d) + a_{17}u\big)\\
F_q &= hP + b(er + u),\\
F_u &= c\,F_r|_{u=0} + br\left\{cf\big(d(a_1a_6 + a_2e) + a_3q\big) + h\big(F_c|_{u=0} + a_5cdf + a_{13}ceg\big)\right\}.
\end{aligned}$$

In F_u, the notation $F_r|_{u=0}$ and $F_c|_{u=0}$ refers to the polynomials displayed above.

The associated Laurent exchange polynomials are $\widehat{F}_v = F_v/M_v$, where

v	h	b	c	d	e	f	g	r	q	u
M_v	def	e	f	1	1	1	1	bg	1	bd

Thus $v' = F_v/(vM_v)$; for example, $be\,b' = F_b$ and $rbg\,r' = F_r$.

Each F_v is independent of v and has no mutable monomial factor. In the order $h, b, c, d, e, f, g, r, q, u$, the polynomials are primitive and linear in $a_0, a_{10}, a_8, r, q, u, u, a_{15}, a_7, a_5$, respectively. The linear coefficients are monomials coprime to the constant terms, so Gauss's lemma proves irreducibility over $\mathbb{C}$. The LP normalization gives the monomials M_v in the table.

A.4. **The first mutations.** With the same coordinate convention their formulas are

$$\begin{aligned}
h' &= x_2q - a_7(a_2x_{14} + a_1y_{13}) - (a_1a_6 + a_2x_6)x_{12}, \\
b' &= x_{12}, & c' &= x_4x_8 - a_3a_9, \\
d' &= x_{17}, & e' &= y_{13}, \\
f' &= x_4y_7 - a_4y_{10} - a_3x_9, & g' &= x_{14}y_7 - a_{14}x_6, \\
r' &= x_{14}(y_2y_7 - x_3y_{10} - x_1x_8 + a_3x_{11}) - x_6(a_{13}x_1 + a_{14}y_2), \\
q' &= x_{11}y_0 - a_{11}x_1, \\
u' &= x_9(x_1y_{16} - a_2x_{14} - a_1y_{13}) - a_9x_1x_{17}.
\end{aligned}$$

Substitution of the Laurent coordinate expressions gives $vM_vv' = F_v$ in all ten directions. These identities also follow in R_7 from the injective Laurent chart constructed below.

A.5. **The upper coordinate ring.** We first establish the initial Laurent chart. Set

$$L_0 = S_7[h^{\pm1}, b^{\pm1}, c^{\pm1}, d^{\pm1}, e^{\pm1}, f^{\pm1}, g^{\pm1}, r^{\pm1}, q^{\pm1}, u^{\pm1}].$$

We compute Laurent expressions in L_0 for the 56 coordinates from the cyclic quadratic relations. The complete expressions are in `verification/freudenthal_source_coordinates.json`, and the adjacent-chart divisibilities are recorded in `verification/freudenthal_source_divisibilities.json`. We check these identities over $\mathbb{Q}$. The first three noninitial coordinates are

$$x_{17} = \frac{r + a_{17}c}{d}, \qquad y_{13} = \frac{q + a_6f}{e}, \qquad y_7 = \frac{u + a_{14}eg + a_3cf + a_4df}{fg}.$$

The expressions satisfy the 144 cyclic translates of the defining quadrics in (A.1), which span the 133-dimensional space of quadratic relations. They therefore define a homomorphism $R_7 \to L_0$. The eighteen frozen coordinates, seven initial linear coordinates, and three functions in (A.2) map to the 28 independent indeterminates of L_0. The image has transcendence degree 28, so the homomorphism is injective because R_7 is a 28-dimensional domain. After inverting the ten mutable functions, we obtain $R_7[(hbcdefgrqu)^{-1}] = L_0$. Every denominator is a monomial in the ten mutable coordinates; in particular, every frozen exponent is nonnegative. We also verify all 204 negative-slice divisibilities of (2.1), including the slices with poles of order two. Thus R_7 is contained in all ten adjacent Laurent rings. We now check the geometric hypotheses of Lemma 2.4.

By Lemma 2.5, R_7 is factorial and each initial degree-one variable is prime. For the higher-degree variables, we use a grading by degree and the seven Cartan weights. All 56 degree-one weights are distinct. A factor of a homogeneous torus semi-invariant is again a homogeneous semi-invariant: the connected torus cannot nontrivially permute its finitely many irreducible factors. Thus a reducible degree-two or degree-three variable has a degree-one coordinate factor.

Since r, q, u are units in the initial Laurent ring, such a degree-one factor must be a unit there as well. The Laurent expressions show that the only degree-one coordinates invertible in L_0 are h, b, c, d, e, f, g; the frozen coordinates remain polynomial. For each such coordinate v, we test $r/v, q/v, u/v$ in the relevant complete degree-one or degree-two weight space of R_7. Every test is negative. The degree-two spaces used here are spanned by at most six coordinate products, and are complete because R_7 is generated in degree one. It follows that

r, q, u are irreducible, hence prime. Their degree-and-weight vectors are pairwise distinct, so the ten initial variables are nonassociate.

The same graded calculation proves $v \nmid v'$ for all ten first mutations. For nine of them the quotient has a weight absent from the required homogeneous degree. The remaining quotient h'/h is tested against its complete six-product quadratic weight space and does not belong to it. The required weight-space calculations are over $\mathbb{Q}$. Lemma 2.4 now applies over S_7.

Proposition A.1. *The rank-ten seed* (A.3) *gives* $\mathcal{A}(\Sigma_7^\circ) \subseteq \mathcal{U}(\Sigma_7^\circ) = R_7$. *Every cluster variable is regular on the Freudenthal cone.*

A.6. **Infinite type.**

Proposition A.2. *The rank-ten Freudenthal seed* (A.3) *is of infinite type. More precisely, after mutating first at h and then at g, freezing the other eight mutable variables leaves a rank-two LP pattern of exchange degrees* $(1, 4)$.

Proof. Write $H = h'$ and $G = g'$ for the two new variables. These are new independent cluster coordinates; all the other original variables keep their names. Set

$$\begin{aligned} K &= u + a_3cf + a_4df, \\ \mathsf{A} &= (a_{16}d + a_{17}b)P + dq(a_{11}a_{16} + a_{15}b), \\ \mathsf{B} &= b\big((a_{16}d + a_{17}b)u + dq(a_0f + a_1a_{14})\big), \\ \mathsf{C} &= deq(a_0f + a_1a_{14}), \\ \mathsf{D} &= a_{17}efq(a_3c + a_4d) + du\big(f(a_1a_6 + a_2e) + a_1q\big). \end{aligned}$$

The initial equations have the useful decompositions

$$F_h|_{r=0} = g\mathsf{C} + \mathsf{D}, \qquad F_r = gh\mathsf{A} + g\mathsf{B} + a_{17}bqK.$$

Since $\widehat{F}_h = F_h/(def)$, the first mutation gives

$$F_r^{(h)} = g^2\mathsf{AC} + g(\mathsf{AD} + defH\mathsf{B}) + a_{17}bdefqHK.$$

In the resulting seed $\widehat{F}_g = F_g = K$. Substitution $g = K/G$, clearing G^2, and removal of the single common factor K gives the ordinary exchange polynomial

$$\mathcal{Q}(d) = K\mathsf{AC} + G(\mathsf{AD} + defH\mathsf{B}) + a_{17}bdefqHG^2. \tag{A.4}$$

There is no further factor of K in (A.4). Indeed, set $u = -2$ and all other variables to 1; then $K = 0$, whereas $\mathcal{Q} = -57$.

Both mutations leave $F_d = r + a_{17}c$ unchanged. In the final seed the associated exchanges in these two directions are

$$\widehat{F}_d = r + a_{17}c, \qquad \widehat{F}_r = \frac{\mathcal{Q}(d)}{befGq}.$$

All the factors b, e, f, G, q become coefficient units when the eight variables H, b, c, e, f, G, q, u are frozen. In particular the active exchange degrees are unchanged. From (A.4) we obtain

$$\begin{aligned} [d^4]\mathcal{Q} &= a_4a_{12}a_{16}e^2fq(a_0f + a_1a_{14}) \neq 0, \\ [d^0]\mathcal{Q} &= a_3a_{17}^2bcefGq(a_6a_{11}f + a_7bf + a_9a_{14}e + a_{10}ef + a_{13}ce) \neq 0. \end{aligned}$$

Hence the two degrees are exactly $(1, 4)$.

The freezing operation, including its coefficient-unit adjustments, is the one proved compatible with LP mutation in [12, Proposition 3.7]. By [12, Proposition 6.2 and Theorem 6.4], the rank-two degree pair $(1, 4)$ has infinitely many denominator vectors: its real-root system has Cartan matrix

$$\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}.$$

These variables are also cluster variables of the original rank-ten pattern. This proves infinite type. □

Here *infinite type* means infinitely many cluster variables; it does not assert infinite mutation type, which concerns the exchange data up to the usual equivalence.

Remark A.3. The infinite-type assertion concerns the auxiliary seed (A.3). It does not exclude a different positive finite-type rank-ten structure as in [4, Conjecture 6.1]. Positivity throughout this infinite mutation class and mutation equivalence of the Coxeter-translated presentations are separate questions.

A.7. **Comparison of the Freudenthal charts.** To relate the auxiliary seed to the primary seed, start with the ordered cluster

$$\mathbf{x}^\circ = (x_1, x_{11}, x_9, y_{10}, x_6, x_{14}, x_4, r^\circ, q^\circ, u^\circ),$$

where

$$r^\circ = x_{17}y_{10} - a_{17}x_9, \qquad q^\circ = x_6y_{13} - a_6x_{14},$$
$$u^\circ = x_{14}x_4y_7 - a_{14}x_4x_6 - a_3x_{14}x_9 - a_4x_{14}y_{10}.$$

Mutation words act from left to right, with positions kept fixed.

Proposition A.4. *The word* $(4, 9, 8, 10, 4)$ *in the auxiliary original order gives the primary seed of Theorem 4.2, including its stated global polynomial coordinates. It fixes each frozen coordinate and identifies both the generated and upper LP algebras.*

Proof. Apply the five LP mutations over the original coefficient ring, using the associated exchange Laurent polynomials. Direct substitution, also in the reverse direction, identifies the final cluster with the polynomial functions r, q, u of the primary seed.

Irreducibility of the primary original exchanges can also be checked directly by primitive linearity. In the mutable order, one may use

$$a_1, a_{11}, a_4, a_0, a_1, a_1, a_4, a_2, a_0, a_2.$$

The leading coefficient is a monomial coprime to the constant coefficient in each case. All normalization tests keep the frozen variables polynomial.

Mutation-equivalent seeds have the same complete mutation class, since LP mutation is involutive up to the fixed unit convention. They therefore have the same generated algebra and the same intersection of Laurent rings. The assertions about functions are substitutions in this common field. □

This argument concerns the original coordinate-ring seeds only. The mirror is reconstructed and compared with the intrinsic algebra directly in Appendix D; no mutation of an auxiliary mirror is used there.

Appendix B. Spinor mirror calculations

We prove the recurrence and triangular-coordinate identities used in Sections 6 and 7. Throughout, $m = n - 1$ and the characters and the sets $\mathcal{I}_a$ are those of (6.1)–(6.5).

