A BOUNDARY LIMIT FOR SUZUKI'S ARITHMETIC CANONICAL SYSTEM

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Abstract. For the shifted Riemann xi-function, we prove the large-radius boundary limit of Suzuki's arithmetic canonical system. The proof applies to every positive shift for which the reflected quotient is pole-free in the upper half-plane and the two entire functions have no common zero. It therefore applies unconditionally to shifts at least one half, including Suzuki's original range greater than one. We give the weak-kernel construction needed at the smaller shifts, two integral identities for its positive determinant ratio, and decay estimates faster than every fixed exponential in logarithmic radius. The hypotheses on the shift are retained throughout.

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1. Statement and normalization

Write

$$\xi(s)=\tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s),\qquad s_0=\tfrac{1}{2}+\omega,\qquad E(z)=\xi(s_0-iz),$$

and put $F^\sharp(z)=\overline{F(\bar z)}$ for an entire function F. Fix $\omega>0$ and assume

$$E^\sharp/E \text{ is pole-free in } \mathbb{C}_+,\qquad E \text{ and } E^\sharp \text{ have no common zero.} \tag{1.1}$$

The functional equations for ξ give $E^\sharp(z)=E(-z)$ and

$$\Theta(z):=\frac{E^\sharp(z)}{E(z)}=\frac{\xi(1/2-\omega-iz)}{\xi(1/2+\omega-iz)}.$$

Both conditions in (1.1) hold for $\omega\ge 1/2$, since the zeros of ξ lie in $0<\Re s<1$. For smaller shifts they are assumptions.

Here is a definition of the arithmetic system which also specifies the meaning of its determinant ratio when its kernel is weakly singular. Let μ be the Möbius function and set

$$c_\omega(n)=n^\omega\sum_{d\mid n}\frac{\mu(d)}{d^{2\omega}},$$

$$g_\omega(x)=\frac{2\pi^\omega}{\Gamma(\omega)}\left[x^{2-\omega}(1-x^2)^{\omega-1}-\omega x^{\omega-1}\int_{x^2}^1 v^{1/2-\omega}(1-v)^{\omega-1}\,dv\right]\quad(0<x<1),$$

$$h_\omega(x)=\frac{1}{x}\sum_{n\le x}c_\omega(n)g_\omega(n/x)\quad(x>1).$$

Set $g_\omega=0$ on $(1,\infty)$, $h_\omega=0$ on $(0,1)$, and

$$k(x)=e^{x/2}h_\omega(e^x),\qquad k(x)=0\quad(x<0). \tag{1.2}$$

Values at the isolated singularities have no effect on any integral below. For $t\ge 0$, let K_t be the integral operator on $L^2(-t,t)$ with kernel $k(x+y)$. For odd $n\ge 1$ define

$$\tau_n(t)=\int_{(-t,t)^n}\prod_{j=1}^n k(x_j+x_{j+1})\,dx_1\cdots dx_n,\qquad x_{n+1}=x_1,\qquad m(t)=\exp\left(2\sum_{\substack{n\ge1\\ n\text{ odd}}}\frac{\tau_n(t)}{n}\right). \tag{1.3}$$

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At $t = 0$ the integrals are zero. We prove below that the series is locally uniformly convergent and m is positive and continuous. Define the entire pair $(\mathcal{A}_t, \mathcal{B}_t)$ by

$$\begin{aligned} \partial_t \mathcal{A}_t(z) &= z m(t)^2 \mathcal{B}_t(z), \qquad \partial_t \mathcal{B}_t(z) = -z m(t)^{-2} \mathcal{A}_t(z), \\ (\mathcal{A}_0, \mathcal{B}_0) &= \left(\frac{E + E^\sharp}{2}, \frac{i(E - E^\sharp)}{2} \right). \end{aligned} \tag{1.4}$$

The variable t is logarithmic radius: $t = \log a$.

