BOREL RUZIEWICZ PROBLEM

Abstract. We prove that normalized surface measure is the unique finitely additive rotation-invariant probability on the Borel subsets of S^n, $n \geq 2$. We also prove that Lebesgue measure is the unique finitely additive measure on bounded Borel subsets of $\mathbb{R}^d$, $d \geq 3$, invariant under orientation-preserving rigid motions and normalized on the unit cube. The main step is the annihilation of Borel null sets. A spectral gap first gives bounded divergence certificates for compact null sets, chosen in a jointly Borel way. A shared random threshold then combines these certificates to cover arbitrary null sets. For Euclidean space, the construction is applied to families of spheres about two centers; spherical averaging completes the identification of the measure.

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1. Introduction

Let λ_n denote normalized surface measure on S^n. The classical Ruziewicz problem asks whether finite additivity, nonnegativity, normalization, and rotation invariance characterize λ_n on the Lebesgue σ-algebra. Margulis [7] and Sullivan [11] proved this for $n \geq 4$, and Drinfeld [3] settled the cases $n = 2, 3$.

The corresponding question on Borel sets has an additional difficulty. A finitely additive probability defined on the Borel σ-algebra need not be given on all subsets of null sets, and an invariant extension to the Lebesgue completion is not among the hypotheses. Nor can one immediately regard it as a mean on $L^\infty(\lambda_n)$: that passage requires the measure to vanish on Borel null sets. Establishing this property is the principal task of the present paper.

We use the term *charge* for a nonnegative finitely additive function on a Borel algebra. Charges on compact spaces are normalized to have total mass one unless stated otherwise. Our first result is the following.

Theorem A. *For every $n \geq 2$, normalized surface measure is the unique $\mathrm{SO}(n+1)$-invariant probability charge on $\mathcal{B}(S^n)$. The same conclusion holds under invariance by any prescribed finite symmetric set $S \subseteq \mathrm{SO}(n+1)$ with a spectral gap on $L^2_0(S^n, \lambda_n)$.*

Here a spectral gap means that, for some $\kappa > 0$,

$$\max_{s \in S} \|u \circ s^{-1} - u\|_2 \geq \kappa \|u - \lambda_n(u)\|_2 \qquad (u \in L^2(\lambda_n)). \tag{1.1}$$

Finite sets of rotations with this property exist in every dimension $n \geq 2$ by the classical work cited above. To make the argument more self-contained, Appendix A gives an elementary construction using quaternion counting, multiplicity estimates, and averaging about two axes. This belongs to the established circle of arithmetic approaches to spectral gaps; see Gamburd–Jakobson–Sarnak [4] and Bourgain–Gamburd [1].

The Euclidean result likewise requires no reflections. Write $\mathcal{B}_b(\mathbb{R}^d)$ for the algebra of bounded Borel sets and $\mathcal{L}^d$ for Lebesgue measure.

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Theorem B. *Let $d \geq 3$. If $\mu : \mathcal{B}_b(\mathbb{R}^d) \to [0, \infty]$ is finitely additive, invariant under $x \mapsto gx + v$ for $g \in \mathrm{SO}(d)$ and $v \in \mathbb{R}^d$, and satisfies $\mu([0,1]^d) = 1$, then $\mu(A) = \mathcal{L}^d(A)$ for every bounded Borel set A.*

Margulis [8] proved the analogous uniqueness theorem on bounded Lebesgue measurable sets. We give a direct proof of the identification step once Borel null sets have been annihilated.

1.1. **Context.** Conley, Jackson, Marks, Seward, and Tucker-Drob [2] explicitly state the Borel Ruziewicz question, referring to Question 11.13 in Wagon's book. They also construct a continuous action on another Polish space which admits an invariant Borel probability charge but no invariant countably additive probability. Thus the spherical theorem requires a property beyond the mere absence of countably additive invariant probabilities on a null part of the action.

Grabowski, Máthé, and Pikhurko [5] study the Borel question in connection with measurable equidecompositions. Their uniqueness theorem on sets which are both Lebesgue measurable and have the Baire property, and the further proof by Mycielski and Tomkowicz [9], concern a larger algebra than the Borel sets. The existence of an invariant extension to that algebra cannot be inferred from those uniqueness statements. Unger's lectures [12] describe the Borel problem and approaches through bounded flows.

The classical functional-analytic part of the argument is the implication from spectral gap to uniqueness of an invariant mean on L^∞; see Rosenblatt [10]. Our proof uses that principle only after null-set annihilation. All assertions about divergence before that step hold at every point, rather than merely almost everywhere.

1.2. **Outline.** Fix finitely many rotations with a spectral gap. Smoothing a singular countably additive probability shows that some generator moves it by a fixed amount. A separation argument then produces a bounded Borel field whose divergence is almost one on any given compact null set. The bound does not depend on that set. A countable search over finite formulas makes the fields jointly Borel in a compact parameter.

For a general null set, choose closed covers with rapidly decreasing measure. The word distance to the complement of a finite thickening of each cover changes by at most one along a generator. Comparing this distance with a shared random threshold selects a compact null set containing the base point. Neighboring points usually select the same compact set. Averaging the corresponding fields yields positive divergence on a Borel set containing the original null set, and nonnegative divergence everywhere else. Integration against an invariant charge forces the set to have charge zero.

In Euclidean space, polar coordinates first provide suitable covers outside an exceptional null set of radii. A second center converts those exceptional radii into null spherical sections. The parameter version of the same construction therefore annihilates every bounded null set. Finally, three spherical averagings turn a bounded Borel function of bounded support into a continuous compactly supported function, where ordinary Haar uniqueness applies.

