THE ZARANKIEWICZ NUMBER $z(13, 15; 3, 4)$

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Abstract. We establish $z(13, 15; 3, 4) = 114$: a 13×15 binary matrix with 115 ones necessarily contains an all-one submatrix on three rows and four columns. The lower bound is attained by a previously published construction of Rosero. For the upper bound, we exclude dense eight-row column neighborhoods by a finite classification and a complete extension check. The resulting local inequality reduces a hypothetical counterexample to 6,019 pairs of degree sequences. Each pair is excluded by an exact integer certificate for a system of neighborhood-type equations. The proof is computer-assisted; its mathematical reductions, finite search domains, classification representatives, and certificate conditions are given explicitly.

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1. Statement and notation

Place dots at intersections of thirteen horizontal rows and fifteen vertical columns. Require that every choice of three rows and four columns leaves at least one of their twelve intersections empty. How many dots can be placed? We prove that the answer is 114.

For positive integers m, n, s, t, let $z(m, n; s, t)$ denote the largest number of ones in an $m \times n$ binary matrix that contains no all-one submatrix on s distinct rows and t distinct columns. The selected rows and columns need not be consecutive. The order of the two parts matters: throughout this paper the forbidden submatrix has *three rows and four columns*.

Theorem 1.1. *The exact value of the oriented Zarankiewicz number is*

$$z(13, 15; 3, 4) = 114.$$

The general matrix problem appears in Guy's treatment of the Zarankiewicz problem [2]. In the notation of OEIS A006625 [3], Theorem 1.1 gives $a(13) = 115$. The construction establishing the lower bound is due to Rosero [4]; the upper bound is the result proved here.

If A is a binary matrix, write $|A|$ for its number of ones and $A_i \subseteq [n]$ for the support of its ith row, where $[n] = \{1, \dots, n\}$. The avoidance condition is equivalent to

$$|A_i \cap A_j \cap A_k| \leq 3 \qquad (i, j, k \text{ distinct}). \tag{1}$$

Two matrices are equivalent if one is obtained from the other by independent row and column permutations. This equivalence preserves all avoidance and extension questions considered below.

A superscript in a degree sequence indicates multiplicity; for example, $(8^7, 7)$ means seven entries equal to eight and one entry equal to seven.

The proof has two finite components. First, a classification of 8×14 matrices with 63 ones, together with extension checks, improves one local neighborhood bound from 64 to 62. Second, exact linear infeasibility certificates exclude every remaining pair of global degree sequences. These computations enter as explicit finite lemmas; the surrounding implications are proved below.

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2. A construction with 114 ones

The following matrix is Rosero's 114-one construction [4]:

$$A_0 = \begin{pmatrix} 011010110001111 \\ 101101111001100 \\ 010100101111110 \\ 101110000111101 \\ 100010110010100 \\ 111100111010001 \\ 011101001101011 \\ 110001011011101 \\ 111011100111000 \\ 000111110011011 \\ 110111100100101 \\ 011111010010110 \\ 001001101110111 \end{pmatrix}. \tag{2}$$

Here each string is one row and its digits are the matrix entries in column order. Twelve rows have weight nine and one has weight six, so $|A_0| = 12 \cdot 9 + 6 = 114$. Direct intersection of the $\binom{13}{3} = 286$ triples of displayed rows gives

$$\#\{\{i,j,k\} : |A_{0,i} \cap A_{0,j} \cap A_{0,k}| = r\} = \begin{cases} 11, & r = 0, \\ 0, & r = 1, \\ 72, & r = 2, \\ 203, & r = 3. \end{cases}$$

Consequently A_0 satisfies (1), proving $z(13, 15; 3, 4) \geq 114$.

