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THE QUANTUM KAZHDAN–LUSZTIG CONJECTURE FOR QUANTUM LOOP ALGEBRAS

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Abstract. We prove the quantum Kazhdan–Lusztig conjecture for finite-dimensional type 1 representations of untwisted quantum loop algebras at every non-root-of-unity complex parameter.

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1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra and let $q \in \mathbb{C}^\times$ be a parameter which is not a root of unity. Let I be the Dynkin vertex set and let $Y_{i,a}$, for $i \in I$ and $a \in \mathbb{C}^\times$, be formal variables. A dominant monomial is a product $m = \prod_{i,a} Y_{i,a}^{u_{i,a}}$ with nonnegative integer exponents, only finitely many of which are nonzero. The category $\mathscr{C}$ of finite-dimensional type 1 representations of the untwisted quantum loop algebra $U_q(L\mathfrak{g})$ has a commutative Grothendieck ring. The q-character homomorphism of Frenkel–Reshetikhin embeds this ring into a Laurent polynomial ring and records the generalized eigenvalues of the Drinfeld Cartan currents [12]. Its image admits a noncommutative deformation $\mathcal{K}_t$, called the quantum Grothendieck ring. This deformation has a standard basis $\{E_t(m)\}$ and a bar-invariant canonical basis $\{L_t(m)\}$, both indexed by dominant monomials [20, 14].

Let $L(m)$ be the finite-dimensional simple type 1 module with highest ℓ-weight m. Hernandez's quantum Kazhdan–Lusztig conjecture identifies the specialization of $L_t(m)$ at $t = 1$ with the

q-character of the simple module with highest monomial m [20, Conjecture 7.3]. We prove this identification.

Theorem 1.1. *Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra, let $q \in \mathbb{C}^\times$ be not a root of unity, and let $m = \prod_{i\in I, a\in\mathbb{C}^\times} Y_{i,a}^{u_{i,a}}$ be any dominant monomial with finite support. Then*

$$L_t(m)\big|_{t=1} = \chi_q L(m). \tag{1}$$

The convention for the bar and for the ordered standard character is given in Section 2. In the notation of the original formulation [20, Conjecture 7.3], equation (1) is $\pi_+(L_t(m)) = L(m)$, after identifying the Grothendieck ring with its q-character image.

1.1. **The character problem.** Chari–Pressley classified the finite-dimensional simple type 1 representations by their Drinfeld polynomials [3]. Factoring these polynomials gives the dominant monomial m which labels $L(m)$. This classification separates the parametrization of simple modules from the calculation of their characters and of tensor-product multiplicities. The latter questions depend on the spectral parameters: tensor products can become reducible when ratios of parameters take particular powers of q.

In 1999, Frenkel–Reshetikhin introduced the q-character to retain the spectral information lost by the ordinary weight character [12]. Its monomials encode the joint generalized eigenvalues of the commuting Drinfeld Cartan currents. Injectivity of the q-character map makes identities of these Laurent polynomials identities in the Grothendieck ring. Frenkel–Mukhin described its image by rank-one conditions and established the downward root-monomial structure of simple characters [11, Theorems 4.1 and 5.1].

The analogy with Kazhdan–Lusztig theory concerns the transition from standard objects to simple objects. In their 1979 paper, Kazhdan and Lusztig constructed the bar-invariant basis of a Hecke algebra and its transition polynomials and proposed the corresponding character formula [27]. For quantum loop algebras, ordered tensor products of fundamental representations play the role of standard objects. Their ordinary characters are products of fundamental q-characters. The quantum Grothendieck ring refines this product by a parameter t, and the transition from its standard basis to its canonical basis supplies the corresponding Kazhdan–Lusztig polynomials. More precisely, write

$$E_t(m) = \sum_{n\le m} P_{n,m}(t) L_t(n), \qquad P_{m,m}(t) = 1.$$

If $M(m)$ denotes the ordered tensor product of fundamental modules with highest monomial m, the conjecture gives

$$[M(m) : L(n)] = P_{n,m}(1).$$

Thus the canonical transition determines ordinary tensor-product composition multiplicities, as in Hernandez's original formulation [20, §7.2].

1.2. **Geometric and algebraic t-characters.** For simply-laced $\mathfrak{g}$, Nakajima realized finite-dimensional representations through quiver varieties and related standard-to-simple multiplicities to intersection cohomology [29]. Varagnolo–Vasserot constructed quantum Grothendieck rings using perverse sheaves on quiver varieties [35]. Nakajima's subsequent theory of t-analogues of q-characters established the canonical specialization formula in types A, D, and E [30, Theorem 8.1(2)]. The grading in this construction records the cohomological information governing the transition between the two bases.

Hernandez's algebraic construction extended q,t-characters and canonical bases to arbitrary finite type [20]. His Conjecture 7.3, formulated in the 2002 preprint and published in 2004, asks that these canonical elements specialize to the characters of the corresponding simple

representations. The construction already establishes the specialization of standard characters [20, Theorem 6.2]; the conjecture identifies the resulting canonical labels with the ordinary simple labels. Hernandez further studied monomials in the non-simply-laced case [21]. The fundamental character calculations in that work and the Hernandez–Schedler tables [26] provide the exceptional fundamental input used in Section 18. His proof of the Kirillov–Reshetikhin conjecture and the associated T-systems supplies another family of ordinary character formulas [22].

1.3. **Progress on the conjecture.** Hernandez–Leclerc introduced, in simply-laced type, monoidal subcategories whose Grothendieck rings can be studied by cluster algebras [24]. Hernandez–Oya then constructed quantum Grothendieck-ring isomorphisms between distinguished categories of types B_n and A_{2n-1} and proved the specialization formula in the corresponding type-B categories [25]. Fujita–Hernandez–Oh–Oya extended these comparisons to all non-simply-laced types, proved positivity of the transition polynomials and structure constants, and established the specialization conjecture for all simple modules of type B [14, Corollary 11.4(2)].

One consequence of the same work is a uniform statement in every finite type. For integral spectral parameters Y_{i,q^k} with $k \in [a,b] \cap \mathbb{Z}$, the specialization formula holds when

$$b - a < r^\vee h^\vee,$$

where $r^\vee$ is the lacing number and $h^\vee$ is the dual Coxeter number [14, Corollary 9.8(2)]; see [17, Theorem 2.11 and Remark 2.12] for this interval formulation. Further quantum cluster comparisons prove the formula for reachable simple modules and positivity of the monomial coefficients of simple (q,t)-characters [15, Corollaries 6.10 and 6.12]. Here reachability refers to the simple modules represented by cluster monomials in the corresponding monoidal categorification.

Fujita–Qin introduced freezing operators compatible with ordinary and deformed characters [17]. For classical types, they embed a fixed Dynkin diagram into diagrams of larger rank, apply the spectral-width result there, and freeze back to the original diagram. This proves the conjecture in all classical types and, in particular, settles type C [17, Theorem 2.13]. Together with Nakajima's theorem, these results cover every finite Dynkin type except G_2 and F_4. The argument below proves the specialization formula for every dominant monomial in these two types.

There are also geometric and filtration constructions relevant to this problem. Varagnolo–Vasserot constructed actions of non-symmetric quantum loop groups in critical K-theory [36] and related shifted quantum affine representations to Coulomb branches and weighted-flag algebras [37]. Neguţ described quantum loop representations by shuffle quotients and quiver moduli [31, 32]. Fujita–Hernandez constructed monoidal Jantzen filtrations from R-matrices and identified the resulting deformation with the quantum Grothendieck ring in simply-laced type [13]. In the shifted setting, Geiss–Hernandez–Leclerc extended cluster structures to non-simply-laced types and expressed cluster monomials as limits of cluster expansions in classes of positive prefundamental representations [18]. For a broader account of quantum affine representation theory and its connections with cluster algebras, see Hernandez's survey [23].

1.4. **The geometric filtration.** The central construction is attached to an actual simple module. Write its highest Cartan rational function as a quotient whose pole and zero multiplicities are $\boldsymbol{m}$ and $\boldsymbol{l}$, and put $\boldsymbol{p} = \boldsymbol{m} - \boldsymbol{l}$. At a fixed spectral dimension vector $\boldsymbol{\nu}$, a shuffle realization identifies the module with a quotient of critical cohomology by the kernel of a negative-framing Euler operator. The vectors $\boldsymbol{m}, \boldsymbol{l}, \boldsymbol{\nu}$ are indexed by spectral vertices. For finite dimension vectors put

$\mu \cdot \nu = \sum_v \mu_v \nu_v$ and $E(\mu, \nu) = \sum_{v \to w} \mu_v \nu_w$, with the arrows of Section 4. The center is

$$d_{\boldsymbol{p}}(\boldsymbol{\nu}) = \boldsymbol{p} \cdot \boldsymbol{\nu} + E(\boldsymbol{\nu}, \boldsymbol{\nu}) - \boldsymbol{\nu} \cdot \boldsymbol{\nu}. \tag{2}$$

A polynomial of degree r has centered degree $2r - d_{\boldsymbol{p}}(\boldsymbol{\nu})$. This gives a finite filtration on the actual module and a Laurent polynomial $S_t(m)$ which specializes, by construction, to $\chi_q L(m)$.

Several comparisons are needed to extract representation-theoretic information from this filtration. First, we prove the shuffle-kernel statement directly at the numerical parameter q. A normal-order calculation for high current modes and finite-jet interpolation identify the annihilator of the highest vector with a common residue kernel. The generic shuffle theorem is transferred to numerical parameters by a local ideal argument, retaining the factors at orthogonal colors.

Second, ordinary flag towers determine the cohomological residue kernel. A supported nearby-cycle comparison, a resolution of the graph of stable restriction, and dimensional reduction identify the critical image. Keeping the framing parameters equivariant gives a finite free source quotient of the Euler map. Its deformed residue tests identify the generic highest-vector kernel, and the integral quotient supplies the standard module at coincident parameters.

Third, we use the Coulomb-branch realization of Varagnolo–Vasserot [37]. Explicit fundamental modules give the ordinary quantum loop action, including its constants and determinant branches. An integral generation argument identifies the geometric standard lattice with the ordered tensor-product lattice. This compares the standard and intrinsic filtrations in one ordinary flag algebra. Each current is a minuscule correspondence multiplied by a Riemann–Roch unit. An exact inverse using source Chern classes recovers the homogeneous correspondence and its Ext degree. Finite spectral projectors give the ordinary current map onto a common finite graded quotient.

The resulting finite quotient of the ordinary flag algebra has a nonnegative basic grading with semisimple degree zero. Its normalized simple characters are bar-invariant. Hyperbolic restriction computes the graded standard character as $\overline{E_t(m)}$, whereas homogeneous generation places every non-head composition factor in strictly positive external degree. We obtain

$$\overline{E_t(m)} = S_t(m) + \sum_{n<m} p_{m,n}(t) S_t(n), \qquad p_{m,n}(t) \in t\mathbb{N}[t]. \tag{3}$$

A finite-height induction first proves $S_t(m) \in \mathcal{K}_t$. Applying bar to (3) then gives the triangularity required for canonical basis uniqueness. This proves $S_t(m) = L_t(m)$ before any specialization is taken.

1.5. **Organization and conventions.** Section 2 fixes the algebraic conventions. Section 3 proves the numerical shuffle and residue results. The subsequent geometric and Coulomb constructions are followed by the comparison of intrinsic and standard filtrations. We then normalize the simple characters, establish the exceptional fundamental input, calculate the standard characters, and prove Theorem 1.1. An appendix records a finite certificate for the only nonthin fundamental coefficients used in the proof.

We use $\mathbb{N} = \{0, 1, 2, \ldots\}$. All vector spaces and cohomology groups are over $\mathbb{C}$, unless a rational model is explicitly specified. Polynomial degree is half cohomological degree. Critical cohomology uses the monodromic vanishing-cycle convention with the suspension normalization stated below. Completions are taken at specified finite spectral supports; statements about completed actions are used only on finite continuous modules. All dimension vectors have finite support, including dimension vectors on infinite spectral quivers.

2. Quantum characters and canonical bases

2.1. **Highest monomials.** Let I be the vertex set of the finite Dynkin diagram, let $C = (c_{ij})$ be its Cartan matrix, and choose the minimal positive symmetrizer (d_i). Thus $b_{ij} = d_i c_{ij} = d_j c_{ji}$ and $q_i = q^{d_i}$. Set

$$\mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1} : i \in I,\ a \in \mathbb{C}^\times], \qquad \mathcal{M}_+ = \langle Y_{i,a} : i \in I,\ a \in \mathbb{C}^\times \rangle.$$

Elements of $\mathcal{M}_+$ are called dominant monomials. Products here and below have finite support. The monomial $m = \prod Y_{i,a}^{u_{i,a}}$ represents the rational highest ℓ-weight

$$\psi_i^m(z) = q_i^{\sum_a u_{i,a}} \prod_a \left(\frac{1 - a q_i^{-1}/z}{1 - a q_i / z} \right)^{u_{i,a}}. \tag{4}$$

The finite-dimensional simple objects of $\mathscr{C}$ are the $L(m)$ with $m \in \mathcal{M}_+$. A module is of type 1 if each Cartan generator k_i acts diagonally with eigenvalues in $q_i^{\mathbb{Z}}$. The coefficient of m in $\chi_q L(m)$ is one.

The root monomials are

$$A_{i,a} = Y_{i,aq_i^{-1}} Y_{i,aq_i} \prod_{j \neq i} \prod_{s=1}^{-c_{ji}} Y_{j,aq^{-c_{ji}+1-2s}}^{-1}. \tag{5}$$

For monomials n, m, write $n \leq m$ if $n = m \prod A_{i,a}^{-v_{i,a}}$ with $v_{i,a} \in \mathbb{N}$. These root factorizations are unique for finite support. For example, uniqueness follows by the invertibility of the quantum Cartan matrix as a formal Laurent series, or by successively removing the smallest spectral degree on the finite support in each $q^{\mathbb{Z}}$-orbit. All monomials in $\chi_q L(m)$ satisfy $n \leq m$. Let ω_i and α_i denote the fundamental weights and simple roots. We put $\mathrm{wt}(Y_{i,a}) = \omega_i$, so that $\mathrm{wt}(A_{i,a}) = \alpha_i$.

2.2. **The quantum torus.** Choose one representative for each $q^{\mathbb{Z}}$-orbit of parameters. In one such orbit write $Y_{i,k} = Y_{i,bq^k}$. Define the quantum Cartan matrix by

$$C_{ii}(z) = z^{d_i} + z^{-d_i}, \qquad C_{ij}(z) = \frac{z^{c_{ij}} - z^{-c_{ij}}}{z - z^{-1}} \quad (i \neq j).$$

Write $C(z)_{ij}^{-1} = \sum_{r \in \mathbb{Z}} \widetilde{c}_{ij}(r) z^r$, using its expansion at zero. The skew form on the monomial lattice is determined by

$$\begin{aligned} \gamma(Y_{i,k}, Y_{j,l}) = {} & \widetilde{c}_{ij}(k - l - d_i) - \widetilde{c}_{ij}(k - l + d_i) \\ & - \widetilde{c}_{ij}(l - k - d_i) + \widetilde{c}_{ij}(l - k + d_i). \end{aligned} \tag{6}$$

The form between distinct parameter orbits is zero. This convention is independent of the chosen representatives. The based quantum torus $\mathcal{Y}_t$ is the free $\mathbb{Z}[t^{\pm 1/2}]$-module on the ordinary monomials, with product

$$m * n = t^{\gamma(m,n)/2} mn. \tag{7}$$

The bar is the antilinear anti-involution fixing every based monomial and sending $t^{1/2}$ to $t^{-1/2}$. In particular $\overline{x * y} = \overline{y} * \overline{x}$. The quantum Cartan identity gives

$$\gamma(A_{i,k}, Y_{j,l}) = 2\delta_{ij}(\delta_{k-l,d_i} - \delta_{k-l,-d_i}), \tag{8}$$

$$\gamma(A_{i,k}, A_{j,l}) = 2(\delta_{k-l,b_{ij}} - \delta_{k-l,-b_{ij}}). \tag{9}$$

For completeness, put $D(z) = \mathrm{diag}(z^{d_i} - z^{-d_i})$. The identities

$$C(z)^{\mathsf{T}} D(z) = D(z) C(z), \qquad (D(z)C(z))_{ij} = z^{b_{ij}} - z^{-b_{ij}}$$

show that $D(z)C(z)^{-1}$ is symmetric. Its coefficient of z^r in position (i,j) is $\widetilde{c}_{ij}(r-d_i)-\widetilde{c}_{ij}(r+d_i)$. The exponent array of $A_{i,k}$ is the kth translate of the ith column of $C(z)$. Since $C(z)=C(z^{-1})$, pairing this array with (6) amounts to multiplying on the left by $C(z)^{\mathsf{T}}$. The resulting matrix is $D(z)$. Subtracting its coefficients at $k-l$ and $l-k$ gives (8). Pairing with the second root array gives $D(z)C(z)$ and hence (9). These identities fix the grading normalization below.

2.3. **The quantum Grothendieck ring.** For each i, let $\mathcal{K}_{i,t}$ be the subalgebra generated by the elements $Y_{j,k}^{\pm 1}$ for $j\neq i$ and the rank-one characters $Y_{i,k}(1+A_{i,k+d_i}^{-1})$, in the based-monomial convention, and take the corresponding generators in every parameter orbit. Then

$$\mathcal{K}_t=\bigcap_{i\in I}\mathcal{K}_{i,t}.$$

This is a bar-stable subalgebra of $\mathcal{Y}_t$. The algebraic character theory [20, 21, 17] gives a distinguished element $F_t(m)\in\mathcal{K}_t$ for each dominant m, with unique dominant monomial m and coefficient one. All its remaining monomials are below m. These properties determine it.

Write $m=Y_1\cdots Y_\ell$, ordering each orbit by decreasing spectral index. The corresponding fundamental characters at equal spectral index commute [17, §2.7], and factors from different orbits can be placed in any order. Define

$$E_t(m)=t^{-\frac{1}{2}\sum_{r<s}\gamma(Y_r,Y_s)}F_t(Y_1)*\cdots*F_t(Y_\ell). \tag{10}$$

The $E_t(m)$ form a basis of $\mathcal{K}_t$. The canonical element $L_t(m)$ is characterized by

$$\overline{L_t(m)}=L_t(m),\qquad L_t(m)-E_t(m)\in\sum_{n<m}t^{-1}\mathbb{Z}[t^{-1}]E_t(n). \tag{11}$$

Each expansion has finite support. Existence and uniqueness are the canonical-basis theorem of [20]; the precise convention above is [17, Proposition 2.6]. For a single fundamental variable,

$$F_t(Y_{i,a})=E_t(Y_{i,a})=L_t(Y_{i,a}). \tag{12}$$

Their specialization to the fundamental q-characters, and the specialization of every $E_t(m)$ to the corresponding product, are known independently of Theorem 1.1 [20, Theorem 6.2].

2.4. **Finite support and height.** Let $\rho^\vee$ be half the sum of the positive coroots. For a dominant monomial set

$$h(m)=\langle \mathrm{wt}(m),2\rho^\vee\rangle. \tag{13}$$

It is a nonnegative integer, and if $n=m\prod A_{i,a}^{-v_{i,a}}<m$ is dominant, then

$$h(m)-h(n)=2\sum_{i,a}v_{i,a}>0. \tag{14}$$

Thus induction over dominant monomials will always mean induction on $h(m)$. Every individual module, filtration, or character used below has finite support.

If m involves finitely many parameter orbits, a dominant $n\leq m$ involves no new orbit. Indeed, on an orbit absent from m, its weight would have the form $-\sum_i v_i\alpha_i$ with all $v_i\geq 0$. Pairing with $2\rho^\vee$ gives a nonpositive number, whereas dominance gives a nonnegative number, with equality only for the trivial monomial. Hence all v_i on that orbit vanish. This observation will permit the final induction to handle arbitrary complex spectral parameters.

3. Numerical shuffle ideals and highest-vector kernels

Throughout this section $\mathfrak{g}$ is of type G_2 or F_4 and q is a fixed non-root-of-unity complex number. We work directly with ordinary Drinfeld generators $e_{i,r}, f_{i,r}$, $r \in \mathbb{Z}$, and currents

$$e_i(z) = \sum_{r\in\mathbb{Z}} e_{i,r} z^{-r}, \qquad f_i(z) = \sum_{r\in\mathbb{Z}} f_{i,r} z^{-r}, \qquad \varphi_i^{\pm}(z) = \sum_{r\geq 0} \varphi_{i,r}^{\pm} z^{\mp r}, \quad \varphi_{i,0}^{\pm} = k_i^{\pm 1}.$$

Set $c_i = q_i - q_i^{-1}$. Define the logarithmic Cartan modes $p_{j,r}$, $r \neq 0$, by

$$\varphi_j^{\pm}(z) = k_j^{\pm 1} \exp\left(\sum_{r\geq 1} \frac{p_{j,\pm r}}{r z^{\pm r}} \right). \tag{15}$$

The Drinfeld relations then give

$$[p_{j,r}, f_{i,s}] = -(q^{r b_{ij}} - q^{-r b_{ij}}) f_{i,s+r}, \qquad r \neq 0, \quad s \in \mathbb{Z}; \tag{16}$$

see [31, (2.12), (3.12)].

3.1. **Two shuffle conventions.** Put

$$\zeta_{ij}^{\mathrm{rat}}(z) = \frac{z - q^{-b_{ij}}}{z - 1}. \tag{17}$$

For a negative word $W = f_{i_1,s_1} \cdots f_{i_N,s_N}$ its rational shuffle representative is

$$F_W^{\mathrm{rat}} = \mathrm{Sym}_{\mathrm{colors}} \left[\prod_{a=1}^{N} z_a^{s_a} \prod_{a<b} \zeta_{i_b i_a}^{\mathrm{rat}}(z_b/z_a) \right]. \tag{18}$$

Here Sym is a sum over color-preserving permutations. The binary shuffle product includes the factors $(\prod n_i! \prod n_i'!)^{-1}$, so iterating degree-one factors gives exactly (18). Its direction is that of the negative current algebra.

Fix an order on I, and for a dimension vector $\boldsymbol{n}$ put

$$D_{\boldsymbol{n}} = \prod_{i<j} \prod_{a=1}^{n_i} \prod_{b=1}^{n_j} (1 - z_{ia}/z_{jb}).$$

The polynomial representative is $F^{\mathrm{pol}} = D_{\boldsymbol{n}} F^{\mathrm{rat}}$. Its kernel is

$$\zeta_{ij}^{\mathrm{pol}}(z) = \begin{cases} q^{-b_{ij}} - z, & i < j, \\ (q^{-b_{ii}} - z)/(1 - z), & i = j, \\ 1 - q^{-b_{ij}} z^{-1}, & i > j. \end{cases} \tag{19}$$

For every color order one has the exact identity

$$\frac{F^{\mathrm{pol}}}{\prod_{a<b} \zeta_{i_b i_a}^{\mathrm{pol}}(z_b/z_a)} = \frac{F^{\mathrm{rat}}}{\prod_{a<b} \zeta_{i_b i_a}^{\mathrm{rat}}(z_b/z_a)}. \tag{20}$$

Thus the change of convention is valid even at a zero of $D_{\boldsymbol{n}}$. The image of the word map in the polynomial convention is denoted $\mathcal{S}_{\boldsymbol{n}}(q)$.

The quadratic and Serre relations give a homomorphism from the ordinary negative current algebra to this image. These checks are rational identities in the variables and q, hence remain valid at our fixed parameter.

3.2. **High-index highest rows.** Let v be an actual highest vector, with highest rational function $\boldsymbol{\psi}$. Write $\psi_j(z) = \sum_{r\geq 0} \psi_{j,r} z^{-r}$ for its expansion at infinity. For a word of total weight zero, let $\mathrm{HC}_{\boldsymbol{\psi}}$ denote its coefficient on v. The operator CT extracts the coefficient of $z_1^0 \cdots z_N^0$ in the nested expansion $1 \ll |z_1| \ll \cdots \ll |z_N|$.

Lemma 3.1. *Suppose $W = f_{i_1,s_1} \cdots f_{i_N,s_N}$ with every $s_a > 0$, and $E = e_{j_1,d_1} \cdots e_{j_N,d_N}$ with every $d_a \geq 0$ and matching root content. Label the variables in the order $j_1, \ldots, j_N$. Then*

$$\mathrm{HC}_{\boldsymbol{\psi}}(EW) = \left(\prod_{a=1}^{N} c_{j_a}^{-1} \right) \mathrm{CT} \left[\frac{F_W^{\mathrm{rat}}(\boldsymbol{z}) \prod_{a=1}^{N} z_a^{d_a} \psi_{j_a}(z_a)}{\prod_{a<b} \zeta_{j_b j_a}^{\mathrm{rat}}(z_b/z_a)} \right]. \tag{21}$$

The same formula holds in the polynomial convention.

Proof. For $a = q^{-b_{ij}}$, the Cartan-current relation reads

$$\varphi_j^+(y) f_i(x) = f_i(x) \varphi_j^+(y) R_{ij}(x/y), \qquad R_{ij}(u) = \frac{a-u}{1-au}.$$

Write $R_{ij}(u) = \sum_{r\geq 0} \rho_{ij}(r) u^r$. Explicitly,

$$\rho_{ij}(0) = a, \qquad \rho_{ij}(r) = (a^2-1)a^{r-1} \quad (r>0).$$

Comparison of coefficients gives the finite identity

$$\varphi_{j,k}^+ f_{i,s} = \sum_{r=0}^{k} \rho_{ij}(r) f_{i,s+r} \varphi_{j,k-r}^+. \tag{22}$$

In particular, this operation only raises the indices of the negative letters.

Commute the rightmost positive letter e_{j_N,d_N} through W. The terminal term kills v. A contribution occurs at every position k with $i_k = j_N$. Since $d_N + s_k > 0$, its commutator is $c_{j_N}^{-1} \varphi_{j_N,d_N+s_k}^+$; there is no negative Cartan coefficient. Moving it through the suffix by (22) gives

$$c_{j_N}^{-1} \sum_{\substack{r_l \geq 0 \ (l>k) \\ \sum_{l>k} r_l \leq d_N + s_k}} \left(\prod_{l>k} \rho_{i_l j_N}(r_l) \right) \psi_{j_N, d_N+s_k-\sum_{l>k} r_l} W_{k,\boldsymbol{r}} v. \tag{23}$$

Here $W_{k,\boldsymbol{r}}$ deletes the kth letter and replaces each suffix index s_l by $s_l + r_l$. Its remaining negative indices are still positive.

Expand the symmetrization in (18) as the sum over bijections from word positions to variables of the same color. Partition these bijections according to the position k sent to z_N. The pair factor with a position before k cancels the corresponding denominator in (21). For a position after k, the remaining factor is

$$\frac{\zeta_{i_l j_N}^{\mathrm{rat}}(z_{\sigma(l)}/z_N)}{\zeta_{j_N i_l}^{\mathrm{rat}}(z_N/z_{\sigma(l)})} = R_{i_l j_N}(z_{\sigma(l)}/z_N).$$

Taking the z_N constant term therefore gives exactly (23), apart from the displayed $c_{j_N}^{-1}$. The remaining assignments give the shuffle representative of $W_{k,\boldsymbol{r}}$. This proves the formula by induction on N. Repeated colors cause no extra factor: the $n_{j_N}!$ bijections split into n_{j_N} disjoint sets of size $(n_{j_N}-1)!$.

At every step the inequality in (23) bounds the summation. The formula uses finitely many coefficients of $\boldsymbol{\psi}$ and only the scalar denominators c_i. It is consequently an identity at the given numerical parameter. Finally (20) proves the polynomial version. □

3.3. **Invertible simultaneous mode shifts.** Let $L = L(m)$ be a finite-dimensional simple ordinary module with highest vector v_m. The span $W_i = \operatorname{span}\{f_{i,r}|_L : r \in \mathbb{Z}\}$ is a finite-dimensional subspace of $\operatorname{End} L$. By (16), the operators

$$T_i = -\frac{\operatorname{ad}(p_{i,1})}{q_i^2 - q_i^{-2}}, \qquad T_i^{-1} = \frac{\operatorname{ad}(p_{i,-1})}{q_i^2 - q_i^{-2}}$$

send $f_{i,r}$ to $f_{i,r+1}$ and $f_{i,r-1}$, respectively. Their denominator is nonzero, and their compositions fix every spanning generator of W_i. Thus they are well-defined inverse linear maps on W_i.

Fix $\beta = \sum_i n_i\alpha_i$. Form the finite direct sum V_β of ordered tensor products $W_{i_1} \otimes \cdots \otimes W_{i_N}$ with this root content. Evaluation on a highest vector gives a surjection

$$\operatorname{ev} : V_\beta \longrightarrow L_{\operatorname{wt}(m)-\beta}.$$

Let $A_{i,r}$ be the sum of the rth powers of the slot shifts of color i. Let $p_{j,r}^{\operatorname{top}}$ be the scalar by which $p_{j,r}$ acts on v_m. The logarithmic Cartan relations give

$$(p_{j,r}^{\operatorname{top}} - p_{j,r}) \operatorname{ev} = \operatorname{ev} \sum_i (q^{rb_{ij}} - q^{-rb_{ij}}) A_{i,r}. \tag{24}$$

Lemma 3.2. *For $\mathfrak{g}$ of type G_2 or F_4, simultaneous increase of all indices in a negative word of root content β is realized after evaluation by an invertible operator P_β on $L_{\operatorname{wt}(m)-\beta}$.*

Proof. Put $x = q^r$ and $B_r = (x^{b_{ij}} - x^{-b_{ij}})$. Up to reversing the numbering of the Dynkin diagram, direct determinants are

$$\begin{aligned} \det B_r^{G_2} &= x^{-8}(x-1)^2(x+1)^2\Phi_3(x)\Phi_6(x)\Phi_{24}(x), \\ \det B_r^{F_4} &= x^{-12}(x-1)^4(x+1)^4\Phi_4(x)^2\Phi_{36}(x). \end{aligned} \tag{25}$$

Here Φ_s is the sth cyclotomic polynomial. The symmetric matrices used are respectively

$$\begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

To compute the determinants, write $s_j = x^j - x^{-j}$. In type G_2 the determinant is $s_6 s_2 - s_3^2$. In type F_4, expansion of successive tridiagonal minors gives

$$\begin{aligned} \Delta_1 &= s_4, & \Delta_2 &= s_4^2 - s_2^2, \\ \Delta_3 &= s_2\Delta_2 - s_2^2\Delta_1, & \Delta_4 &= s_2\Delta_3 - s_1^2\Delta_2. \end{aligned}$$

Factoring these Laurent polynomials, using $\Phi_{24}(x) = x^8 - x^4 + 1$ and $\Phi_{36}(x) = x^{12} - x^6 + 1$, gives (25). Thus B_r is invertible for every $r > 0$. Solving (24) realizes the slot power sums $A_{i,r}$ as actual Cartan operators after evaluation. Newton identities, using $1 \le r \le n_i$, realize the product of the slot shifts of each color, and hence their total product T, by an operator P_β with $P_\beta \operatorname{ev} = \operatorname{ev} T$. The kernel of ev is T-stable. Since T is invertible and the space is finite-dimensional, this kernel is actually T-invariant. Its quotient operator P_β is invertible, as claimed. □

3.4. **The common residue kernel.** For a fixed color order $\boldsymbol{i} = (i_1, \ldots, i_N)$ set

$$H_{\boldsymbol{i}}(F) = \frac{F^{\operatorname{rat}}(\boldsymbol{z})}{\prod_{a<b} \zeta^{\operatorname{rat}}_{i_b i_a}(z_b/z_a)} \prod_{a=1}^{N} \psi_{i_a}(z_a). \tag{26}$$

Starting with the smallest nested contour, shrink successive contours and retain every residue at a nonzero constant point. Such points include the poles of ψ_{i_a} and the points obtained from

earlier residues by the shifts prescribed by a remaining denominator. An ordered constant-point test is one of the resulting iterated residue functionals applied to $GH_{\boldsymbol{i}}(F)\prod_a \mathrm{d}z_a/z_a$, where G is an arbitrary polynomial in the labeled variables. At a repeated pole we take the residue of the complete rational function at that value, including all coincident denominator factors. Let $\Theta(\boldsymbol{\psi})_{\boldsymbol{n}}$ be their common kernel, taking all color orders and all such G.

Theorem 3.3. *Let $L(m)$ be an actual finite-dimensional simple module in type G_2 or F_4, with highest vector v_m and highest rational function $\boldsymbol{\psi}$. Evaluation on v_m factors through the numerical shuffle word map. The induced map on its image has kernel $\mathcal{S}_{\boldsymbol{n}}(q)\cap\Theta(\boldsymbol{\psi})_{\boldsymbol{n}}$, and gives an isomorphism*

$$L(m)\cong\bigoplus_{\boldsymbol{n}}\mathcal{S}_{\boldsymbol{n}}(q)\Big/\big(\mathcal{S}_{\boldsymbol{n}}(q)\cap\Theta(\boldsymbol{\psi})_{\boldsymbol{n}}\big). \tag{27}$$

The statement is an equality of annihilator kernels and is valid at coincident spectral parameters and repeated poles.

Proof. We first show that the numerical shuffle kernel acts trivially. Let u be in this kernel with root content β. Increase all indices in all its words by the same sufficiently large integer M, so the new indices are positive. Denote the resulting element by $u[M]$. Then

$$F_{u[M]}=(z_1\cdots z_N)^M F_u=0,\qquad u[M]v_m=P_\beta^M uv_m.$$

Lemma 3.1 shows that every positive word of matching root content with nonnegative indices kills the highest coefficient of $u[M]v_m$.

These rows detect every vector in the relevant weight space. Indeed, if $w\neq 0$, simplicity gives an element taking w to the highest line. In its triangular decomposition, a term with a further negative part of positive root content would first have to raise w to a weight strictly above the highest weight. Such a term is zero. Thus a positive-current word suffices. On the finite-dimensional span of positive-current operators, the same normalized adjoint argument gives an invertible mode shift. Every positive mode is therefore a linear combination of arbitrarily high modes of its color. Replacing every factor in the detecting word proves the assertion for nonnegative indices. We obtain $u[M]v_m=0$ and then $uv_m=0$ by Lemma 3.2. Applying the same argument to uw for every negative word w proves that the shuffle kernel kills the whole module.

It remains to compare the kernel on the small shuffle image with the residue tests. For fixed $\boldsymbol{n}$ the denominator in (26) has bounded order, independent of the numerator. The unshifted cross-color denominators cancel against the inverse shuffle denominator. The remaining intervariable poles are ordered:

$$z_b=q^{-b_{i_ai_b}}z_a\quad(a<b).$$

Consequently the first, smallest contour can meet only fixed poles of ψ_{i_1} and possibly zero. Poles expressed in terms of later independent variables lie outside that contour. After substitution at a nonzero constant, only finitely many new shifted constants occur at the next step. Induction gives finitely many ordered tuples. Each residue is supported on a finite jet at its tuple. Resonant diagonals increase the finite jet order by at most the order of their fixed denominator and do not produce an infinite-order functional.

Zero coordinates can be eliminated uniformly for a given numerator. For every nonempty set S of coordinates tending to zero, the rational integrand has a finite lower bound on its total degree in those coordinates. Factors internal to S have fixed homogeneous degree and the factors joining S to a nonzero coordinate have nonnegative power series expansions. Multiplication by P^D, where $P=\prod_a z_a$, increases the bound by $D|S|$. For large D all residues with a zero coordinate vanish.

On the finite jet algebra at any nonzero tuple, P is a unit. The Chinese remainder theorem on the labeled polynomial ring separates all distinct ordered tuples and prescribes arbitrary jets there. It follows that all constant-point tests vanish if and only if all nested constant terms with sufficiently high nonnegative powers $z_1^{d_1}\cdots z_N^{d_N}$ vanish. The finite-jet assertion also handles repeated coordinates in their single local jet algebra.

Choose a negative preimage u of F and shift its modes as above. Multiplication by P^M is invertible on the nonzero jet algebras, so the residue tests vanish for F_u if and only if they vanish for $F_{u[M]}$. By Lemma 3.1 and positive-tail detection, the latter condition is $u[M]v_m = 0$. By Lemma 3.2 it is equivalent to $uv_m = 0$. This proves the kernel equality and the isomorphism. □

The kernel argument also applies over a characteristic-zero coefficient field when the spectral poles split and the highest-weight module is absolutely simple. Indeed, all normal-order identities and invertible mode shifts are finite linear-algebra calculations. The contour calculation is equivalently the formal iterated-residue expansion in the specified order, and Chinese remainders apply to its finite jet algebras. One may extend to an algebraic closure to detect the highest rows and then descend the equality of kernels by faithful scalar extension. We will use this version over $\mathbb{C}((a))$.

