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SMALE'S MEAN VALUE CONSTANT (1981) EXCEEDS 1.1559, AND OTHER MISCELLANEA

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Abstract. We prove that Smale's mean value constant exceeds 1.1559837, improving the bound 1.1018 of Jatar and Ng; after the first version of this paper, the author proved that it equals 4. Smale proved in 1981 that for every complex polynomial P of degree at least 2 and every point z with $P'(z) \neq 0$, some critical point ζ of P satisfies $|P(z)-P(\zeta)| \leq K\,|z-\zeta|\,|P'(z)|$ with $K = 4$, and he conjectured that $K = 1$ suffices; K^* is the best constant, called Smale's mean value constant. That conjecture was disproved in 2026. Jatar and Ng showed that K^* equals the supremum of $1/B(0)$ over the *admissible pairs* (W, B): W is a normalised univalent map of the unit disk, B is a finite Blaschke product with $B(0) > 0$, and the derivative of the product WB has no zero in the disk. We construct such a pair explicitly, with B of degree two, from a polynomial of degree 18 with Gaussian rational coefficients, and we verify the required properties by a computation in exact arithmetic. Writing $\mathcal{B}_d$ for this supremum restricted to $\deg B \leq d$, we prove, partly with certified computation, that $1.1363375 < \mathcal{B}_1 < 1.177801$, $\mathcal{B}_2 < 100/61$, and $\mathcal{B}_d \leq 4 - 0.00597/(d \log d)$ for every $d \geq 2$, which improves the bound of Jatar and Ng for $d \geq 452$. If a pair attains $\mathcal{B}_d$ with $d \geq 1$, then W does not extend holomorphically to any neighbourhood of the closed disk and is not rational. If $1/B_n(0) \to 4$ along a sequence of admissible pairs (W_n, B_n) (such sequences exist, since $K^* = 4$), then after rotation W_n tends to the Koebe function, as Jatar and Ng observed, and the zeros a of B_n, weighted by $\log(1/|a|)$, have an explicit limit distribution. Every admissible pair (W, B) with W meromorphic on a neighbourhood of the closed disk has $\deg B \leq \mathrm{wt}(W)$, a weighted count of the points of the circle where $|W|$ is stationary and $\arg W$ decreases.[1,2]

AI Disclosure. This work was done overwhelmingly by AI models, steered by the author in a continuous critical dialogue, variously with Google's Gemini, Anthropic's Claude and OpenAI's ChatGPT. The author takes full responsibility for the paper.

1. Introduction

We prove that Smale's mean value constant K^*, defined below, exceeds 1.1559837. The best lower bound proved before was 1.1018, due to Jatar and Ng [23]. We also prove other results on the pairs of a univalent map and a finite Blaschke product through which Jatar and Ng recast the problem. After the first version of this paper, the author proved that $K^* = 4$ [18, Thm. 1.1].

Smale's mean value problem. Let P be a complex polynomial of degree $n \geq 2$ and let z be a point with $P'(z) \neq 0$. Smale [41] proved that some critical point ζ of P satisfies

$$\left|\frac{P(z) - P(\zeta)}{z - \zeta}\right| \leq K\,|P'(z)|$$

with $K = 4$, and he asked for the best constant. After a normalisation (Section 2.1) the best constant in degree n is $K(n) = \sup_{p \in \mathcal{P}_n} m(p)$, where $\mathcal{P}_n$ is the set of polynomials of degree n with $p(0) = 0$ and $p'(0) = 1$, and $m(p)$ is the minimum of $|p(b)/b|$ over the critical points b of

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[1]This is an expanded version of the following focused version [19]: https://hexagonmath.org/2611.00258

[2]Additionally, Theorem 1.1 is proved in Lean 4 from Lean's three standard axioms alone, and checked with `lake comparator`: https://github.com/david44hello/papers_supporting_materials/tree/smale-mean-value-constant-v2.90/smale-mean-value-constant

p. We write $K^* = \sup_n K(n)$ for Smale's mean value constant. The polynomial $z - z^n/n$ gives $K(n) \geq (n-1)/n$, and Smale asked whether $K^* = 1$, or even $K(n) = (n-1)/n$ (Smale's (1981) mean value conjecture); in his list of problems for the next century [42] it is the first of the problems in the addendum.

Every known improvement of 4 is a bound for $K(n)$ that depends on n. The conjecture holds for $n \leq 4$ (Tischler [43]; see also [33, Sect. 7.2]), and Crane announced a computer-assisted proof for $n = 5$ [8, Sect. 1]. Beardon, Minda and Ng [4] proved $K(n) \leq 4^{(n-2)/(n-1)}$, Conte, Fujikawa and Lakic [6] proved $K(n) \leq 4(n-1)/(n+1)$, Fujikawa and Sugawa [17] combined the two ideas, and Crane [8] proved $K(n) < 4 - 2.263/\sqrt{n}$ for $n \geq 8$. All these bounds tend to 4 as $n \to \infty$, so none of them improves the bound $K^* \leq 4$ for the constant itself; in fact $K^* = 4$ [18]. Crane [7] showed that in every degree there is an extremal polynomial all of whose critical points attain the minimum in $m(p)$, and $K(n)$ is nondecreasing in n (see [7] and [23, proof of Cor. 4.6]). For the dual problem see [15, 22].

Recent results. In September 2026 a proof that $K^* > 1$ was made public [1]; it shows that some p has $m(p) > 1$, but does not make the degree of p explicit. So Smale's conjecture is false. Adamczewski and Colbrook [2] explain the mechanism behind that proof and give an explicit polynomial p of degree 29 984 with $m(p) > 1.0000004$, verified by interval arithmetic, so $K(n) > 1$ for every $n \geq 29\,984$. In [23] Jatar and Ng recast the problem in univalent function theory. A translation of the problem into one about finite Blaschke products goes back to Sheil-Small [40, Ch. 10]; see also Ng and Zhang [28] and Dubinin [14]. Jatar and Ng proved that

$$K^* = \sup\Big\{\frac{1}{B(0)}\Big\},$$

the supremum over pairs (W, B) of a normalised univalent map W of the unit disk $\mathbb{D}$ (that is, $W \in \mathcal{S}$: W is holomorphic and injective, $W(0) = 0$, $W'(0) = 1$) and a finite Blaschke product B with $B(0) > 0$ such that the derivative of the product WB has no zero in $\mathbb{D}$ [23, Thm. 1.2]; Jatar and Ng call such pairs *admissible*. They also proved that K^* equals the supremum Λ_{JN} of $\min_{\partial\Omega} |\Phi(z)/z|$ over bounded analytic Jordan domains $\Omega \ni 0$ and Φ locally univalent on a neighbourhood of $\overline{\Omega}$ with $\Phi(0) = 0$, $\Phi'(0) = 1$ [23, Thm. 4.4]; we call this the *domain formulation*, and $\min_{\partial\Omega} |\Phi(z)/z|$ the *ratio* of the pair. Writing $\mathcal{B}_d$ for the supremum over $\deg B \leq d$, they proved

$$1.1018586788\ldots = \sqrt{8081/6656} \leq \mathcal{B}_1 < \sqrt{7/5} = 1.1832159566\ldots,$$

so that $K^* > 1.1018$ [23, Thm. 1.3], and $\mathcal{B}_d < 4 - \frac{529}{576}\frac{(\log 2)^2}{d^2}$ for every $d \geq 1$. For a second explicit pair, $\mathcal{B}_1 > 1.1030$ is stated without proof in [23, Rem. 6.7]; Corollary 8.3 proves it. They also give a winding criterion for admissibility, show that every degree of B occurs and that each $\mathcal{B}_d$ is attained, and show that a starlike W carries no nonconstant B. Jatar and Ng [23, Sect. 9] ask whether $\mathcal{B}_d$ is strictly increasing, whether $\mathcal{B}_1 = K^*$, and whether $K^* = 4$. The last two questions are now answered: $K^* = 4$ [18], and so $\mathcal{B}_1 < K^*$ by the upper bound for $\mathcal{B}_1$ displayed above.

The main result.

Theorem 1.1. *Let J_{I} be the rational number*

$$J_{\mathrm{I}} = \frac{267952521936226337027481600174795544467587176908574201599 7}{200518481134769781691297951341495479358548025660497985536 0}.$$

Then $K^* \geq \sqrt{J_{\mathrm{I}}} = 1.15598372919\ldots$, so $K^* > 1.1559837$, and $K(n) > 1.1559837$ for all large n.

Here $J_{\mathrm{I}} = 1/B(0)^2$ for the explicit admissible pair of Theorem 8.8. Theorem 1.1 is Theorem 8.8 and Corollary 8.12; Figure 1 shows the known range before and after it. The bound comes from one explicit admissible pair, with B of degree two, built from a polynomial Q of degree 18 whose derivative has a zero of order 5 on the circle, by a construction of Jatar and Ng [23, Lemma 6.2] that we extend from one zero of B to several (Lemma 8.4). The required properties are checked in exact

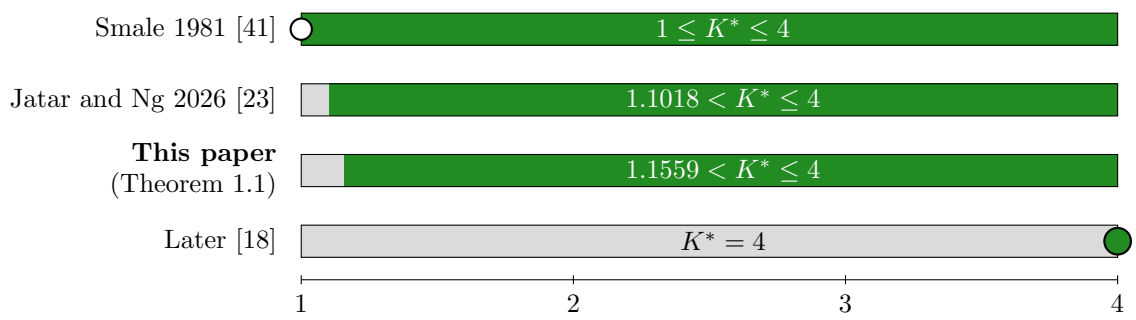


FIGURE 1. What is known about K^*, to scale. In each row the green part is where K^* may still lie; the grey part is ruled out. Smale proved $K^* \le 4$ in 1981 and conjectured $K^* = 1$ (white dot); that conjecture was disproved in 2026 [1, 2]. After the first version of this paper, the author proved that $K^* = 4$ [18] (green dot).

arithmetic (Lemma 8.6 and Appendix A). Besides the cited results of Jatar and Ng, Dieudonné and Crane, the proof of Theorem 1.1 needs only Section 2, Lemma 7.11, Section 8 up to Theorem 8.8, Corollary 8.12 and Appendix A. The proof of Theorem 1.1 alone is also given in [19].

Other results. The other results are about the pairs themselves. A *Blaschke model* is a pair (W, B) of a map $W \in \mathcal{S}$ that is univalent on a neighbourhood of the closed disk and a finite Blaschke product B such that $(WB)'$ has no zero on the closed disk (Definition 2.8); after B is multiplied by a unimodular constant, it is admissible. The *gain* of a pair (W, B) is $1/|B(0)|$, and the gains of Blaschke models have supremum K^* (Theorem 2.9). A pair is *extremal* if it attains some $\mathcal{B}_d$ with $d \ge 1$. Items (1)–(3) are the main results after Theorem 1.1. Table 1 compares the best or closest earlier results with ours. To our knowledge, the results stated without a source are new. Section 11 collects open problems.

(1) *Degrees one and two* (Section 8). $1.1363375 < \mathcal{B}_1 < 1.177801$ and $1.1559837 < \mathcal{B}_2 < 100/61 < 1.639345$. The lower bounds come from explicit pairs (Theorems 8.7 and 8.8); the upper bounds come from the homotopy argument of [23, Thm. 7.4], run on other circles and checked in interval arithmetic (Theorems 8.15 and 8.19).
(2) *Every degree* (Section 7.9). $\mathcal{B}_d \le 4 - 0.00597/(d \log d)$ for every $d \ge 2$ (Theorem 7.37), by Dubinin's separating transformation [13]; this improves [23, Thm. 1.3] for $d \ge 452$.
(3) *Extremal pairs do not extend holomorphically past the circle* (Sections 7.4–7.6). The map W of an extremal pair does not extend holomorphically to any neighbourhood of the closed disk (Corollary 7.14), and it is not rational (Corollary 7.25). In particular no extremal pair is a Blaschke model (Corollary 7.10).
(4) *Counting from the circle* (Section 7.7). If W is meromorphic on a neighbourhood of the closed disk, the zeros of $(WB)'$ in the disk can be counted from the circle, and every admissible pair has $\deg B \le \mathrm{wt}(W)$, a number read off W on the circle (Theorem 7.26).
(5) *Pockets* (Section 3). For a Blaschke model, a *pocket* is a point $e^{i\theta}$ of the circle where $\frac{d}{d\theta}|W(e^{i\theta})| = 0$ and $\frac{d}{d\theta}\arg W(e^{i\theta}) < 0$ (Definition 3.1). At a pocket the *depth* is $x = -\big(\frac{d}{d\theta}\arg W(e^{i\theta})\big)^{-1} > 0$; let $\sigma = \sum_j P_{a_j}$ be the sum of the Poisson kernels $P_{a_j}(w) = (1-|a_j|^2)/|w-a_j|^2$ of the zeros a_j of B. The winding number in the criterion of [23, Sect. 8] is a signed count of the pockets with $\sigma x > 1$, when no pocket has $\sigma x = 1$ (Theorem 3.4).
(6) *Ownership* (Section 5). Each zero a of B in a Blaschke model *owns* a pocket at a local minimum of $|W|$: there $\sigma x > 1$, but $(\sigma - P_a)x \le 1$. Distinct zeros own distinct pockets (Theorem 5.1).

(7) *Gains tending to* 4 (Section 4). Let (W_n, B_n) be admissible pairs with gains tending to 4. After rotations, W_n tends to the Koebe function and B_n tends to $(1-w)^2/4$ (Theorem 4.2); the zeros a of B_n, weighted by $\log(1/|a|)$, have an explicit limit distribution (Proposition 4.4).

(8) *Symmetric pairs* (Section 6). Every n-fold symmetric Blaschke model comes from the n-th root transform of [23, Rem. 8.6] (Theorem 6.2). The transform keeps the gain, but it gives a Blaschke model only under an explicit condition (Lemma 6.1).

(9) *Zeros of B are not small* (Section 7.3). Every zero of B in a Blaschke model has modulus greater than 0.36967 (Corollary 7.8).

(10) *Fixed degree* (Section 9). Take a pair of the construction of [23, Lemma 6.2] built from a polynomial of degree at most n. If it has fewer than n boundary defects (zeros of $(WB)'$, critical points of W, and pairs of points where the curve $W(e^{i\theta})$ meets itself, all on the circle and counted with weights), then a nearby pair of the same kind has a larger gain (Theorem 9.1).

(11) *Adding zeros near the circle* (Section 10). If $W(w) \not\equiv w$ is meromorphic on a neighbourhood of the closed disk, we count exactly the zeros of $(WB)'$ in the disk after zeros of B are added on fixed radii near the circle, with W kept; and we show that one zero of B added near the circle on the radius through a zero of $(WB)'$ on the circle makes the pair inadmissible (Proposition 10.1).

(12) *Other exact pairs, and Dubinin's conjecture* (Section 8). Table 2 lists eighteen pairs with exact rational data (Theorem 8.9). Remark 8.13 notes that, by [1] and [7], Dubinin's conjecture [14, (1.5)], which would give $K(n) = (n-1)/n$, fails in every large degree, and it shows how a polynomial p of degree n with $m(p) > (n-1)/n$ gives Blaschke products that violate it.

Table 1: The best or closest result known before this paper, next to each new result.

Best result before this paper	**New result**
$K^* > 1.1018$ [23, Thm. 1.3]; $K^* > 1.1030$, stated without proof in [23, Rem. 6.7]	$K^* > 1.1559837$, and $K(n) > 1.1559837$ for all large n
$1.1018 < \mathcal{B}_1 < \sqrt{7/5} = 1.1832159\ldots$ [23, Thm. 1.3]; $\mathcal{B}_1 > 1.1030$ [23, Rem. 6.7]	$1.1363375 < \mathcal{B}_1 < 1.177801$
$\mathcal{B}_2 < 3.8897$ [23, Thm. 1.3]	$1.1559837 < \mathcal{B}_2 < 100/61 < 1.639345$
$\mathcal{B}_d < 4 - 529(\log 2)^2/(576d^2)$ [23, Thm. 1.3]; the new bound is better for $d \geq 452$	$\mathcal{B}_d \leq 4 - 0.00597/(d \log d)$ for every $d \geq 2$
[23, Rem. 8.11] left open whether a pair that attains $\mathcal{B}_d$ can be regular (a Blaschke model)	If (W, B) attains $\mathcal{B}_d$ with $d \geq 1$, then W does not extend holomorphically to any neighbourhood of the closed disk and is not rational
—	Every admissible pair whose W is meromorphic on a neighbourhood of the closed disk has $\deg B \leq \mathrm{wt}(W)$, a number read off W on the circle
after rotations W_n tends to the Koebe function, all zeros of B_n tend to the circle and $\deg B_n \to \infty$ [23, Rem. 7.3]; such pairs exist if $K^* = 4$, a question of [23, Sect. 9] that is now answered: $K^* = 4$ [18]	If admissible pairs (W_n, B_n) have $1/B_n(0) \to 4$, then after rotations the zeros a of B_n, weighted by $\log(1/\lvert a\rvert)$, have an explicit limit distribution
—	A local maximiser of the gain among the pairs built from a polynomial of degree at most n has at least n weighted boundary defects

Best result before this paper (continued)	**New result**
the winding criterion of [23, Sect. 8]	A pocket formula for the winding number of $(WB)'/(W'B)$ along the circle, and the ownership of pockets by the zeros of B
every zero has $\lvert a\rvert \geq 1/3$ [23, Thm. 7.1]	Every zero of B in a Blaschke model has modulus greater than 0.36967
the transform, and this correspondence for polynomials $w\,p(w^n)$, in [23, Rem. 8.6]	Every n-fold symmetric pair with $B(0) \neq 0$ whose map is univalent past the circle comes from the n-th root transform

Supporting files. The supporting archive holds the exact data of the eighteen pairs of Table 2 and the programs, with their outputs, for the computer steps of Corollaries 7.8 and 8.3, Section 7.8, Theorems 8.7, 8.8, 8.9, 8.15 and 8.19, and Corollary 9.14. It is deposited at

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Terms. The terms below are introduced in this paper or used in a special sense; the last column gives the page of the definition.

Term	**Meaning**	**p.**
admissible pair (W, B)	$W \in \mathcal{S}$, B a finite Blaschke product with $B(0) > 0$, and $(WB)' \neq 0$ in $\mathbb{D}$	2
$\mathcal{B}_d$	the supremum of $1/B(0)$ over admissible pairs with $\deg B \leq d$	2
gain	$1/\lvert B(0)\rvert$	8
extremal pair	an admissible pair that attains some $\mathcal{B}_d$ with $d \geq 1$	3
$m(p)$, $K(n)$, K^*	the constants of Smale's problem	6
domain pair, ratio, Λ, Λ_{JN}	(Ω, Φ) of the domain formulation; $\min_{\partial\Omega} \lvert\Phi(z)/z\rvert$; its supremum over all domain pairs, resp. over those of [23, Thm. 4.4]	6
Blaschke model	a pair with (M1)–(M3): W univalent past the circle, $(WB)' \neq 0$ on the closed disk	8
regular pair	Jatar and Ng's name for a Blaschke model with $B(0) > 0$	8
triple; σ, v, γ	(W, B, w_0) with a base point w_0; on the circle $\sigma = wB'/B$, $v = (W - W(w_0))/(wW')$, $\gamma = 1 + \sigma v$	8
feasible	$((W - W(w_0))B)'$ has no zero on the closed disk	9
back-facing point	a boundary point ζ of $W(\mathbb{D})$ at which $\operatorname{Re} v < 0$	9
pocket; depth, sign	a point of the circle where $v < 0$; $x = -v$, and the sign change ε of $\operatorname{Im} v$ there	9
on, off (pocket)	$\sigma x > 1$, resp. $\sigma x < 1$	9
frame; base-0 depth	the choice of the base point; the depth x_0 seen from base point 0	9
K_0 (Sections 4 and 7)	the gain $1/\lvert B(0)\rvert$ of a pair normalised by $B(0) > 0$	12
Ω, A, Φ (Sections 4–7.3)	$W(\mathbb{D})$; the zero set of B; $\Phi(z) = zB(W^{-1}(z))$ on Ω, so that $\Phi \circ W = WB$	12
E_b (Sections 4–7.3)	the component of $\{\lvert\Phi\rvert < r_{\mathrm{in}}\}$ that contains the zero b of Φ	12
exit point	a boundary point of $W(\mathbb{D})$ where an inverse branch of $\Phi = zB(W^{-1}(z))$ reaches the boundary	15
owned point	a point of the circle that W maps to an exit point of the branch at $W(a_j)$	17
n-fold symmetric pair	$W(e^{2\pi i/n}w) = e^{2\pi i/n}W(w)$ and the zeros of B are invariant under $w \mapsto e^{2\pi i/n}w$	19
F (from Section 7.4 on)	$F = WB$, so that $F = \Phi \circ W$	23

Term	Meaning	p.
w, z	W, B and F have variable w throughout; for the pairs built from a polynomial, Q and H have variable $z = \varphi(w)$ (Sections 8, 9, and where Sections 7 and 10 use these pairs); z also denotes points of $\Omega = W(\mathbb{D})$	40
Crit, Cont; contact pair	critical points of W on $\mathbb{T}$; points $\zeta \neq \zeta'$ of $\mathbb{T}$ with $W(\zeta) = W(\zeta')$	24
slit plane	$\mathbb{C}$ minus a non-empty finite union of analytic arcs, closed in $\mathbb{C}$	27
Z_-	the points of $\mathbb{T}$ where $wW'/W < 0$: $\lvert W\rvert$ is stationary and $\arg W$ decreases	31
index ι_ζ; Z_χ	the contribution of a point ζ of $\mathbb{T}$ with $wF'/F \leq 0$ to the count of zeros of F'	33
$\mathrm{wt}(W)$	a sum over Z_-, read off W; it bounds $\deg B$	33
flat maximum	a point of Z_- where $\log\lvert W\rvert$ has a strict local maximum and its derivative vanishes to order at least 3	35
$\mathcal{L}_n$, $\mathcal{G}_n$	pairs of the construction of [23, Lemma 6.2] with polynomials of degree at most n	48
boundary defects; n_H	zeros of $(WB)'$, critical points and contact pairs of W on $\mathbb{T}$; $n_H - 2$ counts them with weights	48
margin	the distance from the unit circle to the nearest zero of a polynomial	62
$\mathcal{D}_c$	the Dieudonné polynomials of a polynomial H, $-1 \leq c \leq 1$	40
χ-negative, χ-positive; χ-depth	a zero x of $\frac{d}{d\theta}\log\lvert W\rvert$ on $\mathbb{T}$ with $\chi(x) < 0$, resp. $\chi(x) > 0$, where $\chi = wF'/F$; $\eta(x) = -\chi(x)$	56
hook	a slit that turns back toward 0	61
other letters	a letter used only inside one proof, remark or computation is defined there; $[z^j]P$ is the coefficient of z^j in a polynomial P	

2. Preliminaries

2.1. **The constant.**

Definition 2.1. For $n \geq 2$ let $\mathcal{P}_n$ be the set of complex polynomials p of degree n with $p(0) = 0$ and $p'(0) = 1$. Every critical point b of $p \in \mathcal{P}_n$ is non-zero, and we put

$$m(p) := \min_{p'(b)=0} \frac{|p(b)|}{|b|}, \qquad K(n) := \sup_{p\in\mathcal{P}_n} m(p), \qquad K^* := \sup_{n\geq 2} K(n).$$

For a polynomial P and a point z with $P'(z) \neq 0$, the polynomial $p(u) = (P(z+u) - P(z))/P'(z)$ is normalised, its critical points are the points b with $z + b$ critical for P, and $|P(z) - P(z + b)|/(|b|\,|P'(z)|) = |p(b)|/|b|$. Hence $K(n)$ is the best constant in Smale's inequality in degree n, and K^* is Smale's mean value constant (denoted $\mathcal{M}$ in [23]). Smale proved $K^* \leq 4$ [41]; the polynomial $z - z^n/n$ has $m = (n-1)/n$, so $K^* \geq 1$.

2.2. **Domain pairs.**

Besides the admissible pairs (W, B) of Section 1, we use a second kind of pair, which belongs to the domain formulation.

Definition 2.2. A *domain pair* (Ω, Φ) consists of a bounded domain Ω with $0 \in \Omega$ and $\mathbb{C} \setminus \overline{\Omega}$ connected, and a function Φ holomorphic on an open set containing $\overline{\Omega}$, with $\Phi(0) = 0$, $\Phi'(0) = 1$ and $\Phi'(z) \neq 0$ for every $z \in \overline{\Omega}$. Its *ratio* is

$$\mu(\Omega, \Phi) := \min_{z\in\partial\Omega} \frac{|\Phi(z)|}{|z|},$$

and Λ is the supremum of $\mu(\Omega, \Phi)$ over all domain pairs.

Jatar and Ng [23, Thm. 4.4] proved that K^* equals the supremum Λ_{JN} of the ratio over the pairs in which Ω is a bounded *analytic Jordan* domain containing 0 and Φ is locally univalent near

$\overline{\Omega}$. Such pairs are domain pairs, so $\Lambda_{\mathrm{JN}} \le \Lambda$. The next lemma gives the reverse inequality, so the wider class has the same supremum; Theorem 2.3 is not used below.

Theorem 2.3. $K^* = \Lambda$.

Lemma 2.4. *For every domain pair (Ω, Φ) and every $\varepsilon > 0$ there is a bounded analytic Jordan domain Ω' with $\overline{\Omega} \subset \Omega'$, such that Φ is holomorphic near $\overline{\Omega'}$, $\Phi' \neq 0$ on $\overline{\Omega'}$, and $\min_{\partial\Omega'} |\Phi(z)/z| \ge \mu(\Omega, \Phi) - \varepsilon$.*

Proof. Write $\mu = \mu(\Omega, \Phi)$ and $E := \overline{\Omega}$, a non-degenerate continuum with $\mathbb{C} \setminus E$ connected; let $r_0 := \operatorname{dist}(0, \partial\Omega) > 0$. Fix $\delta_0 \in (0, r_0/2)$ such that $\{z : \operatorname{dist}(z, E) \le \delta_0\}$ lies in the domain of Φ. On the compact set $\{\operatorname{dist}(z, \partial\Omega) \le \delta_0\}$, where $|z| \ge r_0/2$, the function $|\Phi(z)/z|$ is uniformly continuous, and it is $\ge \mu$ on $\partial\Omega$. Choose $\delta \in (0, \delta_0]$ such that, with $V := \{z : \operatorname{dist}(z, E) < \delta\}$, $\Phi' \neq 0$ on $\overline{V}$ and $|\Phi(z)/z| \ge \mu - \varepsilon$ for every $z \in \overline{V} \setminus \Omega$. (Such a z lies within δ of $\partial\Omega$: if its nearest point of E lies in Ω, the segment to it crosses $\partial\Omega$.)

Let g_E be the Green function of $\mathbb{C} \setminus E$ with pole at infinity; it is continuous on $\mathbb{C}$ (every boundary point of $\mathbb{C} \setminus E$ is regular, as E is a non-degenerate continuum [34, Sect. 4.2]), zero on E and positive off E, and $\operatorname{cap}(E) > 0$. Let $\mathcal{F}_n$ be Fekete polynomials of E; their zeros, the Fekete points, lie in E. Then $\|\mathcal{F}_n\|_E^{1/n} \to \operatorname{cap}(E)$, and $\frac{1}{n} \log |\mathcal{F}_n(z)| \to \log \operatorname{cap}(E) + g_E(z)$ locally uniformly on $\mathbb{C} \setminus E$ [34, 35]. Let $g_1 > 0$ be the minimum of g_E on $\{\operatorname{dist}(z, E) = \delta\}$, fix $\tau > 0$ with $3\tau < g_1$, then n so large that $\|\mathcal{F}_n\|_E^{1/n} < \operatorname{cap}(E) e^{\tau}$ and $|\mathcal{F}_n|^{1/n} > \operatorname{cap}(E) e^{g_1 - \tau}$ on $\{\operatorname{dist}(z, E) = \delta\}$, and put $q := \mathcal{F}_n / (\operatorname{cap}(E) e^{2\tau})^n$. Then $|q| < 1$ on E, and $|q| > 1$ on $\{\operatorname{dist}(z, E) \ge \delta\}$ ($\log |q|$ is harmonic there, positive on its boundary and tends to $+\infty$). Moving τ slightly, we may assume that no critical value of q has modulus 1.

Put $\Omega' := \{|q| < 1\}$. Every component of Ω' contains a zero of q (minimum modulus principle), and all zeros of q lie in the connected set $E \subset \Omega'$; so Ω' is a domain, and $E \subset \Omega' \subset V$. As $q' \neq 0$ on $\{|q| = 1\}$, this set is a finite union of disjoint analytic Jordan curves, each point of which is a limit of points with $|q| > 1$. The set $\{|q| > 1\}$ is connected (a bounded component H would have $|q| = 1$ on ∂H, hence $|q| \le 1$ on H by the maximum principle), so its closure $\mathbb{C} \setminus \Omega'$ is connected and Ω' is simply connected. A bounded simply connected plane domain has connected boundary [27]; so $\partial\Omega'$ is one analytic Jordan curve and Ω' is its interior. Finally $\overline{\Omega'} \subset \overline{V}$ lies in the domain of Φ with $\Phi' \neq 0$ there, and $\partial\Omega' \subset \overline{V} \setminus \Omega$ (as $|q| < 1$ on E), where $|\Phi(z)/z| \ge \mu - \varepsilon$. □

Proof of Theorem 2.3. $K^* = \Lambda_{\mathrm{JN}} \le \Lambda$ by [23, Thm. 4.4], and $\Lambda \le \Lambda_{\mathrm{JN}}$ by Lemma 2.4. □

Remark 2.5 (Domain pairs). Jatar and Ng [23, Sect. 9] mention the estimate $\Lambda_{\mathrm{JN}} > 1.00114$ and ask for larger bounds through Λ_{JN}. By Corollary 9.14 and the paragraph after Theorem 8.8, the Blaschke models behind Theorem 1.1 give pairs of the kind defining Λ_{JN} with ratio $\sqrt{J_{\mathrm{I}}} > 1.1559837$.

We also use the inverse-branch argument behind Smale's bound, written for pairs.

Proposition 2.6. $\mu(\Omega, \Phi) \le 4$ *for every domain pair.*

Proof. Let G be the inverse germ of Φ at 0, so $G(0) = 0$ and $G'(0) = 1$. Call $r > 0$ *good* if G extends holomorphically to $D_r = \{|w| < r\}$ with $G(D_r) \subset \Omega$; then $\Phi(G(w)) = w$ on D_r by the identity theorem, so G is injective there. Extensions to two disks agree on the smaller one, so G is defined on D_ρ, $\rho := \sup\{r \text{ good}\} > 0$. Koebe's one-quarter theorem for $G(rw)/r$ gives $G(D_r) \supset D_{r/4}$, and $G(D_r)$ lies in the bounded set Ω; so $\rho < \infty$.

Step 1: $\overline{G(D_\rho)}$ meets $\partial\Omega$. If not, $Q_0 := \overline{G(D_\rho)}$ is a compact subset of Ω. Fix ζ with $|\zeta| = \rho$ and take $w_n \to \zeta$ in D_ρ with $G(w_n) \to z_0 \in Q_0$ (a subsequence). Then $\Phi(z_0) = \zeta$ and $\Phi'(z_0) \neq 0$, so Φ maps an open set V_0 with $z_0 \in V_0 \subset \Omega$ biholomorphically onto a disk $D(\zeta, s)$; let $G_\zeta = (\Phi|_{V_0})^{-1}$. On the convex set $D(\zeta, s) \cap D_\rho$ the set $\{w : G(w) \in V_0\}$ is open and non-empty, and there $G = G_\zeta$, because both take values in V_0, where Φ is injective, and $\Phi(G(w)) = w = \Phi(G_\zeta(w))$. By the identity theorem $G = G_\zeta$ on $D(\zeta, s) \cap D_\rho$. Cover the circle $|w| = \rho$ by finitely many such disks; two

overlapping disks centred on the circle meet inside D_ρ, so the extensions agree on overlaps. This extends G to $D_{\rho+\varepsilon}$ with image in Ω, contrary to the choice of ρ.

Step 2. Take $z^* \in \overline{G(D_\rho)} \cap \partial\Omega$, $z^* = \lim G(w_n)$ with $w_n \in D_\rho$, $w_n \to w^*$ (a subsequence). As Φ is continuous on $\overline{\Omega}$, $\Phi(z^*) = w^*$, and $|w^*| = \rho$ (if $|w^*| < \rho$, then $z^* = G(w^*)$ would lie in Ω). Since $z^* \notin \Omega$ and $G(D_\rho) \supset D_{\rho/4}$, we have $|z^*| \geq \rho/4$. Therefore $\mu(\Omega, \Phi) \leq |\Phi(z^*)|/|z^*| \leq \rho/(\rho/4) = 4$. □

Remark 2.7. If $\mu(\Omega, \Phi) > 1$, then $\Phi(z)/z$ has zeros in Ω: otherwise $\Phi(z)/z$ is zero-free on Ω with value 1 at 0, and the minimum principle gives $\min_{\partial\Omega} |\Phi(z)/z| \leq 1$.

2.3. **Blaschke models.** Let $\mathbb{D}$ be the open unit disk, $\mathcal{S}$ the class of univalent W on $\mathbb{D}$ with $W(0) = 0$, $W'(0) = 1$, and $B_a(w) = (w - a)/(1 - \bar{a}w)$ for $a \in \mathbb{D}$.

Definition 2.8. A *Blaschke model* is a pair (W, B) such that

(M1) $W \in \mathcal{S}$, and W extends to a holomorphic univalent map on a disk $\{|w| < R\}$ for some $R > 1$;
(M2) $B = cB_{a_1} \cdots B_{a_k}$ with $|c| = 1$, $k \geq 0$ and $a_j \in \mathbb{D}$ (a finite Blaschke product);
(M3) $(WB)'$ has no zero on the closed disk $\{|w| \leq 1\}$.

Its *gain* is $1/|B(0)|$.

In the language of [23], the Blaschke models with $B(0) > 0$ are exactly the *regular* pairs (admissible pairs whose W extends to a univalent function on a neighbourhood of the closed disk, with $(WB)' \neq 0$ there); a Blaschke model differs from a regular pair only in that $B(0)$ need not be positive. Jatar and Ng [23, Thm. 1.2] proved that K^* is the supremum of $1/B(0)$ over the pairs with $(WB)' \neq 0$ in the open disk, and [23, Lemma 2.1] that regular pairs give the same supremum. Hence:

Theorem 2.9 ([23, Thm. 1.2, Lemma 2.1])**.** *K^* is the supremum of the gains $1/|B(0)|$ over all Blaschke models.*

Sections 3–5 use this closed-disk form, with an arbitrary base point w_0:

Corollary 2.10. *K^* is the supremum of $1/|B(w_0)|$ over triples (W, B, w_0) with W holomorphic and univalent on a disk $\{|w| < R\}$, $R > 1$, $w_0 \in \mathbb{D}$, B as in* (M2)*, and $\big((W - W(w_0))B\big)'$ zero-free on $\{|w| \leq 1\}$.*

Proof of Corollary 2.10. Given (W, B, w_0), let $\varphi_{w_0}(u) = (u+w_0)/(1+\bar{w}_0 u)$ and $G = (W - W(w_0))B$. Since $G'(w_0) = W'(w_0)B(w_0) \neq 0$, $B(w_0) \neq 0$. Put $\widetilde{W} = (W \circ \varphi_{w_0} - W(w_0))/c_1$ with $c_1 = W'(w_0)\varphi'_{w_0}(0)$, and $\widetilde{B} = B \circ \varphi_{w_0}$, a finite Blaschke product of the same degree. Then $\widetilde{W}$ satisfies (M1), because φ_{w_0} maps a disk $\{|u| < R''\}$, $R'' > 1$, into $\{|w| < R\}$, and $(\widetilde{W}\widetilde{B})' = G'(\varphi_{w_0})\varphi'_{w_0}/c_1$ has no zero on $\{|u| \leq 1\}$. So $1/|B(w_0)| = 1/|\widetilde{B}(0)| \leq K^*$ by Theorem 2.9; the case $w_0 = 0$ gives the converse. □

2.4. **The winding criterion.** Let W be holomorphic and univalent on a disk $\{|w| < R\}$, $R > 1$, let $w_0 \in \mathbb{D}$, and let B be as in (M2) with $\deg B = k$; we call (W, B, w_0) a *triple* (the condition of Corollary 2.10 on $((W - W(w_0))B)'$ is not assumed). Put $G = (W - W(w_0))B$, so that $G' = W'N$ with

$$N = B + \frac{W - W(w_0)}{W'}\, B'.$$

On the circle $N = B\gamma$, where

$$\gamma = 1 + \sigma v, \qquad \sigma = \frac{wB'}{B} = \sum_{j=1}^{k} \frac{1 - |a_j|^2}{|w - a_j|^2} \ \ (\geq 0,\ > 0 \text{ if } k \geq 1), \qquad v = \frac{W - W(w_0)}{wW'}.$$

The following is the argument principle, as used in [23, Sect. 8] (see the display before their Lemma 8.1; for $w_0 = 0$ their g is our $1/v$); we record it in the form used below.

