AN ERGODIC CLASSIFICATION OF SMOOTH TYPE SUBSHIFTS OVER ODD TWO-LETTER ALPHABETS

WILL HOWE, GRANT HUFFMAN, JENSEN KOHLMEYER, AND LIAM KRUER

Abstract. Fix distinct positive odd integers $\alpha < \beta$. We classify the invariant probability measures of the smooth subshifts obtained by prescribing the successive run types. Every such subshift is uniquely ergodic if $\alpha \geq 3$ or $\{\alpha, \beta\} = \{1, 3\}, \{1, 5\}$. For $\alpha = 1$ and $\beta \geq 7$, unique ergodicity holds exactly for directives containing infinitely many zeros; every other directive gives precisely two ergodic measures. The proof combines a two-dimensional frequency recursion with recognized run decompositions and explicit control of the density of decomposition boundaries.

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1. Statement and scope

Let S denote the left shift on a space of bi-infinite words. If every maximal constant run of a word x is finite, its derivative Δx is the word of successive run lengths. Index it so that $(\Delta x)_0$ is the length of the run containing x_0. A word on $\{\alpha, \beta\}$ is *smooth* if every iterated derivative is defined and is again a word on $\{\alpha, \beta\}$. Write X for the space of these words.

The recoding of $x \in X$ records its runs with the four letters

$$\mathsf{A} = \alpha^\alpha, \qquad \mathsf{B} = \beta^\alpha, \qquad \mathsf{C} = \alpha^\beta, \qquad \mathsf{D} = \beta^\beta.$$

Here c^r means a run of color c and length r. Let $Y = \mathrm{rec}(X)$, with the letter at zero encoding the run containing x_0. Denote the color and length of a recoded letter by col and ℓ, respectively. Thus

$$\mathrm{col}(\mathsf{A}) = \mathrm{col}(\mathsf{C}) = \alpha, \quad \mathrm{col}(\mathsf{B}) = \mathrm{col}(\mathsf{D}) = \beta, \quad \ell(\mathsf{A}) = \ell(\mathsf{B}) = \alpha, \quad \ell(\mathsf{C}) = \ell(\mathsf{D}) = \beta.$$

In a word $y \in Y$, form the maximal consecutive groups having the same value of ℓ. Each group has α or β letters, and the common length alternates from one group to the next. Since these numbers are odd, the first colors of consecutive groups also alternate. We say that y, and its underlying ordinary word, has *type zero* if a group of common length α starts in color α, and *type one* if it starts in color β. The choice is the same at every such group. Reading a group as one run gives the recoding of the derivative; denote this operation on Y also by Δ. For $\tau \in \{0, 1\}^{\mathbb{N}}$, put

$$Y_\tau = \{y \in Y : \mathrm{type}(\Delta^j y) = \tau_j \text{ for every } j \geq 0\}, \qquad X_\tau = \{x \in X : \mathrm{rec}(x) \in Y_\tau\}.$$

The type construction and unique ergodicity for $\{1, 3\}$ are due to Jamet, Marcovici, Poirier and de la Rue [1, Theorem 3.1, Corollary 3.2]. Their Problem 2 asks about the corresponding frequency question for general odd two-letter alphabets. We use their run recoding and prove the following classification. The statement concerns odd alphabets only; it makes no assertion about the classical alphabet $\{1, 2\}$.

Theorem 1.1. *Let $\alpha < \beta$ be positive odd integers and let $\tau \in \{0, 1\}^{\mathbb{N}}$. The spaces X_τ and Y_τ are nonempty compact subshifts. Their ergodic classification is as follows.*

(i) *If $\alpha \geq 3$, or if $(\alpha, \beta) = (1, 3)$ or $(1, 5)$, both subshifts are uniquely ergodic for every τ.*

(ii) *Suppose $\alpha = 1$ and $\beta = 2m + 1$ with $m \geq 3$. Both subshifts are uniquely ergodic if and only if τ contains infinitely many zeros. If τ is eventually one, each subshift has exactly two ergodic invariant Borel probability measures.*