B.1. **Normalized cube recurrence.** We normalize each interval exchange by its first summand. Extend the character notation by

$$\rho_{s,t} = \chi_{R_{st}} \quad (s \le t \le m-2), \qquad \rho_{s,s-1} = \chi_{B_{s-1}} \quad (s \le m-2), \qquad \rho_{m-1,m-2} = \chi_Q,$$

and set $\xi_s = \chi_{E_s}$ for $2 \le s \le m-2$ and $\xi_{m-1} = \chi_P$. Introduce the monomials

$$V_{i,m-1} = \xi_i, \qquad V_{ij} = \rho_{i,j-1}\rho_{ij}V_{i+1,j+1} \quad (j \le m-2). \tag{B.1}$$

Empty indexed families are omitted, in particular when $n = 4$. If A_{ij} is the first summand of the interval exchange F_{ij}, then cancellation of monomials in the interval coordinates gives

$$\begin{aligned} N_{ii} &:= F_{ii}/A_{ii} = 1 + \rho_{i,i-1} + V_{ii}, \\ N_{ij} &:= F_{ij}/A_{ij} = 1 + \rho_{i,j-1} + V_{ij} + \rho_{i,j-1}V_{i+1,j} \quad (i < j). \end{aligned} \tag{B.2}$$

At the terminal positions, these specialize to

$$N_{m-1,m-1} = 1 + \chi_P + \chi_Q, \qquad N_{i,m-1} = 1 + \rho_{i,m-2} + \xi_i + \rho_{i,m-2}\xi_{i+1}.$$

To prove (B.1), set $D_{it} = \frac{X_{it}X_{i-1,t+1}}{X_{i-1,t}X_{i,t+1}}$. Then $\rho_{it} = D_{it}/D_{i+1,t+1}$ and $\xi_i = D_{i,m-2}D_{i,m-1}$, so the products telescope. The same cancellation, with $X_{i,i-1} = h$, applies when $i = j$ and gives the four ratios in (B.2).

For $0 \le a \le b \le c \le m$, set $Z(a,b,c) = q_{[1,a]\cup[b+1,c]}$. For $1 \le a < b < c \le m$, the cube with common set $[1,a-1]\cup[b+1,c-1]$ replaces $Z(a-1,b-1,c-1)$ by $Z(a,b,c)$. Processing triples lexicographically gives a sequence of $\binom{m}{3}$ mutations to the complementary-interval chart.

Lemma B.1. *In the positive χ chamber,*

$$\operatorname{lead} Z(a,b,c) = L(a,b,c) := \frac{X_{1a}X_{b-a+1,c}}{X_{b-a+1,b}}. \tag{B.3}$$

The quotient $P_{abc} = Z(a,b,c)/L(a,b,c)$ is a polynomial in the χ characters, with constant term one and linear part $\sum_{\rho\in\mathcal{I}_{(b-a+1,b)}} \chi_\rho$.

Proof. The third product in the cube recurrence gives the leading term in (B.3). The ratios of the other two leading products are the nonconstant monomials M_1, M_2 in (B.5) below. Induction therefore gives P_{abc} as a formal power series with constant term one. Homogeneity and the integral basis put it in $\mathbb{Z}[\chi^{\pm 1}]$ by the Laurent phenomenon for the spinor LPA. The intersection of these two rings is $\mathbb{Z}[\chi]$. The axis calculation in Lemma B.2 gives the linear part. □

For degenerate triples ($a = 0$, $a = b$, or $b = c$), put $P_{abc} = 1$.

Lemma B.2. *For every mirror position ρ and every interval t,*

$$N_t|_{\chi_\sigma=0\ (\sigma\neq\rho)} = 1 + \mathbf{1}_{t=a(\rho)}\chi_\rho.$$

Moreover N_t has total degree at most one in the variables of its set $\mathcal{I}_t$. For every strict triple $a < b < c$,

$$P_{abc}|_{\chi_\sigma=0\ (\sigma\neq\rho)} = 1 + \mathbf{1}_{a(\rho)=(b-a+1,b)}\chi_\rho. \tag{B.4}$$

Proof. The exchange assertions follow from (B.2). If $j < m-1$, every nonconstant summand is divisible by the one assigned character $\rho_{i,j-1}$, and contains it just once. The remaining factors in the two longer ribbons have other indices. If $j = m-1$, the terminal formulas prove the assertion directly when an exchange polynomial occurs twice.

For the cube assertion put $d = b-a+1$ and $\ell = m-c$. In the normalized cube recurrence write

$$P_{abc}P_{a-1,b-1,c-1} = M_1P_{a-1,b,c-1}P_{a,b-1,c} + M_2P_{a,b-1,c-1}P_{a-1,b,c} + P_{a-1,b-1,c}P_{a,b,c-1}, \tag{B.5}$$

where $M_2 = \prod_{t=b-1}^{c-2}\rho_{d,t}$ and

$$M_1 = \begin{cases} \xi_d \prod_{t=b-1}^{m-3}\rho_{d,t}, & \ell = 0, \\ \xi_{d+\ell}\left(\prod_{t=b-1}^{c-2}\rho_{d,t}\right)\left(\prod_{k=0}^{\ell-1}\rho_{d+k,c-1+k}\right)\left(\prod_{k=0}^{\ell-2}\rho_{d+k+1,c-1+k}\right), & \ell \ge 1. \end{cases}$$

These are identities of monomials in the initial interval variables. Every ribbon of length at least two contains at least two distinct characters: the horizontal indices differ, and a terminal ξ differs from every ρ factor.

We prove (B.4) by the lexicographic cube induction. If $c > b+1$, both ribbons vanish on any single axis. The old factor $P_{a-1,b-1,c-1}$ and $P_{a-1,b-1,c}$ have the same assigned target $(d, b-1)$, or are both degenerate. Their axis restrictions therefore cancel, leaving exactly the restriction of $P_{a,b,c-1}$, whose target is (d,b). If $c = b+1 < m$, only $M_2 = \rho_{d,b-1}$ can survive on a single axis. On its own axis all four accompanying factors have restriction one, and it gives $1 + \rho_{d,b-1}$. On every other axis the old and third product factors cancel as before.

Finally, if $c = b + 1 = m$, the only possible surviving ribbons are the two distinct characters ξ_d and $\rho_{d,m-2}$. Each contributes one on its own assigned axis and zero on the other. This gives exactly the terminal repeated case and completes the induction for every n. □

For $t = (u, v) \in \mathcal{M}$, write $\psi_t = q_{[m]\setminus[v-u+2,v]} = X^{h_t} P_t(\chi)$. Its leading term is $X_{1,v-u+1} X_{u,m}/X_{uv}$; hence $h_t^{\mathrm{mut}} = -e_t$. Lemma B.2 applies to P_t.

The full adjacent support is determined by the following tropical identity.

Lemma B.3. *For every strict triple $a < b < c$ and every $s \in \mathcal{M}$,*

$$-\min_{\alpha \in \operatorname{supp} P_{abc}} \langle \alpha, \mu_s \rangle = \mathbf{1}_{s=(b-a+1,b)}.$$

In particular the defect of ψ_t at μ_s is $\mathbf{1}_{t=s}$.

Proof. We use induction in the normalized recurrence. A character $\rho_{u,v-1}$ has value

$$-\mathbf{1}_{s=(u,v)} + \mathbf{1}_{s=(u+1,v)} + \mathbf{1}_{s=(u-1,v-1)} + \mathbf{1}_{s=(u,v+1)}$$

at μ_s, with inadmissible intervals omitted. A terminal character ξ_u has value $-\mathbf{1}_{s=(u,m-1)} + \mathbf{1}_{s=(u-1,m-2)}$. Put $T = \mathbf{1}_{s=(d,b)}$, $O = \mathbf{1}_{s=(d,b-1)}$, $A = \mathbf{1}_{s=(d+1,b)}$, $B = \mathbf{1}_{s=(d-1,b-1)}$, and $\varepsilon = \mathbf{1}_{c>b+1}$. If λ_i is the value of M_i, the three terms on the right of (B.5) have values

$$\lambda_1 - \varepsilon A - B, \qquad \lambda_2 - A - B, \qquad -O - \varepsilon T.$$

Cancellation along the horizontal ribbon gives

$$\lambda_2 - A - B = -T + \mathbf{1}_{s=(d,c)} + \sum_{v=b+1}^{c-1} \mathbf{1}_{s=(d+1,v)} + \sum_{v=b}^{c-2} \mathbf{1}_{s=(d-1,v)}.$$

For the other ribbon, every assigned negative node after its first is cancelled by a denominator contribution of the preceding node. The two off-ribbon nodes $(d-1, b-1)$ and, when $\varepsilon = 1$, $(d+1, b)$ receive positive contributions from the first node. The terminal one-factor case has $\varepsilon = 0$. Consequently $\lambda_1 \geq -T + B + \varepsilon A$. If $\varepsilon = 1$, the third term has value $-O - T$ and the others are no smaller. If $\varepsilon = 0$, the second and third terms attain the same minimum $-O - T$ between them, since O and T cannot both be one. Subtracting the old value $-O$ proves the asserted new value $-T$. Subtraction-free tropicalization of the cube recurrence makes this an identity of Newton minima; coefficientwise polynomial positivity is not being assumed. □

B.2. **The adjacent supports.** The complementary coordinate ψ_t has mutable leading gradient $-e_t$. Its associated slope inequality is a coordinatewise bound $0 \leq d_t \leq \delta_{\psi_t}(\eta)$ for a correction to η; see (B.6)–(B.7) below. Lemma B.3 gives $\delta_{\psi_t}(\mu_s) = \mathbf{1}_{t=s}$.

Lemma B.4. *For $\eta = \mu_a$, impose* (B.6) *for* all *regular functions $f \in \mathcal{O}(U_n)$. The allowable corrections are exactly*

$$d = 0 \quad \text{or} \quad d = e_a.$$

The resulting mirror support is $\{\mu_a, \mu_a + c_a\}$, and its normalized coefficient-one function is

$$z^{\mu_a}(1 + z^{c_a}) = \frac{\mathcal{E}_a}{\prod_{\rho \in \mathcal{I}_a} Z_\rho}.$$

Proof. The complementary coordinates and Lemma B.3 restrict the bounds (B.7) to $\{0, e_a\}$. The zero correction always satisfies the inequalities.

We show that no other regular function can remove e_a. If $(h_f)_a \geq 0$, its inequality is automatic. Suppose $(h_f)_a = -k < 0$. Adjacent-chart Laurent regularity says that the coefficient of X_a^{-k} in f is divisible by F_a^k in the Laurent ring of the other interval coordinates. Let A_a be the least monomial of F_a. For any $\rho \in \mathcal{I}_a$, the monomial $A_a \chi_\rho$ is another term of F_a, by (6.1)–(6.2). Newton polytopes of products add. Since least exponents add under multiplication, the divisible slice therefore has the following Newton-polytope point (which need not be an uncancelled support monomial): $h_f + k\chi_\rho$. Here χ_ρ denotes its exponent vector in the interval character lattice. Since $\langle \chi_\rho, \mu_a \rangle = -1$ in dual coordinates, $\delta_f(\mu_a) \geq k = -(h_f)_a$.

Thus e_a satisfies every regular-function inequality. Finally (6.7) gives the stated Laurent polynomial. □

B.3. **Finite expansions and primitive contacts.** The initial and adjacent tori cover the codimension-one points of U_n: its initial coordinates are nonassociate primes, and the localized chart at a is $X_a w_a = F_a$. Each centre $F_a = 0$ is smooth on the spectator torus, since a corner coordinate occurs linearly with a monomial coefficient. The punctured spinor cone is smooth, and boundary inversion removes its vertex. The initial logarithmic volume form extends across every rank-one chart, since the exchange replaces its wedge by its negative. It is therefore a nowhere vanishing form on U_n. These facts verify the smooth affine log Calabi–Yau hypotheses for the intrinsic algebra.

The common birational model with a toric compactification is the blowup of (X_a, F_a) along each mutable-coordinate ray, with the strict transform of its toric boundary removed. By [10, Lemma 8.21], skeletal curves avoid the fixed codimension-two discrepancy locus. Thus every unmarked toric contact is over one of these centres and has direction c_a, with positive integral multiplicity. In particular all realized cylinder bends lie in

$$\Pi = C_{\mathrm{mut}}\mathbb{Z}^{r_0}_{\geq 0}, \qquad r_0 = \binom{n-2}{2}.$$

The columns are independent and saturated, so Π is a free pointed toric monoid. This verifies [10, Assumption 20.27] for the given blowup model, including its nonbinomial centres.

