Finite-preparation response under heavy-tailed catastrophes: exact amplitudes and separation from Poisson coverage

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Abstract

We determine the terminal response of a one-dimensional binary recovery process to a finite preparation history under independently renewed Poisson interval catastrophes. For regularly varying interval weights $\nu_r \sim r^{-p}L(r)$, $p > 3$, and fixed small positive intensities, we prove an exact leading response when preparation and observation durations grow proportionally. The main result expresses the response as an explicit functional of the actual recovery-front speeds accumulated along each fixed finite-stage preparation history and of the selected target-phase open density. Its weak-intensity second variation is a strictly positive quadratic form in every nonzero fixed signed temporal modulation, including equal-dose redistributions. A separate four-update theorem identifies local cooperation through an explicit positive equal-dose contrast for every nonzero interval law with finite first moment. These positive contrasts separate the process from conditional Poisson Boolean coverage with fixed grain kernels, a fixed environment distribution and affine intensity control, whose contrast is nonpositive at every finite time. As an effective extension, relative power-law tail bounds yield a conservative computable sufficient threshold valid at every later integer time. The core proof combines nonstationary front transfer, uniform background forgetting and exact resetting at intensity switches.

Keywords: interacting particle systems; spatial catastrophes; regularly varying tails; recovery fronts; finite preparation; Boolean coverage

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1. Introduction

Heavy-tailed spatial errors can sustain long relaxation even when each error is eventually removed. Multidimensional heavy-tailed Poisson random-grain fields have Gaussian, intermediate and stable scaling limits [24]; anisotropic and nonlinear extensions are developed in [1,2]. In an interacting recovery process, the surrounding errors also change the speed at which a large damaged interval disappears. We ask how this change enters the response to a preparation history followed by a common target environment.

The main result is the finite-stage history-response theorem (Theorem 4.3). With the intensities fixed, the long-time limit expresses the response through the selected target-phase open density and the actual recovery speeds accumulated after each root is born. Theorem 4.1 evaluates this functional explicitly for a constant preparation intensity. The history functional records how later preparation stages change the lifetime of earlier damage.

The process is a one-dimensional binary growth rule with independently renewed Poisson interval catastrophes. Open sites propagate to their right neighbours; each disaster row removes connected intervals of unbounded length. The deterministic update is a Stavskaya-type eroder [13]. The original two-neighbour rule becomes the self/left rule on a space-time parity sublattice under $h_t(i) = \xi_t(2i - t)$, where ξ_t denotes the original excitation indicator; complementing states gives the corresponding recovery rule. Stability under local independent errors is studied in noisy-eroder theory [3,4,18]. Recent random Toom-contour work treats intrinsic update randomness and discusses why deterministic-map edge speeds do not directly define an effective random edge speed [19, Section 1.6]. For the connected-interval errors here, we construct the actual stochastic recovery fronts and identify their contribution to the preparation response.

Spatial disasters also occur in contact processes [5] and branching random walks [17]. Heavy-tailed temporal cure gaps change the survival and complete-convergence behavior of renewal contact processes [16]. Complementary survival results establish a finite critical infection rate for continuous bounded-support laws and decreasing-hazard densities [21], and more recently for all non-degenerate arithmetic and all non-atomic interarrival laws [22]. Renewal dormancy yields $r_t = o((\log t)^{1+\varepsilon})$ almost surely for every $\varepsilon > 0$ under the regularly varying wake-up assumption with index $-\alpha$, $0 < \alpha < 1$, with no dormant recovery or dormant-to-dormant infection and the initial state specified in [20, Theorem 2.2]. In those renewal models the heavy tails are temporal, whereas our disaster rows are independent in time and the interval sizes are heavy-tailed in space. Poisson hail provides an interacting-grain setting in which overlapping arrivals change workload lifetimes, with stability controlled by spatial and service-time tails [14,15]. Here one mark closes a long binary interval, recovery propagates at finite speed, and the response after a finite preparation history depends on the interval's subsequent recovery in the interacting background.

The nearest results separate into three different limit problems. Surgailis [2, Theorem 2] studies nonlinear spatial integrals when grain intensity grows with the spatial scale. At fixed intensity $M = 1$, Proposition 2 and Corollary 2 give stable distributional limits for nonlinear spatial integrals and Boolean covered volume, respectively [2, (4.13)–(4.14)]. Here the long-time observable is a terminal single-site preparation-history response in an interacting recovery process, with all positive intensities fixed. Regular variation fixes its tail scale, while the actual stochastic recovery-front speeds accumulated through the preparation and target stages determine its coefficient.

For interacting arrivals, the Poisson-hail results [15, Theorems 1–3] give low-rate workload stability, instability examples and uniqueness of the translation-invariant equilibrium under their moment assumptions. Mountford and Wang [23, Theorems 1.1–1.2] refine the stability region using separate critical moments for spatial radii and service times, including stable systems with infinite propagation speed. These results concern workload stability; the response here depends on finite-speed binary recovery through a prescribed preparation history. Random Toom contours [19, Section 1.6] distinguish deterministic edge speeds from effective speeds of a random evolution. Here Lemmas 5.3 and 5.5 construct the actual fronts for connected-interval errors and transfer their limits to a predictable nonstationary exterior; Lemmas 6.2 and 7.1 then control forgetting and reset these fronts through each preparation switch. These are the dynamic inputs that identify the coefficient in Theorem 4.3.

The master-equation analysis of Kovacs protocols [6] and the Green-function expansion of nonlinear response [7] describe preparation effects and perturbative response. In this paper, the fixed-positive-intensity history limit is established before taking the weak-intensity variation. The latter yields the explicit positive quadratic energy, including equal-dose directions, in Theorem 4.4.

Theorem 4.4 is the positive-energy consequence of this functional. Its weak-intensity second variation is strictly positive in every nonzero fixed signed direction, including redistributions with zero integrated modulation. This identifies the temporal arrangement of forcing through the energy of its cumulative history. The fixed-history and fixed-direction conditions are retained in both limits.

Theorem 4.5 supplies an independent local witness after four updates under only a finite first interval moment. Theorem 4.6 extends the long-time sign result to an effective all-future threshold when quantitative tail data are supplied.

Frozen conditional Poisson coverage provides the reference obtained by calibrating each grain's recovery kernel once and holding it fixed across preparation arms. We also fix the external environment distribution and make the birth intensity affine in the control. The Poisson void formula then gives a nonpositive symmetric contrast at every finite time. The recovery process has a positive contrast in the proved regimes: changing the preparation also changes subsequent erosion in the interacting background. Section 8.3 gives the Bernoulli and nonlinear-calibration examples that delimit this sign comparison.

The amplitude theorem requires front transfer in a predictable nonstationary exterior, forgetting uniform in the interval cutoff, and exact two-ray resets at intensity switches. The conditional path-resampling

construction adapts van den Berg, Häggström and Kahn [9, Theorems 2 and 4, Section 3.1]; Appendix A proves its connected-interval and constrained-boundary forms. Last [8, Section 3] supplies the expansion under finite nonnegative perturbations of a Poisson measure. Evaluating its add-one term here requires the actual single-root insertion kernel; the front-transfer, forgetting and reset estimates identify its long-time limit in the interacting background.

The effective extension adds complete-path screening of multiple long roots (Lemma 6.3) and a covariance estimate for unresolved terminal openness (Lemma 6.4). Stopping-set exploration [12] supplies the revealment framework; the proof uses complete reverse rows and centered fresh boundary attachments. The technical supplement retains the full quantitative argument and certificates. Proposition 4.7 explains why the threshold requires full tail data.

Sections 2–4 define the process and state the response, sign, four-update and effective-threshold results. Sections 5–8 prove the response and finite-time results. Appendices A–B provide the conditional correlation, phase, compression and spatial-tail estimates. Online Resource 1 (*Effective threshold supplement*) gives the complete quantitative proof of Theorem 4.6. Appendix A.1–A.3 shares the conditional correlation construction, interval-insertion count, finite-history positive-phase bound and local stationary-flux argument with the author's unpublished manuscript *An exact moment criterion for survival under renewed spatial catastrophes* (version 1, 21 October 2026). The needed arguments are reproduced here. The companion studies the survival moment criterion and stationary dilute density; the present paper establishes the actual-front response amplitude, finite-stage history functional, signed temporal energy, four-update equal-dose witness and effective sign onset.

2. Model, preparation protocol and notation

Let $H_n \subset \mathbb{Z}$ denote the holes at integer time n. The update is

$$H_{n+1} = (H_n \cap (H_n + 1)) \cup D_{n+1}, \qquad H_n + 1 = \{x + 1 : x \in H_n\}. \tag{1}$$

At each origin z and length $r \geq 2$, independent Poisson events with mean $a\nu_r$ create $[z, z + r - 1]$ in D_n. Rows are independent. Without events, an isolated hole interval loses one site each step. Events that sustain adjacent holes can reduce the effective recovery-front speed.

Fix a nonzero law

$$\nu_r \sim r^{-p} L(r), \quad p > 3, \quad \chi = \sum_{r \geq 2} \nu_r \, r(r+1)/2 < \infty, \qquad \gamma = p - 3 > 0, \tag{2}$$

where L is eventually positive and slowly varying. Write μ_a for the stationary law obtained by moving the fully open initial row into the remote past, $\beta_a = \mu_a(0 \in H)$ for its hole density, and $u_a = 1 - \beta_a$ for its open density. The quantity χ is the forcing-weighted space–time area of an isolated interval under unit erosion. Intensities and lattice time are dimensionless.

Choose a fixed preparation ratio $H > 0$, and set $N_T = \lfloor HT \rfloor$. Each arm starts without holes at time $-N_T$. During rows $-N_T + 1, \ldots, 0$, the three old intensities are $a = \rho, 2\rho, 3\rho$. During rows $1, \ldots, T$, all arms use fresh target intensity ρ. Let $q_{a,H}(\rho, T)$ be the final single-site hole probability. The primary observable is

$$\boxed{W^{\mathrm{fin}}(\rho, H, T) = q_{3\rho,H}(\rho, T) - 2q_{2\rho,H}(\rho, T) + q_{\rho,H}(\rho, T).} \tag{3}$$

The reference arm follows the same finite schedule. The contrast depends on the three marginal probabilities and is unchanged by coupling the arms.

3. The finite-time Poisson sign constraint

Theorem 3.1 (frozen conditional coverage).

In the competing class, a Poisson birth carries an independently marked grain with frozen shape, lifetime and motion. Conditional on a frozen external environment E, the controlled births remain Poisson and their marks remain independent. The environment is exogenous, with a fixed distribution across arms. Intensity control multiplies the old birth measure linearly, and all arms begin with an empty grain configuration at the common reset.

At the observation point, write $M_{E,H}(T) < \infty$ for the total conditional covering-grain mean at unit intensity and $0 \le N_{E,H}(T) \le M_{E,H}(T)$ for the part due to old births. The reference arm has conditional void probability $e^{-\rho M_{E,H}(T)}$. The other arms therefore give

$$\boxed{W^{\mathrm{fin}}_{\mathrm{grain}}(\rho, H, T) = -\mathbb{E}_E\left[e^{-\rho M_{E,H}(T)}\left(1 - e^{-\rho N_{E,H}(T)}\right)^2\right] \le 0.} \tag{4}$$

Strict negativity holds if $\Pr(N_{E,H}(T) > 0) > 0$. If no old grain can contribute, the contrast is zero.

The identity holds for each conditional environment before averaging, so it permits frozen external heterogeneity and arbitrary grain lifetimes with finite covering means. Grain parameters may be calibrated once and then held fixed across the arms. Section 8.2 proves the identity.

4. The history-response theorem and its consequences

Theorem 4.3 is the main result for a fixed finite-stage history. We first give its constant-history evaluation in Theorem 4.1, followed by the positive constant-history contrast. Theorem 4.4 extracts the signed temporal energy from the history functional; Theorem 4.5 is an independent finite-time witness, and Theorem 4.6 is the effective extension. All intensities are held fixed in the long-time response limits.

4.1. Constant-history amplitude and its sign consequence

Theorem 4.1 (finite-preparation response).

For a positive intensity a satisfying $a\chi \le 1/32$, let u_a and $\kappa_a = 2v_a - 1 > 0$ be the selected stationary open density and the relative recovery-front speed. The density u_a belongs to the phase defined in Section 2. For initial holes $[1, \infty)$ and an open left exterior, v_a is the limiting displacement per step of the left endpoint of the right-infinite hole component (Lemma 5.3).

For fixed $H > 0$ and fixed positive a, ρ satisfying $\max(a, \rho)\chi \le 1/32$, the terminal response satisfies

$$\frac{q_{a,H}(\rho, T) - q_{\rho,H}(\rho, T)}{A_\nu(T)} \longrightarrow C_\rho[F_{\rho,H}(a) - F_{\rho,H}(\rho)], \tag{5}$$

where

$$\begin{aligned} A_\nu(T) &= \sum_{r \ge T} \nu_r \frac{(r - T + 1)(r - T + 2)}{2} \sim D_p T^{-\gamma} L(T), \\ D_p &= [(p-1)(p-2)(p-3)]^{-1}, \qquad C_\rho = u_\rho \kappa_\rho^{-\gamma}, \\ F_{\rho,H}(a) &= \frac{a}{\kappa_a}\left[1 - \left(1 + \frac{\kappa_a H}{\kappa_\rho}\right)^{-\gamma}\right]. \end{aligned} \tag{6}$$

The finite-age factor in (6) counts roots born during the preparation interval.

Applying (5) to the two fixed old intensities gives

$$\frac{W^{\mathrm{fin}}(\rho, H, T)}{A_\nu(T)} \longrightarrow \mathcal{W}^{\mathrm{fin}}(\rho, H) = C_\rho[F(3\rho) - 2F(2\rho) + F(\rho)]. \tag{7}$$

The case $a = \rho$ in (5) is the exact zero identity.

Theorem 4.2 (eventual sign separation). Define the preparation factor

$$\psi_\gamma(x) = 1 - (1 + x)^{-\gamma} - \gamma x(1 + x)^{-\gamma - 1}, \qquad x > 0. \tag{8}$$

The weak-forcing front estimate is

$$\kappa_a^{-1} = 1 + 2\chi a + e(a), \qquad |e(a)| \le 10\chi^2 a^2, \qquad a\chi \le 1/32. \tag{9}$$

Applying a Taylor bound to an auxiliary function of the inverse speed gives

$$F(3\rho) - 2F(2\rho) + F(\rho) = 4\chi\rho^2\psi_\gamma(H/\kappa_\rho) + \mathcal{E}_{\rho,H}, \quad |\mathcal{E}_{\rho,H}| \le 44K_\gamma\chi^2\rho^3, \tag{10}$$

with $K_\gamma = 10 + (1369/512)\gamma(\gamma+1)$.

Suppose the intensity satisfies

$$0 < \chi\rho \le \frac{\psi_\gamma(H)}{22K_\gamma}, \tag{11}$$

Since $0 < \psi_\gamma(H) < 1$ and $K_\gamma \ge 10$, this implies $3\chi\rho < 3/220 < 1/32$. Hence all three intensities satisfy the front bound and

$$\mathcal{W}^{\mathrm{fin}}(\rho, H) \ge 2C_\rho \chi \rho^2 \psi_\gamma(H) > 0. \tag{12}$$

Thus, for every fixed $H > 0$ and fixed strength obeying (11), a finite $T_0(H, \rho)$ exists after which the cooperative contrast is positive for all integer observation times. This reverses the competing class's finite-time sign.

4.2. Consequences for constant preparation

At fixed $H > 0$, taking time to infinity before strength tends to zero yields

$$\boxed{\lim_{\rho\downarrow 0} \rho^{-2} \lim_{T\to\infty} \frac{W^{\mathrm{fin}}(\rho, H, T)}{A_\nu(T)} = 4\chi\psi_\gamma(H).} \tag{13}$$

The function ψ_γ increases from zero to one. For short preparation ratios, $\psi_\gamma(H) \sim \gamma(\gamma+1)H^2/2$; for large ratios it tends to one. The preparation factor therefore increases continuously from zero to its saturated value.

When the response denominator is positive, the finite-reference ratio

$$R^{\mathrm{fin}} = \frac{q_{3\rho,H} - q_{\rho,H}}{2(q_{2\rho,H} - q_{\rho,H})} \tag{14}$$

obeys $R^{\mathrm{fin}}_{\mathrm{grain}} \le 1$ in the competing class. In the budget (11), the cooperative denominator is eventually positive, its limiting ratio exceeds one, and

$$\lim_{\rho\downarrow 0} \frac{R^{\mathrm{fin},*}_{\mathrm{coop}}(\rho) - 1}{\rho} = \frac{2\chi\psi_\gamma(H)}{1 - (1+H)^{-\gamma}}. \tag{15}$$

The ratio cancels $A_\nu(T)$ and C_ρ. Both observables use the finite reference arm directly; bounds on γ and χ determine the sufficient intensity budget (11).

4.3. The history functional and its positive-energy consequence

The response extends to a fixed finite number of preparation stages. Let $a : [0, H] \to (0, \infty)$ be piecewise constant on a fixed partition $0 = h_0 < \cdots < h_m = H$, with value a_i on (h_{i-1}, h_i). The variable z is age before the switch, so the first interval describes the most recent preparation stage. During the preparation row at time $-j$, $0 \le j < \lfloor HT \rfloor$, use intensity $a(j/T)$; values at the finitely many partition endpoints may be assigned to either adjoining stage. All arms start fully open at $-\lfloor HT \rfloor$ and use intensity ρ for the final T rows. Denote the terminal hole probability by $q_{[a]}(\rho, T)$, and let $[\rho]$ denote the constant preparation profile.