Theorem 1.1. *Under* (1.1)*, with the definitions above,*

$$\mathcal{A}_t(z) \longrightarrow \xi(s_0), \qquad \mathcal{B}_t(z) \longrightarrow 0 \quad (t \longrightarrow \infty) \tag{1.5}$$

locally uniformly for $z \in \mathbb{C}$. Moreover,

$$\int_0^\infty m(t)^{-2}\, dt = \frac{\xi'(s_0)}{\xi(s_0)}, \tag{1.6}$$

$$\int_0^\infty m(t)^2 \int_t^\infty m(u)^{-2}\, du\, dt = \frac{\xi''(s_0)}{2\xi(s_0)}. \tag{1.7}$$

For every $R > 0$ and $N > 0$, as $t \to \infty$,

$$\sup_{|z| \le R} |\mathcal{B}_t(z)| = O_{R,N}(e^{-Nt}), \qquad \int_t^\infty m(u)^{-2}\, du = O_N(e^{-Nt}). \tag{1.8}$$

For $\omega > 1$, $m(t)$ is Suzuki's Fredholm determinant ratio at radius $a = e^t$, and (1.4) *is his normalized arithmetic system.*

Suzuki constructed the system for $\omega > 1$ and posed its large-radius limit immediately after Theorem 2.3 of [1]. The kernel and the original determinant construction are his. The reproducing-kernel chain used below is also part of his general framework [2]. We include the approximation argument needed to define the same arithmetic system under (1.1) for all $\omega > 0$. The assertion concerns this boundary limit, and makes no assertion that (1.1) holds at all positive shifts.

2. The causal operator and its finite compressions

We use the Fourier convention $\mathcal{F}f(z) = \int_{\mathbb{R}} f(x) e^{izx}\, dx$; its L^2 norm is $\sqrt{2\pi}$ times that of f. Inner products are linear in the first variable. Reflection of a Fourier function means $RF(z) = F(-z)$.

Lemma 2.1. *The function k is real, is supported in $[0, \infty)$, and belongs locally to L^q for some $q > 1$. For every sufficiently large σ,*

$$\int_0^\infty |k(x)| e^{-\sigma x}\, dx < \infty, \qquad \int_0^\infty k(x) e^{izx}\, dx = \Theta(z) \quad (\Im z = \sigma). \tag{2.1}$$

Under (1.1)*, Θ is inner in $\mathbb{C}_+$. The operator*

$$J = \mathcal{F}^{-1} M_\Theta R \mathcal{F} \tag{2.2}$$

is a real, complex-linear, self-adjoint unitary involution. On compactly supported functions its kernel, in the distributional and local integral senses, is $k(x + y)$.

Proof. Near $x = n$ the only new summand in h_ω is $O(|x - n|^{\omega - 1})$; the remaining finitely many summands are locally bounded. Choose $q > 1$ such that $q(\omega - 1) > -1$ when $\omega < 1$. This proves the local assertion, including after the change of variables in (1.2).

For completeness, the Mellin identity underlying (2.1) can be obtained without an inversion argument. Put $\gamma(s) = \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma(s/2)$. The beta integral and Fubini give

$$\int_0^1 g_\omega(x) x^s\, \frac{dx}{x} = \frac{\gamma(s - \omega)}{\gamma(s + \omega)}, \qquad \sum_{n \ge 1} \frac{c_\omega(n)}{n^s} = \frac{\zeta(s - \omega)}{\zeta(s + \omega)} \quad (\Re s > 1 + \omega).$$

The same calculation with absolute values is convergent on this half-plane: the Dirichlet series is absolutely convergent and the beta integrals converge absolutely. Substitution $x \mapsto n/x$ in the definition of h_ω yields $\int_1^\infty h_\omega(x)x^{1-s}\,dx/x = \xi(s-\omega)/\xi(s+\omega)$, with absolute convergence. Taking $s = 1/2 - iz$ proves (2.1). This is the transform calculation of [1, Proposition 2.1].

The hypotheses imply that E has no zero in $\mathbb{C}_+$ and no real zero. Use the genus-zero paired product for the even entire function $v \mapsto \xi(1/2+v)$. Truncate its zeros in finite sets invariant under $v \mapsto -v$ and conjugation. All zeros of the corresponding shifted polynomials E_j lie in the lower half-plane, because the same is true of E. Each factor of $E_j^\sharp/E_j$ is, up to a constant of modulus one, $(z-\bar\lambda)/(z-\lambda)$ with $\Im\lambda < 0$. Consequently $|E_j^\sharp/E_j| \le 1$ in $\mathbb{C}_+$. The paired products converge locally uniformly, so $|\Theta| \le 1$ there. It is nonconstant and has unimodular real boundary values, hence is inner and satisfies $|\Theta| < 1$ in $\mathbb{C}_+$.

