hexagon:2611.00326v1

Marked-point modules and central nilpotents at roots of unity

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Abstract

We study extensions in the two-marked-point representation of the periodic Temperley–Lieb algebra at $q = e^{i\pi/p}$, $p \in \mathbb{Z}_{\geq 3}$. For every positive even size N, we construct a submodule Z_N extending $T_N = W_{N;2,\varepsilon}$ by $W_N = W_{N;0,\varepsilon q}$, where $\varepsilon = \pm 1$. The dual trees of noncrossing matchings give an explicit injection $\Theta_N : T_N \to W_N$. Both central braid transfer elements have nilpotent part $g\Theta_N\pi_N$ on Z_N, where π_N is the quotient map and $g = x + x^{-1} + \varepsilon(y + y^{-1})$ for the marked-point parameters $x, y \in \mathbb{C}^\times$. The extension splits exactly when $g = 0$, and the formula determines the central Jordan forms at every even size. In the window $2 \leq N < 2p - 2$, Z_N is a joint central summand and, when $g \neq 0$, has simple layers $T_N/S_N/T_N$. An explicit intertwiner and annular localization identify its two-layer quotient with a periodic XXZ sector. The associated basic image algebras have dimensions five and four; the quotient preserves first extensions between simples and kills a nonzero Yoneda product in degree two. An annular operator detects the extension with coefficient $2\varepsilon g$. Beyond the isolated window, we determine a five-simple image algebra and compute a 43-dimensional joint central summand at $p = 3, N = 6$, on which the two central nilpotents generate a four-dimensional algebra. Exact matrix calculations and verification code accompany these finite computations.

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1 Introduction and main results

Periodic Temperley–Lieb representations describe connectivity sectors, twisted spin chains, and bulk operator insertions. At roots of unity, their structure is not determined by their simple composition factors: the extensions between those factors and the nilpotent action of central elements must also be specified. Two-marked-point link states provide a setting in which these data can be calculated together.

Graham and Lehrer developed the standard modules and their successor homomorphisms; Erdmann and Green studied finite quotients and extensions [1, 2]. Pinet and Saint-Aubin determined the Loewy structures of periodic XXZ sectors, including the root-of-unity cases [8]. Together with the standard-module structure, their result supplies the two opposite extensions used in our isolated-channel comparison.

Ikhlef and Morin-Duchesne introduced two-marked connectivity modules and analyzed periodic fusion [3, 4]. Their non-generic analysis for q away from roots of unity produces a three-layer module and a parameter-dependent central Jordan action. At a root of unity, additional standard constituents can enter the same central spectrum, and a generic spectral decomposition need not specialize term by term. We give a direct construction of the first marked channel at the root of unity itself. Its maps are described by two-node diagrams and the incidence matrices of finite trees.

Fix

$$q = e^{i\pi/p}, \qquad p \in \mathbb{Z}_{\geq 3}, \qquad \beta = -q - q^{-1}, \qquad \varepsilon \in \{1, -1\}, \qquad w = \varepsilon q,$$

and let $x, y \in \mathbb{C}^\times$. Set

$$f = \varepsilon(q + q^{-1}), \qquad g = x + x^{-1} + \varepsilon(y + y^{-1}).$$

For $N = 2n$, put $C_n = (n+1)^{-1}\binom{2n}{n}$ and abbreviate

$$W_N = W_{N;0,w}, \qquad T_N = W_{N;2,\varepsilon}, \qquad Y_N = Y_{0,0,x,y,[z,w]}(N).$$

The last module is independent of z.

Theorem 1.1 (The first marked channel)**.** *For every even $N \geq 2$, there is an explicitly based submodule $Z_N \subset Y_N$ and a natural injective standard homomorphism $\Theta_N : T_N \to W_N$ such that*

$$0 \longrightarrow W_N \longrightarrow Z_N \xrightarrow{\pi_N} T_N \longrightarrow 0, \qquad (F - f)|_{Z_N} = (\overline{F} - f)|_{Z_N} = g\Theta_N\pi_N.$$

This sequence splits exactly when $g = 0$. If $g \neq 0$, the Jordan form of each central element is

$$J_2(f)^{\oplus nC_n} \oplus J_1(f)^{\oplus C_n}.$$

If $2 \leq N < 2p - 2$, then Z_N is the joint generalized (f, f)-eigenspace in Y_N, T_N is simple, and W_N has simple head S_N and radical $\Theta_N T_N$. For $g \neq 0$, the module Z_N is uniserial with layers $T_N/S_N/T_N$.

Here and below, layers in a displayed quotient notation are read from head to socle. The first part of the theorem holds at arbitrarily large even sizes with p fixed. The size restriction is used to identify simple layers and to separate the channel as a summand of the full marked module. The proof is given in Section 3.

The three-layer module changes the image algebra in a precise way. Write $P_N = Z_N$ in the nonsplit isolated case and $E_N = P_N/\operatorname{soc} P_N$.

Theorem 1.2 (Finite images and the lifting obstruction)**.** *Under the isolated nonsplit assumptions of Theorem 1.1, one has*

$$E_N \cong X^{+}_{N;0,\varepsilon q}.$$

Let B_N be the image algebra on P_N and R_N its quotient acting on $W_N \oplus E_N$. Their basic algebras are

$$B_0 = \mathbb{C}Q/(ba), \qquad R_0 = \mathbb{C}Q/(ab, ba), \qquad Q: \; S \xrightarrow{a} T \xrightarrow{b} S,$$

where products are compositions, with the rightmost arrow traversed first. The basic kernel $\mathbb{C}ab$ is central and has square zero. Inflation from R_N to B_N is an isomorphism on first extensions between simple modules. If $\alpha \in \mathrm{Ext}^1(S_N, T_N)$ and $\xi \in \mathrm{Ext}^1(T_N, S_N)$ are the two nonzero classes, then

$$\alpha \circ \xi \neq 0 \text{ over } R_N, \qquad \alpha \circ \xi = 0 \text{ over } B_N.$$

The latter equality is realized by $0 \to W_N \to P_N \to T_N \to 0$.

The comparison with the XXZ module is proved through a full annular corner and an explicit two-node intertwiner. If $s = \dim S_N = C_n$ and $t = \dim T_N = nC_n$, the physical image algebras have dimensions

$$\dim B_N = s^2 + 2st + 2t^2, \qquad \dim R_N = (s+t)^2.$$

Their quotient kernel has dimension t^2 and is generated by a central nilpotent. For $t > 1$, only its one-dimensional scalar subspace is central; in the basic algebra the entire kernel is central.

We also compute a normalized annular extension datum. Cup and cap maps from two nodes to N transport $e_1 - e_0$ to an operator annihilating both endpoints of $0 \to S_N \to E_N \to T_N \to 0$. Its action on a specified lift is $2\varepsilon g$ times a specified nonzero lower vector. For $p = 5, N = 6, x = 1, y = 2, \varepsilon = 1$, the central coefficient is $9/2$ and this annular coefficient is 9.

For a specified standard–XXZ generating list at $p = 3, N = 6$, we determine a five-simple image algebra of dimension 328, including its quiver, quadratic relations, projective covers and second extensions. Adding the inverse-twist XXZ sector changes the image dimension to 397 and changes its Morita type. The non-isolated marked module exhibits a further change. For $p = 3, N = 6, x = 1, y = 2, w = q$, exact diagrammatic matrices give a 43-dimensional joint central summand. On it, F has Jordan form

$$J_3(1)^{\oplus 2} \oplus J_2(1)^{\oplus 17} \oplus J_1(1)^{\oplus 3}.$$

The restrictions of the two central elements generate

$$\mathbb{C}[u, v]/(u^3, uv, v^2), \qquad u = F - 1, \quad v = F - \overline{F}.$$

The 35-dimensional depth-one submodule of Theorem 1.1 persists inside this larger summand. The matrices, exact rank calculations, and an independent verification program are provided with the paper.

Ikhlef and Morin-Duchesne construct a distributive, commutative, associative fusion bifunctor on families of periodic modules [6]. Our cup and cap maps act on families in the same diagrammatic category. The uncoiled quotients of Langlois-Rémillard and Morin-Duchesne [7] give another class of finite algebras; our image-algebra arguments use faithfulness and module structure, without a cellularity assumption. The ADE construction of [5] associates local operators to irreducible constituents. Here the annular coefficient is defined on a specified extension with fixed endpoint vectors; comparison with a continuum logarithmic coupling requires additional normalization and limiting data.

Section 2 fixes the diagrammatic and twist conventions. Section 3 constructs the marked extension and computes its central action. Section 4 identifies its finite image algebras and their extension groups. Sections 5 and 6 treat the five-simple image and the exact marked-module calculations. Section 7 discusses higher channels and module families beyond the isolated window.

2 Annular diagrams, twists and standard modules

Let $\mathscr{D}_\beta$ be the complex linear annular Temperley–Lieb category on nonnegative even objects. A morphism $M \to N$ is a noncrossing connectivity diagram in an annulus with M nodes on its inner boundary and N nodes on its outer boundary. Contractible closed curves are evaluated as β; essential closed curves are retained as diagrams. Composition is concatenation. Write $\mathsf{A}_N = \mathrm{End}_{\mathscr{D}_\beta}(N)$. This diagrammatic convention includes $N = 0, 2$ without importing relations valid only at larger sizes.

For $N \geq 4$, the local cup–cap diagrams e_i and the rotation Ω obey

$$e_i^2 = \beta e_i, \qquad e_i e_{i\pm1} e_i = e_i, \qquad e_i e_j = e_j e_i \quad (i - j \not\equiv \pm 1 \pmod N), \qquad \Omega e_i \Omega^{-1} = e_{i-1}.$$

Indices belong to $\mathbb{Z}/N\mathbb{Z}$. We use the enlarged periodic diagram algebra, also denoted $\mathrm{EPTL}_N(\beta)$ in the marked-module literature.

An annular module family is a collection of spaces M_N equipped with the action of every morphism in $\mathscr{D}_\beta$. If $I : M \to N$ adds $k = (N - M)/2$ adjacent cups and $C : N \to M$ is its reflected cap diagram, then

$$CI = \beta^k 1_M. \tag{2.1}$$

Thus I is injective on every such family when $\beta \neq 0$. This elementary contraction will be used both for the marked modules and for the spin-chain family.

2.1 Standard labels and actual twists

The standard module $W_{N;d,z}$ has d defects, $0 \leq d \leq N$ even, and twist $z \in \mathbb{C}^\times$. Its dimension is

$$\dim W_{N;d,z} = \binom{N}{(N-d)/2}.$$

For $d = 0$ the noncontractible-loop value is $z + z^{-1}$ and $W_{N;0,z} \cong W_{N;0,z^{-1}}$. The standard and simple label set therefore identifies $(0, z)$ with $(0, z^{-1})$. A spin-chain label $X^+_{N;d,z}$ retains the actual complex number z. We impose no inversion identification on this latter label set.