Proposition 2.11. *G' has no zero on $\{|w| \le 1\}$ if and only if $\gamma \neq 0$ on the circle and* $\operatorname{wind}(\gamma) = -k$.

Proof. $W' \neq 0$ near the closed disk, so the zeros of G' there are those of N, which is holomorphic near the closed disk. On the circle $|B| = 1$, so $N \neq 0$ there iff $\gamma \neq 0$, and then the number of zeros of N in $\mathbb{D}$ is $\operatorname{wind}(N) = \operatorname{wind}(B) + \operatorname{wind}(\gamma) = k + \operatorname{wind}(\gamma)$. □

3. Pocket calculus

Throughout this section (W, B, w_0) is a triple as in Section 2.4, with $k = \deg B \ge 1$, zeros $a_1, \dots, a_k$ and $z_0 = W(w_0)$; v, σ, γ and N are as in Section 2.4. For $a \in \mathbb{D}$ and $|w| = 1$ we write $P_a(w) = (1 - |a|^2)/|w - a|^2$, so that $\sigma = \sum_j P_{a_j}$ on the circle, and ω for the measure $P_{w_0}(e^{i\theta})\, d\theta/2\pi$, the harmonic measure of z_0 in $\Omega = W(\mathbb{D})$ pulled back to the circle. The triple is *feasible* if $((W - z_0)B)'$ has no zero on $\{|w| \le 1\}$; by Proposition 2.11 this holds if and only if $\gamma \neq 0$ on the circle and $\operatorname{wind}(\gamma) = -k$. For a feasible triple, $B(w_0) \neq 0$, since $G'(w_0) = W'(w_0)B(w_0) \neq 0$.

3.1. **Pockets.** On the circle, $w = e^{i\theta}$, the point $\zeta = W(w)$ runs along $\partial\Omega$ in the positive sense with tangent $iwW'(w)$, so the outward unit normal is $n = wW'/|W'|$ and

$$v = \frac{\zeta - z_0}{|W'|\, n}, \qquad \frac{d}{d\theta}|\zeta - z_0|^2 = 2|W'|^2 \operatorname{Im} v.$$

Thus $\alpha := |\arg v| \in [0, \pi]$ is the angle between $\zeta - z_0$ and the outward normal at ζ, and $\operatorname{Re} v < 0$ if and only if $\alpha > \pi/2$; we then call ζ *back-facing*.

Definition 3.1. A *pocket* is a point w of the circle at which $v(w)$ is a negative real number; its *depth* is $x(w) := -v(w) > 0$, and Z denotes the set of pockets. The *sign* $\varepsilon(w) \in \{+1, 0, -1\}$ of a pocket is the sign change of $\operatorname{Im} v$ at w as θ increases: $+1$ if $\operatorname{Im} v$ passes from positive to negative (then $|\zeta - z_0|$ has a local maximum along $\partial\Omega$ at $\zeta = W(w)$), -1 if it passes from negative to positive (a local minimum), and 0 if it does not change sign. A pocket is *on* for σ if $\sigma x > 1$ and *off* if $\sigma x < 1$.

Lemma 3.2 (frames)**.** *Let $\varphi_{w_0}(u) = (u + w_0)/(1 + \bar{w}_0 u)$, and let $\widetilde{W}$, $\widetilde{B}$ be as in the proof of Corollary 2.10, with quantities $\widetilde{v}$, $\widetilde{\sigma}$ for the triple $(\widetilde{W}, \widetilde{B}, 0)$. On the circle,*

$$\widetilde{v}(u) = P_{w_0}(\varphi_{w_0}(u))\, v(\varphi_{w_0}(u)), \qquad \widetilde{\sigma}(u) = \sigma(\varphi_{w_0}(u))/P_{w_0}(\varphi_{w_0}(u)),$$

and the same holds for each P_{a_j}. Hence the products σx and $P_{a_j} x$, the set of pockets and their signs do not depend on the frame*, that is, on whether we work with (W, B, w_0) or with $(\widetilde{W}, \widetilde{B}, 0)$. The* base-0 depth *$x_0 := P_{w_0}\, x = 2\pi|\zeta - z_0|\, d\omega/ds$ (s = arc length on $\partial\Omega$) is the depth in the frame $(\widetilde{W}, \widetilde{B}, 0)$.*

Proof. Write $\varphi = \varphi_{w_0}$. On the circle $u\varphi'(u)/\varphi(u) = |\varphi'(u)| = 1/P_{w_0}(\varphi(u))$, since φ is an orientation-preserving automorphism of the disk. Then $\widetilde{v}(u) = (W(\varphi(u)) - z_0)/(uW'(\varphi(u))\varphi'(u)) = v(\varphi(u))\, \varphi(u)/(u\varphi'(u))$, and $\widetilde{\sigma}(u) = u\varphi'(u)B'(\varphi(u))/B(\varphi(u)) = \sigma(\varphi(u))\, u\varphi'(u)/\varphi(u)$; the factor is positive, so the pockets and the signs are preserved (the parametrisation keeps its orientation). The same computation applies to each factor B_{a_j}, since $B_{a_j} \circ \varphi$ is a unimodular multiple of $B_{\varphi^{-1}(a_j)}$. Finally $d\omega/ds = P_{w_0}/(2\pi|W'|)$ and $x = |\zeta - z_0|/|W'|$ at a pocket. □

Lemma 3.3. *If Ω is a disk centred at z_0, there are no pockets. Otherwise* $\operatorname{Im} v$ *has finitely many zeros on the circle; in particular Z is finite and every sign is defined.*

Proof. By Lemma 3.2 we may take $w_0 = 0$, $W \in \mathcal{S}$. Then $v = W/(wW')$ is holomorphic and zero-free near the closed disk, with $v(0) = 1$. If $\operatorname{Im} v = 0$ on the whole circle, the harmonic function $\operatorname{Im} v$ vanishes in $\mathbb{D}$, so $v \equiv 1$, $wW' = W$ and $W(w) = w$: Ω is the unit disk and $v = 1$ has no pocket. Otherwise the real-analytic function $\operatorname{Im} v$ on the circle is not identically zero and has finitely many zeros. □

3.2. **The pocket formula.**

Theorem 3.4 (pocket formula)**.** *Let $\hat{\sigma} > 0$ be any continuous function on the circle with $\hat{\sigma}x \neq 1$ on Z. Then $1 + \hat{\sigma}v$ has no zero on the circle and*

$$\operatorname{wind}(1 + \hat{\sigma}v) = \sum_{w \in Z,\ \hat{\sigma}(w)x(w) > 1} \varepsilon(w).$$

Proof. Put $\hat{\gamma} = 1 + \hat{\sigma}v$. Then $\hat{\gamma} = 0$ means $v = -1/\hat{\sigma}$, a pocket with $\hat{\sigma}x = 1$, which is excluded. Since $\hat{\sigma} > 0$, $\hat{\gamma}$ is real exactly where v is real: where $v > 0$ we have $\hat{\gamma} > 1$, and at a pocket $\hat{\gamma} = 1 - \hat{\sigma}x$, which is negative if and only if $\hat{\sigma}x > 1$. So $\hat{\gamma}$ meets the negative real axis only at the finitely many pockets with $\hat{\sigma}x > 1$, and its winding number about 0 is the sum, over these points, of the local crossing numbers: $+1$ if $\operatorname{Im}\hat{\gamma}$ passes from positive to negative, -1 for the opposite passage, and 0 if it does not change sign. As $\operatorname{Im}\hat{\gamma} = \hat{\sigma}\operatorname{Im} v$, this local number is ε. □

With Proposition 2.11: a triple is feasible if and only if no pocket has $\sigma x = 1$ and the signs of the on pockets add up to $-k$. So, for σ with $\sigma x \neq 1$ on Z, feasibility depends on σ only through the set of pockets that σ switches on.

Corollary 3.5. *Let $w_0 = 0$. For every $c > 0$ such that $1/c$ is not a depth,* $\operatorname{wind}(1 + cv) \geq 0$. *Consequently, in any frame, for every $T > 0$ that is not a base-0 depth,*

$$\sum_{w \in Z,\ x_0(w) > T} \varepsilon(w) \ \geq\ 0.$$

In particular the signs of the pockets of largest base-0 depth add up to a non-negative number, and a unique deepest pocket has $\varepsilon \geq 0$ (a local maximum of $|\zeta - z_0|$, or a point where $\operatorname{Im} v$ does not change sign).

Proof. $1 + cv = (wW' + c(W - z_0))/(wW')$. The numerator is holomorphic near the closed disk, has no zero on the circle and vanishes at $w = 0$; the denominator has exactly one zero in the closed disk, at 0. So $\operatorname{wind}(1 + cv)$ is the number of zeros of the numerator in $\mathbb{D}$ minus one, which is ≥ 0. Apply Theorem 3.4 with $\hat{\sigma} = c = 1/T$ in the frame $(\widetilde{W}, \widetilde{B}, 0)$ of Lemma 3.2; for the last claim take T just below the largest base-0 depth. □

In a frame with $w_0 \neq 0$ the numerator $wW' + c(W - z_0)$ takes the value $w_0 W'(w_0) \neq 0$ at w_0, while wW' still vanishes at 0, and the argument gives only wind ≥ -1; this is why Corollary 3.5 uses base-0 depths.

Proposition 3.6 (guard)**.** *Let the triple be feasible and let $\mathcal{O}$ be the set of its on pockets. Then some pocket outside $\mathcal{O}$ has $\varepsilon = +1$ and base-0 depth at least* $\min_{\mathcal{O}} x_0$.

Proof. By Theorem 3.4, $\sum_{\mathcal{O}} \varepsilon = -k \leq -1$, so $\mathcal{O} \neq \emptyset$. Let $x_s = \min_{\mathcal{O}} x_0$ and choose $T < x_s$ such that no base-0 depth lies in $[T, x_s)$. Every pocket of $\mathcal{O}$ has $x_0 > T$, so by Corollary 3.5

$$\sum_{x_0 > T,\ w \notin \mathcal{O}} \varepsilon = \sum_{x_0 > T} \varepsilon - \sum_{\mathcal{O}} \varepsilon \ \geq\ 0 + k \ \geq 1,$$

and a pocket outside $\mathcal{O}$ with $x_0 > T$, hence $x_0 \geq x_s$, has $\varepsilon = +1$. □

Proposition 3.6 gives one off pocket of sign $+1$ at least as deep as the shallowest on pocket; it does not give one for each on pocket of sign -1.

3.3. **Independence.**

Lemma 3.7 (pocket independence)**.** *Let $Z = Z_0 \cup Z_1 \cup \cdots \cup Z_k$ be a partition, and write $P_j = P_{a_j}$. Assume*

(I1) *for each $j \geq 1$: $\sum\{\varepsilon(w) : w \in Z_j,\ P_j(w)x(w) > 1\} = -1$;*

(I2) *for each $j \geq 1$ and $w \in Z_j$: $P_j x \neq 1$, and $P_j x < 1$ implies $\sigma x < 1$; for $w \in Z_0$: $\sigma x < 1$.*

Then the triple is feasible, with gain $1/|B(w_0)| = \prod_j 1/|B_{a_j}(w_0)|$.

Proof. Since $\sigma \geq P_j$, a pocket of Z_j with $P_j x > 1$ is on; by (I2) all other pockets are off. So $\sigma x \neq 1$ on Z, the on pockets are those of the sets $\{w \in Z_j : P_j x > 1\}$, and by (I1) and Theorem 3.4 $\mathrm{wind}(\gamma) = -k$. Proposition 2.11 gives feasibility. □

By Lemma 3.2 the hypotheses do not depend on the frame. (I2) holds if, for each $j \geq 1$, $P_j x \neq 1$ on Z_j and $\sum_{l \neq j} P_{a_l}(w) < 1/x(w) - P_j(w)$ at every $w \in Z_j$ with $P_j(w)x(w) < 1$, and if $\sigma < 1/x$ on Z_0; here $P_{a_l}(w) \leq 2(1 - |a_l|)/|w - a_l|^2$. Lemma 3.7 combines zeros, and their gains multiply, provided that for each j adding the kernels P_{a_l}, $l \neq j$, to P_j switches on no pocket of Z_j that is off for P_j. Section 5 proves a converse direction: in every Blaschke model each zero switches on a pocket of its own.

3.4. **The gain identity and the sine bound.**

Proposition 3.8 (gain identity)**.** (a) *If the triple is feasible, then*

$$\log \frac{1}{|B(w_0)|} = -\int \log|\gamma|\, d\omega.$$

(b) *For every B as in* (M2) *with $B(w_0) \neq 0$, feasible or not,*

$$-\int \log|\gamma|\, d\omega = \log \frac{1}{|B(w_0)|} - \sum_b g_{\mathbb{D}}(w_0, b),$$

where b runs over the zeros of N in $\mathbb{D}$ with multiplicity and $g_{\mathbb{D}}$ is the Green function of the disk. So $-\int \log|1 + \sigma v|\, d\omega \leq \log(1/|B(w_0)|)$, with equality if and only if N has no zero in $\mathbb{D}$.

Proof. $N = B + ((W - z_0)/W')B'$ is holomorphic near the closed disk, $N(w_0) = B(w_0)$, and $|N| = |B||\gamma| = |\gamma|$ on the circle. (b) is Jensen's formula for N at w_0, written with the harmonic measure ω of w_0 (zeros of N on the circle make $\log|\gamma|$ integrable and contribute nothing). (a) is the case without zeros in the closed disk. □

Theorem 3.9 (sine bound)**.** *Let*

$$\mathcal{I}(\Omega, z_0) := \int_{\{\alpha > \pi/2\}} \log \frac{1}{\sin\alpha}\, d\omega,$$

the harmonic measure of the back-facing part of $\partial\Omega$, seen from z_0, weighted by $\log(1/\sin\alpha)$. Then $\mathcal{I}(\Omega, z_0) < \infty$, and every feasible triple on (W, w_0), of any degree, satisfies $\log(1/|B(w_0)|) \leq \mathcal{I}(\Omega, z_0)$.

Proof. Since $\sigma > 0$: where $\mathrm{Re}\, v \geq 0$ we have $|1 + \sigma v| \geq \mathrm{Re}(1 + \sigma v) \geq 1$, and where $\mathrm{Re}\, v < 0$, $|1 + \sigma v|$ is at least the distance from 0 to the line $1 + \mathbb{R}v$, which is $|\mathrm{Im}\, v|/|v| = \sin\alpha$. Insert this into Proposition 3.8(a). Finiteness: $\log(1/\sin\alpha) = \log|v| - \log|\mathrm{Im}\, v|$, $|v|$ is bounded above and below on the circle, and $\log|\mathrm{Im}\, v|$ is integrable because $\mathrm{Im}\, v$ is real-analytic with finitely many zeros (Lemma 3.3; if Ω is a disk about z_0 the back-facing set is empty). □

Remark 3.10. (1) If Ω is star-shaped with respect to z_0, then $\mathrm{Re}\, v \geq 0$ everywhere, $\mathcal{I}(\Omega, z_0) = 0$, and no feasible triple on (W, w_0) has $1/|B(w_0)| > 1$; in particular none has $k \geq 1$. (2) If $\alpha < \pi$ on the whole of $\partial\Omega$, there is no pocket; then $\mathrm{wind}(\gamma) = 0$ for every σ, and no feasible triple with $k \geq 1$ exists on (W, w_0), however large $\mathcal{I}(\Omega, z_0)$ is. So $e^{\mathcal{I}}$ can be far from the largest gain on (W, w_0).

4. Near-extremal structure

In this section we work in the base-0 frame: (W, B) satisfies (M1) and (M2) with $k = \deg B \geq 1$ and zero set A, normalised by $B(0) > 0$, the gain is $K_0 := 1/|B(0)|$, $\Omega = W(\mathbb{D})$ and $r_{\text{in}} := \operatorname{dist}(0, \partial\Omega) \geq 1/4$ (Koebe). A statement for Blaschke models assumes (M3) as well; several statements need $(WB)' \neq 0$ only on the open disk, and we say so. Put

$$\Phi(z) := z\, B(W^{-1}(z)),$$

holomorphic near $\overline{\Omega}$, with $\Phi(0) = 0$, $\Phi'(0) = B(0)$, $\Phi'(W(w))W'(w) = (WB)'(w)$ and $|\Phi(z)| = |z|$ on $\partial\Omega$. Lemma 4.1, Theorem 4.2, Proposition 4.4, Corollary 4.6 and the first statement of Proposition 4.7 hold for all admissible pairs: there $W \in \mathcal{S}$ need not extend past the circle, and Φ is defined on Ω only. On the circle, $v = W/(wW')$, $\sigma = wB'/B = |B'| > 0$ and $\gamma = 1 + \sigma v$ (Section 2.4 with $w_0 = 0$); "mean" is the integral against $d\theta/2\pi$. Since v is holomorphic and zero-free near the closed disk with $v(0) = 1$, the mean of $\log|v|$ is 0. The Koebe function is $\kappa(w) = w/(1-w)^2$; it maps $\mathbb{D}$ onto $\mathbb{C} \setminus (-\infty, -1/4]$, and the tip $-1/4 = \kappa(-1)$ corresponds to $\theta = \pi$.

4.1. **The inverse branch and the limit.**

Lemma 4.1 (inverse branches)**.** *Let (W, B) be an admissible pair. For every zero b of Φ in Ω, i.e. $b \in \{0\} \cup W(A)$, let E_b be the component of $\{z \in \Omega : |\Phi(z)| < r_{\text{in}}\}$ that contains b. Then Φ maps E_b bijectively onto $D(0, r_{\text{in}})$. In particular E_b contains no other zero of Φ, and the components E_b are pairwise disjoint.*

Proof. $\Phi' \neq 0$ in Ω. Let $z_n \in E_b$ leave every compact subset of E_b. We show that $|\Phi(z_n)| \to r_{\text{in}}$ along a subsequence; applied to every subsequence, this gives $|\Phi(z_n)| \to r_{\text{in}}$. Pass to a subsequence that converges on the Riemann sphere, to z. If $z \in \Omega$, then $z \notin E_b$ and $|\Phi(z)| = r_{\text{in}}$, since $|\Phi(z)| < r_{\text{in}}$ would put a disk about z into the component E_b. Otherwise the points $w_n = W^{-1}(z_n)$ tend to the circle (a limit point $w \in \mathbb{D}$ would give $z = W(w) \in \Omega$), so $|B(w_n)| \to 1$, and $|\Phi(z_n)| = |z_n|\,|B(w_n)|$ tends to $|z| \geq r_{\text{in}}$ if $z \in \partial\Omega$ and to ∞ if $z = \infty$. As $|\Phi| < r_{\text{in}}$ on E_b, the case $z = \infty$ does not occur, and in the other cases $|\Phi(z_n)| \to r_{\text{in}}$. So $\Phi : E_b \to D(0, r_{\text{in}})$ is proper, and it is a local homeomorphism, hence a covering of the simply connected disk by the connected set E_b, i.e. a bijection. Since Φ is injective on E_b and vanishes at b, E_b contains no other zero of Φ; two components containing different zeros are therefore different, hence disjoint. □

For $b = 0$ this is the argument in the proof of [23, Thm. 7.1].

Theorem 4.2. *Let (W_n, B_n) be admissible pairs with gains $K_{0,n} = 1/B_n(0) \to 4$. Then, after rotations $W_n \mapsto e^{i\theta_n} W_n(e^{-i\theta_n} w)$, $B_n \mapsto B_n(e^{-i\theta_n} w)$,*

$$W_n \to \kappa \quad \text{and} \quad B_n \to (1-w)^2/4 \qquad \text{locally uniformly in } \mathbb{D},$$

$\Phi_n/B_n(0) \to \kappa^{-1}$ locally uniformly on $\kappa(\mathbb{D})$, the zeros of B_n leave every compact subset of $\mathbb{D}$, and $\deg B_n \to \infty$.

Proof. We first pass to a subsequence; the last step removes it. Write $r_n := r_{\text{in}}(\Omega_n)$ and let $g_n : D(0, r_n) \to E_0$ be the inverse of Lemma 4.1 at $b = 0$.

Schwarz step. $h_n(u) := W_n^{-1}(g_n(r_n u))$ is univalent from $\mathbb{D}$ into $\mathbb{D}$, with $h_n(0) = 0$ and $h_n'(0) = r_n/B_n(0) = r_n K_{0,n} > 0$. By Schwarz's lemma $r_n K_{0,n} \leq 1$, so $r_n \leq 1/K_{0,n} \to 1/4$, while $r_n \geq 1/4$; hence $r_n \to 1/4$ and $h_n'(0) \to 1$. By compactness of $\mathcal{S}$, $W_n \to W \in \mathcal{S}$ along a subsequence. Points $\zeta_n \in \partial\Omega_n$ with $|\zeta_n| = r_n$ have a limit point ζ with $|\zeta| = 1/4$, and $\zeta \notin W(\mathbb{D})$: otherwise, by Hurwitz's theorem, W_n would take the value ζ_n in $\mathbb{D}$ for large n, although $\zeta_n \notin \Omega_n$. So $\operatorname{dist}(0, \partial W(\mathbb{D})) \leq 1/4$, which is the case of equality in Koebe's one-quarter theorem: W is a rotation of κ, and suitable rotations θ_n make it κ.

Limit. The self-maps h_n fix 0 and $h_n'(0) \to 1$, so by normality and Schwarz's lemma $h_n \to \mathrm{id}$. From $\Phi_n(g_n(r_n u)) = r_n u$ and $h_n(u) \neq 0$ for $u \neq 0$,

$$B_n(h_n(u)) = \frac{r_n u}{W_n(h_n(u))} \longrightarrow \frac{u}{4\kappa(u)} = \frac{(1-u)^2}{4} \qquad (u \neq 0),$$

and also at $u = 0$ by Vitali's theorem, both sides being bounded and holomorphic. The family (B_n) is equicontinuous on compact sets, so $B_n \to (1-w)^2/4$. The limit has no zero in $\mathbb{D}$, so by Hurwitz the zeros of B_n leave every compact set. If $\deg B_n = d$ along a subsequence, then for every $R < 1$ all zeros eventually satisfy $|a| > R$, so $B_n(0) = \prod |a_j| \geq R^d$, which contradicts $B_n(0) \to 1/4$ when $R^d > 1/4$; hence $\deg B_n \to \infty$. Finally, by Carathéodory's kernel theorem $W_n^{-1} \to \kappa^{-1}$ locally uniformly on $\kappa(\mathbb{D})$, so $\Phi_n(z) = z B_n(W_n^{-1}(z)) \to z(1-u)^2/4 = u/4$ with $u = \kappa^{-1}(z)$, and $B_n(0) \to 1/4$.

The whole sequence. The limits do not depend on the subsequence, so it can be dropped: choose θ_n to minimise $\max_{|w| \leq 1/2} |e^{i\theta} W_n(e^{-i\theta} w) - \kappa(w)|$. Every subsequence has a further subsequence along which the two steps above apply with some rotations θ_n'; along it the minimum tends to 0, and since no rotation other than the identity fixes κ, $\theta_n - \theta_n' \to 0$ modulo 2π, so the conclusions hold with θ_n. Hence they hold for the whole sequence. □

The Schwarz step alone gives $K_0 \leq 1/r_{\mathrm{in}} \leq 4$ for each (W, B) with $(WB)' \neq 0$ on $\mathbb{D}$ (compare [23, Thm. 7.1]: $B(0) \geq r_{\mathrm{in}} \geq 1/4$), with equality only if $r_{\mathrm{in}} = 1/4$, i.e. if W is a rotation of κ, which is excluded by (M1). So every single Blaschke model has gain strictly below 4; Section 7 gives quantitative forms.

Remark 4.3. The statements of Theorem 4.2 on W_n, on the zeros of B_n and on $\deg B_n$ are in [23, Rem. 7.3].

Proposition 4.4 (Green density)**.** *Under Theorem 4.2, the measures* $\Theta_n := \sum_j \log(1/|a_j|)\, \delta_{a_j/|a_j|}$ *on the circle (zeros of* B_n*, with multiplicity) have mass* $\log K_{0,n} \to \log 4$ *and converge weak-* to* $\Theta(s)\, ds$ *with*

$$\Theta(s) = -\frac{1}{\pi} \log\Big|\sin\frac{s}{2}\Big|.$$

Proof. As $|a| \to 1$, $g_{\mathbb{D}}(w, a) = (1 + o(1)) \log(1/|a|)\, (1 - |w|^2)/\big|a/|a| - w\big|^2$ uniformly for w in a compact subset of $\mathbb{D}$. So the Poisson integral of Θ_n tends to $\lim_n \log(1/|B_n(w)|) = \log(4/|1-w|^2)$ for every $w \in \mathbb{D}$. Every weak-$*$ limit of (Θ_n) has mass $\log 4$ and this Poisson integral, and a measure is determined by its Poisson integral. The positive harmonic function $\log(4/|1-w|^2)$ is the Poisson integral of $-2\log|\sin(s/2)|\, ds/2\pi$. □

4.2. σ away from the tip.

Lemma 4.5. *Let* $0 \leq \rho < 1$ *and* $t \in \mathbb{R}$*. If every zero* a *of* B *satisfies* $X(a) := (1-|a|^2)(1-\rho^2)/|1 - \bar{a}\rho e^{it}|^2 \leq 1/2$*, then*

$$\log \frac{1}{|B(\rho e^{it})|} \leq 4(1-\rho^2)\, \sigma(e^{it}).$$

Proof. With $w = \rho e^{it}$ we have $1 - |B_a(w)|^2 = X(a)$, so $g_{\mathbb{D}}(w, a) = -\frac{1}{2}\log(1 - X(a)) \leq X(a)$ for $X(a) \leq 1/2$. Put $b = \bar{a} e^{it}$. Then $|1-b| \leq |1 - \rho b| + (1-\rho) \leq 2|1-\rho b|$, since $|1 - \rho b| \geq 1 - \rho$; so $X(a) \leq 4(1-\rho^2)(1-|a|^2)/|1-b|^2 = 4(1-\rho^2) P_a(e^{it})$. Sum over the zeros. □

Corollary 4.6. *Under Theorem 4.2,* $\sigma_n \to \infty$ *uniformly on every closed arc* I *of the circle that avoids* $\theta = \pi$*.*

Proof. Let $\delta = \min(\mathrm{dist}(I, \pi), \pi/2)$ and $c = \cos(\delta/2) < 1$. On I, $|1 - \rho e^{it}| \leq 2|\sin(t/2)| + (1-\rho) \leq 2c + (1-\rho)$, and for $1 - \rho \leq 2(\sqrt{c} - c)$ this gives $\log 4 - 2\log|1 - \rho e^{it}| \geq -\log c > 0$. Since $X(a) \leq (1-|a|^2)(1+\rho)/(1-\rho)$ and the zeros leave compact sets, Lemma 4.5 applies for each

such ρ once n is large, and $\log(1/|B_n(\rho e^{it})|) \to \log 4 - 2\log|1-\rho e^{it}|$ uniformly in t. Hence $\liminf_n \min_I \sigma_n \geq -\log c/(4(1-\rho^2))$ for every $\rho \in [1-2(\sqrt{c}-c), 1)$; let $\rho \to 1$. □

Proposition 4.7 (the limit of the criterion function)**.** *Under Theorem 4.2, after its rotations, let $N_n = B_n + (W_n/W_n')B_n'$ be the function N of Section 2.4 for (W_n, B_n) and $w_0 = 0$. Then*

$$N_n \;\longrightarrow\; N_\infty := \frac{(1-w)^3}{4(1+w)} \qquad \textit{locally uniformly in } \mathbb{D}.$$

If moreover every (W_n, B_n) is a Blaschke model, then, locally uniformly in $w \in \mathbb{D}$,

$$\int_0^{2\pi} \frac{1-|w|^2}{|e^{i\theta}-w|^2} \log|\gamma_n(e^{i\theta})| \,\frac{d\theta}{2\pi} \;\longrightarrow\; \log|N_\infty(w)| = 3\log|1-w| - \log|1+w| - \log 4.$$

Proof. $W_nB_n \to \kappa(1-w)^2/4 = w/4$ locally uniformly, so $(W_nB_n)' \to 1/4$ and $W_n' \to \kappa'$ locally uniformly. Since $\kappa'(w) = (1+w)/(1-w)^3$ has no zero in $\mathbb{D}$, W_n' is bounded away from 0 on each compact set for large n, and $N_n = (W_nB_n)'/W_n' \to 1/(4\kappa') = N_\infty$. For a Blaschke model, $(W_nB_n)' = W_n'N_n$ has no zero on the closed disk, so N_n is zero-free on a neighbourhood of it and $\log|N_n|$ is harmonic there. The Poisson formula, with $|N_n| = |B_n||\gamma_n| = |\gamma_n|$ on the circle, shows that the left-hand side equals $\log|N_n(w)|$; and $\log|N_n| \to \log|N_\infty|$ locally uniformly, as N_∞ has no zero in $\mathbb{D}$. □

Remark 4.8. At $w = 0$ the second statement is the gain identity (Proposition 3.8), since $N_n(0) = B_n(0) \to 1/4$. The limit has a pole at the tip $w = -1$, where κ' vanishes, while for Blaschke models every N_n is bounded on the closed disk; so for them the convergence is not uniform on $\mathbb{D} \cap U$ for any neighbourhood U of -1. The function $(1-w)^3/(1+w)$ is outer, so $\log|N_\infty|$ is the Poisson integral of $-G$, where

$$G(s) := -\log|N_\infty(e^{is})| = \log\frac{|\cos(s/2)|}{|\sin(s/2)|^3} = \log 4 + \log|\kappa'(e^{is})|,$$

and the mean of G is $\log 4$. Thus the Poisson averages of $\log|\gamma_n|$ converge to those of $-G$; here G is positive for $|s| < 1.944\ldots$ and tends to $-\infty$ at the tip. The measures $\log|\gamma_n(e^{i\theta})|\,d\theta$ themselves have no weak-$*$ convergent subsequence: by Corollary 4.11 the means of the positive and negative parts $\ell_n^{\pm}$ of $\ell_n = -\log|\gamma_n|$ both tend to infinity, so the total variation tends to infinity, while a weak-$*$ convergent sequence of measures is bounded.

Lemma 4.9. *For every finite Blaschke product B with $B(0) \neq 0$, $\sigma \geq (K_0-1)/(K_0+1)$ on the whole circle. Consequently every pocket of depth greater than $(K_0+1)/(K_0-1)$ is on; for a Blaschke model this threshold exceeds $5/3$.*

Proof. $P_a \geq (1-|a|)/(1+|a|) = \tanh(\ell_a/2)$ with $\ell_a = \log(1/|a|)$, and $\ell \mapsto \tanh(\ell/2)$ is concave with value 0 at 0, hence subadditive; so $\sigma \geq \tanh(\sum_j \ell_{a_j}/2) = \tanh(\frac{1}{2}\log K_0) = (K_0-1)/(K_0+1)$. For a Blaschke model $K_0 < 4$. □

4.3. **The cost of the gain.**

Lemma 4.10 (cost)**.** *For every $\sigma > 0$, $|1+\sigma v| \geq \max(1, \sigma|v|/2)$ if $\operatorname{Re} v \geq 0$, and $|1+\sigma v| \geq \sin\alpha \max(1, \sigma|v|/2)$ if $\operatorname{Re} v < 0$. Hence every Blaschke model satisfies*

$$\log K_0 \;\leq\; \mathcal{I}(\Omega, 0) - \operatorname{mean}\log^+\frac{\sigma|v|}{2} \;\leq\; \mathcal{I}(\Omega, 0) + \log 2 - \operatorname{mean}\log\sigma.$$

Proof. If $\operatorname{Re} v \geq 0$, then $|1+\sigma v| \geq \max(1, \sigma|v|)$. If $\operatorname{Re} v < 0$, then $|1+\sigma v| \geq \sin\alpha$ (proof of Theorem 3.9), and $|1+\sigma v| \geq \sigma|v| - 1 \geq \sigma|v|/2$ when $\sigma|v| \geq 2$. Take logarithms and use Proposition 3.8(a); then $\log^+ \geq \log$ and $\operatorname{mean}\log|v| = 0$. □

Corollary 4.11. *Let (W_n, B_n) be Blaschke models with $K_{0,n} \to 4$. Then* $\operatorname{mean}\log\sigma_n \to \infty$, *hence $\mathcal{I}(\Omega_n, 0) \to \infty$. Moreover, with $\ell_n = -\log|\gamma_n|$, the mean of $\ell_n^- = \log^+|\gamma_n|$ is at least* $\operatorname{mean}\log\sigma_n - \log 2 \to \infty$, *and the mean of the positive part ℓ_n^+ is at least* $\log K_{0,n} + \operatorname{mean}\log\sigma_n - \log 2$.

Proof. These quantities do not change under rotations or unimodular factors, so Theorem 4.2 and Corollary 4.6 apply. Thus $\sigma_n \to \infty$ uniformly on a fixed closed arc I avoiding π, and $\sigma_n \geq (K_{0,n}-1)/(K_{0,n}+1) \to 3/5$ off I (Lemma 4.9); so $\operatorname{mean}\log\sigma_n \to \infty$, and Lemma 4.10 gives $\mathcal{I}(\Omega_n, 0) \to \infty$. Pointwise $\log^+|\gamma| \geq \log(\sigma|v|/2)$ (if $\sigma|v| \geq 2$, then $|\gamma| \geq \sigma|v| - 1 \geq \sigma|v|/2 \geq 1$; otherwise the right side is negative). Take means, use $\operatorname{mean}\log|v| = 0$ and $\operatorname{mean}\ell_n = \log K_{0,n}$ (Proposition 3.8). □

Proposition 4.12 (where $|\gamma| < 1$)**.** *For every Blaschke model, $|\gamma(w)| < 1$ implies* $\operatorname{Re} v(w) < 0$ *and $|W'(w)| > \sigma(w)/8$. In particular, for Blaschke models as in Theorem 4.2, the positive part of $-\log|\gamma_n|$ on a closed arc I avoiding π is carried by a part of $\partial\Omega_n$ where the harmonic density $d\omega/ds = 1/(2\pi|W_n'|)$ tends to 0 uniformly.*

Proof. $|1+\sigma v| < 1$ means $\sigma|v|^2 < -2\operatorname{Re} v$, i.e. $\operatorname{Re} v < 0$ and $\sigma|v| < -2\cos\alpha \leq 2$. As $|W| \geq r_{\text{in}} \geq 1/4$ on the circle, $|W'| = |W|/|v| > \sigma/8$. Then apply Corollary 4.6. □

Remark 4.13. For $v = v_W$, by Propositions 2.11 and 3.8, the supremum of $\log K_0$ over the Blaschke models on W with $\deg B \geq 1$ is the supremum of $-\operatorname{mean}\log|1+\sigma v|$ over $\sigma = \sum_{j\leq k} P_{a_j}$ with $k \geq 1$, $1+\sigma v \neq 0$ on the circle and $\operatorname{wind}(1+\sigma v) = -k$.

4.4. **Exit points and an identity at the first exit.** For a Blaschke model, every zero $b \in \{0\} \cup W(A)$ of Φ in Ω is simple ($\Phi' \neq 0$ on $\overline{\Omega}$), so Φ has a holomorphic inverse g_b near 0 with $g_b(0) = b$. Let R_b be the supremum of the $r > 0$ such that g_b extends holomorphically to $D(0,r)$ with values in Ω. Since $\Phi(g_b(w)) = w$ there, g_b is injective on $D(0, R_b)$.

Lemma 4.14 (exit points)**.** *Let (W, B) be a Blaschke model and $b \in \{0\} \cup W(A)$. Then $r_{\text{in}} \leq R_b < \infty$. Call $z^* \in \partial\Omega$ an* exit point *of b if $z^* = \lim g_b(w_n)$ for some $w_n \in D(0, R_b)$ with $|w_n| \to R_b$. Exit points exist, and every exit point z^* satisfies $|z^*| = |\Phi(z^*)| = R_b$ and the following properties, where $q^* := \Phi(z^*)$, $u^* := q^*/R_b$, $w^* := W^{-1}(z^*)$ and $h_b(u) := W^{-1}(g_b(R_b u))$.*

(i) *g_b extends holomorphically to a neighbourhood of q^* with $g_b(q^*) = z^*$; so h_b is a holomorphic map of $\mathbb{D}$ into $\mathbb{D}$, holomorphic at u^*, with $h_b(u^*) = w^*$ and $h_b(0) = W^{-1}(b)$.*

(ii) *$|z| \geq |z^*|$ for all $z \in \partial\Omega$ near z^*: the point z^* is a local minimum of $|z|$ along $\partial\Omega$.*

(iii) *$u^* h_b'(u^*)/h_b(u^*) = |h_b'(u^*)| > 0$, $v(w^*)$ is real, and*

$$\sigma(w^*) + \frac{1}{v(w^*)} = \frac{1}{|h_b'(u^*)|}.$$

Proof. The translated pair $(\Omega - b,\ \Phi(b+\cdot)/\Phi'(b))$ is a domain pair, and its inverse germ at 0 is $w \mapsto g_b(\Phi'(b)w) - b$. Proposition 2.6 and its Steps 1–2 give $R_b < \infty$ and an exit point; for any exit point z^*, a subsequence of the w_n tends to some q^*, and z^* is a limit of $g_b(w_n)$ with $w_n \to q^*$, $|q^*| = R_b$ and $\Phi(z^*) = q^*$; then $|z^*| = |\Phi(z^*)| = R_b$, and $R_b \geq r_{\text{in}}$ as $z^* \in \partial\Omega$.

