hexagon:2611.00321v1

OSTERWALDER–SCHRADER AXIOMS FOR SCALAR QED$_3$ IN THE HIGGS PHASE

github.com/marikgoldstein/scalar-qed-existence*

Abstract. We study scalar quantum electrodynamics in three dimensions (QED$_3$) in its Higgs phase: the three-dimensional lattice abelian Higgs model in unitary gauge, with the Villain action, on lattices whose spacing tends to zero with canonically scaled couplings. At each spacing we construct infinite-volume states, as limit points of the periodic states and as limits of states in finite boxes; they carry the lattice symmetries, reflection positivity and bounds uniform in the volume. For the smeared curvature and bounded functions of the density, the generating functionals converge along a subsequence of spacings to a translation-invariant limit functional with the continuity required for the Osterwalder–Schrader reconstruction. For the limits built on the box states, and on the periodic states along dyadic sequences of sides, the limit curvature field satisfies the axioms OS0–OS3, with invariance under the full Euclidean group. The cluster axiom OS4 is proved in a weak form, on almost every ergodic component. The tools are Gaussian domination of the Villain curvature, Ginibre inequalities and reflection positivity.

Contents

1. Introduction 1
2. Setting and the scheme 5
3. Gaussian domination of the Villain curvature 12
4. The estimate 14
5. Symmetries of the fine state 14
6. The passage from the estimate to the limit functional 15
7. Existence and continuity of the limit functional 16
8. Unbounded density functions and general normalisations 18
9. The remaining Osterwalder–Schrader axioms for the limit state 30
10. Correlation inequalities for the Villain fine model 43
11. The existence of the fine infinite-volume state 53
12. Rotation invariance 61
13. Rigidity of the Villain–Higgs Gibbs measures on $\varepsilon_N\mathbb{Z}^3$ 96
14. Diagonal reflection positivity of the torus limit points 131
15. Rotation covariance of the moments from reflection positivity 159
Appendix A. Probability measures on spaces of distributions 186
References 204

1. Introduction

1.1. **The model and the problem.** The abelian Higgs model, scalar quantum electrodynamics, is the Euclidean field theory of a real vector field A_μ and a complex scalar field ϕ on $\mathbb{R}^3$ with

*This work and its companion are machine generated and have been contributed by M. Goldstein and A. Guevara, based on the results described in https://github.com/marikgoldstein/scalar-qed-existence on October 10th, 2026.

formal action

$$(1.1)\quad S(A,\phi)=\int_{\mathbb{R}^3}\Big[\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}+\tfrac{1}{2}\big|(\partial_\mu+ieA_\mu)\phi\big|^2+\lambda\big(|\phi|^2-\rho_0^2\big)^2\Big]\,dx,\qquad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu,$$

with the charge $e>0$, the quartic coupling $\lambda>0$ and $\rho_0^2=m_H^2/(8\lambda)$, $m_H>0$. In three dimensions e^2 and λ have the dimension of a mass, and the only ultraviolet counterterm is a mass counterterm for the scalar field. The potential is minimal on the circle $|\phi|=\rho_0$. Writing $\phi=\rho e^{i\chi}$ with $\rho=|\phi|$ and absorbing the phase χ into the vector field by a gauge transformation (the unitary gauge), one obtains the integrand

$$\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}+\tfrac{1}{2}(\partial_\mu\rho)^2+\tfrac{1}{2}e^2\rho^2A_\mu A_\mu+\lambda\big(\rho^2-\rho_0^2\big)^2,$$

in which, to second order about $\rho=\rho_0$, the modulus ρ has the mass m_H and the vector field the mass $e\rho_0$. It is the simplest gauge theory in which the gauge field becomes massive through its coupling to a charged scalar field with a non-zero vacuum value [1, 12, 22]. In three Euclidean dimensions its action is the free-energy functional of Ginzburg and Landau for a superconductor [18]: the scalar field is the order parameter of the condensate, the gauge field is the vector potential of the magnetic field, and the functional integral is the partition function of the two with their thermal fluctuations [21]. The model has two phases. In the Coulomb phase the scalar field has no vacuum value and the gauge field is massless; this is the normal state, which a magnetic field penetrates. In the Higgs phase the scalar field has a non-zero vacuum value and the gauge field is massive; this is the superconducting state, in which the magnetic field is expelled (the Meissner effect). The two phases require different constructions, and we treat them in two papers: the present paper is devoted to the Higgs phase, and the companion paper [34] to the Coulomb phase. One-loop perturbation theory predicts a fluctuation-induced first-order transition at very small quartic coupling [9, 21]; no claim about the phase diagram is made here.

The gauge field is non-compact. The lattice gauge action is of Villain's form [33] with a *closed* integer 2-form, that is, the compact Villain model with every monopole removed. On a finite lattice with free boundary conditions a closed integer 2-form is exact, and for observables of integer charge this is the model with a real-valued gauge field on the bonds and a quadratic gauge action [5]; on a torus the sum over the closed forms includes the flux classes. The distinction matters in three dimensions: in the compact pure gauge theory monopoles produce a mass gap and confinement at every coupling [20], while the non-compact pure gauge theory is the free field.

In three dimensions the model is superrenormalisable, and its lattice approximations are among the basic examples of a gauge theory with matter [6, 7, 8, 2, 3, 4, 24, 25, 31]. The problem treated here is due to King [24, 25]: with the renormalisation-group method of Bałaban [2, 3, 4] he constructed the continuum limit of the three-dimensional model in finite volume and then its infinite-volume limit, established its translation invariance and reflection positivity, and addressed its rotation invariance through a comparison of two rotated lattices. The Osterwalder–Schrader axioms for his state, rotation invariance included, are established in the companion paper [34] by a different route; the present paper treats the Higgs phase. The framework of the present paper is that of Brydges, Fröhlich and Seiler [6, 7, 8] for the lattice abelian Higgs model: the gauge-invariant fields, and the axioms obtained through lattice approximations; these authors completed the two-dimensional case. The method is reflection positivity and Gaussian domination [14, 13], together with Ginibre's inequalities [17]. We consider the model on the cubic lattices $\varepsilon_N\mathbb{Z}^3$ of spacing $\varepsilon_N=a_0 5^{-N}$, $N=N_0,N_0+1,\dots$ (the *levels*; $a_0>0$ is a fixed length and $N_0\geq 0$ a fixed integer), in unitary gauge and with the Villain form of the gauge action [33]. The variables are the radial part $\rho_x=\rho_{0,0}+\tilde\rho_x\geq 0$ of the scalar field at the sites, written about a vacuum value $\rho_{0,0}>0$, a gauge field B_b on the bonds, and a closed 2-form v_p on the plaquettes with values in $(2\pi/\gamma_N)\mathbb{Z}$. The gauge-invariant fields are the scalar density and the

Villain curvature $\mathfrak{f}_p = (dB)_p + v_p$. The Boltzmann weight is a product of a site factor, which contains a quartic potential and a bare mass counterterm δm_B, of a hopping factor on the bonds, and of the Gaussian factor $e^{-\mathfrak{f}_p^2/2}$ on the plaquettes; it is written out in Section 2. The couplings are scaled with the spacing according to their canonical dimensions: the charge is $\gamma_N = \varepsilon_N^{1/2} e$, the quartic coupling is $\lambda_0 = \varepsilon_N \lambda$, and $\rho_{0,0}^2 = \varepsilon_N m_H^2/(8\lambda)$, with fixed $e, \lambda, m_H > 0$. Every result of this paper holds for arbitrary positive values of these lattice couplings at each level N and for every real δm_B (for the coefficient of the hopping factor see Section 2 and Lemma 2.7); the scaling above is the case of interest.

The problem is to construct limits of the gauge-invariant fields as $N \to \infty$ and to verify for them the axioms of Osterwalder and Schrader [28, 29]: regularity (OS0), Euclidean invariance (OS1), reflection positivity (OS2), symmetry (OS3) and clustering (OS4). The observables are the smeared fields

$$\Phi_N(\sigma) = \varepsilon_N^{3/2} \sum_p f_{\mu\nu(p)}(x_p)\, \mathfrak{f}_p + \varepsilon_N^3 \sum_z h(z)\, g_{\mathrm{d}}(\tilde{\rho}_z), \qquad \sigma = (f, h),$$

where f is a real C_c^∞ 2-form on $\mathbb{R}^3$, $h \in C_c^\infty(\mathbb{R}^3; \mathbb{R})$ and g_{d} is a bounded Borel function (Definition 2.5), and the objects of study are the generating functionals $Z_N(\sigma) = \omega_N(e^{i\Phi_N(\sigma)})$ in an infinite-volume state ω_N of level N. Translation invariance and reflection positivity about the coordinate planes are present on the lattice and pass to the limit. Invariance under rotations is not a symmetry of any of the lattice theories and has to be recovered in the limit. The two main points of the paper are the construction of the infinite-volume states at fixed spacing without an expansion, and the Gaussian domination of the Villain curvature, which holds for all values of the couplings and gives the bounds, uniform in the volume and in the spacing, on which the continuum limit and its regularity rest.

1.2. **Main results.** Two families of infinite-volume states occur: limit points ω_N of the states of the periodic lattices (tori), and the limits ω_N^{box} of the states of finite boxes outside of which the matter field is set to zero (the *killed-box* states, Definition 11.9). We write S and S^{box} for limit functionals built on them.

(1) Infinite-volume states. For every level N, along a subsequence of any prescribed sequence of sides, the torus states converge to a state ω_N given by a Borel probability measure on the configurations of the gauge-invariant variables. The state ω_N is invariant under the lattice translations, the reflections in the lattice planes $\{x_j = c\}$ and the cubic group; it is reflection positive about these planes; and it satisfies the bounds of the torus states that are uniform in the volume (Theorem 11.5). The killed-box states converge along every exhausting sequence of boxes, without the passage to a subsequence, to a state ω_N^{box} with the same properties, which lies below every ω_N on a cone of observables (Theorem 11.11, Lemma 12.20). We do not know whether the two families coincide in general.

(2) Gaussian domination and correlation inequalities. On every torus the Villain curvature is dominated by the Gaussian field with the identity covariance, $\langle \exp \sum_p a_p \mathfrak{f}_p \rangle_L^{\mathrm{fine}} \le \exp\left(\frac{1}{2}\sum_p a_p^2\right)$, with the constant one for all values of the couplings (Theorem 3.1). The torus states and the killed-box states satisfy Ginibre's inequalities on a cone of observables, with the monotonicity in the couplings and in the box that follows from them (Theorem 10.8).

(3) The limit functional. There are a sequence $N_1 < N_2 < \cdots$ and a linear functional S on the algebra $\mathcal{A}$ generated by the symbols e_σ such that $Z_{N_m}(\sigma) \to S(e_\sigma)$ for every test pair σ. The functional S is invariant under the translations of $\mathbb{R}^3$ and under the cubic group, and $t \mapsto S(\Theta a \cdot \tau_t b)$ is continuous on $[0, \infty)$ for a, b supported in the upper half space $\{x_3 > 0\}$, where

Θ is the reflection in the plane $\{x_3 = 0\}$ and τ_t the translation by te_3 (Theorem 7.1). The same holds for the killed-box family (Lemma 12.20).

(4) The axioms OS0–OS3 without the rotations. For every limit functional, the generating functional of the curvature extends continuously to the Schwartz 2-forms and is the characteristic functional of a unique Borel probability measure μ on a space of distributions (Theorem 9.13). Its moments, the Schwinger functions S_n, are tempered distributions with $|S_n(f_1, \dots, f_n)| \leq e^n (n!)^{1/2} \prod_i R(f_i)$ for real f_i, R being the seminorm of Definition 4.1; this gives the growth condition of [29] (Theorem 9.4). The functional S is invariant under all translations, under the reflections in the coordinate planes and under the cubic group about every point (Proposition 9.6), it is reflection positive about every coordinate plane (Proposition 9.9), and the Schwinger functions are symmetric (Proposition 9.12).

(5) Rotation invariance. Let S^{box} be a limit functional built on the killed-box states. For the curvature, S^{box} is invariant under the orthogonal group $\mathrm{O}(3)$, hence under the Euclidean group; its measure is invariant under the Euclidean group; and the axioms OS0–OS3 hold in full (Theorem 12.2). For a bounded density function g_{d}, the joint functional of the curvature and the density is invariant under the Euclidean group, coincident points included (Theorem 12.3). The same statements hold for every limit functional S built on torus limit points ω_N whose sides run along a dyadic sequence $L_n = 2^n L_*$ (Definition 14.1, Theorem 12.4). No rotation statement is made for torus limit points along other sequences of sides.

(6) Clustering. The measure μ of the limit curvature has an ergodic decomposition in which almost every component is invariant and ergodic under the translations of $\mathbb{R}^3$, is reflection positive about the coordinate planes, clusters in the mean, and has a unique vacuum in the reconstruction along each coordinate direction: the weak form of OS4 (Theorem 9.15).

1.3. **Methods.** *Uniform bounds and the limit.* The Gaussian domination of Theorem 3.1 bounds the smeared curvature uniformly in the volume and in the level, in the normalisation $\varepsilon_N^{3/2}$ in which the plaquette action is a Riemann sum of the Maxwell action; the density term is bounded because g_{d} is. This gives an estimate of $Z_N(\sigma) - Z_N(\sigma')$ by a seminorm of $\sigma - \sigma'$ that does not depend on N (Section 4), and a diagonal argument with the lattice symmetries gives the limit functional (Sections 5–7). The exponential moments make the generating functional of the curvature entire, which gives the Schwinger functions and their bounds; the measure is obtained from the Bochner–Minlos theorem [19] and the measure theory of Appendix A.

Infinite volume at fixed spacing. The torus states are reflection positive about the coordinate planes (Theorem 9.7). From this we derive bounds on the local moments of the scalar density that are uniform in the volume (Lemma 11.2); with the Gaussian domination they give tightness, and every limit point inherits the symmetries and the reflection positivity (Section 11). For the killed boxes, Ginibre's inequalities (Section 10) make the expectations of a cone of observables monotone in the box, and the values on the cone determine the state; this gives the convergence of the box states.

Rotations. Reflection positivity about sufficiently many planes forces rotation covariance: this is the content of Theorem 15.1, whose proof (Section 15) proceeds through an analytic continuation in the angle of a planar rotation, the positivity of the continued reflection pairing, a convexity argument and an arithmetic lemma. In proving the rotation invariance we implement the original strategy of [35]; in a subsequent manuscript we shall provide a different strategy, adapting the proof from the companion paper [34] (Remark 12.1). On a cubic torus only the coordinate reflections have all their walls pointwise fixed, and no torus carries the nine families with pointwise-fixed walls (Lemma 12.13). A cubic box has nine symmetry planes, three coordinate and six diagonal ones, and its killed-box state is reflection positive about each of them

(Theorem 12.16); the box limit is then reflection positive about every lattice plane of the nine families (Lemma 12.19), and the limit theory built on it satisfies the hypotheses of Theorem 15.1. This gives the covariance away from coincident points; the coincident points are covered by the closedness of the limit curvature and a theorem on bounded multilinear forms (Theorems 12.26 and 12.34; Section 12). For the torus limit points the reflection positivity about the diagonal planes is proved in Section 14: the proof runs through an adapted torus with a family of diagonal reflection walls, a perturbed law on it that is reflection positive across every wall, a convexity inequality for the perturbed free energy, a surface bound that removes the perturbation in the limit, and the rigidity theorem of Section 13, applied to a comparison state. The dyadic sides enter here.

What is not addressed. We do not address the non-degeneracy of the limit (that S is not the state of a constant field) or its non-triviality; the non-triviality will be the subject of a subsequent manuscript. We do not prove the uniqueness of the infinite-volume state at fixed spacing, that is, the existence of the limit (2.2); for continuous observables it is equivalent to the agreement of all torus limit points (Corollary 11.13). We do not address the exponential clustering of the limit, nor unbounded density functions beyond the cases of Section 8. No renormalisation-group analysis is used in this paper.

1.4. **Related work.** The axioms are those of [28, 29]; the Bochner–Minlos theorem is used in the form of [19]. Reflection positivity on the lattice, Gaussian domination and the estimates derived from them go back to [14, 13]; the correlation inequalities of Section 10 are inequalities of Ginibre's type [17] for the present model. The lattice abelian Higgs model and its continuum limit in two dimensions were studied in [6, 7, 8]; in three dimensions the ultraviolet problem was treated by renormalisation-group methods in [2, 3, 4, 24, 25]. Gibbs measures and their symmetries are treated in [16].

The companion paper [34] establishes the Osterwalder–Schrader axioms for the same model in the Coulomb phase. Its proof of rotation invariance is of a different kind and is original to that paper: the continuum limit is built on two lattices in rotated position and the two are compared on periodic tori carrying a wall in the scalar mass; the walled-torus families are shown to have a common continuum limit, and the infinite-volume state of King's construction [24, 25] is identified as their lower envelope by a closing estimate resting on heavy-shell bounds, the shell mass being carried inside the covariance. The invariance is proved for the curvature sector on all of King's coupling range, and for Wick-square sources on a tube below a threshold.

Added note. For further information on the efforts leading to the production of this manuscript we refer to the GitHub repository `https://github.com/marikgoldstein/scalar-qed-existence`.

1.5. **Organisation.** Sections 2–7 construct the limit functional; Section 8 extends it to unbounded density functions and other normalisations, in the cases verified there. Section 9 proves the axioms without the rotations and the statements on clustering. Sections 10 and 11 contain the correlation inequalities and the infinite-volume states, which the preceding sections use through Definition 2.3. Sections 12–15 prove the rotation invariance. Appendix A contains the measure theory.

2. Setting and the scheme

Throughout, $a_0 > 0$ is a fixed length, $N_0 \geq 0$ is a fixed integer, and the *levels* are the integers $N \geq N_0$. The lattice $\varepsilon_N \mathbb{Z}^3$ of spacing $\varepsilon_N = a_0 5^{-N}$ is called the *fine lattice* of level N; the model defined on it and on its tori is the *fine model*, and its states are the *fine states*. The Villain fine model at level N lives on periodic lattices $\tilde{T}_L = (\varepsilon_N \mathbb{Z}/L\mathbb{Z})^3$ of spacing $\varepsilon_N = a_0 5^{-N}$,

$L/2 \in \varepsilon_N\mathbb{Z}$, with the variables $\tilde\rho_x \in [-\rho_{0,0},\infty)$ (sites), $B_b \in (-\pi/\gamma_N, \pi/\gamma_N]$ (bonds, $B_{-b} = -B_b$), $v_p \in (2\pi/\gamma_N)\mathbb{Z}$ (plaquettes, $v_{-p} = -v_p$) and the weight

$$W = \prod_x g_{\mathrm{ct}}(\tilde\rho_x) \cdot \prod_b h(\tilde\rho_{b_-}, \tilde\rho_{b_+}, B_b) \cdot \prod_p w\big((dB)_p + v_p\big), \qquad w(f) = e^{-f^2/2},$$

where $g_{\mathrm{ct}}(\tilde\rho) := g(\tilde\rho)\, e^{-\frac12\delta m_B(\rho_{0,0}+\tilde\rho)^2}$ is the site factor (its second factor is the bare mass counterterm, explained below), with $g(\tilde\rho) = (\rho_{0,0}+\tilde\rho)e^{-V(\tilde\rho)}$, $V(\tilde\rho) = \lambda_0(\rho^2 - \rho_{0,0}^2)^2$ and $\rho = \rho_{0,0} + \tilde\rho$; the hopping factor is

$$h(r, r', \vartheta) = \exp\big(-\tfrac12\kappa_N |e^{i\gamma_N\vartheta}(\rho_{0,0}+r') - (\rho_{0,0}+r)|^2\big),$$

and the sum over the Villain variables runs over all closed v ($dv = 0$ on every cube), every flux class included. Here $(dB)_p := \sum_{b\in\partial p} B_b$ and $(dv)_c := \sum_{p\in\partial c} v_p$ with the signs of the boundaries.

Here $\gamma_N := \varepsilon_N^{1/2} e$ is the phase constant of the hopping factor, $\lambda_0 := \varepsilon_N\lambda$ the quartic coupling and $\rho_{0,0} > 0$ the vacuum value, $\rho_{0,0}^2 := \varepsilon_N\rho_0^2 = \varepsilon_N m_H^2/(8\lambda)$, with $\kappa_N = 1$. The parameters $e, \lambda, m_H > 0$ are fixed. In angle units $\theta_b = \gamma_N B_b$ the plaquette weight reads $e^{-((d\theta)_p + \gamma_N v_p)^2/(2\gamma_N^2)}$, i.e. the Villain weight with the inverse coupling $1/\gamma_N^2$. The spacing ε_N and the phase constant γ_N are different quantities ($\gamma_N^2/\varepsilon_N = e^2$); the Riemann normalisations below use ε_N only.

The factor $e^{-\frac12\delta m_B\rho_x^2}$ of g_{ct}, $\rho_x := \rho_{0,0} + \tilde\rho_x$, is the bare mass counterterm, with $\delta m_B \in \mathbb{R}$ one real parameter of either sign, which may depend on N (all constants below that involve the site factor are at fixed N, Remark 2.1); every statement below is proved for every real δm_B. The site integral and all its Gaussian moments stay finite because the quartic dominates: $\lambda_0(\rho^2 - \rho_{0,0}^2)^2 \ge \frac12\lambda_0\rho^4 - \lambda_0\rho_{0,0}^4$, so

$$g(\tilde\rho)\, e^{-\frac12\delta m_B\rho^2}\, e^{q\rho^2} \le \rho\, e^{\lambda_0\rho_{0,0}^4}\, e^{-\frac12\lambda_0\rho^4 + (q + \frac12|\delta m_B|)\rho^2} \qquad (\rho = \rho_{0,0}+\tilde\rho \ge 0,\ q \ge 0), \tag{2.1}$$

and the right side is integrable over $\tilde\rho \in [-\rho_{0,0},\infty)$ (Remark 2.2).

We write $\langle\cdot\rangle_L^{\mathrm{fine}}$ for the normalised expectation with respect to the weight W, the Lebesgue measure on the variables $\tilde\rho_x$ and B_b and the sum over the closed v (the fine torus state), and $\mathfrak{f}_p = (dB)_p + v_p$ for the Villain curvature. One would like to define the infinite-volume state of level N as the limit of the torus states,

$$\omega_N(F) = \lim_{L\to\infty} \langle F\rangle_L^{\mathrm{fine}} \qquad \text{for every bounded local gauge-invariant } F \tag{2.2}$$

($N \ge N_0$). We do not assume that this limit exists. The fine infinite-volume state is defined instead as a weak limit point of the torus states (Definition 2.3), which exists for all positive couplings by Theorem 11.5 of Section 11; when the limit (2.2) exists, ω_N is that limit. We have set $\kappa_N = 1$; everything below holds for every $\kappa_N \ge 0$, $\lambda_0 > 0$, $\rho_{0,0} > 0$, every real δm_B, every $\gamma_N > 0$ and every ε_N: the estimates use no property of the couplings, and the symmetry and compactness arguments use only the properties of the limit point listed in Theorem 11.5. The one exception concerns κ_N: Sections 13 and 14 are written for $\kappa_N = 1$. By Lemma 2.7 their main results (Theorems 13.1 and 14.3 and Corollary 14.35) hold for every $\kappa_N > 0$, and so do the statements that rest on them, namely the rotation statements for the torus limit points along dyadic sequences of sides (Theorem 12.4, and Corollary 12.38(b) for the limits of its first sentence); for all these statements the value $\kappa_N = 0$ is excluded.

Remark 2.1 (the coefficients). With the exception stated just before this remark (for which see Lemma 2.7), every statement of this paper holds for arbitrary positive values of the coefficients γ_N, λ_0, $\rho_{0,0}$, for every $\kappa_N \ge 0$ and for every real δm_B (Remark 2.2): the coefficients enter the proofs as parameters only, and a constant depends on them only where this is stated (the constants of Lemma 11.2 and the finiteness majorants inside the proofs of Theorem 3.1, Lemma 9.2 and

Theorem 9.7, all at fixed N). In particular the constant of Theorem 3.1, which is uniform in N, is exactly 1 in the lattice normalisation used there (the curvature $\mathfrak{f}_p = (dB)_p + v_p$ with the plaquette weight $e^{-\mathfrak{f}_p^2/2}$), for every value of the coefficients.

Remark 2.2 (the bare mass counterterm). For every real δm_B the site factor $g_{\mathrm{ct}}(\tilde\rho_x) = g(\tilde\rho_x)e^{-\frac12\delta m_B\rho_x^2}$ has three properties: it is a positive function of the one site variable, the same at every site; its integral against $e^{q\rho^2}$ is finite for every $q \geq 0$, by (2.1); and on $\tilde\rho \in [0,1]$ it is bounded below by $\rho_{0,0}e^{-\lambda_0(2\rho_{0,0}+1)^2}e^{-\frac12|\delta m_B|(\rho_{0,0}+1)^2} > 0$. Every argument of this paper that involves the site factor uses of it only these three properties. This includes the majorant $(\int g)^{|\mathcal{X}|}$ of Lemma 9.2.

In the proof of Theorem 7.1 the site factor enters as follows. The proof of Theorem 3.1 integrates it into $\Omega(m)$ and uses, in Step 4, only $\Omega(m) \geq 0$ and $\Omega(m) < \infty$; Steps 1–3 concern the plaquette sum and the gauge integral, in which no site factor appears. Lemma 5.1 uses that the site factor is the same at every site. Proposition 4.2 uses Theorem 3.1 and a pointwise bound, and not the site factor, and (2.4) concerns ι_N alone. The existence of the limit point and its properties (a)–(e) (Theorem 11.5) rest on Theorem 3.1; on the lattice symmetries, which use that the site factor is the same at every site; on the reflection positivity of the torus states (Theorem 9.7), whose proof uses the positivity of the site factor, its invariance under the reflections and its integrability; and on the uniform density moments of Lemma 11.2, whose proof uses the finiteness of the site integrals and a positive lower bound of the site factor on $\tilde\rho \in [0,1]$. The site integrals G_q^{ct} and the positive lower bound c_g^{ct} of the full site factor g_{ct} on $\tilde\rho \in [0,1]$ are the constants of Lemma 11.2 (Section 11; see also Remark 11.4), and Lemma 11.2 and all its consequences hold with these constants. The counterterm therefore changes no statement: every statement holds for every real δm_B, and the counterterm enters only the constants that depend on the coefficients (Remark 2.1).

The later sections use the site factor g_{ct} in the same sense. It enters the measure $\mu^\circ := g_{\mathrm{ct}}\, e^{-3\kappa_N\rho^2}\, d\rho$ and the weight W_Λ of Sections 10 and 11 (Definition 10.2); the weight restated in Subsection 12.2; the function $\widetilde{g} := g_{\mathrm{ct}}\, e^{-3\kappa_N\rho^2}$ and the site integral $G_0^{\mathrm{ct}} := \int g_{\mathrm{ct}}$ of Section 12; and the proof of Theorem 9.7, which uses that g_{ct} is integrable. Each of those arguments uses only the positivity, the position independence (which gives the reflection invariance) and the exponential moments of the site measure, which g_{ct} has by (2.1).

Definition 2.3 (the fine infinite-volume state: a limit point of the torus states)**.** Let $\Omega_N := \mathbb{R}^{\mathcal{P}_N} \times [-\rho_{0,0}, \infty)^{\mathcal{X}_N}$ be the configuration space of the gauge-invariant variables $(\mathfrak{f}_p)_{p\in\mathcal{P}_N}$, $(\tilde\rho_x)_{x\in\mathcal{X}_N}$ of the infinite lattice $\varepsilon_N\mathbb{Z}^3$ ($\mathcal{P}_N$ its plaquettes, one orientation chosen for each, $\mathcal{X}_N$ its sites; Subsection 11.1), with the product topology and its Borel σ-algebra. A function on Ω_N is local if it depends on the variables of finitely many cells; the bounded Borel local functions are the bounded local gauge-invariant functions of (2.2). For each $N \geq N_0$ fix a sequence of sides $L_n = L_n^{(N)} \to \infty$ with $L_n/2 \in \varepsilon_N\mathbb{Z}$ and a Borel probability measure μ_N on Ω_N such that the finite-dimensional laws of the gauge-invariant variables under the torus states $\langle\cdot\rangle_{L_n}^{\mathrm{fine}}$ converge weakly to those of μ_N, and put

$$\begin{aligned} &\omega_N(F) := \int F\, d\mu_N \quad \text{for every bounded Borel local } F; \\ &\text{then} \quad \omega_N(F) = \lim_{n\to\infty} \langle F\rangle_{L_n}^{\mathrm{fine}} \quad \text{for every bounded continuous local } F. \end{aligned} \tag{2.3}$$

Such a pair $((L_n), \mu_N)$ exists for every N, every choice of positive couplings and any real δm_B, along a subsequence of any prescribed sequence of sides (Theorem 11.5, whose proof uses Theorem 3.1, the reflection positivity of the torus states and a bound on the local moments of the density). The sides L_n are taken along a dyadic sequence (Definition 14.1); by Theorem 11.5

a limit point exists along a subsequence of it, and every statement below holds for each such limit point (Remark 2.4). Throughout, ω_N is the state (2.3) of such a limit point, and "$L \to \infty$" for expectations at level N means $n \to \infty$ along (L_n).

The properties of ω_N used below are those of Theorem 11.5: (a) the limit relation (2.3) for bounded continuous local F, for bounded Borel local F with μ_N-null discontinuity set, and for continuous local F dominated by $Ce^{\sum_x q_x \rho_x^2}$ ($\rho_x = \rho_{0,0} + \tilde{\rho}_x$); (b) ω_N is a state on the bounded Borel local functions ($\omega_N(1) = 1$, positive, $|\omega_N(F)| \le \|F\|_\infty$), with the Gaussian domination (3.1) and the uniform density moment bounds of the torus states; (c) μ_N is invariant under the translations by $\varepsilon_N \mathbb{Z}^3$, the reflections in the planes $\{x_j = c\}$, $c \in \frac{1}{2}\varepsilon_N \mathbb{Z}$, the cubic group about every site and the charge conjugation $(\mathfrak{f}, \tilde{\rho}) \mapsto (-\mathfrak{f}, \tilde{\rho})$; (d) ω_N is reflection positive about every such plane, on the bounded Borel functions of cells in the closed half space and on their $L^2(\mu_N)$ closure; (e) inequalities between torus expectations of bounded continuous local functions pass to ω_N, and $\int F \, d\mu_N \le \liminf_n \langle F \rangle_{L_n}^{\mathrm{fine}}$ for continuous local $F \ge 0$.

If the limit (2.2) exists for every bounded continuous local F, over all sides or over a cofinal class of sides, then all limit points along sides of that class coincide and ω_N, with (L_n) taken in the class, is that limit (Corollary 11.13, whose limits may be taken over any cofinal class of sides). Limit points in a wider sense, subnet limits of the torus states on all bounded Borel local functions, are treated in Proposition 7.2 and Remark 7.3. The fine infinite-volume state is used on nothing more than the functions of the curvature and the density.

Remark 2.4 (the dyadic sides). The sides of Definition 2.3 are taken along a dyadic sequence (Definition 14.1) because the reflection positivity of ω_N at the six diagonal site planes through 0 (Theorem 14.3), on which the rotation statements of Section 12 rest, is proved for the limit points along such sequences. Theorem 11.5 provides a limit point along a subsequence of every prescribed sequence of sides, in particular of every dyadic sequence. The choice of dyadic sides is not a hypothesis, but it is a restriction of the result: the rotation statements of Section 12 for the torus states are stated for the limit points along such sequences only, and no rotation statement is made in this paper for torus limit points along other sequences of sides.

Test pairs and the algebra. A *test pair* is $\sigma = (f, h)$ with f a real C_c^∞ 2-form on $\mathbb{R}^3$ and $h \in C_c^\infty(\mathbb{R}^3; \mathbb{R})$; $\mathcal{T}$ is their real vector space, τ_t the translation by te_3, θ the time reflection: $\theta\sigma$ is the push-forward of $\sigma = (f, h)$ under $x \mapsto (x_1, x_2, -x_3)$, the 2-form transforming as a 2-form (its components with one index 3 change sign). The algebra $\mathcal{A} = \mathbb{C}[\mathcal{T}]$ is the group algebra with basis e_σ, $e_\sigma e_{\sigma'} = e_{\sigma+\sigma'}$, $e_\sigma^* = e_{-\sigma}$, $\Theta e_\sigma = e_{-\theta\sigma}$ (Θ antilinear), $\tau_t e_\sigma = e_{\tau_t \sigma}$, and $\mathcal{A}_+$ the span of the e_σ supported in $\{x_3 > 0\}$.

Definition 2.5 (the scheme)**.** The *scheme* realises e_σ on the fine lattice as $\iota_N(e_\sigma) = e^{i\Phi_N(\sigma)}$, extended linearly to $\mathcal{A}$, where

$$\Phi_N(\sigma) = c_N \sum_p f_{\mu\nu(p)}(x_p)\, \mathfrak{f}_p + c'_N \sum_z h(z)\, \mathfrak{w}_z, \qquad \mathfrak{w}_z = g_{\mathrm{d}}(\tilde{\rho}_z),$$

with real numbers c_N, c'_N (the lattice normalisations) and a real Borel function g_{d} of the scalar density (the density function). We fix them as follows, and every statement below refers to this choice, the *pinned scheme*, except in Section 8, which treats general normalisations and density functions:

$$c_N = \varepsilon_N^{3/2}, \qquad c'_N = \varepsilon_N^3, \qquad g_{\mathrm{d}} \text{ a fixed bounded real Borel function on } \mathbb{R}, \quad \|g_{\mathrm{d}}\|_\infty < \infty.$$

Here x_p is the centre of the plaquette p and $f_{\mu\nu(p)}$ the component of f along the orientation of p; the sums run over the plaquettes and sites of the fine lattice (finitely many terms, since σ has compact support), so $\Phi_N(\sigma)$ is a real local function of finitely many gauge-invariant variables,

additive in σ, unbounded in the curvature part and bounded in the density part. We write $Z_N(\sigma) := \omega_N(e^{i\Phi_N(\sigma)})$ and $M(\sum_j c_j e_{\sigma_j}) := \sum_j |c_j|$; since the exponentials are unimodular,

$$|\iota_N(a)| \le M(a) \quad \text{pointwise, for every } a \in \mathcal{A} \text{ and } N. \tag{2.4}$$

Remark 2.6. (i) Definition 2.5 makes sense for every real Borel g_{d} and all real c_N, c'_N; outside Section 8 only the pinned scheme is used. (ii) $c_N = \varepsilon_N^{3/2}$ is the Riemann normalisation of the curvature: the plaquette weight $e^{-\mathfrak{f}_p^2/2}$ corresponds to the continuum Maxwell term when $\mathfrak{f}_p = \varepsilon_N^{3/2} F_{\mu\nu}(x_p)(1 + O(\varepsilon_N))$, so that $\Phi_N(f, 0) \to \int f_{\mu\nu} F_{\mu\nu}$. (iii) The choice of a bounded g_{d} is discussed in Remark 7.5: it makes the density part of the estimate trivial.

Lemma 2.7 (rescaling the hopping coefficient)**.** *Fix a level N and coefficients $\kappa_N > 0$, $\lambda_0 > 0$, $\rho_{0,0} > 0$, $\gamma_N > 0$ and $\delta m_B \in \mathbb{R}$, and put*

$$\lambda_0^\star := \kappa_N^{-2}\lambda_0, \qquad \rho_{0,0}^\star := \kappa_N^{1/2}\rho_{0,0}, \qquad \delta m_B^\star := \kappa_N^{-1}\delta m_B. \tag{2.5}$$

The given model *is the model of this section at level N with the coefficients $(\kappa_N, \lambda_0, \rho_{0,0}, \delta m_B, \gamma_N)$; the* rescaled model *is the same model with the coefficients $(1, \lambda_0^\star, \rho_{0,0}^\star, \delta m_B^\star, \gamma_N)$, so that its hopping coefficient is 1. We mark the objects of the rescaled model with a star: its site variables are $\tilde\rho_x^\star \in [-\rho_{0,0}^\star, \infty)$, with $\rho_x^\star := \rho_{0,0}^\star + \tilde\rho_x^\star$, its site factor is $g_{\mathrm{ct}}^\star$ and its hopping factor is $h^\star$. Let r be the map that multiplies every site variable by $\kappa_N^{-1/2}$ and leaves the other variables unchanged,*

$$r(\tilde\rho^\star, B, v) := (\kappa_N^{-1/2}\tilde\rho^\star, B, v) \quad \text{on the lattice variables,}$$
$$r(\mathfrak{f}, \tilde\rho^\star) := (\mathfrak{f}, \kappa_N^{-1/2}\tilde\rho^\star) \quad \text{on the gauge-invariant variables.}$$

Since $\kappa_N^{-1/2}\rho_{0,0}^\star = \rho_{0,0}$, it is a homeomorphism of the configuration space of the rescaled model onto that of the given model, on every torus, in every box and on $\varepsilon_N\mathbb{Z}^3$; and $\tilde\rho_x \circ r = \kappa_N^{-1/2}\tilde\rho_x^\star$, $\rho_x \circ r = \kappa_N^{-1/2}\rho_x^\star$, while B_b, v_p and $\mathfrak{f}_p = (dB)_p + v_p$ are unchanged. Then:

(a) *(*The weight.*) For all $s, s' \in [-\rho_{0,0}^\star, \infty)$ and every real ϑ,*

$$g_{\mathrm{ct}}(\kappa_N^{-1/2}s) = \kappa_N^{-1/2}\, g_{\mathrm{ct}}^\star(s), \qquad h(\kappa_N^{-1/2}s, \kappa_N^{-1/2}s', \vartheta) = h^\star(s, s', \vartheta),$$

and $(\kappa_N\rho_x^2) \circ r = (\rho_x^\star)^2$, $(\kappa_N\rho_{b_-}\rho_{b_+}) \circ r = \rho_{b_-}^\star\rho_{b_+}^\star$. The plaquette weight, the ranges of B_b and v_p and the constraint $dv = 0$ are the same in the two models.

(b) *(*Finite volume.*) Let $\langle\cdot\rangle_L^\star$ and $\langle\cdot\rangle_\Lambda^\star$ be the torus states and the killed-box states (Definition 11.9) of the rescaled model. For every torus $\tilde T_L$, every box Λ and every bounded Borel function F of the lattice variables,*

$$\langle F\rangle_L^{\mathrm{fine}} = \langle F \circ r\rangle_L^\star, \qquad \langle F\rangle_\Lambda = \langle F \circ r\rangle_\Lambda^\star.$$

(c) *(*Limit points.*) A pair $((L_n), \mu_N^\star)$ is a limit point of the torus states of the rescaled model in the sense of Definition 2.3 if and only if $((L_n), r_*\mu_N^\star)$ is one for the given model. Thus the fine infinite-volume states of the given model are the states*

$$\omega_N(F) = \omega_N^\star(F \circ r) \qquad (F \text{ bounded Borel local}),$$

with $\omega_N^\star$ a fine infinite-volume state of the rescaled model along the same sides; in particular ω_N is a limit point along a dyadic sequence (Definition 14.1) if and only if $\omega_N^\star$ is. In the same way the measures of the killed-box limits of Theorem 11.11 satisfy $\mu_N^{\mathrm{box}} = r_*\mu_N^{\star,\mathrm{box}}$.

(d) *(*Symmetries and reflections.*)* *The map r commutes with the action of every element of the group $\mathcal{G}_N$ of lattice symmetries and with the charge conjugation (Subsection 11.1). For every set of cells, $F \mapsto F \circ r$ maps the bounded Borel functions of the gauge-invariant variables of these cells in the given model onto those in the rescaled model, and for $\mu_N = r_* \mu_N^\star$ it is an isometry of $L^2(\mu_N)$ onto $L^2(\mu_N^\star)$. Consequently, for ω_N and $\omega_N^\star$ as in (c), for every reflection $\theta \in \mathcal{G}_N$ and each closed half space bounded by its plane, each of the following four statements holds for $\omega = \omega_N$ if and only if it holds for $\omega = \omega_N^\star$: $\omega(\overline{F \circ \theta} \cdot F) \geq 0$ for every bounded Borel function F of finitely many cells in the half space; $(F, G) \mapsto \omega(\overline{F \circ \theta} \cdot G)$ is a positive semidefinite Hermitian form on these functions; $\omega(\overline{F \circ \theta}) = \overline{\omega(F)}$ for these functions; the first three statements extend to the L^2 closure of these functions.*

(e) *(*The scheme.*)* *Let c_N, c'_N be real numbers and g_{d} a real Borel function (Definition 2.5, Remark 2.6(i)), and put*

$$g_{\mathrm{d}}^\star(t) := g_{\mathrm{d}}(\kappa_N^{-1/2} t) \qquad (t \in \mathbb{R}),$$

a real Borel function with $\|g_{\mathrm{d}}^\star\|_\infty = \|g_{\mathrm{d}}\|_\infty$. Let $\Phi_N^\star(\sigma)$ be the function of Definition 2.5 in the rescaled model, with the same c_N, c'_N and with the density function $g_{\mathrm{d}}^\star$. Then $\Phi_N(\sigma) \circ r = \Phi_N^\star(\sigma)$ for every test pair σ, the curvature part being unchanged, and for ω_N, $\omega_N^\star$ as in (c)

$$Z_N(\sigma) = \omega_N\big(e^{i\Phi_N(\sigma)}\big) = \omega_N^\star\big(e^{i\Phi_N^\star(\sigma)}\big) =: Z_N^\star(\sigma) \qquad \text{for every test pair } \sigma.$$

The function $g_{\mathrm{d}}^\star$ depends on the level through κ_N. If κ_N is the same at every level, then $g_{\mathrm{d}}^\star$ is one density function for all levels, and every limit functional of the given models (Theorem 7.1) is a limit functional of the rescaled models with the density function $g_{\mathrm{d}}^\star$, along the same levels and with the states $\omega_N^\star$ of (c).

(f) *(*Gibbs measures.*)* *A probability measure $\nu^\star$ on the configurations $(\tilde\rho^\star, B, v)$ of $\varepsilon_N \mathbb{Z}^3$ is a Gibbs measure of the rescaled model, for the specification of Subsection 13.1 (the weight (13.1) read with the hopping coefficient of the model), if and only if $r_* \nu^\star$ is a Gibbs measure of the given model. The two measures are invariant under the same translations of $\varepsilon_N \mathbb{Z}^3$, and $r_* \nu^\star$ satisfies the conditions (I1)–(I3) of Theorem 13.1 if and only if $\nu^\star$ does.*

(g) *(*The results proved at $\kappa_N = 1$.*)* *Theorem 13.1, Theorem 14.3 and Corollary 14.35, which Sections 13 and 14 prove for $\kappa_N = 1$ and arbitrary $\lambda_0 > 0$, $\rho_{0,0} > 0$, $\gamma_N > 0$ and real δm_B, hold as stated for every $\kappa_N > 0$.*

Proof. (a) Let $s \in [-\rho_{0,0}^\star, \infty)$ and $\rho^\star := \rho_{0,0}^\star + s$. By (2.5), $\rho := \rho_{0,0} + \kappa_N^{-1/2} s = \kappa_N^{-1/2} \rho^\star$ and

$$\lambda_0(\rho^2 - \rho_{0,0}^2)^2 = \lambda_0 \kappa_N^{-2}\big((\rho^\star)^2 - (\rho_{0,0}^\star)^2\big)^2 = \lambda_0^\star\big((\rho^\star)^2 - (\rho_{0,0}^\star)^2\big)^2, \qquad \tfrac{1}{2}\delta m_B \rho^2 = \tfrac{1}{2}\delta m_B^\star (\rho^\star)^2.$$

Since $g_{\mathrm{ct}}(\tilde\rho) = \rho\, e^{-\lambda_0(\rho^2 - \rho_{0,0}^2)^2} e^{-\frac{1}{2}\delta m_B \rho^2}$ with $\rho = \rho_{0,0} + \tilde\rho$, and the prefactor is $\rho = \kappa_N^{-1/2}\rho^\star$, the first identity follows. For the second,

$$\kappa_N \big| e^{i\gamma_N \vartheta}(\rho_{0,0} + \kappa_N^{-1/2} s') - (\rho_{0,0} + \kappa_N^{-1/2} s) \big|^2 = \big| e^{i\gamma_N \vartheta}(\rho_{0,0}^\star + s') - (\rho_{0,0}^\star + s) \big|^2,$$

and $-\frac{1}{2}$ times the right side is the exponent of $h^\star$, whose hopping coefficient is 1. The two remaining identities follow from $\rho_x \circ r = \kappa_N^{-1/2} \rho_x^\star$. The plaquette weight w, the intervals $(-\pi/\gamma_N, \pi/\gamma_N]$, the lattices $(2\pi/\gamma_N)\mathbb{Z}$ and the constraint $dv = 0$ involve no site variable and no coefficient other than γ_N, which is the same in the two models.

(b) Let n be the number of sites of $\tilde T_L$ and $W^\star$ the weight of the rescaled model. By (a), $W \circ r = \kappa_N^{-n/2} W^\star$: each site factor contributes $\kappa_N^{-1/2}$, and the hopping and plaquette factors

contribute no constant. The substitution $\tilde\rho_x = \kappa_N^{-1/2}\tilde\rho^\star_x$ at every site maps $[-\rho^\star_{0,0},\infty)$ onto $[-\rho_{0,0},\infty)$, with $d\tilde\rho_x = \kappa_N^{-1/2} d\tilde\rho^\star_x$; the variables B_b, v_p and their reference measures are untouched. Hence, for every bounded Borel F,

$$\sum_v \int F\, W \prod_x d\tilde\rho_x \prod_b dB_b = \kappa_N^{-n} \sum_v \int (F\circ r)\, W^\star \prod_x d\tilde\rho^\star_x \prod_b dB_b,$$

the sums running over the closed v. Dividing by the same identity for $F = 1$ gives the first equality of (b): the constant κ_N^{-n} cancels. For a box Λ with n sites the weight W_Λ of Definition 11.9 satisfies $W_\Lambda \circ r = \kappa_N^{-n/2} W^\star_\Lambda$ by (a), applied to the site factors g_{ct}, to the factors $e^{-3\kappa_N\rho_x^2}$ and to the factors $e^{\kappa_N \rho_{b_-}\rho_{b_+}\cos(\gamma_N B_b)}$ in turn, and the same computation gives the second equality.

(c) Let S be a finite set of plaquettes and sites, and let r_S be the map that multiplies the site variables of S by $\kappa_N^{-1/2}$ and leaves its curvature variables unchanged; it is a homeomorphism of the space of the gauge-invariant variables of S in the rescaled model onto that space in the given model, and the variables of S at the configuration $r(\cdot)$ are r_S of the variables of S. By (b) the law of the variables of S under $\langle\cdot\rangle_L^{\mathrm{fine}}$ is the image under r_S of their law under $\langle\cdot\rangle^\star_L$. Weak convergence is preserved by r_S and by its inverse, and the marginal of $r_*\mu^\star_N$ on the variables of S is the image under r_S of the marginal of $\mu^\star_N$. This proves the equivalence. Given a limit point $((L_n),\mu_N)$ of the given model we take $\mu^\star_N := (r^{-1})_*\mu_N$, and then $\omega_N(F) = \int F\, d(r_*\mu^\star_N) = \omega^\star_N(F\circ r)$. Definition 14.1 is a condition on the sides alone, and the two lengths that it fixes are taken the same for the two models. For the killed boxes the same argument applies to the laws under $\langle\cdot\rangle_\Lambda$ along an exhausting sequence of boxes, with Theorem 11.11(b).

(d) An element of $\mathcal{G}_N$ acts on a configuration by permuting the cells, with a change of sign of the curvature variables whose orientation it reverses, and the charge conjugation changes the sign of the curvature variables; r multiplies every site variable by one and the same number and leaves the curvature variables unchanged. These actions commute. The maps r and r^{-1} are Borel and act on the variables of each cell separately, which gives the statement on the functions of a set of cells, and $\int |F\circ r|^2 d\mu^\star_N = \int |F|^2 d\mu_N$. For a reflection $\theta\in\mathcal{G}_N$ we have $\overline{F\circ\theta}\circ r = \overline{(F\circ r)\circ\theta}$, hence by (c)

$$\omega_N\big(\overline{F\circ\theta}\cdot G\big) = \omega^\star_N\big(\overline{(F\circ r)\circ\theta}\cdot(G\circ r)\big), \qquad \omega_N\big(\overline{F\circ\theta}\big) = \omega^\star_N\big(\overline{(F\circ r)\circ\theta}\big),$$

and since $F\mapsto F\circ r$ is onto, is isometric in L^2 and preserves the set of cells on which a function depends, the four statements for ω_N are the four statements for $\omega^\star_N$.

(e) We have $\mathfrak{f}_p\circ r = \mathfrak{f}_p$ and $g_{\mathrm{d}}(\tilde\rho_z)\circ r = g_{\mathrm{d}}(\kappa_N^{-1/2}\tilde\rho^\star_z) = g^\star_{\mathrm{d}}(\tilde\rho^\star_z)$, and the numbers c_N, c'_N and the test pair do not involve the coefficients; so $\Phi_N(\sigma)\circ r = \Phi^\star_N(\sigma)$. The function $e^{i\Phi_N(\sigma)}$ is bounded Borel local, and (c) gives $Z_N(\sigma) = Z^\star_N(\sigma)$. The supremum norms agree because $t\mapsto \kappa_N^{-1/2}t$ is a bijection of $\mathbb{R}$. If κ_N does not depend on N and S is a limit functional of the given models along $N_1<N_2<\cdots$, then $Z^\star_{N_m}(\sigma) = Z_{N_m}(\sigma)\to S(e_\sigma)$, which is (a) of Theorem 7.1 for the rescaled models with the one density function $g^\star_{\mathrm{d}}$; (b) and (c) of that theorem are properties of S alone.

(f) For a finite Λ the kernel $\gamma_\Lambda(\cdot\mid\omega)$ is the normalised integral of the weight (13.1) over the variables carried by $V(\Lambda)$ at a fixed exterior configuration ω; only the factors that contain such a variable matter. We apply (a) to each of these factors (in the hopping factor of a bond with one end outside Λ the exterior site variable is rescaled as well, and the second identity of (a) holds for all values of its arguments) and make the substitution of (b) at the sites of Λ. This gives

$$\gamma_\Lambda(F\mid r\omega^\star) = \gamma^\star_\Lambda(F\circ r\mid\omega^\star)$$

for every bounded Borel F and every configuration $\omega^\star$ of the rescaled model, the constant $\kappa_N^{-|\Lambda|}$ cancelling in the normalisation. With $\nu := r_*\nu^\star$ we obtain $\nu(\gamma_\Lambda F) = \nu^\star(\gamma^\star_\Lambda(F\circ r))$, while

$\nu(F) = \nu^\star(F \circ r)$; as $F \circ r$ runs over all bounded Borel functions, $\nu\gamma_\Lambda = \nu$ for every finite Λ if and only if $\nu^\star\gamma^\star_\Lambda = \nu^\star$ for every finite Λ. The map r commutes with the translations. Condition (I2) concerns the curvature alone, and $\nu(\rho_x^2) = \kappa_N^{-1}\nu^\star((\rho^\star_x)^2)$, $\nu(e^{q\rho_x^2}) = \nu^\star(e^{(q/\kappa_N)(\rho^\star_x)^2})$, where q/κ_N runs over $(0,\infty)$ with q; so (I1) and (I3) hold for ν if and only if they hold for $\nu^\star$.

(g) Sections 13 and 14 prove the three statements for the hopping coefficient 1 and arbitrary values of the other coefficients, hence for the rescaled model. Let ν be a Gibbs measure of the given model as in Theorem 13.1. By (f), $\nu^\star := (r^{-1})_*\nu$ is a Gibbs measure of the rescaled model, invariant under the same sublattice and satisfying (I1)–(I3); by Theorem 13.1 at $\kappa_N = 1$ it is invariant under the translations by $2\varepsilon_N e_j$, $j = 1,2,3$, and by (f) so is ν. Let ω be a limit point of the given model along a dyadic sequence. By (c), $\omega = \omega^\star(\,\cdot \circ r)$ with $\omega^\star$ a limit point of the rescaled model along the same dyadic sequence, for which Theorem 14.3 and Corollary 14.35 hold; the reflections in the diagonal site planes belong to $\mathcal{G}_N$, and (d) gives the statements of that theorem and of that corollary for ω. □

3. Gaussian domination of the Villain curvature

Theorem 3.1 (Gaussian domination, identity covariance)**.** *On every fine torus $\tilde{T}_L$, for every level N, every choice of positive couplings and any real δm_B, and every real plaquette function $a = (a_p)$,*

$$\Big\langle \exp\Big(\sum_p a_p \mathfrak{f}_p\Big)\Big\rangle_L^{\text{fine}} \;\le\; \exp\Big(\tfrac{1}{2}\sum_p a_p^2\Big). \tag{3.1}$$

Consequently, with $X = \sum_p a_p\mathfrak{f}_p$: $\langle X\rangle_L^{\text{fine}} = 0$, $\langle X^2\rangle_L^{\text{fine}} \le \sum_p a_p^2$, and $\langle X^{2n}\rangle_L^{\text{fine}} \le (2n)!\,(e\sum_p a_p^2/2n)^n$ for $n \ge 1$.

Proof. Write $\|a\|^2 = \sum_p a_p^2$ and $Z(a) = \int d\tilde\rho \int dB \sum_{v \text{ closed}} W\, e^{\sum_p a_p\mathfrak{f}_p}$, so that the left side of (3.1) is $Z(a)/Z(0)$; $Z(a) \in (0,\infty)$ for every real a by the bounds in Step 4 below.

Step 1: the plaquette sum. For $x, s, a \in \mathbb{R}$ put $c_u := (\gamma_N/\sqrt{2\pi})\, e^{-\gamma_N^2 u^2/2} > 0$ $(u \in \mathbb{R})$. Completing the square, $-\frac{1}{2}(x+v)^2 + a(x+v) = \frac{1}{2}a^2 - \frac{1}{2}(x-a+v)^2$, and Poisson summation applied to $n \mapsto e^{-(y+2\pi n/\gamma_N)^2/2} e^{2\pi i s n}$ $(y = x-a,\ v = 2\pi n/\gamma_N)$ gives

$$\sum_{v \in (2\pi/\gamma_N)\mathbb{Z}} e^{-(x+v)^2/2 + a(x+v)}\, e^{i\gamma_N s v} = e^{a^2/2} \sum_{m\in\mathbb{Z}} c_{m-s}\, e^{i\gamma_N(m-s)(x-a)}. \tag{3.2}$$

The Fourier coefficient is

$$\int e^{-(y+2\pi u/\gamma_N)^2/2} e^{2\pi i(s-m)u} du = (\gamma_N/2\pi)\int e^{-w^2/2} e^{i\gamma_N(s-m)(w-y)} dw = c_{m-s} e^{i\gamma_N(m-s)y}.$$

All coefficients c_{m-s} are positive, for every real s.

Step 2: the closed-v constraint. For a cube c, $(dv)_c = \sum_{p\in\partial c} \pm v_p \in (2\pi/\gamma_N)\mathbb{Z}$, so $\mathbf{1}[(dv)_c = 0] = \int_0^1 e^{i\gamma_N t_c (dv)_c}\, dt_c$. Hence, with $t = (t_c) \in [0,1]^{\mathcal{C}}$ and $s := d^*t$ the real plaquette function $(d^*t)_p = \sum_{c\ni p} \pm t_c$ (the signs of ∂), $\sum_{v \text{ closed}} G(v) = \int_{[0,1]^{\mathcal{C}}} dt \sum_{\text{all } v} G(v)\, e^{i\gamma_N \sum_p s_p v_p}$ for every absolutely summable G. Applying (3.2) plaquette by plaquette with $x = (dB)_p$,

$$\sum_{v \text{ closed}} \prod_p w(\mathfrak{f}_p) e^{a_p\mathfrak{f}_p} = e^{\|a\|^2/2} \int dt \sum_{m\in\mathbb{Z}^{\mathcal{P}}} \prod_p c_{m_p - s_p}\; e^{i\gamma_N \sum_p (m_p - s_p)(dB)_p}\; e^{-i\gamma_N \sum_p (m_p - s_p) a_p}.$$

Now $\sum_p (m_p - s_p)(dB)_p = \sum_b \big(d^*(m-s)\big)_b B_b = \sum_b (d^*m)_b B_b$, because $d^*s = d^*d^*t = 0$ $(dd = 0)$; and $(d^*m)_b \in \mathbb{Z}$.

Step 3: the gauge integral. The hopping factor depends on B_b only through $\cos(\gamma_N B_b)$:

$$h(r,r',\vartheta) = e^{-\frac{1}{2}\kappa_N(\rho^2+\rho'^2)} e^{\kappa_N \rho\rho' \cos(\gamma_N\vartheta)}$$

with $\rho = \rho_{0,0}+r \geq 0$, $\rho' = \rho_{0,0}+r' \geq 0$ (the variables are $\geq -\rho_{0,0}$). By $e^{z\cos\varphi} = \sum_{q\in\mathbb{Z}} I_q(z) e^{iq\varphi}$ with modified Bessel functions $I_q(z) = I_{-q}(z) \geq 0$ for $z \geq 0$, and $\sum_q I_q(z) = e^z < \infty$,

$$\int \prod_b dB_b \, e^{i\gamma_N \sum_b (d^*m)_b B_b} \prod_b h(\tilde\rho_{b_-}, \tilde\rho_{b_+}, B_b) = \prod_b \frac{2\pi}{\gamma_N} \, e^{-\frac{1}{2}\kappa_N(\rho_{b_-}^2+\rho_{b_+}^2)} \, I_{(d^*m)_b}\big(\kappa_N \rho_{b_-}\rho_{b_+}\big) \;\geq 0,$$

the integral over $(-\pi/\gamma_N, \pi/\gamma_N]$ picking the Fourier mode $q = -(d^*m)_b$ (an integer).

Step 4: assembly. Define

$$\Omega(m) := \int d\tilde\rho \prod_x g(\tilde\rho_x) e^{-\frac{1}{2}\delta m_B \rho_x^2} \prod_b \frac{2\pi}{\gamma_N} e^{-\frac{1}{2}\kappa_N(\rho_{b_-}^2+\rho_{b_+}^2)} I_{(d^*m)_b}(\kappa_N\rho_{b_-}\rho_{b_+}) \geq 0,$$

which does not depend on a or t. Then

$$Z(a) = e^{\|a\|^2/2} \int_{[0,1]^{\mathcal{C}}} dt \sum_{m\in\mathbb{Z}^{\mathcal{P}}} \Big(\prod_p c_{m_p-(d^*t)_p}\Big) \, \Omega(m) \, e^{-i\gamma_N\,(m-d^*t)\cdot a}. \tag{3.3}$$

The interchanges of the sums and integrals are justified by absolute convergence: $\sum_m \prod_p c_{m_p-s_p} = \prod_p \sum_m c_{m-s_p}$, and $\sum_m c_{m-s} = \sum_n e^{-(2\pi n/\gamma_N)^2/2} e^{2\pi i s n}$ (Poisson again) is bounded by $K_{\gamma_N} := \sum_n e^{-(2\pi n/\gamma_N)^2/2} < \infty$ uniformly in s. Moreover,

$$\begin{aligned} 0 \leq \Omega(m) &\leq \int d\tilde\rho \prod_x g(\tilde\rho_x) e^{-\frac{1}{2}\delta m_B \rho_x^2} \prod_b \frac{2\pi}{\gamma_N} e^{-\frac{1}{2}\kappa_N(\rho_{b_-}-\rho_{b_+})^2} \\ &\leq (2\pi/\gamma_N)^{|\mathcal{B}|} \Big(\int_{-\rho_{0,0}}^{\infty} g(r) e^{-\frac{1}{2}\delta m_B(\rho_{0,0}+r)^2} dr\Big)^{|\mathcal{X}|} < \infty \end{aligned}$$

since $I_q(z) \leq e^z$ and by (2.1) at $q=0$; the lattice is finite. Since $Z(a)$ is the integral of a positive weight it is real and positive, so it equals the real part of (3.3):

$$\begin{aligned} Z(a) &= e^{\|a\|^2/2} \int dt \sum_m \Big(\prod_p c_{m_p-(d^*t)_p}\Big) \Omega(m) \cos\big(\gamma_N (m-d^*t)\cdot a\big) \\ &\leq e^{\|a\|^2/2} \int dt \sum_m \Big(\prod_p c_{m_p-(d^*t)_p}\Big) \Omega(m) = e^{\|a\|^2/2} Z(0), \end{aligned}$$

the last equality being (3.3) at $a=0$. This is (3.1).

Step 5: moments. Apply (3.1) to sa, $s\in\mathbb{R}$: $\varphi(s) := \langle e^{sX}\rangle_L^{\text{fine}} \leq e^{s^2\|a\|^2/2}$. Then $\langle e^{|sX|}\rangle_L^{\text{fine}} \leq 2e^{s^2\|a\|^2/2} < \infty$, so all moments of X are finite and φ is smooth with $\varphi'(0) = \langle X\rangle_L^{\text{fine}}$, $\varphi''(0) = \langle X^2\rangle_L^{\text{fine}}$ (dominated convergence). The function $s \mapsto e^{s^2\|a\|^2/2} - \varphi(s)$ is ≥ 0 with a zero at $s=0$, so its first derivative vanishes there and its second derivative is ≥ 0: $\langle X\rangle_L^{\text{fine}} = 0$ and $\langle X^2\rangle_L^{\text{fine}} \leq \|a\|^2$. For the higher moments, $\langle X^{2n}\rangle_L^{\text{fine}} \leq (2n)!\, s^{-2n} \langle \cosh(sX)\rangle_L^{\text{fine}} \leq (2n)!\, s^{-2n} e^{s^2\|a\|^2/2}$ for every $s>0$; the choice $s^2 = 2n/\|a\|^2$ gives the stated bound. □

Remark 3.2. No property of the couplings enters: $\kappa_N \geq 0$, $\lambda_0 > 0$, $\rho_{0,0} > 0$, $\delta m_B \in \mathbb{R}$, $\gamma_N > 0$, ε_N, L are arbitrary; the site factor enters only through $\Omega(m) \geq 0$ and $\Omega(m) < \infty$ in Step 4. The two structural facts used are that the plaquette weight is Gaussian in the curvature (the Villain form) and that in unitary gauge the gauge field enters the matter weight only through $\cos(\gamma_N B_b)$ with a non-negative coefficient, so that all Fourier coefficients are non-negative (the positivity structure used by Brydges, Fröhlich and Seiler [6] for the non-compact field).

4. The estimate

Definition 4.1 (seminorms)**.** For a real C_c^∞ 2-form f and $h \in C_c^\infty(\mathbb{R}^3;\mathbb{R})$ put

$$R(f) := \sup_{N\ge N_0} \Big(\varepsilon_N^3 \sum_p f_{\mu\nu(p)}(x_p)^2\Big)^{1/2}, \qquad E(h) := \sup_{N\ge N_0} \varepsilon_N^3 \sum_z |h(z)|,$$
$$\|\sigma\| := R(f) + \|g_{\mathrm{d}}\|_\infty E(h),$$

the sums over the plaquettes and sites of the fine lattice at level N (Riemann sums of f^2 and $|h|$ at spacing ε_N). A seminorm $\|\cdot\|'$ on $\mathcal{T}$ is *admissible* if $\sigma_n \to \sigma$ in C_c^∞ (supports in a fixed compact set, all derivatives uniformly) implies $\|\sigma_n - \sigma\|' \to 0$.

For an admissible seminorm $\|\cdot\|'$ also $\|\tau_t\sigma - \sigma\|' \to 0$ as $t \to 0$. If K is a compact set containing the supports and K_{+a_0} its a_0-neighbourhood, then $R(f)^2 \le 3\,|K_{+a_0}|\,\|f\|_\infty^2$ and $E(h) \le |K_{+a_0}|\,\|h\|_\infty$ (the open cubes of side ε_N centred at the sites of the lattice that lie in K are pairwise disjoint and lie in $K_{+\sqrt{3}\varepsilon_N/2} \subset K_{+a_0}$, as $\varepsilon_N \le a_0$, so that at most $|K_{+a_0}|/\varepsilon_N^3$ sites lie in K; the centres x_p of the plaquettes of one orientation form a translate of the lattice, so that at most three times as many plaquettes have their centres in K). Hence both are finite seminorms, and $\sigma_n \to \sigma$ in C_c^∞ (supports in a fixed compact set, uniform convergence of all derivatives; uniform convergence alone suffices) implies $\|\sigma_n - \sigma\| \to 0$: the seminorm $\|\cdot\|$ is admissible. (The Riemann sums converge to $\sum_{\mu<\nu}\|f_{\mu\nu}\|_{L^2}^2$ and $\|h\|_{L^1}$ as $N \to \infty$; only finiteness and continuity are used.)

Proposition 4.2 (the estimate)**.** *For every test pair $\sigma = (f,h)$, every level $N \ge N_0$ and every torus side L large enough for the support of σ to embed in $\tilde{T}_L$,*

$$\big(\langle \Phi_N(\sigma)^2\rangle_L^{\mathrm{fine}}\big)^{1/2} \le \|\sigma\|, \qquad \textit{and hence} \qquad \omega_N\big(|\Phi_N(\sigma)| \wedge 2\big) \le \|\sigma\|. \tag{4.1}$$

Proof. Write $\Phi_N(\sigma) = X + Y$ (notation local to this proof) with $X = \sum_p a_p \mathfrak{f}_p$, $a_p = \varepsilon_N^{3/2} f_{\mu\nu(p)}(x_p)$, and $Y = \varepsilon_N^3 \sum_z h(z) g_{\mathrm{d}}(\tilde{\rho}_z)$. By Theorem 3.1, $\langle X^2\rangle_L^{\mathrm{fine}} \le \sum_p a_p^2 = \varepsilon_N^3 \sum_p f_{\mu\nu(p)}(x_p)^2 \le R(f)^2$. Pointwise, $|Y| \le \|g_{\mathrm{d}}\|_\infty \varepsilon_N^3 \sum_z |h(z)| \le \|g_{\mathrm{d}}\|_\infty E(h)$. By Minkowski's inequality in $L^2(\langle\cdot\rangle_L^{\mathrm{fine}})$, $\big(\langle (X+Y)^2\rangle_L^{\mathrm{fine}}\big)^{1/2} \le R(f) + \|g_{\mathrm{d}}\|_\infty E(h) = \|\sigma\|$. Finally, by the Schwarz inequality in $L^2(\mu_N)$ and Minkowski's inequality there,

$$\omega_N(|\Phi_N(\sigma)| \wedge 2) \le \Big(\int \Phi_N(\sigma)^2 d\mu_N\Big)^{1/2} \le \Big(\int X^2 d\mu_N\Big)^{1/2} + \|g_{\mathrm{d}}\|_\infty E(h),$$

the pointwise bound on Y holding on Ω_N as on the torus (Y is a bounded Borel local function; g_{d} need not be continuous). X^2 is a continuous local function ≥ 0 with $\langle X^2\rangle_L^{\mathrm{fine}} \le R(f)^2$ for all large L, so $\int X^2 d\mu_N \le \liminf_n \langle X^2\rangle_{L_n}^{\mathrm{fine}} \le R(f)^2$ by Theorem 11.5(e). Hence $\omega_N(|\Phi_N(\sigma)| \wedge 2) \le R(f) + \|g_{\mathrm{d}}\|_\infty E(h) = \|\sigma\|$. □

5. Symmetries of the fine state

Lemma 5.1 (lattice symmetries of the fine state)**.** *ω_N is invariant under the translations by $\varepsilon_N\mathbb{Z}^3$, under the lattice reflections in the site planes and under the cubic group about every site: $\omega_N(F \circ g) = \omega_N(F)$ for every bounded local gauge-invariant F and every such g. In particular $\omega_N(\iota_N(\tau_t a)) = \omega_N(\iota_N(a))$ for all $a \in \mathcal{A}$ and $t \in \varepsilon_N\mathbb{Z}$.*

Proof. The weight of the fine torus of side L (Section 2) is a product of position-independent factors (the site factor $g(\tilde{\rho}_x)e^{-\frac{1}{2}\delta m_B \rho_x^2}$ included) over the sites, bonds and plaquettes of the periodic lattice $(\varepsilon_N\mathbb{Z}/L\mathbb{Z})^3$, and the sum over the Villain variables runs over all closed v. Every lattice translation, reflection and cubic rotation g of the torus permutes the sites, bonds and

plaquettes, maps closed v to closed v and preserves dB up to the orientation signs absorbed in $B_{-b} = -B_b$, $v_{-p} = -v_p$; the plaquette weight $w(f) = e^{-f^2/2}$ is even and the hopping factor is symmetric under bond reversal, $h(r, r', \vartheta) = h(r', r, -\vartheta)$, so the reversed orientations that g produces do not change the weight. (We write $F \circ g$ for the observable F evaluated on the transformed configuration; the convention is immaterial since the symmetries form groups.) Hence $\langle F \circ g\rangle_L^{\mathrm{fine}} = \langle F\rangle_L^{\mathrm{fine}}$ for every L with $L/2 \in \varepsilon_N\mathbb{Z}$ and every g of the torus; for a fixed local F and a fixed g of $\varepsilon_N\mathbb{Z}^3$ (or a fixed reflection in a site plane, or a fixed cubic rotation about a site) this holds for all large L, in particular along the sides (L_n) of Definition 2.3.

For bounded continuous local F, (2.3) applied to F and to $F \circ g$ (bounded continuous local as well) gives $\omega_N(F \circ g) = \omega_N(F)$; so $g_*\mu_N$ and μ_N have the same integrals of bounded continuous local functions, hence the same finite-dimensional marginals (a Borel probability measure on a Polish space is determined by the integrals of the bounded continuous functions), and $g_*\mu_N = \mu_N$ (the cylinder sets form a π-system generating the Borel σ-algebra). Therefore $\omega_N(F \circ g) = \omega_N(F)$ for every bounded Borel local F; this is Theorem 11.5(c), whose proof is the argument just given.

The last sentence is the case $g =$ translation by te_3, $t \in \varepsilon_N\mathbb{Z}$: $\iota_N(\tau_t a)$ is $\iota_N(a)$ evaluated on the translated configuration, because $\Phi_N(\tau_t\sigma) = \Phi_N(\sigma) \circ g^{-1}$ for a lattice translation (substitute $p \mapsto p - te_3$ in the plaquette sum and $z \mapsto z - te_3$ in the site sum; $\tau_t\sigma$ has its centres at $x_p + te_3$). The same intertwining holds for the lattice reflections and for the cubic group about a site, with the action on test pairs defined as the push-forward of (f, h) (the 2-form transforming as a 2-form), by the same substitution. □

6. The passage from the estimate to the limit functional

Proposition 6.1 (the passage to the limit)**.** *Assume that each ω_N, $N \geq N_0$, is a state on the bounded Borel local functions (linear, positive, $\omega_N(1) = 1$, hence $|\omega_N(F)| \leq \|F\|_\infty$), and assume the unimodularity bound (2.4), the conclusions of Lemma 5.1 and the estimate (4.1) with an admissible seminorm (Definition 4.1). Then there are $N_0 \leq N_1 < N_2 < \cdots$ and a linear $S : \mathcal{A} \to \mathbb{C}$ such that*

(a) $\lim_m \omega_{N_m}(\iota_{N_m}(a)) = S(a)$ *for every* $a \in \mathcal{A}$;
(b) $S(\tau_t a) = S(a)$ *for all* $t \in \mathbb{R}$ *and* $a \in \mathcal{A}$;
(c) *for* $a, b \in \mathcal{A}_+$ *the function* $t \mapsto S(\Theta a \cdot \tau_t b)$ *is continuous on* $[0, \infty)$.

Moreover S is invariant under all space translations and under the cubic group about every point of $\bigcup_m \varepsilon_{N_m}\mathbb{Z}^3$, and $|S(a)| \leq M(a)$.

The state (2.3) of Definition 2.3 is a state on the bounded Borel local functions by Theorem 11.5(b). For unbounded density functions and other normalisations see Section 8 (Proposition 8.4, Theorem 8.5).

Proof. Step 1: equicontinuity. For $\sigma, \sigma' \in \mathcal{T}$ and every N, by additivity of Φ_N, $|e^{ix} - e^{iy}| = |e^{i(x-y)} - 1| \leq |x - y| \wedge 2$, the positivity of the state ω_N and (4.1),

$$|Z_N(\sigma) - Z_N(\sigma')| \leq \omega_N\big(|\Phi_N(\sigma - \sigma')| \wedge 2\big) \leq C\|\sigma - \sigma'\|. \tag{6.1}$$

Here C denotes the constant with which the estimate (4.1) holds for the seminorm; in the form stated in Proposition 4.2, $C = 1$.

Step 2: a countable dense set. Let $K_1 \subset K_2 \subset \cdots$ be compact cubes exhausting $\mathbb{R}^3$. For each n the space $\mathcal{T}_{K_n}$ of test pairs supported in K_n is a separable Fréchet space (a closed subspace of a finite product of spaces $C^\infty_{K_n}(\mathbb{R}^3)$, each separable), so it has a countable dense subset $\mathcal{D}_n$; let $\mathcal{D} = \bigcup_n \mathcal{D}_n$. Every σ lies in some $\mathcal{T}_{K_n}$ and is a C^∞_c-limit, with supports in K_n, of elements of $\mathcal{D}_n$; by admissibility, for every σ and $\delta > 0$ there is $\sigma' \in \mathcal{D}$ with $\|\sigma - \sigma'\| < \delta$.

Step 3: the subsequence. $|Z_N(\sigma)| \le 1$ for all N, σ by (2.4). A diagonal extraction gives $N_1 < N_2 < \cdots$ such that $Z_{N_m}(\sigma')$ converges for every $\sigma' \in \mathcal{D}$. For arbitrary σ and $\delta > 0$ pick $\sigma' \in \mathcal{D}$ with $\|\sigma - \sigma'\| < \delta$; by (6.1), $\limsup_{m,m'} |Z_{N_m}(\sigma) - Z_{N_{m'}}(\sigma)| \le 2C\delta$, so $Z_{N_m}(\sigma)$ is Cauchy and converges, to a limit $Z(\sigma)$. Define $S(e_\sigma) := Z(\sigma)$ and extend linearly to $\mathcal{A}$; this is well defined because the e_σ form a basis of $\mathcal{A} = \mathbb{C}[\mathcal{T}]$ (the group algebra of the additive group $\mathcal{T}$). For $a = \sum_j c_j e_{\sigma_j}$, $\omega_{N_m}(\iota_{N_m}(a)) = \sum_j c_j Z_{N_m}(\sigma_j) \to S(a)$: this is (a), and $|S(a)| \le M(a)$.

Step 4: continuity in the translation parameter. For $\sigma \in \mathcal{T}$ and $t, t' \in \mathbb{R}$, by (6.1) in the limit,

$$|Z(\tau_t \sigma) - Z(\tau_{t'} \sigma)| \le C \|\tau_t \sigma - \tau_{t'} \sigma\|,$$

and $\tau_t \sigma - \tau_{t'} \sigma \to 0$ in C_c^∞ as $t' \to t$ (the supports stay in a fixed compact set and all derivatives converge uniformly), so the right side tends to 0 by admissibility. Hence $t \mapsto Z(\tau_t \sigma)$ is continuous on $\mathbb{R}$, and so is $t \mapsto S(\tau_t a)$ for every $a \in \mathcal{A}$ (finite sums).

Step 5: (b). Fix $a \in \mathcal{A}$ and m. For $t \in \varepsilon_{N_m}\mathbb{Z}$ Lemma 5.1 gives $\omega_{N_{m'}}(\iota_{N_{m'}}(\tau_t a)) = \omega_{N_{m'}}(\iota_{N_{m'}}(a))$ for every $m' \ge m$, because $\varepsilon_{N_{m'}}\mathbb{Z} \supset \varepsilon_{N_m}\mathbb{Z}$ for $m' \ge m$ ($\varepsilon_N = a_0 5^{-N}$ and $N_{m'} \ge N_m$). Letting $m' \to \infty$, $S(\tau_t a) = S(a)$ for $t \in \varepsilon_{N_m}\mathbb{Z}$, for every m; the union $\bigcup_m \varepsilon_{N_m}\mathbb{Z}$ is dense in $\mathbb{R}$, and $t \mapsto S(\tau_t a)$ is continuous (Step 4): so $S(\tau_t a) = S(a)$ for all real t. The same argument with the space translations gives their invariance; for the cubic group about a site of $\varepsilon_{N_m}\mathbb{Z}^3$ no density is needed: Lemma 5.1 and the intertwining of $\iota_{N_{m'}}$ with the push-forward action (the same substitution as in Lemma 5.1, a signed permutation of the plaquettes and sites) give the invariance for every $m' \ge m$, hence in the limit.

Step 6: (c). Let $a, b \in \mathcal{A}_+$ and $t, t' \ge 0$. Since $\Theta a \cdot \tau_t b$ is a finite sum of basis elements and S is linear, it suffices to treat $a = e_{\sigma_1}$, $b = e_{\sigma_2}$: then $\Theta a \cdot \tau_t b = e_{-\theta\sigma_1 + \tau_t \sigma_2}$ and

$$|S(\Theta a \cdot \tau_t b) - S(\Theta a \cdot \tau_{t'} b)| = |Z(-\theta\sigma_1 + \tau_t\sigma_2) - Z(-\theta\sigma_1 + \tau_{t'}\sigma_2)| \le C\|\tau_t\sigma_2 - \tau_{t'}\sigma_2\|$$

by (6.1) in the limit, which tends to 0 as $t' \to t$ by Step 4 (two-sided in t', so right-continuity at $t = 0$ is included). For a general $a = \sum_j c_j e_{\sigma_j}$ the bound carries the factor $M(a)$: the exponentials $\iota_N(\Theta e_{\sigma_j})$ are unimodular, so the estimate (6.1) applies to the second factor alone. □

Remark 6.2. (i) Only the truncated first-moment form (4.1) is used, through (6.1). (ii) Nothing is assumed about the uniqueness of the limit; different subsequences may give different S, each of which satisfies (a)–(c). (iii) The continuity in (c) is uniform on compact t-sets, with the modulus $C\|\tau_t\sigma - \tau_{t'}\sigma\|$. (iv) The cubic invariance of S comes from Lemma 5.1 and the intertwining of ι_N with the cubic action on test pairs; for a finite group no density argument is needed.

7. Existence and continuity of the limit functional

Theorem 7.1 (the limit functional)**.** *Consider the Villain fine model of Section 2 with any positive couplings and any real δm_B, let the fine infinite-volume states ω_N, $N \ge N_0$, be limit points of the torus states in the sense of Definition 2.3, and let the scheme be the one of Definition 2.5. Then there are $N_0 \le N_1 < N_2 < \cdots$ and a linear $S : \mathcal{A} \to \mathbb{C}$ with*

(a) $\lim_{m\to\infty} \omega_{N_m}(\iota_{N_m}(a)) = S(a)$ *for every $a \in \mathcal{A}$, i.e. $Z_{N_m}(\sigma) = \omega_{N_m}(e^{i\Phi_{N_m}(\sigma)}) \to Z(\sigma) = S(e_\sigma)$ for every test pair σ;*

(b) $S(\tau_t a) = S(a)$ *for all $t \in \mathbb{R}$, $a \in \mathcal{A}$;*

(c) $t \mapsto S(\Theta a \cdot \tau_t b)$ *is continuous on $[0, \infty)$ for all $a, b \in \mathcal{A}_+$.*

Moreover S is invariant under all space translations and under the cubic group about every point of $\bigcup_m \varepsilon_{N_m}\mathbb{Z}^3$, and $|S(a)| \le M(a)$.

Proof. The hypotheses of Proposition 6.1 hold: each ω_N is a state on the bounded Borel local functions by Theorem 11.5(b); the bound (2.4) holds for ι_N; Lemma 5.1 gives the lattice symmetries; and Proposition 4.2 is the estimate (4.1) with the admissible seminorm of Definition 4.1. Proposition 6.1 therefore gives (a)–(c), the invariances and the bound $|S(a)| \le M(a)$. □

The states ω_N exist for every choice of positive couplings and any real δm_B by Theorem 11.5; when the limit (2.2) exists, ω_N is that limit. A functional S as in Theorem 7.1 is called a *limit functional*.

Proposition 7.2 (the limit functional for limit points along subnets)**.** *For each $N \geq N_0$ let (L_k) be a sequence of sides with $L_k \to \infty$ and $L_k/2 \in \varepsilon_N\mathbb{Z}$, and let (L_α) be a subnet of it along which the limit*

$$\omega_N(F) := \lim_\alpha \langle F\rangle^{\mathrm{fine}}_{L_\alpha}$$

exists for every bounded Borel local F (a Tychonoff cluster point of the torus states). Then ω_N is a positive normalised linear functional on the bounded Borel local functions, and Theorem 7.1, its last clause included, holds with these ω_N in place of the limit points of Definition 2.3. The same is true for every pointwise limit point of the torus states on the bounded Borel local functions along any subnet of sides $L \to \infty$, $L/2 \in \varepsilon_N\mathbb{Z}$, or along any cofinal class of them.

Proof. The proof of Theorem 7.1 uses of ω_N only the hypotheses of Proposition 6.1, that is, the following three properties.

(i) The state property. It enters at (6.1), in Step 3 and in Step 6 of the proof of Proposition 6.1. In Step 3 it gives $|Z_N(\sigma)| \leq 1$, together with (2.4), which is a pointwise bound on ι_N in which no state enters; in Step 6 it enters as the positivity, with the unimodular prefactor.
(ii) The invariances of Lemma 5.1, which enter in Step 5.
(iii) The bound $\omega_N(|\Phi_N(\sigma)| \wedge 2) \leq \|\sigma\|$ of (4.1), which enters in Step 1.

Every torus state has the properties (i)–(iii) exactly, at every side L large enough for the supports. We show that each of them passes to a pointwise limit on the bounded Borel local functions.

Property (i). Linearity, positivity and $\langle 1\rangle^{\mathrm{fine}}_L = 1$ are equalities and inequalities between the values of the state on fixed functions, and such relations pass to the limit.

Property (ii). Let F be a bounded Borel local function and let g be a translation, a site-plane reflection or a cubic rotation. By the torus argument in the proof of Lemma 5.1, $\langle F\circ g\rangle^{\mathrm{fine}}_L = \langle F\rangle^{\mathrm{fine}}_L$ for all large L. Since $L_\alpha \to \infty$, it follows that $\omega_N(F \circ g) = \omega_N(F)$.

Property (iii). The function $|\Phi_N(\sigma)| \wedge 2$ is bounded Borel local, and for all large L the Schwarz inequality and the torus half of (4.1) give

$$\langle |\Phi_N(\sigma)| \wedge 2\rangle^{\mathrm{fine}}_L \leq (\langle \Phi_N(\sigma)^2\rangle^{\mathrm{fine}}_L)^{1/2} \leq \|\sigma\|,$$

whence $\omega_N(|\Phi_N(\sigma)| \wedge 2) \leq \|\sigma\|$. In the proof of Proposition 4.2 the measure μ_N is used only in the passage through $L^2(\mu_N)$ and in the step $\int X^2 d\mu_N \leq \liminf_n \langle X^2\rangle^{\mathrm{fine}}_{L_n}$ from Theorem 11.5(e); the bound at every L replaces both.

With (i)–(iii), Steps 1–6 of the proof of Proposition 6.1 and the proof of Theorem 7.1 go through with (i) in place of Theorem 11.5(b) and (iii) in place of Proposition 4.2. This gives (a)–(c), the invariance of S and $|S(a)| \leq M(a)$ for the functionals ω_N of the statement. The same proof applies to every pointwise limit point named in the last sentence of the statement. □

Remark 7.3 (limit points along subnets). Proposition 7.2 extends Theorem 7.1 to limit points of the torus states in a wider sense than Definition 2.3. The functional ω_N of the proposition is finitely additive and not necessarily unique. It agrees with a state of Definition 2.3 on the continuous functions. Indeed, along (L_α) the finite-dimensional laws converge weakly (they are tight by Theorem 3.1 and Lemma 11.2) to those of a Borel probability measure μ_N. The limit laws form a cluster point of the laws along (L_k) in the metrisable countable product of the spaces of laws of the finitely many site and plaquette variables of a finite set Λ of cells, hence they are the limit of the laws along a subsequence (L_n) of (L_k). For this pair $((L_n), \mu_N)$ of Definition 2.3, $\omega_N(F) = \int F\, d\mu_N$ for every bounded continuous local F, so the two functionals give the same

$Z_N(\sigma)$ when g_{d} is continuous. For a discontinuous bounded Borel g_{d} the two values may differ, and the subnet functional is covered by the direct proof of Proposition 7.2.

The scheme for which Theorem 7.1 is proved is that of Definition 2.5: $c_N = \varepsilon_N^{3/2}$, $c'_N = \varepsilon_N^3$ and $\mathfrak{w}_z = g_{\mathrm{d}}(\tilde{\rho}_z)$ with g_{d} bounded Borel. The Riemann scaling $c_N = \varepsilon_N^{3/2}$ enters the proof of Proposition 4.2 essentially, and $g_{\mathrm{d}} = \mathrm{id}$ is not covered (Remark 7.5(ii),(iv)). Section 8 treats unbounded density functions and other normalisations in the cases verified there (Theorem 8.5, Proposition 8.10).

Remark 7.4 (regularity of the limit curvature). For real $f_1, \dots, f_k$ and $s \in \mathbb{R}^k$, Theorem 3.1 with $a_p = \varepsilon_N^{3/2} \sum_j s_j f_j(x_p)$ gives the uniform exponential moment bound

$$\Big\langle \exp\Big(\sum_j s_j \Phi_N(f_j, 0)\Big)\Big\rangle_L^{\mathrm{fine}} \le \exp\Big(\tfrac{1}{2} R\Big(\sum_j s_j f_j\Big)^2\Big) \qquad \text{for all } N, L.$$

This bound has the following standard consequences. The laws of $(\Phi_N(f_j, 0))_{j \le k}$ under the torus states are tight with uniformly Gaussian tails, so along the subsequence of Theorem 7.1 the characteristic function $s \mapsto Z(\sum_j s_j f_j, 0)$ is that of a probability law μ_f on $\mathbb{R}^k$ (Lévy's continuity theorem), $\int e^{s\cdot x} d\mu_f(x) \le \exp(\frac{1}{2} R(\sum_j s_j f_j)^2)$, the moments of the torus laws converge to those of μ_f (uniform integrability from the exponential moments), and the even moments of μ_f obey the bounds of Theorem 3.1. The odd moments vanish: the weight is invariant under $(B, v) \mapsto (-B, -v)$ (the hopping factor depends on B through $\cos(\gamma_N B_b)$, w is even, closed v go to closed v), which maps $\mathfrak{f} \mapsto -\mathfrak{f}$. These are finite-family moment bounds for the curvature sector; the full regularity axiom is Theorem 9.4.

Remark 7.5 (the density component). We collect five observations on the density component of the scheme.

(i) With a bounded g_{d} and $c'_N = \varepsilon_N^3$ the density part of the estimate is pointwise and trivial.

(ii) A genuine scalar sector, the centred density $\varepsilon_N^{5/2} \sum_z h(z)(\tilde{\rho}_z - \omega_N(\tilde{\rho}_0))$, is not covered. Section 8 treats unbounded density functions and other normalisations in the cases verified there (Theorem 8.5, Proposition 8.10).

(iii) Any other bounded Borel function of finitely many fine gauge-invariant variables can be added to the scheme in the same way.

(iv) Since (4.1) is used in first-moment form only, the boundedness of g_{d} can be weakened to $\sup_N \omega_N(|g_{\mathrm{d}}(\tilde{\rho}_0)|) < \infty$, by translation invariance and the inequality $(\sum_z u_z) \wedge 2 \le \sum_z (u_z \wedge 2)$ for $u_z \ge 0$. For $g_{\mathrm{d}} = \mathrm{id}$ at $c'_N = \varepsilon_N^3$ this condition is the level-uniform first-moment bound $\sup_N \omega_N(|\tilde{\rho}_0|) < \infty$. The weakening is proved in Section 8 (Theorem 8.5, Remark 8.7).

(v) We do not prove that the limit is non-degenerate, that is, that S is not the state of a constant field.

8. Unbounded density functions and general normalisations

Theorem 7.1 is stated for the scheme of Definition 2.5, that is, for a *bounded* real Borel density function g_{d} and the *pinned* normalisations $c_N = \varepsilon_N^{3/2}$, $c'_N = \varepsilon_N^3$. Definition 2.5 makes sense for every real Borel density function g_{d} and all real normalisations c_N, c'_N. Boundedness and the pinned normalisations therefore enter only through the proof of the properties (a)–(c) of Theorem 7.1. This section proves these three properties without the two restrictions for every scheme that satisfies the conditions (N) and (Y) of Definition 8.1 (Theorem 8.5). Condition (Y) is verified in the cases (V1)–(V4) of Proposition 8.10, and (N) is a condition on the curvature

normalisation alone, which holds for $c_N = \varepsilon_N^{3/2}$: in each of these cases, for every c_N that satisfies (N), all conclusions of Theorem 8.5 hold.

The proof of Theorem 7.1 uses the boundedness of g_{d} and the pinned normalisations at one place only, the estimate of Proposition 4.2 with the seminorm of Definition 4.1. They enter there four times: through the factor $\|g_{\mathrm{d}}\|_\infty$ in the seminorm; through the pointwise bound of the density sum by $\|g_{\mathrm{d}}\|_\infty \varepsilon_N^3 \sum_z |h(z)|$ (on the tori and under the fine state); through the factor ε_N^3 of the density sum, which makes that bound a Riemann sum; and through the weights $a_p = \varepsilon_N^{3/2} f_{\mu\nu(p)}(x_p)$ of the curvature sum, which make the bound of Theorem 3.1 a Riemann sum. The symmetry lemma (Lemma 5.1) and the abstract passage to the limit (Proposition 6.1) do not involve the density function or the normalisations.

We replace the estimate by one in first-moment form (Lemma 8.3) under two conditions, stated in Definition 8.1: (N), a bound on the ratio of the curvature normalisation to its pinned value, and (Y), a level-independent bound for the truncated first moment of the density term. A level-uniform first moment of the normalised density variable, (Y1), suffices for (Y). Theorem 8.5 then gives clauses (a)–(c), the invariances and the bound $|S(a)| \le M(a)$ for every real Borel g_{d} and all real normalisations satisfying (N) and (Y). No convergence of the normalisation ratios is needed, and the pinned scheme is the case of equality (Corollary 8.9).

Two restrictions remain. The first is the pair (N), (Y) itself. For an unbounded g_{d} at the Riemann normalisation $c'_N = \varepsilon_N^3$, clause (Y1) is a level-uniform first-moment bound for $g_{\mathrm{d}}(\tilde\rho_0)$; this paper proves no such bound that holds for every value of the counterterm $\delta m_B(N)$ (the moment bounds that it proves for every value of the counterterm are uniform in the volume at a fixed level only, Subsection 8.5), and none can be derived without information on the level dependence of the counterterm (Remark 8.13). The cases in which (Y1) is verified are listed in Proposition 8.10. The second concerns the other results of this paper: those listed in Remark 8.15(A) extend to the wider class, while the statements on exponential or second moments of the joint field keep the bounded class (Remark 8.15(C)).

8.1. **The extended scheme and the conditions.** Throughout this section g_{d} is an arbitrary real Borel function on $\mathbb{R}$ (finite at every point), and c_N, c'_N, $N \ge N_0$, are arbitrary real numbers. The functions $\Phi_N(\sigma)$, the realisation ι_N and $Z_N(\sigma) = \omega_N(e^{i\Phi_N(\sigma)})$ are defined by the formulas of Definition 2.5, with these g_{d}, c_N, c'_N in place of those of the pinned scheme. We write

$$\zeta_N := c_N\, \varepsilon_N^{-3/2}, \qquad k_N := c'_N\, \varepsilon_N^{-3} \qquad (N \ge N_0)$$

for the ratios of the two normalisations to their pinned values. They are signed real numbers, and the pinned scheme is the case $\zeta_N = k_N = 1$. We split

$$\Phi_N(\sigma) = X_N(f) + Y_N(h),$$
$$X_N(f) := c_N \sum_p f_{\mu\nu(p)}(x_p)\, \mathfrak{f}_p, \qquad Y_N(h) := c'_N \sum_z h(z)\, g_{\mathrm{d}}(\tilde\rho_z),$$

and we use the Riemann sum of Definition 4.1,

$$E_N(h) := \varepsilon_N^3 \sum_z |h(z)| \le E(h).$$

If K is a compact set containing the support of h, then $E(h) \le |K_{+a_0}|\, \|h\|_\infty$, for the reason given after Definition 4.1: every site of $\varepsilon_N\mathbb{Z}^3$ owns a cell of volume ε_N^3, the cells of the sites in K are disjoint and lie in $K_{+\varepsilon_N} \subset K_{+a_0}$. The same bound holds for every translate and every reflection of h, since the volume of the neighbourhood does not change. Hence E is a finite seminorm on $C_c^\infty(\mathbb{R}^3;\mathbb{R})$, free of N.

The scheme is well posed for every real Borel g_{d}. The sums defining $\Phi_N(\sigma)$ are finite, so $\Phi_N(\sigma)$ is a finite real Borel local function on Ω_N, additive in σ, and $e^{i\Phi_N(\sigma)}$ is a bounded Borel local function of modulus one. Therefore $Z_N(\sigma)$ is defined with no integrability condition on g_{d}, and the unimodularity bound (2.4) holds. For a Borel local function $F \geq 0$ we write $\omega_N(F) := \int F\, d\mu_N \in [0,\infty]$. Since the measure μ_N is invariant under the translations by $\varepsilon_N\mathbb{Z}^3$ (Theorem 11.5(c)),

$$\omega_N\big(|g_{\mathrm{d}}(\tilde\rho_z)|\big) = \omega_N\big(|g_{\mathrm{d}}(\tilde\rho_0)|\big) \in [0,\infty] \qquad \text{for every site } z \text{ of the fine lattice.} \tag{8.1}$$

Definition 8.1 (the extended scheme; conditions (N), (Y), (Y1))**.** The *extended scheme* is the scheme of Definition 2.5 with an arbitrary real Borel function g_{d} on $\mathbb{R}$ and arbitrary real normalisations c_N, c'_N, $N \geq N_0$. For it we consider the following conditions. All suprema run over the levels $N \geq N_0$, the same set of levels for which the conditions are imposed and the conclusions drawn (raising N_0 is allowed).

(N) $$\zeta := \sup_{N\geq N_0} |\zeta_N| = \sup_{N\geq N_0} |c_N|\,\varepsilon_N^{-3/2} < \infty;$$

there is a seminorm p on $C_c^\infty(\mathbb{R}^3;\mathbb{R})$, free of N and admissible, such that

(Y) $$\omega_N\big(|Y_N(h)| \wedge 2\big) \leq \mathsf{p}(h) \qquad \text{for all } h \in C_c^\infty(\mathbb{R}^3;\mathbb{R}) \text{ and all } N \geq N_0;$$

(Y1) $$A_1 := \sup_{N\geq N_0} |k_N|\,\omega_N\big(|g_{\mathrm{d}}(\tilde\rho_0)|\big) < \infty.$$

In (Y1) and below, $0\cdot\infty := 0$: at a level with $c'_N = 0$ the density term vanishes. In (Y), *admissible* is the notion of Definition 4.1, read for h alone: $h_n \to h$ in $C_c^\infty(\mathbb{R}^3;\mathbb{R})$ (supports in a fixed compact set, all derivatives uniformly) implies $\mathsf{p}(h_n - h) \to 0$; equivalently, the seminorm $(f,h) \mapsto \mathsf{p}(h)$ on $\mathcal{T}$ is admissible.

Condition (N) is the *bounded curvature ratio*; (Y) is the abstract *density clause*; and (Y1) is the *first-moment clause*. For the pinned normalisations, (N) holds with $\zeta = 1$, and for bounded g_{d} clause (Y1) holds with $A_1 \leq \|g_{\mathrm{d}}\|_\infty$: the scheme of Definition 2.5 is covered.

Lemma 8.2 (the density clauses)**.** (1) (Y1) *implies* (Y) *with* $\mathsf{p} = A_1 E$. (2) *Clause* (Y) *is equivalent to the* L^1 *bound* $\omega_N(|Y_N(h)|) \leq \mathsf{p}(h)$ *for all* h *and all* $N \geq N_0$*, with the same seminorm. In particular* (Y) *forces* $\omega_N(|g_{\mathrm{d}}(\tilde\rho_0)|) < \infty$ *at every level* $N \geq N_0$ *with* $c'_N \neq 0$.

Proof. (1) Pointwise, $|Y_N(h)| \leq |c'_N| \sum_z |h(z)|\,|g_{\mathrm{d}}(\tilde\rho_z)|$, a finite sum of Borel functions ≥ 0. Integrating term by term and using (8.1),

$$\omega_N\big(|Y_N(h)|\big) \leq |c'_N| \sum_z |h(z)|\,\omega_N\big(|g_{\mathrm{d}}(\tilde\rho_0)|\big) = |k_N|\,\omega_N\big(|g_{\mathrm{d}}(\tilde\rho_0)|\big)\,E_N(h) \leq A_1 E(h).$$

The seminorm E is admissible, by the bound $E(h) \leq |K_{+a_0}|\,\|h\|_\infty$.

(2) An L^1 bound implies (Y) trivially. Conversely, apply (Y) to sh, $s > 0$: since p is a seminorm and Y_N is linear in h, $\omega_N(|Y_N(h)| \wedge (2/s)) \leq \mathsf{p}(h)$; let $s \downarrow 0$ and use monotone convergence. For the last assertion fix N with $c'_N \neq 0$ and choose $h \in C_c^\infty$ with $h(0) = 1$ and support in the open ball of radius ε_N about the origin: the origin is the only site of $\varepsilon_N\mathbb{Z}^3$ in the support, so $Y_N(h) = c'_N g_{\mathrm{d}}(\tilde\rho_0)$ and $\omega_N(|g_{\mathrm{d}}(\tilde\rho_0)|) \leq \mathsf{p}(h)/|c'_N| < \infty$. □

8.2. **The estimate in first-moment form and the theorem.**

Lemma 8.3 (the estimate in first-moment form)**.** *Assume* (N) *and* (Y)*. Then for every test pair* $\sigma = (f,h)$ *and every level* $N \geq N_0$*,*

$$\omega_N\big(|\Phi_N(\sigma)| \wedge 2\big) \leq \zeta\, R(f) + \mathsf{p}(h) =: \|\sigma\|_{\mathrm{x}}, \tag{8.2}$$

and $\|\cdot\|_{\mathrm{x}}$ *is an admissible seminorm on* $\mathcal{T}$*, free of* N*. Consequently, for all* $\sigma,\sigma'\in\mathcal{T}$ *and* $N\ge N_0$*,*

$$|Z_N(\sigma)-Z_N(\sigma')|\le\omega_N\big(|\Phi_N(\sigma-\sigma')|\wedge 2\big)\le\|\sigma-\sigma'\|_{\mathrm{x}}. \tag{8.3}$$

Under (N) *and* (Y1) *this holds with* $\|\sigma\|_{\mathrm{x}}=\zeta R(f)+A_1E(h)$*.*

Proof. Write $X=X_N(f)$, $Y=Y_N(h)$. Pointwise on Ω_N, $|X+Y|\wedge 2\le|X|+(|Y|\wedge 2)$: if $|Y|\ge 2$ the right side is at least 2, and otherwise $|X+Y|\le|X|+|Y|$. *Curvature part.* $X=\sum_p a_p\mathfrak{f}_p$ with the real plaquette function $a_p=c_N f_{\mu\nu(p)}(x_p)$ of finite support, and X^2 is a continuous local function ≥ 0. Theorem 3.1, which holds for every real plaquette function, gives for every large side L

$$\langle X^2\rangle_L^{\mathrm{fine}}\le\sum_p a_p^2=\zeta_N^2\,\varepsilon_N^3\sum_p f_{\mu\nu(p)}(x_p)^2\le\zeta^2R(f)^2,$$

hence $\int X^2d\mu_N\le\liminf_n\langle X^2\rangle_{L_n}^{\mathrm{fine}}\le\zeta^2R(f)^2$ by Theorem 11.5(e), exactly as in the proof of Proposition 4.2, and $\omega_N(|X|)\le\zeta R(f)$ by the Schwarz inequality. *Density part.* $\omega_N(|Y|\wedge 2)\le\mathsf{p}(h)$ is (Y). Adding the two parts gives (8.2). The seminorm R is admissible (the paragraph after Definition 4.1) and p is admissible by (Y), so $\|\cdot\|_{\mathrm{x}}$ is. For (8.3) use the additivity of Φ_N, the inequality $|e^{ix}-e^{iy}|\le|x-y|\wedge 2$ and the positivity of ω_N. The last sentence is Lemma 8.2(1). □

Proposition 6.1 has the estimate (4.1) among its hypotheses. The first half of (4.1) is an L^2 bound on the tori; it fails as soon as $g_{\mathrm{d}}(\tilde\rho_0)$ is not square integrable under the torus states, which (Y) does not exclude. The proof of Proposition 6.1 does not use that half, and we restate the proposition with the inequality it uses as its hypothesis.

Proposition 8.4 (the passage to the limit, with (6.1) as hypothesis)**.** *Let* $\Phi_N(\sigma)$*,* $N\ge N_0$*, be real Borel local functions on* Ω_N*, additive in* $\sigma\in\mathcal{T}$*, and* $\iota_N(e_\sigma)=e^{i\Phi_N(\sigma)}$*,* $Z_N(\sigma)=\omega_N(e^{i\Phi_N(\sigma)})$*. Assume:*

- (h1) *each* ω_N*,* $N\ge N_0$*, is a state on the bounded Borel local functions;*
- (h2) *the unimodularity bound* (2.4)*;*
- (h3) *the conclusions of Lemma 5.1 for these* ω_N *and* ι_N*, including the intertwining of* ι_N *with the lattice translations and the cubic group stated in its proof;*
- (h4) *there is an admissible seminorm* $\|\cdot\|'$ *on* $\mathcal{T}$*, free of* N*, with* $\omega_N(|\Phi_N(\sigma)|\wedge 2)\le\|\sigma\|'$ *for all* $\sigma\in\mathcal{T}$ *and* $N\ge N_0$*.*

Then all conclusions of Proposition 6.1 hold: there are $N_0\le N_1<N_2<\cdots$ *and a linear* $S:\mathcal{A}\to\mathbb{C}$ *with clauses* (a), (b), (c) *there;* S *is invariant under all space translations and under the cubic group about every point of* $\bigcup_m\varepsilon_{N_m}\mathbb{Z}^3$*;* $|S(a)|\le M(a)$*; and* $|Z(\sigma)-Z(\sigma')|\le\|\sigma-\sigma'\|'$ *for* $Z(\sigma)=S(e_\sigma)$*.*

Proof. We follow the proof of Proposition 6.1. That proof uses the estimate (4.1) in its Step 1 only, and there only the second half, to obtain (6.1) (see also part (i) of the remark that follows that proof). Under (h4), the argument of Step 1 (the additivity of Φ_N, the inequality $|e^{ix}-e^{iy}|\le|x-y|\wedge 2$ and the positivity of ω_N) gives (6.1) with $\|\cdot\|'$ in place of $C\|\cdot\|$. Steps 2–6 use only (6.1), the admissibility of the seminorm, (2.4) and Lemma 5.1. Indeed, Step 2 uses separability and admissibility, and the diagonal extraction of Step 3 uses $|Z_N|\le 1$ and (6.1). Step 4 uses (6.1) in the limit, and Step 5 uses Lemma 5.1 on the nested lattices $\varepsilon_{N_m}\mathbb{Z}$ together with Step 4. Step 6 uses (6.1) in the limit, applied to the second factor. None of these steps involves a density function, a normalisation or the tori, so they apply without change. The last inequality of the statement is (6.1) in the limit. □

Theorem 8.5 (the limit functional for the extended scheme)**.** *Let the fine model and the fine infinite-volume states* ω_N*,* $N\ge N_0$*, be as in Theorem 7.1 (any positive couplings, any real* δm_B*,*

which may depend on N; limit points of the torus states in the sense of Definition 2.3), and let the scheme be the extended scheme of Definition 8.1, with a real Borel density function g_{d} and real normalisations c_N, c'_N that satisfy (N) and (Y); in particular, that satisfy (N) and (Y1). Then there are $N_0 \le N_1 < N_2 < \cdots$ and a linear $S : \mathcal{A} \to \mathbb{C}$ with

(a) $\lim_{m\to\infty} \omega_{N_m}(\iota_{N_m}(a)) = S(a)$ *for every $a \in \mathcal{A}$, i.e. $Z_{N_m}(\sigma) \to Z(\sigma) = S(e_\sigma)$ for every test pair σ;*
(b) $S(\tau_t a) = S(a)$ *for all $t \in \mathbb{R}$, $a \in \mathcal{A}$;*
(c) $t \mapsto S(\Theta a \cdot \tau_t b)$ *is continuous on $[0,\infty)$ for all $a, b \in \mathcal{A}_+$.*

Moreover S is invariant under all space translations and under the cubic group about every point of $\bigcup_m \varepsilon_{N_m}\mathbb{Z}^3$, $|S(a)| \le M(a)$, and

$$|Z(\sigma) - Z(\sigma')| \le \zeta\, R(f - f') + \mathsf{p}(h - h') \qquad \text{for all test pairs } \sigma = (f,h),\ \sigma' = (f',h'). \tag{8.4}$$

The ratios ζ_N, k_N may have either sign, need not converge and need not be bounded away from zero. Condition (Y) is verified in the cases (V1)–(V4) of Proposition 8.10.

Proof. We verify the hypotheses of Proposition 8.4. (h1) is Theorem 11.5(b); it does not involve the scheme. (h2) holds because $\Phi_N(\sigma)$ is real (Subsection 8.1). (h3): the first assertion of Lemma 5.1, the invariance of ω_N on bounded Borel local functions under the lattice translations, the reflections and the cubic rotations, is a statement about μ_N alone. Its second assertion, the intertwining of ι_N with these maps, is a relabelling of the plaquettes and sites in the two sums that define $\Phi_N(\sigma)$; it uses only that the same function g_{d} sits at every site and that c_N, c'_N do not depend on the position, and $e^{i\Phi_N(\sigma)}$ is a bounded Borel local function for every real Borel g_{d}. So Lemma 5.1 holds verbatim for the extended scheme. (h4) is Lemma 8.3 with $\|\cdot\|' = \|\cdot\|_{\mathrm{x}}$. Proposition 8.4 gives the assertions, and (8.4) is its last inequality. □

8.3. **Uniform integrability, and what survives without integrability.** The proof of Theorem 8.5 uses a first moment, not uniform integrability and not a second moment: the smallness needed for (b) and (c) comes from the test function, through the admissibility of the seminorm, and not from the tails of g_{d}. Uniform integrability adds one statement, the approximation of the limit by limits of bounded-density schemes.

Lemma 8.6 (uniform integrability and truncation)**.** *Assume (N) and*

$$\eta(\ell) := \sup_{N \ge N_0} \omega_N\Big(|k_N g_{\mathrm{d}}(\tilde\rho_0)|\, \mathbf{1}\{|k_N g_{\mathrm{d}}(\tilde\rho_0)| > \ell\}\Big) \longrightarrow 0 \qquad (\ell \to \infty). \tag{UI}$$

For $\ell \in \{1,2,\dots\}$ let $g_{N,\ell} := g_{\mathrm{d}}\mathbf{1}\{|k_N g_{\mathrm{d}}| \le \ell\}$, a Borel function with $|k_N g_{N,\ell}| \le \ell$ (bounded if $k_N \ne 0$), which depends on the level N unless k_N is constant, and let $\Phi_N^{(\ell)}(\sigma) := X_N(f) + c'_N \sum_z h(z) g_{N,\ell}(\tilde\rho_z)$, $Z_N^{(\ell)}(\sigma) := \omega_N(e^{i\Phi_N^{(\ell)}(\sigma)})$ be the truncated scheme *at height ℓ. Then:*

(i) *(Y1) holds, with $A_1 \le \ell + \eta(\ell)$ for every ℓ; so Theorem 8.5 applies.*
(ii) *$|Z_N^{(\ell)}(\sigma) - Z_N(\sigma)| \le E(h)\,\eta(\ell)$ for all $N \ge N_0$, all $\sigma = (f,h)$ and all ℓ.*
(iii) *There are levels $N_0 \le N_1 < N_2 < \cdots$ and linear functionals S, S_ℓ ($\ell = 1,2,\dots$) on $\mathcal{A}$ such that, along (N_m), $Z_{N_m}(\sigma) \to S(e_\sigma)$ and $Z_{N_m}^{(\ell)}(\sigma) \to S_\ell(e_\sigma)$ for every σ and ℓ; S and every S_ℓ have all the properties stated in Theorem 8.5; and*

$$|S(e_\sigma) - S_\ell(e_\sigma)| \le E(h)\,\eta(\ell),$$
$$\sup_{t\ge 0}\big|S(\Theta a\cdot\tau_t b) - S_\ell(\Theta a\cdot\tau_t b)\big| \le \big(|K_{1,+a_0}|\,\|h_1\|_\infty + |K_{2,+a_0}|\,\|h_2\|_\infty\big)\eta(\ell)$$

for $a = e_{\sigma_1}$, $b = e_{\sigma_2}$, $\sigma_i = (f_i,h_i)$ and compact sets $K_i \supset \operatorname{supp} h_i$.

Thus $S = \lim_{\ell\to\infty} S_\ell$ pointwise on $\mathcal{A}$, with the explicit rate $E(h)\eta(\ell)$ on the basis, along one and the same sequence of levels. For the pinned normalisations the truncated density function is the fixed function $g_{\mathrm{d}}\mathbf{1}\{|g_{\mathrm{d}}| \le \ell\}$, and S_ℓ is a limit functional of Theorem 7.1 as stated.

Proof. (i) $|k_N g_{\mathrm{d}}| \le \ell + |k_N g_{\mathrm{d}}|\mathbf{1}\{|k_N g_{\mathrm{d}}| > \ell\}$; integrate, and choose ℓ with $\eta(\ell) < \infty$.

(ii) By $|e^{ix} - e^{iy}| \le |x - y| \wedge 2$, the positivity of ω_N, the translation invariance of μ_N and $|c'_N| = |k_N|\varepsilon_N^3$,

$$\begin{aligned}|Z_N^{(\ell)}(\sigma) - Z_N(\sigma)| &\le |c'_N| \sum_z |h(z)|\, \omega_N\big(|g_{\mathrm{d}}(\tilde\rho_z)|\mathbf{1}\{|k_N g_{\mathrm{d}}(\tilde\rho_z)| > \ell\}\big)\\ &= E_N(h)\, \omega_N\big(|k_N g_{\mathrm{d}}(\tilde\rho_0)|\mathbf{1}\{|k_N g_{\mathrm{d}}(\tilde\rho_0)| > \ell\}\big) \le E(h)\,\eta(\ell).\end{aligned}$$

(iii) The truncated scheme at height ℓ has a bounded density term: $|c'_N \sum_z h(z) g_{N,\ell}(\tilde\rho_z)| \le \ell E_N(h) \le \ell E(h)$ pointwise (if $k_N = 0$ the term vanishes). As in Lemma 8.3, $\omega_N(|\Phi_N^{(\ell)}(\sigma)| \wedge 2) \le \zeta R(f) + \ell E(h)$, an admissible seminorm free of N. Neither Lemma 5.1 nor the proof of Proposition 8.4 uses that the density function is the same at different levels; they use only that it is the same at all sites of one level. Hence hypotheses (h1)–(h4) hold for each truncated scheme, and for the untruncated one by (i). In Step 3 of the proof of Proposition 6.1 the diagonal extraction can be made for countably many schemes at once: there is one sequence (N_m) along which $Z_{N_m}(\sigma')$ and all $Z_{N_m}^{(\ell)}(\sigma')$, $\ell = 1, 2, \dots$, converge for every σ' in the countable dense set $\mathcal{D}$. Steps 3–6 then apply to each scheme along this sequence and give S and S_ℓ with the stated properties. The first bound is (ii) in the limit. For the second, $\Theta a \cdot \tau_t b = e_{-\theta\sigma_1 + \tau_t\sigma_2}$ has density component $-\theta h_1 + \tau_t h_2$, and

$$E(-\theta h_1 + \tau_t h_2) \le E(\theta h_1) + E(\tau_t h_2) \le |K_{1,+a_0}|\|h_1\|_\infty + |K_{2,+a_0}|\|h_2\|_\infty$$

for every t, because reflections and translations do not change the volume of the neighbourhood. □

Remark 8.7 (sufficient conditions). (UI) holds, with $\eta(\ell) \le C\ell^{-\delta}$, if $\sup_{N\ge N_0} \omega_N(|k_N g_{\mathrm{d}}(\tilde\rho_0)|^{1+\delta}) \le C$ for some $\delta > 0$; (UI) holds with $\eta(\ell) = 0$ for $\ell \ge \sup_N |k_N|\, \|g_{\mathrm{d}}\|_\infty$ if g_{d} is bounded and $\sup_N |k_N| < \infty$. Every property of the S_ℓ that is stable under pointwise limits on $\mathcal{A}$ passes to S. For clauses (a)–(c) Lemma 8.6 is contained in Theorem 8.5; what (UI) adds is the approximation statement (iii). Remark 7.5(iv) states the first-moment weakening for the pinned normalisations; Lemmas 8.2, 8.3 and Theorem 8.5 are its proof, in the general normalisation.

Remark 8.8 (no integrability at all). Let g_{d} be any real Borel function and c_N, c'_N any real numbers, with no condition. (1) Since $|Z_N(\sigma)| \le 1$, the family $(Z_N)_{N\ge N_0}$ lies in the product of closed unit discs indexed by $\mathcal{T}$, which is compact. Hence a *subnet* (N_α) has $Z_{N_\alpha}(\sigma) \to Z(\sigma)$ for every σ, and $S(e_\sigma) := Z(\sigma)$ extends to a linear functional on $\mathcal{A}$ with $\omega_{N_\alpha}(\iota_{N_\alpha}(a)) \to S(a)$ and $|S(a)| \le M(a)$. This is not clause (a), which asks for a sequence of levels; $\mathcal{T}$ is uncountable, and this argument produces no subsequence. (2) For every prescribed *countable* set $\mathcal{D}$ of test pairs, a diagonal extraction gives a subsequence along which $Z_N(\sigma)$ converges for all $\sigma \in \mathcal{D}$; Step 3 of the proof of Proposition 6.1 uses only $|Z_N| \le 1$ for this. (3) For the limits of (1) and of (2), Lemma 5.1 gives $Z(\tau_t\sigma) = Z(\sigma)$ for the times $t \in a_0\mathbb{Z}[1/5] = \bigcup_N \varepsilon_N\mathbb{Z}$ (in (2), for $\sigma \in \mathcal{D}$ and $\mathcal{D}$ invariant under these translations): for such t the invariance is exact at every large level, and no estimate is used. (4) The remaining steps are the passage from $\mathcal{D}$ to all test pairs, the continuity of $t \mapsto S(\tau_t a)$, hence clause (b) for all real t, and clause (c). In the proof of Proposition 6.1 each of them uses (6.1) and nothing else, and what gives (6.1) here is (Y). Without (Y) this argument does not give these steps; we do not know whether they can fail for some g_{d}.

8.4. **General normalisations: the pinned scheme as the case of equality.** Theorem 8.5 is stated for all real c_N, c'_N with (N) and (Y). The Riemann scaling of the curvature normalisation, which enters the proof of Proposition 4.2 (Remark 7.3), can thus be weakened to the *bound* (N); the equality $\zeta_N = 1$ is not required. We do not know whether (N) is necessary. The next statement describes the limit. Its degenerate cases (1) and (3) require $\zeta_{N_m} \to 0$ for the ratio $\zeta_N = c_N \varepsilon_N^{-3/2}$, a condition on the *normalisation* of the curvature; they do not occur for the pinned normalisations, where $\zeta_N = k_N = 1$. Case (2) is a condition on the density term, that is, on the ratio $k_N = c'_N \varepsilon_N^{-3}$ and the density function together.

Corollary 8.9 (the limit for general normalisations)**.** *Assume* (N) *and* (Y)*, and let* (N_m)*,* S *be as in Theorem 8.5.*

(1) *If* $\zeta_{N_m} \to 0$*, then* $S(e_{(f,h)}) = S(e_{(0,h)})$ *for all test pairs: the curvature drops out, and* S *is the limit of the density sector alone.*

(2) *If* $\omega_{N_m}(|Y_{N_m}(h)| \wedge 2) \to 0$ *for every* h*, for instance if* $k_{N_m}\omega_{N_m}(|g_{\mathrm{d}}(\tilde\rho_0)|) \to 0$*, then* $S(e_{(f,h)}) = S(e_{(f,0)})$ *for all test pairs: the density drops out, and* S *is the limit of the curvature sector at the normalisation* c_N*.*

(3) *If both hold, then* $S(e_\sigma) = 1$ *for every* σ*, that is,* $S(\sum_j c_j e_{\sigma_j}) = \sum_j c_j$*. Clauses (a)–(c) hold,* $S(\Theta a \cdot b) = S(\Theta a)S(b)$*, and the Hilbert space reconstructed from* S *(Section 9) is one-dimensional.*

(4) *Assume in addition* $\sup_N |k_N| < \infty$ *and* $A_1^{\mathrm{pin}} := \sup_{N \geq N_0} \omega_N(|g_{\mathrm{d}}(\tilde\rho_0)|) < \infty$*, and let* Φ_N^{pin}*,* Z_N^{pin} *belong to the scheme with the same* g_{d} *and the pinned normalisations. Then* $\Phi_N(f,h) = \Phi_N^{\mathrm{pin}}(\zeta_N f, k_N h)$ *exactly. If* $\zeta_{N_m} \to \zeta_\infty$ *and* $k_{N_m} \to k_\infty$*, which holds along a subsequence of every sequence of levels, then*

$$S(e_{(f,h)}) = S^{\mathrm{pin}}\big(e_{(\zeta_\infty f,\, k_\infty h)}\big) \qquad \text{for all test pairs } (f,h),$$

where S^{pin} *is a limit of the pinned scheme, given by Theorem 8.5, along* (N_m) *if* $\zeta_\infty k_\infty \neq 0$ *and along a subsequence of* (N_m) *otherwise. The dilation* $D(f,h) = (\zeta_\infty f, k_\infty h)$ *of test pairs commutes with* θ *and* τ_t *and preserves supports, so it induces an algebra homomorphism of* $\mathcal{A}$ *which commutes with* Θ *and* τ_t *and maps* $\mathcal{A}_+$ *into* $\mathcal{A}_+$*, and* $S = S^{\mathrm{pin}} \circ D$*. The pinned normalisations are the case of equality* $\zeta_N = k_N = 1$*,* $D = \mathrm{id}$*.*

The properties (a)–(c) of Theorem 7.1 do not exclude case (3); the non-degeneracy of the limit is a separate question. The additional assumptions of part (4) hold in case (V1) of Proposition 8.10.

Proof. (1) By (8.3) and the curvature part of the proof of Lemma 8.3, $|Z_N(f,h) - Z_N(0,h)| \leq \omega_N(|X_N(f)| \wedge 2) \leq |\zeta_N| R(f)$; let $N = N_m \to \infty$. (2) Likewise $|Z_N(f,h) - Z_N(f,0)| \leq \omega_N(|Y_N(h)| \wedge 2)$, and $\omega_N(|Y_N(h)|) \leq |k_N| \omega_N(|g_{\mathrm{d}}(\tilde\rho_0)|) E(h)$ by the proof of Lemma 8.2(1). (3) By (1) and (2), $S(e_{(f,h)}) = S(e_{(0,h)}) = S(e_{(0,0)}) = 1$. Then S is the character $\sum_j c_j e_{\sigma_j} \mapsto \sum_j c_j$ of the group algebra, which is multiplicative; the sesquilinear form $(a,b) \mapsto S(\Theta a \cdot b) = S(\Theta a)S(b)$ has rank one. (4) The identity holds because Φ_N is linear in the test pair and $c_N = \zeta_N \varepsilon_N^{3/2}$, $c'_N = k_N \varepsilon_N^3$. The pinned scheme satisfies (N) with $\zeta = 1$ and (Y1) with $A_1 = A_1^{\mathrm{pin}}$, so by (8.3) for it,

$$\big|Z_N^{\mathrm{pin}}(\zeta_N f, k_N h) - Z_N^{\mathrm{pin}}(\zeta_\infty f, k_\infty h)\big| \leq |\zeta_N - \zeta_\infty|\, R(f) + |k_N - k_\infty|\, A_1^{\mathrm{pin}} E(h) \longrightarrow 0$$

along (N_m). Hence $Z_{N_m}^{\mathrm{pin}}(\zeta_\infty f, k_\infty h) \to S(e_{(f,h)})$ for all (f,h). If $\zeta_\infty k_\infty \neq 0$ the dilation D is onto $\mathcal{T}$, so $Z_{N_m}^{\mathrm{pin}}$ converges on all of $\mathcal{T}$, and its limit S^{pin} has the properties of Theorem 8.5 by Steps 3–6 of the proof of Proposition 6.1 along (N_m). Otherwise pass to a subsequence of (N_m) along which the pinned scheme converges (Theorem 8.5, whose proof starts from any infinite

set of levels). The properties of D are read off its definition; bounded ratios have convergent subsequences. □

8.5. **What the available bounds give.** We write $\rho_x = \rho_{0,0} + \tilde\rho_x \geq 0$ for the site field, as in Section 2, and $\rho_{0,0}(N)$ when the level of the vacuum value must be explicit: $\rho_{0,0}(N)^2 = \varepsilon_N \rho_0^2$ decreases with N. (The constant ρ_0 is the continuum vacuum value; to avoid a clash, the site field is always written at a generic site x. All states below are translation invariant, so one-site moments do not depend on x.)

This paper proves no moment bound for the site field that is uniform in the level and holds for every δm_B: the one constant uniform in N is that of Theorem 3.1 (Remark 2.1). Lemma 11.2 and Theorem 11.5(b) give $\langle e^{q\rho_x^2}\rangle_L^{\mathrm{fine}} \leq C_N(q)$ and $\int e^{q\rho_x^2} d\mu_N \leq C_N(q)$, uniformly in the volume at a fixed level, and the constant diverges with the level:

$$C_N(q) \;\geq\; \pi^{24} \exp\big(192/(\varepsilon_N e^2)\big) \qquad \text{for every } q \geq 0 \text{ and every } N. \tag{8.5}$$

Indeed, in the notation of Lemma 11.2, $C_N(q) = (G_q^{\mathrm{ct}}/G_0^{\mathrm{ct}})C_1^8$ with $G_q^{\mathrm{ct}} \geq G_0^{\mathrm{ct}}$ and

$$C_1 \geq G_0^{\mathrm{ct}} (2\pi/\gamma_N)^3 K'^3_{\gamma_N}/c_0 = (G_0^{\mathrm{ct}}/c_g^{\mathrm{ct}})(\pi/c_h)^3 K'^3_{\gamma_N} e^{24/\gamma_N^2} \geq \pi^3 e^{24/\gamma_N^2},$$

because $G_0^{\mathrm{ct}} \geq \int_0^1 g_{\mathrm{ct}}(r)\, dr \geq c_g^{\mathrm{ct}}$, $c_h \leq 1$ and $K'_{\gamma_N} \geq 1$; and $\gamma_N^2 = \varepsilon_N e^2$. The divergence is not only an effect of the factor C_1^8: at $\delta m_B = 0$ the site ratio itself satisfies $G_q^{\mathrm{ct}}/G_0^{\mathrm{ct}} \geq \pi^{-1/2} \exp\big(q^2/(4\varepsilon_N\lambda) - 1\big)$ for every $q > 0$. Indeed, with $u = \rho^2$, $G_q^{\mathrm{ct}} = \frac12 \int_0^\infty e^{-\lambda_0(u-\rho_{0,0}^2)^2 + qu} du$; the exponent is at least $q^2/(4\lambda_0) - 1$ on the interval of length $\lambda_0^{-1/2}$ to the right of its maximum point $u = \rho_{0,0}^2 + q/(2\lambda_0)$, so $G_q^{\mathrm{ct}} \geq \frac12 \lambda_0^{-1/2} e^{q^2/(4\lambda_0)-1}$, while $G_0^{\mathrm{ct}} \leq \frac12 (\pi/\lambda_0)^{1/2}$.

Proposition 8.10 (pairs (g_{d}, c'_N) that satisfy the density clause)**.** *Let c_N satisfy* (N)*. In each of the cases* (V1)–(V3) *condition* (UI) *holds; hence* (Y1)*, Theorem 8.5 and Lemma 8.6 apply. In case* (V4)*,* (Y1) *holds and Theorem 8.5 applies.*

(V1) Bounded on the relevant half line. *g_{d} is bounded on $[-\rho_{\sup}, \infty)$, where $\rho_{\sup} := \sup_{N \geq N_0} \rho_{0,0}(N)$ (on $\mathbb{R}$ if $\rho_{\sup} = \infty$; $\rho_{\sup} = \rho_{0,0}(N_0)$ for the scaling of Section 2), in particular if it is bounded, and $\sup_N |k_N| < \infty$; the Riemann normalisation $c'_N = \varepsilon_N^3$ is included. Then $\eta(\ell) = 0$ for $\ell \geq \sup_N |k_N| \sup_{r \geq -\rho_{\sup}} |g_{\mathrm{d}}(r)|$. Examples, for $\rho_{\sup} < \infty$: e^{-r}, the negative part of r, $\min(r, 1)$.*

(V2) Polynomial growth, from an exponential bound. *$|g_{\mathrm{d}}(r)| \leq C_g(1 + |r|)^{2m}$ with an integer $m \geq 1$. For each $N \geq N_0$ let $t_N > 0$ and $B_N := C_N(t_N)$, the constant of Lemma 11.2, so that $\langle e^{t_N \rho_x^2}\rangle_L^{\mathrm{fine}} \leq B_N$ for all sides $L \geq 4\varepsilon_N$. Put $M_N := 1 + \rho_{0,0}^{2m} + m!\, t_N^{-m} B_N$ and let $|c'_N| \leq C\, \varepsilon_N^3/M_N$. Then*

$$\omega_N\big(|k_N g_{\mathrm{d}}(\tilde\rho_0)|^2\big) \leq C^2 C_g^2\, 2^{8m-2} \tbinom{2m}{m} \quad (N \geq N_0), \qquad \eta(\ell) \leq C^2 C_g^2\, 2^{8m-2} \tbinom{2m}{m}\, \ell^{-1}.$$

(V3) Polynomial growth, from Lemma 8.12 below. *$|g_{\mathrm{d}}(r)| \leq C_g(1 + |r|)^p$ with a real $p \geq 0$, and $|c'_N| \leq C\, \varepsilon_N^3\, (1 + \rho_{0,0} + b_N^{1/2})^{-p}$, with the scale b_N of Lemma 8.12. Then, with $p' := \max(1, \lceil p \rceil)$ and the constants c_j of that lemma, $\omega_N(|k_N g_{\mathrm{d}}(\tilde\rho_0)|^2) \leq C^2 C_g^2\, c_{p'}^{p/p'}$ for $N \geq N_0$, and $\eta(\ell) \leq C^2 C_g^2\, c_{p'}^{p/p'}\, \ell^{-1}$.*

(V4) Some normalisation always works at the first-moment level. *If $a_N := \omega_N(|g_{\mathrm{d}}(\tilde\rho_0)|) < \infty$ for every $N \geq N_0$, which holds whenever $|g_{\mathrm{d}}(r)| \leq C_g e^{qr^2}$, then* (Y1) *holds for every c'_N with $|c'_N| \leq C\varepsilon_N^3/\max(1, a_N)$. Clause* (Y1) *holds at the Riemann normalisation $c'_N = \varepsilon_N^3$ if and only if $\sup_N a_N < \infty$.*

Proof. (V1) $\tilde\rho_0 \geq -\rho_{0,0}(N) \geq -\rho_{\sup}$ on Ω_N, by the definition of $\rho_{\sup}$, so $|k_N g_{\mathrm{d}}(\tilde\rho_0)| \leq \sup_N |k_N| \sup_{r\geq -\rho_{\sup}} |g_{\mathrm{d}}(r)|$ pointwise.

(V2) $B_N \geq 1$ because $e^{t\rho^2} \geq 1$. By convexity, $(1+|r|)^{4m} \leq 2^{4m-1}(1+|r|^{4m})$ and $|\tilde\rho_0|^{4m} \leq (\rho_x + \rho_{0,0})^{4m} \leq 2^{4m-1}(\rho_x^{4m} + \rho_{0,0}^{4m})$ at the site $x=0$. From $\rho^{4m} \leq (2m)!\, t^{-2m} e^{t\rho^2}$ and Theorem 11.5(e), $\int \rho_x^{4m} d\mu_N \leq \liminf_n \langle \rho_x^{4m}\rangle_{L_n}^{\mathrm{fine}} \leq (2m)!\, t_N^{-2m} B_N$. Hence

$$\omega_N\big((1+|\tilde\rho_0|)^{4m}\big) \leq 2^{8m-2}\big(1+\rho_{0,0}^{4m} + (2m)!\, t_N^{-2m} B_N\big),$$
$$M_N^2 \geq 1 + \rho_{0,0}^{4m} + (m!)^2 t_N^{-2m} B_N^2 \geq 1 + \rho_{0,0}^{4m} + (m!)^2 t_N^{-2m} B_N,$$

and the quotient of the two right sides is at most $2^{8m-2}(2m)!/(m!)^2$. With $|k_N| \leq C/M_N$ this is the L^2 bound; a family bounded in L^2 by Q has $\eta(\ell) \leq Q^2/\ell$.

(V3) For $p=0$ the function is bounded and $|k_N| \leq C$. Let $p>0$. On the probability space (Ω_N, μ_N), first by Lyapunov's inequality ($2p \leq 2p'$) and then by Minkowski's inequality in $L^{2p'}$ ($2p' \geq 2$), with $|\tilde\rho_0| \leq \rho_x + \rho_{0,0}$ and Lemma 8.12,

$$\big\|(1+|\tilde\rho_0|)^p\big\|_{L^2} = \big\|1+|\tilde\rho_0|\big\|_{L^{2p}}^p \leq \big\|1+|\tilde\rho_0|\big\|_{L^{2p'}}^p$$
$$\leq \big(1+\rho_{0,0} + c_{p'}^{1/(2p')} b_N^{1/2}\big)^p \leq c_{p'}^{p/(2p')}\big(1+\rho_{0,0}+b_N^{1/2}\big)^p,$$

since $c_{p'} \geq 1$. Multiply by $|k_N| C_g \leq C C_g (1+\rho_{0,0}+b_N^{1/2})^{-p}$.

(V4) $|k_N| a_N \leq C$. If $|g_{\mathrm{d}}(r)| \leq C_g e^{qr^2}$, then $|g_{\mathrm{d}}(\tilde\rho_0)| \leq C_g e^{2q\rho_{0,0}^2} e^{2q\rho_x^2}$ and $a_N \leq C_g e^{2q\rho_{0,0}^2} C_N(2q) < \infty$ by Theorem 11.5(b). The last sentence is the definition of (Y1) at $k_N = 1$. □

Remark 8.11 (the scope of the cases). Proposition 8.10 covers the following density functions and normalisations, and nothing is asserted outside them. Case (V1): density functions bounded on $[-\rho_{\sup}, \infty)$, at every normalisation with $\sup_N |k_N| < \infty$, the Riemann normalisation $c'_N = \varepsilon_N^3$ included. Cases (V2) and (V3): polynomial growth, at the normalisations stated there, which compare with the Riemann normalisation as follows. In (V2), by (8.5), the normalisation is smaller than the Riemann one by at least the factor $(t_N^m/m!)\,\pi^{-24}\exp(-192/(\varepsilon_N e^2))$. In (V3), $b_N \geq (2\varepsilon_N\lambda)^{-1/2}$, so that it is smaller by at least the factor $(2\varepsilon_N\lambda)^{p/4}$; under the additional condition $\sup_N (\delta m_B)_- \lambda_0^{-1/2} < \infty$ on the counterterm, $b_N \leq C' \varepsilon_N^{-1/2}$ and $c'_N = \varepsilon_N^{3+p/4}$ is allowed in (V3). Case (V4): density functions with $a_N < \infty$ at every level, in particular those of growth at most $C_g e^{qr^2}$, at the normalisations $|c'_N| \leq C\varepsilon_N^3/\max(1, a_N)$. For a density function that is unbounded on $[-\rho_{\sup}, \infty)$ the cases do not give the Riemann normalisation for every value of the counterterm $\delta m_B(N)$ (Remark 8.13): for $g_{\mathrm{d}} = \mathrm{id}$ or a polynomial of degree p, clause (Y1) at the Riemann normalisation is the level-uniform moment bound $\sup_N \omega_N(|\tilde\rho_0|^p) < \infty$. In the cases (V2) and (V3) Theorem 8.5 holds, but Proposition 8.10 does not assert that the density term survives in the limit; this question is treated in Remark 8.14.

Lemma 8.12 (all moments of the site field at the scale b_N)**.** *For every level N, every torus side L, every real δm_B, every $\kappa_N \geq 0$, every site x and every integer $j \geq 1$,*

$$\langle \rho_x^{2j}\rangle_L^{\mathrm{fine}} \leq c_j\, b_N^j, \qquad b_N := \rho_{0,0}^2 + \frac{(\delta m_B)_-}{4\lambda_0} + \frac{1}{(2\lambda_0)^{1/2}},$$
$$c_1 = c_2 = 1, \qquad c_j = c_{j-1} + (j-1)c_{j-2} \quad (j \geq 3),$$

where $(\delta m_B)_- := \max(0, -\delta m_B)$; and $\int \rho_x^{2j} d\mu_N \leq c_j b_N^j$. In particular $c_3 = 3$, $c_4 = 6$, and $c_j \geq 1$.

Proof. Fix N, L and x, write $\langle\cdot\rangle = \langle\cdot\rangle_L^{\mathrm{fine}}$ and $m_i := \langle \rho_x^i \rangle$, which is finite for every $i \geq 0$ (Definition 10.1: every function dominated by $Ce^{\sum_y q_y \rho_y^2}$ is integrable) and does not depend on x. As a function of $\rho_x \in [0,\infty)$, with all other variables fixed, the weight is

$$W = \rho_x\, e^{-U(\rho_x)} \prod_{b \ni x} \exp\Big(-\tfrac{1}{2}\kappa_N \big(\rho_x^2 + \rho_{y(b)}^2 - 2\rho_x \rho_{y(b)} \cos\vartheta_b\big)\Big) \times (\text{a factor free of } \rho_x),$$

$$U(\rho) := \lambda_0(\rho^2 - \rho_{0,0}^2)^2 + \tfrac{1}{2}\delta m_B \rho^2,$$

the product over the six bonds b at x, with $y(b)$ the other end of b and $\vartheta_b = \gamma_N B_b$; the Villain sum and the integrals over the bond variables do not involve ρ_x. Let $k \geq 1$. The function $\rho_x^k W$ vanishes at $\rho_x = 0$ and decays at infinity like $e^{-\lambda_0 \rho_x^4}$ times a polynomial, so $\int_0^\infty \partial_{\rho_x}[\rho_x^k W]\, d\rho_x = 0$, and

$$\partial_{\rho_x}\big[\rho_x^k W\big] = \Big((k+1)\rho_x^{k-1} - \rho_x^k U'(\rho_x) - \kappa_N \sum_{b \ni x} \big(\rho_x^{k+1} - \rho_x^k \rho_{y(b)} \cos\vartheta_b\big)\Big) W.$$

Each term on the right is a polynomial in the site fields times a bounded function times W, hence integrable over all variables (Definition 10.1 and the inequality (2.1)); by Fubini's theorem we may integrate over the other variables and divide by the mass of W:

$$\big\langle \rho_x^k U'(\rho_x)\big\rangle = (k+1)\, m_{k-1} - \kappa_N \sum_{b \ni x} \big\langle \rho_x^{k+1} - \rho_x^k \rho_{y(b)} \cos\vartheta_b \big\rangle. \tag{8.6}$$

The hopping sum is non-negative. Since $\rho \geq 0$ and $|\cos| \leq 1$, Hölder's inequality with the exponents $(k+1)/k$ and $k+1$ and the translation invariance of the torus state give

$$\langle \rho_x^k \rho_y \cos\vartheta_b\rangle \leq \langle \rho_x^{k+1}\rangle^{k/(k+1)} \langle \rho_y^{k+1}\rangle^{1/(k+1)} = \langle \rho_x^{k+1}\rangle.$$

The recursion. $U'(\rho) = 4\lambda_0 \rho(\rho^2 - \rho_{0,0}^2) + \delta m_B \rho$, so

$$\langle \rho_x^k U'(\rho_x)\rangle = 4\lambda_0 m_{k+3} - (4\lambda_0 \rho_{0,0}^2 - \delta m_B) m_{k+1} \geq 4\lambda_0 m_{k+3} - \beta m_{k+1}$$

with $\beta := 4\lambda_0 \rho_{0,0}^2 + (\delta m_B)_-$. With (8.6) and $\kappa_N \geq 0$,

$$4\lambda_0\, m_{k+3} \leq (k+1)\, m_{k-1} + \beta\, m_{k+1} \qquad (k \geq 1). \tag{8.7}$$

$j = 1, 2$. For $k = 1$, with $m_0 = 1$ and $m_2 \leq m_4^{1/2} =: s$, (8.7) reads $4\lambda_0 s^2 - \beta s - 2 \leq 0$, hence

$$s \leq \frac{\beta + (\beta^2 + 32\lambda_0)^{1/2}}{8\lambda_0} \leq \frac{\beta}{4\lambda_0} + \frac{(32\lambda_0)^{1/2}}{8\lambda_0} = \rho_{0,0}^2 + \frac{(\delta m_B)_-}{4\lambda_0} + \frac{1}{(2\lambda_0)^{1/2}} = b_N,$$

that is, $m_4 \leq b_N^2$ and $m_2 \leq b_N$.

$j \geq 3$. Take $k = 2j-3$ in (8.7) and use induction:

$$\begin{aligned} m_{2j} &\leq \frac{j-1}{2\lambda_0}\, m_{2(j-2)} + \frac{\beta}{4\lambda_0}\, m_{2(j-1)} \leq \frac{j-1}{2\lambda_0}\, c_{j-2} b_N^{j-2} + \frac{\beta}{4\lambda_0}\, c_{j-1} b_N^{j-1} \\ &\leq \big((j-1)c_{j-2} + c_{j-1}\big) b_N^j, \end{aligned}$$

because $1/(2\lambda_0) \leq b_N^2$ and $\beta/(4\lambda_0) \leq b_N$.

The limit state. ρ_x^{2j} is a continuous local function ≥ 0, so $\int \rho_x^{2j}\, d\mu_N \leq \liminf_n \langle \rho_x^{2j}\rangle_{L_n}^{\mathrm{fine}} \leq c_j b_N^j$ by Theorem 11.5(e). □

The scale b_N is polynomially bounded in ε_N^{-1} when $(\delta m_B)_-$ stays bounded, against the exponential of (8.5), but it is not uniform in the level: $b_N \geq (2\varepsilon_N \lambda)^{-1/2}$. The bound uses the hopping term only through its sign.

Remark 8.13 (a level-uniform moment bound requires information on the counterterm). Theorems 7.1, 11.5 and 8.5 hold for every real δm_B, with any dependence on N. For some such choices the level-uniform moments fail; so they cannot follow from statements that hold for every δm_B. Let $\delta m_B(N) = -c$ for all N, with a fixed $c > 0$, and $\kappa_N = 1$. Then, uniformly in the side L,

$$\langle \rho_x^2 \rangle_L^{\text{fine}} \;\geq\; \frac{3c}{16\,\lambda\,\varepsilon_N} - \frac{6}{c}\log\frac{1}{\varepsilon_N \lambda} - C', \tag{8.8}$$

$$\langle \rho_x \rangle_L^{\text{fine}} \;\geq\; \frac{\big(\langle \rho_x^2 \rangle_L^{\text{fine}}\big)^{3/2}}{b_N} \;\geq\; \frac{3\sqrt{3}}{16}\Big(\frac{c}{\lambda\,\varepsilon_N}\Big)^{1/2}\big(1 - o(1)\big) \qquad (N \to \infty), \tag{8.9}$$

with C' depending on $c, \lambda, e, \rho_0, a_0$ only and $3\sqrt{3}/16 > 0.32$. By Theorem 11.5(a) (both functions are continuous, local and dominated by $e^{\rho_x^2}$) the same bounds hold for $\omega_N(\rho_x^2)$ and $\omega_N(\rho_x)$. Hence the second *and the first* moment of the site variable diverge, $\omega_N(|\tilde\rho_0|) \geq \omega_N(\rho_x) - \rho_{0,0} \to \infty$, and (Y1) fails for $g_{\mathrm{d}} = \mathrm{id}$ at the Riemann normalisation $c'_N = \varepsilon_N^3$: there it is a genuine condition. Moreover, since $\langle \rho_x^2 \rangle_L^{\text{fine}}$ is non-increasing in δm_B (below), (8.8) holds whenever $\delta m_B(N) \leq -c$; so a level-uniform second moment forces $(\delta m_B)_-(N) \to 0$.

Proof of (8.8) *and* (8.9). Fix N and L; the torus has $n := (L/\varepsilon_N)^3$ sites, $3n$ bonds and $3n$ plaquettes. For $d \in \mathbb{R}$ let $Z_L(d)$ be the mass of the weight W with $\delta m_B = d$, $\langle\cdot\rangle_{L,d}$ its state, and $F(d) := n^{-1}\log Z_L(d)$.

Convexity. By the quartic domination (2.1) one may differentiate under the integral: $F'(d) = -\frac{1}{2}\langle \rho_x^2 \rangle_{L,d}$ (translation invariance) and $F''(d) = (4n)^{-1}\operatorname{Var}_{L,d}\big(\sum_y \rho_y^2\big) \geq 0$, the variance being taken under $\langle\cdot\rangle_{L,d}$. So $d \mapsto \langle \rho_x^2 \rangle_{L,d}$ is non-increasing, and

$$F(-c) - F(-c/2) = \tfrac{1}{2}\int_{-c}^{-c/2} \langle \rho_x^2 \rangle_{L,s}\, ds \leq \tfrac{c}{4}\,\langle \rho_x^2 \rangle_{L,-c}.$$

Upper bound for $F(-c/2)$. As in Step 1 of the proof of Lemma 11.2: the hopping factors are at most 1; the sum over the closed v is at most the sum over all v, which is at most $K'^{3n}_{\gamma_N}$; each bond variable ranges over an interval of length $2\pi/\gamma_N$; and, with $u = \rho^2$ and a completed square, the site integral at $d = -c/2$ is

$$\begin{aligned}\int_0^\infty \rho\, e^{-\lambda_0(\rho^2-\rho_{0,0}^2)^2 + \frac{c}{4}\rho^2} d\rho &= \tfrac{1}{2} e^{\frac{c}{4}\rho_{0,0}^2 + \frac{c^2}{64\lambda_0}} \int_0^\infty e^{-\lambda_0(u - \rho_{0,0}^2 - \frac{c}{8\lambda_0})^2} du \\ &\leq \tfrac{1}{2}\Big(\frac{\pi}{\lambda_0}\Big)^{1/2} e^{\frac{c}{4}\rho_{0,0}^2 + \frac{c^2}{64\lambda_0}}.\end{aligned}$$

Hence

$$F(-c/2) \leq \frac{c^2}{64\lambda_0} + \frac{c}{4}\rho_{0,0}^2 + \frac{1}{2}\log\frac{\pi}{4\lambda_0} + 3\log\frac{2\pi}{\gamma_N} + 3\log K'_{\gamma_N}.$$

Lower bound for $F(-c)$. All terms of the sum over v are ≥ 0; keep $v = 0$ and restrict the integral to $\rho_y \in [R, R+1]$ for all sites, $R^2 := \rho_{0,0}^2 + c/(4\lambda_0)$, and $|B_b| \leq 1/(\gamma_N(R+1))$ for all bonds (a subset of the domain). On this set the site factor at $d = -c$ is

$$\rho\, e^{-\lambda_0(\rho^2-\rho_{0,0}^2)^2 + \frac{c}{2}\rho^2} = \rho\, e^{-\lambda_0(\rho^2 - R^2)^2 + \frac{c^2}{16\lambda_0} + \frac{c}{2}\rho_{0,0}^2} \geq R\, e^{\frac{c^2}{16\lambda_0} - 8\lambda_0\rho_{0,0}^2 - 2c - 2\lambda_0},$$

because $0 \leq \rho^2 - R^2 \leq 2R + 1$ and $\lambda_0(2R+1)^2 \leq \lambda_0(8R^2 + 2)$; the hopping factor is at least $e^{-\kappa_N} = e^{-1}$, because $|e^{i\gamma_N B}\rho' - \rho|^2 = (\rho' - \rho)^2 + 2\rho\rho'(1 - \cos\gamma_N B) \leq 1 + (R+1)^2\gamma_N^2 B^2 \leq 2$; and the plaquette factor is at least $e^{-8/(\gamma_N(R+1))^2}$, because $|(dB)_p| \leq 4/(\gamma_N(R+1))$. The set

has volume 1 per site and $2/(\gamma_N(R+1))$ per bond. Hence

$$F(-c) \geq \frac{c^2}{16\lambda_0} + \log R - 8\lambda_0\rho_{0,0}^2 - 2c - 2\lambda_0 - 3 + 3\log\frac{2}{\gamma_N(R+1)} - \frac{24}{\gamma_N^2(R+1)^2}.$$

The difference. The powers of γ_N cancel, $3\log\frac{2}{\gamma_N(R+1)} - 3\log\frac{2\pi}{\gamma_N} = -3\log(\pi(R+1))$, and $\gamma_N^2R^2 \geq \gamma_N^2 c/(4\lambda_0) = e^2c/(4\lambda)$. Therefore

$$F(-c) - F(-c/2) \geq \frac{3c^2}{64\lambda_0} - \Big[3\log(\pi(R+1)) - \log R + \tfrac{1}{2}\log\frac{\pi}{4\lambda_0} + \big(\tfrac{c}{4} + 8\lambda_0\big)\rho_{0,0}^2 + 2c + 2\lambda_0 + 3 + 3\log K'_{\gamma_N} + \frac{96\lambda}{e^2c}\Big].$$

In the bracket,

$$3\log(R+1) - \log R = \log(\rho_{0,0}^2 + \frac{c}{4\lambda_0}) + 3\log(1+\frac{1}{R}) = \log\frac{1}{\lambda_0} + \log(\frac{c}{4} + \lambda_0\rho_{0,0}^2) + 3\log(1+\frac{1}{R}),$$

and $\lambda_0 \leq a_0\lambda$, $\rho_{0,0}^2 \leq a_0\rho_0^2$, $1/R \leq (4a_0\lambda/c)^{1/2}$, while K'_{γ_N} is bounded uniformly in N by the bound given in Lemma 11.2, which increases with $\gamma_N \leq a_0^{1/2}e$. So the bracket is at most $\frac{3}{2}\log\frac{1}{\lambda_0} + C''$ with $C'' = C''(c,\lambda,e,\rho_0,a_0)$, and with the convexity step,

$$\langle\rho_x^2\rangle_{L,-c} \geq \frac{4}{c}\Big(\frac{3c^2}{64\lambda_0} - \frac{3}{2}\log\frac{1}{\lambda_0} - C''\Big) = \frac{3c}{16\lambda_0} - \frac{6}{c}\log\frac{1}{\lambda_0} - \frac{4C''}{c},$$

which is (8.8) ($\lambda_0 = \varepsilon_N\lambda$).

The first moment. By Hölder's inequality with the exponents 3/2 and 3, $\langle\rho_x^2\rangle \leq \langle\rho_x\rangle^{2/3}\langle\rho_x^4\rangle^{1/3}$, so $\langle\rho_x\rangle \geq \langle\rho_x^2\rangle^{3/2}\langle\rho_x^4\rangle^{-1/2} \geq \langle\rho_x^2\rangle^{3/2}/b_N$ by Lemma 8.12. At $\delta m_B = -c$, $b_N = \frac{c}{4\lambda_0}(1+o(1))$ and $\langle\rho_x^2\rangle_{L,-c} \geq \frac{3c}{16\lambda_0}(1-o(1))$, which gives $(\frac{3c}{16\lambda_0})^{3/2}\frac{4\lambda_0}{c}(1-o(1)) = \frac{3\sqrt{3}}{16}(c/\lambda_0)^{1/2}(1-o(1))$, uniformly in L. □

Remark 8.14 (in the reduced normalisations the density can drop out). Let g_{d}, m, t_N, B_N, M_N and c'_N be as in Proposition 8.10(V2). By the convexity inequalities of its proof and Lemma 8.12, $\omega_N(|g_{\mathrm{d}}(\tilde\rho_0)|) \leq C_g 2^{4m-2}(1+\rho_{0,0}^{2m} + c_m b_N^m)$. By (8.5), $|k_N| \leq Ct_N^m\pi^{-24}\exp(-192/(\varepsilon_N e^2))/m!$. So, as soon as $t_N^m(1+b_N^m)\exp(-192/(\varepsilon_N e^2)) \to 0$, Corollary 8.9(2) applies: $S(e_{(f,h)}) = S(e_{(f,0)})$, and the density drops out of S.

In (V3) the situation is different: for $g_{\mathrm{d}}(r) = r^2$, $c'_N = \varepsilon_N^3(1+\rho_{0,0}+b_N^{1/2})^{-2}$ and $\delta m_B(N) = -c$, the normalised mean satisfies $\liminf_N k_N\omega_N(\tilde\rho_0^2) \geq \frac{3}{4}$ by (8.8) and $b_N = \frac{c}{4\lambda_0}(1+o(1))$, so the sufficient condition $k_N\omega_N(|g_{\mathrm{d}}(\tilde\rho_0)|) \to 0$ of Corollary 8.9(2) fails, and so does its hypothesis: for $h \geq 0$, $h \neq 0$, the variable $Y_N(h) \geq 0$ has mean at least $(\frac{3}{4} - o(1))E_N(h)$ and $L^2(\mu_N)$ norm at most $E(h)$ (Minkowski's inequality and Lemma 8.12), so $\omega_N(Y_N(h) \wedge 2)$ stays bounded below by the Paley–Zygmund inequality. We do not know whether the limit density is more than a constant field in this case.

8.6. **Results that hold for the extended class.**

Remark 8.15 (results that extend). The *extended class* is the extended scheme of Definition 8.1 with (N) and (Y). The following three lists sort results of this paper according to their relation to this class: list (A) contains results that hold for the extended class, list (B) results in which the density function does not enter, and list (C) results that are stated for bounded density functions only.

(A) *Results that extend.* The following results hold for the extended class, in the form indicated. They use of the scheme only clauses (a)–(c) or the inequality (6.1), the unimodularity (2.4), and the fact that the density function and the normalisations do not depend on the position.

(A1) Theorem 7.1 holds in the form of Theorem 8.5, Proposition 6.1 in the form of Proposition 8.4, and Lemma 5.1 verbatim.
(A2) Corollary 11.7(i), which restates the chain from the estimate to Theorem 7.1 for the limit points, extends by way of Theorem 8.5.
(A3) Part (i) of Theorem 9.4, the continuity of Z, holds with the seminorm of (8.4).
(A4) The first assertion of Proposition 9.12(b), the evenness $Z(-f,h)=Z(f,h)$ under charge conjugation, extends: in its proof the invariance of μ_N is applied to the function $e^{i\Phi_N(\sigma)}$, which is bounded and Borel for every real Borel g_{d}.
(A5) The inheritance of reflection positivity by S (Proposition 9.9) extends. It follows from clause (a) and the reflection positivity of ω_N on the bounded Borel functions of the closed half space (Theorem 11.5(d)), because for a test pair supported in $\{x_3>0\}$ the sites and plaquettes that enter $\Phi_N(\sigma)$ lie in the closed half space, whatever g_{d} is. For the other coordinate planes the proof of Proposition 9.9 uses Proposition 9.6, which extends as well: its proof uses of the scheme only the invariances of S stated in Theorem 8.5.

(B) *Results that do not involve the density function.* The following results concern the curvature sector $Z(f,0)$ or the fine state alone: Theorem 3.1; parts (ii), (iii) of Theorem 9.4; Theorem 9.13; Theorem 9.15; Sections 10, 11, 13 and 14; Theorem 12.2, which is stated for $g_{\mathrm{d}}=0$; Theorem 12.26; and Theorem 15.1, which is an abstract statement. Those that involve the scheme are stated at the pinned value of c_N. For a general c_N with (N) one has

$$\Phi_N(f,0)=\Phi_N^{\mathrm{pin}}(\zeta_N f,0) \qquad \text{and} \qquad |Z_N^{\mathrm{pin}}(\zeta_N f,0)-Z_N^{\mathrm{pin}}(\zeta_\infty f,0)|\le|\zeta_N-\zeta_\infty|R(f),$$

with no condition on the density. Hence, along a sequence of levels with $\zeta_{N_m}\to\zeta_\infty$, the curvature sector of the limit is that of the pinned scheme composed with $f\mapsto\zeta_\infty f$, and these results apply to it in that form.

(C) *Results that keep the bounded class.* The following results use $\|g_{\mathrm{d}}\|_\infty$ through exponential or second moments of the density component of the joint field.

(C1) The bounds (12.4) of Lemma 12.10 (Subsection 12.2) and Remark 12.9 belong to this list.
(C2) In Section 12, on the killed-box family, the list contains Lemma 12.22; hypothesis (i) for the joint field in the proof of Proposition 12.36, whereas the proposition itself concerns the curvature Schwinger functions and belongs to list (B); Theorem 12.3, with the joint Schwinger functions and the measure constructed in its proof; and the part of Theorem 12.4 that rests on it.

For the results of list (C) the natural substitute for boundedness is a level-uniform exponential moment of $k_N g_{\mathrm{d}}(\tilde\rho_0)$: by Jensen's inequality with the weights $\varepsilon_N^3|h(z)|/E_N(h)$ and the translation invariance of μ_N,

$$\omega_N\big(e^{t|Y_N(h)|}\big)\le\omega_N\big(e^{tE(h)\,|k_N g_{\mathrm{d}}(\tilde\rho_0)|}\big) \qquad (t\ge0).$$

We do not pursue this, and none of the results of list (C) is asserted for unbounded g_{d}.

9. The remaining Osterwalder–Schrader axioms for the limit state

Throughout this section S is a limit functional of Theorem 7.1 along its sequence (N_m), $Z(\sigma)=S(e_\sigma)$ for test pairs $\sigma=(f,h)$, and $\mathcal{Z}(f):=Z(f,0)$ is the curvature sector (f a real C_c^∞ 2-form); $\Phi_N(f):=\Phi_N(f,0)=\varepsilon_N^{3/2}\sum_p f_{\mu\nu(p)}(x_p)\mathfrak{f}_p$, extended complex-linearly to complex f. The fine state ω_N at level N is the state (2.3) of the limit point of Definition 2.3, with its sequence of sides $(L_n)=(L_n^{(N)})$ and its measure μ_N on Ω_N, and with the properties (a)–(e) of Theorem 11.5; every limit "$L\to\infty$" of torus expectations at level N below is the limit $n\to\infty$ along (L_n), and every bound stated "for all large L" is used along (L_n) only. The rotation group is not treated in this section. Invariance under $SO(3)$ is proved in Sections 12–15: for the limits

built on the killed-box states by Theorems 12.2 and 12.3, and for the limits built on the torus limit points along dyadic sequences (Definition 14.1) by Theorem 12.4. Once that invariance is in hand, the measure μ of Theorem 9.13 is invariant under the full orthogonal group and reflection positive about every plane, and its reconstruction in any direction is unitarily equivalent to the one in a coordinate direction (Remark 9.16).

We write $\|f\|_w := \sup_x (1+|x|^2)^2 \max_{\mu<\nu} |f_{\mu\nu}(x)|$ for a 2-form and $\|h\|_w := \sup_x (1+|x|^2)^2 |h(x)|$ for a function. These are continuous seminorms on the Schwartz spaces $\mathcal{S}(\mathbb{R}^3;\Lambda^2)$ and $\mathcal{S}(\mathbb{R}^3)$, dominated with constant 1 by the weighted C^4 norm $|q|_4 = \sup_{x,\,|\gamma|\le 4}(1+|x|^2)^2 |D^\gamma q(x)|$ of Osterwalder and Schrader [29] (Definition A.1(a)), which we use throughout: $\|\cdot\|_w$ is its derivative-free term. For a 2-form u we write $|u|_4 := (\sum_{\mu<\nu} |u_{\mu\nu}|_4^2)^{1/2}$, as in that definition.

9.1. **Regularity (OS0).**

Lemma 9.1 (the seminorms on Schwartz space)**.** *For every $f \in \mathcal{S}(\mathbb{R}^3;\Lambda^2)$, $h \in \mathcal{S}(\mathbb{R}^3)$ and $N \ge N_0$,*

$$\varepsilon_N^3 \sum_p f_{\mu\nu(p)}(x_p)^2 \le 3c_{a_0}\|f\|_w^2, \qquad \varepsilon_N^3 \sum_z |h(z)| \le c_{a_0}\|h\|_w, \qquad c_{a_0} := \pi^2\big(2+\tfrac{3}{2}a_0^2\big)^2.$$

Hence R and E of Definition 4.1, defined by the same formulas, are finite seminorms on the Schwartz spaces with $R(f) \le (3c_{a_0})^{1/2}\|f\|_w$, $E(h) \le c_{a_0}\|h\|_w$, and $\|\cdot\|$ is continuous for the Schwartz topology.

Proof. The centres of the plaquettes of one orientation form a translate of $\varepsilon_N\mathbb{Z}^3$; let Q_p be the cube of side ε_N centred at x_p (disjoint for distinct p of one orientation). For $y \in Q_p$, $|y| \le |x_p| + \frac{\sqrt{3}}{2}\varepsilon_N$, so $1+|y|^2 \le (1+|x_p|^2)(2+\frac{3}{2}a_0^2)$ and $\varepsilon_N^3(1+|x_p|^2)^{-2} \le (2+\frac{3}{2}a_0^2)^2 \int_{Q_p}(1+|y|^2)^{-2}dy$. Summing over one orientation and using $\int_{\mathbb{R}^3}(1+|y|^2)^{-2}dy = \pi^2$ gives, with $f_{\mu\nu}(x)^2 \le \|f\|_w^2(1+|x|^2)^{-4} \le \|f\|_w^2(1+|x|^2)^{-2}$, $\varepsilon_N^3\sum_p f_{\mu\nu(p)}(x_p)^2 \le \|f\|_w^2 \sum_{\mu<\nu}\sum_p \varepsilon_N^3(1+|x_p|^2)^{-2} \le 3c_{a_0}\|f\|_w^2$. The site sum is estimated in the same way, with one lattice in place of the three orientations. □

Lemma 9.2 (the dual representation at complex sources)**.** *The identity (3.3) holds for every complex plaquette function a, with $\|a\|^2 := \sum_p a_p^2$ (no conjugation). In particular, for every real plaquette function b,*

$$\Big\langle \exp\Big(i\sum_p b_p \mathfrak{f}_p\Big)\Big\rangle_L^{\text{fine}} = \frac{e^{-\|b\|^2/2}}{Z(0)} \int_{[0,1]^{\mathcal{C}}} dt \sum_{m\in\mathbb{Z}^{\mathcal{P}}} \Big(\prod_p c_{m_p-(d^*t)_p}\Big)\Omega(m)\, e^{\gamma_N(m-d^*t)\cdot b} \ \in [0,1].$$

Proof. Both sides of (3.3) are entire functions of $a \in \mathbb{C}^{\mathcal{P}}$: the left side because $|e^{\sum a_p\mathfrak{f}_p}| = e^{\sum(\Re a_p)\mathfrak{f}_p}$ is integrable against W locally uniformly in a (the bounds of Step 4 of the proof of Theorem 3.1), the right side because $|e^{-i\gamma_N(m-s)\cdot a}| = e^{\gamma_N(m-s)\cdot\Im a}$ is controlled by the completed square $c_{m-s}e^{\gamma_N(m-s)y} = \frac{\gamma_N}{\sqrt{2\pi}}e^{y^2/2}e^{-(\gamma_N(m-s)-y)^2/2}$ and $\sum_{m\in\mathbb{Z}} e^{-(\gamma_N m-x)^2/2} \le 1+\sqrt{2\pi}/\gamma_N$ ($x \in \mathbb{R}$), which give $\sum_{m\in\mathbb{Z}} c_{m-s}e^{\gamma_N|m-s|\,|y|} \le 2e^{y^2/2}(1+\gamma_N/\sqrt{2\pi})$ for every real s and locally uniformly in y (here $s = d^*t$, $|s_p| \le 2$); so the m-series converges absolutely and locally uniformly in a, while $0 \le \Omega(m) \le (2\pi/\gamma_N)^{|\mathcal{B}|}(\int g_{\text{ct}})^{|\mathcal{X}|}$ and the t-integral runs over a compact cube. No uniformity in L is needed, since the lemma is a statement at fixed N and L. The two sides agree for real a (Theorem 3.1), hence everywhere. At $a = ib$ every factor of the right side is non-negative, and the left side has modulus at most $Z(0)$. □

Lemma 9.3 (analytic generating functions and their limits)**.** *Let $f = (f_1,\dots,f_k)$ be real C_c^∞ 2-forms, $N \ge N_0$, L large enough for their supports to embed in $\tilde{T}_L$, and for $z \in \mathbb{C}^k$ put $\Psi_{N,L}(z) := \langle\exp(\sum_j z_j\Phi_N(f_j))\rangle_L^{\text{fine}}$ and $\rho(z) := R(\sum_j(\Re z_j)f_j) \le \sum_j |\Re z_j| R(f_j)$.*

(a) *$\Psi_{N,L}$ is entire, $|\Psi_{N,L}(z)| \le \Psi_{N,L}(\Re z) \le e^{\rho(z)^2/2}$, $\Psi_{N,L}(-z) = \Psi_{N,L}(z)$, $\Psi_{N,L}(x) \ge 1$ for real x, and $0 \le \Psi_{N,L}(it) \le 1$ for real t.*

(b) *As $n \to \infty$, $\Psi_{N,L_n} \to \Psi_N$ locally uniformly on $\mathbb{C}^k$; Ψ_N is entire, satisfies the bounds and symmetries of (a), and $\Psi_N(it) = Z_N(\sum_j t_j f_j, 0)$ for real t. The laws μ_{N,L_n} of $(\Phi_N(f_j))_{j\le k}$ under the torus states converge weakly to the law $\mu_N^{(f)}$ of $(\Phi_N(f_j))_{j\le k}$ under μ_N, a law on $\mathbb{R}^k$ with $\int e^{z\cdot x} d\mu_N^{(f)}(x) = \Psi_N(z)$ for all $z \in \mathbb{C}^k$.*

(c) *As $m \to \infty$, $\Psi_{N_m} \to \Psi_f$ locally uniformly on $\mathbb{C}^k$; Ψ_f is entire, satisfies the bounds and symmetries of (a), $\Psi_f(it) = \mathcal{Z}(\sum_j t_j f_j)$, and $\mu_{N_m}^{(f)} \to \mu_f$ weakly, where μ_f is a symmetric law on $\mathbb{R}^k$ with $\int e^{z\cdot x} d\mu_f = \Psi_f(z)$.*

(d) *If f' is a subfamily of f, $\Psi_{f'}$ is the restriction of Ψ_f to the coordinates of f' (the other coefficients set to 0); if $f_1 = g + g'$, then $\Psi_{(f_1,f_2,\dots)}(z_1, z_2, \dots) = \Psi_{(g,g',f_2,\dots)}(z_1, z_1, z_2, \dots)$.*

Proof. (a) For $s \in [0,\infty)^k$,

$$\langle e^{\sum_j s_j |\Phi_N(f_j)|}\rangle_L^{\text{fine}} \le \sum_{\epsilon\in\{\pm1\}^k} \langle e^{\Phi_N(\sum_j \epsilon_j s_j f_j)}\rangle_L^{\text{fine}} \le 2^k \max_\epsilon e^{R(\sum_j \epsilon_j s_j f_j)^2/2} < \infty$$

by additivity of Φ_N and (3.1) with $a_p = \varepsilon_N^{3/2} \sum_j \epsilon_j s_j f_j(x_p)$; so $\Psi_{N,L}$ is a finite-dimensional integral, together with a countable absolutely convergent v-sum, of an entire integrand with a locally uniform integrable majorant, hence entire. For the bound, $|e^{\sum z_j \Phi_N(f_j)}| = e^{\Phi_N(g)}$ with $g = \sum_j (\Re z_j) f_j$, and (3.1) gives $\langle e^{\Phi_N(g)}\rangle_L^{\text{fine}} \le e^{\varepsilon_N^3 \sum_p g(x_p)^2/2} \le e^{R(g)^2/2}$. For the evenness, the map $(B, v) \mapsto (-B, -v)$ preserves the reference measure (up to the null set on which some B_b sits at the endpoint π/γ_N of $(-\pi/\gamma_N, \pi/\gamma_N]$) and the weight (the hopping factor depends on B_b only through $\cos(\gamma_N B_b)$, w is even, closed v go to closed v) and maps every $\mathfrak{f}_p$ to $-\mathfrak{f}_p$. Jensen's inequality and $\langle \Phi_N(g)\rangle_L^{\text{fine}} = 0$ (Theorem 3.1) give $\Psi_{N,L}(x) = \langle e^{\Phi_N(g)}\rangle_L^{\text{fine}} \ge 1$. The last clause is Lemma 9.2 with $b_p = \varepsilon_N^{3/2} \sum_j t_j f_j(x_p)$.