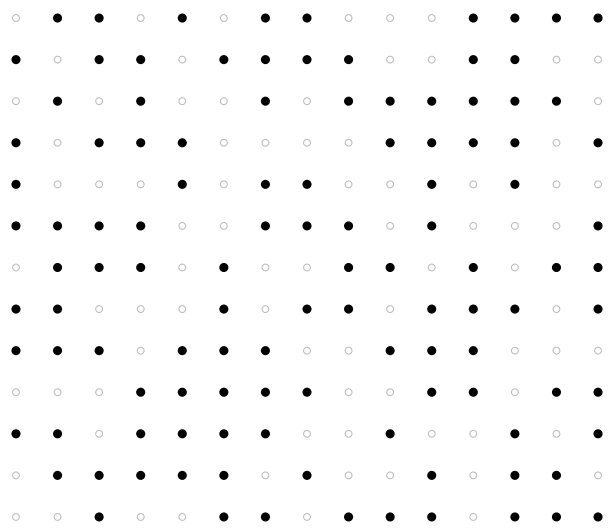


Figure 1. The 114 occupied intersections of (2). Filled circles denote ones; open circles denote zeros. Every three rows have at most three filled columns in common. The construction is due to Rosero [4].

3. Smaller matrices and column neighborhoods

We use the following bounds, denoted by $Z(d)$:

d	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$Z(d)$	0	14	28	30	36	44	50	56	64	70	77	84	91	98

(3)

Thus $z(d,14;3,3) \le Z(d)$. For $6 \le d \le 13$ these are the bounds in Table 4 of Collins, Riasanovsky, Wallace, and Radziszowski [1]. The same table states uniqueness, up to equivalence, of the 8×14 matrix with 64 ones and no all-one 3×3 submatrix.

For completeness, the smaller values in (3) follow from elementary counting. If $q_1, \dots, q_{14}$ are column degrees of a $d \times 14$ matrix avoiding an all-one 3×3 submatrix, then

$$\sum_{j=1}^{14} \binom{q_j}{3} \le 2\binom{d}{3}. \tag{4}$$

For $d = 0, 1, 2$ the area bound suffices. For $d = 3$, at most two columns have degree three, giving at most 30 ones. For $d = 4$, the inequality $q \le 2 + \binom{q}{3}$ for $0 \le q \le 4$ gives at most $28 + 8 = 36$ ones. For $d = 5$, use $3q \le 8 + \binom{q}{3}$ for $0 \le q \le 5$ to obtain $3 \sum q_j \le 112 + 20 = 132$.

We next describe the unique 64-one matrix. Index its rows by $\mathbb{Z}/8\mathbb{Z}$. Its first eight columns have zero sets

$$\{i, i+1, i+3\}, \qquad i \in \mathbb{Z}/8\mathbb{Z}. \tag{5}$$

Its other six columns have one sets

$$\{a, a+4, b, b+4\}, \qquad 0 \le a < b \le 3. \tag{6}$$

Call the resulting matrix G. Its first eight columns have weight five and its last six have weight four, so $|G| = 64$. Each row has weight eight. Intersecting its 56 triples of row supports verifies that it avoids an all-one 3×3 submatrix. By the cited uniqueness theorem, every 64-one 8×14 matrix with this property is equivalent to G. Alternatively, the fact that all its row degrees must be eight follows from $Z(7) = 56$: deletion of any row leaves at most 56 ones.

Lemma 3.1 (Column-neighborhood reduction)**.** *Let A be a 13×15 matrix satisfying* (1)*. Choose a column c and any k rows whose entries in column c are one. On deleting column c from these rows, the resulting $k \times 14$ matrix contains no all-one 3×3 submatrix.*

Proof. Three rows and three columns of ones in the smaller matrix, together with column c, would give three rows and four columns of ones in A. □

When $k = 8$, write the smaller matrix as B. After permuting columns, the eight corresponding rows of A form $(B \mid \mathbf{1})$, obtained by adjoining an all-one column. The other five rows of A are unrestricted in this last column. In particular, every extension search below allows both possible entries there.