For a regular function f with unique minimizing exponent h_f in the positive χ chamber, the principal-divisor slope formula [10, Lemma 15.3 and Theorem 15.7] gives, for a correction d to an incoming label η,

$$h_f^{\mathrm{mut}} d \geq -\delta_f(\eta), \qquad \delta_f(\eta) = \langle h_f C^{-1}, \eta\rangle - \min_{u \in \operatorname{supp} f} \langle u C^{-1}, \eta\rangle. \tag{B.6}$$

The complementary functions therefore give the coordinatewise bound

$$0 \leq d_t \leq \delta_{\psi_t}(\eta) \qquad (t \in \mathcal{M}). \tag{B.7}$$

The same slope formula excludes a nonconstant attached tropical branch rooted in this chamber. Such a branch comes from an unmarked analytic disk with one boundary point and image in U_n. If its vector directed from the root into the disk is $C_{\mathrm{mut}} d$, a regular function has nonnegative slope there: the slope is its number of zeros in the disk. Applying the complementary functions gives $-d_t \geq 0$ for every t. Hence $d = 0$. This argument concerns realized analytic twigs and uses no copositivity assertion about arbitrary combinatorial trees.

The free contact monoid and (B.7), together with Lemma 5.1, give finitely many terms and curve classes before specialization to $q^\beta = 1$. The absence of rooted twigs permits the two- and three-point gluing identities in the intrinsic Laurent expansion and trace theorem of [7], which gives the injective algebra map

$$\Phi_n : \mathscr{M}_n \hookrightarrow \mathscr{S}_n[Z_\rho^{\pm 1}], \qquad M_n = \Phi_n(\mathscr{M}_n).$$

The zero-contact term is monic. For nonnegative mutable labels the box is zero, so the initial variables and both signs of the coefficient characters are straight theta functions in M_n.

B.4. **All primitive facet walls and the adjacent variables.**

Proposition B.5. *At a generic point of the positive-chamber facet* $\mathrm{val}(\chi_\rho) = 0$*, the intrinsic wall function is* $1 + z^{c_{a(\rho)}}$*. Moreover* $\Theta_{\mu_a} = Y_a$ *for every* $a \in \mathcal{M}$*.*

Proof. Take a sufficiently small transverse primitive crossing with incoming direction $-e_\rho$. The axis restriction in Lemma B.2 is the entire local face polynomial of each normalized complementary function. Its gradient changes by one precisely for $t = a(\rho)$. The slope formula for the infinitesimal cylinder consequently gives $0 \leq d_t \leq \mathbf{1}_{t=a(\rho)}$. Thus only the zero-contact configuration and one simple contact at $a = a(\rho)$ can occur. Tangency is automatic because $(c_a)_\rho = 0$.

We count the second configuration directly. Normalize the smooth three-marked domain at $0, \infty, 1$ and write the torus evaluation as $c = (c_\sigma)$. With contact parameter τ, its character map is

$$\chi_\sigma(w) = c_\sigma w^{-\delta_{\sigma\rho}} \left(\frac{w-\tau}{1-\tau}\right)^{(c_a)_\sigma}. \tag{B.8}$$

The same formula in the full basis includes the gauge characters. Every exponent of N_a pairs to zero with c_a, since the interval exchange is independent of X_a. The contact factors therefore cancel in the centre equation. By (B.2), $N_a = A(\chi_{\neq\rho}) + \chi_\rho B(\chi_{\neq\rho})$ with $A(0) = B(0) = 1$, and the incidence equation is

$$A(c_{\neq\rho})\tau + c_\rho B(c_{\neq\rho}) = 0.$$

Give c_ρ valuation $\delta > 0$ sufficiently small relative to the generic positive valuations of the other χ coordinates. The unique solution has valuation δ, and the derivative is a unit. Applying C^{-1} to (B.8), any interval coordinate has the form

$$X_s(w) = X_s(c) w^{-(C^{-1}e_\rho)_s} \left(\frac{w-\tau}{1-\tau}\right)^{\delta_{sa}}.$$

Thus only X_a vanishes at the unmarked contact, and it has a simple zero. Moreover $N_a(w) = A + c_\rho B/w$ has a simple zero there. Consequently F_a/X_a is nonzero at the contact, and the map extends through the rank-one chart.

To identify the incidence solution with a counted cylinder, consider the annulus $0 < \mathrm{val}(w) < 2\delta$, the principal spine has its only bend at $\mathrm{val}(w) = \delta$, as required by [10, Definition 20.2]; the tangential displacement of this root is absorbed in the generic evaluation point. Both exterior tails are toric. There is also a uniform toric-end check: the angular valuations at the two essential ends are $\mathrm{val}(c)$ and $\mathrm{val}(c) + \delta c_a$. All their χ entries are positive after shrinking δ, because $(c_a)_\rho = 0$. If a terminal ray bears another blowup centre, its normalized equation evaluates to $1+$ terms of positive valuation, so the terminal position avoids that centre. Thus both marked ends lie in open essential toric strata. Start with the partial toric fan consisting of the mutable-coordinate and terminal rays, and perform the specified centre blowups. Choose a smooth compactification preserving this partial model, in particular the open strata over all these rays. The simple incidence solution makes evaluation generically etale; [10, Lemmas 3.6 and 3.9] then allow passage to a common resolution while avoiding its fixed codimension-two loci. This establishes the converse from the unique incidence solution to a counted cylinder. The toric zero-contact curve and this primitive simple contact each have count one. Hence the primitive wall function is $1 + z^{c_a}$, with no lattice or covering multiplicity.

Continuation across this wall, followed by the unimodular basis change replacing e_ρ by μ_a, is the first LP mutation. We record why it yields Laurent regularity. The rational transform of any finite initial theta polynomial has as its only nonmonomial denominator a power of $1 + z^{c_a}$. The continued local theta series has nonnegative coefficients by its enumerative definition. If Df is finite, D is a positive polynomial with constant term one, and f is a positive series in a translate of the pointed contact monoid, then f is finite: every term of f already occurs in Df, without cancellation. Thus every Θ_η is Laurent in the adjacent chart. Applying this to every position and the finite LP starfish theorem [6, Theorem 1.3] gives $M_n \subseteq \mathcal{U}_n^\vee$.

Now use the *full* indicator Lemma B.3 in (B.7), with $\eta = \mu_a$. It restricts the theta support to $\{\mu_a, \mu_a + c_a\}$. Hence $\Theta_{\mu_a} = z^{\mu_a}(1 + \lambda z^{c_a})$. In any adjacent position $\rho \in \mathcal{I}_a$ both terms have exponent -1 in Z_ρ. Laurent divisibility by $\mathcal{E}_a$ forces $\lambda = 1$. This recognizes Y_a without a second incidence argument for a full theta label; in particular it also applies when several variables share an exchange polynomial. □

B.5. **Boundary theta functions.** The functions W_b are defined in (6.8)–(6.9).

Proof of the boundary identities in Proposition 7.2. For a distinct frozen coordinate b, the complementary coordinate ψ_t has order zero along $b = 0$. Indeed both are distinct degree-one primes in the spinor coordinate ring; inverting the other initial coordinates cannot create a b factor. Therefore its least b exponent in the initial Laurent chart is zero. The leading numerator of $\psi_{(u,v)}$ is $X_{1,v-u+1}X_{u,m}$, so (B.7) restricts the corrections for Ce_{1k} to zero-one choices on

$$a_1 = (2, k+1),\ a_2 = (3, k+2), \dots,\ a_\ell = (m-k, m-1).$$

For Ce_{im} the corresponding chain is

$$a_1 = (i, m-1),\ a_2 = (i, m-2), \dots,\ a_\ell = (i, i).$$

The boxes for h and X_{1m} are zero.

Let $d_1, \dots, d_\ell \in \{0, 1\}$ be the allowed correction coefficients along either chain. For $j \geq 2$ apply (B.6) to the regular mutation partner F_{a_j}/X_{a_j}. Its least monomial is the first summand of F_{a_j} divided by X_{a_j}. On the chain its mutable gradient is exactly $e_{a_{j-1}} - e_{a_j}$, and its frozen defect at Ce_b is zero. For a prefix chain this follows from the factor $X_{u-1,v-1}/X_{uv}$; for a suffix chain it follows from $X_{u,v+1}/X_{uv}$. The singleton exchange has the same restrictions at the marked end. The other factors are off the chain. Thus $d_1 \geq d_2 \geq \cdots \geq d_\ell$ The allowable supports are precisely the successive prefixes in (6.8)–(6.9).

Write their unknown coefficients as $\lambda_0, \dots, \lambda_\ell$, with $\lambda_0 = 1$. For the edge assigned to $a_j = (u, v)$ choose the mirror position $\rho_{u,v-1}$: it is $R_{u,v-1}$ for $u < v$, B_{u-1} for a nonterminal singleton, and Q for the terminal singleton. Restricting its exponent vector to the chain shows that its only negative slice consists of positions $j-1, j$, both with exponent -1. Their monomial ratio is $z^{c_{a_j}}$. Adjacent Laurentness consequently requires $1 + z^{c_{a_j}}$ to divide $\lambda_{j-1} + \lambda_j z^{c_{a_j}}$ in the Laurent ring in the unchanged coordinates. Since c_{a_j} is primitive and nonzero, this is equivalent to $\lambda_j = \lambda_{j-1}$. Starting with the incoming coefficient one proves $\Theta_{Ce_b} = W_b$ for every individual label.

Proposition B.5 gives the remaining inclusions. After inverting the initial coordinates, or Y_a and the unchanged coordinates in an adjacent seed, the image is the corresponding Laurent ring. Hence these tori are open subsets of the intrinsic mirror. □

B.6. **Adjacent regularity of the boundary sums.** Index the monomials in each boundary chain from zero. On a prefix chain W_{1k}, the negative pairs for B_a, E_a, R_{ab} occur respectively when $k = 1, m-a, b-a+2$, at positions $(a-1, a), (a-2, a-1), (a-2, a-1)$. The pair for P, Q occurs when $k = 1$, at $(m-3, m-2)$. On a suffix chain W_{im}, the pairs for B_a, E_a, R_{ab} occur respectively when $i = a+1, a, a$, at $(m-a-2, m-a-1), (0, 1), (m-b-2, m-b-1)$; the pair for P, Q occurs when $i = m-1$, at $(0, 1)$. Empty indexed families are omitted. The character exponents show that there are no other negative terms. Their pairwise cancellation proves adjacent Laurentness, and the upper-bound theorem gives membership in the full upper LP algebra. Distinctness follows from the chain exponents, or equivalently from the monomial coordinates below.

B.7. **Integral triangular coordinates and their flows.** The vectors v_{ab} and the characters $y_{ab}, \mathfrak{q}$ are those of (6.10).

Lemma B.6. *The characters $\mathfrak{q}$ and y_{ab} form a basis of the character lattice of the initial mirror torus. Their boundary expressions are those of (6.11).*

Proof. Since C is unimodular, we may work in the initial lattice basis. The inverse formulas are

$$e_h = v_{11}, \qquad e_{1m} = \mathbf{1} - v_{mm}, \qquad e_{1j} = v_{j+1,m} - v_{jm} \quad (j < m).$$

For $2 \leq i \leq j \leq m$, put

$$D(i, j) = v_{j-i+1,m+2-i} - v_{j-i+1,m+1-i}.$$

Directly subtracting the two sums in (6.10) gives $D(i, j) = \sum_{s=j}^{m} e_{is}$. Consequently

$$e_{im} = D(i, m), \qquad e_{ij} = D(i, j) - D(i, j+1) \quad (j < m).$$

Every initial basis vector has therefore been expressed integrally in these N vectors. This proves unimodularity.