The successor homomorphisms are controlled by $t = d + 2r > d$ and

$$\begin{array}{ll} \text{(A)} & z^2 = q^t, \quad u = zq^{-r}, \\ \text{(B)} & z^2 = q^{-t}, \quad u = zq^{r}. \end{array} \tag{2.2}$$

At the parameters used here, the standard modules have simple heads and multiplicity-one composition factors described by their successor closures. The maps $W_{N;t,u} \to W_{N;d,z}$ associated with successors are injective. We use these standard facts in the form of [8, Theorem 2.2 and Proposition 3.2]. Actual Loewy arrows must be read from the corresponding module structure; the two successor conditions alone do not prescribe XXZ arrow directions.

For any finite list of actual seed twists, take the direct sum of the specified standard and XXZ modules and form its finite image algebra. Its accessible simple labels are taken in the standard label set, while its generators retain their actual twists. Corollary 4.2 proves the associated block statement. This convention also allows us to specify individual finite images by an explicit list of modules in the non-isolated example.

2.2 The two central braid transfer elements

Set $b_i = q^{1/2}1 + q^{-1/2}e_i$, so that $b_i^{-1} = q^{-1/2}1 + q^{1/2}e_i$. Let $I_{\mathrm{aux}} : N \to N+2$ append an empty adjacent cup, and let $C_{\mathrm{seam}} : N+2 \to N$ cap the last and first nodes and retain the intervening nodes in their cyclic order. With indices $0, \dots, N+1$ on the intermediate boundary, define

$$F_N = C_{\mathrm{seam}} b_0 b_1 \cdots b_{N-1} I_{\mathrm{aux}}, \qquad \overline{F}_N = C_{\mathrm{seam}} b_0^{-1} b_1^{-1} \cdots b_{N-1}^{-1} I_{\mathrm{aux}}. \tag{2.3}$$

The rightmost factor acts first. Since N is even, the total powers of q in the skein expansion are integral. The braid and cup–cap identities allow this closed auxiliary strand to pass across every annular diagram. Consequently these two elements are central and form natural endomorphisms of every diagrammatic module family. These are the central braid transfer elements used in [4, Section 2.3]. Their action on standards is recorded in (3.15).

Formula (2.3) is also the definition used in the accompanying computation. It specifies the actual diagram operator before any spectral decomposition is performed.

3 The first marked channel at every even size

Put $q = \exp(i\pi/p)$ with $p \in \mathbb{Z}_{\geq 3}$, and write

$$c = q + q^{-1}, \qquad \beta = -c, \qquad w = \varepsilon q, \qquad f = \varepsilon c, \qquad \varepsilon \in \{1, -1\}.$$

Thus $c \neq 0$ and $\beta \neq 0$. Fix $x, y \in \mathbb{C}^\times$ and set

$$A = x + x^{-1}, \qquad B = y + y^{-1}, \qquad g = A + \varepsilon B = \frac{(y + \varepsilon x)(\varepsilon x y + 1)}{xy}. \tag{3.1}$$

We use the diagram category and the endomorphism algebras A_N of Section 2. Cup and cap diagrams act between the different sizes in each module family.

For even $N = 2n \geq 2$, abbreviate

$$Y_N = Y_{0,0,x,y,[z,\varepsilon q]}(N), \qquad W_N = W_{N;0,\varepsilon q}, \qquad T_N = W_{N;2,\varepsilon}.$$

The parameter z does not occur in Y_N. The defining loop weights are β, A, B, f according as a removed loop surrounds neither marked point, only the first, only the second, or both. This is the two-marked module of [4, Sections 3.2–3.3].

3.1 A basis described by faces

Let $\mathcal{M}_n$ be the noncrossing perfect matchings of the $2n$ boundary nodes of a disc. For $\pi \in \mathcal{M}_n$, write $\mathcal{F}(\pi)$ for its faces and D_π for its dual graph. The graph D_π is a tree with $n+1$ vertices and n edges. A basis of Y_N is

$$v_{\pi;a,b}, \qquad \pi \in \mathcal{M}_n, \quad a, b \in \mathcal{F}(\pi).$$

The two marked points lie in the ordered pair of faces (a, b). The depth of this state is the distance between a and b in D_π. Let $Y_N^{\leq r}$ be the span of the states of depth at most r. Concatenation cannot increase depth, so these spaces are submodules. The diagonal states $v_{\pi;a,a}$ identify $Y_N^{\leq 0}$ with W_N.

The descriptions by faces and by link states are identical: after forgetting the two marked points, the remaining curves give π, and their isotopy class with the marked points retained is determined by the two faces. In particular,

$$\dim Y_N = (n+1)^2 C_n, \qquad \dim W_N = (n+1) C_n, \qquad C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Lemma 3.1 (The first depth quotient)**.** *For every value of β, there is a decomposition*

$$Y_N^{\leq 1}/W_N \cong W_{N;2,1} \oplus W_{N;2,-1}. \tag{3.2}$$

Modulo W_N, the interchange $v_{\pi;a,b} \mapsto v_{\pi;b,a}$ commutes with the annular action. Its ε-eigenspace is the summand $W_{N;2,\varepsilon}$.

Proof. At depth one the two faces share an edge of D_π. Cut the corresponding arc of π to obtain two defects. The two possible orders of the faces specify the two orders of these defects at the puncture. Changing this order is the inner rotation of the two defects. In the quotient by depth zero, any contraction involving a loop surrounding one marked point vanishes, because it lowers the depth. The surviving concatenations use only the contractible-loop weight β and the inner rotation. Thus this quotient is the two-defect link space with the regular representation of the two-element rotation group at its inner boundary. The two Fourier projections $(1 \pm R)/2$ give the two standard modules in (3.2). This argument uses a change of basis over $\mathbb{C}$ and no rational function of q. □

Choose one orientation (a, b) for each edge of each D_π and put

$$t_{\pi;a,b} = v_{\pi;a,b} + \varepsilon v_{\pi;b,a}, \qquad r_{\pi;a,b} = v_{\pi;a,a} + \varepsilon v_{\pi;b,b}. \tag{3.3}$$

Reversing the chosen orientation multiplies both vectors by ε. Define Z_N to be the inverse image of the ε-summand in (3.2). It is a submodule with the explicit basis

$$\{v_{\pi;a,a}\}_{\pi,a} \;\cup\; \{t_{\pi;a,b}\}_{\pi,\{a,b\}\in E(D_\pi)}. \tag{3.4}$$

In particular,

$$0 \longrightarrow W_N \longrightarrow Z_N \xrightarrow{\pi_N} T_N \longrightarrow 0, \qquad \dim Z_N = (2n+1)C_n. \tag{3.5}$$

We normalize the standard basis vector associated with the cut edge $(\pi; a, b)$ to be $\pi_N(t_{\pi;a,b})$. At object zero we set $T_0 = 0$ and $W_0 = Z_0 = Y_0$, the one-dimensional empty-matching space. These conventions include caps from two nodes to zero in the module families.

3.2 The two-node calculation

At $N = 2$, write the basis as (a, b, u, v), with a, b the two diagonal-face states and u, v the two ordered pairs of distinct faces. Choose the labelling so that $\Omega a = b$, $\Omega u = v$. Direct concatenation gives

$$e_1 = \begin{pmatrix} \beta & f & B & A \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad e_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ f & \beta & A & B \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.6}$$

For the two central braid transfer elements of [4, Section 2.3], the same concatenation gives

$$F_2 = \overline{F}_2 = \begin{pmatrix} f & 0 & A & B \\ 0 & f & B & A \\ 0 & 0 & 0 & c \\ 0 & 0 & c & 0 \end{pmatrix}. \tag{3.7}$$

These matrices are obtained by evaluating the four loop weights directly at the specified root of unity.

Set $r = a + \varepsilon b$ and $t = u + \varepsilon v$. Then

$$\begin{aligned} W_2 = \mathbb{C}a + \mathbb{C}b, \qquad Z_2 = W_2 + \mathbb{C}t, \\ e_0 r = e_1 r = 0, \quad \Omega r = \varepsilon r, \quad e_1 t = \varepsilon g a, \quad e_0 t = g b, \quad \Omega t = \varepsilon t, \\ (F_2 - f)t = (\overline{F}_2 - f)t = gr, \qquad (F_2 - f)W_2 = (\overline{F}_2 - f)W_2 = 0. \end{aligned} \tag{3.8}$$

The radical of W_2 is $\mathbb{C}r$. Its quotient is the one-dimensional module with $e_0 = e_1 = \beta$ and $\Omega = -\varepsilon$.

3.3 Insertion and the central nilpotent

The annular category acts on the families Y_N and on the standard families. The braid push-through identities give, for every annular diagram $D : M \to N$,

$$F_N D = D F_M, \qquad \overline{F}_N D = D \overline{F}_M. \tag{3.9}$$

Every depth-zero or depth-one link state is obtained from a two-node state by such a diagram. One way to see this is to remove cups bounding an unmarked leaf of the dual tree until two boundary nodes remain. For a state of depth at most one, every removed leaf can be chosen away from the path between the marked faces. Reinserting the cups recovers the original state. Equivalently, an edge of D_π can be cut to a two-defect diagram $D : 2 \to N$; with suitable ordering,

$$Da = v_{\pi;a,a}, \quad Db = v_{\pi;b,b}, \quad Du = v_{\pi;a,b}, \quad Dv = v_{\pi;b,a}.$$

For a diagonal state, choose any edge incident to its marked face; such an edge exists for $N \geq 2$. The same insertion realizes the state as Da or Db. This also proves generation at depth zero.

Lemma 3.2 (The first standard inclusion)**.** *There is a natural injective homomorphism*

$$\Theta_N : T_N \longrightarrow W_N, \qquad \Theta_N\big(\pi_N(t_{\pi;a,b})\big) = r_{\pi;a,b}. \tag{3.10}$$

It is the Graham–Lehrer successor homomorphism, normalized at two nodes by $\Theta_2(\pi_2(t)) = r$. In compositions with π_N we view its image inside $W_N \subset Z_N$ and suppress this inclusion.

Proof. At two nodes the vector r is annihilated by both caps and has rotation eigenvalue ε, by (3.8). The cap assertion follows also from $e_j = c_j^\dagger c_j$ and the injectivity of $c_j^\dagger$, since $c_j c_j^\dagger = \beta \mathbf{1}$ and $\beta \neq 0$. The defining presentation of the two-defect standard module therefore allows one to replace its two defects by r. Concatenation gives the indicated natural homomorphism. This is also the specialization of the successor inclusion in [8, Proposition 3.2]; here the insertion has two terms and no quantum-integer denominator.