(i) As $\Phi'(z^*) \neq 0$, Φ is injective on a neighbourhood V_0 of z^* and $(\Phi|_{V_0})^{-1}$ is defined on a disk $D(q^*, s)$. For large n, $g_b(w_n) \in V_0$, so the open set $\{w \in D(0,R_b) \cap D(q^*,s) : g_b(w) \in V_0\}$ is non-empty, and there $g_b = (\Phi|_{V_0})^{-1}$; by the identity theorem this holds on the convex set $D(0,R_b) \cap D(q^*,s)$. This is the identity-theorem part of Step 1 of Proposition 2.6, used at a boundary point (V_0 need not lie in Ω). The rest of (i) follows, as W^{-1} is holomorphic near $\overline{\Omega}$.

(ii) Let $z \in \partial\Omega$ be so close to z^* that $z \in V_0$ and $\Phi(z) \in D(q^*, s)$. If $|\Phi(z)| < R_b$, then $z = (\Phi|_{V_0})^{-1}(\Phi(z)) = g_b(\Phi(z)) \in \Omega$, which is impossible. So $|z| = |\Phi(z)| \geq R_b$.

(iii) $\Phi(g_b(R_b u)) = R_b u$ reads $W(h_b(u))\,B(h_b(u)) = R_b u$. Taking $u\,\frac{d}{du}\log$ at u^* and using $wW'/W = 1/v$ and $wB'/B = \sigma$ on the circle,

$$\frac{u^* h_b'(u^*)}{h_b(u^*)}\Big(\frac{1}{v(w^*)} + \sigma(w^*)\Big) = 1;$$

in particular $h_b'(u^*) \neq 0$. Write $u = e^{s+it}$ and $X = \log|h_b(e^{s+it})|$ near $(0,\theta^*)$, $u^* = e^{i\theta^*}$. Then $X \leq 0$ for $s \leq 0$ and $X(0,\theta^*) = 0$, so $X_t = 0$ and $X_s \geq 0$ there, and by the Cauchy–Riemann equations for $\log h_b$, $uh_b'/h_b = X_s - iX_t$ is real and ≥ 0 at u^*; it is non-zero by the display. Since $|u^*| = |h_b(u^*)| = 1$, it equals $|h_b'(u^*)|$. □

Theorem 4.15 (the first exit)**.** *Let (W,B) be a Blaschke model, $b = 0$, and let z^*, w^*, $u^* = e^{i\theta^*}$, $h = h_0$ be as in Lemma 4.14. Then:*

(i) *$R_0K_0 \leq \min_{a\in A} \hat{c}(|a|) < 1$, with $\hat{c}(r) = 2r\log(1/r)/(1-r^2)$.*

(ii) *Let ν be the Herglotz measure of the positive harmonic function $\log|u/h(u)|$; its mass is $\log(1/(R_0K_0))$. Then*

$$\sigma(w^*) + \frac{1}{v(w^*)} = \frac{1}{1+\mathcal{U}}, \qquad \mathcal{U} := \int \frac{d\nu(\varphi)}{2\sin^2((\varphi-\theta^*)/2)} \;\geq\; \frac{1}{2}\log\frac{1}{R_0K_0} > 0,$$

and hence $\sigma(w^*) + 1/v(w^*) \leq 2/(2+\log(1/(R_0K_0))) \leq (1+R_0K_0)/2 < 1$.

(iii) *Either $v(w^*) > 0$ (front case), and then $\sigma(w^*) < 1 - 1/v(w^*) < 1$; or $v(w^*) < 0$ (back case), and then w^* is an on pocket with $\varepsilon(w^*) = -1$.*

Proof. (i) h maps $\mathbb{D}$ into $\mathbb{D}$ with $h(0) = 0$ and $h'(0) = R_0/B(0) = R_0K_0 > 0$, and h omits every $a \in A$: if $g_0(w) = W(a)$, then $w = \Phi(W(a)) = W(a)B(a) = 0$, but $g_0(0) = 0 \neq W(a)$. With hyperbolic densities of curvature -4, $\lambda_{\mathbb{D}}(z) = 1/(1-|z|^2)$ and, on the punctured disk $\mathbb{D}^* = \mathbb{D}\setminus\{0\}$, $\lambda_{\mathbb{D}^*}(z) = 1/(2|z|\log(1/|z|))$ (cover $\mathbb{D}^*$ by the upper half-plane through $\tau \mapsto e^{i\tau}$); the automorphism B_a maps $\mathbb{D}\setminus\{a\}$ onto $\mathbb{D}^*$ with $|B_a(0)| = |a|$ and $|B_a'(0)| = 1-|a|^2$, so the density of $\mathbb{D}\setminus\{a\}$ at 0 is $(1-|a|^2)/(2|a|\log(1/|a|)) = 1/\hat{c}(|a|)$. Schwarz–Pick gives $h'(0) \leq \hat{c}(|a|)$, and $\hat{c}(r) = \ell/\sinh\ell < 1$ with $\ell = \log(1/r)$.

(ii) h is univalent with no zero besides 0, so $\log(u/h(u))$ is holomorphic in $\mathbb{D}$, with real part ≥ 0 (Schwarz) and value $\log(1/(R_0K_0))$ at 0; hence $\log(u/h(u)) = \int \frac{e^{i\varphi}+u}{e^{i\varphi}-u}\,d\nu(\varphi)$. Apply $u\frac{d}{du}$ at $u = ru^*$, $0 < r < 1$:

$$1 - \frac{ru^*h'(ru^*)}{h(ru^*)} = \int \frac{2ru^*e^{i\varphi}}{(e^{i\varphi}-ru^*)^2}\,d\nu(\varphi).$$

As $r \to 1$, the left side tends to $1 - u^*h'(u^*)/h(u^*)$, the kernel tends to $-1/(2\sin^2((\varphi-\theta^*)/2))$, and its modulus is $2r/((1-r)^2 + 4r\sin^2((\varphi-\theta^*)/2)) \leq 1/(2\sin^2((\varphi-\theta^*)/2))$. This bound is ν-integrable: on an arc I about θ^* on which h is holomorphic and zero-free, $\nu = H\,d\varphi/2\pi$ with $H(\varphi) = \log(1/|h(e^{i\varphi})|)$ (ν is the weak-$*$ limit of $\log|u/h|(re^{i\varphi})\,d\varphi/2\pi$, which converges uniformly on compact subarcs of I); $H \geq 0$ is real-analytic with $H(\theta^*) = 0$, so it has a zero of even order ≥ 2 there, and $H/\sin^2((\varphi-\theta^*)/2)$ is bounded near θ^*; off I the kernel is bounded. (H does not vanish identically near θ^*: otherwise $|z| = R_0$ on an arc of $\partial\Omega$, so $|W(e^{it})| = R_0$ on an arc and hence, by real-analyticity, on the whole circle; then $\Omega = D(0,R_0)$, $W = \mathrm{id}$, and $(wB)'$, the derivative of a Blaschke product of degree $k+1 \geq 2$, has k zeros in $\mathbb{D}$, against (M3).) By dominated convergence $u^*h'(u^*)/h(u^*) = 1 + \mathcal{U}$, so $|h_0'(u^*)| = 1 + \mathcal{U}$ in Lemma 4.14(iii), and $\mathcal{U} \geq \nu(\partial\mathbb{D})/2$ as $\sin^2 \leq 1$. Finally $2/(2+m) \leq (1+e^{-m})/2$ for $m \geq 0$, since $f(m) = (2+m)(1+e^{-m}) - 4$ has $f(0) = 0$ and $f'(m) = 1 - (1+m)e^{-m} \geq 0$; and $m = \log(1/(R_0K_0)) > 0$ by (i).

(iii) By (ii), $\sigma(w^*) + 1/v(w^*) < 1$, which gives the front case. In the back case $\gamma(w^*) = v(w^*)/|h_0'(u^*)| < 0$, so the pocket w^* is on. $\operatorname{Im} v$ is not identically zero (else $W = \mathrm{id}$ and $v = 1$), so its zeros are isolated (Lemma 3.3), and by Lemma 4.14(ii) and $\frac{d}{d\theta}|W(e^{i\theta})|^2 = 2|W'|^2\operatorname{Im} v$, $\operatorname{Im} v$ passes from negative to positive at w^*: $\varepsilon(w^*) = -1$. □

4.5. **The tip.**

Theorem 4.16. (a) *For the limit density of Proposition 4.4,*

$$\int_0^{2\pi} \frac{\Theta(s)\,ds}{2\cos^2(s/2)} = 1.$$

(b) *Along every sequence of Blaschke models with* $K_{0,n} \to 4$, $\liminf_n \min_{\partial\mathbb{D}} \sigma_n \geq 1$.

Proof. (a) With $x = s/2$ the integral is $\frac{1}{\pi}\int_0^\pi (-\log\sin x)\,\frac{dx}{\cos^2 x} = \frac{1}{\pi}\big([-\log(\sin x)\tan x] + \int_0^\pi 1\,dx\big) = 1$, integrating by parts on $(0,\pi/2)$ and $(\pi/2,\pi)$; the bracket vanishes at 0, $\pi/2$ and π.

(b) $\min\sigma_n$ does not change under rotations or unimodular factors, so it suffices to treat a subsequence normalised as in Theorem 4.2. Let $t_n \to \pi$ and $\eta > 0$, and let ψ be continuous on the circle, $0 \leq \psi \leq 1$, $\psi = 0$ on $|s-\pi| \leq \eta$ and $\psi = 1$ on $|s-\pi| \geq 2\eta$. For a zero $a = \rho e^{is}$ with $|s-\pi| \geq \eta$ and for $|t_n - \pi| < \eta/2$,

$$P_a(e^{it_n}) = \frac{1-\rho^2}{1-2\rho\cos(t_n - s) + \rho^2} = (1+o(1))\log\frac{1}{\rho}\cdot\frac{1}{2\sin^2((t_n-s)/2)}$$

uniformly in such a, since all zeros tend to the circle. So $\sigma_n(e^{it_n}) \geq (1+o(1))\int \psi(s)\,(2\sin^2((t_n - s)/2))^{-1}\,d\Theta_n(s)$; the kernel converges uniformly on the support of ψ to $1/(2\cos^2(s/2))$, and $\Theta_n \to \Theta(s)\,ds$ weak-$*$ with bounded masses (Proposition 4.4). Hence

$$\liminf_n \sigma_n(e^{it_n}) \;\geq\; \int \psi(s)\,\frac{\Theta(s)\,ds}{2\cos^2(s/2)},$$

and letting $\eta \to 0$ gives ≥ 1 by (a). A sequence of minimum points of σ_n has a convergent subsequence; if its limit is not π, Corollary 4.6 gives $\sigma_n \to \infty$ there, and otherwise the argument above applies. □

Part (a) is a statement about the limit object; its value 1 is $|B'_\infty(-1)|$ for $B_\infty(w) = (1-w)^2/4$, which is unimodular at the tip $w = -1$, as $B'_\infty(w) = -(1-w)/2$. Part (a) does not say that $\sigma_n(-1) \to 1$: a zero a_n close to -1 adds $P_{a_n}(-1) = (1-|a_n|^2)/|1+a_n|^2$, which part (b) does not bound from above. For comparison, Lemma 4.9 gives only $\sigma \geq (K_0-1)/(K_0+1)$, which tends to $3/5$.

Corollary 4.17. *Let* (W_n, B_n) *be Blaschke models with* $K_{0,n} \to 4$ *and* $k_n = \deg B_n$.

(i) *For every* $\eta > 0$ *and all large* n, (W_n, B_n) *has at least* k_n *on pockets with* $\varepsilon = -1$ *and depth less than* $1+\eta$; *and* $k_n \to \infty$.

(ii) *Along every subsequence of front cases (Theorem 4.15(iii)),* $\min\sigma_n \to 1$ *and* $\sigma_n(w_n^*) \to 1$; *hence* $\mathcal{U}_n \to 0$ *and* $v_n(w_n^*) \to +\infty$, *i.e.* $|W_n'(w_n^*)| \to 0$. *After the rotations of Theorem 4.2,* $w_n^* \to -1$, *the tip.*

Proof. (i) Each Blaschke model has finitely many pockets, so the depths of all the Blaschke models form a countable set; choose $\eta' \in (0,\eta)$ such that $T = 1+\eta'$ is a depth of no Blaschke model of the sequence. By Theorem 4.16(b), every pocket of depth greater than T is on for large n. By Corollary 3.5 the pockets of depth greater than T have signs adding up to ≥ 0, and by Theorem 3.4 the on pockets have signs adding up to $-k_n$. So the on pockets of depth less than T have signs adding up to at most $-k_n$, and at least k_n of them have $\varepsilon = -1$. $k_n \to \infty$ by Theorem 4.2.

(ii) In the front case $\min\sigma_n \leq \sigma_n(w_n^*) \leq 1/(1+\mathcal{U}_n) - 1/v_n(w_n^*) < 1$, and Theorem 4.16(b) gives the limits; then $1/(1+\mathcal{U}_n) \to 1$ and $1/v_n(w_n^*) \to 0$, and $|W_n'(w_n^*)| = |W_n(w_n^*)|/v_n(w_n^*)$ with $|W_n(w_n^*)| = R_0 \leq 1$. Since $\sigma_n(w_n^*) < 1$, Corollary 4.6 forces $w_n^* \to -1$ after the rotations. □

In the back case Theorem 4.15 gives only $|\gamma(w^*)| \leq |v(w^*)|$, and Theorem 4.16(b) gives only $\liminf \min\sigma_n \geq 1$.

5. Ownership

Let (W,B) be a Blaschke model in the base-0 frame, with zeros $a_1,\dots,a_k$. They are simple, since $(WB)'(a_j) = W(a_j)B'(a_j)$ and $(WB)' \neq 0$ on the closed disk by (M3); hence they are distinct. Put $\zeta_j = W(a_j)$ and $P_j = P_{a_j}$. The base points $0,\zeta_1,\dots,\zeta_k$ are distinct zeros of Φ, and each has exit points (Lemma 4.14). We say that a point w^* of the circle is *owned* by a_j if $W(w^*)$ is an exit point of ζ_j.

Theorem 5.1 (ownership)**.** *Let w^* be owned by a_j, and $z^* = W(w^*)$. Then:*

(i) *$v(w^*) < 0$: w^* is a pocket, of depth $x = -v(w^*)$, and $\varepsilon(w^*) = -1$;*
(ii) *$\sigma(w^*)\, x > 1$ and $(\sigma(w^*) - P_j(w^*))\, x \le 1$, i.e. $\sum_{l\neq j} P_l(w^*) \le 1/x < \sigma(w^*)$: the pocket is on, and it would not be on without the kernel of a_j;*
(iii) *$r_{\text{in}} \le R_{\zeta_j} = |z^*| \le |W(a_j)| \prod_{l\neq j} |B_{a_l}(a_j)| \le |W(a_j)|$;*
(iv) *exit points of different base points are different; in particular a point is owned by at most one zero.*

Proof. Let $h = h_{\zeta_j}$ be as in Lemma 4.14 and $\beta = |h'(u^*)|$, so $h(0) = a_j$, $h(u^*) = w^*$ and $\sigma(w^*) + 1/v(w^*) = 1/\beta$, where β is the angular derivative of h at u^*. Julia's lemma [30, 39] at the point 0 gives

$$\beta \;\ge\; \frac{|w^* - h(0)|^2}{1 - |h(0)|^2} = \frac{|w^* - a_j|^2}{1 - |a_j|^2} = \frac{1}{P_j(w^*)}.$$

So $0 < 1/\beta = \sigma(w^*) + 1/v(w^*) \le P_j(w^*)$, i.e. $1/v(w^*) \le -\sum_{l\neq j} P_l(w^*) \le 0$. Since $0 < |v| < \infty$ on the circle, $v(w^*) < 0$; then $1/x = \sigma - 1/\beta < \sigma$ and $1/x \ge \sigma - P_j$. This is (ii) and the first part of (i). The sign: by Lemma 4.14(ii), $|W(e^{i\theta})|^2$ has a local minimum at w^*; $\operatorname{Im} v$ is not identically zero (that would give $W = \mathrm{id}$ and $v = 1 > 0$), so its zeros are isolated, and as $\frac{d}{d\theta}|W(e^{i\theta})|^2 = 2|W'|^2 \operatorname{Im} v$, $\operatorname{Im} v$ passes from negative to positive at w^*. (A zero of $\operatorname{Im} v$ without a sign change is a point where $|W(e^{i\theta})|^2$ is strictly monotone.) (iii) The first inequality is Lemma 4.14. Schwarz–Pick for $f = W^{-1} \circ g_{\zeta_j}$ on $D(0, R_{\zeta_j})$, with $f(0) = a_j$, gives $R_{\zeta_j}|f'(0)| \le 1 - |a_j|^2$, and $f'(0) = 1/(W(a_j)B'(a_j))$, $(1 - |a_j|^2)|B'(a_j)| = \prod_{l\neq j} |B_{a_l}(a_j)|$. (iv) If base points b and c have a common exit point z^*, then $R_b = R_c = |z^*| =: R$, and g_b, g_c both equal $(\Phi|_{V_0})^{-1}$ on $D(0, R) \cap D(\Phi(z^*), s)$ (proof of Lemma 4.14(i)); by the identity theorem $g_b = g_c$ on $D(0, R)$, so $b = g_b(0) = g_c(0) = c$. □

Property (ii) alone does not identify the owner: one pocket can satisfy it for two zeros (for instance $P_1 = P_2 = p$, $\sigma = 2p$ and $1/2 < px \le 1$). Uniqueness (iv) is a property of exit points.

Corollary 5.2 (the count)**.** *Let (W, B) be a Blaschke model and choose one owned point for each zero. These are k distinct on pockets with $\varepsilon = -1$, so the remaining on pockets have signs adding up to 0. If ζ_j has m_j exit points ($m_j \ge 1$, finite), there are at least $\sum_j (m_j - 1)$ on pockets with $\varepsilon = +1$, and at least $\sum_j (m_j - 1) + 1$ if some exit point z^* of the base point 0 is a back case of Theorem 4.15.*

Proof. By Theorem 5.1(i),(ii),(iv), the owned points are distinct on pockets with $\varepsilon = -1$; by Proposition 2.11 and Theorem 3.4 the signs of all on pockets add up to $-k$. Each m_j is finite since the exit points are pockets (Lemma 3.3). In the back case $W^{-1}(z^*)$ is a further on pocket with $\varepsilon = -1$ (Theorem 4.15(iii)), distinct from the owned points by Theorem 5.1(iv). So the on pockets with $\varepsilon = -1$ number at least $\sum_j m_j$ (plus one in the back case), and those with $\varepsilon = +1$ at least that number minus k. □

Corollary 5.3 (shallow pockets off the tip)**.** *Let (W_n, B_n) be Blaschke models as in Theorem 4.2, after its rotations, and let I be a closed arc of the circle that avoids $\theta = \pi$ (possibly a single point). Then*

$$m_n := \min_j \min_{e^{it}\in I} \big(\sigma_n - P_{a_j}\big)(e^{it}) \;\longrightarrow\; \infty,$$

the first minimum being over all zeros a_j of B_n. Consequently every off pocket of (W_n, B_n) in I has depth $x < 1/\min_I \sigma_n \le 1/m_n$, every owned pocket in I has depth $x \le 1/m_n$, and at all these pockets $|W_n'| \ge r_{\text{in}} m_n \ge m_n/4$.

Proof. Let $\delta = \min(\operatorname{dist}(I, \pi), \pi/2)$, $c = \cos(\delta/2)$ and $\rho \in [1 - 2(\sqrt{c} - c), 1)$. As in the proof of Corollary 4.6, $\log 4 - 2\log|1 - \rho e^{it}| \ge -\log c > 0$ for $e^{it} \in I$. With X as in Lemma 4.5, the maximum

of $X(a)$ over the zeros of B_n tends to 0 uniformly in t, since the zeros leave every compact set. Fix j and apply Lemma 4.5 to B_n/B_{a_j}, whose zeros are those of B_n other than a_j; with $w = \rho e^{it}$,

$$\log \frac{1}{|B_n(w)|} - g_{\mathbb{D}}(w, a_j) = \log \frac{1}{|(B_n/B_{a_j})(w)|} \le 4(1-\rho^2)\,(\sigma_n - P_{a_j})(e^{it}),$$

and $g_{\mathbb{D}}(w, a_j) \le X(a_j)$ (proof of Lemma 4.5). Since $\log(1/|B_n(\rho e^{it})|) \to \log 4 - 2\log|1-\rho e^{it}|$ uniformly in t, and neither error term depends on j or on t, we get $\liminf_n m_n \ge -\log c/(4(1-\rho^2))$; let $\rho \to 1$. An off pocket has $\sigma_n x < 1$, and an owned pocket has $(\sigma_n - P_{a_j})x \le 1$ by Theorem 5.1(ii). At a pocket $|v_n| = x$, so $|W_n'| = |W_n|/x$, and $|W_n| \ge r_{\text{in}} \ge 1/4$ on the circle. □

So on each fixed closed arc avoiding the tip, the owned pockets become shallow and sit where the harmonic density $d\omega/ds = 1/(2\pi|W_n'|)$ tends to 0. If the owned pockets accumulate at the tip, the corollary says nothing about them.

Theorem 5.1 gives a necessary condition for feasibility, in the converse direction of Lemma 3.7. That lemma gives a sufficient condition: in a group of pockets with signs adding up to -1, the kernel P_j alone switches the pockets on. Ownership gives less: each zero a_j owns a pocket w at a local minimum of $|W|$ with $\sigma(w)x(w) > 1 \ge (\sigma - P_j)(w)x(w)$. For $k \ge 2$ it does not give $P_j x > 1$.

6. Symmetric pairs

Fix an integer $n \ge 1$, the order of the symmetry. In this section a *pair* (W, B) satisfies (M1) and (M2) of Definition 2.8. We use the criterion of Proposition 2.11 with base point $w_0 = 0$, so that $v_W = W/(wW')$ and $\sigma_B = wB'/B$ on the circle. A pair is *n-fold symmetric* if $W(e^{2\pi i/n}w) = e^{2\pi i/n}W(w)$ and the zero set of B, counted with multiplicities, is invariant under $w \mapsto e^{2\pi i/n}w$. By the uniqueness of the normalised Riemann map, $W(e^{2\pi i/n}w) = e^{2\pi i/n}W(w)$ holds if and only if $W(\mathbb{D})$ is invariant under the rotation by $2\pi/n$.

6.1. **The root transform.** Let $\Psi \in \mathcal{S}$ be holomorphic and univalent on $\{|u| < R\}$, $R > 1$. Then $\Psi(u)/u$ is holomorphic there, equal to 1 at 0, and zero-free, because Ψ is injective and $\Psi(0) = 0$; so $\hat\Psi = (\Psi(u)/u)^{1/n}$, the branch with $\hat\Psi(0) = 1$, exists and has no zero. Put

$$\Psi_n(w) = w\,\hat\Psi(w^n) \qquad (|w| < R^{1/n}), \qquad \text{and} \qquad B(w) = \widetilde B(w^n)$$

for a finite Blaschke product $\widetilde B$. This is the n-th root transform of [23, Rem. 8.6], where its admissibility condition is given in the open-disk setting. We record the closed-disk form in the notation of Proposition 2.11.

Lemma 6.1 ([23, Rem. 8.6])**.** (a) *Ψ_n satisfies* (M1) *with radius $R^{1/n} > 1$, and $\Psi_n(e^{2\pi i/n}w) = e^{2\pi i/n}\Psi_n(w)$.* (b) *$B$ is a finite Blaschke product of degree $n \deg \widetilde B$, and $B(0) = \widetilde B(0)$.* (c) *(Ψ_n, B) is a Blaschke model if and only if $\widetilde\gamma := 1 + n\,\sigma_{\widetilde B}\,v_\Psi$ has no zero on the circle and* $\operatorname{wind}(\widetilde\gamma) = -\deg \widetilde B$.

Proof. (a) $\Psi_n(e^{2\pi i/n}w) = e^{2\pi i/n}\Psi_n(w)$ and $\Psi_n^n = \Psi(w^n)$. If $\Psi_n(w_1) = \Psi_n(w_2)$, then $\Psi(w_1^n) = \Psi(w_2^n)$, so $w_1^n = w_2^n$ and $w_2 = e^{2\pi ij/n}w_1$; then $\Psi_n(w_1) = e^{2\pi ij/n}\Psi_n(w_1)$, so $e^{2\pi ij/n} = 1$ or $\Psi_n(w_1) = 0$, that is $w_1 = 0 = w_2$. Also $\Psi_n'(0) = \hat\Psi(0) = 1$. (b) For a zero $b \ne 0$ of $\widetilde B$ and $\beta^n = b$: $\prod_{j<n}(w - e^{2\pi ij/n}\beta) = w^n - b$ and $\prod_{j<n}(1 - \overline{e^{2\pi ij/n}\beta}\,w) = 1 - \bar b\,w^n$; for $b = 0$ the factor becomes w^n. (c) The logarithmic derivative of $\Psi_n^n = \Psi(w^n)$ gives $v_{\Psi_n}(w) = v_\Psi(w^n)$, and $\sigma_B(w) = n\,\sigma_{\widetilde B}(w^n)$. Hence $\gamma = 1 + \sigma_B v_{\Psi_n} = \widetilde\gamma(w^n)$, which vanishes on the circle if and only if $\widetilde\gamma$ does, and $\operatorname{wind}(\gamma) = n\operatorname{wind}(\widetilde\gamma)$. By Proposition 2.11, (M3) means $\operatorname{wind}(\gamma) = -\deg B = -n\deg\widetilde B$. □

In the open-disk form of [23, Rem. 8.6] the zeros of $\widetilde B$ must also be simple. In (c) no such condition is needed: a multiple zero of $\widetilde B$ is a zero of $(\Psi_n B)'$ in $\mathbb{D}$, and the winding count of Proposition 2.11 includes it.

6.2. **Every symmetric pair is a root transform.** Jatar and Ng note this correspondence for polynomials of the form $w\,p(w^n)$ [23, Rem. 8.6]. We need it at the level of pairs, where the hypothesis $B(0)\neq 0$ enters.

Theorem 6.2. *Let (W,B) be an n-fold symmetric pair with $B(0)\neq 0$. Then $W=\Psi_n$ and $B=\widetilde{B}(w^n)$ for some Ψ as above (with radius R^n) and some finite Blaschke product $\widetilde{B}$.*

Proof. Let W be univalent on $\{|w|<R\}$. By the identity theorem $W(e^{2\pi i/n}w)=e^{2\pi i/n}W(w)$ there, so W^n is invariant under $w\mapsto e^{2\pi i/n}w$ and its Taylor series contains only the powers w^{nm}: $W^n=\Psi(w^n)$ with Ψ holomorphic on $\{|u|<R^n\}$ and $\Psi(u)=u+\cdots$. Ψ is injective: if $\Psi(u_1)=\Psi(u_2)$, choose $w_i^n=u_i$; then $W(w_1)^n=W(w_2)^n$, so $W(w_1)=e^{2\pi ij/n}W(w_2)=W(e^{2\pi ij/n}w_2)$, hence $w_1=e^{2\pi ij/n}w_2$ and $u_1=u_2$. Now $\Psi_n^n=W^n$, and Ψ_n, W have a simple zero at 0 with derivative 1 and no other zero, so Ψ_n/W is holomorphic and zero-free, with n-th power 1 and value 1 at 0; it is the constant 1.

Since $B(0)\neq 0$, every orbit $\{e^{2\pi ij/n}\beta\}$ of a zero of B has exactly n points, all of the same multiplicity. Let $\widetilde{B}$ have one zero β^n per orbit, with that multiplicity. By (b) of Lemma 6.1, B and $\widetilde{B}(w^n)$ have the same zeros with multiplicities, so their quotient is a unimodular constant, which we absorb into $\widetilde{B}$. □

The hypothesis $B(0)\neq 0$ is needed: for $n=2$ the zero set $\{0\}$ of $B(w)=w$ is invariant, but B is not of the form $\widetilde{B}(w^2)$. It holds for every Blaschke model, since $(WB)'(0)=B(0)$.

Corollary 6.3. *For Ψ as above and $t>0$ call a finite Blaschke product $\widetilde{B}$ t-*allowed *for Ψ if $1+t\,\sigma_{\widetilde{B}}\,v_\Psi$ has no zero on the circle and winds $-\deg\widetilde{B}$ times, and let $\operatorname{gain}_t(\Psi)$ be the supremum of $1/|\widetilde{B}(0)|$ over the t-allowed $\widetilde{B}$ ($\deg\widetilde{B}=0$ gives $\operatorname{gain}_t(\Psi)\geq 1$). Then for every integer $n\geq 1$:* (i) $\operatorname{gain}_n(\Psi)\leq K^*$; (ii) *the supremum of the gains of n-fold symmetric Blaschke models equals $\sup_\Psi \operatorname{gain}_n(\Psi)$.*

Proof. (i) By Lemma 6.1(c) each n-allowed $\widetilde{B}$ gives the Blaschke model $(\Psi_n,\widetilde{B}(w^n))$ with the same gain; apply Theorem 2.9. (ii) A symmetric Blaschke model has $B(0)\neq 0$, so by Theorem 6.2 it is $(\Psi_n,\widetilde{B}(w^n))$, and by Lemma 6.1(c) its $\widetilde{B}$ is n-allowed for Ψ, with the same gain. Conversely, for an n-allowed $\widetilde{B}$ the pair $(\Psi_n,\widetilde{B}(w^n))$ is a symmetric Blaschke model, since $B(e^{2\pi i/n}w)=B(w)$. □

Remark 6.4. In the language of Section 3, the pockets of Ψ_n are the n preimages under $w\mapsto w^n$ of those of Ψ, with the same depth and the same sign ($\theta\mapsto n\theta$ is increasing), and the condition of Lemma 6.1(c) is Theorem 3.4 for Ψ with $\hat{\sigma}=n\,\sigma_{\widetilde{B}}$. So a pocket of depth x is on if and only if $n\,x\,\sigma_{\widetilde{B}}>1$ there: for the quotient $\widetilde{B}$, symmetry by $2\pi/n$ acts as if every depth were multiplied by n. When Proposition 3.6 or Lemma 3.7 is applied to the quotient, $\deg\widetilde{B}$ and n times the Poisson kernels of the zeros of $\widetilde{B}$ take the places of $\deg B$ and of the kernels.

7. Bounds for each Blaschke model, and extremal pairs

We keep the notation of Section 4: (W,B) satisfies (M1) and (M2), $k\geq 1$, A is the zero *set* of B, the gain $K_0=1/|B(0)|$ and $r_{\text{in}}=\operatorname{dist}(0,\partial\Omega)\geq 1/4$. In this section we assume $(WB)'\neq 0$ only on the open disk; then $B(0)=(WB)'(0)\neq 0$, so $0\notin A$. Hyperbolic densities λ (of curvature -4), the punctured disk $\mathbb{D}^*$ and the function $\hat{c}(r)=\ell/\sinh\ell$, $\ell=\log(1/r)$, are as in Theorem 4.15(i) and its proof.

7.1. **The punctured Schwarz bound.** The following lemma carries out the suggestion of [23, Rem. 7.3] to use the hyperbolic metric of $\mathbb{D}\setminus A$; compare [4].

Lemma 7.1. $r_{\mathrm{in}} K_0 \le \lambda_{\mathbb{D}}(0)/\lambda_{\mathbb{D}\setminus A}(0) \le \min_{a\in A} \hat{c}(|a|) < 1$. *Hence*

$$K_0 \;\le\; 4\,\frac{\lambda_{\mathbb{D}}(0)}{\lambda_{\mathbb{D}\setminus A}(0)} \;\le\; 4\min_{a\in A}\hat{c}(|a|) \;<\; 4.$$

Proof. Let $g : D(0, r_{\mathrm{in}}) \to E_0$ be the inverse of Lemma 4.1 at $b = 0$. Φ vanishes at every $W(a)$, $a \in A$, and $W(a) \neq 0$; as Φ is injective on E_0 and $\Phi(0) = 0$, E_0 contains no $W(a)$. So $h(u) := W^{-1}(g(r_{\mathrm{in}}u))$ maps $\mathbb{D}$ into $\mathbb{D}\setminus A$, with $h(0) = 0$ and $|h'(0)| = r_{\mathrm{in}}/|B(0)| = r_{\mathrm{in}}K_0$. Schwarz–Pick for the hyperbolic metrics gives $\lambda_{\mathbb{D}\setminus A}(0)\,|h'(0)| \le \lambda_{\mathbb{D}}(0) = 1$. Since $\mathbb{D}\setminus A \subset \mathbb{D}\setminus\{a\}$, $\lambda_{\mathbb{D}\setminus A} \ge \lambda_{\mathbb{D}\setminus\{a\}}$; by the proof of Theorem 4.15(i), $\lambda_{\mathbb{D}\setminus\{a\}}(0) = 1/\hat{c}(|a|)$, and $\hat{c}(|a|) = \ell/\sinh\ell < 1$ with $\ell = \log(1/|a|)$. Finally $r_{\mathrm{in}} \ge 1/4$. □

Corollary 7.2. (a) $\hat{c}$ *increases on* $(0,1)$, *and* $r\,\hat{c}(r)$ *increases from* 0 *to* 1; *let* $\rho_* = 0.31088522\ldots$ *be the root of* $8r^2\log(1/r) = 1 - r^2$, *i.e. of* $r\,\hat{c}(r) = 1/4$.

(b) Every zero: $\min_{a\in A}|a| \ge \rho_*$.

(c) Near 4: *if* $K_0 \ge 4 - \delta$, *then every zero satisfies* $1 - |a| \le \sqrt{3\delta/2}$.

Proof. (a) $\frac{d}{d\ell}\frac{\ell}{\sinh\ell} = (\sinh\ell - \ell\cosh\ell)/\sinh^2\ell < 0$ and $\ell = \log(1/r)$ decreases in r, so $\hat{c}$ increases. Also $r\,\hat{c}(r) = u(r^2)$ with $u(q) = q\log(1/q)/(1-q)$, and $u'(q) = (q - 1 - \log q)/(1-q)^2 > 0$, $u(0+) = 0$, $u(1-) = 1$. (b) $K_0 \ge 1/|a|$ for every zero (each factor $1/|a_j| \ge 1$) and $K_0 \le 4\hat{c}(|a|)$ give $|a|\,\hat{c}(|a|) \ge 1/4$, i.e. $|a| \ge \rho_*$. (c) Put $|a| = 1-\epsilon$. Then $1 - r^2 - 2r\log(1/r) = \sum_{n\ge3} 2\epsilon^n/(n(n-1)) \ge \epsilon^3/3$ and $1 - r^2 \le 2\epsilon$, so $1 - \hat{c}(1-\epsilon) \ge \epsilon^2/6$. From $4 - \delta \le K_0 \le 4\hat{c}(|a|)$ we get $1 - \hat{c}(|a|) \le \delta/4$, so $\epsilon^2 \le 3\delta/2$. □

Lemma 7.1 bounds each Blaschke model below 4; it does not exclude $K^* = 4$, since the zeros may approach the circle along a sequence, and (c) gives a rate at which they must. Indeed $K^* = 4$ [18].

Remark 7.3. The bounds of [23, Thm. 7.1] are stronger than (b) and (c): there $B(0) \ge (1 + |a|)^2/(16|a|)$ for every zero a, so every zero has $|a| \ge 1/3$, and $K_0 \ge 4 - \delta$ forces $(1 - |a|)^2 \le 4|a|\delta/(4-\delta)$. Corollary 7.8 and Theorem 7.37 need only (b) and (c); the constants of Theorem 7.37 are computed from (c).

Lemma 7.4 (Smale at every zero)**.** *Every zero* a *of* B *is simple, and*

$$|W(a)| \prod_{b\in A\setminus\{a\}} |B_b(a)| \;\ge\; r_{\mathrm{in}}\,(1-|a|^2)\,\lambda_X(a) \;\ge\; r_{\mathrm{in}}, \qquad X := \mathbb{D}\setminus(\{0\}\cup(A\setminus\{a\}));$$

equivalently $\sum_{b\neq a} g_{\mathbb{D}}(a,b) \le \log(|W(a)|/r_{\mathrm{in}})$. *Moreover* $|W(a)| \ge r_{\mathrm{in}}/\hat{c}(|a|) > r_{\mathrm{in}}$.