On the real line, $|\Theta| = 1$ and $\Theta(z)\Theta(-z) = 1$. These identities prove unitarity and $J^2 = I$; thus $J = J^*$. Reality follows from $\Theta(-x) = \overline{\Theta(x)}$. Finally (2.1), uniqueness of Fourier–Laplace transforms, and approximation by compactly supported test functions identify the kernel of (2.2) with $k(x+y)$. □

In the rest of the paper P_t denotes multiplication by $\mathbf{1}_{(-\infty,t)}$ on $H = L^2(\mathbb{R})$. We also write $K_t = P_tJP_t$ on P_tH; its nonzero part is the operator already defined on $L^2(-t,t)$.

Lemma 2.2. *For every finite $t \ge 0$, K_t is compact and $\|K_t\| < 1$. The function $t \mapsto K_t$, considered on H, is norm-continuous.*

Proof. Causality makes the kernel vanish unless $x, y \in [-t,t]$. Approximate k in $L^1(0,2t)$ by smooth functions. Young's inequality bounds the operator-norm difference by the L^1 difference. The approximating kernels are Hilbert–Schmidt, proving compactness.

Since J is unitary, $\|K_t\| \le 1$. If equality held, compactness and self-adjointness would give a nonzero $f \in P_tH$ with $K_tf = \pm f$. Equality of norms then implies $Jf \in P_tH$. Causality shows that Jf vanishes below $-t$, and applying the same argument to Jf shows that f also vanishes below $-t$. Their Fourier transforms F and G are nonzero entire functions of exponential type. On the real axis, and therefore meromorphically,

$$G(z) = \frac{E^\sharp(z)}{E(z)}F(-z).$$

Every zero of $E^\sharp$ is a zero of G, with its multiplicity, because E and $E^\sharp$ have no common zero. The number of these zeros in a disk of radius r is of order $r\log r$, by the Riemann–von Mangoldt formula. A nonzero entire function of exponential type has only $O(r)$ zeros in such disks, by Jensen's formula. This is a contradiction. The assertion at $t = 0$ is immediate.

On a bounded range $0 \le t \le T$, $K_t = P_tK_TP_t$. Strong continuity of P_t and compactness of K_T give norm continuity. □

3. Weak kernels and the arithmetic normalization

The details in this section ensure that no differentiability of the singular kernel is assumed in the endpoint argument.

Lemma 3.1. *Fix $T < \infty$. There is an integer $b \ge 2$ such that the kernels of K_t^b, $0 \le t \le T$, are uniformly bounded. If smooth causal kernels k_j converge to k locally in L^q, their corresponding bth powers converge uniformly as kernels on this parameter range. Consequently the series in* (1.3) *converges absolutely and locally uniformly, and m is positive and continuous.*

Proof. Choose b so that $p = b/(b-1) \le q$. Truncate the kernels on a fixed compact interval containing all sums in question. The absolute kernel of a $b$fold product is bounded by a convolution of b copies of this truncation or its reflection. Iterated Young's inequality applies because $b(1-1/p) = 1$,

and gives an L^∞ bound. Expanding a difference of two products into b terms gives the asserted uniform convergence, with a bound proportional to $\|k_j - k\|_p$. In particular these powers converge in Hilbert–Schmidt norm.

For odd n, the linear change of variables $r_j = x_j + x_{j+1}$ has determinant of absolute value two. It follows that the cycle integral is absolutely convergent; its absolute value is at most $\frac{1}{2}\|k\|^n_{L^1(0,2T)}$. The same change of variables proves convergence of each cycle integral under L^1 approximation. It also proves continuity in t: the indicators of the transformed domains converge almost everywhere, and the product of $|k(r_j)|$ is an integrable majorant.