3.5. **Wheel ideals at every numerical parameter.** In the polynomial convention, the multiplicative Serre-wheel condition for $i \neq j$ is

$$F\big|_{z_{i,r}=z_{j,1}q^{b_{ij}+2(r-1)d_i},\ 1\le r\le 1-c_{ij}} = 0. \tag{28}$$

All choices of distinct labeled variables of these colors are included. For $c_{ij} = 0$ this is the condition at $z_{i,1} = z_{j,1}$, which is essential in the augmented polynomial convention.

Fix a spectral tuple $\boldsymbol{x}$. Write its entries as $x_a = b_{c(a)}q^{k_a}$, where the b_c represent distinct $q^{\mathbb{Z}}$-orbits. The associated spectral vertices are $v = (i, b_c q^k)$. There is an arrow

$$(i, b_c q^s) \longrightarrow (j, b_c q^t) \quad\Longleftrightarrow\quad s = t + b_{ij}. \tag{29}$$

Let $E(v,w)$ be the number of such arrows and let ν_v be the multiplicity of v in the tuple. Set $R_{\boldsymbol{\nu}} = \mathbb{C}[y_{v,a}]^{\prod_v \mathfrak{S}_{\nu_v}}$. Every multiplicative wheel through $\boldsymbol{x}$ becomes a diagonal where the corresponding y-variables are equal after writing $z_a = x_a(1+y_a)$. Let $I_{\Gamma,\boldsymbol{\nu}} \subset R_{\boldsymbol{\nu}}$ be the intersection of the ideals of these diagonals. Equivalently its elements vanish on all such lifted Serre wheels. This is a homogeneous ideal, including the diagonals of orthogonal-color pairs.

Theorem 3.4. *At every numerical non-root-of-unity parameter, the completion of $\mathcal{S}_{\boldsymbol{n}}(q)$ at the colored unordered multiset represented by $\boldsymbol{x}$ identifies, in the coordinates $z_a = x_a(1+y_a)$, with the completion of $I_{\Gamma,\boldsymbol{\nu}}$ at the origin. Moreover, $I_{\Gamma,\boldsymbol{\nu}}$ is spanned by the homogeneous additive words*

$$\mathrm{Sym}_{\mathrm{vertices}}\left[G(\boldsymbol{y})\prod_{a<b}(y_b-y_a)^{E(v_a,v_b)-\delta_{v_a,v_b}}\right], \qquad G\in\mathbb{C}[y_1,\dots,y_N], \tag{30}$$

over all vertex orders with multiplicity vector $\boldsymbol{\nu}$. The same assertion holds over $\mathbb{Q}$ for the combinatorial spectral quiver.

Proof. Let $A = \mathbb{C}[z_{ia}^{\pm 1}]$ be the labeled Laurent ring and $R = A^{\prod_i \mathfrak{S}_{n_i}}$. For a fixed color order, the word map $G \mapsto \mathrm{Sym}(GK)$ is R-linear. The apparent equal-color denominators cancel by antisymmetrizing over the corresponding Vandermonde. The image therefore lies in R. Since A is finite over R, each word image is a finite R-module. There are finitely many color orders, so their sum $\mathcal{S}_{\boldsymbol{n}}(q)$ is a finite ideal of R. In particular images commute with flat completion and are closed in the adic topology.

Complete the labeled finite extension over the orbit of $\boldsymbol{x}$. It splits as the product of the completions at its labeled points. In a chosen component full symmetrization becomes symmetrization by the stabilizer $\prod_v \mathfrak{S}_{\nu_v}$. In the coordinates $z_a = x_a(1 + y_a)$ every resonant numerator is $y_b - y_a$ times a unit; every nonresonant factor is a unit; and the same-vertex denominator is the Vandermonde factor. Thus each completed word map is

$$G \longmapsto \mathrm{Sym}(G\kappa_\sigma U_\sigma), \tag{31}$$

where κ_σ is the product in (30) and U_σ is a formal unit. Multiplication by U_σ is an automorphism of the arbitrary labeled input ring. Removing it from the input proves that the completed word ideal depends only on the resonance pattern. If one realizes this argument by polynomial interpolation, one first prescribes the labeled jets to an order larger by the finite Vandermonde division order.

The local wheel ideal has the same property: a wheel component not passing through $\boldsymbol{x}$ disappears on completion, and a component passing through it becomes exactly its linear diagonal. The completed wheel ideal is therefore the completion of the homogeneous ideal $I_{\Gamma,\boldsymbol{\nu}}$.

Now replace the fixed q by an indeterminate q' and replace the orbit representatives by algebraically independent b'_c. The tuple $b'_{c(a)}(q')^{k_a}$ has exactly the same resonance pattern. Indeed, within one orbit all relevant equations are equalities of integer exponents, since the original q is not a root of unity; between different orbits no relevant equality holds. Over $\mathbb{Q}(q', b'_c)$, the generic shuffle theorem [33, Theorem 1.7 and Definition 6.2] identifies the small shuffle ideal with the wheel ideal. Its completion, followed by (31), identifies the completed additive word ideal with the completed diagonal ideal.

Both additive ideals are homogeneous and finitely generated over the invariant polynomial ring defined over $\mathbb{Q}$. Extend this polynomial ring to $\mathbb{Q}(q', b'_c)$, localize at its homogeneous maximal ideal, and complete. Completion is faithfully flat over this local ring. The two graded quotients of the sum of these ideals by either one therefore have zero localization there. A nonzero finitely generated graded module over a nonnegatively graded polynomial ring cannot have this property: graded Nakayama, or its lowest nonzero degree, detects it at the homogeneous maximal ideal. Thus both quotients vanish globally after field extension, and faithful flatness of the field extension proves equality over $\mathbb{Q}$, hence over $\mathbb{C}$. Applying the first two paragraphs again at the original numerical tuple proves the completion statement. All scalar extensions in this argument precede local completion. □

Theorem 3.4 and the finite-jet description of Theorem 3.3 reduce every intrinsic local sector to a homogeneous additive quotient. We next identify this quotient and its grading geometrically.

4. The spectral quiver and cyclic moduli

We give the geometric realization of the local residue quotient. All cohomology groups in this section have rational coefficients; extending scalars to $\mathbb{C}$ will be understood when we compare them with quantum-loop modules. Polynomial degree is half the ordinary cohomological degree. The constructions are made in each finite dimension vector, and therefore do not require a bound on the spectral support.

Let $C = (c_{ij})_{i,j\in I}$ be the Cartan matrix, let (d_i) be its minimal positive symmetrizer, and put $b_{ij} = d_i c_{ij} = d_j c_{ji}$. For one $q^{\mathbb{Z}}$-orbit of spectral parameters define the quiver Γ by

$$\Gamma_0 = I \times \mathbb{Z}, \qquad (i,k) \longrightarrow (j, k - b_{ij}) \quad (i,j \in I). \tag{32}$$

Different spectral orbits give disjoint copies of this quiver. An arrow with $i = j$ will be denoted by $\epsilon_{i,k}$; it goes from (i,k) to $(i, k - 2d_i)$. In particular there is no loop at a spectral vertex. If $c_{ij} = 0$, we retain both arrows between (i,k) and (j,k). These arrows are part of the polynomial

shuffle convention of [32, §3.2], and their quadratic contribution to the potential will be retained throughout.

Here is the potential, including its grading. In the ungraded quiver write $a_{ji} : i \to j$ and $\epsilon_i = a_{ii}$, and use right-to-left composition. The potential is the cyclic expression

$$W = \sum_{i \neq j} a_{ji} \epsilon_i^{-c_{ij}} a_{ij}. \tag{33}$$

Its graded lift W_Γ is the sum, over $i \neq j$ and $k \in \mathbb{Z}$, of the following closed paths:

$$(j,k) \xrightarrow{a_{ij,k}} (i, k - b_{ij}) \xrightarrow{\epsilon_i^{-c_{ij}}} (i, k + b_{ij}) \xrightarrow{a_{ji,k+b_{ij}}} (j,k). \tag{34}$$

The middle path consists of $-c_{ij}$ consecutive ϵ_i-arrows. It has the stated endpoint because $-2d_i(-c_{ij}) = 2b_{ij}$. An exponent zero means the constant path. Thus the two ordered terms for an orthogonal pair are the same cyclic quadratic monomial and have sum $2a_{ji}a_{ij}$. Formula (34) is precisely the graded lift of [32, (91)]. On a finitely supported representation only finitely many of its terms are nonzero.

Fix finitely supported vectors $\boldsymbol{\nu} = (\nu_v)$ and $\boldsymbol{m} = (m_v)$ with nonnegative integer entries, and vector spaces $V_v = \mathbb{C}^{\nu_v}$ and $W_v = \mathbb{C}^{m_v}$. Put

$$\begin{aligned} \operatorname{Rep}_{\boldsymbol{\nu}}(\Gamma) &= \bigoplus_{h: v \to w} \operatorname{Hom}(V_v, V_w), \\ \operatorname{Rep}_{\boldsymbol{\nu},\boldsymbol{m}}(\Gamma) &= \operatorname{Rep}_{\boldsymbol{\nu}}(\Gamma) \oplus \bigoplus_v \operatorname{Hom}(W_v, V_v), \\ G_{\boldsymbol{\nu}} &= \prod_v \operatorname{GL}(V_v). \end{aligned} \tag{35}$$

A framed representation is *cyclic* if the images of the framing maps generate it under all paths in Γ. Its gauge stabilizer is trivial: an automorphism fixing the framing fixes every path-generated vector. The cyclic locus is open, and its geometric quotient

$$X_{\boldsymbol{\nu},\boldsymbol{m}} = \operatorname{Rep}_{\boldsymbol{\nu},\boldsymbol{m}}(\Gamma)^{\mathrm{cyc}} / G_{\boldsymbol{\nu}}$$

is smooth. The trace $f_{\boldsymbol{\nu}} = \operatorname{Tr} W_\Gamma$ is gauge invariant and pulls back to the framed space and descends to $X_{\boldsymbol{\nu},\boldsymbol{m}}$. We write f when the dimension vector is understood.

For finitely supported vectors set

$$E(\boldsymbol{\nu}, \boldsymbol{\mu}) = \sum_{h: v \to w} \nu_v \mu_w, \qquad d_{\boldsymbol{p}}(\boldsymbol{\nu}) = \boldsymbol{p} \cdot \boldsymbol{\nu} + E(\boldsymbol{\nu}, \boldsymbol{\nu}) - \boldsymbol{\nu} \cdot \boldsymbol{\nu}. \tag{36}$$

The complex dimension of $X_{\boldsymbol{\nu},\boldsymbol{m}}$ is $d_{\boldsymbol{m}}(\boldsymbol{\nu})$. Let $\mathcal{V}_v$ be its tautological bundle. For another nonnegative vector $\boldsymbol{l} = (l_v)$, define

$$\boldsymbol{p} = \boldsymbol{m} - \boldsymbol{l}, \qquad \mathcal{E}_{\boldsymbol{l}} = \bigoplus_v \mathcal{V}_v^{\oplus l_v}, \qquad L = \boldsymbol{l} \cdot \boldsymbol{\nu}, \qquad D = e(\mathcal{E}_{\boldsymbol{l}}) \cup -. \tag{37}$$

There is no positivity requirement on $\boldsymbol{p}$. In Chern roots, $e(\mathcal{E}_{\boldsymbol{l}}) = \Delta_{\boldsymbol{l}} := \prod_{v,a} y_{v,a}^{l_v}$, so D has cohomological degree $2L$. Then

$$d_{\boldsymbol{p}}(\boldsymbol{\nu}) = d_{\boldsymbol{m}}(\boldsymbol{\nu}) - L. \tag{38}$$

We use $d_{\boldsymbol{p}}(\boldsymbol{\nu})$ to center the local graded character.

5. Ordinary cohomology and ordered residue tests

We first prove the ordinary kernel theorem for a finite quiver Q. This statement permits oriented cycles and loops. It consequently applies to every finite part of (32). Write $e_{vw} = \#\{v \to w\}$ and

$$R_{\boldsymbol{\nu}} = \mathbb{Q}[y_{v,a} : 1 \le a \le \nu_v]^{\prod_v S_{\nu_v}}, \qquad \deg y_{v,a} = 1.$$

Let $N = \sum_v \nu_v$. For an order $\alpha = (v_1, \dots, v_N)$ with content $\boldsymbol{\nu}$, relabel the roots as $y_1, \dots, y_N$, and define

$$K_{\alpha,\boldsymbol{m}}(y) = \prod_{b=1}^{N} y_b^{-m_{v_b}} \prod_{a<b} (y_b - y_a)^{\delta_{v_a v_b} - e_{v_a v_b}}, \qquad \lambda_{\alpha,\boldsymbol{m}}(P) = [y_1^{-1} \cdots y_N^{-1}]\, P K_{\alpha,\boldsymbol{m}}. \tag{39}$$

Every inverse factor is expanded in y_a/y_b for $a < b$. Equivalently these are constant-point residues, with y_1 taken first. All multipliers below are polynomials in the *labeled* variables; no symmetry is imposed on a multiplier. Set

$$J_{\boldsymbol{m}} = \{F \in R_{\boldsymbol{\nu}} : \lambda_{\alpha,\boldsymbol{m}}(FG) = 0 \text{ for every } \alpha \text{ and every } G \in \mathbb{Q}[y_1, \dots, y_N]\}. \tag{40}$$

The same formula defines $J_{\boldsymbol{p}}$ when some entries of $\boldsymbol{p}$ are negative.

Theorem 5.1. *The tautological evaluation map induces a graded ring isomorphism*

$$R_{\boldsymbol{\nu}}/J_{\boldsymbol{m}} \xrightarrow{\sim} H^{2*}(X_{\boldsymbol{\nu},\boldsymbol{m}}, \mathbb{Q}).$$

The cohomology is finite dimensional and has no odd part.

Proof. We construct smooth projective varieties on which the functionals in (39) are actual integrations. Fix α. At the bth stage add a last kernel line L_b of color $j = v_b$; the old object V' is the cyclic framed quotient by that line. All arrows strictly raise the flag index. On the preceding flag space consider

$$0 \longrightarrow \mathcal{E}_j \longrightarrow W_j \otimes \mathcal{O} \oplus \bigoplus_{h:i\to j} \mathcal{V}'_i \xrightarrow{d_j} \mathcal{V}'_j \longrightarrow 0, \qquad d_j(w, (x_h)) = I_j(w) + \sum_h A_h(x_h). \tag{41}$$

The last map is surjective. Indeed each path-generated vector at j is either a framing vector or is obtained by applying the last incoming arrow of its path. Cyclicity says that these vectors span V'_j. Consequently $\mathcal{E}_j$ is a vector bundle of rank

$$r_b = m_j + \sum_i e_{ij}\nu'_i - \nu'_j.$$

To identify the extensions, choose a splitting $V_j = V'_j \oplus L_b$. The additional framing row and the incoming-arrow rows give a map $W_j \oplus \bigoplus_{h:i\to j} V'_i \to L_b$. Changing the splitting adds ud_j for some $u : V'_j \to L_b$. The quotient of the row data is therefore $\mathrm{Hom}(E_j, L_b)$. The extension is cyclic precisely when this class is nonzero: otherwise the old framed representation splits off, whereas a noncyclic extension has a generated subrepresentation mapping isomorphically to V' and hence provides such a splitting. Dividing by scalar changes of L_b gives

$$F_b = \mathbb{P}_{F_{b-1}}(\mathcal{E}_j), \qquad F_0 = \mathrm{pt}, \tag{42}$$

where projectivization parametrizes one-dimensional quotients. The universal quotient is the actual kernel line L_b. If $r_b = 0$, this stage is empty. Every nonempty final space F_α is a smooth projective iterated projective bundle, and its cohomology is generated by the labeled classes $y_b = c_1(L_b)$.

For a quotient projective bundle $\pi : \mathbb{P}(\mathcal{E}) \to B$ and $t = c_1(\mathcal{O}(1))$, the projective-bundle formula reads

$$\pi_* P(t) = [t^{-1}] \frac{P(t)}{c_t(\mathcal{E})}, \qquad c_t(\mathcal{E}) = t^r - c_1(\mathcal{E}) t^{r-1} + \cdots + (-1)^r c_r(\mathcal{E}),$$

with expansion at infinity. The sign is fixed by $\pi_*(t^{r-1}) = 1$. Applying (41) gives

$$\frac{1}{c_t(\mathcal{E}_j)} = t^{-m_j} \prod_{a<b} (t - y_a)^{\delta_{v_a j} - e_{v_a j}}.$$

Successive integrations consequently yield

$$\int_{F_\alpha} P(y) = \lambda_{\alpha, \boldsymbol{m}}(P). \tag{43}$$

For precision, geometric integration removes y_N first, expanding at infinity relative to the preceding variables. All stages use the single chamber $|y_1| \ll \cdots \ll |y_N|$. Constant-point residues in the reverse order extract the same Laurent coefficient in that chamber. For a polynomial numerator only finitely many expansion terms can contribute: starting with y_1, the desired exponent bounds all powers introduced by factors involving y_1, and induction does the same for the remaining variables. Thus no exchange of an uncontrolled infinite sum is involved. At a rank-zero stage the projective pushforward is the coefficient of t^{-1} in a polynomial and vanishes, which also settles the empty-tower case. If the tower is nonempty, its dimension is

$$d_\alpha = \boldsymbol{m} \cdot \boldsymbol{\nu} + \sum_{a<b} e_{v_a v_b} - \sum_{a<b} \delta_{v_a v_b} - N. \tag{44}$$

The kernel in (39) has degree $-d_\alpha - N$, so the functional sees only the degree-d_α part of its numerator.

We next show that the flag pullbacks jointly detect cohomology. The cyclic GIT quotient is projective over $Y = \mathrm{Rep}_{\boldsymbol{\nu}}(Q)//G_{\boldsymbol{\nu}}$; incoming framing introduces no additional affine invariants. Scaling every arrow with weight one and leaving the framing unchanged contracts Y to its origin. Properness gives a limit as the scaling parameter tends to zero for every point of $X = X_{\boldsymbol{\nu}, \boldsymbol{m}}$. The fixed locus is contained in the projective central fiber, so X is smooth semiprojective. Its core is exactly $C = X \times_Y \{0\}$: a limit at infinity can exist only over the origin, and projectivity of the central fiber gives that limit for every point over the origin. The core theorem [19, Theorem 1.3.1] gives the actual restriction isomorphism

$$H^k(X, \mathbb{Q}) \xrightarrow{\sim} H^k(C, \mathbb{Q}). \tag{45}$$

Every nullcone representation admits a complete flag with zero-arrow vertex-simple factors. One can obtain it from a Hilbert–Mumford one-parameter subgroup taking the arrow representation to zero and then refine its weight filtration by vertex lines. Thus forgetting flags gives a proper surjection $\coprod_\alpha F_\alpha \to C$.

Deligne's proper-surjection theorem [6, Propositions 8.2.5 and 8.2.7] says that the kernel of pullback from a proper variety to a smooth proper variety surjecting onto it has weights at most $k-1$ in degree k. This statement also applies to a finite disjoint union of smooth proper varieties. Here $H^k(C)$ has weights at most k by properness, and at least k by (45) and smoothness of X. The kernel is therefore zero. Equivalently, proper cohomological descent shows that all positive simplicial degrees in a smooth proper hypercover have weights strictly smaller than the total degree, which is precisely the assertion used here. We have proved

$$H^*(X, \mathbb{Q}) \hookrightarrow \bigoplus_\alpha H^*(F_\alpha, \mathbb{Q}). \tag{46}$$

If $\mathrm{ev}(F) = 0$, equation (43) gives every vanishing in (40). Conversely, if $F \in J_{\boldsymbol{m}}$, all its pairings on every F_α vanish. Labeled line classes generate the cohomology of a tower, so the allowed multipliers exhaust all classes against which Poincare duality tests the pullback of F. The pullback is zero on every tower, and (46) gives $\mathrm{ev}(F) = 0$. This proves equality of kernels. Tautological generation, finite dimensionality, and evenness for cyclic moduli of arbitrary finite quivers follow from [10, Theorems 2.21 and 5.5, (59)–(63)]. They complete the proof. □

Lemma 5.2. *The completed kernel of all multiplicative local ordered residue tests is carried to the completed extension of $J_{\boldsymbol{p}}$ by $z_a = x_a e^{y_a}$, where x_a is the prescribed nonzero spectral center. The assertion holds with all framing weights in a common formal parameter as well.*

Proof. A resonant difference becomes a linear difference times a unit: $e^{y_b} - e^{y_a} = (y_b - y_a)U_{ab}$ and $e^{y_a} - 1 = y_a U_a$, with $U_{ab}(0) = U_a(0) = 1$. The residue Jacobian is another unit. For a fixed labeled order their product can be nonsymmetric. Multiplication by it is nevertheless an automorphism of the space of labeled Taylor jets before taking a residue. The finite-jet bound in (44), or the finite principal-part recursion for a prescribed multiplicative tuple, allows its inverse to be replaced by a polynomial to the necessary order. For signed $\boldsymbol{p}$, choose $\boldsymbol{m} \geq \boldsymbol{p}$ with $\boldsymbol{m} \geq 0$; the additional numerator factor is the polynomial $\Delta_{\boldsymbol{m}-\boldsymbol{p}}$, so the same finite principal-part bound applies. Hence the intersections of kernels coincide. Over $\mathbb{Q}[[a]]$ the same argument is applied modulo successive powers of (a, y) and then passed to the inverse limit. All nonunit cross-color factors of the polynomial shuffle convention remain in place; the argument removes only the displayed units. □

Proposition 5.3. *Let $A = \mathbb{G}_m$ act on each framing space with integral weights $c_{v,r}$, and put $a = c_1(\mathbb{C}_1)$. In completed equivariant cohomology the evaluation kernel is the deformed version of (40), obtained by replacing $y_b^{-m_{v_b}}$ with $\prod_{r=1}^{m_{v_b}} (y_b - c_{v_b,r}a)^{-1}$. No difference of framing weights is inverted.*

Proof. The exact sequence (41) and every tower are A-equivariant. Their equivariant integrations give the stated functionals. Evenness of ordinary cohomology implies that $H_A^*(X) \otimes_{\mathbb{Q}[a]} \mathbb{Q}[[a]]$ is finite free over $\mathbb{Q}[[a]]$. Choose finitely many ordinary labeled flag tests separating $H^*(X)$ using Theorem 5.1, and lift them equivariantly. Their pairing matrix has a square minor nonzero modulo a, hence invertible over $\mathbb{Q}[[a]]$. The lifted tests therefore detect every equivariant class. Tautological generators lift and remain surjective by graded Nakayama. This proves both inclusions of kernels, including when several framing factors have the same reduction. □

6. Specialization of a potential with several positive weights

We specify the specialization maps before using them. For a regular function f on a smooth complex variety M, use normalized vanishing cycles

$$\Phi_f(F) = R\Gamma_{\{\mathrm{Re}\, f \geq 0\}} F\Big|_{f^{-1}(0)}, \qquad \Phi_0 = \mathrm{id}.$$

A complex on the zero fiber is extended by zero when regarded as a complex on M. If $m = \dim_{\mathbb{C}} M$ and $\omega_M = \mathbb{Q}_M[2m]$, put

$$H^k_{\mathrm{crit}}(M, f) = H^k R\Gamma(M, \Phi_f \mathbb{Q}_M), \qquad H_i^{\mathrm{crit,BM}}(M, f) = H^{-i} R\Gamma(M, \Phi_f \omega_M) = H^{2m-i}_{\mathrm{crit}}(M, f). \tag{47}$$

All constructions below admit commuting reductive coefficient groups; they are made at finite free Borel approximations and then stabilized in each degree.

Assume that M is smooth and projective over an affine variety B, with an equivariant $\mathbb{G}_m$-action λ for which every point of B has a limit at zero. We also allow $M = B$ to be a

representation with nonnegative point weights. Suppose

$$f = \sum_{d>0} f_d, \qquad f_d(\lambda(u)x) = u^d f_d(x). \tag{48}$$

The sum is finite. Properness implies that every point of M has a limit at zero: apply the valuative criterion to its orbit and its limit in B. Define

$$N = M \times \mathbb{A}^1_t, \qquad \mathcal{W}(x,t) = \sum_{d>0} t^d f_d(x), \qquad u \cdot (x,t) = (\lambda(u)x, u^{-1}t).$$

This action preserves $\mathcal{W}$. Its full attracting set is $C = M \times \{0\}$, and its fixed locus is $M^\lambda \times \{0\} \subset \mathcal{W}^{-1}(0)$. Indeed a point with $t \neq 0$ has no limit under $u \to 0$, whereas a point with $t = 0$ has the required limit by properness. The parameter t is the deformation parameter. Nearby cycles in the following proof are taken for the *potential* map $\mathcal{W} : N \to \mathbb{A}^1_s$; the two parameters have different roles.

Proposition 6.1. *Let $V = \mathcal{W}^{-1}(0)$, let $s_0 : V \hookrightarrow N$, and let $c : C \hookrightarrow V$. The canonical Milnor map induces a quasi-isomorphism of global complexes*

$$\eta_M : R\Gamma(C, \omega_C) \xrightarrow{\sim} R\Gamma(C, c^! \Phi_{\mathcal{W}} \omega_N). \tag{49}$$

This map is natural for pullback between smooth ambient varieties respecting the data and for proper trace. It is compatible with the coefficient-group equivariance.

Proof. Use the dual normalized Milnor triangle

$$\omega_V = s_0^! \omega_N \xrightarrow{\mathrm{can}_{\mathcal{W}}} \Phi_{\mathcal{W}} \omega_N \longrightarrow \Psi_{\mathcal{W}} \omega_N \xrightarrow{+1} . \tag{50}$$

Here Ψ includes the nearby-cycle shift determined by this triangle. The map in (49) is its first arrow after applying $R\Gamma(C, c^! -)$; its cone is $R\Gamma(C, c^! \Psi_{\mathcal{W}} \omega_N)$. Thus the comparison is specified independently of any chosen stratification.

The attracting decomposition of M is filtrable in the present projective-over-affine situation. To obtain the needed finite closed filtration, choose an equivariant relatively ample bundle and an embedding into $\mathbb{P}(V_0) \times B$ by finitely many homogeneous sections. Ordering the projective-coordinate weights gives closed unions of attracting strata; components with the same weight may be grouped. The same construction works if the affine base has weight-zero directions. Write $A_j \subset C$ for a stratum, F_j for its limit locus, $r_j : A_j \to F_j$ for the limit map, and $a_j : A_j \hookrightarrow V$ for its inclusion. Hyperbolic localization for the inverse action has component $(r_j)_* a_j^!$.

Apply the nearby-cycle compatibility of hyperbolic localization [34, Theorem 3.3] to $\mathcal{W} : N \to \mathbb{A}^1_s$. More precisely, base change to the strict henselization of the s-line at zero. The torus action is locally linearizable: the smooth quasi-projective ambient variety admits invariant affine neighborhoods, and this property is retained after that base change. The action on the s-line is trivial; the generic-fiber dualizing complex is equivariant and hence monodromic. Formation of the fixed locus and attracting diagram commutes with base change in s. Every nonzero s-fiber has empty fixed locus, since the full fixed locus of N lies in V. The theorem therefore gives

$$(r_j)_* a_j^! \Psi_{\mathcal{W}} \omega_N = 0, \qquad R\Gamma(A_j, a_j^! \Psi_{\mathcal{W}} \omega_N) = 0.$$

The localization triangles for cohomology with support in the finite closed filtration of C show that the entire global cone of η_M vanishes.

For the coefficient passage, the cited hyperbolic-localization theorem is formulated for etale torsion coefficients. First use $\mathbb{Z}/\ell^n$ at a finite Borel approximation. The comparison theorem for nearby cycles over a smooth complex curve [7, Expose XIV, Theorem 2.8] applies to the separated finite-type map $\mathcal{W}$ and constructible coefficients. Together with constructible comparison for support and direct image, it identifies the resulting vanishing with its Betti

counterpart. The canonical maps above are defined integrally and commute with derived reduction of coefficients. Their cohomology is finitely generated at each finite approximation. A nonzero rational cohomology group of the cone would contain a free integral summand, detected modulo ℓ, contradicting the torsion-coefficient vanishing. The rational statement and then its complex scalar extension follow.

We give the naturality argument since the map needed later is an arbitrary map between smooth ambient spaces. Let $b : M' \to M$ respect the scaling and potentials, put $B' = b \times \mathrm{id}$, and write $d = \dim M' - \dim M$. The central support square is Cartesian, and both central divisors have the same trivial normal parameter line, with the identity map on that line. Local-cohomology exchange gives

$$b^* \Phi_f \longrightarrow \Phi_{f \circ b} b^*.$$

This transformation is defined for every b; its being an isomorphism is needed only for smooth pullback. With dualizing normalization it has BM degree $2d$. Apply the same support exchange to the inclusions $\{\mathcal{W} = 0\} \subset \{\mathrm{Re}\,\mathcal{W} \geq 0\}$. The Milnor canonical map is compatible with these inverse-image support maps, and the purity identification for the central parameter divisor is preserved. Writing $N', C', c', \mathcal{W}'$ for the corresponding data over M', the induced square is

$$\begin{array}{ccc} H_i^{\mathrm{BM}}(M) & \xrightarrow{\eta_M} & \mathbb{H}^{-i}(C, c^! \Phi_{\mathcal{W}} \omega_N) \\ {\scriptstyle b^!}\big\downarrow & & \big\downarrow{\scriptstyle (b \times \mathrm{id})^!} \\ H_{i+2d}^{\mathrm{BM}}(M') & \xrightarrow{\eta_{M'}} & \mathbb{H}^{-i-2d}(C', c'^! \Phi_{\mathcal{W}'} \omega_{N'}). \end{array}$$

The left arrow is the refined l.c.i. pullback between smooth ambient varieties; the right arrow is its supported critical counterpart, with the normalization of [1, §2.2.3(i)]. Equivalently one can factor b into its graph embedding followed by a smooth projection; the graph of $b \times \mathrm{id}$ meets the central parameter divisor transversely, so the refined pullback has an excess bundle of rank zero.

If b is proper, use instead proper compatibility of vanishing cycles and Verdier trace:

$$Rb_* \Phi_{f \circ b} \omega_{M'} \simeq \Phi_f R b_* \omega_{M'} \longrightarrow \Phi_f \omega_M.$$

Proper base change for restriction with support on the central fiber identifies the ambient trace with the trace on that fiber. Naturality of (50) in its coefficient complex gives the commuting square with η for proper pushforward. These are squares of the specified canonical maps, so the inverse of η after global sections is natural too. The attracting filtration establishes invertibility of the canonically defined map η.

For equivariance, let the auxiliary scaling act trivially on a finite free approximation to the coefficient-group classifying space. The fixed loci, attracting diagrams, and closed filtrations descend locally to the associated bundles, and their full attracting set is still the central associated bundle. The preceding argument applies there and is compatible with approximation transitions. □

Over $t \neq 0$ the isomorphism

$$a_M(x, t) = (\lambda(t^{-1})x, t) \tag{51}$$

satisfies $a_M^* \mathcal{W} = f$. Let $\alpha \in H_1^{\mathrm{BM}}(\mathbb{C}^*)$ have boundary 1 at zero, and let ϵ_α be exterior product with its *suspension normalization*: the constant-family boundary satisfies $\partial \epsilon_\alpha(\gamma) = \gamma$. In a raw chain convention in which the boundary is $(-1)^i \gamma$ on a class of BM degree i, this means $\epsilon_\alpha(\gamma) = (-1)^i \gamma \times \alpha$. Define

$$\mathrm{sp}_M = \eta_M^{-1} \partial (a_M)_* \epsilon_\alpha : H_i^{\mathrm{crit,BM}}(M, f) \longrightarrow H_i^{\mathrm{BM}}(M). \tag{52}$$

In cohomological indexing this is a degree-zero map. The boundary is that of the localization triangle for the central support of $\Phi_{\mathcal{W}} \omega_N$. The normalization fixes its sign in every degree.

Corollary 6.2. *The map* sp *commutes with smooth-ambient pullback and proper trace for scaling-equivariant maps. It is linear over the cohomology of a commuting coefficient group. For the zero potential,* $\mathrm{sp}_0 = \mathrm{id}$.

Proof. The first assertions follow from Proposition 6.1 and the corresponding morphisms of localization triangles. The BM pullback shifts have even degree, so the suspension convention introduces no additional sign. To check the final assertion it is enough to prove $(a_M)_*(\gamma \times \alpha) = \gamma \times \alpha$ for ordinary BM homology. Represent α by the properly embedded positive real ray in $\mathbb{C}^*$. On its product with M, the maps

$$H_s(x,t) = (\lambda(t^{-s})x, t), \qquad 0 \le s \le 1,$$

form a proper homotopy from the identity to a_M. Indeed the joint map $(x,t,s) \mapsto (H_s(x,t),s)$ is a homeomorphism, and the parameter interval is compact. Proper homotopy invariance and pushforward from the ray prove the required identity. It is an identity on the product with α, which is exactly what the boundary construction uses. The suspension normalization then gives $\mathrm{sp}_0 = \mathrm{id}$. □

7. A section of critical stable restriction

Let R be a linear representation of a reductive group G, let $U \subset R$ be a nonempty invariant open on which G acts freely, and suppose $X = U/G$ is smooth and projective over $Y = \operatorname{Spec} \mathbb{C}[R]^G$. A commuting reductive flavor group T is allowed. Let $f \in \mathbb{C}[R]^G$ be T-invariant. For specialization assume in addition that R has nonnegative scaling weights, that the scaling preserves U, and that f satisfies (48). These hypotheses hold for (35), with weight one on all arrows and zero on the framing.

Proposition 7.1. *Stable restriction on critical cohomology has a degree-zero,* $H_T^*(\mathrm{pt})$*-linear section*

$$\Psi_f : H^*_{T,\mathrm{crit}}(X,f) \longrightarrow H^*_{G\times T,\mathrm{crit}}(R,f).$$

With the above scaling,

$$\mathrm{sp}_{[R/G]}\, \Psi_f = \Psi_0\, \mathrm{sp}_X\,. \tag{53}$$

In particular, injectivity of specialization on $[R/G]$ *implies injectivity on* X.

Proof. Take the closure of the graph of $U \to X$ in $X \times_Y R$. It is projective over R, and above U it is precisely the smooth open graph. A projective equivariant resolution, isomorphic above this open graph, gives a smooth variety Z and maps

$$X \xleftarrow{p} Z \xrightarrow{q} R, \qquad \dim Z = \dim R, \qquad q^{-1}(U) \simeq U.$$

Characteristic-zero functorial resolution supplies equivariance for the gauge, flavor, and scaling actions simultaneously. The map q is proper birational. Each invariant polynomial, and in particular each f_d, has equal pullbacks by p and q, since they agree on the dense open graph. Thus the total mixed families also agree on Z. Every point of Z has a scaling limit by properness over R; every point of X has one by properness over Y.

Give X the trivial G-action. Define

$$\Psi_f = q_* p^*(1 \otimes -) : H^*_{T,\mathrm{crit}}(X,f) \longrightarrow H^*_{G\times T,\mathrm{crit}}(X,f) \longrightarrow H^*_{G\times T,\mathrm{crit}}(Z,q^*f) \longrightarrow H^*_{G\times T,\mathrm{crit}}(R,f). \tag{54}$$

The pullback is the exchange transformation of the preceding section, and the pushforward is proper trace. Restriction to U, proper base change, and $q^{-1}(U) \simeq U$ identify the composition with pullback along the principal bundle $U \to X$. Smooth descent for vanishing cycles then identifies this restriction with the identity on $H^*_{T,\mathrm{crit}}(X,f)$.

Here is the equivariant degree check. If V_N is a free open approximation to EG of dimension v_N, then $(Z \times V_N)/G$ and $(R \times V_N)/G$ have equal dimension $r + v_N - g$, where $r = \dim R$ and $g = \dim G$. The BM pullback to the first space adds $2v_N$ to a class of BM degree i on X. Converting back to cohomological degree gives $2(r + v_N - g) - (i + 2v_N) = 2\dim X - i$. Thus the section has cohomological degree zero. Flavor approximations add the same dimensions on both sides.

Corollary 6.2 applies to p and q. The gauge-unit arrow is also compatible: it is pullback along the product with a gauge classifying-space approximation, on which scaling is trivial. This proves (53). If $\mathrm{sp}_X(x) = 0$ and ambient specialization is injective, the equation gives $\Psi_f(x) = 0$, and stable restriction gives $x = 0$. □

8. Dimensional reduction and injective specialization

The next identity fixes which dimensional-reduction map is used. Let $\pi : \mathcal{E} \to B$ be a vector bundle on a smooth variety, let $s \in \Gamma(B, \mathcal{E}^\vee)$, and put

$$f(b, v) = \langle v, s(b) \rangle, \qquad Z = Z(s), \qquad S = \mathcal{E}|_Z.$$

Write $k : S \hookrightarrow \mathcal{E}_0 = f^{-1}(0)$ and $i : \mathcal{E}_0 \hookrightarrow \mathcal{E}$. The coefficient group is assumed to preserve the bundle and section. No smoothness or expected-dimension condition is imposed on Z.