Proof. $(WB)'(a) = W(a)B'(a) \neq 0$, as $(WB)' \neq 0$ on $\mathbb{D}$, so $B'(a) \neq 0$ and $W(a) \neq 0$. Let $g_a : D(0, r_{\mathrm{in}}) \to E_{W(a)}$ be the inverse of Lemma 4.1 at $b = W(a)$, and $f := W^{-1} \circ g_a$. Then $f(0) = a$ and $f'(0) = 1/(W'(a)\Phi'(W(a))) = 1/(W(a)B'(a))$. $E_{W(a)}$ contains no other zero of Φ, so f maps $D(0, r_{\mathrm{in}})$ into X, and Schwarz–Pick gives $r_{\mathrm{in}}\lambda_X(a) \le 1/|f'(0)| = |W(a)||B'(a)|$. Now $(1-|a|^2)|B'(a)| = \prod_{b\neq a}|B_b(a)|$, and $(1-|a|^2)\lambda_X(a) \ge 1$ as $X \subset \mathbb{D}$. For the last claim use $X \subset \mathbb{D}^*$: $\lambda_X(a) \ge \lambda_{\mathbb{D}^*}(a) = 1/(2|a|\log(1/|a|))$, together with $|B'(a)| \le 1/(1-|a|^2)$ (Schwarz–Pick for B, $B(a) = 0$). □

7.2. **Any domain pair.**

Proposition 7.5. *Let* (Ω, Φ) *be a domain pair with* $\mu = \mu(\Omega,\Phi) > 0$, $r_{\mathrm{in}} = \mathrm{dist}(0, \partial\Omega)$, *and let* Z_Φ *be the* (*finite*) *set of zeros of* Φ *in* Ω *other than* 0. *Then* $\mu\, r_{\mathrm{in}} \le 1/\lambda_{\Omega\setminus Z_\Phi}(0)$.

Proof. Z_Φ is finite, since $\Phi \not\equiv 0$ has isolated zeros in the compact set $\overline{\Omega}$. On $\partial\Omega$, $|\Phi(z)| \ge \mu|z| \ge \rho := \mu r_{\mathrm{in}}$. The proof of Lemma 4.1, with r_{in} replaced by ρ and its boundary case replaced by the continuity of Φ on $\overline{\Omega}$ (a limit point $z \in \partial\Omega$ of points of E has $|\Phi(z)| \ge \mu|z| \ge \rho$), gives a component E of $\{z \in \Omega : |\Phi(z)| < \rho\}$ containing 0 and an inverse $g : D(0,\rho) \to E$ with

$g(0) = 0$, $g'(0) = 1/\Phi'(0) = 1$; E misses Z_Φ because Φ is injective on E. Schwarz–Pick: $\lambda_{\Omega\setminus Z_\Phi}(0) \le \lambda_{D(0,\rho)}(0) = 1/\rho$. □

Corollary 7.6. *$\mu(\Omega, \Phi) < 4$ for every domain pair with Ω simply connected.*

Proof. Let $W : \mathbb{D} \to \Omega$ be the Riemann map with $W(0) = 0$ and $W'(0) = r(\Omega, 0) > 0$. By conformal invariance $\lambda_{\Omega\setminus Z_\Phi}(0) = \lambda_{\mathbb{D}\setminus W^{-1}(Z_\Phi)}(0)/r(\Omega, 0)$, so Proposition 7.5 gives

$$\mu \;\le\; \frac{r(\Omega, 0)}{r_{\text{in}}} \cdot \frac{\lambda_{\mathbb{D}}(0)}{\lambda_{\mathbb{D}\setminus W^{-1}(Z_\Phi)}(0)},$$

and $r(\Omega, 0) \le 4r_{\text{in}}$ by Koebe's theorem. If $\mu \le 1$ there is nothing to prove. If $\mu > 1$, then $Z_\Phi \neq \emptyset$ by Remark 2.7, and $\lambda_{\mathbb{D}\setminus W^{-1}(Z_\Phi)}(0) \ge 1/\hat{c}(|a|) > 1$ for any $a \in W^{-1}(Z_\Phi)$ (proof of Lemma 7.1); so $\mu < 4$. □

For Blaschke models ($r(\Omega, 0) = 1$, $Z_\Phi = W(A)$, and $\mu = K_0$ after $\Phi \mapsto \Phi/B(0)$) Proposition 7.5 is Lemma 7.1.

7.3. **Removing a disk around each zero.** Lemma 7.1 uses only that the image of the inverse branch of Lemma 4.1 at 0, pulled back to the disk by W^{-1}, misses the points of A. In fact it misses the whole pulled-back image of the branch at each zero, and that image contains a disk of explicit radius.

Lemma 7.7. *For every zero a of B, with $r = |a|$ and*

$$s(r) := r_{\text{in}}\,\frac{(1 - r^2)(1 - r)^2}{4r},$$

one has $s(r) < r$ and

$$r_{\text{in}} K_0 \;\le\; \frac{1}{\lambda_{\mathbb{D}_a}(0)}, \qquad \mathbb{D}_a := \mathbb{D} \setminus \overline{D(a, s(r))}.$$

Proof. For $b \in \{0, W(a)\}$ let $g_b : D(0, r_{\text{in}}) \to E_b$ be the inverse of Lemma 4.1, and put $h_b(u) = W^{-1}(g_b(r_{\text{in}}u))$. The maps h_0 and $h_{W(a)}$ are univalent on $\mathbb{D}$, and their images are disjoint open subsets of $\mathbb{D}$, since $E_0 \cap E_{W(a)} = \emptyset$. As in the proof of Lemma 7.4, $h_{W(a)}(0) = a$ and $|h'_{W(a)}(0)| = r_{\text{in}}/|W(a)B'(a)|$. Since $B(a) = 0$, the Schwarz–Pick lemma gives $|B'(a)| \le 1/(1 - r^2)$, and the growth theorem for $W \in \mathcal{S}$ gives $|W(a)| \le r/(1-r)^2$. Hence $|h'_{W(a)}(0)| \ge 4s(r)$, and by Koebe's one-quarter theorem $h_{W(a)}(\mathbb{D})$ contains the open disk $D(a, s(r))$. The open set $h_0(\mathbb{D})$ misses $h_{W(a)}(\mathbb{D})$, so it misses $D(a, s(r))$ and therefore also its closure. So h_0 maps $\mathbb{D}$ into $\mathbb{D}_a$; in particular $0 = h_0(0)$ is not in the closed disk, that is $s(r) < r$. Finally $h_0(0) = 0$ and $|h'_0(0)| = r_{\text{in}}/|B(0)| = r_{\text{in}}K_0$ (proof of Lemma 7.1), and the Schwarz–Pick lemma for the hyperbolic metrics gives $\lambda_{\mathbb{D}_a}(0)\, r_{\text{in}}K_0 \le \lambda_{\mathbb{D}}(0) = 1$. □

The density $\lambda_{\mathbb{D}_a}(0)$ is explicit. Rotate so that $a = r > 0$, put $x_1 = r - s$, $x_2 = r + s$ with $s = s(r)$, and

$$p = \frac{1 + x_1x_2 - \sqrt{(1 + x_1x_2)^2 - (x_1 + x_2)^2}}{x_1 + x_2}, \qquad q = \frac{x_2 - p}{1 - px_2}.$$

The Möbius map $w \mapsto (w - p)/(1 - pw)$ sends $\partial\mathbb{D}$ to itself and the circle $|w - r| = s$ to $|z| = q$, so $\lambda_{\mathbb{D}_a}(0) = (1 - p^2)\,\lambda_{A_q}(p)$, where $A_q = \{q < |z| < 1\}$ has the density

$$\lambda_{A_q}(t) = \frac{\pi}{2t \log(1/q) \sin\bigl(\pi \log(1/t)/\log(1/q)\bigr)}.$$

Since $s(r)$ is proportional to r_{in} and a larger disk removed gives a larger density, the bound $1/(r_{\text{in}}\lambda_{\mathbb{D}_a}(0))$ of Lemma 7.7 is largest at $r_{\text{in}} = 1/4$; we write $\text{bd}(r)$ for its value there. Every Blaschke model satisfies $K_0 \le \text{bd}(|a|)$ for each of its zeros a.

Corollary 7.8. *Every zero a of B satisfies $|a| > 1/2.7051 = 0.369672\ldots$.*

Proof. $K_0 \geq 1/|a|$, since every factor $1/|a_j|$ of K_0 is at least 1, and $|a| \geq \rho_* = 0.31088\ldots$ by Corollary 7.2(b). On the interval $0.3108 \leq r \leq 1/2.7051$ the function bd is at most $2.7050891 < 2.7051 \leq 1/r$, so no zero has its modulus there. The bound on bd is a ball-arithmetic computation [24] at 200 bits: the interval is cut into 11 pieces, and on a piece $[r_-, r_+]$ the closed disk about the midpoint of radius $s(r_+) - (r_+ - r_-)/2$ lies inside $\overline{D(r, s(r))}$ for every r in the piece, since s is decreasing ($(1-r^2)(1-r)^2$ decreases and $4r$ increases); a smaller disk removed gives a smaller density, so the value of $1/(r_{\mathrm{in}}\lambda)$ at that one disk bounds bd on the piece. □

Corollary 7.8 improves the bound $|a| \geq 1/3$ that [23, Thm. 7.1] gives for every zero. For $\deg B = 1$, Theorem 8.15 gives $|a| > 0.84904$.

7.4. **Pairs that are univalent past the circle are not extremal.** From here on we write F for WB; so $F = \Phi \circ W$, with $\Phi(z) = zB(W^{-1}(z))$ on $W(\mathbb{D})$ as in Section 4. Recall that, for $d \geq 0$, $\mathcal{B}_d$ is the supremum of $1/B(0)$ over the pairs (W, B) with $W \in \mathcal{S}$, B a finite Blaschke product of degree at most d with $B(0) > 0$, and $(WB)' \neq 0$ on $\mathbb{D}$; by [23, Thm. 8.9] each $\mathcal{B}_d$ is attained. The zeros of such a B are simple and non-zero, since a multiple zero, or a zero at 0, would be a zero of $(WB)'$ in $\mathbb{D}$.

Lemma 7.9. *Let (W, B) be as above with $\deg B = k \geq 1$, and assume that W extends to a univalent function on a neighbourhood of the closed disk. Then for every small $\delta > 0$ there is a pair (W_δ, B_δ) of the same kind, with $\deg B_\delta = k$, $B_\delta(0) = (1-\delta)B(0)$ and W_δ again univalent on a neighbourhood of the closed disk. In particular $1/B(0) < \mathcal{B}_k$.*

Proof. Let W be univalent on an open set $U \supset \overline{\mathbb{D}}$, put $F = WB$, and let B_δ be B with one zero a_1 replaced by $(1-\delta)a_1$; the normalising constant depends on $\arg a_1$ only, so $B_\delta(0) = (1-\delta)B(0)$. Since $F'(0) = B(0) \neq 0$, F' has finitely many zeros $w_1, \ldots, w_p$ on the circle, of orders $m_1, \ldots, m_p$, and none in $\mathbb{D}$. Fix a compact neighbourhood $U_1 \subset U$ of the closed disk, free of the poles of B and B_δ, on which the w_i are the only zeros of F'. Let υ_δ be the polynomial of degree at most $1 + \sum_i (m_i + 1)$ with $\upsilon_\delta(0) = \upsilon_\delta'(0) = 0$, $\upsilon_\delta(w_i) = 0$ and

$$(\upsilon_\delta B_\delta)^{(j)}(w_i) = -\big(W(B_\delta - B)/\delta\big)^{(j)}(w_i) \qquad (1 \leq j \leq m_i,\ 1 \leq i \leq p).$$

By Leibniz's rule these conditions form a triangular system for the jet of υ_δ at w_i with diagonal $B_\delta(w_i)$, of modulus 1; so υ_δ exists, depends linearly on the data, and, as $(B_\delta - B)/\delta$ is bounded with its derivatives on U_1 uniformly in δ, it is bounded with its derivatives on U_1 uniformly in δ. Put $W_\delta = W + \delta\upsilon_\delta$ and $F_\delta = W_\delta B_\delta$. Then $F_\delta' - F' = \delta\mathcal{E}_\delta$, where $\mathcal{E}_\delta$ is the derivative of $W(B_\delta - B)/\delta + \upsilon_\delta B_\delta$; it vanishes to order at least m_i at each w_i and is bounded on U_1 uniformly in δ. Hence $\mathcal{E}_\delta/F'$ is holomorphic near the closed disk and, by the maximum principle on small fixed circles about the w_i together with a positive lower bound for $|F'|$ elsewhere on $\overline{\mathbb{D}}$, bounded there by some C_1 independent of δ. So $F_\delta' = F'(1 + \delta\mathcal{E}_\delta/F')$ has no zero in $\mathbb{D}$ once $\delta C_1 < 1$. Finally $W_\delta(0) = 0$, $W_\delta'(0) = 1$, and W_δ is injective on a closed disk E of radius $R > 1$ inside U_1 for small δ. Indeed, let $|W'| \geq c_W > 0$ on E, let C_2 and C_3 bound $|\upsilon_\delta'|$ and $|\upsilon_\delta|$ on E uniformly in δ, and choose $\eta > 0$ with $|W'(\zeta) - W'(\zeta')| < c_W/2$ whenever $\zeta, \zeta' \in E$ and $|\zeta - \zeta'| < \eta$. Let $w \neq w'$ in E. If $|w - w'| < \eta$, integration along the segment from w' to w gives $|W(w) - W(w')| \geq (c_W/2)|w - w'| > \delta C_2|w - w'| \geq \delta|\upsilon_\delta(w) - \upsilon_\delta(w')|$ once $\delta C_2 < c_W/2$. If $|w - w'| \geq \eta$, then $|W(w) - W(w')| \geq c(\eta)$ for some $c(\eta) > 0$, since W is injective on the compact set E, while $\delta|\upsilon_\delta(w) - \upsilon_\delta(w')| \leq 2\delta C_3 < c(\eta)$ for small δ. In both cases $W_\delta(w) \neq W_\delta(w')$. □

If $F' \neq 0$ on the closed disk, no repair is needed ($\upsilon_\delta = 0$); the content of Lemma 7.9 is the repair at the zeros of F' on the circle.

Corollary 7.10. *Let $d \geq 1$. If (W, B) attains $\mathcal{B}_d$, then W does not extend to a univalent function on any neighbourhood of the closed disk. In particular no pair attaining $\mathcal{B}_d$ is a Blaschke model.*

Proof. $\mathcal{B}_d \geq \mathcal{B}_1 > 1$ [23, Thm. 6.1], so $k = \deg B \geq 1$, and $\mathcal{B}_k \leq \mathcal{B}_d$. Lemma 7.9 would give $\mathcal{B}_d = 1/B(0) < \mathcal{B}_k \leq \mathcal{B}_d$. □

Lemma 7.9 gives no explicit lower bound for $\mathcal{B}_k - 1/B(0)$. Corollary 7.10 settles the point left open in [23, Rem. 8.11] (no extremal pair is regular), and shows that for $d \geq 1$ the hypothesis of the last sentence of [23, Prop. 8.8] (a regular pair attaining $\mathcal{B}_d$) is never met.

Lemma 7.11 (Schur–Cohn [25]; compare [23, Lemma 6.4])**.** *Let $p(w) = e_0 + \cdots + e_n w^n$ with $n \geq 1$ (e_n may vanish), $p^*(w) = w^n\overline{p(1/\bar{w})}$ and $\mathcal{R}p = \bar{e}_0\, p - e_n\, p^*$, a polynomial of formal degree $n-1$ with constant term $|e_0|^2 - |e_n|^2$. If $|e_0| > |e_n|$, then p and $\mathcal{R}p$ have the same zeros on the circle, and if p has none there, the same number of zeros in $\mathbb{D}$. Consequently, if $p_0 = p$, $p_j = \mathcal{R}p_{j-1}$ and the constant terms δ_j of p_j ($1 \leq j \leq n$) are all positive, then p has no zero in the closed disk.*

Proof. On the circle $|p^*| = |p|$. If $p(w) = 0$ there, then $\mathcal{R}p(w) = 0$; if $\mathcal{R}p(w) = 0$, then $|e_0||p(w)| = |e_n||p(w)|$, so $p(w) = 0$. If p has no zero on the circle, then $|e_n p^*| < |\bar{e}_0 p|$ there, and Rouché's theorem applies. For the last claim go back from the non-zero constant $p_n = \delta_n$. □

Corollary 8.3, in Section 8, applies Lemma 7.11 to the two degree-one pairs of [23, Thm. 6.1, Rem. 6.7]: their maps W are univalent on a neighbourhood of the closed disk, so these pairs are not extremal.

7.5. **Pairs that are holomorphic past the circle are not extremal.** Lemma 7.9 needs W univalent past the circle. Here W need only be holomorphic past the circle: it may touch itself on the circle or have critical points there. Let $\mathbb{T} = \partial\mathbb{D}$. For $W \in \mathcal{S}$ holomorphic on an open set $U \supset \overline{\mathbb{D}}$ put

$$\mathrm{Crit} = \{\zeta \in \mathbb{T} : W'(\zeta) = 0\}, \qquad \mathrm{Cont} = \{\zeta \in \mathbb{T} : W(\zeta) = W(\zeta') \text{ for some } \zeta' \in \mathbb{T} \setminus \{\zeta\}\}.$$

We call ζ' a *partner* of ζ, and $\{\zeta, \zeta'\}$ a *contact pair*. W is univalent on a neighbourhood of the closed disk if and only if $\mathrm{Crit} \cup \mathrm{Cont}$ is empty (by compactness and part (a) of the next lemma); that is the case of Lemma 7.9.

Lemma 7.12. *Let $W \in \mathcal{S}$ be holomorphic on an open set $U \supset \overline{\mathbb{D}}$.*

(a) $W(\mathbb{T}) \cap W(\mathbb{D}) = \emptyset$.
(b) *Every* $\zeta \in \mathrm{Crit}$ *is a simple zero of W', and $W(\zeta') \neq W(\zeta)$ for $\zeta' \in \mathbb{T} \setminus \{\zeta\}$.*
(c) *Every* $\zeta \in \mathrm{Cont}$ *has exactly one partner ζ', and $W'(\zeta) \neq 0 \neq W'(\zeta')$.*
(d) Crit *and* Cont *are finite.*
(e) *Each* $c \in \mathrm{Crit}$ *has a* partner map β*: a holomorphic involution of a neighbourhood of c with $\beta(c) = c$ such that, for $w \neq w'$ near c, $W(w) = W(w')$ if and only if $w' = \beta(w)$. Each contact pair $\{\zeta_1, \zeta_2\}$ has a partner map τ: a conformal map of a neighbourhood of ζ_1 onto one of ζ_2 with $\tau(\zeta_1) = \zeta_2$ and $W \circ \tau = W$. The* gap functions $\mathrm{gap}(s) = \log|\beta(ce^{is})|$ *and* $\mathrm{gap}(s) = \log|\tau(\zeta_1 e^{is})|$ *are real-analytic near 0, with $\mathrm{gap}(0) = 0$ and $\mathrm{gap}(s) \geq c_{\mathrm{gap}} s^{2j}$ near 0 for some $c_{\mathrm{gap}} > 0$ and an integer $j \geq 1$.*

Proof. (a) If $W(\zeta) = W(w_0)$ with $\zeta \in \mathbb{T}$, $w_0 \in \mathbb{D}$, take a disk Δ_0 about w_0 with closure in $\mathbb{D}$, not containing ζ in its closure. $W(\Delta_0)$ is a neighbourhood of $W(w_0)$, so points $w \in \mathbb{D} \setminus \overline{\Delta_0}$ near ζ have $W(w) = W(\tilde{w})$ with $\tilde{w} \in \Delta_0$, $\tilde{w} \neq w$, against injectivity on $\mathbb{D}$.

(b), (c) Near $\zeta \in \mathbb{T}$ with $W'(\zeta) \neq 0$, $W(\mathbb{D} \cap \Delta)$ contains a sector at $W(\zeta)$ of opening $\pi - \varepsilon$ for small disks Δ about ζ. If $W - W(\zeta)$ has a zero of order $q+1$ at ζ, write $W - W(\zeta) = \hat{x}^{q+1}$ with $\hat{x}$ conformal at ζ, $\hat{x}(\zeta) = 0$; $\hat{x}(\mathbb{D} \cap \Delta)$ contains a sector at 0 of opening $\pi - \varepsilon$, and $u \mapsto u^{q+1}$ is not injective on a sector of opening larger than $2\pi/(q+1)$; so $q \leq 1$. For $q = 1$, $W(\mathbb{D} \cap \Delta)$ contains a sector of opening $2\pi - 2\varepsilon$. The sets $W(\mathbb{D} \cap \Delta)$ for disjoint small disks Δ about the preimages of one value are disjoint. Two sectors of opening $\pi - \varepsilon$ fit at one vertex, three do not, and none fits beside one of opening $2\pi - 2\varepsilon$.

(d), (e) *Local maps.* At $c \in \mathrm{Crit}$ write $W = W(c) + \hat{x}^2$ with $\hat{x}$ conformal near c, $\hat{x}(c) = 0$; then $\beta = \hat{x}^{-1}(-\hat{x})$ is a holomorphic involution near c with $\beta(c) = c$, $\beta'(c) = -1$, and for $w \neq w'$ near c, $W(w) = W(w')$ if and only if $w' = \beta(w)$. At a pair $\{\zeta_1, \zeta_2\}$ of Cont let $\tau = (W|_{\Delta_2})^{-1} \circ W$ near ζ_1, where W is conformal on disjoint disks Δ_1, Δ_2 about ζ_1, ζ_2; $\tau(\zeta_1) = \zeta_2$. Since W is injective on $\mathbb{D}$, $|\beta| \geq 1$ (resp. $|\tau| \geq 1$) on $\overline{\mathbb{D}}$ near c (resp. ζ_1), so $\mathrm{gap}(s) = \log|\beta(ce^{is})|$ (resp. $\log|\tau(\zeta_1 e^{is})|$) is real-analytic near 0, gap ≥ 0, $\mathrm{gap}(0) = 0$.

Reduction to arcs. Crit is finite, as $W' \not\equiv 0$. If some gap function vanishes identically near 0, then β or τ maps an arc of $\mathbb{T}$ into $\mathbb{T}$, and Cont contains an arc. If Cont is infinite, contact pairs (ζ_n, ζ_n') accumulate at some (ζ^*, ζ'^*) with $W(\zeta^*) = W(\zeta'^*)$. If $\zeta^* \neq \zeta'^*$ this is a contact pair and $\zeta_n' = \tau(\zeta_n)$ eventually; if $\zeta^* = \zeta'^*$, then $\zeta^* \in \mathrm{Crit}$ (else W is injective near ζ^*) and $\zeta_n' = \beta(\zeta_n)$. Either way the gap function has zeros $s_n \to 0$, $s_n \neq 0$, so it vanishes identically. It remains to show that Cont contains no arc; then (d) holds, and (e) as well: a real-analytic function gap ≥ 0 with $\mathrm{gap}(0) = 0$ that is not identically 0 starts with a term of even order $2j$ and positive coefficient.

No arc. Suppose the interior O of Cont in $\mathbb{T}$ is not empty. (i) Every boundary point ζ_0 of O in $\mathbb{T}$ is in Crit and is isolated in $\mathbb{T} \setminus O$: take $\zeta_n \in O$, $\zeta_n \to \zeta_0$, with partners $\zeta_n' \to \zeta_0'$. If $\zeta_0' \neq \zeta_0$, then $\zeta_n' = \tau(\zeta_n)$, so gap $\equiv 0$ at ζ_0 and an arc about ζ_0 lies in Cont, against $\zeta_0 \notin O$. So $\zeta_0' = \zeta_0$, and $\zeta_0 \in \mathrm{Crit}$ (else W is injective near ζ_0), $\zeta_n' = \beta(\zeta_n)$, gap $\equiv 0$ at ζ_0, and a punctured arc about ζ_0 lies in Cont, hence in O. Call such points tips; they lie in the finite set Crit. (ii) $\mathbb{T} \setminus O$ consists of tips: otherwise a component arc of the non-empty open set $\mathbb{T} \setminus \overline{O} \neq \mathbb{T}$ has an endpoint; it is a boundary point of O, hence a tip with a punctured neighbourhood in O, while it is a limit of points of that arc, which are not in O. (iii) Let $X = \mathbb{C} \setminus \overline{W(\mathbb{D})}$, a non-empty open set since $W(\mathbb{D})$ is bounded. Its boundary lies in $\overline{W(\mathbb{D})} \setminus W(\mathbb{D}) \subset W(\mathbb{T})$. Let $\zeta_1 \in O$ with partner ζ_2. The partners of points of O near ζ_1 tend to ζ_2 (their limits are partners of ζ_1, by (c)), so they are τ of them, so $|\tau| = 1$ on an arc of $\mathbb{T}$ about ζ_1. Replace Δ_1 by a smaller neighbourhood of ζ_1, invariant under $w \mapsto 1/\bar{w}$, such that $|\tau| = 1$ on the arc $\mathbb{T} \cap \Delta_1$. By the reflection principle, $\tau(1/\bar{w}) = 1/\overline{\tau(w)}$ on Δ_1 (both sides are holomorphic there and agree on that arc). For $w \in \mathbb{D} \cap \Delta_1$ we have $W(\tau(w)) = W(w)$ and $\tau(w) \neq w$, so $\tau(w) \notin \mathbb{D}$ by injectivity and $\tau(w) \notin \mathbb{T}$ by (a): $|\tau(w)| > 1$. Hence $|\tau| < 1$ on $\Delta_1 \setminus \overline{\mathbb{D}}$, so τ maps $\Delta_1 \setminus \overline{\mathbb{D}}$ into $\mathbb{D}$, and, as $W \circ \tau = W$, $W(\Delta_1) \subset W(\mathbb{D}) \cup W(\mathbb{T} \cap \Delta_1)$. As $W(\mathbb{T} \cap \Delta_1)$ is an analytic arc, $W(\Delta_1)$ does not meet X, so $W(\zeta_1)$ is not on the boundary of X. Hence the boundary of X lies in the finite set $W(\mathbb{T} \setminus O)$. As $\mathbb{C}$ minus a finite set is connected and $X \neq \emptyset$, X is the complement of a finite set, which contradicts $W(\mathbb{D}) \cap X = \emptyset$. □

Theorem 7.13. *Let (W, B) be an admissible pair with $\deg B = k \geq 1$, and assume that W extends holomorphically to a neighbourhood of the closed disk. Then for every small $\delta > 0$ there is a pair (W_δ, B_δ) of the same kind with $\deg B_\delta = k$, $B_\delta(0) = (1 - \delta)B(0)$, and W_δ univalent on a neighbourhood of the closed disk. In particular $1/B(0) < \mathcal{B}_k$.*

Corollary 7.14. *Let $d \geq 1$. If (W, B) attains $\mathcal{B}_d$, then W does not extend holomorphically to any neighbourhood of the closed disk.*

Corollary 7.15. *If W is rational without poles on the closed disk, then (W, B) attains $\mathcal{B}_d$ for no $d \geq 1$. This applies to every pair of* [23, Lemma 6.2] *and of its extension, Lemma 8.4: there $W = H(\varphi)/N_b$, with H a polynomial and b, φ and N_b as recalled at the start of Section 8, is a polynomial if $b = 0$ and otherwise has the only pole $-1/\bar{b}$. So it applies to all pairs of Section 8 and to the pairs of Proposition 10.3.*

The corollaries follow from Theorem 7.13 as Corollary 7.10 follows from Lemma 7.9.

Proof of Theorem 7.13. Let $F = WB$, let $w_1, \ldots, w_p$ be the zeros of F' on $\mathbb{T}$, of orders m_i, and let U_1, B_δ be as in the proof of Lemma 7.9. By Lemma 7.12 let $\mathrm{Crit} = \{c_1, \ldots, c_q\}$ and let $\{\zeta_l, \zeta_l'\}$ $(1 \leq l \leq r)$ be the contact pairs. At $c \in \mathrm{Crit}$, $F'(c) = W(c)B'(c) \neq 0$, since $W(c) \neq 0$ by Lemma 7.12(a) and $cB'(c)/B(c) > 0$; so 0, the w_i and the c_j are distinct, and the contact points

are distinct from 0, from the c_j and from each other. A contact point may equal some w_i; let e be the number of such points. Fix pairwise disjoint small disks about the c_j and the contact points.

The plan. We add to W a small multiple $\delta\upsilon_\delta$ of a polynomial, chosen by Hermite interpolation so that, as in Lemma 7.9, $(W_\delta B_\delta)'$ has no zero in $\mathbb{D}$, and so that every defect moves outward: the zero of W_δ' near a point c of Crit is $(1+\delta)c+O(\delta^2)$, and for a contact pair (ζ_1,ζ_2) the point near ζ_2 with the value $W_\delta(\zeta_1)$ is $(1+\delta)\zeta_2+O(\delta^2)$; both lie outside the closed disk. The step 'Local injectivity' then shows that W_δ is injective on the closed disk near each defect, and 'Global injectivity' that it is univalent near the closed disk.

The polynomial υ_δ. Impose: $\upsilon(0)=\upsilon'(0)=0$; at each w_i the conditions of Lemma 7.9 for the derivatives of orders $1,\dots,m_i$ of υB_δ, with $\upsilon(w_i)=0$ unless w_i is a contact point; at each c_j, $\upsilon(c_j)=0$ and $\upsilon'(c_j)=-c_jW''(c_j)$; at each pair, $\upsilon(\zeta_l')=0$ and $\upsilon(\zeta_l)=\zeta_l'W'(\zeta_l')$. The conditions at w_i only make $\mathcal{E}_\delta$ vanish to order m_i at w_i, whatever $\upsilon(w_i)$ is, and given $\upsilon(w_i)$ they form a triangular system for $\upsilon^{(1)}(w_i),\dots,\upsilon^{(m_i)}(w_i)$ with diagonal $B_\delta(w_i)$. So all conditions are Hermite conditions at distinct nodes, n_H in number, where $n_H=2+\sum_i(m_i+1)+2q+2r-e$, and we let υ_δ be the unique interpolant of degree at most n_H-1. Its data are bounded uniformly in δ, so υ_δ is bounded in every $C^m(U_1)$ uniformly in δ. Put $W_\delta=W+\delta\upsilon_\delta$, $F_\delta=W_\delta B_\delta$. As in Lemma 7.9, $F_\delta'\neq 0$ in $\mathbb{D}$ for small δ, $W_\delta(0)=0$ and $W_\delta'(0)=1$; it remains to show that W_δ is univalent near the closed disk. Below, C is a constant independent of δ.

The defects (the points of Crit *and the contact pairs) move apart.* At $c=c_j$, W_δ' has one zero $c_\delta=c-\delta\upsilon_\delta'(c)/W''(c)+O(\delta^2)=(1+\delta)c+O(\delta^2)$ near c. Write $W_\delta=W_\delta(c_\delta)+\hat{x}_\delta^2$. As $W_\delta-W=\delta\upsilon_\delta$ with υ_δ bounded in C^m, and $c_\delta-c=O(\delta)$, the square-root coordinate $\hat{x}_\delta$ and its inverse move by $O(\delta)$ in C^1: $|\hat{x}_\delta-\hat{x}|_{C^1}\le C\delta$ on a fixed disk, and $\beta_\delta=\hat{x}_\delta^{-1}(-\hat{x}_\delta)$ has $|\beta_\delta-\beta|_{C^1}\le C\delta$ and $|\beta_\delta''|\le C$ on a fixed smaller disk; as $\beta_\delta(c_\delta)=c_\delta$ and $\beta_\delta'(c_\delta)=-1$, $\log|\beta_\delta(c)|=2\delta+O(\delta^2)$. At a pair $(\zeta_1,\zeta_2)=(\zeta_l,\zeta_l')$, $\tau_\delta=(W_\delta|_{\Delta_2})^{-1}\circ W_\delta$ has $|\tau_\delta-\tau|_{C^1}\le C\delta$ (likewise: the local inverse of W_δ near ζ_2, where $W'\neq 0$, moves by $O(\delta)$ in C^1) and $\tau_\delta(\zeta_1)-\zeta_2=\delta(\upsilon_\delta(\zeta_1)-\upsilon_\delta(\zeta_2))/W'(\zeta_2)+O(\delta^2)=\delta\zeta_2+O(\delta^2)$, so $\log|\tau_\delta(\zeta_1)|=\delta+O(\delta^2)$.

Local injectivity. Near a point c of Crit, the only point $w'\neq w$ near c with $W_\delta(w')=W_\delta(w)$ is $\beta_\delta(w)$; near a contact pair (ζ_1,ζ_2), the point near ζ_2 with the value $W_\delta(w)$ of a point w near ζ_1 is $\tau_\delta(w)$. So it suffices to show that $|\beta_\delta|>1$ on the closed disk near c and $|\tau_\delta|>1$ on the closed disk near ζ_1: then these points lie outside the closed disk. Let $\phi=\beta$ at $\zeta_0=\zeta_0^*=c$, or $\phi=\tau$ at $\zeta_0=\zeta_1$, $\zeta_0^*=\zeta_2$, and $G_\delta(\rho,s)=\log|\phi_\delta(\rho\zeta_0e^{is})|$ on the box $1-r_b\le\rho\le 1$, $|s|\le r_b$, with r_b fixed. At $\delta=0$, $G(1,s)=\mathrm{gap}(s)\ge c_{\mathrm{gap}}s^{2j}$ by Lemma 7.12(e), and $\partial_\rho G(1,0)=\mathrm{Re}(\bar{\zeta}_0^*\phi'(\zeta_0)\zeta_0)=-\alpha_0<0$: for β, $\alpha_0=1$; for τ, the number $\bar{\zeta}_2\tau'(\zeta_1)\zeta_1$ is real since $\mathrm{gap}'(0)=0$, it is not positive since $|\tau|\ge 1$ inside the disk and $|\tau(\zeta_1)|=1$, and it is not 0, since $\tau'(\zeta_1)=W'(\zeta_1)/W'(\zeta_2)\neq 0$, so $\alpha_0=|\tau'(\zeta_1)|$. Hence $G(\rho,s)\ge c_{\mathrm{gap}}s^{2j}+(\alpha_0/2)(1-\rho)$ on the box for small r_b. Since $|\nabla(G_\delta-G)|\le C\delta$, $G(1,0)=0$ and $G_\delta(1,0)\ge\varpi\delta-C\delta^2$, where $\varpi=2$ at a point of Crit and $\varpi=1$ at a contact pair (the separations computed above), we have $G_\delta(\rho,s)\ge G_\delta(1,0)+G(\rho,s)-C\delta(|s|+1-\rho)$, so

$$G_\delta(\rho,s)\ \ge\ \big(c_{\mathrm{gap}}s^{2j}-C\delta|s|\big)+(\alpha_0/2-C\delta)(1-\rho)+\varpi\delta-C\delta^2\ \ge\ \varpi\delta-C'\delta^{2j/(2j-1)}-C\delta^2\ >\ 0$$

for small δ: the first bracket is at least its minimum over s, which is $-C'\delta^{2j/(2j-1)}$ with $C'=(1-\frac{1}{2j})\,C\,(C/(2jc_{\mathrm{gap}}))^{1/(2j-1)}$, attained at $|s|^{2j-1}=C\delta/(2jc_{\mathrm{gap}})$; the second term is non-negative once $C\delta\le\alpha_0/2$; and $2j/(2j-1)>1$. So $|\beta_\delta|>1$ (resp. $|\tau_\delta|>1$) on the closed disk near c (resp. ζ_1), including the points of $\mathbb{T}$. If $w\neq w'$ in the closed disk near c have $W_\delta(w)=W_\delta(w')$, then $w'=\beta_\delta(w)$ lies outside the closed disk; likewise near a pair, with τ_δ. So W_δ is injective on the closed disk near each defect, and $W_\delta'\neq 0$ there, its only zero near c being $c_\delta\notin\overline{\mathbb{D}}$. Near any other point of the closed disk $W'\neq 0$, and W_δ is injective on a fixed disk about it by the segment argument of Lemma 7.9.

Global injectivity. If $\delta_n\to 0$ and $w_n\neq w_n'$ in the closed disk had $W_{\delta_n}(w_n)=W_{\delta_n}(w_n')$, a subsequence converges to (w^*,w'^*) with $W(w^*)=W(w'^*)$. If $w^*=w'^*$ the local injectivity is

contradicted; otherwise both points lie on $\mathbb{T}$ by Lemma 7.12(a), so they form a contact pair, and the local injectivity at that pair is contradicted. So for small δ, W_δ is injective on the closed disk with $W_\delta' \neq 0$ there, hence univalent on a neighbourhood of it. □

If W is rational, as in Corollary 7.15, then $W(\mathbb{D})$ is a bounded quadrature domain [3], and Lemma 7.12 also follows from the known structure of the boundaries of such domains (finitely many analytic arcs with finitely many cusps and double points); compare Sakai [37] and Gustafsson [21]. Schiffer's boundary variation [16, Ch. 10] shows that extremals of differentiable functionals on $\mathcal{S}$ map onto the plane minus analytic arcs; it does not give Theorem 7.13, since the pairs here are not closed under Schiffer's variations, and the Hermite repair of Lemma 7.9 is what keeps $(WB)' \neq 0$ in $\mathbb{D}$.

Poles on the circle. We now allow poles on the circle; then $W(\mathbb{D})$ is unbounded, as for the Koebe function. Let $W \in \mathcal{S}$ be meromorphic on an open set $U \supset \overline{\mathbb{D}}$; shrinking U, its poles in U lie on $\mathbb{T}$ and form a finite set P. Define Crit and Cont as above with $\mathbb{T} \setminus P$ in place of $\mathbb{T}$. A *slit plane* is a domain $\mathbb{C} \setminus \Gamma$ with Γ a non-empty finite union of analytic arcs, closed in $\mathbb{C}$.