For a direct proof of injectivity, fix π. In the diagonal-face basis of W_N, the columns of this map are $v_a + \varepsilon v_b$ for the edges of the tree D_π. They are linearly independent: a leaf occurs in only one column, so its coefficient first vanishes in any linear dependence; remove that edge and continue by induction. The subspaces for distinct matchings are disjoint. Consequently all columns of Θ_N are independent. □

Theorem 3.3 (The first marked extension at every even size)**.** *For every even $N = 2n \geq 2$ and every integer $p \geq 3$, the submodule $Z_N \subset Y_N$ defined in (3.4) satisfies (3.5) and*

$$(F_N - f)|_{Z_N} = (\overline{F}_N - f)|_{Z_N} = g \Theta_N \pi_N. \tag{3.11}$$

The sequence (3.5) splits if and only if

$$g = 0 \quad \Longleftrightarrow \quad y = -\varepsilon x \quad or \quad y = -\varepsilon x^{-1}. \tag{3.12}$$

If $g \neq 0$, the Jordan form of either central element on Z_N is

$$J_2(f)^{\oplus \binom{N}{N/2-1}} \ \oplus \ J_1(f)^{\oplus C_{N/2}}. \tag{3.13}$$

Proof. The short exact sequence was established in Lemma 3.1. The two sides of (3.11) are natural for annular diagrams, and they agree on the two-node generators by (3.8). Inserting the two-node states into each matching proves the identity on the basis (3.4). More explicitly,

$$(F_N - f)v_{\pi;a,a} = 0, \qquad (F_N - f)t_{\pi;a,b} = g\, r_{\pi;a,b},$$

and the same equations hold for $\overline{F}_N$. Since $\pi_N \Theta_N = 0$, the nilpotent square is zero. When $g \neq 0$, its rank is $\dim T_N$ by Lemma 3.2. A square-zero endomorphism of rank r on a d-dimensional space has r Jordan blocks of size two and $d - 2r$ of size one. This proves (3.13).

Both F_N and $\overline{F}_N$ act by f on each standard module W_N and T_N. A splitting of (3.5) would therefore force their action on Z_N to be scalar. Equation (3.11) excludes this when $g \neq 0$.

If $g = 0$, the vector t at two nodes satisfies $e_0 t = e_1 t = 0$ and $\Omega t = \varepsilon t$. Let $\mathcal{C}_N$ be the span of the edge vectors $t_{\pi;a,b}$. An empty cup or a rotation sends such a vector to another edge vector, up to the orientation factor ε. Under a cap, the two ordered states either remain at depth one with the same contractible-loop coefficient, or both become the same diagonal state. To obtain the latter case, the cap must merge the two marked faces. Reconnecting two different arcs merges faces at dual-tree distance two and therefore cannot merge this adjacent pair. The cap must instead close their separating arc. The two oriented terms then contribute a combined coefficient $A + \varepsilon B = g$ or $B + \varepsilon A = \varepsilon g$, which vanishes. Thus $\mathcal{C}_N$ is preserved by cups, caps and rotation, including caps crossing the seam. These diagrams generate the annular category. Since $\pi_N|_{\mathcal{C}_N}$ is an isomorphism of vector spaces, its inverse is a natural splitting $s_N : T_N \to Z_N$. In the chosen basis it sends $\pi_N(t_{\pi;a,b})$ to $t_{\pi;a,b}$. The factorization in (3.1) proves the stated parameter locus. □

Remark 3.4 (An explicit projector on the first depth space)**.** On $Y_{\bar{N}}^{\leq 1}$ one has

$$(F_N - f)^2 (F_N + f) = 0.$$

Thus

$$Q_N = 1 - \frac{(F_N - f)^2}{4f^2}$$

is an idempotent with image Z_N. This formula has only the nonzero denominator $4f^2$. It is compatible with cup and cap maps at every size.

3.4 The isolated window and the full two-marked module

For completeness, the same cutting argument used for depth one gives

$$Y_{\bar{N}}^{\leq r} / Y_{\bar{N}}^{\leq r-1} \cong \bigoplus_{\omega^{2r}=1} W_{N;2r,\omega}, \qquad 1 \leq r \leq N/2. \tag{3.14}$$

Indeed, cutting the r arcs along the path between the marked faces yields $2r$ defects and a cyclic choice of their origin; the regular action of this cyclic group is diagonalized by the ordinary Fourier

transform. Contractions lowering the depth vanish in the quotient. Consequently (3.14) holds at the present root of unity, even when one of its standard summands is reducible.

On $W_{N;2r,\omega}$ the two central eigenvalues are

$$f_{r,\omega} = \omega q^r + \omega^{-1} q^{-r}, \qquad \bar{f}_{r,\omega} = \omega q^{-r} + \omega^{-1} q^r. \tag{3.15}$$

Lemma 3.5 (Joint central isolation)**.** *Assume $2 \le N < 2p-2$. Among the positive-depth standard summands in (3.14), the only one with joint central character (f, f) is $W_{N;2,\varepsilon}$.*

Proof. If both expressions in (3.15) equal f, then

$$(\omega - \omega^{-1})(q^r - q^{-r}) = 0.$$

Here $1 \le r \le N/2 < p-1 < p$, so $q^r - q^{-r} \neq 0$ and $\omega = \pm 1$. The strict monotonicity of $\cos(\pi r/p)$ for $0 < r < p$ then gives $r = 1$ if $\omega = \varepsilon$, and $r = p-1$ if $\omega = -\varepsilon$. The latter possibility is excluded by the size bound. □

Theorem 3.6 (The first isolated marked summand)**.** *Let $p \ge 3$ be an integer, let N be even, and suppose $2 \le N < 2p-2$. The module Z_N is the joint generalized (f, f)-eigenspace of $(F_N, \overline{F}_N)$ in the full module Y_N; in particular it is a direct summand of Y_N. Put*

$$S_N = L_{N;0,\varepsilon q}.$$

Then T_N is simple and

$$0 \longrightarrow T_N \xrightarrow{\Theta_N} W_N \longrightarrow S_N \longrightarrow 0 \tag{3.16}$$

is non-split. If $g = 0$, then $Z_N \cong W_N \oplus T_N$. If $g \neq 0$, then Z_N is uniserial with successive layers

$$T_N \,/\, S_N \,/\, T_N \quad \text{from head to socle.} \tag{3.17}$$

In particular, $E_N = Z_N/\Theta_N T_N$ is a non-split extension of T_N by S_N.

Proof. Taking a joint generalized central eigenspace is exact on finite-dimensional modules. Apply it to the depth filtration. Lemma 3.5 shows that only W_N and the indicated depth-one standard can survive. Their inverse image is precisely Z_N. The decomposition into joint generalized eigenspaces is a decomposition into A_N-submodules, since the two operators are central.

The Graham–Lehrer successor description, recalled in [8, Theorem 2.2], gives exactly one successor $(2, \varepsilon)$ of $(0, \varepsilon q)$ in this range; the next possible one has through-line number $2p-2$. The standard module T_N has no successor. This proves its simplicity and (3.16), with the radical identified by Lemma 3.2.

The upper arrow in (3.17) is determined by a two-node corner calculation. Choose a cup diagram $I : 2 \to N$ inserting $k = (N-2)/2$ disjoint cups, and let $C : N \to 2$ be its reflected cap diagram. Then

$$CI = \beta^k 1_2, \qquad e = \beta^{-k} IC, \qquad e^2 = e. \tag{3.18}$$

Deleting the cups identifies eW_N with W_2 and eT_N with T_2. The projector of Remark 3.4 commutes with cups and caps, so it also identifies eZ_N with Z_2. The image of e on the diagram family is I times the entire two-node space, because $CI = \beta^k 1_2$. Furthermore

$$e\mathsf{A}_N e \cong \mathsf{A}_2, \qquad b \longmapsto \beta^{-k} IbC.$$

The inverse sends eae to $\beta^{-k}CaI$; both identities follow by contracting CI. If one replaces A_N by its finite-dimensional image on a module M, the resulting corner is the corresponding finite image of A_2 on eM.

Naturality of Θ identifies $e\Theta_N T_N$ with $\mathbb{C}r$. Consequently $eE_N \cong Z_2/\mathbb{C}r$. In this two-dimensional quotient, $b = -\varepsilon a$ and

$$\Omega\bar{a} = -\varepsilon\bar{a}, \quad \Omega\bar{t} = \varepsilon\bar{t}, \quad e_1\bar{t} = \varepsilon g\bar{a}, \quad e_0\bar{t} = -\varepsilon g\bar{a}.$$

For $g \neq 0$, no line of rotation eigenvalue ε is annihilated by both e_0, e_1, so this extension is non-split. If the corresponding extension at size N split, applying the exact corner functor $e(-)$ would split the two-node one, a contradiction.

The two non-split sequences now imply that Z_N has simple head T_N and radical W_N. Indeed, any map $Z_N \to T_N$ kills W_N, while a nonzero map to S_N would descend to a retraction of the lower simple in E_N and split that sequence. The radical of W_N is the simple $\Theta_N T_N$. This proves the asserted uniserial structure. The case $g = 0$ was proved in Theorem 3.3. □

3.5 An explicit annular singular datum

The following annular operator gives an extension datum at each finite size, with endpoint vectors fixed by cup insertion. For all even sizes define

$$D_N = W_N/\Theta_N T_N, \qquad E_N = Z_N/\Theta_N T_N.$$

Then

$$0 \longrightarrow D_N \longrightarrow E_N \longrightarrow T_N \longrightarrow 0, \qquad \dim D_N = C_{N/2}. \tag{3.19}$$

In the isolated window $D_N = S_N$.

Proposition 3.7 (An extension-detecting annular operator)**.** *Let $U = e_1 - e_0 \in \mathsf{A}_2$, and let I, C, k be as in (3.18). Define*

$$U_N = \beta^{-k} IUC \in \mathsf{A}_N.$$

The operator U_N annihilates D_N and T_N. With the lift $I\bar{t} \in E_N$ of $I\pi_2(t) \in T_N$,

$$U_N(I\bar{t}) = 2\varepsilon g\, I\bar{a} \in D_N, \qquad I\bar{a} \neq 0. \tag{3.20}$$

This vector is independent of the chosen lift of $I\pi_2(t)$. Thus, in the indicated normalization, the annular singular coefficient is $2\varepsilon g$, and it is nonzero exactly off the splitting locus.

Proof. On D_2 both e_0, e_1 act by β, and on T_2 both act by zero. Hence U annihilates both modules. Naturality of Θ makes C and I descend to the quotient families. The definition of U_N therefore annihilates D_N and T_N as asserted. At two nodes, the relation $\bar{b} = -\varepsilon\bar{a}$ gives

$$U\bar{t} = \varepsilon g\bar{a} - g\bar{b} = 2\varepsilon g\bar{a}.$$

Multiplying by I and using $CI = \beta^k 1_2$ proves (3.20). The same contraction identity proves that $I\bar{a} \neq 0$. Changing the lift adds an element of D_N, which is annihilated by U_N, so the value is lift-independent. □

For $(p, N, \varepsilon, x, y) = (5, 6, 1, 1, 2)$, these formulas give

$$f = \frac{1+\sqrt{5}}{2}, \qquad g = \frac{9}{2}, \qquad \dim W_6 = 20, \quad \dim T_6 = 15, \quad \dim D_6 = 5, \quad \dim Z_6 = 35.$$

The central Jordan form is $J_2(f)^{\oplus 15} \oplus J_1(f)^{\oplus 5}$, and the coefficient in (3.20) is 9.