Proposition 8.1. *Dimensional reduction gives the specified isomorphism*

$$\delta_f = \mathrm{can}_f\, k_* : H_i^{\mathrm{BM}}(S) \xrightarrow{\sim} H_i^{\mathrm{crit,BM}}(\mathcal{E}, f). \tag{55}$$

If the ambient space $\mathcal{E}$ has a scaling satisfying the hypotheses of Section 6, then

$$\mathrm{sp}_{\mathcal{E}}\, \mathrm{can}_f = i_*, \qquad \mathrm{sp}_{\mathcal{E}}\, \delta_f = (ik)_*. \tag{56}$$

The zero scheme Z, and hence S, need not be invariant under this auxiliary scaling.

Proof. The compactly supported dimensional-reduction theorem [4, Theorem A.1 and Appendix A.2] concerns the natural transformation

$$\Phi_f \mathbb{Q}_{\mathcal{E}} \longrightarrow \mathbb{Q}_{\mathcal{E}_0} \longrightarrow k_* \mathbb{Q}_S.$$

It becomes an isomorphism after $\pi_!$. Its naturality under vector bundle coordinate changes gives the statement for bundles, and its equivariant form follows at finite free approximations [4, Corollary A.9]. Taking compactly supported cohomology and Verdier duality identifies the dual of the second arrow with k_* on BM homology and the dual of the first with the Milnor map can_f. This proves (55) as an equality of maps. There is no BM degree shift when the source is S; replacing it by Z would introduce the vector-bundle shift $2\,\mathrm{rk}\,\mathcal{E}$.

For the specialization identity use the entire family

$$V = \Big\{ \sum_{d>0} t^d f_d = 0 \Big\} \subset \mathcal{E} \times \mathbb{A}^1_t.$$

Its central fiber is all of $\mathcal{E}$ and its open part is identified with $\mathcal{E}_0 \times \mathbb{C}^*$ by (51). The inclusion of this family in the constant ambient family is proper. Its ordinary BM localization boundary therefore commutes with that of $\mathcal{E} \times \mathbb{A}^1_t$. On the open parts the map is the original inclusion i under the same family trivialization, and on the central fiber it is the identity of $\mathcal{E}$. The ambient specialization is the identity by Corollary 6.2. Thus the ordinary zero-fiber specialization is the actual pushforward i_*.

The Milnor canonical morphism gives a map from this ordinary localization triangle to the critical localization triangle for the total potential. On the central support it is exactly $\eta_{\mathcal{E}}$ of (49), and on the open part it is can_f after the trivialization. Applying $\eta_{\mathcal{E}}^{-1}$ therefore gives $\mathrm{sp}_{\mathcal{E}}\, \mathrm{can}_f = i_*$. Composition with k_* proves the second equality. This argument used the whole

potential-zero family, so it requires neither flatness of that family nor scaling invariance of the particular cut-zero scheme. □

We apply the proposition to the spectral potential. For each unordered pair of distinct original colors choose one of the two directions and put all its graded arrows in a cut. Each term of (34) contains precisely one cut arrow. Thus $f_{\boldsymbol{\nu}}$ is fiberwise linear on the bundle of cut-arrow variables over the space of all remaining variables. The latter space consists of the ϵ-variables and the opposite cross-color arrows. The cut equations are linear in those opposite arrows once the ϵ-variables are fixed. This remains true for an orthogonal pair: its equation is $2a = 0$ on the non-cut arrow.

Proposition 8.2. *The specialization*

$$\mathrm{sp}_{\boldsymbol{\nu}} : H^*_{G_{\boldsymbol{\nu}},\mathrm{crit}}(\mathrm{Rep}_{\boldsymbol{\nu}}(\Gamma), f_{\boldsymbol{\nu}}) \longrightarrow R_{\boldsymbol{\nu}}$$

is injective. Its source has no odd cohomology.

Proof. Only finitely many ϵ-arrows occur for a fixed $\boldsymbol{\nu}$. For each original color their components are finite equioriented quivers of type A. Their representations are direct sums of interval representations, so there are finitely many orbits. The stabilizer H of one such representation is connected, with unipotent radical and reductive quotient a product of general linear groups on its interval multiplicity spaces. One sees this directly by writing its automorphism group as the units in the finite-dimensional endomorphism algebra: modulo its Jacobson radical that algebra is a product of matrix algebras, and the kernel of the map on units is a unipotent group. Consequently $H^*(BH, \mathbb{Q})$ is an even polynomial ring and restriction to a maximal torus is injective.

Let $O = G_{\boldsymbol{\nu}}/H$ be an ϵ-orbit. Over O, write $\mathscr{A}_O$ for the vector bundle of non-cut cross-color arrows. The cut equations form an equivariant linear map on this bundle; its rank is constant on the orbit, so its kernel $\mathcal{K}_O$ is a vector subbundle. The source S of dimensional reduction over this orbit is $\mathcal{K}_O$ times the unrestricted cut-arrow bundle, whereas the full representation space over the orbit is $\mathscr{A}_O$ times that same bundle. The normal bundle is therefore $\mathscr{A}_O/\mathcal{K}_O$.

Every weight of this normal bundle is a difference of characters from distinct original colors. Indeed all its variables are cross-color arrows, and the scalar centers for different original colors act independently while preserving the ϵ-representation. Hence no normal weight is zero. Restriction to a maximal torus expresses the Euler class as a product of nonzero linear forms. It is nonzero in the domain $H^*(BH, \mathbb{Q})$.

Include the cut-arrow rank in both ranks and write k for the source rank and a for the target rank over O, with $o = \dim O$. Equivariant Thom isomorphisms identify the stratum map with

$$H^{2(o+k)-i}(BH) \xrightarrow{e(\mathscr{A}_O/\mathcal{K}_O)} H^{2(o+a)-i}(BH).$$

It is injective and both stratum BM groups vanish in odd degrees. Order the finitely many ϵ-orbits so that initial unions are closed, and pull the resulting filtration back to source and target. Their BM localization sequences are short exact in even degree, by induction from the even stratum groups. In a commuting diagram of these sequences, a class in the middle source that maps to zero must restrict to zero on the last stratum, hence come from the preceding filtered source, where induction kills it. Thus the entire map $H_*^{G_{\boldsymbol{\nu}},\mathrm{BM}}(S) \to H_*^{G_{\boldsymbol{\nu}},\mathrm{BM}}(\mathrm{Rep}_{\boldsymbol{\nu}}(\Gamma))$ is injective, with even source. The same argument at finite Borel approximations applies in every stabilized degree.

Finally (56) identifies this map with $\mathrm{sp}_{\boldsymbol{\nu}}\, \delta_f$. Dimensional reduction is an isomorphism, so specialization is injective and the critical groups are even.

□

9. The exact wheel image and the Euler quotient

Let $I_{\Gamma,\boldsymbol{\nu}} \subset R_{\boldsymbol{\nu}}$ be the homogeneous wheel ideal of Theorem 3.4. Thus for each labeled lift of a Serre wheel through the chosen spectral assignment, one imposes the ideal identifying all its local root coordinates, and intersects these ideals before taking invariants. Orthogonal two-variable wheels are included. When no wheel is possible the ideal is $R_{\boldsymbol{\nu}}$. We recall the ordinary Hall word formula in the convention compatible with (39):

$$\kappa_\alpha(y) = \prod_{a<b} (y_b - y_a)^{e_{v_a v_b} - \delta_{v_a v_b}}, \qquad G \longmapsto \mathrm{Sym}_{\prod_v S_{\nu_v}}(G\kappa_\alpha). \tag{57}$$

Although the displayed kernel can have diagonal denominators, its symmetrization is polynomial. After clearing the equal-vertex Vandermonde, the numerator is alternating in every equal-vertex block and hence divisible by that Vandermonde. The equality with the wheel ideal proved in Theorem 3.4 is the exact equality

$$I_{\Gamma,\boldsymbol{\nu}} = \sum_\alpha \mathrm{im}\left(G \mapsto \mathrm{Sym}(G\kappa_\alpha)\right). \tag{58}$$

Theorem 9.1. *Specialization identifies unframed critical cohomology, in every dimension vector, with the wheel ideal:*

$$H^{2*}_{G_{\boldsymbol{\nu}},\mathrm{crit}}(\mathrm{Rep}_{\boldsymbol{\nu}}(\Gamma), f_{\boldsymbol{\nu}}) \xrightarrow[\mathrm{sp}]{\sim} I_{\Gamma,\boldsymbol{\nu}} \subset R_{\boldsymbol{\nu}}. \tag{59}$$

It intertwines Hall multiplication. On the cyclic framed moduli it is injective and has exact image

$$H^{2*}_{\mathrm{crit}}(X_{\boldsymbol{\nu},\boldsymbol{m}}, f) \xrightarrow[\mathrm{sp}]{\sim} (I_{\Gamma,\boldsymbol{\nu}} + J_{\boldsymbol{m}})/J_{\boldsymbol{m}} \subset R_{\boldsymbol{\nu}}/J_{\boldsymbol{m}}. \tag{60}$$

In particular framed critical cohomology is finite dimensional and even.

Proof. Fix a complete vertex flag of type α. The space of representations strictly lowering this flag is a vector bundle over the flag variety. Every cyclic word has trace zero on that space, so its potential is identically zero. Projection to the product of its one-dimensional diagonal representations is smooth, and the map forgetting the flag is proper. The critical push-pull class decorated by a labeled polynomial G is the corresponding Hall word. Indeed the successive invariant-subspace correspondences compose to this complete-flag correspondence; smooth base change and proper composition identify their iterated push-pull maps. All their one-dimensional diagonal potentials are zero.

Specialization on the flag space is the identity. Corollary 6.2 therefore sends this critical Hall word to its ordinary push-pull class $\mathrm{Sym}(G\kappa_\alpha)$. In this formula the arrow blocks give the numerator Euler factors and the flag tangent bundle gives the denominator. Equation (58) proves $I_{\Gamma,\boldsymbol{\nu}} \subset \mathrm{im}\,\mathrm{sp}$.

For the reverse inclusion, (56) places the specialization image in the image of ordinary cohomology with support on $f^{-1}(0)$. Fix a labeled wheel. Choose its indicated one-dimensional lines in the vertex spaces, put nonzero maps on the cycle (34), and put all other arrow blocks zero. The value of f is nonzero: for a nonorthogonal wheel the selected cyclic term is nonzero and all the other terms vanish; for an orthogonal pair its value is twice the product of its two nonzero arrows. A diagonal torus stabilizing this representation equates the characters on the wheel lines and leaves all complementary line characters independent. Pullback from $R_{\boldsymbol{\nu}}$ to this torus is therefore precisely the wheel substitution. Every supported image class restricts to zero, since the representation lies outside $f^{-1}(0)$. Its kernel is the intersection with $R_{\boldsymbol{\nu}}$ of the corresponding labeled wheel-diagonal ideal. Intersecting over all wheels proves $\mathrm{im}\,\mathrm{sp} \subset I_{\Gamma,\boldsymbol{\nu}}$. Proposition 8.2 proves injectivity.

These equalities also prove that the critical algebra is generated by the vertex polynomial classes. To check multiplication, first multiply an arbitrary critical class by one such vertex class.

The vertex potential is zero. The Thom–Sebastiani map with this factor is exterior product with its constant complex, equivalently smooth pullback followed by the Chern decoration. The invariant-line correspondence has a smooth bundle projection and a proper flag-forgetting map, so the same naturality proves compatibility with this multiplication, on either side. Expressing a general class as a sum of vertex words and applying associativity extends the compatibility to every Hall product.

Adding the incoming framing is a vector bundle with pulled-back potential, so its critical cohomology agrees with the unframed group in unshifted cohomological indexing. The section in Proposition 7.1 makes critical stable restriction surjective, and (53) transfers injectivity to the stable quotient. The commutative square with ordinary stable restriction and Theorem 5.1 gives exactly (60). Its target is finite dimensional and even, which proves the last assertion. □

Theorem 9.2. *With $\boldsymbol{p} = \boldsymbol{m} - \boldsymbol{l}$ as in* (37)*, there is a natural graded isomorphism*

$$Q_{\boldsymbol{\nu},\boldsymbol{p}} := H^{2*}_{\mathrm{crit}}(X_{\boldsymbol{\nu},\boldsymbol{m}}, f)/\ker D \;\simeq\; I_{\Gamma,\boldsymbol{\nu}}/(I_{\Gamma,\boldsymbol{\nu}} \cap J_{\boldsymbol{p}}). \tag{61}$$

The source quotient is assigned its source grading. Multiplication by D identifies it with its literal image after the polynomial degree shift $L = \boldsymbol{l} \cdot \boldsymbol{\nu}$. Its centered grading is

$$\deg_{\mathrm{cen}} x = 2\deg_{\mathrm{pol}} x - d_{\boldsymbol{p}}(\boldsymbol{\nu}). \tag{62}$$

Together with the actual simple-kernel identification of Theorem 3.3, (61) *realizes the local ℓ-weight space of the corresponding simple module.*

Proof. The definition of the ordered kernels immediately gives $K_{\alpha,\boldsymbol{p}} = \Delta_{\boldsymbol{l}} K_{\alpha,\boldsymbol{m}}$ and hence

$$J_{\boldsymbol{p}} = (J_{\boldsymbol{m}} : \Delta_{\boldsymbol{l}}). \tag{63}$$

This is an equality of all labeled test kernels, so it remains valid without a genericity assumption on the roots. Under (60), D is multiplication by $\Delta_{\boldsymbol{l}}$. Its kernel on the image of $I_{\Gamma,\boldsymbol{\nu}}$ consists precisely of the classes represented by $I_{\Gamma,\boldsymbol{\nu}} \cap J_{\boldsymbol{p}}$. This proves (61), and multiplication by $\Delta_{\boldsymbol{l}}$ has polynomial degree L. Lemma 5.2 identifies the additive kernels with the multiplicative local kernels on the same labeled tests. Theorem 3.3 identifies those tests with the actual highest-vector annihilator. The resulting isomorphism sends each shuffle word to its evaluation on the highest vector. Finally (38) gives the stated center. □

10. Proper critical support and the weight-one action

We record the support statement needed for duality and for the fundamental-module calculation. It concerns the entire critical locus of the one-sided framed model used here.

Proposition 10.1. *For every finite dimension vector, the reduced critical locus*

$$Z_{\boldsymbol{\nu},\boldsymbol{m}} = \mathrm{Crit}(f : X_{\boldsymbol{\nu},\boldsymbol{m}} \to \mathbb{C})_{\mathrm{red}}$$

is contained in the projective zero fiber of the cyclic GIT morphism and is projective. Scaling the chosen cut arrows with weight one and all other arrows and framing maps with weight zero induces an algebraic $\mathbb{G}_m$-action on $X_{\boldsymbol{\nu},\boldsymbol{m}}$ for which f has weight one.

Proof. We first match the present potential and stability with the simply framed graded W_2 model of [36, §§3.1.5, 3.1.7]. For an orthogonal pair the potential on its two arrow spaces is $2\,\mathrm{Tr}(AB)$. The trace pairing between the rectangular matrix spaces is perfect, so its two derivatives force $A = B = 0$ at every critical point. These equations do not involve any other variable. Removing those zero arrow maps leaves the cyclicity condition unchanged. Thus the entire reduced critical locus of the augmented model is identified with the critical locus after removing these pairs. The extra variables remain part of its smooth ambient space and of the dimension $d_{\boldsymbol{m}}$ in (36).

On a genuine Dynkin edge $\{i,j\}$ put $r_{ij} = -c_{ij} > 0$. The two terms of our potential have plus signs. Choose nonzero constants λ_i so that $\lambda_i^{r_{ij}}/\lambda_j^{r_{ji}} = -1$ on every edge. Such constants exist: root each connected Dynkin tree, choose the constant at its root, and solve for the constant at each new adjacent vertex by taking a nonzero complex root. Replace every graded copy of ϵ_i by $\lambda_i\epsilon_i$, and then rescale one cross-color arrow on each edge to normalize the common coefficient. The resulting two coefficients are $+1$ and -1, according to the chosen orientation. These diagonal linear changes preserve the gauge action, the spectral grading, and stability. The potential has become the signed W_2 potential.

Here is an explicit framing and grading dictionary. Take the vector space dual of every arrow map and every framing map, keeping the vertex index k on the dual vector space:

$$\widehat{V}_{i,k} = V_{i,k}^\vee, \qquad \widehat{W}_{i,k-d_i} = W_{i,k}^\vee.$$

An arrow from (i,k) to $(j,k-b_{ij})$ becomes an arrow from $(j,k-b_{ij})$ to (i,k), of grade b_{ij}. An ϵ_i-arrow becomes one of grade $2d_i$, and the framing becomes an outgoing map of grade $-d_i$. These are the grades of [36, (3.3)]. Transposition reverses each cyclic word; cyclically moving its factors gives the same signed W_2 expression. An incoming cyclic representation becomes an outgoing stable one: a subrepresentation of the dual annihilated by the outgoing framing has annihilator containing the entire framing-generated original representation, and is therefore zero. The converse follows by annihilators in the same way. This is an isomorphism of the relevant smooth moduli after the changes just described. In particular the comparison involves the single outgoing framing of W_2.

Proposition 5.8 of [36] places the entire graded W_2 critical locus in its projective nilpotent fiber. The nilpotence behind this statement is useful here. In its Jacobian algebra the cross-color arrow ideal is nilpotent in finite Dynkin type, and the homogeneous central element $\sum_i \epsilon_i^{t/d_i}$ has degree $2t > 0$, where $t = \max_i d_i$. On a finitely supported graded module this element, and hence each ϵ_i, is nilpotent. These facts make the whole augmentation ideal act nilpotently; see [36, §3.2]. Thus all nonconstant cycle invariants vanish. The graded critical representation maps to the origin of the affine quotient. Reversing the dictionary and restoring the zero orthogonal arrows proves the same assertion for $X_{\boldsymbol{\nu},\boldsymbol{m}}$. Its reduced critical locus is closed in a projective fiber, which proves projectivity.

Finally each term of (34) contains exactly one cut arrow, including each quadratic orthogonal term. The cut action therefore gives $f(u\cdot x) = uf(x)$. For $u \neq 0$, multiplying an arrow by a nonzero scalar leaves invariant-subspace conditions unchanged. It preserves cyclicity, commutes with the gauge action, and descends to the smooth quotient. This proves the stated algebraic action. □

Corollary 10.2. *Put $d = d_{\boldsymbol{m}}(\boldsymbol{\nu})$. The whole critical cohomology of $X_{\boldsymbol{\nu},\boldsymbol{m}}$ is pure and satisfies hard Lefschetz about cohomological degree d. More precisely, for the first Chern class h of an ample line bundle restricted to $Z_{\boldsymbol{\nu},\boldsymbol{m}}$,*

$$h^r : H^{d-r}_{\mathrm{crit}}(X_{\boldsymbol{\nu},\boldsymbol{m}}, f) \xrightarrow{\sim} H^{d+r}_{\mathrm{crit}}(X_{\boldsymbol{\nu},\boldsymbol{m}}, f) \qquad (r \geq 0),$$

after the corresponding Tate twist. Forgetting Tate twists, Verdier duality gives a perfect pairing

$$H^k_{\mathrm{crit}}(X_{\boldsymbol{\nu},\boldsymbol{m}}, f) \otimes H^{2d-k}_{\mathrm{crit}}(X_{\boldsymbol{\nu},\boldsymbol{m}}, f) \longrightarrow \mathbb{Q}. \tag{64}$$

Proof. The variety is smooth quasi-projective, and the potential has positive weight one under the cut action. Euler's identity gives $f = 0$ on its critical locus. Proposition 10.1 shows that this entire locus is projective, so it is a projective union of complete critical components. These are the hypotheses of [5, Corollary 3.2]; weight one removes the nonunipotent monodromy summand. The result gives purity of the global critical cohomology and hard Lefschetz. In our convention

$\Phi_f(\mathbb{Q}_X[d])$ is perverse, so the center is d, the dimension of the smooth ambient quotient. This also retains the contributions of the orthogonal quadratic arrow pairs.

Normalized vanishing cycles commute with Verdier duality. Since X is smooth of dimension d, $\mathbb{D}(\Phi_f\mathbb{Q}_X) = \Phi_f\mathbb{Q}_X[2d]$ after forgetting Tate twists. The support is proper, so ordinary and compactly supported hypercohomology agree. Verdier duality consequently yields (64). □

11. Flavor parameters and homogeneous generation

We retain an external torus $A = \mathbb{G}_m$ acting on the framing spaces and trivially on the unframed arrow representation. Choose also A-representations U_v of dimensions l_v, and replace the Euler bundle by

$$\mathcal{E}_{\boldsymbol{l},A} = \bigoplus_v \operatorname{Hom}(U_v, \mathcal{V}_v).$$

If the weights of U_v are $c'_{v,r}$, its equivariant Euler class is

$$\Delta_{\boldsymbol{l},A} = \prod_{v,a}\prod_{r=1}^{l_v}(y_{v,a} - c'_{v,r}a), \qquad \deg a = \deg y_{v,a} = 1. \tag{65}$$

Let

$$H_A = H^{2*}_{A,\mathrm{crit}}(X_{\boldsymbol{\nu},\boldsymbol{m}}, f), \qquad D_A = \Delta_{\boldsymbol{l},A} \cup -, \qquad \mathcal{Q}_A = H_A/\ker D_A.$$

These definitions retain all flavor factors, including factors which coincide at $a = 0$.

Theorem 11.1. *For every finitely supported dimension vector there is a homogeneous surjection of $\mathbb{Q}[a]$-modules*

$$I_{\Gamma,\boldsymbol{\nu}}[a] \longrightarrow H_A. \tag{66}$$

The modules H_A and $\mathcal{Q}_A$ are finite free over $\mathbb{Q}[a]$. The unframed critical Hall algebra is generated by its homogeneous vertex classes. Its framed module, and the quotient by the Euler kernel, are generated by their dimension-zero vacuum. These statements hold with the source grading on $\mathcal{Q}_A$.

Proof. Since A acts trivially on the unframed representation space, Theorem 9.1 identifies its equivariant critical cohomology with $I_{\Gamma,\boldsymbol{\nu}} \otimes \mathbb{Q}[a]$. Adding the incoming framing is an equivariant vector bundle with pulled-back potential. Unshifted critical cohomology is invariant under that pullback. Proposition 7.1, with flavor group A, makes stable restriction surjective. Every map has cohomological degree zero and is $\mathbb{Q}[a]$-linear, so this is precisely the homogeneous surjection (66).

The ordinary critical group is finite dimensional and even by Theorem 9.1. The equivariant spectral sequence therefore degenerates: both the base $H^*(BA)$ and the fiber critical cohomology are even. Homogeneous lifts of a basis of the fiber give a free basis over $\mathbb{Q}[a]$ by the same degree induction as in the Leray–Hirsch argument. There are no local-system twists since A is connected. Thus H_A is finite free. Multiplication by (65) is homogeneous, and $\mathcal{Q}_A \simeq \operatorname{im} D_A$ as ungraded $\mathbb{Q}[a]$-modules. Its image is a submodule of the free module H_A, so is torsion free; finite torsion-free modules over the polynomial ring in one variable are free. The grading gives a homogeneous basis, obtained for example by homogeneous lifts of a basis modulo a and degree induction.

For generation, (58) states that every polynomial wheel class is a sum of Hall words in vertex polynomials. Each vertex polynomial has its critical lift, and injectivity in Theorem 9.1 shows that these words are the critical classes themselves. Tensoring with $\mathbb{Q}[a]$ gives the equivariant assertion.

The Hall action on cyclic framed moduli uses an unframed invariant kernel and an old cyclic framed quotient. At old dimension zero the flag is unique. The correspondence then consists

exactly of adding an incoming framing to the unframed representation and restricting to the cyclic open. Its action on the vacuum is therefore the map (66); there is no further framing Euler factor. Surjectivity proves framed vacuum generation.

Finally, on a Hall correspondence with unframed kernel $\mathcal{U}$ and old framed quotient $\mathcal{V}'$, the tautological exact sequence $0 \to \mathcal{U} \to \mathcal{V} \to \mathcal{V}' \to 0$ gives

$$e(\mathcal{E}_{\boldsymbol{l},A}(\mathcal{V})) = e(\mathcal{E}_{\boldsymbol{l},A}(\mathcal{U}))e(\mathcal{E}_{\boldsymbol{l},A}(\mathcal{V}')).$$

Let b be an unframed Hall class of dimension vector $\boldsymbol{u}$, and write F_b for its convolution action. The projection formula gives

$$D_{\boldsymbol{\nu}+\boldsymbol{u},A} F_b(x) = F_{b\, e(\mathcal{E}_{\boldsymbol{l},A}(\mathcal{U}))}(D_{\boldsymbol{\nu},A}x). \tag{67}$$

If $D_{\boldsymbol{\nu},A}x = 0$, the right side vanishes, so the Euler kernel is Hall stable. Passing to the quotient preserves homogeneous vacuum generation. □

Corollary 11.2. *Let $R = \mathbb{Q}[[a]]$ and $\mathcal{Q}_R = \mathcal{Q}_A \otimes_{\mathbb{Q}[a]} R$, with the completed total-degree filtration F^d. Every class in $F^d\mathcal{Q}_R$ has a wheel representative of total degree at least d. Moreover*

$$F^d\mathcal{Q}_R \cap a\mathcal{Q}_R = aF^{d-1}\mathcal{Q}_R. \tag{68}$$

These statements concern the source quotient grading. The natural map

$$\mathcal{Q}_A/a\mathcal{Q}_A \longrightarrow H^{2*}_{\mathrm{crit}}(X,f)/\ker D_0$$

is a homogeneous surjection; it need not be an isomorphism.

Proof. Choose a homogeneous free basis $e_1, \dots, e_s$ of $\mathcal{Q}_A$ with degrees d_j. Homogeneity and surjectivity of (66), followed by the quotient map, provide homogeneous lifts $f_j \in I_{\Gamma,\boldsymbol{\nu}}[a]$ of the same degrees. Every element of $F^d\mathcal{Q}_R$ has a unique expression

$$\sum_j a^{\max(0,d-d_j)} u_j(a) e_j, \qquad u_j(a) \in R.$$

Replacing e_j by f_j gives the required representative. The same expression shows that divisibility by a lowers the total filtration index by exactly one, which proves (68). Reduction of the source quotient maps onto the quotient by the reduced Euler kernel, since an element of $\ker D_A$ reduces to an element of $\ker D_0$. □

12. Coulomb realizations of ordinary modules

In this section $\mathfrak{g}$ has type G_2 or F_4, and $q \in \mathbb{C}^\times$ is not a root of unity. We construct an action of a fixed Coulomb algebra on every ordinary tensor product of fundamental modules, including tensor products with coincident spectral parameters. We shall subsequently use the same action on their simple subquotients.

We number the vertices with the long vertices first. Thus

$$C_{G_2} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \qquad (d_1,d_2) = (3,1),$$

$$C_{F_4} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad (d_1,d_2,d_3,d_4) = (2,2,1,1). \tag{69}$$

Put $q_i = q^{d_i}$ and $b_{ij} = d_i c_{ij} = d_j c_{ji}$. A framing zero at $a \in \mathbb{C}^\times$ will mean a factor $1 - a/u$ of a framing polynomial. This convention distinguishes a framing zero from the underlying flavor coordinate, which differs from it by q_i^2 at vertex i.

12.1. **The ordinary current map over the flavor ring.** Let $\boldsymbol{a} = (a_i)$ be the gauge ranks, $\boldsymbol{l} = (l_i)$ the framing ranks, and assume that

$$l_i = \sum_j c_{ji} a_j. \tag{70}$$

Let S be a complex algebra carrying invertible framing coordinates $z_{i,t}$, $1 \le t \le l_i$. We write $\mathscr{C}_S$ for the base change to S of the equivariant K-theoretic Coulomb algebra with symmetrizers and with these ranks. More precisely, it is the convolution algebra of the classical BFN space, using the determinant covers of the gauge groups and fixing the loop-rotation parameter to be q. The BFN space parametrizes a gauge lattice together with a matter section regular in both the original and the new lattice. Its convolution is defined by composing lattice modifications and pulling back the common matter section. We use the construction of [37, §§3.1–3.2], including its finite-dimensional approximations. All the specializations used below are taken in this algebra before considering its modules.

The gauge coefficient ring is

$$B_S = S\big[w_{i,t}^{\pm 1}, D_i^{\pm 1}\big]^{\prod_i \mathfrak{S}_{a_i}} \Big/ \left(D_i^2 - \prod_t w_{i,t} \right)_{i \in I}. \tag{71}$$

The symmetric groups fix the D_i. Set

$$W_i(u) = \prod_{t=1}^{a_i} (1 - w_{i,t}/u), \qquad Z_i(u) = \prod_{t=1}^{l_i} (1 - q_i^2 z_{i,t}/u).$$

The Cartan current in the geometric map is the rational function

$$\Psi_i(u) = \left(\prod_j D_j^{c_{ji}} \right) Z_i(u) \frac{\displaystyle\prod_{j \ne i} \prod_{r=1}^{-c_{ji}} W_j(q_j^{-c_{ji}-2r} u)}{W_i(u) W_i(q_i^{-2} u)}. \tag{72}$$

An empty product is 1. Its expansions at infinity and zero are the positive and negative Cartan series. Equality of these two expansions as rational functions follows either from this formula or from the geometric positive and negative A-polynomials: their additional power of u is $l_i - \sum_j c_{ji} a_j$, which is zero by (70).

The minuscule classes give the root currents. In the conventions of [9, Theorem 7.1 and §7(i)] and [37, (4.2.8), Lemmas 5.1.1–5.1.2], the ordinary root generators before imposing the central relation are

$$e_{i,r} = E_{i,r}^{\mathrm{geom}} / (q_i - q_i^{-1}), \qquad f_{i,r} = F_{i,r}^{\mathrm{geom}} / (q_i - q_i^{-1}), \qquad r \in \mathbb{Z}.$$

The flavor coordinate in the first convention is q_i times the coordinate $z_{i,t}$ used here. We use the extended simply connected current presentation, in which the leading positive and negative Cartan coefficients are independent units, and impose the ordinary relation after the following calculation.

Lemma 12.1. *In $\mathscr{C}_S$ one has*

$$\Psi_i(\infty) = \prod_j D_j^{c_{ji}}, \qquad \Psi_i(0) = K_i \prod_j D_j^{-c_{ji}}, \qquad K_i = q_i^{2l_i - 2a_i} \prod_{j \ne i} q_j^{-a_j c_{ji}} \prod_{t=1}^{l_i} z_{i,t}. \tag{73}$$

Suppose $\kappa_i \in S^\times$ satisfy $\kappa_i^2 K_i = 1$. Multiplying $\Psi_i^{\pm}$ and every $e_{i,r}$ by κ_i, while leaving every $f_{i,r}$ unchanged, gives an ordinary quantum loop algebra homomorphism

$$\Phi_S : U_q(L\mathfrak{g}) \otimes_{\mathbb{C}} S \longrightarrow \mathscr{C}_S. \tag{74}$$

Proof. The value at infinity is immediate. At zero the two denominator factors at vertex i contribute $D_i^4 q_i^{2a_i}$ before including the leading determinant. For a neighboring vertex j, the sum of the shifts in its numerator factors is

$$\sum_{r=1}^{-c_{ji}} (c_{ji} + 2r) = -c_{ji}.$$

The powers of the gauge determinants therefore give the inverse of the determinant at infinity. The remaining scalar is the stated K_i. The sign is $(-1)^{l_i - \sum_{j \neq i} c_{ji} a_j} = (-1)^{2a_i} = 1$. Thus the product of the leading positive and negative coefficients is K_i in the whole coefficient ring.

The substitution

$$\varphi_i^{\pm} \longmapsto \kappa_i \varphi_i^{\pm}, \qquad e_{i,r} \longmapsto \kappa_i e_{i,r}, \qquad f_{i,r} \longmapsto f_{i,r} \quad (r \in \mathbb{Z})$$

preserves every relation of the extended current presentation. Indeed, the Cartan–root and Serre relations are homogeneous under it, and both sides of each $e_{i,r}, f_{i,s}$ commutator are multiplied by κ_i. The leading central product in the image becomes $\kappa_i^2 K_i = 1$. Consequently the map factors through the ordinary presentation. □

We shall need surjectivity before specializing a deformation of the framing parameters. The following argument also gives precise control of the gauge coefficients at a numerical value of q.

Proposition 12.2. *For the paired framings constructed below, the map* (74) *is surjective over their Laurent parameter rings, and after every base change of those rings. Its image contains the whole ring B_S.*

Proof. Both Cartan matrices in (69) have determinant 1. The leading Cartan coefficients and their inverses therefore recover all $D_i^{\pm 1}$ by the integral inverse of C^{T}, with the known scalar units κ_i removed. To recover the remaining gauge coefficients, put $s_{j,n} = \sum_t w_{j,t}^n$ for $n > 0$, and let $h_{i,n}$ be the coefficient of u^{-n} in

$$n \log\!\left(\frac{\Psi_i(u)}{\Psi_i(\infty) Z_i(u)} \right).$$

The logarithm is expanded at infinity. Each of its coefficients is a finite polynomial in the current coefficients already obtained. Put $h_n = (h_{i,n})_{i \in I}$ and $s_n = (s_{j,n})_{j \in I}$. Formula (72) gives

$$h_n = B(q^n) s_n, \qquad B_{ii}(x) = 1 + x^{2d_i}, \quad B_{ij}(x) = -\sum_{r=1}^{-c_{ji}} x^{d_j(c_{ji} + 2r)} \quad (i \neq j). \tag{75}$$

In the present numbering the matrices and their determinants are

$$B_{G_2}(x) = \begin{pmatrix} 1 + x^6 & -x^3 - x - x^{-1} \\ -x^3 & 1 + x^2 \end{pmatrix}, \qquad \det B_{G_2}(x) = x^8 - x^4 + 1 = \Phi_{24}(x). \tag{76}$$

$$B_{F_4}(x) = \begin{pmatrix} 1 + x^4 & -x^2 & 0 & 0 \\ -x^2 & 1 + x^4 & -1 - x^2 & 0 \\ 0 & -x^2 & 1 + x^2 & -x \\ 0 & 0 & -x & 1 + x^2 \end{pmatrix}, \tag{77}$$

$$\det B_{F_4}(x) = x^{12} - x^6 + 1 = \Phi_{36}(x).$$

Here Φ_n denotes the nth cyclotomic polynomial. These determinants at q^n are nonzero complex numbers. Inverting them and using Newton identities recovers every coefficient of $W_j(u)$. The inverse of its last coefficient is already supplied by D_j^{-2}. This proves that B_S is in the image without inverting any flavor function.

Retain the flavor ring in the generation argument of [37, §4.2.2]. Its magnetic filtration is indexed by dominant magnetic charges $\gamma = (\gamma_{i,r})$. Each graded piece is the Laurent coefficient ring invariant under the stabilizer of γ, multiplied by the structure-sheaf class of the corresponding magnetic stratum. In the abelian algebra, one removes a positive coordinate of a nonzero charge by a positive unit step. The framing factor in this product is 1: the two steps have the same sign in that coordinate and the intermediate lattice does not cross the framing hyperplane. If there is no positive coordinate, one removes a negative coordinate by a negative unit step; the same statement holds. This is also obtained directly by comparing the three intersections of matter lattices in [37, (3.2.21), (3.2.25)].

The coefficients in these one-step products are consequently Laurent polynomials in gauge variables and in q, with no flavor factor. The no-common-zero argument of [37, Lemma 4.2.4], at a fixed non-root-of-unity q, supplies a Bezout identity in the gauge Laurent ring. More explicitly, for the set of positive coordinates $\Gamma = \{(i, r) : \gamma_{i,r} > 0\}$ the coefficients involve only the variables indexed by Γ. Every factor has the form $1 - q^e w_{i,r} w_{j,s}^{-1}$. For the finite-type orientations in [37, §2.2.1], all the exponents e have the same weak sign, and exponent zero can occur at only one vertex. At a putative common zero, choose a vanishing factor at every coordinate. Following the resulting arrows in the finite set Γ produces a cycle. Since the graph has no loops and zero exponents occur at only one vertex, that cycle has a nonzero sum of exponents of the same sign. Multiplying its equalities would give $q^e = 1$ for a nonzero integer e, a contradiction. The sign-reversed argument handles all-negative charges. The Laurent Nullstellensatz now supplies the Bezout identity in the gauge ring. This identity is unchanged after tensoring with the flavor ring. Induction on the magnetic charge generates the abelian magnetic classes. The nil-Hecke operators and the full spherical idempotent used in [37, Lemmas 4.2.3 and 4.2.5] are flavor equivariant, so the same induction generates the spherical algebra over that ring. This proves that B_S and the dressed minuscule classes generate $\mathscr{C}_S$ before flavor specialization.