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In the uniquely ergodic cases the frequency of every finite word converges uniformly over all long intervals and all words of the given subshift. For $\alpha = 1$, $\beta = 2m+1$, $m \geq 3$, and $\tau = 1^\infty$, the two recoded letter-frequency vectors, in the order $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$*, are*

$$\left(\frac{\lambda-1}{2\lambda}, \frac{1}{2}, \frac{1}{2\lambda}, 0\right), \qquad \left(\frac{1}{4}, \frac{m+2}{4m}, \frac{1}{4}, \frac{m-2}{4m}\right), \qquad \lambda = \frac{1+\sqrt{1+4m}}{2}. \tag{1.1}$$

The corresponding ordinary-symbol means are λ and m.

2. Recognized expansions and bounded towers

Set $p = (\alpha-1)/2$ and $q = (\beta-1)/2$. The two expansions associated with the types are

	A	B	C	D
φ_0	$\mathsf{A}(\mathsf{BA})^p$	$\mathsf{D}(\mathsf{CD})^p$	$\mathsf{A}(\mathsf{BA})^q$	$\mathsf{D}(\mathsf{CD})^q$
φ_1	$\mathsf{B}(\mathsf{AB})^p$	$\mathsf{C}(\mathsf{DC})^p$	$\mathsf{B}(\mathsf{AB})^q$	$\mathsf{C}(\mathsf{DC})^q$

(2.1)

and $|\varphi_i(a)| = \ell(a)$. The expansions in (2.1) agree with the general-alphabet construction of [1, Section 5.2].

Lemma 2.1. *The sets X and Y are nonempty compact subshifts. Finite type-prefix conditions are clopen relative to these spaces. For every $i \in \{0,1\}$, expansion by φ_i maps Y into its type-i part and satisfies $\Delta\varphi_i(y) = y$ when the origin is a group beginning. Conversely, every type-i word is a shift of $\varphi_i(\Delta y)$, with the group boundaries uniquely determined. Every X_τ and Y_τ is nonempty.*

Proof. The condition that the first derivative is defined and has its letters in $\{\alpha, \beta\}$ is closed: runs longer than β are forbidden, and the lengths of all shorter complete runs must belong to the alphabet. On this closed set the derivative, with the specified origin convention, is continuous. Indeed, a finite window in its output is determined by a finite window in the input, because runs have length at most β. Induction proves that having the first r derivatives in the alphabet is a closed condition. It is nonempty: start with any bi-infinite word on the alphabet and integrate r times, using alternating run colors. These closed sets are nested, so compactness gives $X \neq \varnothing$. The derivative of a translate differs from a translate of the derivative only by its origin index. Hence X is shift invariant.

Recoding is continuous on X, again because runs have bounded length. Every translate of a recoding is obtained by moving the origin to a suitable run of its ordinary word. Consequently Y is compact and shift invariant. In a recoded word the colors alternate. The list of its lengths is the ordinary derivative, so its maximal equal-length groups have sizes α or β. Their common lengths alternate; their first colors alternate because their sizes are odd. This proves that the type is well-defined. The group containing zero, together with its neighboring groups, determines the type in a bounded window. Iterating the same argument proves the assertion about finite prefixes.

If an upper letter has color α and length r, its lower group has common length α and has r alternating letters. Its initial color is selected by the type. The same description, with the opposite initial color, applies to an upper letter of color β. This is exactly (2.1). The last color in each group is its first color, so adjacent groups fit with alternating colors. Decoding the expansion therefore gives an ordinary word whose derivative is the ordinary word underlying the input. In particular it is smooth. The lower groups are recognized by changes in ℓ, and decoding them recovers the input.