Remark 3.8 (Fixed-parameter size growth)**.** The module family Z_N and all identities of Theorem 3.3 exist for every even N at fixed p. For $N \geq 2p-2$, Z_N need not be a direct summand of the full Y_N, and D_N and T_N need not be simple. The depth-one construction supplies a compatible sequence of actual modules beyond the isolated window. A continuum limit would additionally require a topology on the relevant states, choices of normalization, and control of a physical evolution or dilation operator; none of these follows from the central Jordan identities alone.

Lemma 3.9 (The two-node XXZ comparison)**.** *If $g \neq 0$, then $E_2 \cong X^{+}_{2;0,\varepsilon q}$. In the basis $(\bar{a}, \bar{t}/(\varepsilon g))$ of E_2, the generator matrices are*

$$e_1 = \begin{pmatrix} \beta & 1 \\ 0 & 0 \end{pmatrix}, \qquad e_0 = \begin{pmatrix} \beta & -1 \\ 0 & 0 \end{pmatrix}, \qquad \Omega = \begin{pmatrix} -\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}.$$

In the spin basis $(|+-\rangle, |-+\rangle)$, the XXZ matrices are

$$e_1 = \begin{pmatrix} -q & 1 \\ 1 & -q^{-1} \end{pmatrix}, \qquad \Omega = \begin{pmatrix} 0 & \varepsilon q \\ \varepsilon q^{-1} & 0 \end{pmatrix}.$$

An intertwiner sends

$$\bar{a} \longmapsto (-q, 1), \qquad \frac{\bar{t}}{\varepsilon g} \longmapsto \frac{(q,1)}{q-q^{-1}}.$$

Proof. The displayed vectors are eigenvectors of Ω with eigenvalues $-\varepsilon$ and ε. Direct multiplication gives $e_1(-q,1) = \beta(-q,1)$ and $e_1(q,1) = (q-q^{-1})(-q,1)$. The relation $e_0 = \Omega e_1 \Omega^{-1}$ checks the remaining generator. The denominator is nonzero for $p \geq 3$. □

4 Image algebras and extension obstructions

This section distinguishes an extension of two simple modules from a lift of that extension through a nonsimple module. The distinction is essential in the marked-point construction. Adding the three-layer module considered below preserves the first extension groups between simple modules and removes a nonzero second-extension obstruction.

4.1 Blocks detected by a faithful module

For a finite-length module M, write $\operatorname{Supp} M$ for the set of isomorphism classes of its simple composition factors. Let $\mathscr{A}$ be an arbitrary complex algebra, let M be a finite-dimensional $\mathscr{A}$-module, and put

$$A(M) = \mathscr{A} / \operatorname{Ann}_{\mathscr{A}} M = \operatorname{im}\big(\mathscr{A} \longrightarrow \operatorname{End}_{\mathbb{C}} M\big).$$

Thus M is faithful as an $A(M)$-module. A decomposition into indecomposable summands will be denoted by $M = \bigoplus_{r=1}^{h} M_r$. Define a graph on $\operatorname{Supp} M$ by joining two distinct simple classes whenever they both occur in some M_r.

Theorem 4.1 (Block decomposition of a finite image algebra)**.** *Let $A = A(M)$. Every simple A-module occurs in M. The connected components of the graph just defined are precisely the blocks of A. More explicitly, if M_C is the sum of the summands supported on a component C, then*

$$A \simeq \prod_{C} A(M_C).$$

The full subcategory of A-modules generated by M under subquotients, finite direct sums and extensions is the whole category of finite-dimensional A-modules.

Proof. Choose a complete family of primitive orthogonal idempotents of A. For any such idempotent e, the functor $e(-)$ is exact and is nonzero on the corresponding simple class. If that class did not occur in M, then $eM = 0$. This would contradict the faithfulness of M.

For each graph component C, let e_C be the sum of all idempotents in the chosen family whose corresponding simple class belongs to C. Exactness of $e_C(-)$ shows that e_C acts as the identity on M_C and as zero on M_D for $D \neq C$. Consequently its action on M is an A-linear projection. Faithfulness gives $[e_C, a] = 0$ for every $a \in A$. The e_C are therefore central orthogonal idempotents with sum 1, and $A = \prod_C Ae_C$. The faithful action of Ae_C on M_C identifies this factor with $A(M_C)$.

An indecomposable module belongs to a single block. Therefore the simple classes in the support of each M_r belong to the same block, and the same holds along every path in the graph. No factor Ae_C can split further. Finally, all simple A-modules are subquotients of M, and induction on composition length proves the last assertion. □

At fixed size N, choose a list of standard seed labels. Define the *accessible resonance graph* to be the undirected graph whose vertices are the labels in their successor closures with defect number at most N and whose edges are direct successions between those labels. For zero-defect labels, retain the identification $(0, z) \sim (0, z^{-1})$. Each seed closure is connected in this graph.

Corollary 4.2 (The reduced resonance blocks)**.** *Suppose that the standard generators are indecomposable, that their composition factors are their successor closures, and that each accompanying XXZ generator has the same composition factors as the corresponding standard generator. After choosing actual XXZ twist labels, the blocks of the image algebra on their direct sum are the connected components of the accessible resonance graph. No additional block-separation or projectivity hypothesis is needed.*

Proof. All generators are supported on individual resonance components. Conversely, the endpoints of a successor edge accessible from a seed both occur in the indecomposable standard module of that seed. Thus the support graph in Theorem 4.1 has exactly the resonance components. □

4.2 Multiplicity-free generators and the centre

Lemma 4.3. *Every finite-dimensional indecomposable module whose simple composition factors have multiplicity at most one has endomorphism algebra $\mathbb{C}$.*

Proof. The generalized eigenspaces of a module endomorphism are submodules. Indecomposability therefore implies that an endomorphism has a single eigenvalue and is a scalar plus a nilpotent endomorphism. Suppose that a nonzero nilpotent f exists, and choose $r \geq 1$ with $f^r \neq 0$ and $f^{r+1} = 0$. Since $2r \geq r + 1$, one has

$$0 \neq \operatorname{im} f^r \subseteq \ker f^r, \qquad M/\ker f^r \simeq \operatorname{im} f^r.$$

Any simple composition factor of $\operatorname{im} f^r$ occurs at least twice in M, a contradiction. □

Theorem 4.4 (Reduced centre)**.** *Assume that M is a direct sum of multiplicity-free modules. If b is the number of blocks of $A(M)$, then*

$$Z(A(M)) \simeq \mathbb{C}^b.$$

In particular, every central element acts semisimply on M, and the centre of each block is $\mathbb{C}$. A module on which a central element has a nonzero nilpotent part cannot factor through this image algebra if that element has the corresponding scalar value on the selected block.

Proof. Decompose the generators into indecomposables. Each resulting summand is multiplicity-free and hence is a brick by Lemma 4.3. A central element acts by a scalar on each summand. If two summands share a simple composition factor, their scalars agree, since the induced scalar on that simple factor is the same. The scalars are consequently constant on each support component. Theorem 4.1 identifies these components with the blocks. Faithfulness now embeds the centre into $\mathbb{C}^b$, and the central block idempotents show that this embedding is onto. □

4.3 Two opposite extensions

Throughout the remainder of this section, S and T are nonisomorphic finite-dimensional simple $\mathscr{A}$-modules, with $s = \dim S$ and $t = \dim T$. A notation such as S/T means a nonsplit length-two module with head S and socle T.

Theorem 4.5 (The image of two opposite extensions)**.** *Suppose that $W = S/T$ and $E = T/S$ are $\mathscr{A}$-modules, and set $R = A(W \oplus E)$. Then R has two simple modules and is Morita equivalent to*

$$R_0 = \mathbb{C}Q/(ab, ba), \qquad Q: \quad S \xrightarrow{a} T \xrightarrow{b} S.$$

Products of arrows are read as compositions, with the rightmost arrow traversed first. Thus ba is the path from S back to S. The radical of R has square zero, W and E are the respective indecomposable projective covers of S and T, and

$$\dim R = (s+t)^2, \qquad Z(R) = \mathbb{C}.$$

Proof. The two simples occur in the faithful module $W \oplus E$, so there are no others. If $J = \operatorname{rad} R$, then J^2 annihilates both length-two modules, and faithfulness gives $J^2 = 0$.

Pass to a basic corner eRe, choosing one primitive idempotent for each simple class. The functor $e(-)$ is a Morita equivalence, and eW, eE each have one-dimensional spaces at the two vertices. A radical diagonal corner acts as zero on both modules, since each simple occurs only once. The corner $e_T J e_S$ acts only on eW, and embeds into its one-dimensional space of maps from the S-space to the T-space. It is nonzero because W is nonsplit. The opposite corner is likewise one-dimensional and is detected by E. These are all the radical corners. Hence $eRe = R_0$.

The projectives $R_0 e_S$ and $R_0 e_T$ are the two displayed length-two modules. Under Morita equivalence this proves the projective-cover claim. The regular R-module is a direct sum of s copies of W and t copies of E, which gives $\dim R = (s+t)^2$. The centre follows either directly from the quiver or from Theorem 4.4. □

4.4 A three-layer module and its image algebra

Theorem 4.6 (The three-layer image algebra)**.** *Let P be a uniserial $\mathscr{A}$-module with successive composition factors T, S, T, from head to socle, and put $B = A(P)$. Let*

$$W = \operatorname{rad} P = S/T, \qquad E = P/\operatorname{soc} P = T/S.$$

Then the following statements hold.

(i) *The basic algebra of B is*

$$B_0 = \mathbb{C}Q/(ba), \qquad Q: \quad S \xrightarrow{a} T \xrightarrow{b} S.$$

It has basis e_S, e_T, a, b, ab, radical cube zero, and centre $\mathbb{C}1 \oplus \mathbb{C}ab$ with $(ab)^2 = 0$.