Write $\kappa = \|K_T\| < 1$. For $n \geq 2b$, the factorization $K_t^n = K_t^b K_t^{n-2b} K_t^b$ gives

$$\|K_t^n\|_1 \leq C_T \kappa^{n-2b}.$$

Here $\|\cdot\|_1$ is trace norm. The cycle integral equals $\operatorname{tr} K_t^n$: prove this first for smooth kernels and then use both the trace-norm convergence of the factorization and the convergence of the odd cycle integral just proved. This geometric bound proves uniform summability of the odd tail. The finitely many remaining cycles are continuous. The exponential in (1.3) is therefore finite, positive, and continuous. □

For use below let $\phi_\pm(t,\cdot)$ be the solution on $(-t,t)$ of

$$(I \pm K_t)\phi_\pm(t,\cdot) = k(t+\cdot), \tag{3.1}$$

and extend it to all x by

$$\phi_\pm(t,x) = k(x+t) \mp \int_{-t}^{t} k(x+y)\phi_\pm(t,y)\,dy. \tag{3.2}$$

For large $\Im z$ define

$$F_+(t,z) = e^{izt} + \int_t^\infty \phi_+(t,x)e^{izx}\,dx, \qquad F_-(t,z) = e^{izt} - \int_t^\infty \phi_-(t,x)e^{izx}\,dx. \tag{3.3}$$

Proposition 3.2. *Equation* (3.1) *has a unique $L^1(-t,t)$ solution, and the integrals in* (3.3) *converge absolutely on a sufficiently high horizontal line. The solution of* (1.4) *satisfies*

$$\mathcal{A}_t(z) = \tfrac{1}{2}m(t)E(z)F_+(t,z), \qquad \mathcal{B}_t(z) = \tfrac{i}{2m(t)}E(z)F_-(t,z) \tag{3.4}$$

on that line. When $\omega > 1$, these are Suzuki's formulas and $m(t) = \det(I+K_t)/\det(I-K_t)$ in the classical Fredholm sense.

Proof. The bounded-kernel assertion in Lemma 3.1 shows that $K_t^b : L^1(-t,t) \to L^2(-t,t)$ is bounded. The finite resolvent identity

$$(I \pm K_t)^{-1} f = \sum_{j=0}^{b-1} (\mp K_t)^j f + (I \pm K_t)^{-1}_{L^2} (\mp K_t)^b f \tag{3.5}$$

defines a bounded inverse on L^1. Uniqueness follows as well: an L^1 null vector is, after applying K_t^b, an L^2 null vector. The weighted L^1 estimate in Lemma 2.1 and Fubini applied to (3.2) prove the asserted absolute convergence.

Choose real smooth nonnegative functions ρ_j, of integral one, supported in $(0, 1/j)$, and put $k_j = \rho_j * k$. These kernels are smooth and causal, converge locally in L^q, and converge in the weighted L^1 norm of (2.1). On every fixed range $0 \leq t \leq T$, their compressions converge uniformly in operator norm to K_t and hence eventually have norm at most some $\kappa_T < 1$. Lemma 3.1 and its proof show that their odd-cycle ratios m_j converge uniformly to m.

Formula (3.5), the convergence of the bth powers as maps $L^1 \to L^2$, and the uniform L^2 resolvent bound show that the inverses converge on L^1, uniformly for $0 \leq t \leq T$. The sources $k_j(t+\cdot)$ converge in L^1 uniformly in this range. Thus $\phi_{\pm,j}$ converge in $L^1(-t,t)$ uniformly in t. Using (3.2)

and weighted Fubini, their tails converge in weighted L^1 as well. Consequently $F_{\pm,j} \to F_\pm$ uniformly on a high line, locally in t.