1.3. **Notation.** For a finite symmetric family S of homeomorphisms, write $\Gamma = \langle S \rangle$ and $\Gamma_r = \{g : |g|_S \leq r\}$. The family may have repeated elements; its indices are paired by an involution representing inversion. A field is a family of bounded Borel functions F_s, with divergence

$$D_S F(x) = \sum_{s \in S} \bigl(F_s(x) - F_s(s^{-1}x)\bigr).$$

If μ is invariant, then $\mu(D_S F) = 0$. In particular a field satisfying $D_S F \geq a\mathbf{1}_N$ everywhere, with $a > 0$, certifies $\mu(N) = 0$. We write $B_b(X)$ for bounded real Borel functions; all its norms are pointwise supremum norms. Total variation is normalized so that mutually singular probabilities have distance two. We assume standard measure theory and functional analysis; our descriptive-set-theoretic conventions follow [6].

2. A gap for singular probabilities

Let X be compact metric, m a fully supported Borel probability, and S a finite nonempty symmetric family of m-preserving isometries. Suppose (1.1) holds with m in place of λ_n.

Proposition 2.1. *Set $c = \kappa^2/2$. For every countably additive Borel probability $\nu \perp m$ there exist $s \in S$ and $g \in C(X, \mathbb{R})$ such that*

$$\|g\|_\infty \le 1, \qquad \int (g \circ s - g)\, d\nu \ge c. \tag{2.1}$$

Proof. Let $a_r(x,y) = \max(0, 1 - d(x,y)/r)$ and normalize it in the second coordinate:

$$p_r(x,y) = \frac{a_r(x,y)}{\int a_r(x,z)\, dm(z)}, \qquad h_r(y) = \int p_r(x,y)\, d\nu(x).$$

The denominator is positive and bounded away from zero for each $r > 0$. The resulting smoothing preserves mass, commutes with the generators, and has continuous densities. Its adjoint sends a bounded Borel function of norm at most one to a continuous function of norm at most one.

Suppose (2.1) fails. Symmetry of the test class then gives $\|s_*(h_r m) - h_r m\|_{\mathrm{TV}} \le c$ for every s. For $u_r = \sqrt{h_r}$,

$$\|U_s u_r - u_r\|_2^2 \le \|h_r \circ s^{-1} - h_r\|_1 \le c, \qquad \|u_r\|_2 = 1.$$

The spectral gap implies $m(u_r)^2 \ge 1/2$.

On the other hand, singularity forces $m(u_r) \to 0$. To see this, choose a compact null set K with $\nu(K) > 1 - \eta$, and an open $U \supseteq K$ with $m(U) < \eta$. For sufficiently small r, the smoothing of $\nu|_K$ is carried by U, so $\int_{X \setminus U} h_r\, dm < \eta$. Cauchy–Schwarz on U and its complement gives $m(\sqrt{h_r}) \le 2\sqrt{\eta}$. This contradicts the preceding lower bound. □

In Sections 3–4 we only use the conclusion (2.1), with a fixed $c > 0$. The generators may there be homeomorphisms preserving m; they need not be isometries.

3. Compact certificates and Borel choice

For a compact set $K \subseteq X$, let $\mathcal{E}(K)$ consist of finite sums

$$\sum_{j=1}^{a} f_j \prod_{g \in W_j} \mathbf{1}_{gK}, \qquad f_j \in C(X, \mathbb{R}), \tag{3.1}$$

where W_j is a finite list of group words. Empty products are allowed, so $C(X) \subseteq \mathcal{E}(K)$. No support assertion outside ΓK is imposed on this class.

Lemma 3.1. *If $m(K) = 0$, then for every $\varepsilon > 0$ there is a field $q \in \mathcal{E}(K)^S$ such that*

$$\max_s \|q_s\|_\infty < 2/c, \qquad D_S q \ge \mathbf{1}_K - \varepsilon \quad \textit{everywhere.}$$

Proof. Work in the Banach space of all bounded functions on X. Let W be the downward closure of the divergences of fields in $\mathcal{E}(K)^S$ with component norms less than $2/c$. This set is convex. If the conclusion fails, W is disjoint from the open ε-ball about $\mathbf{1}_K$. Separation gives a bounded linear functional Φ and $a \ge 0$ such that

$$\Phi(D_S q) \le a < \Phi(\mathbf{1}_K) \qquad (\max_s \|q_s\|_\infty < 2/c). \tag{3.2}$$

The functional is positive, since W contains every negative ray.

Put $\Lambda_j = \Gamma_j K$ and $A_j = \Phi(\mathbf{1}_{\Lambda_j})$. Then $0 < A_0 \le A_j \le \Phi(1)$. For each j, Riesz representation applied to $f \mapsto \Phi(f \mathbf{1}_{\Lambda_j})$ gives a countably additive measure of mass A_j, carried by the closed null set Λ_j. After normalization, (2.1) supplies s and g, $\|g\|_\infty \le 1$, with

$$cA_j \le \Phi((g \circ s)\mathbf{1}_{\Lambda_j}) - \Phi(g \mathbf{1}_{\Lambda_j}). \tag{3.3}$$

The support statement follows directly from the cutdown: a continuous function vanishing on Λ_j has integral zero.