(b) The family $(\Psi_{N,L_n})_n$ is locally uniformly bounded on $\mathbb{C}^k$ by (a). By Montel's theorem every subsequence of (L_n) has a further subsequence along which Ψ_{N,L_n} converges locally uniformly to an entire function; on $i\mathbb{R}^k$ every such limit equals $t \mapsto \omega_N(e^{i\sum_j t_j \Phi_N(f_j)}) = \int e^{i\sum_j t_j \Phi_N(f_j)} d\mu_N$, by (2.3) (Theorem 11.5(a)) applied to the bounded continuous local function $e^{i\sum_j t_j \Phi_N(f_j)}$ (a continuous function of the finitely many $\mathfrak{f}_p$ in the supports). An entire function F on $\mathbb{C}^k$ vanishing on $i\mathbb{R}^k$ vanishes identically (its Taylor coefficients at 0 are, up to powers of i, the derivatives of the real-analytic function $t \mapsto F(it)$ at 0), so all subsequential limits agree, and $\Psi_{N,L_n} \to \Psi_N$ locally uniformly; the bounds and symmetries of (a) pass to the limit. The characteristic function of $\mu_N^{(f)}$ is $t \mapsto \int e^{it\cdot\Phi_N(f)} d\mu_N = \Psi_N(it)$, and the characteristic functions $t \mapsto \Psi_{N,L_n}(it)$ of μ_{N,L_n} converge to it, so $\mu_{N,L_n} \to \mu_N^{(f)}$ weakly by Lévy's continuity theorem (alternatively, μ_{N,L_n} is the image of the law of the variables in the supports under the continuous linear map $\Phi_N(f)$, and Theorem 11.5(a) gives the weak convergence of that law). For $z \in \mathbb{C}^k$ the continuous functions $e^{z\cdot x}$ are uniformly integrable for $(\mu_{N,L_n})_n$, because $\sup_n \int e^{2(\Re z)\cdot x} d\mu_{N,L_n} = \sup_n \Psi_{N,L_n}(2\Re z) \le e^{2\rho(z)^2}$; hence $\int e^{z\cdot x} d\mu_N^{(f)} = \lim_n \Psi_{N,L_n}(z) = \Psi_N(z)$.

(c) The same argument applies: $(\Psi_{N_m})_m$ is locally uniformly bounded, and on $i\mathbb{R}^k$, $\Psi_{N_m}(it) = Z_{N_m}(\sum_j t_j f_j, 0) \to Z(\sum_j t_j f_j, 0)$ by Theorem 7.1(a). The limit law μ_f has the even characteristic function $\Psi_f(i\,\cdot)$, so it is symmetric.

(d) On the torus both identities are the additivity of Φ_N; they pass to the limits. □

Theorem 9.4 (OS0: regularity of the limit curvature)**.** (i) *(Continuity.)* $|Z(\sigma) - Z(\sigma')| \le \|\sigma - \sigma'\|$ *for all test pairs; in particular* $|\mathcal{Z}(f) - \mathcal{Z}(f')| \le R(f - f') \le (3c_{a_0})^{1/2} \|f - f'\|_w$,

so $\mathcal{Z}$ extends uniquely to a continuous function on $\mathcal{S}(\mathbb{R}^3;\Lambda^2)$, with the same bound; $0\le\mathcal{Z}(f)\le 1$ for every real f, and $\mathcal{Z}(0)=1$.

(ii) *(Entire extension.) For real C_c^∞ 2-forms $f_1,\dots,f_k$ the function Ψ_f of Lemma 9.3 is entire on $\mathbb{C}^k$ with $\Psi_f(it)=\mathcal{Z}(\sum_j t_j f_j)$ and*

$$|\Psi_f(z)|\le\Psi_f(\Re z)\le\exp\Big(\tfrac12\Big(\sum_j|\Re z_j|R(f_j)\Big)^2\Big);$$

for $k=1$ the generating functional $\zeta\mapsto\mathcal{Z}(\zeta f):=\Psi_{(f)}(i\zeta)=\int e^{i\zeta x}d\mu_f(x)$ is entire with $|\mathcal{Z}(\zeta f)|\le e^{|\zeta|^2R(f)^2/2}$.

(iii) *(Schwinger functions.) $S_n(f_1,\dots,f_n):=\partial_{z_1}\cdots\partial_{z_n}\Psi_f(0)=\int x_1\cdots x_n\,d\mu_f(x)$ is well defined (independent of the family containing $f_1,\dots,f_n$), symmetric and multilinear, real for real f_i, and*

$$|S_n(f_1,\dots,f_n)|\le e^n(n!)^{1/2}\prod_{i=1}^n R(f_i)\le\big(e\sqrt{3c_{a_0}}\big)^n(n!)^{1/2}\prod_{i=1}^n\|f_i\|_w\qquad(f_i\text{ real}).$$

S_n extends to a continuous symmetric multilinear form on $\mathcal{S}(\mathbb{R}^3;\Lambda^2_{\mathbb{C}})^n$, with the factor 2^n in the bound for complex test functions, and is a tempered distribution. In the product form $|S_n(f_1,\dots,f_n)|\le\sigma_n\prod_i|f_i|_s$ on product test functions this is the growth condition (E0′) of Osterwalder and Schrader with $s=4$ (for the three component fields, mixed entries included) and $\sigma_n=(2e\sqrt{3c_{a_0}})^n(n!)^{1/2}\le\alpha_\beta(n!)^\beta$ for every $\beta>\frac12$, with the norm $|\cdot|_4$ of Definition A.1(a) (the norm of [29] with $p=4$; the growth condition is that of [29] with $s=4$). $S_1=0$, and $S_2(f,f)\le R(f)^2$.

(iv) *(The lattice moments converge.) The torus Schwinger functions $\langle\prod_i\Phi_N(f_i)\rangle_L^{\mathrm{fine}}=\partial_{z_1}\cdots\partial_{z_n}\Psi_{N,L}(0)$ obey the bound of (iii) with $(\varepsilon_N^3\sum_p f_i(x_p)^2)^{1/2}$ in place of $R(f_i)$, and converge to $S_n(f_1,\dots,f_n)$ as $L\to\infty$ along (L_n) and then $m\to\infty$ along N_m.*

Proof. (i) The continuity bound is (6.1) in the limit, with $C=1$ by Proposition 4.2. The extension to $\mathcal{S}$ follows by uniform continuity and the density of C_c^∞, with Lemma 9.1, and $\mathcal{Z}(f)=\Psi_{(f)}(i)\in[0,1]$ by Lemma 9.3(a),(c). (ii) is Lemma 9.3(c), with $\Re(i\zeta)=-\Im\zeta$ for the last bound.

(iii) By Lemma 9.3(d) the derivative does not depend on the family, and multilinearity in each entry follows from the second identity of (d) (and from $\Psi_{(cf)}(z)=\Psi_{(f)}(cz)$); symmetry is the symmetry of mixed partial derivatives; reality follows from Ψ_f being real on $\mathbb{R}^k$ (a limit of real numbers), so that its Taylor coefficients at 0 are real. For the bound let first $R(f_i)>0$ for all i and normalise $R(f_i)=1$ by multilinearity; on the polydisc $|z_i|\le r$, $\sum_j|\Re z_j|R(f_j)\le nr$, so $|\Psi_f|\le e^{n^2r^2/2}$ there, and Cauchy's estimate for the mixed first derivative gives $|S_n|\le r^{-n}e^{n^2r^2/2}=(en)^{n/2}$ at $r=n^{-1/2}$; $n^n\le e^n n!$ gives $e^n(n!)^{1/2}$. If $R(f_i)=0$ then $\Phi_N(f_i)=0$ for every N, Ψ_f does not depend on z_i and $S_n=0$. The second bound is Lemma 9.1.

The form is continuous for the seminorm $\|\cdot\|_w$ in each entry, so it extends from C_c^∞ to $\mathcal{S}$; for complex $f_i=f_i'+if_i''$ multilinearity gives 2^n terms, each bounded with $\|f_i'\|_w,\|f_i''\|_w\le\|f_i\|_w$; the nuclear theorem ([15, Chapter I], [30, Chapter V]) turns a continuous multilinear form on $\mathcal{S}^n$ into a tempered distribution on $\mathbb{R}^{3n}$ (one for each choice of components). On scalar test functions $\|\cdot\|_w\le|\cdot|_4$ with constant 1, since $\|\cdot\|_w$ is the derivative-free term of the weighted C^4 norm recalled at the start of the section. (For another weighted-C^4 convention a constant C enters, which multiplies σ_n by C^n and leaves the class $\beta>\frac12$ unchanged.) This gives (E0′) in product form with $s=4$ and the stated σ_n; $(e')^n(n!)^{1/2}\le\alpha_\beta(n!)^\beta$ for a constant e' exactly when $\beta>\frac12$. $S_1=0$ and $S_2(f,f)\le R(f)^2$ are the first two moments of Theorem 3.1, passed to μ_f by the convergence of moments (uniform integrability from the exponential moments).

(iv) The torus bound is the Cauchy estimate above with the torus bound of Lemma 9.3(a), and the derivatives at 0 converge under the locally uniform convergences of Lemma 9.3(b),(c), by the Weierstrass convergence theorem. □

Remark 9.5. The constants are absolute or depend on a_0 alone: only Theorem 3.1 (identity covariance), the Riemann-sum count of Lemma 9.1 and the subsequence convergence of Theorem 7.1(a) enter; no property of the couplings is used. The joint field with the density component and its mixed Schwinger functions are treated in Section 12: for the killed-box family in Lemma 12.22 and Theorem 12.3, and for the torus limit points along dyadic sequences in Theorem 12.4.

9.2. **Euclidean invariance without rotations (OS1).** A Euclidean motion $\gamma x = Ox + y$ with $O \in O_h$ (the cubic group, the 48 signed permutation matrices) acts on test pairs by push-forward, $(\gamma\sigma) = (\gamma_* f, h \circ \gamma^{-1})$ with the 2-form transforming as a 2-form, and on $\mathcal{A}$ by $\gamma e_\sigma := e_{\gamma\sigma}$; the time reflection θ of Section 2 (the paragraph before Definition 2.5) is the case $O = \mathrm{diag}(1, 1, -1)$, and τ_{te_3} the case $O = 1$, $y = te_3$.

Proposition 9.6 (OS1, rotations excepted)**.** *$S(\gamma a) = S(a)$ for every $a \in \mathcal{A}$ and every Euclidean motion γ whose linear part lies in O_h: all translations of $\mathbb{R}^3$, the reflections in all coordinate planes $\{x_j = c\}$ ($j = 1, 2, 3$, $c \in \mathbb{R}$) and the cubic group about every point. Equivalently $Z(\gamma\sigma) = Z(\sigma)$ for every test pair.*

Proof. Translations. Theorem 7.1 states the invariance under all space and time translations; we write the argument for a direction e_j. For $\sigma \in \mathcal{T}$ and $t, t' \in \mathbb{R}$, (6.1) in the limit gives $|Z(\tau_{te_j}\sigma) - Z(\tau_{t'e_j}\sigma)| \le \|\tau_{te_j}\sigma - \tau_{t'e_j}\sigma\|$, and $\tau_{t'e_j}\sigma \to \tau_{te_j}\sigma$ in C_c^∞ as $t' \to t$, so $t \mapsto Z(\tau_{te_j}\sigma)$ is continuous by the admissibility of $\|\cdot\|$ (Definition 4.1). For $t \in \varepsilon_{N_m}\mathbb{Z}$ and $m' \ge m$, Lemma 5.1 (the translation by te_j is a lattice translation at level $N_{m'}$, and $\Phi_N(\tau_{te_j}\sigma) = \Phi_N(\sigma) \circ g^{-1}$ by the substitution of its proof) gives $\omega_{N_{m'}}(\iota_{N_{m'}}(\tau_{te_j}a)) = \omega_{N_{m'}}(\iota_{N_{m'}}(a))$; letting $m' \to \infty$, $S(\tau_{te_j}a) = S(a)$ on the dense set $\bigcup_m \varepsilon_{N_m}\mathbb{Z}$, hence for all t by continuity. Composing the three directions gives all translations of $\mathbb{R}^3$.

The cubic group. Theorem 7.1 gives the invariance under the cubic group about every point $x_0 \in \bigcup_m \varepsilon_{N_m}\mathbb{Z}^3$ (Step 5 of the proof of Proposition 6.1, which rests on Lemma 5.1 and on the intertwining of ι_N with the push-forward, a signed permutation of the plaquettes and sites). The cubic group about an arbitrary point x is $\tau_{x-x_0} G_{x_0} \tau_{x_0-x}$, where G_{x_0} is the cubic group about x_0, and S is translation invariant. The reflection in $\{x_j = c\}$ is the element $O = 1 - 2e_j e_j^{\mathsf{T}}$ of the cubic group about any point of that plane. Finally, every motion with linear part in O_h is $\tau_y \circ O$ with O in the cubic group about $0 \in \varepsilon_{N_m}\mathbb{Z}^3$, and $\gamma e_\sigma = e_{\gamma\sigma}$ is a group action (the push-forward is covariant), so the invariances proved give $S(\gamma a) = S(a)$ for the whole group. □

Invariance under $SO(3)$ is the subject of Sections 12–15 (Theorems 12.2, 12.3 and 12.4).

9.3. **Reflection positivity (OS2).** *Reflections of the torus.* Fix N and L ($L/2 \in \varepsilon_N\mathbb{Z}$) and $c \in \frac{1}{2}\varepsilon_N\mathbb{Z}$. The map $R_c : x \mapsto (x_1, x_2, 2c - x_3)$ is an involutive symmetry of $\tilde{T}_L$ fixing pointwise the two planes $P_c = \{x_3 = c\}$ and $P_{c+L/2}$; it permutes the sites, the oriented bonds, the oriented plaquettes and the cubes, commuting with the boundary operator. It acts on configurations by $(R_c\xi)_\kappa := \xi_{R_c\kappa}$ for every cell κ, with the orientation conventions $B_{-b} = -B_b$, $v_{-p} = -v_p$; then $\tilde{\rho}_x \circ R_c = \tilde{\rho}_{R_c x}$, $\mathfrak{f}_p \circ R_c = \mathfrak{f}_{R_c p}$, closed v go to closed v, and the weight and the reference measure are invariant (as in the proof of Lemma 5.1, since w is even and $h(r, r', \vartheta) = h(r', r, -\vartheta)$), so $\langle F \circ R_c\rangle_L^{\mathrm{fine}} = \langle F\rangle_L^{\mathrm{fine}}$. Let $\Lambda_c^+ := \{x \in \tilde{T}_L : x_3 \in [c, c + L/2]\}$ (coordinates taken in a fundamental interval) and $\Lambda_c^- := R_c\Lambda_c^+$; a cell lies in $\Lambda_c^\pm$ if all its vertices do. $\mathfrak{A}_L^+(c)$ is the algebra of bounded measurable functions of the variables $(\tilde{\rho}_x, B_b, v_p)$ of the cells lying in Λ_c^+, and $\Theta_c F := \overline{F \circ R_c}$. When the support of F is small compared with L, its cells lie in Λ_c^+ exactly when they lie in $\{x_3 \ge c\}$.

Theorem 9.7 (reflection positivity of the fine torus state)**.** *For every N, every $L \geq 4\varepsilon_N$ with $L/2 \in \varepsilon_N\mathbb{Z}$, every $c \in \frac{1}{2}\varepsilon_N\mathbb{Z}$ and all $F, G \in \mathfrak{A}_L^+(c)$,*

$$\langle \Theta_c F \cdot G\rangle_L^{\mathrm{fine}} = \overline{\langle \Theta_c G \cdot F\rangle_L^{\mathrm{fine}}}, \qquad \langle \Theta_c F \cdot F\rangle_L^{\mathrm{fine}} \geq 0.$$

Both statements extend to all Λ_c^+-measurable F, G with $\langle |F|^2\rangle_L^{\mathrm{fine}}, \langle |G|^2\rangle_L^{\mathrm{fine}} < \infty$. By the cubic symmetry the same holds for the reflections in the planes $\{x_j = c\}$, $j = 1, 2$, with the slabs taken in the j-th coordinate.

Proof. The Hermitian property is the R_c-invariance of the state: $\langle \overline{F \circ R_c}\, G\rangle_L^{\mathrm{fine}} = \langle \overline{F}\,(G \circ R_c)\rangle_L^{\mathrm{fine}} = \overline{\langle \overline{G \circ R_c}\, F\rangle_L^{\mathrm{fine}}}$. For the positivity we treat $c = 0$ and $c = \varepsilon_N/2$; the other c follow by the lattice translations, which are symmetries of the torus state and conjugate R_c to one of these. We write the sum over the Villain variables as an integral against the counting measure on $((2\pi/\gamma_N)\mathbb{Z})^{\mathcal{P}}$ with the closedness constraint $\prod_{\mathrm{cubes}} \mathbf{1}[(dv)_c = 0]$ as a factor of the weight; so the weight is a product of one factor per site, bond, plaquette and cube, and the constraint is local.

Site plane, $c = 0$. Call a cell a plane cell if all its vertices lie in $P_0 \cup P_{L/2}$. Every other cell lies in exactly one of $\Lambda_0^{\pm}$ (a cell has x_3-extent at most ε_N and $0, L/2 \in \varepsilon_N\mathbb{Z}$), and every cube, with its six faces, lies in one of them. R_0 fixes every plane cell pointwise and maps the non-plane cells of Λ_0^+ bijectively onto those of Λ_0^-. Write $\xi = (\xi_0, \xi_+, \xi_-)$ for the variables of the plane cells, of the other cells of Λ_0^+, of the other cells of Λ_0^-; the reference measure is the product $d\xi_0\, d\xi_+\, d\xi_-$, and R_0 carries $d\xi_-$ to $d\xi_+$. Let $W_+(\xi_0, \xi_+)$ be the product of the factors of the non-plane cells and the cubes of Λ_0^+, $W_-(\xi_0, \xi_-)$ the same for Λ_0^-, and $W_0(\xi_0)$ the product of the factors of the plane cells. Then $W = W_0 W_+ W_-$ and $W_-(\xi) = W_+(R_0\xi)$, because R_0 maps the cells and cubes of Λ_0^- onto those of Λ_0^+ and preserves every factor. For $F \in \mathfrak{A}_L^+(0)$, $F(\xi) = F(\xi_0, \xi_+)$ and $(F \circ R_0)(\xi) = F(\xi_0, \eta)$ with $\eta := (R_0\xi)_+$, a measure-preserving bijective function of ξ_-. Hence, by Fubini's theorem (the integrand is dominated by $W\|F\|_\infty^2$),

$$\begin{aligned}\langle \Theta_0 F \cdot F\rangle_L^{\mathrm{fine}} &= \frac{1}{Z}\int d\xi_0\, W_0(\xi_0) \int d\xi_+ W_+(\xi_0, \xi_+) F(\xi_0, \xi_+) \int d\xi_- W_-(\xi_0, \xi_-) \overline{F(\xi_0, (R_0\xi)_+)} \\ &= \frac{1}{Z}\int d\xi_0\, W_0(\xi_0)\, \big|\mathcal{G}_F(\xi_0)\big|^2 \geq 0,\end{aligned}$$

with $\mathcal{G}_F(\xi_0) := \int d\xi_+ W_+(\xi_0, \xi_+) F(\xi_0, \xi_+)$: after the substitution $\eta = (R_0\xi)_+$ and $W_-(\xi_0, \xi_-) = W_+(\xi_0, \eta)$, the last integral is $\overline{\mathcal{G}_F(\xi_0)}$.

Bond mid-plane, $c = \varepsilon_N/2$. Now no site lies on the fixed planes, and the sites are partitioned between $\Lambda^+ := \{x_3 \in [\varepsilon_N, L/2]\}$ and $\Lambda^- := R\Lambda^+$, $R := R_{\varepsilon_N/2}$ (these are the slabs $\Lambda^{\pm}_{\varepsilon_N/2}$ of the general notation; we drop the subscript in this case). The crossing cells, those lying in neither slab, are the time-like bonds $(x, x + \varepsilon_N e_3)$ with $x_3 \in \{0, L/2\}$, the time-like plaquettes with vertices at $x_3 \in \{0, \varepsilon_N\}$ or $\{L/2, L/2 + \varepsilon_N\}$, and the cubes between these slices; R maps every crossing cell to itself with reversed orientation, so on them $B_b \mapsto -B_b$, $v_p \mapsto -v_p$, and it maps the cells and cubes of Λ^+ bijectively onto those of Λ^-. Write $\xi = (\xi_+, \xi_-, \xi_\times)$ with $\xi_\times = (B_b, v_p)_{\mathrm{crossing}}$, and $W = W_+(\xi_+) W_-(\xi_-) W_\times$, where $W_\pm$ collect the factors of the cells and cubes of $\Lambda^\pm$ (the faces of a cube of $\Lambda^\pm$ lie in $\Lambda^\pm$), $W_- = W_+ \circ R$, and

$$W_\times := \prod_{b\times} h(\tilde\rho_{b_-}, \tilde\rho_{b_+}, B_b) \prod_{p\times} w(\mathfrak{f}_p) \prod_{c\times} \mathbf{1}[(dv)_c = 0], \qquad K(\xi_-, \xi_+) := \int \prod_{b\times} dB_b \sum_{(v_p)_{p\times}} W_\times,$$

the products over the crossing bonds, plaquettes and cubes; K depends on $\xi_\pm$ only through the two boundary slices.

As in the first case, with $\eta := (R\xi)_+$ and $\widetilde{K}(\eta,\xi_+) := K(R\eta,\xi_+)$ ($R\eta$ the Λ^--configuration obtained from η by reflection),

$$\langle \Theta F\cdot F\rangle_L^{\text{fine}} = \frac{1}{Z}\int d\eta\, d\xi_+\, W_+(\eta)W_+(\xi_+)\,\widetilde{K}(\eta,\xi_+)\,\overline{F(\eta)}F(\xi_+),$$

and it suffices to show that $\widetilde{K}$ is a kernel of positive type, $\widetilde{K}(\eta,\xi) = \int d\nu(k)\,\overline{A_k(\eta)}A_k(\xi)$ with a positive measure ν and $\int d\nu(k)|A_k(\eta)||A_k(\xi)|$ bounded uniformly in (η,ξ): then the double integral equals $\int d\nu(k)|\int d\xi\, W_+(\xi)A_k(\xi)F(\xi)|^2 \geq 0$ by Fubini's theorem. Here F is bounded and W_+ is integrable, since g_{ct} is integrable, $h\leq 1$, each B_b ranges over a bounded interval, the indicators are at most 1, and each Villain sum $\sum_{v_p} w(\mathfrak{f}_p)$ is at most K_{γ_N} by (3.2) at $a=0$ and Step 4 of the proof of Theorem 3.1.

Each crossing cell has one end in each of the two slabs; these ends are the two endpoints of a crossing bond, the two spatial edges of a crossing plaquette, and the top and bottom faces of a crossing cube. We write $\uparrow$ for the end of a crossing cell in Λ^+ and $\downarrow$ for its end in Λ^-; for a crossing bond b the two ends are written b_+ and b_- accordingly (at the slices $L/2$, $L/2+\varepsilon_N$ this interchanges $b_\pm$). Thus the spatial edges of a crossing plaquette p with vertices at $x_3\in\{0,\varepsilon_N\}$ are $p^\uparrow$ (at $x_3=\varepsilon_N$) and $p^\downarrow$ (at $x_3=0$), and the top and bottom faces of a crossing cube c between these slices are $c^\uparrow$ and $c^\downarrow$; at the slices $L/2$, $L/2+\varepsilon_N$ the Λ^+ edge or face, which carries the label $\uparrow$, is the lower one (at $x_3=L/2$). The $\uparrow$ variables belong to ξ_+, the $\downarrow$ variables to ξ_-.

We expand K by the computation of Theorem 3.1 restricted to the crossing cells. Write $\mathbf{1}[(dv)_c=0] = \int_0^1 dt_c\, e^{i\gamma_N t_c (dv)_c}$ for the crossing cubes, put $t_c:=0$ for the others and $s_p := (d^*t)_p$. The two cubes containing a crossing plaquette are crossing, and (3.2) at $a=0$ gives $\sum_{v_p} w(\mathfrak{f}_p)e^{i\gamma_N s_p v_p} = \sum_{m_p\in\mathbb{Z}} c_{m_p-s_p} e^{i\gamma_N(m_p-s_p)(dB)_p}$ with $c_u>0$. The four plaquettes containing a crossing bond b are crossing, and $\sum_{p\ni b}\pm(m_p-s_p) = (d^*m)_b - (d^*d^*t)_b = (d^*m)_b =: q_b\in\mathbb{Z}$, so the integral over B_b gives $(2\pi/\gamma_N)e^{-\frac12\kappa_N(\rho_{b_-}^2+\rho_{b_+}^2)} I_{q_b}(\kappa_N\rho_{b_-}\rho_{b_+})$ (Step 3 of the proof of Theorem 3.1). What remains of the phases involves the spatial edges of the crossing plaquettes and the top and bottom faces of the crossing cubes: for every crossing plaquette p and every crossing cube c,

$$(dB)_p = \pm(B_{p^\uparrow}-B_{p^\downarrow}) + (\text{crossing bonds}), \qquad (dv)_c = \pm(v_{c^\uparrow}-v_{c^\downarrow}) + (\text{crossing plaquettes}).$$

Hence

$$K(\xi_-,\xi_+) = \int dt \sum_m \Big(\prod_{p\times} c_{m_p-s_p}\Big)\prod_{b\times}\frac{2\pi}{\gamma_N} e^{-\frac12\kappa_N(\rho_{b_-}^2+\rho_{b_+}^2)} I_{q_b(m)}(\kappa_N\rho_{b_-}\rho_{b_+}) \prod_{p\times} e^{\pm i\gamma_N(m_p-s_p)(B_{p^\uparrow}-B_{p^\downarrow})}\prod_{c\times} e^{\pm i\gamma_N t_c(v_{c^\uparrow}-v_{c^\downarrow})}.$$

The reflection R maps $p^\downarrow$ to $p^\uparrow$, $c^\downarrow$ to $c^\uparrow$ and b_- to b_+, with the spatial orientations preserved; so in $\widetilde{K}(\eta,\xi) = K(R\eta,\xi)$ every $\downarrow$ variable of $R\eta$ is the corresponding $\uparrow$ variable of η. Each factor is then of positive type in (η,ξ): the phases are $\overline{a(\eta)}a(\xi)$ with $a = e^{\pm i\gamma_N(m_p-s_p)B_{p^\uparrow}}$ or $a = e^{\pm i\gamma_N t_c v_{c^\uparrow}}$; the factor $e^{-\frac12\kappa_N(\rho^2+\rho'^2)}$ splits; and

$$I_q(\kappa\rho\rho') = \sum_{k\geq 0}\frac{(\kappa/2)^{2k+|q|}}{k!\,(k+|q|)!}\rho^{2k+|q|}\rho'^{2k+|q|}$$

is a series with positive coefficients of products $\varphi(\rho_{b_+}(\eta))\varphi(\rho_{b_+}(\xi))$ of one real function.

Products, non-negative combinations and integrals over t of kernels of this form are of this form (for products, by the Schur product theorem), which gives $\widetilde{K} = \int d\nu(k)\overline{A_k}\otimes A_k$ with

$k = (t, m, (k_b))$ and

$$d\nu = dt \prod_p c_{m_p - s_p} \prod_b \frac{2\pi}{\gamma_N} \frac{(\kappa_N/2)^{2k_b+|q_b|}}{k_b!(k_b+|q_b|)!};$$

ν is a finite measure, of mass at most $(2\pi e^{\kappa_N}/\gamma_N)^{|b\times|} K_{\gamma_N}^{|p\times|}$, since $\sum_{k\geq 0} \frac{(\kappa/2)^{2k+|q|}}{k!(k+|q|)!} = I_{|q|}(\kappa) \leq e^{\kappa}$. The absolute series is the same expansion with the phases replaced by 1, i.e.

$$\int dt \sum_m \prod_p c_{m_p - s_p} \prod_b \frac{2\pi}{\gamma_N} e^{-\frac{1}{2}\kappa_N(\rho^2+\rho'^2)} I_{q_b}(\kappa_N \rho\rho') \leq (2\pi/\gamma_N)^{|b\times|} K_{\gamma_N}^{|p\times|},$$

by $e^{-\frac{1}{2}\kappa(\rho^2+\rho'^2)} I_q(\kappa\rho\rho') \leq e^{-\frac{1}{2}\kappa(\rho-\rho')^2} \leq 1$ and $\sum_m c_{m-s} \leq K_{\gamma_N}$ (Step 4 of the proof of Theorem 3.1). This is the required bound.

The L^2 extension. Bounded Λ_c^+-measurable functions are dense in the Λ_c^+-measurable part of $L^2(\langle\cdot\rangle_L^{\text{fine}})$, and $|\langle \Theta_c F \cdot F\rangle_L^{\text{fine}} - \langle \Theta_c G \cdot G\rangle_L^{\text{fine}}| \leq \|F-G\|_2(\|F\|_2 + \|G\|_2)$ by the Schwarz inequality and the R_c-invariance of the state; both sides of the Hermitian identity are $\|\cdot\|_2$-continuous in (F, G) as well. □

Corollary 9.8 (reflection positivity of the fine state at the lattice planes)**.** *For $N \geq N_0$ and $c \in \frac{1}{2}\varepsilon_N\mathbb{Z}$ let $\mathfrak{A}_N^+(c)$ be the bounded Borel local functions of the gauge-invariant variables $(\mathfrak{f}_p, \tilde{\rho}_x)$ of the cells lying in $\{x_3 \geq c\}$, and $\Theta_c F := \overline{F \circ R_c}$. Then $\omega_N(\Theta_c F \cdot G) = \overline{\omega_N(\Theta_c G \cdot F)}$ and $\omega_N(\Theta_c F \cdot F) \geq 0$ for $F, G \in \mathfrak{A}_N^+(c)$; $\omega_N \circ R_c = \omega_N$; and ω_N is invariant under τ_{2a_0}, indeed under $\varepsilon_N\mathbb{Z}^3$ (Lemma 5.1). In particular the fine state is reflection positive about the site planes $t = 0$ and $t = a_0$ ($a_0 = 5^N\varepsilon_N \in \varepsilon_N\mathbb{Z}$) and invariant under R_0, $R_{a_0} = \tau_{2a_0} R_0$.*

Proof. For n large the cells of F and G lie in Λ_c^+ of $\tilde{T}_{L_n}$, and their set S together with $R_c S$ lies in a cube of side $< L_n$, so that the torus expectations are those of the fixed functions (Subsection 11.1); on these cells the torus reflection R_c agrees with the reflection of the infinite lattice; $\Theta_c F \cdot G$, $\Theta_c F \cdot F$ and $F \circ R_c$ are bounded local functions of the gauge-invariant variables $(\mathfrak{f}_p \circ R_c = \mathfrak{f}_{R_c p}$, $\tilde{\rho}_x \circ R_c = \tilde{\rho}_{R_c x})$, the same functions of the same finitely many cells for all large n. If F and G are continuous, so are these functions, and (2.3) (Theorem 11.5(a)), applied to their real and imaginary parts, passes the identities of Theorem 9.7 and the invariance $\langle F \circ R_c\rangle_{L_n}^{\text{fine}} = \langle F\rangle_{L_n}^{\text{fine}}$ to ω_N; positive semidefiniteness on finite families follows by applying the inequality to linear combinations.

Let now F, G be bounded Borel functions of the cells of a finite set S. The bounded continuous functions of the variables of S are dense in L^2 of the S-marginal of μ_N (a finite Borel measure on a Polish space is Radon), a bounded Borel F is the $L^2(\mu_N)$-limit of continuous $F^{(i)}$ with $\|F^{(i)}\|_\infty \leq \|F\|_\infty$ (by truncation), and both sides of each identity are continuous in the $L^2(\mu_N)$-norms of F and G, by the Schwarz inequality and $\|F \circ R_c\|_{L^2(\mu_N)} = \|F\|_{L^2(\mu_N)}$ (the R_c-invariance of μ_N, Theorem 11.5(c)). The invariance $\omega_N \circ R_c = \omega_N$ on the bounded Borel class and the invariance under $\varepsilon_N\mathbb{Z}^3$ are Theorem 11.5(c) (Lemma 5.1). The corollary is Theorem 11.5(c),(d) for $j = 3$, whose proof is the argument just given. □

Proposition 9.9 (OS2 for S: inheritance)**.** *For $j \in \{1, 2, 3\}$ and $c \in \mathbb{R}$ let $\Theta_{j,c}$ be the reflection in $\{x_j = c\}$ on $\mathcal{A}$ ($\Theta_{j,c} e_\sigma := e_{-\gamma\sigma}$, γ the reflection) and $\mathcal{A}_+^{(j,c)}$ the span of the e_σ with $\operatorname{supp}\sigma \subset \{x_j > c\}$; $\Theta_{3,0} = \Theta$, $\mathcal{A}_+^{(3,0)} = \mathcal{A}_+$. Then $\langle a, b\rangle := S(\Theta_{j,c} a \cdot b)$ is a positive semidefinite Hermitian sesquilinear form on $\mathcal{A}_+^{(j,c)}$, and $S(\Theta_{j,c} a) = \overline{S(a)}$ on $\mathcal{A}$.*

Proof. For $a = \sum_i \gamma_i e_{\sigma_i}$, $b = \sum_j \delta_j e_{\sigma'_j}$ in $\mathcal{A}_+$ and $N = N_m$ with ε_N smaller than the distance of the supports from the plane, $\iota_N(a)$ and $\iota_N(b)$ are functions of the cells in $\{x_3 \geq 0\}$, and $\Phi_N(\sigma) \circ R_0 = \Phi_N(\theta\sigma)$: R_0 permutes the plaquettes with the orientation sign, $\mathfrak{f}_p \circ R_0 = \mathfrak{f}_{R_0 p}$, and

the push-forward of a 2-form carries exactly this sign (-1 on the components with one index 3, whose plaquettes R_0 reverses), while $\tilde\rho_z \circ R_0 = \tilde\rho_{R_0 z}$. Hence $\iota_N(\Theta a \cdot b) = \Theta_0 \iota_N(a) \cdot \iota_N(b)$, and by Theorem 7.1(a), $S(\Theta a \cdot b) = \lim_m \omega_{N_m}(\Theta_0 \iota_{N_m}(a)\, \iota_{N_m}(b))$. Each term is a positive semidefinite Hermitian form in (a, b) by Corollary 9.8, and these properties of the Gram matrices $(\langle a_i, a_j\rangle)$ are closed under pointwise limits. Likewise $S(\Theta a) = \lim \omega_{N_m}(\Theta_0 \iota_{N_m}(a)) = \lim \overline{\omega_{N_m}(\iota_{N_m}(a))} = \overline{S(a)}$ by the reflection invariance of ω_N. For (j, c) arbitrary choose a Euclidean motion γ with linear part in O_h mapping $\{x_3 > 0\}$ onto $\{x_j > c\}$; then $\mathcal{A}_+^{(j,c)} = \gamma \mathcal{A}_+$, $\Theta_{j,c} = \gamma \Theta \gamma^{-1}$, and Proposition 9.6 gives $S(\Theta_{j,c} a \cdot b) = S(\Theta(\gamma^{-1} a) \cdot \gamma^{-1} b)$. □

Theorem 9.10 (OS2 in matrix form, complex coefficients)**.** *Let $j \in \{1, 2, 3\}$ and $c \in \mathbb{R}$, let $f_1, \dots, f_k$ be real C_c^∞ 2-forms supported in $\{x_j > c\}$, γ the reflection in $\{x_j = c\}$, $F := (\gamma_* f_1, \dots, \gamma_* f_k, f_1, \dots, f_k)$ and Ψ_F its entire function on $\mathbb{C}^{2k}$ (Lemma 9.3). Then for all $z^{(1)}, \dots, z^{(n)} \in \mathbb{C}^k$ and $\lambda \in \mathbb{C}^n$,*

$$\sum_{i,l=1}^{n} \bar\lambda_i \lambda_l \, \Psi_F\big(\overline{z^{(i)}}, z^{(l)}\big) \;\geq\; 0.$$

At purely imaginary $z^{(i)} = i t^{(i)}$ this is $\sum_{i,l} \bar\lambda_i \lambda_l Z(\sigma_l - \gamma\sigma_i) \geq 0$, $\sigma_i := (\sum_r t_r^{(i)} f_r, 0)$, the positivity of Proposition 9.9 on the exponentials.

Proof. It suffices to take $(j, c) = (3, 0)$. Indeed, let γ_1 be a Euclidean motion with linear part in O_h carrying $\{x_3 > 0\}$ onto $\{x_j > c\}$ (as in the proof of Proposition 9.9), so that $\gamma_1^{-1} \gamma \gamma_1 = \theta$, put $g_r := (\gamma_1^{-1})_* f_r$, supported in $\{x_3 > 0\}$, and $F' := (\theta_* g_1, \dots, \theta_* g_k, g_1, \dots, g_k) = ((\gamma_1^{-1})_* F_1, \dots, (\gamma_1^{-1})_* F_{2k})$. Then $\Psi_{F'} = \Psi_F$ on $\mathbb{C}^{2k}$: both are entire (Lemma 9.3(c)) and agree on $i\mathbb{R}^{2k}$, where $\Psi_{F'}(it) = \mathcal{Z}((\gamma_1^{-1})_* \sum_r t_r F_r) = \mathcal{Z}(\sum_r t_r F_r) = \Psi_F(it)$ by Proposition 9.6, hence on $\mathbb{C}^{2k}$ (identity theorem, as in the proof of Lemma 9.3(b)); so the inequality for F is the inequality for F' at $(3, 0)$. On the torus, for ε_N smaller than the distance of the supports from the plane and L large, $A_i := \exp(\sum_r z_r^{(i)} \Phi_N(f_r))$ is Λ_0^+-measurable with $\langle |A_i|^2 \rangle_L^{\text{fine}} < \infty$ (Lemma 9.3(a)), and $\Theta_0 A_i = \exp(\sum_r \overline{z_r^{(i)}} \Phi_N(\gamma_* f_r))$ because $\Phi_N(f) \circ R_0 = \Phi_N(\gamma_* f)$ is real. So $\langle \Theta_0 A_i \cdot A_l \rangle_L^{\text{fine}} = \Psi_{N,L}^F(\overline{z^{(i)}}, z^{(l)})$, where $\Psi_{N,L}^F$ denotes the function $\Psi_{N,L}$ of Lemma 9.3 for the family F, and the L^2 form of Theorem 9.7 applied to $A := \sum_l \lambda_l A_l$ gives the inequality at level (N, L). It passes to the limit $L \to \infty$ along (L_n) and then to $m \to \infty$ by the locally uniform convergences of Lemma 9.3(b),(c) at the finitely many points $(\overline{z^{(i)}}, z^{(l)})$. □

Proposition 9.11 (OS2 in polynomial form)**.** *Let $P = \sum_\alpha c_\alpha \prod_{i \leq n_\alpha} \Phi(f_{\alpha i})$ be a formal polynomial in the smeared curvature with real C_c^∞ sources supported in $\{x_j > c\}$, γ the reflection in $\{x_j = c\}$, $\theta P := \sum_\alpha \bar c_\alpha \prod_i \Phi(\gamma_* f_{\alpha i})$, and let S assign to a monomial its Schwinger function (Theorem 9.4(iii)). Then*

$$S(\theta(P) P) = \sum_{\alpha,\beta} \bar c_\alpha c_\beta \, S_{n_\alpha + n_\beta}(\gamma_* f_{\alpha 1}, \dots, \gamma_* f_{\alpha n_\alpha}, f_{\beta 1}, \dots, f_{\beta n_\beta}) \;\geq\; 0.$$

Proof. Again it suffices to take $(j, c) = (3, 0)$: with γ_1 as in the previous proof, differentiating at 0 the identity $\Psi_{F'} = \Psi_F$ (for the family F of all sources and their reflections named below) gives $S_n((\gamma_1^{-1})_* u_1, \dots, (\gamma_1^{-1})_* u_n) = S_n(u_1, \dots, u_n)$ for all $u_1, \dots, u_n$ in F (Theorem 9.4(iii)), which carries the sum of the statement to the one for the sources $(\gamma_1^{-1})_* f_{\alpha i}$, supported in $\{x_3 > 0\}$, and the reflection θ. On the torus, with ε_N and L as in the previous proof, $P_N := \sum_\alpha c_\alpha \prod_i \Phi_N(f_{\alpha i})$ is Λ_0^+-measurable and square integrable (all moments are finite by Theorem 3.1), and $\Theta_0 P_N = \sum_\alpha \bar c_\alpha \prod_i \Phi_N(\gamma_* f_{\alpha i})$; Theorem 9.7 gives $\langle \Theta_0 P_N \cdot P_N \rangle_L^{\text{fine}} \geq 0$. This expectation is $\sum_{\alpha,\beta} \bar c_\alpha c_\beta$ times the mixed moment of the family F of all sources and their reflections, a mixed derivative

at 0 of $\Psi^F_{N,L}$ (differentiation under the integral, justified by the exponential moments), and these derivatives converge to those of Ψ_F at 0, the Schwinger functions, by Theorem 9.4(iv). □

9.4. **Symmetry (OS3).**

Proposition 9.12 (OS3: symmetry)**.** *(a) Each S_n is symmetric and multilinear, and $S_n(\bar f_1,\dots,\bar f_n)=\overline{S_n(f_1,\dots,f_n)}$; S_n is real on real test functions. (b) $Z(-f,h)=Z(f,h)$ for every test pair; $\Psi_f(-z)=\Psi_f(z)$; and $S_n=0$ for odd n.*

Proof. (a) is Theorem 9.4(iii), which states the symmetry, the multilinearity and the reality on real test functions; the conjugation identity $S_n(\bar f_1,\dots,\bar f_n)=\overline{S_n(f_1,\dots,f_n)}$ follows from the complex-multilinear extension (the 2^n-term expansion in the proof of (iii)) and that reality.

(b) The symmetry used is the charge conjugation $C\colon(\tilde\rho,B,v)\mapsto(\tilde\rho,-B,-v)$ of the fine torus measure, a bijection of the torus configuration space with three properties: (1) it preserves the reference measure (up to the null set on which some B_b sits at the endpoint π/γ_N of $(-\pi/\gamma_N,\pi/\gamma_N]$); (2) it preserves the weight, because the site factor g_{ct} does not see B, the hopping factor of the model of Section 2, $h(r,r',\vartheta)=\exp(-\frac12\kappa_N(\rho^2+\rho'^2-2\rho\rho'\cos(\gamma_N\vartheta)))$, is even in ϑ and so depends on B_b only through $\cos(\gamma_N B_b)$, the plaquette weight w is even, and d is linear, so closed v go to closed v (the sum runs over all closed v); (3) it maps $\mathfrak f_p$ to $-\mathfrak f_p$ and fixes $\tilde\rho$, hence $\Phi_N(f,h)\circ C=\Phi_N(-f,h)$. So C preserves the fine torus measure, $\langle F\circ C\rangle^{\mathrm{fine}}_L=\langle F\rangle^{\mathrm{fine}}_L$ for every bounded Borel F, in particular $\langle e^{i\Phi_N(f,h)}\rangle^{\mathrm{fine}}_L=\langle e^{i\Phi_N(-f,h)}\rangle^{\mathrm{fine}}_L$ for all N, L.

On Ω_N the charge conjugation is $C\colon(\mathfrak f,\tilde\rho)\mapsto(-\mathfrak f,\tilde\rho)$, $\Phi_N(f,h)\circ C=\Phi_N(-f,h)$, and μ_N is C-invariant by Theorem 11.5(c). The proof of that theorem for C passes the torus invariance just proved to the limit point through (2.3), and Lemma 11.1 takes the torus step from the present proof, so that the argument is not circular. Hence

$$Z_N(-f,h)=\int e^{i\Phi_N(-f,h)}d\mu_N=\int e^{i\Phi_N(f,h)}d(C_*\mu_N)=Z_N(f,h)$$

for every N and every bounded Borel $g_{\mathrm d}$, and Theorem 7.1(a) gives $Z(-f,h)=Z(f,h)$. The evenness of Ψ_f is Lemma 9.3(a),(c), and the odd derivatives of an even entire function vanish at 0. (The density component carries no such sign: the argument gives no relation between $Z(f,-h)$ and $Z(f,h)$.) □

9.5. **The measure.** We use the field space $X:=H^{-r}_{\mathrm{loc}}(\mathbb R^3;\Lambda^2)$ (throughout this section X denotes this space) and the Hermite–Sobolev space $\mathcal H^{-r}(\mathbb R^3;\Lambda^2)$ (the dual of the completion of $\mathcal S$ under $\|u\|_r^2=\sum_{\alpha,c}\lambda_\alpha^r\langle h_{\alpha,c},u\rangle^2$, $h_{\alpha,c}$ the Hermite basis, $\lambda_\alpha=2|\alpha|+4$), with the push-forward actions of the Euclidean motions, $(\gamma\Phi)(u)=\Phi(\gamma^{-1}\cdot u)$, the local σ-algebras $\Sigma(U)$ generated by $\Phi\mapsto\Phi(u)$, $u\in C^\infty_c(U;\Lambda^2)$ real, and the restriction map $\iota:\mathcal H^{-r}\to X$, exactly as in Appendix A (probability measures on spaces of distributions; Definition A.3). Of its abstract results we use in particular Lemma A.8 (the Hermite carrier), Proposition A.9 (uniqueness), Proposition A.10 (the verification of (H1)–(H3)) and Lemma A.11 (transfer); they are proved there and do not involve the lattice.

Theorem 9.13 (the measure of the limit curvature)**.** *Let $r\ge14$ be even.*

(a) *There is exactly one Borel probability measure μ^H on $\mathcal H^{-r}(\mathbb R^3;\Lambda^2)$ whose mixed moments on real C^∞_c 2-forms are the Schwinger functions S_n. Its characteristic functional on C^∞_c, indeed on $\mathcal S$, is $\mathcal Z$: $\int e^{i\Psi(u)}d\mu^H(\Psi)=\mathcal Z(u)$; its moments on $\mathcal S$ are the continuous extensions of the S_n and obey the bounds of Theorem 9.4(iii); the law of $(\Psi(f_1),\dots,\Psi(f_k))$ under μ^H is the law μ_f of Lemma 9.3.*

(b) $\mu := \iota_*\mu^H$ *is the unique Borel probability measure on* X *with characteristic functional* $\mathcal{Z}$ *on* C_c^∞*, and also the unique one whose mixed moments on* C_c^∞ *are the* S_n*;* μ^H *is the unique measure on* $\mathcal{H}^{-r}$ *with image* μ*.*

(c) μ *and* μ^H *are invariant under all translations, all coordinate reflections and the cubic group about every point.*

Proof. The proof applies the results of Appendix A; we verify their hypotheses in turn. *Positive definiteness.* For real $u_1, \dots, u_n \in C_c^\infty$ and $c \in \mathbb{C}^n$, $\sum_{l,l'} \bar{c}_l c_{l'} \mathcal{Z}(u_{l'} - u_l) = \lim_m \omega_{N_m}(|\sum_l c_l e^{i\Phi_{N_m}(u_l)}|^2) \geq 0$ by additivity of Φ_N and Theorem 7.1(a); $\mathcal{Z}(0) = 1$.

Continuity. $|\mathcal{Z}(u) - \mathcal{Z}(u')| \leq (3c_{a_0})^{1/2}\|u - u'\|_w$ (Theorem 9.4(i)), and $\|\cdot\|_w$ is a continuous seminorm on $\mathcal{S}$; so $\mathcal{Z}$ extends to a continuous positive definite function on $\mathcal{S}(\mathbb{R}^3; \Lambda^2)$ (positive definiteness is closed under pointwise limits). By the Bochner–Minlos theorem [19], in the form stated after Definition A.3 ($\mathcal{S}$ is nuclear) there is exactly one probability measure μ' on $\mathcal{S}'(\mathbb{R}^3; \Lambda^2)$ with its cylinder σ-algebra and $\int e^{i\Phi(u)} d\mu' = \mathcal{Z}(u)$ for real $u \in \mathcal{S}$.

Moments. For $u_1, \dots, u_n \in C_c^\infty$ the law of $(\Phi(u_l))_l$ under μ' has the characteristic function $t \mapsto \mathcal{Z}(\sum_l t_l u_l) = \Psi_u(it)$, so it is the law μ_u of Lemma 9.3(c), whose mixed moments are the $S_n(u_1, \dots, u_n)$ and whose exponential moments are finite (Theorem 9.4(iii)). For $u_l \in \mathcal{S}$ take cut-offs $\chi_R = \chi(\cdot/R)$, $\chi \in C_c^\infty$, $\chi = 1$ on the unit ball, $0 \leq \chi \leq 1$. Then $\Phi(\chi_R u_l) \to \Phi(u_l)$ for every $\Phi \in \mathcal{S}'$, and

$$\int \prod_l \Phi(\chi_R u_l)^2 d\mu' = S_{2n}(\chi_R u_1, \chi_R u_1, \dots) \leq \sigma_{2n} \prod_l \|\chi_R u_l\|_w^2 \leq \sigma_{2n} \prod_l \|u_l\|_w^2$$

uniformly in $R \geq 1$ (the seminorm $\|\cdot\|_w$ carries no derivative), so the products are uniformly integrable and $\int \prod_l \Phi(u_l) d\mu' = \lim_R S_n(\chi_R u_1, \dots) = S_n(u_1, \dots, u_n)$ by the continuity of S_n on $\mathcal{S}^n$.

The two-point bound and the carrier. $\int \Phi(u)^2 d\mu' = S_2(u,u) \leq R(u)^2 \leq 3c_{a_0}\|u\|_w^2 \leq 3c_{a_0}C^2\|u\|_{10}^2$ on $\mathcal{S}$, where $\|u\|_w \leq |u|_4$ (its derivative-free term) and $|u|_4 \leq C\|u\|_{10}$ (Lemma A.4(b), with the norm $|u|_4$ of a 2-form as in Definition A.1(a)). The Hermite carrier, Lemma A.8, whose hypothesis ($r \geq 14$ even, μ' a probability measure on $\mathcal{S}'$ with its cylinder σ-algebra, $\int \Phi(u)^2 d\mu' \leq C_2\|u\|_{10}^2$ for every real $u \in \mathcal{S}$) holds with $C_2 = 3c_{a_0}C^2$, gives μ^H, the image of μ' under the map Π of that lemma: a Borel probability measure on $\mathcal{H}^{-r}$ with the same characteristic functional and the same mixed moments on $\mathcal{S}$ as μ'.

The factorial moment bound and uniqueness. Proposition A.9 gives uniqueness (a) by the characteristic functional on real C_c^∞, on X or on $\mathcal{H}^{-r}$, and (b) by the mixed moments on C_c^∞, when these exist and $|\int \Phi(u)^n d\mu_1| \leq c(n+1)K^n n!$ for all $n \geq 0$ and every real $u \in C_c^\infty$, with $c, K \geq 0$ depending on u. It applies with $c = 1$ and $K = eR(u)$, since

$$|\int \Phi(u)^n d\mu^H| = |S_n(u, \dots, u)| \leq e^n (n!)^{1/2} R(u)^n \leq (n+1)K^n n!.$$

This gives the uniqueness in (a); for (b), $\mu := \iota_*\mu^H$ has characteristic functional $\mathcal{Z}$ and moments S_n on C_c^∞ (the restriction map is the identity on C_c^∞), both uniqueness clauses hold on X by the same proposition, and Lemma A.11(a) (μ^H is the only Borel probability measure on $\mathcal{H}^{-r}$ whose image under ι is $\iota_*\mu^H$) gives the last clause.

(c) The invariance of μ follows from Proposition A.10(a) (if γ is a Euclidean motion and $\mathcal{Z}(\gamma \cdot u) = \mathcal{Z}(u)$ for all real u, then $\gamma_*\mu = \mu$) and Proposition 9.6, and that of μ^H from Lemma A.11(b). The law of $(\Psi(f_j))_j$ is μ_f as in the moment step. □

9.6. **Clustering in the mean and the vacuum: the weak form of OS4.** *Reconstruction from S.* By Proposition 9.9 the form $\langle a, b\rangle := S(\Theta a \cdot b)$ is positive semidefinite and Hermitian on $\mathcal{A}_+$. Let $\mathcal{H}_\infty$ be the completion of $\mathcal{A}_+$ modulo the null space of this form, $[a] \in \mathcal{H}_\infty$ the class of

a, and $\Omega := [1]$. For $t \geq 0$ the translation τ_t maps $\mathcal{A}_+$ into itself, and by Theorem 7.1(b),(c), $|S(a)| \leq M(a)$ and the argument of Remark 9.14 the map $[a] \mapsto [\tau_t a]$ is a strongly continuous self-adjoint contraction semigroup on $\mathcal{H}_\infty$; we write it e^{-tH}, with $H \geq 0$ self-adjoint and $e^{-tH}\Omega = \Omega$.

Reconstruction from μ. For $j \in \{1,2,3\}$ let $L^2_{j,+} := L^2(\Sigma(\{x_j > 0\}), \mu)$, $U(a)G := G \circ \tau_{-a}$, $\Theta_j G := G \circ \theta_j$ (θ_j the coordinate reflection on X), and $\langle F, G\rangle_j := \mathbb{E}_\mu[\overline{\Theta_j F}\, G]$ on $L^2_{j,+}$. By Theorem 9.15, μ satisfies (H1)–(H3) of Subsection A.1, so the reconstruction in a coordinate direction applies (Proposition A.23; see Remark 9.14): $\mathcal{H}_j$ is the completion of $L^2_{j,+}$ modulo the null space of $\langle\cdot,\cdot\rangle_j$, $\Omega_j = [1]$, $[U(te_j)G] = e^{-tH_j}[G]$ ($t \geq 0$) with $H_j \geq 0$ self-adjoint, $[U(y)G] = V_j(y)[G]$ with $V_j(y)$ unitary for $y \perp e_j$, and $\mathbb{E}_\mu[FG] = \langle[\overline{\Theta_j F}], [G]\rangle_j$ for $F \in L^2(\Sigma(\{x_j < 0\}), \mu)$, $G \in L^2_{j,+}$. For $j = 3$, $\mathcal{H}_3$ is the curvature sector of $\mathcal{H}_\infty$: the map $[e_{(f,0)}] \mapsto [e^{i\Phi(f)}]$, $\operatorname{supp} f \subset \{x_3 > 0\}$, preserves the forms $(S(\Theta e_{(f,0)} e_{(f',0)}) = \mathcal{Z}(f' - \theta f) = \mathbb{E}_\mu[\overline{e^{i\Phi(f)} \circ \theta_3} e^{i\Phi(f')}])$ and has dense range (the exponential algebra is dense in $L^2_{3,+}$, by Lemma A.7), so it extends to a unitary of the closed e^{-tH}-invariant subspace $\mathcal{H}^{\mathrm{curv}}_\infty \ni \Omega$ generated by the curvature exponentials onto $\mathcal{H}_3$, carrying Ω to Ω_3 and e^{-tH} to e^{-tH_3}.

Remark 9.14 (the reconstruction). The two statements used in the preceding paragraph, the reconstruction proposition (Proposition A.23) and the density lemma (Lemma A.7), are elementary consequences of (H1)–(H3) of Subsection A.1. They are proved in Appendix A; we recall the reasons. The relation $\Theta_j U(a) = U(\rho_j a)\Theta_j$ and the invariance of μ give $\langle U(te_j)F, U(te_j)G\rangle_j = \langle F, U(2te_j)G\rangle_j$ and $\langle F, U(te_j)G\rangle_j = \langle U(te_j)F, G\rangle_j$ on $L^2_{j,+}$ for $t \geq 0$ ($U(te_j)L^2_{j,+} \subset L^2_{j,+}$ by (H2)). With $\|[G]\|_j^2 = \mathbb{E}_\mu[\overline{\Theta_j G}\, G] \leq \|G\|^2_{L^2(\mu)}$ (the Schwarz inequality and the θ_j-invariance of μ) and the usual iteration of $\|[U(te_j)G]\|_j^2 = \langle G, U(2te_j)G\rangle_j$, the map $[G] \mapsto [U(te_j)G]$ is a self-adjoint positive contraction semigroup on $\mathcal{H}_j$, strongly continuous because U is strongly continuous on $L^2(\mu)$ (bounded continuous functions are dense, X being Polish, and dominated convergence applies); this is e^{-tH_j} with $H_j \geq 0$. For $y \perp e_j$, $U(y)$ maps $L^2_{j,+}$ onto itself by (H2), since $\{x_j > 0\} + y = \{x_j > 0\}$.

The transverse translations. For $y \perp e_j$, Θ_j commutes with $U(y)$ by (A.3), so $\langle U(y)F, U(y)G\rangle_j = \mathbb{E}[\overline{U(y)\Theta_j F}\, U(y)G] = \langle F, G\rangle_j$ by invariance of μ. Thus $V_j(y)$ preserves the null space of $\langle\cdot,\cdot\rangle_j$, is isometric on the quotient and has the isometric inverse $V_j(-y)$; it extends to a unitary operator, and $y \mapsto V_j(y)$ is a representation, strongly continuous by the same estimate as for e^{-tH_j}. The operators e^{-tH_j} and $V_j(y)$ commute because the $U(a)$ do, and $e^{-tH_j}\Omega_j = [U(te_j)1] = [1] = \Omega_j$, $V_j(y)\Omega_j = \Omega_j$.

The identity $\mathbb{E}_\mu[FG] = \langle[\overline{\Theta_j F}], [G]\rangle_j$ is the definition of the form, Θ_j being an involution commuting with conjugation and $\overline{\Theta_j F} \in L^2_{j,+}$ by (H2). The span of the $e^{i\Phi(u)}$, $u \in C_c^\infty(\{x_j > 0\}; \Lambda^2)$ real, is a unital $*$-algebra of bounded functions generating $\Sigma(\{x_j > 0\})$, so the functional monotone class theorem puts every bounded $\Sigma(\{x_j > 0\})$-measurable function in its $L^2(\mu)$-closure; this is the density lemma.

Theorem 9.15 (the weak form of OS4)**.** *μ on X (and μ^H on $\mathcal{H}^{-r}$) satisfies the hypotheses* (H1)–(H3) *of Subsection A.1:*

- (H1) *X is Polish, the translations act jointly continuously, the coordinate reflections θ_j are involutive homeomorphisms with $\theta_j\tau_a = \tau_{\rho_j a}\theta_j$, and μ is invariant under every τ_a and every θ_j;*
- (H2) *the local σ-algebras $\Sigma(U)$ have the monotonicity, exhaustion, countable generation and covariance properties stated in* (H2) *of Subsection A.1;*
- (H3) *$\mathbb{E}_\mu[\overline{F \circ \theta_{j,c}}\, F] \geq 0$ for every j, $c \in \mathbb{R}$ and $F \in L^2(\Sigma(\{x_j > c\}), \mu)$, $\theta_{j,c} = \tau_{ce_j}\theta_j\tau_{-ce_j}$;*

and μ is invariant under the cubic group about every point. Consequently the results of Subsection A.6 apply to μ: the ergodic decomposition $\mu = \int \mu_\xi \, d\nu(\xi)$ (Theorem A.18) with ν-a.e. component translation invariant and ergodic for $\mathbb{R}^3$ (the weak form of OS4), invariant under $\theta_1, \theta_2, \theta_3$, reflection positive about every coordinate plane and satisfying (H2)*; clustering in the mean and under diagonal separation on ν-almost every component (Theorems A.26 and A.28, Corollary A.29(ii)); the reconstruction in each coordinate direction from ν-almost every component, with a unique joint vacuum (Proposition A.24(iii), Corollary A.29(iii)); the regrouping of the components into cubic-invariant ones (Theorem A.20); and the growth bound for the Schwartz moments of the components of μ^H (Theorem A.21), whose moment hypothesis* (A.6) *($\int \Psi(h_{\alpha,c})^{2n} d\mu^H \le c_S (2n)! (C_0 \|h_{\alpha,c}\|_{10})^{2n}$) holds with $c_S = 1$, $C_0 = eC\sqrt{3c_{a_0}}$ (C the constant of Lemma A.4(b)) by Theorem 9.4(iii).*

Proof. In (H1) the properties of X and of the actions are structural for the push-forward actions (Lemma A.5(a) and Lemma A.6(a)), and the invariance of μ is Theorem 9.13(c). (H2) is structural (Lemma A.6(b), which rests on Lemma A.5(b),(c)). (H3) holds by Proposition A.10(c), whose two hypotheses (μ is $\theta_{j,c}$-invariant; $\sum_{k,l \le n} \bar{a}_k a_l \mathcal{Z}(u_l - \rho_{j,c} \cdot u_k) \ge 0$ for all n, all real $u_1, \dots, u_n \in C_c^\infty(\{x_j > c\}; \Lambda^2)$ and all $a \in \mathbb{C}^n$) are Theorem 9.13(c) and Theorem 9.10 at purely imaginary coefficients (i.e. Proposition 9.9 on the exponentials $e_{(u,0)}$). The invariance of μ under the cubic group about every point is Theorem 9.13(c), and the structural hypotheses on the cubic action of the regrouping result (the standing assumption stated before Theorem A.20, namely that the cubic group acts by homeomorphisms with $\rho_j = \theta_j$, $g\tau_a g^{-1} = \tau_{ga}$ and $\Sigma(U) \mapsto \Sigma(gU)$) are identities for the push-forward action (Definition A.3, Lemma A.6).

The conclusions are the theorems of Subsection A.6, which assume nothing but (H1)–(H3) (and the cubic invariance for the regrouping, the moment bound for the growth clause). Between X and $\mathcal{H}^{-r}$ the hypotheses are carried by Lemma A.11(c), and the conclusions by Proposition A.19 (the decomposition of an image law) and Corollary A.22, ι being a Borel isomorphism onto a Borel set of full measure that intertwines the actions and the local σ-algebras (Lemma A.5(d), Lemma A.11(a), Remark A.12). □

Remark 9.16 (consequences of rotation invariance). Suppose that S is invariant under $SO(3)$, hence, together with Proposition 9.6, under the whole Euclidean group. Then μ is $O(3)$-invariant (Proposition A.10(a) again) and reflection positive about every plane: the half-space algebra and reflection of a tilted plane are the rotated ones. Moreover the reconstruction in any direction n is unitarily equivalent to the one in direction e_3, with the same Hamiltonian spectrum. Of the rotation invariance, both statements use only the invariance of μ under the Euclidean group; we give the argument. Let g be a Euclidean motion, acting on X by $(g\Phi)(u) = \Phi(g^{-1} \cdot u)$. By the invariance of μ, $G \mapsto G \circ g^{-1}$ is a unitary of $L^2(\mu)$, and it maps $L^2(\Sigma(V), \mu)$ onto $L^2(\Sigma(gV), \mu)$ for every open set V, because $\Sigma(V)$ is generated by the functions $\Phi \mapsto \Phi(u)$, $u \in C_c^\infty(V; \Lambda^2)$ real.

(i) *Reflection positivity about a plane* Π. Choose g such that $V_\Pi := g\{x_3 > 0\}$ is one of the two open half spaces of Π; then $\theta_\Pi := g\theta_3 g^{-1}$ is the reflection in Π. For $F \in L^2(\Sigma(V_\Pi), \mu)$ put $F' := F \circ g$, a function in $L^2(\Sigma(\{x_3 > 0\}), \mu)$. By the invariance of μ and by (H3) of Theorem 9.15, $\mathbb{E}_\mu[\overline{F \circ \theta_\Pi} \, F] = \mathbb{E}_\mu[\overline{F' \circ \theta_3} \, F'] \ge 0$.

(ii) *The reconstruction in a direction n.* For a unit vector n let g be a rotation with $ge_3 = n$. The unitary $G \mapsto G \circ g^{-1}$ carries $L^2_{3,+}$ onto $L^2(\Sigma(\{x \cdot n > 0\}), \mu)$, the reflection Θ_3 to the reflection in the plane $\{x \cdot n = 0\}$ and the translations $U(te_3)$ to $U(tn)$, by $g\tau_a g^{-1} = \tau_{ga}$. Hence it carries the reconstruction of μ in the direction e_3 (the paragraph before Remark 9.14) to the reconstruction of μ in the direction n, which is defined in the same way from the half space $\{x \cdot n > 0\}$, and the Hamiltonian H_3 to its Hamiltonian: the two reconstructions are unitarily equivalent, with the same spectrum.

Without rotations, what is missing from the full axiom OS1 is exactly the rotation invariance, and from OS2 the reflection positivity about non-coordinate planes; OS0, OS3, the measure and the weak form of OS4 (Theorem 9.15) do not depend on it. The rotation invariance is proved in Sections 12–15: for the killed-box states by Theorems 12.2 and 12.3, and for the torus limit points along dyadic sequences (Definition 14.1) by Theorem 12.4, whose proof uses the reflection positivity at the six diagonal site planes through 0 (Theorem 14.3).

10. Correlation inequalities for the Villain fine model

This section proves the four correlation inequalities (K1)–(K4) that Section 11 uses: the first and second inequalities of Ginibre [17] (for abelian Higgs models see also [6], [10]) on a cone of observables, the monotonicity of the cone expectations in every coupling, and the monotonicity in the volume for the killed boxes together with the comparison of a killed box with the torus. The proofs use the duplicate-variable argument, adapted to the Villain form of the plaquette weight by a duplication identity of that weight (Lemma 10.4). Throughout, N is fixed, $\varepsilon := \varepsilon_N$ is the spacing and γ_N the phase constant; the couplings $\kappa_N \geq 0$, $\lambda_0 > 0$, $\rho_{0,0} > 0$ and the bare mass counterterm $\delta m_B \in \mathbb{R}$ of Section 2 (of either sign) are arbitrary. We write $\rho_x := \rho_{0,0} + \tilde{\rho}_x \in [0,\infty)$ for the modulus, $\theta_b := \gamma_N B_b \in (-\pi,\pi]$ for the angle of a bond and $n_p := \gamma_N v_p/2\pi \in \mathbb{Z}$ for the Villain integer of a plaquette, so that the curvature is

$$\mathfrak{f}_p = (dB)_p + v_p = \gamma_N^{-1}\big((d\theta)_p + 2\pi n_p\big).$$

The coboundaries are the standard ones: for a bond $b = (x, x+\varepsilon e_i)$ and a plaquette $p = (x; ij)$, $i < j$, with the four bonds $(x,i), (x+\varepsilon e_i, j), (x+\varepsilon e_j, i), (x,j)$, $(da)_{x;ij} = a_{x,i} + a_{x+\varepsilon e_i,j} - a_{x+\varepsilon e_j,i} - a_{x,j}$; for a cube $c = (x; 123)$, $(dn)_{x;123} = \sum_{(ijk)\ \text{cyclic}} \big(n_{x+\varepsilon e_i;jk} - n_{x;jk}\big)$ with $n_{x;kj} := -n_{x;jk}$; $dd = 0$. A cochain is closed if its coboundary vanishes on every cube (or plaquette) of the lattice in question; v closed means n closed.

10.1. The class of weights, the states, the cone.

Definition 10.1 (finite lattices; the class of weights)**.** A *finite lattice* G is a finite disjoint union $G = G_1 \sqcup \cdots \sqcup G_r$ ($r \geq 1$) of *components*, each of which is either a fine torus $\tilde{T}_L$ or a *box* $\Lambda \subset \varepsilon\mathbb{Z}^3$, a product of three finite intervals of sites; the components are distinct copies, and the case $r = 1$, a single torus or box, is the case of interest. The cells of a component G_i are: the sites $\mathcal{X}(G_i)$; the bonds $\mathcal{B}(G_i)$ (for a box, the bonds with both ends in Λ); the plaquettes $\mathcal{P}(G_i)$ (for a box, those with all four bonds in $\mathcal{B}(\Lambda)$); the cubes $\mathcal{C}(G_i)$ (for a box, those with all six faces in $\mathcal{P}(\Lambda)$). The cells of G are the disjoint unions $\mathcal{X}(G) := \bigsqcup_i \mathcal{X}(G_i)$, $\mathcal{B}(G)$, $\mathcal{P}(G)$, $\mathcal{C}(G)$ of those of its components; every cell of G lies in exactly one component, and every face of a cube of G lies in the component of that cube.

The variables are $\rho_x \in [0,\infty)$ ($x \in \mathcal{X}(G)$), $B_b \in (-\pi/\gamma_N, \pi/\gamma_N]$ ($b \in \mathcal{B}(G)$) and $v \in (2\pi/\gamma_N)\mathbb{Z}^{\mathcal{P}(G)}$ closed on $\mathcal{C}(G)$, i.e. with $(dv)_c = 0$ for every cube $c \in \mathcal{C}(G)$; since every cube lies in one component, v is closed on $\mathcal{C}(G)$ exactly when each restriction $v|_{\mathcal{P}(G_i)}$ is closed on $\mathcal{C}(G_i)$, and the closed cochains of G are the products of the closed cochains of its components.

Given finite positive Borel measures μ_x on $[0,\infty)$ with

$$\int_0^\infty e^{\alpha\rho^2}\mu_x(d\rho) < \infty \qquad \text{for every } \alpha \in \mathbb{R}, \tag{10.1}$$

coefficients $\kappa_b \geq 0$ ($b \in \mathcal{B}(G)$) and $\beta_p > 0$ ($p \in \mathcal{P}(G)$), the *weight* $W = W_G(\mu, \kappa, \beta)$ is the measure

$$\prod_{x\in\mathcal{X}(G)} \mu_x(d\rho_x) \prod_{b\in\mathcal{B}(G)} e^{\kappa_b \rho_{b_-}\rho_{b_+}\cos\theta_b}\, dB_b \sum_{v\ \text{closed}} \prod_{p\in\mathcal{P}(G)} e^{-\beta_p \mathfrak{f}_p^2/2}$$

and $\langle F\rangle := \int F\,W/\int W$ is its *state*. For $r \geq 2$ every factor of W is indexed by a cell of one component and the sum over the closed v factorises over the components, so $W = W_{G_1}\otimes\cdots\otimes W_{G_r}$ is the product of the weights of the components (each with the restriction of μ, κ, β to its cells), and the state of G is the product state: functions of the variables of different components are independent under $\langle\cdot\rangle$.

The mass $Z := \int W$ is finite and positive: for each p, $\sum_{n\in\mathbb{Z}} e^{-\beta_p(y+2\pi n)^2/(2\gamma_N^2)} \leq K_p := \sum_{m\in\mathbb{Z}} e^{-\beta_p(\pi m)^2/(2\gamma_N^2)}$ for every real y (the term with $y+2\pi n \in (-\pi,\pi]$ is at most 1, and the m-th term on either side of it has $|y+2\pi n| \geq 2\pi|m| - \pi \geq \pi|m|$), so the v-sum is at most $\prod_p K_p$ uniformly in B; and $e^{\kappa_b\rho\rho'} \leq e^{\kappa_b(\rho^2+\rho'^2)/2}$ with (10.1) bounds the ρ-integral. Every F with $|F| \leq C\,e^{\sum_x q_x\rho_x^2}$ is W-integrable, by the same two bounds.

Let $\mu^\circ(d\rho) := g_{\mathrm{ct}}(\rho-\rho_{0,0})\,e^{-3\kappa_N\rho^2}\,d\rho$ on $[0,\infty)$, where $g_{\mathrm{ct}}(\tilde\rho) := g(\tilde\rho)\,e^{-\frac12\delta m_B(\rho_{0,0}+\tilde\rho)^2}$, $g(\tilde\rho) = (\rho_{0,0}+\tilde\rho)e^{-\lambda_0(\rho^2-\rho_{0,0}^2)^2}$, is the site factor of Section 2 with its bare mass counterterm, $\delta m_B \in \mathbb{R}$ of either sign. The measure μ° is finite and positive and satisfies (10.1) for every $\alpha \in \mathbb{R}$ and every real δm_B, because the quartic dominates ((2.1) at $q = |\alpha|$, times $e^{-3\kappa_N\rho^2} \leq 1$): for $\rho \geq 0$,

$$g_{\mathrm{ct}}(\rho-\rho_{0,0})\,e^{-3\kappa_N\rho^2}\,e^{\alpha\rho^2} \leq \rho\,e^{\lambda_0\rho_{0,0}^4}\,e^{-\frac12\lambda_0\rho^4+(|\alpha|+\frac12|\delta m_B|)\rho^2},$$

by $\lambda_0(\rho^2-\rho_{0,0}^2)^2 \geq \frac12\lambda_0\rho^4 - \lambda_0\rho_{0,0}^4$ and $e^{\alpha\rho^2}e^{-\frac12\delta m_B\rho^2} \leq e^{(|\alpha|+\frac12|\delta m_B|)\rho^2}$; the right side is integrable over $[0,\infty)$ since $\lambda_0 > 0$.

Definition 10.2 (torus states and killed-box states)**.** (a) The *torus state* $\langle\cdot\rangle_L^{\mathrm{fine}}$ of Section 2 is the state of $W_{\tilde T_L}(\mu^\circ,\kappa_N,1)$: every site of the torus is an end of six bonds, and the hopping factor h of each bond contributes the diagonal factor $e^{-\kappa_N\rho^2/2}$ to each of its ends, so $\prod_x g_{\mathrm{ct}}\prod_b h = \prod_x g_{\mathrm{ct}}(\tilde\rho_x)e^{-3\kappa_N\rho_x^2}\prod_b e^{\kappa_N\rho_{b_-}\rho_{b_+}\cos\theta_b}$.

(b) For a box Λ the *killed-box state* $\langle\cdot\rangle_\Lambda$ is the state of $W_\Lambda(\mu^\circ,\kappa_N,1)$: the weight

$$W_\Lambda = \prod_{x\in\Lambda} g_{\mathrm{ct}}(\tilde\rho_x)e^{-3\kappa_N\rho_x^2} \prod_{b\in\mathcal{B}(\Lambda)} e^{\kappa_N\rho_{b_-}\rho_{b_+}\cos(\gamma_N B_b)} \prod_{p\in\mathcal{P}(\Lambda)} e^{-\mathfrak{f}_p^2/2}$$

in the variables $\tilde\rho_x$ $(x\in\Lambda)$, B_b $(b\in\mathcal{B}(\Lambda))$, v_p $(p\in\mathcal{P}(\Lambda))$, the sum over the v closed on the cubes of Λ. It is the weight of Section 2 with the matter killed outside Λ ($\rho_y = 0$ for $y\notin\Lambda$, which turns the hopping factor of every bond leaving Λ into the diagonal factor $e^{-\kappa_N\rho_x^2/2}$ of its inner end, so that every site of Λ carries the diagonal halves of all six of its bonds) and with no gauge or Villain variables outside $\mathcal{B}(\Lambda)$, $\mathcal{P}(\Lambda)$.

A function is *supported in* Λ if it depends only on $\mathfrak{f}_p$ $(p\in\mathcal{P}(\Lambda))$, B_b $(b\in\mathcal{B}(\Lambda))$ and ρ_x $(x\in\Lambda)$.

Definition 10.3 (the cone $\mathcal{K}_N$ and the enlarged cone $\mathcal{K}_N^+$)**.** For a real finitely supported plaquette function $a = (a_p)$, an integer finitely supported bond function $k = (k_b)$, a polynomial P in finitely many ρ_x with non-negative coefficients and a finitely supported $q = (q_x)$ with $q_x \geq 0$, put

$$\Theta_{a,k,P,q} := \cos\Big(\sum_p a_p\mathfrak{f}_p + \sum_b k_b\theta_b\Big)\,P(\rho)\,e^{\sum_x q_x\rho_x^2}, \qquad \Theta_{a,P,q} := \Theta_{a,0,P,q}.$$

The *cone* $\mathcal{K}_N$ is the set of finite non-negative linear combinations of the $\Theta_{a,P,q}$, and the *enlarged cone* $\mathcal{K}_N^+$ that of the $\Theta_{a,k,P,q}$; $\mathcal{K}_N \subset \mathcal{K}_N^+$. The *generators* of $\mathcal{K}_N$ are the $\Theta_{a,1,q}$ (this is the sense of Section 11; the polynomial factors are in the cone but are not needed to generate the state). Both cones are closed under products ($\cos u\cos v = \frac12[\cos(u+v)+\cos(u-v)]$, and products of polynomials with non-negative coefficients and of the exponentials stay in the class). In particular, for every site x, bond b and plaquette p and every $q \geq 0$,

$$\rho_x^2 \in \mathcal{K}_N, \qquad e^{q\rho_x^2} \in \mathcal{K}_N, \qquad \rho_x^2 e^{q\rho_x^2} \in \mathcal{K}_N, \qquad \cos\Big(\sum_p a_p\mathfrak{f}_p\Big) \in \mathcal{K}_N, \qquad \rho_{b_-}\rho_{b_+}\cos\theta_b \in \mathcal{K}_N^+$$

($a=0$, $P=\rho_x^2$ or $P=1$; the last with $a=0$, $k=\delta_b$, $P=\rho_{b_-}\rho_{b_+}$).

Every element of $\mathcal{K}_N^+$ is continuous, is local, and satisfies $|\Theta|\le Ce^{\sum_x q'_x\rho_x^2}$ for some $C<\infty$ and finitely supported $q'\ge 0$; hence it is integrable for every weight of Definition 10.1. On a finite lattice G the cone $\mathcal{K}_N$ and the enlarged cone $\mathcal{K}_N^+$ *of* G are defined by the same formulas with a, k, P, q indexed by the plaquettes, bonds and sites of G (cells of different components may occur in one element: the functions are evaluated on the cells of G, and only elements whose plaquettes, bonds and sites are cells of G are considered). The energy densities $\mathfrak{f}_p^2$ are not in $\mathcal{K}_N^+$; their covariances with the cone have a sign of their own, (K3) of Theorem 10.8.

10.2. **The duplication identity and three lemmas.** Fix a finite lattice G and write $\mathcal{B}=\mathcal{B}(G)$, $\mathcal{P}=\mathcal{P}(G)$, $\mathcal{C}=\mathcal{C}(G)$ (G may have several components; every statement of this subsection is about the cells of G, with closedness imposed cube by cube, and holds component by component). Two independent copies (ρ,θ,n) and (ρ',θ',n') of the variables are called the *duplicate variables*; F' denotes the function F evaluated in the primed copy. The new variables are

$$u_x:=\tfrac12(\rho_x+\rho'_x),\quad v_x:=\tfrac12(\rho'_x-\rho_x),\qquad \alpha_b:=\tfrac12(\theta_b+\theta'_b),\quad \eta_b:=\tfrac12(\theta_b-\theta'_b), \tag{10.2}$$

so that $\rho=u-v$, $\rho'=u+v$ with $u_x\ge|v_x|$, and $\theta=\alpha+\eta$, $\theta'=\alpha-\eta$.

Lemma 10.4 (duplication identity of the Villain weight)**.** *Let σ range over $\{0,1\}^{\mathcal{P}}$ and, for each σ, let $\mathsf{K}_\sigma:=\{k\in\mathbb{Z}^{\mathcal{P}}:\ 2dk+d\sigma=0 \text{ on } \mathcal{C}\}$ (possibly empty). If G has the components G_i, write $\mathsf{K}_\sigma^{(i)}$ for the set defined by the same condition for $\sigma|_{\mathcal{P}(G_i)}$ on the cubes of G_i; then $k\in\mathsf{K}_\sigma$ exactly when each restriction $k|_{\mathcal{P}(G_i)}$ lies in $\mathsf{K}_\sigma^{(i)}$, because the condition is imposed cube by cube and every cube lies in one component: $\mathsf{K}_\sigma=\prod_i\mathsf{K}_\sigma^{(i)}$. The map $(k,l,\sigma)\mapsto(n,n'):=(k+l+\sigma,\ k-l)$ is a bijection from $\{(k,l,\sigma):\ k,l\in\mathsf{K}_\sigma\}$ onto the pairs of closed integer cochains (n,n'), with inverse $\sigma_p=(n_p+n'_p)\bmod 2$, $k=\frac12(n+n'-\sigma)$, $l=\frac12(n-n'-\sigma)$. Under it, with*

$$\begin{aligned} X_p&:=(d\alpha)_p+\pi\sigma_p+2\pi k_p, \qquad Y_p:=(d\eta)_p+\pi\sigma_p+2\pi l_p,\\ \xi_p&:=X_p/\gamma_N, \qquad \upsilon_p:=Y_p/\gamma_N, \end{aligned} \tag{10.3}$$

one has $(d\theta)_p+2\pi n_p=X_p+Y_p$ and $(d\theta')_p+2\pi n'_p=X_p-Y_p$, i.e. $\mathfrak{f}_p=\xi_p+\upsilon_p$, $\mathfrak{f}'_p=\xi_p-\upsilon_p$, and

$$\begin{aligned} \sum_{n,n'\ \text{closed}}\ \prod_{p\in\mathcal{P}} e^{-\beta_p(\mathfrak{f}_p^2+\mathfrak{f}_p'^2)/2}&=\sum_\sigma\sum_{k\in\mathsf{K}_\sigma}\sum_{l\in\mathsf{K}_\sigma}V_\sigma(\alpha,k)\,V_\sigma(\eta,l),\\ V_\sigma(\alpha,k)&:=\prod_{p\in\mathcal{P}}e^{-\beta_pX_p^2/\gamma_N^2}\ge 0, \end{aligned} \tag{10.4}$$

where $V_\sigma(\eta,l)$ is the same function evaluated at (η,l). Moreover $\sum_{k\in\mathbb{Z}^{\mathcal{P}}}V_\sigma(\alpha,k)\le\prod_p K_p$ and $\sum_{k\in\mathbb{Z}^{\mathcal{P}}}|X_p|^jV_\sigma(\alpha,k)\le C_j<\infty$ uniformly in α and σ, for every $j\ge0$ and p.

Proof. Given closed n,n', the integers $n_p+n'_p$ and $n_p-n'_p$ have the same parity, so $\sigma_p:=(n_p+n'_p)\bmod 2\in\{0,1\}$ makes $k:=\frac12(n+n'-\sigma)$ and $l:=\frac12(n-n'-\sigma)$ integer cochains; $2dk=dn+dn'-d\sigma=-d\sigma$ and $2dl=-d\sigma$ on $\mathcal{C}$, so $k,l\in\mathsf{K}_\sigma$, and $k+l+\sigma=n$, $k-l=n'$. Conversely, for $k,l\in\mathsf{K}_\sigma$ the cochains $n:=k+l+\sigma$ and $n':=k-l$ are integer and closed: $dn=dk+dl+d\sigma=-\frac12d\sigma-\frac12d\sigma+d\sigma=0$ and $dn'=dk-dl=0$; and the two maps are inverse to each other ($(n+n')\bmod 2=(2k+\sigma)\bmod 2=\sigma$). Then $(d\theta)_p+2\pi n_p=(d\alpha)_p+(d\eta)_p+2\pi(k_p+l_p+\sigma_p)=X_p+Y_p$ and $(d\theta')_p+2\pi n'_p=(d\alpha)_p-(d\eta)_p+2\pi(k_p-l_p)=X_p-Y_p$. Since $(X+Y)^2+(X-Y)^2=2X^2+2Y^2$, $e^{-\beta_p[(X_p+Y_p)^2+(X_p-Y_p)^2]/(2\gamma_N^2)}=e^{-\beta_pX_p^2/\gamma_N^2}e^{-\beta_pY_p^2/\gamma_N^2}$, which is (10.4) after the reindexing of the double sum.

For the uniform bounds, observe that for all real y_p

$$\sum_{k\in\mathbb{Z}^{\mathcal{P}}}\prod_{p} e^{-\beta_p(y_p+2\pi k_p)^2/\gamma_N^2} = \prod_{p}\sum_{k\in\mathbb{Z}} e^{-\beta_p(y_p+2\pi k)^2/\gamma_N^2} \le \prod_{p} K_p,$$

because each summand is at most the summand $e^{-\beta_p(y_p+2\pi k)^2/(2\gamma_N^2)}$ of Definition 10.1, whose sum over k is at most K_p (here K_p is the constant of the halved stiffness). With a factor $|y_p+2\pi k_p|^j$ in one plaquette, the one-dimensional sum $\sum_k |y+2\pi k|^j e^{-\beta_p(y+2\pi k)^2/\gamma_N^2}$ is bounded uniformly in y by the same comparison with $\pi|m|$, since the function $s\mapsto s^j e^{-\beta_p s^2/\gamma_N^2}$ is bounded on $[0,\infty)$ and summable along $s=\pi m$. The identity (10.4) is a product over the plaquettes of G of per-plaquette factors and the bijection is defined plaquette by plaquette, with closedness imposed cube by cube; so for a lattice with several components it is the product of the identities of the components, with K_σ the product set above. □

Lemma 10.5 (the angle domain)**.** *Let F be an integrable function on $(-\pi,\pi]^{\mathcal{B}}\times(-\pi,\pi]^{\mathcal{B}}$ which is 2π-periodic in each of its $2|\mathcal{B}|$ arguments. Then*

$$\iint_{(-\pi,\pi]^{2\mathcal{B}}} F(\theta,\theta')\,d\theta\,d\theta' = \iint_{(-\pi,\pi]^{2\mathcal{B}}} F(\alpha+\eta,\alpha-\eta)\,d\alpha\,d\eta.$$

Proof. Regard F as a function on the compact group $\mathbb{T}:=(\mathbb{R}/2\pi\mathbb{Z})^{2\mathcal{B}}$ (here $\mathcal{B}$ may be any finite set). The map $\Psi(\alpha,\eta):=(\alpha+\eta,\alpha-\eta)$ is a continuous homomorphism $\mathbb{T}\to\mathbb{T}$, and it is onto: $(\theta,\theta')=\Psi(\frac12(\theta+\theta'),\frac12(\theta-\theta'))$. The image under Ψ of the Haar measure m of $\mathbb{T}$ is translation invariant: for $y=\Psi(x)$ and Borel A, $\Psi_*m(A+y)=m(\Psi^{-1}(A)+x)=m(\Psi^{-1}(A))=\Psi_*m(A)$, using $\Psi^{-1}(A+\Psi(x))=\Psi^{-1}(A)+x$. By the uniqueness of the Haar measure Ψ_*m is a multiple of m, and the multiple is 1 because both have total mass $m(\mathbb{T})$. Hence $\int F\circ\Psi\,dm=\int F\,dm$, which is the claim in the coordinates $(-\pi,\pi]^{2\mathcal{B}}$ of $\mathbb{T}$. □

Lemma 10.6 (closed cochains on a box are exact; extension)**.** *Let Λ be a box and $n\in\mathbb{Z}^{\mathcal{P}(\Lambda)}$ closed on $\mathcal{C}(\Lambda)$. Then there is $a\in\mathbb{Z}^{\mathcal{B}(\Lambda)}$ with $da=n$ on $\mathcal{P}(\Lambda)$. Consequently, if G is any finite lattice whose cells contain those of Λ (for instance a box $\Lambda'\supset\Lambda$, or a torus $\tilde{T}_L$ in which the sites, bonds, plaquettes and cubes of Λ are cells, i.e. Λ a translate of a sub-box of the fundamental box Λ_L of* (K4) *of Theorem 10.8), then every n closed on $\mathcal{C}(\Lambda)$ is the restriction to $\mathcal{P}(\Lambda)$ of a cochain closed on $\mathcal{C}(G)$, and two such extensions differ by an element of the group $\mathsf{K}(G,\Lambda):=\{k\in\mathbb{Z}^{\mathcal{P}(G)}:\ k \text{ closed on } \mathcal{C}(G),\ k=0 \text{ on } \mathcal{P}(\Lambda)\}$.*

Proof. Let $\Lambda=\prod_{i=1}^3\{m_i,m_i+\varepsilon,\dots,M_i\}$. Put $a_{x,1}:=0$ for every bond in direction 1. On the face $x_1=m_1$ put $a_{x,3}:=0$ and $a_{x,2}:=-\sum_{m_3\le y_3<x_3} n_{(m_1,x_2,y_3);23}$, so that $(da)_{x;23}=a_{x,2}-a_{x+\varepsilon e_3,2}=n_{x;23}$ there. For $x_1>m_1$ put $a_{x,j}:=a_{(m_1,x_2,x_3),j}+\sum_{m_1\le y_1<x_1} n_{(y_1,x_2,x_3);1j}$ for $j=2,3$. Then $(da)_{x;1j}=a_{x+\varepsilon e_1,j}-a_{x,j}=n_{x;1j}$ for $j=2,3$ on every plaquette of $\mathcal{P}(\Lambda)$. For the 23-plaquettes let $D_x:=(da)_{x;23}-n_{x;23}$; then $D_x=0$ at $x_1=m_1$, and

$$D_{x+\varepsilon e_1}-D_x = n_{x;12}+n_{x+\varepsilon e_2;13}-n_{x+\varepsilon e_3;12}-n_{x;13}-\big(n_{x+\varepsilon e_1;23}-n_{x;23}\big) = -(dn)_{x;123}=0$$

by the formulas for da and dn (the four differences of a are the 12- and 13-increments just computed); so $D\equiv0$. For the extension, extend a by zero to $\mathcal{B}(G)$ and put $\tilde{n}:=da$ on $\mathcal{P}(G)$: it is closed ($dd=0$), and on a plaquette of $\mathcal{P}(\Lambda)$, whose four bonds lie in $\mathcal{B}(\Lambda)$, it equals n. If $\tilde{n}_1,\tilde{n}_2$ are two closed extensions, $\tilde{n}_1-\tilde{n}_2$ is closed and vanishes on $\mathcal{P}(\Lambda)$. □

Lemma 10.7 (the endpoint of the interpolation)**.** *Let Λ be a box and G any finite lattice whose cells contain those of Λ (for instance a box $\Lambda'\supset\Lambda$, or a torus $\tilde{T}_L$ in which the sites, bonds, plaquettes and cubes of Λ are cells, as in Lemma 10.6), and let μ_x ($x\in\mathcal{X}(G)$), $\kappa_b\ge0$ ($b\in\mathcal{B}(G)$), $\beta_p>0$ ($p\in\mathcal{P}(G)$) be as in Definition 10.1. For $t\in(0,1]$ let $\langle\cdot\rangle_t$ be the state of the weight*

$W_G(\mu,\kappa^{(t)},\beta^{(t)})$ with $\kappa_b^{(t)}=\kappa_b$ for $b\in\mathcal{B}(\Lambda)$, $\kappa_b^{(t)}=t\kappa_b$ for $b\notin\mathcal{B}(\Lambda)$, $\beta_p^{(t)}=\beta_p$ for $p\in\mathcal{P}(\Lambda)$, $\beta_p^{(t)}=t\beta_p$ for $p\notin\mathcal{P}(\Lambda)$, and let $\langle\cdot\rangle_\Lambda$ be the state of $W_\Lambda(\mu,\kappa,\beta)$ (the restrictions of μ,κ,β to the cells of Λ). Then for every Θ supported in Λ with $|\Theta|\le Ce^{\sum_x q_x\rho_x^2}$,

$$\lim_{t\downarrow 0}\langle\Theta\rangle_t=\langle\Theta\rangle_\Lambda.$$

Proof. Write $\mathcal{P}_{\mathrm{new}}:=\mathcal{P}(G)\setminus\mathcal{P}(\Lambda)$, $\mathcal{B}_{\mathrm{new}}$, $\mathcal{X}_{\mathrm{new}}$ likewise, and $\mathsf{K}:=\mathsf{K}(G,\Lambda)$, a subgroup of $\mathbb{Z}^{\mathcal{P}_{\mathrm{new}}}$, free of some rank $r\ge 0$. For each n° closed on $\mathcal{C}(\Lambda)$ fix one extension $E(n^\circ)$ closed on $\mathcal{C}(G)$ (Lemma 10.6); by that lemma the cochains closed on $\mathcal{C}(G)$ are exactly the $E(n^\circ)+k$, $k\in\mathsf{K}$, each once. On $\mathbb{R}^{\mathcal{P}_{\mathrm{new}}}$ use the inner product $\langle z,w\rangle_\beta:=\sum_{p\in\mathcal{P}_{\mathrm{new}}}\beta_p z_p w_p/\gamma_N^2$ and the norm $Q(z):=\langle z,z\rangle_\beta$; let V be the real span of K ($\dim V=r$) and $z=z_V+z_\perp$ the orthogonal decomposition. For $y\in\mathbb{R}^{\mathcal{P}_{\mathrm{new}}}$ put

$$S_t(y):=\sum_{k\in\mathsf{K}}\prod_{p\in\mathcal{P}_{\mathrm{new}}}e^{-t\beta_p(y_p+2\pi k_p)^2/(2\gamma_N^2)}=\sum_{k\in\mathsf{K}}e^{-tQ(y+2\pi k)/2}=e^{-tQ(y_\perp)/2}\sum_{k\in\mathsf{K}}e^{-tQ(y_V+2\pi k)/2},$$

since $2\pi k\in V$.

Poisson summation on the lattice $2\pi\mathsf{K}$ of the Euclidean space $(V,\langle\cdot,\cdot\rangle_\beta)$ (the r-dimensional form of the identity behind (3.2); for $r=0$ the sum is the single term 1) gives

$$\sum_{k\in\mathsf{K}}e^{-tQ(y_V+2\pi k)/2}=\frac{(2\pi/t)^{r/2}}{\mathrm{vol}(V/2\pi\mathsf{K})}\sum_{\lambda\in(2\pi\mathsf{K})^*}e^{-Q(\lambda)/(2t)}e^{i\langle\lambda,y_V\rangle_\beta}=:c_{\mathsf{K}}\,t^{-r/2}\big(1+\varepsilon_t(y_V)\big),$$

with $(2\pi\mathsf{K})^*:=\{\lambda\in V:\ \langle\lambda,2\pi k\rangle_\beta\in2\pi\mathbb{Z}\text{ for all }k\in\mathsf{K}\}$ the dual lattice (the convention that goes with the characters $e^{i\langle\lambda,\cdot\rangle_\beta}$ in the identity) and $c_{\mathsf{K}}:=(2\pi)^{r/2}/\mathrm{vol}(V/2\pi\mathsf{K})\in(0,\infty)$, whose value cancels in the quotient at the end of the proof. With $\delta^2:=\min\{Q(\lambda):\lambda\in(2\pi\mathsf{K})^*\setminus\{0\}\}>0$ and $t\le 1$, $|\varepsilon_t(y_V)|\le\sum_{\lambda\neq0}e^{-Q(\lambda)/2}e^{-\delta^2(1/t-1)/2}=:\bar\varepsilon_t$, which is finite and tends to 0 as $t\downarrow0$, uniformly in y.

Now fix the old variables ρ_x ($x\in\Lambda$), B_b ($b\in\mathcal{B}(\Lambda)$), n° closed on $\mathcal{C}(\Lambda)$ and the new variables ρ_x ($x\in\mathcal{X}_{\mathrm{new}}$), B_b ($b\in\mathcal{B}_{\mathrm{new}}$), and put $y_p:=(d\theta)_p+2\pi E(n^\circ)_p$ for $p\in\mathcal{P}_{\mathrm{new}}$. Every closed cochain of G is $E(n^\circ)+k$ with $k=0$ on $\mathcal{P}(\Lambda)$, so on $\mathcal{P}(\Lambda)$ the curvature is $\mathfrak{f}_p=\gamma_N^{-1}((d\theta)_p+2\pi n_p^\circ)$, independent of k; hence Θ (supported in Λ) and the plaquette factors of $\mathcal{P}(\Lambda)$ do not depend on k, and the sum over k of the new plaquette factors is $S_t(y)$. Therefore, with

$$W^\circ:=\prod_{x\in\Lambda}\mu_x(d\rho_x)\prod_{b\in\mathcal{B}(\Lambda)}e^{\kappa_b\rho_{b_-}\rho_{b_+}\cos\theta_b}dB_b\prod_{p\in\mathcal{P}(\Lambda)}e^{-\beta_p\mathfrak{f}_p^2/2}$$

(the weight of $\langle\cdot\rangle_\Lambda$ at the given n°),

$$\frac{t^{r/2}}{c_{\mathsf{K}}}\int\Theta\,W_G(\mu,\kappa^{(t)},\beta^{(t)})=\int\prod_{x\in\mathcal{X}_{\mathrm{new}}}\mu_x(d\rho_x)\prod_{b\in\mathcal{B}_{\mathrm{new}}}dB_b\sum_{n^\circ}\int\Theta\,W^\circ\prod_{b\in\mathcal{B}_{\mathrm{new}}}e^{t\kappa_b\rho_{b_-}\rho_{b_+}\cos\theta_b}\,e^{-tQ(y_\perp)/2}\big(1+\varepsilon_t(y_V)\big).$$

The last three factors tend to 1 as $t\downarrow0$, for every fixed value of the variables (y does not depend on t), and their product is bounded by $(1+\bar\varepsilon_1)\prod_{b\in\mathcal{B}_{\mathrm{new}}}e^{\kappa_b(\rho_{b_-}^2+\rho_{b_+}^2)/2}$. The remaining integrand, $|\Theta|\,W^\circ$ times this bound times the new site measures, is integrable: the new B_b range over a bounded set, $\sum_{n^\circ}\prod_{p\in\mathcal{P}(\Lambda)}e^{-\beta_p\mathfrak{f}_p^2/2}\le\prod_p K_p$, and the ρ-integral is finite by (10.1) and $|\Theta|\le Ce^{\sum q_x\rho_x^2}$. By dominated convergence the right side tends to $(2\pi/\gamma_N)^{|\mathcal{B}_{\mathrm{new}}|}\prod_{x\in\mathcal{X}_{\mathrm{new}}}\mu_x([0,\infty))\int\Theta\,W_\Lambda(\mu,\kappa,\beta)$. The same holds with Θ replaced by 1; the common factor is in $(0,\infty)$, and the quotient of the two limits is $\langle\Theta\rangle_\Lambda$. □

10.3. **The inequalities.**

Theorem 10.8 (Ginibre's inequalities for the Villain fine model)**.** *Let $\langle\cdot\rangle$ be the state of a weight $W_G(\mu,\kappa,\beta)$ of Definition 10.1 on a finite lattice G, a finite disjoint union of fine tori and boxes with the product weight (for instance the disjoint union of two copies of one torus carrying the same weight, whose state is the product of two independent copies of one state of a torus weight); in particular a torus state $\langle\cdot\rangle_L^{\mathrm{fine}}$ or a killed-box state $\langle\cdot\rangle_\Lambda$ of Definition 10.2, for every $\gamma_N>0$, every $\kappa_N\geq 0$, $\lambda_0>0$, $\rho_{0,0}>0$, every real δm_B and every side L. For $\Theta,\Theta'\in\mathcal{K}_N^+$ whose cells are cells of G (in particular for $\Theta,\Theta'\in\mathcal{K}_N$):*

(K1) $\langle\Theta\rangle\geq 0$;

(K2) $\langle\Theta;\Theta'\rangle:=\langle\Theta\Theta'\rangle-\langle\Theta\rangle\langle\Theta'\rangle\geq 0$;

(K3) *$\langle\Theta\rangle$ is a continuously differentiable function of each $\kappa_b\in[0,\infty)$, of each $\beta_p\in(0,\infty)$ and of $q\in[0,\infty)$ under $\mu_x\mapsto e^{q\rho^2}\mu_x$ (one site x), the other parameters held fixed, and its partial derivatives are continuous functions of all the parameters $(\kappa_b)_b$, $(\beta_p)_p$, q jointly (so that $\langle\Theta\rangle$ is continuously differentiable along every continuously differentiable path of parameters), with*

$$\partial_{\kappa_b}\langle\Theta\rangle=\langle\Theta;\rho_{b_-}\rho_{b_+}\cos\theta_b\rangle\geq 0,\qquad \partial_{\beta_p}\langle\Theta\rangle=-\tfrac{1}{2}\langle\Theta;\mathfrak{f}_p^2\rangle\geq 0,\qquad \partial_q\langle\Theta\rangle=\langle\Theta;\rho_x^2\rangle\geq 0; \tag{10.5}$$

hence $\langle\Theta\rangle$ is non-decreasing in each of them. For the torus states and the killed-box states this means that the expectation of Θ is non-decreasing in each hopping coefficient $\kappa_b\geq 0$ (the coefficient of $\rho_{b_-}\rho_{b_+}\cos(\gamma_N B_b)$, bond by bond, the diagonal factors held fixed), in each plaquette stiffness $\beta_p>0$ ($w_p(f)=e^{-\beta_p f^2/2}$) and in $q\geq 0$ under the change of the site weight $g_{\mathrm{ct}}(\tilde\rho_x)\mapsto g_{\mathrm{ct}}(\tilde\rho_x)e^{q\rho_x^2}$.

(K4) *For boxes $\Lambda\subset\Lambda'$ and Θ supported in Λ: $\langle\Theta\rangle_\Lambda\leq\langle\Theta\rangle_{\Lambda'}$. Identify the sites of $\tilde T_L$ with the box $\Lambda_L:=\{0,\varepsilon,\dots,L-\varepsilon\}^3$, its bonds and plaquettes with the pairs (site, direction) and triples (site, two directions) based at those sites, so that $\mathcal{B}(\Lambda_L)\subset\mathcal{B}(\tilde T_L)$ and $\mathcal{P}(\Lambda_L)\subset\mathcal{P}(\tilde T_L)$. Then for every box $\Lambda\subset\Lambda_L$ and Θ supported in Λ: $\langle\Theta\rangle_\Lambda\leq\langle\Theta\rangle_{\Lambda_L}\leq\langle\Theta\rangle_L^{\mathrm{fine}}$. In particular, if Λ lies in a cube of side less than L (a fortiori if Λ together with its nearest-neighbour sites does), a translation of the torus (which leaves $\langle\cdot\rangle_L^{\mathrm{fine}}$ invariant) brings Λ into Λ_L, and $\langle\Theta\rangle_\Lambda\leq\langle\Theta\rangle_L^{\mathrm{fine}}$.*

The proof occupies the rest of this subsection. We derive (K1) from the positivity of the Fourier coefficients of the weight, as in the proof of Theorem 3.1, then (K2) and the three signs of (K3) from the duplication identity, and finally (K4) from (K3) and Lemma 10.7. By linearity it suffices to treat $\Theta=\Theta_{a,k,P,q}$ and $\Theta'=\Theta_{a',k',P',q'}$. Every step of the proofs of (K1)–(K3) is cell by cell: the site, bond and plaquette factors are indexed by the cells of G, the closure constraint is imposed cube by cube, the duplication identity is a product over the plaquettes and the angle domain a product over the bonds; so the proofs apply without change when G has several components, and (K1)–(K3) hold on every finite lattice of Definition 10.1. (K4) is a statement about boxes and the torus, and the lattice G of Lemma 10.7 in its proof is a single box or a single torus.

Proof of (K1). Let $\Theta=\Theta_{a,k,P,q}$ and $\Phi:=\sum_p a_p\mathfrak{f}_p+\sum_b k_b\theta_b$. Put $Z(\Theta):=\int P(\rho)e^{\sum_x q_x\rho_x^2}e^{i\Phi}\,W$, so that $\langle\Theta\rangle=\operatorname{Re}Z(\Theta)/Z$; we show $Z(\Theta)\geq 0$.

The plaquette sum. For $\beta>0$, real x,s,a and $c_u^\beta:=(\gamma_N/\sqrt{2\pi\beta})\,e^{-\gamma_N^2u^2/(2\beta)}>0$,

$$\sum_{v\in(2\pi/\gamma_N)\mathbb{Z}} e^{-\beta(x+v)^2/2}e^{ia(x+v)}e^{i\gamma_N sv}=e^{-a^2/(2\beta)}\sum_{m\in\mathbb{Z}}c_{m-s}^\beta\,e^{i\gamma_N(m-s)x}\,e^{\gamma_N(m-s)a/\beta}. \tag{10.6}$$

This is Poisson summation for $n \mapsto e^{-\beta(x+2\pi n/\gamma_N)^2/2} e^{ia(x+2\pi n/\gamma_N)} e^{2\pi i s n}$ (a Schwartz function of $n \in \mathbb{R}$), whose Fourier coefficient at m is, after the substitution $w = x + 2\pi u/\gamma_N$,

$$\frac{\gamma_N}{2\pi} e^{i\gamma_N(m-s)x} \int e^{-\beta w^2/2} e^{i(a+\gamma_N(s-m))w} dw = \frac{\gamma_N}{\sqrt{2\pi\beta}} e^{i\gamma_N(m-s)x} e^{-(a+\gamma_N(s-m))^2/(2\beta)},$$

and $(a+\gamma_N(s-m))^2 = a^2 - 2a\gamma_N(m-s) + \gamma_N^2(m-s)^2$. (For $\beta = 1$ this is (3.2) with a replaced by ia.) The closure constraint is resolved as in the proof of Theorem 3.1: $\mathbf{1}[(dv)_c = 0] = \int_0^1 e^{i\gamma_N t_c (dv)_c} dt_c$ for every cube $c \in \mathcal{C}$, so with $t \in [0,1]^{\mathcal{C}}$ and $s := d^*t$, $(d^*t)_p = \sum_{c \ni p} \pm t_c$ (at most two cubes, $|s_p| \le 2$),

$$\sum_{v \text{ closed}} \prod_p e^{-\beta_p \mathfrak{f}_p^2/2} e^{i a_p \mathfrak{f}_p} = e^{-\sum_p a_p^2/(2\beta_p)} \int_{[0,1]^{\mathcal{C}}} dt \sum_{m \in \mathbb{Z}^{\mathcal{P}}} \prod_p c_{m_p - s_p}^{\beta_p} e^{\gamma_N (m_p - s_p) a_p/\beta_p} \; e^{i \sum_b (d^*m)_b \theta_b},$$

by (10.6) plaquette by plaquette with $x = (dB)_p$ and $\gamma_N \sum_p (m_p - s_p)(dB)_p = \sum_b (d^*(m-s))_b \theta_b = \sum_b (d^*m)_b \theta_b$, because $d^*s = d^*d^*t = 0$; here $(d^*m)_b \in \mathbb{Z}$.

The gauge integral. By $e^{z\cos\varphi} = \sum_{j \in \mathbb{Z}} I_j(z) e^{ij\varphi}$ with $I_j(z) = I_{-j}(z) \ge 0$ for $z \ge 0$ and $\sum_j I_j(z) = e^z$, for every bond b and integer $j_b := (d^*m)_b + k_b$,

$$\int_{(-\pi/\gamma_N, \pi/\gamma_N]} dB_b \; e^{i j_b \theta_b} \, e^{\kappa_b \rho_{b_-} \rho_{b_+} \cos\theta_b} = \frac{2\pi}{\gamma_N} \, I_{j_b}\big(\kappa_b \rho_{b_-} \rho_{b_+}\big) \; \ge 0.$$

Assembly. Hence, with

$$\Omega_{k,P,q}(m) := \int \prod_x \mu_x(d\rho_x) \, P(\rho) e^{\sum_x q_x \rho_x^2} \prod_b \frac{2\pi}{\gamma_N} I_{(d^*m)_b + k_b}(\kappa_b \rho_{b_-} \rho_{b_+}) \ge 0,$$

$$Z(\Theta) = e^{-\sum_p a_p^2/(2\beta_p)} \int_{[0,1]^{\mathcal{C}}} dt \sum_{m \in \mathbb{Z}^{\mathcal{P}}} \Big(\prod_p c_{m_p - s_p}^{\beta_p} e^{\gamma_N (m_p - s_p) a_p/\beta_p} \Big) \, \Omega_{k,P,q}(m) \; \ge 0. \tag{10.7}$$

The interchanges are justified by absolute convergence:

$$\sum_m \prod_p c_{m_p - s_p}^{\beta_p} e^{\gamma_N (m_p - s_p)|a_p|/\beta_p} = \prod_p \sum_{m \in \mathbb{Z}} c_{m - s_p}^{\beta_p} e^{\gamma_N (m - s_p)|a_p|/\beta_p}$$

is bounded uniformly in $|s_p| \le 2$ (a Gaussian sum against an exponential), and

$$0 \le \Omega_{k,P,q}(m) \le \int \prod_x \mu_x(d\rho_x) P(\rho) e^{\sum q_x \rho_x^2} \prod_b \frac{2\pi}{\gamma_N} e^{\kappa_b(\rho_{b_-}^2 + \rho_{b_+}^2)/2} < \infty$$

by $I_j(z) \le e^z$ and (10.1); the lattice is finite. So $Z(\Theta)$ is real and non-negative, and $\langle \Theta \rangle = Z(\Theta)/Z \ge 0$. □

Proof of (K2) and of the signs in (K3). Write $\Theta_1 := \Theta = F(\rho)\cos\Phi_1$ and $\Theta_2 := \Theta' = G(\rho)\cos\Phi_2$ with $F(\rho) := P(\rho) e^{\sum_x q_x \rho_x^2}$, $G(\rho) := P'(\rho) e^{\sum_x q'_x \rho_x^2}$, $\Phi_1 := \sum_p a_p \mathfrak{f}_p + \sum_b k_b \theta_b$, Φ_2 likewise with (a', k'); from now on a prime denotes the duplicate copy of Section 10.2. For integrable F_1, F_2,

$$Z^2 \langle F_1; F_2 \rangle = \tfrac{1}{2} \iint (F_1 - F_1')(F_2 - F_2') \, W \, W', \tag{10.8}$$

since the right side is $\frac{1}{2}\big(2Z \int F_1 F_2 W - 2 \int F_1 W \int F_2 W\big)$.

The integrand in the new variables. Expand $F(\rho) = \sum_m c_m \rho^m$ (multi-indices m, $c_m \ge 0$, convergent everywhere; P has non-negative coefficients and $e^{q\rho^2}$ an entire series with non-negative coefficients). In the variables (10.2), $F(u+v) = \sum_m c_m \prod_x \sum_{j_x \le m_x} \binom{m_x}{j_x} u_x^{m_x - j_x} v_x^{j_x}$; let $F_{\mathrm{e}}(u,v)$ be the sum of the terms with $\sum_x j_x$ even and $F_{\mathrm{o}}(u,v)$ that of the terms with $\sum_x j_x$ odd. Both

are power series in (u, v) with non-negative coefficients, absolutely convergent and dominated by $F(u + |v|)$, and

$$F(\rho') = F(u + v) = F_{\mathrm{e}} + F_{\mathrm{o}}, \qquad F(\rho) = F(u - v) = F_{\mathrm{e}} - F_{\mathrm{o}},$$

because $v \mapsto -v$ changes the sign of exactly the odd terms. Likewise $G = G_{\mathrm{e}} \mp G_{\mathrm{o}}$.

In the parametrisation of Lemma 10.4, $\Phi_1 = \Xi_1 + H_1$ and $\Phi_1' = \Xi_1 - H_1$ with

$$\Xi_1 := \sum_p a_p \xi_p + \sum_b k_b \alpha_b \quad \text{(a function of } (\alpha, k, \sigma)\text{)},$$

$$H_1 := \sum_p a_p \upsilon_p + \sum_b k_b \eta_b \quad \text{(the same function of } (\eta, l, \sigma)\text{)},$$

by $\mathfrak{f} = \xi + \upsilon$, $\mathfrak{f}' = \xi - \upsilon$, $\theta = \alpha + \eta$, $\theta' = \alpha - \eta$; and Ξ_2, H_2 likewise. Hence, by the addition formulas,

$$\begin{aligned} \Theta_1 + \Theta_1' &= 2F_{\mathrm{e}} \cos \Xi_1 \cos H_1 + 2F_{\mathrm{o}} \sin \Xi_1 \sin H_1, \\ \Theta_1 - \Theta_1' &= -2F_{\mathrm{o}} \cos \Xi_1 \cos H_1 - 2F_{\mathrm{e}} \sin \Xi_1 \sin H_1, \end{aligned} \tag{10.9}$$

and the same for Θ_2.

For a bond $b = (x, y)$,

$$\rho_x \rho_y \cos \theta_b + \rho_x' \rho_y' \cos \theta_b' = 2(u_x u_y + v_x v_y) \cos \alpha_b \cos \eta_b + 2(u_x v_y + v_x u_y) \sin \alpha_b \sin \eta_b,$$

so the pair of hopping factors is

$$e^{\kappa_b(\rho_x \rho_y \cos \theta_b + \rho_x' \rho_y' \cos \theta_b')} = \sum_{m, m' \geq 0} \frac{(2\kappa_b)^{m+m'}}{m!\, m'!} (u_x u_y + v_x v_y)^m (u_x v_y + v_x u_y)^{m'} \big(\cos^m \alpha_b \sin^{m'} \alpha_b\big)\big(\cos^m \eta_b \sin^{m'} \eta_b\big),$$

a series each term of which is the product of a non-negative coefficient, a polynomial in u, v with non-negative coefficients and a factor $T(\alpha_b)T(\eta_b)$ with the same trigonometric monomial T in both copies; the absolute values of the terms sum to at most $e^{2\kappa_b(u_x + |v_x|)(u_y + |v_y|)} \leq e^{\kappa_b(\rho_x^2 + \rho_x'^2 + \rho_y^2 + \rho_y'^2)}$. The pair of Villain weights is (10.4), and the site measures $\prod_x \mu_x(d\rho_x)\mu_x(d\rho_x')$ are invariant under $\rho_x \leftrightarrow \rho_x'$, i.e. under $v_x \mapsto -v_x$ with u fixed, for each x separately.

The positive structure. By (10.9),

$$(\Theta_1 - \Theta_1')(\Theta_2 - \Theta_2') = 4\big[F_{\mathrm{o}} \cos \Xi_1 \cos H_1 + F_{\mathrm{e}} \sin \Xi_1 \sin H_1\big]\big[G_{\mathrm{o}} \cos \Xi_2 \cos H_2 + G_{\mathrm{e}} \sin \Xi_2 \sin H_2\big],$$

which consists of four terms, each the product of a series in u, v with non-negative coefficients and a factor $\tau(\alpha, k, \sigma)\, \tau(\eta, l, \sigma)$, where τ is a product of two of the functions $\cos \Xi_i, \sin \Xi_i$, the same function in both copies. Multiplying out with the hopping series and (10.4), the full integrand of (10.8) becomes

$$\sum_\iota c_\iota\, M_\iota(u, v) \sum_\sigma \sum_{k, l \in \mathsf{K}_\sigma} \tau_\iota(\alpha, k, \sigma) V_\sigma(\alpha, k)\; \tau_\iota(\eta, l, \sigma) V_\sigma(\eta, l), \tag{10.10}$$

with $c_\iota \geq 0$, M_ι a monomial $\prod_x u_x^{a_x} v_x^{b_x}$ and τ_ι a product of cosines and sines of the α_b, Ξ_1, Ξ_2. The series converges absolutely, uniformly on the angle domain, and is integrable, because $|\tau_\iota| \leq 1$,

$$\sum_\iota c_\iota |M_\iota| \leq 4F(u + |v|)G(u + |v|) \prod_b e^{\kappa_b(\rho_{b_-}^2 + \rho_{b_-}'^2 + \rho_{b_+}^2 + \rho_{b_+}'^2)}$$

with $F(u + |v|) = F(\max(\rho, \rho')) \leq P(\rho + \rho') e^{\sum_x q_x(\rho_x^2 + \rho_x'^2)}$ (the maximum taken site by site; P has non-negative coefficients), $\sum_\sigma \sum_{k, l \in \mathbb{Z}^{\mathcal{P}}} V_\sigma(\alpha, k) V_\sigma(\eta, l) \leq 2^{|\mathcal{P}|} (\prod_p K_p)^2$ by Lemma 10.4, and the site measures satisfy (10.1).

The integral. The function of (θ,θ') obtained from the integrand of (10.8) at fixed (ρ,ρ') by the sums over n,n' is 2π-periodic in every θ_b and θ'_b: the shift $\theta_b\mapsto\theta_b+2\pi$ together with $n\mapsto n-d\delta_b$ (a bijection of the closed cochains, $dd=0$) leaves $\mathfrak{f}$, $\cos\theta_b$ and $e^{ik\cdot\theta}$, hence Θ_i and W, unchanged. By Lemma 10.5 (with $dB_b=d\theta_b/\gamma_N$) and Fubini, $2Z^2\langle\Theta_1;\Theta_2\rangle$ equals the integral of (10.10) over $\prod_x\mu_x(d\rho_x)\mu_x(d\rho'_x)\,\gamma_N^{-2|\mathcal{B}|}d\alpha\,d\eta$ on $[0,\infty)^{2\mathcal{X}}\times(-\pi,\pi]^{2\mathcal{B}}$, and term by term (absolute convergence) this is

$$2Z^2\langle\Theta_1;\Theta_2\rangle=\gamma_N^{-2|\mathcal{B}|}\sum_\iota c_\iota\Big(\int M_\iota(u,v)\prod_x\mu_x(d\rho_x)\mu_x(d\rho'_x)\Big)\sum_\sigma\Big(\sum_{k\in\mathsf{K}_\sigma}\int_{(-\pi,\pi]^{\mathcal{B}}}\tau_\iota(\alpha,k,\sigma)V_\sigma(\alpha,k)\,d\alpha\Big)^2.$$

The angular factor is a square. The radial factor is $\prod_x\iint u_x^{a_x}v_x^{b_x}\mu_x(d\rho_x)\mu_x(d\rho'_x)$: it vanishes if some b_x is odd (the symmetry $v_x\mapsto-v_x$) and is ≥0 otherwise ($u_x\ge0$, $v_x^{b_x}\ge0$). Every term is ≥0; this is (K2).

The signs of (K3). The functions $\rho_{b_-}\rho_{b_+}\cos\theta_b$ and ρ_x^2 are in $\mathcal{K}_N^+$ (Definition 10.3), so $\langle\Theta;\rho_{b_-}\rho_{b_+}\cos\theta_b\rangle\ge0$ and $\langle\Theta;\rho_x^2\rangle\ge0$ are instances of (K2). For the plaquette energy, $\mathfrak{f}_p^2-\mathfrak{f}_p'^2=(\xi_p+\upsilon_p)^2-(\xi_p-\upsilon_p)^2=4\xi_p\upsilon_p$, so by (10.9)

$$(\Theta-\Theta')(\mathfrak{f}_p^2-\mathfrak{f}_p'^2)=-8\big[F_{\mathrm{o}}\cos\Xi_1\cos H_1+F_{\mathrm{e}}\sin\Xi_1\sin H_1\big]\,\xi_p\,\upsilon_p,$$

which has the structure of (10.10) with $\tau_\iota(\alpha,k,\sigma)$ replaced by $\tau_\iota(\alpha,k,\sigma)\,\xi_p$ (again the same function of (α,k,σ) and of (η,l,σ)) and an overall sign $-$; the absolute convergence holds with the second bound of Lemma 10.4 ($j=1$). The same computation expresses $Z^2\langle\Theta;\mathfrak{f}_p^2\rangle$ as $-4\gamma_N^{-2|\mathcal{B}|}$ times a sum over ι of products of $c_\iota\ge0$, a non-negative radial factor and a sum over σ of squares; hence $\langle\Theta;\mathfrak{f}_p^2\rangle\le0$. □

Proof of the derivative formulas of (K3). Let λ be one of the parameters κ_b, β_p, q (the last for $\mu_x\mapsto e^{q\rho_x^2}\mu_x$ at one site x), and let W_λ be the weight; then $\partial_\lambda W_\lambda=E_\lambda W_\lambda$ with $E_{\kappa_b}=\rho_{b_-}\rho_{b_+}\cos\theta_b$, $E_{\beta_p}=-\frac12\mathfrak{f}_p^2$, $E_q=\rho_x^2$. On a compact interval of parameters ($\kappa_b\in[0,\bar\kappa]$, $\beta_p\in[\beta_0,\bar\beta]$ with $\beta_0>0$, $q\in[0,\bar q]$) the functions $|\Theta|W_\lambda$ and $|\Theta E_\lambda|W_\lambda$ are dominated by one integrable function: $e^{\kappa_b\rho\rho'\cos\theta_b}\le e^{\bar\kappa\rho\rho'}$ and $\rho\rho'e^{\bar\kappa\rho\rho'}$ is integrable by (10.1); $\sum_{n_p\in\mathbb{Z}}(1+\mathfrak{f}_p^2)e^{-\beta_p\mathfrak{f}_p^2/2}\le\sum_{n_p}(1+\mathfrak{f}_p^2)e^{-\beta_0\mathfrak{f}_p^2/2}$ is bounded uniformly in $(dB)_p$ (Lemma 10.4, second bound); $(1+\rho_x^2)e^{q\rho_x^2}\le(1+\rho_x^2)e^{\bar q\rho_x^2}$ is μ_x-integrable.

So $\lambda\mapsto\int\Theta W_\lambda$ and $\lambda\mapsto Z_\lambda$ are continuously differentiable with derivatives $\int\Theta E_\lambda W_\lambda$ and $\int E_\lambda W_\lambda$ (differentiation under the integral, dominated convergence), the partial derivatives being continuous in all parameters jointly (the same domination over a compact box of parameters), and the quotient rule gives $\partial_\lambda\langle\Theta\rangle=\langle\Theta E_\lambda\rangle-\langle\Theta\rangle\langle E_\lambda\rangle=\langle\Theta;E_\lambda\rangle$. This is (10.5); the signs were proved above, and a continuously differentiable function with non-negative derivative on an interval is non-decreasing. Every intermediate weight belongs to the class of Definition 10.1, and for the torus states and the killed-box states the three parameters are exactly the hopping coefficient of a bond with its diagonal factors fixed, the stiffness of a plaquette and the site factor $e^{q\rho_x^2}$. □

Proof of (K4). Let $\Lambda\subset\Lambda'$ be boxes and $\Theta\in\mathcal{K}_N^+$ supported in Λ. Take $G=\Lambda'$ with the parameters of the model ($\mu_x=\mu^\circ$, $\kappa_b=\kappa_N$, $\beta_p=1$) and let $\langle\cdot\rangle_t$, $t\in(0,1]$, be the interpolated states of Lemma 10.7: the hopping coefficients of the bonds of $\mathcal{B}(\Lambda')\setminus\mathcal{B}(\Lambda)$ at $t\kappa_N$ and the stiffnesses of the plaquettes of $\mathcal{P}(\Lambda')\setminus\mathcal{P}(\Lambda)$ at t, everything else fixed; $\langle\cdot\rangle_1=\langle\cdot\rangle_{\Lambda'}$. By (K3) the function $\varphi(t):=\langle\Theta\rangle_t$ is continuously differentiable on $(0,1]$ (the parameters depend linearly on t,

and the partial derivatives are jointly continuous by (K3)) with

$$\varphi'(t) = \sum_{b\in\mathcal{B}(\Lambda')\setminus\mathcal{B}(\Lambda)} \kappa_N \langle \Theta; \rho_{b_-}\rho_{b_+}\cos\theta_b\rangle_t - \frac{1}{2}\sum_{p\in\mathcal{P}(\Lambda')\setminus\mathcal{P}(\Lambda)} \langle \Theta; \mathfrak{f}_p^2\rangle_t \;\geq 0.$$

Hence φ is non-decreasing on $(0,1]$, and $\langle\Theta\rangle_{\Lambda'} = \varphi(1) \geq \lim_{t\downarrow 0}\varphi(t) = \langle\Theta\rangle_\Lambda$ by Lemma 10.7 (whose conditions are met: $\Lambda' \supset \Lambda$ as in Lemma 10.6, $|\Theta| \leq Ce^{\sum q'_x\rho_x^2}$, and the parameters on the cells of Λ are those of $\langle\cdot\rangle_\Lambda$).

For the torus, the identification of the statement makes the sites, bonds and plaquettes of Λ_L cells of $\tilde{T}_L$ (the bonds (x,i) with $x_i \leq L-2\varepsilon$ and the plaquettes $(x;ij)$ with $x_i, x_j \leq L-2\varepsilon$), the cubes of Λ_L cubes of the torus, and Λ_L is a box; so Lemma 10.6 and Lemma 10.7 apply to $G = \tilde{T}_L \supset \Lambda_L$. The torus state is the state of $W_{\tilde{T}_L}(\mu^\circ, \kappa_N, 1)$ and the killed-box state of Λ_L that of $W_{\Lambda_L}(\mu^\circ, \kappa_N, 1)$ (Definition 10.2: the same site measure, since in both every site carries the diagonal halves of six bonds). The argument of the first paragraph, with the bonds of $\mathcal{B}(\tilde{T}_L)\setminus\mathcal{B}(\Lambda_L)$ and the plaquettes of $\mathcal{P}(\tilde{T}_L)\setminus\mathcal{P}(\Lambda_L)$ interpolated, gives $\langle\Theta\rangle_{\Lambda_L} \leq \langle\Theta\rangle_L^{\mathrm{fine}}$ for every Θ supported in Λ_L; combined with the first clause for $\Lambda \subset \Lambda_L$ this is $\langle\Theta\rangle_\Lambda \leq \langle\Theta\rangle_{\Lambda_L} \leq \langle\Theta\rangle_L^{\mathrm{fine}}$. Finally, if Λ lies in a cube of side $s < L$ ($s \in \varepsilon\mathbb{Z}$; the nearest-neighbour sites of Λ play no role), then $s \leq L-\varepsilon$ and the cube has at most L/ε sites along each axis, so some translate $y+\Lambda$ ($y \in \varepsilon\mathbb{Z}^3$) lies in Λ_L; the torus weight is invariant under the translations of $\tilde{T}_L$ and the killed-box weight is carried along by them, so $\langle\Theta\rangle_L^{\mathrm{fine}} = \langle\Theta_y\rangle_L^{\mathrm{fine}}$ and $\langle\Theta\rangle_\Lambda = \langle\Theta_y\rangle_{y+\Lambda}$ for the translate $\Theta_y \in \mathcal{K}_N^+$ of Θ, supported in $y+\Lambda$. The argument of the first paragraph uses only (K3) and Lemma 10.7, which hold for every weight of Definition 10.1; it therefore gives the same comparison for any such weight on a finite lattice G whose cells contain those of Λ and whose parameters on the cells of Λ are those of the Λ-weight. □

Remark 10.9 (the hypotheses used by each clause). (i) (K1) uses only the positivity of the Fourier coefficients of the weight: the coefficients $c_u^\beta > 0$ of the Villain plaquette weight (its Gaussian form, through Poisson summation) and the Bessel coefficients $I_j \geq 0$ of the hopping factor, which depends on the gauge field only through $\cos(\gamma_N B_b)$ with the non-negative coefficient κ_b; the site measures, the polynomial P and the factor $e^{\sum q_x\rho_x^2}$ enter as non-negative weights, and the closure constraint only through the phases $e^{i\gamma_N t_c (dv)_c}$. It holds for every $\gamma_N > 0$ and gives the first inequality alone: positivity of the Fourier coefficients does not imply (K2) (the weight $1+\cos t\cos 2t$ on the circle has non-negative Fourier coefficients, and $\langle\cos t;\cos 3t\rangle = 0 - \frac{1}{4}\cdot\frac{1}{4} = -\frac{1}{16}$).

(ii) (K2)–(K4) rest on the duplication identity (10.4), which uses that the Villain integers are summed over *all* closed cochains: the closed integer cochains form a group, so that $n+n'$ and $n-n'$ are closed again and the parametrisation of Lemma 10.4 exists; a sum restricted to one nonzero flux class of the torus (a coset, not a group) does not have this property. The sum over all v without the closure constraint has it as well. The phases of the enlarged cone carry integer bond coefficients k_b so that the integrand is a function on the angle torus (Lemma 10.5), and the frequencies a_p of the curvature are arbitrary real numbers because $\mathfrak{f}_p$ is a real variable.

(iii) The argument of (K2) applied to $\Theta+\Theta'$ in (10.9) gives (K1) a second time.

(iv) Nothing in (K1)–(K3) uses the form of the site measure beyond (10.1); the killed-box convention (every site with all six diagonal halves) is what makes the site measure the same in every volume, which (K4) needs.

(v) For the torus states and the killed-box states every statement holds for every $\gamma_N > 0$, $\kappa_N \geq 0$, $\lambda_0 > 0$, $\rho_{0,0} > 0$, $\delta m_B \in \mathbb{R}$ and every L. These are statements at a fixed level N, which suffices for Section 11, where the limit $L\to\infty$ is taken at fixed N.

(vi) Definition 10.1 admits lattices with several components, and Theorem 10.8 covers them because the proofs of (K1)–(K3) are cell by cell.

Remark 10.10 (the bare mass counterterm). The site measure $\mu^\circ = g_{\mathrm{ct}}(\rho - \rho_{0,0})e^{-3\kappa_N\rho^2}d\rho$, $g_{\mathrm{ct}} = g\,e^{-\frac{1}{2}\delta m_B\rho^2}$ (Definition 10.2), carries the bare mass counterterm of Section 2, $\delta m_B \in \mathbb{R}$ of either sign. Every statement of this section holds for every real δm_B: Definition 10.1, Lemma 10.7 and the proofs of (K1)–(K3) use of the site measures only that they are finite positive measures on $[0,\infty)$ with (10.1) (item (iv) of Remark 10.9), which μ° has by (2.1). The proof of (K4) uses in addition only that μ° is the same measure at every site of every box and of the torus (the counterterm is a per-site factor, and the killed-box convention keeps it at every site); no monotonicity, no identity and no sign of a derivative of g is used.

11. The existence of the fine infinite-volume state

The relation (2.2) would define the fine infinite-volume state ω_N as the limit of the torus states; in this paper ω_N is a limit point of the torus states (Definition 2.3), and the existence of the limit is not assumed. The results of this section are the following. First, limit points of the torus states exist along subsequences of sides, and every limit point carries all the lattice symmetries, the reflection positivity and the uniform bounds of the torus states (Theorem 11.5); the only new estimate is a bound on the local moments of the scalar density, uniform in the volume, which we derive from the reflection positivity of the torus states (Lemma 11.2). Second, no statement of this paper requires the existence of the limit (2.2): the chain from the estimate to the limit functional and the statements of Section 9 are proved for limit points of this kind (Corollary 11.7).

Third, on the cone of observables for which Ginibre's inequalities hold, the killed-box states converge, every torus limit point dominates the box limit, and the values on the cone determine the state (Theorem 11.11, Lemma 11.12); hence (2.2), in its continuous form (ii) of Corollary 11.13, is equivalent to the agreement of the torus limit points on the cone generators (the form (i) implies it).

Throughout, $N \geq N_0$ is fixed, $\varepsilon := \varepsilon_N$ is the spacing, γ_N the phase constant, and the couplings $\kappa_N \geq 0$, $\lambda_0 > 0$, $\rho_{0,0} > 0$ and the bare mass counterterm $\delta m_B \in \mathbb{R}$ (Remark 2.2) are arbitrary: nothing in this section depends on their values. All constants may depend on N through $(\varepsilon, \gamma_N, \kappa_N, \lambda_0, \rho_{0,0}, \delta m_B)$; this is sufficient, since the limit $L \to \infty$ is taken at fixed N.

11.1. **Configuration space, local functions, the symmetries of the torus states.** Let $\mathcal{P}_N$ be the set of plaquettes of $\varepsilon\mathbb{Z}^3$, one orientation chosen for each (the opposite orientation carries $\mathfrak{f}_{-p} = -\mathfrak{f}_p$), and $\mathcal{X}_N$ its set of sites. The configuration space of the gauge-invariant variables is

$$\Omega_N := \mathbb{R}^{\mathcal{P}_N} \times [-\rho_{0,0}, \infty)^{\mathcal{X}_N}, \qquad \xi = \big((f_p)_{p\in\mathcal{P}_N}, (\tilde\rho_x)_{x\in\mathcal{X}_N}\big), \qquad \rho_x := \rho_{0,0} + \tilde\rho_x \geq 0,$$

with the product topology and its Borel σ-algebra; Ω_N is a countable product of Polish spaces, hence Polish, and its Borel σ-algebra is the product σ-algebra, generated by the cylinder sets. For a finite set S of plaquettes and sites put $E_S := \mathbb{R}^{\mathcal{P}_S} \times [-\rho_{0,0}, \infty)^{\mathcal{X}_S}$ and let $\pi_S \colon \Omega_N \to E_S$ be the projection. A function F on Ω_N is *local with support* S if $F = F_S \circ \pi_S$; it is bounded Borel local, bounded continuous local, etc., according to F_S. The bounded Borel local functions are the "bounded local gauge-invariant functions" of (2.2).

If S lies in a cube of side less than L, the quotient map $\varepsilon\mathbb{Z}^3 \to \tilde T_L$ is injective on the cells of S, and the gauge-invariant variables of $\tilde T_L$ at the image cells define a measurable map from the torus configurations to E_S; its law under $\langle\cdot\rangle_L^{\mathrm{fine}}$ is a Borel probability measure ν_L^S on E_S, and for F local with support S we write $\langle F\rangle_L^{\mathrm{fine}} := \int F_S\, d\nu_L^S$. This is the meaning of $\langle F\rangle_L^{\mathrm{fine}}$ for local F throughout; by the translation invariance of the torus state it does not depend on the translate of S used. Let $\mathcal{G}_N$ be the group generated by the translations by $\varepsilon\mathbb{Z}^3$, the reflections $R_{j,c}$ in the planes $\{x_j = c\}$, $j \in \{1,2,3\}$, $c \in \frac{1}{2}\varepsilon\mathbb{Z}$, and the cubic group about every site, acting on cells and on Ω_N by $(g\xi)_\kappa := \xi_{g\kappa}$ with the orientation conventions (f changes sign on the plaquettes whose

orientation g reverses: this is the time-like sign on the curvature); and let $C\colon (f,\tilde\rho)\mapsto(-f,\tilde\rho)$ be the charge conjugation.

Lemma 11.1 (symmetries of the torus states)**.** *For every $g\in\mathcal{G}_N\cup\{C\}$ and every bounded Borel local F, $\langle F\circ g\rangle_L^{\mathrm{fine}}=\langle F\rangle_L^{\mathrm{fine}}$ for all L with $L/2\in\varepsilon\mathbb{Z}$ so large that S and gS lie in a cube of side less than L.*

Proof. For the translations and the cubic group this is the proof of Lemma 5.1; for the reflections $R_{j,c}$ it is the paragraph before Theorem 9.7 (every $R_{j,c}$ with $c\in\frac12\varepsilon\mathbb{Z}$ is an involutive symmetry of $\tilde T_L$ for $L/2\in\varepsilon\mathbb{Z}$, and the weight and the reference measure are invariant); for C it is the proof of Proposition 9.12(b). □

11.2. **Tightness: the uniform local moments.** For the curvature the uniform bound is Theorem 3.1: $\langle e^{s\mathfrak{f}_p}\rangle_L^{\mathrm{fine}}\le e^{s^2/2}$ for all real s, hence $\langle e^{s|\mathfrak{f}_p|}\rangle_L^{\mathrm{fine}}\le 2e^{s^2/2}$. For the density we use the reflection positivity of the torus states about the bond mid-planes.

Lemma 11.2 (uniform local moments of the density)**.** *Let $g_{\mathrm{ct}}(\tilde\rho):=g(\tilde\rho)\,e^{-\frac12\delta m_B(\rho_{0,0}+\tilde\rho)^2}$ be the full site factor of the model of Section 2, the bare mass counterterm included ($\delta m_B\in\mathbb{R}$ of either sign; Remark 2.2). Put, for $q\ge0$,*

$$G_q^{\mathrm{ct}}:=\int_{-\rho_{0,0}}^{\infty} g_{\mathrm{ct}}(r)\,e^{q(\rho_{0,0}+r)^2}\,dr<\infty,$$

$$K'_{\gamma_N}:=\sup_{x\in\mathbb{R}}\sum_{v\in(2\pi/\gamma_N)\mathbb{Z}} e^{-(x+v)^2/2}\le 1+2\sum_{n\ge1}e^{-2\pi^2(n-\tfrac12)^2/\gamma_N^2},$$

$$c_g:=\rho_{0,0}e^{-\lambda_0(2\rho_{0,0}+1)^2},\quad c_g^{\mathrm{ct}}:=c_g\,e^{-\frac12|\delta m_B|(\rho_{0,0}+1)^2},\quad c_h:=e^{-\kappa_N(1+(\rho_{0,0}+1)^2)/2},$$

$$c_0:=c_g^{\mathrm{ct}}\,(2c_h/\gamma_N)^3e^{-24/\gamma_N^2},\quad C_1:=\max\Big\{1,\ \frac{G_0^{\mathrm{ct}}(2\pi/\gamma_N)^3K'^3_{\gamma_N}}{c_0}\Big\}.$$

(K'_{γ_N} is a constant distinct from the K_{γ_N} of the proof of Theorem 3.1.) Then for every $q\ge0$, every site x and every $L\ge4\varepsilon$ with $L/2\in\varepsilon\mathbb{Z}$,

$$\langle e^{q\rho_x^2}\rangle_L^{\mathrm{fine}}\ \le\ C_N(q):=\frac{G_q^{\mathrm{ct}}}{G_0^{\mathrm{ct}}}\,C_1^8. \tag{11.1}$$

Proof. Write $\mathcal{X}_L,\mathcal{B}_L,\mathcal{P}_L$ for the sites, bonds and plaquettes of $\tilde T_L$, so that $|\mathcal{B}_L|=|\mathcal{P}_L|=3|\mathcal{X}_L|$, and $M:=L/\varepsilon$ (an even integer ≥4). The integral G_q^{ct} is finite for every $q\ge0$ and every real δm_B by the inequality (2.1): $\lambda_0(\rho^2-\rho_{0,0}^2)^2\ge\frac12\lambda_0\rho^4-\lambda_0\rho_{0,0}^4$ and $e^{-\frac12\delta m_B\rho^2}\le e^{\frac12|\delta m_B|\rho^2}$ give $g_{\mathrm{ct}}(r)\,e^{q\rho^2}\le\rho\,e^{\lambda_0\rho_{0,0}^4}\,e^{-\frac12\lambda_0\rho^4+(q+\frac12|\delta m_B|)\rho^2}$, $\rho=\rho_{0,0}+r\ge0$, which is integrable over $r\in[-\rho_{0,0},\infty)$ since $\lambda_0>0$.

Step 1: crude bounds on partition functions. For a set $A\subset\mathcal{X}_L$ and $q\ge0$ let $Z(A,q):=\int d\tilde\rho\int dB\sum_{v\ \text{closed}}W\prod_{y\in A}e^{q\rho_y^2}$, so that $Z=Z(\emptyset,0)$ and $\langle\prod_{y\in A}e^{q\rho_y^2}\rangle_L^{\mathrm{fine}}=Z(A,q)/Z$. All terms are non-negative ($g_{\mathrm{ct}}\ge0$). Since $h\le1$, since the sum over closed v is at most the sum over all v, and $\sum_{v_p}w((dB)_p+v_p)\le K'_{\gamma_N}$ for every value of $(dB)_p$,

$$Z(A,q)\le\Big(\int d\tilde\rho\prod_x g_{\mathrm{ct}}(\tilde\rho_x)\prod_{y\in A}e^{q\rho_y^2}\Big)\Big(\frac{2\pi}{\gamma_N}\Big)^{|\mathcal{B}_L|}K'^{|\mathcal{P}_L|}_{\gamma_N}=(G_q^{\mathrm{ct}})^{|A|}(G_0^{\mathrm{ct}})^{|\mathcal{X}_L|-|A|}\Big(\frac{2\pi}{\gamma_N}\Big)^{3|\mathcal{X}_L|}K'^{3|\mathcal{X}_L|}_{\gamma_N}.$$

For a lower bound on Z keep only the term $v\equiv0$ (which is closed) and restrict the integration to $\tilde\rho_x\in[0,1]$ for all x and $B_b\in[-1/\gamma_N,1/\gamma_N]\subset(-\pi/\gamma_N,\pi/\gamma_N]$ for all b. There $g_{\mathrm{ct}}(\tilde\rho_x)\ge c_g^{\mathrm{ct}}$ (the exponent $(\rho_{0,0}+r)^2-\rho_{0,0}^2$ lies in $[0,2\rho_{0,0}+1]$ for $r\in[0,1]$, so $g(r)\ge c_g$, and $\frac12\delta m_B(\rho_{0,0}+r)^2\le$

$\frac12|\delta m_B|(\rho_{0,0}+1)^2)$, $|(dB)_p|\le 4/\gamma_N$ so $w((dB)_p)\ge e^{-8/\gamma_N^2}$, and, with $\rho,\rho'\in[\rho_{0,0},\rho_{0,0}+1]$ and $|\gamma_N B_b|\le 1$,

$$|e^{i\gamma_N B_b}\rho'-\rho|^2=(\rho-\rho')^2+2\rho\rho'(1-\cos\gamma_N B_b)\le 1+\rho\rho'\gamma_N^2B_b^2\le 1+(\rho_{0,0}+1)^2,$$

so $h\ge c_h$. Hence $Z\ge (c_g^{\rm ct})^{|\mathcal{X}_L|}(2c_h/\gamma_N)^{|\mathcal{B}_L|}e^{-8|\mathcal{P}_L|/\gamma_N^2}=c_0^{|\mathcal{X}_L|}$.

Step 2: doubling by reflection. Fix x and q, put $F:=e^{q\rho_x^2}$, and for $n=(n_1,n_2,n_3)$ with $n_j=2^{k_j}\le M$ let $B(n):=\{x+\varepsilon(m_1,m_2,m_3):0\le m_j<n_j\}$ (a box of $|B(n)|=n_1n_2n_3$ distinct sites of $\tilde T_L$) and $\Pi_n:=\prod_{y\in B(n)}e^{q\rho_y^2}$. We claim $\big(\langle F\rangle_L^{\rm fine}\big)^{|B(n)|}\le\langle\Pi_n\rangle_L^{\rm fine}$. For $n=(1,1,1)$ this is an identity.

Let it hold for n with $n_j\le M/2$ for some j, and let n' be n with n_j doubled. Take the bond mid-plane $\{x_j=c\}$, $c:=x_j-\varepsilon/2$, and the slab Λ_c^+ of Theorem 9.7 in the j-th coordinate, whose sites have $x_j'\in[x_j,\,x_j-\varepsilon/2+L/2]$; $B(n)\subset\Lambda_c^+$ because $(n_j-1)\varepsilon\le L/2-\varepsilon/2$. The reflection $R_{j,c}$ maps the sites of $B(n)$ onto the adjacent box $\{x_j'\in\{x_j-\varepsilon,\dots,x_j-n_j\varepsilon\}\}$, disjoint from $B(n)$, and $B(n)\cup R_{j,c}B(n)$ is a translate of $B(n')$. The function Π_n is Λ_c^+-measurable with $\langle\Pi_n^2\rangle_L^{\rm fine}=Z(B(n),2q)/Z<\infty$ by Step 1, so the L^2 clause of Theorem 9.7 applies: the form $(A,A')\mapsto\langle\Theta_cA\cdot A'\rangle_L^{\rm fine}$ is positive semidefinite and Hermitian, and the Schwarz inequality for it, with $A=\Pi_n$ and $A'=1$, gives

$$\big(\langle\Theta_c\Pi_n\rangle_L^{\rm fine}\big)^2\le\langle\Theta_c\Pi_n\cdot\Pi_n\rangle_L^{\rm fine}\,\langle\Theta_c1\cdot1\rangle_L^{\rm fine}=\langle(\Pi_n\circ R_{j,c})\,\Pi_n\rangle_L^{\rm fine}=\langle\Pi_{n'}\rangle_L^{\rm fine},$$

by the reflection and translation invariance of the torus state (Lemma 11.1; $\Pi_n\circ R_{j,c}=\prod_{y\in R_{j,c}B(n)}e^{q\rho_y^2}$ is real). Also $\langle\Theta_c\Pi_n\rangle_L^{\rm fine}=\langle\Pi_n\circ R_{j,c}\rangle_L^{\rm fine}=\langle\Pi_n\rangle_L^{\rm fine}$. Hence $\big(\langle F\rangle_L^{\rm fine}\big)^{|B(n')|}=\Big(\big(\langle F\rangle_L^{\rm fine}\big)^{|B(n)|}\Big)^2\le\big(\langle\Pi_n\rangle_L^{\rm fine}\big)^2\le\langle\Pi_{n'}\rangle_L^{\rm fine}$, which is the claim for n'.

Now take $k_1=k_2=k_3=k:=\lfloor\log_2 M\rfloor$: every doubling on the way starts from a box with $n_j\le 2^{k-1}\le M/2$, the final box has $n_j=2^k\in(M/2,M]$ and $|B(n)|=2^{3k}>M^3/8=|\mathcal{X}_L|/8$.

Step 3: assembly. By Steps 1 and 2, with n the final box,

$$\langle\Pi_n\rangle_L^{\rm fine}=\frac{Z(B(n),q)}{Z}\le\Big(\frac{G_q^{\rm ct}}{G_0^{\rm ct}}\Big)^{|B(n)|}\Big(\frac{G_0^{\rm ct}(2\pi/\gamma_N)^3K'^3_{\gamma_N}}{c_0}\Big)^{|\mathcal{X}_L|}\le\Big(\frac{G_q^{\rm ct}}{G_0^{\rm ct}}\Big)^{|B(n)|}C_1^{|\mathcal{X}_L|},$$

so

$$\langle F\rangle_L^{\rm fine}\le\big(\langle\Pi_n\rangle_L^{\rm fine}\big)^{1/|B(n)|}\le(G_q^{\rm ct}/G_0^{\rm ct})\,C_1^{|\mathcal{X}_L|/|B(n)|}\le(G_q^{\rm ct}/G_0^{\rm ct})\,C_1^8,$$

since $C_1\ge1$. □

Remark 11.3. (i) The bound is a multiple-reflection bound of the chessboard kind [13], in a weak form that suffices here: it has the exponent 8 in place of 1 and concerns one observable only, so that no limiting argument on the number of reflections is needed; the ingredients are the reflection positivity about the bond mid-planes in the three directions (Theorem 9.7) and the per-cell bounds of Step 1. (ii) The same bound gives $\langle\prod_{y\in A}e^{q\rho_y^2}\rangle_L^{\rm fine}\le\prod_{y\in A}\big(\langle e^{|A|q\rho_y^2}\rangle_L^{\rm fine}\big)^{1/|A|}\le C_N(|A|q)$ for every finite set A of sites by Hölder's inequality, and the bound $\langle e^{\sum_x q_x\rho_x^2}\rangle_L^{\rm fine}\le\max_x C_N(|X|q_x)$ for finitely many sites X by the convexity of the exponential, $e^{\sum_{x\in X}q_x\rho_x^2}\le|X|^{-1}\sum_{x\in X}e^{|X|q_x\rho_x^2}$. (iii) The constants depend on N; this is sufficient, because the limit $L\to\infty$ is taken at fixed N.

Remark 11.4 (the bare mass counterterm in this section). The site factor of the model is $g_{\rm ct}=g\,e^{-\frac12\delta m_B\rho^2}$, $\delta m_B\in\mathbb{R}$ of either sign (Remark 2.2). Every statement of this section holds for every real δm_B, with the constants $G_q^{\rm ct}$, $c_g^{\rm ct}$ (hence c_0, C_1, $C_N(q)$), at fixed N. Indeed, the proofs use only three properties of the site factor. The first is that $g_{\rm ct}\ge0$, $G_q^{\rm ct}<\infty$ for every $q\ge0$ and $g_{\rm ct}\ge c_g^{\rm ct}$ on $\tilde\rho\in[0,1]$; it is used in Step 1 of the proof of Lemma 11.2. The second is that $g_{\rm ct}$ is one positive function of the one site variable, the same at every site and independent of f. It is used in Lemma 11.1, hence in Theorem 11.5(c); in Step 2 of the proof of Lemma 11.2 and in

Theorem 11.5(d), through Theorem 9.7, whose proof uses the positivity, the reflection invariance and the integrability of the site factor; and in Theorem 3.1, which uses only that the site integral is non-negative and finite (Remark 2.2). The third, for Definition 11.9 and Theorem 11.10, is that $g_{\mathrm{ct}}(\rho-\rho_{0,0})e^{-3\kappa_N\rho^2}d\rho$ is a finite positive measure on $[0,\infty)$ with $\int e^{\alpha\rho^2}<\infty$ for every $\alpha\in\mathbb{R}$ (by (2.1)), which is the condition (10.1) that Section 10 imposes on the site measure. No monotonicity of g, no moment identity and no sign of a derivative of g is used.

11.3. **Limit points of the torus states.**

Theorem 11.5 (limit points, with every symmetry)**.** *Fix N and a sequence of sides $L_n'\to\infty$ with $L_n'/2\in\varepsilon\mathbb{Z}$. There are a subsequence (L_n) and a Borel probability measure μ_N on Ω_N with the following properties.*

(a) *(*Convergence.*) For every finite S, $\nu^S_{L_n}\to(\pi_S)_*\mu_N$ weakly. Equivalently, with*

$$\omega_N(F):=\int F\,d\mu_N\qquad\text{for every bounded Borel local } F,$$

one has $\omega_N(F)=\lim_n\langle F\rangle^{\mathrm{fine}}_{L_n}$ for every bounded continuous local F. The same limit relation holds for every bounded Borel local F whose set of discontinuities is μ_N-null, and for every continuous local F with $|F|\le Ce^{\sum_{x\in X}q_x\rho_x^2}$ for some finite X, $C<\infty$, $q\ge0$.

(b) *(*State; uniform bounds.*) ω_N is linear and positive on the bounded Borel local functions, $\omega_N(1)=1$, $|\omega_N(F)|\le\|F\|_\infty$; and for every real finitely supported a and every $q\ge0$, x,*

$$\int e^{\sum_p a_pf_p}\,d\mu_N\le e^{\frac12\sum_p a_p^2},\qquad\int e^{q\rho_x^2}\,d\mu_N\le C_N(q).$$

(c) *(*Symmetries.*) $g_*\mu_N=\mu_N$ for every $g\in\mathcal{G}_N\cup\{C\}$; hence $\omega_N(F\circ g)=\omega_N(F)$ for every bounded Borel local F. In particular the conclusion of Lemma 5.1 holds for ω_N.*

(d) *(*Reflection positivity.*) For $j\in\{1,2,3\}$ and $c\in\frac12\varepsilon\mathbb{Z}$ let $\mathfrak{A}^+_N(j,c)$ be the bounded Borel functions of the variables of finitely many cells lying in $\{x_j\ge c\}$ (all vertices in the closed half space) and $\Theta_{j,c}F:=\overline{F\circ R_{j,c}}$. Then $(F,G)\mapsto\omega_N(\Theta_{j,c}F\cdot G)$ is a positive semidefinite Hermitian form on $\mathfrak{A}^+_N(j,c)$, and $\omega_N(\Theta_{j,c}F)=\overline{\omega_N(F)}$. Both statements extend to functions of those cells in $L^2(\mu_N)$.*

(e) *(*Inequalities pass.*) If F,G are bounded continuous local functions with $\langle F\rangle^{\mathrm{fine}}_L\le\langle G\rangle^{\mathrm{fine}}_L$ for all large L, then $\omega_N(F)\le\omega_N(G)$; and $\int F\,d\mu_N\le\liminf_n\langle F\rangle^{\mathrm{fine}}_{L_n}$ for every continuous local $F\ge0$.*

Moreover (a)–(e) hold for every pair $((L_n),\mu_N)$, $L_n\to\infty$ with $L_n/2\in\varepsilon\mathbb{Z}$, for which the weak convergence of (a) holds.

Proof. Tightness. Let S be finite and $K_R:=[-R,R]^{\mathcal{P}_S}\times[-\rho_{0,0},R]^{\mathcal{X}_S}$, a compact subset of E_S. By Theorem 3.1 and Markov's inequality, $\nu^S_L(|f_p|>R)\le e^{-R}\langle e^{|\mathfrak{f}_p|}\rangle^{\mathrm{fine}}_L\le 2e^{1/2-R}$; by Lemma 11.2, $\nu^S_L(\tilde\rho_x>R)\le e^{-(\rho_{0,0}+R)^2}\langle e^{\rho_x^2}\rangle^{\mathrm{fine}}_L\le C_N(1)e^{-R^2}$. Hence $\sup_L\nu^S_L(E_S\setminus K_R)\le 2|\mathcal{P}_S|e^{1/2-R}+|\mathcal{X}_S|C_N(1)e^{-R^2}\to0$ as $R\to\infty$: the family $(\nu^S_L)_L$ is tight on the Polish space E_S, hence relatively compact for the weak topology (Prokhorov).

Diagonal extraction and consistency. Let S_k be the set of cells of $\varepsilon\mathbb{Z}^3$ lying in the cube $[-ka_0,ka_0]^3$, $k\ge1$; every finite S is contained in some S_k. A diagonal argument gives a subsequence (L_n) of (L_n') along which $\nu^{S_k}_{L_n}\to\nu^{S_k}_\infty$ weakly for every k. For $k\le k'$ the S_k-marginal of $\nu^{S_{k'}}_{L_n}$ is $\nu^{S_k}_{L_n}$, and the image of a weakly convergent sequence under the continuous projection $E_{S_{k'}}\to E_{S_k}$ converges weakly to the image of the limit; so the S_k-marginal of $\nu^{S_{k'}}_\infty$ is $\nu^{S_k}_\infty$: the family $(\nu^{S_k}_\infty)_k$ is consistent. By Kolmogorov's extension theorem (a countable product of Polish

spaces) there is a unique Borel probability measure μ_N on Ω_N with $(\pi_{S_k})_*\mu_N = \nu_\infty^{S_k}$ for all k, hence $(\pi_S)_*\mu_N = \lim_n \nu_{L_n}^S$ weakly for every finite S.

(a). For F bounded continuous local with support S, $\langle F\rangle_{L_n}^{\mathrm{fine}} = \int F_S\, d\nu_{L_n}^S \to \int F_S\, d(\pi_S)_*\mu_N = \omega_N(F)$ by the definition of weak convergence. For bounded Borel F_S with a $(\pi_S)_*\mu_N$-null discontinuity set this is the portmanteau theorem. For continuous F_S with $|F_S| \le Ce^{\sum_{x\in X} q_x\rho_x^2}$: by Remark 11.3(ii), $\sup_L \int e^{2\sum_x q_x\rho_x^2} d\nu_L^S < \infty$, so (F_S) is uniformly integrable for $(\nu_{L_n}^S)_n$, and weak convergence together with uniform integrability gives the convergence of the integrals.

(b). Linearity, positivity, normalisation and the bound are those of the integral against a probability measure. The function $e^{\sum_p a_p f_p}$ is continuous and non-negative, so by the portmanteau theorem for lower semicontinuous functions bounded below (apply weak convergence to $\min(e^{\sum a_p f_p}, m)$ and let $m\to\infty$) $\int e^{\sum a_p f_p} d\mu_N \le \liminf_n \langle e^{\sum_p a_p \mathfrak{f}_p}\rangle_{L_n}^{\mathrm{fine}} \le e^{\frac12 \sum a_p^2}$ by (3.1); likewise $\int e^{q\rho_x^2} d\mu_N \le \liminf_n \langle e^{q\rho_x^2}\rangle_{L_n}^{\mathrm{fine}} \le C_N(q)$ by (11.1). This also proves the second clause of (e); the first clause of (e) is (a) applied to F and G.

(c). Let $g \in \mathcal{G}_N \cup \{C\}$ and F bounded continuous local. Then $F\circ g$ is bounded continuous local, and by Lemma 11.1 $\langle F\circ g\rangle_{L_n}^{\mathrm{fine}} = \langle F\rangle_{L_n}^{\mathrm{fine}}$ for all large n; by (a), $\int F\circ g\, d\mu_N = \int F\, d\mu_N$, i.e. $\int F\, d(g_*\mu_N) = \int F\, d\mu_N$ for every bounded continuous local F. A Borel probability measure on the Polish space E_S is determined by the integrals of the bounded continuous functions, so $g_*\mu_N$ and μ_N have the same finite-dimensional marginals; the cylinder sets form a π-system generating the Borel σ-algebra of Ω_N, so $g_*\mu_N = \mu_N$. The conclusion of Lemma 5.1 for ω_N follows, since $\varepsilon\mathbb{Z}^3$, the reflections in the site planes and the cubic group about every site lie in $\mathcal{G}_N$.

(d). Let first F, G be bounded *continuous* functions of the variables of finitely many cells lying in $\{x_j \ge c\}$. For n large these cells lie in the slab Λ_c^+ of $\tilde T_{L_n}$ (in the j-th coordinate), so Theorem 9.7 (with the cubic symmetry for $j\in\{1,2\}$) gives $\langle \Theta_{j,c}F\cdot G\rangle_{L_n}^{\mathrm{fine}} = \overline{\langle \Theta_{j,c}G\cdot F\rangle_{L_n}^{\mathrm{fine}}}$ and $\langle \Theta_{j,c}F\cdot F\rangle_{L_n}^{\mathrm{fine}} \ge 0$; the functions $\Theta_{j,c}F\cdot G$, $\Theta_{j,c}F\cdot F$ and $F\circ R_{j,c}$ are bounded continuous local, so by (a) the identities hold for ω_N, and $\omega_N(\Theta_{j,c}F) = \overline{\omega_N(F)}$ by (c). Positive semidefiniteness of the form on finite families follows by applying the inequality to linear combinations. For the extension let S be a finite set of cells lying in $\{x_j \ge c\}$. The bounded continuous functions on E_S are dense in $L^2(E_S, (\pi_S)_*\mu_N)$ (a finite Borel measure on a Polish space is Radon), and a bounded Borel F_S can be approximated by continuous $F_S^{(i)}$ with $\|F_S^{(i)}\|_\infty \le \|F_S\|_\infty$ (truncate). Both sides of the identities are continuous in the $L^2(\mu_N)$-norms of F and G, by the Schwarz inequality and $\|F\circ R_{j,c}\|_{L^2(\mu_N)} = \|F\|_{L^2(\mu_N)}$ (the $R_{j,c}$-invariance of μ_N, (c)). This gives (d) for bounded Borel F, G and for F, G in $L^2(\mu_N)$ measurable with respect to the cells of S.

The last clause. The proofs of (b)–(e) above use only (a) and the torus statements along (L_n). □

Remark 11.6 (uniqueness and the choice of sides). (i) Different sequences of sides may give different limit points; the theorem claims no uniqueness, and (2.2) in its continuous form (ii) of Corollary 11.13 is exactly the statement that all limit points coincide, the form (i) implying it. (ii) Any cofinal class of sides may be prescribed. (iii) The proof of Theorem 11.5 uses only Theorem 3.1, Theorem 9.7, Lemma 11.1 and Lemma 11.2, which hold for arbitrary positive couplings and every real δm_B.