Lemma 7.16. *The orders of the poles of W add up to at most 2, and $W(\mathbb{T} \setminus P) \cap W(\mathbb{D}) = \emptyset$.*

Proof. Near a pole p of order m, $V = 1/W = \hat{x}^m$ with $\hat{x}$ conformal at p, $\hat{x}(p) = 0$, and V is injective on $\mathbb{D} \cap \Delta$. As in Lemma 7.12(b), $\hat{x}(\mathbb{D} \cap \Delta)$ contains a sector at 0 of opening $\pi - \varepsilon$, and $u \mapsto u^m$ is injective on it; so $m(\pi - \varepsilon) \leq 2\pi$ and $m \leq 2$. Then $V(\mathbb{D} \cap \Delta_p)$ contains a sector of opening $m(\pi - \varepsilon)$. These sets are disjoint for different poles, so $\sum_p m_p(\pi - \varepsilon) \leq 2\pi$ for every $\varepsilon > 0$. The second statement is proved as Lemma 7.12(a). □

At a double pole p write $1/W = \hat{x}^2$ with $\hat{x}$ conformal, and put $\beta_\infty = \hat{x}^{-1}(-\hat{x})$. At two simple poles p_1, p_2 put $\tau_\infty = (V|_{\Delta_2})^{-1} \circ V$ near p_1, so $\tau_\infty(p_1) = p_2$. A lone simple pole has no partner, as V is injective near it. As in the proof of Lemma 7.12, $\mathrm{gap}_\infty(s) = \log|\beta_\infty(pe^{is})|$, resp. $\log|\tau_\infty(p_1 e^{is})|$, is real-analytic near 0, $\mathrm{gap}_\infty \geq 0$ and $\mathrm{gap}_\infty(0) = 0$.

Lemma 7.17. *Either* Cont *is finite, and every gap function of Lemma 7.12*(e) *(defined in the same way on $\mathbb{T} \setminus P$) and every* gap_∞ *is at least $c_{\mathrm{gap}} s^{2j}$ near 0, for some $c_{\mathrm{gap}} > 0$ and $j \geq 1$; or $W(\mathbb{D}) = \mathbb{C} \setminus \Gamma$ is a slit plane with $\Gamma = W(\mathbb{T} \setminus P)$, all but finitely many points of Γ have exactly two preimages in $\mathbb{T} \setminus P$, every component of Γ is unbounded, and P consists of one double pole or of two simple poles.*

Proof. The proof has two parts. First, if Cont is infinite, or some gap or gap_∞ vanishes identically, then Cont contains an arc. Second, if Cont contains an arc, let O be its interior: we show in turn that $\mathbb{T} \setminus O$ is a finite set of tips, that $\Gamma = \mathbb{C} \setminus W(\mathbb{D})$ equals $W(\mathbb{T} \setminus P)$, and that the partner map pairs the arcs of O, so that all but finitely many points of Γ have exactly two preimages and every component of Γ is unbounded.

Parts (b), (c) of Lemma 7.12 and the local maps β, τ are local at finite values, so they hold on $\mathbb{T} \setminus P$. Crit is finite, as $|W'| \to \infty$ at the poles, so the zeros of W' on $\mathbb{T} \setminus P$ do not accumulate. If Cont is infinite, or some gap or gap_∞ vanishes identically, then Cont contains an arc. For contact pairs (ζ_n, ζ_n') with bounded values this is the reduction in the proof of Lemma 7.12. If the values tend to ∞, both points tend to poles: to one double pole, with $\zeta_n' = \beta_\infty(\zeta_n)$, or to the two simple poles, with $\zeta_n' = \tau_\infty(\zeta_n)$ up to order (a lone simple pole is excluded, as V is injective near it); then $\mathrm{gap}_\infty \equiv 0$, and β_∞ or τ_∞ maps an arc of $\mathbb{T}$ into $\mathbb{T}$.

Let $O \neq \emptyset$ be the interior of Cont in $\mathbb{T}$. Every boundary point of O is a tip, that is, isolated in $\mathbb{T} \setminus O$: for a point not in P as in Lemma 7.12 (the limit partner has a finite value, so it is not a pole); at a double pole, or at one of two simple poles, the partners of nearby points of O tend to a pole, so $\mathrm{gap}_\infty \equiv 0$ and a punctured arc about the pole lies in O. A lone simple pole is not a boundary point of O, since a neighbourhood of it in $\mathbb{T}$ misses Cont. As in Lemma 7.12, $\mathbb{T} \setminus O$ consists of finitely many tips; in particular there is no lone simple pole. With $X = \mathbb{C} \setminus \overline{W(\mathbb{D})}$, the

proof of Lemma 7.12 shows that the boundary of X lies in the finite set $W(\text{tips} \setminus P)$ (the step (iii) is local at a finite value), so $X = \emptyset$: a non-empty open set with finite boundary is the complement of a finite set, and it would meet $W(\mathbb{D})$; here $\overline{W(\mathbb{D})} \setminus W(\mathbb{D}) \subset W(\mathbb{T} \setminus P)$, since a finite limit of values $W(w_n)$ comes from points w_n that tend to a point of the closed disk other than a pole. Hence $\Gamma := \mathbb{C} \setminus W(\mathbb{D}) = \overline{W(\mathbb{D})} \setminus W(\mathbb{D}) = W(\mathbb{T} \setminus P)$, by Lemma 7.16, and $\Gamma = W(O) \cup W(\text{tips} \setminus P)$. As $W(\mathbb{D})$ is dense, it is unbounded, so $P \neq \emptyset$; since poles are not in Cont, O lies in $\mathbb{T} \setminus P \neq \mathbb{T}$, and every component of O is an open arc. The partner of a point of O lies in Cont and is not a tip (a tip in Crit has no partner, and a pole is not in $\mathbb{T} \setminus P$), so it lies in O, and the partner map is a continuous involution of O without fixed points, as $\text{Crit} \cap \text{Cont} = \emptyset$; since an open interval has no such involution (a continuous involution of an interval is monotone; an increasing one is the identity, and a decreasing one has a fixed point), it maps each component arc A of O onto another one, A'. So W is injective on A, and $W(A) = W(A')$ is an analytic arc whose points have exactly two preimages. Γ is not empty, since $W(\mathbb{D})$ misses $W(\mathbb{T} \setminus P)$. By Lemma 7.16, P is one double pole or two simple poles. Finally $\Gamma \cup \{\infty\}$, the complement of the simply connected domain $W(\mathbb{D})$ in the sphere, is connected, and Γ has finitely many components; a bounded one would be compact and would split $\Gamma \cup \{\infty\}$. □

Theorem 7.18. *Let (W, B) be an admissible pair with $\deg B = k \geq 1$, let W be meromorphic on a neighbourhood of the closed disk, and let* Cont *be finite. Then for every small $\delta > 0$ there is a pair (W_δ, B_δ) of the same kind with $\deg B_\delta = k$ and $B_\delta(0) = (1-\delta)B(0)$. In particular $1/B(0) < \mathcal{B}_k$.*

Corollary 7.19. *Let $d \geq 1$. If (W, B) attains $\mathcal{B}_d$ and W extends meromorphically to a neighbourhood of the closed disk, then $W(\mathbb{D})$ is the plane minus a non-empty finite union Γ of analytic arcs, each traced twice by W on the circle, every component of Γ is unbounded, and W has a double pole or two simple poles on the circle.*

For the corollary, Theorem 7.18 shows that Cont is infinite (as in the proof of Corollary 7.10), and Lemma 7.17 gives the rest.

Proof of Theorem 7.18. We follow the proof of Theorem 7.13; near the points of $\mathbb{T} \setminus P$ nothing changes. To the conditions on υ_δ we add, at each pole p_∞, $\upsilon^{(l)}(p_\infty) = 0$ for $0 \leq l < 2j$, where j is the exponent of gap_∞ in Lemma 7.17 ($\text{gap}_\infty(s) \geq c_{\text{gap}} s^{2j}$) at a double pole, resp. at p_1 for two simple poles, and $j = 0$ for a lone simple pole. The poles are distinct from the other nodes, and $W_\delta = W + \delta\upsilon_\delta$ has the same poles with the same orders.

$F'_\delta \neq 0$ *in* $\mathbb{D}$. Near a pole p_∞ of order m, with $R_\delta = (B_\delta - B)/\delta$,

$$\frac{\mathcal{E}_\delta}{F'} = \frac{R_\delta + (W/W')R'_\delta + (\upsilon_\delta B_\delta)'/W'}{B + (W/W')B'},$$

where $W/W' = O(|w - p_\infty|)$ and $1/W' = O(|w - p_\infty|^{m+1})$. As $B(p_\infty) \neq 0$, $\mathcal{E}_\delta/F'$ is bounded on a fixed disk about p_∞ uniformly in δ; on the rest of the closed disk the proof of Lemma 7.9 applies. So $F'_\delta \neq 0$ in $\mathbb{D}$ for small δ.

Injectivity near the poles. Put $V_\delta = 1/W_\delta = Vu_\delta$ with $u_\delta = 1/(1 + \delta\upsilon_\delta V)$. On $|w - p_\infty| \leq r$, $|u_\delta - 1| \leq C\delta r^{2j+m}$ and $|u'_\delta| \leq C\delta r^{2j+m-1}$. At a simple pole, $V'(p_\infty) \neq 0$; fix a disk of radius at most r about p_∞ on which $|V' - V'(p_\infty)| < |V'(p_\infty)|/4$. For small δ, $|V'_\delta - V'(p_\infty)| < |V'(p_\infty)|/2$ there, and integration along the segment from w' to w gives $|V_\delta(w) - V_\delta(w')| \geq |V'(p_\infty)|\,|w - w'|/2 > 0$ for $w \neq w'$ in it, as in the proof of Lemma 7.9; so V_δ is injective there. At a double pole, $\hat{x}_\delta = \hat{x} u_\delta^{1/2}$ is conformal with $|\hat{x}_\delta - \hat{x}|_{C^1} \leq C\delta r^{2j+1}$, and $\beta_{\infty,\delta} = \hat{x}_\delta^{-1}(-\hat{x}_\delta)$ fixes p_∞. At two simple poles $\tau_{\infty,\delta}(p_1) = p_2$, and $V(\tau_{\infty,\delta})u_\delta(\tau_{\infty,\delta}) = Vu_\delta$ gives $|\tau_{\infty,\delta} - \tau_\infty|_{C^1} \leq C\delta r^{2j}$, since by the bounds on u_δ near p_1 and near p_2 the map V_δ and its local inverse near p_2 differ from V and its inverse by $O(\delta r^{2j})$ in C^1 (this uses the condition at both poles). With $G_\delta(\rho, s) = \log|\phi_\delta(\rho\zeta_0 e^{is})|$ for $\phi = \beta_\infty$ or τ_∞, as in the proof of Theorem 7.13, we get $G_\delta(1, 0) = G(1, 0) = 0$. The maps $\beta_{\infty,\delta}$ and β_∞ fix p_∞, and the bound on $\hat{x}_\delta$ gives $|\beta_{\infty,\delta} - \beta_\infty|_{C^1} \leq C\delta r^{2j+1}$ on $|w - p_\infty| \leq r$; integrating from p_∞

along a segment gives $|\beta_{\infty,\delta}(w)-\beta_\infty(w)| \le C\delta|w-p_\infty|^{2j+2}$. Likewise $\tau_{\infty,\delta}(p_1)=\tau_\infty(p_1)=p_2$ gives $|\tau_{\infty,\delta}(w)-\tau_\infty(w)| \le C\delta|w-p_1|^{2j+1}$. As $|w-\zeta_0| \le C(|s|+1-\rho)$ at $w=\rho\zeta_0 e^{is}$, and $\log|\cdot|$ is Lipschitz near the circle, $|G_\delta - G| \le C\delta(|s|+1-\rho)^{2j+1}$: the exponent is $2j+2$ or $2j+1$, and $|s|+1-\rho \le 2$, since $|s| \le 1$ and $0 \le 1-\rho \le 1$ on the box. So $(|s|+1-\rho)^{2j+1} \le 2^{2j+1}\big(|s|^{2j+1}+(1-\rho)^{2j+1}\big) \le 2^{2j+1}\big(s^{2j}+(1-\rho)\big)$. Since $G(\rho,s) \ge c_{\mathrm{gap}}s^{2j}+(\alpha_0/2)(1-\rho)$ on a small box,

$$G_\delta(\rho,s) \;\ge\; (c_{\mathrm{gap}}-C'\delta)s^{2j}+(\alpha_0/2-C'\delta)(1-\rho) \;>\; 0 \qquad \text{for } (\rho,s)\neq(1,0)$$

and small δ. So W_δ is injective on the closed disk minus P near each pole.

Global injectivity. If $W_{\delta_n}(w_n)=W_{\delta_n}(w_n')\to\infty$ with $w_n\neq w_n'$ in $\mathbb{D}$, then w_n, w_n' tend to poles, against the previous step; bounded values are treated as in the proof of Theorem 7.13. So $W_\delta \in \mathcal{S}$. □

Theorem 7.18 does not say that W_δ is univalent past the circle; near a double pole it cannot be. Whether an extremal W is meromorphic past the circle is open. For unbounded quadrature domains compare Shapiro [38] and Sakai [37].

7.6. **Rational extremals.** Corollary 7.15 excludes a rational W without poles on the closed disk, and Corollary 7.19 says that if a rational W with poles on the circle belongs to a pair that attains $\mathcal{B}_d$ with $d \ge 1$, then $W(\mathbb{D})$ is a slit plane. We show that a rational map onto a slit plane has straight slits (Proposition 7.20), that such a map carries no pair with non-constant B (Corollary 7.24, from the sign rule of Theorem 7.22), and so that no extremal pair has a rational W (Corollary 7.25). Let $\mathcal{J}(w)=w+1/w$, the Joukowski map; it maps $\mathbb{D}$ conformally onto the sphere minus $[-2,2]$. Proposition 7.20 below follows from the classical fact that a rational (resp. entire) Schwarz function forces a circle or a line (resp. a line); see [10] and [9, pp. 104–106]. The part proved here is that the Schwarz function of a rational univalent W with dense image is a Möbius map, which gives the form (iii) of W.

Proposition 7.20. *Let W be a rational function that is univalent in $\mathbb{D}$, with no pole in $\mathbb{D}$, such that $W(\mathbb{D})$ is dense in $\mathbb{C}$. Let P be the set of poles of W on $\mathbb{T}$, and $\Gamma=\mathbb{C}\setminus W(\mathbb{D})$.*

(i) *$\Gamma = W(\mathbb{T}\setminus P)$; it is non-empty and closed, with empty interior and of locally finite length.*
(ii) *Γ lies on a line: it is one closed ray, or two disjoint closed rays on one line that point away from each other.*
(iii) *$W=\Sigma\circ\mathcal{J}\circ\psi$, where ψ is an automorphism of $\mathbb{D}$ and Σ is a Möbius map whose pole lies in $[-2,2]$; in particular $\deg W = 2$.*

Conversely, every such $\Sigma\circ\mathcal{J}\circ\psi$ is univalent in $\mathbb{D}$, and its image is $\mathbb{C}$ minus one or two closed rays on a line.

The image of a rational W in Corollary 7.19 is a slit plane, and a slit plane is dense in $\mathbb{C}$: a finite union of analytic arcs that is closed in $\mathbb{C}$ has empty interior. So Proposition 7.20 applies to it.

Proof. Replacing W by $(W-W(0))/W'(0)$, an affine change that keeps the hypotheses and the conclusions, we may assume $W\in\mathcal{S}$. As $W(\mathbb{D})$ is dense, it is unbounded, so $P\neq\emptyset$. Put $\Omega=W(\mathbb{D})$.

(i) Let $z_0\in\Gamma$ and $W(w_n)\to z_0$ with $w_n\in\mathbb{D}$. A limit point ζ of (w_n) is not in $\mathbb{D}$, since $z_0\notin\Omega$, and not in P, since $W\to\infty$ there; so $\zeta\in\mathbb{T}\setminus P$ and $W(\zeta)=z_0$. Conversely $W(\mathbb{T}\setminus P)\cap\Omega=\emptyset$ by Lemma 7.16. So $\Gamma=W(\mathbb{T}\setminus P)$. It is closed, as Ω is open; it is non-empty, since the inverse of a conformal map of $\mathbb{D}$ onto $\mathbb{C}$ would be a bounded non-constant entire function; and it has empty interior, as Ω is dense. For a compact set $X\subset\mathbb{C}$, the points $\zeta\in\mathbb{T}\setminus P$ with $W(\zeta)\in X$ stay away from P and form a compact subset of $\mathbb{T}\setminus P$, on which W is analytic; so $\Gamma\cap X$ has finite length.

The Schwarz function. The function $W^*(w)=\overline{W(1/\bar w)}$ is rational, so $S_\Omega = W^*\circ W^{-1}$ is meromorphic on Ω, with finitely many poles; they lie in the open set Ω, at positive distance from Γ. Let $z_0\in\Gamma$ and $z_n\to z_0$ with $z_n\in\Omega$, and put $w_n=W^{-1}(z_n)$. As in (i), every limit point ζ of (w_n) lies in $\mathbb{T}\setminus P$ with $W(\zeta)=z_0$; W^* is continuous at ζ, and $W^*(\zeta)=\overline{W(\zeta)}$ since $1/\bar\zeta=\zeta$. So

$S_\Omega(z_n) \to \overline{z_0}$, whichever preimage the w_n approach, and S_Ω extends continuously to $\mathbb{C}$ minus its poles, with $S_\Omega(z) = \bar{z}$ for $z \in \Gamma$.

S_Ω is meromorphic on $\mathbb{C}$. S_Ω is continuous on $\mathbb{C}$ minus finitely many poles and holomorphic off the closed set Γ, which has σ-finite length by (i). By a theorem of Besicovitch [5], a function that is continuous on an open set and holomorphic off a closed subset of σ-finite length is holomorphic there. So S_Ω is meromorphic on $\mathbb{C}$.

S_Ω is a Möbius map. S_Ω is not constant, as $S_\Omega(z) = \bar{z}$ on Γ, which contains an arc. Put $S^*_\Omega(\xi) = \overline{S_\Omega(\bar{\xi})}$, meromorphic on $\mathbb{C}$. For $z \in \Gamma$, $\overline{S_\Omega(z)} = z$, so $S^*_\Omega(S_\Omega(z)) = \overline{S_\Omega(z)} = z$; as Γ has non-isolated points, $S^*_\Omega \circ S_\Omega$ is the identity. So S_Ω is injective on $\mathbb{C}$ minus its poles. It has at most one pole, since near two poles the values near ∞ would be taken twice (open mapping theorem). It has no essential singularity at ∞: if R exceeds the moduli of all poles, then by the Casorati–Weierstrass theorem $S_\Omega(\{|z| > R\})$ is dense in $\mathbb{C}$, so it meets the non-empty open set $S_\Omega(\{|z| < R\} \setminus \{\text{poles}\})$, against injectivity. So S_Ω is rational and injective, that is, a Möbius map.

(ii) $A(z) = \overline{S_\Omega(z)}$ is an antiholomorphic map of the sphere with $A \circ A = S^*_\Omega \circ S_\Omega = \mathrm{id}$ and $A(z) = z$ on Γ. An antiholomorphic involution of the sphere is conjugate by a Möbius map to $z \mapsto \bar{z}$, whose fixed set is a circle, or to $z \mapsto -1/\bar{z}$, which has no fixed point. So Γ lies on a circle C_0. As Ω is simply connected, the complement $\Gamma \cup \{\infty\}$ of Ω in the sphere is connected; so the non-empty closed set Γ is unbounded, and ∞ lies in the closed set C_0: C_0 is a line together with ∞. Now $\Gamma \cup \{\infty\}$ is a connected closed proper subset of C_0 (the plane minus a line is not connected) with more than one point, that is, a closed arc of C_0 through ∞. If ∞ is an end of the arc, Γ is one closed ray; otherwise Γ is the line minus a bounded open interval, that is, two closed rays.

(iii) A Möbius map Σ maps $[-2, 2]$ onto the arc $\Gamma \cup \{\infty\}$ (ends to ends and one inner point to one inner point), so $\Omega = \Sigma(\text{sphere} \setminus [-2, 2])$, and the pole of Σ lies in $[-2, 2]$. Then $\Sigma \circ \mathcal{J}$ and W are conformal maps of $\mathbb{D}$ onto Ω, so $W = \Sigma \circ \mathcal{J} \circ \psi$ with ψ an automorphism of $\mathbb{D}$, and $\deg W = \deg \mathcal{J} = 2$. The converse is the same computation read backwards. □

Remark 7.21. (a) If a rational W, univalent in $\mathbb{D}$ and possibly with a pole there, has $W(\mathbb{D})$ dense in the sphere, then the sphere minus $W(\mathbb{D})$ is a closed circular arc and $W = \Sigma \circ \mathcal{J} \circ \psi$ with ψ an automorphism of $\mathbb{D}$ and Σ a Möbius map whose pole need not lie in $[-2, 2]$: apply Proposition 7.20 to the composition of W with a Möbius map that sends a point outside $W(\mathbb{D})$ to ∞. (b) For one ray (the pole of Σ at ± 2) W has one double pole on $\mathbb{T}$, since $\mathcal{J}(w) - 2 = (w-1)^2/w$; for two rays (the pole of Σ in $(-2, 2)$) it has two simple poles on $\mathbb{T}$, since $\mathcal{J}(w) = x$ has two roots on $\mathbb{T}$ for $-2 < x < 2$. This is the alternative of Corollary 7.19. (c) Examples: the Koebe function, $1/(\mathcal{J}(w) - 2)$, equal to $1/(2\cos t - 2)$ at $w = e^{it}$, omits the ray $(-\infty, -1/4]$; $w/(1 + w^2) = 1/\mathcal{J}(w)$, equal to $1/(2\cos t)$ on $\mathbb{T}$, omits $(-\infty, -1/2] \cup [1/2, \infty)$; $w/(1 - w^2)$, equal to $i/(2\sin t)$ on $\mathbb{T}$, omits two rays on the imaginary axis. (d) So no rational map of $\mathbb{D}$ onto a slit plane has a curved slit.

When all poles of W lie on the circle, the Schwarz function S_Ω of the proof of Proposition 7.20 has no pole; compare Sakai's null quadrature domains [36], and Shapiro [38] for unbounded quadrature domains.

A sign rule. Let $W \in \mathcal{S}$ be meromorphic on an open set $U \supset \overline{\mathbb{D}}$, with its poles in U forming a finite set $P \subset \mathbb{T}$, as before Lemma 7.16. By that lemma $W(\mathbb{T} \setminus P)$ misses $W(\mathbb{D}) \ni 0$, so $W \neq 0$ on $\mathbb{T} \setminus P$. Put $g = wW'/W$; it is holomorphic near the closed disk minus P and zero-free in $\mathbb{D}$, with $g(0) = 1$. On $\mathbb{T} \setminus P$ write $W(e^{it}) = e^{u(t) + i\vartheta(t)}$ locally, so that

$$g(e^{it}) = \vartheta'(t) - iu'(t), \qquad u = \log|W|.$$

u is real-analytic on $\mathbb{T} \setminus P$ and tends to $+\infty$ at P, so it is constant on no arc of $\mathbb{T} \setminus P$, unless $P = \emptyset$ and $|W|$ is constant on $\mathbb{T}$; then $W(\mathbb{D})$ is a disk about 0 and $W(w) = w$. Near a pole p of order m, $g = -mp/(w - p) + O(1)$, so along the circle

$$g(pe^{it}) = \frac{im}{2} \cot\frac{t}{2} + O(1) \qquad (t \to 0).$$

Hence the set

$$Z_- = \{\zeta \in \mathbb{T} \setminus P : \ g(\zeta) \in (-\infty, 0)\}$$

stays away from P; it consists of zeros of u', as $\operatorname{Im} g = -u'$, so it is finite. At $\zeta = e^{it} \in Z_-$ the map $t \mapsto \arg W(e^{it})$ turns clockwise and $|W|$ is stationary. For a pair (W, B) put $F = WB$, $\sigma = wB'/B$, which is holomorphic near $\mathbb{T}$ and positive there (the function σ of Section 2.4), and

$$\chi = g + \sigma = \frac{wF'}{F}.$$

Theorem 7.22. *Let W be as above, and suppose that $u''(t) > 0$ at every $e^{it} \in Z_-$; that is, every stationary point of $|W|$ on the circle at which $\arg W$ turns clockwise is a nondegenerate minimum of $|W|$. Then there is no admissible pair (W, B) with B non-constant.*

If W is starlike, then $\operatorname{Re} g > 0$ in $\mathbb{D}$, so $Z_- = \emptyset$; for such W, meromorphic past the circle, Theorem 7.22 recovers the observation of [23, Sect. 9] that a starlike W admits no non-constant factor B.

Proof. Let (W, B) be admissible with $\deg B = k \geq 1$; the zeros $a_1, \dots, a_k$ of B are simple and non-zero (Section 7.4). On $\mathbb{T} \setminus P$, $F \neq 0$, so the zeros of χ there are the zeros of F'. At such a point $\zeta = e^{it}$, $\operatorname{Im} \chi = -u'(t) = 0$ and $\operatorname{Re} \chi = \vartheta'(t) + \sigma(\zeta) = 0$, so $g(\zeta) = -\sigma(\zeta) < 0$ and $\zeta \in Z_-$; there $\frac{d}{dt}\chi(e^{it})$ has imaginary part $-u''(t) \neq 0$, so the zero is simple. At a zero c of W' on $\mathbb{T} \setminus P$, $\chi(c) = \sigma(c) > 0$.

Let $\mathbb{T}_\rho$ be the circle with inward semicircular indents of radius ρ at the points of P and at the zeros of χ on $\mathbb{T} \setminus P$, where 2ρ is less than the distance between any two points of $P \cup Z_-$, and no zero of B lies within 2ρ of $\mathbb{T}$. Inside $\mathbb{T}_\rho$, χ has no zero, as $F' \neq 0$ in $\mathbb{D}$ and $\chi(0) = 1$, and it has exactly k poles, all simple, at the a_j, as $W \neq 0$ in $\mathbb{D} \setminus \{0\}$. By the argument principle $\operatorname{wind}(\chi \circ \mathbb{T}_\rho) = -k < 0$. For small ρ the curve $\chi \circ \mathbb{T}_\rho$ misses 0 and meets $(-\infty, 0)$ at finitely many points (see below), and its winding number is the sum of the local crossing numbers there: $+1$ for a passage from $\operatorname{Im} \chi > 0$ to $\operatorname{Im} \chi < 0$, -1 for the opposite passage, and 0 if the sign does not change.

(a) On the arcs of $\mathbb{T}$ outside the indents, $\operatorname{Im} \chi = -u'$, as σ is real. So χ meets $(-\infty, 0)$ only at zeros of u' where $\vartheta' + \sigma < 0$; there $g = \vartheta' < -\sigma < 0$, so the point lies in Z_- and is a nondegenerate minimum of u. So u' passes from negative to positive, and $\operatorname{Im} \chi$ from positive to negative: the crossing number is $+1$.

(b) At an indent about a zero ζ of χ, put $A_\zeta = i\zeta\chi'(\zeta)$, so that $\chi(\zeta e^{it}) = A_\zeta t + O(t^2)$ and $\operatorname{Im} A_\zeta = -u'' < 0$. On the inward semicircle $w - \zeta$ runs from $-i\zeta\rho$ through $-\zeta\rho$ to $i\zeta\rho$, clockwise about ζ, so $\chi = \chi'(\zeta)(w - \zeta) + O(\rho^2)$ runs from $-A_\zeta\rho$ through $iA_\zeta\rho$ to $A_\zeta\rho$: it starts in the upper half-plane, passes through the right half-plane ($\operatorname{Re}(iA_\zeta) = -\operatorname{Im} A_\zeta > 0$) and ends in the lower half-plane. It meets the positive axis, not $(-\infty, 0)$: the indent contributes 0.

(c) At an indent about a pole p of order m, write $w - p = \rho p e^{i\phi}$ with $\phi \in [\pi/2, 3\pi/2]$. As σ is bounded near p, $\chi = -(m/\rho)e^{-i\phi} + O(1)$, so $\arg \chi = \pi - \phi + O(\rho)$ lies in $[-\pi/2, \pi/2]$ up to $O(\rho)$, and where $\operatorname{Im} \chi$ is small, ϕ is near π and $\operatorname{Re} \chi$ is near $m/\rho > 0$. The indent does not meet $(-\infty, 0]$: it contributes 0.

So $\operatorname{wind}(\chi \circ \mathbb{T}_\rho)$ is the number of crossings in (a), which is at least 0, against $-k < 0$. □

Remark 7.23. For Blaschke models, Theorem 7.22 is contained in the pocket formula: with base point 0, $v = 1/g$, so the pockets are the points of Z_-, a minimum of u is a pocket of sign -1 and a maximum one of sign $+1$; and by Proposition 3.6 a feasible triple has a pocket of sign $+1$. The method, the argument principle for wF'/F along the circle, is that of [23, Sect. 8] and of Proposition 2.11. What Theorem 7.22 adds is the treatment of the boundary: indents at poles of W on the circle and at zeros of $(WB)'$ there; at critical points of W on the circle (for instance the ends of slits) $\chi = \sigma > 0$, so no indent is needed. This does not contradict [23, Lemma 8.1, Thm. 8.7(b)], which build admissible pairs with non-constant B from nondegenerate minima in Z_-: if W has no pole or critical point on the circle, for instance if it is univalent past the circle,

then g is holomorphic and zero-free near the closed disk, $\operatorname{wind}(g) = 0$, and under the hypothesis of Theorem 7.22 each point of Z_- adds $+1$ to this winding number, so $Z_- = \emptyset$. So the theorem has content only when W has poles or critical points on the circle.

Corollary 7.24. *Let $W = \Sigma \circ \mathcal{J} \circ \psi \in \mathcal{S}$ be as in Proposition 7.20*(iii)*, so that $W(\mathbb{D})$ is $\mathbb{C}$ minus one closed ray, or minus two disjoint closed rays on one line. Then there is no admissible pair (W, B) with B non-constant.*

Proof. W is rational, so the setting above applies, and $\Gamma = W(\mathbb{T} \setminus P)$ by Proposition 7.20(i). At a zero of W' on $\mathbb{T} \setminus P$ (an end of a ray) $g = 0$, so it is not in Z_-. At the other points of $\mathbb{T} \setminus P$, the map $t \mapsto W(e^{it})$ is a local diffeomorphism onto a ray $\{z_e + se^{i\theta} : s \geq 0\}$, so locally $W(e^{it}) = z_e + s(t)e^{i\theta}$ with $s'(t) \neq 0$. The function $s \mapsto |z_e + se^{i\theta}|^2$ is a strictly convex quadratic, positive as $0 \notin \Gamma$, so $\ell(s) = \log|z_e + se^{i\theta}|$ has at most one stationary point, and there $\ell'' = 1/|z_e + se^{i\theta}|^2 > 0$. As $u = \ell \circ s$, a zero of u' is a stationary point of ℓ, and there $u'' = \ell''\,(s')^2 > 0$. So every point of Z_- is a nondegenerate minimum of u, and Theorem 7.22 applies. □

Corollary 7.24 needs straight slits: Jatar and Ng [23, Sect. 9] mention a pair with $\deg B = 1$ and gain $\sqrt{61/58}$, built from a bent radial slit.

Corollary 7.25. *Let $d \geq 1$. No pair that attains $\mathcal{B}_d$ has a rational W.*

Proof. Let (W, B) attain $\mathcal{B}_d$ with W rational. B is not constant, as $1/B(0) = \mathcal{B}_d \geq \mathcal{B}_1 > 1$ by [23, Thm. 6.1]. $W \in \mathcal{S}$ is rational, so it is meromorphic on $\mathbb{C}$, and by Corollary 7.19, $W(\mathbb{D})$ is a slit plane, which is dense in $\mathbb{C}$. By Proposition 7.20, $W = \Sigma \circ \mathcal{J} \circ \psi$ with the pole of Σ in $[-2, 2]$, and Corollary 7.24 gives a contradiction. □

So an extremal W, which exists for each $d \geq 1$ by [23, Thm. 8.9], is not rational. By Corollaries 7.14 and 7.19 it does not extend holomorphically past the circle, and if it extends meromorphically, then it is a non-rational map onto a slit plane; by Theorem 7.22, Z_- then has a point at which u is not a nondegenerate minimum.

7.7. **Counting the zeros of $(WB)'$ from the circle.** Theorem 7.22 has a quantitative form: an exact count of the zeros of $(WB)'$ in $\mathbb{D}$, read off the boundary, for every finite Blaschke product B. Let $W(w) \not\equiv w$, P, u, ϑ, g and Z_- be as in Section 7.6. Let B be a finite Blaschke product of degree $k \geq 1$, with zeros anywhere in $\mathbb{D}$ (repetitions allowed). Put $F = WB$, $\sigma = wB'/B$ and $\chi = g + \sigma = wF'/F$, and let $N_{\mathbb{D}}(F')$ be the number of zeros of F' in $\mathbb{D}$, counted with multiplicity. For a Blaschke model with base point 0, the pockets of Section 3 are the points of Z_-, and a pocket is on if and only if $\chi > 0$ there (see the text after Theorem 7.26). On $\mathbb{T} \setminus P$ we have $\operatorname{Im}\chi = -u'$, as σ is real there. For $\zeta \in \mathbb{T} \setminus P$ write

$$\chi(\zeta e^{it}) = c_\zeta t^{m_\zeta} + O(t^{m_\zeta + 1}) \qquad (t \in \mathbb{R},\ t \to 0,\ c_\zeta \neq 0).$$

As $F(\zeta) \neq 0, \infty$, $m_\zeta \geq 0$ is the order of the zero of F' at ζ. If u' vanishes at ζ, let $k_\zeta \geq 1$ be the order of this zero; it is finite, since u is constant on no arc of $\mathbb{T} \setminus P$. Let $\varepsilon_\zeta = +1$, -1 or 0 according as u has a strict local maximum, a strict local minimum, or neither at ζ. In the last case u' has one sign s_ζ on both sides of ζ. So k_ζ is odd if and only if $\varepsilon_\zeta \neq 0$. Since $\operatorname{Im}\chi = -u'$, we have $k_\zeta \geq m_\zeta$, and c_ζ is real if $k_\zeta > m_\zeta$. The set

$$Z_\chi = \{\zeta \in \mathbb{T} \setminus P :\ \chi(\zeta) \in (-\infty, 0]\}$$

lies in Z_-, since there $u' = 0$ and $\vartheta' = \chi - \sigma < 0$. It contains the zeros of F' on $\mathbb{T} \setminus P$, and no zero of W' (there $\chi = \sigma > 0$). The index defined next is the contribution of ζ to minus the winding number of χ along the circle indented into the disk at the points of Z_χ and P; the indices add up

to $k - N_{\mathbb{D}}(F')$ (Theorem 7.26(i)). For $\zeta \in Z_\chi$ define the *index*

$$\iota_\zeta = \begin{cases} m_\zeta/2 + \varepsilon_\zeta & \text{if } m_\zeta \text{ is even and } c_\zeta < 0, \\ m_\zeta/2 & \text{if } m_\zeta \text{ is even and } c_\zeta \notin (-\infty, 0), \\ (m_\zeta + \varepsilon_\zeta)/2 & \text{if } m_\zeta \text{ is odd and } \varepsilon_\zeta \neq 0, \\ (m_\zeta + s_\zeta \operatorname{sgn} c_\zeta)/2 & \text{if } m_\zeta \text{ is odd and } \varepsilon_\zeta = 0 \text{ (then } c_\zeta \text{ is real).} \end{cases}$$

If $m_\zeta = 0$, that is $\chi(\zeta) < 0$, then $\iota_\zeta = \varepsilon_\zeta$: -1 at a minimum of u and $+1$ at a maximum. Finally put

$$\operatorname{wt}(W) = \sum_{\zeta \in Z_-} \frac{k_\zeta + \varepsilon_\zeta}{2}.$$

Every term is a non-negative integer, and $\operatorname{wt}(W)$ depends on W only.

Theorem 7.26. *Let W and B be as above.*

(i) $N_{\mathbb{D}}(F') = k - \sum_{\zeta \in Z_\chi} \iota_\zeta$. *In particular $F' \neq 0$ in $\mathbb{D}$ if and only if the indices add up to k.*

(ii) $\iota_\zeta \leq (k_\zeta + \varepsilon_\zeta)/2$ *for every $\zeta \in Z_\chi$. Hence every admissible pair (W, B) satisfies* $\deg B \leq \operatorname{wt}(W)$.