Use (3.2) with the single nonzero component $-tg\mathbf{1}_{s\Lambda_j}$ at index s^{-1}, where $t = 3/(2c) < 2/c$. It gives

$$t(\Phi((g \circ s)\mathbf{1}_{\Lambda_j}) - \Phi(g\mathbf{1}_{s\Lambda_j})) \le a.$$

For $j = n+1$, both Λ_j and $s\Lambda_j$ contain Λ_n and lie in Λ_{n+2}. Positivity therefore bounds the change between their cutdowns by $A_{n+2} - A_n$. Combining with (3.3) yields

$$\frac{3}{2c}(cA_{n+1} - (A_{n+2} - A_n)) \le a < A_0 \le A_{n+1}.$$

Thus $A_{n+2} - A_n > cA_0/3$. Iterating on the even indices contradicts $A_j \le \Phi(1)$. □

Proposition 3.2. *Let P be compact metrizable and $\mathcal{K} \subseteq P \times X$ closed. For every $\varepsilon > 0$ there is a jointly Borel family $F_s^p(x)$ such that*

$$0 \le F_s^p(x) \le 4/c \quad (p \in P, s \in S, x \in X), \qquad D_S F^p(x) \ge 1 - \varepsilon \quad (x \in \mathcal{K}_p, m(\mathcal{K}_p) = 0).$$

Proof. Fix a countable dense sequence in $C(X, \mathbb{R})$ and enumerate fields of the form (3.1) with coefficients in that sequence. A code is valid at p when its component norms are at most $2/c$ and its divergence is at least $1 - \varepsilon$ on $\mathcal{K}_p$.

For a fixed code, these conditions are Borel in p. Indeed, the translated membership sets in $P \times X$, including the pullbacks appearing in divergence, are closed. Their finite Boolean combinations partition the product into σ-compact pieces on which the code evaluations and their divergences are continuous. The violation set for either a component bound or the divergence inequality is therefore σ-compact. Its projection to P is σ-compact, hence Borel. The component bounds are tested on all of $P \times X$, and the divergence condition only on $\mathcal{K}$.

Lemma 3.1, used with $\varepsilon/2$, followed by sufficiently close approximation of its finitely many continuous coefficients, gives a valid code whenever the fiber is null. Choose the least valid code, and choose zero when none exists. This is a jointly Borel field q^p, bounded by $2/c$ in each component. Reorient it by

$$F_s^p = (q_s^p)^+ + (q_{s^{-1}}^p)^- \circ s.$$

This preserves divergence and gives the asserted nonnegative bound. □

4. The random-threshold kernel

Lemma 4.1. *Let ν be a finite Borel measure on a compact metric space. If $\nu^*(N) = 0$ and $\eta_r > 0$, there are closed sets C_r with $\nu(C_r) \le \eta_r$ and $N \subseteq \limsup_r C_r$.*

Proof. Choose open $O_j \supseteq N$ with $\nu(O_j) < 2^{-j}$. Closed shells defined by the intervals $[1, \infty)$ and $[1/(i+1), 1/i]$ in the distance to the complement cover O_j with multiplicity at most two. If O_j is the whole space use it as a single closed set. Enumerating all shells gives a sequence (D_a) of summable measures whose limsup contains N. Group it into successive finite blocks, starting each block far enough out that the remaining measure sum is at most η_r. □

Proposition 4.2. *Let $C_r \subseteq X$ be closed, and let L_r be positive integers such that $r \le L_r$ and*

$$\sum_{r \ge 0} L_r^{-1} \le \frac{c}{16|S|}.$$

If $m(\Gamma_{2L_r} C_r) \to 0$, then $\limsup_r C_r$ is contained in a Borel set Z carrying a bounded Borel field H with $D_S H \ge \frac{1}{2}\mathbf{1}_Z$ everywhere.

Proof. Put $E_r = \Gamma_{2L_r} C_r$ and let

$$t_r(x) = \min(2L_r, \min\{|g| : gx \notin E_r\}),$$

with $\min \emptyset = \infty$. Its superlevel sets are closed, $|t_r(sx) - t_r(x)| \le 1$, and $t_r = 2L_r$ on C_r. Define

$$Z = \{x : \forall n\ \exists r \ge n, t_r(x) \ge L_r + n\}.$$

This set is Borel and invariant. It contains $\limsup C_r$, since $L_r \ge r$.

For $\omega \in [0,1)$ set

$$\theta_r(\omega) = \lfloor L_r \omega \rfloor, \qquad b_r(x,\omega) = \mathbf{1}\{\theta_r(\omega) < t_r(x)\}.$$

Use the compact parameter space $P = \{0,1\}^{\mathbb{N}}$ and the closed family

$$\mathcal{K} = \{(b,y) : b_r = 1 \Rightarrow y \in E_r \text{ for every } r\}.$$

For every x, its selected fiber contains x. If $x \in Z$, then $b_r(x,\omega) = 1$ for infinitely many r, so $m(\mathcal{K}_{b(x,\omega)}) = 0$ by $m(E_r) \to 0$.