Theorem 4.3 (finite-stage history response). Suppose $\max(\rho, \|a\|_\infty)\chi \le 1/32$. Put

$$K_a(z) = \int_0^z \kappa_{a(s)}\, ds, \qquad \mathcal{J}_\rho[a] = \gamma u_\rho \int_0^H \frac{a(z)}{[\kappa_\rho + K_a(z)]^{\gamma+1}}\, dz.$$

Then

$$\lim_{T\to\infty} \frac{q_{[a]}(\rho, T) - q_{[\rho]}(\rho, T)}{A_\nu(T)} = \mathcal{J}_\rho[a] - \mathcal{J}_\rho[\rho].$$

Writing $K_i = \sum_{j\le i} \kappa_{a_j}(h_j - h_{j-1})$, $K_0 = 0$ and $k = \kappa_\rho$, the functional has the finite-sum expression

$$\mathcal{J}_\rho[a] = u_\rho \sum_{i=1}^{m} \frac{a_i}{\kappa_{a_i}} \left[(k + K_{i-1})^{-\gamma} - (k + K_i)^{-\gamma}\right].$$

For a constant profile, this is $C_\rho F_{\rho,H}(a)$ from (6). The cumulative speed $K_a(z)$ records the subsequent preparation history experienced by a root born at age z. For two stages, with the more recent stage ending at

age h_1, a root born at $z > h_1$ has $K_a(z) = \kappa_{a_1} h_1 + \kappa_{a_2}(z - h_1)$. Its limiting remaining width is $(s - \kappa_\rho - K_a(z))_+$ for scaled initial length s: the denominator in the history functional accounts for all erosion after birth.

The following theorem extracts the positive energy carried by a signed temporal modulation. Its coefficient follows from the history functional and the first-order change of the actual recovery speed.

Theorem 4.4 (signed temporal response). Let f_b, h be fixed piecewise-constant functions on a common finite partition of $[0, H]$, with $f_b - |h| \geq \varepsilon > 0$. Define $f_\pm = f_b \pm h$ and

$$\Xi_h(\rho, T) = q_{[\rho f_+]}(\rho, T) - 2q_{[\rho f_b]}(\rho, T) + q_{[\rho f_-]}(\rho, T), \qquad P_h(z) = \int_0^z h(s)\, ds.$$

For intensities satisfying the phase budget of Theorem 4.3, let $\Xi_h^*(\rho) = \lim_{T\to\infty} \Xi_h(\rho, T)/A_\nu(T)$. Set

$$\begin{aligned} Q_\gamma[h] &= \int_0^H \frac{h(z) P_h(z)}{(1+z)^{\gamma+2}}\, dz \\ &= \frac{1}{2}\left[\frac{P_h(H)^2}{(1+H)^{\gamma+2}} + (\gamma+2) \int_0^H \frac{P_h(z)^2}{(1+z)^{\gamma+3}}\, dz \right]. \end{aligned}$$

For every h nonzero on a set of positive measure, $Q_\gamma[h] > 0$, and

$$\boxed{\lim_{\rho \downarrow 0} \rho^{-2} \Xi_h^*(\rho) = 4\chi\gamma(\gamma+1) Q_\gamma[h] > 0.}$$

Equivalently, the quadratic form has kernel

$$Q_\gamma[h] = \frac{1}{2} \int_0^H \int_0^H \frac{h(s) h(z)}{[1 + \max(s, z)]^{\gamma+2}}\, ds\, dz.$$

The coefficient is independent of the baseline history f_b. It remains strictly positive when $\int_0^H h = 0$, in which case the three profiles have equal integrated forcing.

An explicit sufficient intensity bound is available. Define

$$\begin{aligned} B_* &= \max\{1, \|f_+\|_\infty, \|f_b\|_\infty, \|f_-\|_\infty\}, \\ b_{\gamma,H} &= 1 - (1+H)^{-\gamma}, \\ c_\gamma &= (\gamma+1)\left[4 + \frac{9}{2}(\gamma+2) \left(\frac{10}{9} \right)^{\gamma+3} \right]. \end{aligned}$$

If

$$0 < \chi\rho \leq \frac{\gamma(\gamma+1) Q_\gamma[h]}{2 c_\gamma B_*^3 b_{\gamma,H}},$$

then all stages lie in the positive-phase budget and

$$\Xi_h^*(\rho) \geq 2 u_\rho \chi\gamma(\gamma+1)\rho^2 Q_\gamma[h] > 0.$$

Consequently $\Xi_h(\rho, T) > 0$ for all sufficiently large T, with the history and the positive intensity fixed.

For the frozen conditional Poisson class of Section 3, allow the same affine temporal control $\rho(f_b + \theta h)$, $\theta \in \{-1, 0, 1\}$. Conditional on E, write the finite covering mean as $\Lambda_E + \theta\Delta_E$, including the common target period in Λ_E. The birth measures must be nonnegative for each arm; Δ_E may have either sign. At every finite observation time,

$$\boxed{\Xi_{h,\mathrm{grain}}(\rho, T) = -4\mathbb{E}_E\left[e^{-\Lambda_E} \sinh^2(\Delta_E/2) \right] \leq 0.}$$

The kernels, environment distribution and affine intensity calibration are fixed across these arms.

A concrete equal-dose protocol takes $H = 1$, $f_b = 2$, and $h = 1$ for recent ages $0 \leq z < 1/2$, $h = -1$ for older ages $1/2 \leq z \leq 1$. Table 1 lists the preparation stages in chronological order for even T.

Table 1: Equal-dose preparation profiles in chronological order.

Arm	Older $T/2$ rows	Recent $T/2$ rows	Integrated preparation intensity
$+$	ρ	3ρ	$2\rho T$
Baseline	2ρ	2ρ	$2\rho T$
$-$	3ρ	ρ	$2\rho T$

The cumulative intensity is equal at every origin and for every interval length after multiplication by ν_r. At $p = 4$, so $\gamma = 1$, this protocol gives

$$\lim_{\rho\downarrow 0} \rho^{-2} \lim_{\substack{T\to\infty \\ T \text{ even}}} \frac{\Xi_h(\rho, T)}{A_\nu(T)} = \frac{2\chi}{9}.$$

The sign separation therefore persists when the preparation arms redistribute forcing in time while preserving its total intensity.

4.4. A four-update witness of local cooperation

The history theorem describes a long-time response. The following four-update result isolates its local cooperation mechanism in a finite causal cone under the weaker assumption of a finite first interval moment.

Theorem 4.5 (four-update separation). Let $\nu_r \geq 0$, $r \geq 2$, with

$$0 < S_0 := \sum_{r\geq 2} \nu_r, \qquad S_1 := \sum_{r\geq 2} r\nu_r < \infty.$$

Starting fully open, run the three chronological intensity schedules

$$(\rho, 3\rho, \rho, \rho), \qquad (2\rho, 2\rho, \rho, \rho), \qquad (3\rho, \rho, \rho, \rho).$$

The first two rows are preparation; the last two are the common target period. Write their terminal hole probabilities as q_+, q_b, q_- and set $\Xi_2 = q_+ - 2q_b + q_-$. Define

$$C_\nu = 4S_0^2 + (S_0 - \nu_2)^2, \qquad M_\nu = 6S_1 + 12S_0.$$

Then, for every $\rho \geq 0$,

$$\left|\Xi_2(\rho) - C_\nu\rho^2\right| \leq \frac{8}{3} M_\nu^3 \rho^3.$$

In particular,

$$\boxed{\Xi_2(\rho) > 0 \quad \text{if} \quad 0 < \rho < \frac{3C_\nu}{8M_\nu^3}.}$$

Each arm has cumulative preparation intensity 4ρ at every origin and length. The frozen conditional coverage class of Section 3, with the affine signed control of Theorem 4.4, has a nonpositive contrast for these same schedules.

For the regularly varying law $\nu_r = r^{-4}$, the elementary bounds $S_0 \leq 5/48$, $S_1 \leq 1/4$ imply the explicit certificate

$$\Xi_2(\rho) > \rho^2 \left(\frac{1}{64} - 72\rho\right) \quad (0 < \rho < 1/4608),$$

and hence

$$\boxed{\Xi_2(10^{-4}) > \frac{337}{4 \cdot 10^{12}} = 8.425 \times 10^{-11}.}$$

Thus a finite-time sign witness is available after four total updates. The heavy-tail results quantify how the response persists on the scale $A_\nu(T)$ when the preparation and target durations grow together. Section 8.5 proves the finite-time bound directly from event pairs; it is independent of the recovery-front and conditional-correlation estimates.

4.5. Effective extension and required tail data

Normalize the law and intensity by $\widehat{\nu} = \nu/\chi$ and $b = \rho\chi$. The normalized forcing moment is one. The qualitative response theorems allow general regular variation. A computable all-future threshold uses the following effective tail data.

Theorem 4.6 (effective sign onset). Fix the finite-stage profiles of Theorem 4.4 with $Q_\gamma[h] > 0$. Suppose the full normalized law is specified and, for supplied $R_* \geq 2$ and $c > 0$,

$$|\widehat{\nu}_r - cr^{-p}| \leq cr^{-p}\omega(r), \qquad r \geq R_*, \quad p > 3,$$

where ω is a specified nonincreasing function tending to zero. A supplied integer modulus Ω satisfies $\Omega(\varepsilon) \geq R_*$ and $\omega(r) \leq \varepsilon$ for $r \geq \Omega(\varepsilon)$. The finite prefix, tail parameters and profile data admit certified evaluation, and the discrete stage schedule is effectively specified.

Set $c_+ = c[1+\omega(R_*)]$, $E = \gamma(\gamma+1)Q_\gamma[h]$ and $S = cD_pE$. For each admissible tuning tuple, the certificate in Supplement Sections S1 and S9 supplies an explicit budget $b_0 > 0$. For every fixed $0 < b \leq b_0$, it computes a finite threshold $T_0(b)$ such that

$$\boxed{\Xi_h(\rho, T) \geq \frac{1}{2}Sb^2T^{-\gamma} > 0 \quad \text{for every integer } T \geq T_0(b).}$$

The direct capped-kernel certificate applies for every $\gamma > 0$. At $\gamma = 1$, a separate adaptive cap and one-sided martingale estimate give the certificate in S9.3. Exact power tails correspond to $\omega = 0$. For a fixed b, a finite collection of admissible tuples permits the smallest certified $T_0(b)$ to be selected among the tuples whose budgets admit that b.

For the equal-dose profiles of Section 4.3, the normalized law r^{-4}/χ_4 at $b = 10^{-14}$ has the certificate $T_0(b) \leq 10^{20}$, where $\chi_4 = \sum_{r\geq 2} r^{-4}r(r+1)/2$. Supplement S11 retains all sixteen certificates, including the normalized $r^{-4}(1+1/r)$ law, parameter tuples and scalar conditions. These are conservative sufficient all-future bounds. The finite-time witness of Theorem 4.5 has its separate intensity budget.

Proposition 4.7 (no summary-only threshold). Fix $p > 3$, $\chi > 0$ and $0 < \rho\chi \leq 10^{-3}$. For every prescribed integer $N \geq 1$, there exists a law with moment exactly χ and a positive exact r^{-p} tail for which the constant three-arm contrast at $H = 1, T = N$ is negative. Consequently no finite threshold depending only on (p, χ, ρ, H) works uniformly over these laws. The intensity can also satisfy the sufficient budget of Theorem 4.2, so every such law still has eventual positive contrast. The construction in Supplement S10 concerns the constant three-arm family.

5. Recovery fronts in a nonstationary background

The core response argument constructs the actual fronts in this section, controls the short background in Sections 6.1–6.3, and identifies the insertion kernel in Section 7. The proof of Theorem 4.3 in Section 8.4 applies these inputs successively across the fixed preparation stages. Sections 6.4–6.5 give the sharper root-remainder and gate-covariance estimates used by the effective extension; the qualitative history limit uses the coarser remainder estimate in Section 7.2.

5.1. Boolean-row attachment

Write $S_1 = \sum_r r\nu_r$ and $b = a\chi$. The small-intensity phase estimates in Appendix A give

$$u_a = 1 - \beta_a > 0, \qquad b \leq \beta_a \leq \frac{1 - \sqrt{1-4b}}{2}. \tag{16}$$

Throughout the proof, constants may depend on the fixed length law and fixed positive intensities. A superscript K denotes deletion of marks longer than K.

Lemma 5.1 (mean attachment in one row). Let m_a be the mean number of sites attached to the left of a deterministic right ray by one fresh Boolean disaster row. Define

$$B_n = \sum_r \nu_r(r-n)_+, \qquad n \geq 0.$$

Then

$$m_a = \sum_{n\ge 0}(e^{aB_n} - 1), \qquad b \le m_a \le be^{2b/3}. \tag{17}$$

Proof. Put $w = e^{-aS_1}$. Conditional on a site being uncovered, every interval affecting sites strictly to its right starts to its right. Revealing successive origin batches therefore makes uncovered sites a renewal process. If τ is its next uncovered-site gap, its renewal mass is we^{aB_n}, including $n = 0$. For $M(z) = \sum_{n\ge 0}(e^{aB_n} - 1)z^n$ and $F(z) = \mathbb{E}z^\tau$, the renewal equation gives

$$\frac{1}{1 - F(z)} = \frac{w}{1 - z} + wM(z).$$

Thus $Q(z) = w(1-F(z))/(1-z) = [1+(1-z)M(z)]^{-1}$. The coefficients satisfy $[z^n]Q(z) = w\Pr(\tau > n)$ and $\sum_n [z^n]Q(z) = 1$. A stationary covered block of length n can be partitioned by its last uncovered site on the left, giving its probability as $\sum_{j\ge n} w\Pr(\tau > j)$. Thus Q is also the probability generating function of the fixed-boundary attachment length, and its mean is $Q'(1-) = w\mathbb{E}[\tau(\tau-1)/2] = M(1)$. The corresponding tail convolution is derived in Supplement S4.2. The derivative is justified by $\sum_n B_n = \chi < \infty$ and monotonicity of B_n. Alternatively, truncate lengths, differentiate, and use monotone convergence. Finally,

$$aB_n \le e^{aB_n} - 1 \le ae^{aS_1}B_n, \qquad S_1 \le \frac{2}{3}\chi,$$

which proves (17). □

5.2. Adaptive boundary drift

Let L_t be the left endpoint of the right-infinite hole component when the initial holes are $[1, \infty)$. Set $L_0 = 1$ and $J_t = L_t + 1 - L_{t+1} \ge 0$.

Lemma 5.2 (drift at the actual boundary). For every boundary value x of positive probability,

$$m_a \le \mathbb{E}[J_t \mid L_t = x] \le m_a/u_a. \tag{18}$$

Proof. Let H_t^0 be the full-open finite-history baseline and H_t^S the process whose initially open sources are $S = (-\infty, 0]$, coupled through the same rows. Fix a boundary value x and put $z = x - 1$. Write

$$O_z = \{z \notin H_t^S\}, \qquad A_z = \{z \notin H_t^0\}, \qquad R_x = \{[x, t] \subset H_t^S\}.$$

Here R_x is certain when $x = t+1$. Since $[t+1, \infty) \subset H_t^S$, the boundary event is exactly $\{L_t = x\} = O_z \cap R_x$.

For a hole set B, let $\widetilde{B}$ agree with B strictly left of z and force z, x to be open. Attach the hole ray $[x+1, \infty)$ and add an independent fresh row D of intensity $a\nu$. Let $\mathcal{A}_x(\widetilde{B}, D)$ be its left attachment length, measured from x:

$$\{\mathcal{A}_x(\widetilde{B}, D) \ge n\} = \{[x - n + 1, x] \subset \widetilde{B} \cup D\}, \quad n \ge 1.$$

For $M < \infty$, define

$$f_M(B) = \mathbb{E}_D[\mathcal{A}_x(\widetilde{B}, D) \wedge M].$$

This is bounded and increasing in the hole coordinates strictly left of z. When $B = E(H_t^0)$, it depends only on the old coordinates strictly left of z. The new row is independent of these old coordinates.

On O_z, the baseline and seeded old fields agree at every $y < z$. To see this, take a baseline path to y and a seeded path to z. Their pointwise minimum is an open path to y with an allowed source in S. Both old fields are open at z, so their eroded fields agree on $(-\infty, x]$ and have z, x open. On the full boundary event $O_z \cap R_x$, the eroded seeded ray begins at $x + 1$. Independence of D then gives

$$\mathbb{E}[J_t \wedge M \mid L_t = x] = \mathbb{E}[f_M(E(H_t^0)) \mid O_z, R_x].$$

Apply Lemma A.6 with checkpoint row 0, cap 0, and endpoint $z = x - 1$. On O_z, its part (i) identifies the seeded and full-source fields strictly left of z. The function $H \mapsto f_M(E(H))$ is increasing and reads only old hole coordinates strictly left of z: the eroder at such a site reads that site and its immediate left neighbour, and the two boundary coordinates have been clamped open. The event R_x is increasing in seeded holes strictly right of z. Parts (ii)–(iii), or directly (A6.2), therefore give

$$\mathbb{E}[f_M(E(H_t^0)) \mid O_z, R_x] \le \mathbb{E}[f_M(E(H_t^0)) \mid O_z] \le \mathbb{E}[f_M(E(H_t^0)) \mid A_z].$$

The fresh row is averaged in f_M before applying the comparison.