For a minuscule charge at vertex i, the partial invariant ring is generated over the full invariant ring by the one selected root w. It satisfies the total characteristic polynomial $\prod_t (T - w_{i,t})$. Since the constant coefficient of this polynomial is a unit, w^{-1} is also a polynomial in w with coefficients in B_S. All its powers occur among the integer current modes, up to invertible determinant and numerical factors. They therefore supply all minuscule dressings. The determinant cover only base changes this assertion. Hence the ordinary current image contains every generator of $\mathscr{C}_S$, which proves surjectivity. Each step was an identity over the flavor ring, so it remains valid after base change. □

Lemma 12.3. *The geometric difference-operator realization of $\mathscr{C}_S$ is injective at every numerical non-root-of-unity q used here. In particular, polynomial bimodule identities established in that realization hold in the specialized Coulomb algebra.*

Proof. The magnetic filtration and its associated graded description [37, (3.2.17)] are obtained by the equivariant affine-cell filtration. They commute with the present specializations: each piece is an invariant Laurent coefficient ring times one magnetic class. At fixed q these pieces are free over the external flavor coefficient ring and torsion-free with respect to the gauge Laurent Euler factors used in torus localization. Such a denominator preserves the nonzero leading magnetic term of a nonzero element. Gauge localization is therefore injective. Torus localization identifies the resulting algebra with its difference-operator image, as in [37, §3.2.4 and §5.1]. This proves the assertion directly at the specialized parameter. □

12.2. **Six fundamental cup modules.** We next specify all finite data needed to construct the fundamental modules. The unfolded diagrams are D_4 and E_6. Number their vertices by the

following ordered lists of pairs, and define their edges by the displayed sets:

	unfolded vertices $1, 2, \ldots$	edges
D_4	$(1,2),(1,4),(1,6),(2,1)$	$\{\{1,4\},\{2,4\},\{3,4\}\}$
E_6	$(1,2),(1,4),(2,2),(2,4),(3,2),(4,1)$	$\{\{1,4\},\{2,3\},\{3,5\},\{4,5\},\{5,6\}\}$.

(78)

Write $\widetilde{C}$ for the simply laced Cartan matrix determined by this graph, and ϖ_c for its fundamental weights. If an original gauge root has color i and value q^k, its unfolded color is $(i, k \bmod 2d_i)$, with residues represented by $1, \ldots, 2d_i$. Its weighted-flag height is

$$h = k + \sigma_i, \qquad \sigma = (0,-2) \text{ in } G_2, \quad \sigma = (0,0,0,0) \text{ in } F_4. \tag{79}$$

The short-vertex shift in G_2 is the change of grading in [37, (3.1.72)]. A framing zero q^K at vertex i gives a red strand of color $(i, K - 2d_i \bmod 2d_i)$ at height $K - 2d_i + \sigma_i$.

Set $H = 12$ in type G_2 and $H = 18$ in type F_4. For the fundamental of color a we use framing zeros

$$q^{s-d_a}, \qquad q^{s+d_a+H}, \tag{80}$$

both at color a. Table 1 gives s, the red heights, and their unfolded colors. In its last two columns are the folded and unfolded gauge ranks. The red weights are $\lambda_L = \varpi_{c_L}$ and $\lambda_R = \varpi_{c_R}$.

Type, a	s	(c_L, c_R)	ℓ_L	ℓ_R	$\boldsymbol{a}$	$\widetilde{\boldsymbol{a}}$
$G_2, 1$	1	$(2,2)$	-8	10	$(4,2)$	$(1,2,1,2)$
$G_2, 2$	0	$(4,4)$	-5	9	$(6,4)$	$(2,2,2,4)$
$F_4, 1$	0	$(1,2)$	-6	16	$(4,6,4,2)$	$(2,2,3,3,4,2)$
$F_4, 2$	0	$(3,4)$	-6	16	$(6,12,8,4)$	$(3,3,6,6,8,4)$
$F_4, 3$	1	$(5,5)$	-2	18	$(8,16,12,6)$	$(4,4,8,8,12,6)$
$F_4, 4$	0	$(6,6)$	-3	17	$(4,8,6,4)$	$(2,2,4,4,6,4)$

TABLE 1. The red pairs and gauge ranks.

We give a complete rule for the highest loading. Read the admissible blocks in decreasing height. A color $c = (i, r)$ occurs at height h if $h - \sigma_i \equiv r \pmod{2d_i}$. The allowed interval is

$$\ell_L < h \le \ell_R. \tag{81}$$

The right endpoint is included because a black strand at a red height precedes the red in the weighted convention. At a common height read in decreasing order of (ϵ'_i, i, r), where

$$\epsilon' = (1,0) \text{ for } G_2, \qquad \epsilon' = (0,1,0,1) \text{ for } F_4.$$

Initialize the Dynkin-label vector v to the unit vector at c_R. At each allowed block c set

$$A_{h,c} = \max(0, v_c), \qquad v_j \longleftarrow v_j - A_{h,c}\widetilde{C}_{jc} \quad \text{for all } j. \tag{82}$$

A zero power is an actual step of this rule and is retained.

Tables 2 and 3 give every allowed power. A dash means that the height lies outside (81). For D_4 there is one allowed color at each height, printed in its own column. For E_6, at even heights the entries are triples in the color order $(4,5,2)$ if $h \equiv 0 \pmod 4$ and $(3,5,1)$ if $h \equiv 2 \pmod 4$; at odd heights the single color is 6.

Let e_c denote the standard basis vector at a vertex c, and let w_0 be the longest Weyl group element of the unfolded root system. All entries and their endpoints can be checked using only subtraction of columns of the graph matrix (78). The endpoint in each column is $-e_{c_L}$. The number $n_{i,k}$ of gauge roots at q^k is, by definition, the entry $A_{k+\sigma_i,(i,k \bmod 2d_i)}$. Thus the two

h	color	$a=1$	$a=2$
10	2	1	$--$
9	4	1	1
8	1	1	1
7	4	0	0
6	3	1	1
5	4	1	1
4	2	1	2
3	4	0	1
2	1	0	1
1	4	0	0
0	3	0	1
-1	4	0	1
-2	2	0	0
-3	4	0	0
-4	1	0	0
-5	4	0	$--$
-6	3	0	$--$
-7	4	0	$--$

TABLE 2. The complete greedy powers for G_2.

h	$a=1$	$a=2$	$a=3$	$a=4$
18	$--$	$--$	$(0,1,0)$	$--$
17	$--$	$--$	1	1
16	$(0,0,1)$	$(1,1,0)$	$(1,1,0)$	$(0,1,0)$
15	0	1	0	0
14	$(1,1,0)$	$(1,1,1)$	$(2,1,1)$	$(1,0,0)$
13	1	0	1	0
12	$(1,1,0)$	$(2,1,1)$	$(2,2,2)$	$(1,1,1)$
11	0	1	1	1
10	$(1,0,1)$	$(2,2,1)$	$(3,2,1)$	$(1,1,1)$
9	0	1	1	0
8	$(1,1,1)$	$(2,1,1)$	$(3,2,1)$	$(2,1,0)$
7	1	0	1	1
6	$(1,1,0)$	$(2,1,1)$	$(2,1,2)$	$(1,1,1)$
5	0	1	0	0
4	$(1,0,0)$	$(1,1,1)$	$(2,1,1)$	$(1,0,1)$
3	0	0	1	0
2	$(0,0,1)$	$(1,0,0)$	$(1,1,0)$	$(1,1,0)$
1	0	0	0	1
0	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$
-1	0	0	0	0
-2	$(0,0,0)$	$(0,0,0)$	$--$	$(0,0,0)$
-3	0	0	$--$	$--$
-4	$(0,0,0)$	$(0,0,0)$	$--$	$--$
-5	0	0	$--$	$--$

TABLE 3. The complete greedy powers for F_4.

tables specify the gauge multisets completely, including repeated roots. Summing their entries by original or unfolded color gives Table 1, and direct multiplication gives

$$C^{\mathsf{T}}\boldsymbol{a} = 2e_a, \qquad \widetilde{C}\widetilde{\boldsymbol{a}} = e_{c_L} + e_{c_R}. \tag{83}$$

In particular $\lambda_L = -w_0\lambda_R$, as is also read directly from the D_4 and E_6 diagrams.

Here is a further check of the rational highest weight, stated in a form that makes its verification finite. Work in the free abelian group on symbols $[i,k]$. Start with $[a, s-d_a] + [a, s+d_a+H]$. For each of the $n_{j,k}$ roots subtract $[j,k] + [j,k+2d_j]$ and add

$$\sum_{i:c_{ji}<0} \sum_{r=1}^{-c_{ji}} [i, k + d_j c_{ji} + 2d_j r]. \tag{84}$$

For each of the six columns the result is $[a, s-d_a] - [a, s+d_a]$. This follows by inserting the full lists in the two tables and canceling equal symbols. Formula (84) is exactly the divisor of (72). Hence on the prescribed loading

$$\frac{\Psi_i(u)}{\Psi_i(\infty)} = \begin{cases} (1-q^{s-d_a}/u)/(1-q^{s+d_a}/u), & i = a, \\ 1, & i \neq a. \end{cases} \tag{85}$$

The leading determinant and normalizing exponents are listed in Table 4. If $S_j = \sum_k k n_{j,k}$, they are respectively $t_i = \frac{1}{2}\sum_j c_{ji} S_j$ and $\tau_i = d_i\delta_{ia} - t_i$. In particular $\kappa_i = q^{\tau_i}$ gives the ordinary highest constant $q_i^{\delta_{ia}}$. Evaluating (85) at zero gives $\kappa_i \Psi_i(0) = q_i^{-\delta_{ia}}$ on this loading. Since K_i is independent of the loading by Lemma 12.1, this also verifies $\kappa_i^2 K_i = 1$ for the displayed normalizers.

Type, a	$(t_i)_i$	$(\tau_i)_i$
$G_2, 1$	$(1, 4)$	$(2, -4)$
$G_2, 2$	$(-12, 12)$	$(12, -11)$
$F_4, 1$	$(9, -4, 3, 0)$	$(-7, 4, -3, 0)$
$F_4, 2$	$(0, 1, 6, 0)$	$(0, 1, -6, 0)$
$F_4, 3$	$(0, -12, 18, 0)$	$(0, 12, -17, 0)$
$F_4, 4$	$(0, -6, 4, 9)$	$(0, 6, -4, -8)$

TABLE 4. Leading Cartan constants and ordinary normalization.

Proposition 12.4. *For each row of Table 1, the algebra $\mathscr{C}_{\mathbb{C}}$ with the indicated gauge ranks and paired framing has a finite-dimensional simple module whose pullback by $\Phi_{\mathbb{C}}$ is the ordinary fundamental module $L(Y_{a,q^s})$.*

Proof. We first construct the actual module of the unfolded flag algebra. Mirror the horizontal strand convention of [37, §2.1] to the convention of [38, §7.1]. The ordered red pair is then $(\lambda, \lambda^*) = (\lambda_R, \lambda_L)$. Put $\beta = \lambda + \lambda^*$ in the positive root lattice and take the weight-zero tensor-product algebra $T = T_0^{\lambda,\lambda^*}$. Let J be generated by the idempotents with a black strand beyond the second red strand in this convention. The first-red violating relation already excludes black strands on the other outside region. Black diagrams between the red strands give a surjection

$$R^{\lambda}(\beta) \twoheadrightarrow T/J. \tag{86}$$

To check surjectivity, cut a diagram at a height where a black strand crosses a red. The diagram factors through an excluded idempotent and is zero in the quotient. Every surviving diagram

is therefore a black cyclotomic diagram between the red strands. Its first-red relation is the defining cyclotomic relation, proving the claim.

The cyclotomic algebra in (86) is the lowest-weight block of V_λ. It has one simple isomorphism class because that weight space is one dimensional. The extremal divided-power construction of [38, Proposition 7.2] gives a simple projective in this block. A finite-dimensional algebra with one simple class whose simple is projective is a matrix algebra. On the other hand, [38, Lemma 7.3] supplies a nonzero cup simple annihilated by every idempotent generating J. Thus $T/J \neq 0$, and the surjection from the matrix algebra is an isomorphism. This identifies the actual between-red restriction of the cup.

Read this lowest-weight cyclotomic module from the right red in the original convention. At an extremal weight v, the maximal nonzero lowering divided power of color c is $\max(0, \langle v, \alpha_c^\vee \rangle)$. If the pairing is negative the extremal vector is already at the bottom of that string; if it is zero both directions vanish. A maximal nonzero step gives the next extremal simple projective. The rule (82) is exactly this construction, and the displayed endpoint is the lowest weight. The corresponding loading is therefore supported and nonzero. Any lexicographically larger loading has a first block where it takes more than the maximal nonzero power following the same prefix. That prefix is zero, so the whole loading is zero. This proves that the displayed loading is the unique greatest supported loading. The inclusion of every zero block in the tables makes this comparison over all admissible loadings.

We now pass from complete words to weighted flags. A block of multiplicity A uses F_c^A in the complete-word description and $F_c^{(A)}$ in the extremal construction. The complete-to-weighted flag map has fiber $\mathrm{Flag}(\mathbb{C}^A)$; for several blocks these fibers multiply. The flag inside a common-height block is unrestricted: the strict lowering condition and the prescribed even-before-odd order force a same-height arrow to be zero before choosing these additional flags. The matrix factor in [37, (2.1.35)–(2.1.42)] consequently has size $\prod A!$, equal to the dimension of the cohomology of these fibers. The multiplicity of the divided-power extremal projective is one, so the greatest *weighted* idempotent space is one dimensional.

It remains to check that the positive currents annihilate this space. Let $\Gamma_i \subset \operatorname{Spec} B_{\mathbb{C}} \times \operatorname{Spec} B_{\mathbb{C}}$ be the finite closed correspondence replacing one root of color i by q_i^2 times that root, with the compatible change of determinant. It is the symmetric quotient of the finite union of the labeled shift graphs. In the difference realization, a positive current is a sum of terms $f_t(w)\mathbf{D}_{i,t}^{-1}$. Every element of the ideal of Γ_i, acting by the two coefficient-ring actions, annihilates these terms. By Lemma 12.3, it annihilates the current in the actual algebra. If two generalized gauge characters are outside Γ_i, one element of that ideal has a nonzero value there. On their finite generalized block it is an invertible scalar plus a nilpotent. The corresponding current block is therefore zero. This proves the support assertion also when roots within a block coincide.

The relation $\mathbf{D}_{i,t}^{-1} w_{i,t} = q_i^{-2} w_{i,t} \mathbf{D}_{i,t}^{-1}$ shows that a positive current takes a source root to q_i^2 times that root. It strictly raises its height and hence the loading. It thus kills the greatest loading. There are only finitely many allowed heights and a fixed total number of black strands. The cup restriction has finite complete-word spaces, so its nonzero weighted even corner is finite dimensional and simple. The completed equivalence of [37, Theorem 4.3.1] gives an actual finite-dimensional $\mathscr{C}_{\mathbb{C}}$-module from this corner.

Finally, the ordinary map is surjective by Proposition 12.2. Its pullback is therefore simple, and the greatest line is an ordinary highest ℓ-weight line. Equations (85) and (73), with Table 4, identify that ℓ-weight with Y_{a,q^s}. Since the simple module is generated by this line, the ordinary Cartan–root relations make its leading Cartan operators semisimple with type-1 eigenvalues. The classification by highest ℓ-weight proves the assertion. □

Every gauge rank in Table 1 is even. If all gauge roots and framing coordinates of one row are multiplied by $b \in \mathbb{C}^\times$, the chosen determinant is multiplied by $b^{a_i/2}$, and

$$K_i(b) = b^{2\delta_{ia}} K_i(1), \qquad \kappa_i(b) = b^{-\delta_{ia}} \kappa_i(1). \tag{87}$$

These are Laurent functions of b. We thus obtain the ordinary fundamental $L(Y_{a,bq^s})$ for every b, with a Laurent-algebraic ordinary current map. Products of paired framings have the product normalization, since the constants K_i multiply.

12.3. **Separated parameters and completed convolution.** Fix finitely many of the preceding fundamental factors, indexed by r, with spectral scalars b_r, and first suppose that

$$b_r/b_s \notin q^{\mathbb{Z}} \quad (r \neq s). \tag{88}$$

Use the sums of their gauge ranks and the unions of their framing multisets to define the total Coulomb algebra. Let Ω_r be the collection of all integral loadings with the ranks and residue refinements of the rth factor. Let Ω be their product, regarded as loadings of the total multiset. Completions below are two-sided completions at these gauge characters, retaining every source and target idempotent.

Lemma 12.5. *At these completed loadings, the determinant cover of the total gauge group admits a compatible branch on which $D_i = \prod_r D_{i,r}$. The branch is compatible with every current correspondence between the loadings in Ω.*

Proof. The total monic characteristic polynomial factors uniquely into its prescribed spectral clusters in the completed coefficient ring. This is Hensel factorization: the resultants between distinct cluster polynomials have nonzero constant terms by (88). Each cluster determinant is a unit. A square root of a unit in this complete local complex algebra is uniquely determined by its nonzero constant term, since the derivative of $X^2 - d$ at either root is nonzero.

At a loading γ, prescribe the constant term

$$D_{i,r}(\gamma) = b_r^{a_{i,r}/2} q^{\frac{1}{2}\sum_k k n_{i,r,k}}. \tag{89}$$

The even ranks make the power of b_r integral. Fix the half-character convention for the powers of q once and for all. A current replacing k by $k + 2d_i$, or by $k - 2d_i$, multiplies this value by q_i, or by q_i^{-1}, respectively. These are the determinant commutation factors of the minuscule correspondence. A closed path returning to the same multiset returns to the same sum of exponents, so these choices are consistent on every path. Uniqueness of the formal square roots extends the consistency to the completed rings on both sides of a correspondence. Prescribing the global constant as their product gives $D_i = \prod_r D_{i,r}$. □

Proposition 12.6. *With the branch of Lemma 12.5, there is an isomorphism of completed convolution algebras*

$$\widehat{\mathscr{C}}_\Omega \;\simeq \widehat{\bigotimes_r} \widehat{\mathscr{C}}_{r,\Omega_r}. \tag{90}$$

It respects every source and target idempotent and multiplies the Cartan rational functions of the factors.

Proof. We give the fixed-point and multiplication argument, including its dependence on the spectral scalars. At any finite lattice bound, a coefficient of a gauge, matter, or framing map from cluster s to cluster r can be fixed by the defining semisimple operator only if

$$(b_r/b_s)q^n = 1$$

for an integer n. The symmetrizers change the integer n and do not change the scalar ratio; this follows directly from the ramified loop characters in [37, (3.1.60), (3.1.64)]. Thus there are no fixed maps between different clusters.

The splitting of fixed lattices holds as a statement of schemes. Between two lattice bounds, the loop quotient is finite dimensional. The semisimple operator has disjoint eigenvalue sets for distinct clusters. Polynomial spectral projectors decompose every invariant submodule functorially, over any test algebra, into its cluster summands. Consequently the fixed lattice functor is the product of the factor functors. The decompositions commute with increasing the lattice bounds. The fixed gauge centralizer, matter bundles, framing bundles, and classical BFN correspondences are therefore products. In particular for every pair $\gamma, \eta \in \Omega$,

$$\mathcal{Z}_{\gamma,\eta} = \prod_r \mathcal{Z}_{\gamma_r,\eta_r}. \tag{91}$$

The threefold correspondence and all projections defining convolution split in the same way.

Within one cluster the scalar b_r cancels from every fixed equation, leaving its integral weighted-flag model. Simultaneous scalar multiplication of the gauge and framing spaces acts trivially on the internal Hom spaces and flags. Translating that scalar character from b_r to 1 in the completed equivariant coefficient ring changes only the scalar parameter. Formula (89) supplies its lift to the determinant cover. Thus the reduction to the integral model is compatible with equivariant K-theory, in addition to the underlying fixed varieties.

Apply semisimple equivariant localization to each finite approximation. Cross-cluster normal characters give Euler factors $1 - (b_r/b_s)q^n$, whose constant terms are nonzero, hence units in the chosen completion. Localization only requires the given semisimple element and its diagonalizable Zariski closure. It places no restriction on the absolute value of q and requires no integral cocharacter containing every b_r.

To check multiplication, write the restriction as $r_\Omega = \sigma^* \iota_*^{-1}$, as in [37, §4.3.2]. If r_2 is torus localization of the original correspondence and r_1 is torus localization of its fixed correspondence, the Cartesian base-change diagram for the threefold correspondence gives

$$r_2 = r_1 r_\Omega.$$

The two torus localizations are algebra homomorphisms, and r_1 is injective. The last assertion follows on every pair of blocks from the product of the factor embeddings. It follows that r_Ω is multiplicative. The moving cross-cluster normal factors have already entered ι_*^{-1} in this identity. Thus their Euler factors are part of the multiplication comparison.

Finally apply the modified equivariant Riemann–Roch map and Borel–Moore Kunneth to (91). This gives (90) on each pair of source and target blocks, compatibly with convolution. The determinant comparison holds on the branch already chosen. Equivalently, the product of the factor determinant covers maps to the restricted total cover by a finite central isogeny acting trivially on the varieties; on the chosen branch it gives the same rational equivariant cohomology. The Hensel factorization of the characteristic polynomial is unique at the unordered total multiset, so it introduces no permutation factor. Total W_i, D_i, and Z_i are the products of their factor counterparts. Formula (72) proves the asserted compatibility of Cartan currents. □

12.4. **Descent through collisions.**

Theorem 12.7. *Let $m = \prod_{r=1}^{N} Y_{a_r,A_r}$ be any dominant monomial in type G_2 or F_4, with arbitrary $A_r \in \mathbb{C}^\times$. Associate to its rth factor the spectral translate of Table 1 with $b_r = A_r q^{-s_r}$. For the total gauge ranks and paired framing obtained by their sums and unions, respectively, there is a surjective ordinary map*

$$\Phi_m : U_q(L\mathfrak{g}) \twoheadrightarrow \mathscr{C}_m.$$

Every ordered ordinary tensor product $\bigotimes_r L(Y_{a_r,A_r})$ factors through this map. Its simple subquotients factor through the same map. These assertions also hold over the Laurent deformation family separating the factors and over its completion at any prescribed collision.

Proof. First assume (88). The external tensor product of the finite cup modules is a finite-dimensional smooth module of the right side of (90). It has finite idempotent support. Transporting it by that isomorphism and restricting to the uncompleted algebra gives an actual $\mathscr{C}_m$-module N. This restriction is well defined: a word of finitely many minuscule generators has finitely many root-shift possibilities from any fixed source block. The finite support therefore uses only finite sums.

The Cartan compatibility in Proposition 12.6 gives

$$\chi_q N = \prod_r \chi_q L(Y_{a_r,A_r}). \tag{92}$$

The leading normalized Cartan operators are tensor products of the leading operators of the factors. They are semisimple with type-1 eigenvalues. Fundamental modules are special, and all variables of their characters lie in their spectral q-orbit [22, Theorem 2.11 and Corollary 2.13]. Distinct factors in the present situation have disjoint such orbits. Hence the right side of (92) has just one dominant monomial, namely m, with coefficient 1: a negative power in any factor cannot be canceled by another factor. Every finite-dimensional simple composition factor contributes its dominant highest monomial with positive multiplicity. Thus both N and the ordinary tensor product have a single composition factor, of multiplicity one. They are isomorphic to the ordinary simple of highest monomial m. The actual ordinary tensor therefore factors through Φ_m at separated parameters.

For arbitrary initial parameters, put

$$S = \mathbb{C}[u, u^{-1}], \qquad b_r(u) = b_r(1)u^{c_r},$$

where the integers c_r are pairwise distinct. The spectral automorphism on each fundamental representation and the ordinary coproduct give an S-free tensor module M_S whose action matrices are Laurent polynomials in u. The framing coordinates and the normalization units (87) are likewise Laurent monomials. Proposition 12.2 gives a surjection $\Phi_S : U_q(L\mathfrak{g}) \otimes S \twoheadrightarrow \mathscr{C}_S$ over this same ring.

For each pair $r \neq s$, the failure of separation at a complex point u_0 means $u_0^{c_r-c_s} \in b_s(1)b_r(1)^{-1}q^{\mathbb{Z}}$. The union of the excluded sets is countable. Its complement is infinite and Zariski dense in $\mathbb{C}^\times$. If $x \in \ker \Phi_S$, its specialization annihilates M_{u_0} at every point of this complement by the separated construction. Every entry of its action matrix on M_S is a Laurent polynomial, and vanishes at all these points. It is identically zero. We have proved

$$(\ker \Phi_S) M_S = 0.$$

Surjectivity now gives an action of $\mathscr{C}_S$ on the actual tensor family. Specializing at $u = 1$ proves the assertion at the prescribed collision, in any order of the factors. Completing at $u = 1$, or making the formal substitution $u = \exp(a)$, gives the corresponding action over $\mathbb{C}[[a]]$.

More generally, if a surjective algebra map $U_R \to A_R$ acts on an R-torsion-free lattice whose generic action factors through $A_R \otimes_R \operatorname{Frac}(R)$, its kernel kills the lattice. This proves the same assertion for every ordinary stable lattice in the tensor family used later. Right exactness under specialization ensures that the special fiber is an $A_R/\mathfrak{m}A_R$-module. Finally, a U-stable subspace of a module factoring through a surjection $U \to A$ is A-stable. The assertion for all simple subquotients follows. □

The generalized gauge support of a subquotient is contained in that of the tensor module. Indeed, the finite gauge spectral projectors are polynomials in the coefficient ring, obtained by

Chinese remainders; their functors are exact. Moreover the highest gauge multiset is determined by the ordinary Cartan series and the fixed framing: the logarithmic inversion (75) recovers all its power sums and hence its characteristic polynomials. Thus the highest ordinary line and its gauge loading agree in every construction above.

13. The weighted-flag algebra and geometric currents

We record the precise comparison needed for the filtrations in the subsequent sections. In particular, the comparison keeps the ordinary quantum loop action just constructed.

13.1. **The fixed weighted-flag model.** Fix a gauge loading and a framing semisimple character. For a single spectral component, the unfolded vector space is graded by the colors in (78). Orient each edge from parity 0 to parity 1, using the lifted parities ϵ' specified before (82). A weighted flag is an increasing sequence $F^{\leq h}$ of unfolded-graded subspaces, equal to 0 for $h \ll 0$ and to the whole gauge space for $h \gg 0$. A flag of type γ has graded block dimensions prescribed by its root multiplicities, at the heights (79). Let Flag_γ be its partial flag variety. Write the unfolded representation space as

$$N = \bigoplus_c \mathrm{Hom}(\widetilde{W}_c, \widetilde{V}_c) \oplus \bigoplus_{c \to d} \mathrm{Hom}(\widetilde{V}_c, \widetilde{V}_d),$$

and denote a point by (A, B). Give the framing space its red-height grading. Then set

$$X_\gamma = \{(A, B, F) : F \in \mathrm{Flag}_\gamma, B(F^{\leq h}) \subset F^{<h}, \quad A(\widetilde{W}^h) \subset F^{\leq h} \text{ for every } h\}.$$

These are linear equations in the representation coordinates at each flag. Thus X_γ is a vector bundle over Flag_γ. Its proper projection to N is denoted by π_γ. The unfolded gauge group is $\underline{G} = \prod_c \mathrm{GL}(\widetilde{V}_c)$.

The weighted-flag convolution algebra is

$$\mathscr{A}^{\mathrm{fl}} = \bigoplus_{\gamma,\eta} \mathrm{Ext}^*_{\underline{G}} \left((\pi_\eta)_* \mathbb{C}_{X_\eta}[\dim X_\eta], (\pi_\gamma)_* \mathbb{C}_{X_\gamma}[\dim X_\gamma] \right). \tag{93}$$

Its product is convolution on $X_\gamma \times_N X_\eta$. The indicated shifts are the selfdual normalization and fix its grading. We use the locally unital algebra, with idempotents e_γ for all loadings. Source coefficient classes act on the right of a correspondence, and target coefficient classes act on the left.

Proposition 13.1. *The semisimple completion of the Coulomb algebra acting in Theorem 12.7 is identified with the completion of* (93) *by localization and the modified Riemann–Roch map. Write this isomorphism as* Θ*. It preserves source and target idempotents and the completed gauge coefficient rings. For arbitrary finitely many multiplicative spectral components the fixed spaces are products of these unfolded models, and their convolution algebra is the corresponding completed tensor product.*

Proof. For the selected integral component this is the construction of [37, Theorem 4.3.1 and Lemmas 4.3.2–4.3.5]. The fixed loop stabilizer is identified, by evaluation at 1, with the parabolic preserving the weighted flag. Its action on a graded quotient factors through the corresponding Levi factor. The fixed matter equations give exactly the vector bundles X_γ and their common-flag correspondences. The modified Chern character of [37, Lemma 2.1.2] is multiplicative for their convolution.

We check the component extension. The parity vectors for gauge roots are $(0, 1)$ in type G_2 and $(0, 0, 0, 1)$ in type F_4. Components can be represented by roots bq^k with these parities, with b taken modulo $q^{2\mathbb{Z}}$. A one-step spectral translate gives the other parity class. Same-color gauge shifts are $2d_i$; on an edge the parity of the fixed matter shift is the difference of the

displayed parities. Consequently gauge, matter, and framing fixed equations do not connect distinct components. Their lattice functors and all their twofold and threefold correspondences split, by the spectral-projector argument of Proposition 12.6. Inside a component the common scalar cancels from the equations. The moving normal Euler factors between components have nonzero constants and are units in its semisimple completion. The same $r_2 = r_1 r_\Omega$ argument proves multiplicativity, followed by modified Riemann–Roch and Borel–Moore Kunneth. The determinant branches are selected by Lemma 12.5. This proves the asserted extension, including the two parity classes inside one $q^{\mathbb{Z}}$-orbit. □

13.2. **The minuscule current object.** Consider a negative current that takes one color-i root from q^k to q^l, where $l = k - 2d_i$. A scalar spectral translate does not change the following calculation. Denote this transition by v. If the source block dimensions at k, l are α, β, the target dimensions are $\alpha - 1, \beta + 1$. Since the same unfolded color has heights congruent modulo $2d_i$, there is no intermediate step of that color between them. The common flag refinement S_v is the projective bundle of lines in the source block of dimension α. The moved line determines the target flag uniquely. Equivalently S_v is the projective bundle of one-dimensional quotients of the target block of dimension $\beta + 1$.

Let Z_v consist of the common-refinement flags together with the representation maps satisfying both weighted flag conditions. It is a constant-rank vector bundle over S_v. To see the constant rank, the unfolded group is transitive on the common-refinement flag base, and the two flag conditions are equivariant linear equations in the representation coordinates. Thus Z_v is smooth. It has a closed incidence embedding

$$j : Z_v \hookrightarrow X_t \times_N X_s.$$

Write L for the moved line and $x = c_1(L)$ in the logarithmic line convention for the current. The source and target of this embedding are labeled s and t throughout.

The geometric negative and positive current objects are the tautological line sheaves on the classical minuscule spaces:

$$\mathscr{Q}_i = (V_{i,\mathcal{O}}/L') \otimes \mathcal{O}_{\mathcal{R}_{\omega_{i,1}}}, \qquad \mathscr{S}_i = (L'/V_{i,\mathcal{O}}) \otimes \mathcal{O}_{\mathcal{R}_{\omega_{i,-1}}}. \tag{94}$$

Here $V_{i,\mathcal{O}} = V_i[[z_i]]$, L' is the gauge lattice, and $\mathcal{R}_{\omega_{i,\pm 1}}$ is the classical BFN stratum for the positive or negative first minuscule coweight at i. Integer current modes are powers of these line sheaves, with fixed determinant and numerical factors. This description determines the localization of the actual current class.

Lemma 13.2. *On a finite approximation, let B be the minuscule Grassmannian, T_B its ambient matter vector bundle, and $E \subset T_B$ the classical minuscule BFN subbundle. For a fixed component $B_0 \subset B$, write*

$$T_B|_{B_0} = T_0 \oplus T_m, \qquad E|_{B_0} = E_0 \oplus E_m$$

for fixed and moving parts. The localized class is

$$r_\gamma[\mathcal{O}_E] = [\mathcal{O}_{E_0}] \frac{\lambda_{-1}((T_m/E_m)^*)}{\lambda_{-1}(N^*_{B_0/B})}. \tag{95}$$

The quotient in this formula is a regular unit at the selected semisimple completion.

Proof. The minuscule Grassmannian B is a projective space. The classical minuscule space is the constant-rank subbundle $E_b = N_{\mathcal{O}} \cap gN_{\mathcal{O}}$, as in [37, (3.1.23)–(3.1.26)]. Proper naturality of localization allows the current class, supported on this minuscule piece, to be localized on B. The inclusion $B_0 \hookrightarrow B$ is a regular immersion of smooth varieties. Since E is flat over B, this first restriction gives $\mathcal{O}_{E|_{B_0}}$, and inverting fixed pushforward contributes exactly $\lambda_{-1}(N^*_{B_0/B})^{-1}$.

Put $Q = T_B|_{B_0}/E|_{B_0}$. Its fixed and moving parts are $Q_0 = T_0/E_0$ and $Q_m = T_m/E_m$. Resolve $\mathcal{O}_E$, as a sheaf on the ambient bundle, by the Koszul complex of Q^* and pull it back to T_0. The fixed equations Q_0^* remain a regular sequence and resolve $\mathcal{O}_{E_0}$. The moving equations become zero and contribute the exterior class $\lambda_{-1}(Q_m^*)$. This is also the derived pullback in the BFN construction: associativity of derived tensor products gives

$$\mathcal{O}_E \otimes^L_{\mathcal{O}_{\mathfrak{R}}} \left(\mathcal{O}_{\mathfrak{R}} \otimes^L_{\mathcal{O}_{T_B}} \mathcal{O}_{T_0}\right) \simeq \mathcal{O}_E \otimes^L_{\mathcal{O}_{T_B}} \mathcal{O}_{T_0}.$$

Here $\mathfrak{R}$ is the derived BFN space, and the original class is the classical sheaf (94). Combining the base and ambient restrictions proves (95). Every Euler factor in the displayed quotient is moving, hence has the form $1 - \xi\exp(\eta)$ with $\xi \neq 1$. In the integral component $\xi = q^b$, $b \neq 0$; between different components its constant is likewise nontrivial. It is therefore a unit. Both normal quotients are finite dimensional. Enlarging the finite approximation adds common free directions, which cancel from the formula. □

Proposition 13.3. *Let $F_{v,r}$ denote the local negative current tested against the rth power of the logarithm of its spectral variable, centered at the source root, for $r \geq 0$. Under Θ one has*

$$\Theta(F_{v,r}) = j_*([Z_v]x^r U_v), \qquad U_v \in \widehat{H}^*_{\underline{G}}(Z_v)^\times. \tag{96}$$

The unit U_v has nonzero constant term and is independent of r. The analogous statement holds for the reversed positive correspondence with its own unit.

Proof. The fixed classical subbundle E_0 in Lemma 13.2, induced from its flag parabolic to $\underline{G}$, is exactly the common-refinement incidence Z_v. A line in the source block is the entire minuscule datum, so there is no additional component or covering degree in this identification. Its representation equations are the two flag conditions defining Z_v.

Use the symmetric modified Riemann–Roch convention. Let χ_v denote the product of the restricted line and determinant characters. The image of the localized current is the incidence fundamental class times

$$U_v = \chi_v \operatorname{ch}\left(\frac{\lambda_{-1}((T_m/E_m)^*)}{\lambda_{-1}(N^*_{B_0/B})}\right) \frac{\operatorname{Td}(TZ_v)}{\sqrt{p_s^* \operatorname{Td}(TX_s)\, p_t^* \operatorname{Td}(TX_t)}}. \tag{97}$$

Here $p_s : Z_v \to X_s$ and $p_t : Z_v \to X_t$ are the source and target projections, and ch is the completed equivariant Chern character; the fixed mode-independent powers of the tautological line are included in the first factor. This convention sends the diagonal structure sheaf to the diagonal fundamental class and preserves convolution. Every Todd series has constant term 1, the line and determinant characters have nonzero constant terms, and the middle factor is a unit by Lemma 13.2.