Apply Proposition 3.2 with $\varepsilon = 1/4$. Define

$$H_s(x) = \int_0^1 F_s^{b(x,\omega)}(x)\, d\omega \quad (x \in Z), \qquad H_s(x) = 0 \quad (x \notin Z).$$

All parameter maps are Borel, so these functions are Borel and bounded. Fix $x \in Z$. Using the same ω at x and $s^{-1}x$, the integrand of its divergence is

$$D_S F^{b(x,\omega)}(x) + \sum_s \big(F_s^{b(x,\omega)}(s^{-1}x) - F_s^{b(s^{-1}x,\omega)}(s^{-1}x)\big).$$

The first term is at least $3/4$. A summand vanishes when the parameters agree and is otherwise at least $-4/c$. At coordinate r the parameters can differ only when θ_r equals the smaller of the two neighboring distances, an event of measure at most L_r^{-1}. The union bound gives

$$D_S H(x) \ge \frac{3}{4} - \frac{4|S|}{c} \sum_r L_r^{-1} \ge \frac{1}{2}.$$

Invariance of Z makes the divergence zero outside Z. No integration against a finitely additive charge has been used in this construction. □

θ_r	0	1	2	3
$t_r = 2$	1	1	0	0
$t_r = 3$	1	1	1	0

FIGURE 1. The selected bit $\mathbf{1}\{\theta_r < t_r\}$ for $L_r = 4$. Neighboring distances disagree for only one threshold. The same random number ω is used in every coordinate and at every base point. Independence between coordinates is not required.

Theorem 4.3. *Every invariant probability charge on X annihilates every m-null Borel set.*

Proof. Choose $L_r = N_0 2^r$ with N_0 large enough for Proposition 4.2. Since $|\Gamma_a| \le (|S|+1)^a$, Lemma 4.1 provides C_r whose limsup contains the given null set and such that $m(\Gamma_{2L_r} C_r) \le 1/(r+1)$. The kernel proposition gives $D_S H \ge \mathbf{1}_Z/2$. Invariance and positivity imply $0 = \mu(D_S H) \ge \mu(Z)/2$. □

5. Invariant means and the spherical theorem

We record the parameter version of the classical uniqueness argument, since it will be used again for Euclidean space. Let (X, m) be compact metric with a probability-preserving finite nonempty symmetric family S satisfying (1.1). Let T be compact metric and α a finite Borel measure on T. The lifted action fixes T.

Lemma 5.1. *Let φ be an invariant mean on bounded Borel functions on $T \times X$ which annihilates $(\alpha \otimes m)$-null Borel sets. For every bounded Borel f,*

$$\varphi(f) = \varphi(Ef), \qquad Ef(t,x) = \int_X f(t,y)\, dm(y).$$

Proof. Normalize α, the zero-measure case being incompatible with $\varphi(1) = 1$. Null-set annihilation lets us work in L^∞. Probability densities are weak-$*$ dense in its positive dual unit sphere. For a fixed f, separation and the equality of weak and norm closures of convex sets in $(L^1)^S \times \mathbb{R}$ give densities w_j with

$$\sum_s \|U_s w_j - w_j\|_1 \to 0, \qquad \int (f - Ef) w_j \to \varphi(f - Ef).$$

Set $u_j = \sqrt{w_j}$ and $a_j = Eu_j$. Applying the fiberwise gap in sum form and integrating in t gives

$$\kappa^2 \|u_j - a_j\|_2^2 \le \sum_s \|U_s u_j - u_j\|_2^2 \le \sum_s \|U_s w_j - w_j\|_1 \to 0.$$

Since $\|u_j\|_2 = 1$ and $\|a_j\|_2 \le 1$, $\|w_j - Ew_j\|_1 \le 2\|w_j - a_j^2\|_1 \le 4\|u_j - a_j\|_2$. Self-adjointness of E now gives $\int (f - Ef) w_j = \int f(w_j - Ew_j) \to 0$. □

Proof of Theorem A. For a prescribed finite spectral-gap family, Proposition 2.1 and Theorem 4.3 annihilate Borel null sets. The charge therefore descends to an invariant mean on $L^\infty(\lambda_n)$. Lemma 5.1, with a one-point parameter space, identifies it with the spherical integral. Existence of a spectral-gap family, established in Appendix A, gives the assertion for all rotations. □

6. Euclidean space

6.1. **Rapid fiberwise covers.** Keep X, m, S as in the preceding section, and assume in addition the continuous-test singular gap (2.1) for a fixed $c > 0$. For compact metric T and closed $C_r \subseteq T \times X$, write $C_r(t) = \{x : (t,x) \in C_r\}$. Say these sets give a *rapid fiberwise cover* of $N \subseteq T \times X$ if

$$N \subseteq \limsup_r C_r, \qquad A^{2^r} m(C_r(t)) \longrightarrow 0 \quad \text{for every } (t,x) \in N \text{ and every integer } A \ge 1. \tag{6.1}$$

Proposition 6.1. *Every invariant probability charge for the lifted action on $T \times X$ annihilates a Borel set admitting a rapid fiberwise cover.*

Proof. Choose $L_r = N_0 2^r$ as before, and let G be the Borel set of parameters for which the decay condition in (6.1) holds. The lifted thickenings satisfy

$$m((\Gamma_{2L_r} C_r)(t)) \le (|S| + 1)^{2L_r} m(C_r(t)) \longrightarrow 0 \qquad (t \in G).$$

Use the construction of Proposition 4.2 with compact parameter space $\{0,1\}^{\mathbb{N}} \times T$. Its compact fibers are

$$\bigcap_{b_r = 1} (\Gamma_{2L_r} C_r)(t),$$

and their full membership family is closed. For kernel points over G, the selected codes have infinitely many active coordinates, so their compact fibers are null. The same countable disagreement estimate gives a bounded Borel certificate on the invariant part of the kernel set over G; extend by zero elsewhere. It contains N, and integrating the certificate proves the assertion. □