For the Palm bound, put $B = E(H_t^0)$ and $\Gamma_x = \{z, x \notin B\}$. Then $A_z \subset \Gamma_x$. The field B is stationary and ergodic in space, with open density $u_B > 0$. On Γ_x, the clamping used in f_M makes no change. For $n \ge 1$ let

$$N_n(B) = \#\{y \in [x-n+1, x] : y \notin B\}, \qquad c_k = \Pr([1,k] \subset D).$$

Single-row compression gives

$$f_M(B) \le \sum_{n \ge 1} c_{N_n(B)} \qquad \text{on } \Gamma_x.$$

Under the open-site Palm law at x, enumerate open sites as

$$X_0 = x > X_1 > X_2 > \cdots, \qquad G_j = X_j - X_{j+1}.$$

The event Γ_x is $\{x \notin B, G_0 = 1\}$. Exactly G_j positive integers n have $N_n(B) = j+1$, namely

$$x - X_j + 1 \le n \le x - X_{j+1}.$$

Consequently the following is a pathwise identity under this Palm law:

$$\sum_{n \ge 1} c_{N_n(B)} = \sum_{j \ge 0} G_j c_{j+1}.$$

Point-shift invariance and Palm inversion give $\mathbb{E}[G_j \mid x \notin B] = 1/u_B$ for every j. Hence

$$\mathbb{E}[\mathbf{1}_{\Gamma_x} G_j] \le \mathbb{E}[\mathbf{1}_{\{x \notin B\}} G_j] = 1.$$

Tonelli's theorem and $\sum_{k \ge 1} c_k = m_a$ imply

$$\mathbb{E}[\mathbf{1}_{\Gamma_x} f_M(B)] \le m_a, \qquad \mathbb{E}[f_M(B) \mid A_z] \le \frac{m_a}{\Pr(A_z)} \le \frac{m_a}{u_a}.$$

Combining the two conditional comparisons and sending $M \to \infty$ proves the upper bound in (18). For the lower bound, conditional on any old boundary value, fresh-row coverage of the n sites immediately left of the eroded ray forces $J_t \ge n$. The fresh row has its unconditional translated law, so summing its coverage probabilities gives $\mathbb{E}[J_t \mid L_t = x] \ge m_a$. □

5.3. The empty-exterior shape and its speed

Lemma 5.3 (functional front limit). There exists v_a such that, for every fixed $C < \infty$,

$$\frac{1}{T}\mathbb{E} \max_{0 \le t \le CT} |L_t - 1 - v_a t| \longrightarrow 0, \qquad 1 - m_a/u_a \le v_a \le 1 - m_a. \tag{19}$$

Moreover, $v_a^K \downarrow v_a$ as $K \to \infty$.

Proof. At each fixed T, the entire hole field is contained in the original ray together with the union of the first T disaster rows. The latter union is a Boolean row of intensity Ta. Lemma 5.1 implies finite mean attachment, so L_T is integrable. Put $d_T = \mathbb{E}(L_T - 1)$. Restarting only the contained ray at a block boundary deletes other holes and increases the future endpoint. Thus $d_{s+t} \le d_s + d_t$. Lemma 5.2 bounds d_T/T between the two values in (19); Fekete's lemma gives its finite limit v_a.

For a fixed block length n, perform the ray-only restart after every n rows. The successive restarted displacements are iid with mean d_n, because each new slab is independent of the preceding rows and spatial translation does not change its law. The restarted endpoint dominates L_{jn}. The strong law and the deterministic bound $L_{t+1} \le L_t + 1$ yield

$$\limsup_{t \to \infty} (L_t - 1)/t \le d_n/n \quad \text{a.s.}$$

Taking the countable infimum over n gives the upper bound v_a. Since $(L_t - 1)/t \le 1$, its positive deviation from v_a tends to zero in expectation. Convergence of its mean then gives the same for its negative deviation, proving L^1 convergence.

For $s \in [j, k]$, the one-sided slope bound gives

$$(L_k - 1 - v_a k) - (1 - v_a)(k - s) \le L_s - 1 - v_a s \le (L_j - 1 - v_a j) + (1 - v_a)(s - j).$$

A finite time grid proves the maximum version in (19). Under common thinning, $L_T^K \downarrow L_T$, with integrable lower bound L_T and upper bound $T+1$. Hence $d_T^K \to d_T$. Monotonicity gives $v_a^K \ge v_a$, while $\limsup_K v_a^K \le d_T/T$ for each T. Taking the infimum over T proves the speed continuity. □

Corollary 5.4 (quantitative weak-intensity speed). For $b = a\chi \le 1/32$,

$$1 - 2b - 4b^2 \le \kappa_a = 2v_a - 1 \le 1 - 2b, \qquad |\kappa_a^{-1} - (1 + 2b)| \le 10b^2. \tag{20}$$

Proof. The small root in (16) implies $\beta_a \le b + (5/2)b^2$ and also $\beta_a \le 14b/13$ throughout this budget. Thus

$$\frac{e^{2b/3}}{u_a} \le \frac{1}{(1 - 2b/3)(1 - 14b/13)} \le 1 + 2b.$$

Combine this with (17)–(19). The resulting lower bound for κ_a is at least $239/256 > 9/10$; division of $1 - \kappa_a(1 + 2b)$ by κ_a gives the stated inverse bound. In particular $u_a, \kappa_a \to 1$ as $a \downarrow 0$. □

5.4. Transfer to a predictable nonstationary exterior

Lemma 5.5 (front in a dominated exterior). Fix a dominating intensity a_+ with $a_+\chi \le 1/32$. Let the initial baseline holes B_T be coupled below a stationary a_+ hole field, and let an insertion location q_T be measurable with respect to the past, with $|q_T| \le KT$ with probability tending to one. All future rows are independent of that past. If their intensity is $a \le a_+$ and their length cutoff $m_T \to \infty$, the left boundary of the inserted right ray satisfies

$$\sup_{0 \le t \le CT} |L_t - q_T - v_a t|/T \longrightarrow 0 \quad \text{in probability.} \tag{21}$$

The right boundary of a left ray has velocity $1 - v_a$.

Proof. Fix $\varepsilon > 0$ and an integer N. The stationary dominating field is a spatially ergodic factor of independent origin histories, with positive open density. Divide a deterministic interval containing all permitted q_T into finitely many cells of length proportional to εT. With probability tending to one, every cell contains at least N open sites. Hence the interval immediately left of any permitted q_T contains N baseline-open sites, chosen by a rule that reads only the past.

For a finite open seed, whenever a descendant survives, its extreme descendants equal the corresponding half-line extremes. Indeed, an extra half-line path ending beyond a surviving seed path must cross it; exchanging prefixes makes its endpoint seed-reachable. The same crossing argument identifies the full-source baseline between the seed extremes.

For the extinction comparison, use reflected propagation $B_-C = C \cup (C - 1)$, killing $K_D C = C \setminus D$, ordinary propagation-then-kill $F_D = K_D B_-$, and actual backward update $G_D = B_- K_D$. Row by row, $B_- F_D = G_D B_-$. Thus

$$G_{D_t} \cdots G_{D_1}(B_- C) = B_- F_{D_t} \cdots F_{D_1} C.$$

Since $B_- A$ is empty exactly when A is empty, ordinary extinction from C equals actual backward extinction from $B_- C$. Finite-history duality and its increasing limit identify the latter probability with $p_a(B_- C)$. Compression and monotone coupling therefore give, for $|C| = N$,

$$\Pr_C^a(F \text{ eventually dies}) = p_a(B_- C) \le h_a(|B_- C|) \le h_{a_+}(N).$$

The last bound tends to zero as $N \to \infty$ and remains an upper bound under every deletion of future marks. Reflection gives the same statement for forward increments $0, 1$. For each selected source, the relevant half-line endpoints in the truncated future are sandwiched between those for the full a law and any fixed length cutoff $K_0 \le m_T$. Lemma 5.3, followed by $K_0 \to \infty$, gives their common velocity. The sources are selected before the future rows, so a union bound over the fixed N sources is valid. One surviving source supplies

a lower bracket for the ray's rightmost open point; the completely open left half-line supplies the upper bracket. Their initial positions differ by at most εT. Let $T \to \infty$, then $N \to \infty$, then $\varepsilon \downarrow 0$.

For fixed N and ε, condition on the entire past after selecting the sources. Each selected source has the same translated future half-line law. The probability that any of the N endpoint bounds fails is therefore at most Ne_T, where $e_T \to 0$ by Lemma 5.3 and the cutoff sandwich. The probability that every selected family dies is at most $h_{a_+}(N)$. The past event that the required open sites exist has probability tending to one. These three bounds justify the order $T \to \infty$, then $N \to \infty$, then $\varepsilon \downarrow 0$, even for a past-dependent insertion location. The reflected boundary statement follows under $x \mapsto t - x$, which preserves allowed increments and the Poisson interval law. The finite-grid argument gives the supremum over bounded scaled times in (21). □

6. Forgetting the short-disaster background

6.1. Backward ancestors and a uniform restart

For a finite target C, define the actual backward update

$$G_D(C) = (C \setminus D) \cup ((C \setminus D) - 1). \tag{22}$$

Write $U_t(C)$ for its iteration, and $p_a(C)$ for the all-hole probability of C in the stationary a phase. Set $p_a(\varnothing) = 1$. Finite-history path reversal gives $p_{a,t}(C) = \Pr_C(U_t = \varnothing)$. Compression and stationary harmonicity also give

$$p_a(C) = \Pr_C(\tau < \infty), \qquad p_a(C) - p_{a,t}(C) = \Pr_C(t < \tau < \infty), \tag{23}$$

where τ is the extinction time. To see the stationary identity, $p_a(U_t)$ is a bounded martingale. On survival, the finite population visits each bounded cardinality range only finitely often: a supported mark length gives a positive one-row killing probability uniformly over each such range. Thus $|U_t| \to \infty$ on survival, and compression bounds $p_a(U_t)$ by $h_a(|U_t|) \to 0$. On extinction it equals one.

Lemma 6.1 (thin surviving ancestors). For all length truncations of a fixed law with the stated budget there is a common sequence $\eta_n \to 0$ such that

$$\sup_{0<|C|<\infty} \Pr_C(0 < |U_{8n}| < \lceil n/2 \rceil) \le \eta_n. \tag{24}$$

Proof. Consider first an ordinary propagation-then-kill process from one site, with reflected increments $0, -1$. On survival, prefix exchange identifies its two extremes with the half-line extremes and identifies its interior with the full-source baseline. At time $8n$, intervals of length n located at displacements $[-5n, -4n)$ and $[-3n, -2n)$ lie strictly inside the typical endpoint envelope, since $v_a > 0.96$. The finite-history full-source baseline contains the corresponding stationary open field. Spatial ergodicity and $u_a > 15/16$ imply that each port contains at least $n/2$ open sites with probability tending to one. Consequently the probability of survival without these dense outputs is at most $\delta_n \to 0$.

The endpoint and baseline good events improve when marks are deleted. For an upper endpoint use a fixed-cutoff half-line as in Lemma 5.3; this gives the same error bound for all truncations. The actual ancestor process first kills, then expands. Its identity

$$G_{D_T} \cdots G_{D_1} C = B_- F_{D_T} \cdots F_{D_2}(C \setminus D_1), \quad B_- A = A \cup (A - 1), \quad F_D = K_D B_-,$$

where $K_D A = A \setminus D$, transfers the single-source statement with one fewer intermediate row. The final expansion only adds sites.

Given any nonempty finite C, select at most N sites before revealing the next slab. If $|C| < N$, extinction of every selected family is extinction of C. Otherwise compression bounds the probability that all selected descendants vanish by $h_{a_+}(N)$. A union bound over the N selected single-source bad events gives

$$\Pr_C(0 < |U_{8n}| < \lceil n/2 \rceil) \le h_{a_+}(N) + N\delta_n.$$

Choose $N = N(n) \to \infty$ sufficiently slowly. This construction is conditional on each predictable input set and therefore can be restarted at successive independent slabs. □

6.2. Cross-intensity control

For $a \le a_+$ define

$$f_{a_+}(C) = p_{a_+}(C)\mathbf{1}_{\{C\neq\varnothing\}}, \qquad R_{a,a_+}(t) = \sup_{C \text{ finite}} \mathbb{E}^a_C f_{a_+}(U_t). \tag{25}$$

The same definitions with subscript m use the common length cutoff m for both intensities.

Lemma 6.2 (uniform history bound and cutoff improvement). There is a constant C independent of m such that

$$R_{a,a_+,m}(t) \le CA_\nu(t). \tag{26}$$

If $m_T = \lfloor \delta T \rfloor$ and $\lambda > 32\delta$, then

$$R_{a,a_+,m_T}(\lfloor \lambda T \rfloor) = o(A_\nu(T)). \tag{27}$$

Proof. Under common thinning, the a ancestor set contains the a_+ ancestor set. Since p_{a_+} decreases under target inclusion, stationary harmonicity shows that f_{a_+} is superharmonic for the a ancestor chain. This remains true after deleting its empty-state contribution. Split the population after $8n$ rows into empty, large and small nonempty sets. Superharmonicity and compression pay $h_{a_+,m}(\lceil n/2 \rceil)$ on the large branch. Retaining the remaining-time cost on the small branch and using Lemma 6.1 gives

$$R_m(t+8n) \le h_{a_+,m}(\lceil n/2 \rceil) + \eta_n R_m(t). \tag{28}$$

Appendix B gives $h_{a_+,m}(k) \le h_{a_+}(k) \le C_0 A_\nu(k)$. For large L, choose $n = \lfloor L/16 \rfloor$, $t = L - 8n$. Regular variation bounds $A_\nu(t)/A_\nu(L)$ and $A_\nu(\lceil n/2 \rceil)/A_\nu(L)$ by constants. Choose constants C_1, C_2 such that $h_{a_+,m}(\lceil n/2 \rceil) \le C_1 A_\nu(L)$ and $A_\nu(t) \le C_2 A_\nu(L)$ for all sufficiently large L, uniformly in m. Increase that threshold until $\eta_n C_2 \le 1/2$. If $R_m(t) \le K A_\nu(t)$, (28) then gives $R_m(L) \le (C_1 + K/2) A_\nu(L)$. Taking $K \ge 2C_1$ and enlarging it over the finitely many smaller L closes strong induction, with one constant for all cutoffs.

For $L = \lfloor \lambda T \rfloor$, the target block in (28) has length asymptotic to $\lambda T/32$, strictly larger than m_T. The subthreshold cutoff estimate in Appendix B makes its probability $o(A_\nu(T))$. The second term is $\eta_n O(A_\nu(t)) = o(A_\nu(T))$. This proves (27). □

6.3. The finite reference arm

Choose

$$0 < \delta < \min\{1/64, H/64\}, \qquad m_T = \lfloor \delta T \rfloor. \tag{29}$$

Run the finite preparation protocol with only marks of length at most m_T. Let $b_{a,T}$ denote its terminal hole probability. Take $a_+ = \max(a, \rho)$. Under a common construction from the past, its old field and the target stationary cutoff field are both between the all-open field and the stationary a_+ cutoff field. With the same target future, the probability that the two bounding fields differ at the terminal site is bounded by $R_{\rho,a_+,m_T}(T)$. Therefore

$$|b_{a,T} - b_{\rho,T}| = o(A_\nu(T)), \qquad b_{a,T}, b_{\rho,T} \longrightarrow 1 - u_\rho. \tag{30}$$

For the last convergence, the target cutoff stationary densities converge to $1 - u_\rho$ by increasing coupling, and (27) bounds the finite-history discrepancy. The bound applies directly to the entire finite ancestor set. Applying it to the old segment uses $H > 32\delta$.

6.4. Complete-path control of the long-root remainder

Lemma 6.3 (all-count extremal-path bound). In a finite terminal cone, condition on the unchanged short-mark background. Insert independent Poisson long interval roots of finite measure μ, and let $F(\zeta)$ be the terminal hole indicator. Put

$$\mathcal{E}(\zeta) = F(\zeta) - F(\varnothing) - \sum_{e\in\zeta} [F(\{e\}) - F(\varnothing)].$$

If row-t roots have homogeneous per-origin intensity $a_t w_r$, define $\mathsf{P} = \sum_t a_t \sum_{r>m} r w_r$. Then

$$-\tfrac{1}{2}\mathsf{P}^2 \le \mathbb{E}\mathcal{E} \le \mathsf{P}^2.$$

For the three-arm schedules and tail bound of Theorem 4.6, the weighted remainder has absolute value at most $3P_\delta^2 b^2 T^{-2\gamma}$, with P_δ specified in Supplement S1. The bound includes roots that are individually ineffective.

Proof. Fix an integer observation time and condition on the short-mark background, with cutoff $m = \lfloor \delta T \rfloor$. Retain every longer root in the deterministic terminal causal cone. Their independent Poisson measure is denoted by μ. It has finite mass because the cone has finitely many rows and $\sum_r r\widehat{\nu}_r < \infty$. The long-root weights may be the actual law or cr^{-p} in the effective-tail hybrid of Supplement S8.