Proof. (i) Let $\mathbb{T}_\rho$ be the circle with inward semicircular indents of radius ρ at the points of P and at the zeros of χ on $\mathbb{T} \setminus P$. F' is meromorphic near the closed disk with poles only at P, and $|F'| \to \infty$ at P, so it has finitely many zeros in $\mathbb{D}$. For small ρ the region inside $\mathbb{T}_\rho$ contains all of them, all zeros of B, and no pole of W. Since wF' and F are holomorphic there,

$$\operatorname{wind}(\chi \circ \mathbb{T}_\rho) = \operatorname{wind}(wF' \circ \mathbb{T}_\rho) - \operatorname{wind}(F \circ \mathbb{T}_\rho) = (1 + N_{\mathbb{D}}(F')) - (1 + k).$$

Around each point x of $Z_\chi \cup P$ let $\mathcal{A}_x$ be the part of $\mathbb{T}_\rho$ formed by the arc of $\mathbb{T}$ from $xe^{-i\eta}$ to the indent at x (if any), the indent, and the arc to $xe^{i\eta}$. On the rest of $\mathbb{T}_\rho$, χ avoids $(-\infty, 0]$, so the principal argument $\operatorname{Arg}\chi$ is continuous there. Put

$$\delta_x = \frac{1}{2\pi}\big(\operatorname{Arg}\chi(xe^{i\eta}) - \operatorname{Arg}\chi(xe^{-i\eta}) - \Delta_{\mathcal{A}_x} \arg\chi\big),$$

an integer. The terms Arg telescope around $\mathbb{T}_\rho$, so $\operatorname{wind}(\chi \circ \mathbb{T}_\rho) = -\sum_x \delta_x$. Each δ_x is constant in ρ and η, and we evaluate it as $\rho, \eta \to 0$.

At a pole p of order m_p, $\mathcal{A}_p$ avoids $(-\infty, 0]$: on the circle $|\operatorname{Im}\chi| \to \infty$ at p, since $g(pe^{it}) = \frac{im_p}{2}\cot\frac{t}{2} + O(1)$, and on the indent see part (c) of the proof of Theorem 7.22. So $\delta_p = 0$.

Let $\zeta \in Z_\chi$, $m = m_\zeta$, $c = c_\zeta$. On the indent $w - \zeta = \rho\zeta e^{i\theta}$, with θ decreasing from $-\pi/2$ to $-3\pi/2$, we have $\chi = c\rho^m e^{im(\theta - \pi/2)}(1 + O(\rho))$, so $\arg\chi$ decreases by $m\pi + O(\rho)$. On the two arcs of the circle a continuous branch of $\arg\chi$ changes by $O(\eta)$. At the ends, $\chi(\zeta e^{\pm i\eta}) = c_\pm \eta^m(1 + O(\eta))$ with $c_\pm = c(\pm 1)^m$. If $c_\pm \notin (-\infty, 0)$, then $\operatorname{Arg}\chi(\zeta e^{\pm i\eta}) \to \operatorname{Arg} c_\pm$. If $c_\pm < 0$, then c is real, so $k_\zeta > m$. Then $\operatorname{Im}\chi(\zeta e^{\pm i\eta}) = -u'$ is of order η^{k_ζ}, with the sign opposite to the sign $s_\pm$ of u' on that side, and $\operatorname{Arg}\chi(\zeta e^{\pm i\eta}) \to -s_\pm\pi$. So $\delta_\zeta = (m\pi + A_+ - A_-)/2\pi$, with $A_\pm$ these limits. At an extremum $\varepsilon_\zeta = s_- = -s_+$; otherwise $s_- = s_+ = s_\zeta$.

- If m is even, then $c_+ = c_- = c$. If $c \notin (-\infty, 0)$, then $A_+ = A_-$ and $\delta_\zeta = m/2$. If $c < 0$, then $\delta_\zeta = m/2 + (s_- - s_+)/2 = m/2 + \varepsilon_\zeta$.
- If m is odd and $c \notin \mathbb{R}$, then $A_+ - A_- = \operatorname{Arg} c - \operatorname{Arg}(-c) = \pi \operatorname{sgn}\operatorname{Im} c$. Also $u' = -\operatorname{Im}(c)t^m + \dots$ changes sign, so $\varepsilon_\zeta = \operatorname{sgn}\operatorname{Im} c$ and $\delta_\zeta = (m + \varepsilon_\zeta)/2$.
- If m is odd and $c > 0$, then $\delta_\zeta = (m + s_-)/2$. If m is odd and $c < 0$, then $\delta_\zeta = (m - s_+)/2$. At an extremum both equal $(m + \varepsilon_\zeta)/2$; otherwise they equal $(m + s_\zeta \operatorname{sgn} c)/2$.

So $\delta_\zeta = \iota_\zeta$, and (i) follows.

(ii) Write $m = m_\zeta$, $\varepsilon = \varepsilon_\zeta$; recall that $m \leq k_\zeta$ and that k_ζ is odd if and only if $\varepsilon \neq 0$. If m is even and $\varepsilon = \pm 1$, then $m \leq k_\zeta - 1$, so $\iota_\zeta \leq m/2 + \max(\varepsilon, 0) \leq (k_\zeta + \varepsilon)/2$. If m is even and $\varepsilon = 0$, then $\iota_\zeta = m/2 \leq k_\zeta/2$. If m is odd and $\varepsilon \neq 0$, the claim is $m \leq k_\zeta$. If m is odd and $\varepsilon = 0$, then k_ζ

is even, so $m \le k_\zeta - 1$ and $\iota_\zeta \le (m+1)/2 \le k_\zeta/2$. For an admissible pair $N_{\mathbb{D}}(F') = 0$; all terms $(k_\zeta + \varepsilon_\zeta)/2$ are non-negative, and $Z_\chi \subset Z_-$, so $k = \sum_{Z_\chi} \iota_\zeta \le \sum_{Z_\chi} (k_\zeta + \varepsilon_\zeta)/2 \le \operatorname{wt}(W)$. □

The term $(k_\zeta + \varepsilon_\zeta)/2$ vanishes exactly at nondegenerate minima ($k_\zeta = 1$, $\varepsilon_\zeta = -1$), so Theorem 7.22 is the case $\operatorname{wt}(W) = 0$ of Theorem 7.26(ii). For Blaschke models (W univalent past the circle and $F' \neq 0$ on the closed disk), Theorem 7.26(i) is the pocket formula, Theorem 3.4, in another form. With base point 0, the pockets are the points of Z_-; a pocket is on if and only if $\chi > 0$ there, and off if and only if $\chi < 0$; and the signs of all pockets add up to 0 (Remark 7.27). So $k = \sum_{\text{off}} \varepsilon$, and (ii) is immediate there. Theorem 7.26 adds the zeros of F' on the circle, of any order, together with poles and critical points of W on the circle, and the bound by $\operatorname{wt}(W)$.

Remark 7.27. If W is univalent past the circle, then $P = \emptyset$, and g is holomorphic and zero-free near the closed disk, so $\operatorname{wind}(g|_{\mathbb{T}}) = 0$. The proof of Theorem 7.26(i), applied to g in place of χ (the case $B = 1$; the proof does not use $k \ge 1$, and here the special points are exactly the points of Z_- and all have $m_\zeta = 0$), gives $\operatorname{wind}(g|_{\mathbb{T}}) = -\sum_{Z_-} \varepsilon_\zeta$. Hence $\sum_{Z_-} \varepsilon_\zeta = 0$ and $\operatorname{wt}(W) = \frac{1}{2} \sum_{Z_-} k_\zeta$.

Corollary 7.28. *Let $W \in \mathcal{S}$ be univalent on a neighbourhood of the closed disk, and suppose that $u'' \neq 0$ at every point of Z_-. Then $\operatorname{wt}(W)$ is the number of points of Z_- at which u has a minimum, and also the number at which it has a maximum. It is the largest degree of a finite Blaschke product B with (W, B) admissible.*

Proof. By Remark 7.27 with $k_\zeta = 1$, Z_- has $n = \operatorname{wt}(W)$ minima and n maxima of u. The minima satisfy the conditions (26) of [23, Lemma 8.1], so for $n \ge 1$ that lemma gives a regular admissible pair of degree n; for $n = 0$ only $B = 1$ is possible. No larger degree is possible, by Theorem 7.26(ii). □

The lower bound in Corollary 7.28 is [23, Lemma 8.1, Thm. 8.7(b)]; the upper bound is Theorem 7.26(ii).

Corollary 7.29. *Let $W \in \mathcal{S}$, $W(w) \not\equiv w$, be holomorphic on a neighbourhood of the closed disk, and let $N_{u'}$ be the number of zeros of u' on $\mathbb{T}$, counted with multiplicity. Then $\deg B \le N_{u'}/2 - 1$ for every admissible pair (W, B). If W is a polynomial of degree $n \ge 2$, then $N_{u'} \le 2n - 2$, so $\deg B \le n - 2$.*

Proof. Let ζ^* be a point where u is maximal on $\mathbb{T}$. By the maximum principle $|W| \le |W(\zeta^*)|$ on the closed disk, so the radial derivative $\operatorname{Re} g = \vartheta'$ is ≥ 0 at ζ^*, and $\zeta^* \notin Z_-$. As u is not constant, ζ^* is a strict local maximum, with $(k_{\zeta^*} + 1)/2 \ge 1$. Strict local maxima and minima of u alternate on the circle, so the sum of $(k_\zeta + \varepsilon_\zeta)/2$ over all zeros of u' is $N_{u'}/2$. Hence $\operatorname{wt}(W) \le N_{u'}/2 - 1$, and Theorem 7.26(ii) applies. For a polynomial W of degree n, $u' = -\operatorname{Im}(wW'\overline{W})/|W|^2$ on the circle, and there $wW'\overline{W}$ is a combination of the terms $w^j \bar{w}^l = e^{i(j-l)t}$ with $1 \le j, l \le n$; so $\operatorname{Im}(wW'\overline{W})$ is a real trigonometric polynomial of degree at most $n - 1$, not identically 0. So $N_{u'} \le 2n - 2$. □

For $n = 2$ the bound says that a univalent quadratic polynomial carries no non-constant B; indeed $w + aw^2$ with $|a| \le 1/2$ is starlike. For the maps $W = (H \circ \varphi)/N_b$ of [23, Lemma 6.2] where H is a polynomial of degree n with $H(b) = 0$, $b \in \mathbb{D} \setminus \{0\}$, $\varphi(w) = (w + b)/(1 + \bar{b}w)$ and $N_b = (1 - |b|^2)H'(b)$ (Section 8 recalls them), the zeros of u' on the circle are those of $\operatorname{Im}(zH'\overline{H})$, in the variable $z = \varphi(w)$ of H, moved by φ (as N_b is a constant); this trigonometric polynomial has degree at most n, and the same argument gives $\deg B \le n - 1$.

Remark 7.30. Pairs XI and X of Table 2 (Section 8) are admissible with F' vanishing on the circle, a case that Theorem 3.4 does not cover. As in Section 8, let $z = \varphi(w)$ be the variable of Q; then $\chi = wF'/F = |\varphi'|\, zQ'(z)/Q(z)$ on the circle, where $w\varphi'/\varphi = |\varphi'| > 0$, and φ preserves the orientation of the circle; so the indices can be computed in the variable z. At $z = 1$, the zero of Q' of order 5 is a maximum of u with $k_\zeta = 5$, and c is not real, so the index is $(5+1)/2 = 3$. The other

points of Z_χ are two minima of u with $\chi < 0$, each of index -1; for pair X, also the point $\zeta_2 \neq 1$ of the circle (Table 2), a simple zero of Q' at a minimum of u, of index 0. So $3 - 1 - 1 = 1 = \deg B$. So these two pairs place a zero of order 5 of $(WB)'$ at a flat maximum of $|W|$ (Section 7.8), where $\arg W$ turns clockwise. That this zero is a strict maximum of u with $k_\zeta = 5$ is proved in Section 7.8.

Remark 7.31. Let W be meromorphic past the circle with $W(\mathbb{D}) = \mathbb{C} \setminus \Gamma$, as for every extremal W of Corollary 7.19. By the sector count of Lemmas 7.12 and 7.17, a finite point of Γ is either an end (one preimage, a zero of W', not in Z_-) or lies on an analytic arc of Γ (two preimages, on the two edges). At such a point, $u' = 0$ exactly when Γ is tangent to the circle about 0 through it. Then ϑ' has opposite signs on the two edges, so exactly one preimage ζ lies in Z_-, with k_ζ and ε_ζ those of the distance to 0 along Γ. So $\deg B$ is at most the sum, over these tangency points, of $(k_\zeta + \varepsilon_\zeta)/2$. A nondegenerate local minimum of the distance to 0 along Γ counts 0, and a local maximum counts at least 1. An extremal pair with $\deg B = k \geq 1$ whose W is meromorphic past the circle therefore needs a slit that bends back toward 0, or touches a circle about 0 to higher order, with total weight at least k. For straight slits the sum is 0 (Corollary 7.24). A slit with a corner is not covered: W is not meromorphic past the circle at the preimages of a corner.

7.8. **Flat maxima and a Schwarzian on the circle.** Corollary 7.28 assumes that $u'' \neq 0$ at every point of Z_-. Here we look at a *flat maximum*: a point $\zeta \in Z_-$ at which u has a strict local maximum with $k_\zeta \geq 3$. Its term in $\mathrm{wt}(W)$ is $(k_\zeta + 1)/2 \geq 2$. By the definition of the index, at such a point $\iota_\zeta \geq 2$ holds only if $m_\zeta \geq 3$, or $m_\zeta = 2$ and $c_\zeta < 0$; and $\iota_\zeta = (k_\zeta + 1)/2$ holds only if $m_\zeta = k_\zeta$, or $m_\zeta = k_\zeta - 1$ and $c_\zeta < 0$. We show that one number, computed from W alone, rules this out when it is negative: then $\iota_\zeta \leq 1$ for every B.

For a positive C^2 function f on an arc of $\mathbb{T}$, with derivatives taken in the angle ($f'(\zeta) = \frac{d}{dt} f(\zeta e^{it})|_{t=0}$), put

$$\Upsilon(f) = \frac{f''}{f} - \frac{3}{2}\Big(\frac{f'}{f}\Big)^2 - \frac{1 - f^2}{2}.$$

If Y is a real function of the angle with $Y' = f$, then $\Upsilon(f) = \mathrm{Sch}\, Y + (Y'^2 - 1)/2$, where $\mathrm{Sch}\, Y = Y'''/Y' - \frac{3}{2}(Y''/Y')^2$ is the Schwarzian derivative. Let $P_a(e^{it}) = (1 - |a|^2)/|e^{it} - a|^2$ be the Poisson kernel of $a \in \mathbb{D}$; on the circle, $\sigma = wB'/B = \sum_j P_{a_j}$, where the a_j are the zeros of B.

Lemma 7.32. (a) (*Classical; see* [29].) $\Upsilon(P_a) = 0$ *on* $\mathbb{T}$ *for every* $a \in \mathbb{D}$.

(b) *Let* B *be a finite Blaschke product of degree* $k \geq 1$, *and* $\sigma = wB'/B$ *on* $\mathbb{T}$. *If* $k = 1$, *then* $\Upsilon(\sigma) = 0$; *if* $k \geq 2$, *then* $\Upsilon(\sigma) > 0$ *at every point of* $\mathbb{T}$.

(c) *Let* φ *be an automorphism of* $\mathbb{D}$, *and let* ψ *be a real function with* $\varphi(e^{i\theta}) = e^{i\psi(\theta)}$. *If* $f > 0$ *and* $\tilde{f} = (f \circ \psi)\,\psi'$, *then* $\Upsilon(\tilde{f}) = \psi'^2\,(\Upsilon(f) \circ \psi)$.

Proof. (a) After a rotation, $a = r \in [0, 1)$. With $D(t) = 1 + r^2 - 2r\cos t$ we have $P_a = (1 - r^2)/D$, $P_a'/P_a = -D'/D$ and $P_a''/P_a = -D''/D + 2D'^2/D^2$, so

$$D^2\,\Upsilon(P_a) = -DD'' + \tfrac{1}{2}D'^2 - \tfrac{1}{2}\big(D^2 - (1 - r^2)^2\big).$$

Here $D' = 2r\sin t$, $D'' = 2r\cos t$ and $D^2 - (1 - r^2)^2 = 4r(r - \cos t)(1 - r\cos t)$, and the right side expands to 0.

(b) On the circle $\sigma = \sum_{j=1}^k f_j$ with $f_j = P_{a_j} > 0$. By (a), $f_j'' = \frac{3}{2}f_j'^2/f_j + \frac{1}{2}f_j(1 - f_j^2)$. Hence

$$\begin{aligned}\sigma\,\Upsilon(\sigma) &= \sum_j f_j'' - \frac{3}{2}\frac{(\sum_j f_j')^2}{\sigma} - \frac{\sigma(1 - \sigma^2)}{2} \\ &= \frac{3}{2}\Big(\sum_j \frac{f_j'^2}{f_j} - \frac{(\sum_j f_j')^2}{\sum_j f_j}\Big) + \frac{1}{2}\Big(\big(\sum_j f_j\big)^3 - \sum_j f_j^3\Big).\end{aligned}$$

The first bracket is non-negative by the Cauchy–Schwarz inequality. The second is 0 if $k = 1$ and positive if $k \geq 2$.

(c) Let $Y' = f$; then $\tilde{f} = (Y \circ \psi)'$. The chain rule $\operatorname{Sch}(Y \circ \psi) = \psi'^2\,(\operatorname{Sch} Y \circ \psi) + \operatorname{Sch}\psi$ gives

$$\Upsilon(\tilde{f}) = \psi'^2\,(\operatorname{Sch} Y \circ \psi) + \operatorname{Sch}\psi + \tfrac{1}{2}\big(\psi'^2 (Y' \circ \psi)^2 - 1\big) = \psi'^2\,(\Upsilon(f) \circ \psi) + \Upsilon(\psi').$$

On the circle $\psi' = |\varphi'|$ is the Poisson kernel $P_{\varphi^{-1}(0)}$, so $\Upsilon(\psi') = 0$ by (a). □

Remark 7.33. Let H be a polynomial with no zero on $\mathbb{T}$, let φ be an automorphism of $\mathbb{D}$, and let $W = C\,(H \circ \varphi)$ with a constant $C \neq 0$. With ψ as in Lemma 7.32(c), $\log|W| = \log|H| \circ \psi + \log|C|$ on the circle, and $wW'/W = \psi'\,(zH'/H) \circ \varphi$ on the circle, with $\psi' = w\varphi'/\varphi > 0$ (H has variable $z = \varphi(w)$, as in Section 8). So the zeros of u' for W and for H correspond under ψ, with the same orders and types, ϑ' keeps its sign, and by Lemma 7.32(c) the number $\Upsilon(-\vartheta')$ at corresponding points of Z_- has the same sign. We use this for $W = (H \circ \varphi)/N_b$.

Corollary 7.34. *Let W, B and χ be as in Section 7.7, let $\zeta \in Z_\chi$ with $m_\zeta \geq 2$, and let $f = -\vartheta'$ near ζ (so $f > 0$ there).*

(i) *If $m_\zeta \geq 3$, then $\Upsilon(f)(\zeta) = \Upsilon(\sigma)(\zeta) \geq 0$, with equality if and only if $k = 1$.*
(ii) *If $m_\zeta = 2$ and $c_\zeta < 0$, then $\Upsilon(f)(\zeta) > 0$.*

Proof. On $\mathbb{T} \setminus P$ we have $\operatorname{Re}\chi = \operatorname{Re} g + \sigma = \sigma - f$. As $\chi(\zeta e^{it}) = c_\zeta t^{m_\zeta} + O(t^{m_\zeta+1})$, the function $\sigma - f$ vanishes to order at least m_ζ at $t = 0$. If $m_\zeta \geq 3$, then f, f', f'' agree with σ, σ', σ'' at ζ, so $\Upsilon(f)(\zeta) = \Upsilon(\sigma)(\zeta)$, and Lemma 7.32(b) gives the rest. If $m_\zeta = 2$ and $c_\zeta < 0$, then $f = \sigma$, $f' = \sigma'$ and $f'' = \sigma'' - 2c_\zeta$ at ζ, so $\Upsilon(f)(\zeta) = \Upsilon(\sigma)(\zeta) - 2c_\zeta/\sigma(\zeta) > 0$. □

In particular, if $\deg B = 1$, every zero of F' on the circle of order at least 3 imposes the equation $\Upsilon(-\vartheta')(\zeta) = 0$ on W alone, whatever the type of ζ. Every one-zero pair of Table 2 (Theorem 8.9, in Section 8) whose Q' has a zero of order 5 on the circle has a zero of F' of order 5 there, and so satisfies it. For pairs XI, X and IX, in the variable z of Remark 7.30, $zQ'/Q = c\,t^5 + O(t^6)$ at $z = e^{it}$ with $c = iR(1)/Q(1)$, where $Q' = (z-1)^5 R$; and c_ζ is a positive multiple of c. In exact rational arithmetic over $\mathbb{Q}(i)$, $\operatorname{Re}(R(1)/Q(1)) > 0$ for all three pairs, so $\operatorname{Im} c_\zeta > 0$: the zero is at a strict local maximum of u with $k_\zeta = 5$.

Proposition 7.35. *Let W be as in Section 7.7, let $\zeta \in Z_-$ be a point at which u has a strict local maximum with $k_\zeta \geq 3$, and put $f = -\vartheta'$ near ζ.*

(i) *If $\Upsilon(f)(\zeta) < 0$, then $\iota_\zeta \leq 1$ for every finite Blaschke product B with $\zeta \in Z_\chi$, and every admissible pair (W, B) satisfies $\deg B \leq \operatorname{wt}(W) - (k_\zeta - 1)/2 < \operatorname{wt}(W)$.*
(ii) *If $\Upsilon(f)(\zeta) \leq 0$, then still $\deg B \leq \operatorname{wt}(W) - (k_\zeta - 1)/2$ for every admissible pair (W, B).*

Proof. (i) At a maximum $\varepsilon_\zeta = 1$, and by the definition of the index, $\iota_\zeta \geq 2$ forces $m_\zeta \geq 3$, or $m_\zeta = 2$ and $c_\zeta < 0$. In both cases Corollary 7.34 gives $\Upsilon(f)(\zeta) \geq 0$. So $\iota_\zeta \leq 1$. For an admissible pair $N_{\mathbb{D}}(F') = 0$, and Theorem 7.26 gives

$$k = \sum_{\xi \in Z_\chi} \iota_\xi \leq 1 + \sum_{\xi \in Z_- \setminus \{\zeta\}} \frac{k_\xi + \varepsilon_\xi}{2} = \operatorname{wt}(W) - \frac{k_\zeta + 1}{2} + 1;$$

if $\zeta \notin Z_\chi$, the same bound holds without the term 1.

(ii) If $\Upsilon(f)(\zeta) = 0$ and $k \geq 2$, Corollary 7.34 excludes $\iota_\zeta \geq 2$ (for $k \geq 2$ both of its cases give $\Upsilon(f)(\zeta) > 0$), and the proof of (i) applies. If $k = 1$, the bound reads $1 \leq \operatorname{wt}(W) - (k_\zeta - 1)/2$, which holds because ζ alone contributes $(k_\zeta + 1)/2$ to $\operatorname{wt}(W)$. □

Proposition 7.36. *The hypothesis $u'' \neq 0$ on Z_- in Corollary 7.28 cannot be dropped. More precisely, there is $W \in \mathcal{S}$, univalent on a neighbourhood of the closed disk, of the form $W = (H \circ \varphi)/N_b$, where H is a polynomial of degree 14, $b \in \mathbb{D}$, $H(b) = 0$, $\varphi(w) = (w + b)/(1 + \bar{b}w)$ and $N_b = (1 - |b|^2)H'(b)$, and a point $\zeta \in Z_-$ at which u has a strict local maximum with $k_\zeta = 5$ and $\Upsilon(-\vartheta')(\zeta) < 0$. For every such W, every admissible pair (W, B) has $\deg B \leq \operatorname{wt}(W) - 2$.*

The proof uses pair IX of Theorem 8.9 and Lemmas 8.5, 8.6 and 9.9; it is given in Section 9, after Lemma 9.9.

So at a flat maximum the bound of Theorem 7.26(ii) can lose $(k_\zeta - 1)/2$, and if $\Upsilon(-\vartheta')(\zeta) < 0$, this loss occurs for every B.

7.9. **A bound for each degree.** In the proof of [23, Thm. 7.1] the inverse branch of F fixing 0 omits every zero of B, and Pick's estimate at one zero gives $\mathcal{B}_d < 4 - \frac{529}{576}\frac{(\log 2)^2}{d^2}$ [23, Thm. 1.3, Cor. 8.10]; [23, Rem. 7.3] suggests that the hyperbolic geometry of $\mathbb{D}$ minus all the zeros does better. We confirm this for large d. Here we use Lemma 7.1: the map h of its proof is univalent and omits all of A. By [23, Lemma 2.1], $\mathcal{B}_d$ is also the supremum of K_0 over the Blaschke models with $\deg B \le d$: regular admissible pairs are Blaschke models, the regularisation keeps $\deg B$, and $B \equiv 1$ gives $1/B(0) = 1$.

We use the following constants. Fix

$$\delta_0 = 10^{-3}, \quad \epsilon_{\max} = \sqrt{3\delta_0/2}, \quad \ell_{\max} = \log\frac{1}{1-\epsilon_{\max}}, \quad c_2 = \frac{(2-\epsilon_{\max})^2(1-\ell_{\max}/2)^2}{1+4\pi^2},$$

$$c_3 = \frac{4}{\log(1+c_2)}, \quad c_0 = 1 + c_3 \log\frac{1}{1-\ell_{\max}/2}, \quad c_1 = 2\log\cosh\frac{1}{2}, \quad \hat{M}_0 = \log(4-\delta_0),$$

$$q_d = c_3 \log\frac{4d}{\hat{M}_0} + c_0, \qquad Y_d = \frac{c_1 \hat{M}_0^2}{4d\, q_d}.$$

Here $c_2 > 0.091311$, $c_3 < 45.778$, $c_0 < 1.9132$, $c_1 > 0.240229$ and $\hat{M}_0 > 1.386044$. The constants enter as follows: δ_0 is the threshold $K_0 \ge 4 - \delta_0$ above which Lemma 7.40 is used; $\epsilon_{\max}$ and $\ell_{\max}$ bound $1 - |a|$ and $\log(1/|a|)$ for the zeros a of such a Blaschke model (Corollary 7.2(c)); c_2, c_3 and c_0 enter the bound $\mathrm{mass}(a) \le \ell_a\, q(\ell_a)$ for the mass of a group of zeros in the proof of Lemma 7.40, with $\ell_a = \log(1/|a|)$ and $q(\ell) = c_3 \log(2/\ell) + c_0$, and $q_d = q(\hat{M}_0/(2d))$; $c_1 = 2\log\cosh\frac{1}{2}$ comes from Lemma 7.39 with $y = \frac{1}{2}$; and $\hat{M}_0 \le \log K_0 = \sum_{a\in A} \log(1/|a|)$.

Theorem 7.37. *For every $d \ge 2$, $\mathcal{B}_d \le 4 - 5.97 \cdot 10^{-3}/(d \log d)$, and $\liminf_{d\to\infty}(4 - \mathcal{B}_d)\, d \log d \ge 1.005 \cdot 10^{-2}$. More precisely, $\mathcal{B}_d \le 4 - 3.99\, Y_d$ for every $d \ge 1$.*

The bound $4 - 5.97 \cdot 10^{-3}/(d \log d)$ is stronger than [23, Thm. 1.3] for every $d \ge 452$, and the bound $4 - 3.99\, Y_d$ for every $d \ge 299$.

Proposition 7.38 and Lemma 7.39 below follow Dubinin's proof of [13, Thm. 2.1]: his separating transformation, with the maps $\Xi \circ p_j$, $\Xi(z) = z/(1+z^2)$ as in the proof of Lemma 7.39 (the map p_j of Proposition 7.38 sends the bisector of $\mathcal{C}_j$ to the positive axis), and then a Koebe step. Since we need sectors of different openings, we include the proofs. The bounds $K(n) \le 4^{(n-2)/(n-1)}$ [4], $K(n) \le 4(n-1)/(n+1)$ [6] and $K(n) < 4 - 2.263/\sqrt{n}$ for $n \ge 8$ [8] of Section 1 concern polynomials of degree n; they do not bound $\mathcal{B}_d$, since the pairs with $\deg B \le d$ are not limited to pairs that come from polynomials of bounded degree.

For a domain $U \ni 0$ in the Riemann sphere with a classical Green function g_U let $r(U, 0)$ be its inner radius at 0: $g_U(z, 0) = \log(1/|z|) + \log r(U, 0) + o(1)$ as $z \to 0$. For simply connected U this is the conformal radius, also when $\infty \in U$. We have $r(U, 0) \le r(U', 0)$ if $U \subset U'$, and r does not change under a conformal map that fixes 0 with derivative 1 there.

Proposition 7.38 (after Dubinin [13, Lemma 1.2])**.** *Let U be a domain with $0 \in U \subset \mathbb{D}$ and a classical Green function. Let $n \ge 2$, $\theta_1 < \cdots < \theta_n < \theta_1 + 2\pi$, let $\mathcal{C}_j = \{\theta_j < \arg w < \theta_{j+1}\}$ ($\theta_{n+1} = \theta_1 + 2\pi$) have opening $\pi\alpha_j$, so that $0 < \alpha_j < 2$ and $\sum_j \alpha_j = 2$, and let $p_j(w) = -i(e^{-i\theta_j} w)^{1/\alpha_j}$, which maps $\mathcal{C}_j \cap \mathbb{D}$ conformally onto the right half-disk and extends continuously and injectively to the sides. Let Ω_j be the component containing 0 of the open set*

$$O_j = p_j(U \cap \mathcal{C}_j) \cup \left\{-\bar{\zeta} : \zeta \in p_j(U \cap \mathcal{C}_j)\right\} \cup \{p_j(w) : w \in U \cap \partial\mathcal{C}_j,\ w \ne 0\} \cup \{0\}.$$

Then $\log(1/r(U,0)) \geq \frac{1}{2}\sum_j \alpha_j^2 \log(1/r(\Omega_j,0))$.

Proof. For a domain $V \ni 0$ with a classical Green function and small $\rho > 0$, let $\mathrm{cap}(\rho, V)$ be the infimum of the Dirichlet integrals $\int |\nabla v|^2$ over the Lipschitz functions v with compact support in V and $v = 1$ on $\{|z| \leq \rho\}$. Then

$$\mathrm{cap}(\rho, V) = \frac{2\pi}{\log(1/\rho) + \log r(V,0) + o(1)} \qquad (\rho \to 0).$$

Indeed, for $t > 0$ the function $\min(g_V(\cdot,0)/t, 1)$ has Dirichlet integral $2\pi/t$, the least one for the level set $\{g_V \geq t\}$ in place of the disk; the capacity increases with the set; and for small ρ the disk $\{|z| \leq \rho\}$ lies between the level sets with $t = \log(1/\rho) + \log r(V,0) \pm o(1)$.

Let $\epsilon > 0$ be small, $\hat{L} = \log(1/\epsilon)$ and $\eta > 0$, and let u be a Lipschitz function with compact support in U, $u = 1$ on $\{|w| \leq \epsilon\}$, and $\int |\nabla u|^2 \leq \mathrm{cap}(\epsilon, U) + \eta$. The sides of the sectors have measure zero, so $\int |\nabla u|^2 = \sum_j \int_{U \cap \mathcal{C}_j} |\nabla u|^2$. Fix j, and define u_j on O_j by $u_j = u \circ p_j^{-1}$ on $p_j(U \cap \mathcal{C}_j)$ and on the image of $U \cap \partial\mathcal{C}_j \setminus \{0\}$, by $u_j(-\bar{\zeta}) = u_j(\zeta)$, and by $u_j(0) = 1$. Away from 0 the map p_j extends conformally across each side of $\mathcal{C}_j$ and maps it into the imaginary axis, which the reflection $\zeta \mapsto -\bar{\zeta}$ fixes; so O_j is open and u_j is Lipschitz. As $u = 1$ near 0 and $|p_j(w)| = |w|^{1/\alpha_j}$, u_j has compact support in O_j and $u_j = 1$ on $\{|\zeta| \leq \epsilon^{1/\alpha_j}\}$. The Dirichlet integral is invariant under the conformal map p_j and under $\zeta \mapsto -\bar{\zeta}$, so

$$\int_{O_j} |\nabla u_j|^2 = 2 \int_{U \cap \mathcal{C}_j} |\nabla u|^2.$$

Since Ω_j is open and closed in O_j, the restriction of u_j to Ω_j, extended by 0, competes in $\mathrm{cap}(\epsilon^{1/\alpha_j}, \Omega_j)$, and $\Omega_j \subset \mathbb{D}$ has a classical Green function. Hence

$$\mathrm{cap}(\epsilon, U) + \eta \;\geq\; \frac{1}{2}\sum_j \mathrm{cap}(\epsilon^{1/\alpha_j}, \Omega_j).$$

Let $\eta \to 0$ and use the asymptotics above, with $\log(1/\epsilon^{1/\alpha_j}) = \hat{L}/\alpha_j$:

$$\frac{2\pi}{\hat{L} + \log r(U,0) + o(1)} \;\geq\; \sum_j \frac{\pi\alpha_j}{\hat{L} + \alpha_j \log r(\Omega_j,0) + o(1)} \qquad (\hat{L} \to \infty).$$

Since $1/(\hat{L} + b + o(1)) = 1/\hat{L} - b/\hat{L}^2 + o(1/\hat{L}^2)$ for fixed b, and $\sum_j \alpha_j = 2$, both sides equal $2\pi/\hat{L}$ plus terms of order $1/\hat{L}^2$:

$$\frac{2\pi}{\hat{L}} - \frac{2\pi \log r(U,0)}{\hat{L}^2} + o\Big(\frac{1}{\hat{L}^2}\Big) \;\geq\; \frac{2\pi}{\hat{L}} - \frac{\pi \sum_j \alpha_j^2 \log r(\Omega_j,0)}{\hat{L}^2} + o\Big(\frac{1}{\hat{L}^2}\Big).$$

Multiplying by $\hat{L}^2/(2\pi)$ and letting $\hat{L} \to \infty$ gives $\log(1/r(U,0)) \geq \frac{1}{2}\sum_j \alpha_j^2 \log(1/r(\Omega_j,0))$. Dubinin [13] proves the inequality with level sets of g_U; the transformation was introduced in [12] (see [13, p. 54]). □

Lemma 7.39 (one sector)**.** *Let U be a simply connected domain with $0 \in U \subset \mathbb{D}$, let $a \in \mathbb{D} \setminus U$, $a \neq 0$, and $0 < \alpha < 2$. Apply Proposition 7.38 to U and a sector $\mathcal{C}$ of opening $\pi\alpha$ whose bisector passes through a, and let Ω be the corresponding domain. Then $\log(1/r(\Omega,0)) \geq \log\cosh y$, where $y = \log(1/|a|)/\alpha$, with equality when U is $\mathbb{D}$ minus the radial slit from a to the circle.*

Proof. This is the Koebe step of Dubinin's proof of [13, Thm. 2.1], in our normalization. Here $p(w) = (w|a|/a)^{1/\alpha}$ and $p(a) = x = e^{-y}$. The set $X = \widehat{\mathbb{C}} \setminus U$ is a continuum, and $E = X \cap \mathcal{C} \cap \mathbb{D}$ is a proper open subset of X containing a. By the boundary bumping theorem [26, Thm. 5.6], applied to $X \setminus E$, the closure of the component of E containing a meets a side of $\mathcal{C}$ or the circle; let K' be its image under p. This is a continuum in the closed right half-disk, it contains x and a point of the circle or of the imaginary axis, and $K' \cap \mathbb{D}$ and $\{-\bar{z} : z \in K'\} \cap \mathbb{D}$ miss Ω.