Lemma 6.2. *Let α be a finite Borel measure on compact metric T. Every $(\alpha \otimes m)$-null set $N \subseteq T \times X$ has a closed limsup cover (C_r) satisfying (6.1) over an α-conull Borel set of parameters.*

Proof. By Lemma 4.1, choose C_r with $(\alpha \otimes m)(C_r) \le 2^{-r}(r+1)^{-2^r}$. Tonelli shows that

$$\sum_r (r+1)^{2^r} m(C_r(t)) < \infty$$

outside an α-null Borel set. Its summands tend to zero, and each fixed A is eventually at most $r+1$. □

6.2. **Two centers.** Fix $d \ge 3$, put $X = S^{d-1}$, and choose spectral-gap rotations $S \subseteq \mathrm{SO}(d)$. For a center p and $R > 0$, use the polar map $\pi_p : [0,R] \times S^{d-1} \to \mathbb{R}^d$, $\pi_p(t,\xi) = p + t\xi$. We say a set has a rapid cover about p if its polar preimage has a cover as in (6.1).

Proposition 6.3. *Let $p \ne q$ and let $N \subseteq \bar{B}(p,R) \cap \bar{B}(q,R)$ be Borel and Lebesgue-null, with $p, q \notin N$. Then $N = N_1 \cup N_2$ for Borel sets N_i having rapid covers about p and q, respectively.*

Proof. Polar coordinates make $\pi_p^{-1}(N)$ null for the product of Lebesgue measure on $[0,R]$ and spherical probability. Apply Lemma 6.2, and denote its bad radii by $B \subseteq [0,R]$. Put

$$N_2 = N \cap \{|x-p| \in B\}, \qquad N_1 = N \setminus N_2.$$

The first set of good fibers gives the required cover of N_1.

The set $B^{(2)} = \{t^2 : t \in B\}$ is null. Choose closed $J_r \subseteq [0,R^2]$ with $B^{(2)} \subseteq \limsup J_r$ and $|J_r| \le 2^{-r}(r+1)^{-2^r}$. The closed sets

$$C_r' = \{(t,\xi) : |q + t\xi - p|^2 \in J_r\}$$

cover the polar preimage of N_2 infinitely often. Let $\ell = |q-p|$ and $e = (q-p)/\ell$. For $t > 0$,

$$|q + t\xi - p|^2 = t^2 + \ell^2 + 2t\ell\langle \xi, e\rangle.$$

The coordinate $\langle \xi, e\rangle$ on S^{d-1} has density proportional to $(1-u^2)^{(d-3)/2}$ on $(-1,1)$, bounded by a constant M_d. Consequently

$$\lambda_{d-1}(C_r'(t)) \le \frac{M_d}{2t\ell}|J_r|.$$

For each fixed positive t this implies the required rapid decay. The fiber $t = 0$ contains no point of the polar preimage of N_2, because $q \notin N$. □

Let μ satisfy the assumptions of Theorem B. It is finite on bounded sets, since finitely many translates of the unit cube cover any such set. Singletons have charge zero by placing arbitrarily many distinct translates inside a fixed cube. For a bounded null Borel N, remove two centers, choose R, and apply Proposition 6.3. The restriction of the polar map to $(0,R] \times S^{d-1}$ is a Borel bijection onto the punctured closed ball intertwining the lifted generators with rotations about the center. If the ball has positive charge, normalize its restriction and transport it to the polar product, extending by zero on the zero-radius fiber; if it has zero charge the conclusion is immediate. Proposition 6.1 gives

$$\mu(N) = 0 \qquad (N \text{ bounded, Borel, and Lebesgue-null}). \tag{6.2}$$

6.3. **Spherical averaging.** Let L denote integration against μ on bounded Borel functions of bounded support. For $p \in \mathbb{R}^d$ put

$$E_p f(x) = \int_{S^{d-1}} f(p + |x-p|\xi)\, d\lambda_{d-1}(\xi).$$

Spherical averaging preserves boundedness, Borel measurability, bounded support, and the Lebesgue integral.

Lemma 6.4. *For every bounded Borel function f of bounded support, $L(f) = L(E_p f)$.*

Proof. Work in a ball about p containing both supports. When its charge is positive, normalize the restriction and pass to polar coordinates. By (6.2), the resulting invariant mean annihilates null sets for the product of radial Lebesgue measure and spherical measure. Lemma 5.1 gives the identity. The center is harmless because its charge is zero. This argument does not interchange integration against the charge with integration over the sphere. □

Lemma 6.5. *If $p \neq q$ and f is bounded Borel with bounded support, then $E_pE_qE_pf$ is continuous and has compact support.*

Proof. Write $E_pf(x) = h(|x-p|^2)$ with h bounded Borel. For $\ell = |q-p|$, the second average at distance $t > 0$ from q is

$$b(t) = \int_{-1}^{1} h(t^2 + \ell^2 + 2t\ell u) g_d(u)\, du,$$

where g_d is the bounded coordinate density used above. Under the affine substitution $v = t^2+\ell^2+2t\ell u$, its density varies continuously in L^1 for $t > 0$. This follows by dominated convergence: in a neighborhood of a positive t, the transformed densities have uniformly bounded supports and bounds and converge pointwise away from the endpoints. Hence E_qE_pf is continuous off q.