An integer current mode differs from the next by multiplication by the character of the single tautological line. At a fixed spectral center, change from its Laurent character to its logarithm. Center the line character first, including the q_i^2 shift between the displayed current variable and that character. The local test of order r is then precisely x^r. All remaining factors in (97) are independent of r. Riemann–Roch for the closed incidence embedding proves (96). □

13.3. **Source coefficients and the exact inverse.**

Lemma 13.4. *The equivariant cohomology of Z_v is generated by the Chern classes of the source flag blocks and the moved-line class x. Every homogeneous class has a polynomial lift in these generators of the same nonnegative degree.*

Proof. The map $Z_v \to S_v$ is a vector bundle, and $S_v \to \mathrm{Flag}_s$ is the projective bundle of lines in the source block of rank α. Hence

$$H^*_{\underline{G}}(Z_v) = H^*_{\underline{G}}(S_v) = H^*_{\underline{G}}(\mathrm{Flag}_s)[x]/(x^\alpha - c_1 x^{\alpha-1} + \cdots + (-1)^\alpha c_\alpha), \tag{98}$$

with the chosen line convention. The flag coefficient ring is the polynomial ring in the Chern classes of its graded blocks. The target block classes add no generators: removing L from the source block divides its total Chern polynomial by $1 + x$, and adding L to the lower block multiplies that block polynomial by $1 + x$. The relation in (98) is homogeneous. This proves the lifting assertion degree by degree. □

These source coefficients are actual completed gauge Cartan elements. The characteristic polynomial of W_i has a unique Hensel factorization by distinct spectral centers; its denominators are differences of distinct constants. This separates blocks, including blocks of repeated roots, without separating the individual roots of a repeated block. Writing the roots of a block as $q^k \exp(x_b)$ identifies their symmetric logarithmic polynomials with its Chern classes. This is the coefficient-ring part of Θ.

The polynomial $W_i(u)$ satisfies the following identity in the Coulomb algebra:

$$W_i(u)F_j(z) = F_j(z)W_i(u)\left(\frac{1 - q_i^{-2}z/u}{1 - z/u}\right)^{\delta_{ij}}. \tag{99}$$

Indeed, a negative minuscule shift acts on one root, and its delta function is evaluated after that shift. The relation $\mathbf{D}_{i,t}w_{i,t} = q_i^2 w_{i,t}\mathbf{D}_{i,t}$ together with that delta function gives the displayed ratio. Other colors commute. Lemma 12.3 transfers the identity from difference operators to $\mathscr{C}_S$ itself.

Iterating (99) through a negative word gives, for the logarithmic power sum of order $r > 0$ in the block (i, k), the operator

$$P^{\mathrm{top}}_{i,k,r} + \sum_{\text{lowering variables at } (i,k+2d_i)} y_b^r - \sum_{\text{lowering variables at } (i,k)} y_b^r. \tag{100}$$

This is an identity on words and survives every module quotient. At the undeformed highest line $P^{\mathrm{top}}_{i,k,r} = 0$. In a formal spectral deformation $q^k \exp(c_b a)$ its value is $\sum_b (c_b a)^r$. Thus every expression in (100) has polynomial degree r when $\deg a = \deg y_b = 1$. Newton identities give the actual degree-r action of the block Chern class c_r. This calculation includes the nilpotent Cartan part of the action.

Corollary 13.5. *Write, degree by degree in the incidence coefficient ring,*

$$U_v^{-1} = \sum_{a \geq 0} x^a p_a,$$

where p_a are completed source Cartan classes, using Lemma 13.4. Then the homogeneous decorated minuscule class satisfies the exact identity

$$j_*([Z_v]x^r) = \sum_{a \geq 0} \Theta(F_{v,r+a})p_a. \tag{101}$$

The source coefficients occur on the right. On every finite-dimensional smooth module the identity is a finite operator identity.

Proof. The projection formula for j identifies right multiplication by p_a with pullback from the source. Multiplying (96) by these coefficients and summing cancels U_v, giving (101). Every homogeneous term of U_v^{-1} has a polynomial lift of nonnegative degree by (98). The local Cartan maximal ideals act nilpotently on a finite-dimensional smooth module. Its current action also

depends on finitely many jets at finitely many reduced spectral centers. Chinese remainders realize each needed logarithmic jet by a finite linear combination of Laurent current modes. Consequently both the coefficient expansions and the local mode expansions truncate on the module. The same argument for the reversed correspondence uses its own source ring. □

13.4. **The geometric degree.** Let $n_j(h)$ be the number of current gauge roots at q^h of color j, and let $w_i(h)$ be the multiplicity of the underlying flavor coordinate q^h. Thus a framing zero at q^K contributes to $w_i(K-2d_i)$. For the move at (i,k) define

$$p'_{i,k} = n_i(k) + n_i(k-2d_i) - w_i(k-2d_i) - \sum_{j:c_{ji}<0} \sum_{b=1}^{-c_{ji}} n_j(k-b_{ij}-2d_jb). \tag{102}$$

It is the pole order at this center of the current Cartan rational function (72); cancellation of numerator and denominator zeros is included in this signed order.

Proposition 13.6. *In the selfdual Ext grading of* (93)*,*

$$\deg j_*([Z_v]x^r) = 2r + 1 - p'_{i,k}. \tag{103}$$

The reversed minuscule correspondence has the same degree.

Proof. The degree of its undecorated fundamental class is

$$\dim X_s + \dim X_t - 2\dim Z_v.$$

The source-to-target change in flag dimension is $\alpha - \beta - 1$, while the common-refinement flag has dimension $\dim \mathrm{Flag}_s + \alpha - 1$. The flag contribution to the displayed difference is therefore $1 - \alpha - \beta$.

We compute the remaining contribution from the linear matter conditions. In a basis compatible with the common refinement, each coordinate of a representation map is independently allowed or required to be zero. The contribution of a coordinate to $\mathrm{rk}\, X_s + \mathrm{rk}\, X_t - 2\,\mathrm{rk}\, Z_v$, after removing the flag dimensions, is 1 precisely when it is allowed in exactly one flag condition. It is zero when it is allowed in both or forbidden in both. A framing coordinate changes its condition exactly at height $k-2d_i$, giving $w_i(k-2d_i)$. A coordinate between color j and the moved line changes its condition precisely when its source root has exponent

$$k - b_{ij} - 2d_jb, \qquad 1 \le b \le -c_{ji}.$$

These are the $-c_{ji}$ ramified matter shifts in (72). Summing the changed coordinates therefore gives

$$w_i(k-2d_i) + \sum_{j:c_{ji}<0} \sum_{b=1}^{-c_{ji}} n_j(k-b_{ij}-2d_jb).$$

Explicitly, the changed neighboring exponents are as follows, in the numbering of Section 12:

Type	Moved color i	Neighbor j and exponents
G_2	1	$j=2:\ k+1,\ k-1,\ k-3$
G_2	2	$j=1:\ k-3$
F_4	1	$j=2:\ k-2$
F_4	2	$j=1:\ k-2;\quad j=3:\ k,\ k-2$
F_4	3	$j=2:\ k-2;\quad j=4:\ k-1$
F_4	4	$j=3:\ k-1$

For example, at the long G_2 node the three values are $k-(-3)-2b$ for $b=1,2,3$. The height correction $\sigma_2=-2$ converts the unfolded flag inequalities to these original root exponents. The other rows follow by the same substitution in $k-b_{ij}-2d_jb$.

Adding the flag contribution, with $\alpha=n_i(k)$ and $\beta=n_i(k-2d_i)$, gives $1-p'_{i,k}$. A power x^r has cohomological degree $2r$. This proves (103). Upon reversing the step the two multiplicities become $\alpha-1,\beta+1$; their sum is unchanged, as are the framing and neighboring terms. The same formula therefore applies to the reverse correspondence. □

The remaining input for a filtered action is the lower degree bound for the local negative insertion and positive removal operators. Together with (100), those bounds will imply that every decorated minuscule class acts with its geometric degree.

14. Euler quotients and integral standard modules

In this section and through Section 20, we treat the types G_2 and F_4. We use the polynomial realization of the negative shuffle algebra. In particular, the arrows between orthogonal colours that occur in this realization are retained in the bilinear form E. Their quadratic contribution cancels from the centered grading, as explained at the end of Section 19. All completions in this section are completions of a specified finite module over a specified Noetherian coefficient ring.

14.1. **The intrinsic quotient.** Fix a dominant monomial n, and write its highest Cartan functions as $\boldsymbol{\psi}=(\psi_i(z))_{i\in I}$. At each spectral vertex v, let m_v and l_v be the pole and zero multiplicities in the framing presentation of $\boldsymbol{\psi}$. We retain a pole and a zero separately if they have the same centre. Thus

$$p_v=m_v-l_v,\qquad d_p(\nu)=p\cdot\nu+E(\nu,\nu)-\nu\cdot\nu.$$

Here $E(\nu,\mu)=\sum_{v\to w}\nu_v\mu_w$, and $\nu\cdot\mu=\sum_v\nu_v\mu_v$. We abbreviate $E(\nu,e_v)$ and $E(e_v,\nu)$ to $E(\nu,v)$ and $E(v,\nu)$, respectively. The polynomial degree of a logarithmic Chern root is one. The corresponding cohomological degree is two.

Let $\mathcal{H}_\nu$ be the critical cohomology group of the framed stable space in dimension ν, with the unshifted convention of the preceding geometric construction. Multiplication by the numerator Euler class is the homogeneous map

$$D_\nu=\prod_v\prod_{b=1}^{\nu_v}y_{v,b}^{l_v},\qquad L_\nu=\deg D_\nu=l\cdot\nu. \tag{104}$$

Set

$$N(n)_\nu=\mathcal{H}_\nu/\ker D_\nu,\qquad N(n)=\bigoplus_\nu N(n)_\nu. \tag{105}$$

We give the quotient the grading of the *source* of D_ν. The map to $\operatorname{im}D_\nu$, with its inherited grading, raises polynomial degree by L_ν. The decreasing normalized filtration is

$$F^hN(n)_\nu=\bigoplus_{2d-d_p(\nu)\geq h}N(n)_\nu[d]. \tag{106}$$

The bracket $[d]$ denotes polynomial degree d, not a shift of a complex. For a finite filtered module $V=\bigoplus_\mu V_\mu$ whose filtration respects its generalized ℓ-weight spaces, we use

$$\operatorname{ch}_t\operatorname{gr}_F V=\sum_{\mu,h}\dim(F^hV_\mu/F^{h+1}V_\mu)t^h\mu. \tag{107}$$

Here μ is the monomial of the corresponding ℓ-weight.

Proposition 14.1. *Evaluation on an ordinary highest vector induces an isomorphism*

$$N(n) \xrightarrow{\sim} L(n), \qquad [F] \longmapsto F v_n.$$

In particular, (106) *is a finite, exhaustive, separated filtration on the actual ordinary simple module. It respects the generalized ℓ-weight decomposition, and its highest line has degree zero.*

Proof. The homogeneous evaluation in Theorem 11.1 maps the additive wheel ideal onto $\mathcal{H}_\nu$. Theorem 9.2 identifies its Euler source quotient with the quotient by all local ordered residue tests. The local completion assertion of Theorem 3.4 identifies this map with the completed numerical small-shuffle evaluation. The elementwise assertion of Theorem 3.3 says precisely that a negative shuffle element annihilates v_n if and only if its evaluation is killed by D_ν. It therefore identifies the two quotients, with the displayed map on every word. Surjectivity is also immediate from highest generation. The quotient map transports the ordinary action.

The simple $L(n)$ is finite dimensional. Each homogeneous quotient in (105) is finite dimensional, and only finitely many are nonzero by the preceding identification. Its grading is bounded on both sides. The direct sum filtration is therefore finite. The sector $\nu = 0$ is $\mathbb{C}$, its Euler class is one, and $d_p(0) = 0$. The spectral decomposition used in evaluation is the generalized Cartan decomposition, which proves the remaining assertions. □

14.2. **A framing deformation and its source quotient.** Write

$$m = \prod_{r=1}^{N} Y_{i_r,b_r}.$$

Choose pairwise distinct integers c_r, and move the two framing parameters belonging to factor r together by $b_r \mapsto b_r \exp(c_r a)$. Let

$$R_0 = \mathbb{C}[a], \qquad R = \mathbb{C}[[a]], \qquad K = \mathbb{C}((a)).$$

The framing torus $A = \mathbb{G}_m$ acts on both framing lines of factor r with weight c_r, and a is its equivariant parameter. Although the multiplicative parameters are formal, their logarithmic framing weights are the homogeneous linear functions $c_r a$. The homogeneous equivariant groups of Theorem 11.1 will be denoted $\mathcal{H}_{0,\nu}$. They are finite graded free R_0-modules. Define

$$D_{0,\nu} = \prod_{v,b} \prod_{\substack{r \\ \text{numerator frame of } r \text{ at } v}} (y_{v,b} - c_r a), \qquad \mathcal{Q}_{0,\nu} = \mathcal{H}_{0,\nu} / \ker D_{0,\nu}. \tag{108}$$

The degree of $D_{0,\nu}$ is again $L_\nu = l \cdot \nu$. Put $H_{R,\nu} = \mathcal{H}_{0,\nu} \otimes_{R_0} R$, $Q_{R,\nu} = \mathcal{Q}_{0,\nu} \otimes_{R_0} R$, and $D_{R,\nu} = D_{0,\nu} \otimes_{R_0} R$. We write

$$\mathcal{H}_0 = \bigoplus_\nu \mathcal{H}_{0,\nu}, \qquad \mathcal{Q}_0 = \bigoplus_\nu \mathcal{Q}_{0,\nu}, \qquad H_R = \bigoplus_\nu H_{R,\nu}, \qquad D_R = \bigoplus_\nu D_{R,\nu}.$$

Lemma 14.2. *The module $\mathcal{Q}_{0,\nu}$ is finite graded free over R_0. Formation of its source quotient commutes with completion:*

$$Q_{R,\nu} = H_{R,\nu} / \ker D_{R,\nu} \simeq \operatorname{im} D_{R,\nu},$$

where the last isomorphism has polynomial degree L_ν. If $e_1, \dots, e_s$ is a homogeneous R_0-basis of $\mathcal{Q}_{0,\nu}$, of degrees $d_1, \dots, d_s$, then its polynomial filtration after completion is

$$F^d_{\mathrm{pol}} Q_{R,\nu} = \bigoplus_{j=1}^{s} a^{\max(0, d-d_j)} R e_j. \tag{109}$$

In particular,

$$F^d_{\mathrm{pol}} Q_{R,\nu} \cap a Q_{R,\nu} = a F^{d-1}_{\mathrm{pol}} Q_{R,\nu}. \tag{110}$$

Proof. The image of the homogeneous map $D_{0,\nu}$ is a finite torsion-free module over the principal ideal domain $\mathbb{C}[a]$. It is therefore free. The source quotient is isomorphic to this image, with the indicated degree shift. Since its kernel is homogeneous, a homogeneous basis can be chosen. Flatness of completion preserves the kernel, image, and quotient of the finite-module map. Expanding a vector in the homogeneous basis proves (109). Intersecting each summand with aRe_j gives

$$a^{\max(1,d-d_j)}Re_j = a\,a^{\max(0,d-1-d_j)}Re_j,$$

which proves (110). □

Proposition 14.3. *The direct sum $Q_R = \bigoplus_\nu Q_{R,\nu}$ is a finite free R-module. It carries the actual ordinary $U_R = R \otimes_{\mathbb{C}} U_q(L\mathfrak{g})$ action with highest Cartan function $\boldsymbol{\psi}(a)$, and is generated by its highest vector.*

Proof. First fix the ordinary root degree. Only finitely many spectral dimension vectors occur in the framed stable space of this degree. Indeed, a cyclic representation of total dimension d is generated by framing paths of length at most $d-1$: the spans of paths of length at most j form an increasing sequence, and equality at one stage implies equality thereafter. Before stabilization their dimensions strictly increase. There are finitely many framing grades and arrow shifts, so this bounds the set of grades that can occur.

For this finite set of sectors, the homogeneous wheel evaluation and the local numerical completion theorem give simultaneous surjectivity of negative-word evaluation modulo a. To see that this is a simultaneous assertion, take the invariant Laurent coordinate ring of the fixed ordinary root degree. Distinct reduced coloured multisets give comaximal maximal ideals. The Chinese remainder theorem prescribes the required finite local jets at all of them. Each resulting small-shuffle element is a finite linear combination of negative words. Choose finitely many such words whose evaluated images span the special fibre. Their determinant on a chosen basis has nonzero reduction, so their images generate the completed finite free module over R. The same conclusion applies to its source quotient Q_R.

Let $V_R(\boldsymbol{\psi}(a))$ be the ordinary highest ℓ-weight induced module, with its distinguished highest vector. Negative evaluation defines a surjection

$$f_R : V_R(\boldsymbol{\psi}(a)) \longrightarrow Q_R.$$

At this stage the map is a map for the negative algebra and the Cartan algebra. Put $\mathcal{N}_R = \ker f_R$. We identify its generic kernel through the equivariant residue presentation. For a fixed sector let

$$B_{R,\nu} = H_A^{2*}(X_{\nu,m}) \otimes_{\mathbb{C}[a]} R,$$

and let $I_{\nu,R}$ be the $(a,\mathbf{y})$-adic completion of $I_{\Gamma,\nu} \otimes_{\mathbb{C}} \mathbb{C}[a]$. Write $J_{m(a)}$ for the kernel of ordinary completed evaluation into $B_{R,\nu}$. Proposition 5.3 identifies this kernel with the ordered tests retaining all factors $y - c_r a$. The unframed critical image is $I_{\nu,R}$, since A acts trivially on the unframed arrow space. The equivariant section of Proposition 7.1 makes critical stable restriction surjective and transfers injectivity of specialization to the stable quotient. Thus its specialization image is precisely $\mathrm{ev}_R(I_{\nu,R}) \subset B_{R,\nu}$.

Specialization commutes with Chern multiplication. If $\Delta_{l(a)}$ denotes the product in (108), the ordered kernels give

$$J_{p(a)} = (J_{m(a)} : \Delta_{l(a)}), \qquad \ker(I_{\nu,R} \longrightarrow Q_{R,\nu}) = I_{\nu,R} \cap J_{p(a)}.$$

Indeed, a representative F maps to zero exactly when $\Delta_{l(a)}\,\mathrm{ev}_R(F) = 0$ in $B_{R,\nu}$. This proves the deformed kernel identity for the same word evaluation that defines f_R.

The ordinary modules $B_{R,\nu}$ for the finitely many sectors are finite free over R. Their labeled flag integrations span a finite R-submodule of the dual of their direct sum. Choose finitely many

tests generating that submodule and apply them after multiplication by $\Delta_{l(a)}$. Flat localization $R \to K$ commutes with their common kernel. Over K, the parameters $b_r \exp(c_r a)$ belong to distinct $q^{\mathbb{Z}}$-orbits: an equality of two such parameters up to a power of q would first identify their constant terms and then force $c_r = c_s$. Chinese remainders separate the finitely many resulting pole centers and prescribe their finite jets, with repeated poles at one center kept in a single local algebra. On these jets the change $z = x \exp(y)$ of Lemma 5.2 identifies the deformed additive tests with the multiplicative tests for $\boldsymbol{\psi}(a)$. All localization here is performed on finite modules; the local jet calculations use only finitely many inverses in K.

The generic ordinary tensor product is absolutely simple. To see this algebraically, first use the Laurent family with parameters $b_r u^{c_r}$. Choose a complex value of u for which their $q^{\mathbb{Z}}$-orbits are disjoint. At this value the product of the fundamental characters has a unique dominant monomial, of coefficient one, so the tensor product is simple. Burnside's theorem supplies finitely many ordinary algebra words whose matrices form a basis of its full endomorphism algebra. The determinant of these word matrices is a nonzero Laurent polynomial in u. It remains nonzero under the embedding $\mathbb{C}(u) \hookrightarrow K$, $u \mapsto \exp(a)$, and after extension to an algebraic closure of K. The generic word matrices therefore generate the full matrix algebra there.

The characteristic-zero version of Theorem 3.3 now identifies the generic deformed residue kernel with the highest-vector annihilator of this tensor product. Equality descends from the algebraic closure to K by faithful scalar extension. Hence $\mathcal{N}_R \otimes_R K$ is stable under the full ordinary algebra.

If $h \in \mathcal{N}_R$ and $u \in U_R$, then $f_R(uh)$ vanishes after extension to K. The target is torsion-free, so $f_R(uh) = 0$ already over R. Thus $\mathcal{N}_R$ is U_R-stable, and the full ordinary action descends to Q_R. This construction proves all ordinary current relations on the quotient at once.

The generic tensor product is finite dimensional. Outside its finitely many ordinary root degrees the torsion-free source quotients become zero over K, and hence are zero over R. Together with the finiteness in each root degree proved above, this shows that Q_R is finite free. The construction makes it generated by its highest vector. □

The choice of a logarithmic Euler representative in (108) does not change this construction. An actual multiplicative Euler factor equals its linear factor times a formal unit. The product of the numerator units is an invertible multiplication operator on the target cohomology. It has the same kernel as the linear Euler product. Units in labelled shuffle kernels are instead absorbed into arbitrary labelled test functions before symmetrization. These are two different operations, each valid in its stated module.

Theorem 14.4. *Order the fundamental factors in decreasing spectral order within each $q^{\mathbb{Z}}$-orbit, and let $M_R(m)$ be their ordinary ordered tensor product with the above framing deformation. There is a highest-vector-preserving U_R-isomorphism*

$$Q_R \xrightarrow{\sim} M_R(m). \tag{111}$$

Consequently the actual ordinary standard module is

$$Q(m) := Q_R/aQ_R = H_R/(\ker D_R + aH_R) \simeq M(m). \tag{112}$$

Its polynomial filtration is the grading of $\mathcal{Q}_0/a\mathcal{Q}_0$, and its normalized sector degree is $2d - d_p(\nu)$.

Proof. For $a = 0$, the decreasingly ordered tensor product of fundamental modules is highest ℓ-weight; the ordering criterion of [2, Section 6.3] permits equal spectral exponents. Factors in distinct $q^{\mathbb{Z}}$-orbits can be placed in any order. Hence a finite set of negative words applied to its highest vector spans its special fibre. The tensor product is finite free over R, so Nakayama's lemma gives

$$M_R(m) = U_R v_{\mathrm{top}}.$$

Proposition 14.3 gives the identical equality for Q_R. Over K both modules are the same simple highest ℓ-weight module. Normalize its isomorphism by sending highest vector to highest vector. In that common K-module both lattices are literally $U_R v_{\mathrm{top}}$. They are therefore equal. The highest-sector Euler class is one, which fixes this normalization.

Reduction of this integral isomorphism proves (112). Lemma 14.2 proves the assertion about the quotient filtration. □

Remark 14.5. The fibre in (112) retains classes that appear in positive powers of the deformation parameter in the Euler image. It need not be $\operatorname{im} D_0$. For example, multiplication by a on R has source quotient R, and its special fibre has dimension one; the image of its specialization is zero. The isomorphism $Q_R \simeq \operatorname{im} D_R$, including its sectorwise Euler degree shift, is the appropriate comparison throughout.

15. Current operators and local filtrations

15.1. **The positive formula in all integer modes.** We first work in the rational shuffle realization. Put

$$\zeta_{ij}^{\mathrm{rat}}(z) = \frac{z - q^{-b_{ij}}}{z-1}, \qquad c_j = q_j - q_j^{-1}, \qquad a_{ij} = q^{-b_{ij}}.$$

The hypothesis on q implies $c_j \neq 0$. If v is an ordinary highest vector, its Cartan series are the expansions at infinity and zero of the same rational function $\psi_j(z)$:

$$\psi_j(z)|_\infty = \sum_{h\geq 0} \psi_{j,h}^{+} z^{-h}, \qquad \psi_j(z)|_0 = \sum_{h\geq 0} \psi_{j,h}^{-} z^{h}.$$

Both expansions are regular at their indicated endpoints. For a Laurent polynomial $g(z) = \sum_d g_d z^d$, write $e_j[g] = \sum_d g_d e_{j,d}$.

Proposition 15.1. *On an actual highest-generated module obtained by the evaluation above, the positive action is*

$$e_j[g](Fv) = \frac{1}{c_j} \sum_{\substack{p \text{ finite} \\ p\neq 0}} \operatorname{Res}_{z=p} \frac{g(z) F(z,\mathbf{z}) \psi_j(z)}{\prod_b \zeta_{ji_b}^{\mathrm{rat}}(z/z_b)} \frac{\mathrm{d}z}{z}\, v. \tag{113}$$

The sum is over all finite nonzero poles over the rational function field of the remaining variables. In particular it includes moving poles. The formula holds for all integer modes and includes the contraction $(\varphi_{j,0}^{+} - \varphi_{j,0}^{-})/c_j$ at mode zero. Its complete value is a finite small-shuffle expression.

Proof. The ordinary Cartan–negative commutation factor is

$$R_{ij}(z) = \frac{\zeta_{ij}^{\mathrm{rat}}(z)}{\zeta_{ji}^{\mathrm{rat}}(z^{-1})} = \frac{a_{ij} - z}{1 - a_{ij} z}. \tag{114}$$

Its two expansions have coefficients

$$\begin{aligned} \rho_{ij}^{+}(0) &= a_{ij}, \quad \rho_{ij}^{+}(h) = (a_{ij}^2 - 1) a_{ij}^{h-1} \quad (h>0), \\ \rho_{ij}^{-}(0) &= a_{ij}^{-1}, \quad \rho_{ij}^{-}(h) = (a_{ij}^{-2} - 1) a_{ij}^{-(h-1)} \quad (h>0). \end{aligned}$$

Taking a fixed coefficient gives finite identities

$$\varphi_{j,k}^{+} f_{i,s} = \sum_{h=0}^{k} \rho_{ij}^{+}(h) f_{i,s+h} \varphi_{j,k-h}^{+}, \tag{115}$$

$$\varphi_{j,k}^{-} f_{i,s} = \sum_{h=0}^{k} \rho_{ij}^{-}(h) f_{i,s-h} \varphi_{j,k-h}^{-}. \tag{116}$$

These are the ordinary double relations, in the conventions of [31, Equations (2.6)–(2.11), (3.7), (3.10), and (3.13)].

Consider a word $W = f_{i_1,s_1} \cdots f_{i_N,s_N}$, with all $s_b \in \mathbb{Z}$, and a positive mode $e_{j,d}$, with $d \in \mathbb{Z}$. Contract the occurrence k with $i_k = j$, and set $h_k = d + s_k$. For $h_k > 0$ its contribution is $c_j^{-1}\varphi^+_{j,h_k}$; for $h_k < 0$ it is $-c_j^{-1}\varphi^-_{j,-h_k}$; for $h_k = 0$ it is $c_j^{-1}(\varphi^+_{j,0} - \varphi^-_{j,0})$. Move this Cartan coefficient through the suffix $b > k$, using (115) or (116). For $h_k > 0$, the resulting contribution is

$$\frac{1}{c_j} \sum_{\substack{u_b \geq 0 \ (b>k) \\ \sum_{b>k} u_b \leq h_k}} \left(\prod_{b>k} \rho^+_{i_b j}(u_b) \right) \psi^+_{j,h_k - \sum u_b} W^+_{k,\mathbf{u}} v, \tag{117}$$

where $W^+_{k,\mathbf{u}}$ is obtained by deleting the k-th letter and replacing s_b by $s_b + u_b$ in its suffix. For $h_k < 0$ use a minus sign, ρ^-, ψ^-, and the replacements $s_b - u_b$. For $h_k = 0$, the deleted word is multiplied by

$$\frac{1}{c_j} \left(\psi^+_{j,0} \prod_{b>k} a_{i_b j} - \psi^-_{j,0} \prod_{b>k} a^{-1}_{i_b j} \right). \tag{118}$$

Every sum in this calculation is finite.

The shuffle representative of W is

$$F_W = \mathrm{Sym}_{\mathrm{col}} \left[\prod_b z_b^{s_b} \prod_{b<c} \zeta^{\mathrm{rat}}_{i_c i_b}(z_c/z_b) \right],$$

where symmetrization is a sum. Distinguish a variable z of colour j. Partition the assignments in this symmetrization according to the occurrence sent to z. Pair factors from earlier letters cancel against the denominator below, and factors from later letters give (114). This proves

$$\frac{F_W(z,\mathbf{z})}{\prod_b \zeta^{\mathrm{rat}}_{j i_b}(z/z_b)} = \sum_{k : i_k = j} \mathrm{Sym}_{\mathrm{remaining}} \left[z^{s_k} \prod_{b \neq k} z_b^{s_b} \prod_{\substack{b<c \\ b,c \neq k}} \zeta^{\mathrm{rat}}_{i_c i_b}(z_c/z_b) \prod_{b>k} R_{i_b j}(z_b/z) \right]. \tag{119}$$

Multiply by $z^d \psi_j(z)$. The constant coefficient at infinity of the k-th term is (117), before the factor c_j^{-1}; it is zero if $h_k < 0$. Its constant coefficient at zero is the analogous negative sum and is zero if $h_k > 0$. If $h_k = 0$, the difference of the two constant coefficients is exactly (118). Thus

$$e_{j,d}(F_W v) = \frac{1}{c_j}(\mathrm{CT}_\infty - \mathrm{CT}_0) \left[\frac{z^d F_W(z,\mathbf{z}) \psi_j(z)}{\prod_b \zeta^{\mathrm{rat}}_{j i_b}(z/z_b)} \right] v.$$

The residue theorem on the projective line over the field of the remaining variables turns this into (113). The endpoint calculation is already a finite word calculation; hence the full residue sum belongs to the small shuffle image. Repeated colours give no extra factor: the assignments of n_j variables split into n_j groups with $(n_j - 1)!$ assignments, exactly the sum over contracted occurrences. Finally, linearity gives every Laurent polynomial test. □

For the polynomial realization, use the augmentation $F^{\mathrm{pol}} = D_{\boldsymbol{n}} F^{\mathrm{rat}}$ and the kernel (19). For a distinguished colour-j variable,

$$\frac{D_{\boldsymbol{n}+e_j}(z,\mathbf{z})}{D_{\boldsymbol{n}}(\mathbf{z})} = \prod_{i_b > j} (1 - z/z_b) \prod_{i_b < j} (1 - z_b/z) = \prod_b \frac{\zeta^{\mathrm{pol}}_{j i_b}(z/z_b)}{\zeta^{\mathrm{rat}}_{j i_b}(z/z_b)}.$$

Multiplying (113) by the target augmentation therefore gives the exact identity

$$e_j[g]F^{\mathrm{pol}} = \frac{1}{c_j} \sum_{\substack{p \text{ finite} \\ p \neq 0}} \mathrm{Res}_{z=p} \frac{g(z)F^{\mathrm{pol}}(z,\mathbf{z})\psi_j(z)}{\prod_b \zeta^{\mathrm{pol}}_{ji_b}(z/z_b)} \frac{\mathrm{d}z}{z}. \tag{120}$$

At coincident orthogonal colours the augmentation is a nonunit. Its factor remains in the denominator and contributes to the augmented-arrow pole count.

15.2. **An integral formula for a whole pole cluster.**

Lemma 15.2. *Let $B = \mathbb{C}[[a, y_1, \dots, y_s]]$, $J = (a, \mathbf{y})$, and let $s_1, \dots, s_H \in J$ be homogeneous linear forms, with repetitions allowed. For $H > 0$, set $D(x) = \prod_{h=1}^{H}(x - s_h)$. If $P(x) = \sum_{k\geq 0} P_k x^k \in B[[x]]$, then the whole-cluster residue is the convergent integral expression*

$$\mathcal{R}_D(P) = \sum_{k \geq H-1} P_k h_{k-H+1}(s_1, \dots, s_H), \tag{121}$$

where h_j is the complete homogeneous symmetric polynomial. It agrees with the sum of residues of $P(x)\,\mathrm{d}x/D(x)$ at all roots whenever this sum is computed after splitting the roots. If P has total degree at least d, its residue has total degree at least $d - H + 1$. Thus an input error of order N loses at most $H - 1$ orders. For $H = 0$ a regular cluster has zero residue.

Proof. The formal expansion

$$\frac{1}{D(x)} = x^{-H} \prod_{h=1}^{H} (1 - s_h/x)^{-1} = x^{-H} \sum_{j\geq 0} h_j(\mathbf{s}) x^{-j}$$

shows that (121) is its coefficient of x^{-1} after multiplication by P. A monomial $x^k a^\alpha \mathbf{y}^\beta$ contributes a homogeneous term of degree

$$k + \alpha + |\beta| - H + 1$$

or contributes zero. Only finitely many monomials can affect any fixed bounded output degree. This proves convergence and the degree bound. For distinct roots the coefficient formula is the residue theorem, and for repeated roots it follows either by the same theorem with multiplicities or by polynomial specialization. All coefficients of the displayed formula are polynomial in the s_h. No difference $s_h - s_{h'}$ has been inverted. □

For example, for a polynomial P,

$$\mathcal{R}_{(x-ca)(x-y)}(P) = \frac{P(ca) - P(y)}{ca - y}.$$

The right side is a polynomial divided difference. It is defined when $ca - y$ is not a unit, including at $ca = y$, by (121).

For the transition from lowering sector $\nu + e_v$ to ν, write $v = (j, s)$, $z = bq^s \exp(x)$ within its $q^{\mathbb{Z}}$-orbit. The remaining variables at vertex w are $z_{w,b} = b_w q^{s_w} \exp(y_{w,b})$. The notation b_w here denotes the orbit representative. In a fixed cluster all framing poles are $x = c_r a$ and all moving poles are $x = y_{w,b}$. Factoring (120) gives, up to a nonzero scalar,

$$\frac{x^r F(x, \mathbf{y}, a)\, N(x, \mathbf{y}, a)}{D(x, \mathbf{y}, a)}\, U(x, \mathbf{y}, a)\, \mathrm{d}x, \tag{122}$$

where U is a regular formal unit and

$$\deg D = m_v + E(\nu, v), \qquad \deg N = l_v + \nu_v. \tag{123}$$

The same-vertex reciprocal Vandermonde contributes the ν_v numerator factors. The incoming-arrow denominator includes the orthogonal-colour factors from the augmentation. A coinciding numerator and denominator framing remains in N/D; neither is moved to U. Every other exponential difference is its linear difference times a unit. Since $\mathrm{d}z/z = \mathrm{d}x$, there is no additional Jacobian degree. Lemma 15.2 consequently gives the polynomial lower degree

$$\Delta_v^+(r,\nu) = r - p_v - E(\nu, v) + \nu_v + 1. \tag{124}$$

For fixed source and target, the loss in a Taylor approximation is bounded by $\max(m_v + E(\nu, v) - 1, 0)$. In particular, this bound does not depend on the modes or lengths of word representatives.

15.3. **Homogeneous lifts and simultaneous approximation.** The next lemma is the strictness needed to apply this estimate to a quotient class.

Lemma 15.3. *Every class of polynomial filtration at least d in $Q_{R,\nu}$ has a lift in the completed additive wheel ideal with no monomial of total degree below d. At $a = 0$ the same assertion holds for $N(n)_\nu$.*

Proof. The geometric evaluation is a homogeneous surjection

$$I_{\Gamma,\nu}[a] \longrightarrow \mathcal{H}_{0,\nu} \longrightarrow \mathcal{Q}_{0,\nu}$$

by Theorem 11.1. Every homogeneous class therefore has a homogeneous polynomial lift of the same degree: take that degree component of any lift. Choose homogeneous lifts f_j of the basis e_j in Lemma 14.2. A vector in (109) has an expansion

$$\sum_j a^{\max(0,d-d_j)} u_j(a) e_j, \qquad u_j(a) \in R.$$

The expression with f_j in place of e_j is the required lift. It has no total degree below d. The same component argument proves the assertion on the finite fixed fibre. □

Lemma 15.4. *Fix an ordinary root degree. Let $\mathcal{T}$ be a finite set of reduced coloured multisets. At every $\gamma \in \mathcal{T}$ prescribe a completed additive wheel element. For every N these elements can be matched, modulo order N in their joint $(a, \mathbf{y})$-adic expansions, by one numerical small-shuffle element over R. It is a finite R-linear combination of actual negative words.*

For a fixed finite set of target sectors, $\mathcal{T}$ can be chosen to contain every possible source multiset contributing to a positive-current residue, whether or not that source is in the support of the module. This choice is independent of the mode indices of the approximating words.

Proof. Let

$$A = \mathbb{C}[z_{i,b}^{\pm 1}]^{\prod_i S_{n_i}},$$

and let $S \subset A$ be the numerical polynomial small-shuffle image. It is an A-ideal: multiplication by an invariant Laurent polynomial can be performed on the unrestricted labelled input of every symmetrization. Same-colour alternating divisibility shows that these maps land in A. In particular S is a finite module over the Noetherian ring A. Set $A_R = R \otimes_{\mathbb{C}} A$ and $S_R = R \otimes_{\mathbb{C}} S$. For the maximal ideal $\mathfrak{m}_\gamma = (a, \text{invariant coordinates at } \gamma)$, let Γ_γ be the spectral quiver of the multiset γ, let I_{Γ_γ} be its wheel ideal in the corresponding dimension vector, and let $P_\gamma = \prod_v \mathfrak{S}_{\nu_v(\gamma)}$ be the stabilizer of a labeled representative. Theorem 3.4 gives

$$S_R \widehat{(A_R)_{\mathfrak{m}_\gamma}} = I_{\Gamma_\gamma} \mathbb{C}[[a, \mathbf{y}]]^{P_\gamma}. \tag{125}$$

The completion on the left is taken *after* adjoining R; it is the joint completion. Flatness of this completion applies because S_R is finite.