11.4. **Limit points along arbitrary sequences of sides.**

Corollary 11.7 (the results hold for every limit point)**.** *For each $N \ge N_0$ let ω_N be the state of a limit point as in Theorem 11.5, along any sequence of sides $(L_n^{(N)})_n$, and write $Z_N(\sigma) := \omega_N(e^{i\Phi_N(\sigma)}) = \int e^{i\Phi_N(\sigma)} d\mu_N$.*

(i) *Proposition 4.2, Lemma 5.1, Proposition 6.1 and Theorem 7.1 hold for these ω_N; the density function g_{d} of Definition 2.5 may be any bounded Borel function.*

(ii) *Every statement of Section 9 holds for these ω_N, with "$L \to \infty$" read as "$n \to \infty$ along $L_n^{(N)}$": Lemma 9.3, Theorem 9.4, Proposition 9.6, Corollary 9.8, Propositions 9.9, 9.11 and 9.12, Theorems 9.10, 9.13 and 9.15.*

Proof. (i) *Proposition 4.2.* The only steps of its proof that use ω_N are the final ones (from "Finally" on), and they rest on Theorem 11.5(e). Indeed, write $\Phi_N(\sigma) = X + Y$ as in that proof, where X is the curvature part, a continuous function of finitely many f_p, and Y is the density part, a bounded Borel function with $|Y| \le \|g_{\mathrm{d}}\|_\infty E(h)$ pointwise. The Schwarz inequality in $L^2(\mu_N)$ and Minkowski's inequality give

$$\omega_N(|\Phi_N(\sigma)| \wedge 2) \le \Big(\int \Phi_N(\sigma)^2 d\mu_N\Big)^{1/2} \le \Big(\int X^2 d\mu_N\Big)^{1/2} + \|g_{\mathrm{d}}\|_\infty E(h),$$

and, since X^2 is continuous and non-negative, Theorem 11.5(e) and Theorem 3.1 give

$$\int X^2 d\mu_N \le \liminf_n \langle X^2\rangle_{L_n}^{\mathrm{fine}} \le R(f)^2.$$

So (4.1) holds. The continuity of g_{d} is not needed, since ω_N is the integral against μ_N.

Lemma 5.1 is Theorem 11.5(c). *Proposition 6.1* uses of ω_N only that it is a state, which gives $|Z_N(\sigma)| \le 1$ and the positivity in Step 1, Lemma 5.1 and (4.1). *Theorem 7.1* follows from these statements.

(ii) *Lemma 9.3(b).* The family $(\Psi_{N,L_n})_n$ is locally uniformly bounded by part (a) of that lemma. By Montel's theorem it has locally uniformly convergent subsequences, and every subsequential limit agrees on $i\mathbb{R}^k$ with

$$t \mapsto \int e^{i\sum_j t_j \Phi_N(f_j)} d\mu_N = \lim_n \langle e^{i\sum_j t_j \Phi_N(f_j)}\rangle_{L_n}^{\mathrm{fine}},$$

by Theorem 11.5(a) applied to a bounded continuous function of finitely many f_p. Two entire functions that agree on $i\mathbb{R}^k$ coincide, so $\Psi_{N,L_n} \to \Psi_N$ locally uniformly with $\Psi_N(it) := \int e^{it\cdot\Phi_N(f)} d\mu_N$, and the law $\mu_N^{(f)}$ of $(\Phi_N(f_j))_j$ under μ_N is the weak limit of the torus laws, with $\int e^{z\cdot x} d\mu_N^{(f)} = \Psi_N(z)$ by the same uniform integrability as in the proof of that lemma. Every later use of the lemma (Theorem 9.4(ii)–(iv), Theorem 9.10, Proposition 9.11) takes the limit along (L_n).

Proposition 9.6 uses of ω_N only Lemma 5.1, Proposition 6.1 and Theorem 7.1, which hold for these ω_N by (i). In *Corollary 9.8*, the identities for bounded Borel F, G of cells in $\{x_3 \ge c\}$ and the invariances are Theorem 11.5(d),(c), and the extension to the bounded Borel class comes from the L^2 clause of (d). *Proposition 9.9* uses Theorem 7.1(a) and Corollary 9.8. For *Proposition 9.12(b)*, $\Phi_N(f,h)\circ C = \Phi_N(-f,h)$, so $Z_N(-f,h) = Z_N(f,h)$ by the C-invariance of μ_N, for every bounded Borel g_{d}; the rest of its proof is unchanged. *Theorem 9.13* and *Theorem 9.15* use only the statements already covered. □

Remark 11.8 (the existence statement that is used). The statements of this paper use the following existence statement and no more: *for each N there are a sequence of sides $L_n \to \infty$, $L_n/2 \in \varepsilon\mathbb{Z}$, and a probability measure μ_N on Ω_N such that the finite-dimensional laws of the gauge-invariant variables under the torus states of sides L_n converge weakly to those of μ_N*; ω_N is then the integral against μ_N. The symmetries, the reflection positivity and the uniform bounds come with it (Theorem 11.5). No averaging over the symmetry group is needed, because every lattice symmetry is a symmetry of every torus state. The existence of the limit (2.2), and a fortiori its form for bounded Borel F (a setwise convergence), is not used in any statement of this paper; it is needed only to speak of *the* fine state.

11.5. **The cone class: the box limit, the floor, and the reduction of the limit (2.2) to the cone.**

Definition 11.9 (the cone; the killed boxes)**.** For a real finitely supported $a = (a_p)_{p\in\mathcal{P}_N}$, a polynomial P in finitely many ρ_x with non-negative coefficients and a finitely supported $q = (q_x)_{x\in\mathcal{X}_N}$ with $q_x \geq 0$ put

$$\Theta_{a,P,q} := \cos\Big(\sum_p a_p f_p\Big)\, P(\rho)\, e^{\sum_x q_x\rho_x^2},$$

and let $\mathcal{K}_N$ be the set of finite non-negative linear combinations of such functions. The *generators* are the $\Theta_{a,1,q}$.

For a finite box $\Lambda \subset \varepsilon\mathbb{Z}^3$ (a product of three intervals of sites) let $\mathcal{B}(\Lambda)$ be the bonds with both ends in Λ and $\mathcal{P}(\Lambda)$ the plaquettes with all four bonds in $\mathcal{B}(\Lambda)$. The *killed-box state* $\langle\cdot\rangle_\Lambda$ is the normalised expectation of the weight

$$W_\Lambda := \prod_{x\in\Lambda} g_{\mathrm{ct}}(\tilde\rho_x) e^{-3\kappa_N\rho_x^2} \prod_{b\in\mathcal{B}(\Lambda)} e^{\kappa_N\rho_{b_-}\rho_{b_+}\cos(\gamma_N B_b)} \prod_{p\in\mathcal{P}(\Lambda)} w\big((dB)_p + v_p\big)$$

in the variables $\tilde\rho_x$ ($x\in\Lambda$), B_b ($b\in\mathcal{B}(\Lambda)$), v_p ($p\in\mathcal{P}(\Lambda)$), the sum over the v with $dv = 0$ on every cube of Λ. It is the weight of the model of Section 2 with the matter killed outside Λ ($\rho_y = 0$ for $y\notin\Lambda$, which turns the hopping factor of every bond leaving Λ into the diagonal factor $e^{-\kappa_N\rho_x^2/2}$ of its inner end, so that every site of Λ carries the diagonal halves of all six of its bonds) and with no gauge or Villain variables outside $\mathcal{B}(\Lambda)$, $\mathcal{P}(\Lambda)$. A function is supported in Λ if it depends only on f_p, $p\in\mathcal{P}(\Lambda)$, and ρ_x, $x\in\Lambda$. A sequence of boxes $(\Lambda_n)_n$ is *exhausting* if every finite subset of $\varepsilon\mathbb{Z}^3$ lies in Λ_n for all large n.

The cone $\mathcal{K}_N$ is closed under products ($\cos u\cos v = \frac12[\cos(u+v)+\cos(u-v)]$), consists of continuous local functions, and each element satisfies $|\Theta| \leq Ce^{\sum_x q'_x\rho_x^2}$ for some $C<\infty$ and finitely supported $q'\geq 0$; so Theorem 11.5(a) gives $\omega_N(\Theta) = \lim_n \langle\Theta\rangle_{L_n}^{\mathrm{fine}}$ for every limit point and every $\Theta\in\mathcal{K}_N$. Gaussian domination holds for the box states by Lemma 12.10.

Theorem 11.10 (Ginibre's inequalities for the Villain fine model)**.** *For the torus states and for the killed-box states, and for $\Theta,\Theta'\in\mathcal{K}_N$:*

(K1) $\langle\Theta\rangle \geq 0$; (K2) $\langle\Theta;\Theta'\rangle := \langle\Theta\Theta'\rangle - \langle\Theta\rangle\langle\Theta'\rangle \geq 0$;

(K3) $\langle\Theta\rangle$ *is non-decreasing in each hopping coefficient $\kappa_b \geq 0$ (the coefficient of $\rho_{b_-}\rho_{b_+}\cos(\gamma_N B_b)$, bond by bond, the diagonal factors held fixed), in each plaquette stiffness $\beta_p > 0$ ($w_p(f) = e^{-\beta_p f^2/2}$) and in $q\geq 0$ under the change of the site weight $g_{\mathrm{ct}}(\tilde\rho_x) \mapsto g_{\mathrm{ct}}(\tilde\rho_x)e^{q\rho_x^2}$; explicitly $\partial_{\kappa_b}\langle\Theta\rangle = \langle\Theta;\rho_{b_-}\rho_{b_+}\cos(\gamma_N B_b)\rangle \geq 0$ and $\partial_{\beta_p}\langle\Theta\rangle = -\frac12\langle\Theta;f_p^2\rangle \geq 0$;*

(K4) *for boxes $\Lambda\subset\Lambda'$ and Θ supported in Λ: $\langle\Theta\rangle_\Lambda \leq \langle\Theta\rangle_{\Lambda'}$; and if Λ together with its nearest-neighbour sites lies in a cube of side less than L: $\langle\Theta\rangle_\Lambda \leq \langle\Theta\rangle_L^{\mathrm{fine}}$.*

These statements are proved, by the duplicate-variable argument of Ginibre [17] adapted to the Villain form of the plaquette weight, in Section 10 (Theorem 10.8); we restate them here and do not repeat the proofs. For (K4) the proof interpolates the couplings of the cells of Λ' (or of $\tilde T_L$) outside Λ from their values to 0 and uses (K3), the endpoint being the box state. The classical input is positivity of the Fourier coefficients of the weight, which gives (K1) alone; (K2)–(K4) need the duplication identity of the Villain weight.

Theorem 11.11 (the box limit on the cone)**.** (a) *For $\Theta\in\mathcal{K}_N$ the net $\Lambda\mapsto\langle\Theta\rangle_\Lambda$, over the boxes containing the support of Θ ordered by inclusion, is non-decreasing and bounded;*

hence

$$\omega_N^{\mathrm{box}}(\Theta) := \lim_{\Lambda\uparrow\varepsilon\mathbb{Z}^3}\langle\Theta\rangle_\Lambda = \sup_\Lambda\langle\Theta\rangle_\Lambda$$

exists, along every exhausting sequence of boxes, and $\omega_N^{\mathrm{box}}(\Theta) \le \omega_N(\Theta)$ *for every limit point* ω_N *of Theorem 11.5 (along any sequence of sides).*

(b) *The box states converge: for every finite* S *the laws* ν_Λ^S *of the variables of* S *under* $\langle\cdot\rangle_\Lambda$ *converge weakly, along every exhausting sequence of boxes, to the marginal of one Borel probability measure* μ_N^{box} *on* Ω_N*;* $\omega_N^{\mathrm{box}}(F) := \int F\,d\mu_N^{\mathrm{box}}$ *extends* ω_N^{box} *from* $\mathcal{K}_N$ *to all bounded Borel local* F *and all* F *as in Theorem 11.5(a) (with* μ_N^{box}*-null in place of* μ_N*-null discontinuity sets);* μ_N^{box} *is invariant under* $\mathcal{G}_N$ *and* C*, and* $\int e^{\sum a_p f_p}d\mu_N^{\mathrm{box}} \le e^{\frac12\sum a_p^2}$*,* $\int e^{q\rho_x^2}d\mu_N^{\mathrm{box}} \le C_N(q)$*.*

Proof. (a) Monotonicity is (K4). Boundedness: for $\Theta = \Theta_{a,P,q}$ and a box Λ containing its support, $|\Theta| \le Ce^{\sum_x q'_x\rho_x^2}$ with $q' > q$ on the sites of P and $q' = q$ elsewhere, and $e^{\sum_x q'_x\rho_x^2} \in \mathcal{K}_N$ is supported in Λ, so by (K4) with a torus of side larger than Λ and its neighbours, and Remark 11.3(ii), $\langle|\Theta|\rangle_\Lambda \le C\langle e^{\sum_x q'_x\rho_x^2}\rangle_L^{\mathrm{fine}} \le C\max_x C_N(|X|q'_x) =: C(\Theta)$, uniformly in Λ; the same bound holds for non-negative combinations. A bounded non-decreasing net of reals converges to its supremum, and the values along any exhausting sequence of boxes converge to the same supremum (every box lies in all late members of the sequence, so by monotonicity the limit inferior along the sequence is at least the supremum over all boxes). For the comparison with a limit point: given a box Λ containing the support of Θ, (K4) gives $\langle\Theta\rangle_\Lambda \le \langle\Theta\rangle_{L_n}^{\mathrm{fine}}$ for all large n, and the right side converges to $\omega_N(\Theta)$ (Theorem 11.5(a)); so $\langle\Theta\rangle_\Lambda \le \omega_N(\Theta)$ for every Λ, and $\omega_N^{\mathrm{box}}(\Theta) \le \omega_N(\Theta)$.

(b) Tightness of $(\nu_\Lambda^S)_\Lambda$: by (K4), $\langle e^{\rho_x^2}\rangle_\Lambda \le \sup_L\langle e^{\rho_x^2}\rangle_L^{\mathrm{fine}} \le C_N(1)$, and $\langle e^{s\mathfrak{f}_p}\rangle_\Lambda \le e^{s^2/2}$ by Theorem 3.1 for the box states (Lemma 12.10). The argument of the proof of Theorem 11.5 applies and gives, along every exhausting sequence, a subsequence and a limit measure μ with the properties (a), (b) of that theorem (with "box" for "torus"), and μ is C-invariant because every box state is. By (a) of the present theorem and the uniform integrability of Theorem 11.5(a), $\int\Theta\,d\mu = \lim\langle\Theta\rangle_\Lambda = \omega_N^{\mathrm{box}}(\Theta)$ along the subsequence, for every $\Theta \in \mathcal{K}_N$. By Lemma 11.12 below, any two such limit measures coincide; hence the whole net converges, to one measure μ_N^{box}, and the extension statement follows.

For the invariance under a translation τ, note that τ maps an exhausting sequence of boxes to an exhausting sequence of boxes and $\langle F\circ\tau\rangle_\Lambda = \langle F\rangle_{\tau^{-1}\Lambda}$, so the limits agree. Under a reflection or a cubic rotation g about a site the same holds: g maps boxes to boxes and exhausting sequences of boxes to exhausting sequences of boxes, and the box weights are covariant, the weight of $g\Lambda$ being the image of W_Λ under g. Indeed every factor of W_Λ is indexed by a cell of Λ, is the same function at every cell, and is even in B_b and in f_p; a cell map commutes with the coboundaries up to the orientation signs, so the closedness of v is preserved; the diagonal factor $e^{-3\kappa_N\rho_x^2}$ is the same at every site. In each case $\int F\circ g\,d\mu_N^{\mathrm{box}} = \int F\,d\mu_N^{\mathrm{box}}$ for every bounded continuous local F, and $g_*\mu_N^{\mathrm{box}} = \mu_N^{\mathrm{box}}$ follows as in the proof of Theorem 11.5(c). The bounds are Theorem 11.5(e) for the box sequence. □

Lemma 11.12 (the cone determines the state)**.** *Let* μ, μ' *be Borel probability measures on* Ω_N*, both* C*-invariant, with* $\int e^{\sum_x q_x\rho_x^2}d\mu < \infty$ *and* $\int e^{\sum_x q_x\rho_x^2}d\mu' < \infty$ *for every finitely supported* $q \ge 0$*, and suppose* $\int\Theta_{a,1,q}\,d\mu = \int\Theta_{a,1,q}\,d\mu'$ *for all generators. Then* $\mu = \mu'$*.*

Proof. Fix finite sets $P \subset \mathcal{P}_N$, $X \subset \mathcal{X}_N$, and for $a \in \mathbb{R}^P$, $z \in \mathbb{C}^X$ put $G(a,z) := \int e^{i\sum_{p\in P} a_p f_p}e^{\sum_{x\in X} z_x\rho_x^2}d\mu$ and G' likewise for μ'. For z in a compact subset of $\mathbb{C}^X$ the integrand is dominated by $e^{\sum_x q_x\rho_x^2}$ with $q_x := \max\{0, \max\Re z_x\}$, the maximum over the compact

set, which is integrable; so $G(a,\cdot)$ is entire on $\mathbb{C}^X$ (differentiation under the integral), and so is $G'(a,\cdot)$. By C-invariance the imaginary part $\int \sin(a\cdot f)e^{z\cdot\rho^2}d\mu$ vanishes for real z, so for $z=q\in[0,\infty)^X$ the hypothesis gives $G(a,q)=\int\Theta_{a,1,q}d\mu=\int\Theta_{a,1,q}d\mu'=G'(a,q)$. Two entire functions on $\mathbb{C}^X$ that agree on the set $[0,\infty)^X$, which has non-empty interior in $\mathbb{R}^X$, coincide (identity theorem, one variable at a time). At $z=it$, $t\in\mathbb{R}^X$, $G(a,it)$ is the characteristic function of the random vector $\big((f_p)_{p\in P},(\rho_x^2)_{x\in X}\big)$ under μ; so the joint law of this vector is the same under μ and μ'. Since $\rho_x\ge 0$, $\tilde\rho_x=\sqrt{\rho_x^2}-\rho_{0,0}$ is a measurable function of ρ_x^2, and the joint law of $\big((f_p)_{p\in P},(\tilde\rho_x)_{x\in X}\big)$ is the same under μ and μ'. This holds for all finite P,X, so all finite-dimensional marginals agree, and $\mu=\mu'$ (π-λ theorem on the cylinder sets). □

Corollary 11.13 (equivalent forms of the existence of the limit)**.** *Fix N. The following are equivalent, the limits being over all L with $L/2\in\varepsilon\mathbb{Z}$ (or over any cofinal class of such sides):*

(ii) $\lim_{L\to\infty}\langle F\rangle_L^{\text{fine}}$ *exists for every bounded continuous local F;*
(iii) $\lim_{L\to\infty}\langle\Theta_{a,1,q}\rangle_L^{\text{fine}}$ *exists for every generator;*
(iv) *all limit points of Theorem 11.5, along all sequences of sides, are one measure μ_N.*

Each is implied by (2.2) as stated (the limit for every bounded Borel local F), which we call (i). When they hold, $\omega_N(F)=\lim_L\langle F\rangle_L^{\text{fine}}$ for every bounded continuous local F, for every F as in Theorem 11.5(a), and for every bounded Borel local F whose discontinuity set is μ_N-null; and $\omega_N^{\text{box}}\le\omega_N$ on $\mathcal{K}_N$.

Proof. (i)⇒(ii) is trivial. (ii)⇒(iii): under (ii) the laws ν_L^S converge weakly as $L\to\infty$ (a tight family whose integrals of bounded continuous functions converge), and the generators are uniformly integrable by Lemma 11.2, so their expectations converge. (iii)⇒(iv): two limit points μ_N,μ_N' satisfy $\int\Theta_{a,1,q}d\mu_N=\lim_L\langle\Theta_{a,1,q}\rangle_L^{\text{fine}}=\int\Theta_{a,1,q}d\mu_N'$ (Theorem 11.5(a)), are C-invariant and have all exponential moments (Theorem 11.5(b),(c)); Lemma 11.12 gives $\mu_N=\mu_N'$. (iv)⇒(ii): if the bounded continuous local F had $\langle F\rangle_L^{\text{fine}}\not\to\int Fd\mu_N$, a subsequence of sides would keep $|\langle F\rangle_L^{\text{fine}}-\int Fd\mu_N|\ge\delta$; Theorem 11.5 applied to it gives a limit point, equal to μ_N by (iv), along a further subsequence of which $\langle F\rangle_L^{\text{fine}}\to\int Fd\mu_N$, a contradiction. The last sentence is Theorem 11.5(a) and Theorem 11.11(a). □

Remark 11.14 (a floor). On the cone the monotonicity (K3), (K4) runs one way: enlarging the volume, raising a hopping coefficient, a plaquette stiffness or a site factor $e^{q\rho^2}$ raises every cone expectation. The killed box is therefore *minimal* among the finite-volume states compared in (K4), and its limit is a floor under every torus limit point (Theorem 11.11(a)).

Remark 11.15 (summary of the section). Limit points of the torus states exist, with every lattice symmetry, reflection positivity about all coordinate planes through sites and bond midpoints, the Gaussian domination of the curvature and uniform local moments of the density (Theorem 11.5). They suffice for Theorem 7.1 and for the statements of Section 9 listed in Corollary 11.7. The killed-box states converge to a state ω_N^{box} with the same symmetries, which lies below every torus limit point on the cone $\mathcal{K}_N$ (Theorem 11.11); and (2.2), in its continuous form (ii) of Corollary 11.13, is equivalent to the agreement of the torus limit points on the generators of the cone, the form (i) implying it.

12. Rotation invariance

Section 9 left one piece of the Euclidean invariance open: the invariance of the limit theory under the rotations of $\mathbb{R}^3$. For the limit theory built on the killed-box states of Definition 11.9 we prove it here. The corresponding statement for the limits built on the torus limit points of Definition 2.3 is Theorem 12.4. Section 12.1 states the theorems and gives a guide to the proof;

Section 12.2 collects the preliminaries; Sections 12.3–12.11 prove the ingredients and Section 12.12 assembles them. The rigidity theorem, the diagonal reflection positivity of the torus limit points and the general theorem on random fields are the subjects of Sections 13, 14 and 15.

Remark 12.1. In proving the rotation invariance, in this section and in Sections 13, 14 and 15, we implement the original strategy of [35]. In a subsequent manuscript we shall provide a different strategy, adapting the proof from the companion paper [34].

12.1. **Setting and the theorems.** Test pairs are $\sigma = (f, h)$ as in Section 2; the transport $O \cdot \sigma$ of a test pair by $O \in \mathrm{O}(3)$, the automorphism γ_O of $\mathcal{A}$, the Euclidean group $E(3)$ and its action $\gamma \cdot \sigma$ are those of Definition 12.7 and Proposition 12.11 below (the 2-form transported as a 2-form, h as a scalar). ω_N^{box} is the killed-box limit of Theorem 11.11 (it exists along every exhausting sequence of boxes), μ_N^{box} its measure on Ω_N. By Lemma 12.20 below, Theorem 7.1 and the statements of Section 9 hold with ω_N^{box} in place of ω_N, in the form given in that lemma; we call S^{box} any limit of Theorem 7.1 so obtained along a sequence $N_1 < N_2 < \cdots$, write $Z^{\mathrm{box}}(\sigma) := S^{\mathrm{box}}(e_\sigma)$, $Z^{\mathrm{box}}(f) := Z^{\mathrm{box}}(f, 0)$, S_n^{box} for its Schwinger functions (Theorem 9.4(iii)) and μ^{box} for its measure (Theorem 9.13, Lemma 12.12).

Theorem 12.2 (rotation invariance for the killed-box states)**.** *Let the couplings be any $\kappa_N \geq 0$, $\lambda_0 > 0$, $\rho_{0,0} > 0$, $\gamma_N > 0$ and any real δm_B (notation of Remark 2.1), and take $g_{\mathrm{d}} \equiv 0$ in Definition 2.5 (the curvature scheme, Remark 12.9). Let S^{box} be a limit along $N_1 < N_2 < \cdots$. Then*

$$Z^{\mathrm{box}}(R \cdot f) = Z^{\mathrm{box}}(f) \quad \text{for every } R \in \mathrm{O}(3) \text{ and every real } C_c^\infty \text{ 2-form } f,$$
$$\text{i.e.} \qquad S^{\mathrm{box}} \circ \gamma_R = S^{\mathrm{box}};$$

hence (with Theorem 7.1) $S^{\mathrm{box}}(e_{\gamma\cdot\sigma}) = S^{\mathrm{box}}(e_\sigma)$ for all $\gamma \in E(3)$; $\gamma_* \mu^{\mathrm{box}} = \mu^{\mathrm{box}}$ for all $\gamma \in E(3)$ (Proposition 12.11); μ^{box} is reflection positive across every affine plane of $\mathbb{R}^3$, on L^2 of either open side, and the reconstruction in any direction is unitarily the one in direction e_3 (Remark 9.16). With OS0 (Theorem 9.4), OS3 (Proposition 9.12) and the measure (Theorem 9.13), S^{box} satisfies OS0–OS3 in full.

Theorem 12.3 (bounded functions of the density)**.** *Let g_{d} be any bounded Borel function (the joint scheme (f, h) of Definition 2.5). Then the curvature marginal of S^{box} is as in Theorem 12.2, and every mixed Schwinger function $S_n(\sigma_1, \ldots, \sigma_n)$ of the joint field (the moments $\int \prod_i \Phi(\sigma_i) d\mu$ of the measure μ of Lemma 12.22) satisfies*

$$S_n(R \cdot \sigma_1, \ldots, R \cdot \sigma_n) = S_n(\sigma_1, \ldots, \sigma_n) \qquad \text{for all } R \in \mathrm{O}(3)$$

whenever the supports of $\sigma_1, \ldots, \sigma_n$ are pairwise disjoint (the Osterwalder–Schrader form of Euclidean covariance, away from coincident points). The coincident points are covered as well (Lemma 12.37, Corollary 12.38(a)): the mixed Schwinger functions extend to bounded multilinear forms, on $L^2(\mathbb{R}^3; \Lambda^2_{\mathbb{C}})$ in the curvature entries and on $L^1(\mathbb{R}^3; \mathbb{C})$ in the density entries, the displayed identity holds for all such entries with no condition on the supports, and

$$Z^{\mathrm{box}}(R \cdot \sigma) = Z^{\mathrm{box}}(\sigma) \quad \text{for every } R \in \mathrm{O}(3) \text{ and every test pair } \sigma = (f, h),$$
$$\text{i.e.} \qquad S^{\mathrm{box}} \circ \gamma_R = S^{\mathrm{box}} \quad \text{on } \mathcal{A};$$

hence (Theorem 7.1, Proposition 12.11) $S^{\mathrm{box}}(e_{\gamma\cdot\sigma}) = S^{\mathrm{box}}(e_\sigma)$ for all $\gamma \in E(3)$ and all test pairs, and the joint measure of Lemma 12.22 is invariant under $E(3)$.

Theorem 12.4 (rotation invariance for the torus limit points)**.** *Let ω_N be a cubic-torus limit point (Definition 2.3, whose sides run along a dyadic sequence, Definition 14.1) and S a limit functional of Theorem 7.1. Then Theorems 12.2 and 12.3 hold for S for $\kappa_N > 0$ at every level*

(by Lemma 2.7; the value $\kappa_N = 0$ is excluded), with the other couplings and the conclusions as stated there.

Remark 12.5. The reflection positivity of ω_N at the six diagonal site planes $\{x_i \pm x_j = 0\}$ through 0, in the sense of Theorem 11.5(d), is Corollary 14.35 at $c = 0$ (Theorem 14.3 with the Hermitian form, the conjugation identity and the L^2 extension) for the limit points along dyadic sequences; it is proved (for $\kappa_N \neq 1$ with Lemma 2.7) through Theorem 13.1 and Lemma 14.14. Every other step of the proof below holds for the torus limit points as it stands.

Guide to the proof. The mechanism is the one of Remark 9.16 read backwards: reflection positivity in enough planes forces rotation covariance. On the cubic torus the reflection argument of Theorem 9.7 applies at the coordinate planes only (the second wall of the torus at a diagonal plane is a glide, not a mirror; no period lattice carries both a coordinate and a diagonal family of pointwise-fixed walls with the same coordinate j, that is $\{x_j = c\}$ and $\{x_j \pm x_k = c\}$, Lemma 12.13), whereas the killed box is reflection positive at a symmetry plane of each of the nine families of lattice planes $\{x_j = c\}$, $\{x_j \pm x_k = c\}$ (Theorem 12.16). The box limit ω_N^{box} of Theorem 11.11 inherits the reflection positivity, at every lattice plane of the nine families (Lemma 12.19), the continuum passage of Sections 2–7 and 9 runs on the box family verbatim, and a general theorem on random fields with the Gaussian bound, translation invariance and reflection positivity in the nine planes (Theorem 12.31, proved for all n in Section 15, its case $n \leq 2$ also in Section 12.10) gives the rotation covariance of the Schwinger functions away from coincident points. The closedness of the limit curvature (Theorem 12.26) and an elementary theorem on bounded multilinear forms (Theorem 12.34) remove the coincident part.

For a bounded density function the coincident points of the density entries are treated in Lemma 12.37 and Corollary 12.38. For the limits built on the torus limit points the assembly is the same (Theorem 12.4): the reflection positivity at the six diagonal planes through the origin, which the reflection argument on the cubic torus does not supply, is the subject of Section 14 and rests on the rigidity theorem of Section 13.

Structure of the proof. Theorem 12.31 applied to the measure μ of Lemma 12.22 (of which μ^{box}, read on $\mathcal{S}'$, is the curvature marginal), whose hypotheses are supplied by Lemma 12.22 (the Gaussian bound), Proposition 9.6 for the box family together with Lemma 12.12 (translations) and Lemma 12.21 (reflection positivity in the nine plane families), gives the covariance away from coincidences (Theorem 12.3). Theorem 12.34 with Lemma 12.22 (moments bounded on L^2) and Theorem 12.26 (closed field) removes the coincident-point part (Proposition 12.36). The uniqueness from moments of Theorem 9.13 gives the invariance of Z^{box}; Proposition 12.11 and Remark 9.16 give the rest.

Notation for the planes. We write $\varepsilon = \varepsilon_N$. The nine families of lattice planes are $\{\phi = c\}$ with $\phi(x) = x_j$ ($j \in \{1,2,3\}$, $c \in \frac{1}{2}\varepsilon\mathbb{Z}$: site planes for $c \in \varepsilon\mathbb{Z}$, bond mid-planes otherwise) and $\phi(x) = x_i + \theta x_j$ ($i < j$, $\theta \in \{\pm 1\}$, $c \in \varepsilon\mathbb{Z}$: the diagonal site planes). For such a plane P, R_P is the orthogonal reflection ($R_P x = x - 2(x_j - c)e_j$, resp. $R_P x = x - (x_i + \theta x_j - c)(e_i + \theta e_j)$); it maps $\varepsilon\mathbb{Z}^3$ onto itself and lies in the group $\mathcal{G}_N$ of Section 11.1 (a signed permutation of the coordinates about a site composed with a lattice translation). The closed side is $P^+ := \{\phi \geq c\}$; a cell (site, bond, plaquette, cube) lies in P^+ if all its vertices do. Through the origin the nine planes are $\{x_j = 0\}$ and $\{x_j \pm x_k = 0\}$, and the corresponding reflections are the signed permutation matrices $1 - 2e_j e_j^{\mathsf{T}}$ and the matrices exchanging e_j and $\mp e_k$. A box is a product of three intervals of sites of $\varepsilon\mathbb{Z}^3$ (Definition 11.9); a *cube of a box* Λ is a unit cube of $\varepsilon\mathbb{Z}^3$ with all eight vertices in Λ (then its twelve bonds lie in $\mathcal{B}(\Lambda)$ and its six faces in $\mathcal{P}(\Lambda)$), and the closedness constraint of Definition 11.9 is imposed on these cubes. A plane P is a *symmetry plane* of Λ if $R_P\Lambda = \Lambda$; for the cube of sites $\Lambda = \alpha + \varepsilon\{0,\dots,M\}^3$, $\alpha \in \varepsilon\mathbb{Z}^3$, the symmetry planes among

the nine families are the three coordinate planes $\{x_j = \alpha_j + M\varepsilon/2\}$ and the six diagonal planes $\{x_i - x_j = \alpha_i - \alpha_j\}$, $\{x_i + x_j = \alpha_i + \alpha_j + M\varepsilon\}$, $i < j$.

12.2. **Preliminaries: lattices on tori, the transport of test pairs, Gaussian domination, Euclidean invariance.** This section collects what the proofs below take from the lattice models and from the limit functional: the Villain model of a lattice on a torus and the transport of test pairs (Definitions 12.6, 12.7, Lemma 12.8); the choice $g_{\mathrm{d}} \equiv 0$ (Remark 12.9); Gaussian domination on every lattice, torus and killed box (Lemma 12.10); Euclidean invariance from invariance under O(3), for the functional and for its measure (Proposition 12.11, Lemma 12.12). They are proved from the statements of Sections 2–11 alone; Proposition 12.11 carries the invariance of S under O(3) as its hypothesis. Notation: for every level N, $\Lambda_N := \varepsilon_N\mathbb{Z}^3$ is the fine lattice, and σ_ν is the reflection $x_\nu \mapsto -x_\nu$ ($\nu = 1, 2, 3$).

Definition 12.6 (lattices, tori and the model)**.** For $O \in \mathrm{O}(3)$ the *lattice* $\Lambda := O\Lambda_N$ carries the *frame* $f_\alpha := Oe_\alpha$, $\alpha = 1, 2, 3$. Its cells are the sites $x \in \Lambda$; the bonds $(x;\alpha)$ from x to $x + \varepsilon_N f_\alpha$, with the reversed bond $-(x;\alpha)$ from $x + \varepsilon_N f_\alpha$ to x; the plaquettes $p = (x;\alpha,\beta)$, $\alpha < \beta$, the squares with corners x, $x + \varepsilon_N f_\alpha$, $x + \varepsilon_N f_\beta$, $x + \varepsilon_N(f_\alpha + f_\beta)$, with centre $c(p) := x + \frac{\varepsilon_N}{2}(f_\alpha + f_\beta)$, oriented boundary $\partial p = (x;\alpha) + (x + \varepsilon_N f_\alpha;\beta) - (x + \varepsilon_N f_\beta;\alpha) - (x;\beta)$ and reversed plaquette $-p$; and the cubes $x + \varepsilon_N[0,1]^3$ in the frame, with the usual oriented boundary (six plaquettes with signs), so that $\partial\partial = 0$. A *period lattice* for Λ is a lattice $P \subset \Lambda$ of rank 3 whose non-zero vectors have length at least $5\varepsilon_N$; $P \subset \Lambda$ is the same as $\Lambda + p = \Lambda$ for $p \in P$, and translation by P permutes the cells of Λ, preserving the frame indices, the orientations and the boundary operators. The quotient Λ/P is a finite cubical complex (two cells of one cube are never identified: a translate of a cell by $q \in P \setminus \{0\}$ lies at distance $|q| \geq 5\varepsilon_N > \sqrt{3}\varepsilon_N$ from it), and $T := \mathbb{R}^3/P$ is its torus.

The *Villain model of* Λ *on* T is the model of Section 2 (notation of Remark 2.1) on the cell complex Λ/P: the variables $\tilde\rho_x \in [-\rho_{0,0}, \infty)$ at the sites, $B_b \in (-\pi/\gamma_N, \pi/\gamma_N]$ on the bonds with $B_{-b} = -B_b$, and $v_p \in (2\pi/\gamma_N)\mathbb{Z}$ on the plaquettes with $v_{-p} = -v_p$, of Λ/P; $(dB)_p := \sum_{b\in\partial p} B_b$ and $(dv)_c := \sum_{p\in\partial c} v_p$ with the signs of the boundaries; v is *closed* if $(dv)_c = 0$ for every cube. The weight is

$$W = \prod_x g_{\mathrm{ct}}(\tilde\rho_x) \cdot \prod_b h(\tilde\rho_{b_-}, \tilde\rho_{b_+}, B_b) \cdot \prod_p w\big((dB)_p + v_p\big)$$

with the same factors g_{ct}, h, w and coefficients $\gamma_N, \lambda_0, \rho_{0,0}, \kappa_N, \delta m_B$ as in Section 2, the products over the sites, the unoriented bonds and the unoriented plaquettes of Λ/P, and the sum over the Villain variables runs over all closed v. Only the cell complex differs from Section 2. $\langle\cdot\rangle_{\Lambda,T}$ denotes the normalised expectation and $\mathfrak{f}_p := (dB)_p + v_p$ the curvature. For $\Lambda = \Lambda_N$ and $P = L\mathbb{Z}^3$, $L \in \varepsilon_N\mathbb{Z}$, this is the fine torus state $\langle\cdot\rangle_L^{\mathrm{fine}}$ of the model of Section 2 (on $\tilde T_L = \Lambda_N/L\mathbb{Z}^3$).

The weight is well defined on unoriented cells: $h(r, r', \vartheta) = h(r', r, -\vartheta)$, so the factor of a bond does not depend on its orientation; w is even, so neither does the factor of a plaquette; and the closedness of v does not depend on the orientations of the plaquettes and cubes. Consequently the model depends only on the point set Λ and on P, not on the frame: replacing (f_α) by $(\pm f_{\pi(\alpha)})$, π a permutation, relabels the cells and reverses some orientations, which changes the signs of the variables on the reversed bonds and plaquettes and nothing else.

Definition 12.7 (sampling of a test pair on a lattice; transport)**.** For a test pair $\sigma = (f, h)$ let $\rho(\sigma)$ be the smallest $\rho \geq 0$ with $\operatorname{supp} f \cup \operatorname{supp} h \subset \bar B_\rho$, the closed Euclidean ball of radius ρ about 0. Let $\Lambda = O\Lambda_N$ with frame (f_α) and let P be a period lattice for Λ whose non-zero

vectors have length $> 2\rho(\sigma)$. The *sampling of* σ *on* Λ, read on $T = \mathbb{R}^3/P$, is

$$\Phi_N^{\Lambda,T}(\sigma) := \varepsilon_N^{3/2} \sum_{p=(x;\alpha,\beta)} f_{c(p)}(f_\alpha, f_\beta)\, \mathfrak{f}_{\bar p} + \varepsilon_N^3 \sum_{z\in\Lambda} h(z)\, g_{\mathrm{d}}(\tilde\rho_{\bar z}), \tag{12.1}$$

where $f_c(u,v)$ is the 2-form f at the point c evaluated on the vectors u, v, the first sum runs over the plaquettes of Λ in $\mathbb{R}^3$ with $\alpha < \beta$, the second over the sites of Λ, and $\bar p, \bar z$ are the images in Λ/P; finitely many terms are non-zero. For the fine lattice Λ_N itself, $f_{c(p)}(e_\mu, e_\nu) = f_{\mu\nu}(c(p))$ and (12.1) is $\Phi_N(\sigma)$ of Definition 2.5 with $x_p = c(p)$. For a rotated lattice $R^{-1}\Lambda_N$, $R \in \mathrm{O}(3)$, the 2-form is sampled as

$$f_{c(p)}\big(R^{-1}e_\alpha, R^{-1}e_\beta\big) \qquad \text{at the centres } c(p) = R^{-1}\big(y + \tfrac{\varepsilon_N}{2}(e_\alpha + e_\beta)\big),\ y \in \Lambda_N,$$

of its own plaquettes, and the density at its own sites $R^{-1}y$.

The *transport* of a test pair by $O \in \mathrm{O}(3)$ is the push-forward

$$(O\cdot f)_x(u,v) := f_{O^{-1}x}(O^{-1}u, O^{-1}v), \qquad (O\cdot h)(x) := h(O^{-1}x), \qquad O\cdot\sigma := (O\cdot f, O\cdot h);$$

$(O,\sigma) \mapsto O\cdot\sigma$ is an action of O(3) by linear maps of $\mathcal{T}$, with $\operatorname{supp}(O\cdot\sigma) = O\operatorname{supp}\sigma$ and $\rho(O\cdot\sigma) = \rho(\sigma)$; for $O = \sigma_3$ it is the time reflection θ of Section 2, and for O in the cubic group about 0 it is the push-forward action of Lemma 5.1. For $O \in \mathrm{O}(3)$ let γ_O be the $*$-automorphism of $\mathcal{A}$ with $\gamma_O e_\sigma := e_{O\cdot\sigma}$ (well defined because $O\cdot$ is additive on $\mathcal{T}$).

The sampling (12.1) has three properties. (i) It does not depend on the frame: for the square with corner x spanned by $u = f_\alpha$, $v = f_\beta$, the term $f_c(u,v)\,\mathfrak{f}_{(x;u,v)}$ is unchanged if the same square is described as $(x; v, u)$ (both factors change sign, f being alternating and $\mathfrak{f}$ odd under reversal) or as $(x + \varepsilon_N u; -u, v)$ (again both change sign, the boundary being reversed); so (12.1) is a function of the point set Λ, of the 2-form and of the plaquette variables alone. (ii) Since the non-zero vectors of P are longer than $2\rho(\sigma)$ and the sampled cells have their centres in $\bar B_{\rho(\sigma)}$, distinct sampled cells have distinct images in Λ/P (two cells differing by $q \in P\setminus\{0\}$ have centres at distance $|q| > 2\rho(\sigma)$): $e^{\mathrm{i}\Phi_N^{\Lambda,T}(\sigma)}$ is the bounded local gauge-invariant observable $e^{\mathrm{i}\Phi_N^{\Lambda}(\sigma)}$ of the infinite lattice Λ, read on the torus. (iii) It is real and additive in σ, so (2.4) holds for every lattice and torus.

Lemma 12.8 (passive covariance)**.** *Let $O \in \mathrm{O}(3)$, let $\Lambda = O'\Lambda_N$ be a lattice with frame (f_α), P a period lattice for Λ and $T = \mathbb{R}^3/P$. Then $O\Lambda$ is the lattice with frame (Of_α), OP is a period lattice for it, and O induces an isomorphism of cubical complexes $\Lambda/P \to O\Lambda/OP$. The induced bijection of configurations,*

$$\tilde\rho'_{Ox} := \tilde\rho_x, \qquad B'_{Ob} := B_b, \qquad v'_{Op} := v_p,$$

maps closed v onto closed v', preserves the weight W, and carries the Villain model of Λ on T onto the Villain model of $O\Lambda$ on $OT := \mathbb{R}^3/OP$: for every bounded Borel function F of finitely many cell variables of Λ/P, writing F^O for the same function of the corresponding variables of $O\Lambda/OP$ (that is, $F^O(\tilde\rho', B', v') = F(\tilde\rho, B, v)$),

$$\langle F^O\rangle_{O\Lambda,OT} = \langle F\rangle_{\Lambda,T}. \tag{12.2}$$

For test pairs $\sigma_1, \dots, \sigma_k$ with $2\rho(\sigma_j)$ smaller than the non-zero vectors of P, the sampled fields correspond, $\Phi_N^{O\Lambda,OT}(O\cdot\sigma_j)(\tilde\rho', B', v') = \Phi_N^{\Lambda,T}(\sigma_j)(\tilde\rho, B, v)$, so that for every bounded Borel G on $\mathbb{R}^k$

$$\Big\langle G\big(\Phi_N^{O\Lambda,OT}(O\cdot\sigma_1), \dots, \Phi_N^{O\Lambda,OT}(O\cdot\sigma_k)\big)\Big\rangle_{O\Lambda,OT} = \Big\langle G\big(\Phi_N^{\Lambda,T}(\sigma_1), \dots, \Phi_N^{\Lambda,T}(\sigma_k)\big)\Big\rangle_{\Lambda,T}. \tag{12.3}$$

Proof. O is an isometry of $\mathbb{R}^3$ with $O\Lambda = OO'\Lambda_N$, $O(x+\varepsilon_N f_\alpha) = Ox + \varepsilon_N O f_\alpha$ and $O\big(x + \frac{\varepsilon_N}{2}(f_\alpha + f_\beta)\big) = Ox + \frac{\varepsilon_N}{2}(Of_\alpha + Of_\beta)$: it maps the site x, the bond $(x;\alpha)$, the plaquette $(x;\alpha,\beta)$ and the cube at x of Λ to the cells with the same labels of $O\Lambda$ in the frame (Of_α), and $c(Op) = Oc(p)$. The boundary operators are defined by the labels, so O commutes with ∂, signs included. $OP \subset O\Lambda$ is a lattice of rank 3 whose non-zero vectors have the same lengths as those of P, and $O(x+p) = Ox + Op$, so O descends to a bijection of the cells of Λ/P onto those of $O\Lambda/OP$ commuting with ∂. Hence the map of configurations is a bijection which preserves the product Lebesgue measure in $(\tilde\rho, B)$ and the counting measure in v, satisfies $(dB')_{Op} = (dB)_p$ and $(dv')_{Oc} = (dv)_c$, so maps closed v onto closed v', and satisfies $W(\tilde\rho', B', v') = W(\tilde\rho, B, v)$, because W is the product of the same position-independent factors over corresponding cells. Therefore the normalisations agree and (12.2) holds. For the sources, $\mathfrak{f}'_{Op} = \mathfrak{f}_p$, $(O\cdot f)_{c(Op)}(Of_\alpha, Of_\beta) = f_{O^{-1}Oc(p)}(f_\alpha, f_\beta) = f_{c(p)}(f_\alpha, f_\beta)$ and $(O\cdot h)(Oz) = h(z)$; summing over the plaquettes and sites of $O\Lambda$, which are the images of those of Λ, gives the correspondence of the sampled fields, and (12.3) is (12.2) for $F = G(\Phi_N^{\Lambda,T}(\sigma_1), \dots)$. The support condition is the same on both sides since $\rho(O\cdot\sigma_j) = \rho(\sigma_j)$. □

Remark 12.9 (the density source and the choice $g_{\mathrm{d}} \equiv 0$). Definition 2.5 allows every bounded Borel g_{d}, in particular $g_{\mathrm{d}} \equiv 0$. With that choice $\Phi_N(\sigma) = \varepsilon_N^{3/2}\sum_p f_{\mu\nu(p)}(x_p)\mathfrak{f}_p$ depends on f alone: the density terms disappear from (12.1) and from the seminorms ($\|\sigma\| = R(f)$); the limit theory is that of the smeared curvature, and Theorem 7.1 applies to this scheme as to any other.

Lemma 12.10 (Gaussian domination on every lattice and torus)**.** *Let $\Lambda = O\Lambda_N$ be a lattice, P a period lattice for it and $T = \mathbb{R}^3/P$. For every real function $a = (a_p)$ on the plaquettes of Λ/P,*

$$\Big\langle \exp\Big(\sum_p a_p \mathfrak{f}_p\Big)\Big\rangle_{\Lambda,T} \le \exp\Big(\tfrac12\sum_p a_p^2\Big).$$

The same inequality holds for the killed-box states $\langle\cdot\rangle_{\Lambda'}$ of Definition 11.9, Λ' a finite box of Λ_N, for every real a on the plaquettes of the box, and more generally for the normalised expectation of every weight that is a product of the factors g_{ct}, h (or, for a bond with one end outside, the diagonal half $e^{-\kappa_N\rho^2/2}$ at its inner end) and w over the sites, bonds and plaquettes of a finite cubical complex with $\partial\partial = 0$, the sum over the v closed on its cubes. Let σ be a test pair with $2\rho(\sigma)$ smaller than the non-zero vectors of P, and define the seminorms of Λ by $R_\Lambda(f)^2 := \sup_{N'\ge N_0}\varepsilon_{N'}^3\sum_p f_{c(p)}(f_\alpha,f_\beta)^2$ (over the plaquettes of the lattice $O\Lambda_{N'}$), $E_\Lambda(h) := \sup_{N'\ge N_0}\varepsilon_{N'}^3\sum_{z\in O\Lambda_{N'}}|h(z)|$, $\|\sigma\|_\Lambda := R_\Lambda(f) + \|g_{\mathrm{d}}\|_\infty E_\Lambda(h)$. Consequently, for all real s and all N,

$$\big\langle e^{s\Phi_N^{\Lambda,T}(\sigma)}\big\rangle_{\Lambda,T} \le e^{|s|\,\|g_{\mathrm{d}}\|_\infty E_\Lambda(h)}\, e^{s^2 R_\Lambda(f)^2/2}, \qquad \big\langle \Phi_N^{\Lambda,T}(\sigma)^2\big\rangle_{\Lambda,T}^{1/2} \le \|\sigma\|_\Lambda, \tag{12.4}$$

and $|\langle e^{\mathrm{i}\Phi_N^{\Lambda,T}(\sigma)}\rangle_{\Lambda,T} - \langle e^{\mathrm{i}\Phi_N^{\Lambda,T}(\sigma')}\rangle_{\Lambda,T}| \le \|\sigma-\sigma'\|_\Lambda$. The seminorms obey $R_\Lambda(f) \le \sqrt{3|K_{+a_0}|}\,\|f\|_\infty$ and $E_\Lambda(h) \le |K_{+a_0}|\,\|h\|_\infty$ for a compact K containing the supports; here $\|f\|_\infty$ is the supremum over $\mathbb{R}^3$ of the Euclidean norm $(\sum_{\mu<\nu} f_{\mu\nu}^2)^{1/2}$, so that $|f_c(u,v)| \le \|f\|_\infty$ for orthonormal u, v.

Proof. Steps 1–4 of the proof of Theorem 3.1 use only that the weight is a product over the sites, bonds and plaquettes of a finite cubical complex with $\partial\partial = 0$ (whence $d^*d^* = 0$ in Step 2), the Fourier expansion of the hopping factor bond by bond (Step 3) and the sum over closed v; nothing refers to the frame or to the period lattice. They apply word for word to the complex Λ/P and give the first inequality. They apply equally to the killed boxes of Definition 11.9 and to the general weights described in the statement: the diagonal halves $e^{-\kappa_N\rho_x^2/2}$ attached to the sites do not depend on the gauge variables, so the Fourier expansion of Step 3 is applied bond by bond to the remaining hopping factors, and Steps 1, 2 and 4 use only the complex and the

plaquette weight w. The cells of a box Λ' form such a complex: every cube of $\mathcal{C}(\Lambda')$ has its six faces in $\mathcal{P}(\Lambda')$ and every plaquette of $\mathcal{P}(\Lambda')$ its four bonds in $\mathcal{B}(\Lambda')$, Definition 10.1; and every bond of $\mathcal{B}(\Lambda')$ carries the hopping factor with both matter variables.

The exponential moment bound follows as in the proof of Proposition 4.2: $\Phi_N^{\Lambda,T}(\sigma) = X + Y$ with $X = \sum_p a_p \mathfrak{f}_p$, $a_p = \varepsilon_N^{3/2} f_{c(p)}(f_\alpha, f_\beta)$, $|Y| \le \|g_{\mathrm{d}}\|_\infty E_\Lambda(h)$ and $\sum_p a_p^2 \le R_\Lambda(f)^2$; the second-moment bound is Step 5 of the proof of Theorem 3.1 with Minkowski's inequality, and the Lipschitz bound is (6.1) with the Schwarz inequality, $|e^{\mathrm{i}x} - e^{\mathrm{i}y}| \le |x - y|$. The bounds on the seminorms are those following Definition 4.1: the open cubes of side $\varepsilon_{N'}$ centred at the sites of the lattice that lie in K are pairwise disjoint and lie in $K_{+\sqrt{3}\varepsilon_{N'}/2} \subset K_{+a_0}$ (as $\varepsilon_{N'} \le a_0$), so there are at most $|K_{+a_0}|/\varepsilon_{N'}^3$ such sites; the centres of the plaquettes of one orientation form a translate of the lattice, so at most three times as many plaquettes have their centres in K. □

Proposition 12.11 (Euclidean invariance from invariance under O(3); invariance of the measure)**.** *Let S be a limit of Theorem 7.1 (for the torus family, or for any family of fine states satisfying its hypotheses), with $Z(\sigma) = S(e_\sigma)$, and assume that S is invariant under the orthogonal group, $S \circ \gamma_R = S$ for all $R \in \mathrm{O}(3)$.*

(a) *Let $E(3) = \mathbb{R}^3 \rtimes \mathrm{O}(3)$ be the Euclidean group, $\gamma = (b, R)$ acting on $\mathbb{R}^3$ by $\gamma x = Rx + b$ and on test pairs by the push-forward $\gamma \cdot \sigma := \tau_b(R \cdot \sigma)$, τ_b the translation by b. Then $S(e_{\gamma\cdot\sigma}) = S(e_\sigma)$ for every $\gamma \in E(3)$ and every test pair σ.*

(b) *Let X be a set of real linear functionals Φ on $\mathcal{T}$, stable under $(\gamma\Phi)(\sigma) := \Phi(\gamma^{-1} \cdot \sigma)$ for $\gamma \in E(3)$, with the σ-algebra generated by the evaluations $\Phi \mapsto \Phi(\sigma)$, and let μ be a probability measure on X with $S(e_\sigma) = \int_X e^{\mathrm{i}\Phi(\sigma)}\, d\mu(\Phi)$ for every σ. Then $\gamma_*\mu = \mu$ for every $\gamma \in E(3)$; in particular μ is $\mathrm{O}(3)$-invariant.*

Proof. (a) $R \cdot (\tau_{b'}\sigma) = \tau_{Rb'}(R \cdot \sigma)$ (substitute $R^{-1}x - b' = R^{-1}(x - Rb')$), so $\gamma \cdot (\gamma' \cdot \sigma) = \tau_b\tau_{Rb'}((RR') \cdot \sigma) = (\gamma\gamma') \cdot \sigma$ with $\gamma\gamma' = (b + Rb', RR')$: this is an action of $E(3)$, and $E(3)$ is generated by the translations and $\mathrm{O}(3)$. S is invariant under all translations by Theorem 7.1 (the time translations by its clause (b), the space translations by its last clause), and under $\mathrm{O}(3)$ by hypothesis; so it is invariant under $E(3)$. (b) $\Phi \mapsto \gamma\Phi$ is measurable, since $(\gamma\Phi)(\sigma)$ is the evaluation at $\gamma^{-1} \cdot \sigma$. For $\sigma_1, \dots, \sigma_k \in \mathcal{T}$ and $s \in \mathbb{R}^k$, by the linearity of Φ and (a),

$$\begin{aligned}\int_X e^{\mathrm{i}\sum_j s_j(\gamma\Phi)(\sigma_j)}\, d\mu(\Phi) &= \int_X e^{\mathrm{i}\Phi(\gamma^{-1}\cdot\sum_j s_j\sigma_j)}\, d\mu(\Phi)\\ &= S\big(e_{\gamma^{-1}\cdot\sum_j s_j\sigma_j}\big) = S\big(e_{\sum_j s_j\sigma_j}\big) = \int_X e^{\mathrm{i}\sum_j s_j\Phi(\sigma_j)}\, d\mu(\Phi).\end{aligned}$$

So the laws of $(\Phi(\sigma_j))_{j\le k}$ under $\gamma_*\mu$ and under μ have the same Fourier transform on $\mathbb{R}^k$ and coincide; the cylinder sets $\{\Phi : (\Phi(\sigma_j))_{j\le k} \in B\}$ form a π-system generating the σ-algebra, on which the two probability measures agree; hence $\gamma_*\mu = \mu$. □

Every limit S^{box} of the killed-box family is invariant under the orthogonal group, by Theorems 12.2 and 12.3, and so is every limit S built on the torus limit points of Definition 2.3 along dyadic sequences of sides, by Theorem 12.4.

Nothing above needs μ to exist; it does, by the following lemma.

Lemma 12.12 (existence of the measure)**.** *Let S be a limit of Theorem 7.1 (for the torus family, or for any family of fine states satisfying its hypotheses), with $Z(\sigma) = S(e_\sigma)$. There is a unique probability measure on the dual $\mathcal{D}'(\mathbb{R}^3; \Lambda^2\mathbb{R}^3 \oplus \mathbb{R})$, with the cylinder σ-algebra, whose characteristic functional is Z; $E(3)$ acts on that dual by the transposed action, and Proposition 12.11(b) applies to this measure when S is invariant under $\mathrm{O}(3)$. Moreover, a probability measure on any*

such space of linear functionals, with the cylinder σ-algebra, is determined by its characteristic functional.

Proof. Z is the pointwise limit of the characteristic functionals $\sigma \mapsto Z_{N_m}(\sigma) = \omega_{N_m}(e^{\mathrm{i}\Phi_{N_m}(\sigma)})$ of the real random variables $\Phi_{N_m}(\sigma)$, which are additive in σ, so Z is positive definite on the additive group $\mathcal{T}$ with $Z(0) = 1$; and Z is continuous on each Fréchet space $\mathcal{T}_K$ of test pairs supported in a compact K, by (6.1) and the admissibility of the seminorm. The Bochner–Minlos theorem, in Minlos' form on $\mathcal{D}'$ ([27], [15, Chapter IV]), then gives the measure and its uniqueness. The argument of Proposition 12.11(b) shows that a probability measure on any such space of linear functionals, with the cylinder σ-algebra, is determined by its characteristic functional. □

12.3. **No torus carries the nine families.**

Lemma 12.13 (reflection walls of a torus)**.** *Let $\Gamma \subset \mathbb{R}^3$ be a lattice of rank 3, n a unit vector, $\theta_n x := x - 2(n\cdot x)n$ and $\pi_n x := (n\cdot x)n$.*

(a) *If $\theta_n\Gamma = \Gamma$, the θ_n-invariant walls of $\mathbb{R}^3/\Gamma$ are the images of the planes $\{n\cdot x = c\}$ with $2c \in n\cdot\Gamma$; such a wall is pointwise fixed iff $2cn \in \Gamma$; all of them are pointwise fixed iff $\pi_n(\Gamma) \subset \Gamma$, i.e. iff $\pi_n(\Gamma) = \Gamma \cap \mathbb{R}n$ (and this inclusion implies $\theta_n\Gamma = \Gamma$).*

(b) *No lattice satisfies $\pi_{e_j}(\Gamma) \subset \Gamma$ and $\pi_n(\Gamma) \subset \Gamma$ with $n = (e_j \pm e_k)/\sqrt{2}$, $k \neq j$.*

Hence no torus $\mathbb{R}^3/\Gamma$ carries a coordinate family and a diagonal family with the same coordinate j (the families $\{x_j = c\}$ and $\{x_j \pm x_k = c\}$, $k \neq j$) with pointwise-fixed walls, a fortiori none carries the nine families. (The cubic lattice $\mathbb{Z}^3$ passes the coordinate test only, the lattice $\mathbb{Z}(e_1 - e_2) + \mathbb{Z}(e_1 + e_2) + \mathbb{Z}e_3$ the test for $(e_1 - e_2)/\sqrt{2}$ but not the one for e_1.)

Proof. (a) The image of $\{n\cdot x = c\}$ in $\mathbb{R}^3/\Gamma$ is the set $\{x : n\cdot x \in c + n\cdot\Gamma\}$; θ_n maps it to $\{n\cdot x \in -c + n\cdot\Gamma\}$, which is the same set iff $2c \in n\cdot\Gamma$. A point x of it is fixed iff $\theta_n x - x = -2(n\cdot x)n \in \Gamma$. Write $n\cdot x = c + n\cdot\gamma$ with $\gamma \in \Gamma$; since $2(n\cdot\gamma)n = \gamma - \theta_n\gamma \in \Gamma$ (here $\theta_n\Gamma = \Gamma$ enters), x is fixed iff $2cn \in \Gamma$, independently of x on the wall. If $\pi_n(\Gamma) \subset \Gamma$ and $2c = n\cdot\gamma$, then $2cn = \pi_n\gamma \in \Gamma$; conversely every $\gamma \in \Gamma$ gives the admissible $c = \frac{1}{2}n\cdot\gamma$ with $\pi_n\gamma = 2cn$, so if all walls are fixed then $\pi_n(\Gamma) \subset \Gamma$. Always $\Gamma \cap \mathbb{R}n \subset \pi_n(\Gamma)$, so $\pi_n(\Gamma) \subset \Gamma$ is the same as $\pi_n(\Gamma) = \Gamma \cap \mathbb{R}n$; and then $\theta_n\gamma = \gamma - 2\pi_n\gamma \in \Gamma$.

(b) Take $j = 1$ and $n = (e_1 - e_2)/\sqrt{2}$; the other cases follow by a signed permutation of the coordinates. Since Γ has rank 3, $\pi_{e_1}(\Gamma) \neq 0$ and $\pi_n(\Gamma) \neq 0$, so under the two hypotheses $\Gamma \cap \mathbb{R}e_1 = p\mathbb{Z}e_1$ and $\Gamma \cap \mathbb{R}n = s\mathbb{Z}(e_1 - e_2)$ with $p, s > 0$, and the hypotheses read: $\gamma_1 \in p\mathbb{Z}$ and $\frac{1}{2}(\gamma_1 - \gamma_2) \in s\mathbb{Z}$ for every $\gamma \in \Gamma$. For $\gamma = s(e_1 - e_2)$ the first gives $s \in p\mathbb{Z}$, $s = kp$; for $\gamma = pe_1$ the second gives $p \in 2s\mathbb{Z}$, $p = 2ls$. Then $p = 2klp$, i.e. $2kl = 1$ with integers $k, l \geq 1$: impossible. The families $\{x_1 = c\}$ and $\{x_1 - x_2 = c\}$ are among the nine, which proves the last sentence; the two examples are checked by the same computation ($\mathbb{Z}^3$: $\pi_{e_1}\mathbb{Z}^3 = \mathbb{Z}e_1 \subset \mathbb{Z}^3$, but $\pi_n(e_1) = \frac{1}{2}(e_1 - e_2) \notin \mathbb{Z}^3$; the second lattice: π_n of its generators lies in $\mathbb{Z}(e_1 - e_2)$, but $\pi_{e_1}(e_1 - e_2) = e_1$ is not in it). □

12.4. **Reflection positivity of the killed-box state at its nine symmetry planes.** *Setting.* Fix N, $\varepsilon := \varepsilon_N$ (the spacing), γ_N the phase constant, κ_N the hopping coefficient (the unindexed symbol κ denotes a cell below), and a cube of sites $\Lambda = \alpha + \varepsilon\{0, \dots, M\}^3$, $\alpha \in \varepsilon\mathbb{Z}^3$, $M \geq 1$; $\mathcal{B}(\Lambda)$, $\mathcal{P}(\Lambda)$ are as in Definition 11.9 and $\mathcal{C}(\Lambda)$ is the set of cubes of Λ. Cells are oriented, with incidence numbers $[\kappa : \sigma] \in \{0, \pm 1\}$, $\partial\kappa = \sum_\sigma [\kappa : \sigma]\sigma$, $\partial\partial = 0$; the variables $\xi = (\tilde\rho_x, B_b, v_p)$ obey $u_{-\sigma} = -u_\sigma$, $(dB)_p = \sum_b [p : b]B_b$, $(dv)_c = \sum_p [c : p]v_p$, $\mathfrak{f}_p = (dB)_p + v_p$. With $\widetilde{g}(r) := g_{\mathrm{ct}}(r)e^{-3\kappa_N\rho^2}$ ($\rho = \rho_{0,0} + r$) and $\widetilde{h}(\rho, \rho', B) := e^{\kappa_N\rho\rho'\cos(\gamma_N B)}$, the weight of Definition 11.9 is

$$W_\Lambda = \prod_{x\in\Lambda} \widetilde{g}(\tilde\rho_x) \prod_{b\in\mathcal{B}(\Lambda)} \widetilde{h}(\rho_{b_-}, \rho_{b_+}, B_b) \prod_{p\in\mathcal{P}(\Lambda)} w(\mathfrak{f}_p) \prod_{c\in\mathcal{C}(\Lambda)} \mathbf{1}[(dv)_c = 0],$$

against the reference measure $m_\Lambda = \prod_x d\tilde\rho_x \prod_b dB_b \prod_p$(counting measure on $(2\pi/\gamma_N)\mathbb{Z}$), $B_b \in (-\pi/\gamma_N, \pi/\gamma_N]$; $Z_\Lambda := \int W_\Lambda\, dm_\Lambda \in (0,\infty)$ (Step 4 below) and $\langle F\rangle_\Lambda := Z_\Lambda^{-1}\int F W_\Lambda\, dm_\Lambda$.

Let Π be a symmetry plane of Λ, φ the affine function $x_j - \alpha_j - M\varepsilon/2$, resp. $x_i \mp x_j - c_{ij}^{\mp}$ with $c_{ij}^- = \alpha_i - \alpha_j$, $c_{ij}^+ = \alpha_i + \alpha_j + M\varepsilon$ (we write Π_{ij}^-, Π_{ij}^+ for these two diagonal planes), so that $\Pi = \{\varphi = 0\}$; $n := \nabla\varphi$, $R = R_\Pi\colon x \mapsto x - 2\varphi(x)n/|n|^2$ the reflection, $H_\pm := \{\pm\varphi \ge 0\}$. R maps $\varepsilon\mathbb{Z}^3$ onto itself ($2\varphi/|n|^2 \in \varepsilon\mathbb{Z}$ on sites, as $c_{ij}^\pm \in \varepsilon\mathbb{Z}$), maps Λ onto Λ ($\alpha_j + \varepsilon m \mapsto \alpha_j + \varepsilon(M-m)$; $(\alpha_i + \varepsilon m_i, \alpha_j + \varepsilon m_j) \mapsto (\alpha_i + \varepsilon m_j, \alpha_j + \varepsilon m_i)$, resp. $(\alpha_i + \varepsilon(M-m_j), \alpha_j + \varepsilon(M-m_i))$), hence maps $\mathcal{B}(\Lambda)$, $\mathcal{P}(\Lambda)$, $\mathcal{C}(\Lambda)$ onto themselves, and commutes with ∂: $[R\kappa : R\sigma] = [\kappa : \sigma]$. It acts on configurations by $(R\xi)_\kappa := \xi_{R\kappa}$, so that $(dB)_p \circ R = (dB)_{Rp}$, $\mathfrak{f}_p \circ R = \mathfrak{f}_{Rp}$, $(dv)_c \circ R = (dv)_{Rc}$; it maps the configuration space $(-\pi/\gamma_N, \pi/\gamma_N]^{\mathcal{B}(\Lambda)}$ onto itself up to the m_Λ-null set where some $B_b = \pi/\gamma_N$, so $F \circ R$ is defined almost everywhere, which suffices.

A cell κ with vertex set $V(\kappa)$ is a *plane cell* if $V(\kappa) \subset \Pi$, a $(+)$*-cell* if $V(\kappa) \subset H_+$ and it is not plane, a $(-)$*-cell* likewise, and *crossing* otherwise. Write $\xi_0, \xi_+, \xi_-, \xi_\times$ for the variables of the four classes and $\mathcal{B}_\times, \mathcal{P}_\times, \mathcal{C}_\times$ for the crossing bonds, plaquettes, cubes (no site crosses). $\mathfrak{A}_\Lambda^+(\Pi)$ is the algebra of bounded Borel functions of (ξ_0, ξ_+), i.e. of the variables of the cells lying in H_+, and $\Theta_\Pi F := \overline{F \circ R_\Pi}$.

Lemma 12.14 (cells at a symmetry plane)**.** *(a) R fixes every plane cell (orientation included), maps the $(+)$-cells bijectively onto the $(-)$-cells, and maps every crossing cell κ onto itself with reversed orientation, $R\kappa = -\kappa$. (b) No face of a crossing cell is a plane cell; every cell having a crossing face is crossing; the faces of a plane or $(+)$-cell are plane or $(+)$-cells. (c) At a coordinate site plane (M even) there are no crossing cells. At a coordinate bond mid-plane (M odd) there are no plane cells; the crossing cells are the e_j-bonds, the e_je_k-plaquettes and the cubes with vertices on both slices $\varphi = \pm\varepsilon/2$. At a diagonal plane Π the plane cells are the sites of Π and the e_k-bonds ($k \ne i,j$) in Π;* no bond crosses*; the crossing cells are the e_ie_j-plaquettes whose vertices carry the φ-values $\{-\varepsilon, 0, 0, \varepsilon\}$ (bisected by Π along a diagonal) and the cubes having such a face; a crossing plaquette has two bonds in H_+ and two in H_-; a crossing cube has two crossing faces and two side faces in each of $H_\pm$; no plaquette is a plane cell.*

Proof. (c) On the sites φ takes values in $\varepsilon\mathbb{Z}$ (site and diagonal planes) or in $\varepsilon\mathbb{Z} + \varepsilon/2$ (mid-plane), and $|\varphi(x + \varepsilon e_l) - \varphi(x)| \in \{0, \varepsilon\}$, so the φ-range of a cell is ε times the number of its directions e_l with $n_l \ne 0$. A cell crosses iff φ takes both signs on $V(\kappa)$. At a coordinate plane the range is at most ε; with values in $\varepsilon\mathbb{Z}$ no cell crosses; with values in $\varepsilon\mathbb{Z} + \varepsilon/2$ a cell crosses iff it has vertices on both slices, and no cell is plane. At a diagonal plane crossing needs the values ε and $-\varepsilon$, hence range 2ε, hence both directions e_i, e_j; for the e_ie_j-plaquette with corner a the values are $\varphi(a), \varphi(a)+\varepsilon, \varphi(a)-\varepsilon, \varphi(a)$ (Π^-), resp. $\varphi(a), \varphi(a)+\varepsilon, \varphi(a)+\varepsilon, \varphi(a)+2\varepsilon$ (Π^+), and it crosses iff they are $\{-\varepsilon, 0, 0, \varepsilon\}$; a cube crosses iff its e_ie_j-faces do. The bonds of a crossing plaquette and the e_ie_k-, e_je_k-faces of a crossing cube take the values $\{0, \varepsilon\}$ or $\{0, -\varepsilon\}$, two of each kind; a plane cell lies in Π, which contains no square and no bond of direction e_i or e_j.

(a) R fixes Π pointwise, hence every vertex of a plane cell and the oriented cell. A $(\pm)$-cell has a vertex in the open side $\{\pm\varphi > 0\}$, which R carries to the other open side, and $R(H_\pm) = H_\mp$; with $R^2 = 1$ the bijection follows. Let κ cross. At a mid-plane, $V(\kappa) = V' \times \{c \mp \varepsilon/2\}$ and R swaps the two slices, fixing the other coordinates. At a diagonal plane the two vertices off Π are $a + \varepsilon e_i$, $a + \varepsilon e_j$ (Π^-, $\varphi(a) = 0$), resp. a, $a + \varepsilon e_i + \varepsilon e_j$ (Π^+, $\varphi(a) = -\varepsilon$), and $R(x) = x - \varphi(x)(e_i \mp e_j)$ exchanges them; for cubes add $\{0, \varepsilon\}e_k$, fixed by R. So $V(\kappa)$ is R-invariant and $R\kappa = \pm\kappa$. R restricted to the affine span E of κ fixes $E \cap \Pi$ pointwise, a hyperplane of E ($E \not\subset \Pi$, and $E \cap \Pi \ne \emptyset$ since κ meets both sides), and moves a vertex: it is the reflection of E in that hyperplane and reverses the orientation, $R\kappa = -\kappa$.

(b) If a face σ of a crossing κ were a plane cell, then $R\sigma = \sigma$ and $[\kappa : \sigma] = [R\kappa : R\sigma] = [-\kappa : \sigma] = -[\kappa : \sigma]$, so $[\kappa : \sigma] = 0$. A cell with a face having vertices on both open sides has such vertices itself. The vertices of a face are vertices of the cell, which gives the last clause. □

Lemma 12.15 (splitting of the coboundaries at a symmetry plane)**.** *Let $\kappa \in \mathcal{P}_\times \cup \mathcal{C}_\times$ and let u be the variables of its faces (B for a plaquette, v for a cube). Put $U_\times(\kappa) := \sum_{\sigma \text{ crossing}} [\kappa : \sigma] u_\sigma$, $U_+(\kappa) := \sum_{\sigma\ (+)} [\kappa : \sigma] u_\sigma$, $U_-(\kappa) := -\sum_{\sigma\ (-)} [\kappa : \sigma] u_\sigma$. Then $(du)_\kappa = U_\times(\kappa) + U_+(\kappa) - U_-(\kappa)$, and $U_+(\kappa) \circ R = U_-(\kappa)$, $U_-(\kappa) \circ R = U_+(\kappa)$. For $t \in \mathbb{R}^{\mathcal{C}_\times}$, $s_p := \sum_{c \in \mathcal{C}_\times} [c : p] t_c$ ($p \in \mathcal{P}_\times$) and every crossing bond b: $\sum_{p \in \mathcal{P}_\times} [p : b] s_p = 0$. We write $B_\times(p)$, $B_\pm(p)$ for $U_\times(p)$, $U_\pm(p)$ when $u = B$, and $V_\pm(c)$ for $U_\pm(c)$ when $u = v$.*

Proof. The first identity is Lemma 12.14(b): no face is plane, so the faces are crossing, (+) or (−). R maps the (+)-faces of κ bijectively onto its (−)-faces (Lemma 12.14(a); $-\kappa$ has the faces of κ), and $[\kappa : R\sigma] = -[R\kappa : R\sigma] = -[\kappa : \sigma]$; so

$$U_+(\kappa)(R\xi) = \sum_{\sigma(+)} [\kappa : \sigma] \xi_{R\sigma} = -\sum_{\sigma(+)} [\kappa : R\sigma] u_{R\sigma}(\xi) = -\sum_{\sigma'(-)} [\kappa : \sigma'] u_{\sigma'}(\xi) = U_-(\kappa)(\xi);$$

apply R once more for the other identity. Finally $\sum_{p \in \mathcal{P}_\times} [p : b] s_p = \sum_c t_c \sum_p [c : p][p : b] = 0$: every plaquette of Λ containing the crossing bond b is crossing (Lemma 12.14(b)), every face of a cube of Λ is a plaquette of Λ, and $\partial\partial = 0$. □

Theorem 12.16 (reflection positivity of the killed-box state at the nine planes)**.** *Let Π be a symmetry plane of the cube Λ (a coordinate plane or a diagonal plane). For all $F, G \in \mathfrak{A}^+_\Lambda(\Pi)$,*

$$\langle F \circ R_\Pi \rangle_\Lambda = \langle F \rangle_\Lambda, \qquad \langle \Theta_\Pi F \cdot G \rangle_\Lambda = \overline{\langle \Theta_\Pi G \cdot F \rangle_\Lambda}, \qquad \langle \Theta_\Pi F \cdot F \rangle_\Lambda \geq 0,$$

and the Hermitian identity and the positivity extend to all (ξ_0, ξ_+)-measurable F, G with $\langle |F|^2 \rangle_\Lambda, \langle |G|^2 \rangle_\Lambda < \infty$.

Proof. Step 0 (invariance). We have $W_\Lambda \circ R = W_\Lambda$, because R permutes Λ, $\mathcal{B}(\Lambda)$, $\mathcal{P}(\Lambda)$, $\mathcal{C}(\Lambda)$; $\widetilde{g}$ is one function of the site variable; $\widetilde{h}$ is symmetric in (ρ, ρ') and even in B; w is even and $\mathfrak{f}_p \circ R = \mathfrak{f}_{Rp}$; $\mathbf{1}[(dv)_c \circ R = 0] = \mathbf{1}[(dv)_{Rc} = 0]$. The reference measure is invariant (a permutation of the coordinates with sign changes of some B_b, v_p; dB on $(-\pi/\gamma_N, \pi/\gamma_N]$ is invariant under $B \mapsto -B$ up to a null set, the counting measure exactly). Hence $\langle F \circ R \rangle_\Lambda = \langle F \rangle_\Lambda$ and $\langle \overline{F \circ R}\, G \rangle_\Lambda = \langle \overline{F}\, (G \circ R) \rangle_\Lambda = \overline{\langle \overline{G \circ R}\, F \rangle_\Lambda}$.

Step 1 (factorisation). Grouping the factors of W_Λ by the class of their cell, $W_\Lambda = W_0(\xi_0) W_+(\xi_0, \xi_+) W_-(\xi_0, \xi_-) W_\times(\xi_+, \xi_-, \xi_\times)$: the faces of a plane or (+)-cell are plane or (+), and, by Lemma 12.14(b), no variable of $W_\times := \prod_{b \in \mathcal{B}_\times} \widetilde{h} \prod_{p \in \mathcal{P}_\times} w(\mathfrak{f}_p) \prod_{c \in \mathcal{C}_\times} \mathbf{1}[(dv)_c = 0]$ is a plane variable. By Step 0 and Lemma 12.14(a), $W_-(\xi_0, \xi_-) = W_+(\xi_0, (R\xi)_+)$, where $\eta := (R\xi)_+$, $\eta_\kappa = \xi_{R\kappa}$ (κ a (+)-cell), is a measure-preserving bijection of the (−)-variables onto the (+)-variables, and $(R\xi)_0 = \xi_0$. For $F \in \mathfrak{A}^+_\Lambda(\Pi)$: $F(R\xi) = F(\xi_0, \eta)$.

Step 2 (the kernel). Let $K(\xi_-, \xi_+) := \int \prod_{b \in \mathcal{B}_\times} dB_b \sum_{(v_p)_{p \in \mathcal{P}_\times}} W_\times$. Write $\mathbf{1}[(dv)_c = 0] = \int_0^1 dt_c\, e^{i\gamma_N t_c (dv)_c}$ for $c \in \mathcal{C}_\times$ (as $(dv)_c \in (2\pi/\gamma_N)\mathbb{Z}$; Step 2 of the proof of Theorem 3.1); by Lemma 12.15 for v, $\sum_c t_c (dv)_c = \sum_{p \in \mathcal{P}_\times} s_p v_p + \sum_c t_c (V_+(c) - V_-(c))$. Fubini's theorem ($\sum_v \prod_p w \leq K'^{|\mathcal{P}_\times|}_{\gamma_N}$, K'_{γ_N} of Lemma 11.2; t ranges over a probability space) and (3.2) at $a = 0$, $x = (dB)_p$, $s = s_p$ (absolutely convergent, $\sum_m c_{m-s} \leq K_{\gamma_N}$ uniformly in s, Step 4 of the proof of Theorem 3.1) give

$$\sum_{(v_p)_{p\times}} \prod_{p\times} w(\mathfrak{f}_p) \prod_{c\times} \mathbf{1}[(dv)_c = 0] = \int dt \sum_m \prod_{p\times} c_{m_p - s_p} e^{i\gamma_N (m_p - s_p)(dB)_p} \prod_{c\times} e^{i\gamma_N t_c (V_+(c) - V_-(c))}.$$

By Lemma 12.15 for B, $(dB)_p = B_\times(p) + B_+(p) - B_-(p)$ and $\sum_{p\times}(m_p - s_p)B_\times(p) = \sum_{b\in\mathcal{B}_\times} q_b B_b$ with $q_b := \sum_p [p : b] m_p \in \mathbb{Z}$ (the s-part drops by the last clause of that lemma). With $e^{z\cos\phi} = \sum_{q\in\mathbb{Z}} I_q(z) e^{iq\phi}$ ($I_q \geq 0$, $\sum_q I_q(z) = e^z$; Step 3 of the proof of Theorem 3.1),

$$\int_{-\pi/\gamma_N}^{\pi/\gamma_N} dB_b\, \widetilde{h}(\rho_{b_-}, \rho_{b_+}, B_b) e^{i\gamma_N q_b B_b} = (2\pi/\gamma_N) I_{q_b}(\kappa_N \rho_{b_-} \rho_{b_+}).$$

Hence

$$K = \int dt \sum_m \prod_{p\times} c_{m_p - s_p} \prod_{b\times} \tfrac{2\pi}{\gamma_N} I_{q_b(m)}(\kappa_N \rho_{b_-}\rho_{b_+}) \prod_{p\times} e^{i\gamma_N (m_p - s_p)(B_+(p) - B_-(p))} \prod_{c\times} e^{i\gamma_N t_c (V_+(c) - V_-(c))}.$$

At a diagonal plane $\mathcal{B}_\times = \emptyset$ (Lemma 12.14(c)) and no Bessel function enters; at a coordinate site plane $K \equiv 1$.

Step 3 (positive type). With $I_q(z) = \sum_{k\geq 0} a_{k,q} z^{2k+|q|}$, $a_{k,q} = 2^{-2k-|q|}/(k!\,(k+|q|)!) \geq 0$, put, for $k = (t, m, (k_b)_{b\times})$,

$$A_k(\xi_+) := \prod_{p\times} e^{i\gamma_N (m_p - s_p) B_+(p)} \prod_{c\times} e^{i\gamma_N t_c V_+(c)} \prod_{b\times} \rho_{b_+}^{2k_b + |q_b|},$$
$$d\nu(k) := dt \prod_{p\times} c_{m_p - s_p} \prod_{b\times} \tfrac{2\pi}{\gamma_N}\, a_{k_b, q_b(m)}\, \kappa_N^{2k_b + |q_b|},$$

a positive measure. By Lemma 12.15 and Lemma 12.14(a) ($Rb = -b$ swaps the ends $b_\pm$ of a crossing bond): $B_-(p) = B_+(p)\circ R$, $V_-(c) = V_+(c)\circ R$, $\rho_{b_-} = \rho_{b_+}\circ R$, functions of $\eta = (R\xi)_+$ alone. Therefore $K(\xi_-, \xi_+) = \widetilde{K}(\eta, \xi_+) := \int d\nu(k)\, \overline{A_k(\eta)} A_k(\xi_+)$, and

$$\int d\nu(k)\, |A_k(\eta)||A_k(\xi)| = \int dt \sum_m \prod_{p\times} c_{m_p - s_p} \prod_{b\times} \tfrac{2\pi}{\gamma_N} I_{q_b}\big(\kappa_N \rho_{b_+}(\eta)\rho_{b_+}(\xi)\big)$$
$$\leq K_{\gamma_N}^{|\mathcal{P}_\times|} \big(\tfrac{2\pi}{\gamma_N}\big)^{|\mathcal{B}_\times|} \prod_{b\times} e^{\frac{\kappa_N}{2}(\rho_{b_+}(\eta)^2 + \rho_{b_+}(\xi)^2)}$$

by $I_q(z) \leq e^z$ and $2\rho\rho' \leq \rho^2 + \rho'^2$.

Step 4 (positivity). Summing $\xi_\times$ first (Tonelli on $W_\Lambda \geq 0$, then $|F| \leq \|F\|_\infty$) and substituting $\eta = (R\xi)_+$ (Step 1),

$$Z_\Lambda \langle \Theta_\Pi F \cdot F\rangle_\Lambda = \int d\xi_0 W_0 \iint d\eta\, d\xi_+\, W_+(\xi_0, \eta) W_+(\xi_0, \xi_+) \int d\nu(k)\, \overline{A_k(\eta) F(\xi_0, \eta)}\, A_k(\xi_+) F(\xi_0, \xi_+).$$

By Step 3 the absolute integrand is at most $\|F\|_\infty^2 K_{\gamma_N}^{|\mathcal{P}_\times|} (2\pi/\gamma_N)^{|\mathcal{B}_\times|}$ times $W_0 W_+(\xi_0,\eta) W_+(\xi_0,\xi_+) e^{\frac{\kappa_N}{2}\sum_{b\times}(\rho_{b_+}(\eta)^2 + \rho_{b_+}(\xi_+)^2)}$, whose integral is, undoing the substitution, $\int W_0 W_+ W_- e^{\frac{\kappa_N}{2}\sum_{b\times}(\rho_{b_-}^2 + \rho_{b_+}^2)}$ over the non-crossing variables. Bounding $\widetilde{h} \leq e^{\frac{\kappa_N}{2}(\rho^2 + \rho'^2)}$ on every non-crossing bond, all bond factors together are at most $\prod_{x\in\Lambda} e^{\frac{\kappa_N}{2}\deg(x)\rho_x^2} \leq \prod_x e^{3\kappa_N \rho_x^2}$ ($\deg(x) \leq 6$ bonds of $\mathcal{B}(\Lambda)$ at x), which cancels the site factors $e^{-3\kappa_N\rho_x^2}$; the indicators are ≤ 1, each dB_b integrates to $2\pi/\gamma_N$, and $\sum_{v_p} w(\mathfrak{f}_p) \leq K'_{\gamma_N}$ per plaquette. So the integral is at most $(G_0^{\mathrm{ct}})^{|\Lambda|} (2\pi/\gamma_N)^{|\mathcal{B}(\Lambda)\setminus\mathcal{B}_\times|} K'^{|\mathcal{P}(\Lambda)\setminus\mathcal{P}_\times|}_{\gamma_N} < \infty$, $G_0^{\mathrm{ct}} := \int g_{\mathrm{ct}}$ (Lemma 11.2, Remark 11.4); the same

bound with all cells gives $Z_\Lambda < \infty$, and $Z_\Lambda > 0$ since $W_\Lambda > 0$ on $\{v \equiv 0\}$. Fubini's theorem now yields

$$Z_\Lambda \langle \Theta_\Pi F \cdot F \rangle_\Lambda = \int d\xi_0\, W_0(\xi_0) \int d\nu(k) \Big| \int d\xi_+\, W_+(\xi_0,\xi_+) A_k(\xi_+) F(\xi_0,\xi_+) \Big|^2 \;\geq 0.$$

Step 5 (L^2). Bounded (ξ_0,ξ_+)-measurable functions are dense in the (ξ_0,ξ_+)-measurable part of $L^2(\langle\cdot\rangle_\Lambda)$, and $|\langle \Theta F \cdot F\rangle_\Lambda - \langle \Theta G \cdot G\rangle_\Lambda| \leq \|F-G\|_2(\|F\|_2 + \|G\|_2)$ by the Schwarz inequality and $\|F \circ R\|_2 = \|F\|_2$ (Step 0); the Hermitian identity is $\|\cdot\|_2$-continuous likewise. □

Remark 12.17 (the crossing kernel). At a diagonal plane the crossing kernel $\widetilde{K}$ of Step 3 of the proof of Theorem 12.16 is of positive type by the Poisson step alone:

$$\widetilde{K}(\eta,\xi) = \int_{[0,1]^{\mathcal{C}_\times}} dt \sum_{m \in \mathbb{Z}^{\mathcal{P}_\times}} \prod_{p \in \mathcal{P}_\times} c_{m_p - s_p}\, \overline{A_{t,m}(\eta)}\, A_{t,m}(\xi),$$
$$A_{t,m} := \prod_{p \in \mathcal{P}_\times} e^{i\gamma_N (m_p - s_p) B_+(p)} \prod_{c \in \mathcal{C}_\times} e^{i\gamma_N t_c V_+(c)},$$

with $c_u = (\gamma_N/\sqrt{2\pi}) e^{-\gamma_N^2 u^2/2} > 0$ (equation (3.2)) and $|A_{t,m}| = 1$. At a coordinate mid-plane the factors $(2\pi/\gamma_N) I_{q_b}(\kappa_N \rho_{b_-} \rho_{b_+})$ of the crossing bonds are added (the Bessel step of Theorem 3.1).

Corollary 12.18 (matrix and exponential forms, complex coefficients)**.** *For (ξ_0,ξ_+)-measurable $A_1,\dots,A_n$ with $\langle |A_i|^2\rangle_\Lambda < \infty$ and $\lambda \in \mathbb{C}^n$: $\sum_{i,l} \bar\lambda_i \lambda_l \langle \Theta_\Pi A_i \cdot A_l\rangle_\Lambda = \langle \Theta_\Pi A \cdot A\rangle_\Lambda \geq 0$ with $A := \sum_l \lambda_l A_l$ (Θ_Π is antilinear); the Gram matrix $(\langle \Theta_\Pi A_i \cdot A_l\rangle_\Lambda)_{i,l}$ is Hermitian positive semidefinite. In particular, for real plaquette functions $a^{(1)},\dots,a^{(k)}$ supported on the plaquettes of $\mathcal{P}(\Lambda)$ lying in H_+, $X_r := \sum_p a_p^{(r)} \mathfrak{f}_p$, $X_r^R := X_r \circ R_\Pi = \sum_p a_p^{(r)} \mathfrak{f}_{R_\Pi p}$ (the reflected source, with the 2-form sign), and $z^{(1)},\dots,z^{(n)} \in \mathbb{C}^k$,*

$$\sum_{i,l=1}^{n} \bar\lambda_i \lambda_l \Big\langle \exp\Big(\sum_r \overline{z_r^{(i)}} X_r^R + \sum_r z_r^{(l)} X_r \Big) \Big\rangle_\Lambda \;\geq\; 0.$$

Proof. $A_i := \exp(\sum_r z_r^{(i)} X_r)$ is (ξ_0,ξ_+)-measurable (the bonds of a plaquette in H_+ lie in H_+), $\Theta_\Pi A_i = \exp(\sum_r \overline{z_r^{(i)}} X_r^R)$ because X_r is real, and $\langle |A_i|^2\rangle_\Lambda = \langle e^{2\sum_r (\mathrm{Re}\, z_r^{(i)}) X_r}\rangle_\Lambda \leq \exp(2\|\sum_r (\mathrm{Re}\, z_r^{(i)}) a^{(r)}\|^2) < \infty$ by Theorem 3.1 for the box states (Lemma 12.10: its proof uses only $\partial\partial = 0$ on the complex and the Fourier positivity of the bond and plaquette factors). Apply the theorem to $A = \sum_l \lambda_l A_l$. With $a_p^{(r)} = \varepsilon^{3/2} f^{(r)}_{\mu\nu(p)}(x_p)$ for real C_c^∞ 2-forms supported in the open side at distance $> \varepsilon$ from Π and inside the cube, this is the box version of Theorem 9.10 at level (N,Λ), which Lemma 12.21 uses; the plaquettes carrying such $a^{(r)}$ lie in H_+ because the crossing plaquettes have their centres on Π. □

12.5. **Reflection positivity of the box limit at every lattice plane of the nine families.**

Lemma 12.19 (reflection positivity of the box limit at the lattice planes of the nine families)**.** *Fix N and a lattice plane $P = \{\phi = c\}$ of one of the nine families (notation of Section 12.1: $c \in \frac{1}{2}\varepsilon\mathbb{Z}$ for $\phi = x_j$, $c \in \varepsilon\mathbb{Z}$ for $\phi = x_i \pm x_j$); write $R := R_P$. Let $\mathfrak{A}_N^+(P)$ be the bounded Borel functions of the gauge-invariant variables $(f_p, \tilde\rho_x)$ of finitely many cells of $\varepsilon\mathbb{Z}^3$ lying in $P^+ = \{\phi \geq c\}$, $\Sigma_N(P^+)$ the σ-algebra generated by all such variables, and $\Theta_P F := \overline{F \circ R}$. Then:*

(i) *$\langle F \circ R\rangle_\Lambda = \langle F\rangle_\Lambda$ for every cube Λ with $R\Lambda = \Lambda$ and every bounded Borel F supported in Λ;*

(ii) *for* $F,G \in \mathfrak{A}_N^+(P)$: $\omega_N^{\mathrm{box}}(\Theta_P F\cdot G) = \overline{\omega_N^{\mathrm{box}}(\Theta_P G\cdot F)}$, $\omega_N^{\mathrm{box}}(\Theta_P F\cdot F)\ge 0$ *and* $\omega_N^{\mathrm{box}}(\Theta_P F)=\overline{\omega_N^{\mathrm{box}}(F)}$; *the form* $(F,G)\mapsto\omega_N^{\mathrm{box}}(\Theta_P F\cdot G)$ *is positive semidefinite Hermitian;*

(iii) *(ii) holds for all* $F,G\in L^2(\Sigma_N(P^+),\mu_N^{\mathrm{box}})$.

The statements do not depend on any exhaustion.

Proof. (i) is Step 0 of the proof of Theorem 12.16 (R is a symmetry of Λ in the sense there; for a lattice plane of the nine families with $R\Lambda=\Lambda$ it is one of the symmetry planes listed in Section 12.1).

(ii) *Symmetric exhaustion.* Let $I_k := \{-k\varepsilon,\dots,k\varepsilon\}$ and $J_k := \{-(k-1)\varepsilon,\dots,k\varepsilon\}$. For $P=\{x_j=c\}$ with $c\in\varepsilon\mathbb{Z}$ let Λ_k be the cube with side I_k in the directions $\neq j$ and $c+I_k$ in the direction j; for $c\in\varepsilon\mathbb{Z}+\frac{\varepsilon}{2}$ take J_k and $c-\frac{\varepsilon}{2}+J_k$; for $\{x_i-x_j=c\}$ take I_k in the directions j,l and $c+I_k$ in the direction i; for $\{x_i+x_j=c\}$ take I_k in the directions j,l and $c-I_k=c+I_k$ in the direction i. In each case $R\Lambda_k=\Lambda_k$ (for $\{x_i-x_j=c\}$: $(Rx)_i=x_j+c\in c+I_k$, $(Rx)_j=x_i-c\in I_k$; for $\{x_i+x_j=c\}$: $(Rx)_i=c-x_j\in c+I_k$, $(Rx)_j=c-x_i\in I_k$; for the mid-planes $c-\frac{\varepsilon}{2}+J_k$ is symmetric about c), $\Lambda_k\subset\Lambda_{k+1}$ and $\bigcup_k\Lambda_k=\varepsilon\mathbb{Z}^3$: an exhausting sequence of cubes, each having P as a symmetry plane.

Continuous F. Let F be a bounded continuous function of the gauge-invariant variables of a finite set S of cells in P^+. For k large, $S\cup RS\subset\Lambda_k$ (all cells of $S\cup RS$ are cells of Λ_k), so F, $F\circ R$ and $\Theta_P F\cdot F$ are functions of the box variables, $F\in\mathfrak{A}^+_{\Lambda_k}(P)$ (a function of $(f_p,\tilde\rho_x)=((dB)_p+v_p,\tilde\rho_x)$ for cells in P^+ is a function of the variables of cells in P^+), and Theorem 12.16 gives $\langle\Theta_P F\cdot F\rangle_{\Lambda_k}\ge 0$. By Theorem 11.11(b) the laws of the variables of $S\cup RS$ under $\langle\cdot\rangle_{\Lambda_k}$ converge weakly to the marginal of μ_N^{box}, and $\Theta_P F\cdot F$ is bounded continuous, so $\omega_N^{\mathrm{box}}(\Theta_P F\cdot F)=\lim_k\langle\Theta_P F\cdot F\rangle_{\Lambda_k}\ge 0$. The Hermitian identity and $\omega_N^{\mathrm{box}}(\Theta_P F)=\overline{\omega_N^{\mathrm{box}}(F)}$ hold for all bounded Borel F,G of finitely many cells: μ_N^{box} is R-invariant ($R\in\mathcal{G}_N$, Theorem 11.11(b)), and $(\overline{F}\cdot(G\circ R))\circ R=\overline{F\circ R}\cdot G$. Positive semidefiniteness on finite families follows by applying the inequality to $\sum_i c_i F_i$.

Bounded Borel functions and L^2. As in the proof of Theorem 11.5(d): for a finite set S of cells in P^+ the bounded continuous functions on E_S are dense in $L^2(E_S,(\pi_S)_*\mu_N^{\mathrm{box}})$ (a finite Borel measure on a Polish space is Radon), a bounded Borel F of the cells of S is the $L^2(\mu_N^{\mathrm{box}})$-limit of continuous $F^{(i)}$ with $\|F^{(i)}\|_\infty\le\|F\|_\infty$ (truncate), and both sides of each identity are continuous in $\|F\|_{L^2}$, $\|G\|_{L^2}$ by the Schwarz inequality and $\|F\circ R\|_{L^2(\mu_N^{\mathrm{box}})}=\|F\|_{L^2(\mu_N^{\mathrm{box}})}$. This gives (ii), and (iii) for functions of finitely many cells. For (iii) in general, the bounded $\Sigma_N(P^+)$-measurable functions that are $L^2(\mu_N^{\mathrm{box}})$-limits of bounded functions of finitely many cells of P^+ form a vector space containing the constants, closed under bounded monotone limits (dominated convergence), and containing the multiplicative class of the bounded functions of finitely many cells, which generates $\Sigma_N(P^+)$; by the functional monotone class theorem it contains every bounded $\Sigma_N(P^+)$-measurable function, and these are dense in $L^2(\Sigma_N(P^+),\mu_N^{\mathrm{box}})$ (truncation); the continuity of both sides concludes the proof of (iii). The statements do not depend on the exhaustion, because μ_N^{box} is one measure (Theorem 11.11(b)). □

12.6. **The continuum passage for the box family.**

Lemma 12.20 (the continuum passage for the box family)**.** *Let (Λ_k) be any exhausting sequence of boxes. The pair $((\Lambda_k),\mu_N^{\mathrm{box}})$ has the properties (a), (b), (c), (d), (e) of Theorem 11.5, with "box" for "torus" and "$k\to\infty$" for "$L\to\infty$", and in addition the reflection positivity (d) at every lattice plane of the nine families (Lemma 12.19). Consequently Proposition 4.2, Lemma 5.1, Proposition 6.1 and Theorem 7.1 hold for ω_N^{box} with any bounded Borel g_{d}; and every statement of*

Section 9 holds for the box family, with Theorem 9.7 replaced by Theorem 12.16 and Corollary 9.8 by Lemma 12.19. S^{box}, Z^{box}, Ψ_f^{box}, S_n^{box}, μ^{box} denote the objects so obtained along (N_m).

Proof. The properties (a)–(e). (a) The weak convergence of all finite-dimensional laws is Theorem 11.11(b), and the Borel clause follows from the portmanteau theorem. For a continuous F with $|F| \le Ce^{\sum_{x\in X} q_x\rho_x^2}$ the uniform integrability needed is

$$\sup_\Lambda \langle e^{2\sum_x q_x\rho_x^2}\rangle_\Lambda \le \max_x C_N(2|X|q_x).$$

This is (K4) of Theorem 11.10, where $e^{\sum_x q_x\rho_x^2} \in \mathcal{K}_N$ and the comparison is with a torus of side larger than Λ and its neighbours, together with Remark 11.3(ii), exactly as in the proof of Theorem 11.11(a).

(b) The functional ω_N^{box} is the integral against a probability measure, and the Gaussian and density bounds are in Theorem 11.11(b).

(c) The invariance under $\mathcal{G}_N$ and C is Theorem 11.11(b), and the conclusion of Lemma 5.1 follows as in the proof of Theorem 11.5(c).

(d) For the planes $\{x_j = c\}$, $c \in \frac{1}{2}\varepsilon\mathbb{Z}$, this is Lemma 12.19, and the additional clause is Lemma 12.19 for the diagonal planes.

(e) The first clause follows from (a); the second is the portmanteau theorem for continuous $F \ge 0$, as in the proof of Theorem 11.5(b).

The statements of Sections 4–7. For Proposition 4.2, Theorem 3.1 for the box states (Lemma 12.10) and Minkowski's inequality give $\langle\Phi_N(\sigma)^2\rangle_\Lambda^{1/2} \le \|\sigma\|$. The bound $\omega_N^{\mathrm{box}}(|\Phi_N(\sigma)| \wedge 2) \le \|\sigma\|$ follows from the Schwarz and Minkowski inequalities, from $\int X^2 d\mu_N^{\mathrm{box}} \le R(f)^2$, which holds by (e) or by Step 5 of the proof of Theorem 3.1 applied to the bound of (b), and from the pointwise bound $|Y| \le \|g_{\mathrm{d}}\|_\infty E(h)$. Lemma 5.1 is (c). Proposition 6.1 uses of the state only the state property, (2.4), Lemma 5.1 and (4.1). Theorem 7.1 follows from these statements, including the invariance under the space translations and under the cubic group about $\bigcup_m \varepsilon_{N_m}\mathbb{Z}^3$ and the bound $|S(a)| \le M(a)$.

The statements of Section 9. Lemma 9.1 is a statement of lattice counting. For Lemma 9.2, the identity (3.3) holds on every finite cubical complex, the killed box included, by the proof of Lemma 12.10, with the same entire continuation.

Part (a) of Lemma 9.3 holds for the box states for the following reasons. The bound follows from Lemma 12.10 on the box. The evenness follows from the substitution $(B, v) \mapsto (-B, -v)$, since W_Λ depends on B through $\cos(\gamma_N B_b)$, w is even and closed v go to closed v. The inequality $\Psi \ge 1$ on $\mathbb{R}^k$ follows from Jensen's inequality and $\langle\Phi_N\rangle_\Lambda = 0$ (Step 5 of the proof of Theorem 3.1), and $0 \le \Psi(it) \le 1$ follows from the dual representation on the box. Part (b) of Lemma 9.3 follows from Montel's theorem and its part (a) for the box states; the limit is identified on $i\mathbb{R}^k$ by the property (a) of the present lemma, and the uniform integrability is obtained as in the torus case. Parts (c) and (d) of Lemma 9.3 hold without change.

Theorem 9.4 follows from (6.1) with Proposition 4.2 and Lemma 9.3. Proposition 9.6 follows from Theorem 7.1 and (c). Proposition 9.9 follows from Theorem 7.1(a) and Lemma 12.19; the argument is written out in Lemma 12.21(i) below.

For Theorem 9.10 and Proposition 9.11, the functions $A_i = \exp(\sum_r z_r^{(i)}\Phi_N(f_r))$ and the polynomial P_N on Ω_N lie in $L^2(\Sigma_N(P^+), \mu_N^{\mathrm{box}})$ by (b). The intertwining $\Phi_N(f) \circ R_P = \Phi_N(\gamma_* f)$ of the proof of Proposition 9.9 gives

$$\Theta_P A_i = \exp(\sum_r \overline{z_r^{(i)}}\Phi_N(\gamma_* f_r)), \qquad \Theta_P P_N = \sum_\alpha \bar{c}_\alpha \prod_i \Phi_N(\gamma_* f_{\alpha i}).$$

Lemma 12.19(iii) gives the positivity at level N, Lemma 9.3(b) identifies $\int \Theta_P A_i \cdot A_l \, d\mu_N^{\mathrm{box}} = \Psi_N^F(\overline{z^{(i)}}, z^{(l)})$ and the moments as derivatives of Ψ_N^F, and Lemma 9.3(c), with Weierstrass' theorem for the derivatives, gives the passage $m \to \infty$.

Proposition 9.12 follows from (c), applied to C. Theorem 9.13 uses only Theorem 9.4, Theorem 7.1(a) and Proposition 9.6, together with the abstract measure-theoretic statements of Appendix A and the Bochner–Minlos theorem, which do not depend on the family. Theorem 9.15 uses only Theorem 9.13(c), Theorem 9.10 and, for the growth bound of its last clause, Theorem 9.4(iii). □

12.7. **Reflection positivity of the limit theory at the nine plane families.**

Lemma 12.21 (reflection positivity of the limit theory at the nine plane families)**.** *Let S^{box} be a limit along (N_m), let P_0 be one of the nine planes through 0, n a unit normal, $c \in \mathbb{R}$, $P_c := P_0 + cn$, γ the reflection R_{P_c} acting on test pairs by push-forward ($\gamma\sigma = (\gamma_* f, h \circ \gamma)$, the 2-form transported as a 2-form), Θ the antilinear map of $\mathcal{A}$ with $\Theta e_\sigma = e_{-\gamma\sigma}$, and $\mathcal{A}_+$ the span of the e_σ with* $\operatorname{supp}\sigma$ *in the open side $\{n \cdot x > c\}$.*

(i) *$\langle a, b\rangle := S^{\mathrm{box}}(\Theta a \cdot b)$ is a positive semidefinite Hermitian sesquilinear form on $\mathcal{A}_+$, and $S^{\mathrm{box}}(\Theta a) = \overline{S^{\mathrm{box}}(a)}$ on $\mathcal{A}$.*

(ii) *For real C_c^∞ 2-forms $f_1, \dots, f_k$ supported in the open side, $F := (\gamma_* f_1, \dots, \gamma_* f_k, f_1, \dots, f_k)$, $z^{(1)}, \dots, z^{(n)} \in \mathbb{C}^k$ and $\lambda \in \mathbb{C}^n$: $\sum_{i,l} \bar\lambda_i \lambda_l \Psi_F^{\mathrm{box}}(\overline{z^{(i)}}, z^{(l)}) \ge 0$.*

(iii) *For real Schwartz pairs $\sigma_1, \dots, \sigma_n$ supported in the closed side $\{n \cdot x \ge c\}$ and $\lambda \in \mathbb{C}^n$: $\sum_{i,l} \bar\lambda_i \lambda_l Z^{\mathrm{box}}(\sigma_l - \gamma\sigma_i) \ge 0$, $Z^{\mathrm{box}}(-\gamma\sigma) = \overline{Z^{\mathrm{box}}(\sigma)}$ and $Z^{\mathrm{box}}(\gamma\sigma) = Z^{\mathrm{box}}(\sigma)$, Z^{box} being the continuous extension of Theorem 9.4(i) to Schwartz pairs.*

Here (ii) is the complex-coefficient form for C_c^∞ sources in the open side; (iii) is the characteristic-function form, with complex coefficients in front of the exponentials, for Schwartz pairs in the closed side; the hypothesis (iii) of Theorem 12.31 is (iii) restricted to the open side.

Proof. Reduction to $c = 0$. $P_c = \tau_y P_0$ with $y = cn$, $\gamma_c = \tau_y \gamma_0 \tau_{-y}$, $\mathcal{A}_+(P_c) = \tau_y \mathcal{A}_+(P_0)$, $\Theta_c = \tau_y \Theta_0 \tau_y^{-1}$, and $S^{\mathrm{box}} \circ \tau_y = S^{\mathrm{box}}$ for every $y \in \mathbb{R}^3$ (Proposition 9.6 for the box family, Lemma 12.20); so (i) for P_c is (i) for P_0 at $\tau_{-y}a, \tau_{-y}b$. For (ii), $\Psi^{\mathrm{box}}_{\tau_y F} = \Psi^{\mathrm{box}}_F$: both are entire on $\mathbb{C}^{2k}$ and agree on $i\mathbb{R}^{2k}$ (Lemma 9.3(c) and the identity theorem). For (iii), $Z^{\mathrm{box}} \circ \tau_y = Z^{\mathrm{box}}$ on Schwartz pairs by continuity.

(i), $c = 0$. $R := R_{P_0}$ is a signed permutation matrix, so $R\varepsilon_N\mathbb{Z}^3 = \varepsilon_N\mathbb{Z}^3$ and P_0 is a lattice plane of its family for every N. For σ supported in $\{n \cdot x \ge \delta\}$ and $\varepsilon_N < \delta$, every cell carrying a term of $\Phi_N(\sigma)$ lies in $\{n \cdot x > 0\}$ (the vertices of a plaquette are within $\varepsilon_N/\sqrt{2}$ of its centre, so $n \cdot x \ge \delta - \varepsilon_N/\sqrt{2} > 0$ on them), and $\Phi_N(\sigma) \circ R = \Phi_N(\gamma\sigma)$: R permutes the oriented plaquettes and the sites, the push-forward of the 2-form carries exactly the orientation sign, and $\tilde\rho_z \circ R = \tilde\rho_{Rz}$ (Definition 12.7 and the properties following it, Lemma 12.8 with $O\Lambda = \Lambda$; this is the intertwining of the proof of Proposition 9.9 for the diagonal reflections as well).

Hence $\iota_N(\Theta a \cdot b) = \Theta_P \iota_N(a) \cdot \iota_N(b)$ for $a, b \in \mathcal{A}_+$, with $\iota_N(a)$ a bounded Borel function of the gauge-invariant variables of cells in P_0^+ (bounded continuous when g_{d} is continuous; for a bounded Borel g_{d} only bounded Borel, which Lemma 12.19(ii) covers). (Lemma 12.19 is stated for the side $\{\phi \ge c\}$; the other side follows from the R-invariance of μ_N^{box}: $\omega_N^{\mathrm{box}}(\Theta_P F \cdot G) = \omega_N^{\mathrm{box}}(\Theta_P(F \circ R) \cdot (G \circ R))$.) By Lemma 12.19(ii) the Gram matrices $(\omega^{\mathrm{box}}_{N_m}(\Theta_P \iota_{N_m}(a_i)\, \iota_{N_m}(a_j)))_{ij}$ are Hermitian positive semidefinite; they converge entrywise to $(S^{\mathrm{box}}(\Theta a_i \cdot a_j))_{ij}$ by Theorem 7.1(a) for the box family (Lemma 12.20), and both properties are closed under limits. $S^{\mathrm{box}}(\Theta a) = \lim_m \overline{\omega^{\mathrm{box}}_{N_m}(\iota_{N_m}(a))} = \overline{S^{\mathrm{box}}(a)}$ by Lemma 12.19(ii).

(ii), $c=0$. This is the step for Theorem 9.10 in the proof of Lemma 12.20, with γ the diagonal or coordinate reflection: the Gram matrices at level N are positive semidefinite by Lemma 12.19(iii) (Corollary 12.18 at the level of the boxes), equal to $(\Psi_N^F(\overline{z^{(i)}}, z^{(l)}))_{il}$ by Lemma 9.3(b), and converge to $(\Psi_F^{\mathrm{box}}(\overline{z^{(i)}}, z^{(l)}))_{il}$ by Lemma 9.3(c).

(iii), $c=0$. For a Schwartz pair $\sigma=(f,h)$ supported in P_0^+ put $\sigma^{(\nu)} := \chi(\cdot/\nu)\,\tau_{n/\nu}\sigma$ with $\chi\in C_c^\infty$, $\chi=1$ on the unit ball and $0\le\chi\le 1$: $\sigma^{(\nu)}\in C_c^\infty$ is supported in $\{n\cdot x\ge 1/\nu\}$, and

$$\|\sigma^{(\nu)}-\sigma\|_w \le \sup_{|x|\ge\nu}(1+|x|^2)^2|\sigma(x)| + \nu^{-1}\,9\sup_y(1+|y|^2)^2|\nabla\sigma(y)| \to 0$$

(mean value theorem; $1+|x|^2\le 3(1+|y|^2)$ for $|x-y|\le 1$; $0\le\chi\le 1$ for the first term). Here $\|\cdot\|_w$ is the derivative-free seminorm of Section 9 applied to each component of the pair. γ is a linear isometry acting on the components by a signed permutation, so $\|\gamma\sigma^{(\nu)}-\gamma\sigma\|_w = \|\sigma^{(\nu)}-\sigma\|_w$. By Theorem 9.4(i) and Lemma 9.1 (its pair version: $|Z(\sigma)-Z(\sigma')|\le\|\sigma-\sigma'\|\le(3c_{a_0})^{1/2}\|f-f'\|_w+\|g_{\mathrm d}\|_\infty c_{a_0}\|h-h'\|_w$, which extends Z^{box} continuously to real Schwartz pairs), $Z^{\mathrm{box}}(\sigma_l^{(\nu)}-\gamma\sigma_i^{(\nu)})\to Z^{\mathrm{box}}(\sigma_l-\gamma\sigma_i)$, while each $\sum_{i,l}\bar\lambda_i\lambda_l Z^{\mathrm{box}}(\sigma_l^{(\nu)}-\gamma\sigma_i^{(\nu)}) = S^{\mathrm{box}}(\Theta a_\nu\cdot a_\nu)\ge 0$ with $a_\nu := \sum_l \lambda_l e_{\sigma_l^{(\nu)}}\in\mathcal{A}_+$, by (i). The two identities pass from C_c^∞ pairs (by (i), and by Proposition 9.6 together with the cubic invariance of Theorem 7.1, γ lying in the cubic group O_h about a point of P_0) to Schwartz pairs by the same continuity. □

12.8. **The Gaussian bound in continuum norms.**

Lemma 12.22 (the Gaussian bound in continuum norms)**.** *Z^{box}, extended to real Schwartz pairs by Theorem 9.4(i) and Lemma 9.1 (pair version, as in the proof of Lemma 12.21(iii)), is positive definite and continuous on $\mathcal{S}(\mathbb{R}^3;\Lambda^2\oplus\mathbb{R})$; let μ be the probability measure on $\mathcal{S}'(\mathbb{R}^3;\Lambda^2\oplus\mathbb{R})$, with its cylinder σ-algebra, such that $\int e^{i\Phi(\sigma)}d\mu = Z^{\mathrm{box}}(\sigma)$ for all real Schwartz pairs (Bochner–Minlos, as in the proof of Theorem 9.13 and in Lemma 12.12; its curvature marginal is the measure of Theorem 9.13 read on $\mathcal{S}'$). Then for every real Schwartz pair $\sigma=(f,h)$,*

$$\int e^{\Phi(f,h)}\,d\mu \;\le\; \exp\Big(\tfrac12\|f\|_{L^2}^2 + \|g_{\mathrm d}\|_\infty\|h\|_{L^1}\Big), \qquad \|f\|_{L^2}^2 := \sum_{\mu<\nu}\|f_{\mu\nu}\|_{L^2}^2.$$

Consequently the curvature Schwinger functions $S_n(f_1,\dots,f_n) = \int\prod_i\Phi(f_i,0)\,d\mu$ (Theorem 9.4(iii), Theorem 9.13(a)) satisfy $|S_n(f_1,\dots,f_n)|\le e^n(n!)^{1/2}\prod_i\|f_i\|_{L^2}$ for real $f_i\in\mathcal{S}(\mathbb{R}^3;\Lambda^2)$, and S_n extends uniquely to a bounded n-linear form on $L^2(\mathbb{R}^3;\Lambda^2_{\mathbb{C}})$, with the same bound on real entries and $2^{n/2}$ times it on complex ones.

Proof. Positive definiteness. We have

$$\sum_{l,l'}\bar c_l c_{l'} Z^{\mathrm{box}}(\sigma_{l'}-\sigma_l) = \lim_m \omega_{N_m}^{\mathrm{box}}(|\sum_l c_l e^{i\Phi_{N_m}(\sigma_l)}|^2)\ge 0$$

by the additivity of Φ_N and Theorem 7.1(a); positive definiteness is preserved under the continuous extension. Continuity on $\mathcal{S}$ is the pair version of Theorem 9.4(i) just mentioned, and $\mathcal{S}(\mathbb{R}^3;\Lambda^2\oplus\mathbb{R})$ is nuclear, so the Bochner–Minlos theorem gives μ.

Level N. $\Phi_N(\sigma)=X+Y$ with $X=\sum_p a_p\mathfrak{f}_p$, $a_p=\varepsilon_N^{3/2}f_{\mu\nu(p)}(x_p)$, and $|Y|\le\|g_{\mathrm d}\|_\infty E_N(h)$ on Ω_N, $E_N(h) := \varepsilon_N^3\sum_z|h(z)|$. By Theorem 11.11(b), $\int e^X d\mu_N^{\mathrm{box}}\le e^{\frac12\sum_p a_p^2}$, so $\int e^{\Phi_N(\sigma)}d\mu_N^{\mathrm{box}}\le\exp(\frac12 R_N(f)^2+\|g_{\mathrm d}\|_\infty E_N(h))$ with $R_N(f)^2 := \varepsilon_N^3\sum_p f_{\mu\nu(p)}(x_p)^2$.

The limit, $\sigma\in C_c^\infty$. The law ν_m of $\Phi_{N_m}(\sigma)$ under $\mu_{N_m}^{\mathrm{box}}$ has the characteristic function $t\mapsto Z_{N_m}(t\sigma)\to Z^{\mathrm{box}}(t\sigma)=\int e^{it\Phi(\sigma)}d\mu$ (Theorem 7.1(a)), which is continuous at 0 by $|Z^{\mathrm{box}}(t\sigma)-1|\le|t|\,\|\sigma\|$; by Lévy's continuity theorem $\nu_m\to\nu$ weakly, ν the law of $\Phi(\sigma)$ under μ. For the

continuous function $e^x \ge 0$, $\int e^x d\nu = \lim_M \int (e^x \wedge M) d\nu \le \liminf_m \int e^x d\nu_m$. The centres of the plaquettes of one orientation form a translate of $\varepsilon_N \mathbb{Z}^3$, so $R_N(f)^2 \to \|f\|_{L^2}^2$ and $E_N(h) \to \|h\|_{L^1}$ (Riemann sums of continuous compactly supported functions). The bound follows.

Schwartz σ. Take $\sigma_\nu := \chi(\cdot/\nu)\sigma \in C_c^\infty$ (χ as in the proof of Lemma 12.21); $\sigma_\nu \to \sigma$ in $\mathcal{S}$, so $\Phi(\sigma_\nu) \to \Phi(\sigma)$ for every $\Phi \in \mathcal{S}'$, and $\|f_\nu\|_{L^2} \to \|f\|_{L^2}$, $\|h_\nu\|_{L^1} \to \|h\|_{L^1}$; Fatou's lemma gives

$$\int e^{\Phi(\sigma)} d\mu \le \liminf_\nu \int e^{\Phi(\sigma_\nu)} d\mu \le \exp\big(\frac{1}{2}\|f\|_{L^2}^2 + \|g_{\mathrm{d}}\|_\infty \|h\|_{L^1}\big).$$

Moments. For real $f_1, \dots, f_n \in \mathcal{S}$ and $z \in \mathbb{C}^n$ put $\Psi(z) := \int e^{\sum_j z_j \Phi(f_j,0)} d\mu$. Since $\int e^{s|\Phi(f,0)|} d\mu \le 2e^{s^2\|f\|_{L^2}^2/2}$ for all real s, the integrand has a locally uniform integrable majorant, so Ψ is entire with $\partial_{z_1} \cdots \partial_{z_n} \Psi(0) = \int \prod_j \Phi(f_j,0)\, d\mu = S_n(f_1, \dots, f_n)$ (the law of $(\Phi(f_j,0))_j$ under μ is μ_f, as in the moment step of the proof of Theorem 9.13), and

$$|\Psi(z)| \le \Psi(\operatorname{Re} z) \le \exp\big(\frac{1}{2}\|\sum_j (\operatorname{Re} z_j) f_j\|_{L^2}^2\big) \le \exp\big(\frac{1}{2}\big(\sum_j |\operatorname{Re} z_j|\, \|f_j\|_{L^2}\big)^2\big).$$

If some $\|f_j\|_{L^2} = 0$ then $f_j = 0$ and $S_n = 0$; otherwise normalise $\|f_j\|_{L^2} = 1$: on the polydisc $|z_j| \le r$, $|\Psi| \le e^{n^2 r^2/2}$, and Cauchy's estimate gives $|S_n| \le r^{-n} e^{n^2 r^2/2} = (en)^{n/2} \le e^n (n!)^{1/2}$ at $r = n^{-1/2}$, by $n^n \le e^n n!$. Real $\mathcal{S}$ is dense in real L^2, so the n-linear S_n extends uniquely by continuity with the same bound; the complex n-linear extension splits each entry into real and imaginary parts, and $\|u\| + \|v\| \le \sqrt{2}\|u + iv\|$. □

12.9. **Closedness of the limit curvature.** *Conventions.* We write $\varepsilon = \varepsilon_N$. The plaquettes are $p = (x; \mu, \nu)$, $\mu < \nu$, with centre x_p and $\partial p = (x;\mu) + (x + \varepsilon e_\mu; \nu) - (x + \varepsilon e_\nu; \mu) - (x; \nu)$ (Definition 12.6); $f_{\mu\nu(p)} = f_{\mu\nu}$. The cubes are $c(y) = y + \varepsilon[0,1]^3$ with centre $x_c = y + \frac{\varepsilon}{2}(1,1,1)$ and oriented boundary

$$\partial c(y) = \sum_{\lambda=1}^{3} \epsilon_{\lambda\mu\nu} \big[(y + \varepsilon e_\lambda; \mu, \nu) - (y; \mu, \nu)\big], \qquad \{\mu < \nu\} = \{1,2,3\} \setminus \{\lambda\},$$

the usual oriented boundary, with $\partial\partial = 0$. $[\kappa : \kappa'] \in \{0, \pm 1\}$ is the coefficient of the cell κ in $\partial\kappa'$ (the smaller cell is written first here; Section 12.4 writes the larger cell first: there $[\kappa : \sigma]$ is the coefficient of σ in $\partial\kappa$). A finite cubical complex $\mathcal{K}$ is a torus $(\varepsilon\mathbb{Z}/L\mathbb{Z})^3$ or a killed box Λ of Definition 11.9, whose cubes are the cubes of Λ in the sense of Section 12.1 (all eight vertices in Λ; all six faces then lie in $\mathcal{P}(\Lambda)$). For functions B on bonds, v, f on plaquettes and a on cubes we put $(dB)_p = \sum_b [b : p] B_b$, $(df)_c = \sum_p [p : c] f_p$, $(d^* a)_p := \sum_{c \in \mathcal{K}} [p : c] a_c$ (the operator of Step 2 of the proof of Theorem 3.1). The Hodge star is $(*du)_{\mu\nu} = \epsilon_{\lambda\mu\nu} \partial_\lambda u$ for $\mu < \nu$, λ the third index, so that $\int \sum_{\mu<\nu} F_{\mu\nu} (*du)_{\mu\nu} = -\int u\, (dF)_{123}$ for smooth F. K_{+t} is the closed t-neighbourhood of a compact set K, and $\|u\|_{C^3} := \max_{|\gamma| \le 3} \|D^\gamma u\|_\infty$.

Lemma 12.23 (lattice Bianchi identity)**.** *Let $\mathcal{K}$ be a finite cubical complex, B real on its bonds, v on its plaquettes with $(dv)_c = 0$ for every cube c of $\mathcal{K}$, and $f := dB + v$. Then for every real a on the cubes of $\mathcal{K}$,*

$$\sum_{p \in \mathcal{P}(\mathcal{K})} (d^* a)_p\, f_p = \sum_{c \in \mathcal{C}(\mathcal{K})} a_c\, (df)_c = 0.$$

Proof. All sums are finite and every face of a cube of $\mathcal{K}$ is a plaquette of $\mathcal{K}$, so $\sum_p \sum_c [p : c] a_c f_p = \sum_c a_c \sum_p [p : c] f_p = \sum_c a_c (df)_c$. Further $(df)_c = (ddB)_c + (dv)_c$, $(dv)_c = 0$ by hypothesis, and $(ddB)_c = \sum_p [p : c] \sum_b [b : p] B_b = \sum_b B_b \sum_p [p : c][b : p] = 0$, since $\sum_p [p : c][b : p]$ is the coefficient of b in $\partial\partial c = 0$. □

Lemma 12.24 (sampling of $*du$)**.** *Let $u \in C_c^\infty(\mathbb{R}^3)$, $K := \operatorname{supp} u$, and $a_c := \varepsilon^{1/2} u(x_c)$ on all cubes of $\varepsilon\mathbb{Z}^3$. For $p = (x;\mu,\nu)$ with third index λ,*

$$(d^*a)_p = \epsilon_{\lambda\mu\nu}\varepsilon^{1/2}\big[u(x_p - \tfrac{\varepsilon}{2}e_\lambda) - u(x_p + \tfrac{\varepsilon}{2}e_\lambda)\big] = -\varepsilon^{3/2}(*du)_{\mu\nu}(x_p) + r_p, \qquad |r_p| \le \tfrac{1}{24}\varepsilon^{7/2}\|\partial_\lambda^3 u\|_\infty,$$

$r_p = 0$ unless $x_p \in K_{+\varepsilon/2}$, and $\sum_p r_p^2 \le \frac{1}{192}|K_{+2a_0}|\,\varepsilon^4\|u\|_{C^3}^2$.

Proof. p is the face $(x;\mu,\nu)$ of $c(x)$, with $[p : c(x)] = -\epsilon_{\lambda\mu\nu}$, and the face $(x;\mu,\nu) = (y+\varepsilon e_\lambda;\mu,\nu)$ of $c(y)$, $y = x - \varepsilon e_\lambda$, with $[p : c(y)] = +\epsilon_{\lambda\mu\nu}$; no other cube contains p. The centres are $x_{c(x)} = x_p + \frac{\varepsilon}{2}e_\lambda$ and $x_{c(y)} = x_p - \frac{\varepsilon}{2}e_\lambda$. This gives the first equality. With $g(t) := u(x_p + te_\lambda)$ and $h = \varepsilon/2$, Taylor's formula gives $g(h) - g(-h) - 2hg'(0) = \frac{h^3}{6}(g'''(\xi_+) + g'''(\xi_-))$, hence

$$|u(x_p + \frac{\varepsilon}{2}e_\lambda) - u(x_p - \frac{\varepsilon}{2}e_\lambda) - \varepsilon\partial_\lambda u(x_p)| \le \frac{\varepsilon^3}{24}\|\partial_\lambda^3 u\|_\infty;$$

multiplying by $\varepsilon^{1/2}$ and using $(*du)_{\mu\nu} = \epsilon_{\lambda\mu\nu}\partial_\lambda u$ gives the second equality and the bound on r_p. If $x_p \notin K_{+\varepsilon/2}$, the three points x_p, $x_p \pm \frac{\varepsilon}{2}e_\lambda$ lie outside K, so $r_p = 0$. The centres of the plaquettes of one orientation form a translate of $\varepsilon\mathbb{Z}^3$; the open cubes Q_p of side ε centred at x_p are pairwise disjoint, and $x_p \in K_{+\varepsilon/2}$ implies $Q_p \subset K_{+\varepsilon/2+\sqrt{3}\varepsilon/2} \subset K_{+2\varepsilon} \subset K_{+2a_0}$ ($\varepsilon_N \le a_0$). So at most $|K_{+2a_0}|/\varepsilon^3$ plaquettes of each of the three orientations have $r_p \neq 0$, and $\sum_p r_p^2 \le 3\,\frac{|K_{+2a_0}|}{\varepsilon^3}\,\frac{\varepsilon^7}{576}\|u\|_{C^3}^2$. □

Proposition 12.25 (closedness at every level)**.** *Fix N, $u \in C_c^\infty(\mathbb{R}^3)$, $s \in \mathbb{R}$, and let μ be either μ_N^{box} (Theorem 11.11(b)) or the measure μ_N of a torus limit point (Definition 2.3, Theorem 11.5); $Z_N(f) := \int e^{i\Phi_N(f,0)}d\mu$ with $\Phi_N(f,0) = \varepsilon^{3/2}\sum_p f_{\mu\nu(p)}(x_p)\mathfrak{f}_p$ (Definition 2.5). With a, r as in Lemma 12.24, $X_0 := \sum_p (d^*a)_p\mathfrak{f}_p$, $X_r := \sum_p r_p\mathfrak{f}_p$ and $\sigma^2 := \sum_p r_p^2$:*

$$X_0 = 0 \ \ \mu\text{-a.s.}, \qquad \int \Phi_N(*du,0)^2\,d\mu \le \sigma^2,$$

$$|Z_N(s\,*du) - 1| \le |s|\,\sigma \le |s|\Big(\frac{|K_{+2a_0}|}{192}\Big)^{1/2}\varepsilon_N^2\,\|u\|_{C^3}.$$

The same bounds hold for the killed-box states $\langle\cdot\rangle_\Lambda$ with Λ containing every site within distance ε of K, and for the torus states $\langle\cdot\rangle_L^{\mathrm{fine}}$ with L large.

Proof. Finite volume. If $a_c \neq 0$ then $x_c \in K$ and the vertices of c lie within $\frac{\sqrt{3}}{2}\varepsilon < \varepsilon$ of x_c, so c is a cube of Λ; every p with $(d^*a)_p \neq 0$ is a face of such a cube, hence lies in $\mathcal{P}(\Lambda)$, and X_0 is supported in Λ in the sense of Definition 11.9; so are $\Phi_N(*du,0)$ (its plaquettes have centres in K, so their vertices are within $\varepsilon/\sqrt{2}$ of K) and $X_r = X_0 + \Phi_N(*du,0)$. The box sum runs over v with $dv = 0$ on every cube of Λ (Definition 11.9), so Lemma 12.23 with $\mathcal{K} = \Lambda$ gives $X_0 = 0$ on every configuration, and $\langle e^{isX_0}\rangle_\Lambda = 1$. On a torus whose side exceeds the diameter of the finite set of cells $\{c : a_c \neq 0\}$ and their faces, these cells embed injectively; Lemma 12.23 with $\mathcal{K}$ the torus and a transported (zero elsewhere) gives $X_0 = 0$ pointwise, so $\langle e^{isX_0}\rangle_L^{\mathrm{fine}} = 1$ for all large L.

Passage. e^{isX_0} is a bounded continuous function of finitely many $\mathfrak{f}_p$; by Theorem 11.11(b) (weak convergence of the box laws along every exhausting sequence) and Theorem 11.5(a), $\int e^{isX_0}d\mu = 1$ for every real s. The law of X_0 under μ has characteristic function 1, so it is δ_0: $X_0 = 0$ μ-a.s.

Remainder. By Theorem 11.11(b), resp. Theorem 11.5(b), $\varphi(t) := \int e^{tX_r}d\mu \le e^{t^2\sigma^2/2}$ for all real t (tr is a real finitely supported plaquette function). Since $|X_r|^k e^{tX_r} \le k!\,(e^{2X_r} + e^{-2X_r})$ for $|t| \le 1$, dominated convergence makes $\varphi \in C^2(-1,1)$ with $\varphi'(0) = \int X_r d\mu$ and $\varphi''(0) = \int X_r^2 d\mu$. The function $\psi(t) := e^{t^2\sigma^2/2} - \varphi(t)$ is ≥ 0 with $\psi(0) = 0$, so $\psi'(0) = 0$ and $\psi''(0) \ge 0$: $\int X_r^2 d\mu \le \sigma^2$ (the argument of Step 5 of the proof of Theorem 3.1). By Lemma 12.24 and Definition 2.5,

$\Phi_N(*du,0)=\sum_p \varepsilon^{3/2}(*du)_{\mu\nu(p)}(x_p)\mathfrak{f}_p=-X_0+X_r$, whence $\int\Phi_N(*du,0)^2d\mu=\int X_r^2d\mu\le\sigma^2$. Finally $|e^{is(-X_0+X_r)}-e^{-isX_0}|\le|s||X_r|$ and $\int e^{-isX_0}d\mu=1$, so by the Schwarz inequality $|Z_N(s*du)-1|\le|s|\int|X_r|d\mu\le|s|\sigma$; Lemma 12.24 bounds σ. In finite volume the same steps run with Theorem 3.1 (valid for the box states, Lemma 12.10) in place of (b). □

Theorem 12.26 (closedness of the limit curvature)**.** *Let S^{box} be a limit of Theorem 7.1 along $N_1<N_2<\cdots$ with ω_N^{box} in place of ω_N (Lemma 12.20), and let S be a limit of Theorem 7.1 for the torus limit points, $\mathcal{Z}(f)=Z(f,0)$. Then for every $u\in C_c^\infty(\mathbb{R}^3)$ and $s\in\mathbb{R}$,*

$$Z^{\mathrm{box}}(s*du)=1,\qquad \mathcal{Z}(s*du)=1,$$

$$\lim_{m\to\infty}\int\Phi_{N_m}(*du,0)^2\,d\mu_{N_m}^{\mathrm{box}}=\lim_{m\to\infty}\int\Phi_{N_m}(*du,0)^2\,d\mu_{N_m}=0.$$

Proof. $s*du$ is a real C_c^∞ 2-form, so $(s*du,0)$ is a test pair. By Theorem 7.1(a) (for the box family, the relation defining S^{box}), $Z^{\mathrm{box}}(s*du)=\lim_m Z_{N_m}^{\mathrm{box}}(s*du)$, and Proposition 12.25 gives $|Z_{N_m}^{\mathrm{box}}(s*du)-1|\le|s|(|K_{+2a_0}|/192)^{1/2}\varepsilon_{N_m}^2\|u\|_{C^3}\to0$, since $\varepsilon_N=a_0 5^{-N}$. The same argument applies to $\mathcal{Z}$ and to the second moments. The density component of $(s*du,0)$ is zero, so g_{d} plays no role: the statement holds for the joint scheme with any bounded Borel g_{d}. □

Corollary 12.27. *(a) Let μ be a probability measure on a set X of linear functionals Φ on the real C_c^∞ 2-forms, with the cylinder σ-algebra and $\int e^{i\Phi(f)}d\mu=Z^{\mathrm{box}}(f)$ for all f (Theorem 9.13(a),(b), Lemma 12.12, Proposition 12.11(b)). Then $\Phi(*du)=0$ μ-a.s. for every $u\in C_c^\infty(\mathbb{R}^3)$; if $X\subset\mathcal{D}'(\mathbb{R}^3;\Lambda^2)$, then μ-a.e. Φ satisfies $\Phi(*du)=0$ for all u simultaneously, i.e. $d\Phi=0$ in the sense of distributions ($*\Phi$ is divergence-free). (b) The Schwinger functions of S^{box} (Theorem 9.4(iii)) satisfy $S_n(f_1,\dots,f_n)=0$ whenever some $f_i=*du$, $u\in C_c^\infty(\mathbb{R}^3)$; likewise for S. (c) (a), (b) and $Z^{\mathrm{box}}(s*du)=1$ hold for $u\in\mathcal{S}(\mathbb{R}^3)$.*

Proof. (a) The characteristic function of the real random variable $\Phi(*du)$ is $s\mapsto Z^{\mathrm{box}}(s*du)=1$, that of δ_0. Let $K_1\subset K_2\subset\cdots$ be compact cubes exhausting $\mathbb{R}^3$; each $C_{K_n}^\infty$ is a separable Fréchet space, so there is a countable $D\subset C_c^\infty$ such that every u is a limit in some $C_{K_n}^\infty$ of elements of D. The union $\mathcal{N}$ of the null sets $\{\Phi(*du)\neq0\}$, $u\in D$, is null; for $\Phi\notin\mathcal{N}$ and $u_k\to u$ in $C_{K_n}^\infty$, $*du_k\to*du$ in $C_{K_n}^\infty(\Lambda^2)$ and $\Phi(*du_k)\to\Phi(*du)$ by the continuity of Φ on test forms, so $\Phi(*du)=0$. For distributions $\langle\Phi,*du\rangle=-\langle(d\Phi)_{123},u\rangle$ by definition, so $d\Phi=0$. (b) By Lemma 9.3(c) and Theorem 9.4(iii), $S_n(f_1,\dots,f_n)=\int x_1\cdots x_n\,d\mu_f(x)$ with μ_f the law on $\mathbb{R}^n$ with characteristic function $t\mapsto Z^{\mathrm{box}}(\sum_j t_jf_j)$ (all exponential moments finite). If $f_1=*du$, the first marginal has characteristic function $t_1\mapsto Z^{\mathrm{box}}(t_1*du)=1$, so $x_1=0$ μ_f-a.s. and $S_n=0$. (c) By Theorem 9.4(i), Z^{box} extends continuously to $\mathcal{S}(\mathbb{R}^3;\Lambda^2)$; for cut-offs $\chi_R=\chi(\cdot/R)$, $*d(\chi_Ru)\to*du$ in $\mathcal{S}$, so $Z^{\mathrm{box}}(s*du)=\lim_R Z^{\mathrm{box}}(s*d(\chi_Ru))=1$; (a) and (b) follow as before, S_n being continuous on $\mathcal{S}^n$ (Theorem 9.4(iii)). □

Remark 12.28. (i) With the opposite orientation of the cubes, $(d^*a)_p=+\varepsilon^{3/2}(*du)_{\mu\nu}(x_p)-r_p$; nothing else changes. (ii) No property of the couplings, of L or of Λ enters: Lemma 12.23 is exact, Lemma 12.24 is calculus, and the only probabilistic input is the Gaussian domination of Theorem 3.1 as carried by Theorems 11.11(b) and 11.5(b).

12.10. **The two-point function: rotation covariance from reflection positivity in nine planes.** *Setting.* Let $T=\Lambda^2\mathbb{R}^3\oplus\mathbb{R}$ or $T=\Lambda^2\mathbb{R}^3$, with its orthogonal O(3)-action π ($\pi(R)=\Lambda^2R\,(\oplus1)$); we write $R^*\chi:=\pi(R)^{-1}\,\chi\circ R$ for the pull-back of $\chi\colon\mathbb{R}^3\to T_{\mathbb{C}}$, $\tau_a\chi:=\chi(\cdot-a)$, and $\|\chi\|:=\|\chi\|_{L^1}+\|\chi\|_{L^2}$. For a unit vector u put $\theta_ux:=x-2(u\cdot x)u$, $\Theta_u:=\theta_u^*$ (so $\Theta_u\chi=\pi(\theta_u)\,\chi\circ\theta_u$), $\Omega_u^+:=\{u\cdot x>0\}$. $\mathcal{N}:=\{e_j,\ (e_j\pm e_k)/\sqrt2:\ j\neq k\}$ is the set of the nine unit normals (up to sign). Let μ be a probability measure on $\mathcal{S}'(\mathbb{R}^3;T)$ with the σ-algebra

generated by the evaluations $\Phi \mapsto \Phi(\chi)$, $\chi \in \mathcal{S}(\mathbb{R}^3;T)$ real; $\Phi(\chi)$ for complex χ is defined by linearity. We consider the following hypotheses:

(i) $\int e^{\Phi(\chi)}d\mu \le e^{C(\|\chi\|_1+\|\chi\|_2^2)}$ for all real $\chi \in \mathcal{S}(\mathbb{R}^3;T)$;
(ii) $\tau_{a*}\mu = \mu$ for all $a \in \mathbb{R}^3$, where $(\tau_a\Phi)(\chi) := \Phi(\tau_{-a}\chi)$;
(iii) $\sum_{a,b} \bar{z}_a z_b \int e^{i\Phi(\chi_a - \Theta_u\chi_b)}d\mu \ge 0$ for every $u \in \mathcal{N}$, every finite family of real $\chi_a \in C_c^\infty(\Omega_u^+;T)$ and all $z_a \in \mathbb{C}$.

We write $S_n(\chi_1,\dots,\chi_n) := \int \prod_i \Phi(\chi_i)\,d\mu$ (complex-multilinear) and $Z(\chi) := \int e^{i\Phi(\chi)}d\mu$. Note that $(R_1R_2)^* = R_2^*R_1^*$.

Theorem 12.29 (rotation covariance of the two-point function)**.** *Under (i)–(iii):*

(a) $\int |\Phi(\chi)|^2 d\mu \le c_0\|\chi\|^2$ *for real* χ, $c_0 := 3C^2 + 15C$; *all* S_n *exist and are continuous for* $\|\cdot\|$ *in each entry.*
(b) μ *is invariant under all translations and under the hyperoctahedral group* B_3 *of signed permutation matrices, acting by* $\Phi \mapsto \Phi \circ g^*$; *in particular* $S_n(g^*\chi_1,\dots,g^*\chi_n) = S_n(\chi_1,\dots,\chi_n)$ *for* $g \in B_3$ *and all* χ_i.
(c) *For* $u \in \mathcal{N}$ *the reconstruction* $(\mathcal{H}_u, H_u \ge 0, V_u, E_u)$ *from* $L^2(\Sigma(\Omega_u^+),\mu)$ *exists, and* $E_u(\{(\omega,p) : \omega < |p\cdot m|\}) = 0$ *whenever* $m \perp u$ *is a unit vector with* $(u+m)/\sqrt{2} \in \mathcal{N}$; *for* $u = e_j$ *this is* $\omega \ge |p|_\infty$.
(d) $S_1(R^*\chi) = S_1(\chi)$ *and* $S_2(R^*\chi_1, R^*\chi_2) = S_2(\chi_1,\chi_2)$ *for every* $R \in \mathrm{O}(3)$ *and all* $\chi_1,\chi_2 \in \mathcal{S}(\mathbb{R}^3;T_{\mathbb{C}})$ *with disjoint supports.*

Proof. Step 1 (moments). Put $X = \Phi(\chi)$ for real χ, $a = C\|\chi\|_1$, $b = C\|\chi\|_2^2$. By (i) at $\pm\lambda\chi$ and Markov's inequality, $\mu(|X| > t) \le 2e^{-\lambda t + a\lambda + b\lambda^2} \le 2e^{-(t-a)^2/4b}$ for $t > a$ ($\lambda = (t-a)/2b$), so $\int X^2 d\mu \le a^2 + 4\int_0^\infty (s+a)e^{-s^2/4b}ds = a^2 + 8b + 4a\sqrt{\pi b} \le 3a^2 + 15b$. Hence $\|\Phi(\chi) - \Phi(\chi')\|_{L^2(\mu)} \le \sqrt{c_0}\|\chi - \chi'\|$ for real χ,χ', and $\|\Phi(\chi)\|_{L^2(\mu)} \le 2\sqrt{c_0}\|\chi\|$ for complex χ; all exponential moments are finite by (i), so all S_n exist (Hölder), and they, as well as Z, are $\|\cdot\|$-continuous; this proves (a). Consequently Φ extends by continuity to $(L^1 \cap L^2)(\mathbb{R}^3;T_{\mathbb{C}})$ with values in $L^2(\mu)$, and S_n to $(L^1\cap L^2)^n$.

Step 2 (reflections). For $\psi_1,\psi_2 \in C_c^\infty(\Omega_u^+;T)$ real the 2×2 matrix $(Z(\psi_a - \Theta_u\psi_b))_{ab}$ is positive semidefinite by (iii), hence Hermitian:

$$Z(\psi_2 - \Theta_u\psi_1) = \overline{Z(\psi_1 - \Theta_u\psi_2)} = Z(\Theta_u\psi_2 - \psi_1) = Z(\Theta_u(\psi_2 - \Theta_u\psi_1)),$$

using $\overline{Z(\chi)} = Z(-\chi)$ and $\Theta_u^2 = 1$. Every real $\chi \in C_c^\infty$ vanishing in a neighbourhood of the plane $\{u\cdot x = 0\}$ is of the form $\psi_2 - \Theta_u\psi_1$, so $Z\circ\Theta_u = Z$ on such χ; for real $\chi \in \mathcal{S}$, the cut-offs $\chi\,\vartheta(u\cdot x/\delta)$ ($\vartheta \in C^\infty(\mathbb{R})$, $\vartheta = 0$ on $[-1,1]$, $\vartheta = 1$ off $[-2,2]$, $0 \le \vartheta \le 1$) converge to χ in $\|\cdot\|$ as $\delta \downarrow 0$ (dominated convergence), and Θ_u is an isometry for $\|\cdot\|$, so $Z\circ\Theta_u = Z$ on $\mathcal{S}$ by (a). Two probability measures on $\mathcal{S}'$ with equal characteristic functionals coincide (the laws of $(\Phi(\chi_j))_{j\le k}$ have the same Fourier transforms, and the cylinder sets form a π-system generating the σ-algebra; the argument of Proposition 12.11(b)), so $\theta_{u*}\mu = \mu$, where $(\theta_u\Phi)(\chi) := \Phi(\Theta_u\chi)$. The nine reflections generate B_3, and (ii) gives the translations; this proves the invariance in (b), and the moment identity follows by the change of variables.

Step 3 (reconstruction). Fix $u \in \mathcal{N}$. Let $\Sigma(\Omega_u^+)$ be the σ-algebra generated by the $\Phi(\chi)$, $\chi \in C_c^\infty(\Omega_u^+;T)$ real, $L^2_{u,+} := L^2(\Sigma(\Omega_u^+),\mu)$, $\Theta_u A := A\circ\theta_u$ (unitary on $L^2(\mu)$ by Step 2) and $\langle A,B\rangle_u := \int \overline{\Theta_u A}\,B\,d\mu$ on $L^2_{u,+}$. On $A = \sum_a z_a e^{i\Phi(\chi_a)}$ the form is $\langle A,A\rangle_u = \sum_{a,b}\bar{z}_a z_b Z(\chi_b - \Theta_u\chi_a) \ge 0$ by (iii); such A form a unital $*$-algebra of bounded functions generating $\Sigma(\Omega_u^+)$, hence dense in $L^2_{u,+}$ by the functional monotone class theorem (Remark 9.14), and the form is L^2-continuous ($|\langle A,B\rangle_u| \le \|A\|_{L^2(\mu)}\|B\|_{L^2(\mu)}$), so it is positive semidefinite on $L^2_{u,+}$. Let $\mathcal{H}_u$ be the completion of $L^2_{u,+}$ modulo the null space, $[A]$ the class of A; the class map is a contraction.

With $U(a)A := A\circ\tau_{-a}$ (so that $U(a)\Phi(\chi)=\Phi(\tau_a\chi)$), U is a strongly continuous unitary group on $L^2(\mu)$ by (ii) and (a) (strong continuity on the dense exponentials by $|e^{ix}-e^{iy}|\le|x-y|$), $\Theta_u U(a)=U(\theta_u a)\Theta_u$, and $U(tu)L^2_{u,+}\subset L^2_{u,+}$ for $t\ge0$. Hence $\langle U(tu)A,U(tu)B\rangle_u=\langle A,U(2tu)B\rangle_u=\langle U(2tu)A,B\rangle_u$ for $t\ge0$ and $\langle U(a)A,U(a)B\rangle_u=\langle A,B\rangle_u$ for $a\perp u$. As in Remark 9.14 ($\|[U(tu)A]\|^2=\langle A,U(2tu)A\rangle_u\le\|[A]\|\,\|[U(2tu)A]\|$, iterated), $[A]\mapsto[U(tu)A]$ defines a strongly continuous self-adjoint contraction semigroup e^{-tH_u} on $\mathcal{H}_u$ with $H_u\ge0$, and $[A]\mapsto[U(a)A]$, $a\perp u$, a strongly continuous unitary group $V_u(a)$ commuting with it; let E_u be their joint spectral measure on $[0,\infty)\times u^\perp$, so that $e^{-tH_u}V_u(a)=\int e^{-t\omega+ip\cdot a}dE_u(\omega,p)$. For $f,f'\in C_c^\infty(\Omega_u^+;T_{\mathbb{C}})$, $t\ge0$, $a\perp u$, since $\overline{\Theta_u\Phi(f)}=\Phi(\overline{\Theta_u f})$,

$$S_2\big(\overline{\Theta_u f},\tau_{tu+a}f'\big)=\langle[\Phi(f)],e^{-tH_u}V_u(a)[\Phi(f')]\rangle_u=\int e^{-t\omega+ip\cdot a}\,d\langle[\Phi(f)],E_u[\Phi(f')]\rangle_u. \tag{12.5}$$

Step 4 (the cone). Let $u\in\mathcal{N}$ and $m\perp u$ a unit vector with $u':=(u+m)/\sqrt2\in\mathcal{N}$ (for $u=e_j$: $m=\pm e_i$, $i\ne j$; for $u=(e_j+e_i)/\sqrt2$: $m=\pm(e_j-e_i)/\sqrt2$, $u'=e_j$ or e_i); put $u'':=(u-m)/\sqrt2\perp u'$. Fix $d>0$ and let A be a bounded $\Sigma(Q_d)$-measurable function, $Q_d:=\{x:0<u\cdot x<d,\ |x-(u\cdot x)u|<d\}$; such classes $[A]$ are dense in $\mathcal{H}_u$: the σ-algebras $\Sigma(Q_d)$ increase to $\Sigma(\Omega_u^+)$ as $d\to\infty$, since every $\chi\in C_c^\infty(\Omega_u^+;T)$ is supported in some Q_d; by the martingale convergence theorem the bounded $\Sigma(Q_d)$-measurable functions are dense in the bounded $\Sigma(\Omega_u^+)$-measurable ones for $L^2(\mu)$, which are dense in $L^2_{u,+}$, and the class map is a contraction. For $a,c\ge0$ put $b(a,c):=2du+\sqrt2\,a\,u'+\sqrt2\,c\,u''=(2d+a+c)u+(a-c)m$ and $G(a,c):=\langle[A],U(b(a,c))[A]\rangle_u=\int\overline{\Theta_uA}\,U(b(a,c))A\,d\mu$.

First representation: $U(b(a,c))=U((2d+a+c)u)U((a-c)m)$ on $L^2_{u,+}$, so $G(a,c)=\int e^{-(2d+a+c)\omega+i(a-c)p\cdot m}d\mu_A(\omega,p)$ with $\mu_A:=\langle[A],E_u(\cdot)[A]\rangle_u$, a finite positive measure. With $z:=\omega-ip\cdot m$ this is $\int e^{-az-c\bar z}e^{-2d\omega}d\mu_A$.

Second representation: put $y_0:=\frac d2(u+m)$, $A^{(1)}:=\Theta_{u'}U(-y_0)\Theta_uA$ and $A^{(2)}:=U(2du-y_0)A$. A point $x\in Q_d$ has $0<u\cdot x<d$ and $|m\cdot x|<d$; the point θ_ux-y_0 has u-component in $(-\frac32d,-\frac12d)$ and m-component in $(-\frac32d,\frac12d)$, hence lies in $\{u'\cdot x<0\}$, which $\theta_{u'}$ maps onto $\{u'\cdot x>0\}$; the point $x+2du-y_0$ has u-component in $(\frac32d,\frac52d)$ and m-component in $(-\frac32d,\frac12d)$, hence lies in $\{u'\cdot x>0\}$; and $\sqrt2au'+\sqrt2cu''$ has non-negative u'-component. So $A^{(1)},A^{(2)}\in L^2_{u',+}$, and $U(\sqrt2au'+\sqrt2cu'')A^{(2)}\in L^2_{u',+}$. By (ii) (translating both factors by $-y_0$) and $\Theta^2_{u'}=1$,

$$\begin{aligned}G(a,c)&=\int\overline{U(-y_0)\Theta_uA}\,U(b(a,c)-y_0)A\,d\mu=\int\overline{\Theta_{u'}A^{(1)}}\,U(\sqrt2au'+\sqrt2cu'')A^{(2)}\,d\mu\\&=\langle[A^{(1)}],e^{-\sqrt2aH_{u'}}V_{u'}(\sqrt2cu'')[A^{(2)}]\rangle_{u'},\end{aligned}$$

which is $\int e^{-\sqrt2a\omega'+i\sqrt2c\,p'\cdot u''}d\varrho_{12}(\omega',p')$ with $\varrho_{12}:=\langle[A^{(1)}],E_{u'}(\cdot)[A^{(2)}]\rangle_{u'}$, a complex measure of total variation at most $\|[A^{(1)}]\|_{u'}\|[A^{(2)}]\|_{u'}\le\|A\|^2_{L^2(\mu)}$. Let ϱ be the image of $e^{-2d\omega}\mu_A$ under $(\omega,p)\mapsto(\omega,p\cdot m)$ and ϱ' the image of ϱ_{12} under $(\omega',p')\mapsto(\sqrt2\omega',\sqrt2\,p'\cdot u'')$. Then $\int e^{-az-c\bar z}d\varrho=\int e^{-a\omega'+icp'}d\varrho'$ for all $a,c\ge0$, and Lemma 12.30 below gives $\varrho(\{\omega<|q|\})=0$, i.e. $\mu_A(\{\omega<|p\cdot m|\})=0$ ($e^{-2d\omega}>0$). By density and the boundedness of $E_u(\{\omega<|p\cdot m|\})$, $E_u(\{\omega<|p\cdot m|\})=0$, which is (c). For $u=e_j$ the admissible m are $\pm e_i$, $i\ne j$, which gives $\omega\ge|p|_\infty$.

Step 5 (planar kernels). Fix a permutation (j,i,l) of $(1,2,3)$ and write $x=(y,z)$ with $y=(x_j,x_i)\in\mathbb{R}^2$ (coordinates in the basis $\hat e_j,\hat e_i$ of the plane) and $z=x_l$. Let $\hat u$ range over the eight planar unit vectors $\pm\hat e_j$, $\pm\hat e_i$, $\pm\hat n$, $\pm\hat m$, $\hat n:=(\hat e_j+\hat e_i)/\sqrt2$, $\hat m:=(\hat e_j-\hat e_i)/\sqrt2$, let $u\in\mathbb{R}^3$ be the corresponding vector (so $\pm u\in\mathcal{N}$, $\Theta_{-u}=\Theta_u$) and $\hat u^\perp$, $u^\perp$ the rotation of $\hat u$, u by $+\pi/2$ in the plane. In each case $(u+u^\perp)/\sqrt2\in\pm\mathcal{N}$, so Step 4 (with $m=u^\perp$; hypothesis (iii) for

$-u$ follows from (iii) for u by the invariance of Z under Θ_u, Step 2) gives: E_u is carried by $\{\omega \geq |p \cdot u^\perp|\}$. For $\phi \in C_c^\infty(\mathbb{R}^2;\mathbb{C})$ and $w \in \mathcal{W} := C_c^\infty(\mathbb{R};T_{\mathbb{C}})$ put $(\phi \otimes w)(x) := \phi(y)w(z)$, and $J_u w := \pi(\theta_u)\bar{w}$, an antilinear involution of $\mathcal{W}$ ($\pi(\theta_u)$ is a real orthogonal involution). Since θ_u fixes z, $\Theta_u(\phi \otimes w) = (\phi \circ \hat{\theta}_u) \otimes \pi(\theta_u)w$ and $\overline{\Theta_u(\phi \otimes w)} = \overline{\phi \circ \hat{\theta}_u} \otimes J_u w$, $\hat{\theta}_u$ the planar reflection.

(5a) The spectral measures and a Laplace bound. Fix $w \in \mathcal{W}$. Let $\psi_k \in C_c^\infty(\mathbb{R}^2)$, $\psi_k \geq 0$, $\int \psi_k = 1$, $\operatorname{supp}\psi_k \subset B(s_k\hat{u}, s_k/2)$, $s_k \downarrow 0$, and $f_k := \psi_k \otimes w \in C_c^\infty(\Omega_u^+;T_{\mathbb{C}})$. Let ν_k be the image of the finite positive measure $\langle[\Phi(f_k)], E_u(\cdot)[\Phi(f_k)]\rangle_u$ under $(\omega,p) \mapsto (\omega, q := p \cdot u^\perp)$; it is carried by the cone $\mathsf{C} := \{\omega \geq |q|\}$. For $\phi \in C_c^\infty(\{\hat{u}\cdot y > 0\})$ put $\mathcal{L}\phi(\omega,q) := \int \phi(y) e^{-(\hat{u}\cdot y)\omega + i(\hat{u}^\perp \cdot y)q} dy$. Identify $y \in \mathbb{R}^2$ with the vector $(\hat{u}\cdot y)u + (\hat{u}^\perp \cdot y)u^\perp \in \mathbb{R}^3$ and write $(\mathcal{L}\phi)(H_u, P\cdot u^\perp) := \int \mathcal{L}\phi(\omega, p\cdot u^\perp)\, dE_u(\omega,p)$. For $\hat{u}\cdot y \geq 0$,

$$[\Phi(\tau_y f_k)] = [U(y)\Phi(f_k)] = e^{-(\hat{u}\cdot y)H_u} V_u((\hat{u}^\perp \cdot y)u^\perp)[\Phi(f_k)],$$

and $\int \phi(y)\tau_y f_k\, dy = (\phi * \psi_k) \otimes w$ as a Bochner integral in $L^1 \cap L^2$, so by Step 1, the continuity of the class map and the spectral calculus,

$$[\Phi((\phi * \psi_k) \otimes w)] = \int \phi(y)[\Phi(\tau_y f_k)]dy = (\mathcal{L}\phi)(H_u, P\cdot u^\perp)[\Phi(f_k)]$$

and

$$\int |\mathcal{L}\phi|^2\, d\nu_k = \|[\Phi((\phi * \psi_k)\otimes w)]\|_u^2 \leq 4c_0 \|(\phi * \psi_k)\otimes w\|^2 \leq K_w\big(\|\phi\|_1^2 + \|\phi\|_2^2\big),$$
$$K_w := 8c_0 \max(\|w\|_1^2, \|w\|_2^2),$$

by Young's inequality ($\|\psi_k\|_1 = 1$), uniformly in k.

Claim: every positive measure ν on C with $\int |\mathcal{L}\phi|^2 d\nu \leq K(\|\phi\|_1^2 + \|\phi\|_2^2)$ for all such ϕ satisfies $\int e^{-\tau\omega} d\nu \leq C_* K(1+\tau^{-2})$ for all $\tau > 0$, with C_* an absolute constant. Indeed, fix $\phi_0 \in C_c^\infty(\mathbb{R}^2)$, $\phi_0 \geq 0$, $\int \phi_0 = 1$, $\operatorname{supp}\phi_0 \subset B(\hat{u}, \frac{1}{8})$, put $A := 8\pi/3$, $\kappa_0 := \frac{9}{4}$ and $\phi_\sigma := \sigma^{-2}\phi_0(\cdot/\sigma)$ ($\sigma > 0$), so that $\|\phi_\sigma\|_1 = 1$, $\|\phi_\sigma\|_2^2 = \sigma^{-2}\|\phi_0\|_2^2$, and $\hat{u}\cdot y \in [\frac{7}{8}\sigma, \frac{9}{8}\sigma]$, $|\hat{u}^\perp \cdot y| \leq \frac{\sigma}{8}$ on $\operatorname{supp}\phi_\sigma$. On $\{\sigma\omega \leq A\} \cap \mathsf{C}$ the phase satisfies $|(\hat{u}^\perp \cdot y)q| \leq \frac{\sigma}{8}\omega \leq \frac{A}{8} = \frac{\pi}{3}$, so $\operatorname{Re}\mathcal{L}\phi_\sigma \geq \frac{1}{2}e^{-\kappa_0\sigma\omega/2}$ there and $\int_{\{\sigma\omega\leq A\}} e^{-\kappa_0\sigma\omega} d\nu \leq 4K(1+\sigma^{-2}\|\phi_0\|_2^2)$. For $\tau > 0$ put $\sigma_m := \tau/(\kappa_0 2^m)$, $m \geq 0$, and split C into $S_0 := \{\tau\omega \leq \kappa_0 A\}$ and $S_m := \{2^{m-1}\kappa_0 A < \tau\omega \leq 2^m\kappa_0 A\}$, $m \geq 1$. On S_m, $\sigma_m\omega \leq A$ and $e^{-\tau\omega} = e^{-\kappa_0\sigma_m\omega}e^{-(2^m-1)\kappa_0\sigma_m\omega} \leq e^{-\kappa_0\sigma_m\omega}e^{-2^{m-2}\kappa_0 A}$ for $m \geq 1$. Summing,

$$\int e^{-\tau\omega} d\nu \leq 4K \sum_{m\geq 0} e^{-2^{m-2}\kappa_0 A}\mathbf{1}_{m\geq 1}(1 + \kappa_0^2 4^m \tau^{-2}\|\phi_0\|_2^2) \leq C_* K(1+\tau^{-2}),$$

the series converging since $\kappa_0 A = 6\pi$. Hence $\int e^{-\tau\omega} d\nu_k \leq C_* K_w(1+\tau^{-2})$ for all k and $\tau > 0$.

(5b) The limit measure. Fix $l \geq 1$. The measures $e^{-\omega/l}\nu_k$ have total mass at most $C_* K_w(1+l^2)$, satisfy

$$(e^{-\omega/l}\nu_k)(\{\omega > R\}) \leq e^{-R/2l}\int e^{-\omega/2l} d\nu_k \leq e^{-R/2l} C_* K_w(1+4l^2),$$

and on $\{\omega \leq R\}$ they are carried by the compact set $\mathsf{C}\cap\{\omega \leq R\}$: they are tight. By Prokhorov's theorem and a diagonal argument there is a subsequence along which $e^{-\omega/l}\nu_k \to \mu_l$ weakly for every l, μ_l a finite measure on C. For $l' < l$ the function $e^{-\omega(1/l'-1/l)}$ is bounded and continuous on C, so $\mu_{l'} = e^{-\omega(1/l'-1/l)}\mu_l$ and $\nu_w := e^{\omega/l}\mu_l$ does not depend on l: a positive measure on C with $\int e^{-\tau\omega} d\nu_w \leq C_* K_w(1+\tau^{-2})$ for every $\tau > 0$ (apply the weak convergence to the bounded continuous function $e^{-(\tau-1/l)\omega}$, $l > 1/\tau$).

For $v \in \mathbb{R}^2$ with $\hat u \cdot v > 0$, along the subsequence,

$$g_k(v) := S_2\big(\overline{\Theta_u f_k}, \tau_v f_k\big) = \int e^{-(\hat u\cdot v)\omega + i(\hat u^\perp\cdot v)q}\, d\nu_k(\omega,q)$$
$$\longrightarrow\ \hat W_w(v) := \int e^{-(\hat u\cdot v)\omega + i(\hat u^\perp\cdot v)q}\, d\nu_w(\omega,q),$$

the first equality by (12.5), the limit by the weak convergence against the bounded continuous function $e^{\omega/l}e^{-(\hat u\cdot v)\omega+i(\hat u^\perp\cdot v)q}$ on $\mathsf C$ for $l > 1/(\hat u\cdot v)$. $\hat W_w$ is continuous on $\{\hat u\cdot v>0\}$ (indeed holomorphic in $\zeta\in\mathbb C^2$ on $\{\mathrm{Re}(\hat u\cdot\zeta) > |\,\mathrm{Im}(\hat u^\perp\cdot\zeta)|\}$, where the integral converges absolutely and locally uniformly, since $|e^{-(\hat u\cdot\zeta)\omega+i(\hat u^\perp\cdot\zeta)q}| \le e^{-(\mathrm{Re}(\hat u\cdot\zeta)-|\,\mathrm{Im}(\hat u^\perp\cdot\zeta)|)\omega}$ on $\mathsf C$), and $|g_k(v)|, |\hat W_w(v)| \le C_* K_w(1+\delta^{-2})$ on $\{\hat u\cdot v\ge\delta\}$.

(5c) Identification. Let $\phi,\phi'\in C_c^\infty(\mathbb R^2;\mathbb C)$ with $\mathrm{supp}\,\phi' - \mathrm{supp}\,\phi \subset \{\hat u\cdot v>0\}$; by compactness $\mathrm{supp}\,\phi' - \mathrm{supp}\,\phi \subset \{\hat u\cdot v\ge\delta\}$ for some $\delta>0$. By (ii), $g_k(y'-y) = S_2(\tau_y\overline{\Theta_u f_k}, \tau_{y'} f_k)$, and $\int\phi(y)\tau_y(\psi_k\circ\hat\theta_u)\,dy = \big((\phi\circ\hat\theta_u)*\psi_k\big)\circ\hat\theta_u$, so, the integrals being Bochner integrals in $L^1\cap L^2$ and S_2 continuous bilinear,

$$\iint \phi(y)\phi'(y')g_k(y'-y)\,dy\,dy' = S_2\Big(\big[((\phi\circ\hat\theta_u)*\psi_k)\circ\hat\theta_u\big]\otimes J_u w,\ (\phi'*\psi_k)\otimes w\Big)$$
$$\longrightarrow\ S_2\big(\phi\otimes J_u w,\ \phi'\otimes w\big),$$

while the left side tends to $\iint\phi(y)\phi'(y')\hat W_w(y'-y)\,dy\,dy'$ by dominated convergence (the bound of (5b) on $\{\hat u\cdot v\ge\delta\}$). Hence

$$\iint \phi(y)\phi'(y')\,\hat W_w(y'-y)\,dy\,dy' = S_2\big(\phi\otimes J_u w,\ \phi'\otimes w\big) \qquad (\mathrm{supp}\,\phi'-\mathrm{supp}\,\phi\subset\{\hat u\cdot v>0\}). \tag{12.6}$$

A continuous function on $\{\hat u\cdot v>0\}$ is determined by these integrals (approximate identities concentrating ϕ at y_0 and ϕ' at y_0+v_0), so $\hat W_w$ does not depend on the subsequence.