If a bounded Borel F is continuous off q, then E_pF is continuous everywhere for $p \neq q$. For $x_j \to x$ the evaluation points $p + |x_j - p|\xi$ converge; the limit equals q for at most one ξ when $x \neq p$, and for no ξ when $x = p$. Spherical measure is atomless, so dominated convergence applies. Bounded support is preserved at each averaging; its closure is compact in finite-dimensional Euclidean space. □

Proof of Theorem B. On $C_c(\mathbb{R}^d)$, the positive translation-invariant functional L is $a\mathcal{L}^d$ for some $a \geq 0$, by Riesz representation and uniqueness of Haar measure. Given bounded Borel A, apply the preceding lemmas to $f = \mathbf{1}_A$ and $h = E_pE_qE_pf$, with $p \neq q$. Then

$$\mu(A) = L(f) = L(h) = a \int h = a\mathcal{L}^d(A).$$

The unit-cube normalization gives $a = 1$. □

Appendix A. An elementary spectral-gap construction

We give an elementary construction of the finite rotation families used in Theorem A. The proof starts on S^3, viewed as the group of unit quaternions, transfers to S^2, and then raises the dimension. The arithmetic and multiplicity arguments are in the spirit of [4, 1]. The existence theorem is classical; our purpose here is to supply the analytic input within the paper.

A.1. **Quaternion words.** Identify $G = S^3$ with the unit quaternions and let m be Haar probability. The units satisfy $i^2 = j^2 = k^2 = ijk = -1$, and $\bar{q}$ denotes quaternionic conjugation. Put

$$\mathcal{A} = \{1 \pm 2i, 1 \pm 2j, 1 \pm 2k\}, \qquad S = \{\alpha/\sqrt{5} : \alpha \in \mathcal{A}\}, \qquad Tf(x) = \tfrac{1}{6}\sum_{s \in S} f(s^{-1}x).$$

The quaternion norm is $N(a + bi + cj + dk) = a^2 + b^2 + c^2 + d^2$. Left multiplication by a unit quaternion is a rotation of $\mathbb{R}^4$. For $l \geq 1$, let $p_l(\eta)$ be the proportion of words of length $2l$ in S whose product is within Euclidean distance η of 1. Words here are counted with multiplicity, without reduction.

Lemma A.1 (Nonconcentration)**.** *There is a constant C such that, for $l \geq 1$ and $0 < \eta \leq 1$,*

$$p_l(\eta) \leq (3/4)^{2l} + C\eta^3 5^{l/20}. \tag{A.1}$$

Proof. We first record the elementary estimate

$$r_3(n) := |\{(a,b,c) \in \mathbb{Z}^3 : a^2 + b^2 + c^2 = n\}| \leq C_0 n^{1/2+1/40} \qquad (n \geq 1). \tag{A.2}$$

Write $r_2(t)$ for the number of representations $t = a^2 + b^2$ with $a, b \in \mathbb{Z}$. Then $r_3(n) \leq (2\sqrt{n} + 1) \max_{0 \leq t \leq n} r_2(t)$. For $t > 0$, primitive representations $a^2 + b^2 = t$ map to square roots of -1 modulo

t by ab^{-1}, with at most four representations per root. To see the fiber bound, if two primitive pairs give the same root, both $aa' + bb'$ and $ab' - a'b$ are divisible by t, while the sum of their squares is t^2; the pairs therefore differ by one of the four Gaussian units. The Chinese remainder theorem bounds the number of roots by $2^{\omega(t)} \le d(t)$, where $\omega(t)$ counts distinct prime factors and $d(t)$ counts positive divisors. Summing over common divisors gives $r_2(t) \le 4d(t)^2$; also $r_2(0) = 1$. Finally $d(t) \le C_\varepsilon t^\varepsilon$: in $d(t)/t^\varepsilon = \prod_p (e_p+1)p^{-\varepsilon e_p}$ the factors are at most one for all sufficiently large primes, and uniformly bounded for each of the finitely many remaining primes. Take $\varepsilon = 1/80$ to obtain (A.2).

For an integral quaternion β of norm 5^L, write $\beta = 5^v\gamma$ with γ not divisible by 5, and define its level to be $j = L - 2v$. Let $N_L(\beta)$ count length-L words in $\mathcal{A}$ with product β. The following elementary divisibility observation controls multiplicities: for primitive γ, at most one $\alpha \in \mathcal{A}$ has $5 \mid \gamma\bar\alpha$. Otherwise 5 divides $\gamma(\bar\alpha - \bar\alpha')$, and multiplication by its conjugate forces $5 \mid \gamma$, since $N(\bar\alpha - \bar\alpha') \in \{8, 16\}$. If $\gamma\bar\alpha = 5\gamma'$, the quaternion γ' is also primitive: multiplying by α would otherwise make γ divisible by 5.

Define nonnegative numbers $a_L(j)$ by

$$a_0(j) = \mathbf{1}_{\{0\}}(j), \qquad a_{L+1}(0) = 6a_L(1), \qquad a_{L+1}(j) = a_L(j-1) + 5a_L(j+1) \quad (j \ge 1).$$

They are the numbers of length-L walks to a fixed vertex at distance j in the 6-regular tree. Direct induction gives

$$\begin{aligned} &a_L(j+2) \le a_L(j), \qquad a_L(j)2^j \le (9/2)^L, \\ &a_L(0) + \sum_{j\ge1} 6\cdot 5^{j-1} a_L(j) = 6^L. \end{aligned} \tag{A.3}$$

For the middle bound, multiplying the recursion by 2^j gives coefficients 2 and $5/2$ away from zero, and coefficient 3 at zero. The final identity counts all walks by their endpoint.