For a horizontal arrow, with $i = m + 1 - b$, the same subtraction gives

$$v_{k+1,b} - v_{kb} = \sum_{r=1}^{i} e_{r,r+k-1} = e_{1k} + \sum_{t=1}^{m-b} e_{t+1,t+k}.$$

As b decreases from m to $k+1$, these are exactly the cumulative exponents of the coordinates defining W_{1k}. The vertical arrow with index a has exponent

$$v_{a,m+2-i} - v_{a,m+1-i} = \sum_{s=i+a-1}^{m} e_{is}.$$

As a decreases from $m + 1 - i$ to 1, these are exactly the cumulative exponents defining W_{im}. The other two boundary functions follow from the first two inverse formulas. □

Call an arrow $y_{a+1,b}/y_{ab}$ horizontal and an arrow $y_{a,b+1}/y_{ab}$ vertical; these names fix the conventions regardless of how the triangular diagram is drawn. Give the initial variables nonnegative real values w_ρ and give every κ_j value zero. Set

$$\lambda_{ab} = w(y_{ab}), \qquad T_{a,d} = \sum_{j=1}^{a} \lambda_{j,j+d} \quad (0 \le d \le m-1,\ 1 \le a \le m-d).$$

Let ϵ_{ab} denote the array supported at the single cell (a, b) with value one.

Lemma B.7. *The contribution to λ from a unit value of each initial variable is as follows:*

$$\begin{array}{c|l} B_i & \epsilon_{1,m-i} - \epsilon_{2,m-i+1} \\ R_{ij} & \epsilon_{j-i+2,m+1-i} - \epsilon_{j-i+3,m+2-i} \\ E_i\ (a = m-i) & \epsilon_{a,a} + \epsilon_{a,a+1} - \epsilon_{a+1,a+2} - \epsilon_{a+2,a+2} \\ P & \epsilon_{11} + \epsilon_{12} - \epsilon_{23} - \epsilon_{33} \\ Q & \epsilon_{12} - \epsilon_{23}. \end{array} \tag{B.9}$$

In particular,

$$T_{m-d,d} = 0, \qquad T_{a,d} \ge 0 \quad (a < m-d). \tag{B.10}$$

More precisely, set

$$p_1 = w_P, \qquad q_1 = w_Q, \qquad p_0 = p_{m-1} = 0,$$
$$p_a = w_{E_{m-a}}, \qquad q_a = w_{R_{m-a,m-2}} \quad (2 \le a \le m-2).$$

Then

$$T_{a,0} = p_{a-1} + p_a \qquad (1 \le a < m), \tag{B.11}$$
$$T_{a,1} = p_a + q_a \qquad (1 \le a < m-1),$$
$$T_{a,d} = \begin{cases} w_{B_{m-d-1}}, & a = 1, \\ w_{R_{m+1-a-d,m-d-1}}, & a \ge 2, \end{cases} \qquad (d \ge 2,\ 1 \le a < m-d). \tag{B.12}$$

The map $w \mapsto \lambda$ is injective.

Proof. We compute the coefficient of each character in v_{ab}. Fix a cell (a, b), put $L = m+1-b$, and write $A_r = \mathbf{1}_{r \le L}$ and $U_d = \mathbf{1}_{a \le d+1}$. The coefficient of e_{rs} in v_{ab} is $[v_{ab} : e_{rs}] = 1 - A_r U_{s-r}$. This remains valid for an empty interval $X_{r,r-1} = h$, since $U_{-1} = 0$. Pairing the exponent vector of $\chi_{R_{ij}}$ with this formula gives

$$(A_{i+1} - A_i)U_{j-i} + (A_{i-1} - A_{i+1})U_{j-i+1} + (A_i - A_{i-1})U_{j-i+2}.$$

It equals 1 exactly at $(a, b) = (j - i + 2, m + 1 - i)$, equals -1 exactly at $(j - i + 3, m + 2 - i)$, and is zero elsewhere. The character χ_{B_i} gives $(A_i - A_{i+2})U_0 + (A_{i+1} - A_i)U_1$, giving its two indicated cells. With $a_0 = m - i$, the character χ_{E_i} gives

$$A_i \mathbf{1}_{a \in \{a_0, a_0+1\}} - A_{i-1} \mathbf{1}_{a \in \{a_0+1, a_0+2\}}.$$

The domain $a \le b$ restricts its nonzero cells to precisely the four shown in (B.9). Substituting $i = m-1$ in this last calculation gives the exponent vector of χ_P, using $X_{m-1,m-2} = h$. Finally the character χ_Q gives $-(\mathbf{1}_{L=m-2})U_1 + (\mathbf{1}_{L=m-2} + \mathbf{1}_{L=m-1})U_0$, which is $\epsilon_{12} - \epsilon_{23}$.

The E_i column is the sum of two successive forward flows on diagonal zero and one forward flow on diagonal one. The P column is the same pattern at $a = 1$, whereas Q has only its diagonal-one flow. Every column has total zero on each diagonal and nonnegative proper partial sums, proving (B.10).

Taking the partial sums of these individual columns gives (B.11)–(B.12). These equations also recover w from λ: first recover the far-diagonal B and R values; recover p_a recursively from $p_0 = 0$ and $p_a = T_{a,0} - p_{a-1}$; finally recover $q_a = T_{a,1} - p_a$. □

Appendix C. Calculations for the Cayley-plane mirror

We describe the change of characters on the Cayley-plane torus, the affine charts, and the contact counts used in Theorem 8.2.

C.1. **The Cayley-plane transition.** In the published order $(x_1, x_2, x_3, x_4, y_3)$, the word $(3, 2, 5)$ gives $(x_1, x_7, z_3, x_4, t_1)$. The first relation identifies $z_3 = (y_3 + a_3x_2 + a_{10}x_4)/x_3$; the next two identify x_7 and $t_1 = x_1x_4x_7x_{10} - a_3a_6a_9a_{12}$. The full seed updates and their inverses give exchange support sizes $(2, 2, 5, 2, 12)$ and degrees $(1, 1, 1, 1, 4)$. The seed belongs to the family J in [4]; the representative in their Table 3 uses z_2, not the z_3 of this chart. This identifies the Newton chart of Subsection 8.1 with a seed in the published mutation class.

C.2. **The Cayley-plane chart and Newton coordinates.** We use the Daisey–Ducat seed $(x_1, x_7, z_3, x_4, t_1)$. Put $\kappa = a_3a_6a_9a_{12}$, and take the following ten characters of the initial torus:

$$\begin{aligned}
\sigma &= t_1/\kappa, & \mu &= x_1x_7/(a_3a_9),\\
\nu &= x_1x_7/(a_6a_{12}), & \rho_1 &= a_4x_1/(a_1a_9),\\
\rho_2 &= a_7x_1x_4/(a_1a_{12}a_9), & \rho_3 &= a_{10}x_1x_7x_4/(a_1a_{12}a_3a_9),\\
\rho_4 &= a_2x_1z_3/(a_1a_{12}a_9), & \rho_5 &= a_{11}x_1x_7z_3/(a_1a_{12}a_6a_9),\\
\rho_6 &= a_8x_1x_7x_4z_3/(a_1a_{12}a_3a_6a_9), & \rho_7 &= a_5x_1^2x_7x_4z_3/(a_1a_{12}^2a_3a_6a_9).
\end{aligned}$$

Their exponent vectors, followed by those of $a_6, \dots, a_{12}$, form a unimodular basis of the seventeen-dimensional character lattice of the Cayley-plane torus. The dual coordinates are

$$(s, u, v, p_1, \dots, p_7; \zeta_6, \dots, \zeta_{12}).$$

Thus s is dual to σ, u to μ, v to ν, and p_i to ρ_i. Duality of lattices is meant here; the two coordinate systems belong to dual tori.

Choose the chamber in which all ten characters of the initial torus have positive valuation. The normalized numerators of the regular functions x_{10}, z_2, y_3, respectively, are

$$N_x = 1 + \sigma, \qquad N_z = (1+\mu)(1+\nu) + \sigma, \qquad N_t = (1+\nu)(1+\rho_1+\rho_2+\rho_3) + \rho_4 + \cdots + \rho_7.$$

Their leading mutable denominators are $x_1x_7x_4$, $x_1x_7z_3$, and t_1. For a label

$$m = (m_\sigma, m_\mu, m_\nu, m_{\rho_1}, \dots, m_{\rho_7}),$$

we write a correction in the five mutable directions as $b = (b_1, b_7, b_z, b_4, b_t) \in \mathbb{Z}^5_{\ge 0}$. Regularity gives

$$b_1 + b_7 + b_4 \le A(m), \qquad b_1 + b_7 + b_z \le B(m), \qquad b_t \le C(m). \tag{C.1}$$

Here

$$\begin{aligned}
A(m) &= \max(0, -m_\sigma),\\
B(m) &= \max(0, -m_\mu, -m_\nu, -m_\mu - m_\nu, -m_\sigma),\\
C(m) &= \max(0, -m_\nu, \{-m_{\rho_i}\}_{i=1}^7, \{-m_\nu - m_{\rho_i}\}_{i=1}^3).
\end{aligned}$$

Indeed these are the tropical defects of the three normalized polynomials paired with the mutable parts of their leading exponents. All five correction coordinates are bounded, with at most

$$(C+1)\sum_{j=0}^{\min(A,B)}(j+1)(A-j+1)(B-j+1)$$

possibilities. Indeed, there are $j+1$ choices for (b_1,b_7) with $b_1+b_7=j$, followed by independent choices for b_4,b_z,b_t. The intrinsic support bound follows from the model for contacts with the mutable-coordinate divisors and the slope inequality proved below.

C.3. **Affine charts and prime divisors.**

C.3.1. *The open set where s is invertible.* Let

$$Z=\operatorname{Spec}\frac{\mathbb{C}[p_1,\dots,p_7,k,u,\bar u,v,\bar v]}{(u\bar u-1-Q,\ v\bar v-1-\Pi k)}.$$

The two equations involve disjoint sets of variables, so Z is a product of two smooth fivefolds of the form $ab=1+c_1c_2c_3c_4$. Such a hypersurface is smooth: a putative singular point would have $a=b=0$, hence all $c_i\neq 0$, while the derivative with respect to each c_i is nonzero. It is integral by primitive linearity, or by localization and the same equation.

The last two equations of (8.4) are the linear system

$$\begin{pmatrix} u & -Tv\\ p_4\bar v & -\bar u\end{pmatrix}\begin{pmatrix} f\\ g\end{pmatrix}=\begin{pmatrix} -(1+R)\\ p_3(1+H)\end{pmatrix}.$$

Its determinant is s. Therefore $Y_6|_{s\neq0}\simeq Z|_{s\neq0}$, and Cramer's rule yields

$$sf=(1+R)\bar u+vL(1+H),\qquad sg=p_4(1+R)\bar v+p_3u(1+H).$$

These imply (8.2)–(8.3) on the initial torus.

C.3.2. *The open set where Pk is invertible.* On this open all p_i and k are units. Set $J=\Pi k$ and $\widetilde g=Tg$. The eight characters (Q,J,R,H,L,p_5,p_6,k) are a unimodular basis of the torus with coordinates $(p_1,\dots,p_7,k)$. The exponent matrix is

$$\begin{pmatrix}
0&0&0&1&1&1&1&0\\
1&1&1&0&0&0&0&1\\
0&1&1&0&0&1&1&0\\
1&1&0&1&0&0&1&0\\
0&0&1&0&1&1&1&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&1
\end{pmatrix},\qquad \det=1.$$

The inverse is

$$\begin{aligned}
p_1&=HLp_6/(QR), & p_2&=Rp_5/L,\\
p_3&=QJ/(Hp_5p_6k), & p_4&=Q^2J/(HLp_5p_6k),\\
p_7&=HLk/(QJ),
\end{aligned}$$

with p_5,p_6,k unchanged.