(5d) Polarisation. The map $(w_1,w_2)\mapsto S_2(\phi\otimes J_u w_1, \phi'\otimes w_2)$ is sesquilinear on $\mathcal W$ (antilinear in w_1), and (12.6) gives its diagonal. By the polarisation identity $s(w_1,w_2) = \frac14\sum_{k=0}^3 i^k s(w_2+i^k w_1, w_2+i^k w_1)$, the complex measure $\nu_{w_1,w_2} := \frac14\sum_k i^k\nu_{w_2+i^k w_1}$ on $\mathsf C$ and its transform $\hat W_{w_1,w_2}(v) := \int e^{-(\hat u\cdot v)\omega+i(\hat u^\perp\cdot v)q}d\nu_{w_1,w_2}$ on $\{\hat u\cdot v>0\}$ satisfy $\int e^{-\tau\omega}d|\nu_{w_1,w_2}|\le C(1+\tau^{-2})$ and $\iint\phi\phi'\hat W_{w_1,w_2}(y'-y) = S_2(\phi\otimes J_u w_1,\phi'\otimes w_2)$ for the pairs of (12.6). For $\alpha,\beta\in T_{\mathbb C}$ and $\eta,\eta'\in C_c^\infty(\mathbb R;\mathbb C)$ put $W^u_{\eta\alpha,\eta'\beta} := \hat W_{J_u(\eta\alpha),\eta'\beta}$ (so $J_u w_1 = \eta\alpha$). Then, for $\mathrm{supp}\,\phi'-\mathrm{supp}\,\phi\subset\{\hat u\cdot v>0\}$,

$$\iint \phi(y)\phi'(y')\,W^u_{\eta\alpha,\eta'\beta}(y'-y)\,dy\,dy' = S_2\big((\phi\otimes\eta)\alpha,\ (\phi'\otimes\eta')\beta\big), \tag{12.7}$$

$W^u_{\eta\alpha,\eta'\beta}$ is complex-bilinear in $(\eta\alpha,\eta'\beta)\in\mathcal W\times\mathcal W$ (the right side is, and the kernel is unique), it is the transform of a complex measure $\nu^u_{\eta\alpha,\eta'\beta}$ on $\mathsf C$ with $\int e^{-\tau\omega}d|\nu^u|\le C(1+\tau^{-2})$, and $\nu^u_{\eta\alpha,\eta'\beta}\ge 0$ whenever $\eta\alpha = J_u(\eta'\beta)$, in particular whenever $\eta'=\eta$ is real and $\alpha=\pi(\theta_u)\bar\beta$.

(5e) One kernel on $\mathbb R^2\setminus\{0\}$. The four open half-planes $\{\hat u\cdot v>0\}$, $\hat u\in\{\pm\hat e_j,\pm\hat e_i\}$, cover $\mathbb R^2\setminus\{0\}$; on an overlap the two kernels agree, since both satisfy (12.7) for the pairs with $\mathrm{supp}\,\phi'-\mathrm{supp}\,\phi$ in the overlap, which determine a continuous function there. So there is one continuous $W_{\eta\alpha,\eta'\beta}$ on $\mathbb R^2\setminus\{0\}$, bilinear in $(\eta\alpha,\eta'\beta)$, with (12.7) in every frame, including the diagonal frames $\pm\hat n,\pm\hat m$ (same argument). It satisfies (12.7) for *all* ϕ,ϕ' with disjoint supports: with $D := \mathrm{dist}(\mathrm{supp}\,\phi,\mathrm{supp}\,\phi')>0$, $\delta<D/4$ and a smooth partition of unity (ϱ_a) of $\mathbb R^2$ with $\mathrm{diam}\,\mathrm{supp}\,\varrho_a<\delta$, write $\phi=\sum_a\varrho_a\phi$, $\phi'=\sum_b\varrho_b\phi'$. For a pair (a,b) of non-zero pieces pick $y_a\in\mathrm{supp}\,\varrho_a\phi$, $y'_b\in\mathrm{supp}\,\varrho_b\phi'$; $v_0 := y'_b - y_a$ has $|v_0|\ge D$, so $|\hat u\cdot v_0|\ge D/\sqrt2 > 2\delta$ for $\hat u=\hat e_j$ or

$\hat{e}_i$, and $\operatorname{supp}\varrho_b\phi' - \operatorname{supp}\varrho_a\phi \subset B(v_0, 2\delta) \subset \{\pm\hat{u}\cdot v > 0\}$. Apply (12.7) in that frame to each pair and sum (both sides are bilinear).

Three consequences, each by the uniqueness of the kernel on open sets: (1) *symmetry*: $S_2(f,f') = S_2(f',f)$ gives $W_{\eta\alpha,\eta'\beta}(v) = W_{\eta'\beta,\eta\alpha}(-v)$; (2) *conjugation*: $S_2(\bar{f},\bar{f}') = \overline{S_2(f,f')}$ (μ is a measure on real fields) gives $W_{\bar{\eta}\bar{\alpha},\bar{\eta}'\bar{\beta}} = \overline{W_{\eta\alpha,\eta'\beta}}$; (3) *dihedral covariance*: for $g \in B_3$ fixing e_l, acting on the plane by $\hat{g}$, (b) and $g^*((\phi\otimes\eta)\alpha) = ((\phi\circ\hat{g})\otimes\eta)\pi(g)^{-1}\alpha$ give $W_{\eta\alpha,\eta'\beta}(\hat{g}v) = W_{\eta\pi(g)^{-1}\alpha,\,\eta'\pi(g)^{-1}\beta}(v)$ for all $v \neq 0$.

Step 6 (the angle). Let R_t be the rotation by t in the (j,i)-plane ($R_te_j = \cos t\, e_j + \sin t\, e_i$, $R_te_i = -\sin t\, e_j + \cos t\, e_i$, $R_te_l = e_l$), $\hat{R}_t$ its planar action, and $n_t := (\cos t, \sin t)$. $T_{\mathbb{C}}$ splits into weight vectors $\pi(R_t)\epsilon = e^{iwt}\epsilon$: $e_j \wedge e_i$ and the scalar component with $w = 0$, and $\epsilon_\pm := (e_j\wedge e_l \mp i e_i \wedge e_l)/\sqrt{2}$ with $w = \pm 1$ (indeed $\Lambda^2 R_t$ rotates the pair $(e_j\wedge e_l, e_i\wedge e_l)$). One computes $\bar{\epsilon}_\pm = \epsilon_\mp$, $\pi(\theta_j)\epsilon_\pm = -\epsilon_\mp$, and, for $n := (e_j + e_i)/\sqrt{2}$ ($\theta_n e_j = -e_i$, $\theta_n e_i = -e_j$), $\pi(\theta_n)\epsilon_+ = i\epsilon_-$, $\pi(\theta_n)\epsilon_- = -i\epsilon_+$.

Fix η,η' and weight vectors a,b with weights w_a, w_b, $w := w_a + w_b \in \{0,\pm1,\pm2\}$, and write $W_{ab} := W_{\eta a,\eta' b}$. By (5e)(3) with $g = R_{\pi/2} \in B_3$: $W_{ab}(\hat{R}_{\pi/2}v) = e^{-iw\pi/2}W_{ab}(v)$. Fix $r > 0$. By (12.7) in the frame $\hat{e}_j$, $F_0(\zeta) := \int e^{-r\cos\zeta\,\omega + ir\sin\zeta\, q}\, d\nu^{e_j}_{\eta a,\eta' b}(\omega,q)$ is holomorphic on the strip $|\operatorname{Re}\zeta| < \pi/2$ and $F_0(t) = W_{ab}(rn_t)$ for real t there: on C, with $\zeta = t + i\beta$, $\operatorname{Re}(-r\cos\zeta\,\omega + ir\sin\zeta\, q) = -r\cos t\,(\omega\cosh\beta + q\sinh\beta) \le -r\cos t\, e^{-|\beta|}\omega$, so the integral converges absolutely and locally uniformly, and $|F_0(t+i\beta)| \le C(1 + (r\cos t)^{-2}e^{2|\beta|})$ by the Laplace bound of (5d).

The functions $\zeta \mapsto e^{-iwk\pi/2}F_0(\zeta - k\pi/2)$, $k \in \mathbb{Z}$, defined on the strips $|\operatorname{Re}\zeta - k\pi/2| < \pi/2$, agree with $W_{ab}(rn_t)$ on the real points of their strips by the quarter-turn covariance, hence agree on the overlaps of consecutive strips by the identity theorem, and define an entire function h with $h(t) = W_{ab}(rn_t)$ for real t, $h(\zeta + \pi/2) = e^{-iw\pi/2}h(\zeta)$ and $M(\beta) := \sup_t |h(t+i\beta)| \le C(1 + 2r^{-2}e^{2|\beta|})$ (the bound on $|t| \le \pi/4$ and the periodicity). Thus $e^{iw\zeta}h(\zeta)$ is entire with period $\pi/2$, so $h(\zeta) = \sum_{k\in\mathbb{Z}} c_k e^{i(4k-w)\zeta}$ with $|c_k| \le M(\beta)e^{(4k-w)\beta}$ for every real β; letting $\beta \to \mp\infty$ shows $c_k = 0$ unless $|4k - w| \le 2$. For $w \in \{0,\pm1\}$ only $k = 0$ survives: $W_{ab}(\hat{R}_t v) = e^{-iwt}W_{ab}(v)$ for all t and $v \neq 0$. For $w = 2$ ($a = b = \epsilon_+$) the survivors are $k \in \{0,1\}$: $h(\zeta) = c_0e^{-2i\zeta} + c_1e^{2i\zeta}$, and for $w = -2$ symmetrically $h = c_0e^{2i\zeta} + c_{-1}e^{-2i\zeta}$.

The sign step, $w = 2$, first for $\eta' = \eta$ real. By (5d), $\nu^{e_j}_{\eta\alpha,\eta\epsilon_+} \ge 0$ for $\alpha = \pi(\theta_j)\bar{\epsilon}_+ = \pi(\theta_j)\epsilon_- = -\epsilon_+$, i.e. $-\nu^{e_j}_{++} \ge 0$, so $-h(i\beta) = -F_0(i\beta) = \int e^{-r(\omega\cosh\beta + q\sinh\beta)}d(-\nu^{e_j}_{++}) \ge 0$ for all real β; as $-h(i\beta) = -c_0e^{2\beta} - c_1e^{-2\beta}$, this forces c_0, c_1 real and ≤ 0. In the frame $\hat{n}$, (5d) gives $\nu^{n}_{\eta\alpha,\eta\epsilon_+} \ge 0$ for $\alpha = \pi(\theta_n)\epsilon_- = -i\epsilon_+$, i.e. $-i\nu^n_{++} \ge 0$. Since $n_t = \cos(t - \frac{\pi}{4})\hat{n} + \sin(t-\frac{\pi}{4})\hat{n}^\perp$, (12.7) in the frame $\hat{n}$ gives $-ih(t) = F^n(t)$ for $t \in (-\frac{\pi}{4}, \frac{3\pi}{4})$, where $F^n(\zeta) := \int e^{-r\cos(\zeta-\pi/4)\omega + ir\sin(\zeta-\pi/4)q}d(-i\nu^n_{++})$ is holomorphic on $|\operatorname{Re}\zeta - \frac{\pi}{4}| < \frac{\pi}{2}$; by the identity theorem $-ih = F^n$ on that strip, and at $\zeta = \frac{\pi}{4} + i\beta$, $F^n(\frac{\pi}{4} + i\beta) = \int e^{-r(\omega\cosh\beta + q\sinh\beta)}d(-i\nu^n_{++}) \ge 0$, while $-ih(\frac{\pi}{4} + i\beta) = -c_0e^{2\beta} + c_1e^{-2\beta}$. Letting $\beta \to -\infty$ gives $c_1 \ge 0$, hence $c_1 = 0$: $h(\zeta) = c_0e^{-2i\zeta}$, which is the covariance $W_{++}(\hat{R}_t v) = e^{-2it}W_{++}(v)$ for $\eta' = \eta$ real.

General η,η'. The coefficient $c_1 = c_1^{\eta\eta'}$ is bilinear in (η,η') (the kernel is, and the Fourier coefficient is a linear functional of h). By (5e)(1), $h^{\eta\eta'}(t) = W_{\eta\epsilon_+,\eta'\epsilon_+}(rn_t) = W_{\eta'\epsilon_+,\eta\epsilon_+}(-rn_t) = h^{\eta'\eta}(t+\pi)$, and $e^{i(4k-2)\pi} = 1$, so $c_1^{\eta\eta'} = c_1^{\eta'\eta}$. For real η,η': $0 = c_1^{\eta+\eta',\eta+\eta'} = c_1^{\eta\eta} + 2c_1^{\eta\eta'} + c_1^{\eta'\eta'} = 2c_1^{\eta\eta'}$; complex η,η' follow by bilinearity. The channel $w = -2$ follows from (5e)(2) ($W_{\bar{\eta}\epsilon_-,\bar{\eta}'\epsilon_-} = \overline{W_{\eta\epsilon_+,\eta'\epsilon_+}}$), or by the same argument. Hence, for all $\alpha,\beta \in T_{\mathbb{C}}$, all η,η', all $t \in \mathbb{R}$ and $v \neq 0$ (expanding α,β in weight vectors),

$$W_{\eta\alpha,\eta'\beta}(\hat{R}_t v) = W_{\eta\pi(R_t)^{-1}\alpha,\ \eta'\pi(R_t)^{-1}\beta}(v). \tag{12.8}$$

Step 7 (assembly). (7a) Planar-disjoint tests. For $f = (\phi\otimes\eta)\alpha$, $f' = (\phi'\otimes\eta')\beta$ with $\operatorname{supp}\phi\cap\operatorname{supp}\phi' = \emptyset$: $R_t^* f = ((\phi\circ\hat R_t)\otimes\eta)\pi(R_t)^{-1}\alpha$ (R_t fixes z), the planar supports of $\phi\circ\hat R_t$, $\phi'\circ\hat R_t$ are disjoint, and (12.7) (for all disjoint planar supports, (5e)), then the substitution $y\mapsto\hat R_t^{-1}y$, $y'\mapsto\hat R_t^{-1}y'$ and (12.8) for $-t$ at $v = y'-y$, give

$$\begin{aligned} S_2(R_t^*f, R_t^*f') &= \iint \phi(\hat R_t y)\phi'(\hat R_t y') W_{\eta\pi(R_t)^{-1}\alpha,\eta'\pi(R_t)^{-1}\beta}(y'-y)\,dy\,dy' \\ &= \iint \phi(y)\phi'(y') W_{\eta\alpha,\eta'\beta}(y'-y)\,dy\,dy' = S_2(f,f'). \end{aligned}$$

For disjoint open $U, U'\subset\mathbb{R}^2$ the finite sums of such products with $\phi\in C_c^\infty(U)$, resp. $\phi'\in C_c^\infty(U')$, are dense in $C_c^\infty(U\times\mathbb{R};T_{\mathbb{C}})$, resp. $C_c^\infty(U'\times\mathbb{R};T_{\mathbb{C}})$, for the C_c^∞ topology, hence for $\|\cdot\|$; R_t^* is a $\|\cdot\|$-isometry and S_2 is $\|\cdot\|$-continuous (Step 1). Hence, for each of the three coordinate planes: *(P)* $S_2(R_t^*\chi_1, R_t^*\chi_2) = S_2(\chi_1,\chi_2)$ for all $t\in\mathbb{R}$ and all $\chi_1,\chi_2\in C_c^\infty(\mathbb{R}^3;T_{\mathbb{C}})$ whose supports have disjoint orthogonal projections onto that plane.

(7b) Small supports. Let $\chi_1,\chi_2\in C_c^\infty(\mathbb{R}^3;T_{\mathbb{C}})$ be supported in the closed balls $\bar B(x_1,\delta)$, $\bar B(x_2,\delta)$ with $|x_2-x_1|>3\delta$, and put $F(R) := S_2(R^*\chi_1, R^*\chi_2) - S_2(\chi_1,\chi_2)$ for $R\in\mathrm{SO}(3)$. F is C^∞ on $\mathrm{SO}(3)$: in a local chart $\vartheta\mapsto R(\vartheta)$ the function $(\vartheta,x)\mapsto(R(\vartheta)^*\chi)(x) = \pi(R(\vartheta))^{-1}\chi(R(\vartheta)x)$ is C^∞ with x-support in a fixed compact set for ϑ in a compact neighbourhood, so $\vartheta\mapsto R(\vartheta)^*\chi$ is C^∞ into $L^1\cap L^2$ (its difference quotients converge uniformly, with uniformly bounded supports), and S_2 is a continuous bilinear form on $(L^1\cap L^2)^2$ (Step 1).

For $a\in\{1,2,3\}$ let $R_s^{(a)} = \exp(sL_a)$ be the rotation by s about e_a, $(L_a)_{bc} = -\epsilon_{abc}$, and X_a the left-invariant vector field $(X_aG)(R) := \frac{d}{ds}G(RR_s^{(a)})|_{s=0}$; then $[X_a,X_b] = X_{[L_a,L_b]} = \pm X_c$ for $\{a,b,c\} = \{1,2,3\}$, and X_1,X_2,X_3 span the tangent space at every point. Let $P_a^\perp$ be the orthogonal projection onto $e_a^\perp$ and $U_a := \{R\in\mathrm{SO}(3) : |P_a^\perp R^{-1}(x_2-x_1)|>2\delta\}$, an open set. For $R\in U_a$ the pair $(R^*\chi_1, R^*\chi_2)$ is supported in $\bar B(R^{-1}x_1,\delta)$, $\bar B(R^{-1}x_2,\delta)$, whose projections along e_a are disjoint, so (P) for the plane $e_a^\perp$ gives $S_2(R_s^{(a)*}R^*\chi_1, R_s^{(a)*}R^*\chi_2) = S_2(R^*\chi_1,R^*\chi_2)$ for all s, i.e. $F(RR_s^{(a)}) = F(R)$ since $(RR_s^{(a)})^* = R_s^{(a)*}R^*$; hence $X_aF = 0$ on U_a. Every $R\in\mathrm{SO}(3)$ lies in at least two of U_1,U_2,U_3: with $w := R^{-1}(x_2-x_1)$, $\sum_a |P_a^\perp w|^2 = 2|w|^2 > 18\delta^2$, and if two of the three terms were $\le 4\delta^2$ we would get $2|w|^2\le 8\delta^2+|w|^2$, i.e. $|w|^2\le 8\delta^2<9\delta^2$. On the open set $U_a\cap U_b$, $X_aF = X_bF = 0$, hence $X_cF = \pm(X_a(X_bF) - X_b(X_aF)) = 0$ ($F\in C^2$); so $dF = 0$ on $U_a\cap U_b$, and every point of $\mathrm{SO}(3)$ lies in such a set. Thus $dF\equiv 0$ on the connected manifold $\mathrm{SO}(3)$, and $F\equiv F(1) = 0$.

(7c) Conclusion. Let $\chi_1,\chi_2\in C_c^\infty(\mathbb{R}^3;T_{\mathbb{C}})$ have disjoint supports, $D := \operatorname{dist}(\operatorname{supp}\chi_1,\operatorname{supp}\chi_2) > 0$, $\delta := D/6$, and let (ϱ_a) be a smooth partition of unity of $\mathbb{R}^3$ with $\operatorname{supp}\varrho_a\subset\bar B(y_a,\delta)$. Then $\chi_1 = \sum_a\varrho_a\chi_1$, $\chi_2 = \sum_b\varrho_b\chi_2$ are finite sums, every pair $(\varrho_a\chi_1,\varrho_b\chi_2)$ of non-zero pieces has $|y_a-y_b|\ge D-2\delta = 4\delta > 3\delta$, and (7b) with the bilinearity of S_2 and of R^* gives $S_2(R^*\chi_1,R^*\chi_2) = S_2(\chi_1,\chi_2)$ for $R\in\mathrm{SO}(3)$. For $R\in\mathrm{O}(3)\setminus\mathrm{SO}(3)$ write $R = \theta_3R'$ with $R'\in\mathrm{SO}(3)$; then $R^* = R'^*\Theta_3$, the supports of $\Theta_3\chi_1,\Theta_3\chi_2$ are disjoint, and (b) gives $S_2(R^*\chi_1,R^*\chi_2) = S_2(\Theta_3\chi_1,\Theta_3\chi_2) = S_2(\chi_1,\chi_2)$. For Schwartz χ_1,χ_2 with disjoint supports the cut-offs $\vartheta(\cdot/\nu)\chi_i$ ($\vartheta\in C_c^\infty$, $\vartheta = 1$ on the unit ball) converge to χ_i in $\|\cdot\|$ with supports inside the original ones, and R^* is a $\|\cdot\|$-isometry; Step 1 concludes.

Finally S_1: for real $\chi\in C_c^\infty$ and $\rho\in C_c^\infty$, $\rho\ge 0$, $\int\rho = 1$, $\rho_R := R^{-3}\rho(\cdot/R)$, (ii) and the Bochner integral give $S_1(\chi*\rho_R) = S_1(\chi)$, while

$$\|\chi*\rho_R - (\textstyle\int\chi)\rho_R\| \le \int|\chi(y)|\,\|\tau_y\rho_R-\rho_R\|\,dy \le R^{-1}(\|\nabla\rho\|_1+\|\nabla\rho\|_2)\int|y||\chi(y)|dy\to 0;$$

applied to $\chi = \rho_{R_1}e_r$ (e_r a basis vector of T) this shows that $m_r := S_1(\rho_Re_r)$ does not depend on R, and then $S_1(\chi) = m(\int\chi)$ with $m := \sum_r m_re_r^*\in T^*$. By (b), $m\circ\pi(g)^{-1} = m$ for $g\in B_3$; the sign flip $g = 1-2e_je_j^{\mathsf T}$ acts by -1 on $e_j\wedge e_k$, so m vanishes on $\Lambda^2\mathbb{R}^3$ and lives on the scalar

component, on which π is trivial; hence $S_1(R^*\chi) = m(\pi(R)^{-1}\int\chi) = S_1(\chi)$ for all $R \in \mathrm{O}(3)$ (and $S_1 = 0$ for $T = \Lambda^2\mathbb{R}^3$). This proves (d). $\square$

The lemma used in Step 4 is the following statement of complex analysis; its proof is independent of everything above. For $h \in L^1(0,\infty)$ put $\tilde h(z) := \int_0^\infty h(a)e^{-az}da$ for $\operatorname{Re} z \geq 0$ (continuous, bounded by $\|h\|_{L^1}$, holomorphic for $\operatorname{Re} z > 0$); powers are principal, $z^\nu := \exp(\nu \operatorname{Log} z)$ for $z \notin (-\infty, 0]$ with $\arg z \in (-\pi, \pi)$.

Lemma 12.30 (two representations of one correlation)**.** *Let ϱ be a finite positive Borel measure on $[0,\infty)\times\mathbb{R}$ and ϱ' a complex Borel measure on $[0,\infty)\times\mathbb{R}$ of finite total variation $\|\varrho'\|$. Suppose that, with $z := \omega - iq$,*

$$\int e^{-az-c\bar z}\, d\varrho(\omega,q) = \int e^{-a\omega'+icq'}\, d\varrho'(\omega',q') \qquad \text{for all } a,c \geq 0.$$

Then $\varrho(\{(\omega,q) : \omega < |q|\}) = 0$.

Proof. An inequality. Let $h \in L^1(0,\infty)$. Multiply the hypothesis by $h(a)\overline{h(c)}$ and integrate over $a, c \geq 0$; on the left $|e^{-az-c\bar z}| = e^{-(a+c)\omega} \leq 1$ and ϱ is finite, on the right the integrand has modulus at most 1 and $\|\varrho'\| < \infty$, so Fubini's theorem applies on both sides. Since $\int_0^\infty \overline{h(c)}e^{-c\bar z}dc = \overline{\tilde h(z)}$ and $\int_0^\infty \overline{h(c)}e^{icq'}dc = \overline{\tilde h(iq')}$,

$$\int |\tilde h(z)|^2\, d\varrho = \int \tilde h(\omega')\,\overline{\tilde h(iq')}\, d\varrho' \leq \|\varrho'\| \sup_{t\geq 0}|\tilde h(t)| \sup_{s\in\mathbb{R}}|\tilde h(is)|. \tag{12.9}$$

The test functions. Fix $\epsilon \in \{1,-1\}$ and an integer $N \geq 1$, and put $\nu := 1 - 4i\epsilon N/\pi$ ($\operatorname{Re}\nu = 1$, $\nu \notin \{0,1,2,\dots\}$). We construct $h_N \in L^1(0,\infty)$ with $\tilde h_N(z) = e^{-N}z^\nu(1+z)^{-3}$ for $\operatorname{Re} z \geq 0$, $z \neq 0$, and $\tilde h_N(0) = 0$. The binomial coefficients $b_k := \binom{\nu}{k}$ satisfy $b_k = (-1)^k\Gamma(k-\nu)/(\Gamma(-\nu)\Gamma(k+1))$ and $|\Gamma(k-\nu)/\Gamma(k+1)| \sim k^{-\operatorname{Re}\nu-1} = k^{-2}$ (Stirling), so $\sum_k |b_k| < \infty$ and $\sum_k b_k(-u)^k$ converges absolutely and uniformly on the closed unit disc. Its sum is $(1-u)^\nu$ for $|u| < 1$ (Taylor series of the holomorphic function $u \mapsto (1-u)^\nu$, $\operatorname{Re}(1-u) > 0$ there), hence also for $|u| \leq 1$, $u \neq 1$, by continuity of $w \mapsto w^\nu$ on the closed right half-plane minus 0; and as $u \to 1$ in the disc, $|(1-u)^\nu| = |1-u|e^{-\operatorname{Im}\nu\arg(1-u)} \leq |1-u|e^{2N} \to 0$, so the sum is 0 at $u = 1$.

For $k \geq 1$ let $\gamma_k(a) := a^{k-1}e^{-a}/(k-1)!$, so $\gamma_k \geq 0$, $\int_0^\infty \gamma_k = 1$, $\int_0^\infty \gamma_k(a)e^{-az}da = (1+z)^{-k}$ for $\operatorname{Re} z \geq 0$, and let $m := \delta_0 + \sum_{k\geq 1}(-1)^k b_k \gamma_k(a)\, da$, a complex measure on $[0,\infty)$ of total variation at most $1+\sum_k |b_k|$. For $\operatorname{Re} z \geq 0$ put $u := (1+z)^{-1}$; then $|u| \leq 1$ (as $|1+z|^2 = 1 + 2\operatorname{Re} z + |z|^2 \geq 1$) and $u = 1$ only for $z = 0$, so $\int e^{-az}dm(a) = \sum_k b_k(-1)^k(1+z)^{-k} = (z/(1+z))^\nu$ for $z \neq 0$ and $\int dm = 0$.

Next let $q(a) := a^{2-\nu}e^{-a}/\Gamma(3-\nu)$, $a > 0$; since $\operatorname{Re}(2-\nu) = 1$, $|q(a)| = ae^{-a}/|\Gamma(3-\nu)|$, so $q \in L^1(0,\infty)$, and $\int_0^\infty q(a)e^{-az}da = (1+z)^{\nu-3}$ (for real $z > -1$ by the Gamma integral, then for $\operatorname{Re} z > -1$ by analytic continuation). Put $h_N := e^{-N}m * q$, i.e. $h_N(a) = e^{-N}(q(a) + \sum_{k\geq 1}(-1)^k b_k \int_0^a \gamma_k(a-b)q(b)db)$; then $h_N \in L^1(0,\infty)$ and, by Fubini's theorem, $\tilde h_N(z) = e^{-N}(z/(1+z))^\nu(1+z)^{\nu-3}$ for $\operatorname{Re} z \geq 0$, $z \neq 0$, and $\tilde h_N(0) = 0$. For $\operatorname{Re} z \geq 0$, $z \neq 0$, the arguments of z and $1+z$ lie in $[-\pi/2, \pi/2]$ and have the same sign ($\operatorname{Im} z = \operatorname{Im}(1+z)$), so $\operatorname{Log} z - \operatorname{Log}(1+z)$ is the principal logarithm of $z/(1+z)$, and $(z/(1+z))^\nu(1+z)^{\nu-3} = \exp(\nu\operatorname{Log} z - 3\operatorname{Log}(1+z)) = z^\nu(1+z)^{-3}$.

Bounds. For $\operatorname{Re} z \geq 0$, $z \neq 0$: $|z^\nu| = |z|\exp(-\operatorname{Im}\nu\arg z)$ with $-\operatorname{Im}\nu = 4\epsilon N/\pi$, so $|\tilde h_N(z)| = e^{-N}|z|\,|1+z|^{-3}\exp(4\epsilon N\arg z/\pi)$. Since $|1+z| \geq \max(1,|z|)$ on the closed right half-plane, $|z||1+z|^{-3} \leq 1$ there. On $(0,\infty)$, $\arg z = 0$, so $\sup_{t\geq 0}|\tilde h_N(t)| \leq e^{-N}$; on $i\mathbb{R}\setminus\{0\}$, $\arg z = \pm\pi/2$, so $\sup_s |\tilde h_N(is)| \leq e^{-N}e^{2N} = e^N$; the product of the two suprema is at most 1. On the sector

$\Sigma_{\epsilon,\delta} := \{z \neq 0 : \operatorname{Re} z \geq 0,\ \epsilon \arg z \geq \pi/4 + \delta\}$, $\delta > 0$: $|\tilde{h}_N(z)|^2 \geq e^{-2N} e^{8N(\pi/4+\delta)/\pi} |z|^2 |1+z|^{-6} = e^{8\delta N/\pi} |z|^2 |1+z|^{-6}$.

Conclusion. By (12.9) with $h = h_N$ and the bounds, $\int_{\Sigma_{\epsilon,\delta}} |z|^2 |1+z|^{-6} d\varrho \leq e^{-8\delta N/\pi} \|\varrho'\|$ for all $N \geq 1$, $\Sigma_{\epsilon,\delta}$ read as a set of points (ω, q) through $z = \omega - iq$. Letting $N \to \infty$, and since the weight is strictly positive on $\Sigma_{\epsilon,\delta}$, $\varrho(\Sigma_{\epsilon,\delta}) = 0$. The union of the sets $\Sigma_{\pm 1,1/k}$, $k \geq 1$, is $\{z \neq 0 : \operatorname{Re} z \geq 0,\ |\arg z| > \pi/4\}$, and for $z = \omega - iq$ with $\omega \geq 0$ the condition $|\arg z| > \pi/4$ is $|q| > \omega$ (for $\omega > 0$, $|\arg z| = \arctan(|q|/\omega)$; for $\omega = 0$, $z \neq 0$ means $q \neq 0$). □

Theorem 12.31 (rotation covariance of all moments away from coincident points)**.** *Let $T = \Lambda^2\mathbb{R}^3$ or $\Lambda^2\mathbb{R}^3 \oplus \mathbb{R}$ with its orthogonal $\mathrm{O}(3)$-action and let μ be a probability measure on $\mathcal{S}'(\mathbb{R}^3; T)$, with the σ-algebra generated by the evaluations, with the properties (i), (ii), (iii) of Section 12.10. Then*

$$S_n(R^*\chi_1, \dots, R^*\chi_n) = S_n(\chi_1, \dots, \chi_n)$$

for all $n \geq 1$, $R \in \mathrm{O}(3)$ and $\chi_1, \dots, \chi_n \in \mathcal{S}(\mathbb{R}^3; T_{\mathbb{C}})$ with pairwise disjoint supports.

Proof. Theorem 12.31 is Theorem 15.1 of Section 15, proved for all n from (i)–(iii) alone; for $n \leq 2$ it is also Theorem 12.29(d). Two conventions of Section 15 differ from those of Section 12.10. Hypothesis (iii) is asked there of finite families of real Schwartz tests supported in one open side of the plane; (iii) above, with C_c^∞ tests, implies it: for real Schwartz χ_a supported in Ω_u^+ the cut-offs $\chi_a \vartheta(\cdot/q)$ ($\vartheta \in C_c^\infty(\mathbb{R}^3)$, $\vartheta = 1$ on $B(0,1)$) lie in $C_c^\infty(\Omega_u^+; T)$ and converge to χ_a in $\|\cdot\|$ as $q \to \infty$, and $|Z(\chi) - Z(\chi')| \leq \int |\Phi(\chi - \chi')|\, d\mu \leq c_0^{1/2} \|\chi - \chi'\|$ for real χ, χ' (Theorem 12.29(a)), so each finite Gram sum of (iii) passes to the limit. The measure there is a Borel probability measure on $\mathcal{S}'$; the proofs of Section 15 use μ only through the random variables $\Phi(\chi)$, $\chi \in \mathcal{S}$ real, and the σ-algebras they generate (Remark 15.3), so they apply to every probability measure on the cylinder σ-algebra, in particular to the measure of Lemma 12.22, with the reading given in that remark. The conclusion there is stated, as above, for Schwartz tests with pairwise disjoint supports (complex ones by multilinearity, the real and imaginary parts keeping the supports). □

12.11. **Bounded multilinear forms vanishing off the diagonal and on gradients.** *Notation.* Let $U \subset \mathbb{R}^3$ be open and $H(U) := L^2(U; \mathbb{C}^3)$ (elements extended by 0 to $\mathbb{R}^3$); $|E|$ denotes the Lebesgue measure and 1_E the multiplication by the indicator; "b vanishes on E" means almost everywhere; an n-linear form D on $H(U)$ ($\mathbb{C}$-linear in each entry) is bounded if $|D(b_1, \dots, b_n)| \leq C \prod_i \|b_i\|_{L^2}$. We consider two hypotheses on D:

- (H1$_U$) $D(b_1, \dots, b_n) = 0$ whenever $b_1, \dots, b_n \in C_c^\infty(U; \mathbb{C}^3)$ have pairwise disjoint supports (for $n = 1$: for all b_1);
- (H2$_U$) $D(b_1, \dots, b_n) = 0$ whenever one entry is ∇u, $u \in C_c^\infty(U; \mathbb{C})$, the others arbitrary in $H(U)$.

For $U = \mathbb{R}^3$ we write (H1) and (H2) for (H1$_U$) and (H2$_U$).

Lemma 12.32 (locality gives a multiplier)**.** *Let β be a bounded bilinear form on $H(U)$ with constant C and $\beta(b_1, b_2) = 0$ whenever $b_1, b_2 \in C_c^\infty(U; \mathbb{C}^3)$ have disjoint supports. Then there is a unique $M \in L^\infty(U; M_3(\mathbb{C}))$ with $\beta(b_1, b_2) = \int_U b_1^{\mathsf{T}} M b_2\, dx$ for all $b_1, b_2 \in H(U)$, and $|M(x)v| \leq \sqrt{3}\, C|v|$ almost everywhere.*

Proof. (i) $(b_1, b_2) \mapsto \beta(\bar{b}_1, b_2)$ is bounded sesquilinear, so $\beta(\bar{b}_1, b_2) = \langle b_1, T b_2 \rangle$ for a unique bounded operator T on $H(U)$, $\|T\| \leq C$; $\langle b_1, T b_2 \rangle = 0$ for $b_1, b_2 \in C_c^\infty(U)$ with disjoint supports ($\bar{b}_1$ has the support of b_1), and likewise for T^*.

(ii) *Claim: if $V \subset U$ is open and $b \in H(U)$ vanishes on V, then Tb vanishes on V.* Let $K \subset V$ be compact. If $U \setminus V = \emptyset$ then $b = 0$. Otherwise $\delta := \operatorname{dist}(K, U \setminus V) > 0$ (a sequence of $U \setminus V$ approaching K would accumulate at a point of $K \subset V$, while $U \setminus V$ is closed in

U). Put $W := \{x \in U : \operatorname{dist}(x, K) < \delta/3\}$ and $V' := \{x \in U : \operatorname{dist}(x, U \setminus V) < \delta/3\}$: open, disjoint, $K \subset W$, $b \in H(V')$. Take $b^{(k)} \in C_c^\infty(V'; \mathbb{C}^3)$ with $b^{(k)} \to b$ in L^2. For $b_1 \in C_c^\infty(W; \mathbb{C}^3)$, $\langle b_1, Tb^{(k)} \rangle = 0$, hence $\langle b_1, Tb \rangle = 0$; $C_c^\infty(W)$ is dense in $H(W)$, so $Tb = 0$ on $W \supset K$; V is a countable union of compact sets.

(iii) By (ii), $1_V T(1 - 1_V) = 0$; by (ii) for T^*, $(1 - 1_V)T1_V = 0$; so $T1_V = 1_V T$ for every open $V \subset U$. The class of Borel $E \subset U$ with $T1_E = 1_E T$ contains the open sets, is closed under complements ($1_{U\setminus E} = 1 - 1_E$) and under countable disjoint unions ($\sum_{k\le m} 1_{E_k} b \to 1_E b$ in L^2, T bounded); by Dynkin's theorem it is the Borel σ-algebra. By linearity and $\|M_\phi\| \le \|\phi\|_\infty$, $TM_\phi = M_\phi T$ for every bounded Borel ϕ.

(iv) Let $U_k := U \cap \{|x| < k\}$ and $m_j^{(k)} := T(1_{U_k} e_j)$. For $k \le k'$, $1_{U_k} m_j^{(k')} = T(1_{U_k} e_j) = m_j^{(k)}$ and $(1 - 1_{U_k}) m_j^{(k)} = 0$, so $m_j^{(k)} = 1_{U_k} m_j$ for one measurable m_j, and $T(1_E e_j) = 1_E T(1_{U_k} e_j) = 1_E m_j$ for Borel $E \subset U_k$. Hence $\int_E |m_j|^2 = \|T(1_E e_j)\|^2 \le \|T\|^2 |E|$ for every bounded Borel $E \subset U$, so $|m_j| \le \|T\|$ a.e. (otherwise integrate over a bounded subset of positive measure of $\{|m_j| > \|T\|\}$). With $M := (m_1|m_2|m_3)$, $|Mv| \le \sum_j |v_j||m_j| \le \sqrt{3}\|T\||v|$, and $Tb = Mb$ on the dense set of finite sums $\sum c_{ij} 1_{E_{ij}} e_j$ (bounded Borel E_{ij}); T and the multiplication by M are bounded, so $T = M$, i.e. $\beta(b_1, b_2) = \langle \bar{b}_1, Mb_2 \rangle = \int_U b_1^{\mathsf{T}} M b_2$. Uniqueness: $\int b_1^{\mathsf{T}} M b_2 = 0$ for all b_1, b_2 gives $M 1_{U_k} e_j = 0$ a.e. for all j, k. □

Lemma 12.33 (gradients kill the multiplier)**.** *If in addition $\beta(b_1, \nabla u) = 0$ for all $b_1 \in H(U)$ and $u \in C_c^\infty(U)$, then $M = 0$ and $\beta = 0$.*

Proof. $b_1 = \overline{M\nabla u} \in H(U)$ gives $\int_U |M\nabla u|^2 = 0$, so $M\nabla u = 0$ a.e. For a ball B with $\bar{B} \subset U$ take $\chi \in C_c^\infty(U)$, $\chi = 1$ near $\bar{B}$, and $u = \chi x_k$: $Me_k = 0$ a.e. on B, $k = 1, 2, 3$. U is a countable union of such balls. □

Theorem 12.34 (bounded multilinear forms vanishing off the diagonal and on gradients)**.** *For every $n \ge 1$ and every open $U \subset \mathbb{R}^3$, a bounded n-linear form on $H(U)$ satisfying ($H1_U$) and ($H2_U$) is zero.*

Proof. We argue by strong induction on n, for all open sets at once. For $n = 1$, $D = 0$ on the dense set $C_c^\infty(U; \mathbb{C}^3)$. For $n = 2$ apply Lemma 12.32 with ($H1_U$), then Lemma 12.33 with ($H2_U$). Let $n \ge 3$, assume the theorem for all $m < n$ and all open sets, and let D be bounded n-linear on $H(U)$, with constant C, satisfying ($H1_U$), ($H2_U$).

Step 1 (one confined group). Let $\emptyset \ne J \subsetneq \{1, \dots, n\}$, F its complement, $K \subset U$ compact, $b_i \in H(U)$ vanishing outside K for $i \in J$ and on K for $i \in F$. Then $D(b_1, \dots, b_n) = 0$. Indeed, let $U' := U \setminus K$ and let E be the $|F|$-linear form on $H(U')$ obtained from D by fixing the entries J at the b_i; E is bounded, $1 \le |F| \le n - 1$, and ($H2_{U'}$) is inherited. For ($H1_{U'}$), let $c_j \in C_c^\infty(U'; \mathbb{C}^3)$, $j \in F$, have pairwise disjoint supports, fix $j_0 \in F$ with $c_{j_0} \ne 0$, $T := \operatorname{supp} c_{j_0} \subset U'$, and let G be the $(n-1)$-linear form on $H(U \setminus T)$ obtained from D by fixing entry j_0 at c_{j_0}. G is bounded, ($H2_{U\setminus T}$) is inherited, and ($H1_{U\setminus T}$) holds because its $n - 1$ arguments together with c_{j_0} are n elements of $C_c^\infty(U; \mathbb{C}^3)$ with pairwise disjoint supports ($H1_U$). By the theorem for $n - 1$, $G = 0$. The c_j, $j \in F \setminus \{j_0\}$, and the b_i, $i \in J$, vanish on T (T is disjoint from K and from the other supports), so $E((c_j)_{j\in F}) = G(\dots) = 0$. By the theorem for $|F|$ on U', $E = 0$ on $H(U')^{|F|}$, and $(b_j)_{j\in F} \in H(U')^{|F|}$.

Step 2 (one confined entry). For $i \ne j$: $D(b_1, \dots, b_n) = 0$ whenever b_i vanishes outside a compact $S \subset U$ and b_j vanishes on S, the other entries arbitrary. Expand $b_l = 1_S b_l + 1_{U\setminus S} b_l$ for $l \notin \{i, j\}$: each of the 2^{n-2} terms is of the form of Step 1 with $K = S$, $J = \{i\} \cup \{l : 1_S b_l \text{ chosen}\}$, $j \notin J$. The statement extends to Borel $S \subset U$: choose compacts $K_m \uparrow$ in S with $|S \setminus \bigcup_m K_m| = 0$; then $1_{K_m} b_i \to b_i$ in L^2 and D is bounded.

Step 3 (the diagonal). Fix $b_3,\dots,b_n \in H(U)$. $\beta(b_1,b_2) := D(b_1,\dots,b_n)$ is bounded with constant $C\prod_{i\ge3}\|b_i\|$ and vanishes for $b_1,b_2 \in C_c^\infty(U)$ with disjoint supports (Step 2 with $(i,j)=(1,2)$, $S=\operatorname{supp} b_1$). By Lemma 12.32, $\beta(b_1,b_2)=\int_U b_1^{\mathsf T} M[b_3,\dots,b_n]\,b_2$ with $M[\cdot]$ multilinear (uniqueness) and $\|M[b_3,\dots,b_n]\|_{L^\infty}\le\sqrt{3}C\prod_{i\ge3}\|b_i\|$. If b_3 vanishes outside a Borel $E\subset U$, Step 2 with $(i,j)=(3,1)$ gives $\beta(b_1,b_2)=0$ for all $b_1\in H(U\setminus E)$, $b_2\in H(U)$, hence $M[b_3,\dots]=0$ on $U\setminus E$. Given $\eta>0$ choose $\delta>0$ with $\int_E|b_3|^2\le\eta^2$ whenever $|E|\le\delta^3$ (absolute continuity of the integral) and let (E_α) be the countable partition of U by half-open cubes of side δ. Then $b_3=\sum_\alpha 1_{E_\alpha}b_3$ in L^2, so $M[b_3,\dots]=\sum_\alpha M[1_{E_\alpha}b_3,\dots]$ in L^∞; the summands vanish outside the disjoint sets E_α and each has L^∞ norm at most $\sqrt{3}C\eta\prod_{i\ge4}\|b_i\|$, so $\|M[b_3,\dots,b_n]\|_{L^\infty}\le\sqrt{3}C\eta\prod_{i\ge4}\|b_i\|$. Letting $\eta\downarrow0$, $M[b_3,\dots,b_n]=0$, i.e. $D(b_1,\dots,b_n)=0$. □

Remark 12.35. For $n\ge3$ the hypothesis (H1) alone does not confine D to the full diagonal: $D(b_1,b_2,b_3)=\langle b_1,b_2\rangle\ell(b_3)$, ℓ a bounded linear functional on $H(\mathbb{R}^3)$, is bounded, vanishes on pairwise disjoint supports and is carried by $\{x_1=x_2\}$; (H2) excludes it, and is needed in every entry ($n=1$ is the case $D(b)=\int b\cdot w$, excluded by the $n=1$ case of (H1)). (H2) enters the induction only through the bilinear case, inside Step 1.

The Hodge map and the transport. For $f\in L^2(\mathbb{R}^3;\Lambda^2_{\mathbb{C}})$ put $\flat f:=(f_{23},f_{31},f_{12})$, so that $f_x(u,v)=\langle\flat f(x),u\times v\rangle$ (bilinear pairing), $\|\flat f\|_{L^2}^2=\sum_{\mu<\nu}\|f_{\mu\nu}\|_{L^2}^2$ (the norm of Lemma 12.22) and $\flat(*du)=\nabla u$. With the transport $(R\cdot f)_x(u,v)=f_{R^{-1}x}(R^{-1}u,R^{-1}v)$ of Definition 12.7 and $(R^{-1}u)\times(R^{-1}v)=\det R\,R^{-1}(u\times v)$,

$$\flat(R\cdot f)(x)=\det R\,R\flat f(R^{-1}x)=:(\rho(R)\flat f)(x),\qquad \rho(R)\nabla u=\det R\,\nabla(u\circ R^{-1}),$$

$\rho(R)$ an isometry of $H(\mathbb{R}^3)$ preserving C_c^∞; thus $R\cdot(*du)=\det R\,*d(u\circ R^{-1})$, and $R\cdot f=(R^{-1})^*f$ is the pull-back by R^{-1} with the tensor action (notation of Section 12.10).

Proposition 12.36 (the coincident part vanishes)**.** *Let S_n be the curvature Schwinger functions of a limit S^{box} of Theorem 7.1 for the box family (Lemma 12.20), with any bounded Borel $g_{\mathrm d}$, extended to $L^2(\mathbb{R}^3;\Lambda^2_{\mathbb{C}})$ (Lemma 12.22). For all $n\ge1$, $R\in\mathrm{O}(3)$ and $f_1,\dots,f_n\in L^2(\mathbb{R}^3;\Lambda^2_{\mathbb{C}})$: $S_n(R\cdot f_1,\dots,R\cdot f_n)=S_n(f_1,\dots,f_n)$. Consequently $Z^{\mathrm{box}}(R\cdot f)=Z^{\mathrm{box}}(f)$ for every real C_c^∞ 2-form f.*

Proof. Put

$$D_R(b_1,\dots,b_n):=S_n(\flat^{-1}\rho(R)b_1,\dots,\flat^{-1}\rho(R)b_n)-S_n(\flat^{-1}b_1,\dots,\flat^{-1}b_n)$$

on $H(\mathbb{R}^3)$; since $\flat^{-1}\rho(R)b=R\cdot\flat^{-1}b$, $D_R=S_n(R\cdot\,\cdot)-S_n(\cdot)$ read on vector fields. D_R is bounded by Lemma 12.22 ($|S_n|\le e^n(n!)^{1/2}\prod\|f_i\|_{L^2}$, with the factor $2^{n/2}$ for complex entries) and the isometries.

Verification of (H1). If $b_i\in C_c^\infty(\mathbb{R}^3;\mathbb{C}^3)$ have pairwise disjoint supports, the $f_i=\flat^{-1}b_i$ are Schwartz 2-forms with pairwise disjoint supports and $R\cdot f=(R^{-1})^*f$. The measure μ of Lemma 12.22 satisfies (i) (with $C=\max(\frac12,\|g_{\mathrm d}\|_\infty)$, as $\|h\|_{L^1}\le\|\chi\|_{L^1}$ and $\|f\|_{L^2}\le\|\chi\|_{L^2}$ for $\chi=(f,h)$), (ii) (Proposition 9.6 for the box family on Schwartz pairs by continuity, and the uniqueness of a measure with given characteristic functional, Lemma 12.12, the hypothesis-free form of the argument of Proposition 12.11(b)) and (iii) (Lemma 12.21(iii) on the open side; the push-forward by a reflection is its pull-back), and its moments on $(f,0)$ are the S_n (Lemma 12.22). So Theorem 12.31 with R^{-1} gives $D_R=0$ on such b_i (real f_i first, complex by multilinearity, the real and imaginary parts keeping the supports).

Verification of (H2). For $b_i=\nabla u$, $\flat^{-1}b_i=*du$ and $\flat^{-1}\rho(R)b_i=\det R\,*d(u\circ R^{-1})$. By Theorem 12.26, $Z^{\mathrm{box}}(s*dw)=1$ for all real s and real $w\in C_c^\infty$; for the measure μ of Theorem 9.13 (box family by Lemma 12.20), $\int e^{is\Phi(*dw)}d\mu=Z^{\mathrm{box}}(s*dw)=1$, so

$\Phi(*dw) = 0$ μ-a.s.; the moments of μ on real C_c^∞ 2-forms are the S_n (Theorem 9.13(a)), hence $S_n(g_1, \dots, *dw, \dots, g_n) = \int \Phi(g_1) \cdots \Phi(*dw) \cdots d\mu = 0$ for real C_c^∞ g_j, then for complex $g_j \in L^2$ (multilinearity, boundedness, density) and complex w (linearity). Both terms of D_R vanish. Theorem 12.34 with $U = \mathbb{R}^3$ gives $D_R = 0$.

For the last claim, $\zeta \mapsto Z^{\mathrm{box}}(\zeta f) = \Psi_{(f)}(i\zeta)$ is entire (Theorem 9.4(ii)); $\Psi_{(f,\dots,f)}(z_1, \dots, z_n) = \Psi_{(f)}(z_1 + \cdots + z_n)$ (at level N the exponent is $(\sum_j z_j)\Phi_N(f)$, and the limits are those of Lemma 9.3(b),(c); compare (d)), so $\Psi_{(f)}^{(n)}(0) = S_n(f, \dots, f)$ (Theorem 9.4(iii)) and $Z^{\mathrm{box}}(f) = \sum_{n \geq 0} \frac{i^n}{n!} S_n(f, \dots, f)$, the same for $R \cdot f$, with equal coefficients. (Equivalently: the image of μ under $\Phi \mapsto \Phi(R^{-1} \cdot \cdot)$ has the same moments on C_c^∞ as μ, with the factorial bound of Theorem 9.4(iii), so it equals μ by the uniqueness from moments in Theorem 9.13(b), and the characteristic functionals agree.) □

The density component at coincident points. In Lemma 12.37, Corollary 12.38 and their proofs, $X := \mathcal{S}'(\mathbb{R}^3; \Lambda^2 \oplus \mathbb{R})$ with its cylinder σ-algebra, $\Phi(\sigma)$ is the evaluation at a real Schwartz pair $\sigma = (f, h)$ (f a 2-form, h a function), and $A(f) := \Phi(f, 0)$, $D(h) := \Phi(0, h)$, so that $\Phi(f, h) = A(f) + D(h)$. The symbols A, D, M, M_R, Δ_R, E, B, B_0, C_m are local to these statements and proofs (the generic form called D in the notation of this section does not occur below). The transport $R \cdot f$, $R \cdot h$, $R \cdot \sigma$ of Definition 12.7 is given by the same formulas on L^2 2-forms and on L^1 functions: $\|R \cdot f\|_{L^2} = \|f\|_{L^2}$ ($\rho(R)$ is an isometry), $\|R \cdot h\|_{L^1} = \|h\|_{L^1}$ ($|\det R| = 1$); it preserves the real Schwartz pairs and C_c^∞, and $\operatorname{supp}(R \cdot \sigma) = R \operatorname{supp} \sigma$. Here $\|f\|_{L^2}^2 := \sum_{\mu<\nu} \|f_{\mu\nu}\|_{L^2}^2$, as in Lemma 12.22.

Lemma 12.37 (an L^1-bounded entry has no coincident-point part)**.** *Let $c \geq 0$ and let μ be a probability measure on X with:*

- (P1) $\int e^{\Phi(f,h)}\, d\mu \leq \exp\left(\frac{1}{2}\|f\|_{L^2}^2 + c\,\|h\|_{L^1}\right)$ *for every real Schwartz pair (f, h);*
- (P2) *for all $m, k \geq 0$ with $m + k \geq 1$, every $R \in \mathrm{O}(3)$, and all real C_c^∞ 2-forms $f_1, \dots, f_m$ and real C_c^∞ functions $h_1, \dots, h_k$ whose $m + k$ supports are pairwise disjoint:* $\int \prod_i A(R \cdot f_i) \prod_j D(R \cdot h_j)\, d\mu = \int \prod_i A(f_i) \prod_j D(h_j)\, d\mu$;
- (P3) $\int e^{isA(*du)}\, d\mu = 1$ *for every real s and every real $u \in C_c^\infty(\mathbb{R}^3)$.*

Then $\int e^{i\Phi(R\cdot\sigma)}\, d\mu = \int e^{i\Phi(\sigma)}\, d\mu$ for every $R \in \mathrm{O}(3)$ and every real Schwartz pair σ. Moreover every mixed moment $M(f_1, \dots, f_m; h_1, \dots, h_k) := \int \prod_i A(f_i) \prod_j D(h_j)\, d\mu$ extends to a bounded multilinear form on $L^2(\mathbb{R}^3; \Lambda^2_{\mathbb{C}})^m \times L^1(\mathbb{R}^3; \mathbb{C})^k$, and $M(R \cdot f_1, \dots, R \cdot f_m; R \cdot h_1, \dots, R \cdot h_k) = M(f_1, \dots, f_m; h_1, \dots, h_k)$ for all such entries, with no condition on the supports.

Corollary 12.38 (bounded functions of the density, coincident points included)**.** *Let g_{d} be any bounded Borel function (the joint scheme of Definition 2.5), with the couplings as in Theorem 12.2.*

- (a) *(The box family.) Every limit S^{box} of Theorem 7.1 for the box family satisfies $S^{\mathrm{box}}(e_{R\cdot\sigma}) = S^{\mathrm{box}}(e_\sigma)$ for every $R \in \mathrm{O}(3)$ and every test pair $\sigma = (f, h)$, that is $S^{\mathrm{box}} \circ \gamma_R = S^{\mathrm{box}}$ on $\mathcal{A}$; the mixed Schwinger functions of the joint field extend to bounded multilinear forms, on $L^2(\mathbb{R}^3; \Lambda^2_{\mathbb{C}})$ in the curvature entries and on $L^1(\mathbb{R}^3; \mathbb{C})$ in the density entries, and the identity of Theorem 12.3 holds for all such entries, with no condition on the supports.*
- (b) *(The torus family.) Let S be a limit of Theorem 7.1 built on cubic-torus limit points ω_N whose sides run along dyadic sequences (Definition 14.1), and $Z(\sigma) = S(e_\sigma)$. Then Z, extended to the real Schwartz pairs as in Lemma 12.22, is the characteristic functional of a probability measure μ on $\mathcal{S}'(\mathbb{R}^3; \Lambda^2 \oplus \mathbb{R})$ with its cylinder σ-algebra, and for every real Schwartz pair (f, h)*

$$\int e^{\Phi(f,h)}\, d\mu \;\leq\; \exp\left(\tfrac{1}{2}\|f\|_{L^2}^2 + \|g_{\mathrm{d}}\|_\infty \|h\|_{L^1}\right)$$

(the display of Lemma 12.22, for the torus family). The statements of (a), with S in place of S^{box}, hold for every limit S on torus limit points that satisfies the covariance identity of Theorem 12.3 on pairwise disjoint supports; for the limits S of the first sentence this identity is proved, for $\kappa_N > 0$, in the proof of Theorem 12.4 (Section 12.12), before the present statement is used there.

In both cases Proposition 12.11 then gives the invariance of the limit functional under $E(3)$ on all test pairs, and the invariance under $E(3)$ of the joint measure (the measure of Lemma 12.22 in (a), the measure μ in (b)).

Proof of Corollary 12.38, given Lemma 12.37. (a) Let μ be the measure of Lemma 12.22: $\int e^{i\Phi(\sigma)}d\mu = Z^{\mathrm{box}}(\sigma)$ for all real Schwartz pairs, and $Z^{\mathrm{box}}(\sigma) = S^{\mathrm{box}}(e_\sigma)$ for test pairs (Theorem 7.1(a)). (P1) with $c = \|g_{\mathrm{d}}\|_\infty$ is the display of Lemma 12.22. (P2) is the identity of Theorem 12.3 on pairwise disjoint supports (proved there from Theorem 12.31, independently of its sentence on the coincident points), for the pairs $\sigma_i = (f_i, 0)$, $\sigma_{m+j} = (0, h_j)$: these are real C_c^∞ pairs with pairwise disjoint supports, $R\cdot\sigma_i = (R\cdot f_i, 0)$, $R\cdot\sigma_{m+j} = (0, R\cdot h_j)$, and the mixed Schwinger functions of the joint field are the moments $\int \prod_i \Phi(\sigma_i)\, d\mu$ of this measure (proof of Theorem 12.3). (P3) is Theorem 12.26: $\int e^{isA(*du)}d\mu = Z^{\mathrm{box}}(s*du, 0) = 1$. By Lemma 12.37, for a test pair σ (then $R\cdot\sigma$ is a test pair), $S^{\mathrm{box}}(e_{R\cdot\sigma}) = \int e^{i\Phi(R\cdot\sigma)}d\mu = \int e^{i\Phi(\sigma)}d\mu = S^{\mathrm{box}}(e_\sigma)$, and by linearity on the basis (e_σ) of $\mathcal{A}$, $S^{\mathrm{box}}\circ\gamma_R = S^{\mathrm{box}}$. The statement on the mixed Schwinger functions is the last sentence of the lemma. Proposition 12.11 applies, its hypothesis being $S^{\mathrm{box}}\circ\gamma_R = S^{\mathrm{box}}$ for all $R \in \mathrm{O}(3)$.

(b) The proof of Lemma 12.22 applies to the torus family with Theorem 11.5(b) in place of Theorem 11.11(b): the two theorems state the same Gaussian bound, and the continuity on pairs (Theorem 9.4(i)) and the convergence (Theorem 7.1(a)) are stated for the torus family. This gives the measure μ, with $\int e^{i\Phi(\sigma)}d\mu = Z(\sigma)$ on the real Schwartz pairs (Bochner–Minlos), and the display; so (P1) holds with $c = \|g_{\mathrm{d}}\|_\infty$. (P3) is Theorem 12.26 for $\mathcal{Z}(f) = Z(f, 0)$: $\int e^{isA(*du)}d\mu = Z(s*du, 0) = 1$. Now let S be a limit on a torus limit point that satisfies the covariance identity of Theorem 12.3 on pairwise disjoint supports. The mixed Schwinger functions in the conclusion of Theorem 12.3 for S are the moments of this same measure μ (their definition in the proof of Theorem 12.3, carried over by the proof of Theorem 12.4); so (P2) holds, and (P1), (P2), (P3) are properties of one measure. Lemma 12.37 applies, and the rest is as in (a). □

Proof of Lemma 12.37. Throughout, μ satisfies (P1)–(P3), "a.s." means μ-almost surely, and $C_m := 2e^{1/2}m!$ for $m \geq 1$, $C_0 := 1$.

Step 1: the joint $L^2 \times L^1$ bound. (1a) *For every real Schwartz h, $|D(h)| \leq c\|h\|_{L^1}$ a.s.* By (P1) for the pair $(0, sh)$, s real, $\int e^{sD(h)}d\mu \leq e^{c|s|\|h\|_{L^1}}$. For $\eta > 0$ and $s > 0$, by the Chebyshev inequality, $\mu\big(D(h) \geq c\|h\|_{L^1} + \eta\big) \leq e^{-s(c\|h\|_{L^1}+\eta)} \int e^{sD(h)}d\mu \leq e^{-s\eta}$, which tends to 0 as $s \to \infty$. So $D(h) < c\|h\|_{L^1} + \eta$ a.s. for every $\eta = 1/n$, hence $D(h) \leq c\|h\|_{L^1}$ a.s.; the same for $-h$, D being linear.

(1b) *For every real Schwartz 2-form f and every integer $p \geq 1$, $\int |A(f)|^p d\mu \leq C_p\|f\|_{L^2}^p$.* If $f = 0$ then $A(f) = 0$. Otherwise, for $t > 0$ and real x, $|x|^p \leq p!\, t^{-p}e^{t|x|} \leq p!\, t^{-p}(e^{tx} + e^{-tx})$, and (P1) for the pairs $(tf, 0)$ and $(-tf, 0)$ gives $\int e^{\pm tA(f)}d\mu \leq e^{t^2\|f\|_{L^2}^2/2}$; so $\int |A(f)|^p d\mu \leq 2\, p!\, t^{-p}e^{t^2\|f\|_{L^2}^2/2}$, and $t := 1/\|f\|_{L^2}$ gives $2e^{1/2}p!\,\|f\|_{L^2}^p$.

(1c) *For real Schwartz $f_1, \dots, f_m$, $h_1, \dots, h_k$ the product $\prod_i A(f_i) \prod_j D(h_j)$ is μ-integrable, and $|M(f_1, \dots, f_m; h_1, \dots, h_k)| \leq C_m \prod_i \|f_i\|_{L^2} \prod_j \big(c\|h_j\|_{L^1}\big)$.* By (1a), outside a null set (a finite union), $|\prod_j D(h_j)| \leq \prod_j c\|h_j\|_{L^1}$. By the Hölder inequality with m exponents equal to m and (1b) with $p = m$, $\int \prod_i |A(f_i)|\, d\mu \leq \prod_i \big(\int |A(f_i)|^m d\mu\big)^{1/m} \leq C_m \prod_i \|f_i\|_{L^2}$ (for $m = 0$ there is no curvature factor, and the constant is $C_0 = 1$).

(1d) *Extension.* M is real-multilinear on real Schwartz entries, every $\Phi \in X$ being linear. Extend it $\mathbb{C}$-multilinearly to complex Schwartz entries ($A(f) := A(\operatorname{Re} f) + iA(\operatorname{Im} f)$, likewise D): expanding into 2^{m+k} real terms, $|M| \le B_0 \prod_i \|f_i\|_{L^2} \prod_j \|h_j\|_{L^1}$ with $B_0 := 2^{m+k} C_m c^k$. The complex Schwartz functions are dense in $L^2(\mathbb{R}^3;\Lambda^2_{\mathbb{C}})$ and in $L^1(\mathbb{R}^3;\mathbb{C})$, so M extends uniquely to a $\mathbb{C}$-multilinear form on $L^2(\mathbb{R}^3;\Lambda^2_{\mathbb{C}})^m \times L^1(\mathbb{R}^3;\mathbb{C})^k$ with the same bound. This is the first half of the last sentence of the lemma.

(1e) $M_R(f;h) := M(R\cdot f_1,\dots,R\cdot f_m; R\cdot h_1,\dots,R\cdot h_k)$ is $\mathbb{C}$-multilinear with the same bound ($R\cdot$ is linear, isometric on L^2 and on L^1, and commutes with real and imaginary parts), and on real Schwartz entries it is $\int \prod_i A(R\cdot f_i) \prod_j D(R\cdot h_j)\, d\mu$. Put $\Delta_R := M_R - M$; then $|\Delta_R(f;h)| \le B \prod_i \|f_i\|_{L^2} \prod_j \|h_j\|_{L^1}$ with $B := 2B_0 = 2^{m+k+1} C_m c^k$.

Step 2: closedness in every curvature entry. Let $m \ge 1$, $1 \le i \le m$ and $w \in C^\infty_c(\mathbb{R}^3;\mathbb{C})$. If $f_i = *dw$, then M and M_R, hence Δ_R, vanish at $(f_1,\dots,f_m;h_1,\dots,h_k)$, for all $f_l \in L^2(\mathbb{R}^3;\Lambda^2_{\mathbb{C}})$ ($l \ne i$), all $h_j \in L^1(\mathbb{R}^3;\mathbb{C})$ and all $R \in \mathrm{O}(3)$. Real w first. By (P3) the law of the real random variable $A(*dw)$ has the characteristic function $s \mapsto \int e^{isA(*dw)} d\mu = 1$, so it is the unit mass at 0: $A(*dw) = 0$ a.s. For real Schwartz f_l and h_j the integrand of M contains the factor $A(*dw)$, is integrable by (1c) and vanishes a.s.; so $M = 0$ on real Schwartz other entries, then on complex Schwartz ones (multilinearity), then on all of L^2 and L^1 (density and the bound of (1d)). For complex $w = w_1 + iw_2$: $*dw = *dw_1 + i*dw_2$ and linearity in the entry i. For M_R: $R\cdot(*dw) = \det R * d(w\circ R^{-1})$ and $w\circ R^{-1} \in C^\infty_c(\mathbb{R}^3;\mathbb{C})$, so

$$M_R(\dots,*dw,\dots;\dots) = \det R \cdot M(R\cdot f_1,\dots,*d(w\circ R^{-1}),\dots,R\cdot f_m; R\cdot h_1,\dots,R\cdot h_k) = 0$$

by the first part.

Step 3: entries carried by pairwise disjoint Borel sets. Let $E_1,\dots,E_m,G_1,\dots,G_k$ be pairwise disjoint Borel subsets of $\mathbb{R}^3$, $f_i \in L^2(\mathbb{R}^3;\Lambda^2_{\mathbb{C}})$ with $f_i = 0$ a.e. outside E_i, and $h_j \in L^1(\mathbb{R}^3;\mathbb{C})$ with $h_j = 0$ a.e. outside G_j. Then $\Delta_R(f_1,\dots,f_m;h_1,\dots,h_k) = 0$ for every $R \in \mathrm{O}(3)$. (1) *Smooth entries.* For real C^∞_c entries with pairwise disjoint supports, $\Delta_R = 0$ is (P2). For complex C^∞_c entries with pairwise disjoint supports, expand in real and imaginary parts: each of them has its support inside the support of the entry, so every term vanishes.

(2) *Compact carriers.* Suppose in addition that the E_i and G_j are compact. Pairwise disjoint compact sets have pairwise positive distances; let $d > 0$ be the smallest (if fewer than two of the sets are non-empty there is no such distance: then any $r > 0$ serves below). Let η_r, $0 < r < d/3$, be a standard mollifier supported in the ball of radius r. Then $f_i * \eta_r$ (componentwise) and $h_j * \eta_r$ are C^∞_c, supported in the closed r-neighbourhoods of E_i and G_j, which are pairwise disjoint (distances $\ge d - 2r > 0$), and $f_i * \eta_r \to f_i$ in L^2, $h_j * \eta_r \to h_j$ in L^1 as $r \to 0$. By (1), Δ_R vanishes on the mollified entries, and Δ_R is continuous for these norms (1e); so $\Delta_R(f;h) = 0$.

(3) *Borel carriers.* The finite Borel measures $|f_i|^2 dx$ and $|h_j|\, dx$ on $\mathbb{R}^3$ are inner regular: for every $\epsilon > 0$ there are compact $K_i \subset E_i$ and $K'_j \subset G_j$ with $\|f_i - f_i 1_{K_i}\|_{L^2} < \epsilon$ and $\|h_j - h_j 1_{K'_j}\|_{L^1} < \epsilon$. The entries $f_i 1_{K_i}$, $h_j 1_{K'_j}$ are carried by the pairwise disjoint compact sets K_i, K'_j, so Δ_R vanishes on them by (2); letting $\epsilon \to 0$ and using (1e) again, $\Delta_R(f;h) = 0$. This step requires neither a positive distance between the carriers nor a smooth partition of unity; it rests on the joint bound alone.

Step 4: the curvature entries, the density entries being spectators. Let $K \subset \mathbb{R}^3$ be compact, $G_1,\dots,G_k$ pairwise disjoint Borel subsets of K, and $h_j \in L^1(\mathbb{R}^3;\mathbb{C})$ with $h_j = 0$ a.e. outside G_j ($k \ge 0$; for $k = 0$ any compact K, the empty set included). Then $\Delta_R(f_1,\dots,f_m;h_1,\dots,h_k) = 0$ for every $R \in \mathrm{O}(3)$ and all $f_1,\dots,f_m \in L^2(\mathbb{R}^3;\Lambda^2_{\mathbb{C}})$ that vanish a.e. on K. The case $m = 0$ is Step 3. Let $m \ge 1$, fix R and the h_j, and put $U := \mathbb{R}^3 \setminus K$, an open set. With the Hodge map $\flat$ (an isometry of $L^2(\mathbb{R}^3;\Lambda^2_{\mathbb{C}})$ onto $L^2(\mathbb{R}^3;\mathbb{C}^3)$, $\flat(*du) = \nabla u$) define on $H(U)$, its elements extended

by 0 to $\mathbb{R}^3$,

$$E(b_1,\dots,b_m) := \Delta_R\big(\flat^{-1}b_1,\dots,\flat^{-1}b_m;h_1,\dots,h_k\big),$$

the difference of $M\big(R\cdot\flat^{-1}b_1,\dots,R\cdot\flat^{-1}b_m;R\cdot h_1,\dots,R\cdot h_k\big)$ and $M\big(\flat^{-1}b_1,\dots,\flat^{-1}b_m;h_1,\dots,h_k\big)$. E is an m-linear form on $H(U)$, $\mathbb{C}$-linear in each entry, and bounded: $|E(b_1,\dots,b_m)| \le \big(B\prod_j\|h_j\|_{L^1}\big)\prod_i\|b_i\|_{L^2}$ by (1e).

Verification of (H1$_U$) *for* E. Let $b_1,\dots,b_m\in C_c^\infty(U;\mathbb{C}^3)$ have pairwise disjoint supports. The sets $\operatorname{supp} b_i$ are compact subsets of U, so they are disjoint from K, hence from every G_j; the $m+k$ sets $\operatorname{supp} b_1,\dots,\operatorname{supp} b_m,G_1,\dots,G_k$ are pairwise disjoint Borel sets carrying the entries $\flat^{-1}b_i$ and h_j, and Step 3 gives $E(b_1,\dots,b_m)=0$ (for $m=1$: $E(b_1)=0$ for all $b_1\in C_c^\infty(U;\mathbb{C}^3)$).

Verification of (H2$_U$) *for* E. Let one entry be $b_i=\nabla u$, $u\in C_c^\infty(U;\mathbb{C})$, the others arbitrary in $H(U)$. Then $\flat^{-1}b_i = *du$ with $u\in C_c^\infty(\mathbb{R}^3;\mathbb{C})$, and Step 2 gives $\Delta_R(\dots,*du,\dots;h_1,\dots,h_k)=0$ (both terms vanish separately).

By Theorem 12.34 with $n=m$ and this U, $E=0$ on $H(U)^m$. If f_i vanishes a.e. on K, then $b_i := (\flat f_i)|_U$ is in $H(U)$, its extension by 0 is $\flat f_i$, and $\Delta_R(f;h)=E(b_1,\dots,b_m)=0$. (Theorem 12.34 is applied once, on $U=\mathbb{R}^3\setminus K$, to the difference E of two forms of the unrotated arguments; (H2$_U$) holds for each of the two terms, (H1$_U$) for the difference only. The density entries are fixed inside the definition of E.)

Step 5: no mass on coincidences of an L^1-bounded entry. $\Delta_R(f_1,\dots,f_m;h_1,\dots,h_k)=0$ *for every* $R\in\mathrm{O}(3)$, *all* $m,k\ge0$ *with* $m+k\ge1$, *all* $f_i\in L^2(\mathbb{R}^3;\Lambda^2_{\mathbb{C}})$ *and all* $h_j\in L^1(\mathbb{R}^3;\mathbb{C})$ (the second half of the last sentence of the lemma). For $h\in L^1$ and $f\in L^2$ put $\omega_h^1(t):=\sup\{\|h1_E\|_{L^1}:\ E \text{ Borel},\ |E|\le t\}$ and $\omega_f^2(t):=\sup\{\|f1_E\|_{L^2}:\ E \text{ Borel},\ |E|\le t\}$; both tend to 0 as $t\to0$ (absolute continuity of the integrals of $|h|$ and of $|f|^2$). If $k=0$, the claim is Step 4 with $K=\emptyset$. Let $k\ge1$. Fix $\delta>0$ and let $\mathcal{Q}_\delta$ be the partition of $\mathbb{R}^3$ into the half-open cubes $\delta(z+[0,1)^3)$, $z\in\mathbb{Z}^3$. For $h\in L^1$, $h=\sum_Q h1_Q$ with $\sum_Q\|h1_Q\|_{L^1}=\|h\|_{L^1}$, so the series converges in L^1. Δ_R is linear and continuous in each density entry (1e); expanding the entries one after the other,

$$\Delta_R(f;h_1,\dots,h_k)=\sum_{(Q_1,\dots,Q_k)\in\mathcal{Q}_\delta^k}\Delta_R\big(f;h_11_{Q_1},\dots,h_k1_{Q_k}\big), \tag{12.10}$$

and the series converges absolutely (so in any order): the sum of the moduli of its terms is at most

$$B\prod_i\|f_i\|_{L^2}\sum_{(Q_1,\dots,Q_k)}\prod_j\|h_j1_{Q_j}\|_{L^1}=B\prod_i\|f_i\|_{L^2}\prod_j\|h_j\|_{L^1}.$$

(a) *Tuples with a repeated cube* (only if $k\ge2$). For a pair $i<j$ the tuples with $Q_i=Q_j$ contribute at most

$$B\prod_l\|f_l\|_{L^2}\Big(\sum_Q\|h_i1_Q\|_{L^1}\|h_j1_Q\|_{L^1}\Big)\prod_{l\notin\{i,j\}}\|h_l\|_{L^1}\ \le\ B\prod_l\|f_l\|_{L^2}\,\omega_{h_i}^1(\delta^3)\prod_{l\ne i}\|h_l\|_{L^1},$$

since $\|h_i1_Q\|_{L^1}\le\omega_{h_i}^1(\delta^3)$ for every cube and $\sum_Q\|h_j1_Q\|_{L^1}=\|h_j\|_{L^1}$. Hence the sum over the tuples with a repeated cube is at most $a(\delta):=B\prod_l\|f_l\|_{L^2}\sum_{i<j}\omega_{h_i}^1(\delta^3)\prod_{l\ne i}\|h_l\|_{L^1}$ in modulus.

(b) *Tuples of pairwise distinct cubes.* Fix such a tuple and let K be the union of the closures of $Q_1,\dots,Q_k$: K is compact, $|K|=k\delta^3$, and the Q_j are pairwise disjoint Borel subsets of K. Write $f_i=f_i1_K+f_i1_{\mathbb{R}^3\setminus K}$ in every curvature entry and expand the term of (12.10) into 2^m terms. The term in which every entry carries $f_i1_{\mathbb{R}^3\setminus K}$ vanishes by Step 4 (with $G_j:=Q_j$). Each of the other 2^m-1 terms carries f_i1_K in at least one entry i and entries of norm $\le\|f_l\|_{L^2}$ in the others, so by (1e) it is at most $B\,\omega_{f_i}^2(k\delta^3)\prod_{l\ne i}\|f_l\|_{L^2}\prod_j\|h_j1_{Q_j}\|_{L^1}$ in modulus. Hence the term of (12.10) is at most $(2^m-1)\,B\,W(\delta)\prod_j\|h_j1_{Q_j}\|_{L^1}$ in modulus, where $W(\delta):=\max_i\omega_{f_i}^2(k\delta^3)\prod_{l\ne i}\|f_l\|_{L^2}$ does not depend on the tuple. Summing over the tuples of distinct cubes, the sum is at most

$b(\delta) := (2^m - 1)\, B\, W(\delta) \prod_j \|h_j\|_{L^1}$ in modulus. (For $m = 0$ every such term vanishes by Step 3, and $b = 0$.)

So $|\Delta_R(f;h)| \le a(\delta)+b(\delta)$ for every $\delta > 0$; the left side does not depend on δ, and $a(\delta)+b(\delta) \to 0$ as $\delta \to 0$. Hence $\Delta_R(f;h) = 0$.

Step 6: from the moments to the characteristic functional. Let $\sigma = (f,h)$ be a real Schwartz pair and $R \in \mathrm{O}(3)$; $R\cdot\sigma = (R\cdot f, R\cdot h)$ is a real Schwartz pair. By (P1) for σ and for $-\sigma$,

$$\int e^{|\Phi(\sigma)|} d\mu \le \int \big(e^{\Phi(\sigma)} + e^{-\Phi(\sigma)}\big) d\mu \le 2\exp\big(\tfrac{1}{2}\|f\|_{L^2}^2 + c\|h\|_{L^1}\big) < \infty,$$

and the same holds for $R\cdot\sigma$, with the same right side. For real x, $\big|\sum_{n\le N}(ix)^n/n!\big| \le e^{|x|}$; so by dominated convergence $\int e^{i\Phi(\sigma)} d\mu = \sum_{n\ge 0} \frac{i^n}{n!} \int \Phi(\sigma)^n d\mu$, an absolutely convergent series ($\sum_n \frac{1}{n!}\int |\Phi(\sigma)|^n d\mu = \int e^{|\Phi(\sigma)|} d\mu$), and the same for $R\cdot\sigma$. Since $\Phi(\sigma) = A(f)+D(h)$, $\int \Phi(\sigma)^n d\mu = \sum_{j=0}^n \binom{n}{j} M(f,\dots,f;h,\dots,h)$ with j entries f and $n-j$ entries h, every term being finite by (1c); likewise $\int \Phi(R\cdot\sigma)^n d\mu = \sum_{j=0}^n \binom{n}{j} M_R(f,\dots,f;h,\dots,h)$. By Step 5 ($\Delta_R = 0$, with the entries f and h repeated: coincident supports in every entry) the two sums are equal for every $n \ge 1$; for $n = 0$ both are 1. Hence $\int e^{i\Phi(R\cdot\sigma)} d\mu = \int e^{i\Phi(\sigma)} d\mu$. This argument uses no uniqueness theorem for a moment problem: the two characteristic functionals are compared coefficient by coefficient under one measure, the series converging by the exponential bound (P1). □

12.12. **Proofs of the theorems.**

Proof of Theorem 12.2. By Lemma 12.20 the box family satisfies the hypotheses of Theorem 7.1 and of every statement of Section 9 for the couplings of the theorem and any bounded Borel g_{d}, so S^{box}, its Schwinger functions S_n and its measure μ^{box} exist, and Lemmas 12.21, 12.22 and Theorem 12.26 hold for them. Proposition 12.36, whose inputs are these three together with Theorem 12.31 and Theorem 12.34, gives $Z^{\mathrm{box}}(R\cdot f) = Z^{\mathrm{box}}(f)$ for every $R \in \mathrm{O}(3)$ and every real C_c^∞ 2-form f. With the choice $g_{\mathrm{d}} \equiv 0$, $\Phi_N(\sigma)$ depends on f alone (Remark 12.9), so $Z^{\mathrm{box}}(f,h) = Z^{\mathrm{box}}(f,0)$ for every test pair, and $Z^{\mathrm{box}}(R\cdot\sigma) = Z^{\mathrm{box}}(\sigma)$ for every test pair σ; by linearity on the basis (e_σ) of $\mathcal{A}$, $S^{\mathrm{box}} \circ \gamma_R = S^{\mathrm{box}}$ for all $R \in \mathrm{O}(3)$.

Proposition 12.11, whose hypothesis is the invariance of S under all translations (Theorem 7.1, here for the box family) and under O(3) (just proved), applies to S^{box}. Its part (a) gives $S^{\mathrm{box}}(e_{\gamma\cdot\sigma}) = S^{\mathrm{box}}(e_\sigma)$ for all $\gamma \in E(3)$, and its part (b) gives $\gamma_*\mu^{\mathrm{box}} = \mu^{\mathrm{box}}$ for every $\gamma \in E(3)$ and every measure representing Z^{box} on a set of linear functionals with the cylinder σ-algebra. This holds in particular for the measure of Theorem 9.13, either through Proposition A.10(a) or directly through (b) applied to μ on X, whose Borel σ-algebra is generated by the evaluations (Lemma A.5(c)), and Lemma A.11(b); it holds also for the Minlos measure of Lemma 12.12.

Remark 9.16 now applies to S^{box}: μ^{box} is reflection positive across every affine plane, on L^2 of either open side (the half-space algebra and the reflection of a tilted plane are the rotated ones, and μ^{box} is $E(3)$-invariant), and the reconstruction in any direction is unitarily the one in direction e_3. OS0 is Theorem 9.4, OS1 is Proposition 9.6 together with the O(3)-invariance just proved, OS2 is Proposition 9.9 and Theorem 9.10 together with the reflection positivity across every plane, OS3 is Proposition 9.12, and the measure is Theorem 9.13, all for the box family by Lemma 12.20. □

Proof of Theorem 12.3. Let g_{d} be bounded Borel. *The curvature marginal.* Lemmas 12.20, 12.21, 12.22, Theorem 12.26 and Proposition 12.36 hold for any bounded Borel g_{d} (the density component of $(f,0)$ and of $(s*du,0)$ is zero), so $Z^{\mathrm{box}}(R\cdot f) = Z^{\mathrm{box}}(f)$ for all $R \in \mathrm{O}(3)$ and all real C_c^∞ 2-forms f, i.e. the curvature functional $f \mapsto Z^{\mathrm{box}}(f,0)$ and its measure (the curvature

marginal of μ of Lemma 12.22, which is the measure of Theorem 9.13) have all the properties of Theorem 12.2, by the proof just given.

The joint field. Let μ be the measure of Lemma 12.22 on $\mathcal{S}'(\mathbb{R}^3;\Lambda^2\oplus\mathbb{R})$, with $\int e^{i\Phi(\sigma)}d\mu = Z^{\mathrm{box}}(\sigma)$ for real Schwartz pairs. It satisfies (i) of Section 12.10 with $C=\max(\frac12,\|g_{\mathrm{d}}\|_\infty)$ (Lemma 12.22: $\|h\|_{L^1}\le\|\chi\|_{L^1}$ and $\|f\|_{L^2}\le\|\chi\|_{L^2}$ for $\chi=(f,h)$); (ii) by Proposition 9.6 for the box family (translation invariance of Z^{box} on test pairs, extended to Schwartz pairs by the continuity of Theorem 9.4(i) in its pair version) and the uniqueness of a probability measure on $\mathcal{S}'$ with a given characteristic functional (Lemma 12.12, the hypothesis-free form of the argument of Proposition 12.11(b)); (iii) by Lemma 12.21(iii) for the planes through 0 and tests in the open side (for a reflection the push-forward of a test pair is its pull-back with the tensor action, $\Theta_u=\theta_u^*$).

Its moments $\int\prod_i\Phi(\sigma_i)d\mu$ are the mixed Schwinger functions $S_n(\sigma_1,\dots,\sigma_n)$ of the joint field: for real Schwartz pairs the law of $(\Phi(\sigma_i))_i$ under μ has the characteristic function $t\mapsto Z^{\mathrm{box}}(\sum_i t_i\sigma_i)$, which is the limit of the characteristic functions of $(\Phi_{N_m}(\sigma_i))_i$ under $\mu^{\mathrm{box}}_{N_m}$ (Theorem 7.1(a)), and the moments pass to the limit by the uniform exponential bounds of Lemma 12.22 at level N (the density part is bounded by $\|g_{\mathrm{d}}\|_\infty E_N(h)$, the curvature part by Theorem 11.11(b)). The Schwinger functions of the joint field are defined as these moments, which for $h_i\equiv0$ are the S_n of Theorem 9.4(iii). Theorem 12.31 with $T=\Lambda^2\mathbb{R}^3\oplus\mathbb{R}$ and R^{-1} ($R\cdot\sigma=(R^{-1})^*\sigma$) gives $S_n(R\cdot\sigma_1,\dots,R\cdot\sigma_n)=S_n(\sigma_1,\dots,\sigma_n)$ for all $R\in\mathrm{O}(3)$ whenever the supports of the σ_i are pairwise disjoint. The density entries carry no closedness identity, so Theorem 12.34 does not apply to them; their coincident-point parts are removed by the L^1 bound of Lemma 12.22 instead (Lemma 12.37, Corollary 12.38(a)). □

Proof of Theorem 12.4. We run the proofs of Theorems 12.2 and 12.3 for the torus limit points ω_N themselves, and go through their ingredients in turn. Theorem 7.1 and Section 9 hold for the torus limit points as stated.

Lemma 12.21 needs of the level-N states only the reflection positivity of Theorem 11.5(d) at the lattice planes of the nine families through 0: its part (i) uses it through Lemma 12.19(ii), and its part (ii) uses it, in the L^2 form, through Lemma 12.19(iii). At the coordinate planes this reflection positivity is Theorem 11.5(d) itself. At the six diagonal site planes through 0 it is Corollary 14.35 at $c=0$, the Hermitian form and the L^2 closure included, and Theorem 14.3 for the bounded Borel functions. Here the sides of Definition 2.3 run along a dyadic sequence (Definition 14.1), as that corollary requires. Section 14 proves these statements for $\kappa_N=1$; for $\kappa_N>0$ they are Lemma 2.7(g). Explicitly, at each level N the state ω_N is $F\mapsto\omega_N^\star(F\circ r)$, where r multiplies the site variables by $\kappa_N^{-1/2}$ and $\omega_N^\star$ is a limit point, along the same dyadic sequence, of the torus states of the model with hopping coefficient 1 and the coefficients (2.5) (Lemma 2.7(c)); the statements hold for $\omega_N^\star$, and they pass to ω_N because r commutes with the lattice reflections and preserves the functions of the cells of a half space and the L^2 norms (Lemma 2.7(d)). This is the only step of the proof in which the value of κ_N enters, and it is taken level by level: the limit functional S, the scheme and the density function g_{d} are those of the given models, and every other step holds for every $\kappa_N\ge0$. For a diagonal reflection R the intertwining $\Phi_N(\sigma)\circ R=\Phi_N(\gamma\sigma)$ and the R-invariance of μ_N hold by Theorem 11.5(c), since $R\in\mathcal{G}_N$.

Lemma 12.22 uses only Theorem 11.5(b), in place of Theorem 11.11(b). Theorem 12.26 is proved for both families. Theorems 12.31 and 12.34 are statements about the limit law and about multilinear forms, and Proposition 12.36 and the assembly of these ingredients in the two proofs use nothing else.

Finally, Corollary 12.38(b) uses, besides Lemma 12.37, only the display of Lemma 12.22 in its torus form, which is stated in that corollary, Theorem 12.26, Theorem 12.34 and the covariance

on pairwise disjoint supports, which the paragraph on the joint field in the proof of Theorem 12.3 has just given for the torus limit points. Hence the conclusions of Theorems 12.2 and 12.3 hold for S. □

13. Rigidity of the Villain–Higgs Gibbs measures on $\varepsilon_N\mathbb{Z}^3$

This section proves a rigidity theorem (Theorem 13.1): a Gibbs measure of the fine Villain–Higgs specification on $\varepsilon_N\mathbb{Z}^3$ (the model of Section 2, in unitary gauge) which is invariant under some finite-index sublattice of $\varepsilon_N\mathbb{Z}^3$ and has the three regularity properties (I1)–(I3) below is invariant under the translations by $2\varepsilon_N e_j$, $j = 1, 2, 3$. The main ingredient is the statement that two consecutive layers determine the Gibbs measure (Theorem 13.30, Subsection 13.9), proved on $\varepsilon_N\mathbb{Z}^3$. Two elementary steps, a conjugation of reflections by coordinate transpositions and a composition of two parallel reflections, are proved in Subsection 13.10 (Proposition 13.39); the relative-entropy estimate on which the argument rests is the tangent inequality of Lemma 13.9. Throughout, N is fixed, $\varepsilon := \varepsilon_N$, and all constants may depend on N, ε, the couplings and the finite sets involved.

13.1. Statement.

The model on $\varepsilon_N\mathbb{Z}^3$. Sites $x \in \varepsilon\mathbb{Z}^3$ carry the radial field $\tilde\rho_x \in [-\rho_{0,0}, \infty)$, $\rho_x := \rho_{0,0} + \tilde\rho_x$; links b carry $B_b \in \mathrm{FD} := (-\pi/\gamma_N, \pi/\gamma_N]$, $B_{-b} = -B_b$; plaquettes p carry the Villain 2-form $v_p \in (2\pi/\gamma_N)\mathbb{Z}$, $v_{-p} = -v_p$, closed ($dv = 0$ on every cube). Here $\gamma_N > 0$ is the phase constant of Section 2, a general coefficient (Remark 2.1), and $\kappa_N = 1$ is the value of the hopping coefficient for which this section and the next are written (Lemma 2.7(g) extends Theorem 13.1 to every $\kappa_N > 0$); the local weight is that of Section 2:

$$W = \prod_x g_{\mathrm{ct}}(\tilde\rho_x) \prod_b h(\tilde\rho_{b_-}, \tilde\rho_{b_+}, B_b) \prod_p w\big((dB)_p + v_p\big), \tag{13.1}$$
$$w(f) = e^{-f^2/2}, \qquad h(r, r', \vartheta) = \exp\big(-\tfrac{1}{2}\kappa_N |e^{i\gamma_N\vartheta}(\rho_{0,0} + r') - (\rho_{0,0} + r)|^2\big),$$

$h(r, r', \cdot)$ being $2\pi/\gamma_N$-periodic and $g_{\mathrm{ct}}e^{q\rho^2}$ integrable on $[-\rho_{0,0}, \infty)$ for every $q \geq 0$ by (2.1). The curvature is $f_p := (dB)_p + v_p$; a plaquette is temporal if it contains a time link (x_3 is the time coordinate, e_3 the time direction), spatial otherwise. We write $c := 2\pi/\gamma_N$ for the Villain period and $\mathbb{T} := \mathbb{R}/c\mathbb{Z}$.

For a finite $\Lambda \subset \varepsilon\mathbb{Z}^3$ let $V(\Lambda)$ be the set of sites in Λ together with the bonds and plaquettes meeting Λ. The specification $\gamma_\Lambda(\cdot \mid \omega)$ is the normalised integral of (13.1) over the variables carried by $V(\Lambda)$ (the v_p summed over the closed integer 2-forms with the exterior v fixed), the exterior configuration ω held fixed. The specification $\{\gamma_\Lambda\}$ is consistent on the cofinal family $\{V(\Lambda)\}$, and a probability measure ν on the configurations $(\tilde\rho, B, v)$ of $\varepsilon\mathbb{Z}^3$ is a *Gibbs measure* if $\nu\gamma_\Lambda = \nu$ for every finite Λ (equivalently for the full specification, by consistency); this is the Dobrushin–Lanford–Ruelle (DLR) equation, see [11, 26] and [16, Chapters 1 and 2]. For $a \in \varepsilon\mathbb{Z}^3$, τ_a is the translation of configurations; ν is Γ-invariant, $\Gamma \subset \varepsilon\mathbb{Z}^3$ a sublattice, if $\nu \circ \tau_g = \nu$ for all $g \in \Gamma$. For $\mu \in \{1, 2, 3\}$ and $c \in \varepsilon(\mathbb{Z} + \frac{1}{2})$, $\theta_{\mu,c}$ is the reflection in the mid-plane $\{x_\mu = c\}$ (half-way between two adjacent layers of sites), acting on configurations by transport of the forms. In this section, for a lattice isometry $g \in \varepsilon\mathbb{Z}^3 \rtimes B_3$ or a mid-plane reflection $g = \theta_{\mu,c}$, we also write $\hat g$ for the map of configurations induced by g by transport of the forms, and $\nu \circ \hat g$ for the measure $f \mapsto \nu(f \circ \hat g)$; thus $\hat\theta_{\mu,c}$ and $\theta_{\mu,c}$ denote the same map of configurations.

Theorem 13.1 (rigidity). *Let N be fixed and let ν be a Gibbs measure of the fine Villain–Higgs specification on $\varepsilon\mathbb{Z}^3$, $\varepsilon = \varepsilon_N$, which is invariant under a sublattice $\Gamma \subset \varepsilon\mathbb{Z}^3$ of finite index containing a time translation, and which satisfies*

(I1) *$\nu(f_p^2) < \infty$ and $\nu(\rho_x^2) < \infty$ for every plaquette p and site x (finite local second moments);*

(I2) *joint Gaussian domination of the curvature over all plaquette orientations: there is* $C < \infty$ *with* $\nu\big(\exp\sum_p s_p f_p\big) \le C\exp(|s|^2/2)$ *for every finitely supported real* s *on the plaquettes of* $\varepsilon\mathbb{Z}^3$*, temporal and spatial alike;*

(I3) $\sup_x \nu\big(\exp(q\rho_x^2)\big) < \infty$ *for every* $q > 0$.

Then ν *is invariant under the translations* $\tau_{2\varepsilon e_j}$, $j = 1, 2, 3$.

In Theorem 13.1 the clause "containing a time translation" is redundant: $\Gamma \supset k\varepsilon\mathbb{Z}^3$ with $k = [\varepsilon\mathbb{Z}^3 : \Gamma]$ (Subsection 13.10, Step 2 of the proof of the theorem).

Remark 13.2 (where (I1)–(I3) hold). (I1)–(I3) hold for every cubic-torus limit point ω_N of Theorem 11.5: (I2) is Theorem 3.1 (constant 1, all plaquettes jointly), (I1) and (I3) are Lemma 11.2. They also hold for the comparison states of Section 14 (Lemma 14.20, Proposition 14.22).

13.2. **Phase restoration and the bijection of Gibbs measures.** Let Ω^u be the unitary-gauge configuration space $\{(\tilde\rho, B, v)\}$ of Subsection 13.1 and $\widetilde\Omega := \Omega^u \times \mathbb{T}^{\text{sites}}$, with Higgs phases $\psi_x \in \mathbb{T}$ and normalised Haar measure on each ψ_x as reference measure. The *phase-restored* weight keeps g_{ct}, $w((dB)_p + v_p)$ and the closure constraint, and replaces the hopping factor by $h(\tilde\rho_{b_-}, \tilde\rho_{b_+}, B_b + \psi_{b_+} - \psi_{b_-})$; it is well defined on $\mathbb{T}$ because $h(r, r', \cdot)$ is c-periodic.

Lemma 13.3 (phase-restored specification)**.** *Let* $G := \mathbb{T}^{\text{sites}}$*. For* $\lambda \in G$ *with real lift* $\widetilde\lambda \in \mathrm{FD}^{\text{sites}}$ *put*

$$\begin{aligned} \lambda.(\tilde\rho, \psi, B, v) &:= \big(\tilde\rho,\ \psi - \lambda,\ [B + d\widetilde\lambda]_{\mathrm{FD}},\ v + c\,dn\big), \\ n_b &:= \big(B_b + (d\widetilde\lambda)_b - [B_b + (d\widetilde\lambda)_b]_{\mathrm{FD}}\big)/c \in \mathbb{Z}, \end{aligned} \tag{13.2}$$

where $[\cdot]_{\mathrm{FD}}$ *is the representative in* FD *modulo* c*. Then:*

(a) (13.2) *is independent of the lift, is a group action of* G *on* $\widetilde\Omega$*, keeps* v *closed* ($d\,dn = 0$)*, leaves the phase-restored weight invariant factor by factor and preserves the reference measure.*

(b) *The gauge fixing* $\Phi(\widetilde\omega) := \psi.\widetilde\omega$*, read in* Ω^u*, and the Haar randomisation* $\Psi(\omega, \psi) := (-\psi).(\tilde\rho, 0, B, v)$ *satisfy* $\Phi(\lambda.\widetilde\omega) = \Phi(\widetilde\omega)$*,* $\widetilde\omega = (-\psi).\Phi(\widetilde\omega)$*,* $\Phi \circ \Psi = \mathrm{id}$ *and* $\lambda.\Psi(\omega, \psi) = \Psi(\omega, \psi - \lambda)$*; the* G*-orbits are the fibres of* Φ*, and the* G*-invariant functions on* $\widetilde\Omega$ *are exactly the functions* $f \circ \Phi$*.*

(c) *Let* γ_Λ *be the unitary-gauge specification of Subsection 13.1 and* $\widetilde\gamma_\Lambda$ *the phase-restored one (Haar integral over* ψ_Λ*, phase-restored hopping), both on the cofinal family* $\{V(\Lambda)\}$*.* Φ *is local outside* $V(\Lambda)$*, and for every bounded* F *on* $\widetilde\Omega$

$$\widetilde\gamma_\Lambda(F \mid \widetilde\omega) = \int \mathrm{Haar}(d\psi_\Lambda) \int \gamma_\Lambda\big(d\omega_{\text{int}} \mid \Phi(\widetilde\omega)\big)\, F\big(\Psi(\omega_{\text{int}} \cup \Phi(\widetilde\omega)_{\text{ext}},\ \psi_\Lambda \cup \psi_{\text{ext}})\big); \tag{13.3}$$

in particular $\widetilde\gamma_\Lambda(f \circ \Phi \mid \widetilde\omega) = \gamma_\Lambda(f \mid \Phi(\widetilde\omega))$*.*

(d) *If* $\widetilde\mu$ *is a Gibbs measure for* $\widetilde\gamma$ *then* $\Phi_*\widetilde\mu$ *is a Gibbs measure for* γ*; if* μ *is a Gibbs measure for* γ *then* $\widetilde\mu := \Psi_*(\mu \times \mathrm{Haar}^{\text{sites}})$ *is a* G*-invariant Gibbs measure for* $\widetilde\gamma$ *with* $\Phi_*\widetilde\mu = \mu$*; and* Φ_**,* $\Psi_*(\cdot \times \mathrm{Haar})$ *are mutually inverse bijections between the* G*-invariant Gibbs measures of* $\widetilde\gamma$ *and the Gibbs measures of* γ*.*

(e) $\Phi \circ \tau_a = \tau_a \circ \Phi$ *and* $\Psi \circ (\tau_a \times \tau_a) = \tau_a \circ \Psi$ *for every* $a \in \varepsilon\mathbb{Z}^3$*; hence* μ *is* τ_a*-invariant if and only if* $\widetilde\mu$ *is. The same holds for the lattice reflections* $\theta_{\mu,c}$*, up to the Lebesgue-null endpoint of* FD*, and, without exception, for the transpositions* g *of two coordinates; for such a* g*, moreover,* $\Psi_*\big((\mu \circ \hat g) \times \mathrm{Haar}^{\text{sites}}\big) = \widetilde\mu \circ \hat g$*.*

Proof. (a) Two lifts differ by cm, m integer, which changes $d\widetilde\lambda$ by $c\,dm$ and n by $+dm$, leaving $[B + d\widetilde\lambda]_{\mathrm{FD}}$ and $v + c\,dn$ unchanged; the composition rule follows from the same bookkeeping,

and $d\,dn = 0$ keeps v closed. The hopping factor is invariant since $B + d\psi$ is unchanged modulo c; the plaquette factor since $d[B + d\widetilde{\lambda}]_{\mathrm{FD}} + v + c\,dn = dB + v$; g_{ct} does not see λ. The reference measure is preserved: $B_b \mapsto [B_b + (d\widetilde{\lambda})_b]_{\mathrm{FD}}$ is a rotation of FD, $\psi \mapsto \psi - \lambda$ a Haar shift, and at fixed B, $v \mapsto v + c\,dn$ is a bijection of the closed integer 2-forms.

(b) is read off from (13.2): $\Phi(\widetilde{\omega})$ has $\psi = 0$ and is constant on orbits, $(-\psi).\Phi(\widetilde{\omega})$ restores ψ.

(c) Locality: Φ changes only variables carried by the bonds and plaquettes at the sites where $\psi \neq 0$ enters, and ψ_Λ is integrated out. For (13.3) make the interior change of variables $(B, v)_{\mathrm{int}} \mapsto (\psi.(\cdot))_{\mathrm{int}}$: it is bijective and measure preserving by (a), maps the closure constraint onto itself, turns the phase-restored weight into the unitary-gauge weight W^u_Λ at boundary condition $\Phi(\widetilde{\omega})$, and leaves the normalisation free of ψ. The special case is $F = f \circ \Phi$ with $\Phi \circ \Psi = \mathrm{id}$.

(d) The first claim is the special case of (c) integrated against $\widetilde{\mu}$; the second is (13.3) with Fubini and $\mu\gamma_\Lambda = \mu$, the G-invariance from $\lambda.\Psi(\omega, \psi) = \Psi(\omega, \psi - \lambda)$ and the shift invariance of Haar; the third from $\widetilde{\mu} = \Psi_*(\Phi_*\widetilde{\mu} \times \mathrm{Haar})$ for G-invariant $\widetilde{\mu}$.

(e) Φ and Ψ are given by the same local formula at every site, and $\mathrm{Haar}^{\mathrm{sites}}$ is shift invariant. For a reflection, $[-x]_{\mathrm{FD}} = -[x]_{\mathrm{FD}}$ except at $x = \pi/\gamma_N$, and B_b has a Lebesgue density under every Gibbs measure because every $\gamma_\Lambda(\cdot \mid \omega)$ has one. A transposition g of two coordinates maps every bond from x to $x + \varepsilon e_\alpha$ to a bond of the same form, so that no B_b changes sign; the formula (13.2), which is built from d and from the bond-wise representative $[\cdot]_{\mathrm{FD}}$, therefore commutes with $\hat{g}$ without exception; a plaquette whose orientation g reverses (every plaquette of the plane of the two swapped coordinates is one) carries $-v$ and $-dn$ after the transport, because the Villain form is transported as a 2-cochain and d commutes with the transport, and no representative $[\cdot]_{\mathrm{FD}}$ is taken on the plaquettes, whose variables range over the symmetric set $c\mathbb{Z}$. The last identity follows from $\Psi \circ (\hat{g} \times \hat{g}) = \hat{g} \circ \Psi$ and from the invariance of $\mathrm{Haar}^{\mathrm{sites}}$ under the relabelling of the sites. □

By Lemma 13.3(d)–(e), Theorem 13.1 for the unitary-gauge specification is equivalent to the same statement for the G-invariant Gibbs measures of the phase-restored specification. The hypotheses (I1)–(I3) concern $\tilde{\rho}$ and $f = dB + v$ only, which Φ and Ψ leave unchanged ($\tilde{\rho}$ untouched, f gauge invariant by (a)), so ν satisfies (I1)–(I3) if and only if $\widetilde{\nu} := \Psi_*(\nu \times \mathrm{Haar})$ does. *From here on ν denotes $\widetilde{\nu}$.*

13.3. **Slab, temporal gauge, compact twist and the slab kernel.** Fix two adjacent time layers n and $n+1$ and the slab between them. The lower layer carries $c = (\tilde{\rho}, \psi, B, v)$ (its sites, spatial bonds and spatial plaquettes), the upper layer $c' = (\tilde{\rho}', \psi', B', v')$; the temporal links carry $B_t(x) \in \mathrm{FD}$ (Lebesgue measure on FD, which is Haar) and the temporal plaquettes carry the Villain integers $v^t_b \in c\mathbb{Z}$ (b a spatial bond) with the closure relation $d_s v^t = v - v'$ (d_s the spatial exterior derivative). The slab weight is

$$\prod_x h\big(\rho_x, \rho'_x, B_t(x) + \psi'_x - \psi_x\big) \prod_b w\big(B_b - B'_b + (d_s B_t)_b + v^t_b\big), \tag{13.4}$$

where here and below $h(\rho, \rho', \vartheta)$ abbreviates $h(\tilde{\rho}, \tilde{\rho}', \vartheta)$ with $\rho = \rho_{0,0} + \tilde{\rho}$.

Lemma 13.4 (temporal gauge, slab kernel, compact twist)**.** (a) *(Temporal gauge.) The gauge transformation (13.2) with $\lambda = 0$ on layer n and $\lambda = -B_t$ on layer $n+1$ sends $B_t \mapsto 0$, $\psi' \mapsto \widetilde{\psi}' := \psi' + B_t$, $B' \mapsto u := [B' - d_s B_t]_{\mathrm{FD}}$, $v' \mapsto \widetilde{v}' := v' + c\,d_s n'$, $v^t \mapsto \widetilde{v}^t := v^t - c\,n'$, where n' are the wrap integers of u; one has $B - u + \widetilde{v}^t = B - B' + d_s B_t + v^t$ and $d_s \widetilde{v}^t = v - \widetilde{v}'$. Write $\widetilde{c}' := (-B_t).c'$ and $\Theta_\sigma(\omega) := (c, \widetilde{c}', \widetilde{v}^t)$: the two temporal-gauge layers* together with *the temporal Villain integers.*

(b) *(Slab kernel.) Under every finite-volume slab integral the slab data* $(c, \widetilde{c}', \widetilde{v}^t)$ *have the law*

$$W(c)W(\widetilde{c}')\prod_x h(\rho_x, \widetilde{\rho}'_x, \widetilde{\psi}'_x - \psi_x)\prod_b w(B_b - u_b + \widetilde{v}^t_b)\,\mathbf{1}[d_s\widetilde{v}^t = v - \widetilde{v}']$$

with ψ' *an independent uniform spectator,* $W(c)$ *being the one-layer weight, that is, the product of the factors of the phase-restored weight that belong to the sites, spatial bonds and spatial plaquettes of the layer; summing out* $\widetilde{v}^t$ *gives the two-layer law* $W(c)W(\widetilde{c}')K(c, \widetilde{c}')$ *with*

$$K(c, \widetilde{c}') := \prod_x h\big(\rho_x, \widetilde{\rho}'_x, \widetilde{\psi}'_x - \psi_x\big)\sum_{\widetilde{v}^t:\ d_s\widetilde{v}^t = v - \widetilde{v}'}\ \prod_b w\big(B_b - u_b + \widetilde{v}^t_b\big). \tag{13.5}$$

K *is symmetric,* $K(c, \widetilde{c}') = K(\widetilde{c}', c)$*, and invariant under the diagonal layer gauge action,* $K(\lambda.c, \lambda.\widetilde{c}') = K(c, \widetilde{c}')$*. The gauge-invariant slab functions are the functions of* $(c, \widetilde{c}', \widetilde{v}^t)$ *invariant under the diagonal action.*

(c) *(Compact twist.) Let* θ *be the reflection in the mid-plane of the slab and let* sw *swap the two layers and reverse the sign of* $\widetilde{v}^t$*. Then* $\Theta_\sigma(\theta\omega) = B_t.\big(\mathrm{sw}\,\Theta_\sigma(\omega)\big)$*, the twist being the diagonal action of the compact element* $g = B_t \in \mathbb{T}^{\mathrm{sites}}$*. Consequently, for every diagonal-gauge-invariant* F*,* $F(\Theta_\sigma\theta\omega) = F(\mathrm{sw}\,\Theta_\sigma\omega)$*.*

Proof. (a) is the computation of (13.2) at the two layers; $d\widetilde{\lambda}$ on a temporal link is $-B_t$, which kills B_t, and on a spatial bond of the upper layer it is $-(d_sB_t)_b$. (b) The map $(\psi', B_t) \mapsto (\psi', \widetilde{\psi}')$ preserves Haar on $\mathbb{T}\times\mathbb{T}$, and at fixed phases $(B', v', v^t) \mapsto (u, \widetilde{v}', \widetilde{v}^t)$ is the measure-preserving bijection of Lemma 13.3(a), which preserves the closure set; the weight (13.4) is then ψ'-free and reads as stated. Symmetry of K: $h(r, r', \vartheta) = h(r', r, -\vartheta)$, w is even, and $\widetilde{v}^t \mapsto -\widetilde{v}^t$ maps the coset $\{d_s\widetilde{v}^t = v - \widetilde{v}'\}$ onto $\{d_s\widetilde{v}^t = \widetilde{v}' - v\}$. Diagonal gauge invariance: the diagonal action moves $\widetilde{v}^t$ by the wrap difference of the two layers and maps the coset onto the new one, while each factor is invariant by Lemma 13.3(a). (c) The mid-plane reflection swaps the layers and reverses B_t and v^t; its temporal gauge needs $\lambda = +B_t$ on the new upper layer, which gives $\Theta_\sigma(\theta\omega) = (c', B_t.c, -\widetilde{v}^t) = B_t.(\mathrm{sw}\,\Theta_\sigma(\omega))$. The last claim follows since F is invariant under the diagonal action of B_t. □

Remark 13.5 (the Villain coset sum). The sum in (13.5) runs over the coset of closed integer 1-forms fixed by $v - \widetilde{v}'$; it is bounded above by the free periodised Gaussian $\prod_b w_{\mathrm{per}}(B_b - u_b)$ but is not bounded below uniformly in $(v, \widetilde{v}')$: it decays like $\exp(-\operatorname{dist}(\mathrm{coset}, \mathrm{FD}^S - \mathrm{FD}^S)^2/2)$. This is why the insertion calculus of Subsections 13.4–13.5 works with $\mathbb{Z}$-periodic (frame-blind) insertions and why the Villain sector is treated separately in Subsection 13.7.

13.4. **Averaging identity, integer residual frame, Zak feature map.** Let S be a finite spatial bond set and K the slab kernel (13.5); in Lemma 13.7(c) and its proof one layer of the periodic measure is a finite spatial torus, and S is there the set of all spatial bonds of the layer. The pair coordinates of the slab over S are $B_b \in \mathrm{FD}$ and $u_b := \widetilde{B}'_b - \widetilde{v}^t_b \in \mathbb{R}$, $b \in S$; $(\widetilde{B}', \widetilde{v}^t) \leftrightarrow u$ is a bijection ($\widetilde{B}' = [u]_{\mathrm{FD}}$, $\widetilde{v}^t = [u]_{\mathrm{FD}} - u$) and $f := B - u = B - \widetilde{B}' + \widetilde{v}^t$ is the temporal plaquette curvature of Lemma 13.4(a). Put $h^x(\xi) := (2/\pi)^{|S|/4}e^{-|\xi - x|^2}$ on $\mathbb{R}^S$, so that $\prod_{b\in S} w(B_b - \widetilde{B}'_b + \widetilde{v}^t_b) = e^{-|B-u|^2/2} = \langle h^B, h^u\rangle_{L^2(\mathbb{R}^S)}$: the slab bond factor factorises through the Gaussian dressing, the closure constraint restricting only the integer parts $[u]_{\mathrm{FD}} - u$. For a bounded operator X on $L^2(\mathbb{R}^S)$ the *dressed insertion* and its ratio are

$$k_X(B, u) := \langle h^B, Xh^u\rangle, \qquad r_X := k_X/k_1, \qquad k_1(B, u) = e^{-|B-u|^2/2}. \tag{13.6}$$

X is $\mathbb{Z}$-*periodic* if $X\tau_a = \tau_a X$ for all $a \in c\mathbb{Z}^S$ (τ_a the translation of $L^2(\mathbb{R}^S)$, $(\tau_a g)(\xi) = g(\xi + a)$), and $X^a := \tau_a X\tau_a^*$.

Lemma 13.6 (frame, periodic insertions, gauge covariance)**.** (a) *(Residual frame.) After the compact twist of Lemma 13.4(c) the reflection acts on $(B, \widetilde{B}', \widetilde{v}^t)$ as $(\widetilde{B}', B, -\widetilde{v}^t)$, i.e. on the pair as $R(B,u) = T_a\,\mathrm{sw}(B,u)$ with the diagonal shift T_a by $a := [u]_{\mathrm{FD}} - u = \widetilde{v}^t \in c\mathbb{Z}^S$. R is an involution of $\mathrm{FD}^S \times \mathbb{R}^S$, $k_1 \circ R = k_1$ and $k_X \circ R = \overline{k_{(X^a)^*}}$. The shift $a = \widetilde{v}^t$ (the temporal Villain flux) is called the* frame*; it is part of the configuration, and no averaging over a is available.*

(b) *For $\mathbb{Z}$-periodic X, $X^a = X$ and $k_X \circ R = \overline{k_{X^*}}$: the reflection acts on dressed $\mathbb{Z}$-periodic insertions with no frame.*

(c) *Under the diagonal layer gauge action with $\lambda \in \mathbb{T}^{\mathrm{sites}}$ the pair moves by the diagonal translation $(B,u) \mapsto (B + d\widetilde{\lambda} - cn,\, u + d\widetilde{\lambda} - cn)$; for $\mathbb{Z}$-periodic X the wrap n drops, $\lambda \mapsto X^{d\widetilde{\lambda}}$ is an action of the compact group $\mathbb{T}^{\mathrm{sites}}$, and the gauge average $\bar{X} := \int_{\mathbb{T}^{\mathrm{sites}}} d\lambda\, X^{d\widetilde{\lambda}}$ satisfies $\|\bar{X}\| \le \|X\|$.*

Proof. (a) With $(\widetilde{B}', B, -\widetilde{v}^t)$ the new pair is $([u]_{\mathrm{FD}},\, B + [u]_{\mathrm{FD}} - u) = T_a(u, B)$. Applying R twice: the new frame is $[B + a]_{\mathrm{FD}} - (B + a) = -a$ (as $B \in \mathrm{FD}$), so $R^2 = \mathrm{id}$. $k_1 \circ R = k_1$ since $|u + a - (B + a)| = |B - u|$. For k_X:

$$k_X(R(B,u)) = \langle h^{u+a}, X h^{B+a}\rangle = \langle \tau_a^* h^u, X \tau_a^* h^B\rangle = \langle h^u, X^a h^B\rangle = \overline{\langle h^B, (X^a)^* h^u\rangle}.$$

(b) is (a) with $X^a = X$. (c) The diagonal action of Lemma 13.3(a) shifts both B and $\widetilde{B}'$ by $d\widetilde{\lambda}$ modulo the wrap, and $\widetilde{v}^t$ by the wrap difference, which is the stated diagonal translation of (B,u); k_X transforms into $k_{X^{d\widetilde{\lambda}-cn}} = k_{X^{d\widetilde{\lambda}}}$ for $\mathbb{Z}$-periodic X; the average of a strongly continuous unitary conjugation over a compact group has norm at most $\|X\|$. □

Lemma 13.7 (Zak feature map; averaging identity)**.** *Let $(Zh)(\theta,\xi) := \sum_k e^{-2\pi i\theta\cdot k} h(\xi + ck)$, with $\xi \in \mathrm{FD}^{\mathrm{bonds}}$ and $\theta \in [0,1)^{\mathrm{bonds}}$, be the Zak transform, extended to all $\theta \in \mathbb{R}^{\mathrm{bonds}}$ by periodicity and defined at every θ when h is regular in the sense of (b), and let $\widetilde{d}$ be the adjoint of d_s.*

(a) *The closure constraint has the integral representation $\mathbf{1}[d_s\widetilde{v}^t = v - \widetilde{v}'] = \int_{[0,1)^{\mathrm{plaq}}} d\alpha\, e^{-2\pi i\alpha\cdot(d_s\widetilde{v}^t - v + \widetilde{v}')/c}$. Put $\theta := \widetilde{d}\alpha$. The Zak transform turns the integer frame into a phase, $Zh^{x-cn}(\theta) = e^{2\pi i\theta\cdot n} Zh^x(\theta)$, and the Villain part of (13.5) is a positive kernel in the data (B,v) of the two layers:*

$$K^{\mathrm{Vill}}(c,c') = \langle \Phi(c), \Phi(c')\rangle_{\mathcal{H}}, \qquad \mathcal{H} := \int_{[0,1)^{\mathrm{plaq}}}^{\oplus} L^2(\mathrm{FD}^{\mathrm{bonds}})\, d\alpha, \tag{13.7}$$
$$\Phi(c)(\alpha) := e^{-2\pi i\alpha\cdot v/c}\,(Zh^B)(\widetilde{d}\alpha).$$

The hopping factor enters the kernel as a further tensor factor, through the Bessel factorisation $h(r,r',\vartheta) = e^{-\kappa_N(\rho^2+\rho'^2)/2}\sum_{n\in\mathbb{Z}} I_n(\kappa_N\rho\rho')e^{in\gamma_N\vartheta}$.

(b) *(Fibred operators.) For $y \in \mathbb{R}^S$ let $k(y) := (y - [y]_{\mathrm{FD}})/c \in \mathbb{Z}^S$ be the cell index of y, and for $a, \theta \in \mathbb{R}^S$ let $\tau_a(\theta)$ be the unitary operator on $L^2(\mathrm{FD}^S)$*

$$(\tau_a(\theta)F)(\xi) := e^{2\pi i\theta\cdot k(\xi+a)} F([\xi + a]_{\mathrm{FD}}).$$

A function $g \in L^2(\mathbb{R}^S)$ is regular *if $\sum_{k\in\mathbb{Z}^S} \|g(\cdot + ck)\|_{L^2(\mathrm{FD}^S)} < \infty$; then the series that defines $(Zg)(\theta,\cdot)$ converges in $L^2(\mathrm{FD}^S)$ uniformly in θ, and $\theta \mapsto (Zg)(\theta,\cdot)$ is norm-continuous. A bounded operator X on $L^2(\mathbb{R}^S)$ is* fibred *if there are bounded operators $X(\theta)$ on $L^2(\mathrm{FD}^S)$, $\theta \in \mathbb{R}^S$, its* fibres*, with $\sup_\theta \|X(\theta)\| < \infty$ and $X(\theta + j) = X(\theta)$ for $j \in \mathbb{Z}^S$, such that $\theta \mapsto \langle F, X(\theta)G\rangle$ is continuous for all $F, G \in L^2(\mathrm{FD}^S)$ and*

$$\langle g_1, X g_2\rangle = \int_{[0,1)^S} \big\langle (Zg_1)(\theta,\cdot), X(\theta)(Zg_2)(\theta,\cdot)\big\rangle_{L^2(\mathrm{FD}^S)}\, d\theta \qquad \text{for all } g_1, g_2 \in L^2(\mathbb{R}^S). \tag{13.8}$$

A fibred X satisfies the summability condition *if $\sum_{a\in c\mathbb{Z}^S}|k_X(B,u-a)|<\infty$ for all $B,u\in\mathbb{R}^S$ (Remark 13.8). The following hold.*

(b1) *The fibres of a fibred X are determined by X, $\|X(\theta)\|\le\|X\|$ for every θ, and X is $\mathbb{Z}$-periodic. The fibred operators form a linear space, closed under adjoints, with $X^*(\theta)=X(\theta)^*$, and under limits in the operator norm.*

(b2) *Every translation τ_a, $a\in\mathbb{R}^S$, is fibred with the fibres $\tau_a(\theta)$; these are norm-continuous in θ, they satisfy $\tau_a(\theta)\tau_b(\theta)=\tau_{a+b}(\theta)$, and $(a,\theta)\mapsto\tau_a(\theta)F$ is continuous for every F. Every operator $\sum_{a\in c\mathbb{Z}^S}c_a\tau_a$ with $\sum_a|c_a|<\infty$ is fibred, with the scalar fibres $\sum_a c_a e^{2\pi i\theta\cdot a/c}$. Every operator $\sum_{a\in c\mathbb{Z}^S}\tau_a^*X_0\tau_a$ with X_0 bounded and $X_0=P_KX_0P_K$ for some $K<\infty$, where P_K is the multiplication by $\mathbf{1}_{[-K,K]^S}$, is bounded and fibred. If X is fibred, then so are: the operator $X^a=\tau_aX\tau_a^*$ for $a\in\mathbb{R}^S$, with the fibres $\tau_a(\theta)X(\theta)\tau_a(\theta)^*$; the gauge average $\bar X$ of Lemma 13.6(c), with the fibres $\int d\lambda\,\tau_{d\widetilde\lambda}(\theta)X(\theta)\tau_{d\widetilde\lambda}(\theta)^*$ (weak integrals; here and below, in $X^{d\widetilde\lambda}$ and in these fibres, $d\widetilde\lambda$ stands for its restriction $(d\widetilde\lambda)_S$ to the bonds of S); the operator $M_\sigma^{-1}XM_\sigma$ for $\sigma\in\mathbb{R}^S$, where M_σ is the multiplication by $e^{2\pi i\sigma\cdot k(y)}$, with the fibres $X(\theta+\sigma)$; and the complex conjugate $X^\sharp:=\mathsf{C}X\mathsf{C}$, where C is the complex conjugation, with the fibres $\mathsf{C}X(-\theta)\mathsf{C}$. If S' is a finite bond set disjoint from S and X' is fibred on $L^2(\mathbb{R}^{S'})$, then $X\otimes X'$ is fibred on $L^2(\mathbb{R}^{S\cup S'})$, with the fibres $X(\theta_S)\otimes X'(\theta_{S'})$.*

(b3) *For a fibred X there is exactly one bounded operator $\hat X$ on $\mathcal{H}$ with*

$$\langle G,\hat XF\rangle=\int_{[0,1)^{\mathrm{plaq}}}\big\langle G(\alpha),\big(X((\widetilde{d\alpha})_S)\otimes 1\big)F(\alpha)\big\rangle\,d\alpha\qquad\text{for all } F,G\in\mathcal{H},$$

where the factor 1 acts on the bonds outside S and on the Higgs factor of Lemma 13.21; we say that $\hat X$ has the fibres $X((\widetilde{d\alpha})_S)\otimes 1$. The map $X\mapsto\hat X$ is linear, $(X^*)^\wedge=(\hat X)^*$, $\|\hat X\|\le\|X\|$, and $\langle G,\widehat{\bar X}F\rangle=\int d\lambda\,\langle G,(X^{d\widetilde\lambda})^\wedge F\rangle$ for all $F,G\in\mathcal{H}$. Moreover $(\hat XF)(\alpha)=\big(X((\widetilde{d\alpha})_S)\otimes 1\big)F(\alpha)$ for almost every α, for every $F\in\mathcal{H}$; consequently, for fibred operators X_i on pairwise disjoint bond sets S_i, $\prod_i\hat X_i=(\bigotimes_i X_i)^\wedge$, in any order.

(b4) *Let X be fibred and let g_1,g_2 be regular, with $\sum_{m\in\mathbb{Z}^S}|\langle g_1,X\tau_{cm}g_2\rangle|<\infty$. Then, for every $\theta\in\mathbb{R}^S$,*

$$\big\langle(Zg_1)(\theta,\cdot),X(\theta)(Zg_2)(\theta,\cdot)\big\rangle=\sum_{m\in\mathbb{Z}^S}e^{-2\pi i\theta\cdot m}\langle g_1,X\tau_{cm}g_2\rangle.$$

For a fibred X put $K^X(c,c'):=\langle\Phi(c),\hat X\Phi(c')\rangle$; if X satisfies the summability condition, K^X is the kernel with the insertion (13.6).

(b5) *The gauge action on the feature map is $\Phi(\lambda.c)=R(\lambda)\Phi(c)$, where $(R(\lambda)F)(\alpha):=\tau_{-d\widetilde\lambda}(\widetilde{d\alpha})F(\alpha)$, with the operators $\tau_a(\theta)$ on $L^2(\mathrm{FD}^{\mathrm{bonds}})$ of all the bonds of the layer; the operator $R(\lambda)$ is unitary and independent of the configuration, and $R(\lambda)^*\hat XR(\lambda)=(X^{d\widetilde\lambda})^\wedge$ for every fibred X.*

(c) *(Averaging identity.) Let $L:=L^2(dc)$ be the layer space, Q the multiplication by the one-layer weight $W(c)$, $T:=Q^{1/2}KQ^{1/2}$ the transfer operator on L, and $\mathsf{B}:L\to\mathcal{H}$ the dressing $(\mathsf{B}\psi):=\int W(c)^{1/2}\Phi(c)\psi(c)\,dc$, so that $\mathsf{B}^*\mathsf{B}=T$. Here, and in the proof, the feature map Φ and the space $\mathcal{H}$ include the Higgs tensor factor that reproduces the hopping factor of (13.5) (see (a); it is written out in Lemma 13.21), so that $\langle\Phi(c),\Phi(c')\rangle=K(c,c')$; every $\hat X$ acts as the identity on that factor, and $R(\lambda)$ includes the rotation $U(\lambda)$ of that lemma. Let $R_L(\lambda)\psi:=\psi((-\lambda).c)$ be the gauge representation on L and*

$P_G := \int dG\, R_L(G)$ the projection onto the gauge-invariant layer subspace L_G. Let $X_1, \dots, X_k$ be fibred insertions at distinct heights that satisfy the summability condition of (b), and let $Z_M(X_1, \dots, X_k)$ be the partition function of the periodic measure with M layers and with these insertions, that is, the integral of the weight of this measure in which, for each s, the slab bond factor $k_1(B,u)$ of the slab at the height of X_s, written in the pair coordinates (B,u) of that slab, is replaced by $k_{X_s}(B,u)$; in this integral the sum over the temporal Villain variables $\widetilde{v}^t$ of each slab is taken first, where it converges absolutely by the summability condition and gives the kernel K^{X_s} of (b4), and the remaining integral over the layers converges absolutely by the bound $|K^{X_s}(c,c')| \le \|X_s\| \sqrt{K(c,c)K(c',c')}$ of the proof. Then

$$Z_M(X_1, \dots, X_k) = \mathrm{Tr}_{L_G}\Big(\prod_s \mathsf{B}^* \widehat{X}_s \mathsf{B}\Big). \tag{13.9}$$

Proof. (a) The form $d_s\widetilde{v}^t - v + \widetilde{v}'$ takes its values in $c\mathbb{Z}^{\mathrm{plaq}}$, so the indicator of the closure constraint is equal to $\int_{[0,1)^{\mathrm{plaq}}} e^{-2\pi i \alpha\cdot(d_s\widetilde{v}^t - v + \widetilde{v}')/c} d\alpha$. The indicator is real, so the sign of the exponent is a convention; the sign chosen is the one consistent with (13.7). Write $\widetilde{v}^t = cm$ with $m \in \mathbb{Z}^S$, and note that $\alpha \cdot d_s m = (\widetilde{d}\alpha)\cdot m = \theta \cdot m$. The sum over m is then free of constraints and factorises over the bonds. For one bond, with $x, y \in \mathrm{FD}$,

$$\sum_{m\in\mathbb{Z}} e^{-2\pi i\theta m}\, w(x - y + cm) = \sum_{m\in\mathbb{Z}} e^{-2\pi i\theta m} \langle h^x, h^{y-cm}\rangle_{L^2(\mathbb{R})} = \big\langle (Zh^x)(\theta,\cdot), (Zh^y)(\theta,\cdot)\big\rangle_{L^2(\mathrm{FD})},$$

by the definition of Z, the rearrangement $\sum_{k,k'} = \sum_k \sum_{m=k'-k}$ and the decomposition $\bigcup_k (\mathrm{FD} + ck) = \mathbb{R}$. The remaining phase splits as $e^{2\pi i\alpha\cdot(v-\widetilde{v}')/c} = \overline{e^{-2\pi i\alpha\cdot v/c}}\, e^{-2\pi i\alpha\cdot\widetilde{v}'/c}$, and (13.7) follows. The frame identity is the relation $h^{x-cn}(\xi) = h^x(\xi + cn)$, after a shift of the summation index in the sum that defines Z. The Bessel factorisation is the generating function $e^{z\cos\varphi} = \sum_n I_n(z) e^{in\varphi}$ applied to $|e^{i\gamma_N\vartheta}\rho' - \rho|^2 = \rho^2 + \rho'^2 - 2\rho\rho'\cos(\gamma_N\vartheta)$; it exhibits the hopping factor as a positive-definite kernel in the data (ρ, ψ) of the two layers, which enters the kernel as a tensor factor.

(b) *Preliminaries.* For almost every ξ the function $\theta \mapsto (Zg)(\theta,\xi)$ is the Fourier series of the square-summable sequence $(g(\xi + ck))_k$. By Parseval's identity, which makes the Fourier series an isometric isomorphism of $\ell^2(\mathbb{Z}^S)$ onto $L^2([0,1)^S)$, and by $\bigcup_k (\mathrm{FD}^S + ck) = \mathbb{R}^S$, the map Z is unitary from $L^2(\mathbb{R}^S)$ onto $L^2([0,1)^S \times \mathrm{FD}^S)$; moreover $Z\tau_{ck} g = e^{2\pi i\theta\cdot k} Zg$, and $(Zg)(\theta,\cdot)$ is periodic in θ with the period lattice $\mathbb{Z}^S$. For a regular g the tail of the series is bounded in $L^2(\mathrm{FD}^S)$ by $\sum_{|k|>n} \|g(\cdot + ck)\|_{L^2(\mathrm{FD}^S)}$, uniformly in θ, which gives the uniform convergence and the norm-continuity. *Measurability.* Let $A(\theta)$ be operators on a separable Hilbert space $\mathcal{K}$, bounded uniformly in θ, with $\theta \mapsto \langle F, A(\theta)G\rangle$ continuous for all $F, G \in \mathcal{K}$, and let G_1, G_2 be square-integrable $\mathcal{K}$-valued functions. The finite sums $\sum_j \varphi_j(\theta) F_j$ are dense in the L^2 space of such functions, and a sequence that converges there has a subsequence that converges in $\mathcal{K}$ for almost every θ. For finite sums the function $\theta \mapsto \langle G_1(\theta), A(\theta)G_2(\theta)\rangle$ is a finite sum of products of measurable functions and continuous functions, and the uniform bound on A carries the convergence almost everywhere over to these functions. So $\theta \mapsto \langle G_1(\theta), A(\theta)G_2(\theta)\rangle$ is measurable, of modulus at most $\sup_\theta \|A(\theta)\|\, \|G_1(\theta)\|\, \|G_2(\theta)\|$. In particular the right side of (13.8) is a sesquilinear form in (g_1, g_2) bounded by $\sup_\theta \|X(\theta)\|\, \|g_1\|\, \|g_2\|$.

(b1) Let $F, G \in L^2(\mathrm{FD}^S)$ and $\varphi \in L^2([0,1)^S)$, and insert $g_1 := Z^*(\varphi \otimes F)$ and $g_2 := Z^*(\varphi \otimes G)$ in (13.8): $\int |\varphi(\theta)|^2 \langle F, X(\theta)G\rangle\, d\theta = \langle g_1, X g_2\rangle$, which has modulus at most $\|X\|\, \|\varphi\|^2 \|F\|\, \|G\|$. Let $|\varphi|^2$ run through the normalised indicator functions of the intersections of $[0,1)^S$ with balls that shrink to a point θ_0 of the closed cube: the continuity of $\theta \mapsto \langle F, X(\theta)G\rangle$ gives $|\langle F, X(\theta_0)G\rangle| \le \|X\|\, \|F\|\, \|G\|$, and by periodicity this holds for every $\theta_0 \in \mathbb{R}^S$. If two families of fibres belong to the same X, their difference $D(\theta)$ satisfies $\int |\varphi|^2 \langle F, D(\theta)G\rangle\, d\theta = 0$ for all φ, so

the continuous periodic function $\langle F, D(\theta)G\rangle$ vanishes. By (13.8) and $Z\tau_{ck}g = e^{2\pi i\theta\cdot k}Zg$ one has $\langle g_1, X\tau_{ck}g_2\rangle = \langle \tau_{ck}^* g_1, Xg_2\rangle$, so X is $\mathbb{Z}$-periodic. Linear combinations and adjoints satisfy (13.8) with the same combinations and the adjoints of the fibres. If fibred operators X_n converge to X in norm, the bound, applied to $X_n - X_m$, gives $\|X_n(\theta) - X_m(\theta)\| \le \|X_n - X_m\|$ for every θ; the fibres converge in norm, uniformly in θ, their limits are bounded, periodic and weakly continuous, and (13.8) passes to the limit.

(b2) *Translations.* For g of compact support the sums are finite, and with $\xi + a = [\xi + a]_{\mathrm{FD}} + c\,k(\xi + a)$ a shift of the summation index gives

$$(Z\tau_a g)(\theta, \xi) = e^{2\pi i\theta\cdot k(\xi+a)}(Zg)(\theta, [\xi + a]_{\mathrm{FD}}) = \big(\tau_a(\theta)(Zg)(\theta, \cdot)\big)(\xi).$$

The map $\xi \mapsto [\xi+a]_{\mathrm{FD}}$ is a measure-preserving bijection of FD^S, so $\tau_a(\theta)$ is unitary and both sides are isometric in g; the identity therefore holds for all $g \in L^2(\mathbb{R}^S)$ as an identity in $L^2([0,1)^S \times \mathrm{FD}^S)$, and (13.8) for τ_a follows from the unitarity of Z. The fibres are periodic because k is integer valued. On FD^S one has $|k(\xi+a)| \le |a|/c + \sqrt{|S|}$, hence $\|\tau_a(\theta) - \tau_a(\theta_0)\| \le 2\pi|\theta - \theta_0|\,(|a|/c + \sqrt{|S|})$. For fixed θ the function $\tau_a(\theta)F$ is the restriction to FD^S of the translate by a of the locally square-integrable function $E_\theta F(y) := e^{2\pi i\theta\cdot k(y)}F([y]_{\mathrm{FD}})$, $y \in \mathbb{R}^S$, and translations act continuously on L^2 of bounded sets; the two facts give the continuity of $(a, \theta) \mapsto \tau_a(\theta)F$. Since $\tau_a^* = \tau_{-a}$, the uniqueness in (b1) gives $\tau_a(\theta)^* = \tau_{-a}(\theta)$. The group law follows from $k(y - cj) = k(y) - j$ and $[y - cj]_{\mathrm{FD}} = [y]_{\mathrm{FD}}$ for $j \in \mathbb{Z}^S$, applied to $y = \xi + a + b$ and $j = k(\xi + a)$. For $a = cj$, $j \in \mathbb{Z}^S$, one has $k(\xi + a) = j$ and $[\xi + a]_{\mathrm{FD}} = \xi$ on FD^S, so $\tau_a(\theta) = e^{2\pi i\theta\cdot a/c}$; the series $\sum_a c_a\tau_a$ converges in norm, and (b1) gives the claim on these series.

Localised sums. Let $X = \sum_{a\in c\mathbb{Z}^S} \tau_a^* X_0 \tau_a$ with $X_0 = P_K X_0 P_K$, and let N be a number, depending only on K, c and $|S|$, such that every translate of $[-K, K]^S$ or of FD^S meets at most N of the sets $[-K, K]^S + a$ and at most N of the cells $\mathrm{FD}^S + a$, $a \in c\mathbb{Z}^S$. The function $\tau_a^* X_0\tau_a g$ is supported in $[-K, K]^S + a$ and equals $\tau_a^* X_0 P_K\tau_a g$, so $\|Xg\|^2 \le N\sum_a \|X_0 P_K\tau_a g\|^2 \le N^2\|X_0\|^2\|g\|^2$: the operator X is bounded, and a shift of the summation index shows that it commutes with the τ_{ck}. Put

$$X(\theta)F := \sum_{a\in c\mathbb{Z}^S} \big(\tau_a^* X_0 P_K \tau_a E_\theta F\big)\big|_{\mathrm{FD}^S}, \qquad F \in L^2(\mathrm{FD}^S),$$

a sum with at most N non-zero terms. The map $F \mapsto P_K\tau_a E_\theta F$ has norm at most $N^{1/2}$ and is norm-continuous in θ, as for the translations, because k is bounded on $[-K, K]^S + a$. So the operators $X(\theta)$ are bounded uniformly in θ and norm-continuous in θ, and they are periodic because $E_{\theta+j} = E_\theta$. For g of compact support one has $E_\theta\big((Zg)(\theta, \cdot)\big) = \sum_k e^{-2\pi i\theta\cdot k}\tau_{ck}g$ on $\mathbb{R}^S$, a locally finite sum, and Xg has compact support; therefore, all sums being finite,

$$X(\theta)(Zg)(\theta, \cdot) = \sum_k e^{-2\pi i\theta\cdot k}\big(X\tau_{ck}g\big)\big|_{\mathrm{FD}^S} = \sum_k e^{-2\pi i\theta\cdot k}\big(\tau_{ck}Xg\big)\big|_{\mathrm{FD}^S} = (ZXg)(\theta, \cdot)$$

for every θ. With the unitarity of Z this is (13.8) for g_2 of compact support and all g_1; both sides of (13.8) are bounded sesquilinear forms, so (13.8) holds for all g_2.

Stability. For X^a: the identity for $Z\tau_a$, applied to $\tau_a^* = \tau_{-a}$, and $\tau_{-a}(\theta) = \tau_a(\theta)^*$ turn (13.8) into $\langle g_1, X^a g_2\rangle = \langle \tau_a^* g_1, X\tau_a^* g_2\rangle = \int\langle \tau_a(\theta)^*(Zg_1)(\theta, \cdot), X(\theta)\tau_a(\theta)^*(Zg_2)(\theta, \cdot)\rangle\, d\theta$. The family $\tau_a(\theta)X(\theta)\tau_a(\theta)^*$ is bounded and periodic, and $\langle F, \tau_a(\theta)X(\theta)\tau_a(\theta)^* G\rangle = \langle \tau_a(\theta)^* F, X(\theta)\tau_a(\theta)^* G\rangle$ is continuous in (a, θ), because the vectors $\tau_a(\theta)^* F = \tau_{-a}(\theta)F$ and $\tau_a(\theta)^* G$ are norm-continuous in (a, θ) and $X(\theta)$ is bounded and weakly continuous. For $\bar{X} = \int d\lambda\, X^{d\widetilde{\lambda}}$: the lift $\lambda \mapsto \widetilde{\lambda}$ is Borel, so $(\lambda, \theta) \mapsto \langle F, X^{d\widetilde{\lambda}}(\theta)G\rangle$ is Borel and bounded; the weak integrals over λ define bounded periodic operators, weakly continuous in θ by dominated convergence; and (13.8) for $\bar{X}$ follows from (13.8) for the operators $X^{d\widetilde{\lambda}}$ by Fubini's theorem, the integrand being measurable in (λ, θ)

by the argument of the preliminaries. For M_σ: for g of compact support, $k(\xi + ck) = k$ on FD^S gives $(ZM_\sigma g)(\theta,\cdot) = (Zg)(\theta-\sigma,\cdot)$, both sides are isometric in g, and M_σ is unitary; inserting this in (13.8) for $\langle M_\sigma g_1, XM_\sigma g_2\rangle$ and shifting the integration variable, which the periodicity allows, gives (13.8) for $M_\sigma^{-1}XM_\sigma$ with the fibres $X(\theta+\sigma)$. For $X^\sharp$: $(Z\mathsf{C}g)(\theta,\cdot) = \mathsf{C}(Zg)(-\theta,\cdot)$ and $\langle g_1, \mathsf{C}X\mathsf{C}g_2\rangle = \overline{\langle \mathsf{C}g_1, X\mathsf{C}g_2\rangle}$ give (13.8) with the fibres $\mathsf{C}X(-\theta)\mathsf{C}$, after the substitution $\theta \mapsto -\theta$. For $X \otimes X'$: the family $X(\theta_S)\otimes X'(\theta_{S'})$ is bounded and periodic; the function $\langle \Psi_1, (X(\theta_S)\otimes X'(\theta_{S'}))\Psi_2\rangle$ is continuous for finite sums Ψ_1, Ψ_2 of product vectors and hence, by density and the uniform bound, for all vectors; and (13.8) holds for products $g_i = g_{i,S}\otimes g_{i,S'}$, the Zak transform of a product being the product of the Zak transforms, hence for finite sums of products, and for all g_i because both sides are bounded sesquilinear forms.

(b3) The family $A(\alpha) := X((\widetilde{d}\alpha)_S)\otimes 1$ is bounded by $\sup_\theta \|X(\theta)\| \le \|X\|$ and weakly continuous in α, by the argument just given for tensor products, $\alpha \mapsto (\widetilde{d}\alpha)_S$ being continuous. By the preliminaries the right side of the formula in (b3) is a sesquilinear form on $\mathcal{H}$ bounded by $\|X\|\,\|G\|\,\|F\|$, and a bounded sesquilinear form on a Hilbert space is $\langle G, \hat{X}F\rangle$ for exactly one bounded operator $\hat{X}$, whose norm is at most the bound of the form (Riesz). Linearity and the formula for the adjoint are read off from the form. Testing the form with $G(\alpha) = \varphi(\alpha)G_j$, where φ is square-integrable and G_j runs through a countable dense set of vectors, gives $\langle G_j, (\hat{X}F)(\alpha) - A(\alpha)F(\alpha)\rangle = 0$ for almost every α and every j, which is the pointwise formula. Applied to two operators on disjoint bond sets it gives $(\hat{X}_1\hat{X}_2F)(\alpha) = A_1(\alpha)A_2(\alpha)F(\alpha)$ almost everywhere, and $A_1(\alpha)A_2(\alpha)$ is the fibre of $(X_1\otimes X_2)^\wedge$ by (b2); the product formula follows by induction on the number of factors. The identity for $\widehat{\widetilde{X}}$ is Fubini's theorem for the function $(\lambda,\alpha)\mapsto \langle G(\alpha), (X^{d\widetilde{\lambda}}((\widetilde{d}\alpha)_S)\otimes 1)F(\alpha)\rangle$, which is measurable by the continuity in (a,θ) shown in (b2), the Borel lift and the argument of the preliminaries, and is bounded by $\|X\|\,\|G(\alpha)\|\,\|F(\alpha)\|$.

(b4) The function $u(\theta) := \langle (Zg_1)(\theta,\cdot), X(\theta)(Zg_2)(\theta,\cdot)\rangle$ is periodic, and it is continuous because the vectors $(Zg_i)(\theta,\cdot)$ are norm-continuous and $X(\theta)$ is bounded and weakly continuous. By $Z\tau_{cm}g_2 = e^{2\pi i\theta\cdot m}Zg_2$ and (13.8), $\int_{[0,1)^S} e^{2\pi i\theta\cdot m}u(\theta)\,d\theta = \langle g_1, X\tau_{cm}g_2\rangle$. These Fourier coefficients are absolutely summable by assumption, so the series on the right of the claim converges uniformly to a continuous periodic function with the same Fourier coefficients as u. A continuous periodic function is determined by its Fourier coefficients (Fejér's theorem); this proves the claim for every θ. For the kernel, the functions h^x are regular, and $\langle h^B, X\tau_{cm}h^u\rangle = k_X(B, u-cm)$ because $\tau_a h^u = h^{u-a}$. By the definition of $\hat{X}$,

$$\langle \Phi(c), \hat{X}\Phi(c')\rangle = \int d\alpha\, e^{2\pi i\alpha\cdot(v-\widetilde{v}')/c}\big\langle (Zh^B)(\theta,\cdot), \big(X(\theta_S)\otimes 1\big)(Zh^{\widetilde{B}'})(\theta,\cdot)\big\rangle, \qquad \theta = \widetilde{d}\alpha.$$

The scalar product is the product of its factor over the bonds of S, to which the claim applies, of the factors of the other bonds, which are given by (a), and of the factor of the Higgs variables, which does not depend on α. Under the summability condition the resulting series over $m \in \mathbb{Z}^{\text{bonds}}$ converges absolutely, uniformly in α, and is integrated term by term. Since $\theta\cdot m = \alpha\cdot d_s m$, the integral over α selects the terms with $c\,d_s m = v - \widetilde{v}'$, that is, $\widetilde{v}^t := cm$ with $d_s\widetilde{v}^t = v - \widetilde{v}'$:

$$\langle \Phi(c), \hat{X}\Phi(c')\rangle = \sum_{\widetilde{v}^t:\, d_s\widetilde{v}^t = v-\widetilde{v}'} k_X(B, \widetilde{B}' - \widetilde{v}^t),$$

which is the kernel with the insertion (13.6).

(b5) For the gauge action, the configuration $\lambda.c$ has $B \mapsto B + d\widetilde{\lambda} - cn$ and $v \mapsto v + c\,dn$. By the frame identity of (a),

$$\Phi(\lambda.c)(\alpha) = e^{-2\pi i\alpha\cdot v/c}e^{-2\pi i\alpha\cdot dn}e^{2\pi i\theta\cdot n}(Zh^{B+d\widetilde{\lambda}})(\theta),$$

and $\alpha\cdot dn=\theta\cdot n$, so the two wrap phases cancel; this is where $\mathbb{Z}$-periodicity enters. Since $h^{B+a}=\tau_{-a}h^B$, and since the identity for $Z\tau_a$ of (b2) holds pointwise for the functions h^B, whose series converge absolutely, $(Zh^{B+d\widetilde{\lambda}})(\theta,\cdot)=\tau_{-d\widetilde{\lambda}}(\theta)(Zh^B)(\theta,\cdot)$ for every θ. Hence $\Phi(\lambda.c)=R(\lambda)\Phi(c)$. On functions of (α,ξ) the operator $R(\lambda)$ is $(R(\lambda)F)(\alpha,\xi)=e^{2\pi i(\widetilde{d}\alpha)\cdot k(\xi-d\widetilde{\lambda})}F(\alpha,[\xi-d\widetilde{\lambda}]_{\mathrm{FD}})$; it maps measurable functions to measurable functions, is unitary and does not depend on the configuration. The operator $\tau_{-a}(\theta)$ of all the bonds is the tensor product of the operator $\tau_{-a_S}(\theta_S)$ of the bonds of S with that of the other bonds, and $\tau_{-a_S}(\theta_S)^*=\tau_{a_S}(\theta_S)$. So, with $a=d\widetilde{\lambda}$ and $\theta=\widetilde{d}\alpha$, the form of (b3) gives $\langle G,R(\lambda)^*\hat{X}R(\lambda)F\rangle=\int\langle G(\alpha),(\tau_{a_S}(\theta_S)X(\theta_S)\tau_{a_S}(\theta_S)^*\otimes 1)F(\alpha)\rangle\,d\alpha=\langle G,(X^{d\widetilde{\lambda}})^\wedge F\rangle$, by the fibres of X^a in (b2).

(c) Here the sans-serif letter B denotes the dressing operator, while B remains the link field. The proof has three steps.

The trace structure. The kernel $K=\langle\Phi,\Phi\rangle$ is positive, so $T=Q^{1/2}KQ^{1/2}=\mathsf{B}^*\mathsf{B}\ge 0$ with the dressing B of the statement. The operator B is Hilbert–Schmidt, with $\|\mathsf{B}\|_2^2=\int W(c)K(c,c)\,dc=Z_1<\infty$, because

$$K(c,c)=\prod_x h(\rho_x,\rho_x,0)\sum_{\text{closed }\widetilde{v}^t}\prod_b w(\widetilde{v}^t_b)\le\Big(\sum_{m\in\mathbb{Z}}e^{-c^2m^2/2}\Big)^{|S|}$$

and $\int W\,dc<\infty$; the latter follows from (2.1) for the radial integrals, from the convergence of the Gaussian Villain sums and from the compactness of the ranges of B and ψ. Hence T and every operator $\mathsf{B}^*Y\mathsf{B}$, with Y bounded on $\mathcal{H}$, are trace class. The bound $|K^Y(c,c')|\le\|Y\|\sqrt{K(c,c)K(c',c')}$ makes the layer integrals absolutely convergent, and a cyclic integral of $M\ge 2$ such kernels is the trace of the product of the corresponding operators.

The intertwining relations. One has $\mathsf{B}R_L(\lambda)=R(\lambda)\mathsf{B}$, by the change of variables $c=\lambda.c'$ and the invariance of W. Both R_L and R are strongly continuous unitary representations of the compact abelian group $\mathbb{T}^{\text{sites}}$; the operator $R(\lambda)$ does not depend on the lift, because two lifts of $d\widetilde{\lambda}$ differ by cd_sk, whose Zak phase is $e^{-2\pi i\alpha\cdot d_sd_sk}=1$. Consequently $\int dG\,R_L(G)=P_G$ and $P_G\mathsf{B}^*=\mathsf{B}^*P_G^{\mathcal{H}}$, where $P_G^{\mathcal{H}}:=\int dG\,R(G)$ is the projection onto the R-invariant vectors of $\mathcal{H}$.

The change of variables. Let μ_M be the periodic measure with M layers $c^{(1)},\dots,c^{(M)}$ and temporal links $B_t^{(s)}$ between the layers s and $s+1$, where $M+1\equiv 1$. Put $G_1:=0$, $G_{s+1}:=G_s-B_t^{(s)}$ and $W^{(s)}:=G_s.c^{(s)}$. The map $(B_t^{(s)})_s\mapsto(G_{s+1})_s$ is a triangular bijection of $(\mathbb{T}^{\text{sites}})^M$ that preserves the Haar measure, and the map $c^{(s)}\mapsto W^{(s)}$ preserves $W(c)\,dc$ (Lemma 13.3(a)). The three-dimensional closure $dv=0$ holds if and only if every slab satisfies $d_s\widetilde{v}^{t,(s)}=v^{(s)}-\widetilde{v}^{(s+1)}$, so the Villain sum factorises into the sum over the layer 2-forms times the independent slab coset sums carried by the kernels. The operator $X_s^{-d\widetilde{G}_s}$ is fibred by (b2) and satisfies the summability condition together with X_s, because $k_{X^a}(B,u)=k_X(B+a,u+a)$ for every $a\in\mathbb{R}^S$. By (b), the slab $s<M$ with the insertion X_s contributes

$$K^{X_s^{-d\widetilde{G}_s}}(W^{(s)},W^{(s+1)})=\langle\Phi(W^{(s)}),(X_s^{-d\widetilde{G}_s})^\wedge\Phi(W^{(s+1)})\rangle,$$

and the closing slab carries the holonomy $G:=G_{M+1}$: its kernel is that of $\mathsf{B}^*\hat{X}_M\mathsf{B}\,R_L(G)$. Integrating out the layers gives

$$Z_M(X_1,\dots,X_k)=\int dG_2\cdots dG_{M+1}\ \mathrm{Tr}\,\Big(\prod_s\mathsf{B}^*(X_s^{-d\widetilde{G}_s})^\wedge\mathsf{B}\cdot R_L(G)\Big),$$

with the insertion 1 at the heights that carry no X_s.

Conjugating the integrand by $R_L(g)$ shifts every G_s by g and leaves the trace unchanged; after a renaming of the variables, by the invariance of the Haar measure, there is one independent Haar variable for each insertion. For each of them $\int dg\,(X^{-d\widetilde{g}})^\wedge=\widehat{\widetilde{X}}$, by (b) and Lemma 13.6(c); the

integral over g is exchanged with the trace by dominated convergence for trace-class operators. Finally $\int dG\, R_L(G) = P_G$, and P_G commutes with every $\mathsf{B}^*\widehat{\widehat{X}}\mathsf{B}$, because $\bar{X}$ is gauge invariant. This proves (13.9). For $X_s = 1$ the identity reads $Z_M = \mathrm{Tr}_{L_G}(T^M) = \mathrm{Tr}(P_G T^M) \le \mathrm{Tr}(T^M)$. $\square$

Remark 13.8 (absolute convergence). The Zak form of Lemma 13.7(b) needs $\sum_{\widetilde{v}^t} |k_X(B, \widetilde{B}' - \widetilde{v}^t)| < \infty$; it holds for the bounded class $\mathcal{C}$ of Subsection 13.6 (by Lemma 13.11, $|k_X(B,u)| \le C_X e^{c_X|f|-|f|^2/2}$ with $f = B - u$, and the sum over $\widetilde{v}^t$ moves f through the lattice $c\mathbb{Z}^S$) and for the corrector Z of Lemma 13.12 ($\sum_a \alpha_a < \infty$).

13.5. **The Schatten–Hölder bound, the corrector and the bond pair.** Translations of $L^2(\mathbb{R}^S)$ are written $(\tau_a g)(\xi) := g(\xi + a)$, so that $\tau_a h^u = h^{u-a}$, $k_{\tau_a}(B,u) = e^{-|f+a|^2/2}$ and $r_{\tau_a} = e^{-a\cdot f - |a|^2/2}$, $f = B - u$. We first prove the relative-entropy estimate of this subsection, the tangent inequality of Lemma 13.9, which is used in Lemmas 13.14 and 13.25: it compares a Gibbs measure that is invariant under a sublattice of finite index with the periodic measures of cubic tori whose three sides grow together. Because the closure constraint ties the Villain variables across the boundary of a box (on the torus it holds also on the cubes that straddle the seam), the law of a whole fundamental box under the Gibbs measure need not be absolutely continuous with respect to its law under the torus measure; the comparison is therefore made on the closed cell complex of an inner box, on which both laws are carried by the same configurations, and the shell around that complex is summed out on both sides.

Lemma 13.9 (tangent inequality)**.** *Let ν be a Gibbs measure of the phase-restored specification, invariant under a sublattice $\Gamma' \subset \varepsilon\mathbb{Z}^3$ of finite index, which satisfies* (I1) *and* (I2) *of Theorem 13.1. Let v be a local Borel function (a Borel function of the variables of finitely many cells), bounded below, and let ℓ_v be the sup-norm diameter, in units of ε, of its cell support. For an integer $\ell \ge \max\{8, \ell_v + 2\}$ with $\ell\varepsilon\mathbb{Z}^3 \subset \Gamma'$ let $\varepsilon\mathbb{Z}^3_\ell := \varepsilon\mathbb{Z}^3/\ell\varepsilon\mathbb{Z}^3$ be the cubic torus of side ℓ (ℓ time layers, each a layer torus of side ℓ) and let μ_ℓ be its periodic measure in the phase-restored variables: the probability measure on the configurations of $\varepsilon\mathbb{Z}^3_\ell$ ($\tilde{\rho}_x \in [-\rho_{0,0}, \infty)$ and $\psi_x \in \mathbb{T}$ at the sites, $B_b \in \mathrm{FD}$ on the bonds, $v_p \in c\mathbb{Z}$ on the plaquettes) given by the normalised phase-restored weight of Subsection 13.2 on the cells of $\varepsilon\mathbb{Z}^3_\ell$, with Lebesgue measure on each $\tilde{\rho}_x$ and each B_b and normalised Haar measure on each ψ_x, the sum over the Villain variables running over all closed (v_p) (closed on every cube of $\varepsilon\mathbb{Z}^3_\ell$), every flux class included. Let $V_\ell := \sum_g v \circ \tau_g$ be the sum over the $n_\ell = \ell^3/[\varepsilon\mathbb{Z}^3 : \Gamma']$ distinct Γ'-translates of v on $\varepsilon\mathbb{Z}^3_\ell$ (g running over $\Gamma'/\ell\varepsilon\mathbb{Z}^3$). Then*

$$\nu(v) \;\le\; \liminf_{\ell\to\infty}\; n_\ell^{-1} \log \mu_\ell\big(e^{V_\ell}\big),$$

the lower limit being taken along the integers ℓ with $\ell\varepsilon\mathbb{Z}^3 \subset \Gamma'$, and both sides having values in $(-\infty, +\infty]$. More precisely, let $v \ge -c_v$ with a constant $c_v \ge 0$. There are a constant C_v, depending only on ℓ_v and on the index $[\varepsilon\mathbb{Z}^3 : \Gamma']$, integers $n_\ell^- \le n_\ell$ with $n_\ell - n_\ell^- \le C_v\ell^2$, and a constant C_{TV}, depending only on N, the couplings, the constant C of (I2) *and $m_\nu := \sup_x \nu(\rho_x^2)$, such that for every ℓ as above*

$$n_\ell^-\, \nu(v) \;\le\; \log \mu_\ell\big(e^{V_\ell}\big) + c_v C_v\, \ell^2 + C_{\mathrm{TV}}\, \ell^2 (1 + \log_2 \ell)^2. \tag{13.10}$$

Here $m_\nu < \infty$ by (I1) and the Γ'-invariance of ν (the sites fall into finitely many classes modulo Γ'). Hypothesis (I2) enters in its joint form, with one constant C for all finitely supported s: it bounds the second moments of the fluxes through the boundary spheres of boxes whose size grows with ℓ (Claim 3 in the proof).

Proof. Conventions. Lengths and coordinates are in units of ε, so that $\Gamma' \subset \mathbb{Z}^3$ in coordinates. The symbol v without an index is the local function of the statement; the Villain variable of a plaquette p is always written with its index, v_p, or $v_p(\zeta)$ for a configuration ζ. A *cell* is a site, a

bond, a plaquette or a cube. Every bond b and every plaquette p has a reference orientation, in which B_b, v_p and $f_p = (dB)_p + v_p$ are read, and $[p : b] \in \{0, \pm 1\}$ is the incidence number of b in the boundary of p: $(dB)_p = \sum_b [p : b] B_b$, and likewise $(d\mathfrak{a})_p := \sum_b [p : b]\mathfrak{a}_b$ for a function $\mathfrak{a}$ on the bonds. Fix ℓ as in the statement and put $\ell_- := \ell - 3 \geq 5$ and $\ell_+ := \ell + 1$.

Step 0 (geometry). Let

$$\Lambda^- := \{0, \dots, \ell_-\}^3 \ \subset \ \Lambda := \{-1, \dots, \ell - 2\}^3 \ \subset \ \Lambda^+ := \{-2, \dots, \ell - 1\}^3.$$

Λ has ℓ^3 sites and is a fundamental domain of sites of $\varepsilon\mathbb{Z}^3_\ell$. The *closed complex* of a box consists of its sites and of the bonds, plaquettes and cubes all of whose vertices lie in the box; $\mathfrak{C}^-$ and $\mathfrak{C}^+$ are the closed complexes of Λ^- and of Λ^+. The *boundary sphere* Σ of $\mathfrak{C}^-$ consists of the cells of $\mathfrak{C}^-$ contained in the boundary of the solid cube $[0, \ell_-]^3$; it has $6\ell_-^2$ plaquettes, $12\ell_-^2$ bonds and $6\ell_-^2 + 2$ sites. Σ^+ is the boundary sphere of $\mathfrak{C}^+$, of side ℓ_+. With $V(\Lambda)$ as in Subsection 13.1, the *shells* are $\mathfrak{A} := V(\Lambda) \setminus \mathfrak{C}^-$ and, on the torus, the set $\mathfrak{A}_\ell$ of the sites, bonds and plaquettes of $\varepsilon\mathbb{Z}^3_\ell$ that are not in $\mathfrak{C}^-$ (see (G3)).

(G1) The cubes having a face in $V(\Lambda)$ are the cubes of $\mathfrak{C}^+$: a cube with a vertex in Λ lies in Λ^+, and a cube $\{y_1, y_1 + 1\} \times \{y_2, y_2 + 1\} \times \{y_3, y_3 + 1\}$ in Λ^+ has a vertex in Λ (in each coordinate take $y_i + 1$ if $y_i = -2$ and y_i otherwise). The faces of these cubes that are not in $V(\Lambda)$ are the plaquettes of Σ^+. Indeed, a plaquette of $\mathfrak{C}^+$ in a plane $\{x_i = -2\}$ or $\{x_i = \ell - 1\}$ has no vertex in Λ; and a plaquette of $\mathfrak{C}^+$ with normal direction i whose coordinate x_i lies in $[-1, \ell - 2]$ has a vertex in Λ: if it had none, and (y_j, y_k), $(y_j + 1, y_k + 1)$ are its extreme corners in the two tangential coordinates, then $y_j = -2$ and $y_k = \ell - 2$, or the same with j and k exchanged, and the corner with tangential coordinates $(y_j + 1, y_k) = (-1, \ell - 2)$ would lie in Λ.

(G2) Every cell of $\mathfrak{A}$ has a vertex in $\Lambda \setminus \Lambda^-$: a bond or plaquette of $\mathfrak{A}$ meets Λ and not all its vertices lie in Λ^-, so if it had no vertex in $\Lambda \setminus \Lambda^-$ it would have a vertex in Λ^- and a vertex outside Λ, and these are at sup-distance at least 2. $\Lambda \setminus \Lambda^-$ has $\ell^3 - (\ell - 2)^3 \leq 6\ell^2$ sites, each of which lies in 6 bonds and 12 plaquettes, so $|\mathfrak{A}| \leq 114\,\ell^2$. The same count gives $|\mathfrak{A}_\ell| \leq 114\,\ell^2$.

(G3) In $\varepsilon\mathbb{Z}^3_\ell$ two layers of sites separate opposite faces of Λ^- (the layers $x_i = \ell - 2$ and $x_i = \ell - 1 \equiv -1$), so no bond, plaquette or cube of the torus joins Λ^- to itself around the torus: the cells of $\varepsilon\mathbb{Z}^3_\ell$ with all vertices in Λ^- are the cells of $\mathfrak{C}^-$, and $\mathfrak{C}^-$ is embedded in $\varepsilon\mathbb{Z}^3_\ell$.

(G4) A face of a cube not in $\mathfrak{C}^-$ which belongs to $\mathfrak{C}^-$ lies in Σ (the cube has a vertex outside Λ^- in the direction normal to that face). This holds in $\varepsilon\mathbb{Z}^3$ and in $\varepsilon\mathbb{Z}^3_\ell$.

(G5) A plaquette not in $\mathfrak{C}^-$ contains at most one bond of $\mathfrak{C}^-$, and such a bond lies in Σ (two bonds of one plaquette with all ends in the product set Λ^- force its four corners into Λ^-; and the plaquette leaves Λ^- in a direction normal to the bond). A bond of Σ lies in exactly one plaquette not in $\mathfrak{C}^-$ if it is interior to a face of the cube (of the four plaquettes around it two lie in the face and one in the interior of $\mathfrak{C}^-$), and in exactly two if it lies on an edge of the cube. Likewise a bond of Σ^+ lies in exactly one plaquette of $\mathfrak{C}^+$ that is not in Σ^+ if it is interior to a face, and in none if it lies on an edge. No plaquette contains a bond of Σ and a bond of Σ^+ (their sites are at sup-distance at least 2).