(ii) *The modules W and P are the projective covers of S and T, respectively, in B-mod. Moreover* $\mathrm{End}_B(P) \simeq \mathbb{C}[\eta]/(\eta^2)$, *where η is the composite of the quotient onto the head T with an isomorphism from that head to the socle T and the inclusion into P.*

(iii) *With a suitable vector-space decomposition $P = T_{\mathrm{top}} \oplus S \oplus T_{\mathrm{soc}}$, the faithful image $B \subseteq \mathrm{End}_{\mathbb{C}} P$ is exactly the algebra of matrices*

$$\left\{ \begin{pmatrix} U & 0 & 0 \\ B_1 & V & 0 \\ C & D & U \end{pmatrix} \;\middle|\; \begin{array}{ll} U, C \in \mathrm{Mat}_t(\mathbb{C}), & V \in \mathrm{Mat}_s(\mathbb{C}), \\ B_1 \in \mathrm{Mat}_{s\times t}(\mathbb{C}), & D \in \mathrm{Mat}_{t\times s}(\mathbb{C}) \end{array} \right\}. \tag{4.1}$$

Equivalently, B is the full centralizer in $\mathrm{End}_{\mathbb{C}} P$ of the rank-t square-zero map η. In particular,

$$\dim B = s^2 + 2st + 2t^2, \qquad Z(B) = \mathbb{C}1 \oplus \mathbb{C}\eta.$$

(iv) *The quotient acting faithfully on $W \oplus E$ is*

$$R = B/B\eta B.$$

Its kernel consists of the matrices in (4.1) with only the block C nonzero. The kernel has dimension t^2, has square zero, and is generated by the central element η. For $t > 1$ the kernel is not contained in the centre. At the basic-algebra level the quotient is $B_0 \twoheadrightarrow R_0 = B_0/(ab)$ and its kernel is the central one-dimensional space $\mathbb{C}ab$.

Proof. Pass first to a basic corner, and denote the resulting faithful module by V. Its space at S is one-dimensional, and its space at T is two-dimensional. Choose a vector u lifting the head, a vector v in the middle layer, and a vector w spanning the socle. Elements of the basic algebra preserve the radical series. On $\mathbb{C}u \oplus \mathbb{C}w$ their two diagonal entries agree, since the head and socle are the same simple module. Thus the algebra embeds into the five-dimensional space spanned by the two vertex projections and the maps

$$b : u \longmapsto v, \qquad a : v \longmapsto w, \qquad ab : u \longmapsto w.$$

The first two radical corners are nonzero: otherwise one of the successive extensions in P would split or the radical series would have length at most two. Each corner is one-dimensional, so we may rescale its generator to obtain the displayed maps. Their product ab is nonzero, whereas $ba = 0$. The five maps are linearly independent and exhaust the image. This proves the description of B_0. Its projectives are $B_0 e_S = S/T$ and $B_0 e_T = T/S/T$.

In the basic module, a commuting endomorphism is a common scalar on the three layers, together with an arbitrary multiple of $u \mapsto w$. This proves the endomorphism assertion; Morita equivalence preserves this endomorphism algebra. The same matrix calculation gives the stated centre of B_0.

For completeness, the full matrix description can be obtained directly. Choose a semisimple complement to $\mathrm{rad}\, B$ and decompose the three layers as modules over that complement. The semisimple algebra acts by $\mathrm{Mat}_t(\mathbb{C})$ on both copies of T, with the same matrix on those copies, and by $\mathrm{Mat}_s(\mathbb{C})$ on S. The nonzero radical corners from T to S and from S to T are bimodules for these full matrix algebras. A nonzero such bimodule contains every matrix in the corresponding rectangular block. Products of the two rectangular blocks span every matrix in the lower-left $t \times t$ block. No other blocks are allowed by the radical filtration. This proves (4.1). Commuting an

arbitrary matrix with the map η that identifies the first and third summands gives exactly the same block conditions, proving the centralizer assertion.

Finally, restriction to W and passage to E annihilate precisely the lower-left block C. The resulting quotient has the two opposite length-two extensions as a faithful module and is the algebra of Theorem 4.5. Multiplying the central map η by the repeated block U produces every lower-left matrix C. The description of the kernel, its dimension and its square-zero property follow. Its elements commute with all matrices U only when C is scalar, which proves the final qualification. □

Remark 4.7. For the isolated channel with $N = 6$, $\dim S = 5$ and $\dim T = 15$. The three-layer module has dimension 35, its faithful image algebra has dimension 625, and the image on its two length-two subquotients has dimension 400. The quotient kernel has dimension 225, while its intersection with the centre is one-dimensional.

4.5 First extensions, second extensions and lifting

Write $\alpha \in \operatorname{Ext}^1(S,T)$ for the class of $W = S/T$ and $\xi \in \operatorname{Ext}^1(T,S)$ for the class of $E = T/S$. In products such as $\xi\alpha$, the rightmost extension is traversed first; thus $\xi\alpha \in \operatorname{Ext}^2(S,S)$ and $\alpha\xi \in \operatorname{Ext}^2(T,T)$.

Theorem 4.8 (The change in Yoneda extensions)**.** *Let B and R be the image algebras of Theorem 4.6. Inflation along $B \twoheadrightarrow R$ induces isomorphisms*

$$\operatorname{Ext}^1_R(U,V) \xrightarrow{\sim} \operatorname{Ext}^1_B(U,V) \qquad (U,V \in \{S,T\}).$$

For both algebras the two cross-extension spaces are one-dimensional and the two self-extension spaces vanish. The higher extensions satisfy

$$\dim \operatorname{Ext}^n_R(S,S) = \dim \operatorname{Ext}^n_R(T,T) = \begin{cases} 1, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases}$$

$$\dim \operatorname{Ext}^n_R(S,T) = \dim \operatorname{Ext}^n_R(T,S) = \begin{cases} 0, & n \text{ even}, \\ 1, & n \text{ odd}, \end{cases} \qquad (n \geq 0),$$

whereas

$$\operatorname{Ext}^2_B(S,S) = \mathbb{C}, \qquad \operatorname{Ext}^2_B(S,T) = \operatorname{Ext}^2_B(T,S) = \operatorname{Ext}^2_B(T,T) = 0, \qquad \operatorname{gldim} B = 2.$$

All alternating Yoneda products in degree one are nonzero over R. Over B one has

$$\xi\alpha \neq 0, \qquad \alpha\xi = 0.$$

Consequently inflation is an isomorphism on $\operatorname{Ext}^2(S,S)$ and annihilates $\operatorname{Ext}^2_R(T,T)$; it annihilates all groups in degrees at least three.

Proof. It suffices to work with the basic algebras. For R_0, the projectives are $W = S/T$ and $E = T/S$. The syzygies of the simple modules alternate:

$$\Omega_R S = T, \qquad \Omega_R T = S.$$

The minimal projective resolutions give the displayed dimensions. Splicing the two short exact sequences gives the successive minimal syzygy sequences, so every composable alternating product is nonzero.

For B_0, the projectives are W and P. There are minimal resolutions

$$0 \longrightarrow W \longrightarrow P \longrightarrow T \longrightarrow 0$$

and

$$0 \longrightarrow W \longrightarrow P \longrightarrow W \longrightarrow S \longrightarrow 0.$$

In the second resolution, the map $P \to W$ is the quotient onto T followed by the inclusion of the socle of W. Applying $\mathrm{Hom}(-, S)$ and $\mathrm{Hom}(-, T)$ gives all the asserted dimensions. The projective dimensions of T and S are respectively one and two, so the global dimension is two.

The two inflated first extensions remain nonsplit, because inflation is fully faithful and does not change whether a section exists. Since both cross-extension spaces have dimension one, the first-extension maps are isomorphisms. Apply $\mathrm{Hom}_B(-, S)$ to $0 \to T \to W \to S \to 0$. Projectivity of W shows that the connecting map $\mathrm{Ext}^1_B(T, S) \to \mathrm{Ext}^2_B(S, S)$ is an isomorphism. This connecting map is multiplication by α, and hence $\xi\alpha \neq 0$. The other product is zero because T has projective dimension one. Naturality of Yoneda products proves the assertions about inflation in degree two; the higher-degree assertion follows from global dimension. □

Corollary 4.9 (The three-layer module removes a lifting obstruction)**.** *Over R, the extension ξ has no lift to an extension of T by W. Over B, each class in $\mathrm{Ext}^1_B(T, S)$ has a unique lift in $\mathrm{Ext}^1_B(T, W)$, and a nonzero lift is represented by*

$$0 \longrightarrow W \longrightarrow P \longrightarrow T \longrightarrow 0, \qquad \mathrm{Ext}^1_B(T, W) \simeq \mathbb{C}.$$

Thus adjoining P does not create a new first extension between S and T. It makes an existing first extension liftable through W.

Proof. In the long exact sequence obtained by applying $\mathrm{Hom}(T, -)$ to $0 \to T \to W \to S \to 0$, the obstruction to lifting $\xi \in \mathrm{Ext}^1(T, S)$ lies in $\mathrm{Ext}^2(T, T)$ and is $\alpha\xi$. Over R this class is nonzero by Theorem 4.8; indeed the connecting map is an isomorphism and $\mathrm{Ext}^1_R(T, W) = 0$. Over B the obstruction vanishes. The same exact sequence, using $\mathrm{Hom}(T, S) = 0$ and $\mathrm{Ext}^1_B(T, T) = 0$, shows that $\mathrm{Ext}^1_B(T, W) \to \mathrm{Ext}^1_B(T, S)$ is an isomorphism. The displayed sequence is the asserted lift. □

4.6 Compatibility with a separately constructed XXZ module

The preceding results concern P and its own subquotients. An additional length-two module $X = T/S$ over the original algebra need not be isomorphic to E merely because it has the same Loewy diagram.

Proposition 4.10 (A compatibility criterion)**.** *Let $P = T/S/T$ and let $X = T/S$ be a nonsplit $\mathscr{A}$-module. The following conditions are equivalent:*

(i) $X \simeq P/\operatorname{soc} P$ *as $\mathscr{A}$-modules;*

(ii) $\mathrm{Hom}_{\mathscr{A}}(P, X) \neq 0$*;*

(iii) $\mathrm{Ann}_{\mathscr{A}} P \subseteq \mathrm{Ann}_{\mathscr{A}} X$*;*

(iv) *after identifying the simple endpoints, the extension classes of X and $P/\operatorname{soc} P$ span the same line in $\mathrm{Ext}^1_{\mathscr{A}}(T, S)$.*

If these conditions hold and $W = \operatorname{rad} P$, then

$$A(P \oplus W \oplus X) = A(P), \qquad A(W \oplus X) = A(P)/(\eta).$$

Proof. The first condition implies the second and third. A nonzero map $P \to X$ cannot have image contained in the simple socle S, since $\operatorname{Hom}_{\mathscr{A}}(P, S) = 0$. It is therefore onto. Its kernel has length one and is the unique simple submodule $\operatorname{soc} P$, proving the converse to the second condition.