The defining equations eliminate four unit characters:

$$\begin{aligned}
Q&=u\bar u-1, & J&=v\bar v-1,\\
R&=v\widetilde g-uf-1, & H&=\frac{(u\bar u-1)\bar v f-\bar u\widetilde g-L}{L}.
\end{aligned}\tag{C.2}$$

Thus this open is a localization of

$$\mathbb{C}[u,\bar u,v,\bar v,f,\widetilde g,L^{\pm1},p_5^{\pm1},p_6^{\pm1},k^{\pm1}],$$

obtained by inverting the four numerators in (C.2). It is smooth and integral of dimension ten.

Since $Pk - s = 1$, the two opens above cover Y_6. Their intersection is nonempty. This proves smoothness and integrality globally, not just at generic points. It also proves that the defining ideal has height four and gives the stated complete intersection.

C.3.3. *The coordinate divisors.* For $p_i = 0$ one has $s = -1$, so the quotient lies in the s-open. If $i \leq 3$, then $v\bar{v} = 1$; if $i \geq 4$, then $u\bar{u} = 1$. In either case the remaining product description of Z proves that the quotient is a domain. Thus every p_i is prime.

For $u = 0$, the s-open quotient is a localization of $Z/(u)$, which is integral because $Q = -1$ and the other fivefold factor remains integral. The Pk-open quotient is a localization of the polynomial/Laurent ring above with $u = 0$, hence is integral. These quotient opens have nonempty intersection, so $\mathscr{B}_6/(u)$ is a domain. The same argument applies to v.

Finally, $s = 0$ implies $Pk = 1$, so this entire divisor is in the second open. There its equation is $(u\bar{u} - 1)v\bar{v} - u\bar{u} = 0$. As a polynomial in $\bar{v}$, its coefficient $(u\bar{u} - 1)v$ is coprime to its constant coefficient $-u\bar{u}$. Primitive linearity proves irreducibility, and the localization is nonzero. Thus s is prime. The same descriptions show that the ten primes are pairwise nonassociate.

The torus charts. All fourteen generators of (8.4) are cluster variables for (8.1): u, v, p_i are initial, $k, \bar{u}, \bar{v}, f$ are adjacent variables, and g comes from (u, v, s). The coordinate s is recovered as $Pk - 1$.

By the LP Laurent theorem [12], the ring $\mathscr{B}_6$ embeds in every seed Laurent ring. In particular each initial and adjacent cluster defines an open torus of Y_6: localizing at its coordinate product gives exactly that Laurent ring.

At the prime divisor $s = 0$, f is nonzero at the generic point; at $u = 0$ and $v = 0$, the same holds for $\bar{u}$ and $\bar{v}$; and at each $p_i = 0$, it holds for k. All other initial coordinates are units at the corresponding generic point. Hence these eleven tori meet every codimension-one point. Normality gives

$$\mathscr{B}_6 = L_0 \cap L_s \cap L_u \cap L_v \cap \bigcap_{i=1}^{7} L_{p_i}. \tag{C.3}$$

The Laurent theorem and generation give

$$\mathscr{B}_6 \subseteq \mathcal{A}(\Sigma^\vee) \subseteq \mathcal{U}(\Sigma^\vee) \subseteq L_0 \cap \bigcap L_i = \mathscr{B}_6.$$

This proves the algebra identity in Proposition 8.1.

The volume form and units. The initial logarithmic form

$$\omega = \frac{ds}{s} \wedge \frac{du}{u} \wedge \frac{dv}{v} \wedge \frac{dp_1}{p_1} \wedge \cdots \wedge \frac{dp_7}{p_7}$$

agrees up to sign with the analogous form on every adjacent LP torus: an LP exchange replaces one logarithmic differential by its negative plus a linear combination of the other differentials. Therefore ω has no zero or pole at any codimension-one point. Smoothness and extension of sections of a line bundle across codimension two make it a global nowhere-vanishing volume form.

Localizing $\mathscr{B}_6$ at the ten prime initial coordinates gives a Laurent polynomial UFD. Nagata's criterion then proves that $\mathscr{B}_6$ is a UFD. For units, a unit and its inverse in the initial Laurent ring have the form $\lambda s^a u^b v^c \prod p_i^{d_i}$. Comparing with the adjacent Laurent ring forces each corresponding exponent to vanish, because the exchange polynomial is a nonunit irreducible there. Thus the only units of the core are constants. Restoring the seven coefficient units gives precisely their free unit lattice, modulo constants.

C.4. **The torus factor and the contact directions.** We split off the seven invertible characters and write X for the affine variety obtained from the frozen-inverted Cayley-plane cone by splitting off the seven torus factors $a_6, \ldots, a_{12}$. Smoothness and this product decomposition are justified below. Its initial torus has characters $(\sigma, \mu, \nu, \rho_1, \ldots, \rho_7)$. These

are the Newton coordinates introduced above. The five mutable regular functions, in their original order, are $(x_1, x_7, z_3, x_4, t_1)$; the other five coordinates of the core are invertible regular functions. Relative to the ten-character basis, the five mutable divisor directions are

	σ	μ	ν	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7
n_1	0	1	1	1	1	1	1	1	1	2
n_7	0	1	1	0	0	1	0	1	1	1
n_z	0	0	0	0	0	0	1	1	1	1
n_4	0	0	0	0	1	1	0	0	1	1
n_t	1	0	0	0	0	0	0	0	0	0

(C.4)

They are independent and generate a saturated sublattice. All entries are nonnegative. The normalized centre equations in the corresponding toric blowup charts are

x_1, x_7, x_4	$1+\sigma$
z_3	$(1+\mu)(1+\nu)+\sigma$
t_1	$(1+\nu)(1+\rho_1+\rho_2+\rho_3)+\rho_4+\rho_5+\rho_6+\rho_7.$

(C.5)

The centre $N_x = 1+\sigma$ occurs on three distinct boundary divisors with the same centre equation. Repeated centres must not be replaced by independent binomial walls.

C.5. **Contact interpolation on the Cayley-plane complement.** The Cayley-plane coordinate ring is factorial, and its five initial mutable functions are pairwise nonassociate primes. Its initial and adjacent LP charts therefore cover codimension one. On the open where the other coordinates are invertible, an associated exchange $x_i x_i' = F_i/M_i$ becomes $x_i w_i = F_i, \qquad w_i = M_i x_i'$. This is the blowup of the ideal (x_i, F_i) in the toric chart for the ray n_i, with the strict transform of its toric boundary removed. The centres in (C.5) are smooth. For the second centre one can differentiate with respect to σ; for the third, with respect to ρ_4. The repeated first centre is likewise smooth. Thus these charts give a big open $X^\circ \subset X$, with the stated centre equations.

The punctured Cayley-plane cone is smooth because it is the complement of the zero section in the tautological line bundle. Inverting the boundary coordinates removes its vertex; the torus-factor decomposition then preserves smoothness and splits the volume form. On the initial torus take the standard logarithmic volume form. In a smooth rank-one chart $x_i w_i = F_i$, choose F_i as a local coordinate. The identity

$$\frac{dx_i}{x_i} \wedge dF_i = dx_i \wedge dw_i$$

shows that this form extends without a zero or pole across the generic point of each missing mutable-coordinate divisor. The boundary complement is smooth, and the rank-one charts cover codimension one, so the form extends everywhere and nowhere vanishing. Together with the open torus this gives the log Calabi–Yau and maximal-boundary hypotheses of [10, Section 2].

The unimodular character change defines an action of $H = (\mathbb{G}_m)^7$ on the initial torus: it fixes σ, μ, ν, ρ_i and scales $a_6, \dots, a_{12}$ independently. The normalized exchange numerators (C.5) are invariant, so every initial and adjacent coordinate is an H-eigenfunction. Normality and the codimension-one chart cover identify $\mathcal{O}(U_6)$ with the intersection of these Laurent rings. That intersection is therefore $\mathbb{Z}^7$-graded. The units $a_6, \dots, a_{12}$ have the standard basis degrees; multiplying a homogeneous element by the corresponding inverse unit makes it degree zero. Consequently the coordinate ring is

$$\mathcal{O}(X)[a_6^{\pm 1}, \dots, a_{12}^{\pm 1}], \qquad \mathcal{O}(X) = \mathcal{O}(U_6)^H.$$

Thus the decomposition is an isomorphism of affine varieties. Smoothness descends to X, and the unimodular logarithmic volume form splits as the core form times $\bigwedge_{j=6}^{12} d\log a_j$. For an H-semi-invariant regular function on U_6, we multiply it by the unique Laurent monomial in $a_6, \dots, a_{12}$ that makes its H-degree zero before regarding it as a function on X. We keep the

same symbols for these core functions, including $x_1, x_7, z_3, x_4, t_1, x_{10}, z_2, y_3$. This convention changes neither the core divisor directions nor the normalized centre equations (C.4)–(C.5).

Choose a compactification satisfying the hypotheses of [10]. The common toric-blowup model can be enlarged by the toric rays of this compactification. Away from a fixed codimension-two subset it agrees with X° and with the generic points of the essential boundary divisors. The counted maps lie in the smooth evaluation locus by [10, Lemmas 8.20–8.21]; generic avoidance then follows from [10, Lemma 3.9]. We also exclude the intersections of distinct coordinate divisors. After the curve-class bound below, one generic evaluation point suffices for all cases being counted.

We apply Proposition 5.2. It remains to check the two marked ends and to solve the centre equations with distinct contact parameters; the generic transversality and avoidance conditions are supplied by the cited evaluation results.

The leading-coefficient condition holds in every case below, as the following comparison of end rays shows. For each contact vector b, inspect the two end rays m, $-(m + \sum_i b_i n_i)$. For the cases in (C.6) and the ten positive-coordinate labels, the end rays are not positive multiples of a centre ray n_i, except for the two straight labels $\pm e_\sigma$. Add these finitely many end rays to the partial toric fan. The open toric divisors of the new rays are disjoint from the blowup centres, so nonzero leading angular coefficients suffice there. In the two exceptions the relevant centre numerator is $N_t = 1 +$ terms of positive valuation, which is nonzero. Hence these marked ends also lie in the open essential stratum. This finite ray check is included in the calculation below.

One can first complete this partial toric blowup without altering its open ray divisors. The incidence solutions then give maps with the required boundary contacts in that compactification. Their simple incidence Jacobians make evaluation generically étale. The deformation calculation in [10, Lemma 3.6] gives the trivial logarithmic tangent pullback, and the argument of [10, Lemma 3.9] lets generic such curves avoid every fixed codimension-two locus. Apply this to the centres of a common resolution with a compactification of X. Thus passing from the partial blowup to X adds no maps, removes no generic maps, and introduces no extra contact multiplicity. The necessity part of Proposition 5.2, which suffices for the all-label support bounds, needs no extra toric-end assertion.

C.6. **Finite expansions in the adjacent charts.** Let $\mathcal{C}_+$ be the chamber in which all ten character valuations are positive. For an incoming label m, the total correction is $b_1 n_1 + b_7 n_7 + b_z n_z + b_4 n_4 + b_t n_t$ with $b_i \geq 0$. The three functions x_{10}, z_2, y_3 give exactly the bounds (C.1), with A, B, C defined in the preceding Newton-coordinate calculation. These follow from the principal-divisor slope formula for regular source functions.

These functions also exclude a nonconstant tropical branch attached in $\mathcal{C}_+$. Indeed, such a branch would give $b \geq 0$, $b \neq 0$, and the outgoing slope of every regular function at its root is nonnegative. The three slopes in question are $-b_1 - b_7 - b_4$, $-b_1 - b_7 - b_z$, and $-b_t$, which force $b = 0$.