4. The 63-one classification

Call an avoiding matrix *maximal* if changing any zero to one creates an all-one 3×3 submatrix. If B is 8×14, $|B| = 63$, and B avoids this submatrix, deleting a row shows that every row degree is at least $63 - 56 = 7$. Its degree sequence is therefore obtained by adding a partition of seven to (7^8). There are fifteen possibilities, listed in Table 1.

If the degree sequence is $(8^7, 7)$ and B is not maximal, adding one permissible entry gives a matrix equivalent to G. Thus B is equivalent to a one-entry deletion of G. Every other degree sequence has an entry greater than eight. No matrix with such a sequence can be extended to 64 ones, since all rows in a 64-one avoiding matrix have degree eight. Such a matrix is automatically maximal.

Lemma 4.1 (Finite classification)**.** *Up to equivalence, the maximal* 8×14 *matrices with* 63 *ones and no all-one* 3×3 *submatrix are exactly the seventeen matrices* $B_1, \ldots, B_{17}$ *in Appendix A. Their degree sequences and class counts are given in Table 1.*

TABLE 1. The complete maximal 63-one classification. The last column counts distinct labeled forms satisfying both lexicographic orders specified in the proof.

Row degrees	Equivalence classes	Ordered forms
$(8^7, 7)$	10	1521
$(9, 8^5, 7^2)$	2	37
$(9^2, 8^3, 7^3)$	3	14
$(10, 8^4, 7^3)$	1	9
$(9^3, 8, 7^4)$	0	0
$(10, 9, 8^2, 7^4)$	0	0
$(10, 9^2, 7^5)$	1	4
$(11, 8^3, 7^4)$	0	0
$(10^2, 8, 7^5)$	0	0
$(11, 9, 8, 7^5)$	0	0
$(11, 10, 7^6)$	0	0
$(12, 8^2, 7^5)$	0	0
$(12, 9, 7^6)$	0	0
$(13, 8, 7^6)$	0	0
$(14, 7^7)$	0	0
Total	17	1585

Finite verification of Lemma 4.1. Here are the exact constraints and the completeness argument used for the classification. Fix a nonincreasing degree sequence $p = (p_1, \ldots, p_8)$ in Table 1. A candidate is a binary array $x = (x_{ic})$ satisfying

$$\sum_{c=1}^{14} x_{ic} = p_i \qquad (1 \le i \le 8), \tag{7}$$

$$\sum_{i \in I,\, c \in J} x_{ic} \le 8 \qquad \left(I \in \binom{[8]}{3},\ J \in \binom{[14]}{3}\right). \tag{8}$$

Require columns to be nonincreasing in lexicographic order, and impose the same order on rows within each group of equal row degrees. These restrictions preserve at least one representative of every equivalence class: a lexicographically greatest row-major array in the finite permutation orbit satisfies both orders.

For $p = (8^7, 7)$, also require every zero in row eight to be blocked. Explicitly, whenever $x_{8c} = 0$, there must be distinct $i, j \in [7]$ such that

$$x_{ic} = x_{jc} = 1, \qquad \sum_{q=1}^{14} x_{8q} x_{iq} x_{jq} \ge 2. \tag{9}$$

This is exactly the condition that adding the entry $(8, c)$ creates an all-one 3×3 submatrix. A zero in any other row cannot be added, since the resulting 64-one matrix would have a row of degree nine. Thus (9) is equivalent to maximality here.

The constraints were checked as finite Boolean systems. To specify their encoding without a cardinality convention, an exact row degree p_i is expressed by

$$\bigvee_{c\in S} \neg x_{ic} \quad (|S| = p_i + 1), \qquad \bigvee_{c\in T} x_{ic} \quad (|T| = 15 - p_i).$$

For (8), use the disjunction of the nine negated entries for each I, J. Lexicographic comparisons use prefix-equality variables. Condition (9) uses indicators for the three-entry products and for the choice of the pair i, j; the indicated sum is required to be at least two.