Let $F(\zeta) \in \{0,1\}$ be the terminal hole indicator after inserting the finite root configuration ζ. Write

$$s(e) = F(\{e\}) - F(\varnothing), \qquad \mathcal{E}(\zeta) = F(\zeta) - F(\varnothing) - \sum_{e\in\zeta} s(e).$$

Distinct Poisson atoms are counted separately even when they have identical birth row, origin and length.

If $F(\varnothing) = 1$, monotonicity gives $\mathcal{E} = 0$. Suppose instead that the short background is terminal-open. Its openness satisfies

$$X_{t+1} = (X_t \cup (X_t + 1)) \setminus D_{t+1}.$$

A complete open path runs from the fully open reset row to the terminal site, has increments in $\{0,1\}$, and avoids the short disasters at each row. There are finitely many such paths in the terminal cone. Their pointwise minimum and maximum, denoted by x_t^-, x_t^+, are themselves complete open paths. Each extreme coordinate belongs to a valid path and therefore avoids that row's disasters. For either pointwise extreme, the step is nonnegative and at most one because every path satisfies those two inequalities.

At row t, the vertices of complete paths are exactly

$$A_t = U_t \cap X_t,$$

where U_t reads only rows strictly after t, as in Supplement S3.1. A past path to any point of this intersection concatenates with a future path to the terminal site. Consequently

$$x_t^- = \min A_t, \qquad x_t^+ = \max A_t.$$

This identification permits disconnected A_t.

For a root $e = (t, q, r)$ with interval $I_e = [q, q + r - 1]$, put

$$L_e = \mathbf{1}_{\{x_t^- \in I_e\}}, \qquad R_e = \mathbf{1}_{\{x_t^+ \in I_e\}}.$$

Inserting e alone closes the terminal site exactly when it hits every complete open path at its birth row. An interval contains A_t exactly when it contains both extrema. Thus

$$\boxed{s(e) = L_e R_e.}$$

For a root configuration ζ, define

$$B_\zeta = \sum_{e\in\zeta} L_e R_e, \qquad \mathcal{C}_\zeta = \sum_{\substack{e,f\in\zeta \\ e\neq f}} L_e R_f.$$

The second sum counts ordered pairs of distinct atoms.

If $B_\zeta \geq 1$, a single root already closes the terminal site, so

$$\mathcal{E}(\zeta) = 1 - B_\zeta, \qquad \mathcal{C}_\zeta \geq B_\zeta(B_\zeta - 1).$$

It follows that $-\mathcal{C}_\zeta/2 \leq \mathcal{E}(\zeta) \leq 0$. If $B_\zeta = 0$ and the terminal site remains open, the remainder is zero. If $B_\zeta = 0$ and the terminal site closes, both complete extremal paths have been hit. Their hits must belong to distinct roots, since a root hitting both would contribute to B_ζ. Hence $\mathcal{C}_\zeta \geq 1$ and the remainder equals one. These cases prove

$$\boxed{-\tfrac{1}{2}\mathcal{C}_\zeta \leq \mathcal{E}(\zeta) \leq \mathcal{C}_\zeta.}$$

This inequality includes cooperative configurations in which every individual root is ineffective.

Let the row-t long-root measure have intensity $a_t w_r$ per origin and length. Define the deterministic single-path hitting mean

$$\mathsf{P} = \sum_t a_t \sum_{r>m} r w_r.$$

Conditional on a terminal-open short background, the selected paths are fixed and the independent long roots retain their Poisson law. A fixed site belongs to exactly r integer intervals of length r, and all of these intervals are causal-cone candidates. Therefore

$$\int L_e \, \mu(de) = \int R_e \, \mu(de) = \mathsf{P}.$$

The Poisson second factorial moment identity gives

$$\mathbb{E}[\mathcal{C}_\zeta \mid \mathcal{B}] = \iint L_e R_f \, \mu(de)\mu(df) = \mathsf{P}^2 \qquad (F(\varnothing) = 0).$$

The exclusion $e \neq f$ concerns atom labels; the product measure still includes identical mark coordinates. The remainder is zero on terminal-closed backgrounds. Averaging the samplewise inequality over open backgrounds yields

$$\boxed{-\tfrac{1}{2}\mathsf{P}^2 \le \mathbb{E}\mathcal{E} \le \mathsf{P}^2.}$$

Moreover, the Poisson first moment identity gives the exact first-order expansion

$$q = q^{(0)} + \int \mathbb{E}_{\mathrm{short}} s(e) \, \mu(de) + \mathbb{E}\mathcal{E}.$$

It is the same expansion as (34), with its remainder controlled by effective path pairs.

For the schedules in this paper, the row count is at most CT, $a_t \le Bb$, and $w_r \le c_+ r^{-p}$. For $m \ge \delta T/2$,

$$\mathsf{P} \le b P_\delta T^{-\gamma}, \qquad P_\delta = \frac{BCc_+}{p-2}(\delta/2)^{2-p}.$$

For the three arms, the positive weights use the lower remainder bound and the central negative weight uses the upper bound. Reversing the choices gives the corresponding upper error. Thus

$$\boxed{|\mathbb{E}\mathcal{E}_+ - 2\mathbb{E}\mathcal{E}_b + \mathbb{E}\mathcal{E}_-| \le 3P_\delta^2 b^2 T^{-2\gamma}.}$$

The pair coefficient has order $\delta^{-2(p-2)}$; counting all causal-cone candidates gives order $\delta^{-2(p-1)}$. Root screening is performed in the proof conditional on the unchanged short background.

□

6.5. Resolving the actual ancestor gate

A root is inserted after its birth row disasters. Its age n counts the subsequent rows. The first reverse row kills the terminal singleton before expansion. Hence on terminal openness there are $n-1$ ordinary fresh-attachment increments followed by one final expansion; the deterministic one-row correction is recorded in Supplement S3.3.

Lemma 6.4 (ancestor-gate covariance). Work in normalized coordinates, set $B = B_*$, and use deterministic row rates at most Bb and the phase and contraction budget of Supplement S9. Let O be the actual terminal-open gate for a root of age n, including the independent birth field. Let Z be the sum of the centered capped fresh-row attachments of the two complete auxiliary half-lines, continued through every reverse row. At each predictable edge, erase the old exterior holes before measuring this fresh-row attachment, denoted $J'_{L,i}$ or $J'_{R,i}$ in Supplement S3.3. The extra attachment caused by the old exterior is a separate nonnegative excess debt. The per-side capped second moment is bounded by av_K in a row of intensity a. Then

$$|\mathbb{E}[(1-O)Z]| \le 2b\sqrt{BK_R v_K} \sum_{\substack{j\ge 0 \\ 2^j < n}} 2^{j(1-\gamma)/2}.$$

The sum is zero for $n \le 1$. The proof uses the rate-string history estimate $\mathbb{E} f_+(U_h) \le b \min\{2B, K_R h^{-\gamma}\}$ established in Supplement S2.3, where $f_+(C)$ is the maximum-rate stationary all-hole probability for a nonempty finite target and is zero at the empty target.

Proof. Fix a root age $n \ge 1$. Let $\mathcal{F}_h$ expose the first h complete reverse disaster rows, starting at the terminal singleton, $1 \le h \le n$. The birth field remains unrevealed. Its law and every unread row are independent of $\mathcal{F}_h$. Write

$$O = \mathbf{1}_{\{U_n \cap X_{\mathrm{birth}} \neq \varnothing\}}, \qquad Y = 1 - O, \qquad M_h = \mathbb{E}[Y \mid \mathcal{F}_h].$$

The birth holes are dominated by the stationary maximum-rate cutoff field. Evolving them through the unread rows preserves that domination. Conditional on the known U_h, terminal closure is the event that this intermediate field covers U_h. Therefore

$$M_h = 1 \quad \text{if } U_h = \varnothing, \qquad M_h \le p_+(U_h) \quad \text{otherwise.}$$

Since Y is Bernoulli,

$$\mathrm{Var}(Y \mid \mathcal{F}_h) = M_h(1 - M_h) \le f_+(U_h).$$

The rate-string bound now yields

$$\boxed{\mathbb{E}\,\mathrm{Var}(Y \mid \mathcal{F}_h) \le b \min\{2B, K_R h^{-\gamma}\}.}$$

The bound includes both extinction in unread rows and complete covering at the eventual birth row. Even at $h = n$, M_n can remain nonbinary because the birth field has not been revealed.

Use the complete auxiliary half-lines in Supplement S3–S4. Continue them through every reverse row, including outcomes where the finite ancestor family has died. Let Z_h be the sum of their centered capped fresh-row attachments $J'_{L,i} \wedge K$ and $J'_{R,i} \wedge K$ over reverse rows $2, \dots, h$. The first reverse row kills the terminal singleton and has no attachment increment. Thus $Z_1 = 0$ and $Z_n = Z$.

Conditional on the complete preceding rows, each half-line edge is fixed before the fresh row is revealed. Translation invariance gives its fresh capped-attachment mean $\mathbb{E}(J_{a,m} \wedge K)$, independent of the edge position. The two sides can be dependent within a row. Their centered sum has conditional mean zero and conditional variance at most $4av_K$, since $\mathbb{E}(J_{a,m} \wedge K)^2 \le av_K$. The finite-variance untruncated option in Supplement S4.2 obeys the same estimates. Consequently Z_h is a square-integrable martingale and

$$\mathbb{E}[(Z_k - Z_h)^2] \le 4Bb\, v_K(k - h), \qquad 1 \le h < k \le n.$$

This conditional centering uses the full reverse-row filtration; it imposes no independence between the two sides or between the gate and the attachments.

For a block $[h, k]$, projection onto $\mathcal{F}_k$ and martingale centering give

$$\mathbb{E}[Y(Z_k - Z_h)] = \mathbb{E}[(M_k - M_h)(Z_k - Z_h)].$$

Conditional Jensen gives

$$\mathbb{E}[(M_k - M_h)^2] \le \mathbb{E}\,\mathrm{Var}(Y \mid \mathcal{F}_h).$$

Combining the two estimates,

$$\boxed{|\mathbb{E}[Y(Z_k - Z_h)]| \le 2b\sqrt{Bv_K(k - h) \min\{2B, K_R h^{-\gamma}\}}.}$$

The estimate uses unresolved gate variance at the start of each block.

Take successive endpoints $1, 2, 4, \dots, n$, shortening the last block when n is not a power of two. Then $k - h \le h$ in every block. Summing and using the tail branch of the minimum proves

$$\boxed{|\mathbb{E}[(1 - O)Z]| \le 2b\sqrt{BK_R v_K}\, L_\gamma(n), \qquad L_\gamma(n) = \sum_{\substack{j \ge 0 \\ 2^j < n}} 2^{j(1-\gamma)/2}.}$$

Set $L_\gamma(n) = 0$ for $n = 0, 1$; in these cases the attachment sum is zero. The first-row deterministic correction μ of Supplement S3.3 is retained in its existing payment.

For $N \geq 2$, put $q = (1-\gamma)/2$ and use the continuous upper bound

$$\overline{L}_\gamma(N) = \begin{cases} (2N)^q/(2^q-1), & 0 < \gamma < 1, \\ 1 + \log_2 N, & \gamma = 1, \\ (1-2^q)^{-1}, & \gamma > 1. \end{cases}$$

The finite geometric sum is bounded by this expression for every $n \leq N$. In particular $L_1(n) = \lceil \log_2 n \rceil$ for $n \geq 1$.

□

7. Exact finite-preparation response

7.1. Pathwise resetting at the intensity switch

Lemma 7.1 (two-ray representation and exact reset). Evolve a baseline B and a baseline with an inserted interval $[q, q+r-1]$ using the same future rows. Let L_t and R_t be the respective finite endpoints of the right and left hole rays inserted into the same baseline. Before their first crossing, the interval-inserted process equals

$$B_t \cup [L_t, R_t]. \tag{31}$$

After crossing it equals B_t permanently. Each ray can be restarted at any intermediate row using the actual baseline and its actual endpoint.

Proof. The map $\Phi_D(H) = D \cup [H \cap (H+1)]$ preserves intersections. The interval-inserted process is therefore the intersection of the two ray-inserted processes. For the right ray, take an open path to $L_t - 1$. Any baseline open path starting in the deleted half-line and ending left of L_t must cross this path. Exchanging prefixes makes the endpoint reachable from an allowed source. Thus the right-ray holes equal $B_t \cup [L_t, \infty)$. The left-ray identity is symmetric. Their intersection proves (31). If $L_t > R_t$, every baseline-open endpoint is reachable from at least one exterior source, so the interval process and baseline agree. The common update preserves equality thereafter. □

Insert a root of length $r/T \to s$ at old age zT, with $0 \leq z \leq H$ and origin $x/T \to \xi$. Suppress all other marks longer than m_T. The baseline is dominated by the stationary a_+ field. Lemma 5.5 first applies from the birth row to the switch. At the switch, Lemma 7.1 resets the rays at their past-measurable endpoints with the actual baseline. Applying Lemma 5.5 again gives

$$L_T/T = \xi + v_a z + v_\rho + o_{\mathrm{Pr}}(1), \qquad R_T/T = \xi + s + (1-v_a)z + (1-v_\rho) + o_{\mathrm{Pr}}(1). \tag{32}$$

Define the deterministic scaled endpoints of an old root by

$$\ell_a(\xi, z) = \xi + v_a z + v_\rho, \qquad r_a(\xi, s, z) = \xi + s + (1-v_a)z + (1-v_\rho).$$

The limiting width is $(s - \kappa_a z - \kappa_\rho)_+$. If the final limiting interval has positive width, its width was positive throughout both stages because $\kappa_a, \kappa_\rho > 0$. Thus no earlier crossing changes the coverage criterion. Lemma 7.1 identifies the insertion benefit at the terminal site with

$$\mathbb{E}\big[\mathbf{1}_{\{0 \notin B_T\}} \mathbf{1}_{\{L_T \leq 0 \leq R_T\}}\big],$$

where the second indicator is zero after crossing.

To prove uniformity over the root coordinates, fix δ, M, H and a bounded origin box. If the probability error in (32) failed to vanish uniformly over deterministic root coordinates in that box, choose a sequence attaining a positive error and a subsequence whose scaled age, length and origin converge. Lemma 5.5 applies to the corresponding baseline sequence at the birth row. At the switch its endpoints are past-measurable and tight on the scale T; the exact reset and a second application of that lemma give (32) along the subsequence, a contradiction. The functional time bound includes time zero and makes the displacement during a stage of duration $o(T)$ equal to $o_{\mathrm{Pr}}(T)$. The estimates for Lemma 5.5 use only the dominating stationary law and the fixed future intensities, so they apply uniformly to these baseline sequences.

Let f_e be the terminal hole probability with the single root e inserted into the short-mark background, and let $r_{a,T}(e) = f_e - b_{a,T}$ be its insertion benefit. Outside the two endpoint strips $|\ell_a| \le \varepsilon$ and $|r_a| \le \varepsilon$, the preceding uniform limit and (30) give

$$\sup_e \left| r_{a,T}(e) - u_\rho \mathbf{1}_{\{\ell_a(\xi,z)<0<r_a(\xi,s,z)\}} \right| \longrightarrow 0.$$

Indeed, write I_T for the random interval-coverage indicator and I for its deterministic limit. Then

$$\left| \mathbb{E}[\mathbf{1}_{\{0 \notin B_T\}} I_T] - u_\rho I \right| \le \Pr(I_T \ne I) + |\Pr(0 \notin B_T) - u_\rho|.$$

7.2. Zero- and one-root expansion in the causal cone

Fix a maximum scaled length $M > \delta$. The entire preparation and target history has height $N_T + T = O_H(T)$. A mark of length at most MT relevant to the terminal backward cone has origin in a deterministic interval of length $O_{H,M}(T)$. The total mean of relevant marks with $m_T < r \le MT$ is

$$\Lambda_T = O_{H,M,\delta}\left(T^2 \sum_{r>\delta T} \nu_r \right) = O(A_\nu(T)) \longrightarrow 0. \tag{33}$$

With f_e as in Section 7.1, conditioning on zero, one, or at least two candidate marks gives

$$q_{a,H}^{(M)}(\rho, T) = b_{a,T} + \int (f_e - b_{a,T}) \lambda_a(de) + O(\Lambda_T^2). \tag{34}$$

Indeed, the probability of at least two marks is at most $\Lambda_T^2/2$, and expanding $e^{-\Lambda_T}$ in the zero- and one-mark terms supplies the remaining error. The response lies between zero and one. The count in this expansion is the Poisson number of candidate marks. Lemma 6.3 also controls the entire remainder through pairs hitting the two complete extremal open paths, including individually ineffective cooperative roots. Under the effective-tail envelope of Theorem 4.6, it gives the coefficient $3P_\delta^2 b^2 T^{-2\gamma}$. The $O(\Lambda_T^2)$ estimate above suffices for the general regularly varying limit here.