The surface coboundary. For a plaquette p of Σ let $o_p = \pm 1$ be its orientation relative to the outward normal of the cube. Every bond b of Σ lies in exactly two plaquettes p, p' of Σ, and, Σ being a closed oriented surface,

$$o_p[p : b] = -o_{p'}[p' : b]. \tag{13.11}$$

For a real function $\mathfrak{r}$ on the bonds of Σ put $(d_\Sigma \mathfrak{r})_p := o_p \sum_b [p : b]\mathfrak{r}_b$, a function on the plaquettes of Σ. By (13.11) every $d_\Sigma \mathfrak{r}$ has zero sum, and $\mathcal{L}_\Sigma := d_\Sigma d^*_\Sigma$ is the graph Laplacian of the dual

graph of Σ, whose vertices are the $6\ell_-^2$ plaquettes of Σ, two of them adjacent when they share a bond (the graph is 4-regular and connected). Hence the range of d_Σ is the space of the functions of zero sum, and $\mathcal{L}_\Sigma^{-1}$ denotes the inverse of $\mathcal{L}_\Sigma$ on that space, extended by 0 to the constants. $|\cdot|$ and $\langle\cdot,\cdot\rangle$ are the Euclidean norm and scalar product of real functions on the bonds, or on the plaquettes, of Σ. The same notation is used on Σ^+, with ℓ_+ in place of ℓ_-.

Step 1 (reduction to an entropy bound on the interior complex). Let x_v be a vertex of the cell support of v and $k' := [\varepsilon\mathbb{Z}^3 : \Gamma']$, so that $k'\varepsilon\mathbb{Z}^3 \subset \Gamma'$ and $n_\ell = \ell^3/k'$. The Γ'-translates of v on $\varepsilon\mathbb{Z}^3$ are indexed by the position $z \in x_v + \Gamma'$ of the translate of x_v. If $z \in \{\ell_v, \dots, \ell_- - \ell_v\}^3$, the cell support of the translate lies in $\mathfrak{C}^-$. These translates are pairwise incongruent modulo $\ell\mathbb{Z}^3$ (the differences of their positions have coordinates of modulus $< \ell$), so by (G3) they are distinct terms of V_ℓ; let n_ℓ^- be their number and V^- their sum, a function of the variables of $\mathfrak{C}^-$. Every cube of k'^3 sites contains k'^2 points of each coset of Γ', so $n_\ell^- \geq (\ell - 2\ell_v - 2 - k')^3/k'$ whenever $\ell \geq 2\ell_v + 2 + k'$, and in every case

$$n_\ell - n_\ell^- \leq C_v\, \ell^2, \qquad C_v := 3(2\ell_v + 2 + k')/k'.$$

On $\varepsilon\mathbb{Z}^3_\ell$, $V_\ell - V^-$ is the sum of the other $n_\ell - n_\ell^-$ translates, so $V_\ell \geq V^- - c_v(n_\ell - n_\ell^-) \geq V^- - c_v C_v \ell^2$. By the Γ'-invariance of ν, $\nu(V^-) = n_\ell^-\, \nu(v)$, both sides in $(-\infty, +\infty]$.

Let Ω^- be the space of the configurations ζ of the variables of $\mathfrak{C}^-$: $\tilde\rho_y$ and ψ_y at its sites, B_b on its bonds and v_p on its plaquettes, the Villain variables being closed on the cubes of $\mathfrak{C}^-$ and subject to no other constraint. Let ν^- and μ_ℓ^- be the laws of these variables under ν and under μ_ℓ (through the embedding (G3)), and write $\mathrm{Ent}(\cdot \mid \cdot)$ for the relative entropy of probability measures on Ω^-. If $\nu^- \ll \mu_\ell^-$, the entropy inequality

$$\nu^-(F) \leq \log \mu_\ell^-\big(e^F\big) + \mathrm{Ent}(\nu^- \mid \mu_\ell^-)$$

holds for every bounded measurable F on Ω^-, and it extends to F bounded below by monotone convergence on $\min(F, j)$, $j \to \infty$. For $F = V^-$, since $\mu_\ell^-(e^{V^-}) = \mu_\ell(e^{V^-}) \leq e^{c_v C_v \ell^2} \mu_\ell(e^{V_\ell})$,

$$n_\ell^-\, \nu(v) = \nu^-(V^-) \leq \log \mu_\ell^-\big(e^{V^-}\big) + \mathrm{Ent}(\nu^- \mid \mu_\ell^-) \leq \log \mu_\ell\big(e^{V_\ell}\big) + c_v C_v\, \ell^2 + \mathrm{Ent}(\nu^- \mid \mu_\ell^-). \tag{13.12}$$

So (13.10) follows from the *entropy bound*

$$\nu^- \ll \mu_\ell^- \qquad \text{and} \qquad \mathrm{Ent}(\nu^- \mid \mu_\ell^-) \leq C_{\mathrm{TV}}\, \ell^2 (1 + \log_2 \ell)^2, \tag{13.13}$$

which is proved in Steps 2–8. The statement on the lower limit follows from (13.10). Indeed $n_\ell^{-1} \log \mu_\ell(e^{V_\ell}) \geq -c_v$, because $V_\ell \geq -c_v n_\ell$. If the lower limit is $+\infty$ there is nothing to prove. Otherwise choose a sequence of sides along which $n_\ell^{-1} \log \mu_\ell(e^{V_\ell})$ converges to the lower limit, divide (13.10) by n_ℓ^-, and use that $n_\ell/n_\ell^- \to 1$ and $\ell^2(1 + \log_2 \ell)^2/n_\ell^- \to 0$.

No closure is asked of a configuration across the seam of the torus: ν^- and μ_ℓ^- live on the same space Ω^-, whose Villain variables are closed on the cubes of $\mathfrak{C}^-$ only; the cubes at the seam and the translates of v that wrap around the torus belong to the shell, where they have been summed out (in μ_ℓ^-) or bounded below by $-c_v$ (in $V_\ell - V^-$).

Step 2 (densities on Ω^-; decomposition of the entropy). Let $d\zeta$ be the reference measure on Ω^-: Lebesgue measure on $[-\rho_{0,0}, \infty)$ and normalised Haar measure on $\mathbb{T}$ at every site, Lebesgue measure on FD on every bond, and counting measure on the $c\mathbb{Z}$-valued Villain variables that are closed on the cubes of $\mathfrak{C}^-$. Each factor of the phase-restored weight belongs to one cell and depends on the variables of that cell and of its faces: $g_{\mathrm{ct}}(\tilde\rho_y)$ to the site y, $h(\tilde\rho_{b_-}, \tilde\rho_{b_+}, B_b + \psi_{b_+} - \psi_{b_-})$ to the bond b, $w((dB)_p + v_p)$ to the plaquette p; and each closure constraint belongs to one cube. Let $W^-(\zeta)$ be the product of the factors of the sites, bonds and plaquettes of $\mathfrak{C}^-$.

Shell factors. For $\zeta \in \Omega^-$ and a configuration $\widetilde\omega$ of $\varepsilon\mathbb{Z}^3$, of which only the variables outside $V(\Lambda)$ enter, let $\mathcal{Z}^{\mathrm{sh}}(\zeta, \widetilde\omega)$ be the integral over the variables of the shell $\mathfrak{A}$ ($\tilde\rho$ and ψ at the sites of $\Lambda \setminus \Lambda^-$, B on the bonds of $\mathfrak{A}$, and the Villain variables on the plaquettes of $\mathfrak{A}$, summed over

the values for which every cube of $\mathfrak{C}^+$ not in $\mathfrak{C}^-$ is closed, its faces in Σ carrying the values of ζ and its faces in Σ^+ those of $\widetilde{\omega}$) of the product of the factors of the cells of $\mathfrak{A}$, the variables on $\mathfrak{C}^-$ being those of ζ and the variables outside $V(\Lambda)$ those of $\widetilde{\omega}$. Let $\mathcal{Z}^{\mathrm{sh}}_\ell(\zeta)$ be the same on the torus: the integral over the variables of $\mathfrak{A}_\ell$, the Villain variables being summed over the values for which every cube of $\varepsilon\mathbb{Z}^3_\ell$ not in $\mathfrak{C}^-$ is closed, of the product of the factors of the cells of $\mathfrak{A}_\ell$.

By (G1), (G3) and (G4), the weight of the kernel $\widetilde{\gamma}_\Lambda(\cdot \mid \widetilde{\omega})$ of the phase-restored specification (Lemma 13.3(c)) and the weight of μ_ℓ are $W^-(\zeta)$ times the factors of the shell cells, and the closure constraints split into those of the cubes of $\mathfrak{C}^-$, which are carried by $d\zeta$, and those of the other cubes, which are in the shell factors. Hence the laws of the variables of $\mathfrak{C}^-$ under $\widetilde{\gamma}_\Lambda(\cdot \mid \widetilde{\omega})$ and under μ_ℓ are

$$\widetilde{\gamma}^-_\Lambda(d\zeta \mid \widetilde{\omega}) = \frac{W^-(\zeta)\,\mathcal{Z}^{\mathrm{sh}}(\zeta,\widetilde{\omega})}{\mathcal{Z}_\Lambda(\widetilde{\omega})}\,d\zeta, \qquad \mu^-_\ell(d\zeta) = \frac{W^-(\zeta)\,\mathcal{Z}^{\mathrm{sh}}_\ell(\zeta)}{\mathcal{Z}_\ell}\,d\zeta, \tag{13.14}$$

with $\mathcal{Z}_\Lambda(\widetilde{\omega}) := \int W^-(\zeta)\,\mathcal{Z}^{\mathrm{sh}}(\zeta,\widetilde{\omega})\,d\zeta$ and $\mathcal{Z}_\ell := \int W^-(\zeta)\,\mathcal{Z}^{\mathrm{sh}}_\ell(\zeta)\,d\zeta$. By the Gibbs property of ν for the box Λ, that is, the equation $\nu\widetilde{\gamma}_\Lambda = \nu$ of the definition of a Gibbs measure in Subsection 13.1, read for the phase-restored specification,

$$\nu^- = \int \nu(d\widetilde{\omega})\,\widetilde{\gamma}^-_\Lambda(\cdot \mid \widetilde{\omega}).$$

Both laws in (13.14) have densities with respect to $W^-(\zeta)\,d\zeta$, and by Step 4 the shell factors are strictly positive: $\mathcal{Z}^{\mathrm{sh}}_\ell(\zeta) > 0$ for every $\zeta \in \Omega^-$, and $\mathcal{Z}^{\mathrm{sh}}(\zeta,\widetilde{\omega}) > 0$ for every ζ and ν-almost every $\widetilde{\omega}$. So $\widetilde{\gamma}^-_\Lambda(\cdot \mid \widetilde{\omega})$, ν^- and μ^-_ℓ are mutually absolutely continuous; what is used is $\nu^- \ll \mu^-_\ell$. The relative entropy being convex,

$$\mathrm{Ent}(\nu^- \mid \mu^-_\ell) \le \int \nu(d\widetilde{\omega})\,\mathrm{Ent}\big(\widetilde{\gamma}^-_\Lambda(\cdot \mid \widetilde{\omega}) \mid \mu^-_\ell\big),$$

and by (13.14) the logarithm of the density of $\widetilde{\gamma}^-_\Lambda(\cdot \mid \widetilde{\omega})$ with respect to μ^-_ℓ is $\log \mathcal{Z}^{\mathrm{sh}}(\zeta,\widetilde{\omega}) - \log \mathcal{Z}^{\mathrm{sh}}_\ell(\zeta) + \log \mathcal{Z}_\ell - \log \mathcal{Z}_\Lambda(\widetilde{\omega})$ (the factor W^- cancels). Therefore

$$\begin{aligned}\mathrm{Ent}(\nu^- \mid \mu^-_\ell) \le{}& \int \nu(d\widetilde{\omega}) \int \widetilde{\gamma}^-_\Lambda(d\zeta \mid \widetilde{\omega}) \big[\log \mathcal{Z}^{\mathrm{sh}}(\zeta,\widetilde{\omega}) - \log \mathcal{Z}^{\mathrm{sh}}_\ell(\zeta)\big] \\ &+ \int \nu(d\widetilde{\omega}) \big[\log \mathcal{Z}_\ell - \log \mathcal{Z}_\Lambda(\widetilde{\omega})\big].\end{aligned} \tag{13.15}$$

Since $\mathcal{Z}_\Lambda(\widetilde{\omega})/\mathcal{Z}_\ell = \int \mu^-_\ell(d\zeta')\,\mathcal{Z}^{\mathrm{sh}}(\zeta',\widetilde{\omega})/\mathcal{Z}^{\mathrm{sh}}_\ell(\zeta')$ by (13.14), Jensen's inequality for the logarithm gives

$$\log \mathcal{Z}_\ell - \log \mathcal{Z}_\Lambda(\widetilde{\omega}) \le \int \mu^-_\ell(d\zeta') \big[\log \mathcal{Z}^{\mathrm{sh}}_\ell(\zeta') - \log \mathcal{Z}^{\mathrm{sh}}(\zeta',\widetilde{\omega})\big]. \tag{13.16}$$

It remains to bound the shell factors from above (Step 3) and from below (Step 4) and to bound the means of the lower bounds under ν and under μ_ℓ (Steps 5–7); the splitting of the integrals in (13.15) and (13.16) is justified in Step 8.

Step 3 (upper bounds for the shell factors). Dropping the closure constraints enlarges the sum over the Villain variables of the shell to the free sum. Then, the Villain sums being done first, at fixed B, the integrals over the B_b next and the site integrals last, each shell plaquette contributes $\sum_{v_p \in c\mathbb{Z}} w((dB)_p + v_p) \le K'_{\gamma_N}$, whatever the value of $(dB)_p$; each shell bond contributes $\int_{\mathrm{FD}} h\,dB_b \le 2\pi/\gamma_N$, as $h \le 1$; and each shell site contributes $\int g_{\mathrm{ct}} = G^{\mathrm{ct}}_0$, the Haar measure having mass 1. Here K'_{γ_N} and G^{ct}_0 are the constants of Lemma 11.2. With $C_{\mathrm{sh}} := \max\{1, G^{\mathrm{ct}}_0, 2\pi/\gamma_N, K'_{\gamma_N}\}$,

$$\mathcal{Z}^{\mathrm{sh}}(\zeta,\widetilde{\omega}) \le C_{\mathrm{sh}}^{|\mathfrak{A}|}, \qquad \mathcal{Z}^{\mathrm{sh}}_\ell(\zeta) \le C_{\mathrm{sh}}^{|\mathfrak{A}_\ell|} \qquad \text{for all } \zeta \text{ and } \widetilde{\omega}. \tag{13.17}$$

Step 4 (one closed extension; lower bounds for the shell factors). For $\zeta \in \Omega^-$ put

$$\mathfrak{m}(\zeta)_p := o_p\, v_p(\zeta)/c \qquad \text{for the plaquettes } p \text{ of } \Sigma,$$

an integer function on the plaquettes of Σ. It has zero sum: $\sum_{p\in\Sigma} o_p v_p(\zeta)$ is the sum, over the cubes of $\mathfrak{C}^-$, of the coboundary of the Villain variables of ζ (the interior faces cancel), and every term of that sum vanishes. Likewise, for a configuration $\widetilde{\omega}$ of $\varepsilon\mathbb{Z}^3$ with closed Villain variables (as they are ν-almost surely) put $\mathfrak{m}^+(\widetilde{\omega})_p := o_p\, v_p(\widetilde{\omega})/c$ for the plaquettes p of Σ^+; it has zero sum, by the closure on the cubes of $\mathfrak{C}^+$.

Claim 1 (closed extension). Let $\mathfrak{a}$ be an integer-valued function on the bonds which vanishes off the bonds of Σ and satisfies $d_\Sigma \mathfrak{a} = \mathfrak{m}(\zeta)$. Then the Villain variables equal to $v_p(\zeta)$ on the plaquettes p of $\mathfrak{C}^-$ and to $c\,(d\mathfrak{a})_p$ on the plaquettes p not in $\mathfrak{C}^-$ are closed on every cube, in $\varepsilon\mathbb{Z}^3$ and in $\varepsilon\mathbb{Z}^3_\ell$. Outside $\mathfrak{C}^-$ they equal $\pm c\,\mathfrak{a}_b$ on the one or two plaquettes that contain the bond b of Σ, see (G5), and 0 elsewhere. Two-sided form: if moreover $\mathfrak{a}^+$ is an integer-valued function on the bonds which vanishes off the bonds of Σ^+ and satisfies $d_{\Sigma^+}\mathfrak{a}^+ = \mathfrak{m}^+(\widetilde{\omega})$, then the Villain variables equal to $v_p(\zeta)$ on the plaquettes of $\mathfrak{C}^-$, to $v_p(\widetilde{\omega})$ on the plaquettes outside $V(\Lambda)$ and to $c\,(d(\mathfrak{a}+\mathfrak{a}^+))_p$ on the plaquettes of $\mathfrak{A}$ are closed on every cube of $\varepsilon\mathbb{Z}^3$. On $\mathfrak{A}$ they equal $\pm c\,\mathfrak{a}_b$ on the plaquettes at the bonds b of Σ, $\pm c\,\mathfrak{a}^+_b$ on the plaquette at a bond b interior to a face of Σ^+, and 0 elsewhere, with one exception, the plaquettes of $\mathfrak{A}$ that contain two bonds of Σ^+. A plaquette p of $\mathfrak{A}$ has a vertex $y \in \Lambda$, and its two bonds at y are not in Σ^+, whose sites lie outside Λ; so p contains at most two bonds of Σ^+, its bonds b, b' at the vertex z opposite to y. Both are in Σ^+ if and only if z lies on an edge of the cube $[-2, \ell-1]^3$: two bonds of p in a common face would force y into that face, and if z lies on an edge, y differs from z in the two coordinates that are extremal at z, so that each of b, b' keeps one of them and lies in one of the two faces that meet at the edge. There are 12ℓ such plaquettes, ℓ at each edge, and each of them carries $c\,([p:b]\,\mathfrak{a}^+_b + [p:b']\,\mathfrak{a}^+_{b'})$.

Proof of Claim 1. For a plaquette p of Σ, all of whose bonds are bonds of Σ, $c\,(d\mathfrak{a})_p = c\sum_b [p:b]\mathfrak{a}_b = c\,o_p (d_\Sigma \mathfrak{a})_p = c\,o_p\,\mathfrak{m}(\zeta)_p = v_p(\zeta)$. A cube not in $\mathfrak{C}^-$ has its faces either outside $\mathfrak{C}^-$ or in Σ, by (G4); so the variables equal $c\,d\mathfrak{a}$ on all its faces, and their coboundary on the cube is $c\,d\,d\mathfrak{a} = 0$. The cubes of $\mathfrak{C}^-$ are closed because $\zeta \in \Omega^-$. The values outside $\mathfrak{C}^-$ are read off from (G5). Two-sided form: $d\mathfrak{a}^+$ vanishes on the plaquettes of Σ and $d\mathfrak{a}$ on those of Σ^+, by (G5), and $c\,(d\mathfrak{a}^+)_p = v_p(\widetilde{\omega})$ on the plaquettes of Σ^+ as before. So on every face of a cube of $\mathfrak{C}^+$ not in $\mathfrak{C}^-$ the variables equal $c\,d(\mathfrak{a}+\mathfrak{a}^+)$, and such a cube is closed; by (G1) these are all the cubes that have a face in $\mathfrak{A}$, and every other cube outside $\mathfrak{C}^-$ has all its faces outside $V(\Lambda)$ and is closed because the Villain variables of $\widetilde{\omega}$ are. This proves Claim 1.

Flux classes. An $\mathfrak{a}$ as in Claim 1 exists for every $\zeta \in \Omega^-$: by Step 5 the only condition is that $\mathfrak{m}(\zeta)$ has zero sum, that is, that the total flux of the Villain variables of ζ through the sphere Σ vanishes, and this holds by the closure on the cubes of $\mathfrak{C}^-$. On the torus the extension of Claim 1 vanishes on every plaquette lying in one of the three planes $\{x_i = \ell - 2\}$, $i = 1, 2, 3$, which contain no bond of Σ and no cell of $\mathfrak{C}^-$; so its fluxes through these three planes vanish, and it lies in the class of zero flux. In any case the Villain sum of μ_ℓ runs over all closed Villain variables, every flux class included: the extension is one of its terms, and no condition on the flux class arises.

Lower bounds. Let

$$\mathcal{E}_\Sigma(\mathfrak{m}) := \min\{|\mathfrak{a}|^2 :\ \mathfrak{a} \text{ integer valued on the bonds of } \Sigma,\ d_\Sigma \mathfrak{a} = \mathfrak{m}\}$$

for an integer function $\mathfrak{m}$ of zero sum on the plaquettes of Σ (it is finite by Step 5), and $\mathcal{E}_{\Sigma^+}$ likewise on Σ^+. Restrict the integral defining a shell factor to the one value of the Villain variables of the shell given by Claim 1, with minimising $\mathfrak{a}$ (and $\mathfrak{a}^+$), and to $\tilde{\rho} \in [0, 1]$ at the shell sites; ψ and B are not restricted.

A shell site then contributes $\int_0^1 g_{\rm ct} \geq c_g^{\rm ct}$, because $g_{\rm ct} \geq c_g^{\rm ct}$ on $[0,1]$ (proof of Lemma 11.2, Step 1). A shell bond contributes at least $(2\pi/\gamma_N)\, e^{-\kappa_N(\rho^2+\rho'^2)}$, where ρ, ρ' are the values of ρ at its two ends, because $h \geq e^{-\kappa_N(\rho^2+\rho'^2)}$ for every value of its third argument (as $|e^{i\gamma_N\vartheta}\rho' - \rho|^2 \leq 2\rho^2 + 2\rho'^2$); here $\rho \leq \rho_{0,0}+1$ at an end which is a shell site, a site y of Σ lies on at most 3 shell bonds, and a site y of Σ^+ lies on at most one: only the $6\ell^2$ sites interior to the faces of Σ^+ lie on a shell bond, and the sums over $y \in \Sigma^+$ below run over these. A shell plaquette p contributes $w\big((dB)_p + c(d(\mathfrak{a}+\mathfrak{a}^+))_p\big) \geq \exp\big(-c^2(d(\mathfrak{a}+\mathfrak{a}^+))_p^2 - 4c^2\big)$, because $|(dB)_p| \leq 4\pi/\gamma_N = 2c$; and the sum over the shell plaquettes of $c^2(d(\mathfrak{a}+\mathfrak{a}^+))_p^2$ is at most $2c^2|\mathfrak{a}|^2 + 2c^2|\mathfrak{a}^+|^2$, by the values in Claim 1: a bond of Σ lies in at most two shell plaquettes and a bond of Σ^+ in at most one, by (G5), and $(\alpha+\beta)^2 \leq 2\alpha^2 + 2\beta^2$ on the plaquettes that contain two bonds of Σ^+ (on the torus $\mathfrak{a}^+$ is absent). With $c_{\rm sh} := |\log c_g^{\rm ct}| + |\log(2\pi/\gamma_N)| + 2\kappa_N(\rho_{0,0}+1)^2 + 4c^2$ this gives, for every $\zeta \in \Omega^-$ and every $\widetilde{\omega}$ with closed Villain variables,

(13.18)
$$-\log \mathcal{Z}_\ell^{\rm sh}(\zeta) \leq c_{\rm sh}|\mathfrak{A}_\ell| + 3\kappa_N \sum_{y\in\Sigma} \rho_y(\zeta)^2 + 2c^2\, \mathcal{E}_\Sigma(\mathfrak{m}(\zeta)),$$

(13.19)
$$-\log \mathcal{Z}^{\rm sh}(\zeta,\widetilde{\omega}) \leq c_{\rm sh}|\mathfrak{A}| + 3\kappa_N \sum_{y\in\Sigma} \rho_y(\zeta)^2 + \kappa_N \sum_{y\in\Sigma^+} \rho_y(\widetilde{\omega})^2 + 2c^2\, \mathcal{E}_\Sigma(\mathfrak{m}(\zeta)) + 2c^2\, \mathcal{E}_{\Sigma^+}(\mathfrak{m}^+(\widetilde{\omega})).$$

In particular $\mathcal{Z}_\ell^{\rm sh}(\zeta) > 0$ for every ζ and $\mathcal{Z}^{\rm sh}(\zeta,\widetilde{\omega}) > 0$ for every ζ and ν-almost every $\widetilde{\omega}$, as used in Step 2.

Step 5 (integer functions on the bonds of the sphere with prescribed coboundary).

Claim 2 (rounding). Let $\mathfrak{r}$ be a real function on the bonds of Σ with $d_\Sigma\mathfrak{r}$ integer valued. Then there is an integer-valued function $\mathfrak{a}$ on the bonds of Σ with $d_\Sigma\mathfrak{a} = d_\Sigma\mathfrak{r}$ and $\lfloor \mathfrak{r}_b \rfloor \leq \mathfrak{a}_b \leq \lceil \mathfrak{r}_b \rceil$ for every bond b; hence $|\mathfrak{a}|^2 \leq 2|\mathfrak{r}|^2 + 24\,\ell_-^2$.

Proof of Claim 2. Let Fr be the set of the bonds whose value is not an integer. At every plaquette the signed sum of its four values is an integer, so no plaquette has exactly one bond in Fr: in the dual graph the edge set Fr has no vertex of degree 1 and, if it is not empty, it contains a cycle $p_0, b_1, p_1, \dots, b_m, p_m = p_0$, where b_i is the bond shared by p_{i-1} and p_i and the b_i are distinct. Add $t\, o_{p_i}[p_i : b_i]$ to the value on b_i, $i = 1, \dots, m$. This does not change d_Σ: at p_i the bonds b_i and b_{i+1} contribute

$$t\,(o_{p_i}[p_i : b_i])^2 + t\, o_{p_i}[p_i : b_{i+1}]\, o_{p_{i+1}}[p_{i+1} : b_{i+1}] = t - t = 0,$$

by (13.11) (indices modulo m). Increase t from 0 until the first bond of the cycle reaches an integer: no value has crossed an integer, and Fr has lost a bond. Bonds outside Fr are never changed again. After at most $12\,\ell_-^2$ rounds Fr is empty. Each final value lies between the floor and the ceiling of the original one, so $|\mathfrak{a}_b| \leq |\mathfrak{r}_b| + 1$ on each of the $12\,\ell_-^2$ bonds. This proves Claim 2.

Minimal real energy. For an integer function $\mathfrak{m}$ of zero sum on the plaquettes of Σ, the function $\mathfrak{r} := d_\Sigma^* \mathcal{L}_\Sigma^{-1}\mathfrak{m}$ satisfies $d_\Sigma\mathfrak{r} = \mathcal{L}_\Sigma\mathcal{L}_\Sigma^{-1}\mathfrak{m} = \mathfrak{m}$ and $|\mathfrak{r}|^2 = \langle \mathfrak{m}, \mathcal{L}_\Sigma^{-1}\mathfrak{m}\rangle$. By Claim 2,

$$\mathcal{E}_\Sigma(\mathfrak{m}) \leq 2\,\langle \mathfrak{m}, \mathcal{L}_\Sigma^{-1}\mathfrak{m}\rangle + 24\,\ell_-^2 < \infty. \tag{13.20}$$

For $\mathfrak{m} = \mathfrak{m}(\zeta)$ let $f_p^\Sigma := o_p f_p$ for the plaquettes p of Σ, with $f_p = (dB)_p + v_p$ the curvature of ζ, and let B^Σ be the restriction of B to the bonds of Σ. Then $o_p(dB)_p = (d_\Sigma B^\Sigma)_p$, so that $c\,\mathfrak{m}(\zeta) = f^\Sigma - d_\Sigma B^\Sigma$; in particular f^Σ has zero sum. Since $d_\Sigma^*\mathcal{L}_\Sigma^{-1} d_\Sigma$ is the orthogonal projection of the functions on the bonds of Σ onto the range of d_Σ^*, the identity $c\, d_\Sigma^*\mathcal{L}_\Sigma^{-1}\mathfrak{m}(\zeta) =$

$d^*_\Sigma \mathcal{L}_\Sigma^{-1} f^\Sigma - d^*_\Sigma \mathcal{L}_\Sigma^{-1} d_\Sigma B^\Sigma$ and $|B_b| \le \pi/\gamma_N = c/2$ on the $12\,\ell_-^2$ bonds of Σ give

$$c^2 \,\langle \mathfrak{m}(\zeta), \mathcal{L}_\Sigma^{-1}\mathfrak{m}(\zeta)\rangle \le 2\,\langle f^\Sigma, \mathcal{L}_\Sigma^{-1} f^\Sigma\rangle + 2\,|B^\Sigma|^2 \le 2\,\langle f^\Sigma, \mathcal{L}_\Sigma^{-1} f^\Sigma\rangle + 6\,c^2\ell_-^2. \tag{13.21}$$

Together,

$$\mathcal{E}_\Sigma(\mathfrak{m}(\zeta)) \le \frac{4}{c^2}\,\langle f^\Sigma, \mathcal{L}_\Sigma^{-1} f^\Sigma\rangle + 36\,\ell_-^2. \tag{13.22}$$

The same holds on Σ^+ for $\mathfrak{m}^+(\widetilde{\omega})$, with ℓ_+ in place of ℓ_- and with the curvature $f_p^{\Sigma^+} := o_p f_p$ and the link variables of $\widetilde{\omega}$ on Σ^+.

Step 6 (the mean of the surface energy, where the joint form of (I2) enters; the trace bound).

Claim 3 (mean of the surface energy). Let $\mathbb{P}$ be a probability measure on the configurations of $\varepsilon\mathbb{Z}^3$, or of $\varepsilon\mathbb{Z}^3_\ell$, with closed Villain variables, and suppose that with a constant $C_\mathbb{P}$

$$\mathbb{P}\Big(\Big(\sum_p s_p f_p\Big)^2\Big) \le C_\mathbb{P}\,|s|^2 \qquad \text{for every real function } s \text{ on the plaquettes of } \Sigma. \tag{13.23}$$

Then $\mathbb{P}\big(\langle f^\Sigma, \mathcal{L}_\Sigma^{-1} f^\Sigma\rangle\big) \le C_\mathbb{P}\,\mathrm{Tr}\,\mathcal{L}_\Sigma^{-1}$.

Proof of Claim 3. Let (ϕ_i) be an orthonormal basis of eigenfunctions of $\mathcal{L}_\Sigma$ on the functions of zero sum, with eigenvalues $\varkappa_i > 0$. Then $\langle f^\Sigma, \mathcal{L}_\Sigma^{-1} f^\Sigma\rangle = \sum_i \varkappa_i^{-1}\langle \phi_i, f^\Sigma\rangle^2$, and $\langle \phi_i, f^\Sigma\rangle = \sum_p (\phi_i)_p o_p f_p$ is a sum as in (13.23) with a coefficient function of norm 1. Hence the mean is at most $C_\mathbb{P}\sum_i \varkappa_i^{-1} = C_\mathbb{P}\,\mathrm{Tr}\,\mathcal{L}_\Sigma^{-1}$. This proves Claim 3.

The hypothesis (13.23) holds in the two cases that are needed.

(a) For $\mathbb{P} = \nu$, with $C_\mathbb{P} = 2Ce^{1/2}$, C being the constant of (I2). Indeed, for $|s| = 1$ the inequality $\cosh y \ge 1 + y^2/2$ and (I2), applied to s and to $-s$ (extended by 0 to the other plaquettes of $\varepsilon\mathbb{Z}^3$), give

$$1 + \tfrac12\,\nu\Big(\Big(\sum_p s_p f_p\Big)^2\Big) \le \tfrac12\,\nu\big(e^{\sum_p s_p f_p}\big) + \tfrac12\,\nu\big(e^{-\sum_p s_p f_p}\big) \le Ce^{1/2},$$

so that $\nu((\sum_p s_p f_p)^2) \le 2(Ce^{1/2} - 1)$; the general case follows by scaling s. The same holds on Σ^+. Hypothesis (I2) is used here in its joint form: the spheres Σ and Σ^+ grow with ℓ, so that one constant C valid for all finitely supported s is required.

(b) For $\mathbb{P} = \mu_\ell$, with $C_\mathbb{P} = 1$. Let μ_ℓ^u be the unitary-gauge periodic measure of $\varepsilon\mathbb{Z}^3_\ell$: the normalised weight (13.1) on the cells of the torus, the sum over the Villain variables running over all closed (v_p), every flux class included. As $\ell \ge 8$, the lattice $\ell\varepsilon\mathbb{Z}^3$ is a period lattice for $\varepsilon\mathbb{Z}^3$ in the sense of Definition 12.6 (its non-zero vectors have length at least 8ε), and μ_ℓ^u is the Villain model of $\varepsilon\mathbb{Z}^3$ on the torus $\mathbb{R}^3/\ell\varepsilon\mathbb{Z}^3$ of that definition. By Lemma 12.10, whose proof (Steps 1–4 of the proof of Theorem 3.1) is made for exactly this sum over all closed Villain variables, every flux class included,

$$\mu_\ell^u\big(e^{\sum_p a_p f_p}\big) \le e^{|a|^2/2} \qquad \text{for every real function } a \text{ on the plaquettes of } \varepsilon\mathbb{Z}^3_\ell.$$

Applying this to $a = \pm ts$, $t > 0$, and using $\cosh y \ge 1 + y^2/2$ gives $\frac{t^2}{2}\,\mu_\ell^u((\sum_p s_p f_p)^2) \le e^{t^2|s|^2/2} - 1$, whence $\mu_\ell^u((\sum_p s_p f_p)^2) \le |s|^2$ as $t \downarrow 0$.

Finally, $(\tilde\rho, f)$ has the same law under μ_ℓ and under μ_ℓ^u, by the change of variables of Lemma 13.3(a), which is given by the cell-local formula (13.2) and applies on the torus. In detail: at fixed phases ψ with lift $\widetilde{\psi} \in \mathrm{FD}^{\mathrm{sites}}$ put

$$B'_b := [B_b + (d\widetilde{\psi})_b]_{\mathrm{FD}}, \qquad n_b := \big(B_b + (d\widetilde{\psi})_b - B'_b\big)/c \in \mathbb{Z}, \qquad v'_p := v_p + c\,(dn)_p.$$

Then $(dB')_p + v'_p = \big((dB)_p - c(dn)_p\big) + \big(v_p + c(dn)_p\big) = f_p$; (v'_p) is closed if and only if (v_p) is, and in the same flux class, $c\,dn$ being exact; $h(\cdot,\cdot, B_b + \psi_{b_+} - \psi_{b_-}) = h(\cdot,\cdot,B'_b)$ by the c-periodicity

of h; the map $B \mapsto B'$ is a rotation of each FD and preserves Lebesgue measure; and, at fixed B, $(v_p) \mapsto (v'_p)$ is a bijection of the closed Villain variables. So, conditionally on ψ, the variables $\tilde\rho$, B' and (v'_p) have the law μ^u_ℓ. Hence $(\tilde\rho, f)$ has the same law under μ_ℓ and under μ^u_ℓ, and the two partition functions coincide.

Claim 4 (trace bound). $\operatorname{Tr}\mathcal{L}_\Sigma^{-1} \le 144\,\ell_-^2(1+\log_2\ell_-)^2$.

Proof of Claim 4. $\operatorname{Tr}\mathcal{L}_\Sigma^{-1} = \sum_p \mathcal{L}_\Sigma^{-1}(p,p)$, the sum over the plaquettes of Σ. For a set A of plaquettes of Σ write $\bar{\mathbf{1}}_A := |A|^{-1}\mathbf{1}_A$, and $\mathbf{1}_p$ for the indicator of the plaquette p. Since $\mathcal{L}_\Sigma^{-1}$ annihilates the constants,

$$\mathcal{L}_\Sigma^{-1}(p,p) = \big\langle \mathbf{1}_p - \bar{\mathbf{1}}_\Sigma, \mathcal{L}_\Sigma^{-1}(\mathbf{1}_p - \bar{\mathbf{1}}_\Sigma)\big\rangle = \min\big\{|\mathfrak{r}|^2 :\ \mathfrak{r} \text{ real on the bonds of } \Sigma,\ d_\Sigma\mathfrak{r} = \mathbf{1}_p - \bar{\mathbf{1}}_\Sigma\big\},$$

the minimiser being $d^*_\Sigma\mathcal{L}_\Sigma^{-1}(\mathbf{1}_p - \bar{\mathbf{1}}_\Sigma)$. So it suffices to exhibit, for every plaquette p of Σ, a real $\mathfrak{r}$ with $d_\Sigma\mathfrak{r} = \mathbf{1}_p - \bar{\mathbf{1}}_\Sigma$ and $|\mathfrak{r}| \le (2\sqrt{2}+2)(1+\log_2\ell_-)$.

(SF) *Strip flows.* Let Π be a set of plaquettes of Σ arranged in q rows, each row a chain of m plaquettes in which consecutive ones share a bond (inside one face of the cube, or continuing across edges of the cube in an unfolded strip of faces). Let $\mathfrak{t}$ be a real function on Π whose sum over every row is 0 and whose partial sums along every row have modulus at most $\mathfrak{t}^*$. Then the function $\mathfrak{r}$ which, on the bond between two consecutive plaquettes of a row, equals the partial sum of $\mathfrak{t}$ up to the first of them, with the sign $o_p[p:b]$ of that bond b in that plaquette p, and which vanishes on all other bonds, satisfies $d_\Sigma\mathfrak{r} = \mathfrak{t}$ (by (13.11)) and $|\mathfrak{r}|^2 \le q(m-1)(\mathfrak{t}^*)^2$. In particular, if Π' and Π'' are subsets of Π each consisting of full columns of Π (the i-th plaquettes of the rows forming the i-th column), then $\mathfrak{t} = \bar{\mathbf{1}}_{\Pi'} - \bar{\mathbf{1}}_{\Pi''}$ has row sums 0 and partial sums in $[-1/q, 1/q]$, so that $|\mathfrak{r}|^2 \le m/q$.

Inside the face. Let Σ_1 be the face of the cube that contains p, an $\ell_- \times \ell_-$ array of plaquettes. Put $\ell_j := \min(2^j, \ell_-)$ and $j_* := \lceil \log_2 \ell_- \rceil$, and choose rectangles $\Pi_0 = \{p\} \subset \Pi_1 \subset \cdots \subset \Pi_{2j_*} = \Sigma_1$, where Π_{2j-1} has ℓ_j columns and ℓ_{j-1} rows and Π_{2j} has ℓ_j columns and ℓ_j rows (intervals of columns and of rows containing the previous ones). For the step from Π_{2j-2} to Π_{2j-1}, (SF) on Π_{2j-1} with its $q = \ell_{j-1}$ rows of length $m = \ell_j$ gives $\mathfrak{r}$ with $d_\Sigma\mathfrak{r} = \bar{\mathbf{1}}_{\Pi_{2j-2}} - \bar{\mathbf{1}}_{\Pi_{2j-1}}$ and $|\mathfrak{r}|^2 \le \ell_j/\ell_{j-1} \le 2$. For the step from Π_{2j-1} to Π_{2j}, (SF) with the $q = \ell_j$ columns of Π_{2j} as chains of length $m = \ell_j$ gives $d_\Sigma\mathfrak{r} = \bar{\mathbf{1}}_{\Pi_{2j-1}} - \bar{\mathbf{1}}_{\Pi_{2j}}$ and $|\mathfrak{r}|^2 \le 1$. These are $2j_*$ steps, each of norm at most $\sqrt{2}$.

From the face to the sphere. Let $\Sigma_2, \dots, \Sigma_5$ be the faces adjacent to Σ_1 and Σ_6 the opposite face. For each $i = 2, \dots, 6$ we give $\mathfrak{r}$ with $d_\Sigma\mathfrak{r} = (\bar{\mathbf{1}}_{\Sigma_1} - \bar{\mathbf{1}}_{\Sigma_i})/6$. For $i \le 5$: (SF) on the unfolded $\ell_- \times 2\ell_-$ strip $\Sigma_1 \cup \Sigma_i$ with its ℓ_- rows across the common edge; the partial sums are at most $\ell_-/(6\ell_-^2)$, so $|\mathfrak{r}|^2 \le \ell_-(2\ell_-)(6\ell_-)^{-2} = 1/18$. For $i = 6$: the $\ell_- \times 3\ell_-$ strip $\Sigma_1 \cup \Sigma_2 \cup \Sigma_6$ through the adjacent face Σ_2; $|\mathfrak{r}|^2 \le \ell_-(3\ell_-)(6\ell_-)^{-2} = 1/12$. The sum of the five has $d_\Sigma = \frac{5}{6}\bar{\mathbf{1}}_{\Sigma_1} - \frac{1}{6}\sum_{i=2}^6 \bar{\mathbf{1}}_{\Sigma_i} = \bar{\mathbf{1}}_{\Sigma_1} - \bar{\mathbf{1}}_\Sigma$ and norm at most $4/\sqrt{18} + 1/\sqrt{12} < 2$.

Adding all these functions gives $\mathfrak{r}$ with

$$d_\Sigma\mathfrak{r} = (\mathbf{1}_p - \bar{\mathbf{1}}_{\Pi_1}) + \cdots + (\bar{\mathbf{1}}_{\Pi_{2j_*-1}} - \bar{\mathbf{1}}_{\Sigma_1}) + (\bar{\mathbf{1}}_{\Sigma_1} - \bar{\mathbf{1}}_\Sigma) = \mathbf{1}_p - \bar{\mathbf{1}}_\Sigma$$

and $|\mathfrak{r}| \le 2\sqrt{2}\,j_* + 2 \le (2\sqrt{2}+2)(1+\log_2\ell_-)$. So $\mathcal{L}_\Sigma^{-1}(p,p) \le 24\,(1+\log_2\ell_-)^2$, and Σ has $6\ell_-^2$ plaquettes. This proves Claim 4. The same bound holds on Σ^+ with ℓ_+ in place of ℓ_-.

Step 7 (the means of the lower bounds). By (13.22), Claim 3 and Claim 4, for $\mathbb{P} = \nu$ (with $C_\mathbb{P} = 2Ce^{1/2}$) and for $\mathbb{P} = \mu_\ell$ (with $C_\mathbb{P} = 1$), $\mathfrak{m}$ being the function $\mathfrak{m}(\zeta)$ of the variables of $\mathfrak{C}^-$,

$$\mathbb{P}\big(\mathcal{E}_\Sigma(\mathfrak{m})\big) \le \frac{4}{c^2}\,C_\mathbb{P}\operatorname{Tr}\mathcal{L}_\Sigma^{-1} + 36\,\ell_-^2 \le 576\,C_\mathbb{P}\,c^{-2}\,\ell_-^2(1+\log_2\ell_-)^2 + 36\,\ell_-^2, \tag{13.24}$$

and for $\mathbb{P} = \nu$ the same bound holds for $\mathcal{E}_{\Sigma^+}(\mathfrak{m}^+)$ with ℓ_+ in place of ℓ_-.

For the density,

$$\nu(\rho_y^2) \le m_\nu, \qquad \mu_\ell(\rho_y^2) \le C_\rho := \log\max\Big\{1, \frac{G_1^{\mathrm{ct}}(2\pi/\gamma_N)^3 K'^3_{\gamma_N}}{c_0}\Big\} \tag{13.25}$$

for every site y, with the constants of Lemma 11.2. The first bound is the definition of m_ν. For the second, ρ has the same law under μ_ℓ and under μ_ℓ^u (Step 6(b)), and μ_ℓ^u is invariant under the translations of the torus, which act by automorphisms of the cell complex preserving the cell-local weight and the reference measure. So, by Jensen's inequality,

$$\mu_\ell^u(\rho_y^2) = \ell^{-3}\,\mu_\ell^u\Big(\sum_x \rho_x^2\Big) \le \ell^{-3}\log\mu_\ell^u\big(e^{\sum_x \rho_x^2}\big) = \ell^{-3}\log\frac{\mathcal{Z}_\ell^{(1)}}{\mathcal{Z}_\ell^{(0)}},$$

where $\mathcal{Z}_\ell^{(q)}$, $q = 0, 1$, is the integral of the unitary-gauge weight of the torus with $g_{\mathrm{ct}}(\tilde\rho_x)$ replaced by $g_{\mathrm{ct}}(\tilde\rho_x)e^{q\rho_x^2}$ at every site, the sum running over all closed Villain variables. The crude bounds of Step 1 of the proof of Lemma 11.2 hold on $\varepsilon\mathbb{Z}^3_\ell$, which has three bonds and three plaquettes per site: $\mathcal{Z}_\ell^{(1)} \le \big(G_1^{\mathrm{ct}}(2\pi/\gamma_N)^3 K'^3_{\gamma_N}\big)^{\ell^3}$, because $h \le 1$, the sum over the closed Villain variables is at most the free sum and $\sum_{v_p} w((dB)_p + v_p) \le K'_{\gamma_N}$; and $\mathcal{Z}_\ell^{(0)} \ge c_0^{\ell^3}$, from the term with all Villain variables equal to 0, which is closed, and the integration restricted to $\tilde\rho_x \in [0,1]$ and $|\gamma_N B_b| \le 1$. Reflection positivity is not used in these bounds.

Step 8 (assembly). In the first term of (13.15), ζ and the variables of $\widetilde\omega$ outside $V(\Lambda)$, the only ones on which the integrand depends, have by the Gibbs property their joint law under ν; in particular ζ has the law ν^-. Insert (13.17) for $\mathcal{Z}^{\mathrm{sh}}$ and (13.18). In the second term insert (13.16), then (13.17) for $\mathcal{Z}_\ell^{\mathrm{sh}}$ and (13.19), ζ' having the law μ_ℓ^-. This gives

$$\begin{aligned}\mathrm{Ent}(\nu^- \mid \mu_\ell^-) \le (|\mathfrak{A}| + |\mathfrak{A}_\ell|)(\log C_{\mathrm{sh}} + c_{\mathrm{sh}}) + 3\kappa_N \sum_{y\in\Sigma}\big[\nu(\rho_y^2) + \mu_\ell(\rho_y^2)\big] + \kappa_N \sum_{y\in\Sigma^+}\nu(\rho_y^2)\\ + 2c^2\big[\nu\big(\mathcal{E}_\Sigma(\mathfrak{m})\big) + \mu_\ell\big(\mathcal{E}_\Sigma(\mathfrak{m})\big)\big] + 2c^2\,\nu\big(\mathcal{E}_{\Sigma^+}(\mathfrak{m}^+)\big).\end{aligned} \tag{13.26}$$

By (G2), (13.24) and (13.25), with at most $7\ell^2$ sites of Σ and $6\ell^2$ sites of Σ^+ on shell bonds, $\ell_- \le \ell$ and $\ell_+ \le 2\ell$,

$$\mathrm{Ent}(\nu^- \mid \mu_\ell^-) \le C_{\mathrm{TV}}\,\ell^2(1+\log_2\ell)^2, \qquad C_{\mathrm{TV}} := 228\,(\log C_{\mathrm{sh}} + c_{\mathrm{sh}}) + 7\kappa_N(4m_\nu + 3C_\rho) + C_{\mathrm{fl}},$$
$$C_{\mathrm{fl}} := 39168\,Ce^{1/2} + 1152 + 432\,c^2.$$

Here C_{fl} collects the terms with $\mathcal{E}$: by (13.24),

$$2c^2[\nu(\mathcal{E}_\Sigma(\mathfrak{m})) + \mu_\ell(\mathcal{E}_\Sigma(\mathfrak{m}))] \le (2304\,Ce^{1/2} + 1152)\,\ell_-^2(1+\log_2\ell_-)^2 + 144\,c^2\ell_-^2$$

and $2c^2\nu(\mathcal{E}_{\Sigma^+}(\mathfrak{m}^+)) \le 2304\,Ce^{1/2}\ell_+^2(1+\log_2\ell_+)^2 + 72\,c^2\ell_+^2$, with $\ell_+^2 \le 4\ell^2$ and $(1+\log_2\ell_+)^2 \le 4(1+\log_2\ell)^2$. All the terms being finite, the splittings of the integrals in (13.15) and (13.16) are legitimate: $\log\mathcal{Z}^{\mathrm{sh}}$ and $\log\mathcal{Z}_\ell^{\mathrm{sh}}$ are bounded above by (13.17) and below by integrable functions, and $\mathcal{Z}_\Lambda(\widetilde\omega) \le C_{\mathrm{sh}}^{|\mathfrak{A}|}\int W^-(\zeta)\,d\zeta < \infty$. This is the entropy bound (13.13); by Step 1 it gives (13.10) and the lemma. □

Lemma 13.10 (Schatten–Hölder bound)**.** *For fibred insertions $X_1, \dots, X_k$ at distinct heights that satisfy the summability condition of Lemma 13.7(b) (insertions at the same height on disjoint bond sets tensor, $\|X \otimes X'\| = \|X\|\|X'\|$, the dressing being a product over bonds), the periodic partition function with M layers satisfies*

$$|Z_M(X_1,\dots,X_k)| = |\operatorname{Tr}_{L_G}(\prod_s \mathsf{B}^*\widehat{\bar X}_s\mathsf{B})| \le \prod_s \|\bar X_s\| \cdot \operatorname{Tr}_{L_G}(T^M) \le \prod_s \|X_s\|\, Z_M,$$

uniformly in the number of insertions.

The only input of Lemma 13.10 is $Z_M = \mathrm{Tr}(P_G T^M) \le \mathrm{Tr}(T^M) < \infty$, the finite torus partition function at fixed N ((2.1) and the Gaussian Villain sums).

Proof. The equality is (13.9). With $D := \mathsf{B}\mathsf{B}^* \ge 0$ on $\mathcal{H}$, $\mathrm{Tr}\, D^M = \mathrm{Tr}\, T^M$, D commutes with $P_G^{\mathcal{H}}$, cyclicity moves the trace to $\mathrm{Tr}(\prod_s \widehat{\bar{X}}_s D\, P_G^{\mathcal{H}})$, and Hölder's inequality for Schatten norms [32] with exponents $p_s = M$ gives the bound $\prod_s \|\bar{X}_s\|\ \mathrm{Tr}(P_G^{\mathcal{H}} D^M) = \prod_s \|\bar{X}_s\|\, Z_M$; finally $\|\bar{X}_s\| \le \|X_s\|$ (Lemma 13.6(c)). □

Lemma 13.11 (growth of r_Y)**.** *Let $Y = \sum_{a\in c\mathbb{Z}^S} \tau_a^* Y_0 \tau_a$ with Y_0 bounded and $Y_0 = P_K Y_0 P_K$, $P_K := \mathbf{1}_{[-K,K]^S}(\xi)$. Then Y is bounded, fibred (Lemma 13.7(b2)), $\mathbb{Z}$-periodic, and $|r_Y(B,u)| \le C_Y e^{c_Y|f|}$ with $c_Y = 2K\sqrt{|S|}$, $f = B - u$.*

Proof. $|k_{\tau_a^* Y_0 \tau_a}(B,u)| = |\langle P_K h^{B-a}, Y_0 P_K h^{u-a}\rangle| \le \|Y_0\|\, \|P_K h^{B-a}\|\, \|P_K h^{u-a}\| \le C\|Y_0\| e^{-d_1^2 - d_2^2}$ with d_i the distances of $B-a$, $u-a$ to $Q_K := [-K,K]^S$. Since $|f| \le d_1 + d_2 + 2K\sqrt{|S|}$, $d_1^2 + d_2^2 \ge \frac{1}{2}(d_1+d_2)^2 \ge \frac{1}{2}|f|^2 - 2K\sqrt{|S|}\,|f|$, and the sum over a of the remaining Gaussian factors is bounded uniformly; dividing by $k_1 = e^{-|f|^2/2}$ gives the claim. Boundedness of Y and $\mathbb{Z}$-periodicity are clear from the form of Y (the translates $\tau_a^* P_K$ are pairwise orthogonal for $|a|$ large, and finitely many overlap). □

Lemma 13.12 (corrector)**.** *Put $D := c\sqrt{|S|}/2$ (half-diameter of a cell of $c\mathbb{Z}^S$), $\beta := 2c_Y + D + 1$ and $Z := \delta + \sum_{a\in c\mathbb{Z}^S} \alpha_a \tau_a$ with $\alpha_a := C_Z e^{\beta|a| - |a|^2/2}$, $\delta > 0$. Then Z is bounded ($\sum_a \alpha_a < \infty$), fibred (Lemma 13.7(b2)), $\mathbb{Z}$-periodic and real self-adjoint ($\alpha_a = \alpha_{-a}$), $r_Z(B,u) = \delta + \sum_a \alpha_a e^{-a\cdot f - |a|^2/2}$, and for $C_Z \ge C_Y^2 e^{\beta D + D^2}/4$*

$$\delta + \tfrac{1}{4} r_Y^2 \le r_Z \le \delta + C e^{|f|^2/4 + (\beta/2)|f|}. \tag{13.27}$$

Hence $X_t := 1 + tY + t^2 Z$ has $r_{X_t} = 1 + t r_Y + t^2 r_Z \ge (1 + t r_Y/2)^2 + \delta t^2 > 0$ for every real t, and $\|X_t\| \le 1 + t^2(\|Z\| + \|Y\|^2/2) + O(t^3)$ for real skew-adjoint Y.

Proof. Lower bound: let a^* be the lattice point nearest to $-f$, $|a^* + f| \le D$. With $a^* = -f + e$, $|e| \le D$: $\beta|a^*| \ge \beta|f| - \beta D$ and $-|a^*|^2 - a^*\cdot f = -a^*\cdot e \ge -D|f| - D^2$, so the a^*-term alone gives

$$r_Z \ge \delta + C_Z e^{(\beta - D)|f| - \beta D - D^2} = \delta + C_Z e^{(2c_Y+1)|f| - \beta D - D^2} \ge \delta + \tfrac{1}{4} C_Y^2 e^{2c_Y|f|} \ge \delta + \tfrac{1}{4} r_Y^2$$

(Lemma 13.11). Upper bound: $-|a|^2 - a\cdot f = \frac{1}{4}|f|^2 - |a + f/2|^2$ and $\beta|a| \le \beta|a+f/2| + \frac{1}{2}\beta|f|$, and $\sum_a e^{\beta|a+f/2| - |a+f/2|^2}$ is bounded uniformly in f. The bound on $\|X_t\|$ follows from $\|1 + tY\|^2 = \|1 - t^2 Y^2\| \le 1 + t^2\|Y\|^2$ for real skew-adjoint Y. □

Lemma 13.13 (moments from (I2))**.** *Let ν satisfy (I2) and let $f = (f_p)_{p\in P}$ be the curvature on a finite plaquette set P. Then $\nu(e^{s|f_p|}) \le 2Ce^{s^2/2}$ for all $s \ge 0$, and $\nu(e^{a|f|^2/2}) \le C(1-a)^{-|P|/2}$ for every $a \in (0,1)$: Gaussian moments at every rate $a/2 < 1/2$, jointly over P.*

Proof. The first claim follows from (I2) and $e^{s|f_p|} \le e^{sf_p} + e^{-sf_p}$. For the second claim (Hubbard–Stratonovich) $e^{a|f|^2/2} = (2\pi)^{-|P|/2} \int e^{\sqrt{a}\, s\cdot f - |s|^2/2} ds$, and (I2) under the s-integral gives $\nu(e^{a|f|^2/2}) \le C(2\pi)^{-|P|/2} \int e^{(a-1)|s|^2/2} ds = C(1-a)^{-|P|/2}$. □

For a fibred X that satisfies the summability condition of Lemma 13.7(b), the insertion of X at one slab multiplies the slab weight by $r_X \circ \Theta_\sigma$ (the slab data (B,u) have density $k_1(B,u)$ per bond set, Lemma 13.4(b), and the inserted kernel of Lemma 13.7(b) has k_X in its place), so that for insertions X_s at distinct heights, the sum over the variables $\widetilde{v}^t$ being taken first in the

expectation on the left (for non-negative ratios r_{X_s}, as for the operators X_t of Lemmas 13.14 and 13.25, the expectation is then an ordinary integral, by Tonelli's theorem),

$$\mu_M\Big(\prod_s r_{X_s}\circ\Theta_\sigma^{(s)}\Big) = Z_M(X_1,\dots,X_k)/Z_M. \tag{13.28}$$

Lemma 13.14 ($\nu(r_Y)=0$ for the bond pair)**.** *Let ν be as in Theorem 13.1 (after Lemma 13.3), Y real skew-adjoint of the class of Lemma 13.11, Z the corrector of Lemma 13.12 and $v_t := \log(1+tr_Y+t^2r_Z)\circ\Theta_\sigma$, a local slab function. Then $v_t \ge \log(\delta t^2)$, $v_t \le |t|\,|r_Y| + t^2 r_Z \le C(1+|t|e^{c_Y|f|}) + t^2 C e^{|f|^2/4+(\beta/2)|f|}$, $\nu(v_t)=O(t^2)$, and $\nu(r_Y)=0$, i.e. $\nu\big(k_Y\circ\Theta_\sigma/k_1\circ\Theta_\sigma\big)=0$.*

Proof. The two bounds on v_t are (13.27) and $\log(1+x)\le x$.

Upper bound. Choose a sublattice $\Gamma'\subset\Gamma$ (still of finite index, containing a time translation, ν being Γ'-invariant) such that the translates of the window S at equal height are pairwise disjoint; then same-height insertions tensor and Lemma 13.10 applies: the insertion $X_t = 1+tY+t^2Z$ is fibred by Lemma 13.7(b1)–(b2), Y being a localised sum and Z a summable series of lattice translations, and it satisfies the summability condition by Remark 13.8 (the condition holds for 1 and is stable under sums); so do its tensor products over disjoint bond sets, which are fibred by (b2) and for which the sum factorises. Lemma 13.9 applies to v_t (bounded below; (I1), (I2) and the Γ'-invariance of ν give its hypotheses). The lemma is applied on its cubic tori $\varepsilon\mathbb{Z}^3_\ell$, $\ell\varepsilon\mathbb{Z}^3\subset\Gamma'$: the measure μ_ℓ is the periodic measure μ_M with $M=\ell$ layers over the layer torus of side ℓ (so that $\mu_M=\mu_\ell$, $V_M=V_\ell$ and $n_M=n_\ell$ below), and on $\varepsilon\mathbb{Z}^3_\ell$ the translates of the window S at equal height stay pairwise disjoint, because they are the images of distinct Γ'-translates of S in $\varepsilon\mathbb{Z}^3$ and $\ell\varepsilon\mathbb{Z}^3\subset\Gamma'$. By (13.28) and Lemma 13.10 on these tori, $\mu_M(e^{V_M}) = Z_M(X_t,\dots,X_t)/Z_M \le \|X_t\|^{n_M}$, so $\nu(v_t)\le\log\|X_t\| \le t^2(\|Z\|+\|Y\|^2/2)+O(t^3)$ by Lemma 13.12.

Lower bound. On $A_t := \{|t|\,|r_Y|\le\frac12\}$, $r_Z\ge0$ gives $v_t\ge\log(1+tr_Y)\ge tr_Y - t^2r_Y^2$; off A_t, $v_t\ge\log(\delta t^2)$. Hence, for $|t|$ small,

$$t\,\nu(r_Y) \le \nu(v_t) + t^2\nu(r_Y^2) + |t|\,\nu(|r_Y|\mathbf{1}_{A_t^c}) + \log\big(1/(\delta t^2)\big)\,\nu(A_t^c).$$

Here $\nu(r_Y^2)<\infty$ by Lemma 13.11 and the exponential moments of Lemma 13.13; A_t^c forces $|f|\ge c_Y^{-1}\log(1/(2C_Y|t|))$, so by the Gaussian moments of Lemma 13.13 $\nu(A_t^c)=O(e^{-a'\log^2(1/|t|)})$ is $o(t^k)$ for every k, and $|t|\nu(|r_Y|\mathbf{1}_{A_t^c})\le|t|\nu(r_Y^2)^{1/2}\nu(A_t^c)^{1/2}=o(t^k)$. The right side is $O(t^2)$ for both signs of t, so $\nu(r_Y)=0$. The argument uses no moment of r_Z. □

Remark 13.15 (what Subsection 13.5 uses). Hypotheses (I1) and (I2) are used in Lemma 13.9, the latter in its joint form, with one constant for all finitely supported s. Hypothesis (I2) is used once more in the proof of Lemma 13.14, for $\nu(r_Y^2)<\infty$ and for the tail of A_t. The growth of r_Z at the rate $1/4$ in (13.27) is used only in Lemma 13.25 (for the set A_t of its proof), together with the Gaussian moments at rates $>3/8$ of Lemma 13.13.

13.6. **Bond-sector separation.**

Lemma 13.16 (bond pair law)**.** *Let*

$$\mathcal{C} := \{\sum_{a\in c\mathbb{Z}^S}\tau_a^* X_0\tau_a :\ X_0 \text{ bounded on } L^2(\mathbb{R}^S),\ X_0 = P_K X_0 P_K \text{ for some } K\}.$$

$\mathcal{C}$ is closed under adjoints, complex conjugation and real-linear combinations; every $X\in\mathcal{C}$ is fibred (Lemma 13.7(b2)), hence $\mathbb{Z}$-periodic and bounded, with r_X exponential-linear (Lemma 13.11). Let λ be the law of the pair $(B_S,u_S)=(B_S,\widetilde{B}'_S-\widetilde{v}^t_S)$ under ν (after Θ_σ) and $m:=k_1^{-1}\lambda$, a σ-finite measure on $\mathrm{FD}^S\times\mathbb{R}^S$. Then $m(|k_X|)=\lambda(|r_X|)<\infty$ for $X\in\mathcal{C}$, and $(R_*m)(k_X)=m(k_X)$ for every $X\in\mathcal{C}$, R the twisted reflection of Lemma 13.6(a).

Proof. Decompose $X = A + Y + iA' + iY'$ with A, A' real self-adjoint and Y, Y' real skew-adjoint, all in $\mathcal{C}$. By Lemma 13.6(b), $k_A \circ R = k_A$, $k_{iA'} \circ R = k_{iA'}$, $k_Y \circ R = -k_Y$, $k_{iY'} \circ R = -k_{iY'}$. Lemma 13.14 gives $m(k_Y) = \nu(r_Y) = 0$ for the real skew-adjoint elements, hence $m(k_X \circ R) = m(k_X)$. Finiteness is Lemma 13.11 with the exponential moments of Lemma 13.13. □

Lemma 13.17 (separation on the bond pair)**.** *Let $X_0 = |g_1\rangle\langle g_2|$ with $g_i \in L^2(\mathbb{R}^S)$ supported in $[-K, K]^S$, let $X = \sum_a \tau_a^* X_0 \tau_a \in \mathcal{C}$, and put $G_i := e^{-|\cdot|^2} * g_i$. Let $\hat{m} := \sum_{a \in c\mathbb{Z}^S} (T_a)_* m$ be the unfolding of m to $\mathbb{R}^S \times \mathbb{R}^S$, where T_a is the diagonal shift; the set $\mathrm{FD}^S \times \mathbb{R}^S$ is a fundamental domain, so $m \leftrightarrow \hat{m}$ is a bijection. Then the following hold.*

(i) *Up to the normalisation of h, $k_X(B, u) = \sum_{a \in c\mathbb{Z}^S} G_1(B - a)\overline{G_2(u - a)}$, so that $m(k_X) = \hat{m}(G_1 \otimes \overline{G_2})$.*
(ii) *$\hat{m}(e^{-\delta(|B|^2 + |u|^2)}) \le C\lambda(e^{(1-\delta)|f|^2/2}) < \infty$ for every $\delta \in (0, 1)$.*
(iii) *The measure $\hat{m}$ is determined by the numbers $\hat{m}(G_1 \otimes \overline{G_2})$.*
(iv) *$R_* m = m$ and $\lambda \circ R = \lambda$.*

Proof. (i) The formula for k_X is the sum over a of the terms $\langle h^{B-a}, g_1\rangle\langle g_2, h^{u-a}\rangle$, and the identity $m(k_X) = \hat{m}(G_1 \otimes \overline{G_2})$ follows from it by the definition of $\hat{m}$.

(ii) The identity $|B + a|^2 + |u + a|^2 = 2|a + (B + u)/2|^2 + \frac{1}{2}|f|^2$ gives $\sum_a e^{-\delta(|B+a|^2+|u+a|^2)} \le C_\delta e^{-\delta|f|^2/2}$ uniformly in (B, u). Since $m = k_1^{-1}\lambda$ and $k_1 = e^{-|f|^2/2}$, it follows that

$$\hat{m}(e^{-\delta(|B|^2+|u|^2)}) = \int \lambda(dB\, du)\, e^{|f|^2/2} \sum_a e^{-\delta(|B+a|^2+|u+a|^2)} \le C_\delta \lambda(e^{(1-\delta)|f|^2/2}),$$

and the right-hand side is finite by Lemma 13.13 at the rate $(1 - \delta)/2 < 1/2$.

(iii) Let $(\hat{m} * \mathrm{Gauss})(\xi, \eta) := \int e^{-|\xi-B|^2-|\eta-u|^2} \hat{m}(dB\, du)$ be the Gaussian smoothing of $\hat{m}$; by (ii) it is finite, continuous and bounded on compact sets. By Fubini's theorem, $\hat{m}(G_1 \otimes \overline{G_2}) = \int g_1(\xi)\overline{g_2(\eta)}\,(\hat{m} * \mathrm{Gauss})(\xi, \eta)\, d\xi\, d\eta$. The linear span of the products $g_1 \otimes \overline{g_2}$ is dense in L^2 of every compact subset of $\mathbb{R}^S \times \mathbb{R}^S$, so the numbers $\hat{m}(G_1 \otimes \overline{G_2})$ determine the Gaussian smoothing. The smoothing in turn determines the measure. Indeed, $\hat{m} * \mathrm{Gauss}$ is $e^{-|\xi|^2-|\eta|^2}$ times the two-sided Laplace transform of the finite measure $e^{-|B|^2-|u|^2}\hat{m}(dB\, du)$; here the weight with $\delta = 1$ is reached from the weight with $\delta = \frac{1}{2}$ of (ii) by a Gaussian factor. This Laplace transform is finite everywhere and entire, and its restriction to imaginary arguments is the Fourier transform of the measure. Hence two measures with the same Gaussian smoothing coincide.

(iv) By Lemma 13.16, the measures $(R_* m)^\wedge$ and $\hat{m}$ give the same numbers. The measure $R_* m$ satisfies the same growth bound as m, because $k_1 \circ R = k_1$ and R maps f to $-f$. Hence $(R_* m)^\wedge = \hat{m}$ by (iii), and therefore $R_* m = m$ and $\lambda \circ R = \lambda$. □

The conclusion of this subsection is the law of the *bond* pair only: for every finite S the law under ν of the frame-reduced pair (B_S, u_S) is invariant under the twisted reflection R. The insertions of $\mathcal{C}$ act as $X \otimes 1$ on the Higgs tensor factor of Lemma 13.7(a), so the joint law of $(\tilde{\rho}, \tilde{\rho}', \widetilde{\psi}' - \psi, B, u)$ is not separated by Lemma 13.17; the full statement is Theorem 13.28.

13.7. **Higgs and Villain sectors.**

(A) Higgs sector. Let $V := V(S)$ be the finite site set of the slab window, that is, the set of the end points of the bonds of S; $\mathrm{plaq}(S)$ denotes the set of the spatial plaquettes of the layer all four bonds of which lie in S, and S is *closed under the bonds of its plaquettes* if every spatial plaquette of the layer whose four corners lie in $V(S)$ belongs to $\mathrm{plaq}(S)$ (as for the set of the links of a rectangle).

Lemma 13.18 (Gaussian hopping kernel, no frame)**.** *With $\Phi_x := \rho_x e^{i\gamma_N \psi_x}$ and $\widetilde{\Phi}'_x := \widetilde{\rho}'_x e^{i\gamma_N \widetilde{\psi}'_x}$ one has $|e^{i\gamma_N(\widetilde{\psi}'_x - \psi_x)}\widetilde{\rho}'_x - \rho_x| = |\widetilde{\Phi}'_x - \Phi_x|$, so the slab Higgs factor is $\prod_x h = \prod_x e^{-|\widetilde{\Phi}'_x - \Phi_x|^2/2} =$*

$c_V\langle h^{\Phi}, h^{\widetilde{\Phi}'}\rangle_{L^2(\mathbb{C}^V)}$ *with* $h^{\Phi}(\Xi) := e^{-|\Xi-\Phi|^2}$*, and the layer reference measure* $\prod_x g_{\mathrm{ct}}(\tilde\rho_x)\, d\tilde\rho_x\, d\psi_x$ *is* $\prod_x \gamma_N^{-1} e^{-V(\tilde\rho_x)-\delta m_B\rho_x^2/2}\, d^2\Phi_x$ *(polar Lebesgue measure; the factor* ρ *of* $g(\tilde\rho) = \rho e^{-V}$ *is the Jacobian). Here* Φ *ranges over all of* $\mathbb{C}^V$*.*

Proof. $|e^{i\gamma_N\vartheta}\rho' - \rho|$ is invariant under the common rotation by $e^{i\gamma_N\psi}$, and $d^2\Phi = \gamma_N\rho\, d\rho\, d\psi$ on $[0,\infty)\times\mathbb{T}$. The Gaussian identity is that of Subsection 13.4 with c_V the normalisation constant. □

Lemma 13.19 (Higgs class and ratio)**.** *Let* $\mathcal{C}_H := \{X_H$ *bounded on* $L^2(\mathbb{C}^V) = L^2(\mathbb{R}^{2V}) : X_H = P_K X_H P_K\}$ *with* $P_K := \mathbf{1}_{\{|\Xi_x|\le K\ \forall x\}}$*,* $k_{X_H}(\Phi,\widetilde{\Phi}') := \langle h^{\Phi}, X_H h^{\widetilde{\Phi}'}\rangle$*,* $k_1^H = c_V e^{-|f_H|^2/2}$ *with* $f_H := \widetilde{\Phi}' - \Phi$*, and* $r_{X_H} := k_{X_H}/k_1^H$*. Then* $|r_{X_H}| \le C_K\|X_H\| e^{c_Y|f_H|}$ *with* $c_Y = 4\sqrt{V}\,K$*: exponential-linear in* f_H*.*

Proof. As in Lemma 13.11 with the ball $\{|\Xi_x| \le K\ \forall x\}$ in place of Q_K: its Euclidean half-diameter is $\sqrt{V}K$ (attained at $\Phi_1 = \cdots = \Phi_V = K$), so the same computation gives the bound with $2\sqrt{V}K$ in place of c_Y, and a fortiori with the constant $c_Y = 4\sqrt{V}K$ of the statement. □

Lemma 13.20 (Higgs corrector)**.** *For* $a \in \mathbb{Z}^{2V}$*,* $r_{\tau_a}(f_H) = e^{a\cdot f_H - |a|^2/2}$*. Let* $Z_H := \delta + \sum_{a\in\mathbb{Z}^{2V}} \alpha_a\tau_a$*,* $\alpha_a := C_Z e^{\beta|a|-|a|^2/2}$*,* $D_H := \sqrt{2V}/2$*,* $\beta \ge 2c_Y + 3D_H$ *and* $C_Z \ge \frac14 C_Y^2 e^{\beta D_H + D_H^2}$*. Then* Z_H *is bounded and real self-adjoint,* $r_{Z_H} \ge \delta + C_Z e^{(\beta-3D_H)|f_H| - \beta D_H - D_H^2} \ge \delta + \frac14 r_Y^2$ *whenever* $|r_Y| \le C_Y e^{c_Y|f_H|}$*, so* $1 + t r_Y + t^2 r_{Z_H} \ge (1 + t r_Y/2)^2 + \delta t^2 > 0$*, and* $r_{Z_H} \le C e^{|f_H|^2/4 + \beta|f_H|/2}$ *(rate* $1/4$*).*

Proof. The proof of Lemma 13.12 applies word for word on the lattice $\mathbb{Z}^{2V}$ (cell half-diameter D_H; the nearest lattice point a^* to $-f_H$ has $|a^* + f_H| \le D_H$). □

Lemma 13.21 (full feature map and averaging identity)**.** *Let* $\mathcal{H}_{\mathrm{full}} := \mathcal{H} \otimes L^2(\mathbb{C}^V)$ *and* $\Phi_{\mathrm{full}}(c) := \Phi(c) \otimes h^{\Phi(c)}$*. Then* $K(c,\widetilde{c}') = \langle \Phi_{\mathrm{full}}(c), \Phi_{\mathrm{full}}(\widetilde{c}')\rangle$ *is the slab kernel (13.5) (the Gaussian factorisation of Lemma 13.18 replacing the Bessel one; the same positive kernel, so* $T = \mathsf{B}^*\mathsf{B}$ *and* $Z_M = \mathrm{Tr}(P_G T^M)$ *are unchanged). The diagonal gauge action rotates* $\Phi_x \mapsto e^{i\gamma_N\lambda_x}\Phi_x$*; with* $(U(\lambda)g)(\Xi) := g(e^{-i\gamma_N\lambda}\Xi)$*, unitary and configuration independent,* $\Phi_{\mathrm{full}}(\lambda.c) = (R(\lambda)\otimes U(\lambda))\Phi_{\mathrm{full}}(c)$*, and Lemma 13.7(c) holds for every bounded* X *on* $\mathcal{H}_{\mathrm{full}}$ *with* $\bar X := \int d\lambda\, (R\otimes U)(\lambda)^* X (R\otimes U)(\lambda)$*, where for such insertions* $Z_M(X_1,\dots,X_k)$ *denotes the partition function of the periodic measure in which the slab kernel* $K(c,\widetilde{c}')$ *at the height of* X_s *is replaced by* $\langle \Phi_{\mathrm{full}}(c), X_s \Phi_{\mathrm{full}}(\widetilde{c}')\rangle$*; no summability condition enters, and the layer integrals converge absolutely because this kernel is bounded by* $\|X_s\|\sqrt{K(c,c)K(\widetilde{c}',\widetilde{c}')}$*. So does Lemma 13.10. For* $X \otimes X_H$*:* $k = k_X k_{X_H}$ *and* $r = r_X r_{X_H}$*.*

Proof. The kernel identity is Lemma 13.7(a) with the hopping factor written by Lemma 13.18; the covariance of h^{Φ} under rotations is $h^{e^{i\gamma_N\lambda}\Phi} = U(\lambda)h^{\Phi}$; the proof of Lemma 13.7(c) used only the covariance $\Phi(\lambda.c) = R(\lambda)\Phi(c)$ and the measure preservation of Lemma 13.3(a). □

What the Higgs sector uses. (I3) gives, by $|f_{H,x}|^2 \le 2\rho_x^2 + 2\widetilde{\rho}_x'^2$ and Hölder over the $2V$ factors, $\nu(e^{a\sum_{x\in V}|f_{H,x}|^2}) < \infty$ for every a (every rate, in particular the rate $1/4$ of Lemma 13.20), and $\nu(\rho_x^2) < \infty$ for Lemma 13.9.
(B) Villain sector: the layer integers v individually. In the Zak form (13.7) the kernel of every α-decomposable insertion, that is, of every $\mathbb{Z}$-periodic $X \in \mathcal{C}$ and of every multiplication by a function $m(\alpha)$, depends on the Villain integers of the two layers only through $\widetilde{v}' - v$. The translations in α of the next lemma carry, in addition, a character of v itself.

Lemma 13.22 (α-translations carry the character of the lower layer's v)**.** *For $s \in [0,1)^{\mathrm{plaq}(S)}$ let $(T_sF)(\alpha) := F(\alpha - s)$, a unitary operator on $\mathcal{H}$, and put $\sigma := \widetilde{d}s$. Let $k(y) := (y - [y]_{\mathrm{FD}})/c \in \mathbb{Z}^S$ be the cell index of y, and let $u = \widetilde{B}' - \widetilde{v}^t$ be the pair coordinate. Then*

$$k_{T_s}(c,\widetilde{c}') = e_s(v)\,\langle h^B, M_s h^u\rangle_{L^2(\mathbb{R}^S)}, \qquad e_s(v) := e^{+2\pi i s\cdot v/c}, \quad M_s := \text{multiplication by } \chi_s(y) := e^{+2\pi i\sigma\cdot k(y)}, \tag{13.29}$$

and, for fibred X (Lemma 13.7(b)) such that $\sum_{a\in c\mathbb{Z}^S}|\langle h^B, M_s X h^{u-a}\rangle| < \infty$ for all $B,u\in\mathbb{R}^S$, in particular for $X\in\mathcal{C}$, $k_{T_sX}(c,\widetilde{c}') = e_s(v)\langle h^B, M_s X h^u\rangle$. Moreover $|r_{T_s}|\le 1$ and, for $X\in\mathcal{C}$, $|r_{T_sX}| \le Ce^{c_Y|f|}$ as in Lemma 13.11. Thus the kernel of T_sX is the character $e_s(v)$ of the closed $c\mathbb{Z}$-valued Villain 2-form of the lower layer times a pair-space insertion with the same growth bound. The operator T_s is bounded, so Lemmas 13.10 and 13.21 apply to it. In pair coordinates, $\chi_s(u) = e^{-2\pi i\sigma\cdot\widetilde{v}^t/c}$.

Proof. By the definitions of Φ and of T_s,

$$\langle \Phi(c), T_s\Phi(\widetilde{c}')\rangle = \int d\alpha\, e^{2\pi i\alpha\cdot v/c} e^{-2\pi i(\alpha-s)\cdot\widetilde{v}'/c}\langle Zh^B(\theta), Zh^{\widetilde{B}'}(\theta-\sigma)\rangle_{L^2(\mathrm{FD}^S)}.$$

The computation of Lemma 13.7(a), with the extra phase, gives

$$\langle Zh^B(\theta), Zh^{\widetilde{B}'}(\theta-\sigma)\rangle = \sum_m e^{-2\pi i(\theta-\sigma)\cdot m}\langle h^B, M_s h^{\widetilde{B}'-cm}\rangle,$$

because a point $\eta\in\mathrm{FD}^S + ck$ has the cell index $k(\eta)=k$ and $e^{2\pi i\sigma\cdot(k+m)} = e^{2\pi i\sigma\cdot m}\chi_s(\eta)$. Since $\theta\cdot m = \alpha\cdot d_s m$ and $\sigma\cdot m = s\cdot d_s m$, the integral over α selects $cm = \widetilde{v}^t$ with $d_s\widetilde{v}^t = v - \widetilde{v}'$, and the remaining phase is $e^{2\pi i s\cdot(\widetilde{v}'+d_s\widetilde{v}^t)/c} = e_s(v)$. This is (13.29). For fibred X as in the statement the same computation applies. Since $T_s^* = T_{-s}$, the form that defines $\hat{X}$ gives $\langle\Phi(c), T_s\hat{X}\Phi(\widetilde{c}')\rangle = \int d\alpha\,\langle\Phi(c)(\alpha), (X(\theta-\sigma)\otimes 1)\Phi(\widetilde{c}')(\alpha-s)\rangle$ with $\theta = \widetilde{d}\alpha$. Lemma 13.7(b4), applied at the point $\theta-\sigma$ to $g_1 := M_s^{-1}h^B$, which is regular with $(Zg_1)(\theta-\sigma,\cdot) = (Zh^B)(\theta,\cdot)$, and to $g_2 := h^{\widetilde{B}'}$, gives $\langle (Zh^B)(\theta,\cdot), X(\theta-\sigma)(Zh^{\widetilde{B}'})(\theta-\sigma,\cdot)\rangle = \sum_m e^{-2\pi i(\theta-\sigma)\cdot m}\langle h^B, M_sXh^{\widetilde{B}'-cm}\rangle$ for every θ; the assumption makes this series absolutely convergent, so that it may be exchanged with the integral over α. For $X\in\mathcal{C}$ the assumption follows from the growth bound proved next, because $\sum_{a\in c\mathbb{Z}^S} e^{c_Y|f+a|-|f+a|^2/2} < \infty$.

Since $|\chi_s| = 1$ and $h\ge 0$, one has $|r_{T_s}|\le 1$. For the growth bound of r_{T_sX}, each localised piece $\langle\tau_a h^B, (\tau_a M_s\tau_a^*)X_0\tau_a h^u\rangle$ has $|\tau_a M_s\tau_a^*| = 1$, and M_s commutes with P_K, both being multiplications; the proof of Lemma 13.11 therefore applies. The character e_s and the multiplier χ_s carry the same sign in the exponent; both signs may be reversed together with the sign convention of (13.7). □

Lemma 13.23 (quasi-periodicity)**.** *For $a\in c\mathbb{Z}^S$ one has $\chi_s(y+a) = e^{2\pi i\sigma\cdot a/c}\chi_s(y)$. Consequently, for $X_0 = |g_1\rangle\langle g_2|$ and $X = \sum_a \tau_a^* X_0\tau_a$,*

$$k_{T_sX}(c,\widetilde{c}') = e_s(v)\sum_{a\in c\mathbb{Z}^S} e^{2\pi i s\cdot(d_s a)/c}\, G_1^s(B-a)\,\overline{G_2(u-a)}, \qquad G_1^s := e^{-|\cdot|^2} * (\chi_s g_1), \quad G_2 := e^{-|\cdot|^2} * g_2. \tag{13.30}$$

The map $g_1\mapsto\chi_s g_1$ is a bijection of the compactly supported L^2 functions, so that, for fixed s, the functions G_1^s range over the same set as the functions G_1 of Lemma 13.17.

Proof. The cell index satisfies $k(y+a) = k(y) + a/c$, which gives the first claim. Hence $\tau_a M_s\tau_a^*$ is the multiplication by $\chi_s(\cdot+a) = e^{2\pi i\sigma\cdot a/c}\chi_s$, and

$$\langle h^B, M_s\tau_a^* X_0\tau_a h^u\rangle = e^{2\pi i\sigma\cdot a/c}\langle h^{B-a}, \chi_s g_1\rangle\overline{\langle h^{u-a}, g_2\rangle}.$$

Summation over a and Lemma 13.22 give (13.30), since $\sigma \cdot a = s \cdot d_s a$ by adjointness. □

13.8. **Joint separation and the exchange theorem.**

Lemma 13.24 (full class; Hermitian relation)**.** *Let $\mathcal{C}_{\mathrm{full}}$ be the complex span of $\{T_s(X \otimes X_H) : s \in [0,1)^{\mathrm{plaq}(S)},\ X \in \mathcal{C},\ X_H \in \mathcal{C}_H\}$ on $\mathcal{H}_{\mathrm{full}}$. The class $\mathcal{C}_{\mathrm{full}}$ is closed under adjoints; indeed $T_s^* = T_{-s}$ and*

$$(T_s(X \otimes X_H))^* = (X^* \otimes X_H^*)T_{-s} = T_{-s}\big(T_s \hat{X}^* T_{-s} \otimes X_H^*\big),$$

with $T_{-s}\hat{X}T_s = (M_s^{-1} X M_s)^\wedge \in \mathcal{C}$ and $T_s\hat{X}T_{-s} = (M_s X M_s^{-1})^\wedge \in \mathcal{C}$. It is also closed under $J := J_\alpha \otimes J_H$, where $(J_\alpha F)(\alpha) := \overline{F(-\alpha)}$ and J_H is the complex conjugation on $L^2(\mathbb{C}^V)$. One has $J\Phi_{\mathrm{full}}(c) = \Phi_{\mathrm{full}}(c)$, and therefore $k_{JXJ} = \overline{k_X}$ and $k_X \circ \mathrm{sw} = \overline{k_{X^*}}$ for every bounded X on $\mathcal{H}_{\mathrm{full}}$. The ratios satisfy $|r_X| \le C e^{c_Y(|f|+|f_H|)}$ for $X \in \mathcal{C}_{\mathrm{full}}$. Finally, $\mathcal{C}_{\mathrm{full}}$ contains, with each of its elements X, the operators $(X + JXJ)/2$ and $(X - JXJ)/(2i)$, and, with each of its elements W, the operators $(W + W^*)/2$ and $(W - W^*)/2$; these are the J-real and the J-imaginary part of X and the self-adjoint and the skew-adjoint part of W.

Proof. By Lemma 13.22, in the form $Z(M_s X g)(\theta) = X(\theta - \sigma) Z g(\theta - \sigma)$, the operator $T_{-s}\hat{X}T_s$ corresponds under the Zak transform to $M_s^{-1} X M_s$: by Lemma 13.7(b2) the operator $M_s^{-1} X M_s$ is fibred with the fibres $X(\theta + \sigma)$, and these are the fibres of $T_{-s}\hat{X}T_s$, because $T_{-s}^* = T_s$ shifts α by s in the form that defines $\hat{X}$. The function χ_s commutes with P_K, both being multiplications, so $M_s^{-1} X_0 M_s$ is again bounded and P_K-localised, and $M_s^{-1} X M_s \in \mathcal{C}$; the same holds with $-s$ in place of s. This gives the closure under adjoints. The class is also closed under J. Indeed, $J_\alpha T_s J_\alpha = T_{-s}$ and $J_\alpha \hat{X} J_\alpha = \widehat{X^\sharp}$, where $X^\sharp \in \mathcal{C}$ is the complex conjugate of X (Lemma 13.16; the Zak transform turns the complex conjugation of $L^2(\mathbb{R}^S)$ into the map $F(\theta) \mapsto \overline{F(-\theta)}$, and the fibres of $X^\sharp$ are the complex conjugates of the operators $X(-\theta)$, by Lemma 13.7(b2)), and $J_H X_H J_H \in \mathcal{C}_H$ because P_K is a real multiplication. Hence $J\,T_s(X \otimes X_H)\,J = T_{-s}(X^\sharp \otimes J_H X_H J_H)$, and the complex span is preserved because J is antilinear. The identity $J\Phi_{\mathrm{full}} = \Phi_{\mathrm{full}}$ holds because h is real and $\overline{Z h^B(-\theta)} = Z h^B(\theta)$. It gives

$$k_{JXJ}(c, \widetilde{c}') = \overline{\langle J\Phi_{\mathrm{full}}(c), X J \Phi_{\mathrm{full}}(\widetilde{c}')\rangle} = \overline{\langle \Phi_{\mathrm{full}}(c), X \Phi_{\mathrm{full}}(\widetilde{c}')\rangle}$$

and

$$k_X(\mathrm{sw}(c, \widetilde{c}')) = \langle \Phi_{\mathrm{full}}(\widetilde{c}'), X \Phi_{\mathrm{full}}(c)\rangle = \overline{k_{X^*}(c, \widetilde{c}')}.$$

No frame appears in the second identity, because the feature map of a layer depends on the data of that layer only. The bound on the ratios follows from Lemmas 13.11, 13.19 and 13.22. The last assertion follows from the closure of $\mathcal{C}_{\mathrm{full}}$ under adjoints, under J and under complex linear combinations. □

Lemma 13.25 ($\nu(r_Y) = 0$ for $\mathcal{C}_{\mathrm{full}}$)**.** *Let ν be as in Theorem 13.1 (after Lemma 13.3), let the bond set S be closed under the bonds of its plaquettes, and let $Y \in \mathcal{C}_{\mathrm{full}}$ be real skew-adjoint, with $|r_Y| \le C_Y e^{c_Y(|f|+|f_H|)}$. Then $\lambda(r_Y) = \nu(r_Y \circ \Theta_\sigma) = 0$.*

Proof. Let $Z := \delta + Z_b \otimes Z_H$, where Z_b is the corrector of Lemma 13.12 and Z_H that of Lemma 13.20. The operator Z is bounded and real self-adjoint, and $r_Z = \delta + r_{Z_b}(f)\, r_{Z_H}(f_H) \ge \delta + \frac{1}{4} r_Y^2$; the product of the two correctors is taken here because it dominates $e^{2c_Y(|f|+|f_H|)}$, which a sum would not do. Consequently $1 + t r_Y + t^2 r_Z \ge (1 + t r_Y/2)^2 + \delta t^2 > 0$. Moreover $r_Z \le C e^{(|f|^2+|f_H|^2)/4 + c(|f|+|f_H|)} \le C e^{3Q/8}$ with $Q := |f|^2 + |f_H|^2$, because $e^{c(|f|+|f_H|)} \le e^{Q/8+4c^2}$. Put $X_t := 1 + tY + t^2 Z$, so that $\|X_t\| \le 1 + O(t^2)$, and $v_t := \log(1 + t r_Y + t^2 r_Z) \circ \Theta_\sigma$, so that $|v_t| \le |\log(\delta t^2)| + C(1 + Q)$.

Upper bound. We apply Lemma 13.9, whose hypotheses follow from (I1) and (I2), with the periodised sum over a coarser sublattice $\Gamma' \subset \Gamma$, which exists because S and its site set $V(S)$

are finite, for which the translates of S and of $V(S)$ at equal height are pairwise disjoint, and then (13.28) with Lemmas 13.21 and 13.10; the sums over $\vec{v}^t$ in the kernels of Y and of Z converge absolutely, by Lemma 13.22 for the terms $T_s(X \otimes X_H)$ of Y and by Remark 13.8 for Z_b; for the products over the pairwise disjoint translates of the window at one height the kernel is the product of the kernels of the windows and the sum factorises (Lemma 14.27(b),(c), whose proof does not use the present lemma; its hypotheses hold because the translates of S are closed under the bonds of their plaquettes, as S is, and are pairwise disjoint in bonds and in sites). This gives $\nu(v_t) \le \log \|X_t\| = O(t^2)$. As in the proof of Lemma 13.14, Lemma 13.9 is applied on its cubic tori $\varepsilon\mathbb{Z}^3_\ell$ with $\ell\varepsilon\mathbb{Z}^3 \subset \Gamma'$, where μ_ℓ is the periodic measure with $M = \ell$ layers over the layer torus of side ℓ; on these tori the translates of S and of $V(S)$ at equal height stay pairwise disjoint, because $\ell\varepsilon\mathbb{Z}^3 \subset \Gamma'$.

Lower bound. The lower bound uses no moment of r_Z. On the set $A_t := \{|t|\,|r_Y| \le \frac12,\ t^2 r_Z \le \frac12\}$ the inequality $r_Z \ge 0$ gives $v_t \ge \log(1 + t r_Y) \ge t r_Y - t^2 r_Y^2$, and off A_t one has $v_t \ge \log(\delta t^2)$. The complement of A_t satisfies

$$A_t^c \subset \{|f| + |f_H| \ge c_Y^{-1} \log(1/(2C_Y|t|))\} \cup \{Q \ge \tfrac{16}{3}\log(1/|t|) - C\}.$$

The first set has a super-polynomially small ν-measure, by the Gaussian moments. For the second set we use that $\nu(e^{aQ}) < \infty$ for every $a < \frac12$: the bond factor is controlled by Lemma 13.13 at a rate $a' < \frac12$, the Higgs factor by (I3) at every rate, and the two are combined by Hölder's inequality with an exponent $p \downarrow 1$ on the bond factor. Hence $\nu(A_t^c) \le C|t|^{16a/3}$. For $a \in (\frac38, \frac12)$ this gives $\nu(A_t^c)\log(1/(\delta t^2)) = o(t^2)$ and $|t|\nu(|r_Y|\mathbf{1}_{A_t^c}) \le |t|\nu(r_Y^2)^{1/2}\nu(A_t^c)^{1/2} = o(t^2)$. As in the proof of Lemma 13.14, it follows that $t\nu(r_Y) \le O(t^2)$ for both signs of t, so $\lambda(r_Y) = 0$.

The second set calls for a rate $> \frac38$, which the constant $\frac12$ in the exponent of (I2) gives. The condition $t^2 r_Z \le \frac12$ in A_t is not needed for the lower bound (compare the proof of Lemma 13.14), so this requirement is a feature of the present proof. □

Lemma 13.26 (Hermitian property)**.** *Let S be closed under the bonds of its plaquettes, and let $m := K^{-1}\lambda$ on the S-local slab data, where K^{-1} denotes the division by the single-term unfolded kernel $w(B-u)e^{-|f_H|^2/2} > 0$. The S-local slab data are $B \in \mathrm{FD}^S$, $u \in \mathbb{R}^S$, the closed form $v \in (c\mathbb{Z})^{\mathrm{plaq}(S)}$ and $\Phi, \widetilde{\Phi}' \in \mathbb{C}^V$; the form $\vec{v}' = v - d_s([u]_{\mathrm{FD}} - u)$ is determined by them. Then $m(|k_X|) = \lambda(|r_X|) < \infty$ and $m(k_X \circ \mathrm{sw}) = m(k_X)$ for every $X \in \mathcal{C}_{\mathrm{full}}$.*

Proof. The finiteness follows from Lemma 13.24, Lemma 13.13 and (I3). For the identity, write $X = X_r + iX_i$ with $X_r := (X + JXJ)/2$ and $X_i := (X - JXJ)/(2i)$, which both commute with J, and write a J-real operator W as $W = A + Y$, where $A := (W + W^*)/2$ is real self-adjoint and $Y := (W - W^*)/2$ is real skew-adjoint. All these operators belong to $\mathcal{C}_{\mathrm{full}}$ (Lemma 13.24). For a J-real W the kernel k_W is real, since $k_{JWJ} = \overline{k_W}$, and

$$k_W \circ \mathrm{sw} = \overline{k_{W^*}} = k_{W^*} = k_A - k_Y,$$

so that $m(k_W \circ \mathrm{sw}) = m(k_W) - 2\lambda(r_Y) = m(k_W)$ by Lemma 13.25. Applying this to X_r and to X_i gives the claim. □

Lemma 13.27 (separation, the layer integers v included)**.** *The measure m of Lemma 13.26 satisfies $\mathrm{sw}_* m = m$ on the full S-local slab data.*

Proof. Fix $s \in [0,1)^{\mathrm{plaq}(S)}$. For $X_0 = |g_1\rangle\langle g_2|$, $X_H = |g_1^H\rangle\langle g_2^H|$ and $X = T_s(\sum_a \tau_a^* X_0 \tau_a \otimes X_H)$, Lemmas 13.23 and 13.19 give

$$k_X = e_s(v)\Big[\sum_{a \in c\mathbb{Z}^S} e^{2\pi i s\cdot(d_s a)/c} G_1^s(B-a)\overline{G_2(u-a)}\Big] G_1^H(\Phi)\overline{G_2^H(\widetilde{\Phi}')}.$$

The swapped kernel is $k_X \circ \mathrm{sw} = e_s(\widetilde{v}')e^{2\pi i\sigma\cdot a_0/c}\langle h^u, M_s X h^B\rangle \cdots$, where $a_0 := [u]_{\mathrm{FD}} - u$ is the frame and the dots stand for the Higgs factors. Since $\widetilde{v}' = v - d_s a_0$, the quasi-periodicity phase of M_s (Lemma 13.23) converts $e_s(\widetilde{v}')$ into $e_s(v)$. Thus both sides of the Hermitian identity carry the same character.

Define the twisted unfolding $\hat{m}_s := \sum_{a\in c\mathbb{Z}^S} e^{2\pi i s\cdot(d_s a)/c}(T_a)_*\big(e_s(v)\, m\big)$, where T_a is the diagonal shift of (B,u), which fixes $v, \Phi, \widetilde{\Phi}'$. It is a complex measure on $\mathbb{R}^S \times \mathbb{R}^S \times \mathbb{C}^V \times \mathbb{C}^V$ with $|\hat{m}_s| \le \hat{m} := \sum_a (T_a)_* m$. The measure $\hat{m}$ satisfies the following growth bound for every $\delta \in [\frac{3}{4}, 1)$; such a weight dominates the Gaussian smoothing at rate 1:

$$\hat{m}\big(e^{-\delta(|B|^2+|u|^2+|\Phi|^2+|\widetilde{\Phi}'|^2)}\big) \le C_\delta \lambda\big(e^{|f|^2/4}\big)^{1/2} \lambda\big(e^{|f_H|^2/4}\big)^{1/2} < \infty,$$

by Lemma 13.13 at the joint rate $1/4$, used in the Cauchy–Schwarz form, and by (I3). A joint rate $1/8$ would suffice with this weight.

The identities of Lemma 13.26, written for the unfolded pair kernels, state that the integral $\int G_1^s \otimes \overline{G_2} \otimes G_1^H \otimes \overline{G_2^H}\, d\hat{m}_s$ is equal to the same integral with respect to $(\mathrm{sw}_* m)_s^\wedge$, for all compactly supported g_1, g_2, g_1^H, g_2^H. The linear span of the products is dense in L^2 of every compact set of $(B, u, \Phi, \widetilde{\Phi}')$, so, as in the proof of Lemma 13.17(iii), the Gaussian smoothings of the weighted measures agree. The argument of Lemma 13.17 applies: the measures are finite complex measures after multiplication by the Gaussian weight, their two-sided Laplace transforms are entire, and the restriction to imaginary arguments is the Fourier transform. It gives $\hat{m}_s = (\mathrm{sw}_* m)_s^\wedge$ as complex measures, for every s.

Now restrict this identity to $B \in \mathrm{FD}^S$. The sets $\mathrm{FD}^S + a$ are disjoint, and $\mathrm{sw}_* m$ is supported on $B \in \mathrm{FD}^S$ as m is, so the term $a = 0$ is selected on both sides:

$$\int e_s(v)\, m(\cdot, dv) = \int e_s(v)\, (\mathrm{sw}_* m)(\cdot, dv)$$

on $\mathrm{FD}^S \times \mathbb{R}^S \times \mathbb{C}^V \times \mathbb{C}^V$, for every s. The characters e_s separate $(c\mathbb{Z})^{\mathrm{plaq}(S)}$, and Fourier inversion in s on bounded sets of the slab data, where both measures are finite, recovers the v-marginal. Hence $\mathrm{sw}_* m = m$. □

Theorem 13.28 (exchange theorem, S-local form)**.** *Let ν be as in Theorem 13.1, passed to the phase-restored specification by Lemma 13.3, and let S be a finite spatial bond set closed under the bonds of its plaquettes. Then ν and $\nu \circ \hat{\theta}$, θ the mid-plane reflection of the slab, agree on all bounded gauge-invariant S-local slab functions (Higgs, Villain and bond sectors jointly, temporal plaquettes included).*

Proof. $K \circ \mathrm{sw} = K$ and $\mathrm{sw}_* m = m$ (Lemma 13.27) give $\mathrm{sw}_* \lambda = \lambda$ on the S-local temporal-gauge slab data $\Theta_\sigma|_S = (c|_S, \widetilde{c}'|_S, \widetilde{v}^t|_S)$. For a diagonal-gauge-invariant S-local F, Lemma 13.4(c) gives $F(\Theta_\sigma \theta \omega) = F(\mathrm{sw}\, \Theta_\sigma \omega)$, hence $\nu(F \circ \Theta_\sigma \circ \theta) = \lambda(F \circ \mathrm{sw}) = \lambda(F) = \nu(F \circ \Theta_\sigma)$. The gauge-invariant S-local slab functions are exactly the $F \circ \Theta_\sigma$ with F diagonal-invariant and S-local (Lemma 13.4(b)). □

Remark 13.29. Theorem 13.28 is stated for S-local functions, for every finite S closed under the bonds of its plaquettes; the passage to the laws of the full two-layer data over a box happens inside the proof of Theorem 13.30 (Lemma 13.37), not here. It applies to Gibbs measures of the unperturbed specification (the comparison states, by Proposition 14.22), and not to the limit points ν_t, $t \neq 0$, of the perturbed laws of Section 14 (Lemma 14.21).

13.9. **Two consecutive layers determine the Gibbs measure.** Throughout this subsection $\widetilde{\gamma}$ is the phase-restored specification of Subsection 13.2 on $\varepsilon\mathbb{Z}^3 = (\text{time } \varepsilon\mathbb{Z}) \times (\text{space } \varepsilon\mathbb{Z}^2)$, layer k is the set of sites at time $k\varepsilon$, and $\mathbb{Z}_g := c\mathbb{Z}$. *Convention.* Every Gibbs measure of $\widetilde{\gamma}$ is carried by configurations satisfying all cube constraints ($dv = 0$ on every cube), so the flux identities of

Lemma 13.32 below hold almost surely; on the infinite layer there is no total flux class, and the proofs below use none. The DLR kernels of finite sub-sets Δ of $V(\Lambda)$ are the Fubini conditionals of $\widetilde{\gamma}_\Lambda$ (Lemma 13.3(c)). The statement holds for all Gibbs measures of $\widetilde{\gamma}$: it requires neither temperedness nor moment bounds nor G-invariance, and it does not depend on (I1)–(I3).

Theorem 13.30 (two consecutive layers determine the Gibbs measure)**.** *Let ν, ν' be Gibbs measures of $\widetilde{\gamma}$ on $\varepsilon\mathbb{Z}^3$ and $n_0 \in \mathbb{Z}$. Suppose that for every finite spatial set S (equivalently every rectangle B) ν and ν' agree on every bounded function of the variables of the slab $[n_0, n_0+1]$ over S, namely the Higgs fields $(\tilde{\rho}, \psi)$ of layers n_0, n_0+1 at the sites of S, the spatial links of both layers inside S and the spatial Villain integers of both layers over the set $P(S)$ of the plaquettes with all four corners in S, the temporal links at the sites of S, and the temporal plaquettes with their Villain integers over the links of S, that is invariant under the gauge transformations supported in $S \times \{n_0, n_0+1\}$ (equivalently, under all gauge transformations). Then $\nu = \nu'$.*

Lemma 13.31 (Gaussian convolution is injective on positive measures)**.** *Let $d \geq 1$, $\Omega \subset \mathbb{R}^d$ nonempty open, and M, M' positive Borel measures on $\mathbb{R}^d$ with $(WM)(x) := \int e^{-|x-y|^2/2} M(dy) = (WM')(x) < \infty$ for Lebesgue-a.e. $x \in \Omega$. Then $M = M'$.*

Proof. Put $g := e^{-|y|^2/2}M$, $g' := e^{-|y|^2/2}M'$, so that $WM(x) = e^{-|x|^2/2} L_g(x)$ with $L_g(x) := \int e^{x\cdot y} g(dy)$. (i) By Hölder, $C := \{L_g < \infty\}$ is convex and contains a.e. point of Ω; a point $x_0 \in \Omega \setminus C$ would give, by separation, a nonempty open subset of Ω disjoint from C, so $\Omega \subset C$, likewise $\Omega \subset C'$, and $O := \operatorname{int}(C \cap C')$ is open, convex and contains Ω. (ii) $F(z) := \int e^{z\cdot y}(g - g')(dy)$ is holomorphic on the tube $\{\operatorname{Re} z \in O\}$ (domination of $e^{\operatorname{Re} z\cdot y}$ by the vertex exponentials of a small cube, Morera, Hartogs). (iii) $L_g = L_{g'}$ a.e. on Ω and both are continuous on O, hence equal on Ω; all real partial derivatives of F vanish at a point of Ω, so $F = 0$ on the connected tube. Fix $x_0 \in O$: $\xi \mapsto F(x_0 + i\xi)$ is the Fourier transform of the finite signed measure $e^{x_0\cdot y}(g - g')(dy)$ on $\mathbb{R}^d$, which therefore vanishes; so $g = g'$ and $M = M'$. (The restriction to $x_0 + i\mathbb{R}^d$ with $x_0 \in O$ is the general form; in every use below $0 \in O$.) □

Lemma 13.31 assumes no moment bound beyond the finiteness almost everywhere; the case of any other Gaussian rate follows by scaling, and a positive factor $f(x)f(y)$ in the kernel can be absorbed.

Rectangles and straddling plaquettes. A *rectangle* is a set of sites $\Lambda = \{a_1, \dots, a_1 + L_1\} \times \{a_2, \dots, a_2 + L_2\}$ in $\varepsilon\mathbb{Z}^2$ (in units of ε; $L_i \geq 1$). We write $\mathrm{Bo}(\Lambda)$ for the set of links with both ends in Λ, $P(\Lambda)$ for the set of plaquettes with all four corners in Λ, $\partial\mathrm{Bo}(\Lambda)$ for the set of the $2(L_1 + L_2)$ boundary links of the rectangle, and $\mathrm{Bo}^\circ(\Lambda) := \mathrm{Bo}(\Lambda) \setminus \partial\mathrm{Bo}(\Lambda)$. A plaquette that is not in $P(\Lambda)$ and has a link in $\mathrm{Bo}(\Lambda)$ is called *straddling*. A straddling plaquette has exactly one link in $\mathrm{Bo}(\Lambda)$, and this link is a boundary link; conversely, every boundary link lies in exactly one straddling plaquette (the corner plaquettes have no link in $\mathrm{Bo}(\Lambda)$). The straddling plaquettes are thus in bijection $p' \leftrightarrow b(p')$ with $\partial\mathrm{Bo}(\Lambda)$; we write $p'(b)$ for the straddling plaquette that contains the boundary link b, and we put $P^+(\Lambda) := P(\Lambda) \cup \{\text{straddling plaquettes}\}$. The three other links of a straddling plaquette are its *collar* links. They belong to $\mathrm{Bo}(\Lambda^+)$, where $\Lambda^+ := \{\text{sites at sup-distance} \leq \varepsilon \text{ from } \Lambda\}$ is again a rectangle.

For $w \in \mathbb{Z}_g^{\mathrm{Bo}(\Lambda)}$ the Stokes identity reads

$$\sum_{p \in P(\Lambda)} (d_s w)_p = \operatorname{circ}(w) := \sum_{b \in \partial\mathrm{Bo}(\Lambda)} \epsilon_b w_b,$$

where $\epsilon_b = \pm 1$ is the boundary orientation. A boundary link has a fixed orientation in its straddling plaquette, so the straddling cube constraints and the flux invariant defined below carry fixed signs ± 1, one for each boundary link. In the lemmas below these signs only relabel $\mathbb{Z}_g$-valued coordinates, and we suppress them from here on.

Transported slab data. Let ν and ν' be Gibbs measures of $\widetilde{\gamma}$. For the slab $[m,n]$ put $\lambda_x^{(k)} := -(B_t^{(m)} + \cdots + B_t^{(k-1)})_x$ for $m \le k \le n$, let $\widetilde{c}_k := \lambda^{(k)}.c_k$ be the transported configuration of layer k, and let $\widetilde{v}^{t,(k)}$ be the correspondingly shifted temporal integers; the temporal links of the slab then become 0. Here c_k is the configuration of layer k in the notation of Subsection 13.3, and we write $\widetilde{\Phi}^{(k)}$, $\widetilde{B}^{(k)}$ and $\widetilde{v}^{(k)}$ for the Higgs field of $\widetilde{c}_k$ (in the complex form of Lemma 13.18), its spatial links and its spatial Villain integers. Layer m, on which $\lambda^{(m)} = 0$, is called the *base layer* of these data; the data with another base layer $k \in [m,n]$ are defined in the same way, with the gauge transformation that vanishes on layer k and removes the temporal links of the slab. For a rectangle B let

$$\Theta_{m,n,B} := \big(\widetilde{\Phi}^{(k)}|_B,\ \widetilde{B}^{(k)}|_{\mathrm{Bo}(B)},\ \widetilde{v}^{(k)}|_{P(B)},\ m \le k \le n;\ \ \widetilde{v}^{t,(k)}|_{\mathrm{Bo}(B)},\ m \le k \le n-1\big),$$

and let $H(m,n)$ be the statement that the laws of $\Theta_{m,n,B}$ under ν and ν' coincide for every rectangle B. Every finite spatial set lies in a rectangle, so by Lemma 13.37 below the hypothesis of Theorem 13.30 is $H(n_0, n_0+1)$.

The DLR equation of a rectangle and the patch kernel. Fix $m \le n-1$ and a rectangle Λ. Let Δ be the following set of variables: on layer n, the Higgs variables $\{\tilde{\rho}_x, \psi_x : x \in \Lambda\}$, the links $\{B_b : b \in \mathrm{Bo}(\Lambda)\}$ and the Villain integers $\{v_p : p \in P^+(\Lambda)\}$; and the temporal integers $\{v_b^{t,(n-1)}, v_b^{t,(n)} : b \in \mathrm{Bo}(\Lambda)\}$ of both adjacent slabs over $\mathrm{Bo}(\Lambda)$, boundary links included. The temporal links do not belong to Δ; they stay exterior.

We collect the transported coordinates in three groups, which are denoted by v, Y and u until the end of this subsection. The first group consists of the layer-n coordinates over Λ,

$$v := (\widetilde{\Phi}^{(n)}|_\Lambda, \widetilde{B}^{(n)}|_{\mathrm{Bo}(\Lambda)}, \widetilde{v}^{(n)}|_{P^+(\Lambda)}).$$

The second group, Y, consists of all coordinates of Θ_{m,n,Λ^+} other than those of v and other than $\widetilde{v}^{t,(n-1)}|_{\mathrm{Bo}(\Lambda)}$. The third group is

$$u := (\widetilde{\Phi}^{(n+1)}|_\Lambda, \widetilde{B}^{(n+1)}|_{\mathrm{Bo}(\Lambda)}, \widetilde{v}^{(n+1)}|_{P^+(\Lambda)}, \widetilde{v}^{t,(n)}|_{\text{collar links}}).$$

Every coordinate of Y and of u is a function of exterior variables, that is, of variables that do not belong to Δ. Indeed, the coordinates of the layers $m,\dots,n-1$ and of the slabs $m,\dots,n-2$ over Λ^+ involve only B_t and layer-k data with $k < n$. The layer-n coordinates outside Λ are $\widetilde{\Phi}$ at the sites of $\Lambda^+ \setminus \Lambda$, $\widetilde{B}_b$ for $b \notin \mathrm{Bo}(\Lambda)$ and $\widetilde{v}_p$ for $p \notin P^+(\Lambda)$, whose FD-shifts involve only links not in $\mathrm{Bo}(\Lambda)$. The slab-$(n-1)$ integers in Y are those on links not in $\mathrm{Bo}(\Lambda)$. Finally, u involves layer $n+1$ and the collar links only. This is the reason for including the straddling plaquettes in Δ. Since (Y,v) is a sub-family of Θ_{m,n,Λ^+}, the statement $H(m,n)$, applied to the rectangle Λ^+, implies that the law of (Y,v) is the same under ν and ν'.

The factors of the weight W that contain a variable of Δ are the following: the factors $g_{\mathrm{ct}}(\tilde{\rho}_x)$, $x \in \Lambda$; the spatial factors h on $\mathrm{Bo}(\Lambda)$ and on the links with one end in Λ; the spatial factors w on $P(\Lambda)$ and on the straddling plaquettes; the temporal factors h at the sites $x \in \Lambda$ and the temporal factors w on $\mathrm{Bo}(\Lambda)$, for both slabs; and the cube constraints on $P^+(\Lambda)$, for both slabs. The integrand of $\widetilde{\gamma}_\Delta$ is gauge invariant, and for a fixed exterior configuration the gauge action is a measure-preserving bijection of the space of the variables of Δ (Lemma 13.3(a)). We apply the gauge transformation $\lambda = (\lambda^{(k)})_k$ and transport the variables $(c_n|_\Lambda, v^t|_{\mathrm{Bo}(\Lambda)})$. For bounded functions A, a, f this gives

$$\widetilde{\gamma}_\Delta\big(A(Y)a(v)f(u)\big)(\omega) = A(Y)f(u)\,Z(Y,u)^{-1}\int a(v')\,c(v';Y)\,K_\Lambda(v',u)\,\lambda_\Lambda(dv'), \tag{13.31}$$

$$\lambda_\Lambda := \mathrm{Leb}(\mathbb{C}^\Lambda) \times \mathrm{Haar}(\mathbb{T}^{\mathrm{Bo}(\Lambda)}) \times \mathrm{counting}(\mathbb{Z}_g^{P^+(\Lambda)}),$$

with $G(z) := e^{-|z|^2/2}$, $c(v';Y) := W_\Lambda(v';Y)K_\Lambda^{\mathrm{low}}(Y,v')$ and

$$K_\Lambda^{\mathrm{low}}(Y,v') := \prod_{x\in\Lambda} G(\Phi'_x - \widetilde{\Phi}_x^{(n-1)}) \sum_{v^t\text{: lower cube constraints on } P^+(\Lambda)} \prod_{b\in \mathrm{Bo}(\Lambda)} w(B'_b - \widetilde{B}_b^{(n-1)} + v_b^t),$$

$$K_\Lambda(v',u) := \prod_{x\in\Lambda} G(\widetilde{\Phi}_x^{(n+1)} - \Phi'_x) \sum_{v^t\text{: upper cube constraints on } P^+(\Lambda)} \prod_{b\in \mathrm{Bo}(\Lambda)} w(\widetilde{B}_b^{(n+1)} - B'_b + v_b^t),$$

where the integration variable is written $v' = (\Phi', B', (v'_p)_{p\in P^+(\Lambda)})$ and $Z(Y,u)$ denotes the integral on the right side of (13.31) with $a=1$. The function $W_\Lambda(v';Y)$ is the product of the factors g_{ct} on Λ and of the spatial factors h and w that contain a variable of Δ, written in transported variables. The exterior layer-n data that enter the factors at the boundary links and plaquettes are coordinates of Y, and the site Jacobian $\prod_x(\gamma_N\rho'_x)^{-1}$ of Lemma 13.18 is absorbed into W_Λ. We call K_Λ the *patch kernel*.

The upper cube constraints, over which the sum in K_Λ runs, read as follows. On $p\in P(\Lambda)$ the constraint is $(d_s v^t)_p = \widetilde{v}_p^{(n+1)} - v'_p$. On a straddling plaquette p' with boundary link $b = b(p')$ it is $v_b^t = \gamma_{p'}(u) - v'_{p'}$, where

$$\gamma_{p'}(u) := \widetilde{v}_{p'}^{(n+1)} - (\text{signed sum of the three collar integers } \widetilde{v}^{t,(n)} \text{ of } u \text{ at } p')$$

is defined by this exterior expression in u. Hence K_Λ depends on u only through

$$\bar{u} := (\widetilde{\Phi}^{(n+1)}|_\Lambda, \widetilde{B}^{(n+1)}|_{\mathrm{Bo}(\Lambda)}, \widetilde{v}^{(n+1)}|_{P(\Lambda)}, \gamma) \in U_\Lambda := \mathbb{C}^\Lambda \times \mathbb{T}^{\mathrm{Bo}(\Lambda)} \times \mathbb{Z}_g^{P(\Lambda)} \times \mathbb{Z}_g^{\partial\mathrm{Bo}(\Lambda)},$$

and so does Z. We write $K_\Lambda(v',\bar{u})$ and $Z(Y,\bar{u})$ accordingly.

Since ν and ν' are Gibbs measures, integrating (13.31) against $\rho\in\{\nu,\nu'\}$ gives

$$\begin{aligned} \rho\big(A(Y)a(v)f(\bar{u})\big) &= \int A(Y)f(\bar{u})\Psi_a(Y,\bar{u})\,Q_\rho(dY\,d\bar{u}), \\ \Psi_a &:= Z^{-1}\int a(v')c(v';Y)K_\Lambda(v',\bar{u})\lambda_\Lambda(dv'), \end{aligned} \tag{13.32}$$

where Q_ρ is the law of $(Y,\bar{u})$ under ρ; under $H(m,n)$ we write Q^Y for the common law of Y under ν and ν'. The function Z is finite everywhere: one has $h,w,G\le 1$, each coset sum is bounded by a product of periodised Gaussians, the sum over v'_{P^+} of the spatial w-factors converges, and $\int g_{\mathrm{ct}} G\, d^2\Phi < \infty$ by (2.1). Moreover Z lies in $(0,\infty)$ ρ-almost surely. The condition $m\le n-1$ ensures that $\widetilde{c}_{n-1}|_\Lambda$ belongs to Y.

Lemma 13.32 (relative flux on a rectangle)**.** *Let $\beta\in\mathbb{Z}_g^{\partial\mathrm{Bo}(\Lambda)}$ and $f\in\mathbb{Z}_g^{P(\Lambda)}$. The set $\Gamma_\Lambda(f;\beta) := \{w\in\mathbb{Z}_g^{\mathrm{Bo}(\Lambda)} :\ w|_{\partial\mathrm{Bo}}=\beta,\ d_s w = f \text{ on } P(\Lambda)\}$ is nonempty if and only if $\sum_{p\in P(\Lambda)} f_p = \mathrm{circ}(\beta)$, and then it is a coset of the countable subgroup $\Gamma_\Lambda(0;0)$.*

Proof. Necessity follows from the Stokes identity $\sum_{p\in P(\Lambda)}(d_s w)_p = \mathrm{circ}(w)$. Sufficiency: let $w_0 := \beta$ on $\partial\mathrm{Bo}$, 0 on Bo°, and $g := f - d_s w_0$, so $\sum_p g_p = 0$. List $P(\Lambda)$ along a boustrophedon path $p_1,\dots,p_M$; consecutive plaquettes share an interior link b_j lying in no other plaquette of $P(\Lambda)$ and entering $(d_s w)_{p_j}$, $(d_s w)_{p_{j+1}}$ with opposite signs. Choose w on b_1 so that $(d_s w)_{p_1} = g_{p_1}$, which changes the requirement at p_2 to $g_{p_2}+g_{p_1}$; continue; at p_M the requirement is $\sum_p g_p = 0$, met with the remaining interior links 0. Then $w_0 + w \in \Gamma_\Lambda(f;\beta)$. The coset statement follows from the linearity of the two conditions. □

Flux invariant. Let $V^+ := \mathbb{Z}_g^{P^+(\Lambda)}$ be the space of the integer components $(v'_p)_{p\in P^+(\Lambda)}$ of the layer-n variable v'; we write v' also for these components. For $v' \in V^+$ put

$$F(v') := \sum_{p\in P(\Lambda)} v'_p - \mathrm{circ}\big((v'_{p'(b)})_{b\in\partial\mathrm{Bo}}\big),$$

where the integers of the straddling plaquettes are read as boundary data, and for $\bar u$ put

$$F(\bar u) := \sum_{p\in P(\Lambda)} \widetilde v_p^{(n+1)} - \mathrm{circ}(\gamma).$$

We call F the *flux invariant*. The sum over the upper cube constraints in $K_\Lambda(v', \bar u)$ runs over a set of the form considered in Lemma 13.32, whose boundary values are $\gamma_{p'} - v'_{p'}$ and whose f is $\widetilde v^{(n+1)} - v'$ on $P(\Lambda)$; we call it the upper coset. By Lemma 13.32 the upper coset is nonempty if and only if $F(\bar u) = F(v')$. In the same way the lower coset, over which the sum in $K_\Lambda^{\mathrm{low}}(Y, v')$ runs, is nonempty if and only if $F(v') = F(Y)$, where $F(Y)$ is the analogous expression in the layer-$(n-1)$ and slab-$(n-1)$ integers; it is a function of Y. Every Gibbs measure of $\widetilde\gamma$ is carried by the configurations that satisfy all cube constraints (the convention stated at the beginning of this subsection), so under a Gibbs measure both constraints hold almost surely, and $F(v) = F(\bar u) = F(Y)$ almost surely. Finally, $c(v'; Y) > 0$ if and only if $F(v') = F(Y)$ and $\rho'_x > 0$ for all x, where the second condition holds λ_Λ-almost everywhere; indeed all factors h, w, g_{ct} are > 0 there.

For the bond variables let

$$\mathfrak{B}_\Lambda := \mathbb{T}^{\mathrm{Bo}(\Lambda)} \times \mathbb{Z}_g^{P(\Lambda)} \times \mathbb{Z}_g^{\partial\mathrm{Bo}(\Lambda)},$$

with elements $(\widetilde B, \widetilde v, \gamma)$ and with the measure $\lambda_{\mathfrak B} := \mathrm{Haar}\times\mathrm{counting}$, and for $(B', v') \in \mathbb{T}^{\mathrm{Bo}(\Lambda)}\times V^+$ put

$$K_{\mathrm{bond},\Lambda}\big((B', v'), (\widetilde B, \widetilde v, \gamma)\big) := \sum_{w\in\Gamma_\Lambda(\widetilde v - v'|_{P(\Lambda)};\, \gamma - v'|_{\partial\mathrm{Bo}})} \prod_{b\in\mathrm{Bo}(\Lambda)} e^{-(\widetilde B_b - B'_b + w_b)^2/2}$$

where the link variables are taken as their FD representatives. Then $K_\Lambda(v', \bar u) = G(\widetilde\Phi - \Phi')K_{\mathrm{bond},\Lambda}$.

Lemma 13.33 (the bond sector of the patch kernel)**.** *Fix $F_0 \in \mathbb{Z}_g$ and let μ, μ' be positive Borel measures on $\mathfrak{B}_\Lambda$ carried by the class $C(F_0) := \{F(\bar u) = F_0\}$. Suppose that for one $v' \in V^+$ with $F(v') = F_0$ and for Haar-almost every $B' \in \mathbb{T}^{\mathrm{Bo}(\Lambda)}$,*

$$(K_{\mathrm{bond},\Lambda}\mu)(B', v') = (K_{\mathrm{bond},\Lambda}\mu')(B', v') < \infty.$$

Then $\mu = \mu'$.

Proof. Let v' be as in the statement and put

$$D := \{(\widetilde B, \widetilde v, \gamma, w) :\ w \in \Gamma_\Lambda(\widetilde v - v'; \gamma - v'_{\partial\mathrm{Bo}})\}.$$

We first show that the map $T(\widetilde B, \widetilde v, \gamma, w) := \widetilde B + w \in \mathbb{R}^{\mathrm{Bo}(\Lambda)}$ is injective on D. Indeed, if $U = T(\widetilde B, \widetilde v, \gamma, w)$, then $\widetilde B = [U]_{\mathrm{FD}}$ and $w = U - [U]_{\mathrm{FD}}$ on all of $\mathrm{Bo}(\Lambda)$; the boundary values of w then give $\gamma_{p'(b)} = w_b + v'_{p'(b)}$, and $\widetilde v = v' + d_s w$ on $P(\Lambda)$. So U determines the point of D.

Next let $M := T_*(\mu \otimes \mathrm{counting}_D)$, where $\mu \otimes \mathrm{counting}_D$ is the measure on D obtained by integrating against μ the counting measures of the cosets, and define M' from μ' in the same way. By Tonelli's theorem,

$$(K_{\mathrm{bond},\Lambda}\mu)(B', v') = \int \prod_b e^{-(U_b - B'_b)^2/2} M(dU) = (WM)(B')$$

on the open box $\mathrm{int}(\mathrm{FD})^{\mathrm{Bo}(\Lambda)}$, in the notation of Lemma 13.31, and the same holds for μ' and M'. By the hypothesis, $WM = WM' < \infty$ almost everywhere on this open box, so Lemma 13.31 gives $M = M'$.

It remains to recover μ from M. The pairs $(\widetilde{v}, \gamma)$ that occur in $C(F_0)$ form a countable set, and for each of them the coset $\Gamma_\Lambda(\widetilde{v} - v'; \gamma - v'_{\partial\mathrm{Bo}})$ is nonempty by Lemma 13.32, because $F(v') = F_0$; we choose an element $\mathrm{gam}(\widetilde{v}, \gamma)$ of it. For a Borel set $A \subset C(F_0)$ put

$$A_{\mathrm{gam}} := \{(\widetilde{B}, \widetilde{v}, \gamma, \mathrm{gam}(\widetilde{v}, \gamma)) : (\widetilde{B}, \widetilde{v}, \gamma) \in A\}.$$

Then $(\mu \otimes \mathrm{counting})(A_{\mathrm{gam}}) = \mu(A)$. The set $T(A_{\mathrm{gam}})$ is Borel, because $T|_D$ is a Borel injection between Polish spaces (the Lusin–Souslin theorem [23]), and $T^{-1}(T(A_{\mathrm{gam}})) = A_{\mathrm{gam}}$. Therefore

$$\mu(A) = M(T(A_{\mathrm{gam}})) = M'(T(A_{\mathrm{gam}})) = \mu'(A).$$

Since μ and μ' are carried by $C(F_0)$, this gives $\mu = \mu'$. One v' suffices in the hypothesis because the carrier of μ and μ' is a single class. □

Lemma 13.34 (local gauge invariance)**.** *For a gauge transformation Λ_G supported on a finite set of sites, $\nu(F \circ \Lambda_G) = \nu(F)$ for every Gibbs measure ν of $\widetilde{\gamma}$ and bounded F.*

Proof. Λ_G moves only the variables of the finite set $V(\mathrm{supp})$ (Higgs phases at the sites, adjacent links, Villain integers of adjacent plaquettes), where it acts as a measure-preserving bijection at fixed exterior with invariant weight and preserved closure (Lemma 13.3(a)); so $\widetilde{\gamma}_{V(\mathrm{supp})}(F \circ \Lambda_G) = \widetilde{\gamma}_{V(\mathrm{supp})}(F)$ pointwise, and one integrates against ν. □

Lemma 13.35 (injectivity of the patch kernel on positive measures)**.** *Let $F_0 \in \mathbb{Z}_g$ and let m, m' be positive Borel measures on $U_\Lambda = \mathbb{C}^\Lambda \times \mathfrak{B}_\Lambda$ carried by $\{F(\bar{u}) = F_0\}$ with $(K_\Lambda m)(v') = (K_\Lambda m')(v') < \infty$ for λ_Λ-a.e. $v' \in \{F(v') = F_0\}$. Then $m = m'$.*

Proof. (i) Higgs sector. For $b' = (B', v')$ put $m_{b'}(d\widetilde{\Phi}) := \int K_{\mathrm{bond},\Lambda}(b', \widetilde{b})\, m(d\widetilde{\Phi}\, d\widetilde{b})$, so that $(K_\Lambda m)(\Phi', b') = (G * m_{b'})(\Phi')$ (Tonelli). By Fubini on λ_Λ restricted to $\{F(v') = F_0\}$, for $\lambda_{\mathfrak{B}}$-a.e. such b' (outside a null set N_1) the hypothesis holds for Lebesgue-a.e. Φ'; Lemma 13.31 with $\Omega = \mathbb{R}^{2|\Lambda|}$ gives $m_{b'} = m'_{b'}$, and $G * m_{b'}$ is finite everywhere because Ω is the whole space (for a general open Ω Lemma 13.31 gives finiteness only on the convex hull O), so $m_{b'}$ is finite on bounded sets.

(ii) Bond sector. For bounded Borel $E \subset \mathbb{C}^\Lambda$ let $\mu_E(d\widetilde{b}) := m(E \times d\widetilde{b})$, carried by $C(F_0)$, with $(K_{\mathrm{bond},\Lambda}\mu_E)(b') = m_{b'}(E) = m'_{b'}(E) < \infty$ for $b' \notin N_1$. Pick $v'_0 \in V^+$ with $F(v'_0) = F_0$ (e.g. F_0 on one plaquette of $P(\Lambda)$, 0 elsewhere); N_1 is $\lambda_{\mathfrak{B}}$-null, so for Haar-a.e. B' the pair (B', v'_0) is outside N_1, and Lemma 13.33 gives $\mu_E = \mu'_E$.

(iii) σ-finiteness and π-system. The pairs $(\widetilde{v}, \gamma)$ that occur in $C(F_0)$ form a countable set; let $(\widetilde{v}_j, \gamma_j)_j$ be an enumeration of it. For each j the coset $\Gamma_\Lambda(\widetilde{v}_j - v'_0; \gamma_j - v'_{0,\partial\mathrm{Bo}})$ is nonempty (Lemma 13.32), so $K_{\mathrm{bond},\Lambda}((B', v'_0), (\widetilde{B}, \widetilde{v}_j, \gamma_j))$ is a positive continuous function of $(B', \widetilde{B})$ on the compact $\overline{\mathrm{FD}}^{\mathrm{Bo}(\Lambda)} \times \overline{\mathrm{FD}}^{\mathrm{Bo}(\Lambda)}$ (continuity on the closed box of FD-representatives, where the Gaussian sum converges uniformly; not on the torus, where the representative jumps), bounded below by $c_j > 0$. Hence $\mu_E(\mathbb{T}^{\mathrm{Bo}(\Lambda)} \times \{(\widetilde{v}_j, \gamma_j)\}) \le (K_{\mathrm{bond},\Lambda}\mu_E)(B', v'_0)/c_j < \infty$ for a B' outside N_1. The sets $\mathrm{ball}(0,k) \times \mathbb{T}^{\mathrm{Bo}(\Lambda)} \times \{(\widetilde{v}_j, \gamma_j) : j \le k\}$ exhaust the carrier with finite m-mass, and m, m' agree on the π-system of products $E \times F$; the π-λ theorem gives $m = m'$. □

Lemma 13.36 (upward step)**.** *$H(m, n)$ implies $H(m, n+1)$ for every $m \le n - 1$.*

Proof. Assume $H(m, n)$. Fix a rectangle Λ and use the notation of (13.31)–(13.32). We show that the laws of $\Theta_{m,n+1,\Lambda}$ under ν and ν' coincide.

Step 1: disintegration. The space U_Λ and the space of Y are Polish. Let $q_\rho(Y, d\bar{u})$ be a regular conditional law of $\bar{u}$ given Y under Q_ρ, and put $m^Y_\rho := Z(Y, \cdot)^{-1} q_\rho(Y, \cdot)$. For Q^Y-almost every Y

this is a positive σ-finite measure carried by $\{F(\bar u) = F(Y)\}$. Indeed, $Z < \infty$ everywhere, and by Lemma 13.32 one has $Z > 0$ if and only if $F(\bar u) = F(Y)$, an equality that holds almost surely.

Step 2: the conditional laws of $\bar u$ given Y coincide. Taking $f = 1$ in (13.32) and using Tonelli's theorem, we obtain

$$\rho(A(Y)a(v)) = \int Q^Y(dY)A(Y) \int a(v')c(v';Y)(K_\Lambda m_\rho^Y)(v')\lambda_\Lambda(dv'),$$

and $\int c\,(K_\Lambda m_\rho^Y)\,d\lambda_\Lambda = \int ZZ^{-1}q_\rho(Y, d\bar u) = 1$. So $\pi_\rho^Y := c(\cdot;Y)(K_\Lambda m_\rho^Y)\lambda_\Lambda$ is a probability measure on the space of v', which is locally compact and second countable. The left side of the display depends only on the law of (Y, v). This law is common to ν and ν' by $H(m, n)$ on Λ^+, the coordinates of Y and u being functions of exterior variables. Applying the display to a countable dense set of functions a in C_c and using the Riesz representation theorem, we get $\pi_\nu^Y = \pi_{\nu'}^Y$ for Y outside a Q^Y-null set. Fix such a Y. Then

$$c(v';Y)(K_\Lambda m_\nu^Y)(v') = c(v';Y)(K_\Lambda m_{\nu'}^Y)(v') < \infty$$

for λ_Λ-almost every v'. On the set where $c > 0$, that is, where $F(v') = F(Y)$ and $\rho' > 0$, this is the hypothesis of Lemma 13.35 with $F_0 = F(Y)$; both measures m_ν^Y and $m_{\nu'}^Y$ are carried by $\{F = F(Y)\}$. On the set where $F(v') \neq F(Y)$ the coset is empty, so that $K_\Lambda(v', \bar u) = 0$ for every $\bar u$ in the carrier, and there is nothing to verify. Lemma 13.35 gives $m_\nu^Y = m_{\nu'}^Y$. Hence $q_\nu(Y, \cdot) = q_{\nu'}(Y, \cdot)$ for Q^Y-almost every Y, and therefore $Q_\nu = Q_{\nu'}$.

Step 3: the joint law of $(Y, v, \bar u)$. By (13.32) and $Q_\nu = Q_{\nu'}$, the number $\rho(A(Y)a(v)f(\bar u))$ is the same for $\rho = \nu$ and $\rho = \nu'$, for all bounded Borel A, a, f. Since the product sets form a π-system, the laws of $(Y, v, \bar u)$ under ν and ν' coincide.

Step 4: the temporal integers over $\mathrm{Bo}(\Lambda)$. For $b \in \partial\mathrm{Bo}(\Lambda)$ with straddling plaquette $p'(b)$ the upper cube constraint reads $\widetilde v_b^{t,(n)} = \gamma_{p'(b)} - \widetilde v_{p'(b)}^{(n)}$, which is a function of $(v, \bar u)$.

Let $\Delta_2 := \{v_b^{t,(n-1)}, v_b^{t,(n)} : b \in \mathrm{Bo}(\Lambda)\}$. The factors of the weight that contain a variable of Δ_2 are the temporal factors w on $\mathrm{Bo}(\Lambda)$ of the two slabs and the cube constraints on $P^+(\Lambda)$ of the two slabs. Every other variable is exterior to Δ_2, and in particular so is all of $(Y, v, \bar u)$: the layer-n links are now exterior, so that the transported straddling integers of layer n are functions of exterior variables, and γ is the exterior expression fixed above. After the gauge transport, which is a bijection of the integer sets, we have for bounded g

$$\widetilde\gamma_{\Delta_2}(g)(\omega) = \mathcal N^{-1} \sum_{w_1 \in \Gamma_\Lambda(\widetilde v^{(n)} - \widetilde v^{(n-1)};\, \widetilde v_{p'}^{(n)} - \widetilde v_{p'}^{(n-1)} - \mathrm{collar}^{(n-1)})} \ \sum_{w_2 \in \Gamma_\Lambda(\widetilde v^{(n+1)} - \widetilde v^{(n)};\, \gamma - \widetilde v_{p'}^{(n)})}$$
$$g(w_1, w_2) \prod_b w(\widetilde B_b^{(n)} - \widetilde B_b^{(n-1)} + w_{1,b})\, w(\widetilde B_b^{(n+1)} - \widetilde B_b^{(n)} + w_{2,b}) =: G^\sharp[g](Y, v, \bar u),$$

with $\mathcal N \in (0, \infty)$ almost surely, because the cosets are nonempty by Lemma 13.32 and the convention stated at the beginning of this subsection, and the sums are bounded by periodised Gaussians. Here $\mathrm{collar}^{(n-1)}$ stands for the signed sum of the three collar integers $\widetilde v^{t,(n-1)}$ at p'. Writing $\widetilde\Delta_2 := (\widetilde v^{t,(n-1)}|_{\mathrm{Bo}(\Lambda)}, \widetilde v^{t,(n)}|_{\mathrm{Bo}(\Lambda)})$ for the transported variables of Δ_2, the DLR equation gives

$$\rho(A(Y, v, \bar u)g(\widetilde\Delta_2)) = \rho(A(Y, v, \bar u)G^\sharp[g](Y, v, \bar u)),$$

and by Step 3 the right side is the same for ν and ν'. Hence the laws of $(Y, v, \bar u, \widetilde v^{t,(n-1)}|_{\mathrm{Bo}(\Lambda)}, \widetilde v^{t,(n)}|_{\mathrm{Bo}(\Lambda)})$ under ν and ν' coincide.

Step 5: conclusion. The family $\Theta_{m,n+1,\Lambda}$ is a sub-family of these coordinates. The layers $m, \dots, n$ on Λ come from (Y, v), because $P(\Lambda) \subset P^+(\Lambda)$ and $\Lambda \subset \Lambda^+$. The slabs $m, \dots, n-2$ on $\mathrm{Bo}(\Lambda)$ come from Y, and the slab $n-1$ on $\mathrm{Bo}(\Lambda)$ comes from Step 4. The layer $n+1$, that is, $\widetilde\Phi$

on Λ, $\widetilde{B}$ on $\mathrm{Bo}(\Lambda)$ and $\widetilde{v}$ on $P(\Lambda)$, comes from $\bar{u}$, and the slab n on $\mathrm{Bo}(\Lambda)$ comes from Step 4. So the laws of $\Theta_{m,n+1,\Lambda}$ under ν and ν' coincide. Since Λ was arbitrary, this proves $H(m,n+1)$. □

Lemma 13.37 (transported slab data and gauge-invariant slab functions)**.** *Let ν,ν' be Gibbs measures of $\widetilde{\gamma}$, let B be a rectangle and let $m<n$. The following are equivalent:*

(i) *the laws of $\Theta_{m,n,B}$ (base layer m) under ν and ν' coincide;*

(ii) *$\nu=\nu'$ on every bounded function of the slab-$[m,n]$ variables over B (sites of B, links of $\mathrm{Bo}(B)$, plaquettes of $P(B)$, temporal links at the sites of B and temporal plaquettes over $\mathrm{Bo}(B)$ between layers m and n) that is invariant under all gauge transformations supported in $B\times[m,n]$;*

(iii) *the statement (i) with any other base layer $k\in[m,n]$.*

Moreover, if $\nu=\nu'$ on all bounded gauge-invariant local functions, then $\nu=\nu'$.

Proof. We first determine how the slab data transform. Under any gauge transformation G, the rule $B_t^{(k)}\mapsto B_t^{(k)}+G^{(k+1)}-G^{(k)}$ gives $\lambda^{(k)}\mapsto\lambda^{(k)}-G^{(k)}+G^{(m)}$, so that $\widetilde{c}_k\mapsto G^{(m)}.\widetilde{c}_k$ for every k. Thus $\Theta=\Theta_{m,n,B}$ transforms by the diagonal action of $G^{(m)}|_B$; here we use that $\mathbb{T}$ is abelian, and the integers shift consistently.

(ii) implies (i). For a bounded Borel function H let

$$\bar{H}(\omega):=\int_{\mathbb{T}^B}H(g.\Theta(\omega))\,\mathrm{Haar}(dg).$$

By the invariance of the Haar measure, $\bar{H}$ is a function of the slab variables over B that is invariant under every gauge transformation. For $g\in\mathbb{T}^B$ let G_g be the gauge transformation that equals g on layer m over B and 0 elsewhere. It is finitely supported and acts on Θ as the diagonal action of g, so Lemma 13.34 gives $\nu(H(\Theta))=\nu(H(g.\Theta))$ for every g, and the same for ν'. Integrating over g and using (ii) for $\bar{H}$, we obtain

$$\nu(H(\Theta))=\nu(\bar{H})=\nu'(\bar{H})=\nu'(H(\Theta)).$$

(i) implies (ii). Let F be as in (ii). For $\lambda=(\lambda_x^{(k)})_{x\in B,\,m\le k\le n}$ let G_λ be the gauge transformation that equals λ on the slab over B and 0 elsewhere. For each fixed value of λ it is finitely supported and F is invariant under it; therefore $F(\omega)=F(G_{\lambda(\omega)}.\omega)$ pointwise. The configuration $G_{\lambda(\omega)}.\omega$ has temporal links 0 on the slab over B and slab data $\Theta(\omega)$. So $F=F^\sharp(\Theta_{m,n,B})$ with

$$F^\sharp(\theta):=F(\text{the configuration with slab data }\theta\text{ and temporal links }0).$$

Hence (i) gives $\nu(F)=\nu'(F)$.

(i) and (iii) are equivalent. Condition (ii) does not mention the base layer, and the two implications just proved hold, with the same proof, for every base layer $k\in[m,n]$.

The last assertion. Let F be a bounded local function, and let $\bar{F}$ be its Haar average over the finitely many sites whose links and plaquettes F depends on. Then $\bar{F}$ is local and gauge invariant, and $\nu(F)=\nu(\bar{F})$ by Lemma 13.34. So if ν and ν' agree on the bounded gauge-invariant local functions, then they agree on all bounded local functions, and hence $\nu=\nu'$. □

Lemma 13.38 (downward step)**.** *$H(m,n)$ implies $H(m-1,n)$ for every $m\le n-1$.*

Proof. Let $\theta:(x,k)\mapsto(x,-k-1)$ be the time reflection in a mid-plane; it is a lattice automorphism of $\varepsilon\mathbb{Z}^3$. The weight is invariant under θ factor by factor: the factors g_{ct} and the spatial factors h, w are relabelled; the temporal factor h satisfies $h(r,r',\vartheta)=h(r',r,-\vartheta)$; the temporal factor w is even; and the maps $B_t\mapsto-B_t$, $v^t\mapsto-v^t$ preserve Leb × Haar × counting and the closure constraint. Consequently $\widetilde{\gamma}_\Delta(F\circ\theta)=(\widetilde{\gamma}_{\theta\Delta}F)\circ\theta$ for every finite Δ, the measure $\theta_*\nu$ is a Gibbs measure, and θ conjugates finitely supported gauge transformations to finitely supported gauge transformations.

The reflection θ maps the slab $[m,n]$ over B onto the slab $[-n-1,-m-1]$ over B and preserves the class of functions in Lemma 13.37(ii). Since condition (ii) of that lemma does not involve the base layer, the equivalence of (i) and (ii) shows that $H(m,n)$ for (ν,ν') is equivalent to $H(-n-1,-m-1)$ for $(\theta_*\nu,\theta_*\nu')$. Since $-n-1\le(-m-1)-1$, Lemma 13.36 for the reflected pair gives $H(-n-1,-m)$, and reflecting back gives $H(m-1,n)$. □

Proof of Theorem 13.30. A function of the slab-$[m,n]$ variables over a rectangle B sees a gauge transformation G only through G at the sites of $B\times[m,n]$, so for such functions invariance under the gauge transformations supported in $B\times[m,n]$ is invariance under all gauge transformations; Lemma 13.37 (ii)⇒(i) turns the hypothesis into $H(n_0,n_0+1)$. Lemma 13.36 gives $H(n_0,n)$ for all $n>n_0$ and Lemma 13.38 gives $H(m,n)$ for all $m\le n_0<n$. Every bounded local F depends on finitely many variables, all contained in the slab $[m,n]$ over some rectangle B; if F is gauge invariant, Lemma 13.37 (i)⇒(ii) with $H(m,n)$ gives $\nu(F)=\nu'(F)$; by the last claim of Lemma 13.37, $\nu=\nu'$. □

13.10. **Conclusion: proof of Theorem 13.1.** The proof uses the following elementary statement.

Proposition 13.39 (covariance; conjugation and composition of reflections)**.** (i) *(Covariance.) The fine Villain–Higgs specification, in unitary gauge and phase-restored alike, is covariant under the lattice isometries $g\in\varepsilon\mathbb{Z}^3\rtimes B_3$ and the mid-plane reflections $\theta_{\mu,c}$, $c\in\varepsilon(\mathbb{Z}+\frac12)$: $\gamma_{g^{-1}\Delta}(f\circ\hat g)=(\gamma_\Delta f)\circ\hat g$. Hence $\nu\circ\hat g$ is Gibbs whenever ν is, and Γ-invariance of ν becomes $g\Gamma$-invariance of $\nu\circ\hat g$.*

(ii) *(Conjugation.) Let ν be a Gibbs measure such that ν and $\nu\circ\hat g$, for $g\in B_3$ the transpositions of the coordinates μ and 3 ($\mu=1,2$), are each invariant under every $\theta_{3,c}$, $c\in\varepsilon(\mathbb{Z}+\frac12)$. Then ν is invariant under every $\theta_{\mu,c}$, $\mu=1,2,3$.*

(iii) *(Composition.) $\theta_{\mu,c}\circ\theta_{\mu,c'}$ maps x_μ to $x_\mu+2(c-c')$ and fixes the other coordinates; with $c-c'=\varepsilon$ it is $\tau_{2\varepsilon e_\mu}$. Hence a measure invariant under all $\theta_{\mu,c}$ is invariant under $\tau_{\pm2\varepsilon e_\mu}$, $\mu=1,2,3$, i.e. under $2\varepsilon\mathbb{Z}^3$.*

Proof. (i) $h(r,r',\vartheta)$ is symmetric in (r,r') and even in ϑ, so with $B_{-b}=-B_b$ the hopping factor is a function of the unoriented bond; w is even and $v_{-p}=-v_p$, so the plaquette factor is a function of the unoriented plaquette; g_{ct} is a site function; the constraint $dv=0$ is mapped to itself by any lattice isometry (cubes to cubes, orientation signs cancel); the reference measures are invariant under the relabelling-with-sign bijection. Hence $Z_{g^{-1}\Delta}(\hat g\xi)=Z_\Delta(\xi)$ and the covariance. The mid-plane reflections map sites to sites (no site on the plane) and bonds to bonds. For the phase-restored specification the phase field is a site field and h is even in ϑ. (ii) $g\theta_{\mu,c}g=\theta_{3,c}$. The $\theta_{3,c}$-invariance of $\nu\circ\hat g$ reads $\nu(f\circ\hat\theta_{3,c}\circ\hat g)=\nu(f\circ\hat g)$ for all bounded f; replacing f by $f\circ\hat g$ and using $\hat g\hat\theta_{3,c}\hat g=\hat\theta_{\mu,c}$ gives $\nu(f\circ\hat\theta_{\mu,c})=\nu(f)$. (iii) is the computation $x_\mu\mapsto 2c-(2c'-x_\mu)$ and $\widehat{g_1g_2}=\hat g_1\hat g_2$. □

Proof of Theorem 13.1. Step 0. Let ν be as in the statement. By Lemma 13.3(d), $\widetilde\nu:=\Psi_*(\nu\times\mathrm{Haar}^{\mathrm{sites}})$ is a G-invariant Gibbs measure of $\widetilde\gamma$ with the same law of $(\tilde\rho,f)$ as ν, so (I1)–(I3) hold for $\widetilde\nu$; by Lemma 13.3(e) $\widetilde\nu$ is Γ-invariant, and ν is τ_a-invariant if and only if $\widetilde\nu$ is. We work with $\widetilde\nu$, called ν.

Step 1 (time reflections). Let $c\in\varepsilon(\mathbb{Z}+\frac12)$, $\theta:=\theta_{3,c}$ and $\nu':=\nu\circ\hat\theta$, a Gibbs measure of $\widetilde\gamma$ by Proposition 13.39(i). Let n_0 be the layer below the plane, so that θ is the mid-plane reflection of the slab $[n_0,n_0+1]$. Let B be a rectangle and $S:=\mathrm{Bo}(B)$ the set of its links (a bond set closed under the bonds of its plaquettes $P(B)$), and let f be a bounded function of the slab-$[n_0,n_0+1]$ variables over B invariant under the gauge transformations supported in $B\times\{n_0,n_0+1\}$, equivalently under all gauge transformations. Such an f is a bounded

gauge-invariant S-local slab function in the sense of Lemma 13.4(b) ($f = F \circ \Theta_\sigma|_S$ with F diagonal-invariant, $\widetilde{v}^t$ included), so Theorem 13.28 (which uses (I1)–(I3) and the Γ-invariance of ν, including a time translation, through Lemma 13.9) gives $\nu(f \circ \hat{\theta}) = \nu(f)$, i.e. $\nu'(f) = \nu(f)$, for every rectangle B. This is the hypothesis of Theorem 13.30, in its form for rectangles (the passage from S-local agreement to the laws of $\Theta_{n_0,n_0+1,B}$ is made in the proof of that theorem, by Lemma 13.37). Theorem 13.30 gives $\nu' = \nu$: ν is $\theta_{3,c}$-invariant for every c.

Step 2 (the other directions). Let $\mu \in \{1,2\}$ and $g \in B_3$ the transposition of the coordinates μ and 3. By Proposition 13.39(i), $\nu \circ \hat{g}$ is a Gibbs measure invariant under $g\Gamma$, which contains a time translation because a finite-index sublattice Γ of $\varepsilon\mathbb{Z}^3$ contains $k\varepsilon\mathbb{Z}^3$, $k = [\varepsilon\mathbb{Z}^3 : \Gamma]$, hence translations along every axis; (I1) and (I3) are invariant under relabelling, and (I2) is stated for all plaquettes, so it holds for $\nu \circ \hat{g}$ (whose temporal plaquettes are ν's plaquettes that contain a link in the direction e_μ). By Lemma 13.3(e), $\nu \circ \hat{g}$ is the phase restoration of the unitary-gauge measure $(\Phi_*\nu) \circ \hat{g}$, which is a Gibbs measure by Proposition 13.39(i), is invariant under $g\Gamma$ and satisfies (I1)–(I3) for the same reasons. So Step 1 applies to $\nu \circ \hat{g}$ and gives its $\theta_{3,c}$-invariance, and Proposition 13.39(ii) gives the invariance of ν under every $\theta_{\mu,c}$, $\mu = 1,2,3$.

Step 3 (translations). By Proposition 13.39(iii), ν is invariant under $\tau_{2\varepsilon e_\mu}$, $\mu = 1,2,3$. (The argument uses only reflections in mid-planes of links, so it does not give the invariance under the translations by ε.)

Step 4. By Step 0 and Lemma 13.3(e) the unitary-gauge measure ν is $\tau_{2\varepsilon e_j}$-invariant, $j = 1,2,3$. □

Remark 13.40 (where each hypothesis enters). The hypotheses of Theorem 13.1 enter its proof as follows. The Gibbs property is used in Theorem 13.28, by way of Lemma 13.9 and the averaging identity through the torus states, and in Theorem 13.30, by way of the DLR equation (13.31) (Lemma 13.36) and Lemma 13.34; in both it is used for ν and for $\nu \circ \hat{g}$. The invariance under a sublattice of finite index is used only in Lemma 13.9, which is applied in Lemmas 13.14 and 13.25 and compares the measure with the Γ'-periodic torus states; it is used for ν and for $\nu \circ \hat{g}$.

Hypothesis (I1), through $\sup_x \nu(\rho_x^2)$, and hypothesis (I2) in its joint form enter the entropy bound of Lemma 13.9; the joint form provides one constant for all finitely supported s, and this bounds the second moments of the fluxes through spheres of growing size. Hypothesis (I2), jointly over all plaquettes, also gives the exponential moments of f, which are used in Lemma 13.16 and for $\nu(r_Y^2)$ in Lemmas 13.14 and 13.25, and its Gaussian moments at every rate $< \frac{1}{2}$. The latter are used at the rate $\frac{1}{4}$ for the growth of $\hat{m}$ in Lemma 13.17 and of $\hat{m}_s$ in Lemma 13.27, and for the tail bound $\nu(A_t^c) \le C|t|^{16a/3}$, $a \in (\frac{3}{8}, \frac{1}{2})$, in Lemma 13.25. The proof of Lemma 13.25 requires a rate $> \frac{3}{8}$, which the constant $\frac{1}{2}$ in the exponent of (I2) provides; this requirement is a feature of that proof, as explained at its end. Hypothesis (I3) gives the Gaussian moments of f_H at every rate, which are used in Lemma 13.20, for the tails in Lemma 13.25 and for the growth in Lemma 13.27; it also gives $\nu(\rho_x^2)$ for Lemma 13.9.

Theorem 13.30 and Subsection 13.2 use none of (I1)–(I3). The proof of Theorem 13.1 uses nothing beyond (I1)–(I3), the Gibbs property and the Γ-invariance; in particular, it uses no temperedness, uniqueness or clustering. The theorem applies to Gibbs measures of the unperturbed specification: in this paper it is applied to the comparison states of Section 14 (Lemma 14.20, Proposition 14.22; compare Remark 13.2). It does not apply to the limit points ν_t, $t \neq 0$, of the perturbed laws (Lemma 14.21).

14. Diagonal reflection positivity of the torus limit points

This section proves the reflection positivity at the diagonal planes for the limit states ω of Definition 2.3, taken along the dyadic sequences of sides introduced in Definition 14.1. The proof runs through an adapted torus with a family of reflection walls, a perturbed twisted law on it

which is reflection positive across every wall, a fold identity and a convexity inequality for the perturbed free energy, a surface bound which removes the perturbation in the limit, and the rigidity theorem of the preceding section, Theorem 13.1, applied to a sourceless comparison state; as in that section, $\kappa_N = 1$ (Lemma 2.7(g) extends Theorem 14.3 and Corollary 14.35 to every $\kappa_N > 0$).

14.1. **Dyadic sequences of sides.**

Definition 14.1 (dyadic sequences of sides)**.** For each level N fix two lengths $L_{\mathrm{IV}} > 0$ and $L_{\min} > 0$ with L_{IV}/a_0 rational, and call a side L *admissible* if $L \in 4L_{\mathrm{IV}}\mathbb{N}$, $L/2 \in a_0\mathbb{Z}$ and $L \geq L_{\min}$. Fix an admissible side L_*. The *dyadic sequence* generated by L_* is the sequence of sides

$$L_n := 2^n L_*, \qquad n \geq 0,$$

and the associated *wall spacings* are

$$h_j := 2^j \frac{L_*}{2}, \qquad j \geq 0, \qquad \text{so that } h_j \text{ divides } \tfrac{L_n}{2} \text{ for every } n \geq j.$$

A *dyadic limit point* is a weak-$*$ limit point of the torus states $(\langle\cdot\rangle^{\mathrm{fine}}_{L_n})_{n\geq 0}$ of Definition 2.3 along a dyadic sequence.

Remark 14.2. Admissible sides exist: writing $L_{\mathrm{IV}}/a_0 = p/q$ with integers $p, q \geq 1$, the side $L = 4qkL_{\mathrm{IV}} = 4pka_0$ is admissible for every integer $k \geq L_{\min}/(4pa_0)$. Every L_n is admissible ($4L_{\mathrm{IV}}\mathbb{N}$ is closed under multiplication by 2, $L_n/2 = 2^{n-1}L_* \in a_0\mathbb{Z}$ for $n \geq 1$, while $L_0/2 = L_*/2 \in a_0\mathbb{Z}$ by assumption, and $L_n \geq L_* \geq L_{\min}$). Definition 2.3 fixes a sequence of sides with $L_n/2 \in \varepsilon_N\mathbb{Z}$ and a limit law along a subsequence of it, any cofinal prescription of sides being allowed (Theorem 11.5, Remark 11.6(ii)); here that prescribed sequence is *chosen* to be a dyadic sequence, which is a choice and not a hypothesis. The divisibility $h_j \mid L_n/2$ ($n \geq j$) survives the passage to any subsequence, so every dyadic limit point keeps it; and (a)–(e) of Theorem 11.5 hold for it along that subsequence (its "Moreover" clause).

Theorem 14.3 (diagonal reflection positivity through the origin)**.** *Let ω be a dyadic limit point. Let P be one of the six diagonal site planes through* 0*,*

$$P = P^{\sigma}_{ij} := \{x \in \mathbb{R}^3 : x_i = \sigma x_j\}, \qquad 1 \leq i < j \leq 3, \quad \sigma \in \{+1, -1\},$$

let θ_P be the reflection in P and $\overline{H}_P$ one of the two closed half spaces bounded by P. Then for every bounded Borel function F of the gauge-invariant variables $(\mathfrak{f}_p, \tilde{\rho}_x)$ of finitely many cells (plaquettes, sites) all of whose vertices lie in $\overline{H}_P$,

$$\omega\big(\overline{F \circ \theta_P} \cdot F\big) \;\geq\; 0,$$

θ_P acting on cells and fields as $R_{j,c}$ does in Theorem 11.5(d) (so $\overline{F \circ \theta_P}$ is the antilinear reflection $\Theta_P F$ of that theorem, now at a diagonal plane).

Theorem 14.3 is proved in Subsection 14.7.

Remark 14.4 (the planes and the observables). The proof of Theorem 12.4 uses, at $c = 0$ (Remark 14.36), the reflection positivity of ω_N at the six diagonal site-plane families $\{x_i \pm x_j = c\}$, $c \in \varepsilon_N\mathbb{Z}$, in the sense of Theorem 11.5(d). Theorem 14.3 is the case $c = 0$ with bounded Borel observables, real or complex, of finitely many cells; Corollary 14.35 in Subsection 14.7 extends it to all $c \in \varepsilon_N\mathbb{Z}$ (by translation invariance of the limit points), to the Hermitian form and to the L^2 closure, which is the form Theorem 11.5(d) uses. Remark 14.36 states the consequence for Theorems 12.2, 12.3 and 12.4. Throughout, θ_P acts on a configuration by pulling back the fields along the reflection in P (a site plane, so that the cells of P are mapped to themselves), and on a function F of the configuration by composition, $\theta_P F := F \circ \theta_P$.

14.2. **The adapted torus and its reflection walls.** *Notation.* Throughout N is fixed, ε_N is the spacing of the fine lattice $\varepsilon_N\mathbb{Z}^3$, the cells are its sites x, bonds b and plaquettes p, the fields are $(\tilde\rho_x, B_b, v_p)$ of Section 2 with $\rho_x = \rho_{0,0} + \tilde\rho_x$ and $\mathfrak{f}_p = (dB)_p + v_p$, and the weight is the local weight W of Section 2 (v closed on every cube). We use the coordinates $u := x_1 - x_2$, $v := x_1 + x_2$ (the symbol v for this coordinate is always written with its argument, $v(x)$, where confusion with the Villain variable is possible) and x_3. For a plane Π, θ_Π is the reflection of cells and fields in Π (the action of Theorem 11.5(d)); a function "of cells in a closed half space" depends on the variables of finitely many cells all of whose vertices lie in it; $\mathcal{A}_+(\Pi)$ denotes the bounded continuous such functions for the closed half space on the + side of Π. The cubic torus $T_L^{\mathrm{cub}} := \varepsilon_N\mathbb{Z}^3/L\mathbb{Z}^3$ is the fine torus $\tilde T_L$ of Section 2; we write $\langle\cdot\rangle_L^{\mathrm{cub}}$ for its state $\langle\cdot\rangle_L^{\mathrm{fine}}$ and Z_L^{cub} for its partition function.

Definition 14.5 (the adapted torus)**.** For L with $L/2 \in \varepsilon_N\mathbb{Z}$ let $\Gamma(L) := \frac{L}{2}(e_1 - e_2)\mathbb{Z} + L(e_1 + e_2)\mathbb{Z} + Le_3\mathbb{Z} \subset \varepsilon_N\mathbb{Z}^3$ and $T_L^{\mathrm{tw}} := \varepsilon_N\mathbb{Z}^3/\Gamma(L)$. Since $|\det\Gamma(L)| = L^3$, T_L^{tw} has $(L/\varepsilon_N)^3$ sites and three times as many bonds and plaquettes. Its state $\langle\cdot\rangle_L^{\mathrm{tw}}$ is the probability law with weight W on the cells of T_L^{tw} (sum over closed v), with partition function Z_L^{tw}; μ_L^{tw} denotes this law.

Lemma 14.6 (symmetries and walls of the adapted torus)**.** *For $c \in \varepsilon_N\mathbb{Z}$ put $r_c(x) := (x_2{+}c,\ x_1{-}c,\ x_3)$ and $s_c(x) := (c - x_2,\ c - x_1,\ x_3)$, and for $c \in \frac12\varepsilon_N\mathbb{Z}$ put $q_c(x) := (x_1,\ x_2,\ 2c - x_3)$. Each maps $\varepsilon_N\mathbb{Z}^3$ onto itself, its linear part fixes $\Gamma(L)$, and it descends to an involutive automorphism of the cell complex T_L^{tw} with $W \circ R = W$ cell by cell and an invariant reference measure. The pointwise fixed walls are: $\{u = c\}$ and $\{u = c + L/2\}$ for r_c; $\{v(x) = c\}$ and $\{v(x) = c + L\}$ for s_c; $\{x_3 = c\}$ and $\{x_3 = c + L/2\}$ for q_c. On T_L^{tw} the coordinates u, $v(x)$, x_3 are defined modulo L, $2L$, L, and $r_{c+L/2} = r_c$. The translations by $\varepsilon_N\mathbb{Z}^3$ act on T_L^{tw} and preserve W. Moreover $r_cr_{c'}$, $s_cs_{c'}$, $q_cq_{c'}$ are the translations by $(c - c')(e_1 - e_2)$, $(c - c')(e_1 + e_2)$, $2(c - c')e_3$.*

Proof. The linear part of r_c is the coordinate swap $(1\,2)$, which sends $\frac{L}{2}(e_1 - e_2)$ to its negative and fixes $e_1 + e_2$, e_3; that of s_c is $-(1\,2)$, which fixes $e_1 - e_2$ and negates $e_1 + e_2$; that of q_c is $\mathrm{diag}(1, 1, -1)$. So each fixes $\Gamma(L)$ and descends to the quotient complex. For the fixed points, $r_cx - x = (c - u)(e_1 - e_2)$ vanishes for $u = c$ and equals $-\frac{L}{2}(e_1 - e_2) \in \Gamma(L)$ for $u = c + L/2$; $s_cx - x = (c - v(x))(e_1 + e_2)$ is 0 for $v(x) = c$ and $-L(e_1 + e_2) \in \Gamma(L)$ for $v(x) = c + L$; $q_cx - x = 2(c - x_3)e_3$ is 0 for $x_3 = c$ and $-Le_3 \in \Gamma(L)$ for $x_3 = c + L/2$. The products are $r_cr_{c'}x = x + (c - c')(e_1 - e_2)$, $s_cs_{c'}x = x + (c - c')(e_1 + e_2)$, $q_cq_{c'}x = x + 2(c - c')e_3$; with $c' = c + L/2$ the first is the translation by $-\frac{L}{2}(e_1 - e_2) \in \Gamma(L)$, the identity of T_L^{tw}, so $r_{c+L/2} = r_c$. Invariance of W is Step 0 of the proof of Theorem 12.16, which uses only that g_{ct} is one function of the site variable, h is symmetric in (ρ, ρ') and even in B, w is even, $\mathfrak{f}_p \circ R = \mathfrak{f}_{Rp}$ and the closedness constraint is R-invariant; the reference measure is invariant, because R acts on the variables by a permutation of the coordinates with sign changes of some B_b, v_p. That both walls of each reflection are pointwise fixed is Lemma 12.13(a) for the lattice $\Gamma(L)$. □

Lemma 14.7 (reflection positivity of the adapted torus at its walls)**.** *Let $L \ge 6\varepsilon_N$, $L/2 \in \varepsilon_N\mathbb{Z}$, and let R be one of r_c ($c \in \varepsilon_N\mathbb{Z}$), s_c ($c \in \varepsilon_N\mathbb{Z}$), q_c ($c \in \frac12\varepsilon_N\mathbb{Z}$), with its two walls Π, Π' of Lemma 14.6 and the two closed halves of T_L^{tw} they bound ($\{c \le u \le c + L/2\}$ and $\{c + L/2 \le u \le c + L\}$ for r_c; likewise in $v(x)$ with L in place of $L/2$, and in x_3). Then for all bounded Borel F, G of cells in one closed half,*

$$\langle F \circ R\rangle_L^{\mathrm{tw}} = \langle F\rangle_L^{\mathrm{tw}}, \qquad \langle\overline{F \circ R}\cdot G\rangle_L^{\mathrm{tw}} = \overline{\langle\overline{G \circ R}\cdot F\rangle_L^{\mathrm{tw}}}, \qquad \langle\overline{F \circ R}\cdot F\rangle_L^{\mathrm{tw}} \ge 0.$$

Proof. For $L \ge 6\varepsilon_N$ no cell meets both walls of R: the u-, $v(x)$- and x_3-ranges of a cell are at most $2\varepsilon_N$, $2\varepsilon_N$, ε_N, while the walls are $L/2$, L, $L/2$ apart. The proof of Theorem 12.16 uses of its box only: (i) that R is an automorphism of the complex fixing its wall(s) pointwise, exchanging the two open halves, with $W \circ R = W$ cell by cell and an invariant reference measure (Step 0;

here Lemma 14.6); (ii) the local classification of Lemma 12.14 of the cells by the sign pattern of $\varphi = u - c$, $v(x) - c$ or $x_3 - c$ on their vertices ($|\varphi| \le 2\varepsilon_N$ on a cell), which is local and holds at every wall of Lemma 14.6; (iii) the splitting Lemma 12.15 and the Poisson and Bessel identities of Steps 1–4 of the proof of Theorem 3.1, again local. With "the two closed halves" as in the statement, Steps 0–4 of the proof of Theorem 12.16 run on T_L^{tw} unchanged and give the three relations. At the diagonal walls $\{u = c\}$, $\{v(x) = c\}$ no bond crosses and the crossing kernel is of positive type by the Poisson step alone; at $\{x_3 = c\}$ with $c \in \varepsilon_N\mathbb{Z} + \varepsilon_N/2$ the Bessel factors of the crossing bonds are added, as in Remark 12.17. □

Remark 14.8 (two walls, two crossing kernels). On the torus each reflection R has two walls, Π and Π', hence two families of crossing cells. Steps 2–3 of the proof of Theorem 12.16 give one kernel of positive type per family, $K_0(\eta,\xi) = \int d\nu(k)\, \overline{A_k(\eta)} A_k(\xi)$ at Π and $K'(\eta',\xi') = \int d\nu'(k')\, \overline{A'_{k'}(\eta')} A'_{k'}(\xi')$ at Π' ($|A_k| = |A'_{k'}| = 1$ at the diagonal walls), and the full crossing weight is their *product*, of positive type as the explicit integral $\int d\nu\, d\nu'\, \overline{(A_k A'_{k'})}(A_k A'_{k'})$ (the Schur product of two kernels of positive type). This is the form in which the fold of Proposition 14.13 writes the weight.