For every finite binary word $t_0 \cdots t_{r-1}$, expand any word of Y by $\varphi_{t_0} \cdots \varphi_{t_{r-1}}$. The result has this type prefix. The associated nonempty compact prefix sets are nested along any directive τ, proving $Y_\tau \neq \varnothing$. Decoding gives the corresponding assertion for X_τ. □

We record the measure-theoretic consequence of recognition explicitly. If Z is a compact subshift and $r : Z \to \{1, \ldots, R\}$ is a continuous integer roof, its discrete tower is

$$Z^r = \{(z,j) : 0 \leq j < r(z)\}, \qquad T(z,j) = \begin{cases} (z, j+1), & j+1 < r(z), \\ (Sz, 0), & j+1 = r(z). \end{cases}$$

Lemma 2.2. *Invariant probability measures of (Z, S) and (Z^r, T) correspond bijectively. The correspondence preserves ergodicity and hence the number of ergodic measures. It sends ν to the measure determined by*

$$\int F\, d\widehat{\nu} = \frac{1}{\int r\, d\nu} \int_Z \sum_{j=0}^{r(z)-1} F(z,j)\, d\nu(z). \tag{2.2}$$

For every directive τ, Y_τ is conjugate to the tower over $Y_{S\tau}$ with roof ℓ, via expansion by φ_{τ_0}. Also X_τ is conjugate to the tower over Y_τ with roof ℓ, via ordinary run decoding.

Proof. Formula (2.2) is invariant because the terms at interior levels telescope under T and the remaining base terms agree by S-invariance. Conversely, the restriction of a T-invariant probability to the base $\{j = 0\}$, normalized to mass one, is invariant under its return map S. Every tower point returns to the base within R steps. Invariance therefore determines all the other levels from the base and gives (2.2). Invariant sets in the tower correspond, up to null sets, to invariant sets in the base, proving the ergodicity assertion.

For expansion, associate (y, j) with the shift by j of the expansion whose zero starts $\varphi_{\tau_0}(y_0)$. The group recognition in Lemma 2.1 provides the inverse: locate the group containing zero, decode the groups, and record zero's offset in that group. Both maps are continuous and intertwine the transformations. Ordinary run boundaries give the same proof for the second conjugacy. □

3. The frequency recursion

Because colors alternate, every limit of recoded letter frequencies has the form

$$(a, b, \tfrac{1}{2} - a, \tfrac{1}{2} - b), \qquad 0 \le a, b \le \tfrac{1}{2}.$$

Write

$$u = 2(a + b) - 1, \qquad v = 2(b - a), \qquad L = \frac{\alpha + \beta}{2}, \qquad B_* = \frac{\beta - \alpha}{2}. \tag{3.1}$$

The frequency square becomes the diamond $\mathcal{D} = \{(u, v) : |u| + |v| \le 1\}$. The mean expansion length is $d = \beta - (\beta - \alpha)(a + b) = L - B_* u$. Counting the letters in (2.1), the first two output frequencies for φ_0 have numerators

$$\frac{\beta + 1}{4} - B_* a, \qquad \frac{\beta - 1}{4} - B_* a$$

and common denominator d. For φ_1 the two numerators are exchanged. Thus the maps in the coordinates (3.1) are

$$H_\sigma(u, v) = \left(\frac{B_* v}{L - B_* u}, \frac{\sigma}{L - B_* u} \right), \qquad \sigma = \begin{cases} -1, & \text{type 0,} \\ +1, & \text{type 1.} \end{cases} \tag{3.2}$$

These formulas follow directly from the substitutions; in particular the two type maps have opposite second coordinates.

The denominator is at least $L - B_* = \alpha > 0$ on $\mathcal{D}$, and

$$L - B_* u - B_* |v| \ge \alpha \ge 1.$$

Hence both maps preserve $\mathcal{D}$. On its positive part $\mathcal{D}_+ = \{u, v \ge 0 : u + v \le 1\}$, H_+ is coordinatewise increasing, and

$$|H_\sigma(u, v)| \le H_+(|u|, |v|) \tag{3.3}$$

coordinatewise. This comparison may be iterated.