Let j be the integer old age of a root at the switch. In a fixed origin box $|x| \le KT$, define the normalized unit-intensity measure

$$\mathfrak{m}_T^{\mathrm{old}} = \frac{1}{A_\nu(T)} \sum_{0 \le j < N_T} \sum_{m_T < r \le MT} \sum_{|x| \le KT} \nu_r \, \delta_{(x/T, r/T, j/T)}.$$

The finite one-row differences caused by the precise endpoint convention have normalized mass $O(T^{-1})$. Regular variation, uniformly for $\delta \le r/T \le M$, implies convergence of these finite measures on the compact box to

$$\frac{1}{D_p} s^{-p} \, d\xi \, ds \, dz, \qquad |\xi| \le K, \quad \delta \le s \le M, \quad 0 \le z \le H.$$

Their total masses are bounded. The actual old-root measure is $a\mathfrak{m}_T^{\mathrm{old}}$ in the a arm and $\rho\mathfrak{m}_T^{\mathrm{old}}$ in the reference arm.

For either arm, the two limiting endpoint surfaces are affine in ξ with slope one. Their ε strips therefore have limiting normalized mass $O_{H,M,\delta}(\varepsilon)$. In the discrete sums their masses are $O(\varepsilon) + o(1)$ as well, by the same Riemann-sum limit or direct counting of $O(\varepsilon T + 1)$ origins per length and birth row. On the complement, the uniform response estimate from Section 7.1 permits replacement of the insertion benefit by u_ρ times the deterministic coverage indicator. Boundedness by one pays for the strips. Taking $T \to \infty$ and then $\varepsilon \downarrow 0$ identifies the full old-root integral. Integration over ξ gives precisely the width $(s - \kappa_a z - \kappa_\rho)_+$.

Target-era roots require the same argument separately. If yT is their age at observation, their deterministic endpoints are

$$\ell_{\mathrm{new}}(\xi, y) = \xi + v_\rho y, \qquad r_{\mathrm{new}}(\xi, s, y) = \xi + s + (1 - v_\rho) y, \quad 0 \le y \le 1.$$

These endpoints do not depend on the old intensity. Their normalized unit-intensity measure converges to $D_p^{-1} s^{-p} \, d\xi \, ds \, dy$. Both arms have target birth intensity ρ, and both no-root terminal baselines have open

density tending to u_ρ by (30). The uniform front-transfer argument therefore gives the same limiting target-root integral in both arms. Their difference is $o(A_\nu(T))$. The zero-root difference is also $o(A_\nu(T))$ by (30), and $\Lambda_T^2/A_\nu(T) \to 0$ by (33). Combining these facts with (34) gives

$$\lim_{T\to\infty} \frac{q_{a,H}^{(M)} - q_{\rho,H}^{(M)}}{A_\nu(T)} = \frac{u_\rho}{D_p}\Bigg[a\int_\delta^M s^{-p}\int_0^H (s-\kappa_a z-\kappa_\rho)_+\,dz\,ds \\ -\rho\int_\delta^M s^{-p}\int_0^H (s-\kappa_\rho z-\kappa_\rho)_+\,dz\,ds\Bigg]. \tag{35}$$

The factor D_p^{-1} comes from $A_\nu(T) \sim D_p T^{3-p} L(T)$.

7.3. Releasing the length cutoff and evaluating the integral

Deleting marks longer than MT changes the terminal site only if one of them hits its finite backward cone. The mean number of such marks is at most

$$C_{H,a,\rho}\, T \sum_{r>MT} (r + C_H T)\nu_r. \tag{36}$$

Dividing by $A_\nu(T)$ and taking the upper limit in T bounds the error by a constant times $M^{2-p} + M^{1-p}$. This tends to zero as $M \to \infty$. The factor T in (36) is the number of time rows.

The lower cutoff disappears since $\delta < \kappa_\rho$ and the integrand vanishes for $s \le \kappa_\rho$. For $k = \kappa_\rho$ and $c = \kappa_a$, Tonelli's theorem gives

$$\begin{aligned} I_H(c,k) &= \int_k^\infty s^{-p}\int_0^H (s-cz-k)_+\,dz\,ds \\ &= \frac{1}{(p-1)(p-2)}\int_0^H (k+cz)^{2-p}\,dz \\ &= D_p\frac{k^{-\gamma}}{c}\left[1-\left(1+\frac{cH}{k}\right)^{-\gamma}\right]. \end{aligned} \tag{37}$$

Substitution in (35) proves Theorem 4.1. The order of limits is fixed parameters, then $T \to \infty$, then $M \to \infty$. The old-age integral is over the fixed interval $[0, H]$. □

8. Proof of the sign separation and its consequences

8.1. Positive cooperative curvature

Let $g(a) = \kappa_a^{-1} \ge 1$ and

$$\Phi(g) = g\left[1-\left(1+\frac{H}{\kappa_\rho g}\right)^{-\gamma}\right].$$

Then $F_{\rho,H}(a) = a\Phi(g(a))$. For $x = H/(\kappa_\rho g)$,

$$\Phi'(g) = \psi_\gamma(x), \qquad \Phi''(g) = -\frac{\gamma(\gamma+1)x^2}{g(1+x)^{\gamma+2}}. \tag{38}$$

For $g \ge 1$, $0 \le \Phi'(g) \le 1$ and $|\Phi''(g)| \le \gamma(\gamma+1)$. Corollary 5.4 implies

$$g(a) = 1 + 2\chi a + e(a), \quad |e(a)| \le 10\chi^2 a^2, \quad 0 \le g(a) - 1 \le (37/16)\chi a.$$

Taylor's inequality applied to this explicit function yields

$$F(a) = a\Phi(1) + 2\chi a^2\Phi'(1) + R(a), \qquad |R(a)| \le K_\gamma \chi^2 a^3. \tag{39}$$

The linear terms cancel in $F(3\rho) - 2F(2\rho) + F(\rho)$. The quadratic coefficient is $2(9-8+1) = 4$, and the absolute cubic error weight is $27 + 16 + 1 = 44$. This proves (10).

Direct differentiation gives

$$\psi'_\gamma(x) = \gamma(\gamma+1)x(1+x)^{-\gamma-2} > 0, \quad \psi_\gamma(0) = 0. \tag{40}$$

Since $\kappa_\rho \le 1$, the leading term in (10) is at least $4\chi\rho^2\psi_\gamma(H)$. The budget (11) leaves at least half of it after paying the error. Equations (7) and (12) imply eventual positivity and prove Theorem 4.2.

Taking $\rho \downarrow 0$ after the time limit, using $u_\rho, \kappa_\rho \to 1$, proves (13). Expansion of (40) at zero gives $\psi_\gamma(H) \sim \gamma(\gamma+1)H^2/2$; its defining formula gives the limit one at infinity. The ratio denominator is eventually positive throughout (11). Indeed, common-row thinning makes $a \mapsto v_a$ and $a \mapsto \kappa_a$ nonincreasing. Thus $g(2\rho) \ge g(\rho)$; because $\Phi > 0$ and $\Phi' \ge 0$,

$$F(2\rho) - F(\rho) = 2\rho\Phi(g(2\rho)) - \rho\Phi(g(\rho)) \ge \rho\Phi(g(\rho)) > 0.$$

Equation (5) transfers this strict limiting difference to the terminal probabilities for all sufficiently large T. For the weak-intensity ratio in (14), (39) gives

$$F(2\rho) - F(\rho) = \rho\Phi(1) + 6\chi\rho^2\Phi'(1) + O(\rho^3),$$

while $F(3\rho) - F(\rho) = 2\rho\Phi(1) + 16\chi\rho^2\Phi'(1) + O(\rho^3)$. Since $\Phi(1) \to 1 - (1+H)^{-\gamma} > 0$, division proves (15).

8.2. The Poisson negative square

Conditional on the environment, use the notation M, N of Section 3. At old intensity a the mean number of grains covering the terminal point is $aN + \rho(M-N)$. The conditional uncovered probability is therefore

$$e^{-\rho M}e^{-(a-\rho)N}.$$

Taking the second difference of the three covered probabilities gives

$$-e^{-\rho M}(1 - 2e^{-\rho N} + e^{-2\rho N}) = -e^{-\rho M}(1 - e^{-\rho N})^2.$$

Expectation proves Theorem 3.1. All integrands are bounded; finiteness of M almost surely suffices. The proof permits arbitrary frozen conditional grain kernels, including age-dependent shapes and motion. For a positive denominator, the ratio bound follows from $W \le 0$. □

8.3. Control calibration and observation accuracy

Linear intensity control is a substantive part of the comparison. For example, a single frozen coverage variable with hole probability $q(a) = 1 - e^{-na^2}$, $n > 0$, arises from a nonlinear birth-intensity calibration. Its three-point contrast is

$$q(3\rho) - 2q(2\rho) + q(\rho) = 2n\rho^2 + O(\rho^4) > 0$$

for small ρ. Thus the control-to-intensity calibration changes the curvature and must be fixed for the sign comparison.

The Poisson birth assumption also matters. Consider two independent Bernoulli persistent grains, each covering the observation point, with preparation probabilities $(\rho, 3\rho)$, $(2\rho, 2\rho)$ and $(3\rho, \rho)$ across the three arms, where $0 < \rho \le 1/3$. Their kernels are fixed and their total probability doses equal 4ρ. The side-arm coverage probabilities are $4\rho - 3\rho^2$ and the middle probability is $4\rho - 4\rho^2$, giving contrast $2\rho^2 > 0$. A positive contrast therefore excludes the specified conditional Poisson class while permitting other independent birth laws.

If the three terminal probabilities are known within deterministic errors $\epsilon_3, \epsilon_2, \epsilon_1$, then

$$|\widehat{W} - W| \le \epsilon_3 + 2\epsilon_2 + \epsilon_1. \tag{41}$$

A certified positive lower bound for W excludes the frozen class under the specified protocol. Equation (41) is an error-propagation identity; constructing such probability bounds from observations requires a sampling design. The asymptotic signal decreases like $\rho^2 T^{-\gamma} L(T)$ at fixed small intensity.

8.4. Finite-stage histories and their quadratic response

Proof of Theorem 4.3. Fix the finite preparation partition and a dominating intensity $a_+ = \max(\rho, a_1, \dots, a_m)$. Use the short-mark cutoff in (29). At every preparation time, the baseline from the fully open start is coupled below the stationary a_+ cutoff field. At the switch, this field and the stationary target cutoff field are bracketed by the same all-open and stationary a_+ fields. Lemma 6.2 over the final T rows therefore gives the counterpart of (30), with terminal openness tending to u_ρ and the zero-root difference $o(A_\nu(T))$.

Insert a root at scaled origin ξ, scaled length s and preparation age z. Apply Lemma 5.5 during each remaining part of a constant-intensity preparation stage. At every stage boundary, Lemma 7.1 restarts the two rays at their actual endpoints in the actual baseline. Each restart reads only the past, so the next independent slab again satisfies Lemma 5.5. The number of restarts is fixed. With $V_a(z) = \int_0^z v_{a(w)}\,dw$, induction over the stages gives

$$L_T/T = \xi + V_a(z) + v_\rho + o_{\mathrm{Pr}}(1), \qquad R_T/T = \xi + s + z - V_a(z) + 1 - v_\rho + o_{\mathrm{Pr}}(1).$$

Thus the final width is $(s - \kappa_\rho - K_a(z))_+$. All segment speeds are positive. A positive final width with fixed margin implies positive widths throughout the earlier segments, and the functional front limits exclude an earlier crossing. A crossing gives permanent equality with the baseline by Lemma 7.1.

The compact-coordinate argument of Section 7.1 applies to finitely many stages. A subsequence fixes the birth stage; a stage whose remaining duration is $o(T)$ has displacement $o_{\mathrm{Pr}}(T)$ by the functional limit including time zero. The functions K_a, V_a are continuous at partition boundaries. Remove their finitely many age-boundary strips as well as the two spatial endpoint strips; their normalized candidate-root masses are $O(\epsilon)+o(1)$. On the complement, insertion benefits converge uniformly to u_ρ times the deterministic coverage indicator. Partition endpoint row conventions alter only finitely many age slices, whose long-root candidate mass is $O(A_\nu(T)/T)$ per slice. Short-mark baseline differences are paid for by the final target-period forgetting bound.

The old-root measure now has density $D_p^{-1}a(z)s^{-p}d\xi\,ds\,dz$. The zero/one-root expansion and target-era root cancellation of Section 7.2 apply with all intensities bounded by a_+. At fixed positive lower cutoff, both arms have the same target-root intensity and limiting insertion benefit. Their target-root contributions cancel before the cutoff is released. Integrating over origins and then lengths gives

$$\frac{u_\rho}{D_p}\int_0^H a(z)\int_0^\infty s^{-p}(s - \kappa_\rho - K_a(z))_+\,ds\,dz = \gamma u_\rho \int_0^H \frac{a(z)}{[\kappa_\rho + K_a(z)]^{\gamma+1}}\,dz.$$

Here $D_p^{-1}/[(p-1)(p-2)] = \gamma$. The lower cutoff contributes zero below κ_ρ; the upper-tail estimate (36) releases the upper cutoff. Subtracting the constant-profile reference proves the limit. Integrating on each constant stage proves the finite-sum expression. □

Proof of Theorem 4.4. For a fixed profile f among f_+, f_b, f_-, put $F_f(z) = \int_0^z f(s)ds$ and $d(z) = 1 + z$. The speed bound (20) gives, uniformly on $[0, H]$,

$$D_f(z) := \kappa_\rho + \int_0^z \kappa_{\rho f(s)}ds = d(z) - 2\chi\rho[1 + F_f(z)] + E_f(z), \qquad |E_f(z)| \le 4B_*^2\chi^2\rho^2 d(z).$$

First assume $\chi\rho B_* \le 1/32$. Then $D_f \ge (9/10)d$ and $|D_f - d| \le 3B_*\chi\rho d$. Taylor's theorem for $x^{-\gamma-1}$ at d yields

$$D_f^{-\gamma-1} = d^{-\gamma-1} + 2(\gamma+1)\chi\rho[1+F_f]d^{-\gamma-2} + r_f, \qquad |r_f| \le c_\gamma B_*^2\chi^2\rho^2 d^{-\gamma-1}.$$

The first contribution to c_γ bounds the E_f term; the second bounds the second derivative between d and D_f. Since $\gamma\int_0^H d^{-\gamma-1}dz = b_{\gamma,H}$,

$$\begin{aligned}\mathcal{J}_\rho[\rho f] &= \gamma u_\rho \rho \int_0^H f(z)d(z)^{-\gamma-1}dz \\ &\qquad + 2\gamma(\gamma+1)u_\rho\chi\rho^2\int_0^H f(z)[1+F_f(z)]d(z)^{-\gamma-2}dz + \mathcal{R}_f, \\ |\mathcal{R}_f| &\le u_\rho c_\gamma B_*^3\chi^2\rho^3 b_{\gamma,H}.\end{aligned}$$

The symmetric difference cancels the first term exactly. Because $F_{f_\pm} = F_{f_b} \pm P_h$, the second difference of $f(1+F_f)$ is $2hP_h$. Therefore

$$\left|\Xi_h^*(\rho) - 4\gamma(\gamma+1)u_\rho\chi\rho^2 Q_\gamma[h]\right| \le 4u_\rho c_\gamma B_*^3\chi^2\rho^3 b_{\gamma,H}.$$

The reference term in Theorem 4.3 cancels from the three-arm contrast. Taking $\rho \downarrow 0$, with $u_\rho \to 1$, proves the coefficient.

Integration by parts using $(P_h^2)' = 2hP_h$ gives the displayed positive-energy representation of Q_γ. If that expression vanishes, continuity forces P_h to vanish identically, hence $h = 0$ almost everywhere. Splitting the square $[0,H]^2$ into its two triangles gives the kernel expression. These are properties of the weak-intensity quadratic coefficient for each fixed history direction.

To verify the sufficient budget, use $|h| \le B_*$ and $|P_h(z)| \le B_* z$ to obtain

$$Q_\gamma[h] \le \frac{B_*^2\psi_\gamma(H)}{\gamma(\gamma+1)} \le \frac{B_*^2 b_{\gamma,H}}{\gamma(\gamma+1)}.$$

Since $c_\gamma > 16$, the stated budget implies $\chi\rho B_* < 1/32$, justifying the phase bound used above. It also makes the cubic error at most half of the positive quadratic term, proving the lower bound and eventual positivity at fixed intensity.

For the conditional Poisson comparison, the three conditional void probabilities are $e^{-\Lambda_E-\Delta_E}$, $e^{-\Lambda_E}$ and $e^{-\Lambda_E+\Delta_E}$. Their covered-probability contrast is $-e^{-\Lambda_E}(e^{\Delta_E}+e^{-\Delta_E}-2)$, which gives the stated identity after averaging. The nonnegative birth measures and finite covering means make all three probabilities well defined. The sign holds before averaging and allows Δ_E to change sign.

For the equal-dose example, $P_h(z) = z$ on $[0,1/2]$ and $P_h(z) = 1-z$ on $[1/2,1]$. At $\gamma = 1$,

$$Q_1[h] = \int_0^{1/2} \frac{z}{(1+z)^3}dz - \int_{1/2}^1 \frac{1-z}{(1+z)^3}dz = \frac{1}{18} - \frac{1}{36} = \frac{1}{36}.$$

The coefficient is consequently $4\chi \cdot 1 \cdot 2/36 = 2\chi/9$. Even observation times make both preparation halves integral, giving exactly equal cumulative intensities in the discrete protocol. □

8.5. Proof of the four-update witness

For a terminal observation after L rows, the disaster sites that can affect the answer on row i form

$$I_i = [i-L, 0] \cap \mathbb{Z}.$$

A length-r mark has $r + L - i$ origins whose interval intersects I_i. The total relevant mark measure is finite under $S_1 < \infty$. It can equivalently be partitioned into the finitely many possible nonempty interval intersections with I_i.