The kernel comparison in (125) can be read on the labelled finite cover. It splits over the finite orbit of labelings of γ. On one component every word map has the form

$$G \longmapsto \mathrm{Sym}_{P_\gamma}(G\kappa_\sigma U_\sigma),$$

where κ_σ is the homogeneous additive word kernel and U_σ is a regular unit. Multiplication by U_σ is an automorphism of the labelled input, regardless of whether that unit is invariant. Hence the completed image equals the completed additive word image. The latter is the wheel ideal by Theorem 3.4. This argument retains every nonunit linear factor in κ_σ.

The ideals for distinct γ have distinct reductions modulo a, and so their powers are pairwise comaximal. The Chinese remainder theorem for the finite module S_R, together with (125), gives the required simultaneous approximation. If it is carried out using labelled polynomials directly, choose the labelled jets to order $N + V$, where V is the total same-vertex Vandermonde degree. Alternating division then leaves order at least N. The integer V is fixed by the root degree. This is the only additional loss in that explicit construction.

For the second assertion, examine (120). Given a target multiset, every finite nonzero pole centre comes either from a pole of a highest Cartan function or from a fixed q-shift of one of its remaining roots. There are finitely many such possibilities. Adjoin each of them to that target, for every possible contracted colour. The resulting finite list contains all source multisets of any positive residue. The input is a Laurent polynomial and hence introduces no further finite nonzero pole; arbitrary integer mode indices introduce poles only at zero or infinity. Thus neither this list nor its maximal pole order changes with the approximating words. □

Proposition 15.5. *The local logarithmic positive current with test x^r, $r \geq 0$, acts on Q_R with polynomial lower degree* (124)*. It is the already constructed ordinary action on Q_R. After specialization the corresponding finite current jet acts on $Q(m)$ and on every intrinsic simple $N(n)$ with the same bound.*

Proof. Take a class in $F^d_{\mathrm{pol}} Q_{R,\nu+e_v}$. Lemma 15.3 gives a wheel lift F of total degree at least d. In Lemma 15.4, include every occupied source sector of this ordinary root degree and every possible source multiset for the retained target sectors. Prescribe F at its selected source multiset and zero at every other multiset in this finite list. Let F_N be the resulting finite word approximants.

Their evaluations converge to the given class in the a-adic topology. Indeed, on a finite free equivariant module the degree topology and the a-adic topology are equivalent: the positive Chern ideal is nilpotent modulo a, so for some M its M-th power sends the module into a times the module; conversely multiplication by a has positive degree. Every fixed ordinary current is R-linear on a finite free module, and is therefore continuous.

For a Laurent test g, its action on F_N is precisely (120), by Proposition 15.1. At a retained target, each possible source cluster belongs to the chosen finite list. The uniform loss in Lemma 15.2 shows that the local residue values converge to the cluster residue of the prescribed lift. Thus this limiting residue is the actual ordinary action, by continuity. This first proves the completed-class formula for ordinary Laurent tests.

Now approximate the logarithmic test x^r in its selected cluster, and zero in the other clusters, by Laurent polynomials. The Chinese remainder theorem is used only between distinct reduced nonzero centres. Roots with the same reduction remain together in one cluster. Applying the preceding uniform estimate to homogeneous lifts of a finite R-basis gives convergence of these operators in $\mathrm{End}_R(Q_R)$. Their limit still belongs to the actual ordinary operator image: that image is an R-submodule of the finite free module $\mathrm{End}_R(Q_R)$, hence is finite and a-adically closed. Formula (122) and Lemma 15.2 give the required lower degree. The filtration steps are closed by (109), so the estimate passes to the limit.

Two different lifts give the same value, since both limiting constructions recover the action defined in Proposition 14.3. On a fixed finite fibre, positive local Chern coordinates are nilpotent. A sufficiently high finite jet of the test is then exact. The same proof, without a, applies to Proposition 14.1. □

16. The fixed convolution grading

16.1. **Normalized current degrees.** The local negative current is particularly explicit. For $v = (i, s)$ and $w = (j, t)$, put

$$K_{vw}(x, y) = \zeta_{ji}^{\mathrm{pol}}(q^{t-s} e^{y-x})$$

when the vertices belong to the same orbit; for different orbits include the quotient of their orbit representatives in the argument. Every factor has a unique decomposition

$$K_{vw}(x, y) = c_{vw}(y - x)^{E(v,w)-\delta_{vw}} U_{vw}(x, y), \qquad c_{vw} \neq 0, \qquad U_{vw}(0, 0) = 1. \tag{126}$$

Here $E(v, w)$ is the number of arrows from v to w. This is read directly from (19). If $i = j$ and $s = t$, its order is -1; if the corresponding arrow is resonant, its order is 1; every other factor has order zero. Distinct orbits give only the last case. No c_{vw} vanishes at the fixed numerical non-root-of-unity parameter.

For a local polynomial F in sector ν, the insertion with test x^r is

$$\mathcal{F}_{v,r} F = \sum_{a=1}^{\nu_v+1} x_{v,a}^r F(\mathbf{x} \setminus x_{v,a}) \prod_{(w,b)\ \mathrm{old}} K_{vw}(x_{v,a}, x_{w,b}). \tag{127}$$

In each summand the old variables at v are all the variables other than $x_{v,a}$. The permutations of these old variables cancel the shuffle normalization, giving exactly this sum and no additional factorial. After a common Vandermonde denominator is introduced, the numerator is alternating in the variables of each repeated vertex. Its divisibility by the Vandermonde holds in each homogeneous degree. Thus the displayed operator is a polynomial formal series, and (126) gives its lower polynomial degree

$$\Delta_v^-(r, \nu) = r + E(v, \nu) - \nu_v. \tag{128}$$

The same finite-jet argument as in Lemma 15.4, with a fixed Vandermonde loss, shows that this is the actual local ordinary negative operator on every quotient under consideration.

Proposition 16.1. *For either sign, the normalized filtration degree of the local current with logarithmic test x^r is bounded below by*

$$\delta(v, r) = 2r + 1 - p'_v, \qquad p'_v = p_v + E(\nu, v) - E(v, \nu), \tag{129}$$

where p'_v is the signed pole order of the current ordinary Cartan function. The formula applies to the transition $\nu \leftrightarrow \nu + e_v$ and to both its directions.

Proof. There is no loop at a spectral vertex. Consequently

$$d_p(\nu + e_v) - d_p(\nu) = p_v + E(\nu, v) + E(v, \nu) - 2\nu_v - 1. \tag{130}$$

Twice (128) minus (130) equals $2r + 1 - p_v - E(\nu, v) + E(v, \nu)$. Twice (124) plus (130) gives the same number. The Cartan commutation factor (114) gives exactly the indicated change of the signed pole divisor after lowering by ν. Since $E(v, v) = 0$, that divisor is unchanged by reversing this single v-step. □

16.2. **Intrinsic sectors in a larger framing.** Fix a standard $M(m)$ and its actual action of the Coulomb algebra $\mathscr{C}_m$ from Theorem 12.7. An ordinary simple constituent $L(n)$ inherits this same action: the ordinary current map is surjective by Proposition 12.2. We transport it to the intrinsic quotient $N(n)$ using Proposition 14.1.

Let $W_i(u)$ be its gauge-root polynomial, normalized by its nonzero constant roots, and let $\eta_{i,k}$ be the multiplicity of the roots bq^k on the highest line of $L(n)$. We suppress the orbit index where it does not affect a formula. The actual Cartan root-movement identity is

$$W_i(u)F_j(z) = F_j(z)W_i(u)\left(\frac{1-q_i^{-2}z/u}{1-z/u}\right)^{\delta_{ij}}. \tag{131}$$

The branch of the determinant cover is the one in the ordinary current map of Theorem 12.7.

Lemma 16.2. *The intrinsic lowering sector ν corresponds to the inherited gauge-root multiplicities*

$$a_{i,k}(\nu) = \eta_{i,k} + \nu_{i,k+2d_i} - \nu_{i,k}. \tag{132}$$

This correspondence is injective on finite-support dimension vectors. Hence the intrinsic filtration respects the actual degree-zero idempotents of the larger weighted convolution algebra.

Proof. Iterate (131) through a negative word. Each lowering at $z = bq^k$ removes a gauge root at bq^k and adds one at bq^{k-2d_i}, giving (132). If two finite-support arrays give the same multiplicities, their difference h satisfies $h_{i,k+2d_i} = h_{i,k}$. A nonzero such periodic array has infinite support. Thus $h = 0$. Every occupied block of $L(n)$ is reached by a negative word, since this module is highest generated. Its intrinsic sectors and inherited occupied idempotents consequently coincide. □

Lemma 16.3. *On the intrinsic quotient with its inherited larger action, every source block Chern class of polynomial degree r has filtration degree at least $2r$. The same assertion holds on the standard family before specialization when* $\deg a = 1$. *On the fixed fibres the ordinary Cartan zero modes remain semisimple of type one on the associated graded.*

Proof. Separate the distinct reduced gauge-root centres by Hensel factorization of $W_i(u)$. The relevant resultants are units; roots within one repeated block are not separated. For $r > 0$, logarithmic power sums in the block at bq^k give from (131) the operator identity

$$P_{i,k,r} = P^{\mathrm{top}}_{i,k,r} + \sum_{\text{lowerings at } (i,k+2d_i)} y_b^r - \sum_{\text{lowerings at } (i,k)} y_b^r. \tag{133}$$

This is first an identity on negative words and therefore on their actual quotient. On an intrinsic fixed fibre the top logarithmic roots are zero, so $P^{\mathrm{top}}_{i,k,r} = 0$. In the deformed standard family they are the corresponding sums of $(c_b a)^r$. In either case the right side has polynomial degree r. Newton identities express all block Chern classes as polynomials of the asserted degree in these power sums. This proves an operator statement, including the nilpotent Chern parts.

For the last assertion, sum (133) over all vertices and all finitely many occupied orbits. The lowering terms telescope. At $a = 0$ the total positive logarithmic power sums therefore vanish. If $N_i = \sum_k \nu_{i,k}$, the nonzero constant part of the gauge determinant, computed from (132), is

$$\det V_i|_\nu = (\det V_i)_{\mathrm{top}} q_i^{-2N_i}. \tag{134}$$

Its logarithmic nilpotent part is zero. On the prescribed square root branch, $D_i = (\det V_i)^{1/2}$ consequently acts as $D_i^{\mathrm{top}} q_i^{-N_i}$. The ordinary current normalization expresses k_i as a fixed framing constant times $\prod_j D_j^{c_{ji}}$. Symmetrizability then gives

$$k_i|_\nu = k_i^{\mathrm{top}}\, q_i^{-\sum_j c_{ij}N_j}. \tag{135}$$

The vanishing logarithmic relations used above are homogeneous Chern relations and pass to the associated graded. Thus (135), with no nilpotent part, holds there as well. □

For a dimension vector τ, write $A^{\tau} = \prod_{i,k} A_{i,bq^k}^{\tau_{i,k}}$, taking the product over all parameter orbits. If $n = mA^{-\tau}$, then

$$p_v^n = p_v^m + E(\tau, v) - E(v, \tau),$$

and direct expansion gives

$$d_{p^m}(\tau + \mu) - d_{p^m}(\tau) - d_{p^n}(\mu) = 2\big(E(\mu, \tau) - \mu \cdot \tau\big). \tag{136}$$

The compatible degree in (129) instead uses the pole order of the actual current Cartan function. That function is the same in the inherited action and in the intrinsic realization.

16.3. **Homogeneous operators.** For an ordinary constituent $L(n)$ of $M(m)$, let

$$\theta_n : N(n) \xrightarrow{\sim} L(n), \qquad [F] \longmapsto F v_n,$$

be the word-evaluation isomorphism of Proposition 14.1. The surjection in Theorem 12.7 gives its inherited $\mathscr{C}_m$-action. Explicitly, for $c \in \mathscr{C}_m$ choose $u \in U_q(L\mathfrak{g})$ mapping to c and set

$$c \cdot v = \theta_n^{-1}(u\theta_n(v)).$$

The kernel of $U_q(L\mathfrak{g}) \to \mathscr{C}_m$ kills $L(n)$, so this formula is independent of the chosen preimage. Hence completed algebra identities act on the same ordinary module used in the intrinsic filtration.

Use the fixed completed comparison

$$\Theta : \widehat{\mathscr{C}_m} \xrightarrow{\sim} \widehat{\mathscr{A}^{\mathrm{fl}}}$$

of Proposition 13.1. For either sign, Proposition 13.3 identifies a local actual current with

$$\Theta(T_{v,r}) = j_*([Z_v]x^r U_v), \qquad U_v(0) \neq 0. \tag{137}$$

The unit is independent of r. The homogeneous fundamental class $[Z_v]$ has actual Ext degree $1 - p'_v$, by Proposition 13.6.

Proposition 16.4. *Every homogeneous minuscule class $j_*([Z_v]x^r)$, of either sign, acts on the above finite fibres with its Ext-degree filtration bound. The statement applies simultaneously to the standard and to the intrinsic filtrations of all its actual simple constituents.*

Proof. The source-cohomology description of Lemma 13.4 expresses the incidence ring using the source block Chern classes and the moved-line class x, with a homogeneous monic projective-bundle relation. Every homogeneous part of U_v^{-1} therefore has a polynomial lift in those classes. Write, degree by degree,

$$U_v^{-1} = \sum_{h \geq 0} x^h p_h.$$

The projection formula, with the source coefficient on the right, gives the exact identity of Corollary 13.5:

$$j_*([Z_v]x^r) = \sum_{h \geq 0} \Theta(T_{v,r+h})\, p_h. \tag{138}$$

For the reversed positive incidence, its own source ring is used in the same calculation.

If a homogeneous part of p_h has polynomial degree b, Lemma 16.3 and Proposition 16.1 give the bound

$$\delta(v, r + h) + 2b = \delta(v, r) + 2(h + b) \geq \delta(v, r).$$

This is exactly the degree of $j_*([Z_v]x^r)$. The series acts by a finite sum on a finite continuous module: the positive central ideal acts nilpotently, and the moved-line powers reduce by the

monic projective-bundle relation. Equivalently, a summand with $h + b$ sufficiently large has lower degree bound $\delta(v, r) + 2(h + b)$ beyond the finite filtration range and acts by zero. Thus passage from the completed identity to this operator bound is valid on the stated finite fibres. The extra top gauge roots of a lower simple have zero logarithm, and (133) already includes their ranks. Finally, its intrinsic p'_v is the same ordinary Cartan divisor as in (137), by (136) and the preceding discussion. □

Theorem 16.5. *There is one finite-dimensional $\mathbb{Z}$-graded quotient $\mathscr{A}_S$ of the known weighted-flag convolution algebra on which the standard $Q(m)$, all its ordinary simple constituents, and their associated graded modules are defined. For every homogeneous $a \in (\mathscr{A}_S)_d$,*

$$aF^hV \subseteq F^{h+d}V \tag{139}$$

for the standard filtration and every intrinsic simple filtration. The degree-zero idempotents retain their ordinary generalized ℓ-weight sectors.

Proof. Let S be the finite union of the gauge-root supports of $Q(m)$ and its finitely many ordinary simple constituents. By Lemma 16.2, it is also the union of their intrinsic supports. In the locally unital graded algebra $\mathscr{A}^{\mathrm{fl}}$, first kill the homogeneous trace ideal

$$J_{\mathrm{out}} = \sum_{\gamma \notin S} \mathscr{A}^{\mathrm{fl}} e_\gamma \mathscr{A}^{\mathrm{fl}}.$$

Then take the corner with $e_S = \sum_{\gamma \in S} e_\gamma$. Each of the stated modules factors through this quotient, since every e_γ outside S acts by zero. Taking the trace quotient first also kills paths that leave the retained support and return to it.

For fixed γ, η, the diagonal unfolded gauge group has finitely many orbits on $\mathrm{Flag}_\gamma \times \mathrm{Flag}_\eta$, indexed by relative positions. Above each orbit, the two flag conditions on matter maps define a constant-rank vector bundle. Its stabilizer is an intersection of two parabolic subgroups. The unipotent radical does not change equivariant cohomology, and its Levi factor is a product of general linear groups. The corresponding Weyl-invariant polynomial ring is finite over $H^*_{\underline{G}}(\mathrm{pt})$. The finite stratum filtration and Noetherianity therefore show that

$$H_*^{\underline{G},\mathrm{BM}}(X_\gamma \times_N X_\eta)$$

is finite over this central ring. The geometric dimension bound bounds the selfdual Ext degrees below; equivariant parameters have positive degree.

There are finitely many retained pairs, so their corner is bounded below and finite over the positively graded central equivariant ring. The trace quotient preserves both properties. Choose a common power of the positive central ideal annihilating the finitely many continuous modules under consideration, and take that homogeneous quotient. It is the required finite-dimensional graded algebra $\mathscr{A}_S$.

The ordinary current map on the finite quotient. Let $\mathscr{C}_m$ be the specialized Coulomb algebra associated with $M(m)$, let B be its gauge coefficient ring, and let $\mathfrak{m}_\gamma \subset B$ be the maximal ideal of a gauge loading γ, including the chosen determinant branch. Its semisimple completion has the two-sided blocks [37, (4.2.18)–(4.2.20)]

$$\widehat{\mathscr{C}_m}(\gamma, \eta) = \varprojlim_k \mathscr{C}_m/(\mathfrak{m}_\gamma^k \mathscr{C}_m + \mathscr{C}_m \mathfrak{m}_\eta^k).$$

The algebra $\mathscr{C}_m$ acts on their direct sum by multipliers. Indeed, coefficient elements are diagonal, and a minuscule generator has only finitely many possible target loadings from a fixed source, and finitely many possible source loadings for a fixed target. The same is true of every finite word. Proposition 12.2 therefore makes the matrix of every $c \in \mathscr{C}_m$ row and column finite, with

its (γ,η) entry equal to the image of c in the displayed two-sided completion. Multiplication of these matrices is the multiplication defining the completed algebra.

Each flag coefficient ring is finite over the central equivariant ring. Modulo the positive central ideal it is finite dimensional and graded, so a power of its positive augmentation ideal lies in that central ideal. These two adic topologies are consequently cofinal. The homogeneous quotient map from the flag algebra to $\mathscr{A}_S$ extends to each completed block, because its kernel contains the chosen power of the positive central ideal. Denote this map, after transport by Θ, by π. For $c \in \mathscr{C}_m$, define

$$\rho_S(c) = \sum_{\gamma,\eta\in S} \pi(c_{\gamma,\eta}) \in \mathscr{A}_S.$$

This is a finite sum. It is a unital algebra homomorphism: in the block formula for a product, a term with an intermediate loading outside S belongs to the completed trace ideal generated by that idempotent and has zero image. The remaining terms give precisely $\rho_S(c)\rho_S(c')$. The completed trace ideal is killed because π is continuous and kills its algebraic generators.

We prove surjectivity on the algebra itself. On the block γ, Θ carries $\mathfrak{m}_\gamma$ to power series with zero constant term in the positive-degree Chern classes of its source flag blocks. Since $\mathscr{A}_S$ is finite dimensional and graded, there is one integer M such that these Mth powers act by zero on both sides of every retained block of $\mathscr{A}_S$. For example, if all nonzero degrees of $\mathscr{A}_S$ lie between d_- and d_+, it suffices to take $2M > d_+ - d_-$. Thus one common order of coefficient jets suffices on the quotient algebra.

Consequently the image of an element of $\widehat{\mathscr{C}_m}(\gamma,\eta)$ depends only on its class modulo $\mathfrak{m}_\gamma^M\mathscr{C}_m + \mathscr{C}_m\mathfrak{m}_\eta^M$. By the definition of that completed block, this class is represented by some $c \in \mathscr{C}_m$. Distinct retained loadings give distinct maximal ideals of B: their colored multisets of gauge roots differ, or their determinant branches differ. The fixed spectral representatives and the non-root-of-unity condition ensure that the weighted-flag label is determined by this multiset. Chinese remainders therefore give elements $p_\gamma \in B$ such that

$$p_\gamma \equiv \delta_{\gamma,\delta} \pmod{\mathfrak{m}_\delta^M} \qquad (\delta \in S).$$

They satisfy $\rho_S(p_\gamma) = e_\gamma$. Thus $p_\gamma c p_\eta$ represents the required element of the (γ,η) block and has zero image in every other retained block. Summing over the finitely many retained pairs proves that ρ_S is surjective. Composition with the ordinary current map gives the surjection

$$U_q(L\mathfrak{g}) \xrightarrow{\Phi_m} \mathscr{C}_m \xrightarrow{\rho_S} \mathscr{A}_S. \tag{140}$$

Its action on $Q(m)$ and on each intrinsic $N(n)$ is the action already constructed: their smooth completed actions factor through the same trace and central-power ideals, and the block maps above are the extensions of those actions. Hence this is one ordinary current map on the quotient algebra, available also on every associated graded module subsequently constructed.

Filtration bounds. The retained idempotents preserve each filtration by Lemma 16.2. Source Chern classes satisfy the required bounds by Lemma 16.3, and decorated minuscule classes satisfy them by Proposition 16.4.

Let $\mathscr{D} \subseteq \mathscr{A}_S$ be generated by the retained idempotents, the homogeneous source-block Chern classes, and all homogeneous decorated minuscule fundamental classes of both signs. For an ordinary integer current mode, its blockwise image under Θ is the full incidence class of Proposition 13.3. Its coefficient is the unit U_v times the corresponding Laurent-line character, expanded in the logarithmic moved-line class. By Lemma 13.4, every homogeneous coefficient has a polynomial lift in that class and the source Chern classes. The projection formula puts the latter coefficients on the right. Only finitely many homogeneous terms can survive in $\mathscr{A}_S$, so each ordinary root-current image belongs to $\mathscr{D}$. The same argument uses the source of the

reversed incidence for a positive current. The Cartan coefficients, including their zero modes and the prescribed determinant branch, are completed coefficient-ring expressions; they likewise truncate to polynomials in source Chern classes and constants on the retained idempotents. Their images also belong to $\mathscr{D}$.

The surjective map $U_q(L\mathfrak{g}) \to \mathscr{A}_S$ gives $\mathscr{D} = \mathscr{A}_S$. Since these generators are homogeneous, a homogeneous element of $\mathscr{A}_S$ is a sum of words of its own degree. The filtration bounds add under composition and are preserved under such sums. This proves (139), and the induced action on each associated graded follows from that identity. □

Corollary 16.6. *For every ordinary simple constituent $L(n)$ of the fixed standard, $\operatorname{gr} N(n)$ is a graded lift of the same simple for the algebra $\mathscr{A}_S$. Its ordinary highest line has degree zero.*

Proof. We use the following elementary fact. If A is a finite-dimensional $\mathbb{Z}$-graded complex algebra, V is a simple ungraded A-module, and F is a finite exhaustive separated filtration satisfying

$$A_d F^h V \subseteq F^{h+d} V,$$

then $\operatorname{gr}_F V$ is isomorphic to V as an ungraded A-module.

Indeed, the Jacobson radical of A is homogeneous: scaling degrees gives algebra automorphisms that preserve the radical. The radical kills V and therefore also kills $\operatorname{gr}_F V$. Put $B = A/\operatorname{rad}(A)$. The grading action of $\mathbb{C}^\times$ preserves the finite set of central primitive idempotents of the semisimple algebra B. Since $\mathbb{C}^\times$ is connected, each of these idempotents is fixed and has degree zero. Choose degree-zero representatives in A of these central idempotents. The block idempotent acting as one on V also acts as one on $\operatorname{gr}_F V$, and every other block idempotent acts as zero on both. If the selected block is $M_d(\mathbb{C})$, then $\dim V = d$. The associated graded has the same dimension d and is a module for the same matrix block, so it is that block's unique simple module. This proves the fact.

Apply it to $A = \mathscr{A}_S$ and $V = N(n)$, using Theorem 16.5. The ordinary restriction of $N(n)$ is $L(n)$, and the same surjective ordinary current map acts on its associated graded. Thus $\operatorname{gr} N(n)$ is a graded lift of that same simple. Its highest line is the one-dimensional degree-zero line of Proposition 14.1. □

16.4. **Leading Hall operators and the standard highest line.** Let $\mathsf{H}_{v,r}$ denote the homogeneous Hall insertion of a line at v decorated by its r-th Chern power. In the conventions used here its flag is

$$0 \longrightarrow U \longrightarrow V \longrightarrow V' \longrightarrow 0, \qquad \dim U = e_v,$$

where U is unframed and V' is the old framed quotient.

Lemma 16.7. *In the transition $\nu \to \nu + e_v$, the lowest polynomial degree part of $\mathcal{F}_{v,r}$ is a nonzero scalar multiple of $\mathsf{H}_{v,r}$, namely*

$$C_{v,\nu} \sum_{a=1}^{\nu_v+1} x_{v,a}^r F(\mathbf{x} \setminus x_{v,a}) \frac{\prod_{v\to w} \prod_{b \text{ old at } w} (x_{w,b} - x_{v,a})}{\prod_{b \neq a} (x_{v,b} - x_{v,a})}, \qquad C_{v,\nu} = \prod_w c_{vw}^{\nu_w} \neq 0. \tag{141}$$

For the actual homogeneous minuscule class,

$$\operatorname{gr}_{\delta(v,r)} j_*([Z_v]x^r) = U_v(0)^{-1} C_{v,\nu}\, \mathsf{H}_{v,r}.$$

Proof. Substitute (126) into (127). The degree-zero parts of all U_{vw} are one, giving exactly (141). Every other term has strictly larger total degree. Alternating divisibility by the common Vandermonde holds degree by degree, so taking the lowest degree commutes with its cancellation.

For the geometric Hall map, preservation of the subrepresentation U imposes the vanishing of $\mathrm{Hom}(U_v, V'_w)$ for each arrow $v \to w$. Its Euler factor is the numerator of (141). The tangent bundle of the Grassmannian of lines contributes $\mathrm{Hom}(U_v, V'_v)$, giving the denominator. The incoming framing component into U is free, so it gives no Euler numerator. The extension pullback is smooth and the flag pushforward is proper. Their usual localization formula is therefore the sum in (141). Critical specialization is injective and compatible with these two maps by Theorem 11.1; hence the same identity holds on the actual critical Hall module.

Finally apply (138). Its summand of minimal normalized degree is the constant term $h = b = 0$, whose coefficient is $U_v(0)^{-1}$. Every other homogeneous summand has degree at least two higher. Combining this with Proposition 16.1 gives the second assertion. □

Lemma 16.8. *The homogeneous Euler kernel is preserved by Hall insertions. In particular, the Hall action on $\mathcal{H}_0$ descends to $\mathcal{Q}_0$ and then to $\mathcal{Q}_0/a\mathcal{Q}_0$.*

Proof. On the Hall flag the tautological exact sequence gives an exact sequence for the numerator framing bundles. Write $\mathcal{E}_{\boldsymbol{l},A}(U)$ for the numerator framing bundle evaluated on the unframed kernel U, as in (67). Its Euler classes satisfy

$$D_{\nu+e_v}|_{\mathrm{Flag}} = e(\mathcal{E}_{\boldsymbol{l},A}(U))\, D_\nu|_{\mathrm{Flag}}.$$

The projection formula consequently gives, for every decoration b,

$$D_{\nu+e_v}\, \mathsf{H}_b(h) = \mathsf{H}_{b\, e(\mathcal{E}_{\boldsymbol{l},A}(U))}(D_\nu h). \tag{142}$$

Thus $D_\nu h = 0$ implies $D_{\nu+e_v}\mathsf{H}_b(h) = 0$. The operators and the Euler kernels are homogeneous over $\mathbb{C}[a]$, so both quotient and specialization preserve the assertion. □

Theorem 16.9. *As a module for the same algebra $\mathscr{A}_S$ of Theorem 16.5, the associated graded standard* $\mathrm{gr}\, Q(m)$ *is generated by its degree-zero highest line under the homogeneous negative minuscule classes. Its semisimple head consists of one copy of the simple with ordinary highest monomial m, with that highest line in degree zero.*

Proof. The equivariant Hall-vacuum generation of Theorem 11.1 generates $\mathcal{H}_0$ over $\mathbb{C}[a]$ by homogeneous Hall words. Lemma 16.8 passes this generation to the source quotient $\mathcal{Q}_0$. Equation (110) identifies its specialized grading with the associated graded of the actual standard $Q(m)$. Hence Hall words generate that graded space from its vacuum.

By Lemma 16.7, each such word is a nonzero scalar multiple of the corresponding word in the homogeneous negative minuscule classes, after fixing its intermediate sectors with the degree-zero idempotents. The scalar is the product of the nonzero constants on its successive transitions. Therefore those known homogeneous classes generate $\mathrm{gr}\, Q(m)$ from the same vacuum, whose normalized degree is zero.

Every simple quotient must contain the image of this generating m-line. If its ordinary highest monomial is n, all its ℓ-weights belong to the support of $Q(m)$, so $n \le m$. Since it also contains the monomial m, highest generation of the simple gives $m \le n$. Thus $n = m$. The m-space of the standard has dimension one, so the semisimple head contains only one such copy. The degree of its highest line fixes the displayed shift. □

17. The grading and duality of simple modules

Fix an ordered standard module $M(m)$. We use the finite graded flag algebra of Theorem 16.5 for this standard and its ordinary simple constituents, with their intrinsic filtrations.

17.1. **Ordinary labels of the finite graded algebra.** Throughout this section we use the single finite graded algebra $\mathscr{A}_S$ of Theorem 16.5, and the surjective ordinary current homomorphism (140). For each retained spectral idempotent e_γ, let n_γ be the ordinary monomial obtained by substituting its constant gauge roots and the fixed determinant branch in the Cartan rational functions. These constants occur on the actual standard or one of its ordinary constituents, so n_γ is an ordinary type 1 ℓ-weight.

Lemma 17.1. *Every ungraded simple $\mathscr{A}_S$-module V restricts, through* (140)*, to a simple finite-dimensional type* 1 *quantum loop module. Its ordinary q-character is*

$$\chi_q V = \sum_{\gamma \in S} \dim(e_\gamma V)\, n_\gamma.$$

If several idempotents determine the same monomial, their dimensions are added in this formula.

Proof. Surjectivity of (140) implies simplicity of the ordinary restriction. It remains to check its Cartan action; the same calculation applies to every surviving simple.

Fix γ. In its completed gauge coefficient ring, the maximal ideal at the chosen constant roots is generated by positive logarithmic Chern coordinates. Their images in $e_\gamma \mathscr{A}_S e_\gamma$ are homogeneous of strictly positive degree and commute. Since $\mathscr{A}_S$ is finite dimensional and graded, the ideal they generate in this commutative coefficient subalgebra is nilpotent: a product of sufficiently many positive-degree coordinates has degree larger than every nonzero graded piece of $\mathscr{A}_S$. Denote this nilpotent ideal by J_γ.

Every coefficient of a Cartan current commutes with e_γ. The fixed Cartan rational formula, expanded at zero or infinity, expresses its restriction to this block as its constant value plus an element of J_γ. The same nilpotent ideal works for all Cartan coefficients. Consequently $e_\gamma V$ is a simultaneous generalized ℓ-weight space with generalized eigencharacter n_γ, or is contained in that space when different retained idempotents have the same constant character. In particular all eigenvalues of each k_i are integral powers of q_i.

The Cartan zero modes on this simple restriction are semisimple. Write the multiplicative Jordan decomposition on V as $k_i = s_i u_i$, with s_i semisimple and u_i unipotent. The relations

$$k_i x^{\pm}_{j,r} k_i^{-1} = q_i^{\pm c_{ij}} x^{\pm}_{j,r}$$

make every root generator an eigenvector of $\mathrm{Ad}(k_i)$. In the multiplicative Jordan decomposition of this adjoint operator, its unipotent part $\mathrm{Ad}(u_i)$ fixes every such eigenvector. It fixes the Cartan generators as well. Thus u_i commutes with the entire ordinary quantum loop action. Schur's lemma makes u_i scalar; being unipotent, it equals the identity. Hence all k_i are semisimple, and their already determined eigenvalues prove type 1. The displayed q-character formula follows from the simultaneous generalized eigenspaces just identified. □

17.2. **The positive basic presentation.** Write the normalized ordinary flag complexes as

$$\mathcal{L}_\gamma = (\pi_\gamma)_* \mathbb{C}_{X_\gamma}[\dim X_\gamma] = \bigoplus_{\alpha,d} \mathrm{IC}_\alpha[-d] \otimes V^d_{\gamma,\alpha}. \tag{143}$$

The sources X_γ are smooth and the maps π_γ are proper. The decomposition theorem gives the displayed finite decomposition in each finite corner. The IC_α are distinct simple perverse complexes, with their local systems included in the index α. Let G be their direct sum. Perverse orthogonality and Schur's lemma give

$$B = \mathrm{Ext}^*(G,G), \qquad B_{<0} = 0, \qquad B_0 = \bigoplus_\alpha \mathbb{C} e_\alpha. \tag{144}$$

The additive categories generated by all shifts of the $\mathcal{L}_\gamma$ and by all shifts of the IC_α agree. Their Ext algebras are therefore graded Morita equivalent. The spaces $V^d_{\gamma,\alpha}$ occur in the projective

generator of this equivalence. They account for the internal grading of a geometric simple and must be retained when discussing composition shifts.

Under this equivalence the trace ideal and the central-power ideal defining $\mathscr{A}_S$ induce homogeneous ideals. They can impose positive-degree relations in addition to deleting basic idempotents. Quotienting a nonnegative algebra by a homogeneous ideal nevertheless preserves a nonnegative grading and semisimple degree zero. We obtain a finite basic algebra B' graded Morita equivalent to $\mathscr{A}_S$, with

$$B'_{<0} = 0, \qquad B'_0 = \bigoplus_{\alpha\in\Lambda} \mathbb{C}e_\alpha, \qquad \operatorname{rad} B' = B'_{>0}. \tag{145}$$

A surviving simple has the same geometric multiplicity spaces in (143); the extra positive relations can change extensions but do not change those spaces.

17.3. **Bar invariance and the highest-line normalization.** Verdier duality fixes every $\mathcal{L}_\gamma$, since the flag source is smooth and the map is proper. It permutes the simple perverse labels by $\alpha \mapsto \alpha^*$ and gives

$$V^d_{\gamma,\alpha^*} \cong (V^{-d}_{\gamma,\alpha})^*. \tag{146}$$

Verdier duality fixes each normalized flag complex and hence each geometric idempotent. It preserves the homogeneous trace ideal in Theorem 16.5. On the equivariant coefficient ring it preserves degree and the augmentation ideal; a possible sign on a Chern class preserves the same ideal. It therefore preserves the chosen power of that ideal and descends to $\mathscr{A}_S$. Thus the basic simple at degree zero with label α is dual to the basic simple at degree zero with label α^*.

Proposition 17.2. *For every dominant monomial n in type G_2 or F_4, the intrinsic character*

$$S_t(n) = \operatorname{ch}_t \operatorname{gr} N(n) \tag{147}$$

is the character of the degree-zero normalized basic simple. In particular,

$$\overline{S_t(n)} = S_t(n), \qquad S_t(n)|_{t=1} = \chi_q L(n). \tag{148}$$

Its highest monomial has coefficient one in degree zero.

Proof. Work in any fixed standard module in which $L(n)$ occurs; one may take $M(n)$. Equation (146) makes the ungraded geometric-idempotent dimensions of the two normalized basic simples equal. By the current comparison these dimensions are their ordinary q-characters. Lemma 17.1 makes both pullbacks simple type 1 modules with these characters. Injectivity of the q-character map therefore fixes their label. Thus $\alpha^* = \alpha$ on the relevant simples, and their geometric graded characters are palindromic.

The highest ℓ-weight space is one-dimensional. If its degree in the normalized basic simple is r, its coefficient is t^r. Palindromy forces $r = 0$. Every graded lift of the same ungraded $\mathscr{A}_S$-simple is a shift of this lift, because its basic version is a one-dimensional simple concentrated in a single degree. By Corollary 16.6, $\operatorname{gr} N(n)$ is such a lift, and its intrinsic highest line has degree zero. Hence the shift is zero. This proves bar invariance and the normalization. The specialization follows from the actual isomorphism $N(n) \cong L(n)$ and the dimensions of the finite filtration. □

The construction is independent of the larger standard module used to realize $L(n)$. Indeed the intrinsic filtration is defined using the ordinary highest rational function of n; the comparison with a larger framing proves compatibility with its inherited action, and the highest-line normalization removes the possible shift. This normalization applies in every standard module containing $L(n)$.