Definition 3.1. A compatible frequency chain for a directive τ is a sequence $z_j \in \mathcal{D}$, $j \ge 0$, satisfying

$$z_j = H_{\sigma_j}(z_{j+1}), \qquad \sigma_j = 2\tau_j - 1. \tag{3.4}$$

Lemma 3.2. *Limits of empirical letter-frequency vectors in arbitrary intervals whose lengths tend to infinity in words of Y_τ give compatible frequency chains after passage to a subsequence. If every compatible chain has the same value at depth zero, all letter frequencies converge uniformly over these intervals and words.*

Proof. At depth j, discard the two possible partial depth-j tiles at the interval endpoints and take the associated interval in $\Delta^j y$. Every such tile has length at most β^j. Therefore this upper interval has length tending to infinity for fixed j. A diagonal subsequence makes its four empirical letter frequencies converge at every depth. Colors alternate, so the limits have the required form. At two consecutive depths the lost endpoint tiles contribute a bounded error, and the substitution counts give (3.4). If uniform convergence at depth zero failed, there would be a sequence of intervals with frequency vectors staying a fixed distance from the asserted common limit. The same extraction would give a different compatible value, a contradiction. □

4. Uniqueness of compatible frequencies

4.1. **The case $\alpha \geq 3$.** For any two maps and any input in $\mathcal{D}$, the absolute first coordinate of the composition is

$$\frac{B_*}{L(L - B_* u) - B_*^2 v} \leq \frac{B_*}{\alpha L} < \frac{1}{\alpha}.$$

To check the inequality, when $u \geq 0$ use $v \leq 1 - u$ to bound the denominator below by $\alpha\beta - \alpha B_* u \geq \alpha L$. When $u < 0$, use $v \leq 1 + u$ to bound it below by $\alpha\beta$. Put

$$c = 1 - \frac{1}{\alpha}, \qquad d_0 = L - \frac{B_*}{\alpha} = \alpha + cB_*.$$

Every threefold image is contained in

$$R_\alpha = \{(u, v) : |u| \leq 1/\alpha,\ |v| \leq 1/d_0\}.$$

This rectangle is invariant: the second coordinate bound is immediate, and

$$\frac{B_*}{d_0^2} \leq \frac{1}{4\alpha c} \leq \frac{1}{\alpha}.$$

The first inequality follows by maximizing $t/(\alpha + ct)^2$ over $t \geq 0$. On this rectangle the absolute Jacobian is bounded entrywise by

$$J = \begin{pmatrix} B_*^2/d_0^3 & B_*/d_0 \\ B_*/d_0^2 & 0 \end{pmatrix}. \tag{4.1}$$

If $e = B_*^2/d_0^3$, its characteristic polynomial is $r^2 - er - e$, and

$$e \leq \frac{4}{27\alpha c^2} \leq \frac{1}{9}.$$

Thus its positive eigenvalue r is less than one. If w is a positive right eigenvector, the norm $\max_i |z_i|/w_i$ makes both maps r-Lipschitz on R_α, by the mean-value inequality and (4.1). Every point in a compatible chain lies in this rectangle, since it is a threefold image of later points. Comparing two chains through arbitrarily many levels gives distance at most $r^n \operatorname{diam}(R_\alpha)$ at any fixed depth. Their values therefore coincide.

4.2. **The alphabet $\{1,3\}$.** Here $L=2$, $B_*=1$. The same twofold-image estimate gives $|u|\le 1/2$; after one further map $|v|\le 2/3$. The rectangle

$$R_{1,3}=[-1/2,1/2]\times[-2/3,2/3]$$

is invariant, because the image satisfies $|u|\le 4/9$ and $|v|\le 2/3$. Its absolute Jacobian is bounded by

$$\begin{pmatrix} 8/27 & 2/3 \\ 4/9 & 0 \end{pmatrix}.$$

The characteristic polynomial is $r^2-(8/27)r-8/27$; its positive root is less than one. The preceding weighted-norm argument proves uniqueness of compatible chains. This also supplies a direct proof of the frequency contraction needed for the already known $\{1,3\}$ case.