For each of the ten character coordinates, let $\mathcal{C}_i$ be the chamber where that coordinate is negative and all others are positive. Each entire orthant lies in one chamber of the three centre tropicalizations. The unique minimizing terms of the last two centres, and their slopes on b, give the following recession tests:

negative coordinate	x_{10} slope	z_2 slope	y_3 slope
σ	$-b_1 - b_7 - b_4 + b_t$	$-b_1 - b_7 - b_z + b_t$	$-b_t$
μ	$-b_1 - b_7 - b_4$	$-b_z$	$-b_t$
ν	$-b_1 - b_7 - b_4$	$-b_z$	$b_1 + b_7 - b_t$
ρ_i	$-b_1 - b_7 - b_4$	$-b_1 - b_7 - b_z$	$\langle \rho_i, \sum b_j n_j \rangle - b_t$.

Nonnegativity of each triple, together with $b \geq 0$, forces $b = 0$. Hence these chambers also contain no rooted twig and have bounded correction polytopes for every incoming label. The actual contact directions generate a free pointed correction monoid, including the bends of

infinitesimal cylinders. Lemma 5.1 therefore bounds the contributing classes. The specialized local expansions are Laurent polynomials in all eleven chambers. The straight coefficient is one.

In $\mathcal{C}_i$, the theta labels $-e_i$ and e_j for $j \neq i$ are straight. We give the extra argument needed to exclude an alternative bent spine, rather than merely observing that one straight ray stays inside the chamber. Trace a contributing spine backwards from its terminal point. Every intermediate exponent is its incoming label plus a nonnegative combination of (C.4). For $j \neq i$ these exponents have nonnegative j-component. Consequently those coordinates stay at least as large as their strictly positive values at the marked ends. The entire spine lies in $\mathcal{C}_i$, $\mathcal{C}_+$, or their common facet. Neither open chamber permits a bend. Every possible facet bend has zero i-component: on the σ facet the y_3 slope forces $b_t = 0$; on the μ facet the x_{10}, y_3 slopes leave only b_z; on the ν facet the x_{10} slope leaves only b_z, b_t; on a ρ_i facet the x_{10}, z_2 slopes leave only b_t. In each case the remaining directions have zero component normal to that facet. Thus the i-component of the spine direction is unchanged. It is -1 for $-e_i$, and zero for each unchanged coordinate e_j, so the backwards spine cannot reach the facet from its incoming end. There is no bend anywhere. The corresponding local expansions are therefore precisely z^{-e_i} and z^{e_j}. At a generic point of the common facet there is only the one integral hyperplane in question. The wall-crossing automorphism is therefore determined by its value on this one transverse monomial, by [10, Theorem 20.9 and Proposition 20.13].

Here are the required expansions in the positive chamber:

incoming label	Θ_m in $\mathcal{C}_+$	allowed contact types
$-e_\sigma$	E/s	$\varnothing, 1, 7, z, 4, \{4, z\}$
$-e_\mu$	$(1+Q)/u$	$\varnothing, z$
$-e_\nu$	$(1+s+Q)/v$	$\varnothing, z, t$
$-e_{\rho_i}$	$(1+s)/p_i$	$\varnothing, t.$

(C.6)

Every listed incidence scheme has length one. The formula for $-e_\sigma$ follows by placing an x-contact at $-\sigma$ and a z-contact at $-\sigma/((1+\mu)(1+\nu))$; the mixed $\{4, z\}$ contacts are distinct for generic data. The next and last cases have linear centre equations. For the third case the only additional possibility is $\{z, t\}$. With contact parameters r, t, subtraction of the two centre equations factors as

$$(r-t)\left[(1+\mu)(1-t)\left(1+\frac{t(\rho_4+\cdots+\rho_7)}{(1+\rho_1+\rho_2+\rho_3)(1-r)}\right)+\sigma r\right]=0.$$

All evaluation coordinates have positive valuation. Thus $1+\rho_1+\rho_2+\rho_3$ and $1-r$ are units, while $t(\rho_4+\cdots+\rho_7)/((1+\rho_1+\rho_2+\rho_3)(1-r))$ and σr have positive valuation. The bracket consequently has residue 1. Hence $r = t$, and the remaining centre equation gives $r = t = -\nu$. This collision is excluded. This proves (C.6).

The resulting wall functions are

$$E, \qquad 1+Q, \qquad 1+s+Q, \qquad 1+s \quad \text{(the last repeated seven times)}.$$

The source bounds and common compactification verify [7, intrinsic Laurent expansion and trace theorem]. The resulting faithful Laurent realization is compatible with the computed wall transformations. The associated normalizers are monomials in the unchanged variables, so these transformations give exactly the ten adjacent LP Laurent rings. Their straight coordinates form unimodular monomial systems; localizing the intrinsic algebra at them identifies the corresponding open tori. By the intersection calculation (C.3),

$$\Phi : A_{\mathrm{int}}(X)_1 \hookrightarrow \mathscr{B}_6 \subset \mathbb{C}[s^{\pm1}, u^{\pm1}, v^{\pm1}, p_1^{\pm1}, \ldots, p_7^{\pm1}]. \tag{C.7}$$

C.7. **Recognition of generators and markings by Laurent regularity.** The map (C.7) is already Laurent in the initial and ten adjacent charts. We apply [7, finite Laurent interpolation] to the following source-bounded supports, using the notation of §8.1.

A finite refinement of the source bounds. Write a contact vector in the order $\beta = (\beta_1, \beta_7, \beta_z, \beta_4, \beta_t)$. The three previously established bounds are

$$\beta_1 + \beta_7 + \beta_4 \le A(m), \qquad \beta_1 + \beta_7 + \beta_z \le B(m), \qquad \beta_t \le C(m).$$

They alone do not make every normalized regular lift unique. Four additional regular source coordinates suffice: x_3, x_5, x_9, x_{11}. Their mutable leading gradients in source order $(x_1, x_7, z_3, x_4, t_1)$ and the required tropical defects are

	$-h^{\mathrm{mut}}$	$\delta(g)$	$\delta(W_3)$	$\delta(W_9)$	$\delta(W_{12})$
x_3	$(1,1,1,0,1)$	1	2	2	2
x_5	$(1,0,0,0,1)$	1	1	1	2
x_9	$(2,1,1,1,1)$	2	2	3	2
x_{11}	$(1,1,0,1,1)$	2	1	2	2

Coefficient units in the W_i are removed. Substitution in the Cayley-plane quadrics gives this finite table. These source computations precede the mirror comparison. The coefficient equations resulting from the support bounds are reproduced in `verification/e6_recognition_certificates.json`; these calculations use the full ten-character lattice. The seven split unit directions are removed by homogeneity, not by specialization.

Each row supplies the inequality $-h_{x_i}^{\mathrm{mut}}\beta \le \delta_{x_i}(m)$. Together with the three coarse bounds, these inequalities give the following possible numerator supports. In every row the displayed incoming numerator coefficient is normalized to one.

function	possible numerator monomials	size	rank
f	$\operatorname{supp} E$	6	5
$\bar{u}$	$1, Q$	2	1
$\bar{v}$	$1, Q, s, sQ$	4	3
k	$1, s$	2	1
g	$\operatorname{supp} E \cup \{s, sR\}$	8	7
W_3	$\operatorname{supp} E \cup \{s, sQ\}$	8	7
W_9	$\operatorname{supp} E \cup \{s, sR, sQ, sRQ, suvL\}$	11	10
W_{12}	$\operatorname{supp} E \cup \{s, sQ, sR, suvL, s^2\}$	11	10

The first four rows already follow from the coarse bounds; the extra candidate sQ in the $\bar{v}$ row is removed by adjacent divisibility. Their denominators, in the same order, are

$$su, \quad u, \quad vQ, \quad P, \quad svT, \quad sup_3p_6p_7, \quad suP, \quad svp_2p_3p_4p_5p_6p_7^2.$$

Proposition C.1. *Each of these eight finite support spaces has a unique Laurent-regular function with incoming coefficient one. They are exactly the displayed functions $f, \bar{u}, \bar{v}, k, g, W_3, W_9/\zeta_9, W_{12}/\zeta_{12}$, and hence are the corresponding intrinsic theta expansions.*

Proof. The table gives the ranks of the Laurent-divisibility systems. Each rank is one less than the support size, as shown by a maximal minor ± 1. The normalized kernel is the displayed numerator. The exact matrices, minors and kernel vectors are in `verification/e6_recognition_certificates.json`. The finite Laurent interpolation criterion of [7] therefore applies. The eliminations can also be read directly: In each of the last four rows, the s^{-1} slice is divisible by E, fixing its six coefficients. The p_5^{-1} slice fixes the two remaining coefficients of g, and the p_3^{-1} slice does the same for W_3. The p_1^{-1} slice fixes the five remaining coefficients of W_9. For the last row, write

$$E + s(a_0 + a_Q Q + a_R R + a_L uvL) + a_2 s^2.$$

The p_7^{-2} slice requires $(1+s)^2 \mid 1 + a_0 s + a_2 s^2$, hence $a_0 = 2$ and $a_2 = 1$. The v^{-1} and p_2^{-1} slices then give $a_Q = a_R = a_L = 1$. The extra sQ allowed in the support of $\bar{v}$ is eliminated by divisibility by $1 + s + Q$.

Every intrinsic expansion belongs to its indicated space by the source bounds and the already established adjacent Laurentness. Uniqueness therefore identifies the eight

enumerative functions; no candidate expression is substituted for a theta function in setting up the calculation. □

Appendix D. Calculations for the Freudenthal mirror

D.1. **Newton characters and support bounds.** We use the following mutable and frozen orders:

$$(h, b, c, d, e, f, g, r, q, u), \qquad (a_0, \dots, a_{17}).$$

The following twenty characters are expressed in the source seed of Theorem 4.2.

i	χ_i	i	χ_i
1	$bda_4a_{17}/(qa_3a_{16})$	11	$hqa_5/(bdga_4)$
2	$hcra_{13}/(gua_{14})$	12	$fa_{10}/(ca_{13})$
3	$qa_{16}/(hba_{15})$	13	$bfa_7/(cea_{13})$
4	fa_4/r	14	$ra_3/(dga_4)$
5	$da_1a_6/(ea_3a_{16})$	15	$bdga_0/(qa_3a_{16})$
6	$fa_6a_{11}/(cea_{13})$	16	$bega_{15}/(fqa_5)$
7	$a_9a_{14}/(ca_{13})$	17	$dega_{12}/(fqa_5)$
8	$u/(hea_{12})$	18	$da_2/(a_3a_{16})$
9	$u/(hfa_8)$	19	$ga_8/(ca_5)$
10	$bega_{14}/(fqa_6)$	20	$fqa_5/(era_{13})$.

Follow these by $a_0, \dots, a_7$, and let C be the resulting exponent matrix. Exact integral elimination gives $\det C = 1$. The normalized support differences of the fifty-six linear coordinates have Hermite normal form equal to the 20×20 identity matrix. They therefore generate the full lattice of the χ_i.

All fifty-six coordinate expansions are positive. In the chamber $\mathrm{val}(\chi_i) > 0$, each has one coordinatewise least mutable exponent, with coefficient one and a fixed coefficient-unit exponent. Dividing by that term gives a polynomial in the χ_i. They also hold for $f^\sharp = F_f/(hf)$. The 56 expansions contain 6050 terms in total and satisfy all 144 cyclic translates of the defining quadrics.

Let H_ψ be the mutable exponent of the least term of one of these 57 functions, and let N_ψ be its normalized polynomial support in the χ_i. For the mutable part $m \in \mathbb{Z}^{20}$ of a label, set $\delta_\psi(m) = -\min_{a \in N_\psi} \langle a, m \rangle$. A permitted correction $\beta \in \mathbb{Z}^{10}_{\geq 0}$ satisfies $H_\psi \beta \geq -\delta_\psi(m)$ for all ψ. The product $G = x_{15}^2 x_2 y_{14} x_5 y_{11} x_{10} x_{13} x_0^2 y_1^3$ has mutable least exponent $(-1, \dots, -1)$, yielding the finite bound $\sum_{j=1}^{10} \beta_j \leq \delta_G(m)$ We enumerate the lattice points in this simplex and retain those satisfying all the inequalities.