We record the smooth calculation to specify all signs and scale factors. Put $q_j(t) = \phi_{+,j}(t,t) + \phi_{-,j}(t,t)$. Differentiating (3.1) and integrating by parts gives

$$\partial_t \phi_{+,j} = \partial_x \phi_{-,j} - q_j \phi_{+,j}, \qquad \partial_t \phi_{-,j} = \partial_x \phi_{+,j} + q_j \phi_{-,j}.$$

There is no lower boundary term, since the kernel is causal and smooth. The usual differentiable Fredholm determinant formula, or its convergent trace series, gives $m_j'/m_j = q_j$. Integrating the last two identities against e^{izx} over (t,∞) yields

$$\partial_t F_{+,j} = izF_{-,j} - q_j F_{+,j}, \qquad \partial_t F_{-,j} = izF_{+,j} + q_j F_{-,j}.$$

Hence the two expressions on the right of (3.4), with subscript j, solve (1.4) with coefficient m_j. At $t = 0$ their initial values are

$$\tfrac{1}{2}(E + \widehat{\rho}_j E^\sharp), \qquad \tfrac{i}{2}(E - \widehat{\rho}_j E^\sharp), \qquad \widehat{\rho}_j(z) = \int \rho_j(x) e^{izx}\, dx.$$

These are entire and converge locally uniformly to the initial values in (1.4). Continuous dependence for the integral form of the linear differential system, using uniform convergence of m_j and m_j^{-1} on bounded intervals, now proves convergence of these solutions locally uniformly in (t,z). Comparison on the high line proves (3.4).

If $\omega > 1$, k is continuous, and the classical Fredholm determinants of the smooth approximations converge to those of K_t. Their logarithmic ratio is exactly the odd trace series. This proves the final assertion and the agreement of the solution normalizations. The unitary change of variables from the multiplicative formulation is $Uf(x) = e^{x/2} f(e^x)$; its source factor $e^{t/2}$ is the $\sqrt{a}$ factor in Suzuki's tail integrals. □

4. The reproducing identity

Define the closed subspace and its orthogonal projection by

$$V_t = \ker P_t \cap \ker P_t J, \qquad \Pi_t = \mathrm{proj}_{V_t}.$$

For $z \in \mathbb{C}_+$ put $e_z^t(x) = \mathbf{1}_{(t,\infty)}(x) e^{izx}$, $Y_z^t = \Pi_t e_z^t$, and

$$j_t(z,w) = \frac{1}{2\pi} \langle Y_z^t, Y_w^t \rangle. \tag{4.1}$$

In particular,

$$0 \le j_t(iy, iy) \le \frac{e^{-2ty}}{4\pi y} \quad (y > 0). \tag{4.2}$$

Proposition 4.1. *For $t \ge 0$ and $z, w \in \mathbb{C}_+$,*

$$E(z)\overline{E(w)} j_t(z,w) = \frac{\mathcal{B}_t(z)\overline{\mathcal{A}_t(w)} - \mathcal{A}_t(z)\overline{\mathcal{B}_t(w)}}{\pi(z - \bar{w})}. \tag{4.3}$$

Proof. We give a proof that uses only the local integrability already established. Fix t, write $P = P_t$ and $K = PJP$, and, initially on a high line, put

$$u = PJe_z^t, \qquad a = (I - K^2)^{-1} u.$$

The projection has the explicit formula

$$Y_z^t = e_z^t - Ja + Ka. \tag{4.4}$$

Indeed, P and PJ annihilate the right side, and the difference from e_z^t lies in $PH + JPH$. This sum is closed: the Gram matrix of $(b,c) \mapsto b + Jc$ is $\left(\begin{smallmatrix} I & K \\ K & I \end{smallmatrix}\right)$, which is invertible by Lemma 2.2.

The functions in (4.4) have right limits at $x = t$. To see this, the high-line formula

$$Je_z^t(x) = e^{-izx} \int_{x+t}^{\infty} k(r) e^{izr}\, dr$$

is continuous on bounded x intervals, with $k = 0$ on negative arguments. The source u is bounded on $[-t, t]$. Expanding $(I - K^2)^{-1}$ a finite number of times and using Lemma 3.1 shows that a and Ka are bounded on that interval. For $x < -t$, $a = u$ and $Ka = 0$. Finally, translations are continuous in local L^1, so convolution of the local kernel with these bounded functions is continuous near $x = t$.