For each displayed representative, permute rows within equal-degree groups, sort the columns, and retain the forms whose rows also have the required order. Excluding every such form is expressed by the clause

$$\bigvee_{i,c} \ell_{ic}, \qquad \ell_{ic} = \begin{cases} x_{ic}, & \text{the excluded entry is 0,} \\ \neg x_{ic}, & \text{the excluded entry is 1.} \end{cases}$$

The fifteen resulting residual systems are all unsatisfiable. This finite calculation was certified by propositional refutations, checked against the above constraints. Direct checks of the representatives give (7), (8), maximality, and disjoint orbits. The ordered-form counts are those in Table 1. Since any missing class would supply a satisfying assignment of a residual system, none is omitted. □

5. Five-row extensions

For an 8×14 avoiding matrix B, put $b_i = B_i \cup \{15\}$, where $B_i \subseteq [14]$ is its ith row support. Define

$$\mathcal{V}(B) = \{v \subseteq [15] : |v \cap b_i \cap b_j| \le 3 \text{ for all } i < j\}. \tag{10}$$

This set is determined by testing the 2^{15} subsets of $[15]$. Let $M(B) = \max_{v\in\mathcal{V}(B)} |v|$ and $T(B) = 115 - (|B| + 8)$. Any five-row extension with at least 115 ones requires five members of $\mathcal{V}(B)$ with total weight at least $T(B)$. A member of weight below $T(B) - 4M(B)$ cannot occur in such an extension and may be deleted from the list.

Order the remaining list as $v_1, \dots, v_L$ in nonincreasing weight, breaking ties by the binary word. The following recursion specifies the complete extension test. Its state consists of a list S of chosen new rows, a smallest permitted index q, and the current weight $w = \sum_{u\in S} |u|$. Put $h = 5 - |S|$.

(i) If $h = 0$, accept precisely when $w \ge T(B)$.
(ii) For each $j \ge q$, stop the loop if $w + h|v_j| < T(B)$.
(iii) Otherwise, proceed with v_j only if

$$|v_j \cap u \cap b_i| \le 3 \quad (u \in S,\ 1 \le i \le 8),$$
$$|v_j \cap u \cap u'| \le 3 \quad (u, u' \text{ from distinct positions of } S).$$

Recurse at $(S \parallel v_j, j, w + |v_j|)$.

The same index j is retained, so repeated new rows are allowed. They are rejected only if the actual intersection conditions fail. The recursion accepts if one branch accepts and rejects otherwise.

Lemma 5.1 (Complete extension test)**.** *The recursion accepts if and only if* $(B \mid \mathbf{1})$ *extends by five rows to a matrix satisfying* (1) *with at least* 115 *ones.*

Proof. Three old rows satisfy (1), since their intersection in B has size at most two. Two old rows and one new row are checked by (10). One old row and two new rows, and three new rows, are exactly the two tests in step (iii). Thus an accepted list gives a valid extension.

Conversely, order the five rows of any valid extension by their indices in the list. A row deleted by the weight cutoff could not reach the target even if each of the other four rows had weight

$M(B)$. At every state along the branch representing this extension, the remaining rows have weight at most the current list entry. Hence the weight bound in step (ii) cannot discard that branch. The intersection tests also pass, so the branch accepts. □

Lemma 5.2 (Dense neighborhoods do not extend)**.** *If B is an 8×14 avoiding matrix with $|B| \in \{63, 64\}$, then $(B \mid \mathbf{1})$ has no avoiding five-row extension with at least 115 ones.*

Finite verification. If $|B| = 64$, the uniqueness theorem gives $B \cong G$. If $|B| = 63$, Section 4 gives either a maximal representative B_i or a one-entry deletion of G. Apply the complete recursion above to these finite lists. Its output is rejection in all cases, as summarized in Table 2.

TABLE 2. Exhaustive extension tests. All rows allow both values in the adjoined column. The 64 deletions are tested separately, without assuming they are inequivalent.