For the ith primitive facet, use instead the axis degrees

$$d_{\psi i} = \max\{a_i : a \in N_\psi,\ a_j = 0\ (j \neq i)\}.$$

Impose $H_\psi \beta \geq -d_{\psi i}$ and the tangency equation $\sum_j C_{ij} \beta_j = 0$. The weighted sum obtained from G again bounds $\sum_j \beta_j$. Besides zero, the unique permitted vector is listed below. The symbols $e_h, \dots, e_u$ denote the standard basis in the ten original mutable directions.

i	β	i	β
$1, 5, 15, 18$	$e_h + e_u$	2	e_f
$6, 7, 12, 13, 20$	e_u	3	e_d
$11, 14$	$e_g + e_r + e_q$	4	e_g
$16, 17, 19$	e_r	8	$e_b + e_q$
		9	e_c
		10	$e_e + e_f + e_u$.

Mapping these two-point supports by C and clearing their monomial factors gives the ordinary mirror exchanges. Their coefficient-one normalization is proved in Proposition 8.4.

For the boundary label Ce_{a_j}, let m_j be its first twenty coordinates; its last eight coordinates give ϵ_j. The full normalized supports, not their axis restrictions, give:

j	0	1	2	3	4	5	6	7	8
$\delta_G(m_j)$	0	0	0	3	3	2	3	0	2
support size	1	1	1	3	2	3	5	1	3
j	9	10	11	12	13	14	15	16	17
$\delta_G(m_j)$	0	0	0	2	1	3	6	2	0
support size	1	1	1	2	2	5	5	2	1.

For a Laurent polynomial $P = \sum_k v_i^k P_k$ supported in one of these sets, adjacent regularity is equivalent to $E_i^{-k} \mid P_k$ for $k < 0$ in the Laurent ring of the unchanged coordinates. This gives a rational linear system. In every one of the eighteen cases its kernel is one-dimensional. Normalizing at the boundary label makes every coefficient one and yields precisely W_j. The normalizing monomials of the LP exchanges are units in these unchanged-coordinate Laurent rings and are included in the seed checks.

D.2. **Intrinsic comparison in the primary chart.** We use the primary coordinates and Newton characters introduced above. Put $t_i = \chi_i$ and $N = C_{\{1,\dots,20\},\{1,\dots,10\}}$. Its columns n_v, in the order $v = (h, b, c, d, e, f, g, r, q, u)$, are the contact directions of the ten mutable coordinate divisors. These columns span a saturated direct summand: the remaining eight rows of C are the original characters $a_0, \dots, a_7$, so are zero in the mutable columns. The degree-and-weight grading makes every normalized coordinate expansion independent of $a_0, \dots, a_7$. Since the character change is unimodular, those eight invertible characters split off in the localized coordinate ring. Thus $U_{\mathrm{F}} = X_7 \times \mathbb{G}_m^8$ The formulas below concern X_7; we restore the coefficient torus by [7, compatibility with torus products].

D.2.1. *The initial intrinsic chart.* Normalize the ordinary exchange F_v of the original seed by its coordinatewise least monomial in the characters $(t_i, a_0, \dots, a_7)$. The quotient is a polynomial $P_v(t)$ with positive coefficients and constant term one. It is independent of the $a_0, \dots, a_7$ and satisfies $\langle \alpha, n_v \rangle = 0$ for each exponent $\alpha \in \operatorname{supp} P_v$. Thus $P_v = 0$ is the equation of the blowup centre on the toric divisor in direction n_v. For instance,

$$P_h = 1 + t_2t_3t_4t_5t_{10}t_{14}t_{16}t_{20}, \qquad P_b = 1 + t_2t_4t_6t_8t_{10}t_{11}t_{17}, \qquad P_g = 1 + t_4.$$

This prescription and the original exchanges define all ten centres. They need not be expanded into their 87 monomials in the argument.

The coordinate-ring theorem and the logarithmic Jacobian identity for LP mutation give a volume-preserving initial torus. They identify the rank-one charts along the ten coordinate divisors with the corresponding toric blowup charts. Their centres are smooth on the relevant tori: each original exchange is primitive linear in an invertible coordinate with monomial leading coefficient. The charts cover codimension one. Resolving their birational identifications with a smooth compactification gives a common model; its discrepancy loci have codimension at least two.

Let G be the regular product in Proposition 8.3. Its mutable leading exponent is $-\mathbf{1}$. The regular-function slope inequalities therefore bound every contact vector at a fixed incoming label by a finite simplex. They also exclude a nonconstant branch rooted in the positive chamber: its outgoing contact vector would satisfy $-\mathbf{1} \cdot \beta \geq 0$, so $\beta = 0$; a branch with no contact is toric and constant. Every bend is a nonnegative combination of the independent n_v. Lemma 5.1 now bounds the complete numerical curve classes before specialization.

The intrinsic Laurent expansion and trace theorem of [7] now gives finite theta expansions with distinct coefficient-one least terms and a faithful algebra map

$$\Phi : A_{\mathrm{int}}(X_7)_1 \hookrightarrow \mathbb{C}[v_1^{\pm 1}, \dots, v_{20}^{\pm 1}].$$

This theorem includes the multiplication and trace comparison. The defects vanish on nonnegative coordinate labels, so their theta functions are the corresponding monomials.

Inverting the v_i therefore gives an open torus of the intrinsic spectrum. The next step is local coefficient recognition, before assuming any adjacent mirror chart.

D.2.2. *Initial forms and the eight mixed-contact calculations.* At facet i, the inequalities with the axis-degree bounds in Appendix D allow precisely the zero vector and the vector $\beta^{(i)}$ listed there. All coordinates of $\beta^{(i)}$ are zero or one. There are therefore twelve single-contact cases, five pairs, and three triples.

Lemma D.1. *The twelve single-contact directions have coefficient-one nonstraight terms. They are determined by the following six first-mutated source tori:*

mirror directions i	*source variable* v	*least monomial of* F_v	*source normalizer*
2	f	$a_{14}gu$	h
3	d	$a_{15}bgh$	bg
4	g	r	1
9	c	a_8fgh	g
$6, 7, 12, 13, 20$	u	$a_{13}cdeghr$	dg
$16, 17, 19$	r	$a_5cdfghq$	1

The last column records the associated normalizer M_v in $v' = F_v/(vM_v)$; it is retained in the original source ring.

Proof. Let μ_i be the actual first-mutant label and let P_v be F_v divided by its least character monomial. The source formulas and the full unimodular 28×28 matrix give

$$\operatorname{supp}\Theta_{\mu_i} \subseteq \{\mu_i, \mu_i + Ce_v\}, \qquad \min_{\alpha \in \operatorname{supp} P_v} \langle \alpha, \mu_i \rangle = -1.$$

For $g_v = v' \prod_{w \neq v} w$, its least exponent h_v satisfies $\langle h_v, Ce_v \rangle = -1$. The corrected term is therefore uniquely least at the required g_v-value, and [7, coefficient determination from an initial form] gives its coefficient one. The transverse normal pairing is one, so this determines the wall. The exact exponents, including all eighteen frozen characters and the associated source normalizers, are checked in `verification/e7_calibration.json`. □

For directions $1, 5, 8, 10, 11, 14, 15, 18$, the correction term has strictly larger value than the straight term for each of these ten valuations. We determine its coefficient by the following contact calculations.

We compute the remaining local counts over an algebraically closed, complete non-archimedean field of residue characteristic zero. Choose $\delta > 0$ and generic evaluation coordinates c_j with $\nu(c_i) = \delta$ and $\nu(c_j) > 100\delta$ for $j \neq i$. These valuations lie in a sufficiently small transverse neighborhood of a generic point of the facet. We choose the angular components outside the finitely many loci of cancellation in the coefficient sums. Proposition 5.2 gives the character map with evaluation c at 1:

$$t_j(z) = c_j z^{-\delta_{ij}} \prod_{v:\, \beta_v^{(i)} = 1} \left(\frac{z - r_v}{1 - r_v} \right)^{(n_v)_j}. \tag{D.1}$$

At contact v we impose $P_v(t(r_v)) = 0$. Its own factor drops out because P_v is invariant in direction n_v.

The infinitesimal spine has its bend at the chosen facet. Its unmarked contacts consequently have valuation δ. We will in fact check that the incidence equations have exactly one solution in the larger region $0 < \nu(r_v) < 2\delta$ and that this solution has all valuations δ. This excludes extra roots from the local count.

We check the end and compactification conditions of Proposition 5.2. The centre equations extend a solution across its distinct contacts in the rank-one charts. The two essential-end directions are $-e_i$ and $e_i - N\beta^{(i)}$. Neither is a blowup-centre ray; this follows directly from the columns of N. Thus both ends extend to the indicated open toric strata. The simple incidence solutions below give a generically étale evaluation map. The smooth-evaluation and

avoidance results [10, Lemmas 3.6, 3.9 and 8.21], on the common compactification, then give trivial logarithmic tangent pullback and avoidance of the fixed codimension-two loci. The preceding curve-class bound permits one common generic evaluation choice.

On the principal annulus the tropicalization is

$$q - \lambda e_i + \min(\lambda, \delta) N\beta^{(i)}, \qquad q = (\nu(c_j))_j.$$

Tangency $(N\beta^{(i)})_i = 0$ puts the two cuts at $0, 2\delta$ on opposite sides of the facet. There are no unmarked contacts on the two outside annuli. Thus the toric-tail condition holds, and deformation invariance identifies these reduced incidence solutions with the local cylinders of [10, Definition 20.2]. All contact types here are distinct, so the unordered-contact convention in Proposition 5.2 introduces no permutation quotient.

Pairs. The pairs are (h, u) at $i = 1, 5, 15, 18$, and (b, q) at $i = 8$. Write s, t for the first and second contacts. Grouping the two centre polynomials gives

$$1 + Ms^{-\epsilon}\frac{1-t}{s-t} = 0, \qquad U + \left(V + \frac{W}{t}\right)\frac{1-s}{t-s} = 0,$$

where $\epsilon = 1$ for $i = 5, 8$ and is zero otherwise. The coefficients have the form

$$V = LV_0, \qquad W = Lc_iW_0, \qquad M = \begin{cases} L\epsilon_0, & \epsilon = 0, \\ Lc_i\epsilon_0, & \epsilon = 1, \end{cases}$$

where U, V_0, W_0 have residue one and L, ϵ_0 are monomials in the other evaluation coordinates, of valuation greater than 100δ. More precisely,

i	L	ϵ_0
$1, 15, 18$	$c_{14}c_{16}c_{20}c_3c_4$	$c_{10}c_2c_5$
5	$c_{14}c_{16}c_{20}c_3c_4$	$c_{10}c_2$
8	$c_{11}c_{17}c_4$	$c_{10}c_2c_6$.

These factorizations follow by collecting the terms with the same powers of c_i and the other contact factor in P_v.

If $\epsilon = 0$, elimination gives

$$s = (1+M)t - M, \qquad t = -\frac{(1+M)W}{UM + (1+M)V}.$$

Both contacts have valuation δ, their difference is nonzero, and this is the unique solution. If $\epsilon = 1$, the second equation gives $s = \frac{Ut^2+Vt+W}{Ut+Vt+W}$. The first reduces, after removing the inadmissible factor $t - 1$, to

$$UVt^3 + (M(U+V)^2 + UW + V^2)t^2 + 2W(M(U+V) + V)t + (1+M)W^2 = 0.$$

Writing $\ell = \nu(L)$, the coefficient valuations, from constant to cubic, are $2\ell + 2\delta, 2\ell + \delta, \ell + \delta, \ell$. The Newton polygon gives one simple root of valuation δ and two roots of valuation $(\ell+\delta)/2 > 2\delta$. The admissible root gives $\nu(s) = \delta$, and $s - t = -M(1-t)/s \neq 0$. Thus each of the five pair cases contributes one.