Lemma 14.9 (uniform moments and limit points of the adapted torus states)**.** *(a) Theorem 3.1 holds on T_L^{tw} for every $L \ge 8\varepsilon_N$ with $L/2 \in \varepsilon_N\mathbb{Z}$: $\langle e^{\sum_p a_p \mathfrak{f}_p}\rangle_L^{\mathrm{tw}} \le e^{\frac12 \sum_p a_p^2}$ for every real plaquette function a. (b) With the constants G_q^{ct}, C_1 of Lemma 11.2, for every $q \ge 0$, every site x and every $L \ge 6\varepsilon_N$ with $L/2 \in \varepsilon_N\mathbb{Z}$,*

$$\langle e^{q\rho_x^2}\rangle_L^{\mathrm{tw}} \le C'_N(q) := (G_q^{\mathrm{ct}}/G_0^{\mathrm{ct}})\, C_1^{16}.$$

(c) The family $(\langle\cdot\rangle_L^{\mathrm{tw}})_L$ has weak limit points ω^{tw} on Ω_N and on the full configuration space $E := [-\rho_{0,0},\infty)^{\mathcal{X}_N} \times (-\pi/\gamma_N, \pi/\gamma_N]^{\mathcal{B}_N} \times ((2\pi/\gamma_N)\mathbb{Z})^{\mathcal{P}_N}$ of $\varepsilon_N\mathbb{Z}^3$; every limit point is invariant under the reflections of Lemma 14.6 and reflection positive at every plane $\{u = c\}$, $\{v(x) = c\}$ ($c \in \varepsilon_N\mathbb{Z}$) and $\{x_3 = c\}$ ($c \in \frac12\varepsilon_N\mathbb{Z}$), in the sense of Theorem 11.5(d).

Proof. (a) is Lemma 12.10, since T_L^{tw} is a finite cubical complex with $\partial\partial = 0$ carrying the weight W, the product of the local factors of Section 2 over its cells with v closed on its cubes.

(b) *Step 1 of the proof of Lemma 11.2 does not depend on the cell complex.* With $Z(A,q) := \int d\tilde\rho \int dB \sum_{v \text{ closed}} W \prod_{y\in A} e^{q\rho_y^2}$ on T_L^{tw} and $|\mathcal{B}| = |\mathcal{P}| = 3|\mathcal{X}|$ (Definition 14.5), the same crude bounds give

$$Z(A,q) \le (G_q^{\mathrm{ct}})^{|A|} (G_0^{\mathrm{ct}})^{|\mathcal{X}|-|A|} (2\pi/\gamma_N)^{3|\mathcal{X}|} K'^{\,3|\mathcal{X}|}_{\gamma_N} \qquad \text{and} \qquad Z \ge c_0^{|\mathcal{X}|},$$

hence $Z(A,q)/Z \le (G_q^{\mathrm{ct}}/G_0^{\mathrm{ct}})^{|A|} C_1^{|\mathcal{X}|}$.

Doubling. Fix x and q, $F := e^{q\rho_x^2}$. For $n = (n_1,n_2,n_3)$, $n_i = 2^{k_i}$, let $B(n) := \{x + \varepsilon_N m_1(e_1 - e_2) + \varepsilon_N m_2(e_1 + e_2) + \varepsilon_N m_3 e_3 : 0 \le m_i < n_i\}$ (sites with u, $v(x)$, x_3 in arithmetic progressions of spacings $2\varepsilon_N$, $2\varepsilon_N$, ε_N) and $\Pi_n := \prod_{y\in B(n)} e^{q\rho_y^2}$. We claim $(\langle F\rangle_L^{\mathrm{tw}})^{|B(n)|} \le \langle \Pi_n\rangle_L^{\mathrm{tw}}$ whenever the extents $2\varepsilon_N(n_1 - 1)$, $2\varepsilon_N(n_2 - 1)$, $\varepsilon_N(n_3 - 1)$ are less than L, $2L$, L (so that $B(n)$ consists of distinct sites of T_L^{tw}). For $n = (1,1,1)$ it is an identity. Suppose it holds for n and double n_1: the wall $\{u = u(x) - \varepsilon_N\}$ of $r_{u(x)-\varepsilon_N}$ passes through no site of $B(n)$, and $B(n)$ lies in the closed half $\{u(x) - \varepsilon_N \le u \le u(x) - \varepsilon_N + L/2\}$ as soon as $2\varepsilon_N n_1 \le L/2$ (the condition under which we double n_1); its reflection is the adjacent set with $u \in u(x) - 2\varepsilon_N\{1,\dots,n_1\}$, disjoint from $B(n)$, and the union is a translate (by $-\varepsilon_N n_1(e_1 - e_2) \in \varepsilon_N\mathbb{Z}^3$) of $B(n')$, n' the doubled triple. Lemma 14.7 applied to the bounded $\Pi_n \wedge M$ gives

$$(\langle \Pi_n \wedge M\rangle_L^{\mathrm{tw}})^2 \le \langle (\Pi_n \wedge M) \circ R \cdot (\Pi_n \wedge M)\rangle_L^{\mathrm{tw}} \le \langle (\Pi_n \circ R)\, \Pi_n\rangle_L^{\mathrm{tw}} = \langle \Pi_{n'}\rangle_L^{\mathrm{tw}}$$

by the Schwarz inequality for the form of Lemma 14.7 with $G = 1$, the reflection invariance and then the translation invariance of Lemma 14.6; then let $M \to \infty$, by monotone convergence.

Doubling n_2 across the wall of $s_{v(x)-\varepsilon_N}$ (the plane where the coordinate v equals $v(x) - \varepsilon_N$; closed half of v-length L, condition $2\varepsilon_N n_2 \le L$) and n_3 across the wall of $q_{x_3(x)-\varepsilon_N/2}$ (closed half of length $L/2$, condition $\varepsilon_N n_3 \le L/2$) is the same. Doubling each n_i as long as its condition holds ends with $n_1 > L/(4\varepsilon_N)$, $n_2 > L/(2\varepsilon_N)$, $n_3 > L/(2\varepsilon_N)$, and the extents are still below the periods ($n_1 \le L/(2\varepsilon_N)$, $n_2 \le L/\varepsilon_N$, $n_3 \le L/\varepsilon_N$), so the final $A := B(n)$ has $|A| > L^3/(16\varepsilon_N^3) = |\mathcal{X}|/16$ distinct sites. Then

$$\langle F\rangle_L^{\mathrm{tw}} \le (Z(A,q)/Z)^{1/|A|} \le (G_q^{\mathrm{ct}}/G_0^{\mathrm{ct}})\, C_1^{|\mathcal{X}|/|A|} \le C'_N(q).$$

(c) With (a) and (b) in place of Theorem 3.1 and Lemma 11.2, the tightness, diagonal extraction and Kolmogorov steps of the proof of Theorem 11.5 apply verbatim to the laws of the variables of finite sets of cells under $\langle\cdot\rangle_L^{\mathrm{tw}}$, on Ω_N and on E. For the laws on E, the variable B_b ranges over a bounded interval and $v_p = \mathfrak{f}_p - (dB)_p$ is tight because $\mathfrak{f}_p$ is; the limits are taken with B_b in the closed interval $[-\pi/\gamma_N, \pi/\gamma_N]$. Under every limit point the end point $-\pi/\gamma_N$ has measure zero, by the identity of Lemma 14.21(i) for a Δ containing b, whose kernel integrates B_b against Lebesgue measure. It suffices to have that identity for bounded continuous local functions on the closed space, and for these the first step of the proof of that lemma, which at $t = 0$ involves no datum and applies along every sequence of sides, gives it without using that the limit points are carried by E, because the factors of the kernel are continuous functions of the bond variables on the closed interval. Hence the limit points are carried by E, and the convergence holds for the bounded functions that are continuous off this null set.

Each reflection R of Lemma 14.6 satisfies $\langle F \circ R\rangle_L^{\mathrm{tw}} = \langle F\rangle_L^{\mathrm{tw}}$ for bounded continuous local F and all large L (Lemma 14.7); both sides pass to the limit, so ω^{tw} is R-invariant. For bounded continuous F of cells in the closed half space, $\overline{F \circ R}\cdot F$ is bounded continuous local, so $\omega^{\mathrm{tw}}(\overline{F \circ R}\cdot F) = \lim\langle \overline{F \circ R}\cdot F\rangle_L^{\mathrm{tw}} \ge 0$; such F are dense in L^2 of the bounded Borel functions of those cells (Lusin, the marginal being a finite Borel measure on a Polish product), and $F \mapsto \omega^{\mathrm{tw}}(\overline{F \circ R}\cdot F)$ is L^2-continuous because $F \mapsto F \circ R$ is an isometry of $L^2(\omega^{\mathrm{tw}})$ by R-invariance. This is reflection positivity in the sense of Theorem 11.5(d). □

14.3. **The perturbed twisted law: invariance, reflection positivity, fold and convexity.**

Definition 14.10 (datum, wall group, perturbed law)**.** Let $L/2 \in \varepsilon_N\mathbb{Z}$, $L \ge 6\varepsilon_N$. A *datum* is $D = (c_0, c_+, c_3; h; O)$ with $c_0, c_+ \in \varepsilon_N\mathbb{Z}$, $c_3 \in \frac{1}{2}\varepsilon_N\mathbb{Z}$, $h \in \varepsilon_N\mathbb{Z}$ with $h \mid L/2$, and O a real bounded continuous function of the variables of finitely many cells (sites, bonds, plaquettes) all of whose vertices lie in the open cell $C_0 := \{c_0 < u < c_0 + h,\ c_+ < v(x) < c_+ + h,\ c_3 < x_3 < c_3 + h\}$. The *walls* of D are the planes $\{u = c_0 + jh\}$, $\{v(x) = c_+ + jh\}$, $\{x_3 = c_3 + jh\}$, $j \in \mathbb{Z}$; since $h \mid L$ and $h \mid 2L$ they close up on T_L^{tw}. $\mathcal{W}$ is the group generated by the reflections r_{c_0+jh}, s_{c_++jh}, q_{c_3+jh} of Lemma 14.6, a finite group of automorphisms of T_L^{tw}. The *$\mathcal{W}$-cells* are the components of the complement of the walls, open boxes of side h in $(u, v(x), x_3)$; there are $n_c := (L/h)(2L/h)(L/h) = 2L^3/h^3$ of them, and C_0 is one. Put $O_c := O \circ w_c^{-1}$ for a $\mathcal{W}$-cell c, where $w_c \in \mathcal{W}$ is the element with $w_c C_0 = c$, and $O_d := \sum_c O_c$. For real t let

$$Z_t := \langle e^{tO_d}\rangle_L^{\mathrm{tw}} \in \left[e^{-|t|n_c\|O\|_\infty}, e^{|t|n_c\|O\|_\infty}\right], \qquad \mu_L^t := Z_t^{-1} e^{tO_d}\mu_L^{\mathrm{tw}}, \qquad \omega_L^t(F) := \mu_L^t(F).$$

Lemma 14.11 (the wall group acts simply transitively on the cells)**.** *$\mathcal{W}$ permutes the walls of D and the $\mathcal{W}$-cells, and acts simply transitively on the $\mathcal{W}$-cells; so w_c is well defined, $|O_d| \le n_c\|O\|_\infty$, and $O_d \circ w = O_d$ for every $w \in \mathcal{W}$.*

Proof. By Lemma 14.6 the products $r_c r_{c'}$, $s_c s_{c'}$, $q_c q_{c'}$ of two generators of one family are the translations by $(c-c')(e_1-e_2)$, $(c-c')(e_1+e_2)$, $2(c-c')e_3$, which shift u, $v(x)$, x_3 by $2(c-c') \in 2h\mathbb{Z}$

and fix the other two coordinates; each generator maps the wall family it belongs to onto itself and fixes the walls of the other two families setwise. So $\mathcal{W}$ permutes the walls, hence the cells. In each coordinate the action is that of the dihedral group generated by the reflections in the points of $h\mathbb{Z}$ on the circle of length L (resp. $2L$, L), with L/h (resp. $2L/h$, L/h) even because $h \mid L/2$: its translations are the shifts by $2h\mathbb{Z}$ and its reflections have their fixed points in $h\mathbb{Z}$, so no element other than the identity maps an open cell of length h onto itself, and the action on the cells of that circle is simply transitive. The product action on the $\mathcal{W}$-cells is then simply transitive too. Finally $O_{wc} = O \circ w_{wc}^{-1} = O \circ (ww_c)^{-1} = O_c \circ w^{-1}$ gives $O_d \circ w^{-1} = \sum_c O_{wc} = O_d$. □

Lemma 14.12 ($\mathcal{W}$-invariance and reflection positivity of the perturbed law, every real t)**.** *Let D be a datum and $t \in \mathbb{R}$. (i) $\mu_L^t(F \circ w) = \mu_L^t(F)$ for every $w \in \mathcal{W}$ and bounded Borel F; in particular $\omega_L^t(O_c) = \omega_L^t(O)$ for every $\mathcal{W}$-cell c. (ii) Let Π be a wall of D, $\theta_\Pi \in \mathcal{W}$ the reflection in it, and Π' its second wall (Lemma 14.6), also a wall of D since $h \mid L/2$. For every bounded Borel G of cells in one of the two closed halves of T_L^{tw} bounded by Π and Π',*

$$\mu_L^t\big(\overline{G \circ \theta_\Pi} \cdot G\big) \geq 0, \qquad \mu_L^t(G \circ \theta_\Pi) = \mu_L^t(G).$$

Proof. (i) We have

$$Z_t\, \mu_L^t(F \circ w) = \mu_L^{\mathrm{tw}}\big((F \circ w)e^{tO_d}\big) = \mu_L^{\mathrm{tw}}\big((Fe^{tO_d}) \circ w\big) = \mu_L^{\mathrm{tw}}(Fe^{tO_d}) = Z_t\, \mu_L^t(F),$$

by $O_d \circ w = O_d$ (Lemma 14.11) and the $\mathcal{W}$-invariance of μ_L^{tw} (Lemmas 14.6 and 14.7; $\mathcal{W}$ is generated by the reflections). Then $\omega_L^t(O_c) = \mu_L^t(O \circ w_c^{-1}) = \omega_L^t(O)$. (ii) θ_Π exchanges the two open halves bounded by Π and Π' and maps the $\mathcal{W}$-cells of one onto those of the other (a $\mathcal{W}$-cell meets no wall, so it lies in one open half). Hence $O_d = O_+^\Pi + O_+^\Pi \circ \theta_\Pi$ with $O_+^\Pi := \sum_{c \subset H_+} O_c$, the sum over the $\mathcal{W}$-cells of the open half H_+ containing the cells of G: a real bounded function of cells inside H_+. Therefore $e^{tO_d} = e^{tO_+^\Pi} \cdot \big(e^{tO_+^\Pi} \circ \theta_\Pi\big)$ and, with $G' := Ge^{tO_+^\Pi}$, a bounded Borel function of cells in the closed half,

$$\mu_L^t\big(\overline{G \circ \theta_\Pi} \cdot G\big) = Z_t^{-1} \mu_L^{\mathrm{tw}}\big(\overline{G' \circ \theta_\Pi} \cdot G'\big) \geq 0$$

by Lemma 14.7. The invariance is (i). □

Proposition 14.13 (the fold: comparison of the two tori of the same side)**.** *Let D be a datum with $c_0 = 0$ and $L \geq 6\varepsilon_N$, $h \mid L/2$. Define O_d on the cubic torus T_L^{cub} by the same formula $O_d := O_+ + O_+ \circ r_0$, $O_+ := \sum_{c \subset \{0<u<L/2\}} O_c$ (both tori contain the strip $0 \leq u \leq L/2$ with its $\mathcal{W}$-cells and r_0 as a symmetry). Then for every real t*

$$\langle e^{tO_d} \rangle_L^{\mathrm{cub}} \;\leq\; e^{\phi_L}\, Z_t, \qquad \phi_L := \log \frac{Z_L^{\mathrm{tw}}}{Z_L^{\mathrm{cub}}} \;\geq\; 0.$$

Proof. Two quotients of one cylinder. Let $\Gamma_\parallel := L(e_1 + e_2)\mathbb{Z} + Le_3\mathbb{Z} = \Gamma(L) \cap \{u = 0\} = L\mathbb{Z}^3 \cap \{u = 0\}$ (a vector (a, a, b) lies in $L\mathbb{Z}^3$ if and only if $a, b \in L\mathbb{Z}$) and $\mathcal{C} := \varepsilon_N \mathbb{Z}^3/\Gamma_\parallel$, a cylinder on which $u \in \varepsilon_N\mathbb{Z}$ is a global coordinate. Then $T_L^{\mathrm{tw}} = \mathcal{C}/\langle \tau_2 \rangle$ with τ_2 the translation by $\frac{L}{2}(e_1 - e_2)$, and $T_L^{\mathrm{cub}} = \mathcal{C}/\langle \tau_1 \rangle$ with τ_1 the translation by Le_1; both shift u by L. On $\mathcal{C}$ one checks $r_{L/2}\tau_2 = r_0$ and $r_{L/2}\tau_1 = Yr_0$, where Y is the translation by $\frac{L}{2}(e_1 + e_2)$: indeed $r_{L/2}(x_1 + L, x_2, x_3) = (x_2 + L/2, x_1 + L/2, x_3) = Y(x_2, x_1, x_3)$; Y^2 is the translation by $L(e_1 + e_2) \in \Gamma_\parallel$, so Y is an involution of $\mathcal{C}$ preserving u, the strip $S_+ := \{0 \leq u \leq L/2\}$, both walls $\Pi_0 = \{u = 0\}$, $\Pi_{L/2} = \{u = L/2\}$ and the wall structure of D (it shifts $v(x)$ by $L \in h\mathbb{Z}$ and fixes x_3).

Fold variables. On $\mathcal{C}$ let ξ_0, ξ' be the variables of the cells lying in the walls Π_0, $\Pi_{L/2}$ (sites and e_3-bonds in the wall, Lemma 12.14), ξ_+ those of the cells with all vertices in S_+ and not all in a wall, ξ_- the same for $S_- := \{-L/2 \leq u \leq 0\}$; the remaining cells are the crossing cells at Π_0 and at the seam $\Pi_{L/2} \equiv \Pi_{-L/2}$: the e_1e_2-plaquettes bisected by the wall and the cubes having

them as faces; no bond is a crossing cell, and the factors of the crossing cells contain no wall variable. Torus i ($i=2$: T_L^{tw}; $i=1$: T_L^{cub}) is the strip $\{-L/2 \le u \le L/2\}$ of $\mathcal{C}$ with $\Pi_{-L/2}$ glued to $\Pi_{L/2}$ by τ_i; on $\Pi_{-L/2}$, $\tau_2 = r_0$ and $\tau_1 = Y r_0$ (by the identities $r_{L/2}\tau_2 = r_0$, $r_{L/2}\tau_1 = Y r_0$ and $r_{L/2} = \mathrm{id}$ on $\Pi_{L/2}$). Its weight is

$$W_0(\xi_0) W_0'(\xi')\, W_+(\xi_0, \xi', \xi_+)\, W_-(\xi_0, \tau_i^* \xi', \xi_-)\, K_0\, K_i',$$

where W_0, W_0' collect the factors of the wall cells, $W_\pm$ those of the $(\pm)$-cells (which see the wall variables of the walls they touch, the $(-)$-cells seeing $\Pi_{L/2}$ through τ_i), and K_0, K_i' are the products of the factors of the crossing cells at Π_0 and at the seam.

Reflection and kernels. Put $\eta := r_0^* \xi_-$, the $(+)$-configuration obtained by reflecting ξ_- in Π_0. By $W \circ R = W$ cell by cell (Lemma 14.6), $W_-(\xi_0, \tau_2^*\xi', \xi_-) = W_+(\xi_0, \xi', \eta)$ and $W_-(\xi_0, \tau_1^*\xi', \xi_-) = W_+(\xi_0, Y^*\xi', \eta)$. Steps 2–3 of the proof of Theorem 12.16 (Poisson resummation at a diagonal wall, Remark 14.8) write $K_0 = \int d\nu(k)\, \overline{A_k(\eta)} A_k(\xi_+)$ and $K_2' = \int d\nu'(k')\, \overline{A_{k'}'(\eta)} A_{k'}'(\xi_+)$, with positive measures ν, ν' on auxiliary variables and $|A_k| = |A_{k'}'| = 1$, A_k depending on the $(+)$-cells at Π_0 and $A_{k'}'$ on those at $\Pi_{L/2}$; for the cubic torus the seam is glued through $\tau_1^{-1} r_{L/2} = r_0 Y^{-1}$ instead of r_0, so $K_1' = \int d\nu'(k')\, \overline{A_{k'}'(Y^*\eta)} A_{k'}'(\xi_+)$.

The two scalar products. Let $\mathcal{H} := L^2\big(d\xi_0\, W_0\ d\xi'\, W_0'\ d\nu(k)\, d\nu'(k')\big)$ and, for a bounded $F = F(\xi_+)$,

$$\Xi_F(\xi_0, \xi', k, k') := \int d\xi_+\, W_+(\xi_0, \xi', \xi_+)\, A_k(\xi_+) A_{k'}'(\xi_+)\, F(\xi_+),$$
$$(Y\Xi)(\xi_0, \xi', k, k') := \Xi(\xi_0, Y^*\xi', k, Yk'),$$

where Y permutes the wall cells of $\Pi_{L/2}$ and the seam's crossing cells, W_0', the reference measure and ν' are Y-invariant and $A_{Yk'}'(\eta) = A_{k'}'(Y^*\eta)$; so Y is a unitary involution of $\mathcal{H}$. Take $F := e^{tO_+}$, a real bounded function of ξ_+. On both tori $O_d = O_+ + O_+ \circ r_0$ (the $\mathcal{W}$-cells of $\{-L/2 < u < 0\}$ are the r_0-images of those of the strip and $O_{r_0 c} = O_c \circ r_0$, Lemma 14.11; on T_L^{cub} this is the definition), and $O_+ \circ r_0$, as a function of ξ_-, is $O_+(\eta)$. Since $O_+ \circ Y = O_+$ ($Y = s_{c_+ + L/2}\, s_{c_+} \in \mathcal{W}$), the substitution $\xi_- \mapsto \eta$ (measure preserving) and the integration over ξ_+ and η give

$$Z_L^{\mathrm{tw}} \langle e^{tO_d} \rangle_L^{\mathrm{tw}} = \langle \Xi_F, \Xi_F \rangle_{\mathcal{H}}, \qquad Z_L^{\mathrm{cub}} \langle e^{tO_d} \rangle_L^{\mathrm{cub}} = \langle Y\Xi_F, \Xi_F \rangle_{\mathcal{H}}.$$

By the Cauchy–Schwarz inequality and $\|Y\| = 1$, $Z_L^{\mathrm{cub}} \langle e^{tO_d} \rangle_L^{\mathrm{cub}} \le Z_L^{\mathrm{tw}} Z_t$, which is the claim; $t = 0$ gives $Z_L^{\mathrm{cub}} \le Z_L^{\mathrm{tw}}$, i.e. $\phi_L \ge 0$. □

Lemma 14.14 (surface bound for the adapted torus)**.** *Fix the level N and the couplings of the fine Villain–Higgs model. Let $L > 0$ with $L/2 \in \varepsilon_N \mathbb{Z}$ and $L \ge 8\varepsilon_N$, and write $M := L/\varepsilon_N \in 2\mathbb{Z}$, $M \ge 8$. Let Z_L^{cub} be the partition function of the model on the cubic torus $T_L^{\mathrm{cub}} = \varepsilon_N \mathbb{Z}^3 / L\mathbb{Z}^3$, the fine torus $\tilde{T}_L$ of Section 2, and let Z_L^{tw} be the partition function of the same model (same N, same couplings, same local weights) on the adapted torus $T_L^{\mathrm{tw}} = \varepsilon_N \mathbb{Z}^3 / \Gamma(L)$ of the same side L. Then*

$$\phi_L := \log \frac{Z_L^{\mathrm{tw}}}{Z_L^{\mathrm{cub}}} \qquad \textit{satisfies} \qquad 0 \ \le\ \phi_L \ \le\ \Big(\frac{L}{\varepsilon_N}\Big)^2 \Big(5 \log \frac{L}{\varepsilon_N} + A_N\Big),$$

where $A_N < \infty$ depends only on N and the couplings (not on L). In particular $\phi_L = o(L^3)$ as $L \to \infty$ at fixed N.

Proof. The lower bound $\phi_L \ge 0$ is Proposition 14.13 at $t = 0$; here we prove the upper bound. Throughout, N and the couplings are fixed, γ_N is the phase constant, κ_N the hopping coefficient, g_{ct} the site factor and $\rho_x := \rho_{0,0} + \tilde{\rho}_x$, so that the weight on either torus is

$$W = \prod_x g_{\mathrm{ct}}(\tilde{\rho}_x) \prod_b h(\tilde{\rho}_{b_-}, \tilde{\rho}_{b_+}, B_b) \prod_p w((dB)_p + v_p).$$

Here $w(f) = e^{-f^2/2}$, $B_b \in (-\pi/\gamma_N, \pi/\gamma_N]$, $v_p \in (2\pi/\gamma_N)\mathbb{Z}$, the sum runs over the closed v ($dv = 0$ on every cube), and

$$e^{-\kappa_N(\rho^2+\rho'^2)} \;\le\; h(\tilde\rho,\tilde\rho',\vartheta) = \exp\big(-\tfrac12\kappa_N|e^{i\gamma_N\vartheta}\rho' - \rho|^2\big) \;\le\; 1$$
$$(|e^{i\gamma_N\vartheta}\rho' - \rho|^2 \le (\rho+\rho')^2 \le 2(\rho^2+\rho'^2)).$$

For $q \in \mathbb{R}$ put $G_q := \int_{-\rho_{0,0}}^{\infty} g_{\mathrm{ct}}(r)\, e^{q(\rho_{0,0}+r)^2}\, dr$, finite for every real q because the quartic dominates ((2.1); for $q \ge 0$ this is the G_q^{ct} of Lemma 11.2); let K'_{γ_N} and c_0 be the constants of Lemma 11.2 ($\sum_{v\in(2\pi/\gamma_N)\mathbb{Z}} w(f+v) \le K'_{\gamma_N}$ for every real f, and $Z \ge c_0^{|X|}$ on any torus with site set X). The proof has nine steps.

Step 0: seam geometry. Write $u := x_1 - x_2$, $v := x_1 + x_2$ and let $\Gamma'_{\parallel} := L(e_1-e_2)\mathbb{Z} + L(e_1+e_2)\mathbb{Z} + Le_3\mathbb{Z}$ (index 2 in $L\mathbb{Z}^3$ and in $\Gamma(L) = \frac{L}{2}(e_1-e_2)\mathbb{Z} + L(e_1+e_2)\mathbb{Z} + Le_3\mathbb{Z}$). On the finite cylinder $C' := \varepsilon_N\mathbb{Z}^3/\Gamma'_{\parallel}$ (the quotient of the cylinder $\mathcal{C}$ of Proposition 14.13 by the translation $L(e_1-e_2)$: $\Gamma'_{\parallel} = \Gamma_{\parallel} + L(e_1-e_2)\mathbb{Z}$) the coordinate u is defined mod $2L$ and v mod $2L$. Both tori are quotients of C' by a single translation: $T^{\mathrm{tw}} := \varepsilon_N\mathbb{Z}^3/\Gamma(L) = C'/\tau_{\mathrm{tw}}$ with $\tau_{\mathrm{tw}} = \frac{L}{2}(e_1-e_2)\colon (u,v,x_3) \mapsto (u+L, v, x_3)$, and $T^{\mathrm{cub}} := \varepsilon_N\mathbb{Z}^3/L\mathbb{Z}^3 = C'/\tau_{\mathrm{cub}}$ with $\tau_{\mathrm{cub}} = Le_1\colon (u,v,x_3) \mapsto (u+L, v+L, x_3)$. On both tori u is defined mod L, and the *layer* $\{u \equiv c\}$ has exactly M^2 sites, labelled by (v,x_3) with $v \in \varepsilon_N\mathbb{Z}/2L\mathbb{Z}$, $v \equiv c \pmod{2\varepsilon_N}$, and $x_3 \in \varepsilon_N\mathbb{Z}/L\mathbb{Z}$.

An e_1-bond raises u by ε_N, an e_2-bond lowers it by ε_N, an e_3-bond keeps it; so the vertices of any cell (bond, plaquette, cube) occupy at most three consecutive layers. Call a cell *seam* if it has a vertex in the layer $u \equiv 0$ and *interior* if all its vertices lie in the layers $u \in \{\varepsilon_N, \dots, L-\varepsilon_N\}$; every cell is one or the other. The quotient maps $C' \to T^{\mathrm{tw}}$, $C' \to T^{\mathrm{cub}}$ are injective on the strip $\varepsilon_N \le u \le L - \varepsilon_N$ (its τ-image lies in $L + \varepsilon_N \le u \le 2L - \varepsilon_N$), so the interior cells, their incidences and their labels are the same for both tori; the two tori differ only in how the cells touching $u \equiv 0$ are glued to the strip.

Counts. The plane $u \equiv 0$ has M^2 sites. The seam bonds are $4M^2$ bonds of type e_1, e_2 (four at each plane site, each with exactly one plane vertex) and M^2 plane e_3-bonds, $5M^2$ in all. The seam plaquettes are, in terms of the u-coordinate u_x of the base corner x, the e_1e_2-plaquettes with $u_x \in \{-\varepsilon_N, 0, \varepsilon_N\}$, the e_1e_3-plaquettes with $u_x \in \{-\varepsilon_N, 0\}$ and the e_2e_3-plaquettes with $u_x \in \{0, \varepsilon_N\}$, $7M^2$ in all. The seam cubes are the cubes with $u_x \in \{-\varepsilon_N, 0, \varepsilon_N\}$, called $\Sigma_-, \Sigma_0, \Sigma_+$ respectively (middle layer $u \equiv -\varepsilon_N, 0, \varepsilon_N$), $3M^2$ in all. The six faces of the cube with base corner x meet its middle layer: the two e_1e_2-faces (base u_x), the e_1e_3-face at x ($u \in \{u_x, u_x+\varepsilon_N\}$), the e_1e_3-face at $x + \varepsilon_N e_2$ ($u \in \{u_x - \varepsilon_N, u_x\}$), the e_2e_3-face at x ($u \in \{u_x - \varepsilon_N, u_x\}$) and the e_2e_3-face at $x + \varepsilon_N e_1$ ($u \in \{u_x, u_x + \varepsilon_N\}$). Hence a Σ_+-cube has exactly two interior faces (e_1e_3 at x, e_2e_3 at $x + \varepsilon_N e_1$, both with $u \in \{\varepsilon_N, 2\varepsilon_N\}$), a Σ_--cube exactly two (e_1e_3 at $x + \varepsilon_N e_2$, e_2e_3 at x, $u \in \{L - 2\varepsilon_N, L - \varepsilon_N\}$) and a Σ_0-cube none. We call these $4M^2$ interior faces of seam cubes the *data faces*; each data face lies in exactly one seam cube (its other cube has base $u \in \{2\varepsilon_N, L - 2\varepsilon_N\}$ and is interior since $M \ge 4$), and we write c_p for that seam cube and c'_p for the interior one. Every seam plaquette lies in exactly two seam cubes (for an e_1e_2-plaquette the cubes at x and $x - \varepsilon_N e_3$, for an e_1e_3-plaquette those at x and $x - \varepsilon_N e_2$, for an e_2e_3-plaquette those at x and $x - \varepsilon_N e_1$), with opposite incidence signs $[c:p] = -[c':p]$ (two cubes induce opposite orientations on a common face). Every interior plaquette lies in at least one interior cube (again $M \ge 4$).

The seam-cube graph. Let G^i ($i \in \{\mathrm{cub}, \mathrm{tw}\}$) be the multigraph with vertex set the $3M^2$ seam cubes of T^i and edge set the $7M^2$ seam plaquettes, each joining the two seam cubes containing it. Label the Σ_s-cubes by the (v,x_3) of their base corner, lifting Σ_- to base $u = L - \varepsilon_N$ in C'.

The e_1e_2-faces give the column edges $(v, x_3) \sim (v, x_3 + \varepsilon_N)$ in each of the three layers; the e_1e_3- and e_2e_3-faces give $\Sigma_+(v, x_3) \sim \Sigma_0(v \pm \varepsilon_N, x_3)$ in both tori, and $\Sigma_-(v, x_3) \sim \Sigma_0(v \pm \varepsilon_N - \delta_i, x_3)$ with $\delta_{\mathrm{tw}} = 0$, $\delta_{\mathrm{cub}} = L$, because the far faces of the lifted Σ_--cubes have vertices at $u = L$ and are glued to the plane $u = 0$ by τ_i^{-1}. Since $L \in 2\varepsilon_N\mathbb{Z}$ preserves the parity class of v, the map $\Sigma_-(v, x_3) \mapsto \Sigma_-(v - L, x_3)$, identity on $\Sigma_0 \cup \Sigma_+$, is a graph isomorphism $G^{\mathrm{cub}} \cong G^{\mathrm{tw}}$; both graphs are connected: the column edges join all values of x_3, and the paths $(v, 0) \sim (v+\varepsilon_N, +) \sim (v + 2\varepsilon_N, 0)$ join all values of v.

Step 1: seam decomposition. Write ξ_{int} for the interior variables ($\tilde\rho_x$ at the sites of the layers $u \in \{\varepsilon_N, \dots, L - \varepsilon_N\}$, B_b on the interior bonds, v_p on the interior plaquettes) and

$$W_{\mathrm{int}}(\xi_{\mathrm{int}}) := \prod_{x\ \mathrm{int}} g_{\mathrm{ct}}(\tilde\rho_x) \prod_{b\ \mathrm{int}} h \prod_{p\ \mathrm{int}} w\big((dB)_p + v_p\big) \prod_{c\ \mathrm{int}} \mathbf{1}[(dv)_c = 0],$$

the same function of the same variables for both tori (Step 0). The seam variables are $\tilde\rho_x$ at the M^2 plane sites, B_b on the $5M^2$ seam bonds and v_p on the $7M^2$ seam plaquettes. For a seam cube c of T^i every face is a seam plaquette or a data face, so

$$(dv)_c = \mathrm{div}_{G^i} v_{\mathrm{seam}}(c) - q(c), \qquad \mathrm{div}_G v(c) := \sum_{\mathrm{seam}\ p\subset c} [c{:}p]\, v_p, \qquad q(c) := -\sum_{\mathrm{data}\ p\subset c} [c{:}p]\, v_p,$$

where q depends on ξ_{int} only (data faces are interior), $q = 0$ on Σ_0, and div_G is the signed incidence (divergence) operator of the graph G with the edge p oriented from the cube with $[c{:}p] = -1$ to the cube with $[c{:}p] = +1$. Thus v is closed on T^i if and only if it is closed on the interior cubes and $\mathrm{div}_{G^i} v_{\mathrm{seam}} = q$.

For $i \in \{\mathrm{cub}, \mathrm{tw}\}$ define the *seam factor*

$$\Sigma^i(\xi_{\mathrm{int}}) := \int \prod_{u(x)=0} g_{\mathrm{ct}}(\tilde\rho_x)\, d\tilde\rho_x \int \prod_{\mathrm{seam}\ b} dB_b \prod_{\mathrm{seam}\ b} h \sum_{v_{\mathrm{seam}}:\ \mathrm{div}_{G^i} v_{\mathrm{seam}} = q} \prod_{\mathrm{seam}\ p} w\big((dB)_p + v_p\big).$$

Every cell weight of T^i appears exactly once in $W_{\mathrm{int}}\Sigma^i$ and all terms are non-negative, so by Tonelli $Z^i = \int d\xi_{\mathrm{int}}\, W_{\mathrm{int}}\, \Sigma^i(\xi_{\mathrm{int}})$. (The seam weights depend on ξ_{int}: a seam plaquette may contain an interior bond, and a seam bond of type e_1, e_2 has one end in the layer $u = \pm\varepsilon_N$; this dependence is the only place, besides the constraint graph G^i, where the two tori differ, through the gluing δ_i of Step 0.)

The data charges sum to zero. If v is closed on the interior cubes, then

$$0 = \sum_{c'\ \mathrm{int}} (dv)_{c'} = \sum_{c'\ \mathrm{int}} \sum_{p\subset c'} [c'{:}p] v_p.$$

An interior plaquette lying in two interior cubes contributes 0 (opposite signs); one lying in exactly one interior cube is a data face p with interior cube c'_p (its other cube is the seam cube c_p); seam plaquettes lie in no interior cube. Hence

$$0 = \sum_{\mathrm{data}\ p} [c'_p{:}p] v_p = -\sum_{\mathrm{data}\ p} [c_p{:}p] v_p = \sum_c q(c).$$

The same computation with $f := dB + v$ (closed on interior cubes, as $d\,dB = 0$) and with dB alone gives $\sum_c Q(c) = 0$ for $Q(c) := -\sum_{\mathrm{data}\ p\subset c}[c{:}p] f_p$ and $\sum_c \sum_{\mathrm{data}\ p\subset c}[c{:}p](dB)_p = 0$.

Step 2: Jensen. Let $\mu(d\xi_{\mathrm{int}}) := W_{\mathrm{int}}\Sigma^{\mathrm{tw}} d\xi_{\mathrm{int}}/Z_L^{\mathrm{tw}}$, the law of the interior variables under $\langle\cdot\rangle_L^{\mathrm{tw}}$, and $R := \Sigma^{\mathrm{cub}}/\Sigma^{\mathrm{tw}}$. By Step 1, $Z_L^{\mathrm{cub}}/Z_L^{\mathrm{tw}} = \int R\, d\mu$. For every ξ_{int} with $W_{\mathrm{int}} > 0$ (the others form a μ-null set) both seam factors lie in $(0, \infty)$: finiteness is Step 3 and positivity is Step 5

(each applies to either torus, since $G^{\mathrm{cub}} \cong G^{\mathrm{tw}}$ and the cell counts agree). By the concavity of log (Jensen),

$$\phi_L = -\log \int R\, d\mu \;\le\; -\int \log R\, d\mu \;=\; \big\langle \log \Sigma^{\mathrm{tw}} - \log \Sigma^{\mathrm{cub}} \big\rangle_L^{\mathrm{tw}},$$

where $\log R$ is μ-integrable: by Steps 3 and 5,

$$|\log \Sigma^i| \le \mathrm{const} \cdot M^2 + \langle Q, \Delta^+_{G^i} Q\rangle + 2\kappa_N \sum_{u(x)=\pm\varepsilon_N} \rho_x^2 \qquad \text{for both } i,$$

and the right side has finite μ-mean by Step 6 (with $\operatorname{Tr}\Delta^+_{G^{\mathrm{cub}}} = \operatorname{Tr}\Delta^+_{G^{\mathrm{tw}}}$ by the isomorphism of Step 0).

Step 3: upper bound on the seam factor. In Σ^i use $h \le 1$, drop the constraint $\operatorname{div}_{G^i} v_{\mathrm{seam}} = q$ and sum each v_p separately ($\sum_{v_p} w((dB)_p + v_p) \le K'_{\gamma_N}$); each of the $5M^2$ seam bonds contributes $\int dB_b = 2\pi/\gamma_N$ and each of the M^2 plane sites $\int g_{\mathrm{ct}} = G_0$. Hence

$$\log \Sigma^i \;\le\; 5M^2 \log \frac{2\pi}{\gamma_N} + M^2 \log G_0 + 7M^2 \log K'_{\gamma_N} \qquad (i \in \{\mathrm{cub}, \mathrm{tw}\}),$$

uniformly in ξ_{int}.

Step 4: a rounding lemma. Let (V, E) be a finite multigraph without loops (parallel edges allowed), each edge oriented, $D\colon \mathbb{R}^E \to \mathbb{R}^V$ its signed incidence (divergence) operator, and $j \in \mathbb{R}^E$ with $Dj \in \mathbb{Z}^V$. Then there is $n \in \mathbb{Z}^E$ with $Dn = Dj$ and $|n_e - j_e| \le 1$ for every $e \in E$. Indeed, $\phi := j - \lfloor j \rfloor \in [0,1)^E$ has $D\phi = Dj - D\lfloor j\rfloor \in \mathbb{Z}^V$. The set $\{\phi' \in [0,1]^E : D\phi' = D\phi\}$ contains ϕ; choose in it a ϕ' with the fewest non-integer coordinates and let $F := \{e : \phi'_e \notin \{0,1\}\}$. Suppose $F \neq \emptyset$. At a vertex x, $(D\phi')_x$ is an integer and the integer-valued edges contribute an integer, so the signed sum of the values $\phi'_e \in (0,1)$ over the F-edges at x is an integer; a vertex with exactly one F-edge is therefore impossible. Thus every vertex has F-degree 0 or ≥ 2, and F contains a cycle (a closed trail without repeated edges; two parallel edges form one). Let $\gamma \in \{0, \pm 1\}^E$ be its circulation (the sign of each cycle edge according to its orientation along a traversal, 0 off the cycle), so $D\gamma = 0$. For real t the vector $\phi' + t\gamma$ has the same divergence; take $|t|$ minimal such that some cycle edge reaches 0 or 1 (every cycle edge starts in the open interval $(0,1)$). Then $\phi' + t\gamma \in [0,1]^E$ and it has fewer non-integer coordinates than ϕ': a contradiction. Hence $\phi' \in \{0,1\}^E$, and $n := \lfloor j \rfloor + \phi' \in \mathbb{Z}^E$ satisfies $Dn = D\lfloor j\rfloor + D\phi = Dj$ and $n_e - j_e = \phi'_e - \phi_e \in [-1, 1]$.

Step 5: lower bound on the seam factor by one closed extension. Fix ξ_{int} with $W_{\mathrm{int}} > 0$ (so v is closed on the interior cubes) and fix B_{seam}. Put $f := dB + v$ on the interior plaquettes and $Q(c) := -\sum_{\text{data } p \subset c} [c\!:\!p] f_p$ for the seam cubes c; Q depends on ξ_{int} only (a data face is interior, and so are its four bonds), and $\sum_c Q(c) = 0$ by Step 1. Let $G := G^{\mathrm{cub}}$ with the edge orientation of Step 1, D its divergence, $\nabla = D^{\mathsf{T}}$ the gradient, $\Delta := DD^{\mathsf{T}}$ its graph Laplacian and Δ^+ the pseudo-inverse. G is connected, so $\ker\Delta$ is the constants and $\Delta\Delta^+ Q = Q$. Put $\psi := \Delta^+ Q$ and $j^* := \nabla\psi$; then $\operatorname{div} j^* = \Delta\psi = Q$ and $|j^*|^2 = \langle \psi, \Delta\psi\rangle = \langle Q, \Delta^+ Q\rangle$. Since $(d\,dB)_c = 0$ at every seam cube, $\operatorname{div}\big((dB)|_{\mathrm{seam}}\big)(c) = -\sum_{\text{data } p\subset c} [c\!:\!p](dB)_p$, hence $j' := j^* - (dB)|_{\mathrm{seam}}$ has

$$\operatorname{div} j'(c) = Q(c) + \sum_{\text{data } p\subset c} [c\!:\!p](dB)_p = -\sum_{\text{data } p\subset c} [c\!:\!p]\, v_p = q(c) \in \frac{2\pi}{\gamma_N}\mathbb{Z}.$$

Step 4, applied to $(\gamma_N/2\pi)j'$, gives $v^\dagger \in ((2\pi/\gamma_N)\mathbb{Z})^{\mathrm{seam}}$ with $\operatorname{div}_{G^{\mathrm{cub}}} v^\dagger = q$ and $|v^\dagger_p - j'_p| \le 2\pi/\gamma_N$ for every seam plaquette p. By Step 1, $(v_{\mathrm{int}}, v^\dagger)$ is closed on T^{cub}, and $f^\dagger := dB + v^\dagger$

on the seam plaquettes obeys $f^\dagger - j^* = v^\dagger - j'$, so $|f^\dagger - j^*|_\infty \le 2\pi/\gamma_N$ and, on the $7M^2$ seam plaquettes,

$$|f^\dagger|^2 \le 2|j^*|^2 + 2\cdot 7M^2(2\pi/\gamma_N)^2 = 2\langle Q, \Delta^+ Q\rangle + 56\pi^2 M^2/\gamma_N^2,$$

uniformly in B_{seam}. Keep in Σ^{cub} only the term $v_{\mathrm{seam}} = v^\dagger$:

$$\prod_{\text{seam } p} w(f_p^\dagger) = e^{-|f^\dagger|^2/2} \ge \exp\big(-\langle Q, \Delta^+ Q\rangle - 28\pi^2 M^2/\gamma_N^2\big).$$

Bound each seam-bond factor by $h \ge e^{-\kappa_N(\rho^2+\rho'^2)}$: a plane site lies in six seam bonds $(\pm e_1, \pm e_2, \pm e_3)$ and a site of the layers $u = \pm\varepsilon_N$ in exactly two ($-e_1, +e_2$ at $u = \varepsilon_N$; $+e_1, -e_2$ at $u = -\varepsilon_N$), so

$$\prod_{\text{seam } b} h \ge \prod_{u(x)=0} e^{-6\kappa_N \rho_x^2} \prod_{u(x)=\pm\varepsilon_N} e^{-2\kappa_N\rho_x^2}.$$

Integrating over the plane sites ($\int g_{\mathrm{ct}} e^{-6\kappa_N\rho^2} = G_{-6\kappa_N} > 0$) and the seam bonds ($\int dB_b = 2\pi/\gamma_N$), we obtain

$$\log\Sigma^{\mathrm{cub}} \;\ge\; -\langle Q, \Delta^+_{G^{\mathrm{cub}}} Q\rangle - \frac{28\pi^2 M^2}{\gamma_N^2} + 5M^2 \log\frac{2\pi}{\gamma_N} + M^2 \log G_{-6\kappa_N} - 2\kappa_N \sum_{u(x)=\pm\varepsilon_N} \rho_x^2.$$

The same argument with G^{tw} bounds Σ^{tw} from below, which gives $\Sigma^{\mathrm{tw}} > 0$ in Step 2. In particular both constraint sets are non-empty for every interior datum: the only condition for the existence of a closed extension is $\sum_c q(c) = 0$.

Step 6: moments under the twisted state. Write $\langle\cdot\rangle_L^{\mathrm{tw}}$ for the normalised state of weight W on T^{tw}; the interior variables have law μ under it (Step 2).

(a) *Gaussian domination.* By the first clause of Lemma 12.10, for every real function b on the plaquettes of $\varepsilon_N\mathbb{Z}^3/P$, P a period lattice for $\varepsilon_N\mathbb{Z}^3$ (a rank-3 sublattice of $\varepsilon_N\mathbb{Z}^3$ whose non-zero vectors have length $\ge 5\varepsilon_N$), $\langle e^{\sum_p b_p f_p}\rangle \le e^{|b|^2/2}$, with $f_p = (dB)_p + v_p$ (the $\mathfrak{f}_p$ of Section 2). The lattice $\Gamma(L) \subset \varepsilon_N\mathbb{Z}^3$ (as $L/2 \in \varepsilon_N\mathbb{Z}$) has rank 3 and is a period lattice for $L \ge 8\varepsilon_N$: its non-zero vectors $a\frac{L}{2}(e_1 - e_2) + bL(e_1 + e_2) + cLe_3$ have squared length $L^2(a^2/2 + 2b^2 + c^2) \ge L^2/2$, and $L/\sqrt{2} \ge 5\varepsilon_N$ holds for $L \ge 8\varepsilon_N$ (as $L \in 2\varepsilon_N\mathbb{Z}$, $L \ge 5\sqrt{2}\,\varepsilon_N$ means $L \ge 8\varepsilon_N$). Applying the bound to $\pm sb$ and using $\cosh y \ge 1 + y^2/2$ gives $\frac{s^2}{2}\langle(\sum_p b_p f_p)^2\rangle_L^{\mathrm{tw}} \le e^{s^2|b|^2/2} - 1$ for every $s > 0$, whence $\langle(\sum_p b_p f_p)^2\rangle_L^{\mathrm{tw}} \le |b|^2$ ($s \downarrow 0$). Let $S_{cc'} := \langle Q(c)Q(c')\rangle_L^{\mathrm{tw}}$ be the second-moment matrix of the data charges. For $a \in \mathbb{R}^V$ (V the seam cubes), $\sum_c a_c Q(c) = \sum_{\text{data } p} b_p f_p$ with $b_p := -a_{c_p}[c_p : p]$ (c_p the unique seam cube of the data face p), so $\langle a, Sa\rangle \le |b|^2 = \sum_{\text{data } p} a_{c_p}^2 \le 2|a|^2$, since a seam cube has at most two data faces. Thus $S \le 2\,\mathrm{Id}$ as quadratic forms, and, Δ^+ being positive semi-definite,

$$\big\langle\langle Q, \Delta^+ Q\rangle\big\rangle_L^{\mathrm{tw}} = \mathrm{Tr}(\Delta^+ S) = \mathrm{Tr}\,\big((\Delta^+)^{1/2} S (\Delta^+)^{1/2}\big) \le 2\,\mathrm{Tr}\,\Delta^+.$$

(b) *Density moments.* The translations by $\varepsilon_N\mathbb{Z}^3$ act transitively on the sites of T^{tw} by automorphisms of the cell complex which preserve the cell-local weights and the product reference measure, so $\langle\cdot\rangle_L^{\mathrm{tw}}$ is translation invariant and $\langle\rho_x^2\rangle_L^{\mathrm{tw}}$ does not depend on x. Let X be the site set ($|X| = M^3$). By Jensen's inequality ($\langle Y\rangle \le \log\langle e^Y\rangle$), in the notation $Z(A, q) := \int d\tilde\rho \int dB \sum_{v\ \mathrm{closed}} W \prod_{y\in A} e^{q\rho_y^2}$ of Step 1 of the proof of Lemma 11.2,

$$\begin{aligned}\langle\rho_x^2\rangle_L^{\mathrm{tw}} &= M^{-3}\Big\langle \sum_{y\in X}\rho_y^2\Big\rangle_L^{\mathrm{tw}} \le M^{-3}\log\frac{Z(X,1)}{Z_L^{\mathrm{tw}}}\\ &\le \log\frac{G_1(2\pi/\gamma_N)^3 K'^3_{\gamma_N}}{c_0} \le C_N^{(2)} := \log\max\Big\{1, \frac{G_1(2\pi/\gamma_N)^3 K'^3_{\gamma_N}}{c_0}\Big\}.\end{aligned}$$

Here the crude bounds $Z(A,q) \le G_q^{|A|} G_0^{|X|-|A|} (2\pi/\gamma_N)^{3|X|} K'^{3|X|}_{\gamma_N}$ and $Z \ge c_0^{|X|}$ of that Step 1 are used on T^{tw}. Their proof uses only $g_{\mathrm{ct}} \ge 0$, $h \le 1$, the sum over closed v being at most the free sum, $\sum_{v_p} w((dB)_p + v_p) \le K'_{\gamma_N}$, the term $v \equiv 0$ (closed on every complex) with the integration restricted to $\tilde\rho_x \in [0,1]$, $|\gamma_N B_b| \le 1$, and the counts $|\mathcal{B}| = |\mathcal{P}| = 3|X|$, which hold on T^{tw} (three bonds and three plaquettes per site on any quotient of $\varepsilon_N\mathbb{Z}^3$ by a rank-3 lattice). Consequently

$$2\kappa_N \sum_{u(x)=\pm\varepsilon_N} \langle \rho_x^2 \rangle_L^{\mathrm{tw}} \le 2\kappa_N \cdot 2M^2 \cdot C_N^{(2)} = 4\kappa_N M^2 C_N^{(2)}.$$

Step 7: the trace bound. By the isomorphism of Step 0, $\operatorname{Tr}\Delta^+_{G^{\mathrm{cub}}} = \operatorname{Tr}\Delta^+_{G^{\mathrm{tw}}}$, so we bound the latter. Relabel the $\Sigma_\pm$-cubes of $G := G^{\mathrm{tw}}$ by $w := v - \varepsilon_N$ (so that $w \equiv 0 \pmod{2\varepsilon_N}$, like the v of the Σ_0-cubes). With $\delta_{\mathrm{tw}} = 0$ the graph of Step 0 has the vertex set $V = \mathbb{Z}^2_M \times \{-,0,+\}$ ($w \in 2\varepsilon_N\mathbb{Z}/2L\mathbb{Z} \cong \mathbb{Z}_M$, $x_3 \in \varepsilon_N\mathbb{Z}/L\mathbb{Z} \cong \mathbb{Z}_M$), the column edges $(w,x_3,s) \sim (w,x_3+\varepsilon_N,s)$ in each layer s, and the edges $(w,x_3,\pm) \sim (w,x_3,0)$, $(w,x_3,\pm) \sim (w+2\varepsilon_N,x_3,0)$; it has $3M^2 + 4M^2 = 7M^2$ edges. Let $H := T_M \,\square\, P_3$ be the Cartesian product of the $M \times M$ torus grid T_M (vertex set $\mathbb{Z}^2_M$, nearest-neighbour edges in both directions) with the path P_3 on $\{-,0,+\}$ (edges $-\sim 0$, $0 \sim +$), a graph on the same vertex set V. Its column edges and its vertical edges $(w,x_3,\pm) \sim (w,x_3,0)$ are edges of G. Its in-layer w-edges are not, but the two ends of each are joined by a path of two G-edges (x_3 suppressed): $(w,0)-(w,+)-(w+2\varepsilon_N,0)$ in the layer 0, and $(w,\pm)-(w+2\varepsilon_N,0)-(w+2\varepsilon_N,\pm)$ in the layers $\pm$; every non-column G-edge is used by at most two of these paths. With the Dirichlet forms $\mathcal{E}_G(\psi) := \sum_{\{a,b\}\in E(G)} (\psi_a - \psi_b)^2 = \langle \psi, \Delta_G \psi\rangle$ and the inequality $(\psi_a-\psi_b)^2 \le 2(\psi_a-\psi_c)^2 + 2(\psi_c-\psi_b)^2$ on each w-edge,

$$\mathcal{E}_H(\psi) \;\le\; \mathcal{E}_G(\psi) + 2\cdot 2 \sum_{\text{non-column } \{a,b\}\in E(G)} (\psi_a-\psi_b)^2 \;\le\; 5\,\mathcal{E}_G(\psi) \qquad (\psi \in \mathbb{R}^V),$$

$$\text{i.e.} \quad \Delta_H \le 5\Delta_G.$$

Both graphs are connected, so both Laplacians have kernel the constants; on its orthogonal complement $0 < \Delta_H \le 5\Delta_G$ gives $\Delta_G^+ \le 5\Delta_H^+$ (monotonicity of the inverse), hence $\operatorname{Tr}\Delta_G^+ \le 5\operatorname{Tr}\Delta_H^+$. The spectrum of Δ_H is $\{\lambda_T(k) + \mu_j : k \in \mathbb{Z}^2_M,\ j = 1,2,3\}$ with $\lambda_T(k) = 4\sin^2(\pi k_1/M) + 4\sin^2(\pi k_2/M)$ and $\mu_j \in \{0,1,3\}$ the eigenvalues of the Laplacian of P_3. Writing $\tilde k_i \in (-M/2, M/2]$ for the representative of k_i and using $\sin \pi t \ge 2t$ on $[0,\frac12]$, $\lambda_T(k) \ge 16|\tilde k|^2/M^2$. At most $8r$ points k have $|\tilde k|_\infty = r$ ($1 \le r \le M/2$), and $|\tilde k|^2 \ge r^2$ there; the terms with $\mu_j \in \{1,3\}$ are at most $M^2(1+\frac13)$. Hence

$$\operatorname{Tr}\Delta_H^+ = \sum_{(k,j):\ \lambda_T(k)+\mu_j>0} \frac{1}{\lambda_T(k)+\mu_j} \;\le\; \frac{M^2}{16}\sum_{r=1}^{M/2}\frac{8r}{r^2} + \frac43 M^2 \;\le\; \frac{M^2}{2}\Big(1+\log\frac{M}{2}\Big) + \frac43 M^2,$$

and therefore, since $5 - 5\log 2 + \frac{40}{3} = 14.87\ldots \le 15$,

$$2\operatorname{Tr}\Delta^+_{G^{\mathrm{cub}}} = 2\operatorname{Tr}\Delta_G^+ \;\le\; 10\operatorname{Tr}\Delta_H^+ \;\le\; 5M^2\log M + \Big(5 - 5\log 2 + \frac{40}{3}\Big)M^2 \;\le\; 5M^2\log M + 15M^2.$$

Step 8: assembly. Insert in Step 2 the bound of Step 3 for $\log\Sigma^{\mathrm{tw}}$ and the bound of Step 5 for $\log\Sigma^{\mathrm{cub}}$, and take the $\langle\cdot\rangle_L^{\mathrm{tw}}$-mean; the terms $5M^2\log(2\pi/\gamma_N)$ cancel, and Steps 6 and 7 bound the two non-constant terms:

$$\begin{aligned}\phi_L &\le \big\langle \langle Q, \Delta^+_{G^{\mathrm{cub}}} Q\rangle \big\rangle_L^{\mathrm{tw}} + 2\kappa_N \sum_{u(x)=\pm\varepsilon_N} \langle\rho_x^2\rangle_L^{\mathrm{tw}} + M^2\Big[\frac{28\pi^2}{\gamma_N^2} + 7\log K'_{\gamma_N} + \log\frac{G_0}{G_{-6\kappa_N}}\Big] \\ &\le 5M^2\log M + M^2\Big[15 + 4\kappa_N C_N^{(2)} + \frac{28\pi^2}{\gamma_N^2} + 7\log K'_{\gamma_N} + \log\frac{G_0}{G_{-6\kappa_N}}\Big] = M^2\big(5\log M + A_N\big),\end{aligned}$$

with

$$A_N := 15 + 4\kappa_N C_N^{(2)} + \frac{28\pi^2}{\gamma_N^2} + 7\log K'_{\gamma_N} + \log\frac{G_0}{G_{-6\kappa_N}},$$

which depends on N and the couplings only (through γ_N, κ_N, K'_{γ_N}, c_0 and the site moments G_q), not on L. Since $M = L/\varepsilon_N$ and N is fixed, $M^2(5\log M + A_N) = O(L^2\log L) = o(L^3)$. Together with $\phi_L \geq 0$ (Proposition 14.13 at $t = 0$) this proves the lemma. □

Remark 14.15 (the logarithm in the surface bound). The bound of Lemma 14.14 is $O(L^2\log L)$. The logarithm comes from the energy $\langle Q, \Delta^+ Q\rangle$ of the single extension of Step 5: its mean under $\langle\cdot\rangle_L^{\mathrm{tw}}$ is at most $2\,\mathrm{Tr}\,\Delta^+$ by the covariance bound $S \leq 2\,\mathrm{Id}$ of Step 6(a), and $\mathrm{Tr}\,\Delta^+$ is at most of order $M^2\log M$ (Step 7). Of this bound the argument of this section uses only $\phi_L = o(L^3)$.

Proposition 14.16 (Jensen and convexity: the comparison inequality at finite volume)**.** *Let $D = (c_0, c_+, c_3; h; O)$ be a datum on T_L^{tw}, $L \geq 6\varepsilon_N$, $h \mid L/2$. Then for every real t*

$$t\,n_c\,\langle O\rangle_L^{\mathrm{cub}} \;\leq\; \phi_L + t\,n_c\,\omega_L^t(O), \qquad \textit{hence for } t > 0: \qquad \langle O\rangle_L^{\mathrm{cub}} \;\leq\; \omega_L^t(O) + \frac{h^3\,\phi_L}{2\,t\,L^3}.$$

Proof. Reduction to $c_0 = 0$. The translation τ by $-c_0 e_1$ lies in $\varepsilon_N\mathbb{Z}^3$, acts on both tori and preserves both states (Lemma 14.6; Lemma 11.1); it maps D to the datum $(0, c_+ - c_0, c_3; h; O\circ\tau^{-1})$, carries μ_L^t to the perturbed law of that datum, and leaves $\langle O\rangle_L^{\mathrm{cub}}$, Z_t, ϕ_L and $\omega_L^t(O)$ unchanged. So let $c_0 = 0$ and define O_d on T_L^{cub} as in Proposition 14.13.

Jensen on the cubic torus. Each of the n_c summands of O_d on T_L^{cub} is $O\circ w^{-1}$ with w a composition of the maps r_c, s_c, q_c and r_0, each an automorphism of T_L^{cub} preserving W and the reference measure (a signed coordinate permutation about a site composed with a translation of $\varepsilon_N\mathbb{Z}^3$, Lemma 11.1); hence $\langle O\circ w^{-1}\rangle_L^{\mathrm{cub}} = \langle O\rangle_L^{\mathrm{cub}}$ and $\langle O_d\rangle_L^{\mathrm{cub}} = n_c\langle O\rangle_L^{\mathrm{cub}}$. Jensen's inequality and Proposition 14.13 give

$$t\,n_c\langle O\rangle_L^{\mathrm{cub}} = t\langle O_d\rangle_L^{\mathrm{cub}} \leq \log\langle e^{tO_d}\rangle_L^{\mathrm{cub}} \leq \phi_L + \log Z_t.$$

Convexity on the adapted torus. $p(s) := \log Z_s$ is finite ($|O_d| \leq n_c\|O\|_\infty$) and convex in s (Hölder), $p(0) = 0$, and $p'(s) = Z_s^{-1}\mu_L^{\mathrm{tw}}(O_d e^{sO_d}) = \mu_L^s(O_d) = n_c\,\omega_L^s(O)$ by Lemma 14.12(i) (differentiation under the integral sign, O_d bounded). Convexity gives $p(t) \leq p(0) + t\,p'(t) = t\,n_c\,\omega_L^t(O)$. Combining, $t\,n_c\langle O\rangle_L^{\mathrm{cub}} \leq \phi_L + t\,n_c\omega_L^t(O)$; for $t > 0$ divide by $t\,n_c = 2tL^3/h^3$. □

Remark 14.17 (use along the sides). To pass Proposition 14.16 to ω_N the fold must run on the adapted torus of the *same* side L_n as the cubic torus of Definition 2.3, with $h \mid L_n/2$, and the error $h^3\phi_{L_n}/(2tL_n^3)$ must vanish. Under the choice of dyadic sides of Definition 14.1, for a datum with $h = h_j$ one has $h_j \mid L_n/2$ for all $n \geq j$, so the fold runs for all large n along (L_n) and along every subsequence of it, in particular along the one defining a dyadic limit point; and Lemma 14.14 gives $h^3\phi_{L_n}/(2tL_n^3) \to 0$ at fixed h and $t > 0$. No other consequence of Lemma 14.14 is used below.

14.4. **Limits: the comparison states and the hypotheses of the rigidity theorem.** Fix a dyadic sequence (L_n), a datum $D = (c_0, c_+, c_3; h; O)$ with $h = h_j$ one of its wall spacings, and $n \geq j$, so that $h \mid L_n/2$ and D is a datum on every $T_{L_n}^{\mathrm{tw}}$. For a finite set of cells of $\varepsilon_N\mathbb{Z}^3$ and L_n larger than its diameter, the cells embed in $T_{L_n}^{\mathrm{tw}}$ and $\mu_{L_n}^t$ induces a law of their variables; in this sense $(\mu_{L_n}^t)_n$ is a sequence of laws on the full configuration space E of Lemma 14.9(c). We write $\langle\cdot\rangle_t$ for $\mu_{L_n}^t$, $L = L_n$.

Lemma 14.18 (block doubling under the perturbed law)**.** *Let Q be a block of $2^{k_1}\times 2^{k_2}\times 2^{k_3}$ consecutive $\mathcal{W}$-cells (the walls between them included), with $2^{k_1}h \leq L/2$, $2^{k_2}h \leq L$, $2^{k_3}h \leq L/2$, and let $G' \geq 0$ be a bounded Borel function of cells inside the open block Q. Then there is a finite*

$A \subset \mathcal{W}$ with the blocks $w(Q)$, $w \in A$, pairwise disjoint and $|A|\,|Q| > n_c/8$ ($|Q| = 2^{k_1+k_2+k_3}$ the number of cells of Q), such that

$$\langle G' \rangle_t \;\le\; \Big\langle \prod_{w\in A} G' \circ w^{-1} \Big\rangle_t^{1/|A|} \;\le\; \Big(e^{2|t|n_c\|O\|_\infty} \Big\langle \prod_{w\in A} G' \circ w^{-1} \Big\rangle_L^{\mathrm{tw}} \Big)^{1/|A|}.$$

The same holds for unbounded Borel $G' \ge 0$.

Proof. Let Π be one of the two bounding walls of Q in the u-direction, a wall of D; by the hypothesis on the u-extent, Q lies in one of the two closed halves bounded by Π and its second wall. By Lemma 14.12(ii) the form $(G, \tilde G) \mapsto \langle \overline{G \circ \theta_\Pi}\, \tilde G \rangle_t$ is positive semidefinite and Hermitian on the bounded Borel functions of cells in that closed half (Hermitian by the θ_Π-invariance), so the Schwarz inequality with $\tilde G = 1$ gives $\langle G' \rangle_t^2 \le \langle (G' \circ \theta_\Pi)\, G' \rangle_t$, and $(G' \circ \theta_\Pi) G'$ is a function of cells inside the doubled open block $Q \cup \Pi \cup \theta_\Pi(Q)$, a wall-bounded block of twice the cells, whose cells are distinct modulo the torus since its u-extent is at most L. We iterate this doubling, first in u while the current u-extent is at most $L/2$, then in $v(x)$ (period $2L$, halves of length L) while the v-extent is at most L, then in x_3 while the x_3-extent is at most $L/2$. The final block has extents greater than $L/2$, L, $L/2$, so it covers more than $(L/2)\cdot L \cdot (L/2) = L^3/4$ of the volume $2L^3$ of the torus in the coordinates $(u, v(x), x_3)$: with A the set of the 2^m elements of $\mathcal{W}$ (products of the reflections used) carrying Q onto the 2^m constituent blocks, $|A|\,|Q| > n_c/8$ and $\langle G' \rangle_t^{2^m} \le \langle \prod_{w\in A} G' \circ w^{-1} \rangle_t$. The second inequality is the removal of the source: for $H \ge 0$,

$$\langle H \rangle_t = Z_t^{-1} \mu_L^{\mathrm{tw}}(e^{tO_d} H) \le e^{2|t|n_c\|O\|_\infty} \langle H \rangle_L^{\mathrm{tw}}$$

by the bounds on Z_t and e^{tO_d} in Definition 14.10. For unbounded $G' \ge 0$ apply the two inequalities to $G' \wedge M$ and let $M \to \infty$ (monotone convergence). □

Lemma 14.19 (uniform bounds and tightness of the perturbed laws)**.** *For a site x let Q_x be the smallest wall-bounded block of $\mathcal{W}$-cells containing x in its interior ($|Q_x| \in \{1,2,4,8\}$, one cell if x lies on no wall of D, doubled in each coordinate direction in which x lies on a wall). Then for every $q \ge 0$, every real t and all large n,*

$$\langle e^{q\rho_x^2} \rangle_t \;\le\; e^{16|t|\,|Q_x|\,\|O\|_\infty} \frac{G_q^{\mathrm{ct}}}{G_0^{\mathrm{ct}}}\, C_1^{4|Q_x|(h/\varepsilon_N)^3} \;\le\; e^{128|t|\,\|O\|_\infty} \frac{G_q^{\mathrm{ct}}}{G_0^{\mathrm{ct}}}\, C_1^{32(h/\varepsilon_N)^3} =: C_N''(q,t,h),$$

uniformly in x and n. For a finite set P of plaquettes and $s \in \mathbb{R}^P$, with Q_P a block as in Lemma 14.18 whose six bounding walls have every vertex of every $p \in P$ strictly between them (it exists for all large n),

$$\langle e^{\sum_{p\in P} s_p \mathfrak{f}_p} \rangle_t \le e^{16|t|\,|Q_P|\,\|O\|_\infty e^{|s|^2/2}}.$$

Consequently $(\mu_{L_n}^t)_n$ is tight on E and has weak limit points ν_t (along subsequences of (L_n)).

Proof. Sites. Apply Lemma 14.18 to $Q = Q_x$ and $G' = e^{q\rho_x^2}$. The sites wx, $w \in A$, are pairwise distinct (x is interior to Q_x and the blocks $w(Q_x)$ are disjoint), so $\prod_{w\in A} G' \circ w^{-1} = \prod_{y\in A\cdot x} e^{q\rho_y^2}$ with $|A\cdot x| = |A|$, and Step 1 of the proof of Lemma 11.2 on T_L^{tw} (proof of Lemma 14.9(b)) gives

$$\Big\langle \prod_{y\in A\cdot x} e^{q\rho_y^2} \Big\rangle_L^{\mathrm{tw}} \le (G_q^{\mathrm{ct}}/G_0^{\mathrm{ct}})^{|A|} C_1^{|\mathcal{X}|} \qquad \text{with } |\mathcal{X}| = (L/\varepsilon_N)^3.$$

Since $|A| > n_c/(8|Q_x|)$ and $n_c = 2L^3/h^3$, we have $|\mathcal{X}|/|A| < 8|Q_x|(L/\varepsilon_N)^3 h^3/(2L^3) = 4|Q_x|(h/\varepsilon_N)^3$ and $2|t|n_c\|O\|_\infty/|A| < 16|t|\,|Q_x|\,\|O\|_\infty$.

Plaquettes. Apply Lemma 14.18 to $Q = Q_P$ and $G' = e^{\sum_{p\in P} s_p \mathfrak{f}_p}$, a continuous function ≥ 0 of cells inside Q_P ($\mathfrak{f}_p$ depends on p and its bonds). $G' \circ w^{-1} = e^{\sum_{p'} s'_{p'} \mathfrak{f}_{p'}}$, where the sum runs over the plaquettes $p' = wp$ of $w(Q_P)$ and $s' = \pm s$ (the reflections may reverse

orientations), so that $|s'| = |s|$; the p' are distinct over $w \in A$ (disjoint blocks). By Lemma 14.9(a), $\langle \prod_{w\in A} G' \circ w^{-1}\rangle_L^{\mathrm{tw}} \le e^{|A|\,|s|^2/2}$, so

$$\langle G'\rangle_t \le \big(e^{2|t|n_c\|O\|_\infty e^{|A||s|^2/2}}\big)^{1/|A|} \le e^{16|t|\,|Q_P|\,\|O\|_\infty} e^{|s|^2/2}.$$

Tightness. The bounds for $e^{\rho_x^2}$ and $e^{\pm \mathfrak{f}_p}$ give, by Markov's inequality, uniform tail bounds for $\tilde\rho_x$ and $\mathfrak{f}_p$. The variable B_b ranges over a bounded interval and $v_p = \mathfrak{f}_p - (dB)_p$ is tight because $\mathfrak{f}_p$ is; the limits are taken with B_b in the closed interval $[-\pi/\gamma_N, \pi/\gamma_N]$. Under every limit point ν_t each of the two end points $\pm\pi/\gamma_N$ has measure zero. To prove this, let $\Delta = \{b\}$ and let γ_Δ, γ_Δ^t, I_Δ and S_Δ be as in the paragraph on the fine specification that precedes Lemma 14.21. For a bounded continuous local function $f \ge 0$ on the closed space (the space E with the closed intervals $[-\pi/\gamma_N, \pi/\gamma_N]$ for the bond variables) and all large n, integrating the variable B_b first gives $\langle f\rangle_t = \langle \gamma_\Delta^t f\rangle_t \le e^{2|t|S_\Delta}\langle \gamma_\Delta f\rangle_t$, where the inequality holds because $e^{-|t|S_\Delta} \le e^{t\sum_{c\in I_\Delta} O_c} \le e^{|t|S_\Delta}$. The function $\gamma_\Delta f$ is bounded and continuous on the closed space, because the factors of W are continuous functions of the bond variables on the closed interval, and therefore $\nu_t(f) \le e^{2|t|S_\Delta}\nu_t(\gamma_\Delta f)$. For $f_k := \max\{0, 1 - k(\pi/\gamma_N - |B_b|)\}$ the functions $\gamma_\Delta f_k \le 1$ tend to 0 pointwise as $k \to \infty$, since γ_Δ integrates B_b against a density with respect to Lebesgue measure, so that $\nu_t(|B_b| = \pi/\gamma_N) \le \nu_t(f_k) \le e^{2|t|S_\Delta}\nu_t(\gamma_\Delta f_k) \to 0$ by dominated convergence. Hence the limit points are carried by E, and the convergence holds for the bounded functions that are continuous off this null set. The finite-dimensional laws are tight on the Polish factors of E, and Prokhorov's theorem with the diagonal argument of the proof of Theorem 11.5 gives weak limit points on E. □

Lemma 14.20 (joint Gaussian domination of the comparison states)**.** *Let ν_t be a weak limit point of $(\mu_{L_n}^t)_n$. For every finite set P of plaquettes of $\varepsilon_N\mathbb{Z}^3$ and every $s \in \mathbb{R}^P$,*

$$\nu_t\big(e^{\sum_{p\in P} s_p \mathfrak{f}_p}\big) \le e^{16|t|\,|Q_P|\,\|O\|_\infty} e^{|s|^2/2}.$$

If ν_0 is a weak limit point of $(\nu_{1/k})_k$, then $\nu_0\big(e^{\sum_{p\in P} s_p\mathfrak{f}_p}\big) \le e^{|s|^2/2}$ for every finite P and $s \in \mathbb{R}^P$; this is hypothesis (I2) *of Theorem 13.1 with $C = 1$.*

Proof. $G := e^{\sum_p s_p\mathfrak{f}_p}$ is continuous and ≥ 0 on E, so along the subsequence with $\mu_{L_n}^t \to \nu_t$ the Portmanteau theorem (lower semicontinuity for non-negative continuous functions) gives $\nu_t(G) \le \liminf_n \langle G\rangle_t \le e^{16|t||Q_P|\|O\|_\infty} e^{|s|^2/2}$ by Lemma 14.19, and then $\nu_0(G) \le \liminf_k \nu_{1/k}(G) \le e^{|s|^2/2}$. □

The fine specification. For a finite set Δ of cells of $\varepsilon_N\mathbb{Z}^3$ the kernel γ_Δ integrates the variables of Δ against the product of those local factors of W (site, bond and plaquette factors and the closedness indicators of the cubes) that involve a cell of Δ, normalised by $Z_\Delta(\xi_{\mathrm{out}})$, the outside variables held fixed; this is the specification of Theorem 13.1 (unitary gauge). The walls, $\mathcal{W}$-cells and functions O_c are defined on $\varepsilon_N\mathbb{Z}^3$ as in Definition 14.10 (the reflections of Lemma 14.6 act on $\varepsilon_N\mathbb{Z}^3$; the group they generate is again simply transitive on the $\mathcal{W}$-cells, by the dihedral argument of Lemma 14.11 on the line), and γ_Δ^t is the kernel with the extra factor $e^{t\sum_{c\in I_\Delta} O_c}$, I_Δ the finite set of $\mathcal{W}$-cells c for which O_c depends on a variable of Δ; $S_\Delta := |I_\Delta|\,\|O\|_\infty$.

Lemma 14.21 (the comparison states: Gibbs, $\mathcal{W}$-invariant, reflection positive)**.** *Let ν_t be a weak limit point of $(\mu_{L_n}^t)_n$ and ν_0 a weak limit point of $(\nu_{1/k})_k$ (which exists: the bounds of Lemma 14.19 are uniform in $|t| \le 1$). (i) ν_t is carried by closed v, and $\nu_t(f) = \nu_t(\gamma_\Delta^t f)$ for every finite Δ and bounded Borel f on E. (ii) ν_t is invariant under the reflections generating $\mathcal{W}$ and reflection positive at every wall of D in the sense of Theorem 11.5(d) (positive semidefinite Hermitian form, conjugation identity, $L^2(\nu_t)$ extension). (iii) ν_0 is a Gibbs measure of the fine*

specification ($\nu_0(f) = \nu_0(\gamma_\Delta f)$ for all finite Δ, bounded Borel f), carried by closed v, invariant under the reflections generating $\mathcal{W}$, reflection positive at every wall of D, and satisfies $\nu_0(e^{q\rho_x^2}) \le (G_q^{\mathrm{ct}}/G_0^{\mathrm{ct}})\, C_1^{32(h/\varepsilon_N)^3}$ for every site x and $q \ge 0$, and the Gaussian bound of Lemma 14.20.

Proof. (i) For $L = L_n$ larger than the diameter of Δ and of the cells of the O_c, $c \in I_\Delta$, integrating the Δ-variables first under μ_L^t gives $\mu_L^t(f) = \mu_L^t(\gamma_\Delta^t f)$ for bounded continuous local f: the torus weight times e^{tO_d} is the product of the factors involving Δ (those of γ_Δ^t, including $e^{t\sum_{c\in I_\Delta} O_c}$) and factors independent of the Δ-variables. The kernel is Feller: the factors are continuous in the outside variables, $Z_\Delta(\xi_{\mathrm{out}}) \in (0,\infty)$ on the outside configurations that are restrictions of closed configurations (the closed configuration itself provides an admissible term; finiteness by (2.1)), and dominated convergence applies; so $\gamma_\Delta^t f$ is bounded continuous, both sides pass to the limit, and $\nu_t(f) = \nu_t(\gamma_\Delta^t f)$ for bounded continuous local f. For each cube c the set $\{(dv)_c = 0\}$ is clopen in E (finitely many discrete variables), so $\nu_t((dv)_c = 0) = \lim_n \mu_{L_n}^t((dv)_c = 0) = 1$: ν_t is carried by closed v, where the kernel is defined. Now $f \mapsto \nu_t(\gamma_\Delta^t f)$ is a Borel probability measure on E (γ_Δ^t is a Markov kernel) agreeing with ν_t on the bounded continuous local functions; these determine every finite-dimensional marginal (bounded continuous functions determine a finite Borel measure on a metric space), and the cylinder sets form a π-system generating the Borel σ-algebra of the Polish product E, so the two measures coincide: the identity holds for every bounded Borel f.

(ii) For a generating reflection R and bounded continuous local F, $\langle F\circ R\rangle_t = \langle F\rangle_t$ for all large n (Lemma 14.12(i)); both sides converge, so ν_t is R-invariant. For F bounded continuous of cells in a closed half space bounded by a wall Π of D, the cells lie in one closed half of $T_{L_n}^{\mathrm{tw}}$ for all large n, $\overline{F\circ\theta_\Pi}\cdot F$ is bounded continuous local, and $\nu_t(\overline{F\circ\theta_\Pi}F) = \lim_n \langle \overline{F\circ\theta_\Pi}F\rangle_t \ge 0$ by Lemma 14.12(ii). The Hermitian form, the identity $\nu_t(\overline{F\circ\theta_\Pi}) = \overline{\nu_t(F)}$ and the L^2 extension follow as in the proof of Lemma 14.9(c) (polarisation; θ_Π-invariance; Lusin density and L^2-continuity).

(iii) The source bound $e^{-|t|S_\Delta} \le e^{t\sum_{c\in I_\Delta} O_c} \le e^{|t|S_\Delta}$ gives $\|\gamma_\Delta^t f - \gamma_\Delta f\|_\infty \le 2\|f\|_\infty(e^{2|t|S_\Delta}-1)$ (write $\gamma_\Delta^t f - \gamma_\Delta f = \gamma_\Delta^t(f-\gamma_\Delta f)$ and compare the two normalised weights). Hence for bounded continuous local f:

$$\nu_0(f) = \lim_k \nu_{1/k}(f) = \lim_k \nu_{1/k}(\gamma_\Delta^{1/k} f) = \lim_k \nu_{1/k}(\gamma_\Delta f) = \nu_0(\gamma_\Delta f)$$

($\gamma_\Delta f$ bounded continuous by the Feller property), and the extension to bounded Borel f is the determining-class argument of (i). Closedness of v, R-invariance and reflection positivity are closed conditions on bounded continuous local functions and pass from $\nu_{1/k}$ to ν_0 as in (ii). For the per-site bound, $\nu_{1/k}(e^{q\rho_x^2}) \le \liminf_n \langle e^{q\rho_x^2}\rangle_{1/k} \le C_N''(q,1/k,h)$ (Portmanteau, Lemma 14.19), and $\nu_0(e^{q\rho_x^2}) \le \liminf_k \nu_{1/k}(e^{q\rho_x^2}) \le \lim_k C_N''(q,1/k,h) = (G_q^{\mathrm{ct}}/G_0^{\mathrm{ct}})C_1^{32(h/\varepsilon_N)^3}$. □

Proposition 14.22 (the comparison states satisfy the hypotheses of Theorem 13.1)**.** *Let $\nu_0 = \nu_0(D)$ be as in Lemma 14.21(iii). Then ν_0 is a Gibbs measure of the fine specification on $\varepsilon_N\mathbb{Z}^3$, invariant under the sublattice $\Lambda' := h(e_1-e_2)\mathbb{Z} + h(e_1+e_2)\mathbb{Z} + 2he_3\mathbb{Z}$ of $\varepsilon_N\mathbb{Z}^3$, which has finite index (it contains $2h\mathbb{Z}^3$) and contains the time translation $2he_3$, and satisfies* (I1), (I2) *and* (I3) *of Theorem 13.1. Consequently ν_0 is invariant under the translations $\tau_{2\varepsilon_N e_j}$, $j = 1,2,3$.*

Proof. The Gibbs property is Lemma 14.21(iii). For the periodicity, by Lemma 14.6 the products $r_c r_{c'}$, $s_c s_{c'}$, $q_c q_{c'}$ of generating reflections ($c - c' \in h\mathbb{Z}$) are the translations by $h\mathbb{Z}(e_1-e_2)$, $h\mathbb{Z}(e_1+e_2)$, $2h\mathbb{Z}e_3$, under which ν_0 is invariant by Lemma 14.21(iii); $2he_1 = h(e_1-e_2)+h(e_1+e_2)$ and $2he_2 = h(e_1+e_2)-h(e_1-e_2)$ show $2h\mathbb{Z}^3 \subset \Lambda'$. Hypothesis (I2) is Lemma 14.20 with $C = 1$. Hypothesis (I3) is the per-site bound of Lemma 14.21(iii), uniform in x, for every $q > 0$. For (I1), with $s = \pm 1$ on the single plaquette p, (I2) gives $\nu_0(\mathfrak{f}_p^2) \le 2\nu_0(\cosh \mathfrak{f}_p) \le 2e^{1/2}$ (as

$f^2 \le 2\cosh f$), and $\nu_0(\rho_x^2) \le \nu_0(e^{\rho_x^2}) < \infty$ by (I3). Theorem 13.1 then gives the invariance under $\tau_{2\varepsilon_N e_j}$, $j = 1, 2, 3$. □

Proposition 14.23 (the comparison inequality)**.** *Let ω be a dyadic limit point, formed along the subsequence $(L_{n_k})_k$ of the dyadic sequence (L_n), let D be a datum with $h \in \{h_j\}$ whose observable O is a function of the gauge-invariant variables $(\mathfrak{f}_p, \tilde\rho_x)$ of its cells (evaluated under a law on E through $\mathfrak{f}_p = (dB)_p + v_p$), and let ν_t, ν_0 be weak limit points as in Lemma 14.21 formed along subsequences of $(L_{n_k})_k$. Then*

$$\omega(O) \;\le\; \nu_0(O).$$

Proof. Fix $t > 0$. For all large k, $h \mid L_{n_k}/2$ (Remark 14.17) and Proposition 14.16 gives

$$\langle O\rangle^{\mathrm{cub}}_{L_{n_k}} \le \omega^t_{L_{n_k}}(O) + h^3\phi_{L_{n_k}}/(2tL^3_{n_k}).$$

Along the subsequence with $\mu^t_{L_{n_k}} \to \nu_t$ the left side tends to $\omega(O)$ (Definition 2.3, O bounded continuous local), $\omega^t_{L_{n_k}}(O) \to \nu_t(O)$, and the error tends to 0 by Lemma 14.14. So $\omega(O) \le \nu_t(O)$ for every $t > 0$, and $\nu_{1/k}(O) \to \nu_0(O)$ along the subsequence defining ν_0. □

Lemma 14.24 (reflection positivity from one-sided domination)**.** *Let Π be a plane, $\theta := \theta_\Pi$, and let ν be a law on Ω_N or on E such that for every real bounded continuous local O there is a law ν_O, θ-invariant and reflection positive at Π on $\mathcal{A}_+(\Pi)$, with $\nu(O) \le \nu_O(O)$. Then ν is θ-invariant on the bounded continuous local functions and $\nu(\overline{F\circ\theta}\cdot F) \ge 0$ for every real $F \in \mathcal{A}_+(\Pi)$.*

Proof. For real $F \in \mathcal{A}_+(\Pi)$ take $O := -(F\circ\theta)F$, which is real, bounded, continuous and local; then $\nu(-(F\circ\theta)F) \le \nu_O(-(F\circ\theta)F) \le 0$. For a real bounded continuous local G take $O := \pm(G\circ\theta - G)$; then $\pm\nu(G\circ\theta - G) \le \pm\nu_O(G\circ\theta - G) = 0$, so $\nu(G\circ\theta) = \nu(G)$. □

14.5. **A second proof of the tangent inequality for the comparison states.** This subsection gives a direct proof, for the comparison states, of the inequality which the proofs of Lemmas 13.14 and 13.25 take from the tangent inequality, Lemma 13.9; the result is Proposition 14.31. The proof does not use Lemma 13.9 and uses no relative entropy: the comparison states are limits of laws on tori, and on a torus the inequality follows from Jensen's inequality and the transfer calculus of Section 13. Its main ingredients are Lemmas 13.3, 13.7, 13.10 and 13.21 (with Lemmas 13.22, 13.24 and Remark 13.8 for the full class), the insertion identity (13.28), the covariance of the weight under the permutations of the axes (the first part of the proof of Proposition 13.39(i)), the fold at zero source (Proposition 14.13), the lower bound of the partition function in the proof of Lemma 11.2, and Lemmas 14.6, 14.12 and 14.21. Neither Theorem 13.1 nor the upper bound of Lemma 14.14 enters the proof, and no other part of this paper depends on this subsection (Remark 14.32).

Notation. In this subsection t is the parameter of the operators X_t of Lemmas 13.14 and 13.25, as in Section 13; the source strength of Subsection 14.4 is always written $1/k$. The symbol T without an index is the transfer operator of Lemma 13.7(c), and tori carry an index. The symbol L is a side length; the layer space of Lemma 13.7(c) is not given a symbol, and R_L, P_G denote, as there, its gauge representation and its gauge projection.

Tori and partition functions. For a sublattice $\Gamma \subset \varepsilon_N\mathbb{Z}^3$ of finite index all of whose non-zero elements have length at least $4\varepsilon_N$ let $T_\Gamma := \varepsilon_N\mathbb{Z}^3/\Gamma$, let μ_Γ be the probability law with weight W on the cells of T_Γ (sum over closed v) and

$$Z(\Gamma) := \int d\tilde\rho \int dB \sum_{v\ \mathrm{closed}} W$$

its partition function (Lebesgue measure on the $\tilde\rho_x$ and on the $B_b \in (-\pi/\gamma_N, \pi/\gamma_N]$, counting measure on the closed v: the convention of Step 1 of the proof of Lemma 11.2). Thus $Z^{\mathrm{cub}}_L =$

$Z(L\mathbb{Z}^3)$, $Z_L^{\mathrm{tw}} = Z(\Gamma(L))$ and $\mu_L^{\mathrm{tw}} = \mu_{\Gamma(L)}$. For a permutation g of the coordinate axes the relabelling $\hat{g}$ maps the cell complex, the weight, the closure constraint and the reference measure of T_Γ onto those of $T_{g\Gamma}$ (first part of the proof of Proposition 13.39(i)), so that

$$Z(g\Gamma) = Z(\Gamma), \qquad \mu_{g\Gamma} = \mu_\Gamma \circ \hat{g}. \tag{14.1}$$

For $a_1, a_2, a_3 \in \varepsilon_N\mathbb{N}$, each at least $4\varepsilon_N$, we write $Z(a_1, a_2, a_3) := Z(a_1e_1\mathbb{Z} + a_2e_2\mathbb{Z} + a_3e_3\mathbb{Z})$ for the partition function of the product torus with sides a_1, a_2, a_3, which has $a_1a_2a_3/\varepsilon_N^3$ sites; $Z_L^{\mathrm{cub}} = Z(L, L, L)$.

Tori with a twist. With e_3 as time direction, as in Section 13, let $\Gamma_s \subset \varepsilon_N(\mathbb{Z}e_1 + \mathbb{Z}e_2)$ be a sublattice of finite index, $\varepsilon_N(\mathbb{Z}e_1 + \mathbb{Z}e_2)/\Gamma_s$ the *layer torus*, $n_{\mathrm{lay}} := [\varepsilon_N(\mathbb{Z}e_1 + \mathbb{Z}e_2) : \Gamma_s]$ its number of sites, $M \geq 2$ an integer and $\zeta \in \varepsilon_N(\mathbb{Z}e_1 + \mathbb{Z}e_2)$ (the *twist*), and put

$$\Gamma(\Gamma_s, M, \zeta) := \Gamma_s + \mathbb{Z}\,(M\varepsilon_N e_3 + \zeta).$$

We assume, as above, that all non-zero elements of $\Gamma(\Gamma_s, M, \zeta)$ have length at least $4\varepsilon_N$. The torus $T_{\Gamma(\Gamma_s,M,\zeta)}$ consists of M layers, each a copy of the layer torus, and its layer $M+1$ is the layer 1 translated by ζ: in terms of the layer configurations $c^{(s)}$ of Subsection 13.4, $c^{(M+1)} = \tau_\zeta c^{(1)}$ with $(\tau_\zeta c)(y) := c(y - \zeta)$. The *$\zeta$-twisted periodic measure* μ_M^ζ is the probability law with the phase-restored weight of Subsection 13.2 on the phase-restored configurations of $T_{\Gamma(\Gamma_s,M,\zeta)}$. For $\zeta = 0$ it is the periodic measure μ_M with M layers of the proof of Lemma 13.7(c), for the layer torus at hand: the statements and proofs of Lemmas 13.7(c), 13.10, 13.21 and of (13.28) use the layer only as a finite complex (its sites, spatial bonds and spatial plaquettes, the gauge group $\mathbb{T}^{\mathrm{sites}}$ and the one-layer weight, of finite integral) and hold for every layer torus. For general ζ the only change is that the closing slab joins $c^{(M)}$ to $\tau_\zeta c^{(1)}$ instead of $c^{(1)}$.

For insertions X_s at the heights $s \in \{1, \dots, M\}$ ($X_s := 1$ where there is none; fibred in the sense of Lemma 13.7(b), or bounded on $\mathcal{H}_{\mathrm{full}}$ as in Lemma 13.21), $Z_M^\zeta(X_1, \dots, X_M)$ denotes the partition function of μ_M^ζ with the slab kernel K replaced by K^{X_s} (Lemma 13.7(b)), or by $\langle \Phi_{\mathrm{full}}(c), X_s\Phi_{\mathrm{full}}(\bar{c}')\rangle$ for an insertion on $\mathcal{H}_{\mathrm{full}}$ (Lemma 13.21), at the slab s, in the normalisation of Lemma 13.7(c), and $Z_M^\zeta := Z_M^\zeta(1, \dots, 1)$; for $\zeta = 0$, and for insertions that satisfy the summability condition of Lemma 13.7(b) or act on $\mathcal{H}_{\mathrm{full}}$, these are the quantities $Z_M(\cdot)$ and Z_M of (13.9).

Lemma 14.25 (phase restoration on a finite torus)**.** *Let T_Γ be a torus as above and let $\widetilde{\mu}_\Gamma$ be the probability law with the phase-restored weight of Subsection 13.2 on the phase-restored configurations of T_Γ (normalised Haar measure on each phase), with partition function $\widetilde{Z}(\Gamma)$. The formulas of Lemma 13.3 are local and define the gauge action and the Haar randomisation Ψ on the configurations of T_Γ. Then*

(a) *$\widetilde{Z}(\Gamma) = Z(\Gamma)$ and $\widetilde{\mu}_\Gamma = \Psi_*(\mu_\Gamma \times \mathrm{Haar}^{\mathrm{sites}})$;*

(b) *$\Psi \circ (\tau_a \times \tau_a) = \tau_a \circ \Psi$ for $a \in \varepsilon_N\mathbb{Z}^3$; so for every law μ on the configurations of T_Γ the law $\Psi_*(\mu \times \mathrm{Haar}^{\mathrm{sites}})$ is invariant under every translation which leaves μ invariant;*

(c) *if $\Gamma = \Gamma(\Gamma_s, M, \zeta)$, then $\widetilde{\mu}_\Gamma = \mu_M^\zeta$ and $Z(\Gamma) = \varkappa^{n_{\mathrm{lay}}M} Z_M^\zeta$, where the constant $\varkappa \in (0, \infty)$ depends only on the normalisation of the reference measure of the temporal links (in $Z(\Gamma)$ each of the $n_{\mathrm{lay}}M$ temporal links carries the Lebesgue measure of* FD*, of mass c; in (13.9) it carries the normalised Haar measure) and not on Γ_s, M, ζ. ($\varkappa$ is not the hopping constant κ_N.)*

Proof. (a) Let F be a bounded Borel function of the phase-restored configurations (ω, ψ) of T_Γ and fix ψ. By (13.2), $\Psi(\omega, \psi) = (-\psi).(\omega, 0)$ has the phases ψ, and by Lemma 13.3(a) the map $\omega \mapsto \omega^\psi$, the unitary-gauge part of $\Psi(\omega, \psi)$, is a bijection of the configurations with closed v which preserves the reference measure (its inverse is the unitary-gauge part of $\psi.(\,\cdot\,, \psi)$), and the

phase-restored weight is invariant: its value at $\Psi(\omega,\psi)$ is its value at $(\omega,0)$, which is $W(\omega)$ (at zero phases the phase-restored hopping factor is the unitary-gauge one). Hence, with $d\omega$ the reference measure of the unitary-gauge variables and $\widetilde{W}$ the phase-restored weight,

$$\int d\omega\, W(\omega)\, F\big(\Psi(\omega,\psi)\big) = \int d\omega'\, \widetilde{W}(\omega',\psi)\, F(\omega',\psi) \qquad \text{for every } \psi.$$

Integrating over ψ against the normalised Haar measure gives $Z(\Gamma)\,\Psi_*(\mu_\Gamma \times \mathrm{Haar}^{\mathrm{sites}})(F) = \widetilde{Z}(\Gamma)\,\widetilde{\mu}_\Gamma(F)$; $F=1$ gives $\widetilde{Z}(\Gamma)=Z(\Gamma)$, and then the two laws agree.

(b) The identity is Lemma 13.3(e), a statement about the local formulas; it gives $\Psi_*(\mu\times\mathrm{Haar}^{\mathrm{sites}})\circ\tau_a = \Psi_*((\mu\circ\tau_a)\times\mathrm{Haar}^{\mathrm{sites}})$, the Haar measure being translation invariant.

(c) $\widetilde{\mu}_\Gamma$ and μ_M^ζ are both the normalised phase-restored weight on the same torus, written in the layers $c^{(1)},\dots,c^{(M)}$, the temporal links and the temporal Villain integers; the two partition functions differ only by the normalisation of the reference measure of the temporal links, one factor $\varkappa$ for each of the $n_{\mathrm{lay}}M$ temporal links, and $\widetilde{Z}(\Gamma)=Z(\Gamma)$ by (a). □

Lemma 14.26 (twisted averaging identity and twisted Schatten–Hölder bound)**.** *Let Γ_s, $M\ge 2$, ζ be as above, let $T=\mathsf{B}^*\mathsf{B}$, R_L, P_G be the transfer operator, the gauge representation and the gauge projection of Lemma 13.7(c) for the layer torus $\varepsilon_N(\mathbb{Z}e_1+\mathbb{Z}e_2)/\Gamma_s$, and let $(\mathsf{S}_\zeta\psi)(c):=\psi(\tau_\zeta^{-1}c)$ on the layer space. Let X_s, $1\le s\le M$, be insertions at the M heights, each fibred (Lemma 13.7(b)) or bounded on $\mathcal{H}_{\mathrm{full}}$ ($X_s=1$ where there is none); for an insertion on $\mathcal{H}_{\mathrm{full}}$ the symbol $\widehat{\bar{X}_s}$ stands for its gauge average $\bar{X}_s$ of Lemma 13.21, which satisfies $\|\bar{X}_s\|\le\|X_s\|$. Then*

(i) *(trace identity) S_ζ is unitary and commutes with T and with P_G, and*

$$Z_M^\zeta(X_1,\dots,X_M) = \mathrm{Tr}\Big(\prod_{s=1}^{M}\mathsf{B}^*\widehat{\bar{X}_s}\mathsf{B}\cdot P_G\mathsf{S}_\zeta\Big), \qquad \textit{in particular}\quad Z_M^\zeta = \mathrm{Tr}\,\big(P_G T^M\mathsf{S}_\zeta\big); \tag{14.2}$$

(ii) *(Hölder bound) $|Z_M^\zeta(X_1,\dots,X_M)|\le\prod_s\|X_s\|\cdot\mathrm{Tr}(P_GT^M)=\prod_s\|X_s\|\cdot Z_M^0$, where $Z_M^0=Z_M$ is the partition function of the untwisted torus with the same layer torus and the same number of layers; in particular $Z_M^\zeta\le Z_M^0$;*

(iii) *(insertions) if every fibred X_s satisfies the summability condition of Lemma 13.7(b), and every X_s on $\mathcal{H}_{\mathrm{full}}$ is one of the full operators of Lemma 14.27(b) to which Lemma 14.27(c) applies, then, the sum over the variables $\widetilde{v}^t$ being taken first as in (13.28), $\mu_M^\zeta\big(\prod_s r_{X_s}\circ\Theta_\sigma^{(s)}\big)=Z_M^\zeta(X_1,\dots,X_M)/Z_M^\zeta$, hence $\big|\mu_M^\zeta\big(\prod_s r_{X_s}\circ\Theta_\sigma^{(s)}\big)\big|\le\prod_s\|X_s\|\cdot Z_M^0/Z_M^\zeta$.*

Proof. The shift. The one-layer weight $W(c)$, the layer reference measure, the slab kernel K and the gauge action are given by the same local formulas at every cell of the layer torus, so $W(\tau_\zeta c)=W(c)$, the reference measure is τ_ζ-invariant, $K(\tau_\zeta c,\tau_\zeta c')=K(c,c')$ and $\tau_\zeta(\lambda.c)=(\tau_\zeta\lambda).(\tau_\zeta c)$, where $\tau_\zeta\lambda$ is the translated gauge function. Hence S_ζ is unitary on the layer space $L^2(W(c)\,dc)$, commutes with Q and with $T=Q^{1/2}KQ^{1/2}$, and $\mathsf{S}_\zeta R_L(\lambda)\mathsf{S}_\zeta^{-1}=R_L(\tau_\zeta\lambda)$; since $\lambda\mapsto\tau_\zeta\lambda$ is an automorphism of the compact group $\mathbb{T}^{\mathrm{sites}}$, it preserves the Haar measure, and $\mathsf{S}_\zeta P_G\mathsf{S}_\zeta^{-1}=\int dG\,R_L(\tau_\zeta G)=P_G$.

Change of variables. We repeat the proof of Lemma 13.7(c) up to the closing slab. Put $G_1:=0$, $G_{s+1}:=G_s-B_t^{(s)}$ and $W^{(s)}:=G_s.c^{(s)}$ for $1\le s\le M$. In temporal gauge the slab s has the layers $c^{(s)}=(-G_s).W^{(s)}$ and $(-B_t^{(s)}).c^{(s+1)}$. For $s<M$, $(-B_t^{(s)}).c^{(s+1)}=(-B_t^{(s)}-G_{s+1}).W^{(s+1)}=(-G_s).W^{(s+1)}$, and by Lemma 13.7(b) the slab contributes, as in that proof, $\langle\Phi(W^{(s)}),(X_s^{-d\widetilde{G}_s})^\wedge\Phi(W^{(s+1)})\rangle$. For the closing slab, $c^{(M+1)}=\tau_\zeta c^{(1)}=\tau_\zeta W^{(1)}$ (as $G_1=0$), so its upper layer is $(-B_t^{(M)}).\tau_\zeta W^{(1)}=(-G_M).\big(G.\tau_\zeta W^{(1)}\big)$ with the holonomy

$G := G_{M+1} = G_M - B_t^{(M)}$, and it contributes

$$\big\langle \Phi(W^{(M)}), (X_M^{-d\widetilde{G}_M})^{\wedge} \Phi\big(G.\tau_\zeta W^{(1)}\big)\big\rangle.$$

For a bounded operator $\hat{X}$ on $\mathcal{H}$, the kernel $(c, c'') \mapsto W(c)^{1/2}\langle \Phi(c), \hat{X}\Phi(G.\tau_\zeta c'')\rangle W(c'')^{1/2}$ is the integral kernel of $\mathsf{B}^*\hat{X}\mathsf{B}\, R_L(G)\mathsf{S}_\zeta$: the substitution $c' = G.\tau_\zeta c''$ preserves $W(c)\,dc$ and turns $\psi(c'')$ into $\psi(\tau_\zeta^{-1}((-G).c')) = (R_L(G)\mathsf{S}_\zeta\psi)(c')$. Integrating out the layers (the trace-class structure and the absolute convergence are those of that proof; $R_L(G)\mathsf{S}_\zeta$ is unitary) gives

$$Z_M^\zeta(X_1, \dots, X_M) = \int dG_2 \cdots dG_{M+1}\ \operatorname{Tr}\Big(\prod_{s=1}^{M} \mathsf{B}^*(X_s^{-d\widetilde{G}_s})^{\wedge}\mathsf{B} \cdot R_L(G_{M+1})\mathsf{S}_\zeta\Big).$$

Averaging. Conjugate the operator under the trace by $R_L(\lambda)$, $\lambda \in \mathbb{T}^{\text{sites}}$; the trace is unchanged. By $\mathsf{B}R_L(\lambda) = R(\lambda)\mathsf{B}$ and $R(\lambda)^*\hat{X}R(\lambda) = (X^{d\widetilde{\lambda}})^{\wedge}$ the factor s becomes $\mathsf{B}^*(X_s^{-d(\widetilde{G}_s - \widetilde{\lambda})})^{\wedge}\mathsf{B}$ (for $s = 1$: $\mathsf{B}^*(X_1^{d\widetilde{\lambda}})^{\wedge}\mathsf{B}$), and

$$R_L(\lambda)^{-1} R_L(G_{M+1})\mathsf{S}_\zeta R_L(\lambda) = R_L(G_{M+1} - \lambda + \tau_\zeta\lambda)\mathsf{S}_\zeta.$$

Integrate over λ against the Haar measure and, at fixed λ, substitute $G'_s := G_s - \lambda$ $(2 \le s \le M)$ and $G' := G_{M+1} - \lambda + \tau_\zeta\lambda$, a Haar-preserving bijection: $\lambda, G'_2, \dots, G'_M, G'$ are then independent Haar variables, one for each height and one for the holonomy. The integrand is multilinear in the insertions and linear in $R_L(G')$; exchanging the integrals with the trace (dominated convergence for trace-class operators, as in the proof of Lemma 13.7(c)) and using $\int d\lambda\, (X^{\pm d\widetilde{\lambda}})^{\wedge} = \widehat{\widetilde{X}}$ (Lemma 13.7(b), Lemma 13.6(c)) and $\int dG'\, R_L(G') = P_G$ gives (14.2). With $X_s = 1$ for all s it reads $Z_M^\zeta = \operatorname{Tr}(T^M P_G \mathsf{S}_\zeta) = \operatorname{Tr}(P_G T^M \mathsf{S}_\zeta)$. For insertions on $\mathcal{H}_{\text{full}}$ the same computation and the Hölder step below apply with $R \otimes U$ in place of R and with $P_G^{\mathcal{H}} = \int dG\, (R \otimes U)(G)$, since they use of Lemma 13.21 only the covariance of the feature map and the measure preservation.

Hölder. $\|\widehat{\bar{X}_s}\| \le \|\bar{X}_s\| \le \|X_s\|$ (Lemma 13.7(b), Lemma 13.6(c)). $\bar{X}_s$ is gauge invariant, so $\widehat{\bar{X}_s}$ commutes with every $R(\lambda)$ and with the orthogonal projection $P_G^{\mathcal{H}} = \int dG\, R(G)$; P_G commutes with every $\mathsf{B}^*\widehat{\bar{X}_s}\mathsf{B}$, and $P_G\mathsf{B}^* = \mathsf{B}^* P_G^{\mathcal{H}}$ (proof of Lemma 13.7(c)). Hence, with $P_G = P_G^M$,

$$\prod_{s=1}^{M} \mathsf{B}^*\widehat{\bar{X}_s}\mathsf{B} \cdot P_G = \prod_{s=1}^{M} A_s, \qquad A_s := P_G\mathsf{B}^*\widehat{\bar{X}_s}\mathsf{B} = (P_G^{\mathcal{H}}\mathsf{B})^*\,\widehat{\bar{X}_s}\,(P_G^{\mathcal{H}}\mathsf{B}).$$

By Hölder's inequality for Schatten norms [32] (exponent M on each A_s, the operator norm on the unitary S_ζ), $|\operatorname{Tr}(A_1 \cdots A_M \mathsf{S}_\zeta)| \le \prod_s \|A_s\|_M$, and $\|A_s\|_M \le \|\widehat{\bar{X}_s}\|\, \|P_G^{\mathcal{H}}\mathsf{B}\|_{2M}^2$ with

$$\|P_G^{\mathcal{H}}\mathsf{B}\|_{2M}^{2M} = \operatorname{Tr}\big((\mathsf{B}^* P_G^{\mathcal{H}}\mathsf{B})^M\big) = \operatorname{Tr}\big((P_G T)^M\big) = \operatorname{Tr}(P_G T^M).$$

So $|Z_M^\zeta(X_1, \dots, X_M)| \le \prod_s \|X_s\| \cdot \operatorname{Tr}(P_G T^M)$, and $\operatorname{Tr}(P_G T^M) = Z_M = Z_M^0$ by (13.9) with all insertions equal to 1. This is (ii); without insertions it gives $Z_M^\zeta = |Z_M^\zeta| \le Z_M^0$.

Insertions. The identity (13.28) rests on the statement preceding it, that the insertion of X at one slab multiplies the weight of that slab by $r_X \circ \Theta_\sigma$; this concerns one slab at a time and holds for every slab of μ_M^ζ, the closing one included (its slab data are those of the pair $c^{(M)}$, $\tau_\zeta c^{(1)}$). For a full operator X of Lemma 14.27(b),(c) the same statement holds: by (13.29) and Lemma 13.21 its kernel $\langle \Phi_{\text{full}}(c), X\Phi_{\text{full}}(\tilde{c}')\rangle$ is the sum over $\tilde{v}^t$ of the pair kernel k_X, absolutely convergent by Lemma 14.27(c), whose proof does not use the present lemma. This gives the identity in (iii), and the bound follows from (ii). □

Lemma 14.27 (several windows at one height; the full class)**.** *Let $S_1, \dots, S_\ell$ be pairwise disjoint finite spatial bond sets at one slab, and $S := S_1 \cup \cdots \cup S_\ell$.*

(a) *(Bond class.) For each i let $X^{(i)}$ be a fibred operator on $L^2(\mathbb{R}^{S_i})$ (Lemma 13.7(b)). Then $X := \bigotimes_i X^{(i)}$ on $L^2(\mathbb{R}^S) = \bigotimes_i L^2(\mathbb{R}^{S_i})$ is fibred (Lemma 13.7(b2)), hence bounded and $\mathbb{Z}$-periodic, $\|X\| = \prod_i \|X^{(i)}\|$, and*

$$k_X(B,u) = \prod_i k_{X^{(i)}}(B_{S_i}, u_{S_i}), \qquad r_X = \prod_i r_{X^{(i)}};$$

if every $X^{(i)}$ has the absolute convergence required in Remark 13.8, so has X.

(b) *(Full class.) Assume moreover that every S_i is closed under the bonds of its plaquettes and that the site sets $V(S_i)$ are pairwise disjoint. Let V be a finite set of sites of the layer containing $\bigcup_i V(S_i)$ (on a layer torus: all its sites), and let $\mathcal{H}_{\mathrm{full}}$, Φ_{full} be defined as in Lemma 13.21 with this V; that lemma is stated for the site set of one window, and its statement and proof hold for every finite V. For each i let $X^{(i)}$ be a finite linear combination of operators $T_s(X' \otimes X'_H)$ with $s \in [0,1)^{\mathrm{plaq}(S_i)}$, X' fibred on $L^2(\mathbb{R}^{S_i})$ (Lemma 13.7(b)) with $\sum_{a \in c\mathbb{Z}^{S_i}} |\langle h^B, M_s X' h^{u-a}\rangle| < \infty$ for all $B, u \in \mathbb{R}^{S_i}$, as in Lemma 13.22, and X'_H bounded on $L^2(\mathbb{C}^{V(S_i)})$, acting on $\mathcal{H}_{\mathrm{full}}$ as the identity on the bonds outside S_i and on the sites outside $V(S_i)$. Then the $X^{(i)}$ commute, the* full operator *$X := \prod_i X^{(i)}$ satisfies $\|X\| \le \prod_i \|X^{(i)}\|$, and*

$$k_X = \prod_i k_{X^{(i)}}, \qquad r_X = \prod_i r_{X^{(i)}},$$

where k_X, r_X are taken over the union of the windows (the pair coordinates over S, the Villain integers over the plaquettes of the S_i and the Higgs coordinates over $\bigcup_i V(S_i)$) and $k_{X^{(i)}}$, $r_{X^{(i)}}$ over the window S_i.

(c) *(Absolute convergence in the full class.) The operators $X_t = 1 + tY + t^2 Z$ of Lemma 13.25, with $Y \in \mathcal{C}_{\mathrm{full}}$ and $Z = \delta + Z_b \otimes Z_H$, are of the form in (b), and the sum over $\widetilde{v}^t$ of Remark 13.8 converges absolutely for them and for the full operators built from them in (b).*

Proof. (a) The Gaussian $h^x(\xi) = (2/\pi)^{|S|/4} e^{-|\xi - x|^2}$ is a product over the bonds, $h^x = \bigotimes_i h^{x_{S_i}}$, so $\langle h^B, X h^u\rangle = \prod_i \langle h^{B_{S_i}}, X^{(i)} h^{u_{S_i}}\rangle$; for $X^{(i)} = 1$ this is the factorisation of k_1, and the one of $r_X = k_X/k_1$ follows. The translations by $c\mathbb{Z}^S$ are tensor products of translations, which gives the $\mathbb{Z}$-periodicity; the norm of a tensor product is the product of the norms; and the sum over $\widetilde{v}^t \in c\mathbb{Z}^S$ of $|k_X(B, \widetilde{B}' - \widetilde{v}^t)|$ is the product of the sums over the windows. (This is the tensoring used in Lemma 13.10.)

(b) Consider first one term $\Xi_i = T_{s_i}(X_i \otimes X_{H,i})$ for each window. One has $T_s T_{s'} = T_{s+s'}$. For $i' \neq i$, $T_{s_{i'}}$ commutes with the decomposable operator $\hat{X}_i$: its fibres are $X_i((\widetilde{d}\alpha)_{S_i})$ (Lemma 13.7(b)), and $T_{s_{i'}}$ shifts α by $s_{i'}$, hence $\widetilde{d}\alpha$ by $\widetilde{d}s_{i'}$, which vanishes on S_i, because $s_{i'}$ is supported on $\mathrm{plaq}(S_{i'})$, the plaquettes of $\mathrm{plaq}(S_{i'})$ have all their bonds in $S_{i'}$, and $S_{i'} \cap S_i = \emptyset$. The Higgs factors $X_{H,i}$ act on different tensor factors of $L^2(\mathbb{C}^V)$ and commute with all the T_s and $\hat{X}_{i'}$. Hence $\prod_i \Xi_i = T_s\big((\bigotimes_i X_i) \otimes (\bigotimes_i X_{H,i})\big)$ with $s := \sum_i s_i$, in any order; in particular the $X^{(i)}$ commute. By (13.29) and Lemma 13.21,

$$k_{T_s(X' \otimes X'_H)}(c, \widetilde{c}') = e_s(v)\, \langle h^B, M_s X' h^u\rangle\, k_{X'_H}(\Phi, \widetilde{\Phi}').$$

For $s = \sum_i s_i$: $e_s(v) = \prod_i e_{s_i}(v)$, the i-th factor depending on v over $\mathrm{plaq}(S_i)$ only; $\chi_s = \prod_i \chi_{s_i}$, where χ_{s_i} depends on the coordinates of the bonds of S_i only ($\widetilde{d}s_i$ is supported on S_i), so $M_s = \bigotimes_i M_{s_i}$; h^B and h^u factorise over the disjoint bond sets as in (a), and h^Φ, $h^{\widetilde{\Phi}'}$ over the disjoint site sets. Therefore $k_{\prod_i \Xi_i} = \prod_i k_{\Xi_i}$. Both sides are multilinear in the $X^{(i)}$, so

$k_X = \prod_i k_{X^{(i)}}$; with $X^{(i)} = 1$ this is the factorisation of k_1, and $r_X = \prod_i r_{X^{(i)}}$ follows. The norm bound is the submultiplicativity of the operator norm.

(c) $Y \in \mathcal{C}_{\text{full}}$ is a finite linear combination of operators $T_s(X' \otimes X'_H)$ with $X' \in \mathcal{C}$, $X'_H \in \mathcal{C}_H$ (Lemma 13.24), and $Z_b \otimes Z_H = T_0(Z_b \otimes Z_H)$, $1 = T_0(1 \otimes 1)$; so X_t is of the form in (b). For one term, $|k_{T_s(X' \otimes X'_H)}| = |\langle h^B, M_s X' h^u\rangle|\, |k_{X'_H}(\Phi, \widetilde{\Phi}')|$: the Higgs factor does not depend on $\widetilde{v}^t$, the character $e_s(v)$ is unimodular, and M_s is the multiplication by a unimodular function, which commutes with the localisations $\tau_a^* P_K \tau_a$ in the proof of Lemma 13.11 and preserves norms, so the Gaussian bound of that proof, which gives the absolute convergence for the class $\mathcal{C}$ (Remark 13.8), holds for $\langle h^B, M_s X' h^u\rangle$ as for $\langle h^B, X' h^u\rangle$. For the term $Z_b \otimes Z_H$ the sum over $\widetilde{v}^t$ is that of the bond corrector Z_b, which converges by Remark 13.8, and for the term 1 it is a Gaussian sum. For the full operators of (b) the sum over $\widetilde{v}^t \in c\mathbb{Z}^S$ factorises over the windows. □

Lemma 14.28 (free energy of the product tori: halving a side)**.** *Let $a_1, a_2, a_3 \in \varepsilon_N \mathbb{N}$, each at least $4\varepsilon_N$.*

(i) *(Symmetry.) $Z(a_1, a_2, a_3)$ is symmetric under the permutations of (a_1, a_2, a_3).*

(ii) *(Halving.) $Z(a_1, a_2, 2a_3) \le Z(a_1, a_2, a_3)^2$, and likewise in the first and in the second argument.*

(iii) *(Cubic tori.) Let $\mathsf{p}(L) := (\varepsilon_N/L)^3 \log Z_L^{\text{cub}}$ be the free energy per site of the cubic torus of side $L \in \varepsilon_N \mathbb{N}$, $L \ge 4\varepsilon_N$. Then $\mathsf{p}(2L) \le \mathsf{p}(L)$, and $\mathsf{p}(L) \ge \log c_0$ with the constant $c_0 > 0$ of Lemma 11.2 if moreover $L/2 \in \varepsilon_N \mathbb{Z}$. Consequently, along a dyadic sequence (L_n), the sequence $\mathsf{p}(L_n)$, $n \ge 2$, is non-increasing and bounded below, and*

$$\eta_n := \mathsf{p}(L_{n-1}) - \mathsf{p}(L_n) \;\ge\; 0 \qquad (n \ge 3) \qquad \text{tends to } 0 \text{ as } n \to \infty.$$

(iv) *(With the volumes.) For L with $L/2 \in \varepsilon_N \mathbb{N}$, $L/2 \ge 4\varepsilon_N$,*

$$\log Z\big(\tfrac{L}{2}, 2L, L\big) \;\le\; 8 \log Z_{L/2}^{\text{cub}},$$

$$\text{so that} \qquad \log Z\big(\tfrac{L}{2}, 2L, L\big) - \log Z_L^{\text{cub}} \;\le\; \Big(\frac{L}{\varepsilon_N}\Big)^3 \big(\mathsf{p}(L/2) - \mathsf{p}(L)\big).$$

Proof. (i) is (14.1) for the permutations of the coordinate axes.

(ii) By (i) it suffices to treat the third argument. Let $\Gamma_s := a_1 e_1 \mathbb{Z} + a_2 e_2 \mathbb{Z}$ and $M := a_3/\varepsilon_N \ge 4$. The positivity used is that of the transfer operator, as it is given in Lemma 13.7(c) and its proof, for the layer torus $\varepsilon_N(\mathbb{Z}e_1 + \mathbb{Z}e_2)/\Gamma_s$: $T = \mathsf{B}^*\mathsf{B} \ge 0$ is trace class; P_G is an orthogonal projection (the average of a unitary representation of a compact group) which commutes with T (it commutes with every $\mathsf{B}^* \widehat{X} \mathsf{B}$, in particular for $X = 1$); and $Z_{M'} = \operatorname{Tr}(P_G T^{M'})$ for every number $M' \ge 2$ of layers. Hence $T_G := P_G T = P_G T P_G$ is self-adjoint, non-negative and trace class, $\operatorname{Tr}(P_G T^{M'}) = \operatorname{Tr}(T_G^{M'}) = \sum_i \lambda_i^{M'}$ with $\lambda_i \ge 0$ the eigenvalues of T_G, and

$$\operatorname{Tr}(T_G^{2M}) = \sum_i \lambda_i^{2M} \;\le\; \Big(\sum_i \lambda_i^M\Big)^2 = \big(\operatorname{Tr}(T_G^M)\big)^2.$$

By Lemma 14.25(c) with $\zeta = 0$, $Z(a_1, a_2, M'\varepsilon_N) = \varkappa^{n_{\text{lay}} M'} \operatorname{Tr}(T_G^{M'})$ for $M' \ge 4$; the two sides of the claim carry the same power of $\varkappa$, so

$$Z(a_1, a_2, 2a_3) = \varkappa^{2n_{\text{lay}} M} \operatorname{Tr}(T_G^{2M}) \le \varkappa^{2n_{\text{lay}} M} \operatorname{Tr}(T_G^M)^2 = Z(a_1, a_2, a_3)^2.$$

(iii) Three applications of (ii) give $Z(2L, 2L, 2L) \le Z(2L, 2L, L)^2 \le Z(2L, L, L)^4 \le Z(L, L, L)^8$. The torus of side $2L$ has $8(L/\varepsilon_N)^3$ sites, so

$$\mathsf{p}(2L) = \tfrac{1}{8}(\varepsilon_N/L)^3 \log Z_{2L}^{\text{cub}} \le (\varepsilon_N/L)^3 \log Z_L^{\text{cub}} = \mathsf{p}(L).$$

The lower bound is the last line of Step 1 of the proof of Lemma 11.2: $Z_L^{\text{cub}} \ge c_0^{(L/\varepsilon_N)^3}$ for $L \ge 4\varepsilon_N$ with $L/2 \in \varepsilon_N \mathbb{Z}$ (keep the term $v \equiv 0$ and restrict the integrations). Along a dyadic

sequence $L_n = 2L_{n-1}$, and $L_n \geq 4\varepsilon_N$, $L_n/2 \in \varepsilon_N\mathbb{Z}$ for $n \geq 2$ (as $L_* \in \varepsilon_N\mathbb{N}$); so $\mathsf{p}(L_n)$, $n \geq 2$, is non-increasing and bounded below by $\log c_0$, hence convergent, and $\eta_n \to 0$ (indeed $\sum_{n\geq 3}\eta_n \leq \mathsf{p}(L_2) - \log c_0$).

(iv) By (ii) in the second argument, twice, and then in the third:

$$Z(\tfrac{L}{2}, 2L, L) \leq Z(\tfrac{L}{2}, L, L)^2 \leq Z(\tfrac{L}{2}, \tfrac{L}{2}, L)^4 \leq Z(\tfrac{L}{2}, \tfrac{L}{2}, \tfrac{L}{2})^8 = (Z^{\mathrm{cub}}_{L/2})^8.$$

The cubic torus of side $L/2$ has $\frac{1}{8}(L/\varepsilon_N)^3$ sites, so $8\log Z^{\mathrm{cub}}_{L/2} = (L/\varepsilon_N)^3\mathsf{p}(L/2)$, while $\log Z^{\mathrm{cub}}_L = (L/\varepsilon_N)^3\mathsf{p}(L)$. □

Lemma 14.29 (the adapted torus in the three coordinate directions; the twist costs $o(L^3)$)**.** *Let $g_3 := \mathrm{id}$ and, for $j = 1, 2$, let $g_j \in B_3$ be the transposition of the coordinates j and 3. Let L with $L/2 \in \varepsilon_N\mathbb{N}$, $L/2 \geq 4\varepsilon_N$.*

(i) *(Geometry.) $g_j\Gamma(L) = \Gamma(\Gamma_{s,j}, M_j, \zeta_j)$ for $j = 1, 2, 3$, with*

$$\begin{aligned}
j = 3: &\quad \Gamma_{s,3} = \tfrac{L}{2}(e_1 - e_2)\mathbb{Z} + L(e_1 + e_2)\mathbb{Z}, && M_3 = L/\varepsilon_N, && \zeta_3 = 0;\\
j = 1: &\quad \Gamma_{s,1} = Le_1\mathbb{Z} + 2Le_2\mathbb{Z}, && M_1 = L/(2\varepsilon_N), && \zeta_1 = -\tfrac{L}{2}e_2;\\
j = 2: &\quad \Gamma_{s,2} = 2Le_1\mathbb{Z} + Le_2\mathbb{Z}, && M_2 = L/(2\varepsilon_N), && \zeta_2 = -\tfrac{L}{2}e_1.
\end{aligned}$$

For $j = 1, 2$ the untwisted torus with the same layer torus and the same number of layers, $T_{\Gamma(\Gamma_{s,j}, M_j, 0)}$, is the product torus with sides $(L, 2L, \frac{L}{2})$, resp. $(2L, L, \frac{L}{2})$; we write $Z^{\mathrm{rect}}_L := Z(\frac{L}{2}, 2L, L)$ for its partition function (Lemma 14.28(i)).

(ii) *(Twist comparison.) For $j = 1, 2$, in the notation of Lemma 14.26 for $(\Gamma_{s,j}, M_j, \zeta_j)$,*

$$1 \;\leq\; \frac{Z^0_{M_j}}{Z^{\zeta_j}_{M_j}} = \frac{Z^{\mathrm{rect}}_L}{Z^{\mathrm{tw}}_L} \;\leq\; \exp\Big[\Big(\frac{L}{\varepsilon_N}\Big)^3\big(\mathsf{p}(L/2) - \mathsf{p}(L)\big)\Big].$$

(iii) *(The surface term of the fold.) With $\phi_L = \log(Z^{\mathrm{tw}}_L/Z^{\mathrm{cub}}_L)$ as in Proposition 14.13,*

$$0 \;\leq\; \phi_L \;\leq\; \Big(\frac{L}{\varepsilon_N}\Big)^3\big(\mathsf{p}(L/2) - \mathsf{p}(L)\big).$$

Along a dyadic sequence, for all large n: $0 \leq \log Z^{\mathrm{rect}}_{L_n} - \log Z^{\mathrm{tw}}_{L_n} \leq (L_n/\varepsilon_N)^3\eta_n$ and $0 \leq \phi_{L_n} \leq (L_n/\varepsilon_N)^3\eta_n$, with $\eta_n \to 0$ (Lemma 14.28(iii)).

Proof. (i) $\Gamma(L) = \frac{L}{2}(e_1 - e_2)\mathbb{Z} + L(e_1 + e_2)\mathbb{Z} + Le_3\mathbb{Z}$ (Definition 14.5), which is the case $j = 3$. The transposition g_1 gives $g_1\Gamma(L) = \frac{L}{2}(e_3 - e_2)\mathbb{Z} + L(e_3 + e_2)\mathbb{Z} + Le_1\mathbb{Z}$, and $L(e_3 + e_2) = 2\cdot\frac{L}{2}(e_3 - e_2) + 2Le_2$, so $g_1\Gamma(L) = Le_1\mathbb{Z} + 2Le_2\mathbb{Z} + \mathbb{Z}\big(\frac{L}{2}e_3 - \frac{L}{2}e_2\big)$. The transposition g_2 gives $g_2\Gamma(L) = \frac{L}{2}(e_1 - e_3)\mathbb{Z} + L(e_1 + e_3)\mathbb{Z} + Le_2\mathbb{Z}$, and $L(e_1 + e_3) = 2Le_1 - 2\cdot\frac{L}{2}(e_1 - e_3)$, so $g_2\Gamma(L) = 2Le_1\mathbb{Z} + Le_2\mathbb{Z} + \mathbb{Z}\big(\frac{L}{2}e_3 - \frac{L}{2}e_1\big)$. The untwisted tori are $\varepsilon_N\mathbb{Z}^3/(Le_1\mathbb{Z} + 2Le_2\mathbb{Z} + \frac{L}{2}e_3\mathbb{Z})$ and $\varepsilon_N\mathbb{Z}^3/(2Le_1\mathbb{Z} + Le_2\mathbb{Z} + \frac{L}{2}e_3\mathbb{Z})$; all these tori have $(L/\varepsilon_N)^3$ sites.

(ii) By Lemma 14.25(c), (14.1) and (i), $\varkappa^{(L/\varepsilon_N)^3}Z^{\zeta_j}_{M_j} = Z(g_j\Gamma(L)) = Z(\Gamma(L)) = Z^{\mathrm{tw}}_L$ and $\varkappa^{(L/\varepsilon_N)^3}Z^0_{M_j} = Z^{\mathrm{rect}}_L$, which is the equality. The lower bound is Lemma 14.26(ii) without insertions: $Z^{\mathrm{tw}}_L \leq Z^{\mathrm{rect}}_L$. For the upper bound, Proposition 14.13 gives $Z^{\mathrm{cub}}_L \leq Z^{\mathrm{tw}}_L$ (its statement contains $\phi_L \geq 0$; it applies, as $L \geq 8\varepsilon_N$, to the datum $(0, 0, 0; \frac{L}{2}; 0)$ with zero source), and Lemma 14.28(iv) gives $\log Z^{\mathrm{rect}}_L - \log Z^{\mathrm{cub}}_L \leq (L/\varepsilon_N)^3(\mathsf{p}(L/2) - \mathsf{p}(L))$.

(iii) $\phi_L \geq 0$ is part of Proposition 14.13, and by (ii) and Lemma 14.28(iv),

$$\phi_L = \log Z^{\mathrm{tw}}_L - \log Z^{\mathrm{cub}}_L \leq \log Z^{\mathrm{rect}}_L - \log Z^{\mathrm{cub}}_L \leq (L/\varepsilon_N)^3(\mathsf{p}(L/2) - \mathsf{p}(L)).$$

Along a dyadic sequence $L_n/2 = L_{n-1}$, so $\mathsf{p}(L_n/2) - \mathsf{p}(L_n) = \eta_n$, and the hypotheses on $L = L_n$ hold for all large n. □

Lemma 14.30 (bounded Borel local functions converge along the limits)**.** *In the setting of Subsection 14.4, for $k \in \mathbb{N}$ let $\nu_{1/k}$ be the weak limit of $(\mu_{L_n}^{1/k})_n$ along a subsequence of (L_n) (which may depend on k), and let ν_0 be the weak limit of $(\nu_{1/k})_k$ along a subsequence (Lemma 14.21). Let f be a bounded Borel function on E of the variables of finitely many cells. Then $\mu_{L_n}^{1/k}(f) \to \nu_{1/k}(f)$ along the subsequence defining $\nu_{1/k}$, for every k, and $\nu_{1/k}(f) \to \nu_0(f)$ along the subsequence defining ν_0.*

Proof. Let Δ be a finite set of cells containing the cells of f, so that f is a function of the Δ-variables only, and write $\gamma_\Delta^{1/k}$ for the kernel of Subsection 14.4 with the source strength $1/k$. For $L = L_n$ larger than the diameter of Δ and of the cells of the O_c, $c \in I_\Delta$, integrating the Δ-variables first under $\mu_L^{1/k}$ gives $\mu_L^{1/k}(f) = \mu_L^{1/k}(\gamma_\Delta^{1/k} f)$: this is the first step of the proof of Lemma 14.21(i), an application of Fubini's theorem which uses of f only that it is bounded and Borel. The function $\gamma_\Delta^{1/k} f$ depends on the outside variables of finitely many cells only, and it is bounded and continuous: this is the Feller argument of the same proof (the factors of the kernel are continuous in the outside variables, $Z_\Delta(\xi_{\mathrm{out}}) \in (0, \infty)$, dominated convergence), in which f enters only as a bounded Borel function of the integration variables. Hence, by the weak convergence and by Lemma 14.21(i) for the bounded Borel f, along the subsequence defining $\nu_{1/k}$,

$$\mu_{L_n}^{1/k}(f) = \mu_{L_n}^{1/k}(\gamma_\Delta^{1/k} f) \;\longrightarrow\; \nu_{1/k}(\gamma_\Delta^{1/k} f) = \nu_{1/k}(f).$$

For the second limit, $\nu_{1/k}(f) = \nu_{1/k}(\gamma_\Delta^{1/k} f)$ by Lemma 14.21(i), $\|\gamma_\Delta^{1/k} f - \gamma_\Delta f\|_\infty \le 2\|f\|_\infty (e^{2S_\Delta/k} - 1)$ by the comparison of the two normalised weights in the proof of Lemma 14.21(iii) (valid for bounded Borel f), and $\gamma_\Delta f$ is bounded and continuous (the Feller argument without source). So, along the subsequence defining ν_0, $\nu_{1/k}(f) - \nu_{1/k}(\gamma_\Delta f) \to 0$ and $\nu_{1/k}(\gamma_\Delta f) \to \nu_0(\gamma_\Delta f) = \nu_0(f)$, the last equality by Lemma 14.21(iii). □

Proposition 14.31 (the tangent inequality of the comparison states, without relative entropy)**.** *Let $\nu_0 = \nu_0(D)$ be a comparison state as in Lemma 14.21(iii), formed along a dyadic sequence (L_n) with a datum $D = (c_0, c_+, c_3; h; O)$. For $j \in \{1, 2, 3\}$ let g_j be as in Lemma 14.29 and let*

$$\nu^{(j)} := \Psi_*\big((\nu_0 \circ \hat{g}_j) \times \mathrm{Haar}^{\mathrm{sites}}\big)$$

be the phase restoration (Lemma 13.3(d)) of the Gibbs measure $\nu_0 \circ \hat{g}_j$ (Proposition 13.39(i)): for $j = 3$ the phase restoration of ν_0 itself; for $j = 1, 2$ its time slabs are the slabs of ν_0 between adjacent layers orthogonal to e_j. Let $\nu = \nu^{(j)}$ for some $j \in \{1, 2, 3\}$, let $[n_0, n_0 + 1]$ be a slab between two adjacent time layers and $t \neq 0$ real. Then:

(a) *for every finite spatial bond set S, every real skew-adjoint Y of the class of Lemma 13.11 and the corrector Z of Lemma 13.12, the slab function $v_t = \log(1 + t r_Y + t^2 r_Z) \circ \Theta_\sigma$ of Lemma 13.14 satisfies*

$$\nu(v_t) \;\le\; \log \|X_t\|, \qquad X_t = 1 + tY + t^2 Z; \tag{14.3}$$

(b) *for every finite spatial bond set S closed under the bonds of its plaquettes, every real skew-adjoint $Y \in \mathcal{C}_{\mathrm{full}}$ and the corrector $Z = \delta + Z_b \otimes Z_H$ of the proof of Lemma 13.25, the slab function v_t of that proof satisfies (14.3).*

The inequality (14.3) is the upper bound which the proof of Lemma 13.14 obtains from Lemma 13.9, (13.28) and Lemma 13.10, and which the proof of Lemma 13.25 obtains from these and Lemma 13.21; here it is obtained in each direction e_j from the construction of ν_0.

Proof. Fix j and put $g := g_j$; fix the case (a) or (b), S, the slab, Y, Z and $t \neq 0$, and write $X := X_t$ and $v := v_t = \log r_X \circ \Theta_\sigma$. By Lemma 13.12, resp. the proof of Lemma 13.25,

$r_X \geq \delta t^2 > 0$, so $v \geq \log(\delta t^2)$. The function v depends on the phase-restored variables of a finite set Δ_S of cells of the slab (the sites, bonds and plaquettes of the window in the two layers, the temporal links and the temporal plaquettes between them), and it is Borel (r_X is continuous in the pair and Higgs coordinates of the slab for each value of the integer variables, and Θ_σ is a Borel map). Let $h = h_{j_0}$ be the wall spacing of D.

Step 1 (reduction to bounded Borel functions on E). For $m \in \mathbb{N}$ let $v \wedge m := \min(v, m)$ and

$$f_m(\omega) := \int \mathrm{Haar}(d\psi)\,(v \wedge m)\big(\Psi(\hat{g}\omega, \psi)\big),$$

the integral being over the finitely many phases on which the integrand depends. Since Ψ and $\hat{g}$ are local, f_m is a bounded Borel function on E of the variables of finitely many cells, and by the definition of $\nu = \nu^{(j)}$ and Fubini's theorem $\nu(v \wedge m) = \nu_0(f_m)$. As $v \wedge m \uparrow v$ and $v \wedge m \geq \min(\log(\delta t^2), 1)$, monotone convergence gives

$$\nu(v) = \sup_m \nu_0(f_m) \in (-\infty, +\infty]. \tag{14.4}$$

It therefore suffices to prove $\nu_0(f_m) \leq \log \|X\|$ for every m.

Step 2 (the torus laws). Let $k \in \mathbb{N}$, $n \geq j_0$ and $L = L_n$, so large that the cells of f_m embed in T_L^{tw} and Lemma 14.29 applies. Let $\mu' := \mu_L^{1/k} \circ \hat{g}$, a law on the configurations of the torus $T_{g\Gamma(L)}$, and $\widetilde{\mu}' := \Psi_*(\mu' \times \mathrm{Haar}^{\mathrm{sites}})$. The computation of Step 1, on the torus, gives

$$\mu_L^{1/k}(f_m) = \widetilde{\mu}'(v \wedge m) \;\leq\; \widetilde{\mu}'(v). \tag{14.5}$$

We bound $\widetilde{\mu}'(v)$.

Step 3 (a sublattice of periods; the periodised sum). Let $\Lambda' := h(e_1 - e_2)\mathbb{Z} + h(e_1 + e_2)\mathbb{Z} + 2he_3\mathbb{Z}$. By Lemma 14.6 the products of the reflections in two adjacent parallel walls of D are the translations by $h(e_1 - e_2)$, $h(e_1 + e_2)$ and $2he_3$; they belong to $\mathcal{W}$, so $\mu_L^{1/k}$ is invariant under the translations by Λ' (Lemma 14.12(i)). Every non-zero element of Λ' has length at least h. Let d_S be the diameter of Δ_S, choose an integer $q \geq 0$ with $2^q h > d_S + 2\varepsilon_N$, and put

$$\Gamma' := 2^q\, g\Lambda', \qquad I' := [\varepsilon_N\mathbb{Z}^3 : \Gamma'] = 4 \cdot 8^q\,(h/\varepsilon_N)^3.$$

$\Gamma(L) \subset 2^q\Lambda'$ holds if and only if $2^q h$ divides $L/2$, that is, for $L = L_n$ with $n \geq j_0 + q$ ($L_n/2 = 2^{n-j_0}h$); we assume this from now on. Then $g\Gamma(L) \subset \Gamma'$, $n_L := [\Gamma' : g\Gamma(L)] = (L/\varepsilon_N)^3/I'$, and μ' is invariant under the translations by $g\Lambda' \supset \Gamma'$ ($\tau_{ga} \circ \hat{g} = \hat{g} \circ \tau_a$); so is $\widetilde{\mu}'$ (Lemma 14.25(b)).

For $\gamma \in \Gamma'$ the function $v \circ \tau_\gamma$ on the phase-restored configurations of $T_{g\Gamma(L)}$ depends only on the class of γ modulo $g\Gamma(L)$. It is the slab function $\log r_{X^\gamma} \circ \Theta_\sigma^{(s)}$ of the translated slab, of height $s = s(\gamma) \in \{1, \dots, M_j\}$, and of the translated window S^γ, where X^γ is the copy of X on S^γ, an operator of the same class and the same norm. Let

$$V_L := \sum_{\gamma \in \Gamma'/g\Gamma(L)} v \circ \tau_\gamma,$$

a sum of n_L terms, bounded below. The cell sets $\Delta_S + \gamma$ of distinct classes are at distance at least $2\varepsilon_N$ from each other on $T_{g\Gamma(L)}$: if $\Delta_S + \gamma$ and $\Delta_S + \gamma' + \gamma''$, $\gamma'' \in g\Gamma(L)$, are at distance less than $2\varepsilon_N$ in $\varepsilon_N\mathbb{Z}^3$, then the element $\gamma' + \gamma'' - \gamma$ of Γ' has length less than $d_S + 2\varepsilon_N < 2^q h$, hence is zero. In particular the windows S^γ at one height are pairwise disjoint, and so are their site sets. By the invariance of $\widetilde{\mu}'$ (applied to the truncations of v, then monotone convergence),

$$\widetilde{\mu}'(v) = n_L^{-1}\,\widetilde{\mu}'(V_L), \tag{14.6}$$

both sides in $(-\infty, +\infty]$.

Step 4 (Jensen's inequality; removal of the source). For a probability measure, the integral of a function bounded below is at most the logarithm of the integral of its exponential, both

sides in $(-\infty,+\infty]$ (Jensen's inequality for the truncations of the function from above, then monotone convergence on the left). Thus $\widetilde{\mu}'(V_L) \le \log \widetilde{\mu}'(e^{V_L})$. The density of $\mu_L^{1/k}$ with respect to μ_L^{tw} is $Z_{1/k}^{-1} e^{O_d/k} \le e^{2n_c\|O\|_\infty/k}$ (Definition 14.10; this is the removal of the source in the proof of Lemma 14.18), and $\mu_L^{\mathrm{tw}} \circ \hat{g} = \mu_{g\Gamma(L)}$ by (14.1). Hence, for every Borel $F \ge 0$ on the phase-restored configurations of $T_{g\Gamma(L)}$,

$$\widetilde{\mu}'(F) = \int \mu_L^{1/k}(d\omega) \int \mathrm{Haar}(d\psi)\, F\big(\Psi(\hat{g}\omega,\psi)\big) \;\le\; e^{2n_c\|O\|_\infty/k}\, \widetilde{\mu}_{g\Gamma(L)}(F) = e^{2n_c\|O\|_\infty/k}\, \mu_{M_j}^{\zeta_j}(F),$$

by Lemma 14.25(a),(c) and Lemma 14.29(i). With (14.6),

$$\widetilde{\mu}'(v) \;\le\; n_L^{-1}\Big[\frac{2n_c\|O\|_\infty}{k} + \log \mu_{M_j}^{\zeta_j}\big(e^{V_L}\big)\Big], \qquad \frac{n_c}{n_L} = \frac{2L^3}{h^3}\cdot\frac{I'\varepsilon_N^3}{L^3} = 8^{q+1}. \tag{14.7}$$

Step 5 (insertions on the twisted torus). For each height $s \in \{1,\dots,M_j\}$ let $X^{[s]}$ be the full operator of the copies X^γ with $s(\gamma) = s$: their tensor product in the case (a) (Lemma 14.27(a)), their product in the case (b) (Lemma 14.27(b); the windows S^γ are closed under the bonds of their plaquettes, as S is, and by Step 3 they are pairwise disjoint in bonds and in sites); $X^{[s]} := 1$ if there is no such γ. In the case (b) the feature map is that of Lemma 14.27(b) with V the set of all sites of the layer torus, the same at all heights. By Lemma 14.27, $r_{X^{[s]}}$ is the product of the r_{X^γ} with $s(\gamma) = s$, $\|X^{[s]}\|$ is at most the product of their norms, and $X^{[s]}$ has the absolute convergence on which (13.28) rests (Remark 13.8 in the case (a), Lemma 14.27(c) in the case (b)). Hence

$$e^{V_L} = \prod_{\gamma\in\Gamma'/g\Gamma(L)} r_{X^\gamma}\circ\Theta_\sigma^{(s(\gamma))} = \prod_{s=1}^{M_j} r_{X^{[s]}}\circ\Theta_\sigma^{(s)}, \qquad \prod_{s=1}^{M_j}\|X^{[s]}\| \le \|X\|^{n_L},$$

and Lemma 14.26(iii) gives

$$\mu_{M_j}^{\zeta_j}\big(e^{V_L}\big) = \frac{Z_{M_j}^{\zeta_j}(X^{[1]},\dots,X^{[M_j]})}{Z_{M_j}^{\zeta_j}} \;\le\; \|X\|^{n_L}\,\frac{Z_{M_j}^{0}}{Z_{M_j}^{\zeta_j}}.$$

For $j = 3$ the twist ζ_3 vanishes and the last ratio is 1 (this is Lemma 13.10 with (13.28)). For $j = 1,2$ it is at most $\exp[(L/\varepsilon_N)^3\eta_n]$ by Lemma 14.29(ii), $L = L_n$. Since $(L/\varepsilon_N)^3/n_L = I'$, (14.7) becomes

$$\widetilde{\mu}'(v) \;\le\; \log\|X\| + \frac{2\cdot 8^{q+1}\|O\|_\infty}{k} + I'\,\eta_n, \tag{14.8}$$

where q and I' depend on the window, the slab function and h, but not on n, k or m (for $j = 3$ the last term is absent).

Step 6 (the limits: first $n\to\infty$ at fixed k, then $k\to\infty$). By (14.5) and (14.8), $\mu_{L_n}^{1/k}(f_m) \le \log\|X\| + 2\cdot 8^{q+1}\|O\|_\infty/k + I'\eta_n$ for all large n. Let $n\to\infty$ along the subsequence defining $\nu_{1/k}$: by Lemma 14.30 the left side tends to $\nu_{1/k}(f_m)$, and $\eta_n \to 0$ along the whole sequence (Lemma 14.28(iii)), so $\nu_{1/k}(f_m) \le \log\|X\| + 2\cdot 8^{q+1}\|O\|_\infty/k$ for every k. Let $k\to\infty$ along the subsequence defining ν_0: by Lemma 14.30, $\nu_0(f_m) \le \log\|X\|$. This holds for every m, and (14.4) gives $\nu(v) \le \log\|X\|$, which is (14.3). □

Remark 14.32 (a second proof). (i) *A second proof of the invariance used in Proposition 14.33.* For the comparison states, Proposition 14.31 gives a proof of Lemmas 13.14 and 13.25, of Theorem 13.28 and of the conclusion of Theorem 13.1 which does not use the tangent inequality, Lemma 13.9. Indeed, let ν_0 be a comparison state. In the proof of Theorem 13.1 for ν_0 (Subsection 13.10), Lemma 13.9 is applied only to the three measures $\nu^{(j)}$ of Proposition 14.31

(in Step 1 to the phase restoration of ν_0, in Step 2 to those of $\nu_0 \circ \hat{g}_1$ and $\nu_0 \circ \hat{g}_2$), and only through the upper bounds (14.3) in the proofs of Lemmas 13.14 and 13.25 (Remark 13.40); and Proposition 14.31 gives these bounds directly. This applies in particular to the invariance of $\nu_{O'}$ under $\tau_{2\varepsilon_N e_1}$ in step (3) of the proof of Proposition 14.33. For that invariance the case $j = 1$ of Proposition 14.31 suffices: Step 1 of the proof of Theorem 13.1 for $\nu_0 \circ \hat{g}_1$ gives its invariance under the time reflections, the computation in the proof of Proposition 13.39(ii) turns it into the invariance of ν_0 under the reflections $\theta_{1,c}$, and Proposition 13.39(iii) gives the invariance under $\tau_{2\varepsilon_N e_1}$.

(ii) *The surface term.* The convergence which the argument of this section takes from the surface bound, Lemma 14.14, also follows from Lemma 14.29(iii). By that lemma, $0 \le \phi_{L_n} \le (L_n/\varepsilon_N)^3 \eta_n$ with $\eta_n \to 0$ along a dyadic sequence, hence $\phi_{L_n}/L_n^3 \to 0$. In the notation of Remark 14.17, where t is the source strength, this is the statement $h^3 \phi_{L_n}/(2tL_n^3) \to 0$ at fixed h and $t > 0$ of that remark, which is the only consequence of Lemma 14.14 that the argument of this section uses (Remark 14.17, the proof of Proposition 14.23, Remark 14.15). Here it follows from $\phi_L \ge 0$ (Proposition 14.13), the Hölder bound of Lemma 14.26 without insertions and the halvings of Lemma 14.28, with no analysis of the seam. The bound of Lemma 14.14 itself, of the order $L^2 \log L$, is not obtained in this way, since Lemma 14.29(iii) gives no rate for η_n.

(iii) *Restriction to the comparison states.* Proposition 14.31 is a statement about the comparison states only: its proof uses their construction as limits of the torus laws $\mu_{L_n}^{1/k}$ and gives nothing for other Gibbs measures. Section 13 does not depend on it; there the tangent inequality is Lemma 13.9.

14.6. **Straddling: the rigidity step.**

Proposition 14.33 (reflection positivity of the dyadic limit points at the planes $\{u = c_0\}$)**.** *Let ω be a dyadic limit point, $c_0 \in \varepsilon_N \mathbb{Z}$, $\Pi := \{u = c_0\}$ and $\theta := \theta_\Pi = r_{c_0}$. Then $\omega\big((F \circ \theta)\, F\big) \ge 0$ for every real bounded continuous function F of the variables of finitely many cells with all vertices in $\{u \ge c_0\}$ (a function on Ω_N, in the sense of Theorem 11.5(d)), and $\omega(G \circ \theta) = \omega(G)$ for every bounded continuous local G.*

For Theorem 14.3 only $c_0 = 0$ is needed.

Proof. The invariance is Theorem 11.5(c), since r_{c_0} is a signed coordinate permutation about a site composed with a translation of $\varepsilon_N \mathbb{Z}^3$, an element of $\mathcal{G}_N$. Let F be as stated and $O := -(F \circ \theta)F$. This function is real, bounded, continuous and local, a function of the gauge-invariant variables $(\mathfrak{f}_p, \tilde{\rho}_x)$, but of cells on *both* sides of Π (those of F and their reflections), so no datum with the wall Π admits it.

Translate. Let $u_{\min}$ be the least value of u over the vertices of the cells of O and pick $a \in \varepsilon_N \mathbb{Z}$ with $u_{\min} + 2a \in \{c_0 + \varepsilon_N, c_0 + 2\varepsilon_N\}$ (one of two consecutive multiples of ε_N lies in $2\varepsilon_N \mathbb{Z}$); let τ be the translation by $2ae_1 \in \varepsilon_N \mathbb{Z}^3$ (it raises u and $v(x)$ by $2a$ and fixes x_3) and $O' := O \circ \tau^{-1}$, a real bounded continuous function of the gauge-invariant variables of cells with all vertices in $\{u \ge c_0 + \varepsilon_N\}$. Choose $h \in \{h_j\}$ (unbounded, Definition 14.1) larger than each of the u-, $v(x)$- and x_3-ranges of the vertices of the cells of O' plus $2\varepsilon_N$, and put $c_+ := \min v(x) - \varepsilon_N \in \varepsilon_N \mathbb{Z}$, $c_3 := \min x_3 - \varepsilon_N/2 \in \frac{1}{2}\varepsilon_N \mathbb{Z}$ over those vertices. Then the cells of O' lie inside the open cell $C_0 = \{c_0 < u < c_0 + h,\ c_+ < v(x) < c_+ + h,\ c_3 < x_3 < c_3 + h\}$ of the datum $D_{O'} := (c_0, c_+, c_3; h; O')$, whose wall set contains Π. Let $\nu_{O'} := \nu_0(D_{O'})$ be a comparison state of Lemma 14.21(iii), formed along subsequences of the subsequence defining ω.

The chain of relations. The following four relations hold.

(1) $\omega(O) = \omega(O')$, because the law μ_N of ω (Definition 2.3) is invariant under the translations by $\varepsilon_N \mathbb{Z}^3$ (Theorem 11.5(c)) and $2ae_1 \in \varepsilon_N \mathbb{Z}^3$.

(2) $\omega(O') \le \nu_{O'}(O')$, by Proposition 14.23 ($O'$ is a bounded continuous function of the gauge-invariant variables of cells inside C_0; under $\nu_{O'}$ it is evaluated through $\mathfrak{f}_p = (dB)_p + v_p$).
(3) $\nu_{O'}(O') = \nu_{O'}(O)$. Indeed, by Proposition 14.22 (the sourceless $\nu_{O'}$ is Gibbs, Λ'-periodic with the time translation $2he_3$, and satisfies (I1)–(I3)), Theorem 13.1 makes $\nu_{O'}$ invariant under $\tau_{2\varepsilon_N e_1}$, hence under $\tau = \tau_{2ae_1}$ ($a \in \varepsilon_N\mathbb{Z}$). A second proof of this invariance, which does not use Lemma 13.9, is obtained with Proposition 14.31 in the place of Lemma 13.9; see Remark 14.32(i).
(4) $\nu_{O'}(O) = -\nu_{O'}\big((F\circ\theta)F\big) \le 0$, by the reflection positivity of $\nu_{O'}$ at its wall Π (Lemma 14.21(iii)), F being a real bounded continuous function of cells in the closed half space $\{u \ge c_0\}$ (as a function on E through $\mathfrak{f}_p = (dB)_p + v_p$; the reflection acts on $(\tilde\rho, B, v)$ and induces on $\mathfrak{f}$ the action used on Ω_N).

Hence $\omega\big((F\circ\theta)F\big) = -\omega(O) \ge 0$. (Equivalently: $\nu_{O'}$ is the dominating state of Lemma 14.24 for this O, after the translation τ has been undone by the rigidity theorem.) □

14.7. **Conclusion: the six planes through 0 and all their translates.**

Lemma 14.34 (complex form, Hermitian form, Borel functions and the L^2 closure)**.** *Let ω be a dyadic limit point, $c_0 \in \varepsilon_N\mathbb{Z}$, $R := r_{c_0}$ and $\Theta F := \overline{F\circ R}$ for functions F of cells in $\{u \ge c_0\}$. (a) For complex bounded continuous such F, G: $\omega(\Theta F\cdot F) \ge 0$, $\omega(\Theta F) = \overline{\omega(F)}$, and $(F,G) \mapsto \omega(\Theta F\cdot G)$ is a positive semidefinite Hermitian form. (b) The same holds for bounded Borel F, G of finitely many cells in $\{u \ge c_0\}$ and for their $L^2(\mu_N)$-closure.*

Proof. (a) Write $F = F_1 + iF_2$ with F_1, F_2 real. Then

$$\omega(\Theta F\cdot F) = \omega((F_1\circ R)F_1) + \omega((F_2\circ R)F_2) + i\big[\omega((F_1\circ R)F_2) - \omega((F_2\circ R)F_1)\big],$$

and the bracket vanishes by the R-invariance of μ_N (Theorem 11.5(c), $R \in \mathcal{G}_N$): $\omega((F_2\circ R)F_1) = \omega\big(((F_2\circ R)F_1)\circ R\big) = \omega(F_2\,(F_1\circ R))$. So $\omega(\Theta F\cdot F) \ge 0$ by Proposition 14.33, and $\omega(\Theta F) = \overline{\omega(F\circ R)} = \overline{\omega(F)}$. The Hermitian symmetry is the identity

$$\omega(\Theta F\cdot G) = \omega(\overline{F\circ R}\,G) = \omega(\overline{F}\,(G\circ R)) = \overline{\omega(\overline{G\circ R}\,F)} = \overline{\omega(\Theta G\cdot F)},$$

again by R-invariance; a sesquilinear form with Hermitian symmetry and non-negative diagonal is positive semidefinite (polarisation). (b) Bounded Borel functions of finitely many cells are $L^2(\mu_N)$-limits of bounded continuous functions of the same cells (the marginal is a finite Borel measure on a Polish product; Lusin's theorem), and $(F,G) \mapsto \omega(\Theta F\cdot G)$ is L^2-continuous because $F \mapsto F\circ R$ is an isometry of $L^2(\mu_N)$ by R-invariance, so Θ is an antiunitary involution of $L^2(\mu_N)$. The three statements pass to the limits, and then to the $L^2(\mu_N)$-closure. □

Proof of Theorem 14.3. Let $P = P^{\sigma}_{ij} = \{x_i = \sigma x_j\}$, $1 \le i < j \le 3$, $\sigma \in \{\pm 1\}$, and k the third index. Let R' be the signed coordinate permutation with $R'e_1 = e_i$, $R'e_2 = \sigma e_j$, $R'e_3 = e_k$: for $y = R'x$ one has $y_i - \sigma y_j = x_1 - \sigma^2 x_2 = u(x)$, so R' maps $\{u = 0\}$ onto P and $\{u \ge 0\}$ onto the closed half space $\{x_i - \sigma x_j \ge 0\}$. For the other closed half space use $R'\circ(1\,2)$, which maps $\{u = 0\}$ onto P and $\{u \ge 0\}$ onto $\{x_i - \sigma x_j \le 0\}$. Call the chosen map g; it fixes 0, so $g \in \mathcal{G}_N$ (Theorem 11.5(c)), and $\theta_P = g\,\theta_0\,g^{-1}$ with $\theta_0 := r_0$ the reflection in $\{u = 0\}$ (both sides are the involutive isometry of $\varepsilon_N\mathbb{Z}^3$ fixing P pointwise). For F bounded Borel of cells in $\overline{H}_P = g\{u \ge 0\}$, $F\circ g$ is bounded Borel of cells in $\{u \ge 0\}$, and by g-invariance of μ_N and Lemma 14.34(b) at $c_0 = 0$,

$$\omega\big(\overline{F\circ\theta_P}\cdot F\big) = \omega\big((\overline{F\circ\theta_P}\cdot F)\circ g\big) = \omega\big(\overline{(F\circ g)\circ\theta_0}\cdot(F\circ g)\big) \;\ge\; 0,$$

using $\theta_P\circ g = g\circ\theta_0$. As $(i<j,\sigma)$ vary, all six diagonal site planes through 0 are reached. □

Corollary 14.35 (all diagonal site planes; Hermitian form; L^2 closure)**.** *Let ω be a dyadic limit point. For every $c \in \varepsilon_N\mathbb{Z}$ and every plane $\Pi = \{x_i - \sigma x_j = c\}$ $(1 \le i < j \le 3,\ \sigma \in \{\pm 1\})$, with θ_Π the reflection in Π and $\Theta_\Pi F := \overline{F \circ \theta_\Pi}$: $\omega(\Theta_\Pi F \cdot F) \ge 0$ for every bounded Borel F of finitely many cells in either closed half space bounded by Π; $(F, G) \mapsto \omega(\Theta_\Pi F \cdot G)$ is a positive semidefinite Hermitian form on these functions; $\omega(\Theta_\Pi F) = \overline{\omega(F)}$; and all three statements extend to the $L^2(\mu_N)$-closure.*

Proof. With g as in the proof of Theorem 14.3 put $g_c(x) := gx + ce_i$; since $ce_i \in \varepsilon_N\mathbb{Z}^3$, $g_c \in \mathcal{G}_N$ (Theorem 11.5(c)), $g_c\{u = 0\} = \Pi$ ($y = gx + ce_i$ has $y_i - \sigma y_j = u(x) + c$), $g_c\{u \ge 0\}$ is the chosen closed half space and $\theta_\Pi = g_c\theta_0 g_c^{-1}$. The computation of that proof, with g_c in place of g, transports the three statements of Lemma 14.34(b) at $c_0 = 0$ to Π (for the Hermitian form and the conjugation identity: $\omega(\Theta_\Pi F \cdot G) = \omega(\Theta_0(F \circ g_c) \cdot (G \circ g_c))$ and $\omega(\Theta_\Pi F) = \omega(\Theta_0(F \circ g_c))$ by g_c-invariance), and $F \mapsto F \circ g_c$ is an isometry of $L^2(\mu_N)$, so the L^2-closure statement transports too. Beyond Theorem 14.3, the proof uses only the $\varepsilon_N\mathbb{Z}^3$-translation invariance of μ_N. □

14.8. **Consequence for the rotation theorems.**

Remark 14.36 (consequence for the rotation theorems). Let ω_N be a dyadic limit point and S a limit functional of Theorem 7.1 built on it. Then Theorems 12.2 and 12.3 hold for S; this is Theorem 12.4. Indeed, apart from the reflection positivity of ω_N at the six diagonal site-plane families, every step of the proofs of Theorems 12.2 and 12.3 holds for the torus limit points as it stands, with Theorem 11.5(a)–(d) in place of the corresponding statements for the box states: the Gaussian bound of Lemma 12.22 follows from Theorem 11.5(b), the translation invariance and the coordinate reflections are given by Theorem 11.5(c),(d), and the continuum passage is given by Theorem 7.1 and Section 9, which are stated for ω_N. Lemma 12.21 requires reflection positivity only at the nine planes through 0, and the Hermitian identities which it uses follow from Theorem 11.5(c). Theorem 14.3 therefore suffices for the proof, while Corollary 14.35 gives the reflection positivity at all diagonal site planes. Theorem 14.3 rests in turn on the rigidity theorem (Theorem 13.1) and on the surface bound (Lemma 14.14). One and the same dyadic limit point ω_N must be used throughout, in Definition 2.3 and in the sequence (N_m) of Theorem 7.1.

15. Rotation covariance of the moments from reflection positivity

Throughout this section T is $\Lambda^2\mathbb{R}^3$ or $\Lambda^2\mathbb{R}^3 \oplus \mathbb{R}$ with its orthogonal $O(3)$-action π (on the 2-form summand $\pi(g) = \det(g)\, g$ under $\Lambda^2\mathbb{R}^3 \cong \mathbb{R}^3$, on the scalar summand the identity), $T_\mathbb{C}$ its complexification with the Hermitian extension of the inner product, and $\mathcal{S} = \mathcal{S}(\mathbb{R}^3; T)$ the real Schwartz space; $\mathcal{S}'(\mathbb{R}^3; T)$ carries its Borel σ-algebra. For a Borel probability measure μ on $\mathcal{S}'(\mathbb{R}^3; T)$ put $F(\chi)(\phi) := \phi(\chi)$ and $S_n(\chi_1, \dots, \chi_n) := \int \prod_{j=1}^n F(\chi_j)\, d\mu$ whenever the integral converges absolutely. For $g \in O(3)$ the pull-back is $(g^*\chi)(x) := \pi(g)^{-1}\chi(gx)$, and for $a \in \mathbb{R}^3$, $(\tau_a\chi)(x) := \chi(x - a)$. For a reflection Θ in an affine plane, $\Theta^*\chi$ is the pull-back $(\Theta^*\chi)(x) = \pi(\Theta_0)\chi(\Theta x)$, Θ_0 the linear part. The constants of this section depend only on the constant C of hypothesis (i) below (and on the number of fields where this is stated), never on μ otherwise. The symbols $\mathcal{W}$, $\mathcal{N}$, N and μ of this section are local to it (elsewhere in this paper N is the level and $\mathcal{N}$ the set of the nine normals).