Under the third condition, X is a module for $B = A(P)$. Theorem 4.8 shows that $\operatorname{Ext}^1_B(T, S)$ is one-dimensional. Hence its nonsplit length-two modules are all isomorphic to E. The equivalence with the fourth condition is the usual action of $\operatorname{Aut}(S) \times \operatorname{Aut}(T) = (\mathbb{C}^\times)^2$ on extensions, which rescales their classes. The two image-algebra identities now follow from the subquotient descriptions and Theorem 4.6. □

4.7 Full corners and annular localisation

Lemma 4.11 (Fullness on a specified set of simple modules)**.** *Let A be a finite-dimensional algebra and e an idempotent. If $eL \neq 0$ for every simple A-module L, then $AeA = A$. In this case $e(-)$ is a Morita equivalence from A-mod to eAe-mod and reflects isomorphisms, exact sequences, radical and socle filtrations, projectivity and Yoneda extension classes.*

Proof. If A/AeA were nonzero, it would have a simple quotient annihilated by e, contradicting the hypothesis. Thus e is full. The equivalence has quasi-inverse $Ae \otimes_{eAe} (-)$; the stated preservation and reflection properties follow from exactness of an equivalence. □

Lemma 4.12 (The annular two-site corner)**.** *Let A_N be the endomorphism algebra of N in the annular Temperley–Lieb diagram category, with contractible loop value $\beta \neq 0$, and let $N = 2 + 2k$. Choose a cup diagram $I : 2 \to N$ adding k adjacent cups and its reflected cap diagram $C : N \to 2$, so that $CI = \beta^k 1_2$. Then*

$$e = \beta^{-k} IC \in \mathsf{A}_N$$

is an idempotent and

$$j : \mathsf{A}_2 \xrightarrow{\sim} e\mathsf{A}_N e, \qquad j(a) = \beta^{-k} IaC$$

is an algebra isomorphism, with inverse $x \mapsto \beta^{-k} CxI$. If A is the finite image of A_N on a module M, then eAe is exactly the image of A_2 on eM under this identification.

Proof. The equation $CI = \beta^k 1_2$ gives $e^2 = e$ and $j(a)j(b) = j(ab)$. It also verifies the stated inverse. For any $x \in \mathsf{A}_N$,

$$exe = \beta^{-2k} ICxIC = j(\beta^{-k} CxI),$$

which proves surjectivity onto the entire corner. Passing to a representation image gives the final assertion. □

Lemma 4.13 (Localisation of the periodic XXZ sector)**.** *Use the XXZ convention in which the local generator on the ordered basis $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$ has the sole nonzero block*

$$\begin{pmatrix} -q & 1 \\ 1 & -q^{-1} \end{pmatrix}$$

on the middle two basis vectors, and the rotation is

$$\Omega_N |s_1 s_2 \cdots s_N\rangle = z^{-s_1} |s_2 \cdots s_N s_1\rangle, \qquad s_i \in \{1, -1\}.$$

Then the idempotent of Lemma 4.12 satisfies

$$eX^+_{N;0,z} \simeq X^+_{2;0,z}$$

as A_2*-modules. The analogous statement holds for the charge* 2 *sector.*

Proof. Take the cup vector and cap functional

$$c = -q|+-\rangle + |-+\rangle, \qquad \ell(|+-\rangle) = 1, \quad \ell(|-+\rangle) = -q^{-1}, \quad \ell(|++\rangle) = \ell(|--\rangle) = 0.$$

They satisfy $c\ell = e_i$ on the selected adjacent pair and $\ell c = \beta = -q - q^{-1}$. Their two snake contractions are identity maps. The seam matrix $D_z = \mathrm{diag}(z^{-1}, z)$ satisfies $(D_z \otimes D_z)c = c$ and $\ell(D_z \otimes D_z) = \ell$. These identities give the annular diagram action with the stated local generator and rotation.

Place the added cups away from the seam. Their spin-space maps are

$$I_{\mathrm{spin}}(u) = u \otimes c^{\otimes k}, \qquad C_{\mathrm{spin}} = 1 \otimes \ell^{\otimes k}.$$

The image of e is precisely the image of I_{spin}. Every cup has charge zero, so its charge-zero part is identified with the two-dimensional charge-zero space on the active two sites. Cup–cap contraction in $j(a)$ cancels the k factors β and leaves the two-site action of a. In particular the remaining rotation matrix is

$$\begin{pmatrix} 0 & z \\ z^{-1} & 0 \end{pmatrix}.$$

This proves the assertion. In charge 2, the active two-site vector is $|++\rangle$, and the same contractions apply. □

Proposition 4.14 (Transport from an isolated two-site channel)**.** *Let* $M_N = P_N \oplus W_N \oplus X_N$ *be a finite-dimensional* A_N*-module. Assume independently that all its simple composition factors belong to two classes* S_N, T_N*, and that under the annular corner isomorphism one has* $eS_N \simeq S_2$ *and* $eT_N \simeq T_2$*, where* S_2, T_2 *are nonisomorphic simple modules. Suppose in addition that*

$$eP_N \simeq P_2 = T_2/S_2/T_2, \qquad eW_N \simeq \operatorname{rad} P_2, \qquad eX_N \simeq P_2/\operatorname{soc} P_2.$$

Then $P_N = T_N/S_N/T_N$*,* $\operatorname{rad} P_N \simeq W_N$*, and* $P_N/\operatorname{soc} P_N \simeq X_N$*. All conclusions of Theorems 4.6 and 4.8 hold for the joint image on* M_N*.*

Proof. Let $A = A(M_N)$. The support assumption and the nonvanishing of e on the two simple classes imply that e is full by Lemma 4.11. Its corner algebra is the image of A_2 on the three displayed modules. The last two are subquotients of P_2, so this corner algebra is $A(P_2)$. The Morita equivalence transports the radical series of P_2, the identifications of its subquotients, and all homological conclusions to A. In particular W_N and X_N are subquotients of P_N, so the joint image is $A(P_N)$. □

Remark 4.15. The support assumption in Proposition 4.14 must be established before invoking fullness. The equality $eP_N \simeq P_2$ alone does not exclude additional composition factors annihilated by e.

4.8 The isolated marked image and its XXZ quotient

We now apply the image-algebra results to the modules constructed in Section 3. Fix $2 \le N < 2p-2$ and $g \neq 0$, and write $P_N = Z_N$ and $E_N = P_N/\Theta_N T_N$.

Theorem 4.16 (The marked and XXZ comparison)**.** *There is an isomorphism of annular modules*

$$E_N \cong X^+_{N;0,\varepsilon q}.$$

Consequently, if

$$B_N = A(P_N), \qquad R_N = A\big(W_N \oplus X^+_{N;0,\varepsilon q}\big),$$

then $B_N \twoheadrightarrow R_N$ is the quotient of Theorem 4.6, and all conclusions of Theorem 4.8 and Corollary 4.9 apply. In particular, W_N is the projective cover of S_N in both image categories. The projective cover of T_N is P_N over B_N and $X^+_{N;0,\varepsilon q}$ over R_N.

Proof. Form the finite image on $P_N \oplus W_N \oplus X^+_{N;0,\varepsilon q}$. By Theorem 3.6 and the XXZ composition-factor theorem [8, Theorem 2.4], its simple support is exactly $\{S_N, T_N\}$. For the annular idempotent $e = \beta^{-(N-2)/2} IC$, the standard localization and exactness give

$$eT_N \cong T_2, \qquad eW_N \cong W_2, \qquad eS_N \cong S_2.$$

The two simple corner modules are distinct: their rotation eigenvalues are ε and $-\varepsilon$. Thus e is full by Lemma 4.11. The marked localization gives $eP_N \cong P_2$ and $eE_N \cong E_2$. The explicit spin cup–cap calculation in Lemma 4.13 identifies $eX^+_{N;0,\varepsilon q}$ with $X^+_{2;0,\varepsilon q}$. The intertwiner of Lemma 3.9 identifies the last module with E_2. The Morita equivalence of the full corner therefore identifies E_N and the stated XXZ module. Both W_N and this XXZ module are subquotients of P_N, so the joint image is $A(P_N)$. Theorem 4.6 now gives the quotient and the projective-cover statements. □

In the normalization of the two-node vectors a, t, r, the central action is $g\nu_N$, where $\nu_N = \Theta_N \pi_N$ identifies the head and socle of P_N. Its image algebra has centre

$$Z(B_N) = \mathbb{C}1 \oplus \mathbb{C}\nu_N,$$

whereas $Z(R_N) = \mathbb{C}$. The quotient kernel is the ideal generated by ν_N, of dimension $(nC_n)^2$.

Corollary 4.17 (A complete detector in the isolated image)**.** *Fix the endpoint identifications obtained by localization at two nodes. Evaluation of the annular operator U_N of Proposition 3.7 on the specified lifted vector defines a nonzero linear functional on the one-dimensional space $\mathrm{Ext}^1_{B_N}(T_N, S_N)$, and hence identifies that space with $\mathbb{C}$. On the displayed extension class $[E_N]$, its value is $2\varepsilon g$.*

Proof. The operator annihilates the two endpoints, so changing the lift does not change its value. Formation of the upper-right action block of an extension is linear under Baer sum, and evaluation at the fixed corner vectors is linear. Proposition 3.7 gives the nonzero value $2\varepsilon g$ on E_N. The dimension assertion follows from Theorem 4.8. □

This detector is attached to a specified annular operator and specified cup–cap normalization. It supplies an algebraic extension parameter without choosing an ADE field or a continuum normalization.

5 A five-simple image algebra beyond the isolated window

The image algebra depends on the actual twists of the chosen XXZ modules. We therefore specify the generating module explicitly. Throughout this section let $q = e^{i\pi/3}$, $N = 6$, and

$$M = W_{6;0,q} \oplus X^{+}_{6;0,q} \oplus X^{+}_{6;2,1}, \qquad A = \operatorname{im}\big(\mathsf{A}_6 \to \operatorname{End}_{\mathbb{C}} M\big). \tag{5.1}$$

The Graham–Lehrer injection $W_{6;2,1} \hookrightarrow W_{6;0,q}$ shows that adjoining $W_{6;2,1}$ does not change this image. We do not identify the twists q and q^{-1} in (5.1).

Write the five simple classes as

$$a = L_{6;0,q}, \quad b = L_{6;2,1}, \quad c = L_{6;4,-1}, \quad d = L_{6;6,-q}, \quad e = L_{6;6,q^2}.$$

Their dimensions are respectively $1, 13, 4, 1, 1$. The standard and XXZ structure theorems give the following radical layers:

Module	head	first radical layer	second radical layer
$W_{6;0,q}$	a	$b \oplus c$	$d \oplus e$
$X^{+}_{6;0,q}$	b	$a \oplus d \oplus e$	c
$X^{+}_{6;2,1}$	d	b	e

In the first row every middle constituent is joined to both bottom constituents. In the second row the head is joined to all three middle constituents, each of which is joined to the bottom. The third row is a chain. These are the orientations in [8, Theorems 2.2 and 2.4].

5.1 Exact reconstruction of the image

Proposition 5.1 (The image dimension)**.** *The algebra A in (5.1) has dimension* 328.