Let $h(x) = k(t + x)$ on $x < t$, and use bilinear integral pairings in the next three formulas. The boundary values are

$$Y_z^t(t+) = e^{izt} - \langle h, a\rangle, \qquad (JY_z^t)(t+) = Je_z^t(t) + \langle h, Ka\rangle.$$

Fubini in (3.2) and symmetry of the resolvent give

$$\int_t^{\infty} \phi_+(t,x) e^{izx}\, dx = Je_z^t(t) - \langle h, (I-K)a\rangle,$$
$$\int_t^{\infty} \phi_-(t,x) e^{izx}\, dx = Je_z^t(t) + \langle h, (I+K)a\rangle.$$

These identities remain valid with $h \in L^1$. Indeed, the adjoint of the bounded L^1 inverse in (3.5) is the bounded L^∞ inverse of the same symmetric kernel. On $L^2 \cap L^\infty$ it agrees with the L^2 inverse. Thus its application to the bounded source u on $[-t, t]$ gives $(I \mp K)a$, and the asserted duality is legitimate. All kernel interchanges are absolutely convergent by the L^1 kernel and bounded second factor. Thus

$$F_+(t,z) = Y_z^t(t+) + (JY_z^t)(t+), \qquad F_-(t,z) = Y_z^t(t+) - (JY_z^t)(t+). \tag{4.5}$$

We next differentiate the projection without differentiating k. For $\delta > 0$ let $Q_\delta = \mathbf{1}_{(t,t+\delta)}$ and

$$L_\delta : V_t \longrightarrow Q_\delta H \oplus Q_\delta H, \qquad f \longmapsto (Q_\delta f, Q_\delta J f).$$

Then $V_{t+\delta} = \ker L_\delta$ inside V_t, and

$$L_\delta L_\delta^* = I + o(1) \quad \text{in operator norm as } \delta \downarrow 0. \tag{4.6}$$

Here are the details of the latter assertion. Without Π_t in the adjoint synthesis map, the diagonal blocks are I and the off-diagonal blocks are $Q_\delta J Q_\delta = o(1)$. Projection of $b + Jc$ onto $PH + JPH$ has norm at most $C_t \|PJQ_\delta\|(\|b\| + \|c\|)$ by the invertible Gram matrix. Both PJQ_δ and $Q_\delta J Q_\delta$ are compressions of the fixed compact K_{t+1}, with a factor Q_δ tending strongly to zero. Their norms therefore tend to zero, proving (4.6).

For small δ, the projection onto the orthogonal complement of $\ker L_\delta$ in V_t is $L_\delta^*(L_\delta L_\delta^*)^{-1} L_\delta$. Also the projections of e_z^t and $e_z^{t+\delta}$ onto $V_{t+\delta}$ agree by support. The right traces and (4.6) consequently imply

$$\partial_t^+ \langle Y_z^t, Y_w^t\rangle = -Y_z^t(t+)\overline{Y_w^t(t+)} - (JY_z^t)(t+)\overline{(JY_w^t)(t+)}. \tag{4.7}$$

Indeed the observation vectors have norm $O(\delta^{1/2})$, so replacing the inverse Gram operator by I changes their pairing by $o(\delta)$.

Use (4.5) and (3.4) in (4.7). The right derivative of the left side of (4.3) is

$$-\frac{1}{\pi}\left(m(t)^{-2} \mathcal{A}_t(z)\overline{\mathcal{A}_t(w)} + m(t)^2 \mathcal{B}_t(z)\overline{\mathcal{B}_t(w)} \right).$$

Direct differentiation using (1.4) gives the same derivative for its right side. Both sides are continuous in t: for the projections this follows from strong continuity of P_t and norm continuity of K_t. More explicitly, for a fixed $f \in H$ put $c_t = (I - K_t^2)^{-1}(P_t J f - K_t P_t f)$ and $b_t = P_t f - K_t c_t$. The Gram-matrix formula is $\Pi_t f = f - b_t - J c_t$, which is strongly continuous in t. The displayed right derivative is continuous. A continuous function with continuous right derivative obeys the

fundamental theorem of calculus (subtract the integral and use the elementary zero-right- derivative criterion). Thus equality at $t = 0$ implies equality at every t.