Theorem 15.1 (rotation covariance of the moments away from coincident points)**.** *Let μ be a Borel probability measure on $\mathcal{S}'(\mathbb{R}^3; T)$ such that*

(i) $\int e^{F(\chi)}\, d\mu \le e^{C(\|\chi\|_{L^1} + \|\chi\|_{L^2}^2)}$ *for all real $\chi \in \mathcal{S}(\mathbb{R}^3; T)$, with one constant $C < \infty$;*
(ii) *μ is invariant under all translations of $\mathbb{R}^3$;*

(iii) $\sum_{a,b} \bar z_a z_b \int e^{iF(\chi_a - \Theta^*\chi_b)}\, d\mu \ge 0$ *for every reflection* Θ *in one of the nine planes* $\{x_j = 0\}$, $\{x_j \pm x_k = 0\}$ $(1 \le j < k \le 3)$, *all finite families* $\chi_a \in \mathcal{S}$ *supported in one open side of the plane and all* $z_a \in \mathbb{C}$.

Then $S_n(R^*\chi_1, \dots, R^*\chi_n) = S_n(\chi_1, \dots, \chi_n)$ *for all* $n \ge 1$, *all* $R \in O(3)$ *and all* $\chi_1, \dots, \chi_n \in \mathcal{S}(\mathbb{R}^3; T)$ *with pairwise disjoint supports.*

For the two representations of the theorem the planar rotation weights are $0, \pm 1$ (the 2-forms) and 0 (the scalar), and the proof below is written for weights in $\{-2, \dots, 2\}$ wherever the weights enter (Proposition 15.21). The proof occupies the rest of the section: Subsection 15.1 fixes the notation, Subsections 15.2–15.14 prove the intermediate statements in order, and Subsection 15.15 assembles them.

Remark 15.2 (the tests in (iii)). Hypothesis (iii) with real C_c^∞ tests supported in one open side implies (iii) as stated, with Schwartz tests supported in that open side: for such χ_a the cut-offs $\chi_a \vartheta(\cdot/q)$ ($\vartheta \in C_c^\infty(\mathbb{R}^3)$, $\vartheta = 1$ on $B(0,1)$) are compactly supported in the open side and converge to χ_a in the norm N of Subsection 15.1 as $q \to \infty$, and the characteristic functional Z is N-continuous (Lemma 15.6(b)), so each finite Gram sum of (iii) passes to the limit. Conversely the stated form contains the compactly supported one. Only the compactly supported form is ever used below.

Remark 15.3 (the σ-algebra). The proofs of this section use μ only through the random variables $F(\chi)$, $\chi \in \mathcal{S}$ real, and the σ-algebras they generate (the $\Sigma(K)$ and $\Sigma\{u \cdot v > 0\}$ of Subsection 15.1, and in Lemma 15.6(e) the uniqueness of a probability measure with a given characteristic functional on the σ-algebra generated by the evaluations). They apply therefore to a probability measure on $\mathcal{S}'(\mathbb{R}^3; T)$ carrying the cylinder σ-algebra generated by the evaluations, which is the form in which the measures of Sections 9 and 12 are given: "Borel" is then read as "on that σ-algebra", and the image measures used in the proofs (under translations, reflections and the rotation by $\pi/4$ of a coordinate plane) are again of this form, because every evaluation of the image of ϕ is an evaluation of ϕ.

15.1. **Setting and notation.** *Fields.* For $\chi \in (L^1 \cap L^2)(\mathbb{R}^3; T_{\mathbb{C}})$ we put $N(\chi) := \|\chi\|_1 + \|\chi\|_2$, the norms being taken with the Hermitian norm on $T_{\mathbb{C}}$; we note that $\|\operatorname{Re}\chi\|_p \le \|\chi\|_p$. By Lemma 15.6 below, F extends to $(L^1 \cap L^2)(\mathbb{R}^3; T_{\mathbb{C}})$, complex-linearly, with $F(\chi) \in \bigcap_{p<\infty} L^p(\mu)$, $\overline{F(\chi)} = F(\bar\chi)$, and S_n is defined on $(L^1 \cap L^2)^n_{\mathbb{C}}$. $Z(\chi) := \int e^{iF(\chi)}\, d\mu$ is the characteristic functional.

Coordinates. A coordinate pair is an ordered pair $(t, x) = (x_j, x_k)$ of distinct coordinates, $y = x_l$ the third; e_t, e_x, e_y are the unit vectors, and a point of $\mathbb{R}^3$ is written (t, x, y) or (z, y) with $z = (t, x) \in \mathbb{R}^2$ the planar part. $\theta_\circ(t, x, y) := (-t, x, y)$ is the reflection in $\{t = 0\}$, $\theta_c(t, x, y) := (2c - t, x, y)$ the reflection in $\{t = c\}$, $n := (e_t + e_x)/\sqrt{2}$, $n' := (e_t - e_x)/\sqrt{2}$, and θ_n the reflection in $\{t + x = 0\}$ (so $\theta_n n = -n$, $\theta_n n' = n'$, $\theta_\circ n = -n'$). On tests we put $(\theta_c^*\chi)(v) := \pi(\theta_\circ)\chi(\theta_c v)$ and $(\theta_n^*\chi)(v) := \pi(\theta_n)\chi(\theta_n v)$. Since $\pi(\theta_\circ)$ and $\pi(\theta_n)$ are real orthogonal involutions, each θ^* is an involutive isometry of L^1 and of L^2, $\operatorname{Re}\theta_c^* k = \theta_c^* \operatorname{Re} k$, and $\theta_c^* \tau_a \theta_c^* = \tau_{\theta_\circ a}$ for $a \in \mathbb{R}^3$. We put $J_c\chi := \overline{\theta_c^*\chi}$ and $J := J_0$, and for a transverse profile $f \in (L^1 \cap L^2)(\mathbb{R}; T_{\mathbb{C}})$ we put $Jf := \pi(\theta_\circ)\bar f$, so that $J(\alpha f) = \tilde\alpha\, Jf$, where $(\alpha f)(t, x, y) := \alpha(t, x) f(y)$ and $\tilde\alpha(t, x) := \overline{\alpha(-t, x)}$.

Observables. The translations and the reflections act on $\mathcal{S}'$ by $(\tau_a\phi)(\chi) := \phi(\tau_{-a}\chi)$ and $(\theta\phi)(\chi) := \phi(\theta^*\chi)$ for $\theta \in \{\theta_c, \theta_n\}$, and on functions G of ϕ by $U(a)G := G \circ \tau_{-a}$, $\Theta_c G := G \circ \theta_c$, $\Theta_\circ := \Theta_0$ and $\Theta_n G := G \circ \theta_n$. Hence $U(a)F(\chi) = F(\tau_a\chi)$, $\Theta_c F(\chi) = F(\theta_c^*\chi)$, $\Theta_n F(\chi) = F(\theta_n^*\chi)$, $U(a)(GG') = U(a)G\, U(a)G'$, $U(a)U(b) = U(a+b)$, and, from the affine identities $\theta_c\theta_0 = \tau_{2ce_t}$, $\theta_c\tau_{ce_t} = \tau_{ce_t}\theta_0$, $\theta_0\tau_{-ce_t} = \tau_{-ce_t}\theta_c$, $\theta\tau_a = \tau_{\theta_0 a}\theta$ (θ_0 the linear part),

$$\begin{gathered} \Theta_c\Theta_\circ = U(2ce_t), \quad \Theta_c U(ce_t) = U(ce_t)\Theta_\circ, \quad \Theta_\circ U(-ce_t) = U(-ce_t)\Theta_c, \\ \Theta_c U(a) = U(\theta_\circ a)\Theta_c, \quad \Theta_n U(a) = U(\theta_n a)\Theta_n. \end{gathered} \tag{15.1}$$

These identities hold on the generators $F(\chi)$ by the point-map identities, and the $F(\chi)$ separate the points of $\mathcal{S}'$. $\Sigma(O)$ denotes the σ-algebra generated by the $F(\chi)$, $\chi \in \mathcal{S}$ real with $\operatorname{supp}\chi \subset O$ (O open); it coincides with the one generated by the $F(\chi)$, $\chi \in C_c^\infty(O;T)$, since $\chi\,\varphi(\cdot/j) \to \chi$ in $\mathcal{S}$ for a cut-off φ. We put $L^2_c := L^2(\Sigma\{t > c\},\mu)$, $L^2_{\circ,+} := L^2_0$ and $L^2_{n,+} := L^2(\Sigma\{n\cdot v > 0\},\mu)$.

The OS space. We define the forms $\langle G, G'\rangle_c := \mathbb{E}_\mu[\overline{\Theta_c G}\, G']$ on L^2_c, $\langle\cdot,\cdot\rangle_\circ := \langle\cdot,\cdot\rangle_0$, and $\langle G, G'\rangle_n := \mathbb{E}_\mu[\overline{\Theta_n G}\, G']$ on $L^2_{n,+}$. The space $\mathcal{H}_\circ$ is the completion of $L^2_{\circ,+}$ modulo the null space $\mathcal{N}$ of $\langle\cdot,\cdot\rangle_\circ$ (Lemma 15.5); we write $[G]$ or $[G]_\circ$ for the class of G, and $\Omega := [1]$, so that $\langle\Omega, [G]\rangle_\circ = \mathbb{E}_\mu[G]$. The space $\mathcal{H}_n$ and the classes $[G]_n$ are defined likewise for the plane $\{t+x=0\}$. The operators $H \geq 0$ and P are the self-adjoint generators with $[U(te_t + xe_x)G]_\circ = e^{-tH}e^{ixP}[G]_\circ$ for $t \geq 0$, $x \in \mathbb{R}$, and Π is their joint spectral measure on $[0,\infty)\times\mathbb{R}$ (Lemma 15.5); H_n, P_n, Π_n are the same objects for $\{t + x = 0\}$, with n as time and n' as space.

Complex translations. For $z = (t,x) \in \mathbb{C}^2$ we put $q(z) := \operatorname{Re} t - |\operatorname{Im} x|$, $\mathcal{W} := \{q > 0\}$, $\mathcal{C}_+ := \{(t,x) \in \mathbb{R}^2 : t > |x|\}$, $\Theta_\circ z := (-t, x)$ and $\bar z := (\bar t, \bar x)$. The function q is superadditive, $q(z+z') \geq q(z) + q(z')$, and $q(-\Theta_\circ \bar z) = q(z)$. For $q(z) \geq 0$ we put $V(z) := \int e^{-t\omega + ixp}\, d\Pi(\omega,p)$ (Lemma 15.12). We write $R(\theta)$ for the rotation of the (t,x)-plane by θ, acting on $\mathbb{R}^3$ with y fixed, and $\varrho(\theta) := \pi(R(\theta))$ for its tensor action, so that $(R(\theta)^*\chi)(z,y) = \varrho(-\theta)\chi(R(\theta)z, y)$; both extend to complex θ as entire functions (polynomials in $e^{\pm i\theta}$).

Narrow and admissible configurations. A finite planar configuration $(z_1,\dots,z_N) \in (\mathbb{R}^2)^N$ is narrow if for some ordering σ and some unit vector $u \in \mathbb{R}^2$ all differences $z_{\sigma(j+1)} - z_{\sigma(j)}$ lie in the open cone of half-angle $\pi/4$ about u. An ordered tuple of tests $\chi_j = \alpha_j f_j$ with compact planar supports $K_j = \operatorname{supp}\alpha_j$ is admissible if $K_1 \subset \mathcal{C}_+$ and $K_{j+1} - K_j \subset \mathcal{C}_+$ for all j. The symbols K_1, K_6, K_7 and K_8 also denote the constants of this section listed in Remark 15.40: as a numerical factor in an estimate K_n is that constant, and in an inclusion or under a map it is the support $\operatorname{supp}\alpha_n$.

Remark 15.4 (the four planes of a coordinate pair). Every statement of Subsections 15.2–15.13 is proved for one coordinate pair (t,x) from (i), (ii) and hypothesis (iii) restricted to the four planes $\{t=0\}$, $\{x=0\}$, $\{t+x=0\}$, $\{t-x=0\}$ of that pair; where a proof uses fewer planes this is said. Hypothesis (iii) of Theorem 15.1 supplies this four-plane hypothesis for every coordinate pair, since for $(t,x) = (x_j, x_k)$ the four planes are $\{x_j=0\}$, $\{x_k=0\}$, $\{x_j+x_k=0\}$, $\{x_j - x_k = 0\}$, all among the nine. The four-plane form is what Proposition 15.33 needs: it applies Subsections 15.2–15.12 to the law $\tilde\mu$ obtained from μ by the rotation by $\pi/4$ in the (t,x)-plane, which has (iii) at the four planes of the pair but not at the other five. Proposition 15.37 uses Theorem 15.8 for the six ordered coordinate pairs, that is (iii) at the three coordinate planes and at the three planes $\{x_j + x_k = 0\}$, $j<k$ (Theorem 15.8 for the ordered pair (t,x) uses the planes $\{t=0\}$ and $\{t+x=0\}$ only, and the two orderings of a pair share the second); the three planes $\{x_j - x_k = 0\}$ enter through Proposition 15.33 (the plane P_3 of each pair) and through the B_3-invariance of Lemma 15.7 used in step (4) of the assembly. The hypothesis of Theorem 15.1 supplies all of them.

Lemma 15.5 (reconstruction)**.** *Let θ be the reflection in a plane $\{u\cdot v = 0\}$ through 0, with $\theta_*\mu = \mu$, μ translation invariant, and (iii) for θ. Put $L^2_{u,+} := L^2(\Sigma\{u\cdot v > 0\},\mu)$, $\Theta_u G := G\circ\theta$ and $\langle G, G'\rangle_u := \mathbb{E}_\mu[\overline{\Theta_u G}\, G']$. Then $\langle\cdot,\cdot\rangle_u$ is a positive semidefinite Hermitian form on $L^2_{u,+}$; on the Hilbert space $\mathcal{H}_u$ (completion modulo the null space) $\|[G]_u\| \leq \|G\|_{L^2(\mu)}$; the classes of bounded functions in $L^2(\Sigma(K),\mu)$, $K \subset \{u\cdot v > 0\}$ bounded open, are dense in $\mathcal{H}_u$; and there are a self-adjoint $H_u \geq 0$ and, for each unit $m \perp u$, a self-adjoint $P_{u,m}$, all strongly commuting, with joint spectral measure Π_u on $[0,\infty)\times\mathbb{R}$ for each $(H_u, P_{u,m})$, such that $[U(su + s'm)G]_u = \int e^{-s\omega + is'p}\, d\Pi_u(\omega,p)\,[G]_u$ for $s \geq 0$, $s' \in \mathbb{R}$, $G \in L^2_{u,+}$.*

Proof. Positivity: for $G=\sum_b z_b e^{iF(\chi_b)}$ with real $\chi_b\in\mathcal{S}$ supported in $\{u\cdot v>0\}$, $\Theta_u e^{iF(\chi)}=e^{iF(\theta^*\chi)}$, so $\mathbb{E}_\mu[\overline{\Theta_u G}\,G]=\sum_{a,b}\bar z_a z_b Z(\chi_b-\theta^*\chi_a)$, which is $\sum_{a,b}\bar z_a z_b M_{ba}$ with the matrix $M_{ab}=Z(\chi_a-\theta^*\chi_b)$ of (iii); M is positive semidefinite, hence so is $M^{\mathsf T}=\bar M$, and the expectation is ≥ 0. These G form a unital $*$-algebra $\mathcal{A}_u$ of bounded functions generating $\Sigma\{u\cdot v>0\}$; by the functional monotone class theorem the $L^2(\mu)$-closure of $\mathcal{A}_u$ (closed under bounded pointwise sequential limits by dominated convergence) contains every bounded $\Sigma\{u\cdot v>0\}$-measurable function, and bounded functions are dense in $L^2_{u,+}$. The form is $L^2\times L^2$-continuous ($|\mathbb{E}[\overline{\Theta_u G}G']|\le\|G\|_2\|G'\|_2$, Θ_u being an isometry of $L^2(\mu)$ by $\theta_*\mu=\mu$), so it is positive semidefinite on $L^2_{u,+}$; a positive semidefinite sesquilinear form is Hermitian and obeys the Schwarz inequality, and $|\langle G,G\rangle_u|\le\|\Theta_u G\|_2\|G\|_2=\|G\|_2^2$.

For $s\ge 0$, $U(su)L^2_{u,+}\subset L^2_{u,+}$ and, by $\Theta_u U(su)=U(-su)\Theta_u$ and (ii), $\langle U(su)G,G'\rangle_u=\mathbb{E}[\overline{U(-su)\Theta_u G}\,G']=\mathbb{E}[\overline{\Theta_u G}\,U(su)G']=\langle G,U(su)G'\rangle_u$; hence $\|[U(su)G]\|^2=\langle G,U(2su)G\rangle_u\le\|[G]\|\,\|[U(2su)G]\|$ and, iterating, $\|[U(su)G]\|\le\|[G]\|^{1-2^{-m}}\|G\|_{L^2}^{2^{-m}}\to\|[G]\|$. So $[G]\mapsto[U(su)G]$ is a well-defined symmetric contraction semigroup on $\mathcal{H}_u$; it is strongly continuous because U is strongly continuous on $L^2(\mu)$ (for $G=e^{iF(\chi)}$, $U(a)G=e^{iF(\tau_a\chi)}\to G$ pointwise and boundedly as $a\to 0$, ϕ being continuous on $\mathcal{S}$; such G span a dense set; the $U(a)$ are unitary); thus it is e^{-sH_u} with $H_u\ge 0$ self-adjoint (the spectral theorem [30]). For $m\perp u$, $\Theta_u U(s'm)=U(s'm)\Theta_u$ gives $\langle U(s'm)G,U(s'm)G'\rangle_u=\langle G,G'\rangle_u$; the induced operators form a strongly continuous unitary group commuting with e^{-sH_u} (both compositions equal $[U(su+s'm)G]_u$), and $e^{is'P_{u,m}}$ is that group (Stone's theorem [30]). A unitary commuting with the bounded self-adjoint e^{-H_u} commutes with its spectral projections, which are those of H_u; so H_u and $P_{u,m}$ strongly commute, and the joint spectral measure exists, carried by $[0,\infty)\times\mathbb{R}$.

Density of the localised bounded classes: the $\Sigma(K)$ are countably generated and $\Sigma\{u\cdot v>0\}=\bigvee_K\Sigma(K)$ over bounded open $K\subset\{u\cdot v>0\}$, so by martingale convergence and truncation such functions are dense in $L^2_{u,+}$, and the class map is a contraction with dense range. □

For $u=e_t$ the objects of Lemma 15.5 are $\mathcal{H}_\circ$, H, $P:=P_{e_t,e_x}$ and Π; for $u=n$ they are $\mathcal{H}_n$, H_n, $P_n:=P_{n,n'}$ and Π_n. The invariances $\theta_{\circ *}\mu=\mu$, $\theta_{n*}\mu=\mu$ required are supplied by Lemma 15.7 from (i), (ii) and (iii) at the respective plane. The identification $[U(te_t+xe_x)G]_\circ=e^{-tH}e^{ixP}[G]_\circ$ ($t\ge 0$) is the last formula of Lemma 15.5 with $u=e_t$, $m=e_x$.

15.2. **Fields and moments.**

Lemma 15.6 (moments)**.** *Assume (i), and put $K_1:=17(1+C)$. Then:*

(a) $\|F(\chi)\|_{L^p(\mu)}\le K_1(\|\chi\|_1+\sqrt{p}\,\|\chi\|_2)$ *for all real $\chi\in\mathcal{S}$ and $1\le p<\infty$;*

(b) *F extends uniquely to a map $(L^1\cap L^2)(\mathbb{R}^3;T_{\mathbb{C}})\to\bigcap_{p<\infty}L^p(\mu)$, complex-linear, continuous for N in each L^p, with (a) for real arguments, $\|F(\chi)\|_p\le 2K_1(\|\chi\|_1+\sqrt{p}\|\chi\|_2)$ for complex ones, $\overline{F(\chi)}=F(\bar\chi)$, and (i) holds for all real $\chi\in L^1\cap L^2$;*

(c) $|S_n(\chi_1,\dots,\chi_n)|\le(2K_1\sqrt{n})^n\prod_{j=1}^n N(\chi_j)$ *for $\chi_j\in(L^1\cap L^2)_{\mathbb{C}}$; S_n is a symmetric n-linear form, continuous in each entry for N, and its restriction to $\mathcal{S}^n$ is a tempered distribution on $\mathbb{R}^{3n}$;*

(d) $\mathbb{E}_\mu e^{|F(\chi)|}<\infty$ *for real $\chi\in L^1\cap L^2$, and $Z(\chi)=\sum_{n\ge0}i^n S_n(\chi,\dots,\chi)/n!$, the series converging absolutely;*

(e) *for an affine isometry g of $\mathbb{R}^3$ with linear part in $O(3)$ (acting on tests by the pull-back with the tensor action), $g_*\mu=\mu$ on the σ-algebra generated by the evaluations $\phi\mapsto\phi(\chi)$, $\chi\in\mathcal{S}$ (the sense in which the invariance of a measure under g is used in this section; Remark 15.3), if and only if $Z\circ g^*=Z$ on $\mathcal{S}$, if and only if $S_n(g^*\chi_1,\dots,g^*\chi_n)=S_n(\chi_1,\dots,\chi_n)$ for all n and all $\chi_j\in\mathcal{S}$.*

Proof. (a) Let $\chi \neq 0$, $X := F(\chi)$, $a := C\|\chi\|_1$, $b := C\|\chi\|_2^2 > 0$. Hypothesis (i) at $\pm\lambda\chi$ ($\lambda > 0$) and the Chebyshev inequality give $\mu(|X| > s) \le 2e^{-\lambda s + a\lambda + b\lambda^2}$; for $s > a$ the choice $\lambda = (s-a)/(2b)$ yields $\mu(|X| > s) \le 2e^{-(s-a)^2/(4b)}$. Hence

$$\begin{aligned}\mathbb{E}|X|^p &= p\int_0^\infty s^{p-1}\mu(|X|>s)\,ds \le a^p + 2p\int_0^\infty (a+r)^{p-1}e^{-r^2/(4b)}\,dr\\ &\le a^p + 2^p p\,a^{p-1}\sqrt{\pi b} + 2^{p-1}p\,(4b)^{p/2}\Gamma(p/2),\end{aligned}$$

using $(a+r)^{p-1} \le 2^{p-1}(a^{p-1}+r^{p-1})$, $\int_0^\infty e^{-r^2/4b}dr = \sqrt{\pi b}$ and $\int_0^\infty r^{p-1}e^{-r^2/4b}dr = \frac12(4b)^{p/2}\Gamma(p/2)$. Taking p-th roots, with $(u+v+w)^{1/p} \le u^{1/p}+v^{1/p}+w^{1/p}$, $p^{1/p} \le e^{1/e} < 1.45$, $\pi^{1/(2p)} \le \sqrt{\pi} < 1.78$, $a^{(p-1)/p}b^{1/(2p)} \le a + \sqrt{b}$ (Young's inequality) and $\Gamma(p/2)^{1/p} \le 2\sqrt{p}$ (for $p \ge 2$ from $\Gamma(x) \le x^{x-1}$, $x \ge 1$; for $1 \le p \le 2$ from $\Gamma(p/2) \le \sqrt{\pi}$):

$$\|X\|_p \le a + 5.2\,(a+\sqrt{b}) + 2\cdot 1.45\cdot 2\sqrt{p}\cdot 2\sqrt{b} \le 6.2\,a + 16.8\sqrt{p}\,\sqrt{b} \le 7C\|\chi\|_1 + 17\sqrt{C}\sqrt{p}\,\|\chi\|_2,$$

which is (a) since $7C \le K_1$ and $17\sqrt{C} \le K_1$.

(b) F is linear on $\mathcal{S}$ (ϕ is linear), so (a) gives $\|F(\chi)-F(\chi')\|_p \le K_1(1+\sqrt{p})N(\chi-\chi')$; $\mathcal{S}$ is dense in $L^1\cap L^2$ for N, and the extension by uniform continuity is unique; put $F(\chi_1+i\chi_2) := F(\chi_1)+iF(\chi_2)$ for real χ_1,χ_2. The conjugation identity is clear. For real $\chi \in L^1\cap L^2$ and Schwartz $\chi_j \to \chi$ in N, a subsequence of $F(\chi_j)$ converges a.e., and Fatou's lemma gives $\mathbb{E}e^{F(\chi)} \le \liminf_j e^{C(\|\chi_j\|_1+\|\chi_j\|_2^2)} = e^{C(\|\chi\|_1+\|\chi\|_2^2)}$.

(c) The bound follows from Hölder's inequality with exponents $n,\dots,n$ and (b) at $p=n$. Symmetry and multilinearity are clear; continuity in each entry follows from (b); on $\mathcal{S}$ the norm N is dominated by finitely many Schwartz seminorms, so S_n is a separately continuous n-linear form on the Fréchet space $\mathcal{S}$, hence jointly continuous, hence a tempered distribution on $\mathbb{R}^{3n}$ by the kernel theorem [15, Chapter I], [30, Chapter V].

(d) The first assertion follows from $e^{|X|} \le e^X + e^{-X}$ and (b). By (c), $\sum_n |S_n(\chi,\dots,\chi)|/n! \le \sum_n (2K_1\sqrt{n}N(\chi))^n/n! < \infty$ ($n^{n/2}/n! \le e^{n/2}/\sqrt{n!}$), and dominated convergence (the partial sums of e^{iX} are bounded by $e^{|X|}$) gives the expansion.

(e) The cylinder sets $\{\phi : (\phi(\chi_1),\dots,\phi(\chi_N)) \in B\}$, $\chi_j \in \mathcal{S}$, $B \subset \mathbb{R}^N$ Borel, form a π-system generating the σ-algebra generated by the evaluations $\phi \mapsto \phi(\chi)$, $\chi \in \mathcal{S}$, and by the linearity of ϕ the characteristic functional determines the characteristic function $s \mapsto Z(\sum_j s_j\chi_j)$ of each vector $(\phi(\chi_1),\dots,\phi(\chi_N))$, hence its law; so two probability measures with the same characteristic functional on $\mathcal{S}$ coincide on that σ-algebra (Dynkin's π–λ theorem, as in the proof of Proposition A.9(a)), which is the one through which μ is used (Remark 15.3); $\int e^{iF(\chi)}d(g_*\mu) = Z(g^*\chi)$ (with the pull-back convention, $(g\phi)(\chi) = \phi(g^*\chi)$ up to the orientation of the action, which does not affect the equivalence since g^{-1} is again such an isometry). This gives the first equivalence. If $Z\circ g^* = Z$ on $\mathcal{S}$ then, by (d) applied to $\sum_j s_j\chi_j$ and the identification of the coefficient of $s_1\cdots s_n$ (polarisation), $S_n(g^*\chi_1,\dots,g^*\chi_n) = S_n(\chi_1,\dots,\chi_n)$; conversely the invariance of all S_n gives that of Z through (d). □

15.3. **The discrete symmetries.**

Lemma 15.7 (reflections and the one-point function)**.** *Assume (i), (ii), and (iii) for one reflection θ in a plane through 0. Then $\theta_*\mu = \mu$. Consequently, under (iii) for the nine planes, μ is invariant under the group B_3 of signed permutations of the coordinates, and $S_1(\chi) = \langle m, \int\chi\rangle$ with a vector $m \in T$ fixed by $\pi(B_3)$; m has no 2-form component, so $\pi(R)m = m$ and $S_1(R^*\chi) = S_1(\chi)$ for all $R \in O(3)$. Under (iii) for the four planes of a coordinate pair only, μ is invariant under the four reflections and the dihedral group D_4 they generate (the rotations by multiples of $\pi/2$ of*

that plane among them), $S_1(\chi) = \langle m, \int \chi\rangle$ with m fixed by $\pi(D_4)$, and $S_1(R(\theta)^*\chi) = S_1(\chi)$ for every rotation $R(\theta)$ of that plane.

Proof. Let $\psi_1, \psi_2 \in \mathcal{S}$ be real, supported in the open positive side of the plane. The matrix $(Z(\psi_a - \theta^*\psi_b))_{a,b=1,2}$ is positive semidefinite by (iii), hence Hermitian: $Z(\psi_2 - \theta^*\psi_1) = \overline{Z(\psi_1 - \theta^*\psi_2)} = Z(\theta^*\psi_2 - \psi_1) = Z(\theta^*(\psi_2 - \theta^*\psi_1))$, using $\overline{Z(\chi)} = Z(-\chi)$ and $(\theta^*)^2 = 1$. Every real $\chi \in \mathcal{S}$ vanishing on a neighbourhood of the plane is of the form $\psi_2 - \theta^*\psi_1$ (its restrictions to the two open sides), so $Z \circ \theta^* = Z$ on such tests. For general real $\chi \in \mathcal{S}$ let $\chi_\delta := \chi \cdot (1 - \varphi(u \cdot v/\delta))$, $\varphi \in C_c^\infty(\mathbb{R})$ equal to 1 near 0, u the unit normal; $\chi_\delta \in \mathcal{S}$ vanishes near the plane and $N(\chi - \chi_\delta) \to 0$ as $\delta \to 0$ by dominated convergence; Z is N-continuous (Lemma 15.6(b)) and θ^* is an N-isometry, so $Z \circ \theta^* = Z$ on $\mathcal{S}$, and Lemma 15.6(e) gives $\theta_*\mu = \mu$. The nine reflections generate B_3, the four reflections of a pair the dihedral group of order 8 in that plane, and invariance passes to products.

S_1 is a translation-invariant tempered distribution (Lemma 15.6(c) and (ii)), so its first derivatives vanish and $S_1(\chi) = \int \langle m, \chi(v)\rangle dv$ for a vector $m \in T$. For each of the nine reflections, $S_1(\theta^*\chi) = S_1(\chi)$ gives $\langle m, \pi(\theta)\int\chi\rangle = \langle m, \int\chi\rangle$ for all χ, i.e. $\pi(\theta)m = m$ ($\pi(\theta)$ symmetric), hence $\pi(g)m = m$ for $g \in B_3$. On the 2-form summand $\pi(g) = \det(g)g$; for $g = \mathrm{diag}(-1,1,1)$, $\pi(g) = \mathrm{diag}(1,-1,-1)$, which fixes only the first axis, and the three coordinate sign changes together fix only 0: the 2-form component of m vanishes. The scalar summand carries the trivial action, so $\pi(R)m = m$ for all $R \in O(3)$, and $S_1(R^*\chi) = \langle m, \pi(R)^{-1}\int\chi(Rv)\,dv\rangle = \langle \pi(R)m, \int\chi\rangle = S_1(\chi)$. Under the four planes of one pair only, the same argument gives $\pi(g)m = m$ for $g \in D_4$, in particular for the rotation by $\pi/2$ of the plane, whose tensor action is $\varrho(\pi/2) = \sum_a i^{w_a} P_a$ in the weight decomposition $\varrho(\theta) = \sum_a e^{iw_a\theta}P_a$ of the conventions of Subsection 15.9 ($w_a \in \{-2,\dots,2\}$); since $i^w = 1$ with $|w| \le 2$ only for $w = 0$, m lies in the weight-0 subspace, so $\varrho(\theta)m = m$ for all θ and $S_1(R(\theta)^*\chi) = \langle \varrho(\theta)m, \int\chi\rangle = S_1(\chi)$. □

15.4. **The spectral cone.**

Theorem 15.8 (spectral cone)**.** *Assume (i), (ii) and (iii) for the two planes $\{t = 0\}$ and $\{t + x = 0\}$ of a coordinate pair. Then the joint spectral measure Π of (H, P) on $\mathcal{H}_\circ$ satisfies*

$$\Pi\big(\{(\omega,p) : \omega < |p|\}\big) = 0, \qquad \text{hence} \qquad D(H) \subset D(P), \quad \langle\psi, H\psi\rangle \ge \langle\psi, |P|\psi\rangle \quad (\psi \in D(H)).$$

Only the invariances $\theta_{\circ *}\mu = \theta_{n*}\mu = \mu$ (Lemma 15.7 for these two planes), (ii), (iii) at the two planes, and Lemma 15.5 enter; (i) enters only through Lemma 15.7, that is through the N-continuity of Z (Lemma 15.6(b)). Throughout the proof $z(\omega,p) := (\omega - ip)/\sqrt{2}$, so $\mathrm{Re}\, z = \omega/\sqrt{2} \ge 0$ and $\mathrm{Im}\, z = -p/\sqrt{2}$ on the carrier of Π.

Lemma 15.9 (one pairing, two quantisations)**.** *Let $K \subset \{t > 0\}$ be bounded open, $A \in L^2(\Sigma(K), \mu)$, $G(b) := \mathbb{E}_\mu[\overline{\Theta_\circ A}\, U(b) A]$ for $b \in \mathbb{R}^3$, $\alpha := \inf_K n \cdot v$, $\alpha' := \inf_K n' \cdot v$, $s_0 := -\alpha - \alpha'$, and $\nu_A := \langle [A]_\circ, \Pi(\cdot)[A]_\circ\rangle_\circ$ (a positive measure of mass $\le \|A\|^2_{L^2(\mu)}$).*

(a) *For $s, s' \ge 0$: $\langle [U(s'n)A]_\circ, [U(sn)A]_\circ\rangle_\circ = G(sn + s'n')$ and $[U(sn)A]_\circ = \int e^{-s\, z(\omega,p)}\, d\Pi(\omega,p)\, [A]_\circ$.*

(b) *There is a complex Borel measure ρ on $[0,\infty)\times\mathbb{R}$ with $|\rho|([0,\infty)\times\mathbb{R}) \le \|A\|^2_{L^2(\mu)}$ such that $G(sn + s'n') = \int e^{-(s-s_0)\omega' + is'p'}\, d\rho(\omega', p')$ for all $s \ge s_0$, $s' \in \mathbb{R}$.*

Proof. (a) $U(sn)A \in L^2(\Sigma(K + sn)) \subset L^2_{\circ,+}$ for $s \ge 0$, and $[U(sn)A]_\circ = e^{-(s/\sqrt{2})H} e^{i(s/\sqrt{2})P}[A]_\circ = \int e^{-s(\omega - ip)/\sqrt{2}} d\Pi\, [A]_\circ$ by Lemma 15.5. Since $\Theta_\circ U(s'n) = U(-s'n')\Theta_\circ$ by (15.1) ($\theta_\circ n = -n'$), (ii) gives $\langle U(s'n)A, U(sn)A\rangle_\circ = \mathbb{E}[\overline{U(-s'n')\Theta_\circ A}\, U(sn)A] = \mathbb{E}[\overline{\Theta_\circ A}\, U(sn + s'n')A]$.

(b) For $v \in K$, $n \cdot (\theta_\circ v + \alpha' n) = -n' \cdot v + \alpha' \le 0$, so the open set $\theta_\circ K + \alpha' n$ lies in the open half-space $\{n \cdot v < 0\}$; likewise $K - \alpha n \subset \{n \cdot v > 0\}$. Put $A_1 := \Theta_n U(\alpha' n)\Theta_\circ A \in L^2(\Sigma(\theta_n(\theta_\circ K +$

$\alpha' n))) \subset L^2_{n,+}$ and $A_2 := U(-\alpha n)A \in L^2_{n,+}$; $\|A_i\|_{L^2(\mu)} = \|A\|_{L^2(\mu)}$. For $b = sn + s'n'$ and $c := \alpha' n$, $b + c = (s - s_0)n + s'n' - \alpha n$, so by (ii) and $\Theta_n A_1 = U(c)\Theta_\circ A$,

$$G(b) = \mathbb{E}[\overline{U(c)\Theta_\circ A}\, U(b+c)A] = \mathbb{E}[\overline{\Theta_n A_1}\, U((s-s_0)n+s'n')A_2] = \langle [A_1]_n, e^{-(s-s_0)H_n} e^{is'P_n}[A_2]_n\rangle_n$$

for $s \geq s_0$ (Lemma 15.5 with $u = n$, $m = n'$). Set $\rho(\Delta) := \langle [A_1]_n, \Pi_n(\Delta)[A_2]_n\rangle_n$; the Schwarz inequality over finite partitions gives $|\rho| \leq \|[A_1]_n\|\, \|[A_2]_n\| \leq \|A\|^2_{L^2(\mu)}$. □

Lemma 15.10 (test inequality)**.** *In the setting of Lemma 15.9 let* $s_1 := \max(s_0, 0)$, $h \in L^1(\mathbb{R})$ *vanishing on* $(-\infty, s_1)$, *and* $\tilde h(w) := \int h(s)e^{-sw}ds$ *for* $\operatorname{Re} w \geq 0$. *Then*

$$\int |\tilde h(z(\omega,p))|^2\, d\nu_A(\omega,p) \leq \|A\|^2_{L^2(\mu)} \sup_{\omega'\geq 0} e^{s_0\omega'}|\tilde h(\omega')| \sup_{p'\in\mathbb{R}} |\tilde h(ip')|.$$

Proof. Let $\psi_h := \int h(s)[U(sn)A]_\circ\, ds$ (Bochner integral; the integrand is continuous and bounded by $\|[A]_\circ\|$). By Lemma 15.9(a) and Fubini against $\langle \varphi, \Pi(\cdot)[A]_\circ\rangle$ ($|h(s)e^{-sz}| \leq |h(s)|$), $\psi_h = \tilde h(z(H,P))[A]_\circ$, so $\|\psi_h\|^2 = \int |\tilde h(z)|^2 d\nu_A$. Also $\|\psi_h\|^2 = \iint \overline{h(s')}h(s)\, G(sn + s'n')\, ds'ds$, and for $s, s' \geq s_1 \geq s_0$ Lemma 15.9(b) applies with $|e^{-(s-s_0)\omega'}| \leq 1$, so the integrals interchange: $\|\psi_h\|^2 = \int e^{s_0\omega'}\tilde h(\omega')\, \overline{\tilde h(ip')}\, d\rho(\omega', p')$. Bound by $|\rho|$; the first supremum is finite since $|e^{s_0\omega'}\tilde h(\omega')| \leq e^{(s_0-s_1)\omega'}\|h\|_1 \leq \|h\|_1$. □

Lemma 15.11 (the Laplace test)**.** *For* $\kappa \in \mathbb{R}\setminus\{0\}$, $\varepsilon > 0$ *let* $h(s) := (s - s_1)^{-i\kappa}e^{-\varepsilon(s-s_1)}$ *for* $s > s_1$ *and* $h(s) := 0$ *otherwise. Then* $\|h\|_1 = 1/\varepsilon$ *and, for* $\operatorname{Re} w \geq 0$ *(principal branch),*

$$\tilde h(w) = e^{-s_1 w}\,\Gamma(1 - i\kappa)\,(w + \varepsilon)^{-1+i\kappa}, \qquad |\tilde h(w)| = e^{-s_1 \operatorname{Re} w}\,|\Gamma(1 - i\kappa)|\,|w + \varepsilon|^{-1}e^{-\kappa \operatorname{Arg}(w+\varepsilon)}.$$

Proof. Substitute $\sigma = s - s_1$ and use $\int_0^\infty \sigma^{a-1}e^{-\upsilon\sigma}d\sigma = \Gamma(a)\upsilon^{-a}$ for $\operatorname{Re} a > 0$, $\operatorname{Re}\upsilon > 0$ (analytic continuation from real υ), with $a = 1 - i\kappa$, $\upsilon = w + \varepsilon$. □

Proof of Theorem 15.8. Fix K, A as in Lemma 15.9. Insert Lemma 15.11 into Lemma 15.10: $e^{s_0\omega'}|\tilde h(\omega')| \leq |\Gamma(1 - i\kappa)|/\varepsilon$ (as $s_1 \geq s_0$ and $\operatorname{Arg}(\omega' + \varepsilon) = 0$) and $|\tilde h(ip')| \leq |\Gamma(1 - i\kappa)|\varepsilon^{-1}e^{|\kappa|\pi/2}$ (as $|\operatorname{Arg}(\varepsilon + ip')| < \pi/2$); dividing by $|\Gamma(1 - i\kappa)|^2 = \pi\kappa/\sinh(\pi\kappa) > 0$,

$$\int e^{-\sqrt{2}s_1\omega}\,|z + \varepsilon|^{-2}\,e^{-2\kappa \operatorname{Arg}(z+\varepsilon)}\, d\nu_A(\omega,p) \leq \|A\|^2_{L^2(\mu)}\,\varepsilon^{-2}e^{|\kappa|\pi/2}, \qquad z = z(\omega,p). \tag{15.2}$$

For $\delta \in (0, \pi/4)$ and $r > 0$ put $S^{\pm}_{\delta,r} := \{(\omega,p) : |z| \geq r,\ \pm\operatorname{Arg} z \geq \pi/4 + \delta\}$. For $\operatorname{Re} z \geq 0$, $|\partial_\varepsilon \operatorname{Arg}(z + \varepsilon)| = |\operatorname{Im} z|/|z + \varepsilon|^2 \leq 1/|z|$, so $|\operatorname{Arg}(z + \varepsilon) - \operatorname{Arg} z| \leq \varepsilon/r$ on $\{|z| \geq r\}$. Choose $\varepsilon := r\delta/2$ (independent of κ). On $S^-_{\delta,r}$, $\operatorname{Arg}(z + \varepsilon) \leq -\pi/4 - \delta/2$, so for $\kappa > 0$ the integrand of (15.2) is at least $e^{-\sqrt{2}s_1\omega}|z + \varepsilon|^{-2}e^{\kappa(\pi/2+\delta)}$ there, and

$$\int_{S^-_{\delta,r}} e^{-\sqrt{2}s_1\omega}\,|z + \varepsilon|^{-2}\, d\nu_A \leq 4\,\|A\|^2_{L^2(\mu)}\,(r\delta)^{-2}e^{-\kappa\delta} \xrightarrow[\kappa\to\infty]{} 0.$$

The weight is strictly positive, hence $\nu_A(S^-_{\delta,r}) = 0$; letting $\kappa \to -\infty$ gives $\nu_A(S^+_{\delta,r}) = 0$. Since $\operatorname{Re} z = \omega/\sqrt{2}$ and $\operatorname{Im} z = -p/\sqrt{2}$, $\{\omega < p\} = \{\operatorname{Arg} z < -\pi/4\} = \bigcup_{m\geq 2} S^-_{1/m,1/m}$ and $\{\omega < -p\} = \bigcup_{m\geq 2} S^+_{1/m,1/m}$, so $\nu_A(\Delta) = 0$ for $\Delta := \{\omega < |p|\}$, i.e. $\Pi(\Delta)[A]_\circ = 0$. The classes of such A are dense in $\mathcal{H}_\circ$ (Lemma 15.5) and $\Pi(\Delta)$ is bounded, so $\Pi(\Delta) = 0$. For $\psi \in D(H)$, $\int p^2\, d\Pi_\psi \leq \int \omega^2\, d\Pi_\psi < \infty$ and $\int |p|\, d\Pi_\psi \leq \int \omega\, d\Pi_\psi$. □

15.5. **Complex translations.**

Lemma 15.12 (complex translations)**.** *Under the hypotheses of Theorem 15.8, $V(z) := \int e^{-t\omega+ixp}\,d\Pi(\omega,p)$, $z=(t,x)\in\mathbb{C}^2$ with $q(z)\ge 0$, is a contraction on $\mathcal{H}_\circ$, strongly continuous on $\{q\ge 0\}$, operator-norm holomorphic on $\mathcal{W}$, with $V(z)V(z')=V(z+z')$, $V(0)=1$, $V(z)^*=V(-\Theta_\circ\bar z)=V(\bar t,-\bar x)$, $V(z)\Omega=V(z)^*\Omega=\Omega$, $V(t,x)[G]_\circ=[U(te_t+xe_x)G]_\circ$ for real $t\ge 0$, $x\in\mathbb{R}$, and*

$$\big\|\partial_t^a\partial_x^b V(z)\big\| \le \sup_{\omega\ge 0}\omega^{a+b}e^{-q(z)\omega} = \Big(\frac{a+b}{e\,q(z)}\Big)^{a+b} \qquad (z\in\mathcal{W},\ a+b\ge 1). \tag{15.3}$$

Proof. On the carrier $\{\omega\ge|p|\}$ of Π, $|e^{-t\omega+ixp}|=e^{-(\operatorname{Re}t)\omega-(\operatorname{Im}x)p}\le e^{-q(z)\omega}\le 1$, so $V(z)$ is a contraction; the semigroup law and the adjoint formula are those of the symbols; the real case is Lemma 15.5; $V(z)\Omega=\Omega$ since Π has the atom $\{(0,0)\}$ at Ω ($e^{-tH}\Omega=\Omega$ because $U(te_t)1=1$, and $e^{ixP}\Omega=\Omega$ likewise), and $V(z)^*\Omega=\Omega$ by the adjoint formula. Strong continuity on $\{q\ge 0\}$: the symbols are bounded by 1 and converge pointwise, so dominated convergence in the spectral integral applies. Holomorphy: for $q(z)>0$ and $|h|<q(z)/2$, $|e^{-h\omega}-1+h\omega|\le|h|^2\omega^2e^{|h|\omega}/2$ and $\sup_\omega\omega^2e^{-(q(z)-|h|)\omega}<\infty$, so the difference quotients of the symbol in t converge uniformly on the carrier to $-\omega e^{-t\omega+ixp}$; the same for x with $|p|\le\omega$; thus V is norm-holomorphic in each variable separately, hence jointly (Hartogs' theorem for Banach-space-valued functions, or directly from the uniform convergence of the two-variable Taylor remainder of the symbol), with $\partial_t^a\partial_x^bV(z)=\int(-\omega)^a(ip)^be^{-t\omega+ixp}d\Pi$; (15.3) follows from $|p|\le\omega$ and the elementary maximum. □

15.6. **The slab-source bound and the bounded slab insertion.** By Lemma 15.7, hypotheses (i), (ii) and (iii) at $\{t=0\}$ give $\theta_{\circ *}\mu=\mu$; hence, with (ii), every Θ_c is an isometry of $L^2(\mu)$ exchanging $L^2(\Sigma\{t<c\})$ and L^2_c. The following two facts, which hold under (i), (ii) and (iii) at $\{t=0\}$ alone, are used throughout this subsection:

(F1) $\langle G,G\rangle_0\ge 0$ on $L^2_{\circ,+}$ (Lemma 15.5); for $G\in L^2_c$, $G':=U(-ce_t)G\in L^2_{\circ,+}$ and $\overline{\Theta_cG}\,G=U(ce_t)(\overline{\Theta_\circ G'}\,G')$ by (15.1), so $\langle G,G\rangle_c\ge 0$ by (ii). Hence $\langle\cdot,\cdot\rangle_c$ is Hermitian and obeys the Schwarz inequality. For $F_-\in L^2(\Sigma\{t<c\})$ and $G_+\in L^2_c$, $\mathbb{E}[F_-G_+]=\langle\overline{\Theta_cF_-},G_+\rangle_c$, whence

$$|\mathbb{E}[F_-G_+]|^2\le\mathbb{E}[F_-\,\overline{\Theta_cF_-}]\;\mathbb{E}[\overline{\Theta_cG_+}\,G_+],\qquad\text{both factors }\ge 0. \tag{15.4}$$

(F2) For real $a\ge 0$, $b\in\mathbb{R}$ and $G\in L^2_{\circ,+}$: $V(a,b)[G]_\circ=[U(ae_t+be_x)G]_\circ$ (Lemma 15.5).

Lemma 15.13 (localisation)**.** *Let $k\in(L^1\cap L^2)(\mathbb{R}^3;T_{\mathbb{C}})$ vanish a.e. outside $\{t\le c\}$ (resp. $\{t\ge c\}$). Then $F(k)\in L^2(\Sigma\{t<c\})$ (resp. L^2_c), $|e^{F(k)}|=e^{F(\operatorname{Re}k)}$, and $\mathbb{E}\,e^{pF(\operatorname{Re}k)}\le e^{C(p\|\operatorname{Re}k\|_1+p^2\|\operatorname{Re}k\|_2^2)}$ for $p>0$.*

Proof. The function $k_j:=\psi_{1/2j}*\big(1_{\{|v|\le j\}}\tau_{-e_t/j}k\big)$, with ψ_ε a mollifier supported in the ε-ball, lies in $C_c^\infty(\{t<c\};T_{\mathbb{C}})$ and $k_j\to k$ in $L^1\cap L^2$; by Lemma 15.6(b), $F(k_j)\to F(k)$ in $L^2(\mu)$, and $L^2(\Sigma\{t<c\})$ is closed. $F(k)=F(\operatorname{Re}k)+iF(\operatorname{Im}k)$ with real summands gives the modulus; the bound is (i) for $p\operatorname{Re}k$ (Lemma 15.6(b)). □

Theorem 15.14 (slab-source bound)**.** *Assume (i), (ii) and (iii) at $\{t=0\}$. Let $l>0$, $h\in(L^1\cap L^2)(\mathbb{R}^3;T_{\mathbb{C}})$ vanish a.e. outside $\{0\le t\le l\}$, and $A\in L^\infty(\Sigma\{t>0\},\mu)$. Then $e^{F(h)}U(le_t)A\in L^2_{\circ,+}$ and*

$$\big\|[e^{F(h)}U(le_t)A]_\circ\big\|\le\exp\big(C(\|\operatorname{Re}h\|_1+\|\operatorname{Re}h\|_2^2)\big)\,\|[A]_\circ\|,$$

with the constant C of (i).

Proof. Fix a version with $|A| \le \|A\|_\infty$ everywhere; $A' := U(le_t)A \in L^\infty(\Sigma\{t > l\})$, $\Theta_\circ A' \in L^\infty(\Sigma\{t < -l\})$, $|\Theta_\circ A'| \le \|A\|_\infty$; $G := e^{F(h)}A' \in L^2_{\circ,+}$ by Lemma 15.13. Put $\kappa := \|\operatorname{Re} h\|_1 + \|\operatorname{Re} h\|_2^2$.

Step 1 (dyadic sources). Put $c_n := (2^{n+1}-1)l$, $k_0 := h + Jh$, $k_{n+1} := k_n + J_{c_n}k_n$. By induction k_n vanishes a.e. outside $\{-l \le t \le c_n\}$: θ_{c_n} maps $[-l, c_n]$ onto $[c_n, 2c_n + l] = [c_n, c_{n+1}]$. The two summands of k_{n+1} (and of k_0: slabs $[0, l]$ and $[-l, 0]$) have a.e. disjoint supports, and θ_c^* preserves $\|\operatorname{Re}\cdot\|_1$ and $\|\operatorname{Re}\cdot\|_2$; hence $\|\operatorname{Re} k_n\|_1 = 2^{n+1}\|\operatorname{Re} h\|_1$ and $\|\operatorname{Re} k_n\|_2^2 = 2^{n+1}\|\operatorname{Re} h\|_2^2$.

Step 2. Put $E_n := \mathbb{E}\big[e^{F(k_n)}\,\overline{\Theta_\circ A'}\,U(c_n e_t)A\big]$, which is finite by Lemma 15.13. Since $\overline{\Theta_\circ e^{F(h)}} = e^{\overline{F(\theta_0^* h)}} = e^{F(Jh)}$, $\overline{\Theta_\circ G}\,G = e^{F(h+Jh)}\,\overline{\Theta_\circ A'}\,A'$, i.e. $E_0 = \|[G]_\circ\|^2$.

Step 3 (one reflection). Claim: $E_{n+1} \ge 0$ and $|E_n| \le \|[A]_\circ\|\,E_{n+1}^{1/2}$. Let $c = c_n$, $F_- := e^{F(k_n)}\overline{\Theta_\circ A'} \in L^2(\Sigma\{t < c\})$ (Lemma 15.13; $-l < c$), $G_+ := U(ce_t)A \in L^\infty(\Sigma\{t > c\})$, so $E_n = \mathbb{E}[F_- G_+]$ and (15.4) applies. Second factor: $\Theta_c U(ce_t) = U(ce_t)\Theta_\circ$ and (ii) give $\mathbb{E}[\overline{\Theta_c G_+}G_+] = \mathbb{E}[U(ce_t)(\overline{\Theta_\circ A}A)] = \|[A]_\circ\|^2$. First factor: $\overline{\Theta_c F_-} = e^{F(J_c k_n)}\,\Theta_c\Theta_\circ A' = e^{F(J_c k_n)}\,U((2c+l)e_t)A$ with $2c_n + l = c_{n+1}$, so $\mathbb{E}[F_-\overline{\Theta_c F_-}] = E_{n+1}$, and this is $\langle\overline{\Theta_c F_-}, \overline{\Theta_c F_-}\rangle_c \ge 0$.

Step 4 (terminal bound). $|E_n| \le \|A\|_\infty^2\,\mathbb{E}\,e^{F(\operatorname{Re} k_n)} \le \|A\|_\infty^2 \exp(2^{n+1}C\kappa)$ by Lemma 15.13 and Step 1.

Step 5. Iterating Step 3, $E_0 \le \|[A]_\circ\|^{2-2^{1-n}} E_n^{2^{-n}}$ (induction: $2 - 2^{1-n} + 2^{-n} = 2 - 2^{-n}$), hence

$$\|[G]_\circ\|^2 = E_0 \le \|[A]_\circ\|^{2-2^{1-n}}\,\|A\|_\infty^{2^{1-n}}\,e^{2C\kappa} \qquad (n \ge 1).$$

If $\|[A]_\circ\| = 0$ the right side is 0; if $A = 0$ a.e. everything vanishes; otherwise let $n \to \infty$. Thus $\|[G]_\circ\| \le e^{C\kappa}\|[A]_\circ\|$. □

Theorem 15.15 (bounded slab insertion)**.** *Assume (i), (ii) and (iii) at $\{t = 0\}$. Let $\delta > 0$, $\chi \in (L^1 \cap L^2)(\mathbb{R}^3; T_{\mathbb{C}})$ vanish a.e. outside $\{|t| \le \delta\}$, $h := \tau_{\delta e_t}\chi$, and $K_6 := e^{2C}$.*

(a) *For $A \in L^\infty(\Sigma\{t > 0\})$: $F(h)U(2\delta e_t)A \in L^2_{\circ,+}$ and $\|[F(h)U(2\delta e_t)A]_\circ\| \le K_6\,N(\chi)\,\|[A]_\circ\|$. The map $[A]_\circ \mapsto [F(h)U(2\delta e_t)A]_\circ$ is well defined on the dense subspace $\mathcal{D} := \{[A]_\circ : A \in L^\infty(\Sigma\{t > 0\})\}$ and extends uniquely to $B_\delta(\chi) \in \mathcal{B}(\mathcal{H}_\circ)$ with $\|B_\delta(\chi)\| \le K_6(\|\chi\|_1 + \|\chi\|_2)$, complex-linear in χ.*

(b) *$B_\delta(\chi)^* = B_\delta(J\chi)$.*

(c) *$V(r + t, x)\,B_\delta(\chi)\,V(r - t, -x) = B_{\delta+r}(\tau_{(t,x,0)}\chi)$ for real $r \ge |t|$, $x \in \mathbb{R}$.*

(d) *For $A \in L^4(\Sigma\{t > 0\}, \mu)$: $F(h)U(2\delta e_t)A \in L^2_{\circ,+}$ and $B_\delta(\chi)[A]_\circ = [F(h)U(2\delta e_t)A]_\circ$. Consequently, for χ_j vanishing outside $\{|t| \le \delta_j\}$, $j = 1, \dots, m$, with $s_j := \delta_j + 2\sum_{i<j}\delta_i$,*

$$B_{\delta_1}(\chi_1)\cdots B_{\delta_m}(\chi_m)\Omega = \Big[\prod_{j=1}^m F(\tau_{s_j e_t}\chi_j)\Big]_\circ,$$

$$\langle\Omega, B_{\delta_1}(\chi_1)\cdots B_{\delta_m}(\chi_m)\Omega\rangle_\circ = S_m(\tau_{s_1 e_t}\chi_1, \dots, \tau_{s_m e_t}\chi_m).$$

Proof. (a) Let $A' := U(2\delta e_t)A$, $G_m := F(h)^m A'/m!$. By Lemma 15.6(b), $\|G_m\|_{L^2} \le \|A\|_\infty \|F(h)\|_{2m}^m/m! \le \|A\|_\infty(2\sqrt{2}K_1 N(h))^m m^{m/2}/m!$, and $m^{m/2}/m! \le (e/\sqrt{m})^m$, so $g(s) := \sum_m s^m G_m$ converges absolutely in $L^2_{\circ,+}$ for every $s \in \mathbb{C}$ and equals $e^{sF(h)}A' = e^{F(sh)}A'$ a.e. By (F1), $f(s) := [g(s)]_\circ = \sum_m s^m [G_m]_\circ$ is an absolutely convergent power series in $\mathcal{H}_\circ$; termwise integration gives $[G_1]_\circ = \frac{1}{2\pi i}\oint_{|s|=r} f(s)s^{-2}ds$, hence $\|[F(h)A']_\circ\| \le r^{-1}\max_{|s|=r}\|f(s)\|$. The source sh vanishes outside $\{0 \le t \le 2\delta\}$ and $|\operatorname{Re}(sh)| \le |s||h|$ pointwise, so Theorem 15.14 ($l = 2\delta$) gives $\|f(s)\| \le \exp(C(r\|h\|_1 + r^2\|h\|_2^2))\|[A]_\circ\|$ for $|s| = r$. If $N(h) = 0$ there is nothing to prove; otherwise $r := 1/N(h)$ yields $\|[F(h)A']_\circ\| \le e^{2C}N(h)\|[A]_\circ\|$, and $N(h) = N(\chi)$. If $[A]_\circ = [A_2]_\circ$ then $\|[F(h)U(2\delta e_t)(A - A_2)]_\circ\| \le K_6 N(\chi)\|[A - A_2]_\circ\| = 0$. Density: for $G \in L^2_{\circ,+}$, $G_M := G1_{\{|G|\le M\}}$ satisfies $\|[G - G_M]_\circ\| \le \|G - G_M\|_{L^2} \to 0$, and $L^2_{\circ,+}/\mathcal{N}$ is dense in $\mathcal{H}_\circ$. Extend by uniform continuity; linearity in χ is that of F.

(b) For $A, A' \in L^\infty(\Sigma\{t>0\})$: $\langle [A']_\circ, B_\delta(\chi)[A]_\circ\rangle_\circ = \mathbb{E}[\overline{\Theta_\circ A'}\, F(\tau_{\delta e_t}\chi)\, U(2\delta e_t)A]$, while

$$\langle B_\delta(J\chi)[A']_\circ, [A]_\circ\rangle_\circ = \mathbb{E}\big[\overline{\Theta_\circ\big(F(\tau_{\delta e_t}J\chi)U(2\delta e_t)A'\big)}\, A\big] = \mathbb{E}\big[F(J\tau_{\delta e_t}J\chi)\,\overline{U(-2\delta e_t)\Theta_\circ A'}\, A\big],$$

using $\overline{\Theta_\circ F(k)} = F(Jk)$ and $\Theta_\circ U(2\delta e_t) = U(-2\delta e_t)\Theta_\circ$. Here $J\tau_{\delta e_t}J\chi = \theta_0^*\tau_{\delta e_t}\theta_0^*\chi = \tau_{-\delta e_t}\chi$, and (ii) applied with $U(2\delta e_t)$ turns the last expectation into the first. Both operators are bounded and $\mathcal{D}$ is dense.

(c) For $A \in L^\infty(\Sigma\{t>0\})$, (F2) ($r \mp t \geq 0$) gives $V(r-t,-x)[A]_\circ = [A_1]_\circ$, $A_1 := U((r-t)e_t - xe_x)A \in L^\infty(\Sigma\{t>0\})$, so $B_\delta(\chi)[A_1]_\circ = [F(\tau_{\delta e_t}\chi)U(2\delta e_t)A_1]_\circ$ and, by (F2) applied to this element of $L^2_{\circ,+}$, $U(a)(GG') = U(a)G\,U(a)G'$ and $U(a)F(k) = F(\tau_a k)$,

$$V(r+t,x)B_\delta(\chi)V(r-t,-x)[A]_\circ = \big[F(\tau_{(\delta+r+t,x,0)}\chi)\,U(2(\delta+r)e_t)A\big]_\circ = B_{\delta+r}(\tau_{(t,x,0)}\chi)[A]_\circ,$$

because $\tau_{(t,x,0)}\chi$ vanishes outside $\{|t'| \leq \delta + |t|\} \subset \{|t'| \leq \delta + r\}$ and $\tau_{(\delta+r)e_t}\tau_{(t,x,0)} = \tau_{(\delta+r+t,x,0)}$. Conclude by density.

(d) $\|F(h)U(2\delta e_t)A\|_{L^2} \leq \|F(h)\|_4\|A\|_4$ (Hölder and (ii)), and measurability follows from Lemma 15.13. With $A_M := A1_{\{|A|\leq M\}}$: $A_M \to A$ in L^4 and L^2, so $B_\delta(\chi)[A_M]_\circ \to B_\delta(\chi)[A]_\circ$, while $\|[F(h)U(2\delta e_t)(A-A_M)]_\circ\| \leq \|F(h)\|_4\|A-A_M\|_4 \to 0$; since $B_\delta(\chi)[A_M]_\circ = [F(h)U(2\delta e_t)A_M]_\circ$, the identity follows. The iterated formula follows by induction from the right: the vector at each stage is the class of a product of fields whose tests vanish outside $\{t \geq 0\}$ (the shifts $s'_j - \delta_j = 2\sum_{2\leq i<j}\delta_i \geq 0$ at the inner stage, and $s'_j + 2\delta_1 = s_j$ after the outer insertion), which lies in $L^4(\Sigma\{t>0\})$ by Lemma 15.6(b), Hölder and Lemma 15.13; finally $\langle\Omega, [G]_\circ\rangle_\circ = \mathbb{E}[G]$. □

15.7. **The planar point insertion.** In this subsection $e_t = (1,0)$ also denotes the planar unit vector; τ_z translates planar functions by $z \in \mathbb{R}^2$ and tests by $(z,0)$.

Theorem 15.16 (planar point insertion)**.** *Assume the hypotheses of Theorem 15.8, and let K_6 be the constant of Theorem 15.15. Let $f \in (L^1\cap L^2)(\mathbb{R};T_\mathbb{C})$, $(\xi,\eta) \in \mathcal{W}^2$, $q_* := \min(q(\xi),q(\eta))$. Then there is a unique $A_f(\xi,\eta) \in \mathcal{B}(\mathcal{H}_\circ)$ with*

$$V(\xi-\delta_n e_t)\,B_{\delta_n}(\alpha_n f)\,V(\eta-\delta_n e_t) \longrightarrow A_f(\xi,\eta) \quad \textit{in operator norm}$$

for every sequence $\alpha_n \in L^2(\mathbb{R}^2)$ with $\int\alpha_n = 1$, $\sup_n\|\alpha_n\|_1 < \infty$, $\operatorname{supp}\alpha_n \subset \{|z| \leq s_n\}$, $s_n \to 0$, and every choice $\delta_n \in [s_n, q_*]$ (for the n with $s_n \leq q_*$). Moreover, with $K_7 := 14K_6$:

(a) *$f \mapsto A_f(\xi,\eta)$ is linear and $\|A_f(\xi,\eta)\| \leq K_7(\|f\|_1 + \|f\|_2/q_*)$;*
(b) *$(\xi,\eta) \mapsto A_f(\xi,\eta)$ is operator-norm holomorphic on $\mathcal{W}^2$;*
(c) *$V(\kappa)A_f(\xi,\eta)V(\kappa') = A_f(\kappa+\xi,\eta+\kappa')$ for $q(\kappa), q(\kappa') \geq 0$;*
(d) *$A_f(\xi,\eta)^* = A_{Jf}(-\Theta_\circ\bar\eta, -\Theta_\circ\bar\xi)$;*
(e) *$A_f(\xi,a)A_g(b,\eta) = A_f(\xi,a')A_g(b',\eta)$ whenever $a,b,a',b' \in \mathcal{W}$ and $a+b = a'+b'$;*
(f) *for $\alpha \in L^2(\mathbb{R}^2)$ with compact support in $\{|t| < q_*\}$ and every $\delta \in [\sup_{\operatorname{supp}\alpha}|t|, q_*]$, $V(\xi-\delta e_t)B_\delta(\alpha f)V(\eta-\delta e_t) = \int\alpha(z)A_f(\xi+z,\eta-z)\,dz$ (norm-convergent Bochner integral); in particular $B_\delta(\alpha f) = \int\alpha(z)A_f(\delta e_t+z,\delta e_t-z)\,dz$ for α compactly supported in $\{|t|<\delta\}$.*

Proof. Step 0 (calculus of V). By Lemma 15.12, $\|V(w)\| \leq 1$ for $q(w) \geq 0$, $V(w)V(w') = V(w+w')$, $V(w)^* = V(-\Theta_\circ\bar w)$, and V is norm-holomorphic on $\mathcal{W}$ with (15.3).

Step 1 (smeared insertions). For $\beta \in L^2(\mathbb{R}^2)$ with compact support let $t_\beta := \sup\{|t| : (t,x) \in \operatorname{supp}\beta\}$; for $q_* > t_\beta$ and $\delta \in (0,q_*]$ with $\delta \geq t_\beta$ ("admissible") put $T_\beta(\xi,\eta) := V(\xi-\delta e_t)B_\delta(\beta f)V(\eta-\delta e_t)$. (1.i) Independence of δ: for admissible $\delta < \delta'$, Theorem 15.15(c) with $(t,x) = 0$ gives $B_{\delta'}(\beta f) = V((\delta'-\delta)e_t)B_\delta(\beta f)V((\delta'-\delta)e_t)$; the semigroup law finishes. (1.ii) $\|T_\beta(\xi,\eta)\| \leq K_6(\|\beta\|_1\|f\|_1 + \|\beta\|_2\|f\|_2)$ since $\|\beta f\|_p = \|\beta\|_p\|f\|_p$; T_β is linear in β and in f. (1.iii) T_β is norm-holomorphic on $\{(\xi,\eta) : q_* > t_\beta\}$: near a point with $q_* > \delta$ it is a product

of norm-holomorphic factors in independent variables, and near a point with $t_\beta < q_* \le \delta$ one first passes by (1.i) to an admissible $\delta'' < q_*$. Also $V(\kappa)T_\beta(\xi,\eta)V(\kappa') = T_\beta(\kappa+\xi,\eta+\kappa')$ for $q(\kappa), q(\kappa') \ge 0$ (semigroup law; q is superadditive). (1.iv) By Theorem 15.15(b), Step 0 and $-\Theta_\circ\overline{(\eta-\delta e_t)} = -\Theta_\circ\bar\eta - \delta e_t$: $T^f_\beta(\xi,\eta)^* = T^{Jf}_{\tilde\beta}(-\Theta_\circ\bar\eta, -\Theta_\circ\bar\xi)$, where $q(-\Theta_\circ\bar\eta) = q(\eta)$ and $t_{\tilde\beta} = t_\beta$.

(1.v) Convolution identity. Let $\gamma \in L^1(\mathbb{R}^2)$ with compact support and $\sup_{\operatorname{supp}\gamma}|t| \le s < r$, $\delta \ge t_\beta$, $\delta + r \le q_*$, and $\Lambda(z) := V(r+t,x)B_\delta(\beta f)V(r-t,-x)$ for real $z=(t,x)$ with $|t| \le r$. By Theorem 15.15(c), $\Lambda(z) = B_{\delta+r}(\tau_z(\beta f))$, norm-continuous in z (τ_z is strongly continuous on $L^1\cap L^2$ and $B_{\delta+r}$ is bounded linear). The Bochner integral $\int\gamma(z)\tau_z(\beta f)dz$ equals $(\gamma*\beta)f$ in $L^1\cap L^2$ (pair with bounded compactly supported functions and use Fubini), and $B_{\delta+r}$ commutes with Bochner integrals, so $\int\gamma\Lambda = B_{\delta+r}((\gamma*\beta)f)$; as $t_{\gamma*\beta} \le t_\beta + s < \delta + r \le q_*$, $\delta+r$ is admissible for $\gamma*\beta$, and

$$T_{\gamma*\beta}(\xi,\eta) = V(\xi-(\delta+r)e_t)\Big[\int\gamma(z)\Lambda(z)\,dz\Big]V(\eta-(\delta+r)e_t). \tag{15.5}$$

Since $V(\xi-(\delta+r)e_t)V(r+t,x) = V(\xi+z-\delta e_t)$ (both factors have $q \ge 0$) and likewise on the right, $V(\xi-(\delta+r)e_t)\Lambda(z)V(\eta-(\delta+r)e_t) = T_\beta(\xi+z,\eta-z)$ for real z with $|t| \le s$ (here $q(\xi+z) = q(\xi)+t \ge \delta+r-s > t_\beta$), whence

$$T_{\gamma*\beta}(\xi,\eta) = \int\gamma(z)\,T_\beta(\xi+z,\eta-z)\,dz. \tag{15.6}$$

(1.vi) Taylor bounds. Let now $\operatorname{supp}\gamma \subset \{|z| \le s\}$. Λ is norm-C^2 on $\{|t| < r\}$; by the Leibniz rule and (15.3) (the arguments of the two V's have $q = r \pm t \ge r - |z|$) every first derivative is bounded by $2\|B\|/(e(r-|z|))$ and every second derivative by $(4+2+4)\|B\|/(e^2(r-|z|)^2)$, $B := B_\delta(\beta f)$. The mean value inequality and Taylor's formula with integral remainder give, for $|z| \le s < r$,

$$\|\Lambda(z)-\Lambda(0)\| \le \frac{2\sqrt{2}\,|z|\,\|B\|}{e(r-s)}, \qquad \|\Lambda(z)-\Lambda(0)-D\Lambda(0)z\| \le \frac{10\,|z|^2\|B\|}{e^2(r-s)^2}. \tag{15.7}$$

Hence, if $\int\gamma = 0$ and $\int\gamma(z)z_i\,dz = 0$ ($i=1,2$), then $\|\int\gamma\Lambda\| \le 10s^2\|\gamma\|_1\|B\|/(e^2(r-s)^2)$.

Step 2 (existence, bound, holomorphy, (c)–(e)). Let $\rho_\varepsilon := (\pi\varepsilon^2)^{-1}\mathbf{1}_{\{|z|\le\varepsilon\}}$, $\beta_\varepsilon := \rho_\varepsilon * \rho_\varepsilon$; then β_ε is even, $\int\beta_\varepsilon = 1$, its support lies in $\{|z| \le 2\varepsilon\}$, and $\|\beta_\varepsilon\|_2 \le \|\rho_\varepsilon\|_2 = (\pi\varepsilon^2)^{-1/2}$. Fix (ξ,η), $0 < q \le q_*$, $\varepsilon \le q/4$. Then $\beta_\varepsilon - \beta_{\varepsilon/2} = \gamma * \beta'$ with $\gamma := \rho_\varepsilon - \rho_{\varepsilon/2}$ (even, $\int\gamma = 0$, $\|\gamma\|_1 \le 2$) and $\beta' := \rho_\varepsilon + \rho_{\varepsilon/2}$ ($\|\beta'\|_1 = 2$, $\|\beta'\|_2 \le 3\pi^{-1/2}\varepsilon^{-1}$), both supported in $\{|z| \le \varepsilon\}$. Apply (15.5) and (1.vi) with $\delta = \varepsilon$, $r = q-\varepsilon$, $s = \varepsilon$ ($r - s \ge q/2$), $\|V\| \le 1$ and Theorem 15.15(a):

$$\|T_{\beta_\varepsilon} - T_{\beta_{\varepsilon/2}}\| \le \frac{80\varepsilon^2}{e^2q^2}K_6\Big(2\|f\|_1 + \frac{3\|f\|_2}{\sqrt{\pi}\,\varepsilon}\Big) = \frac{160K_6}{e^2}\frac{\varepsilon^2}{q^2}\|f\|_1 + \frac{240K_6}{e^2\sqrt{\pi}}\frac{\varepsilon}{q^2}\|f\|_2. \tag{15.8}$$

Let $\varepsilon_k := 2^{-k}$ and k_0 be least with $2^{-k_0} \le q_*/4$, so $\varepsilon_{k_0} > q_*/8$; $\sum_{k\ge k_0}\varepsilon_k^2 \le q_*^2/12$ and $\sum_{k\ge k_0}\varepsilon_k \le q_*/2$. By (15.8) ($q = q_*$), $(T_{\beta_{\varepsilon_k}}(\xi,\eta))_{k\ge k_0}$ is norm-Cauchy; set $A_f(\xi,\eta) := \lim_k T_{\beta_{\varepsilon_k}}(\xi,\eta)$. With (1.ii), $\|T_{\beta_{\varepsilon_{k_0}}}\| \le K_6(\|f\|_1 + 8\pi^{-1/2}\|f\|_2/q_*)$, so

$$\|A_f(\xi,\eta)\| \le K_6\Big[\Big(1+\tfrac{40}{3e^2}\Big)\|f\|_1 + \tfrac{8+120/e^2}{\sqrt{\pi}}\,\tfrac{\|f\|_2}{q_*}\Big] \le K_6\big(2.81\,\|f\|_1 + 13.68\,\|f\|_2/q_*\big),$$

which is (a); linearity in f from (1.ii).

On $\{(\xi,\eta) : q_* \ge q\}$ the tail of (15.8) is uniform, and each $T_{\beta_{\varepsilon_k}}$ is norm-holomorphic there for $2\varepsilon_k < q$ (1.iii); a locally uniform limit of norm-holomorphic functions is norm-holomorphic: (b). (c) follows from (1.iii) in the limit. (d): $\tilde\beta_\varepsilon = \beta_\varepsilon$ (ρ_ε real and even), $\|Jf\|_p = \|f\|_p$, so (1.iv) passes to the limit, the adjoint being norm-continuous. (e): for 2ε below $q(\xi), q(\eta), q(a), q(b), q(a'), q(b')$,

the semigroup law gives $T^f_{\beta_\varepsilon}(\xi,a)T^g_{\beta_\varepsilon}(b,\eta) = V(\xi-2\varepsilon e_t)B_{2\varepsilon}(\beta_\varepsilon f)V(a+b-4\varepsilon e_t)B_{2\varepsilon}(\beta_\varepsilon g)V(\eta-2\varepsilon e_t)$, the same expression for (a',b'); products of norm-convergent sequences converge.

Step 3 ((f) and uniqueness). Let α be as in (f), $t_\alpha := \sup_{\operatorname{supp}\alpha}|t| < q_*$ ($t_\alpha > 0$ unless $\alpha = 0$, in which case both sides vanish), $\delta = t_\alpha$, $r = q_* - t_\alpha$, $0 < \varepsilon < r/4$. (i) By (15.5) with $(\gamma,\beta) = (\beta_\varepsilon,\alpha)$ (here $\operatorname{supp}\gamma \subset \{|z| \le 2\varepsilon\}$, $s = 2\varepsilon < r$) and $\int\beta_\varepsilon = 1$, $T_\alpha(\xi,\eta) = V(\xi - q_*e_t)\Lambda(0)V(\eta - q_*e_t)$ and (15.7) with $s = 2\varepsilon$: $\|T_{\beta_\varepsilon * \alpha}(\xi,\eta) - T_\alpha(\xi,\eta)\| \le 4\sqrt{2}\,\varepsilon\|B_\delta(\alpha f)\|/(e(r-2\varepsilon)) \to 0$. (ii) By (15.6) with $(\gamma,\beta) = (\alpha,\beta_\varepsilon)$, $\delta' = 2\varepsilon$, $r' = q_* - 2\varepsilon$ (so that $\sup_{\operatorname{supp}\alpha}|t| = t_\alpha < r'$ and $\delta' + r' = q_*$): $T_{\alpha*\beta_\varepsilon}(\xi,\eta) = \int\alpha(z)T_{\beta_\varepsilon}(\xi+z,\eta-z)dz$; here $\min(q(\xi+z), q(\eta-z)) \ge q_* - t_\alpha = r$ on $\operatorname{supp}\alpha$, so by Step 2 with $q = r$ the integrand converges to $A_f(\xi+z,\eta-z)$ uniformly on $\operatorname{supp}\alpha$ along $\varepsilon = \varepsilon_k$; the limit is norm-continuous in z by (b), and $\|\alpha\|_1 < \infty$. Since $\alpha * \beta_\varepsilon = \beta_\varepsilon * \alpha$, (i) and (ii) give $T_\alpha(\xi,\eta) = \int\alpha(z)A_f(\xi+z,\eta-z)dz$, i.e. (f); for $\xi = \eta = \delta e_t$, $q_* = \delta$ and $T_\alpha(\delta e_t,\delta e_t) = B_\delta(\alpha f)$ by (1.i).

Uniqueness: for (α_n) as in the statement and n so large that $s_n < q_*$, (f) and $\int\alpha_n = 1$ give $T_{\alpha_n}(\xi,\eta) - A_f(\xi,\eta) = \int\alpha_n(z)[A_f(\xi+z,\eta-z) - A_f(\xi,\eta)]dz$, of norm at most $\sup_n\|\alpha_n\|_1 \sup_{|z|\le s_n}\|A_f(\xi+z,\eta-z) - A_f(\xi,\eta)\| \to 0$ by (b); with (1.i) the limit depends neither on the approximate identity nor on δ_n. □

15.8. **The ordered holomorphic kernel and the one kernel.** Here Theorems 15.15 and 15.16 are used for the coordinate pair (t,x) and for the pair (x,t) (same transverse coordinate y); the second requires (iii) at $\{x = 0\}$ and $\{x + t = 0\}$, so this subsection uses the four planes of the pair. K_7 is the constant of Theorem 15.16, the same for both pairs, and $K_8 := 2K_7$.

Lemma 15.17 (end entries)**.** *For a profile f, $a,b \in \mathcal{W}$ and $\psi \in \mathcal{H}_\circ$: $\xi \mapsto \langle\Omega, A_f(\xi,a)\psi\rangle$ and $\eta \mapsto A_f(b,\eta)\Omega$ are constant on $\mathcal{W}$.*

Proof. Theorem 15.16(c) with $\kappa' = 0$ gives $A_f(\xi+\kappa,a) = V(\kappa)A_f(\xi,a)$ for $q(\kappa) \ge 0$, so $g(\xi+\kappa) := \langle\Omega, A_f(\xi+\kappa,a)\psi\rangle = \langle V(\kappa)^*\Omega, A_f(\xi,a)\psi\rangle = g(\xi)$. For $\xi,\xi' \in \mathcal{W}$, $g(\xi) = g(\xi+\xi') = g(\xi')$ ($\mathcal{W}$ is a convex cone). Likewise $A_f(b,\eta+\kappa')\Omega = A_f(b,\eta)V(\kappa')\Omega = A_f(b,\eta)\Omega$. □

Lemma 15.18 (unbounded vectors)**.** *If $\chi \in (L^1 \cap L^2)_{\mathbb{C}}$ vanishes outside $\{t \ge 0\}$ then $F(\chi) \in L^2_{\circ,+} \cap \bigcap_{p<\infty} L^p(\mu)$. If χ vanishes outside $\{|t| \le \delta\}$ and $A \in L^2_{\circ,+} \cap L^4(\mu)$, then $F(\tau_{\delta e_t}\chi)U(2\delta e_t)A \in L^2_{\circ,+}$ and $B_\delta(\chi)[A]_\circ = [F(\tau_{\delta e_t}\chi)U(2\delta e_t)A]_\circ$.*

Proof. This follows from Lemma 15.13 and Theorem 15.15(d) (with $U(2\delta e_t)L^2_{\circ,+} \subset L^2_{\circ,+}$). □

Proposition 15.19 (ordered holomorphic kernel)**.** *Let $N \ge 2$, $f_1,\dots,f_N \in (L^1 \cap L^2)(\mathbb{R};T_{\mathbb{C}})$, $\mathcal{Z}_N := \{Z = (z_1,\dots,z_N) \in (\mathbb{C}^2)^N : \Delta_j := z_{j+1} - z_j \in \mathcal{W},\ 1 \le j < N\}$. For $Z \in \mathcal{Z}_N$ and $\kappa \in \mathcal{W}$ put*

$$S_N(Z;f) := \big\langle\Omega,\, A_{f_1}(\kappa,\tfrac{\Delta_1}{2})\,A_{f_2}(\tfrac{\Delta_1}{2},\tfrac{\Delta_2}{2})\cdots A_{f_{N-1}}(\tfrac{\Delta_{N-2}}{2},\tfrac{\Delta_{N-1}}{2})\,A_{f_N}(\tfrac{\Delta_{N-1}}{2},\kappa)\,\Omega\big\rangle_\circ$$

(ordered product). Then: (a) $S_N(Z;f)$ does not depend on κ; (b) it is holomorphic on the open set $\mathcal{Z}_N$ and $S_N(Z+(c,\dots,c);f) = S_N(Z;f)$ for $c \in \mathbb{C}^2$; (c) it is N-linear in $(f_1,\dots,f_N)$; (d) $|S_N(Z;f)| \le K_8^N\prod_j(\|f_j\|_1 + q_*^{-1}\|f_j\|_2)$ with $q_* := \min_j q(\Delta_j)$; (e) if $\chi_j = \alpha_j(t,x)f_j(y)$ with $\alpha_j \in L^2(\mathbb{R}^2)$ of compact support K_j and $\max_{K_j} t < \min_{K_{j+1}} t$ $(1 \le j < N)$, then

$$S_N(\chi_1,\dots,\chi_N) = \int_{(\mathbb{R}^2)^N}\prod_j \alpha_j(z_j)\,S_N(Z;f)\,dZ \qquad \textit{(absolutely convergent).}$$

Proof. (a) follows from Lemma 15.17, applied first to the last entry and then to the first entry with $\psi = A_{f_2}(\cdot)\cdots A_{f_N}(\cdot,\kappa)\Omega$. (b) $Z \mapsto \Delta_j/2$ is linear, each factor is operator-norm holomorphic by Theorem 15.16(b), products in $\mathcal{B}(\mathcal{H}_\circ)$ and $\langle\Omega,\cdot\,\Omega\rangle$ preserve holomorphy; only differences enter. (c) follows from Theorem 15.16(a). (d) Take $\kappa = \Delta_1/2$; $q(\Delta_j/2) = q(\Delta_j)/2 \ge q_*/2$, so

Theorem 15.16(a) gives $\|A_{f_j}(\cdot,\cdot)\| \le K_7(\|f_j\|_1 + 2q_*^{-1}\|f_j\|_2) \le 2K_7(\|f_j\|_1 + q_*^{-1}\|f_j\|_2)$ for every factor, and $\|\Omega\| = 1$.

(e) *Slabs.* Choose $a_0 < a_1 < \cdots < a_N$ with $K_j \subset \{a_{j-1} < t < a_j\}$, put $\delta_j := (a_j - a_{j-1})/2$ and $s_j := (a_{j-1} + a_j)/2 - a_0 = 2\sum_{i<j}\delta_i + \delta_j$. Both sides are invariant under the common shift $\chi_j \mapsto \tau_{-a_0 e_t}\chi_j$ (left: (ii) and Lemma 15.6; right: substitution and (b)), so let $a_0 = 0$. Put $\alpha_j^0 := \tau_{-s_j e_t}\alpha_j$ and $\chi_j^0 := \alpha_j^0 f_j$: compact support in $\{|t| < \delta_j\}$, $\chi_j = \tau_{s_j e_t}\chi_j^0$, $\|\chi_j^0\|_1 = \|\alpha_j\|_1\|f_j\|_1$, $\|\chi_j^0\|_2 = \|\alpha_j\|_2\|f_j\|_2$.

Iteration. Put $G_N := F(\tau_{\delta_N e_t}\chi_N^0)$, $G_j := F(\tau_{\delta_j e_t}\chi_j^0)\,U(2\delta_j e_t)G_{j+1}$. Each G_j is a product of fields with tests vanishing outside $\{t \ge 0\}$, hence in $L^2_{\circ,+} \cap \bigcap_p L^p$ (Lemma 15.18, Hölder), so $B_{\delta_N}(\chi_N^0)\Omega = [G_N]_\circ$ and $B_{\delta_j}(\chi_j^0)[G_{j+1}]_\circ = [G_j]_\circ$. Multiplicativity of U and $U(a)F(\chi) = F(\tau_a\chi)$ give $G_1 = \prod_j F(\tau_{s_j e_t}\chi_j^0) = \prod_j F(\chi_j)$; thus $\langle\Omega, B_{\delta_1}(\chi_1^0)\cdots B_{\delta_N}(\chi_N^0)\Omega\rangle_\circ = \int G_1\,d\mu = S_N(\chi_1,\dots,\chi_N)$.

Insertion. Let $A_j(z) := A_{f_j}(\delta_j e_t + z, \delta_j e_t - z)$, norm-continuous for $|t| < \delta_j$ and bounded by $K_7(\|f_j\|_1 + \|f_j\|_2/(\delta_j - |t|))$, hence bounded on $\operatorname{supp}\alpha_j^0$. By Theorem 15.16(f), $B_{\delta_j}(\chi_j^0) = \int \alpha_j^0(z)A_j(z)\,dz$. For fixed bounded Y and $\psi \in \mathcal{H}_\circ$, $X \mapsto \langle Y^*\Omega, X\psi\rangle$ is a bounded functional on $\mathcal{B}(\mathcal{H}_\circ)$ and passes through Bochner integrals; applied N times from the left,

$$\langle\Omega, \textstyle\prod_j B_{\delta_j}(\chi_j^0)\Omega\rangle_\circ = \int \alpha_1^0(z_1)\cdots\int \alpha_N^0(z_N)\,\langle\Omega, A_1(z_1)\cdots A_N(z_N)\Omega\rangle_\circ\,dz_N\cdots dz_1.$$

The inner function is continuous in Z on $\prod_j\{|t_j| < \delta_j\}$ and bounded on the compact set $\prod_j \operatorname{supp}\alpha_j^0$, and $\alpha_j^0 \in L^1$; Fubini turns the iterated integral into one over $(\mathbb{R}^2)^N$.

Identification. Fix real Z with $|t_j| < \delta_j$, $w_j := z_j + s_j e_t$, $W := (w_j)_j$. Then $a_j := \delta_j e_t - z_j$ and $b_j := \delta_j e_t + z_j$ lie in $\mathcal{W}$, and $a_j + b_{j+1} = (\delta_j + \delta_{j+1})e_t + z_{j+1} - z_j = w_{j+1} - w_j =: \Delta_j \in \mathcal{W}$ (as $s_{j+1} - s_j = \delta_j + \delta_{j+1}$), so $W \in \mathcal{Z}_N$. In $A_1(z_1)\cdots A_N(z_N) = A_{f_1}(b_1,a_1)A_{f_2}(b_2,a_2)\cdots A_{f_N}(b_N,a_N)$ apply Theorem 15.16(e) to the adjacent pairs $(a_j, b_{j+1}) \mapsto (\Delta_j/2, \Delta_j/2)$, $j = 1,\dots,N-1$ in turn (each step touches only those two entries), then Lemma 15.17 to b_1 and a_N: $\langle\Omega, A_1(z_1)\cdots A_N(z_N)\Omega\rangle_\circ = S_N(W;f)$. Substituting $w_j = z_j + s_j e_t$, $\int\prod_j \alpha_j^0(z_j)S_N(Z + (s_j e_t)_j; f)\,dZ = \int\prod_j \alpha_j(w_j)S_N(W;f)\,dW$. □

Charts. For $\sigma \in \mathfrak{S}_N$ we put $Z_\sigma := (z_{\sigma(1)},\dots,z_{\sigma(N)})$, and f_σ likewise. We put $\operatorname{sw}(t,x) := (x,t)$, $D_\sigma^t := \{Z : Z_\sigma \in \mathcal{Z}_N\} = \{\operatorname{Re}(t_{\sigma(j+1)} - t_{\sigma(j)}) > |\operatorname{Im}(x_{\sigma(j+1)} - x_{\sigma(j)})|\ \forall j\}$, $D_\sigma^x := \{Z : (\operatorname{sw}Z)_\sigma \in \mathcal{Z}_N\}$ and $\mathcal{D}_N := \bigcup_\sigma(D_\sigma^t \cup D_\sigma^x)$. We write S'_N for the kernel of Proposition 15.19 for the coordinate pair (x,t), and we put $\mathcal{D}_{\mathbb{R}}^\sharp := \{Z \in (\mathbb{R}^2)^N :$ the t_j pairwise distinct and the x_j pairwise distinct$\}$. Note, with $\mathrm{rev} \in \mathfrak{S}_N$ the order reversal $\mathrm{rev}(j) := N+1-j$, that $D_{\sigma\circ\mathrm{rev}}^t = \{Z : -Z_\sigma \in \mathcal{Z}_N\}$ (the chart with time $-t$) and likewise for x, so the charts with the four time directions $\pm e_t, \pm e_x$ are all among the D_σ^t, D_σ^x.

Proposition 15.20 (one kernel)**.** *(a) $D_\sigma^t \cap D_{\sigma'}^t = \emptyset = D_\sigma^x \cap D_{\sigma'}^x$ for $\sigma \ne \sigma'$; all D_σ^t, D_σ^x, $D_\sigma^t \cap D_{\sigma'}^x$ are convex, open and nonempty; $\mathcal{D}_N$ is connected and $\mathcal{D}_N \cap (\mathbb{R}^2)^N = \{$all t_j distinct$\} \cup \{$all x_j distinct$\}$. (b) $\mathcal{S}_N(Z;f) := S_N(Z_\sigma; f_\sigma)$ on D_σ^t and $:= S'_N((\operatorname{sw}Z)_\sigma; f_\sigma)$ on D_σ^x is consistent and defines a holomorphic function on $\mathcal{D}_N$, N-linear in f, with $|\mathcal{S}_N(Z;f)| \le K_8^N\prod_j(\|f_j\|_1 + q_*^{-1}\|f_j\|_2)$ on each chart (q_* the minimum of q over the successive differences of that chart, computed in its coordinates), such that for product tests $\chi_j = \alpha_j f_j$, $\alpha_j \in L^2(\mathbb{R}^2)$ with compact planar supports pairwise strictly ordered in t or pairwise strictly ordered in x (in any order), $S_N(\chi_1,\dots,\chi_N) = \int\prod_j \alpha_j(z_j)\,\mathcal{S}_N(Z;f)\,dZ$. (c) $\mathcal{S}_N(Z_\pi; f_\pi) = \mathcal{S}_N(Z;f)$ for all $\pi \in \mathfrak{S}_N$, $Z \in \mathcal{D}_N$. (d) $R(\theta)\mathcal{D}_N = \mathcal{D}_N$ and $\mathcal{S}_N(R(\theta)Z; \varrho(\theta)f) = \mathcal{S}_N(Z;f)$ for $\theta \in \frac{\pi}{2}\mathbb{Z}$ (with $\varrho(\theta) = \pi(R(\theta))$ acting pointwise on the profiles); in particular $\mathcal{S}_N(R(-\pi/2)Z; \varrho(-\pi/2)f) = \mathcal{S}_N(Z;f)$.*

Proof. (a) On D^t_σ, $\operatorname{Re} t_{\sigma(1)} < \cdots < \operatorname{Re} t_{\sigma(N)}$, which determines σ; the same for x. Convexity: $\mathcal{W}$ is convex and the defining maps are linear. $O_{\sigma\sigma'} := \{Z \text{ real} : t \text{ ordered by } \sigma,\ x \text{ ordered by } \sigma'\}$ is nonempty, open in $\mathbb{R}^{2N}$ and lies in $D^t_\sigma \cap D^x_{\sigma'}$ (for real Δ, $q(\Delta)$ is the t-component). Every t-chart meets every x-chart, so $\mathcal{D}_N$ is connected; the real points are as stated.

(b) By (a) consistency concerns only $D^t_\sigma \cap D^x_{\sigma'}$. Product tests: if the supports are ordered in t along σ, Proposition 15.19(e) for $(\chi_{\sigma(1)}, \dots, \chi_{\sigma(N)})$, the symmetry of S_N and the substitution $W = Z_\sigma$ give $S_N(\chi_1, \dots, \chi_N) = \int \prod_j \alpha_j(z_j) S_N(Z_\sigma; f_\sigma) dZ$ (the support lies in D^t_σ); for supports ordered in x, the same functions read $(\alpha_j \circ \mathrm{sw})(x,t) f_j(y)$ in the chart (x,t), and Proposition 15.19(e) there, with the substitution $Z \mapsto \mathrm{sw}Z$, gives the formula with $S'_N((\mathrm{sw}Z)_{\sigma'}; f_{\sigma'})$. Let $Z_0 \in O_{\sigma\sigma'}$, $\phi \in C^\infty_c(\mathbb{R}^2)$, $\phi \ge 0$, $\int \phi = 1$, support in the unit disc, $\alpha_j := \epsilon^{-2}\phi((\cdot - z_{0,j})/\epsilon)$. For small ϵ both orderings hold, so $\int \prod_j \alpha_j(z_j) \, [S_N(Z_\sigma; f_\sigma) - S'_N((\mathrm{sw}Z)_{\sigma'}; f_{\sigma'})] \, dZ = 0$; both functions are continuous at Z_0, so $\epsilon \to 0$ gives equality at Z_0. Two holomorphic functions agreeing on the open subset $O_{\sigma\sigma'}$ of $\mathbb{R}^{2N}$ have equal complex Taylor coefficients at its points (real derivatives of the difference vanish), hence agree on the connected open set $D^t_\sigma \cap D^x_{\sigma'}$. Holomorphy, multilinearity and the bound are chartwise (Proposition 15.19(b)–(d)).

(c) $(Z_\pi)_\rho = Z_{\pi\rho}$, so $Z \in D^t_\sigma \iff Z_\pi \in D^t_{\pi^{-1}\sigma}$ and $\mathcal{S}_N(Z_\pi; f_\pi) = S_N((Z_\pi)_{\pi^{-1}\sigma}; (f_\pi)_{\pi^{-1}\sigma}) = S_N(Z_\sigma; f_\sigma)$; the same on x-charts.

(d) $R(\pi/2)(t,x) = (-x,t)$ gives $R(\pi/2)D^t_\sigma = D^x_\sigma$ and $R(\pi/2)D^x_\sigma = D^t_{\sigma\circ\mathrm{rev}}$, so $R(\theta)\mathcal{D}_N = \mathcal{D}_N$ and $R(\theta)\mathcal{D}^\sharp_{\mathbb{R}} = \mathcal{D}^\sharp_{\mathbb{R}}$. $G_\theta(Z) := \mathcal{S}_N(R(\theta)Z; \varrho(\theta)f)$ is holomorphic on $\mathcal{D}_N$. Let $Z_0 \in \mathcal{D}^\sharp_{\mathbb{R}}$, α_j as in (b), $\chi_j := \alpha_j f_j$. Substituting $Z' = R(\theta)Z$ and using (b) (the supports of $\alpha_j \circ R(-\theta)$ are the ϵ-discs about the points of $R(\theta)Z_0 \in \mathcal{D}^\sharp_{\mathbb{R}}$, strictly t-ordered for small ϵ):

$$\int \prod_j \alpha_j(z_j) G_\theta(Z)\, dZ = \int \prod_j (\alpha_j \circ R(-\theta))(z'_j)\, \mathcal{S}_N(Z'; \varrho(\theta)f)\, dZ' \\ = S_N(R(-\theta)^*\chi_1, \dots, R(-\theta)^*\chi_N),$$

since $(R(-\theta)^*\chi_j)(z,y) = \pi(R(\theta))\alpha_j(R(-\theta)z) f_j(y)$. The map $(z,y) \mapsto (R(-\theta)z, y)$, $\theta \in \frac{\pi}{2}\mathbb{Z}$, is a product of two of the four reflections $\{t=0\}$, $\{x=0\}$, $\{t \pm x = 0\}$, so Lemma 15.7 (four-plane form) with Lemma 15.6(e) gives $S_N(R(-\theta)^*\chi) = S_N(\chi)$ for Schwartz tests, and then for $L^1 \cap L^2$ tests by the continuity of S_N and the isometry of the pull-back; so the left side equals $\int \prod_j \alpha_j(z_j) \mathcal{S}_N(Z; f) dZ$. Letting $\epsilon \to 0$, $G_\theta = \mathcal{S}_N$ on $\mathcal{D}^\sharp_{\mathbb{R}}$, which meets every convex chart in a nonempty open subset of $\mathbb{R}^{2N}$; the identity theorem chart by chart gives $G_\theta = \mathcal{S}_N$ on $\mathcal{D}_N$. □

15.9. **The entire angular continuation.** *Conventions.* The map $\varrho(\theta) = \pi(R(\theta))$ is a continuous unitary one-parameter group on $T_{\mathbb{C}}$ with $\varrho(2\pi) = 1$, so $\varrho(\theta) = \sum_a e^{iw_a\theta} P_a$ with integers w_a and mutually orthogonal projections P_a, $\sum_a P_a = 1$. We put $w_{\max} := \max_a |w_a|$, which is 1 for the representations of Theorem 15.1; the proof is written for $w_{\max} \le 2$. For $\zeta \in \mathbb{C}$ we put $\varrho(\zeta) := \sum_a e^{iw_a\zeta} P_a$ and $R(\zeta) := \cos\zeta\, I + \sin\zeta\, J_\circ$ on $\mathbb{C}^2$, where $J_\circ := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so that $R(\zeta)R(\zeta') = R(\zeta+\zeta')$. We write $z \cdot w$ for the bilinear pairing on $\mathbb{C}^2$ and put $E := \{\pm e_t, \pm e_x\}$; for $e \in E$ we put $q_e(z) := \operatorname{Re}(z \cdot e) - |\operatorname{Im}(z \cdot J_\circ e)|$ and $\mathcal{W}_e := \{q_e > 0\}$ (so $\mathcal{W}_{e_t} = \mathcal{W}$, $\mathcal{W}_{-e} = -\mathcal{W}_e$, $\mathcal{W}_{e_x} = J_\circ\mathcal{W}$). We put $\mathcal{P} := (L^1 \cap L^2)(\mathbb{R}; T_{\mathbb{C}})$, and $\mathcal{A}$ is the class of bounded measurable planar factors with compact support. For $N \ge 2$, $e \in E$ and $\sigma \in \mathfrak{S}_N$ we put

$$\mathcal{D}^{e,\sigma}_N := \{Z \in (\mathbb{C}^2)^N : \ z_{\sigma(j+1)} - z_{\sigma(j)} \in \mathcal{W}_e,\ 1 \le j < N\}, \qquad q^{e,\sigma}_*(Z) := \min_{j<N} q_e(z_{\sigma(j+1)} - z_{\sigma(j)}).$$

These are the charts of Proposition 15.20: $\mathcal{D}^{e_t,\sigma}_N = D^t_\sigma$, $\mathcal{D}^{-e_t,\sigma}_N = D^t_{\sigma\circ\mathrm{rev}}$, $\mathcal{D}^{e_x,\sigma}_N = D^x_\sigma$, $\mathcal{D}^{-e_x,\sigma}_N = D^x_{\sigma\circ\mathrm{rev}}$. The facts used from Subsections 15.2–15.8 are:

(K1) the kernel $\mathcal{S}_N$ of Proposition 15.20 on $\mathcal{D}_N = \bigcup_{e,\sigma} \mathcal{D}_N^{e,\sigma}$, holomorphic in Z, N-linear in $f \in \mathcal{P}^N$, invariant under simultaneous permutations of (Z, f);

(K2) $|\mathcal{S}_N(Z; f)| \le K_8^N \prod_j (\|f_j\|_1 + q_*^{e,\sigma}(Z)^{-1}\|f_j\|_2)$ for $Z \in \mathcal{D}_N^{e,\sigma}$;

(K3) if $\chi_j = \alpha_j f_j$ with $\alpha_j \in \mathcal{A}$, $f_j \in \mathcal{P}$, and $\sup_{\operatorname{supp}\alpha_{\sigma(j)}} z\cdot e < \inf_{\operatorname{supp}\alpha_{\sigma(j+1)}} z\cdot e$ for $j < N$, then $S_N(\chi_1,\dots,\chi_N) = \int_{(\mathbb{R}^2)^N} \prod_j \alpha_j(z_j)\, \mathcal{S}_N(Z; f)\, dZ$ (Proposition 15.20(b), the orderings along $-e_t$, $-e_x$ being the reversed orderings along e_t, e_x);

(K4) $S_N(R(\pi/2)^*\chi) = S_N(\chi)$ and $|S_N(\chi)| \le (2K_1\sqrt{N})^N \prod_j N(\chi_j)$ for all $\chi \in ((L^1 \cap L^2)(\mathbb{R}^3; T_{\mathbb{C}}))^N$ (Lemma 15.7 for the four planes of the pair, since $R(\pi/2)$ is the product of two of the four reflections, with Lemma 15.6(c),(e)).

Narrow configurations with data. A configuration $X = (x_1,\dots,x_N) \in (\mathbb{R}^2)^N$ is narrow with data (u,γ) if, after relabelling, $d_j := x_{j+1} - x_j$ satisfies $\angle(d_j, u) \le \gamma < \pi/4$ for all j (u a unit vector); this is the notion of Subsection 15.1. Put $\delta := \min_j |d_j| > 0$, $c := \cos(\gamma + \pi/4) > 0$, $r_0 := \delta c/2$.

Proposition 15.21 (entire angular continuation)**.** *Let $N \ge 2$, X narrow with data (u,γ), $0 < r < r_0$, $m := \delta c - 2r > 0$, $K_j := \bar{B}(x_j, r)$, $K := \prod_j K_j$. Let $\chi_j = \alpha_j f_j$ with $\alpha_j \in \mathcal{A}$, $\operatorname{supp}\alpha_j \subset K_j$, $f_j \in \mathcal{P}$, and $D_N[\chi](\theta) := S_N(R(\theta)^*\chi) - S_N(\chi)$ for $\theta \in \mathbb{R}$.*

(a) *$D_N[\chi]$ is the restriction to $\mathbb{R}$ of a unique entire function, again written $D_N[\chi]$, which is $\pi/2$-periodic (hence 2π-periodic) and satisfies*

$$|D_N[\chi](\theta + i\beta)| \le C_\chi\, e^{(1+w_{\max})N|\beta|}, \qquad C_\chi := 2K_8^N \prod_j \|\alpha_j\|_1 \prod_j \big(\|f_j\|_1 + m^{-1}\|f_j\|_2\big),$$

with C_χ independent of θ and β.

(b) *$c_\ell[\chi] := \frac{1}{2\pi}\int_0^{2\pi} D_N[\chi](\theta) e^{-i\ell\theta} d\theta$ vanishes unless $\ell \in 4\mathbb{Z}$ and $|\ell| \le (1 + w_{\max})N$, and*

$$D_N[\chi](\zeta) = \sum_{\ell \in 4\mathbb{Z},\ 0 < |\ell| \le (1+w_{\max})N} c_\ell[\chi]\,(e^{i\ell\zeta} - 1) \qquad (\zeta \in \mathbb{C}).$$

(c) *(b) holds, with the same index set and N-linear c_ℓ, for all $\chi_j \in (L^1 \cap L^2)(\mathbb{R}^3; T_{\mathbb{C}})$ vanishing outside $K_j \times \mathbb{R}$.*

For $w_{\max} \le 2$ the exponent is $3N$; for the weights $0, \pm 1$ of Theorem 15.1 it is $2N$.

Proof. Claim A. For $d \in \mathbb{R}^2$, a unit vector $e \in \mathbb{R}^2$ and $\zeta = \theta + i\beta$: $\operatorname{Re}(R(\zeta)d\cdot e) = \cosh\beta\,(R(\theta)d\cdot e)$ and $\operatorname{Im}(R(\zeta)d\cdot J_\circ e) = \sinh\beta\,(R(\theta)d\cdot e)$; hence $q_e(R(\zeta)d) = e^{-|\beta|}\, R(\theta)d\cdot e$ whenever $R(\theta)d\cdot e \ge 0$. Indeed $\cos(i\beta) = \cosh\beta$, $\sin(i\beta) = i\sinh\beta$, so $R(i\beta) = \cosh\beta\, I + i\sinh\beta\, J_\circ$ and $R(\zeta) = R(i\beta)R(\theta)$; with $d' := R(\theta)d \in \mathbb{R}^2$, $R(\zeta)d = \cosh\beta\, d' + i\sinh\beta\, J_\circ d'$, and $J_\circ d'\cdot J_\circ e = d'\cdot e$. So $q_e(R(\zeta)d) = \cosh\beta\, d'\cdot e - |\sinh\beta|\,|d'\cdot e| = (\cosh\beta - |\sinh\beta|)\, d'\cdot e = e^{-|\beta|} d'\cdot e$.

Claim B. For every $\phi \in \mathbb{R}$ there is $e(\phi) \in E$ with $\angle(R(\phi)u, e(\phi)) \le \pi/4$, and for all $Z \in K$, $j < N$: $R(\phi)(z_{j+1} - z_j)\cdot e(\phi) \ge m$. Consequently, for $\zeta = \phi + i\beta$, $R(\zeta)Z \in \mathcal{D}_N^{e(\phi),\mathrm{id}} \subset \mathcal{D}_N$ with $q_*^{e(\phi),\mathrm{id}}(R(\zeta)Z) \ge e^{-|\beta|}m$, and the sets $R(\phi)K_1,\dots,R(\phi)K_N$ are strictly increasing in the $e(\phi)$-coordinate with gaps $\ge m$. Indeed, the four directions of E are $\pi/2$ apart, so one of them is within $\pi/4$ of $R(\phi)u$; by the triangle inequality for angles, $\angle(R(\phi)d_j, e(\phi)) \le \angle(d_j, u) + \pi/4 \le \gamma + \pi/4 < \pi/2$, so $R(\phi)d_j\cdot e(\phi) \ge |d_j|\, c \ge \delta c$. For $Z \in K$ write $z_j = x_j + \epsilon_j$, $|\epsilon_j| \le r$: $R(\phi)(z_{j+1} - z_j)\cdot e(\phi) \ge \delta c - 2r = m$. Claim A with $d = z_{j+1} - z_j$, $\theta = \phi$, $e = e(\phi)$ gives $q_{e(\phi)}(R(\zeta)(z_{j+1} - z_j)) = e^{-|\beta|} R(\phi)(z_{j+1} - z_j)\cdot e(\phi) \ge e^{-|\beta|}m > 0$. The ordering statement is the case $\beta = 0$ with $z_j \in K_j$, $z_{j+1} \in K_{j+1}$ arbitrary.

Step 1 (the entire function along the orbit). For $Z \in K$ let $g_Z(\zeta) := \mathcal{S}_N(R(\zeta)Z; \varrho(\zeta)f)$, where $\varrho(\zeta)f := (\varrho(\zeta)f_1,\dots,\varrho(\zeta)f_N)$, $\varrho(\zeta)$ acting pointwise in y. By Claim B, $R(\zeta)Z \in \mathcal{D}_N$ for every

$\zeta \in \mathbb{C}$. By N-linearity,

$$g_Z(\zeta) = \sum_{a_1,\dots,a_N} e^{i(w_{a_1}+\cdots+w_{a_N})\zeta}\, \mathcal{S}_N\big(R(\zeta)Z; P_{a_1}f_1, \dots, P_{a_N}f_N\big),$$

a finite sum; each term is an exponential times the composition of the entire map $\zeta \mapsto R(\zeta)Z \in \mathcal{D}_N$ with a holomorphic function (K1). Hence g_Z is entire, $(\zeta, Z) \mapsto g_Z(\zeta)$ is continuous on $\mathbb{C} \times K$, and $g_Z(\zeta + 2\pi) = g_Z(\zeta)$ (the weights are integers, $R(\zeta + 2\pi) = R(\zeta)$). Bound: $\|\varrho(\zeta)v\|^2 = \sum_a e^{-2w_a\beta}\|P_a v\|^2 \le e^{2w_{\max}|\beta|}\|v\|^2$, so $\|\varrho(\zeta)f_j\|_p \le e^{w_{\max}|\beta|}\|f_j\|_p$; (K2) on $\mathcal{D}_N^{e(\phi),\mathrm{id}}$ with $q_* \ge e^{-|\beta|}m$ gives

$$|g_Z(\zeta)| \le K_8^N \prod_j \big(e^{w_{\max}|\beta|}\|f_j\|_1 + e^{(1+w_{\max})|\beta|}m^{-1}\|f_j\|_2\big) \le G\, e^{(1+w_{\max})N|\beta|},$$

$$G := K_8^N \prod_j \big(\|f_j\|_1 + m^{-1}\|f_j\|_2\big).$$

Step 2 (smearing at every real angle). For real θ, $(R(\theta)^*\chi_j)(z,y) = \varrho(\theta)^{-1}\alpha_j(R(\theta)z)f_j(y)$, i.e. $R(\theta)^*\chi_j = (\alpha_j \circ R(\theta))\,\varrho(-\theta)f_j$, a product test with $\alpha_j \circ R(\theta) \in \mathcal{A}$ supported in $R(-\theta)K_j$. By Claim B with $\phi = -\theta$ these supports are strictly increasing in the $e(-\theta)$-coordinate, so (K3) with $e = e(-\theta)$, $\sigma = \mathrm{id}$ gives $S_N(R(\theta)^*\chi) = \int \prod_j \alpha_j(R(\theta)z_j)\,\mathcal{S}_N(Z; \varrho(-\theta)f)\,dZ$. The substitution $Z = R(-\theta)Z'$ (orthogonal on $(\mathbb{R}^2)^N$) yields

$$S_N(R(\theta)^*\chi) = \int_K \prod_j \alpha_j(z_j')\, g_{Z'}(-\theta)\,dZ', \qquad S_N(\chi) = \int_K \prod_j \alpha_j(z_j)\, g_Z(0)\,dZ \quad (\theta = 0).$$

Hence $D_N[\chi](\theta) = \mathcal{E}(\theta)$ for all real θ, where $\mathcal{E}(\zeta) := \int_K \prod_j \alpha_j(z_j)\,[g_Z(-\zeta) - g_Z(0)]\,dZ$.

Step 3 ($\mathcal{E}$ is entire, bounded, periodic). The integrand is jointly continuous and bounded by $|\prod_j \alpha_j(z_j)|\, 2Ge^{(1+w_{\max})N|\beta|}$, so $\mathcal{E}$ is continuous (dominated convergence) and $\oint_{\partial\Delta} \mathcal{E}\,d\zeta = 0$ for every triangle Δ (Fubini and Cauchy); by Morera's theorem $\mathcal{E}$ is entire, and $|\mathcal{E}(\theta + i\beta)| \le \prod_j \|\alpha_j\|_1\,(Ge^{(1+w_{\max})N|\beta|} + G) \le C_\chi e^{(1+w_{\max})N|\beta|}$. $\mathcal{E}(\zeta + 2\pi) = \mathcal{E}(\zeta)$ by Step 1. Since $R(\theta + \pi/2) = R(\theta)R(\pi/2)$ and $(R_1R_2)^* = R_2^*R_1^*$, (K4) gives $S_N(R(\theta + \pi/2)^*\chi) = S_N(R(\pi/2)^*R(\theta)^*\chi) = S_N(R(\theta)^*\chi)$, so $\mathcal{E}(\theta + \pi/2) = \mathcal{E}(\theta)$ on $\mathbb{R}$, hence on $\mathbb{C}$ by the identity theorem, which also gives the uniqueness of the entire extension. This is (a).

Step 4 (Fourier coefficients). As $\mathcal{E}$ is entire and 2π-periodic, $\mathcal{F}(w) := \mathcal{E}(\zeta)$ for $w = e^{i\zeta}$ is well defined and holomorphic on $\mathbb{C} \setminus \{0\}$ (local inverses of $\zeta \mapsto e^{i\zeta}$ differ by multiples of 2π), with Laurent expansion $\mathcal{F}(w) = \sum_{\ell\in\mathbb{Z}} c_\ell w^\ell$, where for every $\beta \in \mathbb{R}$

$$c_\ell = \frac{1}{2\pi i}\oint_{|w|=e^{-\beta}} \mathcal{F}(w)w^{-\ell-1}dw = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{E}(\theta + i\beta)e^{-i\ell(\theta+i\beta)}d\theta;$$

at $\beta = 0$ this is $c_\ell[\chi]$, and $\mathcal{E}(\zeta) = \sum_\ell c_\ell e^{i\ell\zeta}$ absolutely. Thus $|c_\ell| \le C_\chi e^{(1+w_{\max})N|\beta| + \ell\beta}$ for all β: $\beta \to -\infty$ gives $c_\ell = 0$ for $\ell > (1 + w_{\max})N$, $\beta \to +\infty$ gives $c_\ell = 0$ for $\ell < -(1 + w_{\max})N$. The $\pi/2$-periodicity gives $c_\ell(e^{i\ell\pi/2} - 1) = 0$, so $c_\ell = 0$ for $\ell \notin 4\mathbb{Z}$. $\mathcal{E}(0) = 0$ gives $c_0 = -\sum_{\ell\neq 0} c_\ell$, whence (b).

Step 5 (all tests with the same planar supports). Let $\mathcal{X} := \prod_j (L^1 \cap L^2)(K_j \times \mathbb{R}; T_{\mathbb{C}})$ with the norm N in each factor. $R(\theta)^*$ preserves $\|\cdot\|_1$ and $\|\cdot\|_2$ (orthogonal change of variables, unitary $\varrho(-\theta)$), so (K4) and telescoping give, uniformly in $\theta \in \mathbb{R}$,

$$|D_N[\chi](\theta) - D_N[\chi'](\theta)| \le 2(2K_1\sqrt{N})^N \sum_j N(\chi_j - \chi_j') \prod_{i\neq j} \max(N(\chi_i), N(\chi_i')).$$

Finite sums of product tests αf, $\alpha \in \mathcal{A}$ supported in K_j, are dense in $(L^1 \cap L^2)(K_j \times \mathbb{R}; T_{\mathbb{C}})$: truncate to $|y| \le M$ (convergence in N), and on the bounded set $K_j \times [-M, M]$ the L^2 norm dominates the L^1 norm while simple tensors with bounded factors are L^2-dense. By (b) and N-linearity, $D_N[\chi^{(n)}]$ is, for approximants $\chi^{(n)}$ in these spans, a trigonometric polynomial with frequencies in $\Lambda := \{\ell \in 4\mathbb{Z} : 0 < |\ell| \le (1 + w_{\max})N\} \cup \{0\}$ vanishing at 0; $D_N[\chi]$ is its uniform limit, hence of the same form (the space of such polynomials is finite-dimensional), and $c_\ell[\chi] = \lim_n c_\ell[\chi^{(n)}]$, which is N-linear in χ. This is (c). □

For $N = 1$, $D_1 \equiv 0$ by Lemma 15.7. The threshold is explicit: $r < \frac{1}{2} \min_j |d_j| \cos(\gamma + \pi/4)$, and C_χ depends on the neighbourhood only through $m = \delta c - 2r$. The extension of (a)–(c) from small product neighbourhoods to arbitrary compact supports with successive differences in one open $\pi/4$-cone ("narrow tuples") is Lemma 15.28 below.

15.10. **Continuation of positivity.**

Lemma 15.22 (derivative positivity)**.** *Let $I \subset \mathbb{R}$ be open, $k \in C^\infty(I \times I)$ with $\sum_{j,l} \bar\lambda_j \lambda_l k(x_j, x_l) \ge 0$ for all finite families $x_j \in I$, $\lambda_j \in \mathbb{C}$. Then $\sum_{a,b \le M} \bar\lambda_a \lambda_b\, \partial_1^a \partial_2^b k(x_0, x_0) \ge 0$ for $x_0 \in I$, $M \in \mathbb{N}$, $\lambda \in \mathbb{C}^{M+1}$.*

Proof. For $\phi \in C_c(I)$, $Q(\phi) := \iint \overline{\phi(x)} \phi(x') k(x, x')\, dx\, dx' \ge 0$: the Riemann sums $h^2 \sum_{j,l} \overline{\phi(x_j)} \phi(x_l) k(x_j, x_l)$ over grids of mesh h are of the assumed form and converge to $Q(\phi)$ (continuous integrand on a compact square). Take $\eta \in C_c^\infty(\mathbb{R}; \mathbb{R})$, $\int \eta = 1$, $\eta_\varepsilon := \varepsilon^{-1} \eta((\cdot - x_0)/\varepsilon)$, $\phi_\varepsilon := \sum_{a \le M} \lambda_a (-1)^a \eta_\varepsilon^{(a)}$. Integrating by parts, $Q(\phi_\varepsilon) = \sum_{a,b} \bar\lambda_a \lambda_b \iint \eta_\varepsilon(x) \eta_\varepsilon(x')\, \partial_1^a \partial_2^b k(x, x')\, dx\, dx' \to \sum_{a,b} \bar\lambda_a \lambda_b\, \partial_1^a \partial_2^b k(x_0, x_0)$ as $\varepsilon \to 0$, and $Q(\phi_\varepsilon) \ge 0$. □

Lemma 15.23 (disc propagation of kernel positivity)**.** *Let $O \subset \mathbb{C}$ be open with $O = \{\bar z : z \in O\}$, Φ holomorphic on $O \times O$, $z_1 \in O$, $B(z_1, r) \subset O$, and suppose that $K(z, \bar w) := \Phi(z, \bar w)$ is positive semidefinite on a segment $z_1 + (-\varepsilon, \varepsilon)$, meaning $\sum_{j,l} \bar\lambda_j \lambda_l \Phi(z_j, \bar z_l) \ge 0$ for all finite families z_j in the segment. Then K is positive semidefinite on $B(z_1, r)$.*

Proof. $k(x, x') := \Phi(z_1 + x, \bar z_1 + x') = K(z_1 + x, \overline{z_1 + x'})$ is a smooth positive semidefinite kernel on $(-\varepsilon, \varepsilon)$. Real-direction derivatives of a holomorphic function are its complex derivatives, so Lemma 15.22 gives that $c_{ab} := \frac{1}{a!b!} \partial_\zeta^a \partial_{\bar\xi}^b \Phi(z_1, \bar z_1)$ satisfies $\sum_{a,b \le M} \bar\mu_a \mu_b c_{ab} \ge 0$ for all M, μ. Since $\{\bar w : w \in B(z_1, r)\} = B(\bar z_1, r) \subset O$, Φ is holomorphic on the polydisc $B(z_1, r) \times B(\bar z_1, r)$, hence $\Phi(\zeta, \xi) = \sum_{a,b} c_{ab} (\zeta - z_1)^a (\xi - \bar z_1)^b$, absolutely convergent there. For $z^{(1)}, \dots, z^{(m)} \in B(z_1, r)$, $\lambda \in \mathbb{C}^m$, with $v_b := \sum_l \lambda_l \overline{(z^{(l)} - z_1)}^b$,

$$\sum_{j,l} \bar\lambda_j \lambda_l\, \Phi\big(z^{(j)}, \overline{z^{(l)}}\big) = \sum_{a,b} c_{ab}\, \overline{v_a}\, v_b = \lim_{M \to \infty} \sum_{a,b \le M} c_{ab} \overline{v_a} v_b \ \ge 0,$$

the double series being absolutely convergent and each square partial sum nonnegative. □

Proposition 15.24 (continuation of positivity)**.** *Let $\mathcal{X}$ be a complex vector space, $O \subset \mathbb{C}$ open and connected with $O = \{\bar z : z \in O\}$ and $0 \in O$, and $(\zeta, \xi) \mapsto \langle A, B \rangle_{\zeta, \xi}$ holomorphic on $O \times O$ for all $A, B \in \mathcal{X}$, sesquilinear in (A, B) (antilinear in A). Assume that every $A \in \mathcal{X}$ has a map $[0, \delta_A) \ni s \mapsto A(s) \in L^2_{\circ,+}$ with $\langle A, B \rangle_{r,s} = \langle A(r), B(s) \rangle_\circ$ for $0 < r < \delta_A$, $0 < s < \delta_B$. Then $\langle A, A \rangle_{z, \bar z} \ge 0$ for all $z \in O$, $A \in \mathcal{X}$.*

Proof. Fix A; $\Phi(\zeta, \xi) := \langle A, A \rangle_{\zeta, \xi}$ is holomorphic on $O \times O$. As $0 \in O$ is interior, $O \supset (0, \delta')$ with $0 < \delta' \le \delta_A$, and for $r, s \in (0, \delta')$, $\Phi(r, \bar s) = \Phi(r, s) = \langle A(r), A(s) \rangle_\circ$; thus $\sum_{j,l} \bar\lambda_j \lambda_l \Phi(r_j, \bar r_l) = \big\langle \sum_j \lambda_j A(r_j), \sum_l \lambda_l A(r_l) \big\rangle_\circ \ge 0$ by the positivity of the form $\langle \cdot, \cdot \rangle_\circ$ on $L^2_{\circ,+}$ (Lemma 15.5), i.e. K is

positive semidefinite on $(0,\delta')$. Let $G := \{z \in O :\ K$ is positive semidefinite on some open $N_z \ni z,\ N_z \subset O\}$. G is open; $G \neq \emptyset$ by Lemma 15.23 at $r_1 \in (0,\delta')$ with $r = \operatorname{dist}(r_1, \mathbb{C}\setminus O)$; G is closed in O: if $z^* \in O$, $B(z^*,2\rho) \subset O$ and $z_1 \in G \cap B(z^*,\rho)$, Lemma 15.23 (a segment inside N_{z_1}, the disc $B(z_1,\rho) \subset O$) makes K positive semidefinite on $B(z_1,\rho) \ni z^*$. Connectedness gives $G = O$, and the one-point case at $z \in N_z$ gives $\Phi(z,\bar z) = \langle A, A\rangle_{z,\bar z} \geq 0$. □

15.11. **Positivity of the continued reflection pairing.** *Conventions.* We use $R(\zeta) = \begin{pmatrix} \cos\zeta & -\sin\zeta \\ \sin\zeta & \cos\zeta \end{pmatrix}$ on $\mathbb{C}^2$ and $\varrho(\zeta) = \sum_w e^{iw\zeta} P_w$ as in Subsection 15.9, together with $(R(\theta)^*\chi)(t,x,y) = \varrho(-\theta)\chi(R(\theta)(t,x),y)$ and $J\chi = \pi(\theta_\circ)(\bar\chi \circ \theta_\circ)$, so that $\overline{\Theta_\circ F(\chi)} = F(J\chi)$. In this subsection and the next two a product test is $\chi = \alpha f$ with $\alpha \in C_c^\infty(\mathbb{R}^2;\mathbb{R})$ and $f \in C_c^\infty(\mathbb{R};T)$ real, with planar support $K = \operatorname{supp}\alpha$; then $R(\theta)^*(\alpha f) = (\alpha\circ R(\theta))\,\varrho(-\theta)f$ has planar support $R(-\theta)K$ and $J(\alpha f) = (\alpha\circ\theta_\circ)\,\pi(\theta_\circ)f$ has planar support $\theta_\circ K$. We put $\Sigma_{\pi/4} := \{\zeta \in \mathbb{C} : |\operatorname{Re}\zeta| < \pi/4\}$. For Φ holomorphic on $M \times \bar M$, "$K(z,\bar w) := \Phi(z,\bar w)$ is positive semidefinite on M" means $\sum_{j,l} \bar\lambda_j \lambda_l \Phi(z_j,\bar z_l) \geq 0$ for all finite families $z_j \in M$, $\lambda_j \in \mathbb{C}$. For a tuple $\chi = (\chi_1,\dots,\chi_a)$ we put $J\chi := (J\chi_a,\dots,J\chi_1)$ (reflected, conjugated, reversed), and $D_N[\chi](\zeta)$ is the entire function of Proposition 15.21 where it is defined.

Proposition 15.25 (the continued reflection pairing is positive)**.** *Let $\mathfrak{A}$ be the set of admissible tuples $\chi = (\chi_1,\dots,\chi_a)$, $a \geq 0$, of real product tests (planar supports $K_j \subset \mathcal{C}_+$, $K_{j+1} - K_j \subset \mathcal{C}_+$), and $\mathcal{X}$ the free complex vector space on $\mathfrak{A}$ (the empty tuple is the basis vector Ω). For $\chi,\eta \in \mathfrak{A}$ of lengths a,b and $N := a+b$:*

(a) *there is $\Psi_{\chi,\eta}$ holomorphic on $\Sigma_{\pi/4}\times\Sigma_{\pi/4}$ with $\Psi_{\chi,\eta}(\theta_1,\theta_2) = S_N(R(\theta_1)^*J\chi, R(\theta_2)^*\eta)$ for real $|\theta_i| < \pi/4$, and $\Psi_{\chi,\eta}(\zeta,\zeta) = S_N(J\chi,\eta) + D_N[J\chi,\eta](\zeta)$ on $\Sigma_{\pi/4}$;*

(b) *the sesquilinear extension of $\langle\chi,\eta\rangle_{\zeta,\xi} := \Psi_{\chi,\eta}(-\zeta,\xi)$ is jointly holomorphic on $\Sigma_{\pi/4}^2$ and equals $\langle A(r),B(s)\rangle_\circ$ at $(r,s) \in (0,\pi/4)^2$, where $A(s) := \sum_i c_i \prod_j F(R(s)^*\chi_j^{(i)})$ for $A = \sum_i c_i \chi^{(i)}$;*

(c) *for every $\beta \in \mathbb{R}$ the sesquilinear form $\langle\chi,\eta\rangle_\beta := S_N(J\chi,\eta) + D_N[J\chi,\eta](i\beta)$ is positive semidefinite on $\mathcal{X}$; more generally $\langle A,A\rangle_{z,\bar z} \geq 0$ for all $z \in \Sigma_{\pi/4}$.*

The proof uses, besides (i), (ii) and Lemma 15.7 for S_1, only the plane $\{t=0\}$ (through Lemma 15.5) and Propositions 15.19 and 15.21. Write $\chi^{\mathrm{tot}} := (J\chi_a,\dots,J\chi_1,\eta_1,\dots,\eta_b) = (\alpha_m f_m)_{m\leq N}$ with planar supports K_m^{tot}, block index $\iota(m) = 1$ for $m \leq a$ and 2 for $m > a$, and for $Z' \in \prod_m K_m^{\mathrm{tot}}$ put $Z(\zeta_1,\zeta_2;Z') := (R(-\zeta_{\iota(m)})z'_m)_m$.

Lemma 15.26 (geometry of complex rotations)**.** *(i) $K_{m+1}^{\mathrm{tot}} - K_m^{\mathrm{tot}} \subset \mathcal{C}_+$ for all m (the tuple $(J\chi,\eta)$ is narrow with unit vector e_t). (ii) For real $v \in \mathcal{C}_+$ and $\zeta = \theta + i\beta$ with $|\theta| < \pi/4$: $(R(-\theta)v)_t > 0$ and $q(R(-\zeta)v) = e^{-|\beta|}(R(-\theta)v)_t > 0$. (iii) For $(\zeta_1,\zeta_2) \in \Sigma_{\pi/4}^2$ and $Z' \in \prod K_m^{\mathrm{tot}}$ all successive differences of $Z(\zeta_1,\zeta_2;Z')$ lie in $\mathcal{W}$; for real $|\theta_i| < \pi/4$ the sets $R(-\theta_{\iota(m)})K_m^{\mathrm{tot}}$ are compact, strictly ordered in t, in $\{t<0\}$ for $m \leq a$ and in $\{t>0\}$ for $m > a$.*

Proof. (i) $\theta_\circ\mathcal{C}_+ = -\mathcal{C}_+$; for $m < a$, $K_{m+1}^{\mathrm{tot}} - K_m^{\mathrm{tot}} = -\theta_\circ(K_{a-m+1} - K_{a-m}) \subset \mathcal{C}_+$; for $m = a$, $K_1' - \theta_\circ K_1 \subset \mathcal{C}_+ + \mathcal{C}_+ = \mathcal{C}_+$ (K_1' the first support of η); for $m > a$ by admissibility. (ii) $v = |v|(\cos\phi,\sin\phi)$ with $|\phi| < \pi/4$, so $|\phi - \theta| < \pi/2$. With $w := R(-\theta)v$ real, $R(-i\beta)w = (\cosh\beta\, w_t + i\sinh\beta\, w_x,\ -i\sinh\beta\, w_t + \cosh\beta\, w_x)$, hence $q = w_t(\cosh\beta - |\sinh\beta|) = e^{-|\beta|}w_t$. (iii) Within a block, $\Delta_m = R(-\zeta_\iota)(z'_{m+1} - z'_m)$ with $z'_{m+1} - z'_m \in \mathcal{C}_+$, so (ii) applies. At the junction, $\Delta_a = R(-\zeta_2)z'_{a+1} + R(-\zeta_1)(-z'_a)$ with $z'_{a+1} \in K_1' \subset \mathcal{C}_+$ and $-z'_a \in -\theta_\circ K_1 \subset \mathcal{C}_+$, so the claim follows from (ii) and the superadditivity of q. Real angles: the t-components of all rotated differences are positive and continuous on compact sets, so their minimum is positive; $R(-\theta)\mathcal{C}_+ \subset \{t>0\}$ by (ii). □

Lemma 15.27 (the two-angle pairing)**.** *For $N \geq 2$ put $\Psi_{\chi,\eta}(\zeta_1,\zeta_2) := \int_{\prod K_m^{\mathrm{tot}}} \prod_m \alpha_m(z'_m)\, S_N\big(Z(\zeta_1,\zeta_2;Z');(\varrho(-\zeta_{\iota(m)})f_m)_m\big)\, dZ'$ with the kernel of Proposition 15.19; for $N=1$, $\Psi := S_1(\chi_1^{\mathrm{tot}})$; for $N=0$, $\Psi := 1$. Then (a) of Proposition 15.25 holds, except the last identity.*

Proof. By multilinearity, $S_N(Z;(\varrho(-\zeta_{\iota(m)})f_m)_m) = \sum_{w_1,\dots,w_N} e^{-i\sum_m w_m\zeta_{\iota(m)}} S_N(Z;(P_{w_m}f_m)_m)$, a finite sum; $Z(\cdot;Z')$ is entire with values in $\mathcal{Z}_N$ on $\Sigma^2_{\pi/4}$ (Lemma 15.26(iii)), where $S_N(\cdot;f)$ is holomorphic. So the integrand is holomorphic in $(\zeta_1,\zeta_2)\in\Sigma^2_{\pi/4}$ for each Z' and continuous on $\Sigma^2_{\pi/4}\times\prod K_m^{\mathrm{tot}}$. Hence Ψ is continuous, $\oint_\gamma \Psi\, d\zeta_i = \int \oint_\gamma(\dots)d\zeta_i\, dZ' = 0$ for closed triangles $\gamma\subset\Sigma_{\pi/4}$ (Fubini, Cauchy), so Ψ is separately holomorphic (Morera) and jointly holomorphic (Osgood). For real $|\theta_i|<\pi/4$, $\tilde\chi_m := R(\theta_{\iota(m)})^*\chi_m^{\mathrm{tot}} = (\alpha_m\circ R(\theta_{\iota(m)}))\,\varrho(-\theta_{\iota(m)})f_m$ are product tests with compact planar supports strictly ordered in t (Lemma 15.26(iii)); Proposition 15.19(e) gives $S_N(\tilde\chi) = \int\prod_m \alpha_m(R(\theta_{\iota(m)})z_m)\, S_N(Z;(\varrho(-\theta_{\iota(m)})f_m)_m)\, dZ$, and the substitution $z_m = R(-\theta_{\iota(m)})z'_m$ (Jacobian 1) yields $\Psi(\theta_1,\theta_2)$; $S_N(\tilde\chi) = S_N(R(\theta_1)^*J\chi, R(\theta_2)^*\eta)$ by the symmetry of S_N. For $N=1$, $S_1(R(\theta)^*\psi) = S_1(\psi)$ by Lemma 15.7. □

Lemma 15.28 (narrow tuples)**.** *Let $\psi=(\psi_1,\dots,\psi_N)$, $N\geq 2$, be a tuple of tests $\psi_m\in(L^1\cap L^2)(\mathbb{R}^3;T_{\mathbb{C}})$ vanishing outside $L_m\times\mathbb{R}$, with compact planar sets L_m such that, for some unit vector u and some ordering, all differences $L_{m+1}-L_m$ lie in the open cone of half-angle $\pi/4$ about u (a narrow tuple). Then $D_N[\psi]$ extends to an entire $\pi/2$-periodic function with $|D_N[\psi](\theta+i\beta)|\leq C_\psi e^{(1+w_{\max})N|\beta|}$ and the finite Fourier form of Proposition 15.21(b), with N-linear coefficients $c_\ell[\psi]$. In particular this holds for $\chi^{\mathrm{tot}}=(J\chi,\eta)$, and $\Psi_{\chi,\eta}(\zeta,\zeta) = S_N(\chi^{\mathrm{tot}}) + D_N[\chi^{\mathrm{tot}}](\zeta)$ for $\zeta\in\Sigma_{\pi/4}$.*

Proof. On the compact sets $L_{m+1}-L_m$ inside the open cone, $\angle(d,u)\leq\gamma_0<\pi/4$ and $|d|\geq\delta_0>0$ uniformly, so every configuration $X\in\prod_m L_m$ is narrow with data (u,γ_0) and $r_0(X)\geq r_* := \frac12\delta_0\cos(\gamma_0+\pi/4)$. Take a smooth partition of unity $\{\phi_\nu\}$ of $\mathbb{R}^2$ with $\operatorname{diam}\operatorname{supp}\phi_\nu<r_*$ and expand $D_N[\psi]=\sum_{\nu_1,\dots,\nu_N} D_N[(\phi_{\nu_m}\psi_m)_m]$ (a finite sum, by N-linearity, the L_m being compact); each term has planar supports in balls of radius $<r_*$ about a configuration of $\prod L_m$, so Proposition 15.21(a)–(c) applies to it (with (c) for non-product tests); sums of finitely many entire $\pi/2$-periodic functions with the growth bound and the finite Fourier form are of the same kind, and the coefficients add. For χ^{tot} the hypothesis is Lemma 15.26(i) with $u=e_t$. Both sides of the last identity are holomorphic on the connected set $\Sigma_{\pi/4}$ and agree for real $\zeta\in(-\pi/4,\pi/4)$ by Lemma 15.27 with $\theta_1=\theta_2$ (for $N\leq 1$, $D_N\equiv 0$). □

Lemma 15.29 (curves in the OS space)**.** *For $\chi\in\mathfrak{A}$ and $0\leq s<\pi/4$: $A_\chi(s) := \prod_j F(R(s)^*\chi_j)\in L^2_{\circ,+}$, and $\langle A_\chi(r), A_\eta(s)\rangle_\circ = \Psi_{\chi,\eta}(-r,s)$ for $r,s\in[0,\pi/4)$.*

Proof. $R(s)^*\chi_j$ is a real test supported in $R(-s)K_j\times\mathbb{R}\subset\{t>0\}$ (Lemma 15.26(ii)), so $A_\chi(s)$ is $\Sigma(\{t>0\})$-measurable and in L^2 by Lemma 15.6(b) and Hölder. Since $\theta_\circ R(r) = R(-r)\theta_\circ$ and $\pi(\theta_\circ)\varrho(-r) = \varrho(r)\pi(\theta_\circ)$, $J(R(r)^*\psi) = R(-r)^*(J\psi)$ for real tests and real r. $\Theta_\circ$ is multiplicative, so $\overline{\Theta_\circ A_\chi(r)} = \prod_j F(R(-r)^*J\chi_j)$ and $\langle A_\chi(r),A_\eta(s)\rangle_\circ = \mathbb{E}_\mu\big[\prod_j F(R(-r)^*J\chi_j)\prod_k F(R(s)^*\eta_k)\big] = S_N(R(-r)^*J\chi, R(s)^*\eta) = \Psi_{\chi,\eta}(-r,s)$ by Lemma 15.27. □

Proof of Proposition 15.25. (a) is Lemmas 15.27 and 15.28. (b): $\langle\chi,\eta\rangle_{\zeta,\xi} = \Psi_{\chi,\eta}(-\zeta,\xi)$ is holomorphic on $\Sigma^2_{\pi/4}$ ($-\Sigma_{\pi/4}=\Sigma_{\pi/4}$); finite sums preserve this; both sides of the identity in (b) are sesquilinear in (A,B) and agree on basis pairs by Lemma 15.29. (c): $\Sigma_{\pi/4}$ is open, connected, conjugation-symmetric, $0\in\Sigma_{\pi/4}$, so Proposition 15.24 with $O=\Sigma_{\pi/4}$, $\delta_A=\pi/4$ gives $\langle A,A\rangle_{z,\bar z}\geq$

0 for $z \in \Sigma_{\pi/4}$; at $z = -i\beta$, $\langle\chi,\eta\rangle_{-i\beta,i\beta} = \Psi_{\chi,\eta}(i\beta,i\beta) = S_N(J\chi,\eta) + D_N[J\chi,\eta](i\beta) = \langle\chi,\eta\rangle_\beta$ by Lemma 15.28, for every $\beta \in \mathbb{R}$. □

15.12. **Testing and convexity.** *Conventions.* For a unit vector $u \in \mathbb{R}^2$ let $\Gamma(u) := \{p \in \mathbb{R}^2 : p \cdot u > |p|/\sqrt{2}\}$, the open cone of half-angle $\pi/4$ about u; $\Gamma(e_t) = \mathcal{C}_+$, $\mathcal{C}_- := -\mathcal{C}_+ = \theta_\circ\mathcal{C}_+$, $\mathcal{C}_\pm + \mathcal{C}_\pm \subset \mathcal{C}_\pm$, $R\,\Gamma(u) = \Gamma(Ru)$ for planar rotations R. A tuple is $\chi = (\chi_1,\dots,\chi_N)$, $N \geq 0$, of real product tests $\chi_j = \alpha_j f_j$ with compact planar supports $K_j := \operatorname{supp}\alpha_j$; $S_0 := 1$. R^*, $\tau_v\chi = \chi(\cdot - v)$ and $\chi_\sigma := (\chi_{\sigma(1)},\dots,\chi_{\sigma(N)})$ act entrywise. $J\chi_j = \pi(\theta_\circ)(\bar\chi_j \circ \theta_\circ)$ with $\pi(\theta_\circ)$ a real orthogonal involution, so J maps real product tests to real product tests, $J^2 = 1$, the planar support of $J\chi_j$ is $\theta_\circ K_j$, and $J\chi = (J\chi_a,\dots,J\chi_1)$ for a tuple of length a. A tuple χ is narrow if there are a permutation σ and a unit vector u with $K_{\sigma(j+1)} - K_{\sigma(j)} \subset \Gamma(u)$ for $1 \leq j < N$ (every tuple of length ≤ 1 is narrow); this is the hypothesis of Lemma 15.28. Every configuration (X_j) with $X_j \in K_j$ is then narrow; conversely, for a narrow configuration X with data (σ,u) and $0 < \delta < \frac{1}{2}\min_j \operatorname{dist}(X_{\sigma(j+1)} - X_{\sigma(j)}, \partial\Gamma(u))$, every tuple with $K_j \subset B(X_j,\delta)$ is narrow with the same data. For admissible χ (length a) and η (length b), $(J\chi,\eta)$ is narrow with $u = e_t$, $\sigma = \mathrm{id}$ (Lemma 15.26(i)).

Regular tuples and their coefficients. A tuple χ of length N is regular if $D_N[\chi](\theta) = S_N(R(\theta)^*\chi) - S_N(\chi)$, $\theta \in \mathbb{R}$, is the restriction of an entire function $D_N[\chi](\zeta)$ with $\sup_{\theta\in\mathbb{R}} |D_N[\chi](\theta + i\beta)| \leq C_0 e^{3N|\beta|}$ for all $\beta \in \mathbb{R}$ and some $C_0 < \infty$. The extension is unique (identity theorem) and 2π-periodic, since $D_N[\chi](\zeta + 2\pi) - D_N[\chi](\zeta)$ is entire and vanishes on $\mathbb{R}$ ($R(2\pi) = 1 = \varrho(2\pi)$). Hence $D_N[\chi](\zeta) = \sum_{\ell\in\mathbb{Z}} d_\ell e^{i\ell\zeta}$ (Laurent series in $e^{i\zeta}$, absolutely convergent) with $d_\ell = \frac{1}{2\pi}\int_0^{2\pi} D_N[\chi](\theta + i\beta)e^{-i\ell(\theta+i\beta)}\,d\theta$ for every β; so $|d_\ell| \leq C_0 e^{3N|\beta|+\ell\beta}$ and $(\beta \to \mp\infty)$ $d_\ell = 0$ for $|\ell| > 3N$. Put $c_\ell[\chi] := d_\ell$ $(\ell \neq 0)$; $D_N[\chi](0) = 0$ gives

$$D_N[\chi](\zeta) = \sum_{0<|\ell|\leq 3N} c_\ell[\chi]\,(e^{i\ell\zeta} - 1). \tag{15.9}$$

For $\ell \neq 0$, $c_\ell[\chi] = \frac{1}{2\pi}\int_0^{2\pi} D_N[\chi](\theta)e^{-i\ell\theta}d\theta$ is intrinsic. The facts used below are:

(A) every narrow tuple is regular, and $c_\ell[\chi] = 0$ for $\ell \notin 4\mathbb{Z}$ (Lemma 15.31 below);

(B) for admissible χ,η of lengths a,b and $\beta \in \mathbb{R}$, $\langle\chi,\eta\rangle_\beta := S_{a+b}(J\chi,\eta) + D_{a+b}[J\chi,\eta](i\beta)$ (the empty tuple $\emptyset$ allowed, $\langle\emptyset,\emptyset\rangle_\beta = S_0 = 1$) satisfies $\sum_{i,k}\bar z_i z_k\langle\chi^{(i)},\chi^{(k)}\rangle_\beta \geq 0$ for all finite families of admissible tuples and all $z \in \mathbb{C}^n$ (Proposition 15.25(c));

(C) $S_N(\tau_v\chi) = S_N(\chi)$ for all $v \in \mathbb{R}^3$ (Lemma 15.6(e) and (ii)); S_N is symmetric.

Lemma 15.30 (testing)**.** *Assume* (C).

(a) *If χ is regular, so are χ_σ, $\tau_v\chi$ $(v \in \mathbb{R}^3)$ and $R(\phi)^*\chi$ $(\phi \in \mathbb{R})$, and $c_\ell[\chi_\sigma] = c_\ell[\chi]$, $c_\ell[\tau_v\chi] = c_\ell[\chi]$, $c_\ell[R(\phi)^*\chi] = e^{i\ell\phi}c_\ell[\chi]$. These operations map narrow tuples to narrow tuples.*

(b) *Let χ be narrow of length N with data (σ,u), $u = R(\phi)e_t$, and $a+b = N$. Then there are admissible χ',η' of lengths a,b with $(J\chi',\eta') = \tau_v R(\phi)^*\chi_\sigma$ for some $v \in \mathbb{R}^3$; if χ is regular, so is $(J\chi',\eta')$, with $c_\ell[J\chi',\eta'] = e^{i\ell\phi}c_\ell[\chi]$. In particular, if $\ell \neq 0$ and $c_\ell[\chi] \neq 0$ for a narrow regular χ of length N, then for every $a+b = N$ there are admissible χ',η' of lengths a,b with $c_\ell[J\chi',\eta'] \neq 0$.*

Proof. (a) Permutations: $\prod_j F(\chi_j)$ is a commutative product and $R(\theta)^*$ acts entrywise, so $D_N[\chi_\sigma] = D_N[\chi]$ on $\mathbb{R}$; the entire extensions and their coefficients coincide. Translations: $(R^*\tau_v\chi)(x) = \pi(R)^{-1}\chi(Rx - v) = (R^*\chi)(x - R^{-1}v)$, i.e. $R^*\tau_v = \tau_{R^{-1}v}R^*$; by (C), $S_N(R(\theta)^*\tau_v\chi) = S_N(R(\theta)^*\chi)$ and $S_N(\tau_v\chi) = S_N(\chi)$, so $D_N[\tau_v\chi] = D_N[\chi]$ on $\mathbb{R}$. Rotations: π is a representation and planar rotations commute, so $R(\theta)^*R(\phi)^* = (R(\phi)R(\theta))^* = R(\theta+\phi)^*$ and

$$D_N[R(\phi)^*\chi](\theta) = \big(S_N(R(\theta+\phi)^*\chi) - S_N(\chi)\big) - \big(S_N(R(\phi)^*\chi) - S_N(\chi)\big)$$

$$= D_N[\chi](\theta+\phi) - D_N[\chi](\phi).$$

The right side is the restriction of the entire function $\zeta \mapsto D_N[\chi](\zeta+\phi) - D_N[\chi](\phi)$, bounded by $(C_0 + |D_N[\chi](\phi)|)e^{3N|\beta|}$; so $R(\phi)^*\chi$ is regular with $D_N[R(\phi)^*\chi](\zeta) = \sum_\ell c_\ell[\chi](e^{i\ell(\zeta+\phi)} - e^{i\ell\phi}) = \sum_\ell c_\ell[\chi]e^{i\ell\phi}(e^{i\ell\zeta}-1)$, and uniqueness of Laurent coefficients gives $c_\ell[R(\phi)^*\chi] = e^{i\ell\phi}c_\ell[\chi]$. Finally $R(\phi)^*(\alpha f) = (\alpha \circ R(\phi))\, \varrho(\phi)^{-1} f$ is a real product test ($\varrho(\phi)$ real orthogonal) with planar support $R(\phi)^{-1}K$, so the cone condition is transported to $R(\phi)^{-1}\Gamma(u) = \Gamma(R(\phi)^{-1}u)$; τ_v shifts all K_j by the planar part of v, leaving differences unchanged; χ_σ permutes the K_j, and the ordering composes.

(b) Step 1. $\psi := R(\phi)^*(\chi_\sigma)$ has supports $M_j := R(\phi)^{-1}K_{\sigma(j)}$ with $M_{j+1} - M_j \subset R(\phi)^{-1}\Gamma(u) = \Gamma(e_t) = \mathcal{C}_+$.

Step 2 (splitting). With $\lambda_\pm(t,x) := t \pm x$, $\mathcal{C}_+ = \{\lambda_+ > 0,\ \lambda_- > 0\}$. If $1 \le a < N$: $\lambda_\pm(p') > \lambda_\pm(p)$ for $p \in M_a$, $p' \in M_{a+1}$, so by compactness $m^\pm := \max_{M_a} \lambda_\pm < \min_{M_{a+1}} \lambda_\pm =: m'^\pm$. Let $w \in \mathbb{R}^2$ have $\lambda_\pm(w) = \frac{1}{2}(m^\pm + m'^\pm)$ (if $a = 0$: $\lambda_\pm(w) = m'^\pm - 1$; if $a = N$: $\lambda_\pm(w) = m^\pm + 1$). Then $M_a - w \subset \mathcal{C}_-$ and $M_{a+1} - w \subset \mathcal{C}_+$; since $M_j - M_{j+1} \subset \mathcal{C}_-$ and $\mathcal{C}_\pm + \mathcal{C}_\pm \subset \mathcal{C}_\pm$, induction downward and upward gives $M_j - w \subset \mathcal{C}_-$ $(j \le a)$ and $M_j - w \subset \mathcal{C}_+$ $(j > a)$.

Step 3. Let $v := (-w, 0) \in \mathbb{R}^3$ and $\psi' := \tau_v\psi$, so that ψ'_j has planar support $M_j - w$. Define $\chi'_i := J\psi'_{a+1-i}$ $(1 \le i \le a)$, $\eta'_k := \psi'_{a+k}$ $(1 \le k \le b)$: tuples of real product tests with $J\chi' = (J^2\psi'_1, \dots, J^2\psi'_a) = (\psi'_1, \dots, \psi'_a)$, hence $(J\chi', \eta') = \psi' = \tau_v R(\phi)^*\chi_\sigma$. Admissibility: $\operatorname{supp}\chi'_i = \theta_\circ(M_{a+1-i} - w) \subset \theta_\circ\mathcal{C}_- = \mathcal{C}_+$; $\operatorname{supp}\chi'_{i+1} - \operatorname{supp}\chi'_i = \theta_\circ(-(M_{a+1-i} - M_{a-i})) \subset \theta_\circ\mathcal{C}_- = \mathcal{C}_+$; $\operatorname{supp}\eta'_k = M_{a+k} - w \subset \mathcal{C}_+$ and $\operatorname{supp}\eta'_{k+1} - \operatorname{supp}\eta'_k = M_{a+k+1} - M_{a+k} \subset \mathcal{C}_+$. If χ is regular, (a) gives the regularity of ψ' and $c_\ell[J\chi', \eta'] = c_\ell[\psi'] = e^{i\ell\phi}c_\ell[\chi]$. □

Lemma 15.31 (narrow tuples are regular)**.** *Every narrow tuple χ is regular, with $c_\ell[\chi] = 0$ for $\ell \notin 4\mathbb{Z}$, and $|D_N[\chi](\theta + i\beta)| \le C_0 e^{(1+w_{\max})N|\beta|}$. Moreover, if $c_\ell[\chi] \neq 0$ for a narrow tuple χ of length N, then $c_\ell[\chi^\flat] \neq 0$ for some tuple $\chi^\flat$ of length N with planar supports in balls $\bar B(x_j, r)$, $r < r_0$, about a narrow configuration X as in Proposition 15.21; and conversely every such tuple is narrow.*

Proof. Let χ be narrow with data (σ, u), $u = R(\phi)e_t$. Lemma 15.30(b), Steps 1–3, uses only the support geometry and produces admissible χ', η' with $(J\chi', \eta') = \psi' = \tau_v R(\phi)^*\chi_\sigma$. By Lemma 15.28, ψ' is regular, with the growth bound and $c_\ell[\psi'] = 0$ for $\ell \notin 4\mathbb{Z}$. The proof of Lemma 15.30(a) uses only regularity, so $\chi_\sigma = R(-\phi)^*\tau_{-v}\psi'$ and hence χ are regular with $c_\ell[\chi] = e^{-i\ell\phi}c_\ell[\psi']$, which vanishes off $4\mathbb{Z}$ (and the growth bound transports with a larger constant). Alternatively, and for the second claim: by compactness of $\prod_j (K_{\sigma(j+1)} - K_{\sigma(j)}) \subset \Gamma(u)^{N-1}$ there are $\delta_* > 0$ and $\gamma_* < \pi/4$ bounding all drawn configurations, so $r_0 \ge r_* := \frac{1}{2}\delta_* \cos(\gamma_* + \pi/4)$ uniformly; a smooth planar partition of unity with pieces in balls of radius $< r_*$ centred in the K_j and the N-linearity of D_N express $D_N[\chi]$ as a finite sum of $D_N[\chi^\flat]$ with $\chi^\flat$ as in Proposition 15.21, the coefficients adding, so some piece has $c_\ell[\chi^\flat] \neq 0$. Conversely, for $d \in \Gamma(u)$ at angle $\alpha \le \gamma$ from u, $\operatorname{dist}(d, \partial\Gamma(u)) = |d|\sin(\pi/4 - \alpha) = |d|\cos(\pi/4 + \alpha) \ge 2r_0 > 2r$, so $\bar B(d_j, 2r) \subset \Gamma(u)$ and tuples supported in the balls $\bar B(x_j, r)$ are narrow. □

Consequently, for each $\ell \neq 0$ and each N, $c_\ell[\chi] \neq 0$ for some narrow tuple χ of length N if and only if $c_\ell[\chi] \neq 0$ for some tuple χ of length N supported in the balls of Proposition 15.21; in particular the maximum M_N of Proposition 15.32 is the same for both classes of tuples.

Proposition 15.32 (convexity)**.** *Let*

$$M_N := \max\{|\ell| :\ c_\ell[\chi] \neq 0 \text{ for some narrow tuple } \chi \text{ of length } N\},$$

and $M_N := 0$ if there is none. Then $M_0 = 0$, $M_N \in 4\mathbb{Z}$, $0 \le M_N \le 3N$ (indeed $\le (1 + w_{\max})N$), and $2M_{a+b} \le M_{2a} + M_{2b}$ for all $a, b \ge 0$.

Proof. Bounds. By (15.9) and Lemma 15.31, $c_\ell[\chi]=0$ for $|\ell|>(1+w_{\max})N$, so $M_N\le 3N$ and $M_0=0$; by (A) only $\ell\in 4\mathbb{Z}$ occur, so $M_N\in 4\mathbb{Z}$ (0 included).

Convexity. Let $N=a+b$, $m:=M_N$; if $m=0$ there is nothing to prove. Otherwise pick ℓ_0 with $|\ell_0|=m$ and $c_{\ell_0}[\chi]\neq 0$ for a narrow tuple χ of length N; Lemma 15.30(b) gives admissible χ',η' of lengths a,b with $\gamma:=c_{\ell_0}[J\chi',\eta']\neq 0$. The tuples $(J\chi',\eta')$, $(J\chi',\chi')$, $(J\eta',\eta')$ are narrow, of lengths $N,2a,2b$; hence their coefficients c_ℓ, c_ℓ^A, c_ℓ^B vanish for $|\ell|>m$, $|\ell|>M_{2a}$, $|\ell|>M_{2b}$ respectively, by the definition of M. At $\zeta=i\beta$,

$$\langle\chi',\eta'\rangle_\beta = S_N(J\chi',\eta') + \sum_{0<|\ell|\le m} c_\ell\,(e^{-\ell\beta}-1),$$

$$\langle\chi',\chi'\rangle_\beta = S_{2a}(J\chi',\chi') + \sum_{0<|\ell|\le M_{2a}} c_\ell^A\,(e^{-\ell\beta}-1),$$

and likewise for η' (for $a=0$: $\langle\emptyset,\emptyset\rangle_\beta=1$). With $C_A:=|S_{2a}(J\chi',\chi')|+2\sum_{0<|\ell|\le M_{2a}}|c_\ell^A|$ and C_B likewise, $|e^{-\ell\beta}-1|\le 2e^{|\ell||\beta|}$ gives $|\langle\chi',\chi'\rangle_\beta|\le C_Ae^{M_{2a}|\beta|}$ and $|\langle\eta',\eta'\rangle_\beta|\le C_Be^{M_{2b}|\beta|}$ for $\beta\in\mathbb{R}$.

Lower bound: let $s:=-\operatorname{sgn}\ell_0$, $\beta=s\tau$, $\tau>0$. Then $e^{-\ell_0\beta}=e^{m\tau}$, $e^{\ell_0\beta}=e^{-m\tau}\le 1$, and every other ℓ with $0<|\ell|\le m$ has $|\ell|\le m-4$ (all frequencies lie in $4\mathbb{Z}$, and $m\ge 4$). With $C':=|S_N(J\chi',\eta')|+2\sum_{0<|\ell|\le m}|c_\ell|$,

$$|\langle\chi',\eta'\rangle_{s\tau}| \;\ge\; |\gamma|e^{m\tau}-C'e^{(m-4)\tau} \;\ge\; \tfrac12|\gamma|e^{m\tau} \qquad \text{for } \tau\ge\tau_0:=\max\{0,\tfrac14\log(2C'/|\gamma|)\}.$$

Positivity: by (B) for the family $\{\chi',\eta'\}$, the 2×2 matrix $G_\beta:=(\langle\cdot,\cdot\rangle_\beta)$ satisfies $z^*G_\beta z\ge 0$ for all $z\in\mathbb{C}^2$; thus $z^*(G_\beta-G_\beta^*)z=0$ for all z, so by polarisation G_β is Hermitian, positive semi-definite, and $\det G_\beta\ge 0$: $|\langle\chi',\eta'\rangle_\beta|^2\le\langle\chi',\chi'\rangle_\beta\langle\eta',\eta'\rangle_\beta$. Hence $\frac14|\gamma|^2e^{2m\tau}\le C_AC_B\,e^{(M_{2a}+M_{2b})\tau}$ for all $\tau\ge\tau_0$; letting $\tau\to\infty$ forces $2m\le M_{2a}+M_{2b}$. □

Only two-element families (and the empty tuple when $a=0$ or $b=0$, giving $2M_b\le M_{2b}$) of (B) are used. This subsection uses, beyond (i), (ii) and the four planes of the pair through Propositions 15.21 and 15.25, nothing else.

15.13. **Divisibility by eight and the arithmetic lemma.** *Notation.* Recall that $(g^*\chi)(x)=\pi(g)^{-1}\chi(gx)$, so that $(gh)^*=h^*g^*$, g^* preserves reality, $\|\cdot\|_1$ and $\|\cdot\|_2$, and $\operatorname{supp}g^*\chi=g^{-1}\operatorname{supp}\chi$. We put $(\tau_a^*\chi)(x):=\chi(x+a)=(\tau_{-a}\chi)(x)$, hence $g^*\tau_a^*=\tau_{g^{-1}a}^*g^*$. We put $R_\diamond:=R(\pi/4)$ and $P_0:=\{t=0\}$, $P_1:=\{t+x=0\}$, $P_2:=\{x=0\}$, $P_3:=\{t-x=0\}$, and we write Θ_{P_k} for the reflection in P_k. Then $R_\diamond P_k=P_{k+1}$ (indices mod 4), $R_\diamond\{y=0\}=\{y=0\}$, and $\Theta_{gP}=g\Theta_Pg^{-1}$ for every plane P through 0. We put $\tilde\mu:=T_*\mu$ with $T\colon\mathcal{S}'\to\mathcal{S}'$, $(T\omega)(\chi):=\omega(R_\diamond^*\chi)$ (T is the transpose of the continuous linear bijection $R_\diamond^*$ of $\mathcal{S}$, hence continuous, so $\tilde\mu$ is a Borel probability measure); thus $\tilde Z(\chi)=Z(R_\diamond^*\chi)$ and $\tilde S_N(\chi_1,\dots,\chi_N)=S_N(R_\diamond^*\chi_1,\dots,R_\diamond^*\chi_N)$. Objects attached to $\tilde\mu$ carry a tilde. For a regular tuple ψ put $c_0[\psi]:=S_N(\psi)-\sum_{\ell\neq 0}c_\ell[\psi]$, so that $S_N(R(\theta)^*\psi)=\sum_\ell c_\ell[\psi]e^{i\ell\theta}$ for real θ. $\mathfrak{A}_a$ denotes the admissible tuples of length a; by Lemma 15.31 every pair $(J\chi,\eta)$, $\chi\in\mathfrak{A}_a$, $\eta\in\mathfrak{A}_b$, is regular with frequencies in $4\mathbb{Z}$ and $|\ell|\le M_{a+b}$.

Proposition 15.33 (divisibility by eight)**.** *Let μ satisfy (i), (ii) and (iii) for the four planes $P_0,\dots,P_3$ of the coordinate pair (t,x), and let $\tilde\mu$ be as above.*

1. *$\tilde\mu$ satisfies (i) with the same constant C, (ii), and (iii) for the reflections in P_0,P_1,P_2,P_3 (and in $\{y=0\}$ if μ has (iii) there); the four reflections, hence the dihedral group they generate, which contains $R(\pi/2)$, preserve $\tilde\mu$.*
2. *For every regular tuple ψ (for μ) and all $\zeta\in\mathbb{C}$: ψ is regular for $\tilde\mu$, $\tilde D_N[\psi](\zeta)=D_N[\psi](\zeta+\pi/4)-D_N[\psi](\pi/4)$, entire and 2π-periodic, $|\tilde D_N[\psi](\theta+i\beta)|\le 2C_0e^{3N|\beta|}$, and $\tilde c_\ell[\psi]=e^{i\ell\pi/4}c_\ell[\psi]=(-1)^{\ell/4}c_\ell[\psi]$ for $\ell\in 4\mathbb{Z}\setminus\{0\}$.*

(3) $c_{\pm M_{2a}}[J\chi,\chi]\ge 0$ *for all* $\chi\in\mathfrak{A}_a$, *and if* $M:=M_{2a}>0$ *there is* $\chi\in\mathfrak{A}_a$ *with* $c_{-M}[J\chi,\chi]=c_M[J\chi,\chi]>0$.
(4) $M_{2a}\in 8\mathbb{Z}$ *for all* $a\ge 0$.

Proof. (1) (i): for real Schwartz χ, $\int e^{F(\chi)}d\tilde\mu=\int e^{F(R_\diamond^*\chi)}d\mu\le e^{C(\|R_\diamond^*\chi\|_1+\|R_\diamond^*\chi\|_2^2)}=e^{C(\|\chi\|_1+\|\chi\|_2^2)}$, because $R_\diamond^*\chi$ is real Schwartz and $|(R_\diamond^*\chi)(x)|_T=|\chi(R_\diamond x)|_T$ with $R_\diamond$ measure preserving. (ii): $\tilde S_N(\tau_a^*\chi_1,\dots)=S_N(\tau^*_{R_\diamond^{-1}a}R_\diamond^*\chi_1,\dots)=S_N(R_\diamond^*\chi_1,\dots)=\tilde S_N(\chi_1,\dots)$ by the translation invariance of the S_N (Lemma 15.6(e) for μ); since $\tilde\mu$ satisfies (i), Lemma 15.6(e) applied to $\tilde\mu$ gives $\tau_{a*}\tilde\mu=\tilde\mu$. (iii): let $P\in\{P_0,\dots,P_3\}$, $P':=R_\diamond^{-1}P\in\{P_0,\dots,P_3\}$, χ_a real Schwartz supported in one open side P^+ of P, $z_a\in\mathbb{C}$. From $\Theta_P=R_\diamond\Theta_{P'}R_\diamond^{-1}$, $R_\diamond^*\Theta_P^*=(\Theta_PR_\diamond)^*=(R_\diamond\Theta_{P'})^*=\Theta_{P'}^*R_\diamond^*$, so $\sum_{a,b}\bar z_az_b\int e^{iF(\chi_a-\Theta_P^*\chi_b)}d\tilde\mu=\sum_{a,b}\bar z_az_b\int e^{iF(\chi'_a-\Theta_{P'}^*\chi'_b)}d\mu\ge 0$, with $\chi'_a:=R_\diamond^*\chi_a$ supported in $R_\diamond^{-1}P^+$, one open side of P', by (iii) for μ at P'. The same with $P=P'=\{y=0\}$. Invariance under the four reflections follows from Lemma 15.7 applied to $\tilde\mu$ (or directly: $\tilde S_N(\Theta^*_{P_k}\chi_1,\dots)=S_N(\Theta^*_{P_{k-1}}R_\diamond^*\chi_1,\dots)=S_N(R_\diamond^*\chi_1,\dots)$). $\Theta_{P_1}\Theta_{P_0}=R(\pi/2)$.