Base matrix or family	Bases tested	Target $T(B)$	Accepted
G	1	43	0
One-entry deletions of G	64	44	0
$B_1, \ldots, B_{10}$	10	44	0
B_{11}, B_{12}	2	44	0
B_{13}, B_{14}, B_{15}	3	44	0
B_{16}	1	44	0
B_{17}	1	44	0

As a further description of the 64-one case, the individual-row test gives $|v| \leq 9$ when $15 \notin v$, with exactly four possibilities of weight nine, and $|v| \leq 6$ when $15 \in v$. Thus a row containing the adjoined position would limit the total new weight to $6 + 4 \cdot 9 = 42$. With that position absent, total weight at least 43 requires five weight-nine rows, four weight-nine rows and a row of weight at least seven, or three weight-nine rows and two weight-eight rows. Repetition of a weight-nine row fails an intersection test, and the remaining cases are rejected by the same recursion. □

Corollary 5.3 (Improved local bound)**.** *Define*

$$W(k) = \begin{cases} 62, & k = 8, \\ Z(k), & k \neq 8. \end{cases} \tag{11}$$

In a 115-one 13×15 matrix satisfying (1), any k rows incident with a fixed column contain at most $W(k)$ ones outside that column.

Proof. Lemma 3.1 and (3) give the bound unless $k = 8$. In that case the smaller matrix has at most 64 ones, and Lemma 5.2 excludes 63 and 64. This also applies to any eight-row subset of a larger column neighborhood. □

6. Global degree sequences

Suppose a counterexample A has exactly 115 ones. Let its row degrees be $a_1, \dots, a_{13}$ and its column degrees $d_1, \dots, d_{15}$. Then

$$\sum_{i=1}^{13} a_i = \sum_{j=1}^{15} d_j = 115, \tag{12}$$

$$\sum_{j=1}^{15} \binom{d_j}{3} \le 3\binom{13}{3} = 858, \tag{13}$$

$$\sum_{i=1}^{13} a_i(a_i - 1) \le \sum_{j=1}^{15} W(d_j). \tag{14}$$

Indeed, every triple of rows occurs in at most three column neighborhoods, proving (13). For each column j, Corollary 5.3 gives

$$\sum_{i:A_{ij}=1} (a_i - 1) \le W(d_j).$$

Summing over columns gives (14).

At fixed sum 115, the least possible row moment is

$$2 \cdot 8 \cdot 7 + 11 \cdot 9 \cdot 8 = 904. \tag{15}$$

To see this, replacing two integer degrees $u < v - 1$ by $u + 1, v - 1$ strictly decreases the sum of $a(a - 1)$. Hence the minimizing sequence has two eights and eleven nines.

Lemma 6.1 (Finite list of degree pairs)**.** *There are* 750 *nondecreasing column-degree sequences in* $[0, 13]^{15}$ *satisfying* (12), (13)*, and* $\sum_j W(d_j) \ge 904$*. Together with all nondecreasing row-degree sequences in* $[0, 15]^{13}$ *satisfying* (12) *and* (14)*, they give exactly* 6,019 *pairs. In these pairs, every row degree lies in* $[6, 12]$ *and every column degree in* $[2, 12]$*.*

Finite enumeration. For a nondecreasing list of length ℓ, remaining sum s, previous entry b, and maximum permitted entry D, choose the next entry from

$$b \le v \le \min(D, \lfloor s/\ell \rfloor). \tag{16}$$

Recurse with $(\ell - 1, s - v, v, D)$; discard a state if $s < \ell b$ or $s > \ell D$. Accept an empty list precisely when $s = 0$. For columns begin with $(15, 115, 0, 13)$, accumulating $\binom{v}{3}$ and $W(v)$; discard as soon as the former exceeds 858, and require the latter to be at least 904 at completion. For each accepted column list, generate rows from $(13, 115, 0, 15)$, accumulating $v(v - 1)$ and discarding whenever the sum exceeds the corresponding column bound. All accumulated moment costs are nonnegative, so these prunings are valid. Evaluation gives the stated counts. The initial degree domains include zero; the narrower ranges in the statement are outputs, not assumptions. □

7. Neighborhood types and exact certificates

Fix a pair of sequences in Lemma 6.1. Let N_a be the number of rows of degree a and M_d the number of columns of degree d. Write

$$R = \{a : N_a > 0\}, \qquad D = \{d : M_d > 0\}.$$

These are sets of positive degrees by the finite enumeration.