At $t = 0$, the Fourier image of V_0 is $H^2(\mathbb{C}_+) \ominus \Theta H^2(\mathbb{C}_+)$: Fourier transformation sends right support to $H^2(\mathbb{C}_+)$ and reflection sends it to $H^2(\mathbb{C}_-)$. The Hardy reproducing formula, followed by orthogonal projection, gives

$$j_0(z, w) = \frac{i}{2\pi} \frac{1 - \Theta(z)\overline{\Theta(w)}}{z - \bar{w}}.$$

This is (4.3) for the initial pair in (1.4). We have proved the identity on high lines. Both sides are holomorphic in z and antiholomorphic in w on $\mathbb{C}_+$, so the identity theorem proves it throughout that domain. □

5. Normal convergence and the terminal equation

Put $c = \xi(s_0) > 0$. The positivity follows, for example, from the standard positive theta-integral representation of ξ on the real axis. The differential system gives, for every t,

$$\mathcal{A}_t(0) = c, \qquad \mathcal{B}_t(0) = 0. \tag{5.1}$$

It also preserves real entire coefficients and the respective even and odd parities. On every finite interval its transfer matrix has exponential type in z, as follows by iteration of its integral equation. Hence $\mathcal{A}_t, \mathcal{B}_t$ have order at most one. They have no common zero: otherwise uniqueness of the system backward to $t = 0$ would contradict (1.1).

Lemma 5.1. *For every finite t, $G_t = \mathcal{A}_t - i\mathcal{B}_t$ satisfies the strict Hermite–Biehler inequality $|G_t^\sharp(z)| < |G_t(z)|$ in $\mathbb{C}_+$. Both $\mathcal{A}_t$ and $\mathcal{B}_t$ have only real zeros, and*

$$|\mathcal{A}_t(z)| \le \mathcal{A}_t(i|z|), \qquad |\mathcal{B}_t(z)| \le |\mathcal{B}_t(i|z|)|, \qquad \mathcal{A}_t(iy) \ge c \quad (y \ge 0). \tag{5.2}$$

Proof. The diagonal of (4.3) is nonnegative. It implies $|G_t^\sharp| \le |G_t|$ on $\mathbb{C}_+$. If G_t had an upper-half-plane zero, so would $G_t^\sharp$, contrary to the absence of common zeros. Thus $G_t^\sharp/G_t$ is a Schur function. If equality of its modulus held at an interior point, the maximum principle would make it a unimodular constant. Its value at zero is one, by (5.1), so $\mathcal{B}_t$ would vanish identically. Then $\mathcal{A}_t$ would be zero-free, even, and of order at most one. Hadamard factorization would make it the constant c. Transferring this constant pair back to $t = 0$ would make E of exponential type. This is impossible: Stirling's formula on $E(iy) = \xi(s_0 + y)$ gives $\log|E(iy)| = \frac{1}{2} y \log y + O(y)$.

If $\mathcal{A}_t$ or $\mathcal{B}_t$ had a nonreal zero, the strict inequality or its reflection would fail there. Their zeros are therefore real. For an even real entire function of order at most one, pairing the real zeros gives

$$\mathcal{A}_t(z) = c \prod_n (1 - z^2/a_n^2), \qquad a_n > 0.$$

This follows also by applying the genus-zero factorization to the entire function of z^2, whose order is at most one half. The analogous factorization for $\mathcal{B}_t$ includes its odd monomial at zero. Each factor satisfies $|1 - z^2/a_n^2| \le 1 + |z|^2/a_n^2$. These factorizations prove (5.2). □

For $y > 0$ write

$$a_t(y) = \mathcal{A}_t(iy), \qquad b_t(y) = \mathcal{B}_t(iy)/i.$$

Both quantities are real. Lemma 5.1 gives $a_t(y) \ge c$; its strict inequality, applied at iy, gives $b_t(y) > 0$. The canonical equations become

$$\partial_t a_t(y) = -y m(t)^2 b_t(y), \qquad \partial_t b_t(y) = -y m(t)^{-2} a_t(y). \tag{5.3}$$