4.3. **The critical alphabet $\{1,5\}$.** Here $L=3$, $B_*=2$. We first show that H_+^n converges uniformly on $\mathcal{D}_+$ to $(1/2,1/2)$. For a point of $\mathcal{D}_+$ put

$$p_0=\tfrac12-u,\quad q_0=\tfrac12-v,\quad s_0=p_0+q_0\ge 0.$$

Iteration gives

$$p'=\frac{p/2+q}{1+p},\qquad s'=\frac{s}{1+p}.$$

For $s>0$, the quotient $r=p/s$ satisfies $r'=1-r/2$, while $1/s'=1/s+r$. Summing these recurrences yields

$$\begin{aligned} c_n&=\frac{2}{3}(1-(-1/2)^n),\\ D_n&=1+\frac{2}{3}ns_0+c_n(p_0-2s_0/3),\\ s_n&=\frac{s_0}{D_n},\qquad p_n=\frac{2}{3}s_n+\frac{(-1/2)^n(p_0-2s_0/3)}{D_n}. \end{aligned}$$

The identities extend continuously to $s_0=0$. On $\mathcal{D}_+$, $|p_0-2s_0/3|\le 1/2$ and $0\le c_n\le 1$. Consequently

$$D_n\ge \tfrac12+\tfrac23 ns_0,\qquad 0\le s_n\le\frac{3}{2n}\quad(n\ge 1),$$

and the second term in p_n is bounded in absolute value by 2^{-n}. Both p_n and $q_n=s_n-p_n$ tend uniformly to zero.

By (3.3), every compatible chain lies in $R=[-1/2,1/2]^2$. This rectangle is invariant. On it the absolute Jacobian is bounded by

$$\begin{pmatrix} 1/2 & 1 \\ 1/2 & 0 \end{pmatrix}.$$

Both maps are therefore nonexpanding in $\|(x,y)\|_*=\max(|x|,2|y|)$. They strictly decrease the distance between distinct points. Indeed, on the part $u<1/2$ the derivative norm is strictly smaller than one. Any joining segment with a point in that part has a subsegment of positive length there, so integration of the derivative gives a strict inequality. The only remaining segments lie on $u=1/2$; an input difference $(0,t)$ then has output difference $(t,0)$, decreasing this norm by a factor of two.

For each $\varepsilon>0$, compactness of the pairs in R at distance at least ε gives a number $\rho_\varepsilon<1$ bounding both distance ratios there. If the images of a pair under n successive maps have final distance at least ε, all preceding distances are at least ε by nonexpansion. The final distance is then at most $\rho_\varepsilon^n\operatorname{diam}_*(R)$, a contradiction for large n. All composition diameters shrink to zero, proving uniqueness of compatible chains.

4.4. **The alphabets $\{1, 2m+1\}$, $m \geq 3$.** Now $L = m+1$ and $B_* = m$. A minus map followed by any two maps sends $\mathcal{D}$ into the region

$$|u| \leq \frac{m}{2m+1}, \qquad |v| \leq \frac{1}{m+1}. \tag{4.2}$$

For verification, every single image satisfies $|u_1| \leq m/(m+1)$. After the minus map $v_1 < 0$, so the next first coordinate is negative and the next second coordinate has absolute value at most $(m+1)/(2m+1)$. Applying the third map gives (4.2). The absolute values in (4.2) belong to $\mathcal{D}_+$ with slack $1 - u - v \geq 1/4$.