TABLE 3. Degree enumeration grouped by the value $C = \sum_j W(d_j)$.

C	Column sequences	Row sequences per column	Pairs
904	48	1	48
905	41	1	41
906	70	3	210
907	64	3	192
908	93	5	465
909	78	5	390
910	98	8	784
911	64	8	512
912	85	14	1190
913	37	14	518
914	43	21	903
915	13	21	273
916	12	30	360
917	3	30	90
918	1	43	43
Total	750		6019

Definition 7.1. For $a \in R$, a *row type* is an integer vector $t = (t_d)_{d \in D}$ satisfying

$$0 \le t_d \le M_d, \qquad \sum_{d \in D} t_d = a, \qquad \sum_{d \in D} t_d \binom{d-1}{2} \le 198. \tag{17}$$

Let $\mathcal{T}_a$ denote its finite set of possibilities. For $d \in D$, a *column type* is an integer vector $u = (u_a)_{a \in R}$ satisfying

$$0 \le u_a \le N_a, \qquad \sum_{a \in R} u_a = d, \tag{18}$$

with the following additional requirement. Sort the multiset containing u_a copies of $a - 1$ as $w_1 \ge \cdots \ge w_d$. Require

$$\sum_{i=1}^{k} w_i \le W(k) \qquad (1 \le k \le d). \tag{19}$$

Let $\mathcal{U}_d$ denote the resulting finite set.

Lemma 7.2. *The neighborhood of every degree-a row has a type in $\mathcal{T}_a$, and the neighborhood of every degree-d column has a type in $\mathcal{U}_d$.*

Proof. For a fixed row, each pair of the other twelve rows shares at most three of its incident columns. Thus

$$\sum_{j : A_{ij} = 1} \binom{d_j - 1}{2} \le 3 \binom{12}{2} = 198,$$

which gives (17); the availability constraints and degree sum are immediate. For a fixed column, Corollary 5.3 bounds the sum of $a_i - 1$ over each k-element subset of its neighbors. The largest such sum is the prefix of length k in nonincreasing order. This is exactly (19). □

Associate a nonnegative variable $Y_{a,t}$ with every $t \in \mathcal{T}_a$ and a nonnegative variable $Q_{d,u}$ with every $u \in \mathcal{U}_d$. Counting vertices of each degree and counting incidences in two ways gives the equations

$$\sum_{t \in \mathcal{T}_a} Y_{a,t} = N_a \qquad (a \in R), \tag{20}$$

$$\sum_{u \in \mathcal{U}_d} Q_{d,u} = M_d \qquad (d \in D), \tag{21}$$

$$\sum_{t \in \mathcal{T}_a} t_d Y_{a,t} - \sum_{u \in \mathcal{U}_d} u_a Q_{d,u} = 0 \qquad (a \in R,\ d \in D). \tag{22}$$

An actual matrix provides nonnegative integer values of these variables. It therefore provides a nonnegative real solution as well.