18. Exceptional fundamental characters

Theorem 18.1. *For $\mathfrak{g}$ of type G_2 or F_4, every $i \in I$, and every $a \in \mathbb{C}^\times$, one has*

$$S_t(Y_{i,a}) = F_t(Y_{i,a}). \tag{149}$$

Proof. The ordinary G_2 fundamental characters at the short and long nodes have respectively 7 and 15 distinct monomials, all of multiplicity one [28, Lemma 3.1]. Every nonzero intrinsic graded coefficient is therefore one-dimensional. Proposition 17.2 forces it to be 1. For the formal fundamental character, each coefficient lies in $\mathbb{N}[t^{\pm 1}]$ by [21, Theorem 7.5], and its value at $t = 1$ is the corresponding ordinary multiplicity by [20, Theorem 6.2]. A nonnegative Laurent polynomial with value one at 1 is a single monomial t^r. The bar invariance of (12) forces $r = 0$. Coefficients with value zero vanish. This proves the assertion in type G_2.

For type F_4 use Hernandez's numbering

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \qquad (d_1, d_2, d_3, d_4) = (1, 1, 2, 2). \tag{150}$$

This is the numbering of Section 12 reversed by $i \mapsto 5 - i$. The known fundamental characters have the following multiplicities [21, Sections 7–8], [26]:

Node	Distinct monomials	Ordinary dimension	Multiplicity-two monomials
1	26	26	0
2	283	299	16
3	1532	1703	171
4	53	53	0

There are no other multiplicities. The formal coefficient of a thin monomial is 1, and that of each of the 187 remaining monomials is $t + t^{-1}$. The full list of the latter monomials, together with the calculation needed below, is recorded in Appendix A.

We may translate the spectral parameter and take the highest monomial to be $Y_{i,1}$ in integer spectral notation. Write a monomial in this character uniquely as

$$n = Y_{i,1} \prod_{j,s} A_{j,s}^{-\nu_{j,s}}.$$

The signed pole divisor of (4) has a positive entry at $(i, 1 + d_i)$ and a negative entry at $(i, 1 - d_i)$. Thus

$$L = \nu_{i,1-d_i}, \qquad d_{\boldsymbol{p}}(\boldsymbol{\nu}) = \nu_{i,1+d_i} - \nu_{i,1-d_i} + \sum_{(j,s)\to(k,t)} \nu_{j,s}\nu_{k,t} - \sum_{j,s} \nu_{j,s}^2. \tag{151}$$

For every multiplicity-two monomial, the division and the integer calculation in Appendix A give

$$L = 0, \qquad d_{\boldsymbol{p}}(\boldsymbol{\nu}) \equiv 1 \pmod 2. \tag{152}$$

The geometric quotient in this sector is consequently the whole critical cohomology: the negative-framing bundle has rank zero, so its Euler class is 1 and the Euler operator is the identity. Its total dimension is two by Theorem 3.3 and the critical quotient theorem. The whole critical support is projective, as in the projectivity statement for the spectral W_2 model [36, Proposition 5.8]. Scaling a cut direction gives an algebraic $\mathbb{C}^\times$-action of potential weight one; it preserves cyclic stability. The critical Hard Lefschetz theorem therefore applies to the whole critical support [5, Corollary 3.2]. With perverse-normalized vanishing cycles its center

is the smooth ambient dimension. Since $L = 0$, this is precisely $d_{\boldsymbol{p}}(\boldsymbol{\nu})$ in (151). The injective specialization into ordinary cohomology also shows that the cohomology is even.

These facts determine the two-dimensional graded coefficient completely. Its centered exponents are odd by (152), and it is palindromic by Proposition 17.2. It has the form $t^r + t^{-r}$ with r a positive odd integer. If $r > 1$, the Lefschetz isomorphism from the lower nonzero degree to the upper one factors through a zero intermediate degree, since cup product raises cohomological degree by two. This is impossible. Hence $r = 1$, giving $t + t^{-1}$. Equivalently, the two-dimensional Lefschetz representation is the irreducible representation of highest weight one; two trivial summands have the wrong parity.

For a thin monomial, the graded coefficient is one-dimensional and Proposition 17.2 gives 1. Monomials absent from the ordinary character have zero-dimensional local quotients and contribute nothing. We have therefore identified every coefficient with that of $F_t(Y_{i,1})$. Spectral translation proves (149) for every a. □

19. Restriction quotients and the standard character

We now calculate the grading of the actual standard obtained in Theorem 14.4. The calculation uses ordinary restriction to a closed union of complete attracting strata. The coefficient complex on that closed locus is the restriction of the ambient critical complex.

19.1. **A restriction-quotient theorem.** Let $A = \mathbb{G}_m$ act on a smooth quasiprojective variety M, preserving a regular function f. Suppose that the relevant critical support $C = \mathrm{crit}(f)$ is projective and has an A-linearized ample line bundle. Write $\Phi = \Phi_f \mathbb{Q}_M$, with the unshifted convention $\Phi_0\mathbb{Q} = \mathbb{Q}$ used above. For an ambient fixed component $F_\alpha \subset M^A$, set

$$C_\alpha = C \cap F_\alpha, \qquad \Phi_\alpha = \Phi_{f|F_\alpha}\mathbb{Q}_{F_\alpha}.$$

At a fixed point the derivative of an invariant function vanishes on every nonzero tangent weight, so $C_\alpha = \mathrm{crit}(f|F_\alpha)$. Let $S_\alpha \subset C$ be its critical attracting stratum, with inclusion η_α and limit map $p_\alpha : S_\alpha \to C_\alpha$. Denote by r_α^+ the rank of the positive tangent weight part of $T_M|_{F_\alpha}$. We assume that $H^*(C_\alpha, \Phi_\alpha)$ is even for every α.

Let $Z \subset C$ be closed, invariant, and a union of whole strata S_α. Let $i : Z \hookrightarrow C$ and $j : U = C \setminus Z \hookrightarrow C$. Call a fixed block good if its stratum belongs to Z. Let

$$H_A = H_A^*(C, \Phi), \qquad R_0 = H_A^*(\mathrm{pt}) = \mathbb{Q}[a], \quad \deg_{\mathrm{pol}} a = 1.$$

Coefficients may subsequently be extended to $\mathbb{C}$. Let $D : H_A \to H_A$ be R_0-linear and homogeneous of polynomial degree L. Assume that after inverting a, its action on ordinary fixed-point restriction is invertible on every good block and zero on every bad block.

Theorem 19.1. *The restriction map is a homogeneous surjection between finite graded free R_0-modules, and*

$$\ker D = \ker\big(H_A^*(C, \Phi) \longrightarrow H_A^*(Z, i^*\Phi)\big). \tag{153}$$

Consequently there are natural graded isomorphisms

$$H_A/\ker D \simeq H_A^*(Z, i^*\Phi), \qquad (\mathrm{im}\, D)[-L] \simeq H_A^*(Z, i^*\Phi). \tag{154}$$

Here a polynomial shift $[b]$ raises generator degrees by b. If

$$h_\alpha(u) = \sum_k \dim H^{2k}(C_\alpha, \Phi_\alpha) u^k,$$

then

$$\mathrm{Hilb}\big((H_A/\ker D)/a; u\big) = \sum_{\alpha \ good} u^{r_\alpha^+} h_\alpha(u). \tag{155}$$

These assertions commute with completion at $a = 0$.

Proof. The hyperbolic localization theorem for vanishing cycles gives the sheaf-level identity

$$(p_\alpha)_!(\eta_\alpha)^*\Phi \simeq \Phi_\alpha[-2r_\alpha^+](-r_\alpha^+). \tag{156}$$

In the present convention this follows by commuting vanishing cycles with hyperbolic localization and then applying the smooth constant-sheaf Białynicki–Birula calculation; compatibility with the closed critical support gives the displayed formula on C_α. These are the complex-level statements of [8, Lemma 3.15, Proposition 6.2, Equation (6.8), and Theorem 7.1]. If r_α^- is the negative tangent rank, then $\dim M - \dim F_\alpha = r_\alpha^+ + r_\alpha^-$. Removing the ambient shifts from the perverse-normalized formula changes its cohomological shift to

$$(\dim F_\alpha - \dim M) - (r_\alpha^+ - r_\alpha^-) = -2r_\alpha^+,$$

as in (156). The Tate twist does not affect the cohomological Hilbert polynomial. Since C_α is proper, it gives

$$H_c^k(S_\alpha, \Phi|_{S_\alpha}) \simeq H^{k-2r_\alpha^+}(C_\alpha, \Phi_\alpha). \tag{157}$$

Thus each stratum has even compactly supported cohomology with these coefficients.

Choose a sufficiently high very ample power of the linearized line bundle. Its sections embed C equivariantly into $\mathbb{P}(V)$, where $V = \bigoplus_\lambda V_\lambda$ is a finite weight decomposition. The closed sets

$$C \cap \mathbb{P}\left(\bigoplus_{\mu \geq \lambda} V_\mu\right)$$

give a finite filtration by weights. Each successive difference is a union of attracting strata with one limiting weight. If several fixed components have that weight, their inverse images under the limit map are open and closed in this difference, and cohomology splits as a direct sum. Intersect this filtration with Z and with U. The assumption that Z is a union of whole strata ensures that these differences are still unions of whole good or whole bad strata. Compact-support excision and (157), by induction on this finite filtration, give

$$H_c^{\mathrm{odd}}(Z, i^*\Phi) = H_c^{\mathrm{odd}}(U, j^*\Phi) = 0.$$

They also give, since Z is proper,

$$\operatorname{Hilb} H^*(Z, i^*\Phi) = \sum_{\alpha \text{ good}} u^{r_\alpha^+} h_\alpha(u). \tag{158}$$

Evenness is used here for bad as well as good fixed blocks.

The closed/open triangle

$$j_! j^*\Phi \longrightarrow \Phi \longrightarrow i_* i^*\Phi \xrightarrow{+1}$$

therefore gives short exact sequences in even cohomological degrees:

$$0 \longrightarrow H_c^*(U, j^*\Phi) \longrightarrow H^*(C, \Phi) \xrightarrow{\rho} H^*(Z, i^*\Phi) \longrightarrow 0. \tag{159}$$

Compact support in the first term follows from properness of C. In particular, $H^*(C, \Phi)$ is also even.

Apply the same triangle on finite Borel approximations. For each of its three terms the Borel spectral sequence has

$$E_2^{p,q} = H^p(BA) \otimes H^q(\text{ordinary term}).$$

Both indices are even. A differential of odd length has target in an odd column, and one of even length has target in an odd row. All differentials vanish. The base BA is simply connected, so no nonconstant local system enters this formula. Lifting a homogeneous ordinary basis gives

a free graded R_0-basis. Taking the stable range of the Borel approximations gives an exact sequence of finite graded free modules

$$0 \longrightarrow H^*_{c,A}(U, j^*\Phi) \longrightarrow H^*_A(C, \Phi) \xrightarrow{\rho_A} H^*_A(Z, i^*\Phi) \longrightarrow 0. \tag{160}$$

Compact support here is in the variety direction of the Borel approximation. Reduction modulo a recovers (159).

Write $k : C^A \hookrightarrow C$ and $k_\alpha : C_\alpha \hookrightarrow C$ for the inclusions. Let $K_0 = \mathbb{Q}(a)$. Equivariant localization, naturally applied to ρ_A, uses ordinary restriction of Φ to C^A:

$$H_A \otimes K_0 \simeq H^*_A(C^A, k^*\Phi) \otimes K_0.$$

On a good fixed block, restriction to Z is the identity; on a bad block its target is zero. Therefore

$$(\ker \rho_A) \otimes K_0 = \bigoplus_{\alpha \text{ bad}} H^*_A(C_\alpha, k^*_\alpha \Phi) \otimes K_0 = (\ker D) \otimes K_0.$$

Both kernels are saturated submodules of the finite free module H_A. For $\ker \rho_A$ this follows from the freeness of the quotient in (160). For $\ker D$ it follows because its quotient is isomorphic to $\operatorname{im} D$, a torsion-free submodule of H_A. A saturated submodule is the intersection of its fraction-field span with H_A. The equality of those spans therefore proves (153). The map

$$\rho_A(h) \longmapsto Dh$$

is now well defined, bijective, and homogeneous of degree L. This proves (154). Reduction modulo a, together with (158), proves (155). Flatness of completion preserves each finite-module exact sequence and the actual graded identifications. □

In the geometric Euler case the map in (154) has a direct construction. If $D = e_A(E)$, and Z is the zero set of an equivariant section of E, that section is nowhere zero on U. Thus $D|_U = 0$, and the projection formula gives

$$D\, j_!(\alpha) = j_!((D|_U)\alpha) = 0.$$

Multiplication by D factors through ρ_A integrally. On good fixed blocks its factor is invertible after localization, so the kernel of the factored map is torsion. Its source is free, hence that kernel is zero. This gives the same isomorphism for the section under consideration.

19.2. **Whole good strata for ordered fundamental factors.** We verify the hypotheses above for the Euler quotient (108). It is convenient to use the cocyclic form of the framed geometry, obtained by dualizing every vector space and every arrow. Spectral vertex labels are retained.

Write $m_r = Y_{i_r, bq^{s_r}}$ in one orbit, with

$$s_1 \le \cdots \le s_N.$$

For several orbits choose any total order extending these individual orders. Terms in different orbits have disjoint support. The positive and negative framing vertices of factor r are

$$v_r^+ = (i_r, s_r + d_{i_r}), \qquad v_r^- = (i_r, s_r - d_{i_r}).$$

Let I_r^+ and I_r^- be their graded injective modules in the critical representation category. The arrow $\epsilon_{i_r} : v_r^- \to v_r^+$ in the cocyclic quiver induces a morphism of injective modules

$$\varphi_r : I_r^+ \longrightarrow I_r^-, \qquad K_r = \ker \varphi_r.$$

Put $I^+ = \bigoplus_r I_r^+$, $K = \bigoplus_r K_r$. The critical support is described by submodules $V \subset I^+$ of the prescribed dimension. The proper equivariant comparison with this injective Grassmannian is the one in [36, Proposition 5.8]. Its underlying proper bijection suffices for the support and limiting arguments below.

Lemma 19.2. *The fundamental kernel K_r is supported in spectral grades strictly greater than s_r. In particular,*

$$(K_r)_{v_r^-} = 0, \qquad (K_s)_{v_r^-} = 0 \quad (r < s). \tag{161}$$

Moreover, for every finite-dimensional critical module V,

$$\mathrm{Hom}(V, I_r^-) \simeq (V_{v_r^-})^*. \tag{162}$$

Proof. The graded multiplicities of the fundamental kernel are

$$\dim(K_r)_{(j,s_r+u)} = \begin{cases} \widetilde{c}_{ji_r}(u), & 0 \le u \le r_{\mathfrak{g}} h^\vee, \\ 0, & \text{otherwise}, \end{cases} \qquad r_{\mathfrak{g}} = \max_i d_i.$$

This follows from [16, Equation (5.1), Lemma 5.3, and Equation (5.3)]. Proposition 4.2(4) of that work gives zero coefficient at $u = 0$, proving the strict support assertion. If $r < s$ in the same orbit, then $s_r - d_{i_r} < s_s$; if they belong to different orbits, their supports are disjoint. This proves (161). Finally I_r^- is the coinduced injective at the indicated vertex, so evaluation of a morphism on that vertex gives the adjunction (162). □

Choose distinct cocyclic flavour weights

$$c_1^{\mathrm{out}} > \cdots > c_N^{\mathrm{out}}.$$

The two framings of factor r have the same weight. An attracting limit of a submodule $V \subset I^+$ is a direct sum $\bigoplus_r U_r$, with $U_r \subset I_r^+$. Call this allocation good when $U_r \subset K_r$ for every r. Let Z be the locus $V \subset K$.

Proposition 19.3. *The locus Z is closed and is the union of all whole good critical attracting strata. Goodness is a condition on an entire fixed allocation. For that allocation it is equivalent to*

$$(\nu_r)_{v_r^-} = 0 \quad \textit{for all } r. \tag{163}$$

Proof. Let $F_r I^+ = \bigoplus_{s \le r} I_s^+$. The filtration of an attracting submodule by $V \cap F_r I^+$ has associated graded U_r. Suppose first that its limit is good. The component map $\varphi_r|_V$ vanishes on $V \cap F_r I^+$: its projection to I_r^+ is U_r, which lies in K_r, and the earlier summands have zero r-component. Thus it factors through $V/(V \cap F_r I^+)$. At the vertex v_r^-, the dimension of this quotient is

$$\sum_{s>r} \dim(U_s)_{v_r^-} = 0$$

by (161). The injective adjunction (162) implies that the factored map is zero. This holds for every r, so all components of $\bigoplus_r \varphi_r$ vanish on V, and $V \subset K$.

Conversely, Z is closed and flavour invariant. The attracting limit of any point of Z therefore remains in Z; its summands lie in their corresponding K_r. This proves the equality with the union of whole good strata. It can be checked on the reduced underlying supports through the proper equivariant Grassmannian comparison: an equivariant proper bijection carries each submodule limit to the same critical limit.

Closedness also has an intrinsic algebraic description on the framed moduli space. Let W_r be the one-dimensional framing space, let $j_r : \mathcal{V}_{v_r^+} \to W_r$ be its outgoing framing, and let $\epsilon_{i_r} : \mathcal{V}_{v_r^-} \to \mathcal{V}_{v_r^+}$ be the corresponding arrow in the cocyclic convention. Their composition is a section of $\mathrm{Hom}(\mathcal{V}_{v_r^-}, W_r)$, given by

$$j_r \epsilon_{i_r} : \mathcal{V}_{v_r^-} \longrightarrow W_r.$$

On the critical locus it is exactly $\varphi_r|_V$ under coinduction. Its simultaneous zero set has underlying closed set Z. The vector bundle and its section are therefore defined on the moduli space itself.

If $(\nu_r)_{v_r^-} = 0$, the adjunction shows that every submodule of that dimension in I_r^+ is killed by φ_r, and hence lies in K_r. If $(\nu_r)_{v_r^-} > 0$, no such submodule can lie in K_r, by (161). This proves (163) and proves that goodness does not cut out a proper part of a fixed block. □

Lemma 19.4. *After inverting the equivariant parameter a, the Euler operator is invertible on every good fixed block and is zero on every bad fixed block, acting on the ordinary restriction of the ambient critical complex.*

Proof. At a fixed decomposition $V = \bigoplus_s V^s$, the numerator bundle decomposes into the pieces $\mathrm{Hom}(\mathcal{V}^s_{v_r^-}, W_r)$. If $r \neq s$, this bundle has nonzero flavour weight $c_r^{\mathrm{out}} - c_s^{\mathrm{out}}$. Its Euler class is a nonzero power of $(c_r^{\mathrm{out}} - c_s^{\mathrm{out}})a$ times one plus a nilpotent positive Chern class, and is invertible after inverting a. Positive ordinary cohomology acts nilpotently on the bounded cohomology of the proper fixed support, also with the restricted coefficient complex.

For a self piece $r = s$, condition $(\nu_r)_{v_r^-} = 0$ means that the bundle has rank zero and Euler class one. If its rank is positive, the canonical section s_r is nowhere zero on the whole fixed support: a zero would give a submodule of K_r with a nonzero component at v_r^-, contradicting Lemma 19.2. This section is equivariant of weight zero. It splits off an equivariantly trivial line locally, with a vector bundle quotient of rank one less. The top equivariant Chern class is consequently zero. This assertion applies on the reduced support even if the critical scheme is singular. In a local trivialization, one component of a nowhere-zero section is invertible, giving the locally free quotient just used. It follows that the self Euler factor acts by zero on every coefficient complex restricted to this support. Combining the self and cross factors proves the result. □

All fixed critical blocks, including the bad blocks, have even critical cohomology. Indeed, the fixed ambient variety is the product of the one-framing ambient varieties, and the restricted potential is the sum of their potentials. Thom–Sebastiani and the evenness in Theorem 11.1 prove the assertion. The critical support is projective, and the ambient GIT ample bundle restricts to an ample linearized bundle on it. Proposition 19.3 and Lemma 19.4 therefore verify every hypothesis of Theorem 19.1. In particular, in source degree,

$$\mathcal{Q}_{0,\nu} \simeq H_A^*(Z_\nu, i_\nu^*\Phi). \tag{164}$$

This identifies the retained Euler quotient, including its classes that survive only through the integral family.

19.3. **The tangent shift and the quantum-torus exponent.** We keep track of the chamber and the degree centre. Vertexwise duality from the incoming model to the cocyclic model sends

$$y^{\mathrm{out}} = -y^{\mathrm{in}}, \qquad c_r^{\mathrm{out}} = -c_r^{\mathrm{in}}. \tag{165}$$

It reverses arrows but does not replace a spectral label s by $-s$. Thus the chosen decreasing cocyclic flavour weights are obtained by choosing the increasing incoming deformation weights $c_r^{\mathrm{in}} = -c_r^{\mathrm{out}}$ in (108). The integral construction required only their distinctness. The incoming Euler factor transforms as

$$y^{\mathrm{in}} - c_r^{\mathrm{in}} a = c_r^{\mathrm{out}} a - y^{\mathrm{out}},$$

which is the Euler factor of $\mathrm{Hom}(\mathcal{V}^{\mathrm{out}}_{v_r^-}, W_r^{\mathrm{out}})$. If the variables in both conventions are renamed to the same symbols instead, their Euler polynomials differ by the scalar $(-1)^{L_\nu}$. Neither this scalar nor the coordinate change $a \mapsto -a$ changes a polynomial degree or inverts the character variable t.

Let $\boldsymbol{m}^{(r)}$, $\boldsymbol{l}^{(r)}$, and $\boldsymbol{p}^{(r)} = \boldsymbol{m}^{(r)} - \boldsymbol{l}^{(r)}$ be the framing vectors of factor r, and let $\nu = \sum_r \nu_r$ be a fixed allocation. Define

$$\beta^m_{rs} = \boldsymbol{m}^{(r)} \cdot \nu_s + E(\nu_r, \nu_s) - \nu_r \cdot \nu_s, \tag{166}$$

$$\beta^p_{rs} = \beta^m_{rs} - \boldsymbol{l}^{(r)} \cdot \nu_s. \tag{167}$$

The form E is still that of the incoming computational quiver. The actual cocyclic quiver is its opposite.

Lemma 19.5. *In the chosen cocyclic chamber the positive and negative ambient tangent ranks at the fixed allocation are*

$$r^+ = \sum_{r<s} \beta^m_{rs}, \qquad r^- = \sum_{r<s} \beta^m_{sr}. \tag{168}$$

On a good allocation,

$$\boldsymbol{l}^{(r)} \cdot \nu_s = 0 \quad (r \le s), \qquad L_\nu = \sum_{r<s} \boldsymbol{l}^{(s)} \cdot \nu_r. \tag{169}$$

Proof. The cocyclic flavour action sends an outgoing framing j to $A(t)j$. Gauge action sends it to jg^{-1}. To return a fixed representative to itself one therefore uses the compensating gauge matrix

$$g(t) = \bigoplus_r t^{c^{\mathrm{out}}_r} \mathrm{id}_{V^r}.$$

An arrow variation from V^s to V^r, and a framing variation from V^s to W^r, have weight $c^{\mathrm{out}}_r - c^{\mathrm{out}}_s$. For $r < s$ it is positive. The framing contribution has dimension $\boldsymbol{m}^{(r)} \cdot \nu_s$. Because the arrows are reversed in the cocyclic model, their contribution is $E(\nu_r, \nu_s)$. The gauge tangent contribution to be subtracted is $\nu_r \cdot \nu_s$. Stability makes the gauge tangent map injective; the quotient tangent sequence is exact in each weight. Thus the positive block rank is precisely β^m_{rs}. Reversing r, s gives the negative block. Summing proves (168).

For a good allocation $U_s \subset K_s$. The strict support assertion (161), including the case $r = s$, gives $(\nu_s)_{v^-_r} = 0$ for $r \le s$. Each $\boldsymbol{l}^{(r)}$ is the numerator framing at this vertex. This proves the first assertion of (169). Expanding $L_\nu = \sum_{r,s} \boldsymbol{l}^{(r)} \cdot \nu_s$ then gives the second. □

For a good factor allocation, let C_{r,ν_r} be the critical support of the rth fundamental framed model and let Φ_{r,ν_r} be the restriction of its unshifted vanishing-cycle complex to that support. Set

$$h_{r,\nu_r}(u) = \sum_k \dim H^{2k}(C_{r,\nu_r}, \Phi_{r,\nu_r}) u^k, \qquad d_r = d_{\boldsymbol{m}^{(r)}}(\nu_r).$$

Its self numerator Euler bundle has rank zero. Hence this is also the polynomial Hilbert function of the corresponding intrinsic fundamental quotient, and its centre d_r equals $d_{\boldsymbol{p}^{(r)}}(\nu_r)$. Let $\mathrm{gr}_{\mathrm{pol}}$ denote the associated graded for the uncentered polynomial-degree filtration. Equation (110) identifies $\mathrm{gr}_{\mathrm{pol}} Q(m)$ with $\mathcal{Q}_0/a\mathcal{Q}_0$. Thom–Sebastiani and (155) give

$$h_\nu(u) := \mathrm{Hilb}\big((\mathrm{gr}_{\mathrm{pol}} Q(m))_\nu; u\big) = \sum_{\substack{\nu = \sum_r \nu_r \\ \text{good allocations}}} u^{\sum_{r<s} \beta^m_{rs}} \prod_r h_{r,\nu_r}(u). \tag{170}$$

The ambient dimension expands as

$$d_{\boldsymbol{m}}(\nu) = \sum_r d_r + \sum_{r<s} (\beta^m_{rs} + \beta^m_{sr}), \qquad d_{\boldsymbol{p}}(\nu) = d_{\boldsymbol{m}}(\nu) - L_\nu.$$

Using (169), multiplication by the normalizing centre and the substitution $u = t^2$ yield

$$t^{-d_{\boldsymbol{p}}(\nu)} h_\nu(t^2) = \sum_{\text{good allocations}} t^{\sum_{r<s}(\beta^p_{rs}-\beta^p_{sr})} \prod_r \left[t^{-d_r} h_{r,\nu_r}(t^2) \right]. \tag{171}$$

Every exponent in this formula is determined by the ambient tangent sequence and the source Euler quotient.

Lemma 19.6. *For $M_r = m_r A^{-\nu_r}$, the quantum-torus form in the conventions of the introduction satisfies*

$$\gamma(M_r, M_s) - \gamma(m_r, m_s) = 2(\beta^p_{rs} - \beta^p_{sr}). \tag{172}$$

Proof. The root-monomial identities for the torus form are

$$\gamma(m_r, A^{\nu_s}) = -2\boldsymbol{p}^{(r)} \cdot \nu_s, \qquad \gamma(A^{\nu_r}, A^{\nu_s}) = 2\big(E(\nu_r, \nu_s) - E(\nu_s, \nu_r)\big). \tag{173}$$

They follow by substituting a fundamental monomial and a root monomial into the inverse-Cartan definition of γ; equivalently they are [17, Section 2.6, Equation (5)] in the present spectral notation. Bilinearity and skew symmetry give

$$\begin{aligned} \gamma(M_r, M_s) - \gamma(m_r, m_s) &= -\gamma(m_r, A^{\nu_s}) - \gamma(A^{\nu_r}, m_s) + \gamma(A^{\nu_r}, A^{\nu_s}) \\ &= 2\big(\boldsymbol{p}^{(r)} \cdot \nu_s - \boldsymbol{p}^{(s)} \cdot \nu_r + E(\nu_r, \nu_s) - E(\nu_s, \nu_r)\big). \end{aligned}$$

The symmetric terms $\nu_r \cdot \nu_s$ cancel in $\beta^p_{rs} - \beta^p_{sr}$, proving the formula. □

Theorem 19.7. *For every dominant monomial m, the filtered actual standard constructed above satisfies*

$$\operatorname{ch}_t \operatorname{gr} Q(m) = \overline{E_t(m)}. \tag{174}$$

Proof. By Theorem 18.1, the centered fundamental polynomials

$$\sum_{\nu_r \text{ good}} t^{-d_r} h_{r,\nu_r}(t^2)\, m_r A^{-\nu_r}$$

are the fundamental characters $F_t(m_r)$. Formula (171), together with (172) and the product convention $M * N = t^{\gamma(M,N)/2} MN$, therefore says that

$$\operatorname{ch}_t \operatorname{gr} Q(m) = t^{-\frac{1}{2}\sum_{r<s}\gamma(m_r,m_s)} F_t(m_1) * \cdots * F_t(m_N).$$

This is the product in increasing spectral order, normalized so that its highest monomial has coefficient one. The definition of $E_t(m)$ uses decreasing spectral order [17, Section 2.7, Equation (7)]. The bar is an anti-involution, each $F_t(m_r)$ is bar invariant, and the normalized highest coefficient is preserved. Its bar is consequently the increasing product displayed above, proving (174).

Equal spectral exponents introduce no choice: the fundamental characters at the same exponent commute, as in the statement following Equation (7) of [17]. Distinct $q^{\mathbb{Z}}$-orbits have no arrows or root-torus pairings between them, so the same formulas factor over their finitely many components. Thus the computation applies to arbitrary nonzero complex spectral parameters. □

The quadratic pairs introduced by the polynomial augmentation give the same centered formula. Here is the cancellation explicitly. For an orthogonal pair the extra arrow spaces are $T = \operatorname{Hom}(V_i, V_j)$ and its dual, and their potential is the nondegenerate trace pairing $2\operatorname{Tr}(AB)$. If $\operatorname{rk} T = h$, its critical equations are $A = B = 0$; on this locus stability is exactly stability for the reduced quiver. In a neighborhood of the critical support the added normal directions are therefore the vector bundle $T \oplus T^*$ with this hyperbolic quadratic function. The relative quadratic Thom calculation identifies its unnormalized critical complex with the reduced complex

shifted by $[-2h](-h)$. Thus it multiplies the polynomial Hilbert series by u^h. The smooth ambient dimension has increased by $2h$, and the numerator Euler bundle has not changed. With $u = t^2$, the additional factor t^{2h} is cancelled by the additional centering factor t^{-2h}. This applies to each orthogonal pair and to every fixed allocation. Equivalently, their two opposite arrows change the symmetric part of E but not the antisymmetric form in (173). One may therefore use the augmented computational quiver consistently throughout the formulas or its reduced geometric model.

20. Canonical triangularity and the main theorem

Proposition 20.1. *For every dominant monomial m in type G_2 or F_4, there are polynomials $p_{m,n}(t) \in t\mathbb{N}[t]$, only finitely many nonzero, such that*

$$\overline{E_t(m)} = S_t(m) + \sum_{n<m} p_{m,n}(t) S_t(n). \tag{175}$$

Proof. Let $Q(m)$ be the special fiber of the source-quotient standard lattice. Its ordinary action is the action on $M(m)$, and its associated graded is a module for the same finite graded algebra $\mathscr{A}_S$ as the intrinsic simple subquotients. Each spectral idempotent e_γ has degree zero, so

$$\dim e_\gamma \operatorname{gr} Q(m) = \dim e_\gamma Q(m).$$

Lemma 16.3 gives a type 1 ordinary action on $\operatorname{gr} Q(m)$. The Cartan calculation of Lemma 17.1 then gives $\chi_q(\operatorname{gr} Q(m)) = \chi_q M(m)$. Additivity and injectivity of the q-character imply equality of their ordinary composition multiplicities. Thus every graded simple composition factor is a shift of an intrinsic constituent lift, by Corollary 16.6 and Proposition 17.2. The homogeneous generation theorem 16.9 says that $\operatorname{gr} Q(m)$ is generated by its degree-zero highest line and has the unique simple head $L(m)$. Theorem 19.7 gives

$$\operatorname{ch}_t \operatorname{gr} Q(m) = \overline{E_t(m)}. \tag{176}$$

Pass to the finite basic algebra B' of (145). Proposition 17.2 identifies the intrinsic character of each simple with its normalized basic character; its highest line has degree zero. Thus the simple head of the basic module corresponding to $\operatorname{gr} Q(m)$ is the basic simple with label m at external degree zero. Lift a basis vector of this head homogeneously and apply e_m to the lift. It defines a graded map from $B'e_m$ to the basic module which is surjective modulo the radical, hence surjective by graded Nakayama. The degree-zero part of $B'e_m$ is just its head, and its radical is concentrated in positive degree. Every other graded composition factor of its quotient consequently occurs in a strictly positive external shift. These shifts give the coefficients $p_{m,n}(t) \in t\mathbb{N}[t]$.

All composition labels satisfy $n \le m$, because their highest monomials occur in the ordinary standard character. The coefficient of m in that character is one, already accounted for by the head, so the remaining labels satisfy $n < m$. There are finitely many composition factors. Transferring the character equality back through the graded Morita equivalence retains the internal spaces $V^d_{\gamma,\alpha}$ of (143) and proves (175). □

Theorem 20.2. *For type G_2 or F_4 and every dominant monomial m, the intrinsic graded character belongs to the quantum Grothendieck ring and equals its canonical element:*

$$S_t(m) = L_t(m) \in \mathcal{K}_t. \tag{177}$$

Proof. We use induction on the nonnegative integer $h(m)$ in (13). The trivial monomial has $S_t(1) = E_t(1) = 1$. Suppose the assertion has been established at all smaller heights. Every

lower dominant label in (175) has smaller height by (14). Since $\mathcal{K}_t$ is bar-stable, that equality first gives

$$S_t(m) = \overline{E_t(m)} - \sum_{n<m} p_{m,n}(t) S_t(n) \in \mathcal{K}_t.$$

Now use the bar invariance of Proposition 17.2 and apply bar to (175). We obtain

$$S_t(m) = E_t(m) - \sum_{n<m} p_{m,n}(t^{-1}) S_t(n). \tag{178}$$

By induction, every lower $S_t(n)$ has an expansion in the standard basis with leading term $E_t(n)$ and all remaining coefficients in $t^{-1}\mathbb{Z}[t^{-1}]$. The coefficient $p_{m,n}(t^{-1})$ belongs to this same ideal. Substituting those finite expansions into (178), and using transitivity of the monomial order, yields

$$S_t(m) - E_t(m) \in \sum_{n<m} t^{-1}\mathbb{Z}[t^{-1}] E_t(n).$$

Together with $S_t(m) \in \mathcal{K}_t$ and bar invariance, these are exactly the three properties in (11). Uniqueness proves (177). □

Proof of Theorem 1.1. In types G_2 and F_4, Theorem 20.2 and the actual intrinsic realization give

$$L_t(m)|_{t=1} = S_t(m)|_{t=1} = \chi_q L(m).$$

Every numerical step used only that q is not a root of unity. In particular the determinant inversions (25) and the corresponding Coulomb Cartan inversions have only root-of-unity zeros. The local wheel-ideal comparison in Theorem 3.4 holds at each such numerical parameter, and Theorem 14.4 identifies the lattices before collision specialization.

Suppose the spectral support of m meets several $q^{\mathbb{Z}}$-orbits. The spectral quiver is the disjoint union of the corresponding quivers. Local resonance factors between different orbits are units and the skew form between them is zero. The constructions of the actual action, finite filtrations, ordinary flag algebra, and standard character therefore apply to their finite union. A lower dominant label introduces no new orbit, by the argument following (14). The same finite-height induction proves the equality for arbitrary parameters. This argument applies to the entire finite union of spectral orbits.

For simply-laced types the statement is the theorem of Nakajima [29, 30]; for the remaining classical types it is [17, §2.5 and Theorem 2.13]. To spell out the passage from integral spectral notation, write $m = \prod_O m_O$ over its occupied $q^{\mathbb{Z}}$-orbits. The skew form between distinct orbits vanishes, so $E_t(m) = \prod_O E_t(m_O)$. The product $\prod_O L_t(m_O)$ belongs to $\mathcal{K}_t$ and is bar-invariant. Multiplying the finite triangular expansions shows that its highest term is $E_t(m)$ and that every other standard-basis coefficient lies in $t^{-1}\mathbb{Z}[t^{-1}]$. Canonical uniqueness therefore gives

$$L_t(m) = \prod_O L_t(m_O).$$

The ordinary spectral decomposition $L(m) = \bigotimes_O L(m_O)$ [17, §2.5] now transfers the one-orbit specialization to m. These cases and the two exceptional cases exhaust the finite Dynkin types, proving the theorem. □

Corollary 20.3. *In types G_2 and F_4, the coefficient of a monomial n in $L_t(m)$ is the centered Hilbert polynomial of the corresponding intrinsic critical-cohomology quotient of the actual module $L(m)$. In particular it belongs to $\mathbb{N}[t^{\pm 1}]$, is invariant under $t \mapsto t^{-1}$, and its value at $t = 1$ is the dimension of the ℓ-weight space $L(m)_n$.*

Proof. The intrinsic construction assigns to the local sector with dimension vector $\boldsymbol{\nu}$ its source-quotient filtration, centered by $d_{\boldsymbol{p}}(\boldsymbol{\nu})$. Its graded dimensions are nonnegative integers and their sum is $\dim L(m)_n$. Proposition 17.2 gives palindromy, and Theorem 20.2 identifies this character with $L_t(m)$. □

Appendix A. A finite certificate for the nonthin F_4 coefficients

We give the complete finite input to (152). The numbering is (150), and the highest monomial is $Y_{i,1}$. In the tables below j_s denotes $Y_{j,s}$, and products are ordinary commutative products. The row number is the zero-based number of the monomial in the complete fundamental table [26]; the monomial itself is included so the certificate is independent of those row numbers. Every displayed monomial has ordinary multiplicity two. There are 16 rows for node 2 and 171 rows for node 3; nodes 1 and 4 have no such rows. The last column is $d_{\boldsymbol{p}}(\boldsymbol{\nu})$. In every row $\nu_{i,1-d_i} = 0$.