The map $\Xi(z) = z/(1+z^2) = 1/\mathcal{J}(z)$ sends $\mathbb{D}$ conformally onto $\mathbb{C}\setminus\Sigma$, $\Sigma = (-\infty,-\frac12]\cup[\frac12,\infty)$, with $\Xi(0)=0$, $\Xi'(0)=1$ and $\Xi(-\bar z) = -\overline{\Xi(z)}$; it extends continuously to a map of the closed disk into $\widehat{\mathbb{C}}$ that sends the circle onto $\Sigma\cup\{\infty\}$ and the imaginary axis into itself. So $r(\Xi(\Omega),0) = r(\Omega,0)$, and the continua $\Xi(K')$ and $\{-\bar z : z\in\Xi(K')\}$ miss $\Xi(\Omega)$ and contain $\zeta = \Xi(x) = 1/(2\cosh y)$ and $-\zeta$. Either both meet the connected set $\Sigma\cup\{\infty\}$, or they share a point of the imaginary axis. Hence $\pm\zeta$ lie in one component E' of $\widehat{\mathbb{C}}\setminus\Xi(\Omega)$. The component V of $\widehat{\mathbb{C}}\setminus E'$ containing 0 is simply connected, contains $\Xi(\Omega)$ and omits $\pm\zeta$. Let f map $\mathbb{D}$ conformally onto V with $f(0)=0$, $f'(0)>0$. Then $f_1 = \zeta f/(\zeta - f)$ is holomorphic and univalent, $f_1(0)=0$, $f_1'(0) = f'(0)$, and f_1 omits $-\zeta/2$; by Koebe's one-quarter theorem $f'(0)\le 2\zeta$. So $r(\Omega,0)\le r(V,0)\le 2\zeta = 1/\cosh y$. For the slit domain, Ω is $\mathbb{D}$ minus two symmetric slits and $r(\Omega,0) = 2x/(1+x^2) = 1/\cosh y$. □

Lemma 7.40 (selection)**.** *Let (W,B) be a Blaschke model with $k = \deg B$, and let $K_0 \ge 4-\delta_0$. For $a\in A$ put $\ell_a = \log(1/|a|)$. Then there is $A_0\subset A$ such that the arcs $\{e^{it} : |t-\arg a|<\pi\ell_a\}$, $a\in A_0$, are pairwise disjoint and $\sum_{a\in A_0}\ell_a^2 \ge \hat M_0^2/(4kq_k)$.*

Proof. By Corollary 7.2(c), $\epsilon_a = 1-|a| \le \epsilon_{\max}$ and $\ell_a \le \ell_{\max} < 1/2$ for every $a\in A$, and $\sum_{a\in A}\ell_a = \log K_0 \ge \hat M_0$. Order A by decreasing ℓ_a. Take a into A_0 if its arc misses the arcs already taken. Assign each remaining b to an earlier $a\in A_0$ whose arc meets that of b. Then $|b|\ge|a|$, and the angle t between a and b is less than $2\pi\ell_a$, so $|a-b|^2 \le \epsilon_a^2 + t^2 \le \ell_a^2(1+4\pi^2)$. As $\epsilon \ge \ell - \ell^2/2$, we have $1-|a|^2 = \epsilon_a(2-\epsilon_a) \ge (2-\epsilon_{\max})(1-\ell_{\max}/2)\,\ell_a$, and the same for b; so

$$\frac{|1-\bar a b|^2}{|a-b|^2} = 1 + \frac{(1-|a|^2)(1-|b|^2)}{|a-b|^2} \;\ge\; 1 + c_2\,\frac{\ell_b}{\ell_a}.$$

Hence $g_{\mathbb{D}}(a,b) = \log(|1-\bar a b|/|a-b|) \ge \frac12\log(1+c_2\ell_b/\ell_a) \ge \frac{\ell_b}{2\ell_a}\log(1+c_2)$, the last step by concavity, as $\ell_b\le\ell_a$. By Lemma 7.4 and Koebe's growth bound, $\sum_{b\ne a} g_{\mathbb{D}}(a,b) \le \log(4/\epsilon_a^2)$. So the group of a (n_a points, a included; mass(a) is the sum of their ℓ_b) satisfies, as $\epsilon_a \ge (1-\ell_{\max}/2)\,\ell_a$,

$$\operatorname{mass}(a) - \ell_a \;\le\; \frac{2\ell_a}{\log(1+c_2)}\log\frac{4}{\epsilon_a^2} = c_3\ell_a\log\frac{2}{\epsilon_a} \;\le\; c_3\ell_a\Big(\log\frac{2}{\ell_a} + \log\frac{1}{1-\ell_{\max}/2}\Big);$$

that is, mass$(a)\le \ell_a\,q(\ell_a)$ with $q(\ell) = c_3\log(2/\ell) + c_0$. Also mass$(a)\le n_a\ell_a$. Let $\eta = \hat M_0/(2k)$, so that $q(\eta) = q_k$. The groups with $\ell_a<\eta$ carry less than $k\eta = \hat M_0/2$; for the others, mass$(a)^2 \le n_a\ell_a^2 q_k$. By the Cauchy–Schwarz inequality and $\sum n_a \le k$, $(\hat M_0/2)^2 \le k\,q_k\sum_{a\in A_0}\ell_a^2$. □

Proof of Theorem 7.37. For $d=1$, $\mathcal{B}_1 < \sqrt{7/5} < 4 - 3.99Y_1$ by [23, Cor. 8.10]. For $d=2,3,4$, [23, Thm. 1.3] gives $\mathcal{B}_d < 4 - \frac{529}{576}(\log 2)^2/d^2 \le 4 - 3.99Y_d$ (at $d=4$: $\frac{529}{576}(\log 2)^2/16 > 0.0275$ and $3.99Y_4 < 0.0011$). Let $d\ge 5$ and let (W,B) be a Blaschke model with $k=\deg B\le d$. If $K_0 < 4-\delta_0$, then $K_0 < 4 - 3.99Y_d$, since $3.99Y_5\le\delta_0$ and $Y_d\le Y_5$. Otherwise take A_0 from Lemma 7.40. By the proof of Lemma 7.1, $U = h(\mathbb{D})$ is a simply connected domain with $0\in U\subset\mathbb{D}\setminus A$ and $r(U,0) = r_{\rm in}K_0 \ge K_0/4$. Give each $a\in A_0$ the sector with bisector through a and opening $2\pi\ell_a$, that is $\alpha_a = 2\ell_a < 1$. These sectors are disjoint, so their total opening is at most 2π. If it equals 2π, they are all the sectors, and $n = |A_0|\ge 2$. Otherwise the rest of the circle is a finite union of closed arcs (possibly single points), and we add one sector over the interior of each non-degenerate one; again $n\ge 2$ and every $\alpha<2$. Proposition 7.38 and Lemma 7.39 with $y = 1/2$ give

$$\log\frac{4}{K_0} \ge \log\frac{1}{r(U,0)} \ge \frac12\sum_{a\in A_0} 4\ell_a^2\log\cosh\frac12 = c_1\sum_{a\in A_0}\ell_a^2 \ge \frac{c_1\hat M_0^2}{4kq_k} = Y_k \ge Y_d.$$

As $Y_d\le Y_5 < 0.0002$, $K_0 \le 4e^{-Y_d} \le 4 - 3.99Y_d$. For the consequences, $Y_d\,d\log d$ increases in d and tends to $c_1\hat M_0^2/(4c_3)$, and $Y_d d^2$ increases for $d\ge 2$. A direct evaluation gives $3.99\,Y_5\cdot 5\log 5 = 5.97005\ldots\cdot 10^{-3} > 5.970\cdot 10^{-3}$, while for $d=2,3,4$ the bound of [23, Thm. 1.3] times $d\log d$ exceeds 0.15; further $3.99\,c_1\hat M_0^2/(4c_3) > 1.0056\cdot 10^{-2}$ and $3.99\,Y_{299} > 4.9446\cdot 10^{-6} > 4.9357\cdot 10^{-6} >$

$\frac{529}{576}(\log 2)^2/299^2$; also $5.97\cdot 10^{-3}/(452\log 452) > 2.1603\cdot 10^{-6} > 2.1598\cdot 10^{-6} > \frac{529}{576}(\log 2)^2/452^2$, and $d/\log d$ increases for $d \geq 3$. □

Remark 7.41. For a single radial slit from a to the circle equality holds in Lemma 7.39, and Proposition 7.38 gives $0.2402\,\ell_a^2$ for one slit, against the exact $2\log\cosh(\ell_a/2) \sim \ell_a^2/4$. For d points equally spaced on $|z| = r$ with $r^d = 1/4$ (these points cannot be the zeros of B in a Blaschke model, since then $K_0 = 4$; the example tests only Lemma 7.40), Dubinin's theorem [13, Thm. 3.2] gives $\log(1/r(U,0)) \geq \frac{2}{d}\log\frac{5}{4}$ for every simply connected $U \ni 0$ in $\mathbb{D}$ that misses these points, with equality for the symmetric slit domain; the selection of Lemma 7.40, taking the zeros in their order on the circle, keeps every second zero, and the bound $c_1\sum_{a\in A_0}\ell_a^2$ of the proof of Theorem 7.37 is then about $0.230/d$. The remaining $\log d$ comes from Lemma 7.4, which allows groups of about $\log(1/\ell_a)$ zeros at mutual distance $\approx \ell_a$. Whether $4 - \mathcal{B}_d \geq c/d$ holds is open.

8. The constants in degrees one and two

Jatar and Ng [23, Lemma 6.2] build pairs with one zero from a polynomial. Let Q be a polynomial with two distinct zeros $a, b \in \mathbb{D}$ and with $Q' \neq 0$ in $\mathbb{D}$, and suppose that the polynomial

$$H(z) = \frac{(1-\bar{a}z)\,Q(z)}{z-a}$$

is univalent in $\mathbb{D}$. Put $\varphi(w) = (w+b)/(1+\bar{b}w)$, $N_b = (1-|b|^2)H'(b)$, $W = (H\circ\varphi)/N_b$ and let B be the multiple of $B_a\circ\varphi$ by the constant of modulus 1 that makes $B(0) > 0$. Then (W, B) is admissible, and, as $1/|B(0)| = |1-\bar{a}b|/|a-b|$ and $|1-\bar{a}b|^2 - |a-b|^2 = (1-|a|^2)(1-|b|^2)$,

$$\frac{1}{|B(0)|^2} = J(a,b) := 1 + \frac{(1-|a|^2)(1-|b|^2)}{|a-b|^2}.$$

With $\deg Q = 4$ they obtain $\mathcal{B}_1 \geq \sqrt{8081/6656} = 1.1018\ldots$ [23, Thm. 6.1]. We use the same construction, extended to two zeros (Lemma 8.4), and Lemma 8.6 gives a test in exact arithmetic; the pairs of Table 2 have $6 \leq \deg Q \leq 18$. With one zero and $\deg Q = 18$ we obtain an admissible pair with gain larger than 1.1363375 (Theorem 8.7), and with two zeros and $\deg Q = 18$ one with gain larger than 1.1559837 (Theorem 8.8). Both gains are attained exactly by Blaschke models as well (Corollaries 9.11 and 9.14). Theorem 8.9 and Table 2 give the other exact pairs.

For the univalence of a polynomial H of degree at most n, with coefficient $[z^j]H$ of z^j, we use the Dieudonné polynomials of $H - H(0)$,

$$\mathcal{D}_c(w) = \sum_{j=1}^{n} [z^j]H\, U_{j-1}(c)\, w^{j-1} \qquad (c \in [-1,1]),$$

where U_{j-1} is the Chebyshev polynomial of the second kind. For $c = \cos\theta$ they satisfy $H(ze^{i\theta}) - H(ze^{-i\theta}) = 2i\sin\theta\; z\mathcal{D}_c(z)$, and $\mathcal{D}_1(w) = H'(w)$, $\mathcal{D}_{-1}(w) = H'(-w)$.

Lemma 8.1 (Dieudonné's criterion [11]; see also [23, Lemma 6.3])**.** *H is univalent in $\mathbb{D}$ if no $\mathcal{D}_c$ with $c \in [-1,1]$ has a zero in $\mathbb{D}$.*

Lemma 8.2. *Suppose that no $\mathcal{D}_c$ with $c \in [-1,1]$ has a zero on the circle, and that H' has no zero on the closed disk. Then H is univalent on a disk $\{|z| < r\}$ with $r > 1$.*

Proof. $\mathcal{D}_c(0) = H'(0) \neq 0$, so no $\mathcal{D}_c$ vanishes identically. As no $\mathcal{D}_c$ has a zero on the circle, the number $\frac{1}{2\pi i}\oint_{|w|=1}\mathcal{D}_c'/\mathcal{D}_c\,dw$ of zeros of $\mathcal{D}_c$ in $\mathbb{D}$ is a continuous integer-valued function of $c \in [-1,1]$, hence constant; it is 0 at $c = 1$, where $\mathcal{D}_1 = H'$. So no $\mathcal{D}_c$ has a zero on the closed disk. The set of pairs (c,w) with $c \in [-1,1]$, $|w| \leq 2$ and $\mathcal{D}_c(w) = 0$ is compact, so there is $r \in (1,2)$ such that no $\mathcal{D}_c$ vanishes on $\{|w| \leq r\}$. The Dieudonné polynomials of $H(rz)/r$ are $\mathcal{D}_c(rw)$, so $H(rz)/r$ is univalent in $\mathbb{D}$ by Lemma 8.1. □

Corollary 8.3. *The maps W of the degree-one pairs of* [23, Thm. 6.1] *and* [23, Rem. 6.7] *are univalent on a neighbourhood of the closed disk, so these pairs are not extremal. The pair of* [23, Rem. 6.7] *is admissible, with gain* $\sqrt{1768591361/1453520000} = 1.10307\ldots$*; this proves the bound* $\mathcal{B}_1 > 1.1030$ *stated in* [23, Rem. 6.7].

Proof. For both pairs $Q = (z-1)^4 - u^4$ with $u = -x - iy$ [23, proof of Thm. 6.1], where $(x, y) = (9/16, 3/8)$ for [23, Thm. 6.1] and $(x, y) = (14/25, 3/8)$ for [23, Rem. 6.7]; so $Q' = 4(z-1)^3$ has no zero in $\mathbb{D}$. For each pair let H be the quartic of [23, Sect. 6], with b, φ and N_b as at the start of this section, and $\mathcal{D}_c$ the Dieudonné polynomial of $H - H(0)$, of formal degree 3 with coefficients depending on $c \in [-1, 1]$. Its three quantities $\delta_1, \delta_2, \delta_3$ of Lemma 7.11 are real polynomials in c of degrees 6, 12 and 24; an exact computation with Sturm sequences shows that they have no zero in $[-1, 1]$ and are positive at $c = 0$. By Lemma 7.11 no $\mathcal{D}_c$ has a zero in the closed disk, so by compactness there is $r \in (1, 1/|b|)$ such that no $\mathcal{D}_c$ vanishes on $\{|w| \le r\}$. The Dieudonné polynomials of $H(rz)/r$ are $\mathcal{D}_c(rw)$, so H is univalent on $\{|z| < r\}$, and $W = H(\varphi)/N_b$ [23, Lemma 6.2] is univalent on a neighbourhood of the closed disk. As $Q' \neq 0$ in $\mathbb{D}$, both pairs are admissible by [23, Lemma 6.2]; for the pair of [23, Rem. 6.7], $1/B(0)^2 = 1768591361/1453520000$ exactly, by [23, (15)] with $(x, y) = (14/25, 3/8)$. Lemma 7.9 applies with $k = 1$. □

The construction of [23, Lemma 6.2] extends to several zeros.

Lemma 8.4. *Let $k \ge 1$ and let Q be a polynomial with $Q' \neq 0$ in $\mathbb{D}$ and with $k+1$ distinct zeros $b, a_1, \ldots, a_k \in \mathbb{D}$, and suppose that the polynomial*

$$H(z) = Q(z) \prod_{j=1}^{k} \frac{1 - \bar{a}_j z}{z - a_j}$$

is univalent in $\mathbb{D}$. With φ and $N_b = (1 - |b|^2)H'(b)$ as above, put $\eta = \prod_j B_{a_j}(b)$, $W = (H \circ \varphi)/N_b$ and $B = (|\eta|/\eta) \prod_j B_{a_j} \circ \varphi$. Then (W, B) is admissible, $\deg B = k$*, and*

$$\frac{1}{B(0)^2} = \prod_{j=1}^{k} J(a_j, b).$$

Proof. H is a polynomial since the a_j are distinct zeros of Q, and $H(b) = 0$ since $Q(b) = 0$ and b is no a_j. As H is univalent, $H'(b) \neq 0$; as φ is an automorphism of $\mathbb{D}$, W is univalent in $\mathbb{D}$, with $W(0) = H(b)/N_b = 0$ and $W'(0) = H'(b)\varphi'(0)/N_b = 1$. Each $B_{a_j} \circ \varphi$ is a Blaschke factor, so B is a Blaschke product of degree k, and $B(0) = |\eta| > 0$. Since $H \prod_j B_{a_j} = Q$, we have $(WB)' = (|\eta|/(\eta N_b))\, Q'(\varphi)\, \varphi'$, which has no zero in $\mathbb{D}$. Finally $1/|B_{a_j}(b)|^2 = J(a_j, b)$, by the identity at the start of this section. □

Lemma 8.5. *Let Q, $b, a_1, \ldots, a_k$, H, W and B be as in Lemma 8.4. Suppose that Q' has no zero on the closed disk and that H is univalent on a disk $\{|z| < r\}$ with $r > 1$. Then (W, B) is a Blaschke model with k zeros and gain $\prod_{j=1}^{k} J(a_j, b)^{1/2}$.*

Proof. φ is holomorphic and injective on $\{|w| < 1/|b|\}$ and maps the closed disk onto itself, so by continuity it maps a disk $\{|w| < \varrho\}$ with $1 < \varrho < 1/|b|$ into $\{|z| < r\}$, and W is univalent there. With $W(0) = 0$ and $W'(0) = 1$ from the proof of Lemma 8.4, this is (M1). (M2) is clear. By the proof of Lemma 8.4, $(WB)' = (|\eta|/(\eta N_b))\, Q'(\varphi)\, \varphi'$; it has no zero on the closed disk, since Q' has none there and φ maps the closed disk onto itself. This is (M3). The gain is $1/B(0) = \prod_j J(a_j, b)^{1/2}$ by Lemma 8.4. □

Lemma 8.6 reduces admissibility to four conditions that exact arithmetic can verify.

Lemma 8.6. *Let $k \geq 1$, let Q be a polynomial of degree n, let $b, a_1, \dots, a_k \in \mathbb{C}$, put $H(z) = Q(z)\prod_{j=1}^{k}(1-\bar{a}_j z)/(z-a_j)$, and let $\mathcal{D}_c$ be the Dieudonné polynomials of $H - H(0)$. Suppose that the following hold.*

- (D) *$|b|, |a_1|, \dots, |a_k| < 1$, the $k+1$ points are pairwise distinct, and $Q(b) = Q(a_1) = \cdots = Q(a_k) = 0$.*
- (C) *$Q' = \Pi R$, where R has no zero on the closed disk, and Π is either a non-zero constant or a constant times $(z-\zeta_1)^{m_1}\cdots(z-\zeta_p)^{m_p}$ with $p \geq 1$, distinct ζ_i, $|\zeta_i| = 1$ and $m_i \geq 1$.*
- (R) *Let $\mathcal{D}_c^*(w) = w^{n-1}\overline{\mathcal{D}_c(1/\bar{w})}$, at formal degree $n-1$. The resultant $\mathrm{Res}_w(\mathcal{D}_c, \mathcal{D}_c^*)$, taken at formal degrees, is a polynomial in c of degree at most $2(n-1)^2$, and it has no zero in $[-1,1]$.*
- (S) *H' has no zero on the closed disk.*

Then H is univalent on a disk $\{|z| < r\}$ with $r > 1$; the pair (W, B) of Lemma 8.4 is admissible, with $\deg B = k$ and $1/B(0)^2 = \hat{J} := \prod_{j=1}^{k} J(a_j, b)$; W is univalent on a disk $\{|w| < \varrho\}$ with $\varrho > 1$; and if Π is constant, (W, B) is a Blaschke model. Hence $\mathcal{B}_k \geq \sqrt{\hat{J}}$ and $K^* \geq \sqrt{\hat{J}}$.

Proof. By (D), H is a polynomial of degree at most n, so each $\mathcal{D}_c$ has formal degree $n-1$. The Sylvester matrix of $\mathcal{D}_c$ and $\mathcal{D}_c^*$ has size $2(n-1)$, and each entry is 0, $[z^j]H\, U_{j-1}(c)$ or $\overline{[z^j]H}\, U_{j-1}(c)$ for some $j \leq n$, of degree at most $n-1$ in c; so the resultant is a polynomial in c of degree at most $2(n-1)^2$. Let $c \in [-1,1]$ and suppose that $\mathcal{D}_c(w_0) = 0$ for some w_0 with $|w_0| = 1$. Since $1/\bar{w}_0 = w_0$, we get $\mathcal{D}_c^*(w_0) = w_0^{n-1}\overline{\mathcal{D}_c(w_0)} = 0$, so w_0 is a common zero of $\mathcal{D}_c$ and $\mathcal{D}_c^*$, and the resultant vanishes at c (a common zero makes the Sylvester matrix singular, at any formal degrees). This contradicts (R). So no $\mathcal{D}_c$ with $c \in [-1,1]$ has a zero on the circle, and by (S) and Lemma 8.2 H is univalent on a disk $\{|z| < r\}$ with $r > 1$. By (C), $Q' \neq 0$ in $\mathbb{D}$, so Lemma 8.4 applies: (W, B) is admissible, with $\deg B = k$ and $1/B(0)^2 = \hat{J}$. As in the proof of Lemma 8.5, W is univalent on a disk $\{|w| < \varrho\}$ with $\varrho > 1$. If Π is constant, Q' has no zero on the closed disk, and (W, B) is a Blaschke model by Lemma 8.5. So $\mathcal{B}_k \geq \sqrt{\hat{J}}$, and $K^* \geq \sqrt{\hat{J}}$ by [23, Prop. 4.2], which holds for every admissible pair. □

For $k = 1$, Lemma 8.4 is [23, Lemma 6.2], with $B = (|\eta|/\eta)\, B_a \circ \varphi$. In the theorems below, (C) and (S) are checked by Lemma 7.11, and (R) as follows. The resultant is computed exactly (from residues modulo primes, with a Hadamard bound) at the $2(n-1)^2+3$ integer points $c = 0, 1, \dots, 2(n-1)^2+2$; its coefficients are obtained by exact interpolation through the first $2(n-1)^2+1$ values and checked at the other two. Its imaginary part vanishes at all of these points, and its degree is at most $2(n-1)^2$, so it has real coefficients; Descartes' rule of signs with bisection then shows that it has no zero in $[-1,1]$. With one zero and $\deg Q = 18$ we obtain the following pair.

Theorem 8.7. *Let a, b and Q be the data of pair VI in Appendix B, and let (W, B) be the pair of Lemma 8.4 with $k = 1$ and $a_1 = a$. Then (W, B) is an admissible pair with one zero and $1/B(0) = \sqrt{J_{\mathrm{VI}}}$, where*

$$J_{\mathrm{VI}} = J(a,b) = \frac{28429763493984067075056932 41}{22017021387847780824566988 80}, \qquad \sqrt{J_{\mathrm{VI}}} = 1.13633750601\ldots$$

Hence $\mathcal{B}_1 \geq \sqrt{J_{\mathrm{VI}}}$ and $K^* \geq \sqrt{J_{\mathrm{VI}}} > 1.1363375$.

Proof. The facts (D), (C) with $\Pi = (z-1)^5$ and $R(1) \neq 0$, (R) and (S) of Lemma 8.6, with $k = 1$ and $n = 18$ (so in (R) formal degree 17 and degree at most 578 in c), are checked in exact rational arithmetic over $\mathbb{Q}(i)$. Lemma 8.6 gives the rest. □

Here $(WB)'$ has one zero on the circle, of order 5, at $\varphi^{-1}(1)$, so pair VI is not regular. Since W is univalent on a neighbourhood of the closed disk, Lemma 7.9 (or Corollary 7.15) shows that pair VI is not extremal, so $\mathcal{B}_1 > \sqrt{J_{\mathrm{VI}}}$. By Corollary 9.11, Blaschke models attain this gain.

A larger gain comes from a pair with two zeros, with $\deg Q = 18$ and Q' of the same shape.

Theorem 8.8. *Let b, a_1, a_2 and Q be the data of pair I in Appendix A, and let (W, B) be the pair of Lemma 8.4 with $k = 2$. Then (W, B) is an admissible pair with two zeros and $1/B(0) = \sqrt{J_{\mathrm{I}}}$, where $J_{\mathrm{I}} = J(a_1, b)\, J(a_2, b)$,*

$$J_{\mathrm{I}} = \frac{267952521936226337027481600174795544467587176908574201599 7}{200518481134769781691297951341495479358548025660497985536 0},$$

and $\sqrt{J_{\mathrm{I}}} = 1.15598372919\ldots$; hence $\mathcal{B}_2 \geq \sqrt{J_{\mathrm{I}}}$ and $K^* \geq \sqrt{J_{\mathrm{I}}} > 1.1559837$.

Proof. The facts (D), (C) with $\Pi = (z-1)^5$ and $R(1) \neq 0$, (R) and (S) of Lemma 8.6, with $k = 2$ and $n = 18$ (so in (R) formal degree 17 and degree at most 578 in c), are checked in exact rational arithmetic over $\mathbb{Q}(i)$: (R) as described after Lemma 8.6, with 581 integer points (as in [19, Sect. 5]). By Lemma 8.6, (W, B) is admissible, with $\deg B = 2$ and $1/B(0)^2 = J_{\mathrm{I}}$. Hence $\mathcal{B}_2 \geq \sqrt{J_{\mathrm{I}}}$, and $K^* \geq \sqrt{J_{\mathrm{I}}}$ by [23, Prop. 4.2]. □

Here $(WB)'$ has one zero on the circle, of order 5, at $\varphi^{-1}(1)$, so pair I is not regular; since W is univalent on a neighbourhood of the closed disk, Lemma 7.9 shows that pair I does not attain $\mathcal{B}_2$, so $\mathcal{B}_2 > \sqrt{J_{\mathrm{I}}}$. By Corollary 9.14, Blaschke models with two zeros attain this gain.

Theorems 8.7 and 8.8 use [23, Prop. 4.2], which gives $K^* \geq 1/B(0)$ for every admissible pair (W, B); its proof first replaces (W, B) by the regular pairs of [23, Lemma 2.1]. Since Corollaries 9.11 and 9.14 give Blaschke models with the same gains, the two bounds also follow from [23, Prop. 4.3], whose proof is direct: it gives $K^* \geq \mu(\Omega, \Phi)$ for every domain pair (Ω, Φ) in which Ω is an analytic Jordan domain, and for a Blaschke model (W, B) the domain $W(\mathbb{D})$ and the function $z \mapsto zB(W^{-1}(z))/B(0)$ form such a pair, with ratio $1/|B(0)|$ since $|B| = 1$ on the circle.

Lemma 8.6 applies in the same way to the other pairs of Table 2. For a pair of the table, Π is as in the caption: Π is constant for the Blaschke models, pairs XIII, XIV, XV and XVIII; $\Pi = (z-1)^m$ for the boundary pairs other than pair X, with $m = 5$ or $m = 3$ as in the caption; and $\Pi = (z-1)^5(z-\zeta_2)$, with $\zeta_2 \neq 1$ on the circle, for pair X.

Theorem 8.9. *Let one of the pairs II–V and VII–XVIII of Table 2 be given. Let Q, b and a_1, and also a_2 for pairs II–V, be its exact data (in the supporting archive), let k be the number of the a_j, let $n = \deg Q$, and let Π be as above. Then (D), (C), (R) and (S) of Lemma 8.6 hold. So the pair (W, B) of Lemma 8.4 is admissible, with $\deg B = k$ and $1/B(0) = \sqrt{\hat{J}}$, where $\sqrt{\hat{J}}$ is at least the value in the table ($\sqrt{\hat{J}}$ truncated to seven decimals); and (W, B) is a Blaschke model if Table 2 lists the pair as one.*

Proof. For each of these pairs the facts are checked in exact rational arithmetic over $\mathbb{Q}(i)$, (C) and (S) by Lemma 7.11, and (R) as described after Lemma 8.6. That lemma gives the rest. □

For the boundary pairs, $(WB)'$ has zeros on the circle, at the points $\varphi^{-1}(\zeta)$ with $\Pi(\zeta) = 0$, so they are admissible pairs in the sense of [23], but not Blaschke models. For pairs VII, VIII, IX, XI, XII, XVI and XVII, Corollary 9.11 gives Blaschke models with exactly their gains. For the other boundary pairs, $K^* \geq \sqrt{\hat{J}}$ comes from [23, Prop. 4.2], as in Lemma 8.6.

The next remark relates the number of zeros of B to the zeros of $(WB)'$ on the circle.

Remark 8.10 (A phase count). Let (W, B) be admissible with $\deg B = k$, let $F = WB$ be holomorphic near the closed disk, and let F' have the zeros $\zeta_1, \ldots, \zeta_p$ on the circle, of orders $m_1, \ldots, m_p$ and total order $M = m_1 + \cdots + m_p$. With $\chi = wF'/F$ as in Section 7.7, put $\mathrm{turn}_F := \mathrm{Re}(1 + wF''/F')$ on $\mathbb{T} \setminus \{\zeta_1, \ldots, \zeta_p\}$. In $\mathbb{D}$, F has the $k+1$ zeros of W and B, all simple, and F' has none; $F \neq 0$ on the circle, since $|F| = |W|$ there and W, univalent in $\mathbb{D}$ with $W(0) = 0$, takes no value on the circle that it takes in $\mathbb{D}$. Along each arc I of $\mathbb{T} \setminus \{\zeta_1, \ldots, \zeta_p\}$ (if $p = 0$, cut the circle at any point) take a continuous branch of $\arg \chi(e^{i\theta})$. By the argument principle the changes of these branches add up to $(M - 2k)\pi$: the factor w gives 2π, the factor $1/F$ gives $-2(k+1)\pi$, and each factor $(w - \zeta_i)^{m_i}$

of F' gives $m_i\pi$, since $\arg(e^{i\theta}-\zeta_i)$ gains π along $\mathbb{T}\setminus\{\zeta_i\}$. Moreover

$$\frac{d}{d\theta}\arg\chi(e^{i\theta}) = \operatorname{Re}\Big(1+\frac{wF''}{F'}-\frac{wF'}{F}\Big) = \operatorname{turn}_F - \operatorname{Re}\chi,$$

so at a point where $\arg\chi$ takes one of the values $\pi/2+j\pi$ ($j\in\mathbb{Z}$), which we call *levels*, its derivative is turn_F.

Write the limits of $\arg\chi$ at the start and at the end of I as $l_I+\alpha_I\pi$ and $l'_I+\alpha'_I\pi$, with levels l_I, l'_I and $\alpha_I,\alpha'_I\in[0,1)$. Near ζ_i, χ is a non-zero constant times $(w-\zeta_i)^{m_i}(1+O(w-\zeta_i))$, so the two one-sided limits of $\arg\chi$ at ζ_i differ by a multiple of π, and α of each arc equals α' of the arc before it (for $p=0$, $\alpha=\alpha'$ since the change is $-2k\pi$). So the total change is $\sum_I(l'_I-l_I)+\pi\sum_I(\alpha'_I-\alpha_I)=\sum_I(l'_I-l_I)=(M-2k)\pi$.

Proposition 8.11. *In the setting of Remark 8.10, suppose that* $\operatorname{turn}_F>0$ *on* $\mathbb{T}\setminus\{\zeta_1,\dots,\zeta_p\}$. *Then* $k\le M/2$.

Proof. By Remark 8.10, $\sum_I(l'_I-l_I)=(M-2k)\pi$. On an arc I, once $\arg\chi$ has reached a level l it stays $\ge l$, since it can meet l only with positive derivative. It cannot leave a level downwards at the start of I either: if it tends to the level l there, then $u=l-\arg\chi$ satisfies $u'=-\operatorname{turn}_F+|\chi|\cos\arg\chi\le|\chi|\,|u|$, as $\operatorname{turn}_F>0$ and $|\cos\arg\chi|=|\sin u|\le|u|$, and $|\chi|$ is bounded near the start of I; since $u\to0$ there, Gronwall's inequality gives $u\le0$ on I. So $l'_I\ge l_I$ for every arc, and $M-2k=\sum_I(l'_I-l_I)/\pi\ge0$. □

For the pairs of Lemma 8.4, Q has variable $z=\varphi(w)$. Then F is a constant times $Q\circ\varphi$, so M is the total order of the zeros of Q' on the circle, and, as $w\varphi'/\varphi=|\varphi'|$ on the circle, $\operatorname{turn}_F=|\varphi'|\operatorname{Re}\big(1+zQ''(z)/Q'(z)\big)$ at $z=\varphi(w)$. If $Q'(z)$ is a constant times $(z-1)^mR(z)$, with $R\ne0$ on the closed disk, and $\deg Q=n$, then for $|z|=1$

$$\operatorname{Re}\Big(1+\frac{zQ''(z)}{Q'(z)}\Big)=\frac{n+1}{2}-\frac12\sum_{R(\xi)=0}\frac{|\xi|^2-1}{|z-\xi|^2},$$

with the zeros ξ of R counted with multiplicity. So pairs I, II, III, IV and V of Table 2 have $M-2k=1$, and with one zero of Q' of order 5 on the circle a pair with $k=3$ zeros needs, by Proposition 8.11, a point z of the circle where $\sum_{R(\xi)=0}(|\xi|^2-1)/|z-\xi|^2\ge n+1$.

Corollary 8.12. $K(n)>1.1559837$ *for all sufficiently large* n.

Proof. Take $A=1.1559837$, which is below K^* by Theorem 8.8. Since $K^*=\sup_n K(n)$ (Definition 2.1), some n_0 has $K(n_0)>A$. Since $K(n)$ is nondecreasing in n ([7]; see also [23, proof of Cor. 4.6]), $K(n)>A$ for all $n\ge n_0$. □

The proof gives no explicit degree.

Remark 8.13 (Dubinin's conjecture). Dubinin's conjecture [14, (1.5)] is the following. Let $n\ge2$ and $0<t<1$. Every Blaschke product B of degree n with $B(0)=0$ and $|B'(0)|=t$ has a critical point ζ with $|B(\zeta)/(\zeta B'(0))|\le f_n(t)$, where $f_n(t)<1$ is the same ratio for $z(z^{n-1}-t)/(1-tz^{n-1})$ at its critical points in $\mathbb{D}$ [14, (1.5), Lemma 1]. He remarks that this would imply $K(n)=(n-1)/n$. Since $K(N)>1$ for some N [1] (see also Corollary 8.12) and $K(n)$ is nondecreasing [7], the conjecture fails in every degree $n\ge N$; this is the contrapositive of his remark. Take $p\in\mathcal{P}_n$ with $m(p)>(n-1)/n$, write $p(z)=\alpha z\prod_i(z-a_i)$, put $B^{(R)}(z)=z\prod_i(z-a_i/R)/(1-\bar a_iz/R)$, and let $R\to\infty$. Then $R^nB^{(R)}(z/R)\to p(z)/\alpha$ locally uniformly, so by Hurwitz's theorem the $n-1$ critical points of $B^{(R)}$ in $\mathbb{D}$ have ratios tending to those of p, while $|(B^{(R)})'(0)|\to0$ and $f_n(t)\to(n-1)/n$ as $t\to0$ [14, Lemma 1]. At the critical points outside the closed disk the ratio is at least $1/|(B^{(R)})'(0)|>1$, by Schwarz's lemma. In the same way the inequality [14, (1.2)], which by a result of Sheil-Small [40] quoted there would imply $K^*\le1$, fails for some Blaschke product in every degree $n\ge N$. For large n the ratios of $B^{(R)}$ at all its critical points can be made larger than 1.1559837 (Corollary 8.12).

pair	$\deg Q$	type	$\sqrt{\hat{J}} \geq$
I	18	boundary, two zeros	1.1559837
II	17	boundary, two zeros	1.1518591
III	16	boundary, two zeros	1.1509994
IV	15	boundary, two zeros	1.1466539
V	14	boundary, two zeros	1.1392755
VI	18	boundary	1.1363375
VII	17	boundary	1.1361064
VIII	16	boundary	1.1358030
IX	14	boundary	1.1346063
X	13	boundary	1.1332281
XI	12	boundary	1.1315433
XII	12	boundary	1.1306600
XIII	12	Blaschke model	1.1272406
XIV	8	Blaschke model	1.1196072
XV	8	Blaschke model	1.1158108
XVI	6	boundary	1.1144943
XVII	6	boundary	1.1143766
XVIII	8	Blaschke model	1.1138795

TABLE 2. Exact pairs with one zero, and pairs I, II, III, IV and V with two zeros (Theorems 8.7–8.9); $\sqrt{\hat{J}}$ is truncated to seven decimals. "Blaschke model": Q' has no zero on the closed disk. "Boundary": $Q' = (z-1)^m R$ exactly, with $m = 5$ for pairs I, II, III, IV, V, VI, VII, VIII, IX, XI and XII and $m = 3$ for pairs XVI and XVII, and $Q' = (z-1)^5(z-\zeta_2)R$ with $\zeta_2 \neq 1$ on the circle for pair X; R has no zero on the closed disk, so $Q' \neq 0$ in $\mathbb{D}$. The gains of the boundary pairs VI, VII, VIII, IX, XI, XII, XVI and XVII are also attained by Blaschke models (Corollary 9.11), and that of pair I by Blaschke models with two zeros (Corollary 9.14). The exact data of all eighteen pairs are in the supporting archive.

8.1. **An upper bound in degree one.** For $0 < s < 1$ let B_s be the factor B_a of Section 2.3 with $a = s$, and put

$$B_s(w) = \frac{w-s}{1-sw}, \qquad \sigma_s(w) = \frac{wB_s'(w)}{B_s(w)} = \frac{(1-s^2)w}{(w-s)(1-sw)}, \qquad L(r) = \log\frac{1+r}{1-r},$$

and for $\xi \neq 0$ write $|\operatorname{Log}\xi| = (\log^2|\xi| + \operatorname{Arg}^2\xi)^{1/2}$, with the principal logarithm Log; it is the least modulus of a logarithm of ξ. For $W \in \mathcal{S}$ let ℓ be the holomorphic logarithm of $wW'(w)/W(w)$ with $\ell(0) = 0$. By Grunsky's theorem [20] (see also [16, Ch. 3] and [23, (18)]),

$$|\ell(w)| \leq L(|w|) \qquad (w \in \mathbb{D}). \tag{1}$$

Jatar and Ng [23, Thm. 7.4] prove $|a| > \sqrt{5/7}$ for the zero a of every admissible pair of degree one, by a homotopy on the circle $|w| = |a|$. The same homotopy works on other circles.