Lemma 7.3 (Certificate criterion)**.** *Suppose numbers $\alpha_a, \beta_d, \gamma_{ad}$ satisfy*

$$\alpha_a + \sum_{d \in D} \gamma_{ad} t_d \geq 0 \qquad (a \in R,\ t \in \mathcal{T}_a), \tag{23}$$

$$\beta_d - \sum_{a \in R} \gamma_{ad} u_a \geq 0 \qquad (d \in D,\ u \in \mathcal{U}_d), \tag{24}$$

$$\sum_{a \in R} \alpha_a N_a + \sum_{d \in D} \beta_d M_d < 0. \tag{25}$$

Then (20)–(22) *have no nonnegative solution.*

Proof. Multiply (20), (21), and (22) by α_a, β_d, and γ_{ad}, respectively, and add. The left side is nonnegative by (23) and (24), while the right side is negative by (25). □

Lemma 7.4 (Finite degree obstruction)**.** *For each of the* 6,019 *degree pairs in Lemma 6.1, there exist integer multipliers satisfying* (23)–(25)*.*

Exact finite verification. For each degree pair, the sets in Definition 7.1 are enumerated over their stated integer boxes, retaining precisely the vectors that satisfy the displayed sums and prefix bounds. Integer multipliers are then checked by evaluating the left sides of (23) and (24) for every retained vector and the integer sum in (25). All 6,019 cases pass these tests. No approximate inequality is used: the checked conditions are coefficientwise nonnegativity and a strictly negative integer. The largest absolute multiplier in these certificates is 3334. Table 4 records their support sizes.

TABLE 4. Exact dual certificates, grouped by number of nonzero multipliers.

Support size	1	2	3–7	8–12	13–19	20–29	30–38
Cases	3758	535	311	448	630	285	52

Here is a nontrivial example with all its multipliers displayed. Take

$$(a_i) = (8^5, 9^5, 10^3), \qquad (d_j) = (6^5, 7^2, 8, 9^7).$$

In increasing degree order, let

$$\begin{aligned} (\alpha_8, \alpha_9, \alpha_{10}) &= (6, 0, -3), \\ (\beta_6, \beta_7, \beta_8, \beta_9) &= (0, 0, -2, -3), \\ (\gamma_{ad})_{a\in R, d\in D} &= \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \end{aligned} \tag{26}$$

The nonnegative coefficients can also be seen directly. For a degree-eight row, seven neighbors of degrees eight or nine would contribute at least $21 + 6 \cdot 28 + 10 = 199$ to (17), since only one degree-eight column is available. Thus $t_8 + t_9 \le 6$, as required by its coefficient $6 - t_8 - t_9$. A degree-ten row has at most seven neighbors of degrees six or seven, so $t_8 + t_9 \ge 3$, as required by its coefficient $-3 + t_8 + t_9$. Degree-nine rows have coefficient zero.

For a degree-eight column, the full prefix bound gives $64 - u_8 + u_{10} \le 62$, so its coefficient $-2 + u_8 - u_{10}$ is nonnegative. For a degree-nine column, $u_8 = 0$ would make its largest eight weights at least 64, which is impossible. The smallest weight is therefore seven, and the eight-element prefix bound gives $65 - u_8 + u_{10} \le 62$. Hence its coefficient $-3 + u_8 - u_{10}$ is nonnegative. Degree-six and degree-seven columns have coefficient zero. Finally, the right side is

$$6 \cdot 5 - 3 \cdot 3 - 2 \cdot 1 - 3 \cdot 7 = -2.$$

Thus even this degree pair cannot occur. The same finite coefficient test, with the corresponding integer multipliers, excludes each of the enumerated pairs. □

Proof of Theorem 1.1. Section 2 gives the lower bound 114. Suppose an avoiding 13×15 matrix had at least 115 ones. Delete ones, if needed, to obtain exactly 115; deleting ones cannot create a forbidden submatrix. Corollary 5.3 then applies, so its degree sequences belong to the list in Lemma 6.1. By Lemma 7.2, the matrix gives a nonnegative solution of (20)–(22). Lemma 7.4 and the certificate criterion rule out such a solution. This contradiction proves the upper bound 114. □

Computational proof files

The complete computational material for the upper bound is publicly available in computational proof archive, release v1. The corresponding version-specific source tree is fixed at commit `10385847932cd2b61e25cb4e7d039633d108455b`. The archive includes all fifteen residual SAT inputs and their complete DRAT refutations, the five-row extension inputs and outputs for all 82 bases in Table 2, and all 6,019 integer dual certificates together with their degree sequences. It also supplies the input generators, exact verification scripts, dependency versions, and checking instructions.