Here is the full division procedure for verifying the table. Represent a Laurent monomial by its finite exponent array $u_{j,s}$, and start with

$$r_{j,s} = \delta_{j,i}\delta_{s,1} - u_{j,s}, \qquad \nu_{j,s} = 0.$$

Take the smallest s with a nonzero entry of r. For each color j at that index, put $a = r_{j,s}$, set $\nu_{j,s+d_j} \leftarrow \nu_{j,s+d_j} + a$, and subtract a times the exponent array of $A_{j,s+d_j}$ from r. Repeat until $r = 0$. For the four root monomials used here the exponent arrays are explicitly

$$\begin{aligned} A_{1,s} &= 1_{s-1}1_{s+1}2_s^{-1}, \\ A_{2,s} &= 2_{s-1}2_{s+1}1_s^{-1}3_s^{-1}, \\ A_{3,s} &= 3_{s-2}3_{s+2}2_{s-1}^{-1}2_{s+1}^{-1}4_s^{-1}, \\ A_{4,s} &= 4_{s-2}4_{s+2}3_s^{-1}. \end{aligned}$$

The unique lowest-degree factor of $A_{j,s+d_j}$ is j_s; all its other factors have larger spectral degree. Each step is thus forced, and it verifies uniqueness as well as the factorization. In every listed case all the extracted a are nonnegative and the remainder is zero after finitely many steps. Finally evaluate

$$\nu_{i,1+d_i} - \nu_{i,1-d_i} - \sum_{j,s} \nu_{j,s}^2 + \sum_{j,k=1}^{4} \sum_{s} \nu_{j,s}\nu_{k,s-d_j c_{jk}}. \tag{179}$$

The final sum includes $j = k$ and $c_{jk} = 0$, exactly as required by the augmented spectral quiver.

For example, the first node-2 row is $1_{10}2_7 2_9^{-1}2_{11}^{-1}4_6$. Its nonzero root exponents, all equal to one, are

$$(1,3),(1,7),(2,2),(2,4),(2,6),(2,8),(2,10),(3,4),(3,6),(3,8),(4,6).$$

Formula (179) gives 9, and the exponent at $(2,0)$ is zero. This illustrates the same deterministic calculation used in every row. As checks on the enumeration, the node-2 centers have distribution

$$9^2,\ 11^2,\ 13^2,\ 15^1,\ 17^2,\ 19^1,\ 21^2,\ 23^1,\ 25^2,\ 29^1,$$

where d^r means that the value d occurs r times. Every center in both tables is odd. The corresponding calculation on the full fundamental support has 1894 terminating divisions, $L = 0$ in every row, and 1707 thin coefficients. Only the 187 displayed rows and the known ordinary multiplicities are needed in Theorem 18.1.

A.1. **The fundamental at node 2.**

No.	Monomial	$d_{\boldsymbol{p}}$
70	$1_{10}\,2_7\,2_9^{-1}\,2_{11}^{-1}\,4_6$	9
87	$1_{12}^{-1}\,2_7\,2_9^{-1}\,4_6$	9
89	$1_{10}\,2_7\,2_9^{-1}\,2_{11}^{-1}\,3_8\,4_{10}^{-1}$	11
105	$1_{12}^{-1}\,2_7\,2_9^{-1}\,3_8\,4_{10}^{-1}$	11
109	$1_{10}\,2_7\,3_{12}^{-1}$	13
120	$1_8\,1_{10}\,2_9^{-1}\,3_8\,3_{12}^{-1}$	17
124	$1_{12}^{-1}\,2_7\,2_{11}\,3_{12}^{-1}$	13
142	$2_7\,2_{13}^{-1}$	15
143	$1_8\,1_{12}^{-1}\,2_9^{-1}\,2_{11}\,3_8\,3_{12}^{-1}$	17
151	$1_{10}^{-1}\,1_{12}^{-1}\,2_{11}\,3_8\,3_{12}^{-1}$	21
155	$1_8\,2_9^{-1}\,2_{13}^{-1}\,3_8$	19
168	$1_{10}^{-1}\,2_{13}^{-1}\,3_8$	23
173	$1_8\,2_{11}\,2_{13}^{-1}\,3_{12}^{-1}\,4_{10}$	21
188	$1_{10}^{-1}\,2_9\,2_{11}\,2_{13}^{-1}\,3_{12}^{-1}\,4_{10}$	25
193	$1_8\,2_{11}\,2_{13}^{-1}\,4_{14}^{-1}$	25
206	$1_{10}^{-1}\,2_9\,2_{11}\,2_{13}^{-1}\,4_{14}^{-1}$	29

A.2. **The fundamental at node 3.**

No.	Monomial	$d_{\boldsymbol{p}}$
64	$1_3\,1_9\,2_6\,2_8^{-1}\,2_{10}^{-1}\,4_5$	7
90	$1_5^{-1}\,1_9\,2_4\,2_6\,2_8^{-1}\,2_{10}^{-1}\,4_5$	9
91	$3_5\,3_9^{-1}\,4_5$	13
93	$1_3\,1_{11}^{-1}\,2_6\,2_8^{-1}\,4_5$	7
96	$1_3\,1_9\,2_6\,2_8^{-1}\,2_{10}^{-1}\,3_7\,4_9^{-1}$	9
117	$1_5^{-1}\,1_{11}^{-1}\,2_4\,2_6\,2_8^{-1}\,4_5$	9
125	$1_5^{-1}\,1_9\,2_4\,2_6\,2_8^{-1}\,2_{10}^{-1}\,3_7\,4_9^{-1}$	11
126	$3_5\,3_7\,3_9^{-1}\,4_9^{-1}$	15
128	$1_3\,1_{11}^{-1}\,2_6\,2_8^{-1}\,3_7\,4_9^{-1}$	9
138	$1_3\,1_9\,2_6\,3_{11}^{-1}$	11
152	$1_5^{-1}\,1_{11}^{-1}\,2_4\,2_6\,2_8^{-1}\,3_7\,4_9^{-1}$	11
159	$2_8\,2_{10}\,3_5\,3_9^{-1}\,3_{11}^{-1}$	17
162	$1_3\,1_7\,1_9\,2_8^{-1}\,3_7\,3_{11}^{-1}$	15
165	$1_3\,1_{11}^{-1}\,2_6\,2_{10}\,3_{11}^{-1}$	11
166	$1_5^{-1}\,1_9\,2_4\,2_6\,3_{11}^{-1}$	13
194	$1_5^{-1}\,1_{11}^{-1}\,2_4\,2_6\,2_{10}\,3_{11}^{-1}$	13
208	$1_5^{-1}\,1_7\,1_9\,2_4\,2_8^{-1}\,3_7\,3_{11}^{-1}$	17
209	$1_{11}\,2_8\,2_{12}^{-1}\,3_5\,3_9^{-1}$	17
220	$1_3\,2_6\,2_{12}^{-1}$	13
221	$1_3\,1_7\,1_{11}^{-1}\,2_8^{-1}\,2_{10}\,3_7\,3_{11}^{-1}$	15
237	$1_9\,1_{11}\,2_{10}^{-1}\,2_{12}^{-1}\,3_5$	19
238	$1_7\,1_9\,2_6^{-1}\,2_8^{-1}\,3_5\,3_7\,3_{11}^{-1}$	23
251	$1_5^{-1}\,2_4\,2_6\,2_{12}^{-1}$	15
252	$1_3\,1_9^{-1}\,1_{11}^{-1}\,2_{10}\,3_7\,3_{11}^{-1}$	19
253	$1_5^{-1}\,1_7\,1_{11}^{-1}\,2_4\,2_8^{-1}\,2_{10}\,3_7\,3_{11}^{-1}$	17
257	$1_3\,1_7\,2_8^{-1}\,2_{12}^{-1}\,3_7$	17
281	$1_{13}^{-1}\,2_8\,3_5\,3_9^{-1}$	17
289	$1_5^{-1}\,1_9^{-1}\,1_{11}^{-1}\,2_4\,2_{10}\,3_7\,3_{11}^{-1}$	21
294	$1_7\,1_9\,3_7\,3_9^{-1}\,3_{11}^{-1}\,4_7$	25
296	$1_9\,1_{11}\,2_6\,2_8\,2_{10}^{-1}\,2_{12}^{-1}\,3_9^{-1}\,4_7$	21
298	$1_3\,1_9^{-1}\,2_{12}^{-1}\,3_7$	21
300	$1_5^{-1}\,1_7\,2_4\,2_8^{-1}\,2_{12}^{-1}\,3_7$	19
303	$1_7\,1_{11}^{-1}\,2_6^{-1}\,2_8^{-1}\,2_{10}\,3_5\,3_7\,3_{11}^{-1}$	23

No.	Monomial (continued)	$d_{\boldsymbol{p}}$
320	$1_9\,1_{13}^{-1}\,2_{10}^{-1}\,3_5$	19
332	$1_3\,1_7\,2_{10}\,2_{12}^{-1}\,3_{11}^{-1}\,4_9$	19
351	$1_5^{-1}\,1_9^{-1}\,2_4\,2_{12}^{-1}\,3_7$	23
353	$1_7\,2_6^{-1}\,2_8^{-1}\,2_{12}^{-1}\,3_5\,3_7$	25
359	$1_9^{-1}\,1_{11}^{-1}\,2_6^{-1}\,2_{10}\,3_5\,3_7\,3_{11}^{-1}$	27
361	$1_7\,1_{11}^{-1}\,2_{10}\,3_7\,3_9^{-1}\,3_{11}^{-1}\,4_7$	25
362	$1_{11}^{-1}\,1_{13}^{-1}\,3_5$	21
368	$1_7\,1_9\,3_7\,3_{11}^{-1}\,4_{11}^{-1}$	29
370	$1_9\,1_{11}\,2_6\,2_8\,2_{10}^{-1}\,2_{12}^{-1}\,4_{11}^{-1}$	25
382	$1_3\,1_9^{-1}\,2_8\,2_{10}\,2_{12}^{-1}\,3_{11}^{-1}\,4_9$	23
384	$1_5^{-1}\,1_7\,2_4\,2_{10}\,2_{12}^{-1}\,3_{11}^{-1}\,4_9$	21
394	$1_9\,1_{13}^{-1}\,2_6\,2_8\,2_{10}^{-1}\,3_9^{-1}\,4_7$	21
399	$1_3\,1_7\,2_{10}\,2_{12}^{-1}\,4_{13}^{-1}$	23
414	$1_9^{-1}\,2_6^{-1}\,2_{12}^{-1}\,3_5\,3_7$	29
422	$1_5^{-1}\,1_9^{-1}\,2_4\,2_8\,2_{10}\,2_{12}^{-1}\,3_{11}^{-1}\,4_9$	25
428	$1_9^{-1}\,1_{11}^{-1}\,2_8\,2_{10}\,3_7\,3_9^{-1}\,3_{11}^{-1}\,4_7$	29
431	$1_7\,1_{11}^{-1}\,2_{10}\,3_7\,3_{11}^{-1}\,4_{11}^{-1}$	29
432	$1_{11}^{-1}\,1_{13}^{-1}\,2_6\,2_8\,3_9^{-1}\,4_7$	23
436	$1_7\,2_6^{-1}\,2_{10}\,2_{12}^{-1}\,3_5\,3_{11}^{-1}\,4_9$	27
438	$1_7\,2_{12}^{-1}\,3_7\,3_9^{-1}\,4_7$	27
461	$1_3\,1_9^{-1}\,2_8\,2_{10}\,2_{12}^{-1}\,4_{13}^{-1}$	27
463	$1_5^{-1}\,1_7\,2_4\,2_{10}\,2_{12}^{-1}\,4_{13}^{-1}$	25
469	$1_9\,1_{13}^{-1}\,2_6\,2_8\,2_{10}^{-1}\,4_{11}^{-1}$	25
495	$1_5^{-1}\,1_9^{-1}\,2_4\,2_8\,2_{10}\,2_{12}^{-1}\,4_{13}^{-1}$	29
498	$1_9\,1_{11}^{-1}\,1_{13}^{-1}\,2_6\,2_{10}^{-1}\,4_7$	25
500	$1_9^{-1}\,2_8\,2_{12}^{-1}\,3_7\,3_9^{-1}\,4_7$	31
502	$1_7\,2_8\,2_{10}\,2_{12}^{-1}\,3_9^{-1}\,3_{11}^{-1}\,4_7\,4_9$	29
505	$1_9^{-1}\,2_6^{-1}\,2_8\,2_{10}\,2_{12}^{-1}\,3_5\,3_{11}^{-1}\,4_9$	31
512	$1_9^{-1}\,1_{11}^{-1}\,2_8\,2_{10}\,3_7\,3_{11}^{-1}\,4_{11}^{-1}$	33
514	$1_{11}^{-1}\,1_{13}^{-1}\,2_6\,2_8\,4_{11}^{-1}$	27
526	$1_7\,2_6^{-1}\,2_{10}\,2_{12}^{-1}\,3_5\,4_{13}^{-1}$	31
528	$1_7\,2_{12}^{-1}\,3_7\,4_{11}^{-1}$	31
564	$2_{10}^{-1}\,2_{12}^{-1}\,3_7\,4_7$	35
575	$1_7\,1_9\,1_{11}^{-1}\,1_{13}^{-1}\,2_8^{-1}\,2_{10}^{-1}\,3_7\,4_7$	31
577	$1_9^{-1}\,2_8^2\,2_{10}\,2_{12}^{-1}\,3_9^{-1}\,3_{11}^{-1}\,4_7\,4_9$	33
581	$1_9\,1_{11}^{-1}\,1_{13}^{-1}\,2_6\,2_{10}^{-1}\,3_9\,4_{11}^{-1}$	31
583	$1_9^{-1}\,2_8\,2_{12}^{-1}\,3_7\,4_{11}^{-1}$	35
586	$1_7\,2_8\,2_{10}\,2_{12}^{-1}\,3_9^{-1}\,4_7\,4_{13}^{-1}$	33
591	$1_9^{-1}\,2_6^{-1}\,2_8\,2_{10}\,2_{12}^{-1}\,3_5\,4_{13}^{-1}$	35
622	$1_7\,2_8\,2_{10}\,2_{12}^{-1}\,3_{11}^{-1}\,4_9\,4_{11}^{-1}$	33
648	$2_{10}^{-1}\,2_{12}^{-1}\,3_7\,3_9\,4_{11}^{-1}$	41
656	$2_8\,2_{12}^{-1}\,3_{11}^{-1}\,4_7\,4_9$	37
657	$1_7\,1_9\,1_{11}^{-1}\,1_{13}^{-1}\,2_8^{-1}\,2_{10}^{-1}\,3_7\,3_9\,4_{11}^{-1}$	37
666	$1_7\,1_9\,1_{11}^{-1}\,1_{13}^{-1}\,3_{11}^{-1}\,4_7\,4_9$	33
669	$1_7\,2_8\,2_{10}\,2_{12}^{-1}\,4_{11}^{-1}\,4_{13}^{-1}$	37
672	$1_9^{-1}\,2_8^2\,2_{10}\,2_{12}^{-1}\,3_9^{-1}\,4_7\,4_{13}^{-1}$	37
694	$1_9\,1_{11}^{-1}\,1_{13}^{-1}\,2_6\,2_{12}\,3_{13}^{-1}$	35
696	$1_9^{-1}\,2_8^2\,2_{10}\,2_{12}^{-1}\,3_{11}^{-1}\,4_9\,4_{11}^{-1}$	37
729	$1_{11}\,2_8\,2_{10}^{-1}\,2_{12}^{-2}\,4_7\,4_9$	39
755	$1_9^{-1}\,2_8^2\,2_{10}\,2_{12}^{-1}\,4_{11}^{-1}\,4_{13}^{-1}$	41
764	$2_8\,2_{12}^{-1}\,4_7\,4_{13}^{-1}$	41
765	$1_7\,1_9\,1_{11}^{-1}\,1_{13}^{-1}\,4_7\,4_{13}^{-1}$	37
767	$2_8\,2_{12}^{-1}\,3_9\,3_{11}^{-1}\,4_9\,4_{11}^{-1}$	43
768	$1_7\,1_9\,1_{11}^{-1}\,1_{13}^{-1}\,3_9\,3_{11}^{-1}\,4_9\,4_{11}^{-1}$	39
770	$1_9\,1_{11}^{-1}\,2_6\,2_{14}^{-1}$	37
771	$1_7\,1_9\,1_{11}^{-1}\,1_{13}^{-1}\,2_8^{-1}\,2_{12}\,3_7\,3_{13}^{-1}$	41

No.	Monomial (continued)	$d_{\boldsymbol{p}}$
772	$3_7\,3_{13}^{-1}$	45
815	$1_{11}\,2_8\,2_{10}^{-1}\,2_{12}^{-2}\,3_{11}\,4_7\,4_{13}^{-1}$	45
818	$1_{11}\,2_8\,2_{10}^{-1}\,2_{12}^{-2}\,3_9\,4_9\,4_{11}^{-1}$	45
822	$1_7\,1_9\,1_{11}^{-1}\,2_8^{-1}\,2_{14}^{-1}\,3_7$	43
834	$1_{13}^{-1}\,2_8\,2_{10}^{-1}\,2_{12}^{-1}\,4_7\,4_9$	41
840	$2_8\,2_{10}\,3_{11}^{-1}\,3_{13}^{-1}\,4_9$	47
844	$2_8\,2_{12}^{-1}\,3_9\,4_{11}^{-1}\,4_{13}^{-1}$	47
845	$1_7\,1_9\,1_{11}^{-1}\,1_{13}^{-1}\,2_{10}\,2_{12}\,3_{11}^{-1}\,3_{13}^{-1}\,4_9$	43
849	$1_7\,1_9\,1_{11}^{-1}\,1_{13}^{-1}\,3_9\,4_{11}^{-1}\,4_{13}^{-1}$	43
907	$1_{11}\,2_8\,2_{10}^{-1}\,2_{12}^{-2}\,3_9\,3_{11}\,4_{11}^{-1}\,4_{13}^{-1}$	51
911	$1_{11}\,2_8\,2_{10}^{-1}\,2_{12}^{-1}\,2_{14}\,3_{15}^{-1}\,4_7$	49
916	$2_8\,2_{10}\,3_{13}^{-1}\,4_{13}^{-1}$	51
920	$1_{11}\,2_8\,2_{12}^{-1}\,3_{13}^{-1}\,4_9$	49
928	$1_{13}^{-1}\,2_8\,2_{10}^{-1}\,2_{12}^{-1}\,3_{11}\,4_7\,4_{13}^{-1}$	47
930	$1_{13}^{-1}\,2_8\,2_{10}^{-1}\,2_{12}^{-1}\,3_9\,4_9\,4_{11}^{-1}$	47
950	$1_7\,1_9\,1_{11}^{-1}\,1_{13}^{-1}\,2_{10}\,2_{12}\,3_{13}^{-1}\,4_{13}^{-1}$	47
953	$1_7\,1_9\,1_{11}^{-1}\,2_{10}\,2_{14}^{-1}\,3_{11}^{-1}\,4_9$	45
979	$1_{11}\,1_{15}\,2_8\,2_{10}^{-1}\,2_{12}^{-1}\,2_{16}^{-1}\,4_7$	49
981	$1_9\,1_{11}\,2_{10}^{-1}\,2_{12}^{-1}\,3_9\,3_{13}^{-1}\,4_9$	55
998	$1_{13}^{-1}\,2_8\,2_{10}^{-1}\,2_{14}\,3_{15}^{-1}\,4_7$	51
1001	$1_{13}^{-1}\,2_8\,2_{10}^{-1}\,2_{12}^{-1}\,3_9\,3_{11}\,4_{11}^{-1}\,4_{13}^{-1}$	53
1005	$1_{11}\,2_8\,2_{12}^{-1}\,3_{11}\,3_{13}^{-1}\,4_{13}^{-1}$	55
1016	$1_{11}\,2_8\,2_{10}^{-1}\,2_{12}^{-1}\,2_{14}\,3_9\,3_{15}^{-1}\,4_{11}^{-1}$	55
1017	$1_7\,1_9\,2_{12}^{-1}\,2_{14}^{-1}\,4_9$	49
1026	$1_{13}^{-1}\,2_8\,3_{13}^{-1}\,4_9$	51
1044	$1_7\,1_9\,1_{11}^{-1}\,2_{10}\,2_{14}^{-1}\,4_{13}^{-1}$	49
1058	$1_{11}\,1_{15}\,2_8\,2_{10}^{-1}\,2_{12}^{-1}\,2_{16}^{-1}\,3_9\,4_{11}^{-1}$	55
1060	$1_9\,1_{11}\,2_{10}^{-1}\,2_{12}^{-1}\,3_9\,3_{11}\,3_{13}^{-1}\,4_{13}^{-1}$	61
1074	$1_{11}\,1_{17}^{-1}\,2_8\,2_{10}^{-1}\,2_{12}^{-1}\,4_7$	49
1075	$1_{13}^{-1}\,1_{15}\,2_8\,2_{10}^{-1}\,2_{16}^{-1}\,4_7$	51
1076	$1_7\,1_{11}^{-1}\,2_{10}\,2_{12}^{-1}\,2_{14}^{-1}\,4_9$	51
1078	$1_9\,1_{13}^{-1}\,2_{10}^{-1}\,3_9\,3_{13}^{-1}\,4_9$	57
1079	$1_{11}\,2_8\,2_{14}\,3_{13}^{-1}\,3_{15}^{-1}$	59
1085	$1_{13}^{-1}\,2_8\,3_{11}\,3_{13}^{-1}\,4_{13}^{-1}$	57
1096	$1_{13}^{-1}\,2_8\,2_{10}^{-1}\,2_{14}\,3_9\,3_{15}^{-1}\,4_{11}^{-1}$	57
1101	$1_7\,1_9\,2_{12}^{-1}\,2_{14}^{-1}\,3_{11}\,4_{13}^{-1}$	55
1123	$1_{13}^{-1}\,1_{17}^{-1}\,2_8\,2_{10}^{-1}\,4_7$	51
1137	$1_9^{-1}\,1_{11}^{-1}\,2_8\,2_{10}\,2_{12}^{-1}\,2_{14}^{-1}\,4_9$	57
1138	$1_{11}^{-1}\,1_{13}^{-1}\,3_9\,3_{13}^{-1}\,4_9$	61
1140	$1_9\,1_{11}\,2_{10}^{-1}\,2_{14}\,3_9\,3_{13}^{-1}\,3_{15}^{-1}$	65
1146	$1_{11}\,1_{15}\,2_8\,2_{16}^{-1}\,3_{13}^{-1}$	59
1154	$1_{11}\,1_{17}^{-1}\,2_8\,2_{10}^{-1}\,2_{12}^{-1}\,3_9\,4_{11}^{-1}$	55
1155	$1_{13}^{-1}\,1_{15}\,2_8\,2_{10}^{-1}\,2_{16}^{-1}\,3_9\,4_{11}^{-1}$	57
1156	$1_7\,1_{11}^{-1}\,2_{10}\,2_{12}^{-1}\,2_{14}^{-1}\,3_{11}\,4_{13}^{-1}$	57
1158	$1_9\,1_{13}^{-1}\,2_{10}^{-1}\,3_9\,3_{11}\,3_{13}^{-1}\,4_{13}^{-1}$	63
1160	$1_{13}^{-1}\,2_8\,2_{12}\,2_{14}\,3_{13}^{-1}\,3_{15}^{-1}$	61
1177	$1_7\,1_9\,3_{15}^{-1}$	59
1191	$1_9\,1_{11}\,1_{15}\,2_{10}^{-1}\,2_{16}^{-1}\,3_9\,3_{13}^{-1}$	65
1193	$1_{13}^{-1}\,1_{17}^{-1}\,2_8\,2_{10}^{-1}\,3_9\,4_{11}^{-1}$	57
1204	$1_9^{-1}\,1_{11}^{-1}\,2_8\,2_{10}\,2_{12}^{-1}\,2_{14}^{-1}\,3_{11}\,4_{13}^{-1}$	63
1205	$1_{11}^{-1}\,1_{13}^{-1}\,3_9\,3_{11}\,3_{13}^{-1}\,4_{13}^{-1}$	67
1209	$1_9\,1_{13}^{-1}\,2_{10}^{-1}\,2_{12}\,2_{14}\,3_9\,3_{13}^{-1}\,3_{15}^{-1}$	67
1231	$1_{11}\,1_{17}^{-1}\,2_8\,3_{13}^{-1}$	59
1232	$1_{13}^{-1}\,1_{15}\,2_8\,2_{12}\,2_{16}^{-1}\,3_{13}^{-1}$	61
1239	$1_7\,1_{11}^{-1}\,2_{10}\,3_{15}^{-1}$	61
1256	$1_{13}^{-1}\,1_{17}^{-1}\,2_8\,2_{12}\,3_{13}^{-1}$	61

No.	Monomial (continued)	$d_{\boldsymbol{p}}$
1263	$1_{11}^{-1}\,1_{13}^{-1}\,2_{12}\,2_{14}\,3_9\,3_{13}^{-1}\,3_{15}^{-1}$	71
1276	$1_7\,2_{12}^{-1}\,3_{11}\,3_{15}^{-1}$	67
1277	$1_{15}\,2_8\,2_{14}^{-1}\,2_{16}^{-1}$	65
1278	$1_9\,1_{13}^{-1}\,1_{15}\,2_{10}^{-1}\,2_{12}\,2_{16}^{-1}\,3_9\,3_{13}^{-1}$	67
1284	$1_9\,1_{11}\,1_{17}^{-1}\,2_{10}^{-1}\,3_9\,3_{13}^{-1}$	65
1288	$1_9^{-1}\,1_{11}^{-1}\,2_8\,2_{10}\,3_{15}^{-1}$	67
1305	$1_9\,1_{15}\,2_{10}^{-1}\,2_{14}^{-1}\,2_{16}^{-1}\,3_9$	71
1310	$1_{17}^{-1}\,2_8\,2_{14}^{-1}$	65
1311	$1_9\,1_{13}^{-1}\,1_{17}^{-1}\,2_{10}^{-1}\,2_{12}\,3_9\,3_{13}^{-1}$	67
1312	$1_{11}^{-1}\,1_{13}^{-1}\,1_{15}\,2_{12}\,2_{16}^{-1}\,3_9\,3_{13}^{-1}$	71
1317	$1_9^{-1}\,2_8\,2_{12}^{-1}\,3_{11}\,3_{15}^{-1}$	73
1346	$1_{11}^{-1}\,1_{13}^{-1}\,1_{17}^{-1}\,2_{12}\,3_9\,3_{13}^{-1}$	71
1348	$1_{11}^{-1}\,1_{15}\,2_{14}^{-1}\,2_{16}^{-1}\,3_9$	75
1349	$2_{10}^{-1}\,2_{12}^{-1}\,3_9\,3_{11}\,3_{15}^{-1}$	81
1356	$1_9\,1_{17}^{-1}\,2_{10}^{-1}\,2_{14}^{-1}\,3_9$	71
1360	$1_9\,1_{15}\,2_{12}\,2_{14}^{-1}\,2_{16}^{-1}\,3_{13}^{-1}\,4_{11}$	75
1387	$1_{11}^{-1}\,1_{17}^{-1}\,2_{14}^{-1}\,3_9$	75
1392	$3_{11}\,3_{13}^{-1}\,3_{15}^{-1}\,4_{11}$	85
1394	$1_{11}^{-1}\,1_{15}\,2_{10}\,2_{12}\,2_{14}^{-1}\,2_{16}^{-1}\,3_{13}^{-1}\,4_{11}$	79
1397	$1_9\,1_{17}^{-1}\,2_{12}\,2_{14}^{-1}\,3_{13}^{-1}\,4_{11}$	75
1398	$1_9\,1_{15}\,2_{12}\,2_{14}^{-1}\,2_{16}^{-1}\,4_{15}^{-1}$	79
1424	$1_{11}^{-1}\,1_{17}^{-1}\,2_{10}\,2_{12}\,2_{14}^{-1}\,3_{13}^{-1}\,4_{11}$	79
1431	$3_{11}\,3_{15}^{-1}\,4_{15}^{-1}$	89
1433	$1_{11}^{-1}\,1_{15}\,2_{10}\,2_{12}\,2_{14}^{-1}\,2_{16}^{-1}\,4_{15}^{-1}$	83
1436	$1_9\,1_{17}^{-1}\,2_{12}\,2_{14}^{-1}\,4_{15}^{-1}$	79
1452	$1_{11}^{-1}\,1_{17}^{-1}\,2_{10}\,2_{12}\,2_{14}^{-1}\,4_{15}^{-1}$	83

References

[1] Y. Cao, A. Okounkov, Y. Zhou, and Z. Zhou, *Stable envelopes for critical loci*, arXiv:2512.23929v1 (2025).

[2] V. Chari and A. Moura, *Characters and blocks for finite-dimensional representations of quantum affine algebras*, Int. Math. Res. Not. **2005** (2005), no. 5, 257–298; arXiv:math/0406151v3.

[3] V. Chari and A. Pressley, *Quantum affine algebras and their representations*, in *Representations of Groups*, CMS Conference Proceedings, vol. 16, Amer. Math. Soc., Providence, RI, 1995, pp. 59–78; arXiv:hep-th/9411145.

[4] B. Davison, *The critical CoHA of a quiver with potential*, Q. J. Math. **68** (2017), no. 2, 635–703; arXiv:1311.7172v5.

[5] B. Davison, D. Maulik, J. Schürmann, and B. Szendrői, *Purity for graded potentials and quantum cluster positivity*, Compos. Math. **151** (2015), 1913–1944; arXiv:1307.3379v2.

[6] P. Deligne, *Théorie de Hodge III*, Publ. Math. Inst. Hautes Études Sci. **44** (1974), 5–77.

[7] P. Deligne and N. Katz (eds.), *Groupes de monodromie en géométrie algébrique II (SGA 7 II)*, Lecture Notes in Mathematics, vol. 340, Springer, Berlin, 1973, Exposé XIV.

[8] P. Descombes, *Hyperbolic localization in Donaldson–Thomas theory*, arXiv:2506.22400v1 (2025).

[9] M. Finkelberg and A. Tsymbaliuk, *Multiplicative slices, relativistic Toda and shifted quantum affine algebras*, in *Representations and Nilpotent Orbits of Lie Algebraic Systems*, Progress in Mathematics, vol. 330, Birkhäuser, 2019, pp. 133–304; arXiv:1708.01795v6.

[10] H. Franzen and S. Mozgovoy, *Tautological bases of CoHA modules*, arXiv:2111.10114 (2021).

[11] E. Frenkel and E. Mukhin, *Combinatorics of q-characters of finite-dimensional representations of quantum affine algebras*, Comm. Math. Phys. **216** (2001), no. 1, 23–57; arXiv:math/9911112.

[12] E. Frenkel and N. Reshetikhin, *The q-characters of representations of quantum affine algebras and deformations of $\mathcal{W}$-algebras*, in *Recent Developments in Quantum Affine Algebras and Related Topics*, Contemporary Mathematics, vol. 248, Amer. Math. Soc., Providence, RI, 1999, pp. 163–205; arXiv:math/9810055.

[13] R. Fujita and D. Hernandez, *Monoidal Jantzen filtrations*, Adv. Math. **495** (2026), Paper No. 110963; doi:10.1016/j.aim.2026.110963.

[14] R. Fujita, D. Hernandez, S.-j. Oh, and H. Oya, *Isomorphisms among quantum Grothendieck rings and propagation of positivity*, J. Reine Angew. Math. **785** (2022), 117–185; arXiv:2101.07489v3.
[15] R. Fujita, D. Hernandez, S.-j. Oh, and H. Oya, *Isomorphisms among quantum Grothendieck rings and cluster algebras*, arXiv:2304.02562 (2023).
[16] R. Fujita and K. Murakami, *Deformed Cartan matrices and generalized preprojective algebras I: Finite type*, arXiv:2109.07985v3 (2022), doi:10.1093/imrn/rnac054.
[17] R. Fujita and F. Qin, *Freezing operators in representation theory of quantum loop algebras*, arXiv:2601.00687v3 (September 19, 2026).
[18] C. Geiss, D. Hernandez, and B. Leclerc, *Representations of shifted quantum affine algebras and cluster algebras II: Non simply-laced types and cluster expansions*, arXiv:2610.37184v1 (October 27, 2026).
[19] T. Hausel and F. Rodriguez-Villegas, *Cohomology of large semiprojective hyperkähler varieties*, arXiv:1309.4914 (2013).
[20] D. Hernandez, *Algebraic approach to q,t-characters*, Adv. Math. **187** (2004), no. 1, 1–52; arXiv:math/0212257v5.
[21] D. Hernandez, *Monomials of q and q,t-characters for non simply-laced quantum affinizations*, Math. Z. **250** (2005), no. 2, 443–473; arXiv:math/0404187v3.
[22] D. Hernandez, *The Kirillov–Reshetikhin conjecture and solutions of T-systems*, J. Reine Angew. Math. **596** (2006), 63–87; arXiv:math/0501202v3.
[23] D. Hernandez, *Representations and characters of quantum affine algebras at the crossroads between cluster categorification and quantum integrable models*, in *Proceedings of the International Congress of Mathematicians 2026*, vol. 4, 2026, pp. 532–551; doi:10.1137/25M1802614.
[24] D. Hernandez and B. Leclerc, *Cluster algebras and quantum affine algebras*, Duke Math. J. **154** (2010), no. 2, 265–341; arXiv:0903.1452.
[25] D. Hernandez and H. Oya, *Quantum Grothendieck ring isomorphisms, cluster algebras and Kazhdan–Lusztig algorithm*, Adv. Math. **347** (2019), 192–272; arXiv:1803.06754.
[26] D. Hernandez and T. Schedler, *q,t-Characters of fundamental representations of type F_4*, tables produced by the authors' computer program, 42 pp., `https://webusers.imj-prg.fr/~david.hernandez/f4monomials.pdf`.
[27] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), no. 2, 165–184; doi:10.1007/BF01390031.
[28] J.-R. Li and E. Mukhin, *Extended T-system of type G_2*, SIGMA **9** (2013), Paper No. 054, 28 pp.; arXiv:1208.4821.
[29] H. Nakajima, *Quiver varieties and finite-dimensional representations of quantum affine algebras*, J. Amer. Math. Soc. **14** (2001), no. 1, 145–238; arXiv:math/9912158.
[30] H. Nakajima, *Quiver varieties and t-analogs of q-characters of quantum affine algebras*, Ann. of Math. (2) **160** (2004), no. 3, 1057–1097; arXiv:math/0105173v2.
[31] A. Neguţ, *Category $\mathcal{O}$ for quantum loop algebras*, arXiv:2501.00724v3 (2025).
[32] A. Neguţ, *Quiver moduli and quantum loop algebras*, arXiv:2509.20815v2 (February 19, 2026).
[33] A. Neguţ and A. Tsymbaliuk, *Quantum loop groups and shuffle algebras via Lyndon words*, Adv. Math. **439** (2024), Paper No. 109482, 69 pp.; arXiv:2102.11269v3.
[34] T. Richarz, *Spaces with $\mathbb{G}_m$-action, hyperbolic localization and nearby cycles*, J. Algebraic Geom. **28** (2019), no. 2, 251–289; arXiv:1611.01669v3.
[35] M. Varagnolo and E. Vasserot, *Perverse sheaves and quantum Grothendieck rings*, in *Studies in Memory of Issai Schur*, Progress in Mathematics, vol. 210, Birkhäuser, Boston, MA, 2003, pp. 345–365; arXiv:math/0103182v3.
[36] M. Varagnolo and E. Vasserot, *Non-symmetric quantum loop groups and K-theory*, arXiv:2308.01809v3 (July 20, 2025).
[37] M. Varagnolo and E. Vasserot, *Representations of shifted affine quantum groups and Coulomb branches*, arXiv:2503.06262v3 (December 29, 2025).
[38] B. Webster, *Knot invariants and higher representation theory*, Mem. Amer. Math. Soc. **250** (2017), no. 1191; arXiv:1309.3796v3.

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