Triples of types g, r, q. These occur at $i = 11, 14$. Put $a = c_4$, and use contacts s, r, t of types g, r, q. The g-equation gives $s = (1+a)r - a$. Set $V = (r-t)/(1-t)$ and $\widetilde{V} = (t-r)/(1-r)$. The identities obtained by this substitution in the other two centres are

i	r-equation	q-equation
11	$A + K(B + Y/r)/V = 0$	$1 + c_{11}a(B + Y/t)/(t\widetilde{V}) = 0$
14	$A(r) + K(r)(B + L/r)/V = 0$	$1 + c_{11}a(B + L/t)/\widetilde{V} = 0$.

For $i = 11$, the quantities satisfy

$$A \equiv 1, \qquad K = \frac{a(1 + c_{14}(1+a))}{1+a}, \qquad \nu(B) > 100\delta, \quad \nu(Y/B) > 100\delta.$$

For $i = 14$, they satisfy

$$A(r) = A_0 + T/r, \quad A_0 \equiv 1, \quad K(r) = \frac{a}{1+a}\left(1 + \frac{(1+a)c_{14}}{r}\right),$$

with $\nu(B), \nu(L/B), \nu(T) > 100\delta$. Here congruences mean equality in the residue field. The coefficients are defined by regrouping the normalized source exchange polynomials as in the table. The coefficient groups are obtained directly from the source exchange polynomials, with all eighteen frozen variables independent; see `verification/e7_calibration.json`. The valuation estimates use their monomial factors and generic nonzero leading coefficients.

For both facets, let $H(r)$ be the nonconstant term of the r-equation before division by V, divided by A or $A(r)$. Then $t = \frac{r+H(r)}{1+H(r)}$. Throughout $0 < \nu(r) < 2\delta$, we have $\nu(H(r)) > 2\delta$. Thus t and s have the same valuation as r. At $i = 11$, the q-equation becomes

$$t = -c_{11}\frac{aA}{K}(1+H)\frac{B + Y/t}{B + Y/r}.$$

The factor after $-c_{11}$ has residue one; hence $\nu(r) = \delta$. After writing $r = c_{11}R$, the equation reduces to $R + 1 = 0$ modulo the maximal ideal. All its remaining coefficients, and their first derivatives on the unit annulus, have positive valuation. Hensel's lemma gives exactly one solution.

At $i = 14$, the equation instead becomes

$$1 + \frac{(1+a)c_{14}}{r} = -c_{11}(1+a)A(r)(1+H)\frac{B + L/t}{B + L/r}.$$

Its right side has positive valuation, so again $\nu(r) = \delta$. Writing $r = c_{14}R$ reduces it to $R + 1 = 0$, with the same strict Hensel condition. Thus there is precisely one solution. In both cases $s - r = -a(1-r) \neq 0$. The other gap $t - r$ is nonzero and has strictly larger valuation, so the three contacts are distinct. The elimination and the simple Hensel root also prove reducedness.

The triple e, f, u. This occurs at $i = 10$. Set $z = c_{10}$, $a = c_2$, and let s, t, w be the three contacts. Exact grouping of the centre polynomials gives

$$\begin{aligned}
&(s + zaZ)(s - w) + zaA(1 - w) = 0,\\
&t(t - w) + a(B + zY/t)t(1 - w) = 0,\\
&B + zY/w + (C + zLY/w)V + AV/U + DV^2 = 0,
\end{aligned}$$

where $U = (w - s)/(1 - s)$ and $V = (w - t)/(1 - t)$. Here B has residue one, and a, A, Y, Z, C, D, L have valuation greater than 100δ. Also $\nu(Z/A) > 100\delta$: the polynomial A contains $c_{13} + c_6$, whereas $Z = c_{11}c_{17}c_4c_8(c_{13} + c_6)$. The exact coefficients are obtained as follows: the terms of P_e are $1 + zaZ/s + zaA/(sV_e)$, the terms of P_f are $1 + a(B + zY/t)/V_f$, where $V_e = (s - w)/(1 - w)$ and $V_f = (t - w)/(1 - w)$. The six coefficient groups of P_u are the six terms in the last equation above. This determines the notation without expanding all the centres.

For $0 < \nu(w) < 2\delta$, the first two equations each have one solution in that annulus, with $s = w +$ terms of valuation greater than 2δ and the same assertion for t. Their other quadratic roots have valuation greater than 2δ. Moreover

$$U = \frac{zaA}{s + za(Z + A)}, \qquad V = \frac{a(B + zY/t)}{1 + a(B + zY/t)}.$$

Substitution in the third equation gives $B(1 + w/z)$ plus terms of strictly larger valuation. Thus any solution in the annulus has $\nu(w) = \delta$. For existence and simplicity, put

$$w = zW, \qquad s = w + aAS, \qquad t = w + aT.$$

The first two equations, divided by zaA and at, and the third multiplied by S, reduce to

$$WS + 1 = 0, \qquad T + 1 = 0, \qquad S + T = 0.$$

All omitted coefficients have positive valuation, and the denominators are units at $(W, S, T) = (-1, 1, -1)$. The reduced Jacobian has determinant -1. Multivariate Hensel lifting therefore gives one simple solution. Conversely, the first two equations force S, T to be units for every admissible solution, so this is exhaustive. Finally, $\nu(s - w) = \nu(a) + \nu(A) > \nu(a) = \nu(t - w)$, and the contacts are distinct.

Proof of Proposition 8.4. The support bounds exhaust all possibilities. Lemma D.1 settles the twelve single-contact directions. The five pair and three triple calculations above give one reduced solution each; their interpretation by (D.1) and the toric-tail condition gives the remaining local cylinder coefficients. □

D.2.3. *The first mutations and boundary functions.* By wall-crossing consistency [10, Proposition 20.35], the initial intrinsic expansions continue by these binomial wall transformations. The positive-series argument of Lemma 5.3 proves that the continuations are Laurent polynomials. The image of Φ is consequently contained in the twenty-one-chart intersection, which is the upper LP algebra.

Put $b_i = N\beta^{(i)}$, and let M_i be the normalizing monomial of the ith mirror exchange. Write $\mathbf{m}_i$ for its exponent vector. The first-mutation label is $m_i' = -[b_i]_- - \mathbf{m}_i - e_i$, where $[b]_-$ denotes the coordinatewise minimum of b and zero. The full support inequalities at m_i' have exactly two solutions: 0 and $\beta^{(i)}$. The bound on the sum of their entries is $1, 2$, or 3, according to the number of contacts in $\beta^{(i)}$. Adjacent Laurent divisibility has rank one on this two-dimensional monomial space and fixes the second coefficient to one. Thus every first mutation v_i' is itself a theta function in this chart. There is no additional correction term.

For the eighteen boundary labels, the complete bounds and one-dimensional regularity spaces are given in Appendix D. The same reasoning identifies each intrinsic boundary function with the stated W_j. There is a particularly simple check of uniqueness. Every negative coordinate exponent is -1. In each binomial-divisibility test, the terms group in pairs differing by an odd multiple of the exchange exponent difference, so their coefficients must agree. The resulting graph on the allowed monomials is connected for every boundary function. Thus all coefficients are forced to equal the straight coefficient one. The eighteen coefficient matrices, normalized kernels, and maximal minors equal to ± 1 are given in `verification/e7_boundary_certificates.json`. We use the full normalized supports of the 57 regular source functions. The source inequalities bound the supports, and the Laurent divisibility equations determine their coefficients over $\mathbb{Q}$.

Restoring the coefficient torus, let M be the intrinsic image. We have

$$\mathscr{S}_7[v_i, v_i', W_j] \subseteq M \subseteq \mathcal{U}(\Sigma_7^\vee).$$

Inverting the product of all v_i gives the initial torus. With $d_i = \prod_{j \neq i} v_j$, localization of the starfish intersection gives

$$M[d_i^{-1}] = \frac{\mathscr{S}_7[v_j^{\pm 1} : j \neq i][v_i, V_i]}{(v_i V_i - E_i)},$$

where V_i is the monomial multiple of v_i' that clears its normalizer. These are actual intrinsic open sets. The remaining problem is whether they meet every prime divisor.

D.2.4. *A linear test for all prime divisors.* Write

$$\mathscr{F} = \{v_1', \dots, v_{20}', \overline{W}_0, \dots, \overline{W}_{17}\},$$

where the coefficient unit has been removed from W_j. There are 24 distinct functions in this list. They involve only the mutable coordinates and have at most five terms each. The factors remain irreducible after adjoining the eight independent coefficient characters, on which the valuations vanish. The logarithmic form then includes their eight $d\log$ factors, while the inequalities still involve only the twenty mutable values.

For a homogeneous face factor $q = Av_p + B$, with $A = cv^a$ a monomial, choose $\alpha \in \operatorname{supp} q$. For each $f = v^\ell P(v) \in \mathscr{F}$, let $d = \deg_{v_p} P$ and collect

$$A^d P(v_p = (u - B)/A) = \sum_{j \geq 0} u^j R_j(v_{\neq p}).$$

By [7, divisorial inequalities for a linear centre], regularity forces

$$\langle \ell + e - da + j\alpha, w \rangle \geq -j \qquad (e \in \operatorname{supp} R_j). \tag{D.2}$$

This necessary condition follows from the divisorial valuation formula, with coefficients collected before substitution. We apply it to the following finite list of face factors.

Proof of Proposition 8.5. We use the ordinary exchanges, their associated normalizers, and the eighteen boundary functions from Section 8.2.

First, if every monomial in every $f \in \mathscr{F}$ has nonnegative value, a nonnegative rational combination of these inequalities gives $\sum_i w_i \leq 0$. Hence $w = 0$. Otherwise [7, divisorial valuation formula] gives an irreducible factor q of a negative face. To include every possible face, take *all* nonempty subsets of each support, rather than finding the normal fans. After removing monomial factors, the 100 nonmonomial subset checks give 80 distinct irreducible factors. Each is primitive linear in some v_p, with monomial leading coefficient coprime to its constant coefficient. This proves absolute irreducibility and gives the hypothesis of [7, divisorial inequalities for a linear centre].

Every irreducible factor of a weighted-homogeneous Laurent polynomial is weighted-homogeneous: the connected grading torus cannot permute its finite set of prime factors. For each q, we therefore impose the homogeneity equations $\langle \alpha - \alpha', w \rangle = 0$ for its exponents, and impose (D.2) for every $f \in \mathscr{F}$. The resulting rational inequalities give

$$\sum_i w_i \leq \begin{cases} 0, & \text{for seventy factors,} \\ 1, & \text{for the other ten.} \end{cases}$$

The ten latter factors are exactly the ten distinct mirror exchanges. Write the inequalities as $A_q w \geq b_q$ and the homogeneity equations as $H_q w = 0$. Each certificate supplies rational vectors $\lambda_q \geq 0$ and μ_q with

$$\lambda_q^t A_q + \mu_q^t H_q \leq -(1, \ldots, 1), \qquad -\lambda_q^t b_q \in \{0, 1\}.$$

Since $w \geq 0$, this gives exactly the asserted bound. The file `verification/e7_divisor_certificates.json` contains all matrices and rational multipliers. The ancillary calculation constructs the factors and substitution matrices from the stated Laurent functions, then verifies each identity in rational arithmetic. Since w is integral, the resulting bounds leave only $w = 0$ or a coordinate vector. □

Ancillary calculations. The accompanying files give the Cayley-plane coefficient systems, the Freudenthal boundary systems, the initial-form calculations, the 56 source-coordinate expansions, and the divisorial inequalities. They also contain exact checks of the 144 defining quadrics, the first mutations, the 204 adjacent-chart divisibility conditions, and the primality calculations. The conventions and verification instructions are given in `verification/README.md`.

Declaration of generative AI and AI-assisted technologies

During the development of the paper, the author used ChatGPT to assist in exploring and formulating most proof arguments. The author subsequently checked, revised, and independently verified all arguments and takes full responsibility for the mathematical content of the paper.

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