The positive map is the projective action of

$$P = \begin{pmatrix} 0 & m & 0 \\ 0 & 0 & 1 \\ -m & 0 & m+1 \end{pmatrix}.$$

Its characteristic polynomial is $(t-m)(t^2 - t - m)$. Its other eigenvalues $\lambda_\pm = (1 \pm \sqrt{1+4m})/2$ have modulus less than m. Right and left eigenvectors at m are

$$r = (1/m, 1/m, 1)^{\mathsf{T}}, \qquad l = (-1, -1, 1), \qquad lr = 1 - 2/m > 0.$$

For $z = (u, v) \in \mathcal{D}_+$ of slack at least $1/4$, the coefficient of the leading projection of $(u, v, 1)^{\mathsf{T}}$ is $(1 - u - v)/(1 - 2/m)$, bounded away from zero. Since the three eigenvalues are distinct, diagonalization proves, uniformly on this compact region,

$$H_+^n(z) \longrightarrow (1/m, 1/m). \tag{4.3}$$

Suppose that τ contains infinitely many zeros. In a compatible chain, take a minus map arbitrarily far beyond a fixed depth j. That map and its next two outer maps give (4.2). Apply (3.3) for all remaining outer maps down to depth j. As the chosen minus recedes, (4.3) gives $|u_j|, |v_j| \leq 1/m$. Thus the whole chain belongs to the invariant rectangle $R_m = [-1/m, 1/m]^2$. There the absolute Jacobian is bounded by

$$\begin{pmatrix} 1/m & 1 \\ 1/m & 0 \end{pmatrix}.$$

Its positive eigenvalue is less than one, since its characteristic polynomial has value $1 - 2/m > 0$ at one. The weighted-norm argument again proves uniqueness of compatible chains.

5. From letter frequencies to all cylinders

Individual expansion tiles need not grow: for example, $\varphi_0^n(\mathsf{A}) = \mathsf{A}$ when $\alpha = 1$. What is needed is a bound on the *density* of boundaries in long expanded intervals.

Lemma 5.1. *There is $c > 1$, depending only on the alphabet, such that for every fixed depth k, the upper density of depth-k tile boundaries in long intervals of any smooth recoded word is at most c^{-k}. One can take $c = 3$ when $\alpha \geq 3$, and*

$$c = 1 + \frac{\beta - 1}{2(\beta + 1)}$$

when $\alpha = 1$. The estimate is uniform in the word and its directive.

Proof. If $\alpha \geq 3$, every one-level expansion multiplies length by at least three. If $\alpha = 1$, let W be an interval of n letters in a smooth recoded word. Its length list is a factor of an ordinary smooth word. Runs of 1 in that list have length at most β. Partitioning into blocks of length $\beta + 1$ shows that the list contains at least $\lfloor n/(\beta+1) \rfloor$ entries equal to β. Hence, for either expansion,

$$|\varphi_i(W)| \geq n + (\beta - 1) \left\lfloor \frac{n}{\beta + 1} \right\rfloor \geq cn \qquad (n \geq 2(\beta + 1)).$$

Every intermediate expansion is again a factor of a smooth recoded word, and its length is at least n. Thus a k-fold expansion of such a word has length at least $c^k n$.

For an interval of length N at the bottom, remove its two partial depth-k tiles. They have combined length at most $2\beta^k$. The remaining complete tiles come from an interval of n upper letters. For fixed k, $n \to \infty$ with N, because tile lengths are at most β^k. The length estimate gives $N \geq c^k n$ once n is sufficiently large. There are at most $n+1$ relevant boundaries, proving the claim. □

Proposition 5.2. *If compatible frequency chains for a directive are unique, Y_τ is uniquely ergodic, and every finite-word frequency converges uniformly. The same conclusions hold for X_τ.*

Proof. Fix a recoded word w of length r and an expansion depth k. For $a \in \mathcal{A}$, let $T_k(a)$ be its depth-k tile and let $F_k(a)$ be the number of occurrences of w wholly contained in this tile. In a long bottom interval, complete tiles are indexed by a long interval of the upper row. By Lemma 3.2, the letter frequencies in that row converge uniformly to a common vector $\nu^{(k)}$. Therefore the number of internal occurrences divided by the total length of complete tiles tends uniformly to

$$a_k(w) = \frac{\sum_a \nu^{(k)}(a) F_k(a)}{\sum_a \nu^{(k)}(a) |T_k(a)|}.$$

Endpoint errors vanish for fixed k. Each tile boundary is crossed by at most $r-1$ occurrences. Lemma 5.1 therefore puts all limit points of the full frequency in $[a_k(w), a_k(w) + (r-1)c^{-k}]$. These nested requirements force a single common limit as $k \to \infty$. A contrary sequence of intervals would contradict the same estimates, so convergence is uniform.