Splitting a word by its last letter gives

$$N_{L+1}(\beta) = \sum_{\alpha:\, 5\mid\beta\bar\alpha} N_L(\beta\bar\alpha/5).$$

At positive level, there is at most one predecessor of level $j-1$; all others have level $j+1$. If there is none of level $j-1$, the first inequality in (A.3) bounds six higher-level terms by one lower-level term and five higher-level terms. At level zero all predecessors have level one. Thus

$$N_L(\beta) \le a_L(j). \tag{A.4}$$

Take $L = 2l$. At level zero, a product near 5^l has primitive part 1, so its contribution is at most $(9/2)^{2l}$. At level $2j > 0$, write $P = 5^j$ and $\operatorname{Re}\gamma = P - t$. Nearness to 1 implies

$$1 \le t < \eta^2 P/2, \qquad N(\operatorname{Im}\gamma) = t(2P - t) < \eta^2 P^2.$$

The lower bound $t \ge 1$ follows from primitivity, since P itself is divisible by 5. By (A.2), the number of possible primitive parts is at most

$$C_1\eta^2 P(\eta^2P^2)^{1/2+1/40} \le C_1\eta^3 P^2 5^{l/20}.$$

Each has word multiplicity at most $a_{2l}(2j)$. Since $\sum_{j\ge1} 5^{2j}a_{2l}(2j) \le 6^{2l}$, summation and division by 6^{2l} prove (A.1). □

A.2. **Multiplicity and smoothing on polynomial spaces.** Let $\mathcal{P}_k$ be the restrictions to S^3 of real polynomials of degree at most k, and put $W_k = \mathcal{P}_k \ominus \mathcal{P}_{k-1}$, with W_0 the constants. These spaces are finite dimensional, mutually orthogonal, invariant under all rotations, and have dense algebraic sum in $L^2(G, m)$. Write $R_q f(x) = f(xq)$.

Lemma A.2. *If $k \ge 1$ and $0 \ne F \subseteq W_k$ is invariant under every R_q, then $\dim F \ge (k+1)/2$.*

Proof. For an orthonormal basis $e_1, \dots, e_d$ of F, let $K(x,y) = \sum_a e_a(x)e_a(y)$ and $\Psi(z) = \int K(xz,x)\,dm(x)$. Right invariance of F gives $K(xq,yq) = K(x,y)$, hence Ψ is a class function. It has degree at most k, $\Psi(1) = d$, and is orthogonal to $\mathcal{P}_{k-1}$: after $y = xz$, the inner integral against a polynomial $p(z)$ becomes the pairing of $e_a(y)$ with $p(x^{-1}y)$.

Every unit quaternion is conjugate to $\cos\theta + i\sin\theta$. The restriction of Ψ to this circle is an even trigonometric polynomial, so $\Psi(z) = Q(\operatorname{Re} z)$ with $\deg Q \le k$. The real coordinate of Haar measure has density $(2/\pi)\sqrt{1-t^2}$ on $(-1,1)$. Orthogonality therefore gives $Q = cU_k$, where $U_k(\cos\theta)\sin\theta = \sin((k+1)\theta)$ is the Chebyshev polynomial of the second kind. Since $U_k(1) = k+1$, $c = d/(k+1) > 0$.

On F the operator

$$P_k = \frac{1}{\pi}\int_0^{2\pi} \cos(k\theta) R_{\cos\theta + i\sin\theta}\,d\theta$$

is an idempotent, by the group law and orthogonality of trigonometric functions. Its trace is $\pi^{-1}\int \cos(k\theta)cU_k(\cos\theta)\,d\theta = 2c$. This is a positive integer, being the rank of an idempotent. Hence $2d/(k+1) \ge 1$. □

For $f \in \mathcal{P}_k$ and a unit quaternion q, the restriction of f to each great circle is a trigonometric polynomial of degree at most k. The usual Fourier shift estimate, averaged over right translations, yields

$$\|R_q f - f\|_2 \le \frac{\pi k}{2}\,\|q-1\|\,\|f\|_2. \tag{A.5}$$

Indeed, if $q = \cos\alpha + u\sin\alpha$ with $0 \le \alpha \le \pi$, the shift bound is $k\alpha$ and $\alpha \le (\pi/2)\|q-1\|$.

Lemma A.3. *If $F \subseteq \mathcal{P}_k$, $k \ge 1$, is an eigenspace of T with eigenvalue λ, then*

$$(\dim F)\lambda^{2l} \le 16\pi(\pi k)^3 p_l(2/(\pi k)). \tag{A.6}$$

Proof. Set $a = 1/(\pi k)$, $B = \{q : \|q-1\| < a\}$, $\Phi = \mathbf{1}_B/m(B)$, and $Af = \int R_q f\,\Phi(q)\,dm(q)$. By (A.5), $\|Af\|_2 \ge \|f\|_2/2$ on $\mathcal{P}_k$. The coordinate density above gives $m(B) \ge a^3/(4\pi)$.