For $L = 4$, the preparation cones have widths four and three. Put

$$\tau_\ell = \sum_{r \ge \max(2,\ell)} \nu_r, \qquad A_k = \sum_{r \ge \max(2,k+1)} (r-k)\nu_r.$$

A mark in a row with cone width $k+1$ produces a terminal hole on its own precisely when it covers the entire cone, at unit-intensity rate A_k. A proper interval touching one cone boundary and having clipped length ℓ has rate τ_ℓ. In particular $\tau_1 = \tau_2 = S_0$.

Let $f(e)$ and $f(e,e')$ indicate terminal occupation by holes under one and two relevant marks. Poisson expansion at zero intensity gives the quadratic coefficient in a single arm as

$$\frac{1}{2}\iint c_i c_j [f(e,e') - f(e) - f(e')]\, d\lambda(e) d\lambda(e'),$$

where c_i is its row multiplier and λ is the unit-rate mark measure. Repeated mark types are included. The constant and linear terms cancel in Ξ_2. On the two preparation rows the signed multipliers are $s_1 = -1$, $s_2 = 1$ around the baseline multiplier two; on target rows they are zero. The quadratic contrast therefore retains just the two preparation-row marks, with weight $s_i s_j$.

The pair difference is -1 when both marks succeed individually, 1 when neither succeeds individually but together they succeed, and zero otherwise. Two individually unsuccessful intervals can succeed only by covering opposite cone boundaries and meeting or being adjacent after the earlier interval has eroded to the later row. For cone parameters $k \geq h$, this occurs exactly when their clipped lengths satisfy $\ell + m \geq k + 1$. Define

$$B(k,h) = \sum_{\ell=1}^{k} \sum_{m=1}^{h} \mathbf{1}_{\{\ell+m\geq k+1\}} \tau_\ell \tau_m.$$

There are two boundary orientations. Combining the two same-row terms and the cross-row term gives

$$[\rho^2]\Xi_2 = 2B(3,3) + 2B(2,2) - 4B(3,2) - (A_2 - A_3)^2.$$

To evaluate these finite sums, write $z = S_0$ and $t = S_0 - \nu_2$. Then

$$\begin{aligned} B(3,3) &= z^2 + 4zt + t^2, \qquad B(2,2) = 3z^2, \\ B(3,2) &= z^2 + 2zt, \qquad\quad A_2 - A_3 = t. \end{aligned}$$

Substitution gives $C_\nu = 4z^2 + t^2 > 0$.

For the remainder, the four cone widths are $4, 3, 2, 1$. A width-w cone receives unit-intensity mark mass $S_1 + (w-1)S_0$. The full relevant mark masses with the factor ρ removed are consequently

$$\mu_+ = 6S_1 + 10S_0, \qquad \mu_b = 6S_1 + 11S_0, \qquad \mu_- = 6S_1 + 12S_0 = M_\nu.$$

For any terminal indicator f, its third add-one difference is an alternating sum with four positive and four negative terms in $[0,1]$, so its absolute value is at most four. Differentiating the finite-mass Poisson expectation gives $|q_j'''(\rho)| \leq 4\mu_j^3$ at every $\rho \geq 0$. Taylor's theorem and the arm weights yield

$$|\Xi_2 - C_\nu \rho^2| \leq \frac{2}{3}(\mu_+^3 + 2\mu_b^3 + \mu_-^3)\rho^3 \leq \frac{8}{3} M_\nu^3 \rho^3.$$

This proves the general assertion.

For $\nu_r = r^{-4}$, the integral test gives

$$S_0 \leq 2^{-4} + \int_2^\infty x^{-4} dx = \frac{5}{48}, \qquad S_1 \leq 2^{-3} + \int_2^\infty x^{-3} dx = \frac{1}{4}.$$

Thus $M_\nu \leq 11/4 < 3$, and the cubic error is bounded by $72\rho^3$. Also $C_\nu > 4(1/16)^2 = 1/64$, since $\tau_3 > 0$. The claimed interval of intensities and rational certificate follow. □

8.6. Exact finite-cone evaluation

A finite observation admits an exact finite-state recursion even for an unbounded interval law. This gives probability enclosures for specified protocols and a constructive way to find finite witnesses in the regime of Theorem 4.4.

Proposition 8.1 (finite-cone reduction). Consider L updates from a fully open state, with deterministic row intensities $a_1, \ldots, a_L$ and $S_1 < \infty$. On row i, let I_i be the cone above, of width $w = L - i + 1$. Partition all marks according to their exact nonempty intersection $J \subseteq I_i$. The counts for distinct J and rows are independent Poisson variables, with means $a_i \lambda_w(J)$, where, with $\nu_1 = 0$,

$$\lambda_w(J) = \begin{cases} \nu_{|J|}, & J \text{ touches neither boundary of } I_i, \\ \displaystyle\sum_{r \geq \max(2,|J|)} \nu_r, & J \text{ touches exactly one boundary}, \\ \displaystyle\sum_{r \geq \max(2,w)} (r - w + 1)\nu_r, & J = I_i. \end{cases}$$

These rates are determined by S_0, S_1 and $\nu_2, \ldots, \nu_{L-2}$, with the finite list empty for $L \leq 3$.

Proof. An interior intersection fixes the entire original interval. A proper intersection at one boundary fixes the opposite endpoint and admits one origin for each length at least $|J|$. A length-r interval covers the

whole cone in $r - w + 1$ ways. The three cases partition the original Poisson measure into disjoint sets. The side rate is $S_0 - \sum_{2 \le r < |J|} \nu_r$, and the whole-cone rate is

$$S_1 - (w-1)S_0 + \sum_{2 \le r < w} (w - 1 - r)\nu_r.$$

In the last sum the coefficient at $r = w - 1$ vanishes. Proper side lengths are at most $w - 1$ and interior lengths at most $w - 2$, proving the finite-prefix assertion. Each count matters only through whether it is nonzero, with probability $1 - e^{-a_i \lambda_w(J)}$.

Represent a hole state on the cone by a bit mask b, with bit indices increasing from left to right. On passing to the next, narrower cone, erosion maps it to $b \,\&\, (b \gg 1)$. For a type J with mask m, split each state mass between b and $b \,\mathrm{OR}\, m$ with probabilities $e^{-a_i \lambda_w(J)}$ and $1 - e^{-a_i \lambda_w(J)}$. Apply all independent types and proceed to the next row. Starting at mass one on the empty mask, the final mass on mask one is exactly the terminal hole probability. □

For effective evaluation, assume certified real-number evaluators for ν_r and the row intensities, an effectively specified discrete schedule, and a known bound $\chi \le \bar{\chi} < \infty$. Then

$$\sum_{r > K} \nu_r \le \frac{2\bar{\chi}}{(K+1)(K+2)}, \qquad \sum_{r > K} r\nu_r \le \frac{2\bar{\chi}}{K+2}.$$

The rates are therefore computable to arbitrary certified precision. Nonnegative finite-state transitions with outward-rounded arithmetic give convergent probability enclosures. An explicit error control follows by coupling componentwise upper and lower type means: their terminal probabilities differ by at most one minus the exponential of the negative total difference of their means. This controls rate uncertainty separately from rounding error.

For a protocol satisfying Theorem 4.4, enumerate admissible times $T_1, T_2, \ldots$; even times suffice for its equal-dose example. At stage n, enclose every arm at each of $T_1, \ldots, T_n$ to width at most 2^{-n-2}. Return any contrast interval with positive lower endpoint. Each stage is finite. Eventual positivity ensures that some fixed candidate is strictly positive, and subsequent precision refinement resolves its sign. This procedure terminates with one certified finite observation time. It computes a witness. Theorem 4.6 and its supplement supply an all-future threshold under effective power-law tail data and a stronger explicit intensity budget.

For the four-row law and intensity of Theorem 4.5, outward-rounded evaluation gives the enclosure

$$2.7498046792 \times 10^{-10} \;\le\; \Xi_2(10^{-4}) \;\le\; 2.7498286804 \times 10^{-10}.$$

The computation uses 60 decimal digits. Each infinite sum $S_0 = \sum r^{-4}$ and $S_1 = \sum r^{-3}$ is enclosed using terms through length 10^4 and

$$\frac{(N+1)^{1-s}}{s-1} \le \sum_{r > N} r^{-s} \le \frac{N^{1-s}}{s-1} \qquad (s > 1).$$

The resulting enclosure agrees with the independent rational lower bound in Theorem 4.5. Its role is direct evaluation of the finite-time observable; the long-time amplitude follows from Sections 5–8.4.

9. Discussion

The main history-response theorem identifies how the preparation environment changes the lifetime of a large hole. A root contributes through its birth intensity and the actual recovery speeds accumulated during all later preparation stages and the target period. The selected target-phase open density supplies the remaining gate factor. Theorem 4.1 is the explicit constant-history evaluation of this finite-stage functional; both results hold at fixed small positive intensities before any weak-intensity expansion.

The positive-energy consequence resolves the temporal arrangement of forcing. Equal integrated intensity removes the endpoint term of the quadratic form while retaining a strictly positive cumulative-history energy. The four-update theorem supplies an independent local cooperation witness under a finite first moment, and applies to bounded as well as heavy-tailed interval laws.

The frozen coverage reference keeps recovery kernels calibrated across arms. The interacting process adds feedback from the preparation environment to erosion speed. Under fixed conditional Poisson births, kernels,

environment distribution and affine intensity calibration, the resulting contrasts have opposite signs. Section 8.3 records the Bernoulli and nonlinear-calibration examples that delimit this comparison.

Theorem 4.6 is the effective extension of the long-time sign result. Its full quantitative proof controls the ancestor kernel, cooperative long-root remainder, terminal open gate and short-background history at the quadratic signal scale. The supplied tail modulus makes the threshold computable; Proposition 4.7 explains why summary moment and exponent data alone cannot determine it. The conservative certificates and all their scalar conditions are retained in the technical supplement.

The results treat a fixed finite preparation partition and take the time limit before the weak-intensity limit. Extending the history functional to protocols whose partitions grow with time, or obtaining matching intensity-dependent onset bounds, requires additional uniform estimates. The complete-path and gate-variance lemmas identify concrete geometric and probabilistic inputs for such extensions.

Statements and Declarations

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Data and code availability. This study uses no empirical datasets. Section 8.6 specifies the finite-state probability recursion, inputs and rigorous tail bounds for the reported finite-time enclosure. Online Resource 2 (the reproducibility archive, including its index) supplies the finite-cone implementation, complete effective-threshold certificates and independent fraction-arithmetic audit, with their exact inputs and outputs. Online Resource 1 gives the quantitative probability proof and scalar conditions.

Competing interests. The author is employed by a public fund management institution in China. This research was conducted independently and does not represent the views of the author's employer, any fund, or any client. The study did not use non-public employer or client information and does not constitute investment advice.

Generative AI and AI-assisted technologies; verification status. AI assisted theoretical development, literature synthesis, proof drafting and exposition. The text was primarily generated by AI and then adjusted by the author. The author reports independently understanding some arguments, but not having mastered the full argument. The research records identify OpenAI GPT-6.1 Sol and xAI Grok 4.7; AI assistance also included deterministic computations. This repository preparation and its accompanying provenance statement were also prepared with AI assistance. Available internal records contain AI-assisted scoped proof checks, finite diagnostics and certificate replays. These do not verify the full proof. Completed author-performed line-by-line verification or independent human expert verification of the full manuscript is not asserted. General-γ quantitative constants have not all been independently reconstructed, and the Harris (1960) original-source access gap remains. The author takes responsibility for the work. No AI system is a contributor or author.

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Appendix A. Conditional correlation, the positive phase and spatial tails

This appendix uses a fixed intensity λ and writes $c = \lambda\chi$, $u = u_\lambda$, $\beta = 1 - u$. Write $A_* = \chi$ and $A_k = A_\nu(k)$. Define $h_{j,t}$ as the probability of a hole block of length j after t rows from all-open sites, and $h_j = \lim_t h_{j,t}$. Set $V(r) = r(r+1)/2$, so $\sum_r \nu_r V(r) = A_* = \chi$. Lemma A.1 and Corollary A.2 allow independent interval types with nonidentical absence probabilities, as specified there. The phase, stationary flux and tail calculations in Sections A.2–A.4 use spatially homogeneous rows with the stated intensity $\lambda\nu_r$, and a homogeneous last-row interpolation when needed. For nonstationary fields dominated by this model, the corresponding upper bounds are transferred by monotone coupling.

A.1. Conditional negative association

Lemma A.1. At every finite time, conditional on the origin being open, bounded increasing functions depending on finitely many hole coordinates strictly to its left and strictly to its right are negatively correlated. In a finite model, the statement allows arbitrary independent type-dependent probabilities. In the infinite model used here, the type means in a finite history are bounded by $\lambda\nu_r$, including a final row of intensity $\theta\nu_r$ with $0 \le \theta \le \lambda$.

Proof. First restrict to finitely many interval types in a finite slab containing the relevant backward cones. Set $Y_e = 1$ when disaster type e is absent. The variables are independent with parameters strictly between zero and one. The bottom row is fully occupied. Let Γ be the event that a path reaches terminal site 0, and let μ be the product law conditioned on Γ.

The pointwise minimum and maximum of two allowed paths are still allowed: their increments remain in $\{0, 1\}$. For open paths these operations choose an already open site in each row. Every configuration in Γ therefore has a leftmost path L and a rightmost path R to 0. For a specified path π, a connected disaster interval in its row is strictly left of π, intersects π, or is strictly right of π. Denote these disjoint type sets by $\mathcal{L}_\pi, \mathcal{C}_\pi, \mathcal{R}_\pi$.

Define a right update by resampling the variables in $\mathcal{R}_L$ from their original independent laws. This leaves L unchanged. Indeed, if a new path p appeared partly to the left of L, then $\min(p, L)$ would have been open before the update, because strictly right types cannot affect it. This contradicts leftmostness. Moreover, the event $L = \pi$ depends only on $\mathcal{L}_\pi \cup \mathcal{C}_\pi$: existence of a path further left can be tested among paths everywhere at or left of π. Conditional on the remaining variables and L, the right variables retain their product distribution. The right update thus preserves μ. Resampling $\mathcal{L}_R$ analogously preserves R and μ.

Alternating these kernels gives a finite irreducible, aperiodic chain. To verify irreducibility, set all resampled variables to 1 in a complete right-then-left step, an event of positive probability. After the right step the vertical path at 0 is open: any interval hitting it either hits the old leftmost path and was already absent, or is strictly to its right and has just been removed. The vertical path is the largest allowed path to 0. The left step then removes every remaining disaster strictly to its left; all other types are already absent. Thus every state can reach the all-1 state. Since μ is invariant and has full support on Γ, there can be no transient states or another closed class. Self-loops have positive probability.

Define a comparison between two configurations in Γ by $\widehat{\omega} \preceq \omega$ if (i) $\widehat{L} \le L$ and $\widehat{R} \le R$ rowwise, (ii) $\widehat{Y}_e \ge Y_e$ for types strictly left of $\widehat{L}$, and (iii) $\widehat{Y}_e \le Y_e$ for types strictly right of R. No ordering is imposed on the remaining variables. This is the interval-type version of the comparison in [9, Section 3.1]. For a fixed absence vector y, write the updates explicitly as

$$(T_R^y Y)_e = \begin{cases} y_e, & e \in \mathcal{R}_{L(Y)}, \\ Y_e, & e \notin \mathcal{R}_{L(Y)}, \end{cases}$$

$$(T_L^y Y)_e = \begin{cases} y_e, & e \in \mathcal{L}_{R(Y)}, \\ Y_e, & e \notin \mathcal{L}_{R(Y)}. \end{cases}$$

The same vector y is used for the two configurations; each configuration selects its own resampling set.

A common right update preserves the comparison. Write $\alpha = \widehat{L} \le L = \gamma$ and let p be open after the update in $\widehat{\omega}$. We check $q = \max(p, \gamma)$ row by row in the updated ω. At a site of γ, every covering type is absent because γ remains open. At a site $p_t > \gamma_t$, a covering interval either intersects γ_t, in which case it is absent in ω, or has its entire interval strictly right of γ_t. In the latter case it is selected in both configurations and receives the same entry y_e; openness of p forces that entry to be 1. Connectedness of the interval exhausts these cases. Thus q is open, proving the order of the new rightmost paths. The leftmost paths are unchanged. Types strictly left of α are unchanged in both configurations. A type strictly right of the new rightmost path in ω is also strictly right of γ and α, so it was selected in both configurations and now has equal values. These observations verify all three comparison conditions after the right update.

For a common left update, set $\alpha = \widehat{R} \le R = \gamma$. Given a path p open after the update in ω, check $q = \min(p, \alpha)$ in the updated $\widehat{\omega}$. At a site of α, all covering types are absent. At a site $p_t < \alpha_t$, a covering interval either meets α_t and is absent in $\widehat{\omega}$, or is strictly left of both α_t and γ_t and receives the same entry y_e. This proves that q is open and orders the new leftmost paths. The rightmost paths are unchanged. Every type strictly left of the new $\widehat{L}$ is strictly left of both old rightmost paths, so it was identically resampled. Types strictly right of γ are unchanged in both configurations. Hence this update also preserves all three conditions, including when the two selected type sets differ.