In particular,

$$c \le a_t(y) \le a_0(y), \qquad a_t(y) \text{ decreases in } t. \tag{5.4}$$

The diagonal of (4.3), together with (4.2), gives

$$0 < b_t(y) \leq \frac{|E(iy)|^2}{4c} e^{-2ty}. \tag{5.5}$$

Lemma 5.2. *The functions $\mathcal{A}_t$ converge locally uniformly to an even entire function $\mathcal{A}_\infty$ of order at most one. The functions $\mathcal{B}_t$ converge locally uniformly to zero, with the first estimate in* (1.8)*.*

Proof. For $|z| \leq R$, (5.2) and (5.4) bound $|\mathcal{A}_t(z)|$ by $a_0(R)$. Montel's theorem gives normality. Every subsequential limit has the same values on the positive imaginary axis, by monotonicity. The identity theorem makes the limit unique and proves local uniform convergence of the whole family. The global bound $|\mathcal{A}_\infty(z)| \leq a_0(|z|)$ shows that its order is at most one.

The paired product for $\mathcal{B}_t$ shows that $b_t(y)$ is increasing in $y > 0$. Given R, N, choose $y > \max(R, N/2)$. Then $\sup_{|z|\leq R} |\mathcal{B}_t(z)| \leq b_t(y)$, and (5.5) proves the claimed estimate. □

Integrating the second equation in (5.3) and using (5.5) gives

$$b_t(y) = y \int_t^\infty m(u)^{-2} a_u(y)\, du. \tag{5.6}$$

It follows that

$$\int_t^\infty m(u)^{-2}\, du \leq \frac{b_t(y)}{yc} < \infty. \tag{5.7}$$

Choosing y arbitrarily large proves the second estimate in (1.8). Substitution in the first equation in (5.3), followed by Tonelli, gives

$$C := \int_0^\infty m(v)^2 \int_v^\infty m(u)^{-2}\, du\, dv \leq \frac{a_0(y) - c}{cy^2} < \infty. \tag{5.8}$$

For arbitrary $z \in \mathbb{C}$, boundedness of $\mathcal{A}_u(z)$ and (5.7) justify integrating the second canonical equation from t to infinity. Then (5.8) justifies integrating the first one. We obtain

$$\mathcal{B}_t(z) = z \int_t^\infty m(u)^{-2} \mathcal{A}_u(z)\, du, \tag{5.9}$$

$$\mathcal{A}_t(z) = \mathcal{A}_\infty(z) - z^2 \int_t^\infty m(v)^2 \int_v^\infty m(u)^{-2} \mathcal{A}_u(z)\, du\, dv. \tag{5.10}$$

All integrals converge absolutely, locally uniformly in z.

Proof of Theorem 1.1. Suppose $\mathcal{A}_\infty(z_0) = 0$. Choose T so large that

$$\alpha := |z_0|^2 \int_T^\infty m(v)^2 \int_v^\infty m(u)^{-2}\, du\, dv < 1.$$

The finite number $M = \sup_{u\geq T} |\mathcal{A}_u(z_0)|$ then satisfies $M \leq \alpha M$ by (5.10). Hence $M = 0$. Equation (5.9) also makes $\mathcal{B}_t(z_0) = 0$ for $t \geq T$, contradicting backward uniqueness and the absence of a common initial zero. Thus $\mathcal{A}_\infty$ is zero-free. It is even and of order at most one, so Hadamard factorization makes it constant. Its value at zero is c. This proves (1.5); the decay statements were proved above.

Differentiate (5.9) at zero, using its locally uniform absolute convergence. Since $\mathcal{B}_0'(0) = \xi'(s_0)$ and $\mathcal{A}_u(0) = c$, this gives (1.6). Finally, (5.10) at $t = 0, z = iy$ gives

$$\frac{a_0(y) - c}{y^2} = \int_0^\infty m(v)^2 \int_v^\infty m(u)^{-2} a_u(y)\, du\, dv.$$

For $0 < y \leq y_1$, the factor $a_u(y)$ lies between c and $a_0(y_1)$ and tends to c as $y \downarrow 0$. Dominated convergence, using (5.8), shows that the right side tends to cC. The left side tends to $\xi''(s_0)/2$. This proves (1.7). □

References

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