A compact subshift has an invariant probability measure: take a weak limit of orbit averages, whose invariance follows by telescoping. For any invariant measure, integrating the uniform averages of the cylinder indicator $\mathbf{1}_{[w]}$ makes its cylinder probability equal to the common limit. Cylinder sets determine Borel probabilities, so this measure is unique. Lemma 2.2 gives unique ergodicity of X_τ. Uniform word frequencies on X_τ also follow either by the same bounded-run count or by the usual compactness argument: a sequence of nonconvergent interval averages would have a weak limit different from the unique invariant measure. □

The preceding sections prove every positive unique-ergodicity assertion of Theorem 1.1. We now analyze the remaining directives.

6. The two measures for an eventually one directive

Throughout this section $\alpha = 1$, $\beta = 2m+1$ and $m \geq 3$. First let $\tau = 1^\infty$. The incidence matrix of φ_1, whose columns count the images of $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$, is

$$M = \begin{pmatrix} 0 & 0 & m & 0 \\ 1 & 0 & m+1 & 0 \\ 0 & 1 & 0 & m+1 \\ 0 & 0 & 0 & m \end{pmatrix}. \tag{6.1}$$

Its upper-left block Q has characteristic polynomial $(t+1)(t^2 - t - m)$. If $\lambda = (1 + \sqrt{1+4m})/2$, positive right and left eigenvectors for Q at λ are

$$(\lambda - 1, \lambda, 1)^{\mathsf{T}}, \qquad (1, \lambda, \lambda^2).$$

Its other eigenvalues, -1 and $1-\lambda$, have smaller modulus. Consequently the normalized letter counts of $\varphi_1^n(\mathsf{A}), \varphi_1^n(\mathsf{B}), \varphi_1^n(\mathsf{C})$ converge to the first vector $v^{(0)}$ in (1.1). This follows directly by diagonalization, with positive leading coefficient for each of the three coordinate seeds.

For M, the further eigenvalue m is simple and larger in modulus than the other three eigenvalues. Its right eigenvector $(m, m+2, m, m-2)^{\mathsf{T}}$ is positive. The D count in $\varphi_1^n(\mathsf{D})$ is exactly m^n, so its leading coefficient is nonzero. Normalized counts in these tiles converge to the second vector $v^{(1)}$ of (1.1). All four tile lengths tend to infinity.

Lemma 6.1. *For each finite recoded word w, its frequency inside $\varphi_1^n(a)$ converges as $n \to \infty$. The limit is the same for $a = \mathsf{A}, \mathsf{B}, \mathsf{C}$; denote it by $f_0(w)$. Denote the limit for $a = \mathsf{D}$ by $f_1(w)$. There are invariant probabilities ν_0, ν_1 on $Y_{1\infty}$ having these cylinder probabilities.*

Proof. Every letter occurs in some word of Y. Indeed, the derivative of a smooth ordinary word contains both lengths, since it cannot be constant. Thus each of the two lengths occurs in its recoding, in at least one color; complementing the colors supplies the other color. For a given seed a, choose a smooth row containing it. Expanding that row n times produces a smooth row with n initial type ones and containing $\varphi_1^n(a)$.

Fix an inner depth k. Decompose $\varphi_1^n(a)$ into the depth-k tiles indexed by $\varphi_1^{n-k}(a)$. The normalized upper counts converge to $v^{(0)}$ for the first three seeds and to $v^{(1)}$ for the fourth. The argument of Proposition 5.2, using Lemma 5.1, puts the lower and upper limiting frequencies within an interval of length $(|w| - 1)c^{-k}$ determined only by the corresponding vector and the four inner tiles. Letting $k \to \infty$ proves existence of each limit, and its independence of the first three seeds.