For a single right update from identical input configurations, decreasing the resampled absence variables adds only right-side disasters. The old leftmost path stays open, and the new rightmost path can only move left; all types beyond the larger rightmost path have the required absence order, while types left of the common leftmost path are unchanged. This establishes the comparison after that update. For a single left update, increasing the resampled absences adds only left-side open paths; the rightmost path stays fixed and the leftmost path can only move left, with the required order on the two outer type sets. The common-update preservation just proved propagates either comparison through subsequent updates. Finally, if $\widehat{\omega} \preceq \omega$, every path to a terminal $x < 0$ open in ω yields an open path in $\widehat{\omega}$ by taking its minimum with $\widehat{L}$. An interval covering a strictly left point either intersects $\widehat{L}$ or belongs to its strictly left type set, where condition (ii) applies. Terminal reachability on the left therefore increases, while reachability on the right decreases by the corresponding maximum construction and condition (iii).

Realize the resampling chain on a product space by sampling arrays $Y_{k,e}^R$ and $Y_{k,e}^L$ in advance for every update index k and every retained interval type e. Use an entry only when its type belongs to the currently selected resampling set; otherwise ignore it. All these entries are independent with the original type-specific absence probabilities. If one entry is changed, the two chains have identical histories before its update. At that update either the entry is ignored in both chains or the single-update comparison established above applies. Every subsequent common update preserves that comparison, even when the two resampling sets differ. Thus, after any fixed number of updates, every left increasing hole functional is decreasing in the left-update entries and increasing in the right-update entries. Right increasing functionals have the opposite signs. Complementing the right-update entries places the first functional and the negative of the second in the same coordinatewise decreasing class. The product-measure association inequality gives their nonnegative covariance, equivalently the desired negative correlation. A direct proof for the type-dependent parameters used here is given below. Start at the all-1 configuration and let the finite conditional chain converge to μ to obtain the asserted conditional inequality.

For the infinite model, first fix the time and the finite terminal coordinates of the functions. Their backward cones are finite. Remove types of length greater than K; the total mean of omitted types intersecting the cones is at most a constant times $\lambda T \sum_{r>K} \nu_r(r + C)$, which tends to zero by $S_1 < \infty$. The conditional event has probability at least that of an open vertical path, $e^{-\lambda T S_1} > 0$. Thus the conditional inequality passes to the limit. Deterministic absence variables can be removed or obtained by parameter limits. The same argument applies to the stated type-dependent parameters and last-row interpolation. □

The product-measure correlation bound has its origins in Harris' percolation argument [10]. In the finite independent-coordinate form used above, it follows directly by induction over the independent Bernoulli coordinates. For bounded increasing F, G, condition on the last coordinate Y: the total covariance is

$\mathbb{E}\,\mathrm{Cov}(F, G \mid Y) + \mathrm{Cov}(\mathbb{E}[F \mid Y], \mathbb{E}[G \mid Y])$. The first term is nonnegative by induction; the second is $p(1-p)\Delta F \Delta G \geq 0$, where $p = \Pr(Y = 1)$ and $\Delta F = \mathbb{E}[F \mid Y = 1] - \mathbb{E}[F \mid Y = 0]$, with ΔG defined similarly. Negating one function gives the opposite-monotonicity form. The argument permits different parameters at different coordinates.

The proof adapts the extremal-path resampling strategy of [9] to connected interval errors. For a disconnected error closing $\{-1, 1\}$ with probability $1/2$ while leaving 0 open, the two side-hole indicators have joint probability $1/2$ and product of marginals $1/4$. The conclusion is therefore false for arbitrary spatially correlated row errors.

Corollary A.2 (conditional one-sided association). Under the hypotheses of Lemma A.1, bounded increasing functions of finitely many hole coordinates strictly to the same side of the origin are positively correlated, conditional on the origin being open.

Proof. In the finite-type, finite-slab construction above, realize each alternating left/right resampling update using fresh independent absence variables, sampled in advance for every interval type. The comparison argument in the proof of Lemma A.1 shows that every left-increasing hole functional has the same monotonicity in each resampling variable: it is decreasing in left-update absence variables and increasing in right-update absence variables. Complementing the former variables makes every such functional increasing in one common product vector. Harris' inequality therefore gives nonnegative covariance for any two left-increasing hole functionals. For two right-increasing functionals the monotonicity signs are reversed, and complementing the right-update variables gives the same conclusion. Letting the number of updates tend to infinity gives the result under the conditional law μ. The finite-type truncation and backward-cone approximation used at the end of Lemma A.1 pass the inequality to the infinite interval model; finite-history convergence then gives the stationary version whenever the limiting open density is positive. □

A.2. Finite-history phase construction

Lemma A.3 (full insertion count). In a translation-invariant hole field F, insert $[0, r-1]$. Let L_z be the closed run immediately to the left of z, and R_r the closed run beginning at r. The number of newly closed, nonempty finite intervals is

$$\Delta_r(F) = \sum_{z=0}^{r-1} \mathbf{1}_{\{z \notin F\}} (L_z + 1)(r - z + R_r). \tag{A1}$$

Every newly closed interval has a unique first previously open point z. Its left endpoint has $L_z + 1$ choices and its right endpoint has $r - z + R_r$ choices.

Assume $u_F = \Pr(0 \notin F) > 0$ and the conditional inequality of Lemma A.1. If $h_k(F)$ is the k-interval closure probability and $\zeta = \lim_k h_k(F)$, boundary tail sums give

$$\mathbb{E}[\mathbf{1}_{\{z \notin F\}}(L_z + 1)] = u_F + \beta_F - \zeta \leq 1.$$

For $d = r - z$, split $\{z \notin F, [r, r + b - 1] \subset F\}$ by the last open site in $[z, r-1]$, discard other restrictions, and sum over $b \geq 1$. Each of the d possible boundary sites contributes at most $\beta_F - \zeta$, so

$$\mathbb{E}[\mathbf{1}_{\{z \notin F\}} R_r] \leq d\beta_F, \qquad \mathbb{E}[\mathbf{1}_{\{z \notin F\}}(d + R_r)] \leq d.$$

Conditional negative association, followed by bounded truncation and monotone convergence, now yields

$$\mathbb{E}\Delta_r(F) \leq V(r)/u_F. \tag{A2}$$

Lemma A.4 (finite-history bound). If $c \leq 1/32$, then

$$\sup_t \beta_t \leq \frac{1 - \sqrt{1 - 8c}}{2} < \frac{1}{2}. \tag{A3}$$

Proof. Use the last-row interpolation field

$$F_{t,\theta} = E(H_t) \cup D^{(\theta)}, \qquad 0 \leq \theta \leq \lambda,$$

where the first t rows have intensities $\lambda\nu_r$ and the last row has intensities $\theta\nu_r$. It is exactly a finite-history process of Lemma A.1 with non-identical type parameters; its open density is at least $u_{t+1} > 0$.

Work first on the infinite integer line with only lengths $r \leq K$ retained. This model remains translation invariant. A fixed finite observation and its derivative involve finitely many retained types in a backward cone

of this spatially translation-invariant model. Define $\phi_M(F,x) = \sum_{j=1}^{M} \mathbf{1}_{\{[x,x+j-1]\subset F\}}$. Poisson differentiation of the last row, followed by translation mass counting on this infinite-line law, gives

$$\frac{d}{d\theta}\mathbb{E}\phi_M(F_{t,\theta},0) = \sum_r \nu_r\, \mathbb{E}\Delta_{r,\le M}(F_{t,\theta}) \le A_*/u_{t+1}. \tag{A4}$$

Here $\Delta_{r,\le M}$ counts new intervals of length at most M. The same upper bound holds for all K since removal of long disasters only increases openness. Integrate over θ and let $K \to \infty$ by bounded local convergence. Then let $M \to \infty$ in the nonnegative insertion increments. Starting from zero, induction proves that $\Phi_t := \sum_{j\ge 1} h_{j,t} < \infty$ for each finite t: erosion does not increase it, and each row adds at most c/u_{t+1}. Since erosion replaces $h_{j,t}$ by $h_{j+1,t}$,

$$\Phi_{t+1} - \Phi_t \le c/u_{t+1} - \beta_t. \tag{A5}$$

Adding past rows makes the hole marginals larger, so β_t is nondecreasing. A single new row gives $\beta_{t+1} \le \beta_t + q$ with $q = 1 - e^{-\lambda S_1} \le c$. Independent time-block concatenation gives $u_{a+b} \ge u_a u_b$: choose, by a fixed rule using only the last a rows, an open path to 0 and its initial site z when one exists. The first b rows reach that site with probability u_b, independently. The shared boundary row is checked by the first block only. Thus $\beta_{a+b} \le \beta_a + \beta_b$.

Summing this inequality over complementary time indices, and summing (A5) while retaining $\Phi_T \ge 0$, gives

$$\frac{T-1}{2}\beta_T \le \sum_{t<T} \beta_t \le Tc/u_T, \qquad \beta_T(1-\beta_T) \le \frac{2T}{T-1}c \le 4c \quad (T \ge 2). \tag{A6}$$

If T is the first crossing of $1/2$, then $1/2 \le \beta_T \le 1/2 + c$. Its product with $1-\beta_T$ is at least $1/4 - c^2 \ge 255/1024 > 1/8 \ge 4c$, contradicting (A6). Hence the crossing never occurs. Taking $T \to \infty$ in the sharper version of (A6) gives $\beta(1-\beta) \le 2c$ on the small branch, proving (A3). The constant $1/32$ supplies a uniform positive-density regime. □

A.3. Stationary flux

The stationary field is a translation-equivariant measurable function of the independent columns of marked Poisson variables, hence is spatially ergodic. Equation (A3) implies $u > 0$, so it has no all-closed component and $h_j \to 0$. A stationary field cannot have a closed half-line with a finite endpoint at positive probability: the events specifying its unique endpoint are disjoint translates with equal probabilities. Thus closed runs are finite almost surely. The interpolation fields $F_\theta = E(H^*) \cup D^{(\theta)}$ are coupled below F_λ, which has law H^*; they inherit this property and have open densities $u_\theta \ge u$.

Lemma A.5 (local stationary interpolation). Let $\Delta_{r,j}(F)$ count new closed intervals of length j after inserting $[0, r-1]$. Only the $r+j-1$ windows intersecting the insertion can change, so

$$0 \le \Delta_{r,j}(F) \le r + j - 1.$$

For fixed j, this bound is integrable under ν by $S_1 < \infty$. Approximate H^* by finite histories and truncated lengths. Each local insertion difference converges and is bounded by the same integrable envelope. Dominated convergence passes the last-row interpolation to stationarity, giving

$$q_j := h_j - h_{j+1} = \int_0^\lambda \sum_r \nu_r \mathbb{E}\Delta_{r,j}(F_\theta)\, d\theta. \tag{A7}$$

The conditional inequality also passes to these fields from Lemma A.1, since their open densities are bounded below by $u > 0$.

In a stationary field with $h_j(F) \to 0$, the boundary tail identity sharpens to $\mathbb{E}[\mathbf{1}_{\{z\notin F\}} L_z] = \beta_F$. Substituting it in (A1) yields

$$\mathbb{E}\Delta_r(F) = V(r) + \sum_{z=0}^{r-1} \mathbb{E}[\mathbf{1}_{\{z\notin F\}}(L_z+1)R_r] \ge V(r).$$

For the interpolation fields under consideration, $h_j(F) \to 0$. Combining the identity with (A2) and summing (A7) by Tonelli gives

$$\beta = \int_0^\lambda \sum_r \nu_r \mathbb{E}\Delta_r(F_\theta)\, d\theta, \qquad c \le \beta \le c/u. \tag{A8}$$

The inequality $\beta < 1/2$ selects the small root of $\beta(1-\beta) \le c$, giving the density bound (16). Also $\beta \le 2c$ and $\beta - c \le \beta^2 \le 4c^2$. The phase bound (A3) determines the branch used in the stationary flux inequality.

A.4. Branching comparison and the spatial tail bound

Put $q_0 = u$ and $q_j = h_j - h_{j+1}$ for $j \ge 1$. This is a probability mass function on the nonnegative integers; let Q have this law, so $\Pr(Q \ge k) = h_k$. For an interpolation field F, define $q_a^F = h_a^F - h_{a+1}^F$. This is also the probability that a site is open and the closed run immediately to its left has length at least a.

Assign each new interval of length j in (A1) to its first old open point z. Let its left and right endpoint distances from z be a, b with $a + b = j - 1$, and put $d = r - z$. If $b < d$, its right requirement is automatic. If $b \ge d$, partition by the last old open site in $[z, r-1]$ and discard additional restrictions. Both cases give

$$\Pr(z \notin F,\ d + R_r \ge b + 1) \le \sum_{i=0}^{\min(d-1,b)} q_{b-i}^F.$$

Conditional negative association separates the left run event from this right requirement. Summing over endpoints and z gives

$$\mathbb{E}\Delta_{r,j}(F) \le \frac{1}{u_F} \sum_{i=0}^{r-1} (r-i)(q^F * q^F)_{j-1-i}, \tag{A9}$$

with negative subscripts zero. In the coupling $F_\theta \subset F_\lambda$, Q_θ is stochastically bounded by Q. Applying this stochastic domination to tails, sum (A7) over $j \ge k$ and use (A9) to obtain

$$h_k \le \frac{\lambda}{u} \sum_r \nu_r \sum_{i=0}^{r-1} (r-i) \Pr(Q_1 + Q_2 + i \ge k - 1). \tag{A10}$$

Here Q_1, Q_2 are independent auxiliary copies of Q. Define I by

$$\Pr(I = i) = \frac{\sum_r \nu_r (r-i)_+}{A_*}, \qquad \Pr(I \ge m) = A_{m+1}/A_*.$$

With $a = c/u$, (A10) says $Q \le_{\mathrm{st}} B(1 + I + Q_1 + Q_2)$, all variables on the right independent and B Bernoulli with parameter a. Iterate the distributional upper map on a binary tree. The chance of any active leaf at depth n is at most $(2a)^n$. On its complement the depth-n comparison no longer depends on the leaf copies of Q. Since $2a < 1$, this proves

$$Q \le_{\mathrm{st}} X = \sum_{i=1}^{J} W_i, \qquad W = 1 + I,$$

where J is the number of successful nodes, success having probability a and producing two children. The costs are independent of the tree, and

$$\Pr(J = j) = \mathrm{Catalan}(j) a^j (1-a)^{j+1} \le (4a)^j, \qquad \mathbb{E}J = \frac{a}{1-2a}. \tag{A11}$$

For subexponential W, the fixed-j sum tail is asymptotic to $j \Pr(W \ge k)$. Kesten's bound, Theorem 3.34 of [11], bounds this ratio by $C_\epsilon(1+\epsilon)^j$ and applies also when W has infinite mean. Since $a \le 2c \le 1/16$, choose $\epsilon > 0$ with $4a(1+\epsilon) < 1$. Dominated convergence over J gives

$$\Pr(X \ge k) \sim \mathbb{E}J\, A_k/A_*.$$

It follows that $h_k \le CA_k$ for the regularly varying law in Section 2. The constant is finite for each fixed law and intensity. For every length truncation, monotone coupling gives $h_k^{(m)} \le h_k$, so the same upper envelope applies uniformly in m. Independently eroded old intervals are contained in the actual hole field and give $\beta \ge 1 - e^{-c}$; the stronger bound $\beta \ge c$ follows from the stationary flux calculation above.

A.5. Constrained paths and boundary observables

Lemma A.6 (deterministic checkpoint and half-line edge comparison). Fix a finite time slab with rows $0, \dots, n$, allowed forward increments $0, 1$, and initially open sites at row 0. At every later row each retained disaster type is a connected integer interval; its absence indicator is independent of all other retained types and has its own deterministic parameter. First take finitely many types and a finite spatial slab large enough to contain every backward cone under consideration. Fix a deterministic checkpoint row $r \in \{0, \dots, n\}$ and integer c. A constrained path must have coordinate at most c at row r. Write H^0 and H^c for the terminal hole sets with unrestricted and constrained paths, respectively, coupled using the same types. Thus $r = 0$ is the source restriction $(-\infty, c]$.

For a deterministic endpoint z, let

$$A_z = \{z \notin H^0\}, \qquad O_z = \{z \notin H^c\}.$$

Whenever $\Pr(O_z) > 0$, the following assertions hold.

(i) On O_z, the two terminal fields agree strictly to the left of z.

(ii) Given O_z, bounded increasing functions of constrained hole coordinates strictly left and strictly right of z are negatively correlated.

(iii) For any bounded increasing function f of unrestricted terminal hole coordinates strictly left of z,

$$\mathbb{E}[f(H^0) \mid O_z] \le \mathbb{E}[f(H^0) \mid A_z]. \tag{A6.1}$$

Consequently, if $Z = \sup\{y : y \notin H^c\}$ is the constrained rightmost open site, then, for every z with $\Pr(Z = z) > 0$,

$$\mathbb{E}[f(H^0) \mid Z = z] \le \mathbb{E}[f(H^0) \mid A_z]. \tag{A6.2}$$

Here the edge event uses only the finite interval $z + 1, \dots, c + n - r$, since no constrained path can end to its right.