(2) $R_\diamond^*R(\theta)^*=(R(\theta)R_\diamond)^*=R(\theta+\pi/4)^*$, hence $\tilde S_N(R(\theta)^*\psi)=S_N(R(\theta+\pi/4)^*\psi)$ and, for real θ, $\tilde D_N[\psi](\theta)=[S_N(\psi)+D_N[\psi](\theta+\pi/4)]-[S_N(\psi)+D_N[\psi](\pi/4)]$. The right side is entire in θ, so it is the continuation, with the stated period and bound. Inserting (15.9), $\tilde D_N[\psi](\zeta)=\sum_{\ell\ne0}c_\ell[\psi](e^{i\ell(\zeta+\pi/4)}-e^{i\ell\pi/4})=\sum_{\ell\ne0}e^{i\ell\pi/4}c_\ell[\psi](e^{i\ell\zeta}-1)$; uniqueness of Fourier coefficients gives $\tilde c_\ell=e^{i\ell\pi/4}c_\ell$, and $e^{i\ell\pi/4}=(e^{i\pi})^{\ell/4}=(-1)^{\ell/4}$ for $\ell\in4\mathbb{Z}$.

(3) Step 1 (reality). For a real tuple ψ and real θ the tuple $R(\theta)^*\psi$ is real ($\pi(R(\theta))$ is a real matrix), so $S_N(R(\theta)^*\psi)=\int\prod_j F(R(\theta)^*\psi_j)\,d\mu\in\mathbb{R}$ (F is real on real tests). A real trigonometric polynomial has $c_{-\ell}=\overline{c_\ell}$. For real χ, $J\chi$ is real ($\pi(\theta_\circ)$ is real), so this applies to $(J\chi,\eta)$. Step 2 (leading Gram matrix). Let $N=2a$, $M=M_N$, $\chi^{(1)},\dots,\chi^{(m)}\in\mathfrak{A}_a$ and $G(\beta)_{jk}:=S_N(J\chi^{(j)},\chi^{(k)})+D_N[J\chi^{(j)},\chi^{(k)}](i\beta)$. By Proposition 15.25(c), $G(\beta)\ge0$ for every real β. By (15.9) at $\zeta=i\beta$ and the frequency bound, $G(\beta)_{jk}=\sum_{\ell\in4\mathbb{Z},\,|\ell|\le M}c_\ell[J\chi^{(j)},\chi^{(k)}]e^{-\ell\beta}$. Hence $e^{-M\beta}G(\beta)\to G_{-M}:=(c_{-M}[J\chi^{(j)},\chi^{(k)}])_{jk}$ as $\beta\to+\infty$ and $e^{M\beta}G(\beta)\to G_M$ as $\beta\to-\infty$; the cone of positive semidefinite matrices is closed, so $G_{\pm M}\ge0$. With $m=1$: $c_{\pm M}[J\chi,\chi]\ge0$. Step 3. Let $M>0$. By the definition of M_N some $\ell\in\{M,-M\}$ has $c_\ell[\psi]\ne0$ for a narrow tuple ψ of length N; by Step 1 then $\ell=-M$ does. Lemma 15.30(b) with $(a,b)=(a,a)$ gives $\chi,\eta\in\mathfrak{A}_a$ with $c_{-M}[J\chi,\eta]\ne0$. Step 2 for $(\chi^{(1)},\chi^{(2)})=(\chi,\eta)$: the 2×2 matrix $G_{-M}\ge0$ has a nonzero off-diagonal entry, so $0<|c_{-M}[J\chi,\eta]|^2\le c_{-M}[J\chi,\chi]\,c_{-M}[J\eta,\eta]$, whence $c_{-M}[J\chi,\chi]>0$, and $c_M[J\chi,\chi]=\overline{c_{-M}[J\chi,\chi]}=c_{-M}[J\chi,\chi]$ by Step 1.

(4) Let $M=M_{2a}>0$ (otherwise there is nothing to prove) and χ as in (3). By (1), $\tilde\mu$ satisfies the hypotheses of Subsections 15.2–15.11 for the coordinate pair (t,x) (Remark 15.4 and Lemma 15.34 below), so Proposition 15.25(c) holds for $\tilde\mu$: $\tilde S_N(J\chi,\chi)+\tilde D_N[J\chi,\chi](i\beta)\ge0$ for all $\beta\in\mathbb{R}$ (χ is admissible for $\tilde\mu$ as well, admissibility being a property of the supports). By (2) this reads $S_N(J\chi,\chi)+D_N[J\chi,\chi](\pi/4+i\beta)=c_0[J\chi,\chi]+\sum_{\ell\ne0}(-1)^{\ell/4}c_\ell[J\chi,\chi]e^{-\ell\beta}\ge0$ for all real β. Multiplying by $e^{-M\beta}$ and letting $\beta\to+\infty$ gives $(-1)^{M/4}c_{-M}[J\chi,\chi]\ge0$. As $c_{-M}[J\chi,\chi]>0$, $(-1)^{M/4}=1$, i.e. $M/4$ is even: $M\in8\mathbb{Z}$. □

Lemma 15.34 (the four planes of a pair suffice)**.** *Let* ν *be a Borel probability measure on* $\mathcal{S}'(\mathbb{R}^3;T)$ *satisfying (i), (ii) and (iii) for the four planes* $P_0,\dots,P_3$ *of the coordinate pair* (t,x) *only. Then every statement of Subsections 15.2–15.12 holds for* ν *and this pair, with the same proofs; in particular Proposition 15.25(c) holds for* ν.

Proof. We go through the statements in question and determine the planes and symmetries that their proofs use. Lemma 15.6 uses (i) only. In Lemma 15.7 the first assertion needs (i), (ii) and (iii) for the one plane in question, and it gives for ν the invariance under $\Theta_{P_0},\dots,\Theta_{P_3}$

and under the dihedral group D_4 they generate. The four-plane clause of that lemma gives $S_1(R(\theta)^*\chi) = S_1(\chi)$ for all θ, which is the only property of S_1 used later, namely in Lemmas 15.27 and 15.28 and in the paragraph after the proof of Proposition 15.21. Lemma 15.5 uses (ii), (iii) and the invariance at the one plane u in question, where $u \in \{e_t, n\}$ or, for the pair (x,t), $u \in \{e_x, (e_x+e_t)/\sqrt{2}\}$; these planes are all among $P_0, \dots, P_3$.

Theorem 15.8 uses P_0 and P_1 for the pair (t,x), and P_2 and P_1 for the pair (x,t). Lemma 15.12 uses Theorem 15.8 only. Theorems 15.14 and 15.15 use P_0 and its translates $\{t=c\}$ for the pair (t,x), and P_2 and its translates for the pair (x,t), together with (i) and (ii). Theorem 15.16 uses Theorem 15.8, Lemma 15.12 and Theorem 15.15. Propositions 15.19 and 15.20 use Theorems 15.15 and 15.16 for the pairs (t,x) and (x,t), and they use Lemma 15.7 only through $R(\theta)$ with $\theta \in \frac{\pi}{2}\mathbb{Z}$, which is an element of D_4. Proposition 15.21 uses (K1)–(K3), which come from Proposition 15.20, and (K4), which is the invariance under $R(\pi/2) \in D_4$. Proposition 15.24 is abstract, apart from the positivity of the form at P_0. Proposition 15.25 and its lemmas use the form at P_0, Propositions 15.19 and 15.21, and the identity $S_1(R(\theta)^*\chi) = S_1(\chi)$. Lemmas 15.30 and 15.31 and Proposition 15.32 use (ii) and Propositions 15.21 and 15.25. Thus no proof uses a plane outside $\{P_0, P_1, P_2, P_3\}$ or a symmetry outside D_4 and the translations. □

Lemma 15.35 (arithmetic)**.** *Let $(M_n)_{n\ge 0}$ be non-negative integers with $M_0 = 0$, $M_n \le 3n$, $2M_{a+b} \le M_{2a} + M_{2b}$ for all $a, b \ge 0$, and $M_{2n} \in 8\mathbb{Z}$ for all n. Then $M_n = 0$ for all n.*

Proof. Put $g(n) := M_{2n}$; then $g(0) = 0$, $0 \le g(n) \le 6n$, $g(n) \in 8\mathbb{Z}$. For $c \ge 1$ the hypothesis with $(a,b) = (c-1, c+1)$ gives $2g(c) \le g(c-1) + g(c+1)$, so the increments $d_c := g(c) - g(c-1) \in 8\mathbb{Z}$ satisfy $d_{c+1} \ge d_c$. Suppose $g(k) > 0$ for some k. Then $\sum_{c=1}^{k} d_c > 0$, so some $d_j > 0$, hence $d_j \ge 8$, hence $d_c \ge 8$ for all $c \ge j$ and $g(n) = g(j-1) + \sum_{c=j}^{n} d_c \ge 8(n-j+1)$ for $n \ge j$. At $n = 4j$: $g(4j) \ge 24j + 8 > 24j \ge g(4j)$, a contradiction. Thus $M_{2n} = 0$ for all n, and the hypothesis with $b = 0$ gives $2M_a \le M_{2a} + M_0 = 0$, so $M_a = 0$. □

Corollary 15.36. *Under (i), (ii) and (iii) for the four planes of a coordinate pair, $M_N = 0$ for all N; hence $c_\ell[\chi] = 0$ for all $\ell \ne 0$ and every narrow tuple χ, and $D_N[\chi] \equiv 0$ on $\mathbb{C}$ for every narrow tuple and for every tuple of tests in $(L^1 \cap L^2)(\mathbb{R}^3; T_{\mathbb{C}})$ vanishing outside $K_j \times \mathbb{R}$ with K_j the balls of Proposition 15.21.*

Proof. Proposition 15.32, Proposition 15.33(4) and Lemma 15.35 give $M_N = 0$. Then (15.9) gives $D_N[\chi] \equiv 0$ for narrow tuples (Lemma 15.31), and Proposition 15.21(c) (with Lemma 15.31: the tuples supported in the balls are narrow, so their coefficients vanish) for the non-product tests: by Step 5 of the proof of Proposition 15.21 the coefficients $c_\ell[\chi]$ are N-linear in χ and are limits of the coefficients of finite complex-linear combinations of real smooth product tests supported in the same balls, which vanish, so $c_\ell[\chi] = 0$ for $\ell \ne 0$ and $D_N[\chi] \equiv 0$ on $\mathcal{X}$. □

15.14. **From narrow configurations to all distinct points.** Throughout this subsection $(t,x) = (x_j, x_k)$ is a coordinate pair and $y = x_l$ is the third coordinate. We write $P(x) := (x_j, x_k)$ for the planar projection and $R(\theta)$ for the rotation about e_l; for $R = R(\theta)$ we have $\operatorname{supp} R^*\chi = R^{-1} \operatorname{supp} \chi$ and $R^*\tau_a = \tau_{R^{-1}a} R^*$. We put $\mathcal{S}_N := \mathcal{S}(\mathbb{R}^3; T_{\mathbb{C}})^N$ and $B_r(X) := \prod_m B(x_m, r)$ for $X = (x_m)_m \in (\mathbb{R}^3)^N$, and we say that a tuple χ is supported in $\prod O_m$ if $\operatorname{supp} \chi_m \subset O_m$ for every m.

A *move* from X is a pair (H, a) consisting of an affine coordinate hyperplane $H = \{x \cdot n = c\}$, $n \in \{\pm e_1, \pm e_2, \pm e_3\}$, with *margin* $d_H(X) := \min_m |x_m \cdot n - c| > 0$, and of a vector $a = s_1 n + v_1$ ($v_1 \perp n$). Its *forward group* is $F := \{m : x_m \cdot n > c\}$, and the move takes X to the configuration $X^a := (x_m + a 1_F(m))_m$. For a tuple χ, $s \in \mathbb{R}$ and $v \perp n$ we put $\chi_{(s,v)} := T^*_{-sn-v}\chi := (\tau^{1_F(m)}_{sn+v} \chi_m)_m$; this operation translates only the tests of the forward group. Finally, $V(X; r, \delta)$ denotes the statement that $D_N[\chi](\theta) = 0$ for all $|\theta| < \delta$ and all $\chi \in \mathcal{S}_N$ supported in $B_r(X)$.

The proofs of this subsection use Theorem 15.8 for the pairs (x_j, x_k) and (x_j, x_l), for every j, and hence hypothesis (iii) at the three coordinate planes and at the three planes $\{x_j + x_k = 0\}$, $j < k$ (Remark 15.4).

Proposition 15.37 (from narrow configurations to all distinct points)**.** *Assume (i), (ii) and (iii) for the nine planes, $N \geq 1$.*

(a) *Let $X^0 \in (\mathbb{R}^3)^N$, $H = \{x \cdot n = c\}$, $d := d_H(X^0) > 0$, $O := B_{d/2}(X^0)$, $\varepsilon := d/4$. For $\chi \in \mathcal{S}_N$ supported in O, $(s, v) \mapsto S_N(T^*_{-sn-v}\chi)$ is real-analytic on $(-\varepsilon, \infty) \times n^\perp$, the restriction of a bounded holomorphic function on $\{(s, v) \in \mathbb{C} \times n^\perp_{\mathbb{C}} : \operatorname{Re} s + \varepsilon > |\operatorname{Im} v|_1\}$.*

(b) *Let X have pairwise distinct planar projections $P(x_m)$, $N \geq 2$. There are $k \leq N(N+1)/2$ moves $X = X^0 \to X^1 \to \cdots \to X^k$ with $P(X^k)$ narrow, radii $r_i > 0$ and a $\delta > 0$ such that for $|\theta| < \delta$ and χ supported in $B_{r_i}(X^{i-1})$ both $(s, v) \mapsto S_N(\chi_{(s,v)})$ and $(s, v) \mapsto S_N(R(\theta)^*\chi_{(s,v)})$ are real-analytic on a connected open set $\Omega_{i,\theta}$ containing $(0, 0)$ and (s_i, v_i) (the i-th move).*

(c) *If for every $\varphi \in \mathbb{R}$ there are $r_\varphi, \delta_\varphi > 0$ with $V(R(\varphi)X; r_\varphi, \delta_\varphi)$, then $V(X; r_*, \infty)$ for some $r_* > 0$.*

(d) *If $V(X; r_X, \infty)$ holds at every X with pairwise distinct planar projections, it holds at every $X \notin \Delta := \{X : \exists m \neq m',\ x_m = x_{m'}\}$.*

Proof of (a). By (ii) and Lemma 15.6(e), S_N is translation invariant; by Lemma 15.7 it is θ_j^*-invariant for the coordinate reflections, and $\theta_j^*\tau_a = \tau_{\theta_j a}\theta_j^*$. Applying τ_{-cn} and, if $n = -e_j$, θ_j (which maps the forward group to the forward group), assume $H = \{x_j = 0\}$, $n = e_j$. Let $B := \{m : x^0_m \cdot e_j < 0\}$ and $F := \{m : x^0_m \cdot e_j > 0\}$; for χ supported in O, $\operatorname{supp}\chi_m \subset \{x_j < -d/2\}$ for $m \in B$ and $\subset \{x_j > d/2\}$ for $m \in F$. Put $A := \prod_{m\in B} F(\chi_m)$ and $G := \prod_{m\in F} F(\tau_{-\varepsilon e_j}\chi_m)$; by Lemma 15.6(b), Hölder and Lemma 15.13, $A \in L^2(\Sigma(\{x_j < 0\}), \mu)$ and $G \in L^2_{j,+} := L^2(\Sigma\{x_j > 0\}, \mu)$. Lemma 15.5 at the plane $\{x_j = 0\}$ ($u = e_j$) gives $\mathcal{H}_j$, the contraction $G' \mapsto [G']_j$, e^{-sH_j} with $H_j \geq 0$, the strongly continuous two-parameter unitary group $V_j(v)$, $v \perp e_j$, with $[U(se_j + v)G']_j = e^{-sH_j}V_j(v)[G']_j$ ($s \geq 0$) and $\mathbb{E}_\mu[A'G'] = \langle[\overline{\Theta_j A'}]_j, [G']_j\rangle_j$ for $A' \in L^2(\Sigma\{x_j < 0\})$, $G' \in L^2_{j,+}$ (the definition of the form, $\overline{\Theta_j A'} \in L^2_{j,+}$). For $s > -\varepsilon$, $U((s + \varepsilon)e_j + v)G = \prod_{m\in F} F(\tau_{se_j+v}\chi_m) \in L^2_{j,+}$, hence

$$S_N(\chi_{(s,v)}) = \mathbb{E}_\mu[A\,U((s + \varepsilon)e_j + v)G] = \langle\psi, e^{-(s+\varepsilon)H_j}V_j(v)\phi\rangle_j, \qquad \psi := [\overline{\Theta_j A}]_j,\ \phi := [G]_j.$$

e^{-sH_j} and $V_j(v)$ commute (translations commute on $L^2_{j,+}$), so H_j and the Stone generators P_k, P_l of V_j have a joint spectral measure Π_j on $\mathbb{R}^3$ (the spectral theorem and Stone's theorem [30]) whose (ω, p_k)- and (ω, p_l)-marginals are the joint spectral measures of (H_j, P_k) and (H_j, P_l). By Theorem 15.8 for the coordinate pairs (x_j, x_k) and (x_j, x_l), Π_j is carried by $\Gamma := \{\omega \geq |p_k|\} \cap \{\omega \geq |p_l|\} = \{\omega \geq |p|_\infty\}$. With $\rho := \langle\psi, \Pi_j(\cdot)\phi\rangle_j$, $|\rho|(\mathbb{R}^3) \leq \|A\|_{L^2(\mu)}\|G\|_{L^2(\mu)}$ and $S_N(\chi_{(s,v)}) = \int_\Gamma e^{-(s+\varepsilon)\omega + ip\cdot v}d\rho$. For complex (s, v) with $\operatorname{Re} s + \varepsilon > |\operatorname{Im} v|_1$ and $(\omega, p) \in \Gamma$: $|e^{-(s+\varepsilon)\omega+ip\cdot v}| \leq e^{-(\operatorname{Re} s+\varepsilon-|\operatorname{Im} v|_1)\omega} \leq 1$; the integral is continuous (dominated convergence), holomorphic in each variable (Morera and Fubini), hence holomorphic (Osgood), bounded by $\|A\|_2\|G\|_2$; its real restriction is real-analytic. □

Lemma 15.38 (propagation)**.** *Let (H, a) be a move from X, $r_0 > 0$, $\delta \in (0, \infty]$, and for each $|\theta| < \delta$ let $\Omega_\theta \subset \mathbb{R} \times n^\perp$ be connected and open, containing $(0, 0)$ and (s_1, v_1), such that $(s, v) \mapsto S_N(\chi_{(s,v)})$ and $(s, v) \mapsto S_N(R(\theta)^*\chi_{(s,v)})$ are real-analytic on Ω_θ for all χ supported in $B_{r_0}(X)$. If $V(X^a; r', \delta')$ then $V(X; \min(r_0, r'/2), \min(\delta, \delta'))$.*

Proof. Let χ be supported in $B_r(X)$, $r := \min(r_0, r'/2)$, and $|\theta| < \min(\delta, \delta')$. $\chi_{(s,v)}$ is supported in $B_r(X^{(s,v)})$, and $\max_m |x^{(s,v)}_m - x^a_m| = |(s, v) - (s_1, v_1)| < r'/2$ gives $B_r(X^{(s,v)}) \subset B_{r'}(X^a)$,

so $D_N[\chi_{(s,v)}](\theta) = 0$ on a nonempty open subset of Ω_θ; the real-analytic function $(s,v) \mapsto D_N[\chi_{(s,v)}](\theta)$ vanishes on Ω_θ (identity theorem), in particular at $(0,0)$. (If $F = \emptyset$ then $X^a = X$ and there is nothing to prove.) □

Lemma 15.39 (two classes of moves)**.** *Under the hypotheses of Proposition 15.37, let (H,a) be a move from X, and put $d = d_H(X)$.*

(M1) *If $n = \pm e_l$ and $a = v_1$ is planar, then the hypotheses of Lemma 15.38 hold with $r_0 = d/2$, $\delta = \infty$ and $\Omega_\theta = (-d/4, \infty) \times n^\perp$.*

(M2) *If $n \in \{\pm e_j, \pm e_k\}$, $a = we_l$ with $w \in \mathbb{R}$, and $\varrho_X := \max_m |P(x_m)| > 0$, then the hypotheses of Lemma 15.38 hold with $r_0 = d/4$ and $\delta := d/(2\varrho_X)$.*

Proof. Let F be the forward group of the move and B its complement. Since $R^*\tau_a = \tau_{R^{-1}a}R^*$, we have $R(\theta)^*\chi_{(s,v)} = (R(\theta)^*\chi)_{(s',v')}$ with $R(-\theta)(sn+v) = s'n + v'$, provided H separates $R(-\theta)X$ into the same groups F, B; and $R(\theta)^*\chi$ is supported in $B_r(R(-\theta)X)$ if χ is supported in $B_r(X)$.

(M1) The rotation $R(\theta)$ fixes x_l, so $R(-\theta)X$ has margin d and forward group F, and $s' = s$, $v' = R(-\theta)v$. By Proposition 15.37(a), applied to X and to $R(-\theta)X$, the hypotheses of Lemma 15.38 hold with the stated r_0, δ and Ω_θ.

(M2) For $|\theta| < \delta$ we have $|R(-\theta)x_m - x_m| \le |\theta|\varrho_X < d/2$, so H misses $R(-\theta)X$ with margin $d' > d/2$ and forward group F. With n' a planar unit vector perpendicular to n,

$$R(-\theta)(sn + v_x n' + v_y e_l) = (s\cos\theta \mp v_x \sin\theta)n + (\pm s\sin\theta + v_x\cos\theta)n' + v_y e_l.$$

By Proposition 15.37(a) for $R(-\theta)X$ (with $O' \supset B_{d/4}(R(-\theta)X)$ and $\varepsilon' > d/8$) and the linear substitution, $(s,v) \mapsto S_N(R(\theta)^*\chi_{(s,v)})$ is real-analytic on the half-space $\{s\cos\theta \mp v_x\sin\theta > -d/8\}$ for χ supported in $B_{d/4}(X)$. By Proposition 15.37(a) for X, the map $(s,v) \mapsto S_N(\chi_{(s,v)})$ is real-analytic on $\{s > -d/4\}$. The intersection Ω_θ of these two half-spaces is convex and contains $(0,0)$ and $(0, we_l)$, so the hypotheses of Lemma 15.38 hold with $r_0 = d/4$ and this δ. □

Proof of (b). Step 1. For each pair $m < m'$ choose $n_{mm'} \in \{e_j, e_k\}$ with $x_m \cdot n_{mm'} \ne x_{m'} \cdot n_{mm'}$, which is possible as $P(x_m) \ne P(x_{m'})$, and choose $c_{mm'}$ strictly between these two values and outside $\{x_i \cdot n_{mm'}\}_i$. The hyperplane $H_{mm'} := \{x \cdot n_{mm'} = c_{mm'}\}$ misses X; let $d_{mm'}$ be its margin and $F_{mm'}$ its forward group, which contains exactly one of m, m'. Perform, in any order, the moves $(H_{mm'}, w_{mm'}e_l)$ of type (M2), with coefficients $w_{mm'}$ to be chosen. These moves do not change the planar projections, so every $H_{mm'}$ keeps its margin and its forward group, and by Lemma 15.39 each of these moves satisfies the hypotheses of Lemma 15.38 for $|\theta| < \delta := \min_{m<m'} d_{mm'}/(2\varrho_X)$. The third coordinates after these moves are $y_i' = y_i + \sum_{m<m'} w_{mm'} 1_{F_{mm'}}(i)$. For $i \ne i'$ the set $\{w : y_i' = y_{i'}'\}$ is a proper affine hyperplane of $\mathbb{R}^{N(N-1)/2}$, because the coefficient of $w_{ii'}$ is ± 1. We choose w outside the finite union of these hyperplanes; the resulting configuration $X^{(1)}$ then has pairwise distinct third coordinates.

Step 2. Order the third coordinates of $X^{(1)}$ as $y'_{\sigma(1)} < \cdots < y'_{\sigma(N)}$, and put $a_m := (y'_{\sigma(m-1)} + y'_{\sigma(m)})/2$, $H'_m := \{x_l = a_m\}$ and $z_m := P(x_{\sigma(m)})$; the hyperplane H'_m has margin > 0 and forward group $\{\sigma(m), \dots, \sigma(N)\}$. Perform for $m = 2, \dots, N$ the moves (H'_m, v_m) of type (M1), where $v_m := e_j - (z_m - z_{m-1})$. These moves do not change the third coordinates, so each H'_m keeps its margin and its forward group. The final projection of the point with index $\sigma(m)$ is $z_m + \sum_{i=2}^m v_i = z_1 + (m-1)e_j$. Hence all successive differences equal e_j, and $P(X^k)$ is narrow (with $u = e_j$ and angle $0 < \pi/4$). By Lemma 15.39 all the moves satisfy the hypotheses of Lemma 15.38 for $|\theta| < \delta$, and their number is $k \le N(N-1)/2 + N - 1$. □

Proof of (c). If $V(R(\varphi_0)X; r_0, \delta_0)$ and $\max_m |R(\varphi)x_m - R(\varphi_0)x_m| < r_0/2$, then $B_{r_0/2}(R(\varphi)X) \subset B_{r_0}(R(\varphi_0)X)$, so $V(R(\varphi)X; r_0/2, \delta_0)$ on an open interval around φ_0. Compactness of $[0, 2\pi]$ and 2π-periodicity give $r_*, \delta_* > 0$ with $V(R(\varphi)X; r_*, \delta_*)$ for all φ. Let χ be supported in

$B_{r_*}(X)$, $\Phi(\varphi) := D_N[\chi](\varphi)$. Planar rotations commute, so $R(\varphi+\theta)^* = R(\theta)^*R(\varphi)^*$ and $\Phi(\varphi+\theta) = D_N[R(\varphi)^*\chi](\theta) + \Phi(\varphi)$. $R(\varphi)^*\chi$ is supported in $B_{r_*}(R(-\varphi)X)$, so the first term vanishes for $|\theta| < \delta_*$: $\Phi(\varphi+\theta) = \Phi(\varphi)$ for all φ, $|\theta| < \delta_*$; Φ is constant, $\Phi \equiv \Phi(0) = 0$. □

Proof of (d). Let $X \notin \Delta$, and let $y^{(1)} < \cdots < y^{(q)}$ be the distinct values of the third coordinates y_m of its points. If $q = 1$ the planar projections are distinct. Otherwise let $H_i := \{x_l = (y^{(i)} + y^{(i+1)})/2\}$, with forward group $F_i := \{m : y_m > y^{(i)}\}$, and perform the moves $(H_i, w_i e_j)$, $i < q$, of type (M1); they leave the third coordinates unchanged. The final planar projections are $z'_m = P(x_m) + (\sum_i w_i 1_{F_i}(m))e_j$. If $y_m = y_{m'}$, then $z'_m - z'_{m'} = P(x_m) - P(x_{m'}) \neq 0$. If $y_m = y^{(\alpha)} < y_{m'}$, then

$$\{w : z'_m = z'_{m'}\} \subset \{(P(x_m) - P(x_{m'}))\cdot e_j + \sum_i w_i(1_{F_i}(m) - 1_{F_i}(m')) = 0\},$$

and the set on the right is a proper affine hyperplane, because the coefficient of w_α is -1. Choose w outside the finite union of these hyperplanes. The resulting configuration X' has distinct planar projections, so $V(X'; r_{X'}, \infty)$ holds. Applying Lemma 15.38 to the $q-1$ moves in reverse order, for all θ since the moves of type (M1) have $\delta = \infty$ (Lemma 15.39), we obtain $V(X; r, \infty)$. □

15.15. **Assembly.**

Proof of Theorem 15.1. The case $N = 0$ is trivial and $N = 1$ is Lemma 15.7. Let $N \geq 2$; fix the coordinate pair (x_1, x_2), $l = 3$, and write $R(\theta)$ for the rotations about e_3.

(1) By Corollary 15.36, $D_N[\chi](\theta) = 0$ for all real θ and all $\chi \in \mathcal{S}_N$ with $\operatorname{supp}\chi_m \subset U_m \times \mathbb{R}$, U_m the planar balls of Proposition 15.21 about any narrow configuration Z. Hence $V(X; r, \infty)$ whenever $P(X)$ is narrow and r is small (take r below the radius of the balls).

(2) Let X have distinct planar projections. By Proposition 15.37(b) and (1), $V(X^k; r_k, \infty)$; Lemma 15.38 applied k times gives $V(X; r, \delta)$ for some $r, \delta > 0$. Every $R(\varphi)X$ has distinct planar projections, so Proposition 15.37(c) gives $V(X; r_*, \infty)$; Proposition 15.37(d) gives $V(X; r_X, \infty)$ for all $X \notin \Delta$.

(3) Let $\chi_m \in C_c^\infty(\mathbb{R}^3; T_{\mathbb{C}})$ have pairwise disjoint supports K_m. $K := \prod_m K_m \subset (\mathbb{R}^3)^N \setminus \Delta$ is compact, covered by finitely many $B_{r_{X^{(i)}}/2}(X^{(i)})$; with $r := \min_i r_{X^{(i)}}/2$, each $Y \in K$ has $B_r(Y) \subset B_{r_{X^{(i)}}}(X^{(i)})$ for some i. Smooth partitions of unity on K_m with supports in balls $B(y_m^{(\alpha)}, r)$, $y_m^{(\alpha)} \in K_m$, give $\chi_m = \sum_\alpha \chi_m^{(\alpha)}$; by multilinearity $D_N[\chi](\theta) = \sum D_N[(\chi_1^{(\alpha_1)}, \ldots, \chi_N^{(\alpha_N)})](\theta)$, each tuple supported in some $B_r(Y)$, $Y \in K$, hence each term vanishes: $S_N(R(\theta)^*\chi) = S_N(\chi)$ for all rotations about e_3 and all θ.

(4) Other axes: for a cyclic permutation matrix $Q \in B_3$, $QR_{e_3}(\theta)Q^{-1} = R_{Qe_3}(\theta)$, $(QRQ^{-1})^* = (Q^{-1})^*R^*Q^*$, Q^* preserves pairwise disjointness of supports, and $S_N(Q^*\psi) = S_N(\psi)$ (Lemmas 15.7, 15.6(e)): $S_N((QRQ^{-1})^*\chi) = S_N(R^*Q^*\chi) = S_N(Q^*\chi) = S_N(\chi)$. So (3) holds for the rotations about e_1 and e_2 as well.

(5) Euler angles: every $R \in SO(3)$ is $R_{e_3}(\alpha)R_{e_1}(\beta)R_{e_3}(\gamma)$, so $R^* = R_{e_3}(\gamma)^*R_{e_1}(\beta)^*R_{e_3}(\alpha)^*$, each factor preserving pairwise disjointness: $S_N(R^*\chi) = S_N(\chi)$ on $SO(3)$. Since $O(3) = SO(3) \cup \theta_1 SO(3)$ and $(\theta_1 R)^* = R^*\theta_1^*$, Lemma 15.7 extends the identity to $O(3)$.

(6) Schwartz tests: for $\chi_m \in \mathcal{S}$ with pairwise disjoint supports, $\chi_m^{(q)} := \vartheta(\cdot/q)\chi_m$ ($\vartheta \in C_c^\infty$, $\vartheta = 1$ on $B(0,1)$) have compact pairwise disjoint supports, and $\chi_m^{(q)} \to \chi_m$, $R^*\chi_m^{(q)} \to R^*\chi_m$ in $L^1 \cap L^2$; the continuity of Lemma 15.6(c) gives $S_N(R^*\chi) = \lim_q S_N(R^*\chi^{(q)}) = \lim_q S_N(\chi^{(q)}) = S_N(\chi)$. This is the theorem; the extension from real to complex Schwartz tests is by multilinearity. □

Remark 15.40 (constants and the planes used). The constants of the proof are explicit in terms of the constant C of (i): $K_1 = 17(1+C)$ (Lemma 15.6), $K_6 = e^{2C}$ (Theorem 15.15),

$K_7 = 14K_6$ (Theorem 15.16), $K_8 = 2K_7$ (Proposition 15.19), and the geometric thresholds $r_0 = \frac{1}{2}\min_j |d_j|\cos(\gamma+\pi/4)$ (Proposition 15.21), $\varepsilon = d/4$, $\delta = d/(2\varrho_X)$ (Proposition 15.37). Subsections 15.2–15.13 use, for one coordinate pair, only the four planes of that pair (Lemma 15.34); Subsection 15.14 uses Theorem 15.8 for the six ordered coordinate pairs, i.e. the three coordinate planes and the three diagonal planes $\{x_j + x_k = 0\}$; the three planes $\{x_j - x_k = 0\}$ enter through Proposition 15.33 (the plane P_3 of each pair) and through the B_3-invariance of Lemma 15.7 used in step (4) of the assembly (a 3-cycle is a product of two transposition reflections); step (5) of the assembly uses in addition the reflection θ_1. No property of μ beyond (i)–(iii) enters.

Remark 15.41. The statement of Theorem 15.1 is Theorem A of [35]; in proving it we implement the original strategy of that work (Remark 12.1).

Appendix A. Probability measures on spaces of distributions

This appendix contains the measure theory used in Theorems 9.13 and 9.15: the setting (H1)–(H3) in which reflection positivity of the law of a field is formulated; the field spaces and the action of the Euclidean group on them; Hermite–Sobolev norms; the Borel structure of the field spaces; the density of generating algebras; the passage from a measure on $\mathcal{S}'$ to a measure on a Hermite–Sobolev space; the uniqueness of a measure with a given characteristic functional or with given moments; and the verification of (H1)–(H3) from properties of the characteristic functional. Nothing in it uses the lattice. Remark A.12 states where its hypotheses are met in the main text.

The Bochner–Minlos theorem is used in the form stated after Definition A.3. Definition A.1 fixes two conventions. Subsection A.6 contains the results on the ergodic decomposition and on clustering that are used in Theorem 9.15.

Definition A.1 (conventions: the weighted C^4 norm and the transport of test functions)**.** (a) *The weighted C^4 norm.* For a function q and a 2-form g on $\mathbb{R}^3$: $|q|_4 := \sup_{x,|\gamma|\le 4}(1+|x|^2)^2|D^\gamma q(x)|$ and $|g|_4 := (\sum_{\alpha<\beta}|g_{\alpha\beta}|_4^2)^{1/2}$; on scalar functions this is the norm $|q|_p$ of [29] with $p=4$. For a pair $u=(g,q)$ of a 2-form and a function, $|u|_4 := (|g|_4^2+|q|_4^2)^{1/2}$. (b) *The transport of test functions.* For a function h and a 2-form f on $\mathbb{R}^3$ and for $O \in O(3)$ put $(O\cdot h)(x) := h(O^{-1}x)$ and $(O\cdot f)_x(u,v) := f_{O^{-1}x}(O^{-1}u, O^{-1}v)$; for $b\in\mathbb{R}^3$ put $\tau_b h := h(\cdot - b)$ and $(\tau_b f)_x := f_{x-b}$. For $\nu\in\{1,2,3\}$ and $c\in\mathbb{R}$ let $\rho=\rho_{\nu,c}$ be the map $x\mapsto x-2(x_\nu-c)e_\nu$, and put $\theta h := h\circ\rho$ and $\theta f := \rho^* f$, where $(\rho^* f)_x(u,v) := f_{\rho x}(\rho u,\rho v)$ (the components $f_{\mu\nu}$, $\mu\neq\nu$, change sign).

A.1. **The abstract setting.** *Notation.* The standard basis of $\mathbb{R}^3$ is e_1,e_2,e_3. For $j\in\{1,2,3\}$ and $c\in\mathbb{R}$ put $H^{\pm}_{j,c} := \{x\in\mathbb{R}^3 : \pm(x_j-c)>0\}$ and $\rho_j x := x - 2x_j e_j$, and let $\rho_{j,c}$ be the reflection in the plane $\{x_j=c\}$. A bounded open subset of $\mathbb{R}^3$ is denoted K; $K \Subset U$ means that $\overline{K}\subset U$.

(H1) (*Space, actions, measure.*) X is a Polish space with a jointly continuous action $(a,x)\mapsto \tau_a x$ of $\mathbb{R}^3$ and three involutive homeomorphisms $\theta_1,\theta_2,\theta_3$ of X with $\theta_j\tau_a = \tau_{\rho_j a}\theta_j$. μ is a Borel probability measure on X invariant under every τ_a and every θ_j.

(H2) (*Local structure.*) To every open $U\subset\mathbb{R}^3$ is assigned a sub-σ-algebra $\Sigma(U)$ of the Borel σ-algebra $\mathcal{B}$ of X, such that: $\Sigma(U)\subset\Sigma(U')$ if $U\subset U'$; $\Sigma(U) = \bigvee_{K\Subset U}\Sigma(K)$; $\bigvee_K \Sigma(K) = \mathcal{B}$; each $\Sigma(U)$ is generated by a countable family of bounded continuous functions on X; and, with

$$U(a)G := G\circ\tau_{-a}, \qquad \Theta_j G := G\circ\theta_j, \tag{A.1}$$

$U(a)$ maps $\Sigma(U)$-measurable functions to $\Sigma(U+a)$-measurable ones and Θ_j maps $\Sigma(U)$-measurable functions to $\Sigma(\rho_j U)$-measurable ones.

(H3) (*Coordinate reflection positivity.*) For every j, every $c \in \mathbb{R}$ and every $F \in L^2(\Sigma(H^+_{j,c}), \mu)$,

$$\mathbb{E}_\mu\big[\,\overline{F \circ \theta_{j,c}}\, F\,\big] \;\geq\; 0, \qquad \theta_{j,c} := \tau_{ce_j}\theta_j\tau_{-ce_j}. \tag{A.2}$$

By (H1), $a \mapsto U(a)$ is a unitary representation of $\mathbb{R}^3$ on $L^2(\mu)$, Θ_j is a unitary involution, and

$$\Theta_j U(a) = U(\rho_j a)\Theta_j; \quad \text{in particular } \Theta_j U(te_j) = U(-te_j)\Theta_j, \;\; \Theta_j U(y) = U(y)\Theta_j \;\; (y \perp e_j). \tag{A.3}$$

By translation invariance, (A.2) for $c = 0$ implies it for every c: if $F \in L^2(\Sigma(H^+_{j,c}), \mu)$, then $F_0 := U(-ce_j)F = F \circ \tau_{ce_j}$ is $\Sigma(H^+_{j,0})$-measurable, $\overline{F \circ \theta_{j,c}}\, F = (\overline{F_0 \circ \theta_j}\, F_0) \circ \tau_{-ce_j}$, and τ_{-ce_j} preserves μ, so $\mathbb{E}_\mu[\overline{F \circ \theta_{j,c}}\, F] = \mathbb{E}_\mu[\overline{F_0 \circ \theta_j}\, F_0]$.

Remark A.2 (conventions). On a space of distributions we use $(\tau_a\Phi)(u) := \Phi(\tau_{-a}u)$ with $\tau_b u = u(\cdot - b)$ as in Definition A.1(b): this is the push-forward of Φ by $x \mapsto x + a$. Then $U(a)[\Phi(u)] = \Phi(\tau_a u)$, whose test function is supported in $\operatorname{supp} u + a$, as (H2) requires.

A.2. **Field spaces, the Euclidean action and Hermite norms.** *Fibre and transport.* $\mathcal{V}$ is $\Lambda^2 := \Lambda^2\mathbb{R}^3$ with its Euclidean fibre norm, or $\Lambda^2 \oplus \mathbb{R}$; $(e_c)_{c \leq \dim \mathcal{V}}$ is an orthonormal basis of $\mathcal{V}$, and $\langle u, v\rangle := \int_{\mathbb{R}^3} u(x) \cdot v(x)\, dx$ is the L^2 pairing of $\mathcal{V}$-valued functions. All test functions are real. A Euclidean motion $\gamma \in \mathbb{R}^3 \rtimes O(3)$, $\gamma x = Ox + b$, acts on $\mathcal{V}$-valued functions by $\gamma \cdot u := \tau_b(O \cdot u)$, with the transport $O\cdot$ of Definition A.1(b) on each summand. Thus $(\gamma \cdot u)(x) = \pi(O)u(\gamma^{-1}x)$, where $\pi(O)\omega = \omega(O^{-1}\cdot, O^{-1}\cdot)$ on Λ^2 and $\pi(O) = 1$ on $\mathbb{R}$; π is an orthogonal representation of $O(3)$ on $\mathcal{V}$, because the fibre norm of a 2-form is the Frobenius norm of an antisymmetric matrix, up to a factor. Hence $\gamma \mapsto \gamma\cdot$ is an action of $\mathbb{R}^3 \rtimes O(3)$, $\operatorname{supp}(\gamma \cdot u) = \gamma \operatorname{supp} u$, and $\gamma\cdot$ preserves C_c^∞, $\mathcal{S}$, the L^2 pairing and the Sobolev norms $\|u\|^2_{H^s} := \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2\, d\xi$, $s \in \mathbb{N}_0$; and $\gamma \mapsto \gamma \cdot u$ is continuous into L^2 for $u \in C_c^\infty$ (uniform continuity of u). The coordinate reflection $\sigma_j \cdot u$ is the case $\gamma = \rho_j$, and the reflected source $\rho_{j,c*}u$ is $\rho_{j,c} \cdot u$, which is θu of Definition A.1(b) for $\rho = \rho_{j,c}$.

Hermite norms. $\mathcal{A} := -\Delta + |x|^2 + 1$ acts componentwise. The Hermite functions h_α, $\alpha \in \mathbb{N}_0^3$, form an orthonormal basis of $L^2(\mathbb{R}^3)$, and $\mathcal{A}h_\alpha = \lambda_\alpha h_\alpha$ with $\lambda_\alpha = 2|\alpha| + 4$; put $h_{\alpha,c} := h_\alpha e_c$. For $k \in 2\mathbb{N}_0$ and $u \in \mathcal{S}(\mathbb{R}^3; \mathcal{V})$,

$$\|u\|_k := \|\mathcal{A}^{k/2}u\|_{L^2} = \Big(\sum_{\alpha,c} \lambda_\alpha^k \langle h_{\alpha,c}, u\rangle^2\Big)^{1/2},$$

since $\mathcal{A}$ is symmetric on $\mathcal{S}$. $\mathcal{H}^k$ is the completion of $\mathcal{S}(\mathbb{R}^3; \mathcal{V})$ under $\|\cdot\|_k$, and $\mathcal{H}^{-k}$ is its dual, a separable Hilbert space with $\|\Psi\|^2_{-k} = \sum_{\alpha,c} \lambda_\alpha^{-k}\Psi(h_{\alpha,c})^2$, because $(\lambda_\alpha^{-k/2}h_{\alpha,c})$ is an orthonormal basis of $\mathcal{H}^k$ (Lemma A.4(c)). These are the spaces used in Theorem 9.13.

Definition A.3 (the field spaces)**.** Fix an even integer $r \geq 2$ (in the applications $r \geq 14$). $X := H^{-r}_{\mathrm{loc}}(\mathbb{R}^3; \mathcal{V})$ is the space of real linear functionals Φ on $C_c^\infty(\mathbb{R}^3; \mathcal{V})$ such that, for every $\chi \in C_c^\infty(\mathbb{R}^3)$, $u \mapsto \Phi(\chi u)$ is bounded for $\|\cdot\|_{H^r}$, that is, $\chi\Phi \in H^{-r}(\mathbb{R}^3; \mathcal{V})$, the dual of $H^r(\mathbb{R}^3; \mathcal{V})$. Fix $\chi_n \in C_c^\infty(\mathbb{R}^3)$ with $0 \leq \chi_n \leq 1$, $\chi_n = 1$ on the ball B_n and $\operatorname{supp}\chi_n \subset B_{n+1}$, and give X the seminorms $p_n(\Phi) := \|\chi_n\Phi\|_{H^{-r}}$. Since $\chi_n u = u$ when $\operatorname{supp} u \subset B_n$,

$$|\Phi(u)| \leq p_n(\Phi)\|u\|_{H^r} \qquad (u \in C_c^\infty(B_n; \mathcal{V})). \tag{A.4}$$

For $\gamma \in \mathbb{R}^3 \rtimes O(3)$ put $(\gamma\Phi)(u) := \Phi(\gamma^{-1} \cdot u)$ on X and $(\gamma\Psi)(u) := \Psi(\gamma^{-1} \cdot u)$, $u \in \mathcal{H}^r$, on $\mathcal{H}^{-r}$ (with $\gamma^{-1}\cdot$ extended to $\mathcal{H}^r$ by Lemma A.4(e)). In particular τ_a is $\gamma = (x \mapsto x + a)$, so $(\tau_a\Phi)(u) = \Phi(\tau_{-a}u)$; $\theta_j := \rho_j$, so $(\theta_j\Phi)(u) = \Phi(\sigma_j \cdot u)$; $\theta_{j,c} = \tau_{ce_j}\theta_j\tau_{-ce_j}$ is $\rho_{j,c}$; and $O_h \subset O(3)$ acts as a subgroup. Since $\gamma \mapsto \gamma\cdot$ is an action, so is $\gamma \mapsto \gamma$ on X and on $\mathcal{H}^{-r}$; the relations $\theta_j^2 = 1$, $\theta_j\tau_a = \tau_{\rho_j a}\theta_j$ and $O\tau_a O^{-1} = \tau_{Oa}$ ($O \in O(3)$) are relations in $\mathbb{R}^3 \rtimes O(3)$.

The local σ-algebras are $\Sigma(U) := \sigma(\Phi \mapsto \Phi(u) : u \in C_c^\infty(U;\mathcal{V}))$ on X and $\Sigma^H(U) := \sigma(\Psi \mapsto \Psi(u) : u \in C_c^\infty(U;\mathcal{V}))$ on $\mathcal{H}^{-r}$, for open $U \subset \mathbb{R}^3$. The restriction map is $\iota : \mathcal{H}^{-r} \to X$, $\iota\Psi := \Psi|_{C_c^\infty}$. On $\mathcal{S}'(\mathbb{R}^3;\mathcal{V})$ the cylinder σ-algebra is generated by $\Phi \mapsto \Phi(u)$, $u \in \mathcal{S}(\mathbb{R}^3;\mathcal{V})$.

We recall the Bochner–Minlos theorem [27], [15, Chapter IV], [19] in the form used in the proof of Theorem 9.13. Let $\mathcal{Z} : \mathcal{S}(\mathbb{R}^3;\mathcal{V}) \to \mathbb{C}$ be continuous for the Schwartz topology, positive definite ($\sum_{k,l\le N} \bar a_k a_l \mathcal{Z}(u_l - u_k) \ge 0$ for all N, all $u_1,\dots,u_N \in \mathcal{S}(\mathbb{R}^3;\mathcal{V})$ and all $a \in \mathbb{C}^N$) and normalised ($\mathcal{Z}(0) = 1$). Then there is exactly one probability measure μ' on $\mathcal{S}'(\mathbb{R}^3;\mathcal{V})$ with its cylinder σ-algebra such that $\int e^{\mathrm{i}\Phi(u)}\, d\mu'(\Phi) = \mathcal{Z}(u)$ for all $u \in \mathcal{S}(\mathbb{R}^3;\mathcal{V})$.

Lemma A.4 (Hermite norms)**.** *Let $k \in 2\mathbb{N}$.*

(a) *For $|\delta| + |\gamma| \le k$ and $u \in \mathcal{S}(\mathbb{R}^3;\mathcal{V})$, $\|x^\delta \partial^\gamma u\|_{L^2} \le k^{k/2}\|u\|_k$. Conversely $\|u\|_k \le C_k \sum_{|\delta|+|\gamma|\le k} \|x^\delta\partial^\gamma u\|_{L^2}$.*

(b) *$|u|_4 \le C\|u\|_{10}$ for $u \in \mathcal{S}$, $|\cdot|_4$ the weighted C^4 norm of Definition A.1(a), taken componentwise.*

(c) *For $u \in \mathcal{S}$ the Hermite expansion $\sum_{\alpha,c}\langle h_{\alpha,c}, u\rangle\, h_{\alpha,c}$ converges to u in $\mathcal{S}$. C_c^∞ is dense in $\mathcal{H}^k$, and $(\lambda_\alpha^{-k/2} h_{\alpha,c})$ is an orthonormal basis of $\mathcal{H}^k$.*

(d) *$\|u\|_k \le C_{k,n}\|u\|_{H^k}$ for $u \in C_c^\infty(B_n;\mathcal{V})$, and $\|\chi u\|_k \le C_{k,\chi}\|u\|_{H^k}$ for $\chi \in C_c^\infty(\mathbb{R}^3)$, $u \in \mathcal{S}$.*

(e) *For $\gamma x = Ox + b$ and $u \in \mathcal{S}$, $\|\gamma \cdot u\|_k \le C_k(1+|b|)^k\|u\|_k$, with equality and $C_k = 1$ when $b = 0$. So $\gamma\cdot$ extends to a bounded operator on $\mathcal{H}^k$, and γ acts on $\mathcal{H}^{-k}$ with norm $\le C_k(1+|b|)^k$, unitarily if $b = 0$.*

Proof. (a) In each coordinate, $x_j = (a_j + a_j^*)/\sqrt{2}$ and $\partial_j = (a_j - a_j^*)/\sqrt{2}$ on $\mathcal{S}$, where $a_j h_\alpha = \sqrt{\alpha_j}\, h_{\alpha - e_j}$, $a_j^* h_\alpha = \sqrt{\alpha_j + 1}\, h_{\alpha + e_j}$, and a_j, a_j^* are adjoint to each other on $\mathcal{S}$. So $x^\delta\partial^\gamma$ is a sum of 2^ℓ words of length $\ell := |\delta| + |\gamma|$ in the $a_j^\#$, each with coefficient $\pm 2^{-\ell/2}$. A word w maps h_α to $c_w(\alpha)h_{\alpha+v_w}$ (or to 0) for a fixed shift v_w, with $|c_w(\alpha)| \le (|\alpha| + \ell)^{\ell/2} \le (\ell\lambda_\alpha/2)^{\ell/2}$, since each factor multiplies by at most the square root of 1 plus the current $|\alpha|$, and $|\alpha| + \ell \le \ell(|\alpha| + 2)$. Computing the Hermite coefficients of wu by duality on $\mathcal{S}$, $wu = \sum_{\alpha,c} c_w(\alpha)\langle h_{\alpha,c}, u\rangle\, h_{\alpha+v_w,c}$, distinct α going to distinct basis vectors. Hence $\|wu\|^2 \le (\ell/2)^\ell \sum \lambda_\alpha^\ell \langle h_{\alpha,c}, u\rangle^2 \le (\ell/2)^\ell \|u\|_k^2$ for $\ell \le k$, since $\lambda_\alpha \ge 1$, and

$$\|x^\delta\partial^\gamma u\| \le 2^\ell 2^{-\ell/2}(\ell/2)^{\ell/2}\|u\|_k = \ell^{\ell/2}\|u\|_k \le k^{k/2}\|u\|_k.$$

Conversely, $\mathcal{A}^{k/2}$ is a differential operator with polynomial coefficients; moving derivatives to the right with $\partial_j x_j = x_j\partial_j + 1$ writes it as a finite sum of terms $c\, x^\delta\partial^\gamma$ with $|\delta| + |\gamma| \le k$.

(b) Let $|\gamma| \le 4$ and $w := (1 + |x|^2)^2 D^\gamma u_c$ for a component u_c. The Sobolev embedding $H^2(\mathbb{R}^3) \subset L^\infty$ gives $\|w\|_\infty \le C\sum_{|\beta|\le 2}\|\partial^\beta w\|_{L^2}$, and each $\partial^\beta w$ is a finite sum of terms $x^\delta\partial^{\gamma'}u_c$ with $|\delta| \le 4$, $|\gamma'| \le 6$. Apply (a) with $k = 10$.

(c) For $u \in \mathcal{S}$ and $m \in \mathbb{N}$, $\mathcal{A}^m u \in \mathcal{S}$ and $\langle h_{\alpha,c}, \mathcal{A}^m u\rangle = \lambda_\alpha^m\langle h_{\alpha,c}, u\rangle$, so $\sum \lambda_\alpha^{2m}\langle h_{\alpha,c}, u\rangle^2 = \|\mathcal{A}^m u\|^2 < \infty$. Let $P_N u$ be the sum over $|\alpha| \le N$. Then $u - P_N u \in \mathcal{S}$ and $\|u - P_N u\|_k^2 = \sum_{|\alpha|>N} \lambda_\alpha^k\langle h_{\alpha,c}, u\rangle^2 \to 0$ for every k; by (a), $x^\delta\partial^\gamma(u - P_N u) \to 0$ in L^2 for all δ, γ, and by the Sobolev embedding as in (b), $\sup|x^\delta\partial^\gamma(u - P_N u)| \to 0$: this is convergence in $\mathcal{S}$. C_c^∞ is dense in $\mathcal{S}$, whose topology is finer than that of $\|\cdot\|_k$ by the converse in (a), and $\mathcal{S}$ is dense in $\mathcal{H}^k$. The family $(\lambda_\alpha^{-k/2}h_{\alpha,c})$ is orthonormal for $\|\cdot\|_k$, and its span is dense in $\mathcal{S}$, hence in $\mathcal{H}^k$.

(d) By the converse in (a),

$$\|u\|_k \le C_k \sum \|x^\delta\partial^\gamma u\|_{L^2} \le C_k \sum n^{|\delta|}\|\partial^\gamma u\|_{L^2} \le C_{k,n}\|u\|_{H^k}$$

if $\operatorname{supp} u \subset B_n$. For χu, Leibniz writes $x^\delta\partial^\gamma(\chi u)$ as a sum of terms $(x^\delta\partial^{\gamma-\gamma'}\chi)\,\partial^{\gamma'}u$ with bounded first factors.

(e) If $b = 0$, $\gamma\cdot$ is unitary on L^2 and commutes with $\mathcal{A}$ (Δ and $|x|^2$ are $O(3)$-invariant and $\pi(O)$ acts on the fibre), so $\|O\cdot u\|_k = \|u\|_k$. For a translation, $x^\delta\partial^\gamma(\tau_b u) = \tau_b\big((y+b)^\delta\partial^\gamma u\big)$, and expanding $(y+b)^\delta$ gives

$$\|x^\delta\partial^\gamma\tau_b u\|_{L^2} \le 2^{|\delta|}(1+|b|)^{|\delta|}\max_{\delta'\le\delta}\|y^{\delta'}\partial^\gamma u\|_{L^2};$$

the two halves of (a) give the bound. The last statements follow by density and duality. □

A.3. **Borel structure and the Euclidean action.**

Lemma A.5 (Borel structure of the field spaces)**.** *Let $r \ge 2$ be even, and X, $\mathcal{H}^{-r}$, Σ, Σ^H, ι as in Definition A.3.*

(a) *X is a separable Fréchet space, hence Polish; $\mathcal{H}^{-r}$ is a separable Hilbert space.*

(b) *For every open $U \subset \mathbb{R}^3$ there is a countable set $D_U \subset C_c^\infty(U;\mathcal{V})$ such that every $u \in C_c^\infty(U;\mathcal{V})$ is the limit in H^r of a sequence (u_k) in D_U with supports in a fixed compact subset of U. For any such sequence, $\Phi(u_k)\to\Phi(u)$ for every $\Phi\in X$ and $\Psi(u_k)\to\Psi(u)$ for every $\Psi\in\mathcal{H}^{-r}$. Hence $\Sigma(U)$, respectively $\Sigma^H(U)$, is generated by the countably many bounded continuous functions* $\arctan\Phi(u)$, *respectively* $\arctan\Psi(u)$, *$u\in D_U$.*

(c) *There are countable sets $D_X, D_H \subset C_c^\infty(\mathbb{R}^3;\mathcal{V})$ such that the Borel σ-algebra of X is generated by $\Phi\mapsto\Phi(u)$, $u\in D_X$, and that of $\mathcal{H}^{-r}$ by $\Psi\mapsto\Psi(u)$, $u\in D_H$. In particular both Borel σ-algebras equal $\Sigma(\mathbb{R}^3) = \bigvee_K\Sigma(K)$, respectively $\Sigma^H(\mathbb{R}^3) = \bigvee_K\Sigma^H(K)$.*

(d) *ι is continuous, linear and injective; $\iota\mathcal{H}^{-r}$ is a Borel subset of X, and ι is a Borel isomorphism of $\mathcal{H}^{-r}$ onto it. Moreover $\iota\gamma = \gamma\iota$ for $\gamma\in\mathbb{R}^3\rtimes O(3)$, and $\Sigma^H(U) = \iota^{-1}\Sigma(U)$.*

Proof. (a) X is a locally convex space with the countable family of seminorms p_n, which separates points by (A.4), so it is metrizable, with the metric $d(\Phi,\Phi') := \sum_n 2^{-n}\min(1, p_n(\Phi-\Phi'))$. It is complete: if (Φ_k) is Cauchy, then $\chi_n\Phi_k\to\Xi_n$ in H^{-r} for every n; put $\Phi(u) := \Xi_n(u)$ for $\operatorname{supp} u\subset B_n$, which is consistent because $(\chi_n\Phi_k)(u) = \Phi_k(u)$ for such u; then $|\Phi(\chi u)| \le \|\Xi_n\|_{H^{-r}}\|\chi u\|_{H^r}$ for $\operatorname{supp}\chi\subset B_n$, so $\Phi\in X$, and $p_n(\Phi_k-\Phi) = \sup_{\|g\|_{H^r}\le1}|(\chi_{n+1}\Phi_k - \Xi_{n+1})(\chi_n g)|\to 0$. It is separable, because $\Phi\mapsto(\chi_n\Phi)_n$ is a homeomorphism of X onto a subspace of the separable metrizable space $\prod_n H^{-r}(\mathbb{R}^3;\mathcal{V})$. $\mathcal{H}^{-r}$ has the countable orthonormal basis dual to that of Lemma A.4(c).

(b) Choose compact sets $K_1\subset K_2\subset\dots$ with $K_m\subset\operatorname{int}K_{m+1}$ and $\bigcup_m\operatorname{int}K_m = U$, and a countable subset D_m of $C_c^\infty(\operatorname{int}K_m;\mathcal{V})$ dense in it for $\|\cdot\|_{H^r}$ (a subset of a separable metric space is separable). Put $D_U := \bigcup_m D_m$. A $u\in C_c^\infty(U;\mathcal{V})$ has support in some $\operatorname{int}K_m$, and is the H^r-limit of a sequence in D_m, with supports in K_m. If $K_m\subset B_n$, then $|\Phi(u_k)-\Phi(u)|\le p_n(\Phi)\|u_k-u\|_{H^r}$ by (A.4), and $|\Psi(u_k)-\Psi(u)|\le\|\Psi\|_{-r}C_{r,n}\|u_k-u\|_{H^r}$ by Lemma A.4(d). So every generator $\Phi\mapsto\Phi(u)$ of $\Sigma(U)$ is a pointwise limit of the functions $\Phi\mapsto\Phi(u_k)$, $u_k\in D_U$, and $\Sigma(U) = \sigma(\Phi(u) : u\in D_U) = \sigma(\arctan\Phi(u) : u\in D_U)$; the same holds for $\Sigma^H(U)$. The functions are continuous by (A.4), respectively by $|\Psi(u)|\le\|\Psi\|_{-r}\|u\|_r$.

(c) Let $D\subset C_c^\infty(\mathbb{R}^3;\mathcal{V})$ be countable and dense in the unit ball of $H^r(\mathbb{R}^3;\mathcal{V})$ (C_c^∞ is dense in H^r, which is separable). For $\Xi\in X$, $\chi_n\Xi\in H^{-r}$ is continuous on H^r, so $p_n(\Xi) = \sup_{g\in D}|\Xi(\chi_n g)|$. Put $D_X := \{\chi_n g : n\in\mathbb{N},\ g\in D\}$. For fixed Φ_0, $\Phi\mapsto d(\Phi,\Phi_0)$ is then measurable for $\sigma(\Phi(u) : u\in D_X)$. As X is separable, every open set is a countable union of d-balls with centres in a countable dense set, so the Borel σ-algebra is contained in $\sigma(\Phi(u) : u\in D_X)\subset\Sigma(\mathbb{R}^3)$, and the evaluations are continuous. For $\mathcal{H}^{-r}$, let $D_H\subset C_c^\infty$ be countable and dense in the unit ball of $\mathcal{H}^r$ (Lemma A.4(c)); then $\|\Psi-\Psi_0\|_{-r} = \sup_{g\in D_H}|\Psi(g)-\Psi_0(g)|$, and the same argument applies. Finally, every $u\in C_c^\infty$ is supported in a bounded open K, so $\Sigma(\mathbb{R}^3) = \bigvee_K\Sigma(K)$.

(d) For $\chi\in C_c^\infty(\mathbb{R}^3)$ and $u\in C_c^\infty$, $|\Psi(\chi u)|\le\|\Psi\|_{-r}\|\chi u\|_r\le C_{r,\chi}\|\Psi\|_{-r}\|u\|_{H^r}$ by Lemma A.4(d); so $\iota\Psi\in X$ and $p_n(\iota\Psi)\le C_{r,\chi_n}\|\Psi\|_{-r}$: ι is continuous and linear. It is

injective because C_c^∞ is dense in $\mathcal{H}^r$. Let $D_* := \bigcup_n D_{B_n} \setminus \{0\}$, with D_{B_n} from (b), and $N(\Phi) := \sup_{g\in D_*} |\Phi(g)|/\|g\|_r$, a measurable function on X. If $\Phi = \iota\Psi$, then $N(\Phi) \le \|\Psi\|_{-r}$. If $N(\Phi) < \infty$ and $u \in C_c^\infty(B_n;\mathcal{V})$, choose $u_k \in D_{B_n}$ as in (b); then $\Phi(u_k) \to \Phi(u)$ and $\|u_k\|_r \to \|u\|_r$ by Lemma A.4(d), so $|\Phi(u)| \le N(\Phi)\|u\|_r$; as C_c^∞ is dense in $\mathcal{H}^r$, Φ extends to some $\Psi \in \mathcal{H}^{-r}$ with $\iota\Psi = \Phi$. So $\iota\mathcal{H}^{-r} = \{N < \infty\}$ is Borel. The class of Borel sets $B \subset \mathcal{H}^{-r}$ with ιB Borel is a σ-algebra (ι is injective and $\iota\mathcal{H}^{-r}$ is Borel), and it contains the generators $\{\Psi : \Psi(g) \in A\}$, $g \in D_H$, of (c), whose images are $\{\Phi \in \iota\mathcal{H}^{-r} : \Phi(g) \in A\}$. So ι^{-1} is Borel on $\iota\mathcal{H}^{-r}$. For $u \in C_c^\infty$, $(\iota\gamma\Psi)(u) = \Psi(\gamma^{-1}\cdot u) = (\gamma\iota\Psi)(u)$, because $\gamma^{-1}\cdot u \in C_c^\infty$. Since $(\iota\Psi)(u) = \Psi(u)$, the preimages under ι of the generators of $\Sigma(U)$ are the generators of $\Sigma^H(U)$, so $\Sigma^H(U) = \iota^{-1}\Sigma(U)$. □

Lemma A.6 (the Euclidean action and (H2))**.** *Let $r \ge 2$ be even.*

(a) *For every $\gamma \in \mathbb{R}^3 \rtimes O(3)$, $\Phi \mapsto \gamma\Phi$ is a linear homeomorphism of X, and $\mathbb{R}^3 \rtimes O(3) \times X \to X$, $(\gamma,\Phi) \mapsto \gamma\Phi$, is jointly continuous. The same holds on $\mathcal{H}^{-r}$. In particular X and $\mathcal{H}^{-r}$, with τ and the involutions θ_j, have the structural properties required in* (H1)*, and every $\theta_{j,c}$ is an involutive homeomorphism.*

(b) *(X,Σ) and $(\mathcal{H}^{-r},\Sigma^H)$ satisfy* (H2)*. Moreover, for every $\gamma \in \mathbb{R}^3 \rtimes O(3)$ and open U, $G \mapsto G\circ\gamma^{-1}$ maps $\Sigma(U)$-measurable functions to $\Sigma(\gamma U)$-measurable ones, and likewise for Σ^H.*

Proof. (a) *Equicontinuity on X.* Let $\gamma x = Ox + b$ with $|b| \le \beta$, and let c_n be the norm of multiplication by χ_n on H^r. For $g \in C_c^\infty$ with $\|g\|_{H^r} \le 1$, $\gamma^{-1}\cdot(\chi_n g)$ is supported in $\gamma^{-1}B_{n+1} \subset B_m$, $m := n+1+\lceil\beta\rceil$, and has H^r-norm $\|\chi_n g\|_{H^r} \le c_n$; by (A.4), $p_n(\gamma\Phi) \le c_n p_m(\Phi)$. The same estimate with any $\chi \in C_c^\infty$ in place of χ_n shows $\gamma\Phi \in X$. So each γ is a continuous linear map of X with continuous inverse γ^{-1}, and $\{\gamma : |b| \le \beta\}$ is equicontinuous.

Strong continuity on X. Fix Φ, n and $\epsilon > 0$, and let $|b| \le 1$. For g as above, $w := \gamma^{-1}\cdot(\chi_n g) - \chi_n g$ is supported in B_{n+2} with $\|w\|_{H^r} \le 2c_n$, and $(\gamma\Phi - \Phi)(\chi_n g) = \Phi(w) = \Xi(w)$, $\Xi := \chi_{n+2}\Phi \in H^{-r}$. Choose $\psi \in C_c^\infty(\mathbb{R}^3;\mathcal{V})$ with $\|\Xi - \langle\psi,\cdot\rangle\|_{H^{-r}} \le \epsilon$ (C_c^∞ is dense in H^{-r}, through the L^2 pairing). Since $\gamma\cdot$ preserves the L^2 pairing, $\langle\psi, w\rangle = \langle\gamma\cdot\psi - \psi, \chi_n g\rangle$, so $|\Phi(w)| \le 2c_n\epsilon + c_n\|\gamma\cdot\psi - \psi\|_{L^2}$ uniformly in g, and $\limsup_{\gamma\to\mathrm{id}} p_n(\gamma\Phi - \Phi) \le 2c_n\epsilon$. *Joint continuity.* $\gamma'\Phi' - \gamma\Phi = \gamma'(\Phi' - \Phi) + \gamma\big((\gamma^{-1}\gamma')\Phi - \Phi\big)$; the first term is small by equicontinuity, the second by strong continuity and the continuity of γ.

On $\mathcal{H}^{-r}$. By Lemma A.4(e) each γ is bounded with norm $\le C_r(1+|b|)^r$, with inverse γ^{-1}. The functionals $\langle\psi,\cdot\rangle$, $\psi \in C_c^\infty$, are dense in $\mathcal{H}^{-r}$: for $\Psi \in \mathcal{H}^{-r}$ the pairing with $\psi_N := \sum_{|\alpha|\le N} \Psi(h_{\alpha,c})h_{\alpha,c} \in \mathcal{S}$ differs from Ψ by $\big(\sum_{|\alpha|>N} \lambda_\alpha^{-r}\Psi(h_{\alpha,c})^2\big)^{1/2}$, and $\|\langle\phi,\cdot\rangle\|_{-r} \le \|\phi\|_{L^2}$ because $\|u\|_{L^2} \le \|u\|_r$. On these, $\gamma\langle\psi,\cdot\rangle = \langle\gamma\cdot\psi,\cdot\rangle$ and $\|\gamma\langle\psi,\cdot\rangle - \langle\psi,\cdot\rangle\|_{-r} \le \|\gamma\cdot\psi - \psi\|_{L^2} \to 0$. Local boundedness and an $\epsilon/3$ argument give strong continuity, and joint continuity follows as on X.

(b) Monotonicity is clear. $\Sigma(U) = \bigvee_{K\Subset U}\Sigma(K)$ because every $u \in C_c^\infty(U)$ is supported in some bounded open $K \Subset U$. $\bigvee_K \Sigma(K)$ is the Borel σ-algebra by Lemma A.5(c), and the countable generation by bounded continuous functions is Lemma A.5(b). For a generator $G(\Phi) = \Phi(u)$ of $\Sigma(U)$, $(G\circ\gamma^{-1})(\Phi) = \Phi(\gamma\cdot u)$ with $\gamma\cdot u \in C_c^\infty(\gamma U)$; so γ^{-1} is measurable from $(X,\Sigma(\gamma U))$ to $(X,\Sigma(U))$, and $G\circ\gamma^{-1}$ is $\Sigma(\gamma U)$-measurable for every $\Sigma(U)$-measurable G. With $\gamma = \tau_a$ this gives $U(a)G = G\circ\tau_a^{-1}$, $\Sigma(U+a)$-measurable; with $\gamma = \theta_j = \theta_j^{-1}$ it gives $\Theta_j G$, $\Sigma(\rho_j U)$-measurable. The same argument applies to Σ^H. □

A.4. **Density, the Hermite carrier and uniqueness.**

Lemma A.7 (density of generating algebras)**.** *Let m be a probability measure on a measurable space $(\Omega,\mathcal{F})$, and $\mathfrak{A}$ an algebra of bounded complex $\mathcal{F}$-measurable functions, closed under complex*

conjugation and containing the constants. Then $\mathfrak{A}$ is dense in $L^2(\Omega,\sigma(\mathfrak{A}),m)$. The same holds for a countable algebra $\mathfrak{A}_{\mathbb{Q}}$ over $\mathbb{Q}$[i], closed under conjugation and containing 1, whose complex span is such an algebra.

Proof. Let $\mathcal{L}$ be the closure of $\mathfrak{A}$ in $L^2(\Omega,\sigma(\mathfrak{A}),m)$, a closed complex subspace, and $\mathfrak{A}_{\mathbb{R}}$ the set of real elements of $\mathfrak{A}$. Since $\mathfrak{A}$ is closed under conjugation, $\operatorname{Re} a, \operatorname{Im} a \in \mathfrak{A}_{\mathbb{R}}$ for $a \in \mathfrak{A}$; $\mathfrak{A}_{\mathbb{R}}$ is a real algebra containing the constants, and $\sigma(\mathfrak{A}_{\mathbb{R}}) = \sigma(\mathfrak{A})$. (i) Let $a_1,\dots,a_N \in \mathfrak{A}_{\mathbb{R}}$, $|a_i| \le M$, and $\varphi_1,\dots,\varphi_N$ continuous on $[-M,M]$. By Weierstrass' theorem there are polynomials $p_{i,k} \to \varphi_i$ uniformly on $[-M,M]$; then $\prod_i p_{i,k}(a_i) \in \mathfrak{A}$ converges uniformly to $\prod_i \varphi_i(a_i)$, which therefore lies in $\mathcal{L}$. (ii) For $t_i \in \mathbb{R}$ the continuous functions $\varphi_{i,k}(s) := \min(1, k(s-t_i)^+)$ increase to $\mathbf{1}_{\{s>t_i\}}$; by dominated convergence $\prod_i \varphi_{i,k}(a_i) \to \prod_i \mathbf{1}_{\{a_i>t_i\}}$ in L^2, so $\mathbf{1}_P \in \mathcal{L}$ for every P in $\mathcal{P} := \{\bigcap_{i\le N}\{a_i > t_i\} : N \ge 0,\ a_i \in \mathfrak{A}_{\mathbb{R}},\ t_i \in \mathbb{R}\}$.

(iii) $\mathcal{P}$ is a π-system generating $\sigma(\mathfrak{A}_{\mathbb{R}})$. The class of $B \in \sigma(\mathfrak{A})$ with $\mathbf{1}_B \in \mathcal{L}$ contains Ω, is closed under complements ($\mathbf{1}_{B^c} = 1 - \mathbf{1}_B$), under proper differences ($\mathbf{1}_{B\setminus A} = \mathbf{1}_B - \mathbf{1}_A$ for $A \subset B$, as $\mathcal{L}$ is linear) and under increasing countable unions (dominated convergence; $\mathcal{L}$ is closed). By Dynkin's π–λ theorem it is all of $\sigma(\mathfrak{A})$. So $\mathcal{L}$ contains the simple $\sigma(\mathfrak{A})$-measurable functions, which are dense. For $\mathfrak{A}_{\mathbb{Q}}$: if $a = \sum_k c_k a_k$ with $c_k \in \mathbb{C}$, $a_k \in \mathfrak{A}_{\mathbb{Q}}$, and $q_k \in \mathbb{Q}$[i], then $\|a - \sum_k q_k a_k\|_\infty \le \sum_k |c_k - q_k| \|a_k\|_\infty$; so $\mathfrak{A}_{\mathbb{Q}}$ is dense in its complex span, to which the first part applies. □

Lemma A.8 (the Hermite carrier)**.** *Let $r \ge 14$ be even, and let μ' be a probability measure on $\mathcal{S}'(\mathbb{R}^3;\mathcal{V})$ with its cylinder σ-algebra such that $\int \Phi(u)^2\,d\mu' \le C_2\|u\|_{10}^2$ for every real $u \in \mathcal{S}(\mathbb{R}^3;\mathcal{V})$. Put $M(\Phi) := \sum_{\alpha,c} \lambda_\alpha^{-r}\Phi(h_{\alpha,c})^2$. Then $\int M\,d\mu' \le C_2 \dim\mathcal{V} \sum_\alpha \lambda_\alpha^{10-r} < \infty$. On $\{M<\infty\}$, $\Phi|_{\mathcal{S}}$ extends uniquely to an element $\Pi(\Phi)$ of $\mathcal{H}^{-r}$, with $\|\Pi(\Phi)\|_{-r}^2 = M(\Phi)$. The map Π (put $\Pi := 0$ on $\{M=\infty\}$) is measurable into the Borel σ-algebra of $\mathcal{H}^{-r}$, and $\mu^H := \Pi_*\mu'$ is a Borel probability measure on $\mathcal{H}^{-r}$ under which, for all real $u_1,\dots,u_N \in \mathcal{S}$, the vector $(\Psi(u_l))_{l\le N}$ has the law of $(\Phi(u_l))_{l\le N}$ under μ'. In particular μ^H and μ' have the same characteristic functional and the same mixed moments on $\mathcal{S}$.*

Proof. $\|h_{\alpha,c}\|_{10}^2 = \lambda_\alpha^{10}$, so $\int M\,d\mu' \le C_2\sum_{\alpha,c}\lambda_\alpha^{10-r}$, and $\sum_\alpha \lambda_\alpha^{10-r} = \sum_{n\ge0} \frac{(n+1)(n+2)}{2}(2n+4)^{10-r} < \infty$ because $r - 10 > 3$. Hence $\mu'(M<\infty) = 1$. For $\Phi \in \mathcal{S}'$ and $u \in \mathcal{S}$ the Hermite expansion of u converges in $\mathcal{S}$ (Lemma A.4(c)), so $\Phi(u) = \sum_{\alpha,c}\langle h_{\alpha,c}, u\rangle\,\Phi(h_{\alpha,c})$, and by the Cauchy–Schwarz inequality $|\Phi(u)| \le M(\Phi)^{1/2}\|u\|_r$. As $\mathcal{S}$ is dense in $\mathcal{H}^r$, $\Phi|_{\mathcal{S}}$ extends uniquely to $\Pi(\Phi) \in \mathcal{H}^{-r}$ when $M(\Phi) < \infty$, and $\|\Pi(\Phi)\|_{-r}^2 = \sum \lambda_\alpha^{-r}\Pi(\Phi)(h_{\alpha,c})^2 = M(\Phi)$. The Borel σ-algebra of the separable Hilbert space $\mathcal{H}^{-r}$ is generated by the coordinates $\Psi \mapsto \Psi(h_{\alpha,c})$, since $\|\Psi - \Psi_0\|_{-r}$ is a measurable function of them; and $\Pi(\Phi)(h_{\alpha,c}) = \mathbf{1}_{\{M<\infty\}}\Phi(h_{\alpha,c})$ is measurable. Finally $\Pi(\Phi)(u) = \Phi(u)$ for $u \in \mathcal{S}$ on the set $\{M<\infty\}$ of full measure. □

Proposition A.9 (uniqueness by the characteristic functional and by the moments)**.** *Let $r \ge 2$ be even, let Y be X or $\mathcal{H}^{-r}$, write Φ for its elements, and let μ_1,μ_2 be Borel probability measures on Y.*

(a) *If $\int e^{\mathrm{i}\Phi(u)}\,d\mu_1 = \int e^{\mathrm{i}\Phi(u)}\,d\mu_2$ for every real $u \in C_c^\infty(\mathbb{R}^3;\mathcal{V})$, then $\mu_1 = \mu_2$.*

(b) *Suppose that for all real $u_1,\dots,u_N \in C_c^\infty(\mathbb{R}^3;\mathcal{V})$ the mixed moments $\int \prod_{l\le N}\Phi(u_l)\,d\mu_i$ exist and are equal for $i = 1,2$, and that for every real $u \in C_c^\infty$ there are constants $c, K \ge 0$, depending on u, with $|\int \Phi(u)^n\,d\mu_1| \le c\,(n+1)K^n n!$ for all $n \ge 0$. Then $\mu_1 = \mu_2$.*

Proof. (a) Let $D_Y = \{u_1,u_2,\dots\}$ be the countable set of Lemma A.5(c). The sets $\{\Phi : (\Phi(u_1),\dots,\Phi(u_N)) \in B\}$, $N \in \mathbb{N}$, $B \subset \mathbb{R}^N$ Borel, form a π-system generating the Borel σ-algebra of Y. Under μ_i the characteristic function of $(\Phi(u_1),\dots,\Phi(u_N))$ is $t \mapsto \int e^{\mathrm{i}\Phi(\sum_k t_k u_k)}\,d\mu_i$, by linearity of Φ; so the two laws on $\mathbb{R}^N$ agree, and $\mu_1 = \mu_2$ by Dynkin's π–λ theorem.

(b) Fix u and let $Z := \Phi(u)$, $m_n := \int Z^n\,d\mu_1 = \int Z^n\,d\mu_2$; we may take $K > 0$. Even moments are absolute moments, and the Cauchy–Schwarz inequality gives $\mathbb{E}|Z|^{2n+1} \le (m_{2n}m_{2n+2})^{1/2} \le c\,K^{2n+1}(2n+1)!\,\big((2n+3)(2n+2)\big)^{1/2}$, using $(2n)!\,(2n+2)! = ((2n+1)!)^2\,\frac{2n+2}{2n+1}$. Hence $\mathbb{E}|Z|^n \le c\,(n+2)K^n n!$ for all n, under both measures. Let $\varphi_i(s) := \int e^{\mathrm{i}sZ}\,d\mu_i$. Then $\varphi_i^{(k)}(s) = \int(\mathrm{i}Z)^k e^{\mathrm{i}sZ}\,d\mu_i$, and from $|e^{\mathrm{i}y} - \sum_{k<n}(\mathrm{i}y)^k/k!| \le |y|^n/n!$,

$$\Big|\varphi_i(s+v) - \sum_{k<n}\varphi_i^{(k)}(s)\frac{v^k}{k!}\Big| \le \frac{|v|^n\,\mathbb{E}|Z|^n}{n!} \le c\,(n+2)(K|v|)^n \to 0 \qquad (|v| < 1/K).$$

So each φ_i is the sum of its Taylor series on $(s-1/K, s+1/K)$, for every s. The set of s near which $\varphi_1 = \varphi_2$ is open; it contains 0, because $\varphi_i^{(k)}(0) = \mathrm{i}^k m_k$; and it is closed, because at a point s' of it within $1/(2K)$ of a limit point s all derivatives agree and the two Taylor series at s' agree on a neighbourhood of s. So $\varphi_1 = \varphi_2$ on $\mathbb{R}$; at $s = 1$ this is $\int e^{\mathrm{i}\Phi(u)}\,d\mu_1 = \int e^{\mathrm{i}\Phi(u)}\,d\mu_2$, and (a) applies. □

A.5. **(H1)–(H3) from the characteristic functional.**

Proposition A.10 ((H1)–(H3) on the field space)**.** *Let $r \ge 2$ be even, μ a Borel probability measure on $X = H^{-r}_{\mathrm{loc}}(\mathbb{R}^3;\mathcal{V})$, and $\mathcal{Z}(u) := \int e^{\mathrm{i}\Phi(u)}\,d\mu(\Phi)$ for real $u \in C_c^\infty(\mathbb{R}^3;\mathcal{V})$.*

(a) *If $\gamma \in \mathbb{R}^3 \rtimes O(3)$ and $\mathcal{Z}(\gamma\cdot u) = \mathcal{Z}(u)$ for all real u, then $\gamma_*\mu = \mu$.*

(b) *For real $u_k, u_l \in C_c^\infty$ and $E_k := e^{\mathrm{i}\Phi(u_k)}$, $\overline{E_k\circ\theta_{j,c}}\,E_l = e^{\mathrm{i}\Phi(u_l - \rho_{j,c}\cdot u_k)}$, so $\mathbb{E}_\mu[\overline{E_k\circ\theta_{j,c}}\,E_l] = \mathcal{Z}(u_l - \rho_{j,c}\cdot u_k)$.*

(c) *Let $j \in \{1,2,3\}$ and $c \in \mathbb{R}$. If μ is $\theta_{j,c}$-invariant and $\sum_{k,l\le N}\bar a_k a_l\,\mathcal{Z}(u_l - \rho_{j,c}\cdot u_k) \ge 0$ for all N, all real $u_1,\dots,u_N \in C_c^\infty(H^+_{j,c};\mathcal{V})$ and all $a \in \mathbb{C}^N$, then $\mathbb{E}_\mu[\overline{F\circ\theta_{j,c}}\,F] \ge 0$ for every $F \in L^2(\Sigma(H^+_{j,c}),\mu)$.*

(d) *Consequently, if $\mathcal{Z}$ is invariant under the translations and the coordinate reflections ρ_j, and the condition of (c) holds for every j and every c (or for every j and $c = 0$), then $(X,\tau,\theta,\mu,\Sigma)$ satisfies* (H1)–(H3)*. If moreover $\mathcal{Z}$ is invariant under $\tau_b g \tau_{-b}$ for $g \in O_h$ and $b \in \mathbb{R}^3$, then μ is invariant under O_h about every point.*

Proof. (a) The characteristic functional of $\gamma_*\mu$ is $u \mapsto \int e^{\mathrm{i}(\gamma\Phi)(u)}\,d\mu = \mathcal{Z}(\gamma^{-1}\cdot u) = \mathcal{Z}(u)$; apply Proposition A.9(a). (b) $\theta_{j,c}$ is the action of the involution $\rho_{j,c}$, so $E_k(\theta_{j,c}\Phi) = e^{\mathrm{i}\Phi(\rho_{j,c}\cdot u_k)}$, and Φ is linear. (c) For $F = \sum_l a_l E_l$ with $u_l \in C_c^\infty(H^+_{j,c};\mathcal{V})$, (b) gives $\mathbb{E}_\mu[\overline{F\circ\theta_{j,c}}\,F] = \sum_{k,l}\bar a_k a_l \mathcal{Z}(u_l - \rho_{j,c}\cdot u_k) \ge 0$. These F form an algebra $\mathfrak{A}$ of bounded continuous functions (products of exponentials are exponentials of sums), closed under conjugation ($\overline{E_l} = e^{\mathrm{i}\Phi(-u_l)}$) and containing 1; it generates $\Sigma(H^+_{j,c})$, since $\Phi(u) = \lim_{t\to0} t^{-1}\operatorname{Im} e^{\mathrm{i}t\Phi(u)}$. By Lemma A.7, $\mathfrak{A}$ is dense in $L^2(\Sigma(H^+_{j,c}),\mu)$. If $F_n \to F$ in L^2 with $F_n \in \mathfrak{A}$, then $F_n\circ\theta_{j,c} \to F\circ\theta_{j,c}$ in L^2, as $\theta_{j,c}$ preserves μ, so $\overline{F_n\circ\theta_{j,c}}F_n \to \overline{F\circ\theta_{j,c}}F$ in $L^1(\mu)$, and the inequality passes to the limit. (d) The structural parts of (H1) and all of (H2) are Lemma A.6; the invariances follow from (a), every $\theta_{j,c} = \tau_{ce_j}\theta_j\tau_{-ce_j}$ then preserving μ; (H3) is (c), and the case $c = 0$ suffices by the sentence after (A.3). □

Lemma A.11 (transfer to the Hermite–Sobolev space)**.** *Let $r \ge 2$ be even, μ^H a Borel probability measure on $\mathcal{H}^{-r}$, and $\mu := \iota_*\mu^H$.*

(a) *$\mu(\iota\mathcal{H}^{-r}) = 1$; μ^H is the only Borel probability measure on $\mathcal{H}^{-r}$ whose image under ι is μ; and μ, μ^H have the same characteristic functional and the same mixed moments on C_c^∞.*

(b) *For $\gamma \in \mathbb{R}^3 \rtimes O(3)$: $\gamma_*\mu = \mu$ if and only if $\gamma_*\mu^H = \mu^H$.*

(c) *$(X,\tau,\theta,\mu,\Sigma)$ satisfies* (H1)–(H3) *if and only if $(\mathcal{H}^{-r},\tau,\theta,\mu^H,\Sigma^H)$ does.*

Proof. (a) By Lemma A.5(d), ιB is Borel for every Borel B, so any ν with $\iota_*\nu = \mu$ satisfies $\nu(B) = \nu(\iota^{-1}\iota B) = \mu(\iota B)$; with $B = \mathcal{H}^{-r}$, $\mu(\iota\mathcal{H}^{-r}) = 1$. The functionals agree because

$(\iota\Psi)(u) = \Psi(u)$ for $u \in C_c^\infty$. (b) $\iota\gamma = \gamma\iota$ gives $\gamma_*\mu = \iota_*(\gamma_*\mu^H)$; if $\gamma_*\mu^H = \mu^H$ this is μ, and if $\gamma_*\mu = \mu$ then $\gamma_*\mu^H = \mu^H$ by the uniqueness in (a). (c) The structural parts of (H1) and (H2) hold on both spaces (Lemma A.6), and the invariances correspond by (b). For (H3): $\Sigma^H(U) = \iota^{-1}\Sigma(U)$ (Lemma A.5(d)), so the $\Sigma^H(H^+_{j,c})$-measurable functions are exactly the $F' \circ \iota$ with F' $\Sigma(H^+_{j,c})$-measurable (factorisation through ι). For such $F = F' \circ \iota$, $\int |F|^2 d\mu^H = \int |F'|^2 d\mu$, $F \circ \theta_{j,c} = F' \circ \theta_{j,c} \circ \iota$, and $\mathbb{E}_{\mu^H}[\overline{F \circ \theta_{j,c}} F] = \mathbb{E}_\mu[\overline{F' \circ \theta_{j,c}} F']$. □

Remark A.12 (where the hypotheses are met). For the curvature field, $\mathcal{V} = \Lambda^2$, the hypotheses of Lemma A.4(b), Lemma A.8, Proposition A.9, Proposition A.10(a) and Lemma A.11 are verified in the proof of Theorem 9.13, and those of Proposition A.10(c),(d) are verified in the proof of Theorem 9.15.

Lemma A.11 carries (H1)–(H3) and the invariances between X and $\mathcal{H}^{-r}$. Every statement proved for a measure satisfying (H1)–(H3) therefore applies to μ on X and to μ^H on $\mathcal{H}^{-r}$ alike, and the conclusions correspond under ι, which is a Borel isomorphism of $\mathcal{H}^{-r}$ onto a Borel set of full μ-measure that intertwines the actions of $\mathbb{R}^3 \rtimes O(3)$ and the local σ-algebras.

A.6. **Ergodic decomposition and clustering.** This subsection proves the results on the ergodic decomposition, on the reconstruction in a coordinate direction and on clustering that are used in Section 9: in the reconstruction before Remark 9.14 and in Theorem 9.15. The hypotheses (H1)–(H3) are those of Subsection A.1, and $U(a)$, Θ_j are the operators of (A.1).

Notation. A Borel probability measure m on X invariant under every τ_a is *ergodic* (for the action of $\mathbb{R}^3$) if $m(B) \in \{0,1\}$ for every Borel set B with $\tau_a B = B$ for all $a \in \mathbb{R}^3$. For a sub-σ-algebra $\mathcal{F} \subset \mathcal{B}$, $L^2(\mathcal{F}, m)$ denotes the closed subspace of $L^2(m)$ of classes with an $\mathcal{F}$-measurable representative. For $J \subset \{1,2,3\}$ put $V_J := \operatorname{span}\{e_j : j \in J\}$, with $V_\emptyset = \{0\}$. For $a \in \mathbb{R}^3$ and j we write $a = a_j e_j + a_\perp$ with $a_\perp \perp e_j$. For a measure m and an m-integrable function G we write $m(G) := \int G\, dm$.

Remark A.13 (countable determining algebras). Let $d \le 1$ be a complete metric on a Polish space X, (x_n) a dense sequence, and $\mathcal{D}$ the countable $\mathbb{Q}[\mathrm{i}]$-algebra generated by 1 and the functions $x \mapsto d(x, x_n)$. Its elements are bounded and continuous, and $\sigma(\mathcal{D}) = \mathcal{B}$, because the balls $\{d(\cdot, x_n) < q\}$, $q \in \mathbb{Q}$, generate the open sets of the separable space X. By Lemma A.7, $\mathcal{D}$ is dense in $L^2(m)$ for every Borel probability measure m on X; consequently two Borel probability measures m, m' with $m(G) = m'(G)$ for all $G \in \mathcal{D}$ are equal (for bounded Borel F and $G \in \mathcal{D}$, $|m(F) - m'(F)| \le 2\|F - G\|_{L^2((m+m')/2)}$). We call $\mathcal{D}$ a *countable determining algebra* of X.

A.6.1. *Translations and the mean-ergodic projections.* Throughout Subsections A.6.1 and A.6.2, μ is a measure satisfying (H1)–(H3).

Lemma A.14. *$a \mapsto U(a)$ is strongly continuous on $L^2(\mu)$.*

Proof. For a bounded continuous G, $G(\tau_{-a}x) \to G(x)$ for every x as $a \to 0$, by the continuity of the action, and dominated convergence gives $U(a)G \to G$ in $L^2(\mu)$. Bounded continuous functions are dense in $L^2(\mu)$ (the countable determining algebra of Remark A.13), and $\|U(a)\| = 1$; an $\epsilon/3$ argument gives $U(a)G \to G$ for every $G \in L^2(\mu)$, and the group law gives continuity at every point. □

For $J \subset \{1,2,3\}$ let P_J be the orthogonal projection of $L^2(\mu)$ onto $\{G : U(a)G = G\ \forall a \in V_J\}$, with $P_\emptyset = 1$ and $P_j := P_{\{j\}}$; let $\mathcal{I}_J$ be the σ-algebra of V_J-invariant Borel sets completed by μ-null sets, and $\mathcal{I} := \mathcal{I}_{\{1,2,3\}}$. Put $\Lambda_T^J := [-T,T]^3 \cap V_J$ and $\Lambda_T := [-T,T]^3$.

Lemma A.15. (i) *For $\Lambda_T^J := [-T,T]^3 \cap V_J$ and every $G \in L^2(\mu)$, $|\Lambda_T^J|^{-1} \int_{\Lambda_T^J} U(a)G\, da \to P_J G$ in $L^2(\mu)$ as $T \to \infty$, with $|\Lambda_T^J|$ the Lebesgue measure of Λ_T^J in V_J. The same holds for the one-sided means $T^{-1}\int_0^T U(te_j)G\, dt$, $J = \{j\}$.*

(ii) *P_J commutes with every $U(a)$ and every Θ_j; $P_J P_{J'} = P_{J\cup J'}$; and $P_J = \mathbb{E}[\cdot \mid \mathcal{I}_J]$. In particular $P_{\{1,2,3\}} = \mathbb{E}[\cdot \mid \mathcal{I}]$.*

Proof. (i) For one coordinate j, $t \mapsto U(te_j)$ is strongly continuous by Lemma A.14, and Stone's theorem [30] gives $U(te_j) = e^{\mathrm{i}tA_j}$ with A_j self-adjoint, and $(2T)^{-1}\int_{-T}^{T} e^{\mathrm{i}tA_j}\,dt = \phi_T(A_j)$ with $\phi_T(\lambda) = \sin(T\lambda)/(T\lambda)$, $\phi_T(0) = 1$. Since $|\phi_T| \le 1$ and $\phi_T \to \mathbf{1}_{\{0\}}$ pointwise, the functional calculus of the spectral theorem [30] and dominated convergence give strong convergence to the spectral projection of $\{0\}$, which is the projection onto $\operatorname{Ker} A_j = \{G : U(te_j)G = G\ \forall t\}$. For the one-sided means use $\phi_T(\lambda) = (e^{\mathrm{i}T\lambda} - 1)/(\mathrm{i}T\lambda)$. The mean over Λ_T^J is the product of the commuting one-dimensional means over the coordinates in J. Products of uniformly bounded, strongly convergent families converge strongly to the product of the limits, and a product of commuting orthogonal projections is the orthogonal projection onto the intersection of their ranges, which here is the V_J-fixed space.

(ii) The means commute with every $U(a)$, and with Θ_j by (A.3) and $\rho_j V_J = V_J$; strong limits of operators commuting with a fixed operator commute with it. $P_J P_{J'} = P_{J\cup J'}$ by the intersection statement in (i). It remains to show that the range of P_J is $L^2(\mathcal{I}_J, \mu)$; the orthogonal projection onto L^2 of a sub-σ-algebra is the conditional expectation.

Let $G \in L^2(\mu)$ satisfy $U(a)G = G$ for all $a \in V_J$, and fix a Borel version of G. The map $(a, x) \mapsto G(\tau_{-a}x)$ is Borel on $V_J \times X$. For each $a \in V_J$ the set $\{x : G(\tau_{-a}x) \ne G(x)\}$ is μ-null, so by Fubini's theorem the set $X_0 := \{x : G(\tau_{-a}x) = G(x) \text{ for a.e. } a \in V_J\}$ is Borel and of full measure. Let X_1 be the set of x for which $a \mapsto G(\tau_{-a}x)$ is a.e. constant on V_J, that is,

$$\int_{\Lambda_n^J}\int_{\Lambda_n^J} \min\big(1, |G(\tau_{-a}x) - G(\tau_{-b}x)|\big)\,da\,db = 0 \qquad \text{for all } n \in \mathbb{N}.$$

By Tonelli's theorem X_1 is Borel; $X_0 \subset X_1$; and X_1 is V_J-invariant, because $G(\tau_{-a}\tau_c x) = G(\tau_{-(a-c)}x)$ and a.e. constancy on V_J is preserved under shifts of the variable. For $x \in X_1$ let $\tilde G(x)$ be the a.e. value of $a \mapsto G(\tau_{-a}x)$, that is $\tilde G(x) = \lim_{N\to\infty} |\Lambda_1^J|^{-1}\int_{\Lambda_1^J} t_N\big(G(\tau_{-a}x)\big)\,da$, where t_N truncates real and imaginary parts at level N (for $J = \emptyset$ put $\tilde G := G$); put $\tilde G := 0$ off X_1. Then $\tilde G$ is Borel, $\tilde G \circ \tau_c = \tilde G$ everywhere for $c \in V_J$, and $\tilde G = G$ on X_0, hence μ-a.e. Thus G is $\mathcal{I}_J$-measurable. Conversely, an $\mathcal{I}_J$-measurable function is a.e. equal to a function measurable for the σ-algebra of V_J-invariant Borel sets; such a function is V_J-invariant, hence fixed by every $U(a)$, $a \in V_J$. □

The next lemma is the only place where reflection positivity enters the decomposition: functions invariant under the translations in a direction live at infinity in that direction and are fixed by the corresponding reflection.

Lemma A.16 (localisation of invariant functions)**.** *Let $j \in J \subset \{1, 2, 3\}$ and $W \in L^2(\mu)$ with $U(a)W = W$ for all $a \in V_J$. Then:* (i) *$W \in L^2(\Sigma(H_{j,c}^+), \mu) \cap L^2(\Sigma(H_{j,c}^-), \mu)$ for every $c \in \mathbb{R}$;* (ii) *$\Theta_j W = W$;* (iii) *if $G \in L^2(\Sigma(K), \mu)$, then $P_J G \in L^2(\Sigma(K + V_J), \mu)$.*

Proof. (i) Let B_n be the open ball of radius n about 0. Every bounded open K lies in some B_n, so by (H2) the σ-algebras $\Sigma(B_n)$ increase and their join is $\mathcal{B}$. The family of Borel sets B for which $\mathbf{1}_B$ lies in the closure of $\bigcup_n L^2(\Sigma(B_n), \mu)$ is closed under complements and countable disjoint unions and contains the algebra $\bigcup_n \Sigma(B_n)$; by Dynkin's π–λ theorem it is $\mathcal{B}$, and so $\bigcup_n L^2(\Sigma(B_n), \mu)$ is dense in $L^2(\mu)$. Given $\epsilon > 0$, there are therefore a bounded K and $G \in L^2(\Sigma(K))$ with $\|W - G\|_2 < \epsilon$. For every t, $\|W - U(te_j)G\|_2 = \|U(te_j)(W - G)\|_2 < \epsilon$, and $U(te_j)G \in L^2(\Sigma(K + te_j)) \subset L^2(\Sigma(H_{j,c}^+))$ for t large, by (H2). That space is closed, so it contains W. Letting $t \to -\infty$ gives the other half space.

(ii) $\Theta_\jmath W$ is V_J-invariant by (A.3) and $\rho_\jmath V_J = V_J$, so by (i) both W and $\Theta_\jmath W$ lie in $L^2(\Sigma(H^+_{\jmath,0}))$. Put $F := W - \Theta_\jmath W$. Then $\Theta_\jmath F = -F$, and (A.2) gives $0 \le \mathbb{E}[\overline{\Theta_\jmath F}\, F] = -\mathbb{E}|F|^2$, so $F = 0$.

(iii) The means of Lemma A.15(i) applied to G lie in the closed subspace $L^2(\Sigma(K + V_J))$, since $U(a)G \in L^2(\Sigma(K + a))$ and $K + a \subset K + V_J$ for $a \in V_J$; so does their limit $P_J G$. □

A.6.2. *The ergodic decomposition and inheritance.*

Lemma A.17 (conditional probabilities)**.** *Let X be Polish, m a Borel probability measure on X and $\mathcal{F}_0 \subset \mathcal{B}$ a σ-algebra.* (i) *There are Borel probability measures m_ξ, $\xi \in X$, such that $\xi \mapsto m_\xi(B)$ is $\mathcal{F}_0$-measurable and $\int_C m_\xi(B)\, dm(\xi) = m(B \cap C)$ for all $B \in \mathcal{B}$ and $C \in \mathcal{F}_0$.* (ii) *If (m'_ξ) is a second family with these properties, then $m'_\xi = m_\xi$ for m-a.e. ξ.*

Proof. (i) Let $d \le 1$ be a complete metric on X and (x_n) a dense sequence. Then $\jmath(x) := (d(x, x_n))_n$ is a continuous injection of X into the compact metrisable space $Q := [0,1]^{\mathbb{N}}$; by the Lusin–Souslin theorem [23] the image under $\jmath$ of every Borel subset of X is Borel, so $\jmath$ is a Borel isomorphism onto the Borel set $\jmath X$. Let $\mathcal{C}_0$ be a countable $\mathbb{Q}$-linear subspace of the real space $C(Q)$, dense for the supremum norm and containing 1, and choose $\mathcal{F}_0$-measurable real versions $L_\xi(G)$ of $\mathbb{E}_m[G \circ \jmath \mid \mathcal{F}_0]$, $G \in \mathcal{C}_0$. Off a set $N_0 \in \mathcal{F}_0$ with $m(N_0) = 0$ the countably many relations of $\mathbb{Q}$-linearity, positivity ($L_\xi(G) \ge 0$ for $G \ge 0$) and $L_\xi(1) = 1$ hold. For $\xi \notin N_0$ and $G \in \mathcal{C}_0$, applying positivity to $q \pm G$, $q \in \mathbb{Q}$, $q \ge \|G\|_\infty$, gives $|L_\xi(G)| \le \|G\|_\infty$; so L_ξ extends to a positive linear functional on $C(Q)$ with $L_\xi(1) = 1$, given by a Borel probability measure κ_ξ on Q (Riesz representation theorem). On N_0 put $\kappa_\xi := \jmath_* m$. For $G \in C(Q)$, $\kappa_\xi(G)$ is an $\mathcal{F}_0$-measurable version of $\mathbb{E}_m[G \circ \jmath \mid \mathcal{F}_0]$, by uniform approximation from $\mathcal{C}_0$.

Let $\mathcal{L}$ be the family of Borel sets $A \subset Q$ for which $\xi \mapsto \kappa_\xi(A)$ is an $\mathcal{F}_0$-measurable version of $\mathbb{E}_m[\mathbf{1}_A \circ \jmath \mid \mathcal{F}_0]$. It is closed under complements and countable disjoint unions (monotone convergence, for the measures and for conditional expectations), and it contains every closed set A, since $\mathbf{1}_A$ is the decreasing limit of the continuous functions $\max(0, 1 - n\,\mathrm{dist}(\cdot, A))$. Closed sets form a π-system generating the Borel sets, so by Dynkin's π–λ theorem $\mathcal{L}$ contains every Borel set. In particular $\kappa_\xi(\jmath X) = 1$ off a set $N_1 \in \mathcal{F}_0$ with $m(N_1) = 0$. Put $m_\xi := (\jmath^{-1})_* \kappa_\xi$ off $N_0 \cup N_1$ and $m_\xi := m$ on it. For Borel $B \subset X$, $m_\xi(B) = \kappa_\xi(\jmath B)$ off $N_0 \cup N_1$, an $\mathcal{F}_0$-measurable version of $\mathbb{E}_m[\mathbf{1}_B \mid \mathcal{F}_0]$; this is the claim.

(ii) Let $\mathcal{D}$ be a countable determining algebra of X (Remark A.13). For $G \in \mathcal{D}$, $m_\xi(G)$ and $m'_\xi(G)$ are $\mathcal{F}_0$-measurable versions of $\mathbb{E}_m[G \mid \mathcal{F}_0]$, so they agree off an m-null set; off the union of these countably many null sets m_ξ and m'_ξ agree on $\mathcal{D}$, hence are equal. □

Theorem A.18 (ergodic decomposition with inheritance)**.** *Let μ satisfy* (H1)–(H3)*. Let $\mathcal{I}_0 \subset \mathcal{B}$ be the σ-algebra of Borel sets B with $\tau_a B = B$ for all $a \in \mathbb{R}^3$ (its μ-completion is $\mathcal{I}$), let μ_ξ, $\xi \in X$, be measures as in Lemma A.17 for $m = \mu$ and $\mathcal{F}_0 = \mathcal{I}_0$, a regular conditional probability of μ given $\mathcal{I}$, and put $(\Xi, \nu) := (X, \mu)$. Then $\xi \mapsto \mu_\xi(B)$ is $\mathcal{I}_0$-measurable for every Borel set B, and:*

(a) *$\mu = \int_X \mu_\xi\, d\mu(\xi)$; for $G \in L^1(\mu)$ one has $G \in L^1(\mu_\xi)$ and $\mathbb{E}[G \mid \mathcal{I}](\xi) = \mathbb{E}_{\mu_\xi}[G]$ for μ-a.e. ξ; in particular $\mathbb{E}_\mu[WG] = \int W(\xi)\, \mathbb{E}_{\mu_\xi}[G]\, d\mu(\xi)$ for bounded $\mathcal{I}$-measurable W;*

(b) *for ν-a.e. ξ, μ_ξ is invariant under every τ_a and ergodic for the action of $\mathbb{R}^3$;*

(c) *for ν-a.e. ξ, μ_ξ is invariant under $\theta_1, \theta_2, \theta_3$;*

(d) *for ν-a.e. ξ, μ_ξ satisfies* (H3)*;*

(e) *every μ_ξ satisfies* (H2) *with the same $\Sigma(U)$, and every $G \in L^2(\mu)$ lies in $L^2(\mu_\xi)$ for ν-a.e. ξ.*

In particular ν-almost every μ_ξ satisfies (H1)–(H3) *and is ergodic. By Lemma A.17(ii) the family (μ_ξ) is determined up to a ν-null set of ξ.*

Proof. (a) By Lemma A.17, $\mathbb{E}_{\mu_\xi}[G]$ is an $\mathcal{I}_0$-measurable version of $\mathbb{E}[G \mid \mathcal{I}]$ for bounded Borel G (uniform approximation by simple functions), and for $G \geq 0$ in $L^1(\mu)$ by monotone convergence applied to $\min(G, n)$, which also gives $\int \mathbb{E}_{\mu_\xi}[G]\, d\mu = \mathbb{E}_\mu G < \infty$; complex G are treated by real and imaginary parts. Two Borel versions of G differ on a Borel null set N, and $\int \mu_\xi(N)\, d\mu = \mu(N) = 0$, so the statement does not depend on the version. Let $\mathcal{D}$ be a countable determining algebra of X.

(b) *Invariance.* For $B \in \mathcal{I}_0$ and bounded Borel G, $\mathbb{E}_\mu[\mathbf{1}_B\, G \circ \tau_a] = \mathbb{E}_\mu[(\mathbf{1}_B G) \circ \tau_a] = \mathbb{E}_\mu[\mathbf{1}_B G]$. Hence $\mu_\xi(G \circ \tau_a)$ and $\mu_\xi(G)$ are both versions of $\mathbb{E}[G \mid \mathcal{I}]$, and there is a null set off which $\mu_\xi(G \circ \tau_a) = \mu_\xi(G)$ for all $G \in \mathcal{D}$ and $a \in \mathbb{Q}^3$. Since $a \mapsto \mu_\xi(G \circ \tau_a)$ is continuous (joint continuity of the action and dominated convergence), this holds for all a, so μ_ξ is τ-invariant.

Ergodicity. For $G \in \mathcal{D}$ let $q_G(\xi) := \mu_\xi(G)$, a bounded $\mathcal{I}_0$-measurable version of $\mathbb{E}_\mu[G \mid \mathcal{I}]$. For a bounded $\mathcal{I}_0$-measurable h, $\mu_\xi(h)$ is an $\mathcal{I}_0$-measurable version of $\mathbb{E}[h \mid \mathcal{I}] = h$, so $\mu_\xi(h) = h(\xi)$ for a.e. ξ. Applied to $h = q_G$ and $h = |q_G|^2$ this gives $\mu_\xi(|q_G - q_G(\xi)|^2) = 0$: for a.e. ξ, q_G is μ_ξ-a.s. the constant $\mu_\xi(G)$. Let $\Lambda_T := [-T, T]^3$ and $A_T G(x) := |\Lambda_T|^{-1} \int_{\Lambda_T} G(\tau_{-a} x)\, da$. By Fubini's theorem, $A_T G$ is a version of the mean $|\Lambda_T|^{-1} \int_{\Lambda_T} U(a) G\, da$ of Lemma A.15, in L^2 of μ and of every τ-invariant Borel probability measure (both have the same integrals against bounded Borel functions). So, by (a) and Lemma A.15, $\int \mu_\xi(|A_T G - q_G|^2)\, d\mu(\xi) = \mathbb{E}_\mu |A_T G - q_G|^2 \to 0$. Choose $T_k \to \infty$ such that this is at most 2^{-k} for the first k elements of an enumeration of $\mathcal{D}$. Then for a.e. ξ, $A_{T_k} G \to \mu_\xi(G)$ in $L^2(\mu_\xi)$ for every $G \in \mathcal{D}$. The proof of Lemmas A.14 and A.15(i) uses only the invariance of the measure and the continuity of the action, so for τ-invariant μ_ξ the means $A_{T_k} G$ also tend in $L^2(\mu_\xi)$ to the projection of G onto the τ-fixed vectors of $L^2(\mu_\xi)$. Both this projection and $G \mapsto \mu_\xi(G)\mathbf{1}$ are continuous on $L^2(\mu_\xi)$ and agree on the dense set $\mathcal{D}$, so the projection is $G \mapsto \mu_\xi(G)\mathbf{1}$. If $B \in \mathcal{I}_0$, then $\mathbf{1}_B$ is a fixed vector, hence a.s. constant, so $\mu_\xi(B) \in \{0, 1\}$.

(e) (H2) is a condition on X and Σ alone and does not involve the measure, so it holds for every μ_ξ. Also $\int \mathbb{E}_{\mu_\xi}|G|^2\, d\nu = \mathbb{E}_\mu |G|^2 < \infty$ by (a).

(c) Let $W \geq 0$ be bounded and $\mathcal{I}$-measurable and G bounded Borel. By Lemma A.16(ii) with $J = \{1, 2, 3\}$, $\Theta_j W = W$, so $\mathbb{E}_\mu[W\, \Theta_j G] = \mathbb{E}_\mu[\Theta_j(WG)] = \mathbb{E}_\mu[WG]$ by the θ_j-invariance of μ. By (a), $\int W(\xi)\big(\mathbb{E}_{\mu_\xi}[\Theta_j G] - \mathbb{E}_{\mu_\xi}[G]\big)\, d\nu = 0$ for all such W, hence $\mathbb{E}_{\mu_\xi}[\Theta_j G] = \mathbb{E}_{\mu_\xi}[G]$ for ν-a.e. ξ. Taking $G \in \mathcal{D}$ makes the exceptional set independent of G and gives the θ_j-invariance of μ_ξ for a.e. ξ.

(d) By the translation invariance of μ_ξ it suffices to take $c = 0$. Let $W \geq 0$ be bounded and $\mathcal{I}$-measurable and $F \in L^2(\Sigma(H^+_{j,0}), \mu)$. By Lemma A.16(i)–(ii), $\sqrt{W}$ has a $\Sigma(H^+_{j,0})$-measurable version and $\Theta_j \sqrt{W} = \sqrt{W}$. Hence $\sqrt{W} F \in L^2(\Sigma(H^+_{j,0}), \mu)$ and

$$\mathbb{E}_\mu\big[W\, \overline{F \circ \theta_j}\, F\big] = \mathbb{E}_\mu\big[\overline{(\sqrt{W} F) \circ \theta_j}\, (\sqrt{W} F)\big] \geq 0.$$

With (a), $\mathbb{E}_{\mu_\xi}[\overline{F \circ \theta_j}\, F] \geq 0$ for ν-a.e. ξ. Let $\mathcal{L}$ be the countable $\mathbb{Q}[\mathrm{i}]$-algebra, closed under conjugation, generated by 1 and a countable family of bounded continuous functions generating $\Sigma(H^+_{j,0})$ (H2). The exceptional set can be taken uniform over $F \in \mathcal{L}$. By Lemma A.7, $\mathcal{L}$ is dense in $L^2(\Sigma(H^+_{j,0}), m)$ for every Borel probability measure m, and (A.2) extends from $\mathcal{L}$ to $L^2(\Sigma(H^+_{j,0}), \mu_\xi)$ by the bound $|\mathbb{E}[\overline{F \circ \theta_j}\, G]| \leq \|F\|_2 \|G\|_2$, valid because μ_ξ is θ_j-invariant by (c).

As for the final assertions, (H1) for μ_ξ is (b) and (c), the space and the actions being those of μ. □

Proposition A.19 (the decomposition of an image law)**.** *Let X and X' be Polish spaces with jointly continuous actions τ, τ' of $\mathbb{R}^3$, and $\iota : X' \to X$ a continuous injection with $\iota \circ \tau'_a = \tau_a \circ \iota$ for all a. Let μ' be a τ'-invariant Borel probability measure on X' and $\mu := \iota_* \mu'$. Let $(\mu'_y)_{y \in X'}$*

and $(\mu_x)_{x\in X}$ be families as in Lemma A.17 for $(\mu', \mathcal{I}_0(X'))$ and $(\mu, \mathcal{I}_0(X))$, where $\mathcal{I}_0(\cdot)$ is the σ-algebra of translation-invariant Borel sets. Then $\iota X'$ is a translation-invariant Borel set with $\mu(\iota X') = 1$, and $\mu_{\iota y} = \iota_* \mu'_y$ for μ'-a.e. y. In particular, if μ' and μ satisfy (H1)–(H3)*, the components of μ in Theorem A.18 are, for μ-a.e. x, the images under ι of the components of μ', and they are carried by $\iota X'$.*

Proof. By the Lusin–Souslin theorem [23], ι maps Borel sets to Borel sets, so $\iota X'$ is Borel, and it is invariant because $\tau_a \iota X' = \iota \tau'_a X' = \iota X'$; $\mu(\iota X') = \mu'(X') = 1$. If $B' \in \mathcal{I}_0(X')$, then $\iota B'$ is Borel and $\tau_a \iota B' = \iota \tau'_a B' = \iota B'$, so $\iota B' \in \mathcal{I}_0(X)$; and $\iota^{-1} C \in \mathcal{I}_0(X')$ for $C \in \mathcal{I}_0(X)$. Define $m_x := \iota_* \mu'_{\iota^{-1} x}$ for $x \in \iota X'$ and $m_x := \mu_x$ otherwise. For bounded Borel G on X, $\varphi(y) := \mu'_y(G \circ \iota)$ is $\mathcal{I}_0(X')$-measurable, so for a Borel set $A \subset \mathbb{C}$ the set $\{x : m_x(G) \in A\} = \iota\big(\varphi^{-1}(A)\big) \cup \big((X \setminus \iota X') \cap \{x : \mu_x(G) \in A\}\big)$ lies in $\mathcal{I}_0(X)$. For $C \in \mathcal{I}_0(X)$,

$$\int_C m_x(G)\, d\mu(x) = \int_{\iota^{-1} C} \mu'_y(G \circ \iota)\, d\mu'(y) = \mathbb{E}_{\mu'}\big[(G \circ \iota)\, \mathbf{1}_{\iota^{-1} C}\big] = \mathbb{E}_{\mu}[G\, \mathbf{1}_C].$$

So (m_x) is a family as in Lemma A.17(i) for $(\mu, \mathcal{I}_0(X))$, and by Lemma A.17(ii) $m_x = \mu_x$ for μ-a.e. x; since $\mu = \iota_* \mu'$, this is $\mu_{\iota y} = \iota_* \mu'_y$ for μ'-a.e. y. □

A.6.3. *The decomposition with cubic symmetry.* Assume, in addition to (H1)–(H3), that the cubic group O_h acts on X by homeomorphisms $x \mapsto gx$, with $\rho_j x = \theta_j x$ for the coordinate reflections $\rho_j \in O_h$, $g\, \tau_a\, g^{-1} = \tau_{ga}$, μ invariant under every g, and $G \mapsto G \circ g^{-1}$ mapping $\Sigma(U)$-measurable functions to $\Sigma(gU)$-measurable ones. Let $\mathcal{G} := \mathbb{R}^3 \rtimes O_h$, let $N \cong (\mathbb{Z}/2)^3$ be the subgroup of O_h generated by ρ_1, ρ_2, ρ_3, so that $O_h/N \cong S_3$, and let $\mathcal{I}_{\mathcal{G}} \subset \mathcal{I}$ be the σ-algebra of $\mathcal{G}$-invariant Borel sets, completed by μ-null sets.

Theorem A.20 (decomposition with cubic symmetry)**.** *There are Borel probability measures μ'_x on X, $x \in X$, with $x \mapsto \mu'_x(B)$ $\mathcal{I}_{\mathcal{G}}$-measurable for every Borel B and $\mu = \int \mu'_x\, d\mu(x)$, such that for μ-almost every x:*

- (a) *μ'_x is invariant under $\mathcal{G}$ and ergodic for its action (every $\mathcal{G}$-invariant Borel set has μ'_x-measure 0 or 1), and satisfies* (H1)–(H3)*;*
- (b) *there are an integer $k = k(x) \in \{1, 2, 3, 6\}$ and pairwise distinct measures $m_1, \dots, m_k$, each invariant under the translations and the coordinate reflections, ergodic for $\mathbb{R}^3$ and satisfying* (H1)–(H3)*, such that O_h permutes $\{m_1, \dots, m_k\}$ transitively and $\mu'_x = k^{-1} \sum_{i=1}^k m_i$; the m_i are components μ_{gx}, $g \in O_h$, of the decomposition of Theorem A.18.*

Moreover $k(x) = 1$ for μ-almost every x if and only if almost every component of Theorem A.18 is O_h-invariant.

Proof. Step 1: N acts trivially on $\mathcal{I}$, and O_h preserves $\mathcal{I}$. A bounded $\mathcal{I}$-measurable W is invariant along every coordinate axis, so $W \circ \theta_j = W$ a.s. for $j = 1, 2, 3$ by Lemma A.16(ii); hence $W \circ n = W$ a.s. for $n \in N$. For $g \in O_h$ and a translation-invariant Borel set B, $\tau_a(gB) = g\, \tau_{g^{-1} a} B = gB$; and g maps μ-null sets to μ-null sets. So g maps $\mathcal{I}$ onto itself, and $h \circ g$ is $\mathcal{I}$-measurable whenever h is.

Step 2: conditional expectations. Let μ_x be the components of Theorem A.18, and let μ'_x be a regular conditional probability of μ given $\mathcal{I}_{\mathcal{G}}$: the measures of Lemma A.17 for the σ-algebra of $\mathcal{G}$-invariant Borel sets, whose μ-completion is $\mathcal{I}_{\mathcal{G}}$. For bounded Borel F, $g \in O_h$ and bounded $\mathcal{I}$-measurable W, Step 1 and the g-invariance of μ give $\mathbb{E}\big[W \cdot (\mathbb{E}[F \mid \mathcal{I}] \circ g)\big] = \mathbb{E}\big[(W \circ g^{-1})\, \mathbb{E}[F \mid \mathcal{I}]\big] = \mathbb{E}\big[(W \circ g^{-1})\, F\big] = \mathbb{E}[W \cdot (F \circ g)]$, so $\mathbb{E}[F \circ g \mid \mathcal{I}] = \mathbb{E}[F \mid \mathcal{I}] \circ g$ a.s. Applied to F in a countable determining algebra of X, and then as measures,

$$\mu_{gx} = g_* \mu_x \qquad \text{for } \mu\text{-a.e. } x \text{ and every } g \in O_h. \tag{A.5}$$

By Step 1 and (A.5) with $g \in N$, $n_*\mu_x = \mu_{nx} = \mu_x$ for $n \in N$ and a.e. x, so $S_3 = O_h/N$ acts on the components by $\sigma \cdot \mu_x := g_*\mu_x$, $g \in \sigma$.

For bounded $\mathcal{I}$-measurable h put $\bar h := |O_h|^{-1}\sum_{g\in O_h} h\circ g$. It is a.e. equal to a translation-invariant and O_h-invariant Borel function, hence $\mathcal{I}_{\mathcal{G}}$-measurable, and for bounded $\mathcal{I}_{\mathcal{G}}$-measurable W, which satisfies $W\circ g = W$ a.s., $\mathbb{E}[W\bar h] = |O_h|^{-1}\sum_g \mathbb{E}[(W\circ g)(h\circ g)] = \mathbb{E}[Wh]$. So $\mathbb{E}[h \mid \mathcal{I}_{\mathcal{G}}] = \bar h$. Applied to $h = \mathbb{E}[F \mid \mathcal{I}]$, with $\mathbb{E}[F \mid \mathcal{I}_{\mathcal{G}}] = \mathbb{E}\big[\mathbb{E}[F \mid \mathcal{I}] \mid \mathcal{I}_{\mathcal{G}}\big]$ and (A.5),

$$\mu'_x = \frac{1}{48}\sum_{g\in O_h}\mu_{gx} = \frac{1}{48}\sum_{g\in O_h} g_*\mu_x = \frac{1}{6}\sum_{\sigma\in S_3}\sigma\cdot\mu_x \qquad \text{for } \mu\text{-a.e. } x,$$

first evaluated on a countable determining algebra and then as measures. Grouping equal terms, $\mu'_x = k^{-1}\sum_{i=1}^k m_i$, where $\{m_1,\dots,m_k\}$ is the S_3-orbit of μ_x, so $k = 6/|\mathrm{Stab}(\mu_x)| \in \{1,2,3,6\}$, the orders of the subgroups of S_3 being $1,2,3,6$. Let E be the set of full measure of those x for which μ_x has the properties (b)–(e) of Theorem A.18; since every $g \in O_h$ preserves μ, $\bigcap_{g\in O_h} g^{-1}E$ also has full measure, and for x in it every μ_{gx} has these properties. This gives (b): each m_i is translation invariant, ergodic, θ_j-invariant and satisfies (H2) and (H3), hence (H1)–(H3). Since $k = 1$ exactly when μ_x is fixed by S_3, that is O_h-invariant, the last assertion follows.

Step 3: (a). μ'_x is invariant under the translations and under O_h, as the average over an O_h-orbit of translation-invariant measures ($\tau_a g = g\tau_{g^{-1}a}$). If B is a $\mathcal{G}$-invariant Borel set, then $m_i(B) \in \{0,1\}$ by ergodicity, and $(g_*m_i)(B) = m_i(g^{-1}B) = m_i(B)$, so all $m_i(B)$ are equal by transitivity and $\mu'_x(B) \in \{0,1\}$. Each m_i satisfies (H3), and (H3) is preserved under convex combinations: an $F \in L^2(\Sigma(H^+_{j,c}), \mu'_x)$ lies in every $L^2(m_i)$, since $m_i \le k\mu'_x$. (H2) holds as in Theorem A.18(e), and the $\theta_j = \rho_j$-invariance is part of the O_h-invariance. □

A.6.4. *The growth of the moments of the components.* Let $\mathcal{V}$ be a real Euclidean space of dimension d with orthonormal basis $(e_c)_{c\le d}$. Let h_α, $\alpha \in (\mathbb{N}\cup\{0\})^3$, be the Hermite functions on $\mathbb{R}^3$, $h_{\alpha,c} := h_\alpha e_c$, and $\lambda_\alpha := 2|\alpha| + 4$ the eigenvalues of $-\Delta + |x|^2 + 1$. For even $s \ge 0$ put $\|f\|_s^2 := \sum_{\alpha,c}\lambda_\alpha^s |\langle h_{\alpha,c}, f\rangle|^2$, let $\mathcal{H}^s$ be the completion of $\mathcal{S}(\mathbb{R}^3;\mathcal{V})$ in this norm and $\mathcal{H}^{-s}$ its dual, a separable Hilbert space with $\|\Psi\|_{-s}^2 = \sum_{\alpha,c}\lambda_\alpha^{-s}\Psi(h_{\alpha,c})^2$ and $|\Psi(f)| \le \|\Psi\|_{-s}\|f\|_s$. Note that $\|h_{\alpha,c}\|_{10} = \lambda_\alpha^5$, and $\|f\|_s \le \|f\|_{s'}$ for $s \le s'$ since $\lambda_\alpha \ge 1$. There are $(n+1)(n+2)/2$ multi-indices with $|\alpha| = n$, so

$$w_r := d\sum_\alpha \lambda_\alpha^{10-r} < \infty \qquad \text{for every even } r \ge 14.$$

Theorem A.21 (growth of the moments of the components)**.** *Let $r \ge 14$ be even, μ^H a Borel probability measure on $\mathcal{H}^{-r} = \mathcal{H}^{-r}(\mathbb{R}^3;\mathcal{V})$, and suppose that there are constants $c_S \ge 1$ and $C_0 > 0$ with*

$$\int \Psi(h_{\alpha,c})^{2n}\, d\mu^H(\Psi) \le c_S\, C_0^{2n}\,(2n)!\,\|h_{\alpha,c}\|_{10}^{2n} \qquad \text{for all } \alpha,\ c\le d,\ n\in\mathbb{N}. \tag{A.6}$$

Let (Ξ,ν) be a probability space and μ_ξ, $\xi\in\Xi$, Borel probability measures on $\mathcal{H}^{-r}$ with $\xi\mapsto\mu_\xi(B)$ measurable for every Borel B and $\mu^H = \int \mu_\xi\, d\nu(\xi)$; for instance the components of Theorem A.18 or of Theorem A.20, when μ^H satisfies the hypotheses there. Put $C_1 := w_r C_0^2$. Then for ν-a.e. ξ, for every $N\in\mathbb{N}$ and all $f_1,\dots,f_N \in \mathcal{S}(\mathbb{R}^3;\mathcal{V})$, the product $\prod_i \Psi(f_i)$ is μ_ξ-integrable, and its integral $S_N^\xi(f_1,\dots,f_N) := \mathbb{E}_{\mu_\xi}\prod_i\Psi(f_i)$ satisfies

$$|S_N^\xi(f_1,\dots,f_N)| \le \sigma_N^\xi \prod_{i=1}^N \|f_i\|_r, \qquad \sigma_N^\xi := \big(\mathbb{E}_{\mu_\xi}\|\Psi\|_{-r}^{2N}\big)^{1/2} < \infty.$$

There is $n_0(\xi)$ with $\sigma_N^\xi \le N\, c_S^{1/2}(4C_1)^{N/2}N!$ for $N \ge n_0(\xi)$, and consequently $\sigma_N^\xi \le \alpha_\beta(\xi)(N!)^\beta$ for all N, for every $\beta > 1$, with $\alpha_\beta(\xi) := \sup_N \sigma_N^\xi (N!)^{-\beta} < \infty$.

Proof. Put $M(\Psi) := \|\Psi\|_{-r}^2 = \sum_{\alpha,c} \lambda_\alpha^{-r}\Psi(h_{\alpha,c})^2$, a continuous function on $\mathcal{H}^{-r}$. With the probability weights $p_{\alpha,c} := \lambda_\alpha^{10-r}/w_r$,

$$M = \sum_{\alpha,c} p_{\alpha,c}\, w_r \lambda_\alpha^{-10}\Psi(h_{\alpha,c})^2, \qquad \text{so} \qquad M^n \le w_r^n \sum_{\alpha,c} p_{\alpha,c}\, \lambda_\alpha^{-10n}\Psi(h_{\alpha,c})^{2n}$$

by Jensen's inequality for the convex function $t \mapsto t^n$ on $[0,\infty)$ (applied to the finite partial sums, renormalised, and passed to the limit). By (A.6) and $\|h_{\alpha,c}\|_{10}^{2n} = \lambda_\alpha^{10n}$,

$$\mathbb{E}_{\mu^H} M^n \le w_r^n \sup_{\alpha,c} \lambda_\alpha^{-10n}\, \mathbb{E}_{\mu^H}\Psi(h_{\alpha,c})^{2n} \le c_S\, C_1^n\, (2n)!.$$

Put $M_n(\xi) := \mathbb{E}_{\mu_\xi} M^n \in [0,\infty]$, a measurable function of ξ (monotone limit of $\mu_\xi(\min(M^n,k))$). Since $\int M_n\, d\nu = \mathbb{E}_{\mu^H} M^n < \infty$, $M_n(\xi) < \infty$ for all n and ν-a.e. ξ. By Markov's inequality, $\nu\{M_n > n^2 c_S C_1^n (2n)!\} \le n^{-2}$, which is summable, so by the Borel–Cantelli lemma there is, for ν-a.e. ξ, an $n_0(\xi)$ with $M_n(\xi) \le n^2 c_S C_1^n (2n)!$ for $n \ge n_0(\xi)$. Fix such a ξ. For $f_i \in \mathcal{S} \subset \mathcal{H}^r$, $|\Psi(f_i)| \le M^{1/2}\|f_i\|_r$, so

$$\mathbb{E}_{\mu_\xi} \prod_{i=1}^N |\Psi(f_i)| \le \mathbb{E}_{\mu_\xi}\big[M^{N/2}\big] \prod_i \|f_i\|_r \le M_N(\xi)^{1/2} \prod_i \|f_i\|_r$$

by the Cauchy–Schwarz inequality, and $\sigma_N^\xi = M_N(\xi)^{1/2} < \infty$. For $N \ge n_0(\xi)$, $\sigma_N^\xi \le N\, c_S^{1/2} C_1^{N/2}((2N)!)^{1/2} \le N\, c_S^{1/2}(4C_1)^{N/2}N!$, because $(2N)! \le 4^N (N!)^2$. For $\beta > 1$, $N(4C_1)^{N/2}(N!)^{1-\beta} \to 0$, so $\alpha_\beta(\xi) < \infty$. □

Corollary A.22 (the components of a law on a space of local distributions)**.** *In the setting of Theorem A.21, let X be a Polish space with a jointly continuous action of $\mathbb{R}^3$, let $\mathbb{R}^3$ act on $\mathcal{H}^{-r}$ jointly continuously and leave μ^H invariant, and let $\iota : \mathcal{H}^{-r} \to X$ be a continuous injection with $\iota \circ \tau_a^H = \tau_a \circ \iota$. Put $\mu := \iota_* \mu^H$, and let μ_x and μ_y^H be the components of μ and of μ^H given by Lemma A.17 for the σ-algebras of translation-invariant Borel sets. Then for μ-a.e. x the component μ_x is carried by $\iota\mathcal{H}^{-r}$, and with $\Phi(f) := \Psi(f)$ for $\Phi = \iota\Psi$ and $f \in \mathcal{S}(\mathbb{R}^3;\mathcal{V})$ its moments $S_N^x(f_1,\dots,f_N) := \mathbb{E}_{\mu_x} \prod_i \Phi(f_i)$ satisfy the conclusions of Theorem A.21.*

Proof. By Proposition A.19, $\mu_{\iota y} = \iota_* \mu_y^H$ for μ^H-a.e. y, and Theorem A.21 applies to (μ_y^H) with $(\Xi,\nu) = (\mathcal{H}^{-r},\mu^H)$. Let Y be the Borel set of full μ^H-measure of the y for which both statements hold. Then ιY is Borel (Lusin–Souslin [23]) with $\mu(\iota Y) = 1$, and for $x = \iota y \in \iota Y$ the moments of μ_x in the variables $\Phi(f)$ are those of μ_y^H in the variables $\Psi(f)$. □

A.6.5. *Reconstruction in a coordinate direction. Conventions.* In Subsections A.6.5 and A.6.6, X, the translations τ_a, the coordinate reflections θ_j, the local σ-algebras $\Sigma(U)$ and the measure μ satisfy (H1)–(H3) of Subsection A.1, with $U(a)G := G\circ\tau_{-a}$, $\Theta_j G := G\circ\theta_j$ and $\theta_{j,c} := \tau_{ce_j}\theta_j\tau_{-ce_j}$. So $a \mapsto U(a)$ is a unitary representation of $\mathbb{R}^3$ on $L^2(\mu)$, Θ_j is a unitary involution, and $\Theta_j U(a) = U(\rho_j a)\Theta_j$ (A.3); in particular $\Theta_j U(te_j) = U(-te_j)\Theta_j$ and $\Theta_j U(y) = U(y)\Theta_j$ for $y \perp e_j$. Both $U(a)$ and Θ_j are multiplicative and commute with complex conjugation. The expectation $\mathbb{E}_\mu[G'G]$ is taken without conjugation throughout. The union over $n \in \mathbb{N}$ of the spaces $L^2(\Sigma(B_n),\mu)$, B_n the open ball of radius n, is dense in $L^2(\mu)$: every bounded K lies in some B_n, so $\bigvee_n \Sigma(B_n) = \mathcal{B}$ by (H2), and $\mathbb{E}[G \mid \Sigma(B_n)] \to G$ in $L^2(\mu)$ by the martingale convergence theorem. We call $G \in L^2(\Sigma(K),\mu)$ with K bounded *localised*.

Fix μ satisfying (H1)–(H3) and a direction j. Write $L^2_{j,+} := L^2(\Sigma(H^+_{j,0}), \mu)$, a closed subspace of $L^2(\mu)$, and define

$$\langle F, G\rangle_j := \mathbb{E}_\mu\big[\overline{\Theta_j F}\, G\big], \qquad F, G \in L^2_{j,+}. \tag{A.7}$$

The form is positive semidefinite by (H3) with $c = 0$. It is Hermitian: $\overline{\langle F, G\rangle_j} = \mathbb{E}_\mu[\Theta_j F\, \overline{G}] = \mathbb{E}_\mu[\Theta_j(F\,\overline{\Theta_j G})] = \mathbb{E}_\mu[\overline{\Theta_j G}\, F] = \langle G, F\rangle_j$, since Θ_j is a multiplicative involution and μ is θ_j-invariant. It is bounded, $|\langle F, G\rangle_j| \le \|F\|_2 \|G\|_2$, since Θ_j is unitary.

Proposition A.23 (reconstruction in direction j)**.** *Let $\mathcal{N}_j := \{F \in L^2_{j,+} : \langle F, F\rangle_j = 0\}$, $\mathcal{H}_j$ the completion of $L^2_{j,+}/\mathcal{N}_j$ for the form (A.7), $[\,\cdot\,] : L^2_{j,+} \to \mathcal{H}_j$ the quotient map, and $\Omega_j := [1]$.*

(i) *For $t \ge 0$ and $y \perp e_j$, $T_j(t)[G] := [U(te_j)G]$ and $V_j(y)[G] := [U(y)G]$ are well defined. T_j is a strongly continuous semigroup of self-adjoint contractions, so $T_j(t) = e^{-tH_j}$ with $H_j \ge 0$ self-adjoint. V_j is a strongly continuous unitary representation of $e_j^\perp \cong \mathbb{R}^2$. All these operators commute, and $T_j(t)\Omega_j = V_j(y)\Omega_j = \Omega_j$; in particular $H_j\Omega_j = 0$, and Ω_j is a unit vector.*

(ii) *For $G \in L^2_{j,+}$ and $a = a_j e_j + a_\perp$ with $a_j \ge 0$: $U(a)G \in L^2_{j,+}$ and $[U(a)G] = T_j(a_j)V_j(a_\perp)[G]$.*

(iii) *For $F \in L^2(\Sigma(H^-_{j,0}), \mu)$ and $G \in L^2_{j,+}$: $\overline{\Theta_j F} \in L^2_{j,+}$ and $\mathbb{E}_\mu[FG] = \langle[\overline{\Theta_j F}], [G]\rangle_{\mathcal{H}_j}$.*

Proof. This is the Osterwalder–Schrader construction [28], [19, Chapter 6]; we give the argument in the present setting. Two inequalities are used repeatedly: the Schwarz inequality for the positive semidefinite form (A.7), and $\|[F]\|^2_{\mathcal{H}_j} = \langle F, F\rangle_j \le \|F\|_2^2$.

Invariance of $L^2_{j,+}$. For $G \in L^2_{j,+}$ and a with $a_j \ge 0$, $U(a)G$ is $\Sigma(H^+_{j,0} + a)$-measurable by (H2), and $H^+_{j,0} + a = H^+_{j,a_j} \subset H^+_{j,0}$; so $U(a)G \in L^2_{j,+}$. This gives the first claim of (ii).

The semigroup T_j. For $F, G \in L^2_{j,+}$ and $t \ge 0$, by (A.3) and the invariance of μ,

$$\langle U(te_j)F, G\rangle_j = \mathbb{E}\big[\overline{U(-te_j)\Theta_j F}\; G\big] = \mathbb{E}\big[\overline{\Theta_j F}\; U(te_j)G\big] = \langle F, U(te_j)G\rangle_j;$$

in particular $\langle U(te_j)G, U(te_j)G\rangle_j = \langle G, U(2te_j)G\rangle_j$. So $U(te_j)$ is symmetric for the form, and by the Schwarz inequality for the form, $\langle U(te_j)G, U(te_j)G\rangle_j \le \langle G, G\rangle_j^{1/2}\, \langle U(2te_j)G, U(2te_j)G\rangle_j^{1/2}$. Applying this successively at $t, 2t, \dots, 2^{n-1}t$ and using $\langle U(se_j)G, U(se_j)G\rangle_j \le \|U(se_j)G\|_2^2 = \|G\|_2^2$ at the last step gives

$$\langle U(te_j)G, U(te_j)G\rangle_j \le \langle G, G\rangle_j^{1-2^{-n}}\, \|G\|_2^{2^{1-n}},$$

and $n \to \infty$ gives $\langle U(te_j)G, U(te_j)G\rangle_j \le \langle G, G\rangle_j$. Hence $U(te_j)$ maps $\mathcal{N}_j$ into $\mathcal{N}_j$, $T_j(t)$ is well defined on the quotient and is a contraction there, and it is symmetric by the identity above; it extends to a self-adjoint contraction on $\mathcal{H}_j$. The semigroup law $T_j(t)T_j(s) = T_j(t+s)$ is inherited from U, and $T_j(t) = T_j(t/2)^2 \ge 0$. For the strong continuity, $\|T_j(t)[G] - [G]\|_{\mathcal{H}_j} \le \|U(te_j)G - G\|_2 \to 0$ as $t \downarrow 0$ by Lemma A.14, on the dense set $[L^2_{j,+}]$, and the $T_j(t)$ are uniformly bounded. A strongly continuous semigroup of positive self-adjoint contractions with $T_j(0) = 1$ is e^{-tH_j} with $H_j \ge 0$ self-adjoint (by the spectral theorem, as in [19, Chapter 6]).

The transverse translations. For $y \perp e_j$, Θ_j commutes with $U(y)$ by (A.3), so $\langle U(y)F, U(y)G\rangle_j = \mathbb{E}[\overline{U(y)\Theta_j F}\, U(y)G] = \langle F, G\rangle_j$ by invariance of μ. Thus $V_j(y)$ preserves $\mathcal{N}_j$, is isometric on the quotient and has the isometric inverse $V_j(-y)$; it extends to a unitary operator, and $y \mapsto V_j(y)$ is a representation, strongly continuous by the same estimate as for T_j. The operators $T_j(t)$ and $V_j(y)$ commute because the $U(a)$ do, and $T_j(t)\Omega_j = [U(te_j)1] = [1] = \Omega_j$, $V_j(y)\Omega_j = \Omega_j$. Hence $H_j\Omega_j = 0$, and $\|\Omega_j\|^2 = \mathbb{E}[\overline{\Theta_j 1}\, 1] = 1$. This proves (i); the identity in (ii) is the definition, $[U(a)G] = [U(a_j e_j)U(a_\perp)G] = T_j(a_j)V_j(a_\perp)[G]$.

(iii) F is $\Sigma(H^-_{j,0})$-measurable, so $\Theta_j F$ and $\overline{\Theta_j F}$ are $\Sigma(\rho_j H^-_{j,0}) = \Sigma(H^+_{j,0})$-measurable by (H2). Since Θ_j is an involution commuting with conjugation, $\overline{\Theta_j \overline{\Theta_j F}} = F$, and $\langle \overline{\Theta_j F}, G\rangle_j = \mathbb{E}[FG]$. □

Proposition A.24 (zero-energy vectors, invariant functions and the joint vacuum)**.** *In the setting of Proposition A.23, let Q_j be the orthogonal projection of $\mathcal{H}_j$ onto the joint fixed space of $\{T_j(t)\}_{t\ge 0} \cup \{V_j(y)\}_{y\perp e_j}$, and $P_{\ker H_j}$ that onto $\ker H_j$. For every $G \in L^2_{j,+}$:*

(i) *$P_j G \in L^2_{j,+}$, $P_{\ker H_j}[G] = [P_j G]$, and $\|[P_j G]\|^2_{\mathcal{H}_j} = \mathbb{E}_\mu |P_j G|^2$;*
(ii) *$\mathbb{E}[G \mid \mathcal{I}] \in L^2_{j,+}$ and $Q_j[G] = [\mathbb{E}[G \mid \mathcal{I}]]$;*
(iii) *if μ is ergodic for the $\mathbb{R}^3$-action, the joint fixed space is $\mathbb{C}\Omega_j$;*
(iv) *conversely, if the joint fixed space is $\mathbb{C}\Omega_j$, then μ is ergodic for the $\mathbb{R}^3$-action.*

So the joint vacuum is unique, that is, the joint fixed space is $\mathbb{C}\Omega_j$, if and only if μ is ergodic for the $\mathbb{R}^3$-action.

Proof. (i) $P_j G$ is e_j-invariant, so $P_j G \in L^2_{j,+}$ by Lemma A.16(i) with $J = \{j\}$. The quotient map is bounded and linear on $L^2_{j,+}$, and $t \mapsto U(te_j)G$ is continuous into $L^2_{j,+}$ by Lemma A.14; so the quotient map commutes with the Bochner integrals,

$$\frac{1}{T}\int_0^T T_j(t)[G]\,dt = \Big[\frac{1}{T}\int_0^T U(te_j)G\,dt\Big].$$

As $T \to \infty$ the left side tends to $P_{\ker H_j}[G]$ by the spectral theorem, since $T^{-1}\int_0^T e^{-t\lambda}dt \to \mathbf{1}_{\{\lambda=0\}}$ boundedly on $[0,\infty)$; the right side tends to $[P_j G]$ by the one-sided means of Lemma A.15(i) and the continuity of the quotient map. Finally $\Theta_j P_j G = P_j G$ by Lemma A.16(ii), so $\|[P_j G]\|^2 = \mathbb{E}[\overline{\Theta_j P_j G}\, P_j G] = \mathbb{E}|P_j G|^2$.

(ii) $\mathbb{E}[G \mid \mathcal{I}] = P_{\{1,2,3\}}G$ by Lemma A.15(ii); it is invariant under all translations, so it lies in $L^2_{j,+}$ by Lemma A.16(i). Let k, k' be the two indices other than j, and put $W(a) := T_j(a_j)V_j(a_k e_k + a_{k'} e_{k'})$ for $a_j \ge 0$. The means of $W(a)$ over $a \in [0,T]\times[-T,T]^2$ (coordinates ordered as $(a_j, a_k, a_{k'})$) are the product of three commuting means: the one-sided mean of T_j, which tends strongly to $P_{\ker H_j}$ as in (i), and the two-sided means of the unitary groups $s \mapsto V_j(se_k)$ and $s \mapsto V_j(se_{k'})$, which tend strongly to the projections onto their fixed spaces (by the spectral theorem, as in Lemma A.15(i)). These three projections commute, and a product of commuting, uniformly bounded, strongly convergent families converges strongly to the product of the limits, which is the projection onto the intersection of the three fixed spaces, that is Q_j. On the other hand $W(a)[G] = [U(a)G]$ by Proposition A.23(ii), and the means of $U(a)G$ over the same set are the product of the commuting means $T^{-1}\int_0^T U(te_j)\,dt$, $(2T)^{-1}\int_{-T}^T U(se_k)\,ds$ and $(2T)^{-1}\int_{-T}^T U(se_{k'})\,ds$, applied to G; by Lemma A.15(i) these converge strongly to P_j, P_k and $P_{k'}$, so their product converges strongly to $P_j P_k P_{k'} = P_{\{1,2,3\}}$ (Lemma A.15(ii)). The means of $U(a)G$ lie in $L^2_{j,+}$ and are mapped by the quotient map to the means of $W(a)[G]$ (the quotient map commutes with Bochner integrals, as in (i)); by continuity of the quotient map the limit is $[P_{\{1,2,3\}}G] = [\mathbb{E}[G \mid \mathcal{I}]]$.

(iii) If μ is ergodic, $\mathcal{I}$ is trivial and $\mathbb{E}[G \mid \mathcal{I}] = \mathbb{E}_\mu G$ a.s., so by (ii) $Q_j[G] = \mathbb{E}_\mu[G]\,\Omega_j = \langle \Omega_j, [G]\rangle_{\mathcal{H}_j}\Omega_j$, because $\langle [1], [G]\rangle_{\mathcal{H}_j} = \mathbb{E}[\overline{\Theta_j 1}\, G] = \mathbb{E}_\mu G$. Since $[L^2_{j,+}]$ is dense in $\mathcal{H}_j$, Q_j is the projection onto $\mathbb{C}\Omega_j$.

(iv) Suppose μ is not ergodic: there is a translation-invariant Borel set B with $0 < \mu(B) < 1$. Put $W := \mathbf{1}_B - \mu(B)$. It is invariant under all translations, so $W \in L^2_{j,+}$ and $\Theta_j W = W$ by Lemma A.16(i)–(ii). The vector $[W]$ is fixed by every $T_j(t)$ and $V_j(y)$, since $[U(a)W] = [W]$; it is orthogonal to Ω_j, since $\langle \Omega_j, [W]\rangle_{\mathcal{H}_j} = \mathbb{E}_\mu W = 0$; and $\|[W]\|^2_{\mathcal{H}_j} = \mathbb{E}[\overline{\Theta_j W}\, W] = \mathbb{E}|W|^2 = \mu(B)(1-\mu(B)) > 0$. So the joint fixed space is larger than $\mathbb{C}\Omega_j$. □

A.6.6. *Clustering.*

Definition A.25 (forms of clustering)**.** Let μ satisfy (H1). We say that μ satisfies (CL1), *clustering in the mean*, if the Cesàro averages of $\mathbb{E}_\mu[G'\,U(a)G]$ over the cubes Λ_T converge to $\mathbb{E}_\mu[G']\,\mathbb{E}_\mu[G]$ as $T\to\infty$, and (CL2), *clustering under diagonal separation*, if $\mathbb{E}_\mu[G'\,U(a)G]\to \mathbb{E}_\mu[G']\,\mathbb{E}_\mu[G]$ as $\min_j|a_j|\to\infty$, in each case for all $G,G'\in L^2(\mu)$.

A measure satisfies OS4 in the weak form if it is ergodic for the action of the translation group $\mathbb{R}^3$.

Throughout Subsection A.6.6, μ satisfies (H1)–(H3), and $\mu=\int \mu_\xi\,d\nu(\xi)$ is the decomposition of Theorem A.18. We say that a statement holds *on almost every ergodic component* if it holds for μ_ξ for ν-a.e. ξ.

Theorem A.26 (clustering in the mean)**.** *For all $G,G'\in L^2(\mu)$,*

$$\lim_{T\to\infty}\frac{1}{|\Lambda_T|}\int_{\Lambda_T}\mathbb{E}_\mu\big[G'\,U(a)G\big]\,da=\mathbb{E}_\mu\big[\mathbb{E}[G'\mid\mathcal{I}]\,\mathbb{E}[G\mid\mathcal{I}]\big]=\int_\Xi \mathbb{E}_{\mu_\xi}[G']\,\mathbb{E}_{\mu_\xi}[G]\,d\nu(\xi).$$

If μ is ergodic for the $\mathbb{R}^3$-action, the right side is $\mathbb{E}_\mu[G']\,\mathbb{E}_\mu[G]$; in particular (CL1) *holds on almost every ergodic component. For $\emptyset\neq J\subset\{1,2,3\}$ the first equality holds with Λ_T replaced by Λ_T^J and $\mathcal{I}$ by $\mathcal{I}_J$.*

Proof. The map $H\mapsto\mathbb{E}_\mu[G'H]$ is a continuous linear functional on $L^2(\mu)$, and $a\mapsto U(a)G$ is continuous by Lemma A.14; so the mean of $\mathbb{E}_\mu[G'\,U(a)G]$ over Λ_T^J equals $\mathbb{E}_\mu[G'\,A_T^JG]$, where $A_T^JG:=|\Lambda_T^J|^{-1}\int_{\Lambda_T^J}U(a)G\,da$. By Lemma A.15(i)–(ii), $A_T^JG\to P_JG=\mathbb{E}[G\mid\mathcal{I}_J]$ in $L^2(\mu)$, so the mean tends to $\mathbb{E}_\mu[G'\,\mathbb{E}[G\mid\mathcal{I}_J]]=\mathbb{E}_\mu[\mathbb{E}[G'\mid\mathcal{I}_J]\,\mathbb{E}[G\mid\mathcal{I}_J]]$, the last because $\mathbb{E}[G\mid\mathcal{I}_J]$ is square-integrable and $\mathcal{I}_J$-measurable. For $J=\{1,2,3\}$, the function $H:=\mathbb{E}[G'\mid\mathcal{I}]\,\mathbb{E}[G\mid\mathcal{I}]$ is integrable and, by Theorem A.18(a), $H(\xi)=\mathbb{E}_{\mu_\xi}[G']\,\mathbb{E}_{\mu_\xi}[G]$ for ν-a.e. ξ; since $\nu=\mu$, $\mathbb{E}_\mu H=\int\mathbb{E}_{\mu_\xi}[G']\,\mathbb{E}_{\mu_\xi}[G]\,d\nu(\xi)$. If μ is ergodic, $\mathcal{I}$ is trivial and $\mathbb{E}[G\mid\mathcal{I}]=\mathbb{E}_\mu G$. Almost every ergodic component satisfies (H1)–(H3) (Theorem A.18), so the statement applies to it with $G,G'\in L^2(\mu_\xi)$. □

The next lemma is the step in which reflection positivity enters the clustering: through the reconstruction, the part of a localised function orthogonal to the e_ℓ-invariant functions decays under the semigroup e^{-tH_ℓ}.

Lemma A.27 (decay of the non-invariant part along one axis)**.** *Let $\ell\in\{1,2,3\}$, $R>0$, $G'\in L^2(\Sigma(H^-_{\ell,0}),\mu)$, and $\tilde G\in L^2(\Sigma(D),\mu)$ for an open set $D\subset H^+_{\ell,-R}$. Then*

$$\mathbb{E}_\mu\big[G'\,U(a)(1-P_\ell)\tilde G\big]\to 0\qquad \text{as } a_\ell\to+\infty,\ \text{uniformly in } a_\perp.$$

Proof. Put $s:=2R$ and $Y:=U(se_\ell)\tilde G$. By (H2), Y is $\Sigma(D+se_\ell)$-measurable, and $D+se_\ell\subset H^+_{\ell,R}\subset H^+_{\ell,0}$; so $Y\in L^2_{\ell,+}$. Also $P_\ell Y\in L^2_{\ell,+}$ by Lemma A.16(i). Since P_ℓ commutes with the translations (Lemma A.15(ii)), for $a_\ell\ge s$,

$$U(a)(1-P_\ell)\tilde G=U\big((a_\ell-s)e_\ell+a_\perp\big)(1-P_\ell)Y,$$

and the right side lies in $L^2_{\ell,+}$ by Proposition A.23(ii). By Proposition A.23(ii)–(iii), with $F:=\overline{\Theta_\ell G'}\in L^2_{\ell,+}$, and by Proposition A.24(i), which gives $[(1-P_\ell)Y]=(1-P_{\ker H_\ell})[Y]$,

$$\mathbb{E}_\mu\big[G'\,U(a)(1-P_\ell)\tilde G\big]=\Big\langle[F],\ V_\ell(a_\perp)\,e^{-(a_\ell-s)H_\ell}\,(1-P_{\ker H_\ell})[Y]\Big\rangle_{\mathcal{H}_\ell}.$$

Since $V_\ell(a_\perp)$ is unitary, the modulus is at most $\|[F]\|\,\|e^{-(a_\ell-s)H_\ell}(1-P_{\ker H_\ell})[Y]\|$, which does not depend on $a_\perp$ and tends to 0 as $a_\ell\to\infty$ by the spectral theorem and dominated convergence ($e^{-t\lambda}\mathbf{1}_{\{\lambda>0\}}\to 0$ pointwise and boundedly on $[0,\infty)$). □

Theorem A.28 (clustering under diagonal separation)**.** *For all $G, G' \in L^2(\mu)$,*

$$\lim_{\min_j |a_j| \to \infty} \mathbb{E}_\mu\big[G'\,U(a)G\big] = \mathbb{E}_\mu\big[\mathbb{E}[G' \mid \mathcal{I}]\,\mathbb{E}[G \mid \mathcal{I}]\big].$$

If μ is ergodic for the $\mathbb{R}^3$-action, the limit is $\mathbb{E}_\mu[G']\,\mathbb{E}_\mu[G]$; in particular (CL2) *holds on almost every ergodic component.*

Proof. Step 1 (reductions). Since

$$|\mathbb{E}[G'U(a)G] - \mathbb{E}[G'_nU(a)G_n]| \le \|G' - G'_n\|_2\|G\|_2 + \|G'_n\|_2\|G - G_n\|_2$$

uniformly in a, and the right side of the claim is continuous in (G, G') in the same way, it suffices to take $G \in L^2(\Sigma(K))$ and $G' \in L^2(\Sigma(K'))$ with K, K' bounded (Subsection A.6.5). A sequence $a^{(n)}$ with $\min_j|a_j^{(n)}| \to \infty$ splits into finitely many subsequences on each of which the signs $\sigma_j = \operatorname{sgn} a_j^{(n)}$ are fixed, and it suffices to prove the claim along each. On such a subsequence, replace (G', G, a) by $(\Theta_j G', \Theta_j G, \rho_j a)$ for each j with $\sigma_j = -1$. This leaves the left side unchanged, since $\mathbb{E}[\Theta_j G'\,U(\rho_j a)\Theta_j G] = \mathbb{E}[\Theta_j(G'\,U(a)G)] = \mathbb{E}[G'\,U(a)G]$ by (A.3) and the θ_j-invariance of μ; it leaves the right side unchanged, since $\mathbb{E}[\Theta_j G \mid \mathcal{I}] = \Theta_j\mathbb{E}[G \mid \mathcal{I}]$ by Lemma A.15(ii); and it replaces K, K' by the bounded sets $\rho_j K, \rho_j K'$ (H2). So we may assume $a_j \to +\infty$ for every j. By translation invariance and multiplicativity, $\mathbb{E}[G'\,U(a)G] = \mathbb{E}[(U(b)G')\,U(a)(U(b)G)]$ for every $b \in \mathbb{R}^3$. If $K \cup K' \subset (-r, r)^3$, the choice $b = -(r+1)(e_1 + e_2 + e_3)$ makes $U(b)G'$ measurable for $\Sigma(K' + b)$ with $K' + b$ in the open negative octant $\bigcap_j H^-_{j,0}$, and $U(b)G$ measurable for $\Sigma(K + b)$ with $K + b \subset \{x : |x_j| < R\ \forall j\}$, $R := 2r + 1$. Renaming, we may assume $K' \subset \bigcap_j H^-_{j,0}$ and $K \subset \{|x_j| < R\ \forall j\}$.

Step 2 (decomposition). By Lemma A.15(ii), $P_1P_2P_3 = P_{\{1,2,3\}}$, so

$$1 = P_{\{1,2,3\}} + (1 - P_1) + P_1(1 - P_2) + P_1P_2(1 - P_3).$$

The first term contributes $\mathbb{E}[G'\,U(a)\mathbb{E}[G \mid \mathcal{I}]] = \mathbb{E}[G'\,\mathbb{E}[G \mid \mathcal{I}]] = \mathbb{E}[\mathbb{E}[G' \mid \mathcal{I}]\,\mathbb{E}[G \mid \mathcal{I}]]$ for every a, since $\mathbb{E}[G \mid \mathcal{I}]$ is translation invariant. It remains to show that each of the other three terms tends to 0 as $a_\ell \to +\infty$, uniformly in the other coordinates of a, where ℓ is the index of the factor $(1 - P_\ell)$.

Step 3 (one term). Fix ℓ, let $J := \{1, \dots, \ell - 1\}$ and $\tilde G := P_J G$ ($\tilde G = G$ for $\ell = 1$). Since P_J and P_ℓ commute, the term is $\mathbb{E}[G'\,U(a)(1 - P_\ell)\tilde G]$. By Lemma A.16(iii), $\tilde G \in L^2(\Sigma(K + V_J))$, and $K + V_J \subset H^+_{\ell,-R}$ because $e_\ell \notin V_J$ and $K \subset \{|x_\ell| < R\}$. Since $G' \in L^2(\Sigma(H^-_{\ell,0}))$, Lemma A.27 with $D = K + V_J$ gives the claim.

Finally, if μ is ergodic, $\mathbb{E}[G \mid \mathcal{I}] = \mathbb{E}_\mu G$; and almost every ergodic component satisfies (H1)–(H3) (Theorem A.18), so the statement applies to it with $G, G' \in L^2(\mu_\xi)$. □

Corollary A.29 (the ergodic components)**.** *For ν-a.e. ξ:*

(i) *μ_ξ is translation invariant and ergodic for the $\mathbb{R}^3$-action, invariant under $\theta_1, \theta_2, \theta_3$, reflection positive about every coordinate plane in the sense of* (H3)*, and satisfies* (H2)*; so it satisfies* (H1)–(H3)*, and* OS4 *in the weak form;*

(ii) (CL1) *and* (CL2) *hold for μ_ξ: the Cesàro means over Λ_T of $\mathbb{E}_{\mu_\xi}[G'\,U(a)G]$, and $\mathbb{E}_{\mu_\xi}[G'\,U(a)G]$ itself as $\min_j|a_j| \to \infty$, converge to $\mathbb{E}_{\mu_\xi}[G']\,\mathbb{E}_{\mu_\xi}[G]$, for all $G, G' \in L^2(\mu_\xi)$;*

(iii) *in each coordinate direction j, the reconstruction of Proposition A.23 from μ_ξ has a unique joint vacuum: the joint fixed space of the e^{-tH_j}, $t \ge 0$, and of the transverse translations $V_j(y)$, $y \perp e_j$, is $\mathbb{C}\Omega_j$.*

Proof. (i) is Theorem A.18(b)–(e). (ii) is Theorems A.26 and A.28 for the ergodic measure μ_ξ, and (iii) is Proposition A.24(iii) for it. □

References

[1] P. W. Anderson, Plasmons, gauge invariance, and mass, Phys. Rev. **130** (1963), 439–442.

[2] T. Bałaban, $(\text{Higgs})_{2,3}$ quantum fields in a finite volume. I. A lower bound, Comm. Math. Phys. **85** (1982), 603–626.

[3] T. Bałaban, $(\text{Higgs})_{2,3}$ quantum fields in a finite volume. II. An upper bound, Comm. Math. Phys. **86** (1982), 555–594.

[4] T. Bałaban, $(\text{Higgs})_{2,3}$ quantum fields in a finite volume. III. Renormalization, Comm. Math. Phys. **88** (1983), 411–445.

[5] T. Bałaban, J. Imbrie, A. Jaffe, D. Brydges, The mass gap for Higgs models on a unit lattice, Ann. Physics **158** (1984), 281–319.

[6] D. C. Brydges, J. Fröhlich, E. Seiler, On the construction of quantized gauge fields. I. General results, Ann. Physics **121** (1979), 227–284.

[7] D. C. Brydges, J. Fröhlich, E. Seiler, Construction of quantised gauge fields. II. Convergence of the lattice approximation, Comm. Math. Phys. **71** (1980), 159–205.

[8] D. C. Brydges, J. Fröhlich, E. Seiler, On the construction of quantized gauge fields. III. The two-dimensional abelian Higgs model without cutoffs, Comm. Math. Phys. **79** (1981), 353–399.

[9] S. Coleman, E. Weinberg, Radiative corrections as the origin of spontaneous symmetry breaking, Phys. Rev. D **7** (1973), 1888–1910.

[10] G. F. De Angelis, D. de Falco, F. Guerra, Scalar quantum electrodynamics on lattice. Correlation inequalities and infinite volume limit, Phys. Lett. B **68** (1977), 255–257.

[11] R. L. Dobrushin, The description of a random field by means of conditional probabilities and conditions of its regularity, Theory Probab. Appl. **13** (1968), 197–224.

[12] F. Englert, R. Brout, Broken symmetry and the mass of gauge vector mesons, Phys. Rev. Lett. **13** (1964), 321–323.

[13] J. Fröhlich, R. Israel, E. H. Lieb, B. Simon, Phase transitions and reflection positivity. I. General theory and long range lattice models, Comm. Math. Phys. **62** (1978), 1–34.

[14] J. Fröhlich, B. Simon, T. Spencer, Infrared bounds, phase transitions and continuous symmetry breaking, Comm. Math. Phys. **50** (1976), 79–95.

[15] I. M. Gel'fand, N. Ya. Vilenkin, *Generalized Functions. Vol. 4: Applications of Harmonic Analysis*, Academic Press, New York, 1964.

[16] H.-O. Georgii, *Gibbs Measures and Phase Transitions*, 2nd ed., de Gruyter, Berlin, 2011.

[17] J. Ginibre, General formulation of Griffiths' inequalities, Comm. Math. Phys. **16** (1970), 310–328.

[18] V. L. Ginzburg, L. D. Landau, On the theory of superconductivity (Russian), Zh. Eksp. Teor. Fiz. **20** (1950), 1064–1082.

[19] J. Glimm, A. Jaffe, *Quantum Physics. A Functional Integral Point of View*, 2nd ed., Springer, New York, 1987.

[20] M. Göpfert, G. Mack, Proof of confinement of static quarks in 3-dimensional $U(1)$ lattice gauge theory for all values of the coupling constant, Comm. Math. Phys. **82** (1982), 545–606.

[21] B. I. Halperin, T. C. Lubensky, S.-k. Ma, First-order phase transitions in superconductors and smectic-A liquid crystals, Phys. Rev. Lett. **32** (1974), 292–295.

[22] P. W. Higgs, Broken symmetries and the masses of gauge bosons, Phys. Rev. Lett. **13** (1964), 508–509.

[23] A. S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics 156, Springer, New York, 1995.

[24] C. King, The $U(1)$ Higgs model. I. The continuum limit, Comm. Math. Phys. **102** (1986), 649–677.

[25] C. King, The $U(1)$ Higgs model. II. The infinite volume limit, Comm. Math. Phys. **103** (1986), 323–349.

[26] O. E. Lanford III, D. Ruelle, Observables at infinity and states with short range correlations in statistical mechanics, Comm. Math. Phys. **13** (1969), 194–215.

[27] R. A. Minlos, Generalized random processes and their extension in measure (Russian), Trudy Moskov. Mat. Obšč. **8** (1959), 497–518; English transl.: Generalized random processes and their extension to a measure, in: *Selected Transl. Math. Statist. and Prob.*, Vol. 3, American Mathematical Society, Providence, RI, 1963, pp. 291–313.

[28] K. Osterwalder, R. Schrader, Axioms for Euclidean Green's functions, Comm. Math. Phys. **31** (1973), 83–112.

[29] K. Osterwalder, R. Schrader, Axioms for Euclidean Green's functions. II, Comm. Math. Phys. **42** (1975), 281–305.

[30] M. Reed, B. Simon, *Methods of Modern Mathematical Physics. I. Functional Analysis*, Academic Press, New York, 1972.

[31] E. Seiler, *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics*, Lecture Notes in Physics 159, Springer, Berlin, 1982.

[32] B. Simon, *Trace Ideals and Their Applications*, 2nd ed., Mathematical Surveys and Monographs 120, American Mathematical Society, Providence, RI, 2005.

[33] J. Villain, Theory of one- and two-dimensional magnets with an easy magnetization plane. II. The planar, classical, two-dimensional magnet, J. Physique **36** (1975), 581–590.

[34] *Osterwalder–Schrader axioms for scalar QED$_3$ in the Coulomb phase*, companion paper.

[35] *Rotation invariance from lattice symmetry and reflection positivity*, manuscript RPM1 (Theorem A), public repository *Yang–Mills-LLM-experiment*, version of 30 October 2026.