Proof. The assertion is established by an exact finite matrix calculation over $K = \mathbb{Q}[q]/(q^2 - q + 1)$. We describe the matrices and the finite closure check so that both bounds on the dimension are reproducible.

The basis of $W_{6;0,q}$ is indexed by the three-element subsets of $\{0, 1, \dots, 5\}$ giving the initial endpoints of the annular cups. To reconstruct the cups, read the corresponding balanced parenthesis word cyclically, starting at a minimum of its partial sums, and match parentheses by a stack. Rotation decreases every endpoint by one modulo six. The generator e_1 joins endpoints $0, 1$ and reconnects their former partners. A contractible closed component contributes $\beta = -1$; a noncontractible closed component contributes $q + q^{-1} = 1$. This gives two 20×20 matrices with integer entries.

For the two XXZ summands use the spin words of charge zero and charge two, respectively. The local e_1 matrix is the usual block $\left(\begin{smallmatrix} -q & 1 \\ 1 & -q^{-1} \end{smallmatrix}\right)$ on $|+-\rangle, |-+\rangle$, and rotation is

$$|s_1 \cdots s_6\rangle \longmapsto z^{-s_1} |s_2 \cdots s_6 s_1\rangle$$

with $z = q$ and $z = 1$, respectively. The two sector dimensions are 20 and 15.

Let E, O be the simultaneous matrices for e_1, Ω on the three summands. On each summand $O^6 = 1$, as checked directly; hence $O^{-1} = O^5$, and E, O generate the entire periodic image. Starting with 1, append a vector whenever exact row reduction shows that right multiplication of a previously retained vector by E or O leaves the current span. Coordinates are the concatenated matrix entries

of the three summands, a vector space of dimension $20^2 + 20^2 + 15^2 = 1025$. Arithmetic represents $u + vq \in K$ by the pair $(u, v) \in \mathbb{Q}^2$ and uses

$$(u + vq)(r + sq) = (ur - vs) + (us + vr + vs)q.$$

The resulting basis has 328 vectors with distinct leading coordinates. All 656 right products of those vectors by E, O lie in their span. Thus their span contains the unit, is invariant under both generators, and is exactly the image algebra. Linear independence gives the lower bound 328, while the verified closure gives the upper bound. The calculation also checks all affine Temperley–Lieb defining relations exactly before computing the span. The complete construction, exact arithmetic, closure computation and serialized basis are supplied in the supporting files listed below. □

5.2 Quiver, relations and projective covers

For vertices i, j, denote an arrow from i to j by u_{ij}, and read products with the rightmost arrow first.

Theorem 5.2 (The five-simple algebra)**.** *The algebra A is Morita equivalent to the* 19*-dimensional algebra $B = \mathbb{C}Q/I$ with arrows*

$$a \to b, c; \qquad b \to a, d, e; \qquad c \to d, e; \qquad d \to b, c; \qquad e \to c. \tag{5.2}$$

The nonzero paths of length two are exactly the paths occurring in

$$\begin{aligned} u_{bd}u_{ab} &= u_{cd}u_{ac}, \\ u_{be}u_{ab} &= u_{ce}u_{ac}, \\ u_{ac}u_{ba} &= u_{dc}u_{bd} = u_{ec}u_{be}, \\ u_{be}u_{db}&. \end{aligned} \tag{5.3}$$

All other paths of length two are zero. These quadratic relations imply that every path of length three is zero. In particular,

$$\dim B/\operatorname{rad} B = 5, \qquad \dim \operatorname{rad} B/\operatorname{rad}^2 B = 10, \qquad \dim \operatorname{rad}^2 B = 4.$$

For the full image algebra,

$$\dim A/\operatorname{rad} A = 188, \quad \dim \operatorname{rad} A/\operatorname{rad}^2 A = 85, \quad \dim \operatorname{rad}^2 A = 55.$$

Both centres are $\mathbb{C}$, and both algebras have infinite global dimension.

Proof. Write $J = \operatorname{rad} A$. All three summands of the faithful module have radical length at most three, so $J^3 M = 0$ and hence $J^3 = 0$. Pass to a basic corner. Every nonzero vertex space in each of the three modules over the basic corner has dimension one. The action of J^2 can only map a to d, e in the first module, b to c in the second, and d to e in the third. Each of these four corners is nonzero by the stated radical layers. Faithfulness bounds each corner by one dimension, so the basic J^2 has exactly these four one-dimensional corners. In the full algebra, a one-dimensional basic corner from i to j corresponds to a matrix bimodule of dimension $(\dim i)(\dim j)$. The four dimensions are therefore $1, 1, 52, 1$, whose sum is 55.

The adjacent radical layers of the three modules exhibit every arrow in (5.2). Thus the basic first radical layer contains at least these ten independent corners. Weighted by the dimensions

$1, 13, 4, 1, 1$ of the simple modules, they contribute 85 to the full first radical layer. The semisimple quotient has dimension $1^2 + 13^2 + 4^2 + 1^2 + 1^2 = 188$. Proposition 5.1 gives

$$328 = 188 + 85 + 55,$$

so there can be no further arrows and no extra arrow multiplicities. The ten arrow corners are disjoint from the four second-radical corners, so choosing arrow representatives introduces no quadratic correction terms.

Each composition in the first two lines of (5.3) acts nontrivially on the first module. Each composition in the third line acts nontrivially on the second, and the final path acts nontrivially on the third. The four resulting corners of J^2 are one-dimensional. Rescaling the four arrows $u_{cd}, u_{ce}, u_{dc}, u_{ec}$ makes the proportionality constants in the first three lines equal to one. Every other length-two path acts as zero on all three modules. For example, $u_{ce}u_{dc}$ is zero although the corner from d to e is nonzero: its two arrows act on different generating summands.

These relations kill every cubic path. For instance, extending the path $u_{bd}u_{ab}$ by u_{dc} gives $u_{ac}u_{ba}u_{ab} = 0$; extending $u_{ac}u_{ba}$ by u_{cd} gives $u_{bd}u_{ab}u_{ba} = 0$; and extending $u_{be}u_{db}$ by u_{ec} gives $u_{ac}u_{ba}u_{db} = 0$. The other extensions contain a zero quadratic subpath or reduce to one of these. The quotient therefore has a basis of five idempotents, ten arrows and four length-two paths. Its surjection to the basic algebra is an isomorphism by the dimension count.

The centre is $\mathbb{C}$ by Theorem 4.4, since the support graph is connected. The Cartan matrix displayed below is singular. A finite-dimensional algebra of finite global dimension has an invertible Cartan matrix over $\mathbb{Z}$: finite projective resolutions express every simple class as an integral combination of projective classes. Hence the present algebra has infinite global dimension. □

Corollary 5.3 (Projectives in the five-simple image)**.** *With rows indexed by simple modules and columns by their projective covers in the order a, b, c, d, e, the Cartan matrix is*

$$C_B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The Loewy layers and physical dimensions of the projectives are

Projective	*head*	*first radical layer*	*second radical layer*	*dimension*
P_a	a	$b \oplus c$	$d \oplus e$	20
P_b	b	$a \oplus d \oplus e$	c	20
P_c	c	$d \oplus e$	0	6
P_d	d	$b \oplus c$	e	19
P_e	e	c	0	5

In particular,

$$P_a \simeq W_{6;0,q}, \qquad P_b \simeq X^{+}_{6;0,q}, \qquad P_c \simeq W_{6;4,-1}.$$

The standard module $W_{6;2,1}$ is not projective in this image algebra.

Proof. The columns are the spaces of paths starting at the corresponding vertices, using Theorem 5.2. This gives the Cartan matrix and all layers. Weighting the composition multiplicities by $1, 13, 4, 1, 1$ gives the physical dimensions. Each of the three displayed module isomorphisms follows by mapping

the appropriate projective cover onto the known module with that simple head and comparing dimensions. The standard module $W_{6;4,-1}$ is an A-module because it embeds into $W_{6;0,q}$ by a Graham–Lehrer map. Finally $W_{6;2,1}$ has dimension 15 and head b, whereas its projective cover P_b has dimension 20. □

Corollary 5.4 (Second extensions)**.** *The dimensions of* $\operatorname{Ext}^2_B(i,j)$ *are*

	a	b	c	d	e
a	1	0	0	1	1
b	0	2	2	0	0
c	0	1	2	0	0
d	1	0	0	2	1
e	0	0	0	1	1

.

The same dimensions hold for the full image algebra A.

Proof. The defining ideal is generated by the sixteen independent quadratic relations described in Theorem 5.2. Their components from i to j are the minimal relation spaces, which are dual to $\operatorname{Ext}^2_B(i,j)$. Counting the zero paths and the displayed linear relations between parallel paths gives the table. Morita equivalence preserves extension groups. □

5.3 Supporting calculation

The file `certify_c3_image.py` performs the characteristic-zero calculation in Proposition 5.1; it uses only Python rational arithmetic and NumPy for integer matrix construction. The companion `compute_c3_image.py` contains the cyclic link-state basis construction and also provides an optional finite-field exploration. The dimension certificate uses the rational calculation. The exact retained basis is recorded in `c3_exact_basis.json`, with each coefficient represented as two rational numbers in the basis $1, q$. The small file `c3_exact_result.json` records the dimension and closure count. Running

```
python anc/code/certify_c3_image.py
```

from the archive root reconstructs the matrices, verifies all defining relations, recomputes the independent basis and checks closure under both generators.

5.4 The effect of the inverse twist

Proposition 5.5 (Adjoining the inverse-twist sector)**.** *Let*

$$M' = W_{6;0,q} \oplus X^+_{6;0,q} \oplus X^+_{6;0,q^{-1}} \oplus X^+_{6;2,1}, \qquad A' = A(M').$$

Then

$$\dim A' = 397, \qquad \dim A'_{\mathrm{basic}} = 22.$$

Relative to the quiver of Theorem 5.2, the basic algebra has the two additional arrows $c \to a$ and $e \to b$, each of multiplicity one, and the additional one-dimensional second-radical corner from c to b. Consequently

$$\dim \operatorname{rad} A' / \operatorname{rad}^2 A' = 102, \qquad \dim \operatorname{rad}^2 A' = 107,$$

and $\operatorname{rad}^3 A' = 0$, $Z(A') = \mathbb{C}$. *In particular the two twist lists give non-Morita-equivalent image algebras despite having the same simple labels and resonance component.*

Proof. The additional XXZ module has radical layers c, $a \oplus d \oplus e$, b, with all the indicated adjacent arrows nonzero. This follows from the spin-flip comparison and the XXZ structure theorem in [8, Proposition 2.3 and Theorem 2.4]. It adds the two stated first-radical directions and the second-radical direction $c \to b$. The basic second radical therefore has dimension five; its full dimension is $55 + 4 \cdot 13 = 107$. The two new arrow corners contribute $4 \cdot 1 + 1 \cdot 13 = 17$, in addition to the previous contribution 85.