These files were regenerated on November 16, 2026 from the constraints and representatives displayed in this paper and the integer certificate data in the supplied companion Lean file. The regeneration reproduces the 1,585 ordered forms in Table 1, all fifteen unsatisfiable residual systems, all 82 rejected extension tests, and the degree and certificate counts in Tables 3 and 4. Each DRAT refutation was independently checked with `drat-trim`. The dual certificates were checked with exact Python integer arithmetic over every admissible type.

The archive's `README.md` gives a single verification command that checks file hashes, regenerates the SAT inputs, checks the refutations, reruns the extension searches, and verifies every dual certificate. The included companion `Zarankiewicz1315.lean` retains its supplied toolchain and dependency information. The recorded archive checks concern the computational components listed above. The release ZIP has SHA-256

`07f3d6338b4597eb65bba0bd688fc29bf2919a91647d594ff2f8eff1b39979ff`.

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Appendix A. The seventeen maximal representatives

The matrices below are the representatives in Lemma 4.1. Each displayed binary string is one row, with columns read from left to right. The labels correspond to the families in Table 2. These matrices are used only as 8×14 bases; the all-one fifteenth column is adjoined when applying the extension test.

B_1
```
11111111000000
11111000111000
11000110110110
10100101101101
10010011011011
01001101011110
00001110101011
00110001110101
```

B_2
```
11111111000000
11110000111100
11001110100011
11000001011111
00111100011011
00101011110110
00010111101101
10011001110001
```

B_3
```
11111111000000
11110000111100
11001100110011
10100011101011
01011010010111
00111101100110
00000111111101
01100110001110
```

B_4
```
11111111000000
11110000111100
11001100110011
10100011101011
01101010010111
00111101100110
00000111111101
10010110011010
```

B_5
```
11111111000000
11110000111100
11001100110011
10100011101011
01101010010111
00111101100110
00000111111101
10010110001110
```

B_6
```
11111111000000
11110000111100
11001000111011
10101110100110
01000111010111
00110101101011
00011011011101
01101001001110
```

B_7
```
11111111000000
11110000111100
11001100110011
11000011001111
00111010101011
00110101010111
00001111111100
10100110100110
```

B_8
```
11111111000000
11110000111100
11001100110011
10001011101110
01000111011101
00111010011011
00110101100111
10100110010110
```

B_9
```
11111111000000
11110000111100
11001100110011
10001011101110
01100011011011
00110110100111
00011101011101
01011010101001
```

B_{10}
```
11111111000000
11110000111100
11001100110011
11000011001111
00111100001111
00110011110011
00001111111100
10101010101010
```

B_{11}
```
11111111100000
11110000011110
11001110011001
10101001010111
01100101101101
00111010101011
01010011010011
00011101011100
```

B_{12}
```
11111111100000
11111000011100
11000111011010
10100100110111
01010010101111
00011101111001
00110011010011
00101110001101
```

B_{13}
```
11111111100000
11111000011110
11000110011101
10100101110011
01010011101011
00101011010101
00011101001101
00001110111010
```

B_{14}
```
11111111100000
11111000011110
11000111011100
00110110011011
00101101110101
10010100100111
01001010110011
00011011001101
```

B_{15}
```
11111111100000
11111000011110
11000111011100
00110110011011
00101101110101
10010100100111
10001010110011
00011011001101
```

B_{16}
```
11111111110000
11111000001110
11000111001101
00110100111101
00001011111110
10100010101011
10010001100111
01010001011011
```

B_{17}
```
11111111110000
11111100001110
11000011111110
00110011001101
00101010101011
00100101100111
00011010010111
00010101011011
```

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