Take empirical orbit measures on the tile intervals in the smooth rows just constructed. Their lengths tend to infinity, so every weak limit is invariant. For every fixed k, all sufficiently late measures are supported on the clopen set with first k types equal to one. Thus the limits are supported on $Y_{1\infty}$. Their cylinder probabilities are the limits just proved, giving ν_0 and ν_1. Their D probabilities are 0 and $(m-2)/(4m) > 0$, respectively, so they are distinct. □

Proposition 6.2. *Every invariant probability on $Y_{1\infty}$ is a convex combination of ν_0 and ν_1. These two measures are exactly its ergodic measures.*

Proof. Fix a finite family of cylinder words. For depth d, form the vector of their internal frequencies in each of the four tiles $\varphi_1^d(a)$. By Lemma 6.1, these four vectors tend to two vectors: three tend to the cylinder vector of ν_0, and one tends to that of ν_1. Let $L_d = \min_a |\varphi_1^d(a)|$, which tends to infinity.

In any long interval of any row of $Y_{1\infty}$, the internal-frequency vector from complete depth-d tiles is a convex combination of those four tile vectors, weighted by their lengths. Endpoint errors vanish as the interval grows with d fixed. For a word of length r, the remaining crossing error is at most $(r-1)/L_d$ in the limit. Integrate these interval averages against any invariant probability μ, and then let $d \to \infty$. Its cylinder vector belongs to the segment joining those of ν_0 and ν_1.

Include the one-letter cylinder $[\mathsf{D}]$ in every finite family. Its probability determines the same mixture coefficient in every case, namely $t = \mu([\mathsf{D}])/\nu_1([\mathsf{D}])$. Hence $\mu([w]) = (1-t)\nu_0([w]) + t\nu_1([w])$ for every word w. Cylinders determine the measure, so $\mu = (1-t)\nu_0 + t\nu_1$. The two endpoints are the only extreme invariant probabilities. The standard equivalence of ergodicity and extremality gives the last assertion. For completeness, a nontrivial invariant set splits a measure into two invariant conditional probabilities; conversely, a nontrivial convex splitting of an ergodic measure would give a nonconstant invariant Radon–Nikodym derivative. □

The ordinary means can also be computed exactly. Let

$$\ell = (1, 1, 2m+1, 2m+1), \qquad c_0 = (1, 2m+1, 1, 2m+1)$$

be the length and color row vectors. The sum of the ordinary symbols encoded by a letter is ℓc_0 entrywise. Directly from the substitutions,

$$\ell M = \big(\ell(a)c_0(a)\big)_{a \in \mathcal{A}}.$$

For an eigenvector frequency v with eigenvalue θ, the ordinary mean under its tower measure is therefore

$$\frac{\sum_a \ell(a)c_0(a)v(a)}{\sum_a \ell(a)v(a)} = \frac{\ell M v}{\ell v} = \theta.$$

This gives λ and m for the two measures.

Finally suppose τ consists of a finite prefix followed by 1^∞. Apply Lemma 2.2 once for each prefix symbol. Each recognized expansion preserves the number of ergodic measures, so Y_τ has exactly

two. These also have distinct recoded frequency pairs. Indeed, H_σ is injective on $\mathcal{D}$: its nonzero second coordinate determines u from $v' = \sigma/(L - B_* u)$, and then its first coordinate determines v. Thus a finite composition cannot identify the two vectors. One more application of Lemma 2.2 gives exactly two ergodic measures on X_τ. Every directive with only finitely many zeros is eventually one, so the classification is complete. □

References

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Email address: howew@purdue.edu

Email address: huffma97@purdue.edu

Email address: jenwin@purdue.edu

Email address: kruerl@purdue.edu