The same exact rational span computation as in Proposition 5.1, now with the four matrices of dimensions $20, 20, 20, 15$, gives 397 independent vectors and verifies closure in all 794 generator products. Thus

$$397 = 188 + (85 + 17) + (55 + 52),$$

which rules out any further directions or multiplicities. The basic dimension is $5 + 12 + 5 = 22$. Radical cube zero follows from the radical lengths of the four generating modules, and the centre follows from Theorem 4.4. Morita equivalence preserves the basic algebra and the first-extension quiver, so the change in arrow count proves the final assertion. □

This calculation is reproduced by

```
python anc/code/certify_c3_image.py --include-inverse.
```

Its exact basis and closure summary are recorded in `c3_inverse_exact_basis.json` and `c3_inverse_exact_result.json`. All generated certificate files are written to the archive's `anc/data/` directory by default.

The separate script `anc/code/verify_quiver.py` enumerates the path spaces of the presentation. It verifies the sixteen-dimensional quadratic relation space, the vanishing of all thirty-eight cubic paths, and the Cartan matrix, dimensions of projective covers and second-extension table using exact rational arithmetic.

6 Exact computations of the marked central action

The calculations in this section start from the face basis of Section 3 and the hoop diagrams in (2.3). They provide finite checks of the all-size theorem and determine the larger central component in a non-isolated example. All rank decisions are made in a cyclotomic number field.

6.1 Matrix construction and exact ranks

Vertices and boundary gaps are numbered $0, \dots, N-1$ cyclically. For a matching π, the complementary faces are the cycles of $i \mapsto \pi(i+1 \bmod N)$; a face is stored as its sorted set of gaps. The generator e_i joins $i, i+1$ and reconnects their former partners. The marked faces are carried through this surgery, and each erased loop receives one of the weights β, A, B, f according to the marked points that it encloses. Rotation decreases every node and gap label by one. Each generator column is obtained by this concatenation rule.

For the hoop matrices, the program inserts the auxiliary cup, expands the N crossings, and applies the seam cap. A term with k local cup–cap smoothings in F has the scalar factor q^{n-k}. The exchanged skein coefficients give $\overline{F}$. Thus the construction uses integral powers of q, without choosing a square root of q.

All entries are evaluated in $K = \mathbb{Q}(e^{i\pi/p})$. Table 1 records exact ranks on the full joint primary component $\mathcal{H}$ containing Z_N. Here $x = 1$, $\varepsilon = 1$, $w = q$, and $H = (F - q - q^{-1})|_{\mathcal{H}}$.

p	N	y	$\dim Y_N$	$\dim \mathcal{H}$	$(\operatorname{rank} H, \operatorname{rank} H^2, \operatorname{rank} H^3)$
5	2	2	4	3	$(1,0,0)$
5	4	2	18	10	$(4,0,0)$
5	6	2	80	35	$(15,0,0)$
5	6	-1	80	35	$(0,0,0)$
3	6	2	80	43	$(21,2,0)$

Table 1: Exact ranks from the two-marked link-state action.

In the $p = 5, N = 6, y = 2$ example, $\mathcal{H} = Z_6$, $F = \overline{F}$ on this component, and

$$F|_{\mathcal{H}} \sim J_2(\varphi)^{\oplus 15} \oplus J_1(\varphi)^{\oplus 5}, \qquad \varphi = 2\cos(\pi/5).$$

On the full 80-dimensional space the ranks of the first four powers of $F - \varphi$ are $60, 45, 45, 45$. For $y = -1$, both hoops restrict to $\varphi 1_{\mathcal{H}}$, in agreement with the exact splitting locus. The algebra generated by the hoops on $\mathcal{H}$ is therefore the algebra of dual numbers in the former case and $\mathbb{C}$ in the latter. Additional exported examples check both signs of ε at six sites for $p = 3$ and $p = 5$.

6.2 The non-isolated six-site component

Proposition 6.1. *Let $q = e^{i\pi/3}$ and $Y = Y_{0,0,1,2,[z,q]}(6)$. Then*

$$\mathcal{H} = \ker(F-1)^3 = \ker(\overline{F}-1)^3$$

is the joint generalized $(1,1)$-eigenspace and has dimension 43. On $\mathcal{H}$, put $U = F - 1$, $V = \overline{F} - 1$, and $D = U - V$. One has

$$U^2 = UV = V^2, \qquad U^3 = V^3 = 0, \qquad UD = D^2 = 0,$$

and

$$\operatorname{rank} U = \operatorname{rank} V = 21, \qquad \operatorname{rank} U^2 = 2, \qquad \operatorname{rank} D = 2.$$

The two hoops generate the algebra

$$\mathbb{C}[F|_{\mathcal{H}}, \overline{F}|_{\mathcal{H}}] \cong \mathbb{C}[u,d]/(u^3, ud, d^2), \qquad u \longmapsto U, \quad d \longmapsto D.$$

It has dimension 4, radical dimension 3, and radical-square dimension 1. The Jordan form of $F|_{\mathcal{H}}$ is

$$J_3(1)^{\oplus 2} \oplus J_2(1)^{\oplus 17} \oplus J_1(1)^{\oplus 3}.$$

Proof. The depth filtration and the joint characters in (3.15) give the standard factors

$$W_{6;0,q}, \quad W_{6;2,1}, \quad W_{6;4,-1}, \quad W_{6;6,q^2}, \quad W_{6;6,q^{-2}}$$

in the joint $(1,1)$-component. Their dimensions are $20, 15, 6, 1, 1$, whose sum is 43. The following exact calculation determines the central action and identifies this component as the stated kernels. The full 80×80 matrix $F - 1$ has successive ranks $58, 39, 37, 37$. The same ranks hold for $\overline{F} - 1$ and for the vertically stacked pairs of corresponding powers. The latter equalities identify the two kernels, and stabilization at the third power identifies the primary component. The supplied rank certificates consist of a nonzero minor and a full-column-rank right kernel for each of the first three

powers of $F-1$. The verification program checks these certificates and recomputes all ranks over $K=\mathbb{Q}[q]/(q^2-q+1)$. They therefore establish the stated ranks over $\mathbb{C}$. The exported invariant kernel basis gives the restricted matrices, whose products verify the displayed relations.

For an explicit independence witness, use the supplied ordered 43-dimensional kernel basis. Index the rows by $1, U, D, U^2$ and the columns by the matrix positions $(1,1),(1,2),(1,29),(2,29)$, with one-based indices. The resulting evaluation matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 9/2 & -3/2 & 6q-27/2 \\ 0 & 0 & 0 & 12q-6 \\ 0 & 0 & -81/4 & 0 \end{pmatrix}.$$

Its determinant is $\frac{2187}{4}(2q-1)\neq 0$. The stated quotient algebra has basis $1, u, d, u^2$, so its surjection onto the hoop algebra is an isomorphism. The Jordan multiplicities follow from the ranks $43, 21, 2, 0$ of $1, U, U^2, U^3$. □

The submodule Z_6 has dimension 35 inside this 43-dimensional component. Both hoops coincide on Z_6 and have square-zero shifted action of rank 15 there. The larger component thus contains central nilpotence beyond the first-depth construction. This calculation determines the algebra generated by the two hoops on $\mathcal{H}$; the indecomposable decomposition of $\mathcal{H}$ as a module for the full periodic algebra is a separate question.

6.3 Reproduction and data certificates

The accompanying archive contains supplementary programs and data in `anc/`, including the construction program `anc/code/compute_marked.py`, the separate verification program `anc/code/verify_marked_data.py`, and seven exact matrix files in `anc/data/`. From the archive root, the non-isolated example is reproduced by

```
python -m pip install numpy==2.3.5 sympy==1.14.0 \
  python-flint==0.9.0
python anc/code/compute_marked.py --n 6 --p 3 --exact \
  --out anc/data/exact_p3_n6_nonsplit.json
python anc/code/verify_marked_data.py anc/data/exact_p3_n6_nonsplit.json
```

The verification program reads the exported matrices independently of the construction algorithm. It checks the local and affine Temperley–Lieb relations, the rotation relations, central commutation, the scalar depth-zero action, the primary component, and the invariance and central-action identities of the explicit tree basis. It also checks the field metadata and the scalars f and g against the recorded parameters. All seven examples can be verified with

```
python anc/code/verify_marked_data.py anc/data/exact_*.json
```

Each JSON file records the field polynomial, a coefficient dictionary, the ordered link-state basis, all generator and hoop matrices, an ordered primary kernel basis, the restricted matrices, and the explicit tree basis. A coefficient is a list of rational strings in ascending powers $1, q, \ldots, q^{[K:\mathbb{Q}]-1}$. A sparse entry is `[row,column,coefficient_id]`, with zero-based indices. If K_0 is the exported primary basis matrix, the restricted matrices satisfy $GK_0 = K_0 G_{\mathcal{H}}$ for every generator and both hoops. For each rank certificate the verifier checks the nonzero minor, the matrix–kernel product, and an identity submatrix of the kernel basis. These supply both bounds for the asserted rank.

7 Module families and further questions

The depth-one construction is compatible with every annular diagram. At fixed p, it therefore gives a sequence of actual modules Z_N for all even sizes, with natural inclusions Θ_N and identical central-action formulas. The first singular threshold $N = 2p - 2$ changes the composition series of the standard factors. It does not interrupt the construction of the extension or its square-zero central action. The six-site computation at $p = 3$ illustrates how this family embeds in a larger joint central component with cubic nilpotence.

Several further questions arise from these results. For higher marked channels $w = \varepsilon q^m$, $m > 1$, joint central characters again separate the proposed pair of standard factors whenever $2m \le N < 2p - 2m$. Indeed the argument of Lemma 3.5 forces $\omega = \pm 1$ and then $r = m$ or $r = p - m$. Constructing the corresponding higher-depth analogue of the tree-incidence map requires additional relations between several cut arcs. A useful objective is an explicit map with controlled rank at the root of unity, together with its parameter-dependent splitting equation.

The five-simple image of Section 5 and the 43-dimensional marked component record different central behavior at the same root of unity and system size. Determining the full representation image of that marked component, its indecomposable summands, and its comparison with the five-simple quotients would extend the two-simple analysis of Section 4. The four-dimensional algebra generated by the two hoops supplies a specific constraint on such a determination.

The annular module families can also be compared with the fusion bifunctor of [6]. Natural questions are whether the computed extension classes are preserved under fusion, which finite-image quotients can be imposed after fusion, and how the second-extension obstruction changes under these operations.

A scaling-limit problem can now be formulated on the fixed-parameter family Z_N. Its additional inputs are a topology on selected states, a normalization of the inclusion and quotient maps, and a specified physical evolution or dilation operator. The annular coefficient $2\varepsilon g$ can then be compared with matrix elements of that operator. Establishing this comparison would connect the finite-dimensional extension calculation with bulk logarithmic correlation functions.

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