(iv) Let X be any independent spatially stationary random open set. Define the inward union-hole run

$$R_z(H, X) = \inf\{j \ge 0 : z - j \notin H,\ z - j \in X\},$$

with value infinity if the set is empty. If H^0 is spatially stationary and $p = \Pr(A_z) > 0$, then

$$\mathbb{E}\big[R_z(H^0, X) \mid Z = z\big] \le \frac{\mathbb{E}R_0(H^0, X)}{p}. \tag{A6.3}$$

On the left of (A6.3), H^0 can be replaced by H^c in the interior, and the coordinate z is open. The reflected statements hold for a lower-half-line constraint and its leftmost edge, with inward runs directed right. Independent terminal fields need not themselves satisfy a conditional association property.

For the infinite interval model, the conclusions hold at each finite time provided the type means are bounded by $\bar{a}\nu_\ell$, $\sum_\ell \ell\nu_\ell < \infty$, and every conditioning event used has positive probability. Nonnegative observables are obtained by monotone truncation.

Proof. We first prove the finite-type assertion with parameters strictly between zero and one. All required paths are taken in a finite slab containing their backward cones, so its outer walls do not remove any of them. Minima and maxima of two allowed paths remain allowed, and the checkpoint restriction is preserved by both operations. If a full-source path ends at $y < z$ and a constrained path ends at z, their pointwise minimum is a constrained open path ending at y. The opposite inclusion is automatic. This proves (i).

For (ii), condition the independent type law on the existence of a constrained path to z. Let L, R be its leftmost and rightmost paths. Resample types strictly right of L from their independent original laws, then types strictly left of the resulting R. The right update leaves L fixed: a new path partly left of L would give, by minimum with L, an old constrained path contradicting leftmostness. Moreover, the event specifying L reads only types intersecting or left of it. Thus the right variables retain their product law conditional on that event; the update preserves the conditional measure. The symmetric assertion holds for the left update.

The greatest all-absent constrained path to z is

$$\pi_s = \min\{z, c + (s - r)_+\}, \qquad 0 \le s \le n. \tag{A6.4}$$

Positive probability of O_z ensures $z \le c + n - r$, so $\pi_n = z$; the increments are allowed and $\pi_r \le c$. Every constrained path is below π: before the checkpoint its coordinate is at most both its terminal coordinate

and c; after the checkpoint it is at most both z and $c+s-r$. Force all selected absence variables to one in a right-then-left update. After the right update, π is open. Indeed, an interval hitting π_s either also hits L_s, in which case it was already absent, or is strictly right of L_s, in which case it was just cleared. The new rightmost constrained path is therefore π. The left update clears all types strictly left of π. Types intersecting π are absent; types strictly right of π were cleared in the first update. Thus the all-absent configuration is reachable with positive probability. Invariance with full support, and this reachability from every state, imply irreducibility; self-loops give aperiodicity.

For completeness, the common-update comparison used here retains the exact three conditions of A.1: $\widehat{L} \le L$, $\widehat{R} \le R$, absence variables ordered upward strictly left of $\widehat{L}$, and downward strictly right of R. The checkpoint adds no type to be resampled; it only restricts the class of paths. For a common right update, map an open path p in the hatted configuration to $\max(p, L)$ in the other configuration. At a point lying on L, covering types are absent. At a point strictly right of L, a covering interval either hits L, hence is absent, or is strictly right of both leftmost paths, hence receives the same resampling entry. This proves the order of rightmost paths. Leftmost paths are fixed. Outer types left of $\widehat{L}$ are unchanged, and those beyond the new R were resampled equally. For a common left update the same reasoning uses $\min(p, \widehat{R})$; rightmost paths are fixed and leftmost paths retain their order. All these minima and maxima satisfy the checkpoint. These checks prove preservation of the entire comparison, including the outer type orders.

From identical inputs, increasing a left-update absence entry puts the changed state on the left of this comparison; decreasing a right-update entry has the same effect. Subsequent common updates preserve it. Minimum with $\widehat{L}$ transfers terminal reachability on the left to the hatted state, while maximum with R transfers reachability on the right in the reverse direction. Consequently, a left increasing hole observable decreases in left-update absence entries and increases in right-update entries; a right increasing hole observable has the opposite signs. Realize all updates by independent pre-sampled arrays and start from the all-absent state. Complementing the right-update entries makes a left observable and the negative of a right observable both decreasing functions of one product vector. Harris' inequality proves their negative correlation. Convergence of the finite chain proves (ii).

To prove (iii), run the *unrestricted* conditional chain given A_z, and let P be its leftmost open path to z. Set

$$C = \{P_r \le c\}.$$

Existence of any constrained path is equivalent to C on A_z, so $O_z = A_z \cap C$. The common-update comparison orders P at every row, including row r. Hence $\mathbf{1}_C$ increases in left-update absence entries and decreases in right-update entries. Its signs are opposite those of f. Harris' inequality on the same product arrays, followed by chain convergence, gives

$$\mathbb{E}[f\mathbf{1}_C \mid A_z] \le \mathbb{E}[f \mid A_z] \Pr(C \mid A_z).$$

Division by $\Pr(C \mid A_z) > 0$ proves (A6.1). By (i), on O_z a left unrestricted observable equals its constrained version. Apply (ii) to this observable and the right increasing hole event $\{[z+1, c+n-r] \subset H^c\}$. Combining with (A6.1) proves (A6.2).

For (iv), force the coordinate z of H open, denoting this clamped field by $H^{[z]} = H \setminus \{z\}$. For an integer M, put

$$f_{z,M}(H) = \mathbb{E}_X[\min\{R_z(H^{[z]}, X), M\}].$$

This is a bounded increasing function only of the hole coordinates $z-1, \dots, z-M+1$; it reads no hole coordinate at z or outside the strict interior. Independence of X is used before conditioning, and spatial stationarity makes the translated function the same at every deterministic z. On both A_z and $\{Z = z\}$, clamping has no effect. Therefore (A6.2) gives

$$\begin{aligned}\mathbb{E}[R_z(H^0, X) \wedge M \mid Z = z] &\le \mathbb{E}[f_{z,M}(H^0) \mid A_z] \\ &= p^{-1}\mathbb{E}[\mathbf{1}_{A_z}(R_z(H^0, X) \wedge M)] \\ &\le p^{-1}\mathbb{E}R_0(H^0, X).\end{aligned}$$

Monotone convergence proves (A6.3). Reflection reverses both path direction and the inward side, preserving connected intervals and independence.

Finally, for fixed terminal coordinates their backward cones contain finitely many sites. Removing interval lengths greater than K changes their law with probability at most the total mean of omitted types meeting those cones, bounded by a finite constant times $\bar{a}\sum_{\ell>K}(\ell + C)\nu_\ell$, which tends to zero. A fixed feasible

allowed path has positive probability of being open, giving a positive lower bound for each relevant conditioning probability in the approximation. This passes the bounded inequalities to the infinite-type model. Deterministic parameters follow by parameter limits, and nonnegative observables by monotone truncation. □

Finite-slab convention. In the finite proof, the assertion about the rightmost site is made for the set of reachable sites in the relevant terminal cone, with all deterministic sites to its right declared unreachable. The infinite-lattice half-line edge is then recovered by increasing the finite slab. For an observable at finitely many strict-interior sites and a fixed boundary value, the necessary terminal and backward cones are finite. This avoids treating a finite artificial outer wall as an additional random disaster.

Appendix B. Spatial compression and a subthreshold cutoff estimate

B.1. Compression of a finite target

For a finite target $U = \{x_1 < \cdots < x_q\}$ and a gap $x_{i+1} - x_i \geq 2$, let U' shift the right suffix one site to the left. We prove $p_t(U') \geq p_t(U)$ and hence

$$p_t(U) \leq p_t(\{0, \ldots, q-1\}), \qquad p_\lambda(U) \leq h_\lambda(|U|). \tag{B1}$$

The one-row statement needed for the induction is an order on survivor ranks. A set function f is supermodular if $f(A \cup C) + f(A \cap C) \geq f(A) + f(C)$. On a finite rank set, the disaster row is a pure-death chain with generator

$$(\mathcal{L}_U f)(S) = \lambda \sum_r \nu_r \sum_{\ell \in \mathbb{Z}} [f(S \setminus J_{r,\ell}(U)) - f(S)], \quad J_{r,\ell}(U) = \{j : \ell \leq x_j < \ell + r\}.$$

Its total rate on every nonempty state is at most $\lambda |S| S_1 < \infty$.

Fix S and expand $F(A) = f(S \setminus A)$ in Boolean Möbius coefficients. The number of length-r origins covering every member of a nonempty rank subset T is $(r - x_{\max T} + x_{\min T})_+$. Compression changes this number by one exactly when T crosses the gap and its original diameter is at most r. Group the coefficients by their extreme ranks $j \leq i < k$. The intermediate-rank Möbius sum becomes a mixed difference on the survivor set outside $[j, k]$. Thus

$$\begin{aligned}(\mathcal{L}_{U'} - \mathcal{L}_U) f(S) = \lambda \sum_{\substack{j,k \in S \\ j \leq i < k}} \overline{\nu}_{x_k - x_j} \big[& f(T_{jk}) + f(T_{jk} \cup \{j,k\}) \\ & - f(T_{jk} \cup \{j\}) - f(T_{jk} \cup \{k\}) \big] \geq 0,\end{aligned} \tag{B2}$$

where $T_{jk} = S \setminus [j,k]$ in rank coordinates and $\overline{\nu}_d = \sum_{r \geq d} \nu_r$. The sum is over surviving extreme ranks, so it also covers a deletion of any ranks between them.

From generators to one-row laws. Write $P_U(s)$ for the survivor-rank semigroup with generator $\mathcal{L}_U$. For a deterministic set of deleted ranks J, the map $S \mapsto f(S \setminus J)$ preserves supermodularity: deletion commutes with both union and intersection. The random deletion set in one Boolean row is independent of the initial survivor set. Averaging therefore shows that $P_U(s)$ preserves supermodularity for every $s \geq 0$. The rank state space is finite, and its jump rates are finite by $S_1 < \infty$. Hence the finite-matrix Duhamel identity gives

$$[P_{U'}(1) - P_U(1)] f = \int_0^1 P_{U'}(s)(\mathcal{L}_{U'} - \mathcal{L}_U) P_U(1-s) f \, ds \geq 0$$

for supermodular f. Indeed, the inner semigroup preserves supermodularity, (B2) is pointwise nonnegative, and the outer semigroup preserves pointwise nonnegativity. Thus the one-row survivor law has larger expectation for every supermodular rank function after compression.

Predecessor geometry for a fixed survivor set. Put $\Psi(A) = A \cup (A-1)$ and $U_S = \{x_j : j \in S\}$. Fix S before exposing a disaster row and split its surviving sites across the compressed gap:

$$L = U_S \cap (-\infty, x_i], \qquad R = U_S \cap [x_{i+1}, \infty).$$

Then $U'_S = L \cup (R-1)$. If L or R is empty, $\Psi(U'_S)$ is a translate of $\Psi(U_S)$, including the empty-set case. Suppose both are nonempty and set

$$\ell = \max L, \quad r = \min R, \quad d = r - \ell \geq 2, \qquad A = \Psi(L), \quad B = \Psi(R).$$

The predecessor sets are

$$\Psi(U_S) = A \cup B, \qquad \Psi(U'_S) = A \cup (B-1), \qquad \max A = \ell, \quad \min B = r - 1.$$

There are exactly two boundary cases.

If $d \geq 3$, the gap between A and B has size $d - 1 \geq 2$. Moving B left by one is an allowed single-gap compression of the predecessor target $A \cup B$. The induction hypothesis at time t therefore gives

$$p_t\big(A \cup (B-1)\big) \geq p_t(A \cup B).$$

Internal gaps or deleted ranks inside L and R do not affect this argument.

If $d = 2$, the old predecessor pieces are adjacent: $\min B = \ell + 1$. After the shift they meet at exactly ℓ, which is already in A. Remove the single redundant requirement $\ell + 1$ from B before shifting and put

$$C = A \cup \big(B \setminus \{\ell + 1\}\big).$$

Since $r = \ell + 2$ belongs to B, the remaining right part is nonempty and starts at r; its gap from A is two. Compressing that gap by one produces precisely

$$A \cup \big((B \setminus \{\ell+1\}) - 1\big) = A \cup (B-1) = \Psi(U'_S).$$

The induction hypothesis for this compression, followed by decreasingness under inclusion, gives

$$p_t(\Psi(U'_S)) \geq p_t(C) \geq p_t(A \cup B) = p_t(\Psi(U_S)).$$

This accounts explicitly for the one duplicated predecessor when the pieces meet. Consequently, for every survivor-rank set S,

$$F_{U'}(S) := p_t(\Psi(U'_S)) \geq F_U(S) := p_t(\Psi(U_S)).$$

Closing the time induction. The function $p_t(A) = \Pr(A \subset H_t)$ is decreasing under target inclusion and supermodular. For the latter assertion, for each deterministic hole set H_t the indicator $\mathbf{1}_{\{A \subset H_t\}}$ is supermodular; expectation preserves the inequality. The map Ψ preserves unions and satisfies

$$\Psi(U_{S \cap T}) \subseteq \Psi(U_S) \cap \Psi(U_T).$$

Writing $V = \Psi(U_S)$ and $W = \Psi(U_T)$, decreasingness and supermodularity give

$$\begin{aligned} F_U(S \cup T) + F_U(S \cap T) &\geq p_t(V \cup W) + p_t(V \cap W) \\ &\geq p_t(V) + p_t(W) = F_U(S) + F_U(T). \end{aligned}$$

Thus F_U is a supermodular rank function, as required by the one-row comparison.

Let $\mathcal{S}_U$ and $\mathcal{S}_{U'}$ be the sets of ranks left uncovered by a fresh disaster row at the respective targets. Finite-history reversal gives the exact recursion

$$p_{t+1}(U) = \mathbb{E}\, p_t(\Psi(U \setminus D)) = \mathbb{E} F_U(\mathcal{S}_U).$$

The deterministic predecessor comparison and then the one-row semigroup order yield

$$\begin{aligned} p_{t+1}(U') = \mathbb{E} F_{U'}(\mathcal{S}_{U'}) &\geq \mathbb{E} F_U(\mathcal{S}_{U'}) \\ &\geq \mathbb{E} F_U(\mathcal{S}_U) = p_{t+1}(U). \end{aligned}$$

The base $t = 0$ follows from $H_0 = \varnothing$, with $p_0(\varnothing) = 1$. Translation invariance supplies the equality used in the one-sided survivor cases. This proves the single-gap comparison for every finite target by induction on

time. Closing its gaps one at a time gives the first inequality in (B1); the increasing remote-past limit gives the stationary inequality. The argument is unchanged for any length truncation. Applying the one-row order to the supermodular rank function $\mathbf{1}_{\{S=\varnothing\}}$ also gives the corresponding full-coverage comparison for a single Boolean row.

B.2. Marks shorter than the target by a fixed factor

Lemma B.1. Fix $0 < \delta < 1$, let $m_k = \lfloor \delta k \rfloor$, and write $h_k^{(m_k)}$ for the stationary block probability after deleting all marks longer than m_k. Under the assumptions of Section 2,

$$h_k^{(m_k)} = o(A_\nu(k)). \tag{B3}$$

Proof. The branching comparison of Appendix A applies to the truncated law. Write $\chi_m = \sum_{r \le m} \nu_r r(r+1)/2$ and $A_j^{(m)}$ for its tail sum. It gives

$$h_k^{(m)} \le \Pr\left(\sum_{i=1}^{J_m} W_{m,i} \ge k \right), \qquad 1 \le W_{m,i} \le m, \quad \Pr(W_{m,i} \ge j) = A_j^{(m)}/\chi_m.$$

The costs are iid and independent of the tree; $\Pr(J_m = j) \le (4\alpha_m)^j$ with $\alpha_m = \lambda \chi_m / u_m \le 1/16$. For large m, $\chi_m \ge \chi/2$, so the cost tail is bounded by CA_j uniformly in m.

Choose $\gamma < \alpha < \gamma/\delta$ and $s_k = \alpha \log k/k$. We claim $M_{m_k}(s_k) = \mathbb{E} e^{s_k W_{m_k,1}} \to 1$. The integer tail-sum identity gives

$$M_m(s) - 1 = (e^s - 1) \sum_{j=0}^{m-1} e^{sj} \Pr(W_{m,1} > j).$$

For $j \le 1/s$, this is bounded by a constant times $s \sum_{j \le 1/s} A_{j+1} = o(1)$, since $A_j \to 0$. Split the remaining indices into blocks of width $1/s$. Potter bounds imply that the sum of their contributions is bounded by a term tending to zero at the first blocks plus $Ce^{sm}A_m$. At $m = m_k$, the latter is $O(k^{\alpha\delta - \gamma} L(k)) = o(1)$. This proves the claim. In particular $M_{m_k}(s_k) < 2$ eventually and

$$\mathbb{E}[M_{m_k}(s_k)^{J_{m_k}}] \le \sum_{j \ge 0} (4\alpha_{m_k} M_{m_k}(s_k))^j \le 2.$$

Chernoff's bound gives $h_k^{(m_k)} \le 2k^{-\alpha} = o(A_k)$, proving (B3). □

The same estimate applies when the target length is asymptotic to $\ell T/32$ and the cutoff is δT, provided $32\delta < \ell$, where $\ell > 0$ is a fixed time ratio. This is the strict inequality used in Lemma 6.2.