Left and right translations commute. For an orthonormal basis (e_j) of F, this gives

$$\lambda^{2l}\dim F/4 \le \sum_j \|T^l A e_j\|_2^2 \le \|T^l A\|_{\mathrm{HS}}^2.$$

No invariance of F under A is needed in this inequality. The integral kernel of $T^l A$ is

$$6^{-l}\sum_{w\in S^l} \Phi(x^{-1}wy).$$

Expanding its squared norm, the overlap for words w, w' is at most $m(B)^{-1}$, and vanishes unless $\|w'w^{-1} - 1\| < 2a$. This follows after the substitution $z = x^{-1}wy$, since two points of B have quotient within $2a$ of the identity and the distance is invariant under conjugation. Pairs of length-l words correspond bijectively, as index sequences, to length-$2l$ words by reversing and inverting one half. Thus $\|T^l A\|_{\mathrm{HS}}^2 \le m(B)^{-1}p_l(2a)$, which proves (A.6). □

Proposition A.4. *The six rotations given by left multiplication by S have a spectral gap on $L_0^2(S^3, m)$.*

Proof. The operator T is self-adjoint and preserves every W_k. Let λ_k be its largest eigenvalue there, ignoring zero spaces. The corresponding eigenspace is invariant under all right translations. Lemmas A.2, A.3, and A.1 give

$$\lambda_k^{2l} \le C_2 k^2 (9/16)^l + C_3 k^{-1} 5^{l/20}.$$

For positive λ_k and $l = \lceil 6\log k\rceil$, the right side is $O(k^{-1/2})$. For all sufficiently large $k \ge 3$, it is at most $k^{-1/4}$; since $l \le 7\log k$, this gives $\lambda_k \le e^{-1/56} < 1$. Nonpositive eigenvalues require no separate estimate.

For each remaining k, $\lambda_k < 1$. Indeed, equality would give a nonzero $f \in W_k$ fixed by every generator. The powers of $(1+2i)/\sqrt{5}$ are dense in the i-circle: its order is infinite, as the imaginary coordinate of $(1+2i)^r$ cycles through nonzero values modulo 5. The same holds for the j-circle, and these two circles generate SU(2) by Euler angles. Thus a continuous invariant function is constant, contradicting $f \in W_k$, $k \geq 1$.

We have obtained $\sup_{k\geq 1} \lambda_k < 1$. Orthogonality of the W_k, polynomial density, and

$$\sum_{s\in S} \|U_s f - f\|_2^2 = 12(\|f\|_2^2 - \langle Tf, f\rangle)$$

give the required gap on all of L_0^2. □

A.3. **Transfer and the two-axis argument.** Conjugation of unit imaginary quaternions gives the usual map $\mathrm{SU}(2) \to \mathrm{SO}(3)$. Pullback along its orbit map embeds $L^2(S^2)$ equivariantly and isometrically in $L^2(S^3)$. Thus Proposition A.4 gives a spectral gap on S^2 as well.

Proposition A.5. *If S^n, $n \geq 2$, admits a finite spectral-gap rotation set, then so does S^{n+1}.*

Proof. Extend the rotations by the identity on the last coordinate, and adjoin their conjugates by the orthogonal map interchanging the first and last coordinates. All resulting maps have determinant one. Let P_1 and P_2 be averaging over latitudes about these two axes. The fiberwise spectral gap bounds $\|u - P_1 u\|_2^2 + \|u - P_2 u\|_2^2$ by a constant times the sum of the new generator displacements. We show that these two deviations control variance.

Write ρ for the coordinate law on S^n, and set

$$K\phi(t) = \int \phi(\sqrt{1-t^2}\, s)\, d\rho(s), \qquad -1 \leq t \leq 1.$$

The density of ρ is proportional to $(1-s^2)^{(n-2)/2}$ and is at least some $b > 0$ on $[-1/2, 1/2]$. For $\phi \geq 0$, restricting the two coordinates in K^2 to that interval gives

$$K^2\phi(t) \geq b^2 \int_{-1/4}^{1/4} \phi(v)\, dv. \tag{A.7}$$

To check the uniform bound, the scale of the inner coordinate is $r = \sqrt{1-(1-t^2)s^2} \in [\sqrt{3}/2, 1]$ when $|s| \leq 1/2$. Substitute $v = rs'$ and retain $|v| \leq 1/4$; then $|s'| \leq 1/2$ and the Jacobian $1/r$ is at least one.

For $Q = P_1P_2P_1$, write $P_1 u = \phi_u(x_{n+2})$. The two conditional-coordinate formulas give $Qu = (K^2\phi_u)(x_{n+2})$. Hence (A.7) says $Qu \geq \varepsilon\nu(u)$ for $u \geq 0$, where

$$\varepsilon = b^2/2 \in (0,1), \qquad \nu(u) = 2\int_{-1/4}^{1/4} \phi_u(v)\, dv$$

is a positive normalized functional. The operator $R = (Q - \varepsilon\nu)/(1-\varepsilon)$ is positive and unital. For mean-zero u, Jensen's inequality and preservation of spherical measure by Q give

$$\|Qu\|_2^2 \leq \|Qu - \varepsilon\nu(u)\|_2^2 \leq (1-\varepsilon)(\|u\|_2^2 - \varepsilon\nu(u^2)) \leq (1-\varepsilon)\|u\|_2^2.$$

On the other hand, $\|u - Qu\|_2 \leq 2\|u - P_1u\|_2 + \|u - P_2u\|_2$. Therefore

$$\frac{(1-\sqrt{1-\varepsilon})^2}{5}\|u\|_2^2 \leq \|u - P_1u\|_2^2 + \|u - P_2u\|_2^2.$$

Combining with the fiberwise gaps proves the proposition. In applying those gaps, sum over the generators before integrating in the latitude parameter; no measurable choice of a maximizing generator is required. □

Proposition A.4, the transfer to S^2, and Proposition A.5 give the spectral-gap input for every $n \geq 2$.

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