Lemma 8.14. *Let $0 < s < 1$, $r_0 = (1-\sqrt{1-s^2})/s$, and let $r_0 < \rho < 1$ with $\rho \neq s$. If $|\operatorname{Log}(-\sigma_s(w))| > L(\rho)$ for every w with $|w| = \rho$, then for every $W \in \mathcal{S}$ the derivative $(WB_s)'$ has a zero in $\{|w| < \rho\}$.*

Proof. For $0 \leq t \leq 1$ put $\widehat{P}_t(w) = (w-s)(1-sw)e^{t\ell(w)} + (1-s^2)w$. Since $wW'/W = e^{\ell}$, one has $(WB_s)'(w) = W(w)\widehat{P}_1(w)/(w(1-sw)^2)$, as in the proof of [23, Thm. 7.4], and $W(w)/w$ has no zero in $\mathbb{D}$; so it suffices to find a zero of $\widehat{P}_1$ in $\{|w| < \rho\}$. Let $|w| = \rho$. Then $(w-s)(1-sw) \neq 0$, so

$\widehat{P}_t(w) = 0$ would mean $e^{t\ell(w)} = -\sigma_s(w)$; then $t\ell(w)$ is a logarithm of $-\sigma_s(w)$, and $|\operatorname{Log}(-\sigma_s(w))| \le t|\ell(w)| \le L(\rho)$ by (1), a contradiction. Hence no $\widehat{P}_t$ vanishes on the circle $|w| = \rho$, and by the argument principle the number of zeros of $\widehat{P}_t$ in $\{|w| < \rho\}$ does not depend on t. The polynomial $\widehat{P}_0(w) = -sw^2 + 2w - s$ has the zeros r_0 and $1/r_0$, and $r_0 < \rho < 1 < 1/r_0$; so $\widehat{P}_1$ has exactly one zero in $\{|w| < \rho\}$. □

The case $\rho = s$ is the one used in [23]; there σ_s has a pole on the circle, and the point $w = s$ is handled by $\widehat{P}_t(s) = s(1 - s^2) \neq 0$.

Theorem 8.15. *Let $W \in \mathcal{S}$ and let B be a Blaschke product of degree one with $B(0) > 0$ such that $(WB)'$ has no zero in $\mathbb{D}$. Then the zero a of B satisfies $|a| > 0.84904$. Consequently*

$$\mathcal{B}_1 < \frac{1}{0.84904} = 1.1778008\ldots < 1.177801.$$

Proof. Write $a = se^{i\theta_a}$ with $0 < s < 1$. The function $\widetilde{W}(w) = e^{-i\theta_a} W(e^{i\theta_a} w)$ lies in $\mathcal{S}$, and $B(e^{i\theta_a} w)$ is a unimodular multiple of $B_s(w)$; so $(\widetilde{W} B_s)'$ has no zero in $\mathbb{D}$. For $s \le \sqrt{5/7} = 0.845154\ldots$ this is impossible by [23, Thm. 7.4]. For $s \in [0.845, 0.849]$ we apply Lemma 8.14 with $\rho = 0.827$, and for $s \in [0.849, 0.84904]$ with $\rho = 0.8271$. In both cases $r_0 < \rho < s$, because r_0 is increasing in s and $r_0(0.84904) < 0.5556$; and the hypothesis $|\operatorname{Log}(-\sigma_s(w))| > L(\rho)$ on $|w| = \rho$ holds for every s in the range by the computation of Appendix C. Lemma 8.14 then gives a zero of $(\widetilde{W} B_s)'$ in $\mathbb{D}$, a contradiction. Hence $|a| > 0.84904$, and $1/B(0) = 1/|a| < 1/0.84904$. A pair of degree zero has gain 1. Since $\mathcal{B}_1$ is attained [23, Thm. 8.9], $\mathcal{B}_1 < 1/0.84904$. □

Remark 8.16. Lemma 8.14 uses only (1) at each point of the circle, and not the relations between the values of ℓ at different points. Rotations of the Koebe function have $\ell(w) = 2\operatorname{artanh}(e^{it} w)$ with t real, so they attain equality in (1) only where $\ell(w) = \pm L(|w|)$. Compare Popov's joint value region of $\log(W(w)/w)$ and $\log W'(w)$ on $\mathcal{S}$ [31] (see the survey [32]) and Grunsky's value region of $\log(W(w)/w)$ [20].

Remark 8.17. The proof of Lemma 8.14 works on any Jordan curve in $\mathbb{D}$ that surrounds r_0 and avoids s, with the hypothesis $|\operatorname{Log}(-\sigma_s(w))| > L(|w|)$ on the curve. Numerically, the largest s for which a circle satisfies this hypothesis is about 0.849043 (near $\rho = 0.8271$); on the circle $\rho = s$ the corresponding value is about 0.845899. So Theorem 8.15 is close to the limit of this argument. An admissible pair of degree two with gain greater than 1.177801 would show $\mathcal{B}_2 > \mathcal{B}_1$, one of the questions of [23, Sect. 9].

8.2. **An upper bound in degree two.** We use the notation of Section 8.1, with $|\operatorname{Log}\xi|$ the least modulus of a logarithm of $\xi \neq 0$, and we put $|\operatorname{Log} 0| = +\infty$. Let $B(w) = \lambda \prod_{j=1}^{d} (w - a_j)/(1 - \bar{a}_j w)$ be a Blaschke product of degree $d \ge 1$ with simple nonzero zeros, and put

$$\sigma = \frac{wB'}{B} = \sum_{j=1}^{d} \frac{(1 - |a_j|^2) w}{(w - a_j)(1 - \bar{a}_j w)} = \frac{\check{N}}{\check{R}}, \qquad \check{R} = \prod_{j=1}^{d} (w - a_j)(1 - \bar{a}_j w),$$

$$\check{N} = \sum_{j=1}^{d} (1 - |a_j|^2)\, w \prod_{l \neq j} (w - a_l)(1 - \bar{a}_l w), \qquad \widehat{P}_t = \check{R} e^{t\ell} + \check{N} \quad (0 \le t \le 1).$$

The proof of [23, Thm. 7.4] gives the following statement for any number of zeros and any circle in the disk.

Lemma 8.18. *Let $\Delta = \{|w - w_C| \le \rho\}$ be a closed disk contained in $\mathbb{D}$ whose boundary circle C contains no a_j, and assume $|\operatorname{Log}(-\sigma(w))| > L(|w|)$ for every $w \in C$. If wB has a critical point in the interior of Δ, then for every $W \in \mathcal{S}$ the derivative $(WB)'$ has a zero in the interior of Δ.*

Proof. Zeros are counted with multiplicity. Since $wW'/W = e^\ell$ and $B/\check{R} = \lambda/\prod_j(1-\bar{a}_j w)^2$,

$$(WB)' = \frac{W}{w}\,B\,(e^\ell + \sigma) = \frac{\lambda W}{w}\,\frac{\widehat{P}_1}{\prod_j(1-\bar{a}_j w)^2};$$

here $W(w)/w$ extends holomorphically to $\mathbb{D}$ with value 1 at 0 and has no zero in $\mathbb{D}$, so the zeros of $(WB)'$ in $\mathbb{D}$ are those of $\widehat{P}_1$. Next, $\widehat{P}_0 = \check{R}(1+\sigma)$ and $(wB)' = B(1+\sigma)$, so $\widehat{P}_0 = (wB)'\prod_j(1-\bar{a}_j w)^2/\lambda$, and the zeros of $\widehat{P}_0$ in $\mathbb{D}$ are the critical points of wB. Let $w \in C$. Then $\check{R}(w) \neq 0$, since $w \neq a_j$ and $|1-\bar{a}_j w| \geq 1-|w| > 0$. If $\sigma(w) = 0$, then $\widehat{P}_t(w) = \check{R}(w)e^{t\ell(w)} \neq 0$. Otherwise $\widehat{P}_t(w) = 0$ would mean $e^{t\ell(w)} = -\sigma(w)$, so $t\ell(w)$ would be a logarithm of $-\sigma(w)$ and $|\operatorname{Log}(-\sigma(w))| \leq t|\ell(w)| \leq L(|w|)$ by (1), a contradiction. Hence no $\widehat{P}_t$ vanishes on C. Since $\widehat{P}_t$ is jointly continuous in $(t,w) \in [0,1]\times\mathbb{D}$ and holomorphic in w, and $\Delta \subset \mathbb{D}$, the number of zeros of $\widehat{P}_t$ inside C does not depend on t, and it is positive at $t = 0$. □

The disk must lie in $\mathbb{D}$: ℓ and (1) are defined there only, and only the zeros of $\widehat{P}_0$ in $\mathbb{D}$ are critical points of wB. A zero a_j inside C does no harm, since $\widehat{P}_0(a_j) = \check{N}(a_j) \neq 0$, and a circle through 0 is allowed, since $\sigma(0) = 0$. For $w_C = 0$ and $d = 1$ the lemma is Lemma 8.14; unlike the circle $|w| = s$ of [23, Thm. 7.4], the circles used below keep away from every a_j.

Theorem 8.19. *Let $W \in \mathcal{S}$ and let B be a Blaschke product of degree two with $B(0) > 0$ such that $(WB)'$ has no zero in $\mathbb{D}$. Then the zeros a_1, a_2 of B satisfy $|a_1 a_2| > 61/100$. Consequently*

$$\mathcal{B}_2 < \frac{100}{61} = 1.63934\ldots < 1.639345.$$

Proof. The zeros of B are simple and nonzero: $B(0) \neq 0$, and at a multiple zero a one has $(WB)'(a) = W'(a)B(a) + W(a)B'(a) = 0$. A rotation $W(w) \mapsto e^{-i\theta}W(e^{i\theta}w)$, $B(w) \mapsto B(e^{i\theta}w)$ (as in the proof of Theorem 8.15) and the conjugation $W(w) \mapsto \overline{W(\bar{w})}$, $B(w) \mapsto \overline{B(\bar{w})}$ keep the hypotheses and $|a_1 a_2| = B(0)$. Rotating the zero of larger modulus to the positive axis, we may assume $a_1 = r_1 \in (0,1)$, $a_2 = x_2 + iy_2$ with $y_2 \geq 0$, $0 < |a_2| \leq r_1$ and $a_2 \neq a_1$. Suppose $r_1|a_2| \leq 61/100$. The compact set

$$X_{61} = \{(r_1, x_2, y_2) : 0 \leq r_1 \leq 1,\ x_2^2 + y_2^2 \leq r_1^2,\ y_2 \geq 0,\ r_1\sqrt{x_2^2+y_2^2} \leq 61/100\}$$

is covered by finitely many boxes, products of three closed intervals with rational ends, and each box comes with a rational point w_C and a rational $\rho > 0$ such that, for every parameter in the box: (i) $|w_C| + \rho \leq 199/200$, $|a_2| < 1$, and the circle $C = \{|w - w_C| = \rho\}$ has distance more than $1/200$ from a_1 and from a_2; (ii) $|\operatorname{Log}(-\sigma(w))| > L(|w|)$ for every $w \in C$; and (iii) at one rational point of the box, $\widehat{P}_0$ has $m \geq 1$ zeros inside C. By (i), $\check{R} \neq 0$ on C, since $|w - a_j| > 1/200$ and $|1-\bar{a}_j w| \geq 1-|w| \geq 1/200$, and by (ii), $1+\sigma \neq 0$ on C, since $|\operatorname{Log}(-\sigma)| > L(|w|) \geq 0 = |\operatorname{Log} 1|$; so $\widehat{P}_0 = \check{R}(1+\sigma)$ has no zero on C for any parameter in the box. The coefficients of $\widehat{P}_0$ are polynomials in (r_1, x_2, y_2), so by the argument principle the number of zeros of $\widehat{P}_0$ inside C is $m \geq 1$ on the whole box; by (i) these zeros lie in $\mathbb{D}$, so they are critical points of wB. Lemma 8.18 with the circle of the box containing (r_1, x_2, y_2) gives a zero of $(WB)'$ in $\mathbb{D}$, a contradiction. Hence $|a_1 a_2| > 61/100$. The points of X_{61} with $r_1 = 1$, $a_2 = 0$ or $a_2 = a_1$ are not of this form; they only make X_{61} compact, and all functions used are continuous there by (i). Finally, $1/B(0) = 1/|a_1 a_2| < 100/61$, a pair of degree at most one has gain less than $\sqrt{7/5} = 1.18321\ldots < 100/61$ by [23, Thm. 7.4, Cor. 8.10], and $\mathcal{B}_2$ is attained [23, Thm. 8.9]; so $\mathcal{B}_2 < 100/61$. □

The computation that gives the boxes and checks (i)–(iii) is described in Appendix C.

9. Fixed-degree optima

The pairs of Section 8 with B of degree one come from the construction of [23, Lemma 6.2] with a polynomial of some degree n. Theorem 7.13 improves such a pair, but its perturbation is a polynomial whose degree grows with the number of its Hermite conditions. We show that the improvement can be made without raising the degree as long as the pair has few boundary defects. So a pair that is best among those of degree n must have many defects (Theorem 9.1), in the way a best approximation from an $(n+1)$-dimensional space attains its error at $n+2$ points (Chebyshev's alternation theorem).

We use the notation of Section 8, with z the variable of Q and H and w that of W. Let $\mathcal{L}_n$ be the set of pairs (W, B) given there by a polynomial Q of degree at most n: $a \neq b$ in $\mathbb{D}$, $Q(a) = Q(b) = 0$, $Q' \neq 0$ in $\mathbb{D}$, $H = (1-\bar{a}z)Q/(z-a)$ univalent in $\mathbb{D}$, $W = (H \circ \varphi)/N_b$, and B the multiple of $B_a \circ \varphi$ by the constant of modulus 1 with $B(0) > 0$. Let $\mathcal{G}_n$ be the set of pairs given in the same way, $W = (H \circ \varphi)/N_b$ and B the multiple of $B_a \circ \varphi$ by the constant of modulus 1 with $B(0) > 0$, where H is a polynomial of degree at most n, $a, b \in \mathbb{D}$, $H(b) = 0$, H is univalent in $\mathbb{D}$ and $(HB_a)' \neq 0$ in $\mathbb{D}$; here HB_a need not be a polynomial. This is the first of the two approaches described in [23, Sect. 6]. In $\mathcal{G}_n$, $a \neq b$, since otherwise HB_a has a double zero at b; and $\mathcal{L}_n \subset \mathcal{G}_n$. Both kinds of pairs are admissible, with gain $\sqrt{J(a,b)}$, and $F = WB$ is a constant multiple of $(HB_a) \circ \varphi$, so the zeros of F' on $\mathbb{T}$ are the points $\varphi^{-1}(\xi)$ for the zeros ξ of $(HB_a)'$ on $\mathbb{T}$, with the same orders. W is rational, with the only pole $-1/\bar{b}$ if $b \neq 0$, and a polynomial if $b = 0$; so W is holomorphic past the circle, and Lemma 7.12 applies to it.

For such a pair let $z_1, \dots, z_p$ be the zeros of F' on $\mathbb{T}$, of orders $m_1, \dots, m_p$; let $\mathrm{Crit} = \{c_1, \dots, c_q\}$, and let $\{\zeta_l, \zeta_l'\}$ $(1 \leq l \leq r)$ be the contact pairs of W; and let e be the number of contact points that are also some w_i. Put

$$n_H = 2 + \sum_{i=1}^{p} (m_i + 1) + 2q + 2r - e,$$

the number of Hermite conditions in the proof of Theorem 7.13. We call $n_H - 2$ the weighted number of boundary defects of the pair. Here "local", as in "local maximiser", refers to the topology of the coefficients of the triples (H, a, b) for $\mathcal{G}_n$, resp. (Q, a, b) for $\mathcal{L}_n$.

Theorem 9.1. *Let (W, B) be a pair in $\mathcal{G}_n$, resp. in $\mathcal{L}_n$, with data (H, a, b), resp. (Q, a, b).*

- (G) *If $n_H \leq n + 2$, then for every small $\delta > 0$ there is a pair in $\mathcal{G}_n$ with data (H_δ, a_δ, b) and gain $(1-\delta)^{-1}/B(0)$, where $H_\delta \to H$ coefficientwise and $a_\delta \to a$ as $\delta \to 0$.*
- (L) *If $n_H \leq n + 1$, then for every small $\delta > 0$ there is a pair in $\mathcal{L}_n$ with data (Q_δ, a_δ, b) and gain $(1-\delta)^{-1}/B(0)$, where $Q_\delta \to Q$ coefficientwise and $a_\delta \to a$ as $\delta \to 0$.*

In particular a local maximiser of the gain over $\mathcal{G}_n$ has $n_H \geq n + 3$, and a local maximiser over $\mathcal{L}_n$ has $n_H \geq n + 2$; a local maximiser over $\mathcal{L}_n$ such that F' has no zero on $\mathbb{T}$ has $q + r \geq n/2$.

Corollary 9.2. *Suppose that for each n the gain has a local maximiser (W_n, B_n) over $\mathcal{L}_n$, and that $\sum_i (m_i + 1) \leq \mathcal{X}$ for all n. Then $q_n + r_n \geq (n - \mathcal{X})/2$: the number of boundary critical points plus the number of contact pairs grows at least linearly in n.*

By Lemma 7.12 each single pair has finitely many defects; the growth is what Theorem 9.1 forces. The corollary follows from $n_H \geq n + 2$ with $e \geq 0$, and the last statement of the theorem is the case $p = e = 0$. When $\deg Q = n$, Theorem 9.7 below gives $n_H \geq n + 3$.

Proof of Theorem 9.1. Let $\alpha = \varphi^{-1}(a)$, the zero of B; $\alpha \neq 0$ as $a \neq b$.

Step 1: the conditions. Let U_1 and the conditions be as in the proof of Theorem 7.13, with U_1 also free of $-1/\bar{b}$. Here B_δ is B with its zero α replaced by $(1-\delta)\alpha$, so $B_\delta(0) = (1-\delta)B(0)$, and B_δ is a unimodular multiple of $B_{a_\delta} \circ \varphi$ with $a_\delta = \varphi((1-\delta)\alpha)$, a smooth function of δ with $a_0 = a$. We

impose the Hermite conditions of that proof at the distinct nodes 0, w_i, c_j, ζ_l, ζ_l' except $\upsilon'(0) = 0$: there are $n_H - 1$ of them, with data bounded uniformly in δ.

Step 2: $\upsilon = \mathfrak{p} \circ \varphi$. For a polynomial $\mathfrak{p}$, the derivatives of orders $0, \dots, s$ of $\mathfrak{p} \circ \varphi$ at a node w_0 are an invertible triangular transform (with diagonal $\varphi'(w_0)^k \neq 0$) of those of $\mathfrak{p}$ at $\varphi(w_0)$. So the conditions on υ are $n_H - 1$ Hermite conditions on $\mathfrak{p}$ at the distinct fixed nodes $\varphi(0) = b$ (of order 0 only), $\varphi(w_i)$, $\varphi(c_j)$, $\varphi(\zeta_l)$, $\varphi(\zeta_l')$, with data bounded uniformly in δ.

In case (G), $n_H - 1 \leq n + 1$; let $\mathfrak{p}_\delta$ be the interpolant of degree at most $n_H - 2 \leq n$. As the nodes are fixed and the data bounded, its coefficients are bounded uniformly in δ.

In case (L), $\check{Q} = Q/(z - a)$ is a polynomial of degree at most $n - 1$ with $\check{Q}(b) = 0$, and $H = (1 - \bar{a}z)\check{Q}$. Seek

$$\mathfrak{p}_\delta = g_\delta + (1 - \bar{a}_\delta z)\, r_\delta, \qquad g_\delta = \frac{\bar{a} - \bar{a}_\delta}{\delta}\, \frac{z\check{Q}}{N_b}, \qquad \deg r_\delta \leq n - 1.$$

The coefficients of g_δ are bounded uniformly in δ. On the closed disk $1 - \bar{a}_\delta z \neq 0$, so at each node the multiplication by $1 - \bar{a}_\delta z$ is an invertible triangular map on jets, with inverse bounded uniformly in δ. Subtracting the jets of g_δ and dividing, the conditions on $\mathfrak{p}$ become $n_H - 1 \leq n$ Hermite conditions on r_δ at the same nodes, with bounded data (at b: $r_\delta(b) = 0$, as $g_\delta(b) = 0$). Let r_δ be the interpolant of degree at most $n_H - 2 \leq n - 1$; its coefficients, and those of $\mathfrak{p}_\delta$, are bounded uniformly in δ, and $\deg \mathfrak{p}_\delta \leq n$.

In both cases $\upsilon_\delta = \mathfrak{p}_\delta \circ \varphi$ satisfies the conditions and is bounded in every $C^m(U_1)$ uniformly in δ.

Step 3: the scale. The rest of the proof of Theorem 7.13 uses only these conditions and these bounds; the condition $\upsilon'(0) = 0$ served there only to get $W_\delta'(0) = 1$. Here $W_\delta = W + \delta\upsilon_\delta$ has $W_\delta(0) = 0$ and $W_\delta'(0) = 1 + \delta\upsilon_\delta'(0) \neq 0$ for small δ. So $\widetilde{W}_\delta = W_\delta/(1 + \delta\upsilon_\delta'(0))$ is univalent on a neighbourhood of the closed disk, $\widetilde{W}_\delta \in \mathcal{S}$, and $(\widetilde{W}_\delta B_\delta)'$ has the zeros of $(W_\delta B_\delta)'$, so none in $\mathbb{D}$, for all small $\delta > 0$.

Step 4: the pair stays in the family. Put $H_\delta = H + \delta N_b \mathfrak{p}_\delta$, a polynomial of degree at most n, and $N_\delta = (1 - |b|^2)H_\delta'(b) = N_b(1 + \delta\upsilon_\delta'(0))$. As $\mathfrak{p}_\delta(b) = \upsilon_\delta(0) = 0$, $H_\delta(b) = 0$ and $(H_\delta \circ \varphi)/N_\delta = \widetilde{W}_\delta$. So H_δ is univalent in $\mathbb{D}$, and $(H_\delta B_{a_\delta}) \circ \varphi$ is a non-zero constant multiple of $\widetilde{W}_\delta B_\delta$, so $(H_\delta B_{a_\delta})' \neq 0$ in $\mathbb{D}$. Hence (H_δ, a_δ, b) defines a pair in $\mathcal{G}_n$, namely $(\widetilde{W}_\delta, B_\delta)$, with gain $1/B_\delta(0) = (1 - \delta)^{-1}/B(0)$. In case (L), $\delta N_b g_\delta = (\bar{a} - \bar{a}_\delta) z\check{Q}$, so

$$H_\delta = (1 - \bar{a}z)\check{Q} + (\bar{a} - \bar{a}_\delta)z\check{Q} + \delta N_b(1 - \bar{a}_\delta z)r_\delta = (1 - \bar{a}_\delta z)(\check{Q} + \delta N_b r_\delta),$$

and $Q_\delta := H_\delta B_{a_\delta} = (z - a_\delta)(\check{Q} + \delta N_b r_\delta)$ is a polynomial of degree at most n with $Q_\delta(a_\delta) = Q_\delta(b) = 0$, $Q_\delta' \neq 0$ in $\mathbb{D}$ and $H_\delta = (1 - \bar{a}_\delta z)Q_\delta/(z - a_\delta)$; with $a_\delta \neq b$ for small δ, this is a pair in $\mathcal{L}_n$. No node outside the disk and no condition on a is needed.

Step 5: nearness. $H_\delta - H = \delta N_b \mathfrak{p}_\delta = O(\delta)$ coefficientwise, $a_\delta \to a$, and b is fixed; in case (L), $Q_\delta - Q = (a - a_\delta)\check{Q} + \delta N_b(z - a_\delta)r_\delta = O(\delta)$. □

Remark 9.3. For pair X, read as a pair in $\mathcal{L}_{13}$, Theorem 8.9 and Lemma 8.6 show that H is univalent past the circle, so $q = r = e = 0$, and, by (C), that F' has on $\mathbb{T}$ exactly the two zeros of orders 5 and 1. So $n_H = 2 + 6 + 2 = 10 \leq n + 1 = 14$, and Theorem 9.1(L) gives pairs in $\mathcal{L}_{13}$ near pair X with larger gains, with no explicit constant. Raising the degree of Q by one gains nothing to first order: by Lemma 9.5, an automorphism of the disk removes the new top coefficient to first order and keeps the gain.

Remark 9.4. A pair in $\mathcal{G}_n$ such that F' has no zero on $\mathbb{T}$ can be improved within $\mathcal{G}_n$ by moving a alone: keep H and replace a by a_δ. Then $F' \neq 0$ on the closed disk, and $B_\delta \to B$ in C^1 on the closed disk, so $(WB_\delta)' \neq 0$ there for small δ. So every local maximiser over $\mathcal{G}_n$ has $p \geq 1$, in addition to $n_H \geq n + 3$; in particular the Blaschke models XIII, XIV, XV and XVIII of Table 2 can be improved in this way. In $\mathcal{L}_n$ this argument fails, since H must keep the factor $1 - \bar{a}z$.

Raising the degree of Q by one. Theorem 9.1(L) keeps b fixed. Letting b move as well, we show that a local maximiser with $\deg Q = n$ has a defect count n_H one larger: to first order, raising the degree by one is a change of variables by an automorphism of the disk. For $y \in \mathbb{D}$ put $\varphi_y(w) = (w+y)/(1+\bar{y}w)$, so that $\varphi = \varphi_b$, and write $[z^j]P$ for the coefficient of z^j in a polynomial P.

Lemma 9.5. *Let Q be holomorphic near the closed disk, let $a \neq b$ in $\mathbb{D}$ with $Q(a) = Q(b) = 0$, suppose that $H = (1-\bar{a}z)Q/(z-a)$ has $H'(b) \neq 0$, and let (W, B) be given by the formulas of Section 8, with $B(0) > 0$. Let ψ be an automorphism of $\mathbb{D}$. Then the data $(Q\circ\psi, \psi^{-1}(a), \psi^{-1}(b))$ give the pair $w \mapsto \big(e^{-it}W(e^{it}w),\, B(e^{it}w)\big)$ for some $t \in \mathbb{R}$, and their function H is a unimodular multiple of $H\circ\psi$. In particular the two pairs have the same gain, they are admissible together, and their boundary defects correspond under $w \mapsto e^{-it}w$, with the same orders.*

Proof. Since $HB_a = Q$, $F = WB$ is a constant multiple of $Q\circ\varphi_b$, and B is the Blaschke factor with $B(0) > 0$ whose zero is $\varphi_b^{-1}(a)$; with $W'(0) = 1$, the pair is determined by $Q\circ\varphi_b$ and $\varphi_b^{-1}(a)$. The automorphisms $\psi\circ\varphi_{\psi^{-1}(b)}$ and φ_b both take 0 to b, so $\psi\circ\varphi_{\psi^{-1}(b)} = \varphi_b\circ\rho$ with $\rho(w) = e^{it}w$. So the data $(Q\circ\psi, \psi^{-1}(a), \psi^{-1}(b))$ give $(Q\circ\varphi_b)\circ\rho$ and $\rho^{-1}(\varphi_b^{-1}(a))$, that is, a constant multiple of $F\circ\rho$ and the factor $B\circ\rho$, and $W'(0) = 1$ fixes the constant. Finally $B_a\circ\psi$ is a unimodular multiple of $B_{\psi^{-1}(a)}$, which gives the statement on H. □

Lemma 9.6. *Let (W, B) be as in Theorem 7.13, with the points c_j and the contact pairs $\{\zeta_l, \zeta_l'\}$ of Lemma 7.12, and let U_1 be a compact neighbourhood of the closed disk on which W is holomorphic. Let υ_δ $(0 < \delta < \delta_0)$ be holomorphic on U_1, bounded in every $C^m(U_1)$ uniformly in δ, with $\upsilon_\delta(0) = 0$, and suppose that*

$$\varpi_j(\upsilon_\delta) := 2\,\mathrm{Re}\Big(\bar{c}_j\,\frac{-\upsilon_\delta'(c_j)}{W''(c_j)}\Big) \geq \varpi_0, \qquad \varpi_l(\upsilon_\delta) := \mathrm{Re}\Big(\bar{\zeta}_l'\,\frac{\upsilon_\delta(\zeta_l) - \upsilon_\delta(\zeta_l')}{W'(\zeta_l')}\Big) \geq \varpi_0$$

for all j, l and δ, with $\varpi_0 > 0$ fixed. Then $W + \delta\upsilon_\delta$ is univalent on a neighbourhood of the closed disk for all small δ.

Proof. We use the notation of the proof of Theorem 7.13. There the paragraphs on the defects, on local injectivity and on global injectivity use υ_δ only through these bounds and through $c_\delta = c - \delta\upsilon_\delta'(c)/W''(c) + O(\delta^2)$ and $\tau_\delta(\zeta_1) - \zeta_2 = \delta(\upsilon_\delta(\zeta_1) - \upsilon_\delta(\zeta_2))/W'(\zeta_2) + O(\delta^2)$. In general these give $\log|c_\delta| = \varpi_j\delta/2 + O(\delta^2)$; $\log|\beta_\delta(c)| = \varpi_j\delta + O(\delta^2)$, since $\beta_\delta(c) = 2c_\delta - c + O(\delta^2)$; and $\log|\tau_\delta(\zeta_1)| = \varpi_l\delta + O(\delta^2)$. So $|c_\delta| > 1$ and $G_\delta(1,0) \geq \varpi_0\delta - C\delta^2$, and the inequality of the local injectivity holds with ϖ_0 in place of ϖ; the rest of those paragraphs is unchanged. They do not use the conditions at the w_i, which serve only to keep $F_\delta' \neq 0$ in $\mathbb{D}$: near a w_i that is not a contact point, $W' \neq 0$, since $F'(c) \neq 0$ at every $c \in \mathrm{Crit}$, and the segment argument applies there. □

Theorem 9.7. *Let (W, B) be a pair in $\mathcal{L}_n$ with data (Q, a, b), where $\deg Q = n$, and suppose that $n_H \leq n+2$. Then for every small $\delta > 0$ there is a pair in $\mathcal{L}_n$ with data $(Q^\sharp, a^\sharp, b^\sharp)$ and gain at least $(1-\delta)^{-1}/B(0) - C\delta^2$, with C independent of δ, where $Q^\sharp \to Q$ coefficientwise, $a^\sharp \to a$ and $b^\sharp \to b$ as $\delta \to 0$. In particular, a local maximiser of the gain over $\mathcal{L}_n$ with $\deg Q = n$ has $n_H \geq n+3$; if moreover F' has no zero on $\mathbb{T}$, then $q + r \geq (n+1)/2$.*

Proof. The plan. The proof of Theorem 9.1(L) gives the gain factor $(1-\delta)^{-1}$, but in degree $n+1$ (Step 1). A disk automorphism ψ_δ removes the new top coefficient up to $O(\delta^2)$ (Step 2, with Lemma 9.5); the result is not a polynomial, so Step 3 replaces it by a polynomial $Q^\sharp$ of degree at most n whose derivative has the same zeros on $\mathbb{T}$. Step 4 shows that the new map is still univalent, and Step 5 compares the gains.

Below, C is independent of δ. Put $\xi_i = \varphi(w_i)$; then Q' has on the closed disk exactly the zeros $\xi_i \in \mathbb{T}$, of orders m_i, and $\sum_i m_i \leq n-1$ as $\deg Q' = n-1$.

Step 1: one degree up. The pair lies in $\mathcal{L}_{n+1}$, and there $n_H \le (n+1)+1$. So the proof of Theorem 9.1(L) in $\mathcal{L}_{n+1}$ gives, for small δ, the pair $(\widetilde{W}_\delta, B_\delta)$ in $\mathcal{L}_{n+1}$ with data (Q_δ, a_δ, b), where $a_\delta = \varphi((1-\delta)\alpha)$, $Q_\delta = (z-a_\delta)(\check{Q}+\delta N_b r_\delta)$ with $\deg \check{Q} = n-1$, $\deg r_\delta \le n_H - 2 \le n$ and r_δ bounded, and $\widetilde{W}_\delta = (W+\delta\upsilon_\delta)/(1+\delta\upsilon_\delta'(0))$ with υ_δ bounded in every $C^m(U_1)$, $\upsilon_\delta(0)=0$, $\varpi_j(\upsilon_\delta)=2$ and $\varpi_l(\upsilon_\delta)=1$. So $Q_\delta - Q = O(\delta)$, and the coefficient of degree $n+1$ is $[z^{n+1}]Q_\delta = \delta N_b [z^n] r_\delta$. The conditions at the w_i make $(\widetilde{W}_\delta B_\delta)'$ vanish to order at least m_i at w_i, and its derivative of order m_i there tends to that of F', which is not 0. As $F' \ne 0$ elsewhere on the closed disk, for small δ the zeros of Q_δ' on the closed disk are the ξ_i, with the orders m_i.

Step 2: the gauge. Define s_δ by $\bar{s}_\delta\, n\, [z^n]Q = [z^{n+1}]Q_\delta$, so that $s_\delta = O(\delta)$, and put $\psi_\delta = \varphi_{s_\delta}$. On $D_1 = \{|z| \le 1+\eta\}$, with $\eta \in (0,1)$ fixed below, $\psi_\delta(z) = z + s_\delta - \bar{s}_\delta z^2 + O(\delta^2)$, so $Q_\delta \circ \psi_\delta = \widetilde{Q}_\delta + O(\delta^2)$ in $C^k(D_1)$ for every k, where $\widetilde{Q}_\delta = Q_\delta + (s_\delta - \bar{s}_\delta z^2)Q'$ (by Taylor's formula $Q_\delta \circ \psi_\delta = Q_\delta + (\psi_\delta - z)Q_\delta' + O(\delta^2)$, as $Q_\delta - Q = O(\delta)$; Cauchy's estimates on a slightly larger disk give the derivatives). As $[z^{n+1}](z^2Q') = n\,[z^n]Q$, $\deg \widetilde{Q}_\delta \le n$. By Lemma 9.5, the data $(Q_\delta \circ \psi_\delta, \hat{a}, \hat{b})$ with $\hat{a} = \psi_\delta^{-1}(a_\delta)$ and $\hat{b} = \psi_\delta^{-1}(b)$ give the pair $(\widetilde{W}_\delta, B_\delta)$ turned by $w \mapsto e^{it_\delta} w$, but $Q_\delta \circ \psi_\delta$ is not a polynomial.

Step 3: the same boundary zeros in degree n. As $Q_\delta \circ \psi_\delta$ is not a polynomial, we replace it by a polynomial $Q^\sharp$ of degree at most n whose derivative has the same zeros on $\mathbb{T}$, at a cost $O(\delta^2)$. Put $\hat{z}_i = \psi_\delta^{-1}(\xi_i) \in \mathbb{T}$ and $\Pi = \prod_i (z-\hat{z}_i)^{m_i}$. By Step 1, $(Q_\delta \circ \psi_\delta)' = \Pi Y_\delta$ with Y_δ holomorphic on D_1, and $Y_\delta \to Y_0 = Q'/\prod_i (z-\xi_i)^{m_i}$ uniformly on D_1, by the maximum principle on circles of a fixed radius about the ξ_i. Y_0 has no zero on the closed disk; we fix η so that it has none on D_1, and then Y_δ has none for small δ. Divide: $\widetilde{Q}_\delta' = \Pi A_\delta + V_\delta$ with $\deg V_\delta < \sum_i m_i$. By Step 2, the derivatives of $\widetilde{Q}_\delta' - (Q_\delta \circ \psi_\delta)'$ are $O(\delta^2)$ on D_1; at $\hat{z}_i$, those of orders below m_i are the derivatives of V_δ, and the nodes $\hat{z}_i$ stay apart, so $V_\delta = O(\delta^2)$. Then $\Pi(A_\delta - Y_\delta) = O(\delta^2)$ on D_1, so $A_\delta - Y_\delta = O(\delta^2)$ on $D_2 = \{|z| \le 1 + \eta/2\}$ by the maximum principle again, and A_δ has no zero on D_2. Put

$$Q^\sharp(z) = \int_{\hat{b}}^{z} \Pi A_\delta, \qquad b^\sharp = \hat{b}.$$

Then $\deg Q^\sharp \le n$, $Q^\sharp(b^\sharp) = 0$, and $Q^\sharp - Q_\delta \circ \psi_\delta = O(\delta^2)$ in $C^k(D_2)$, as both vanish at $\hat{b}$ and D_2 is convex. $Q_\delta \circ \psi_\delta$ has a simple zero at $\hat{a}$, with derivative bounded away from 0 as $Q'(a) \ne 0$, so by Rouché's theorem $Q^\sharp$ has a simple zero $a^\sharp = \hat{a} + O(\delta^2)$, and $a^\sharp \ne b^\sharp$. The derivative ΠA_δ of $Q^\sharp$ has no zero in $\mathbb{D}$, and on $\mathbb{T}$ exactly the zeros $\hat{z}_i$, of orders m_i.

Step 4: univalence. If $f(z_0) = 0$, then $f(z)/(z-z_0) = \int_0^1 f'(z_0 + x(z-z_0))\,dx$ on the convex set D_2. So $H^\sharp = (1-\bar{a}^\sharp z)Q^\sharp/(z-a^\sharp)$ differs by $O(\delta^2)$ in $C^{k-1}(D_2)$ from the function $(1-\bar{\hat{a}}z)(Q_\delta \circ \psi_\delta)/(z-\hat{a})$, and $N^\sharp = (1-|b^\sharp|^2)H^{\sharp\prime}(b^\sharp)$ differs by $O(\delta^2)$ from the corresponding constant, which is not 0. Let $U_2 \subset U_1$ be a closed disk about 0 whose interior contains the closed disk, with $\varphi_{\hat{b}}(U_2) \subset D_2$ for small δ. Then $W^\sharp = (H^\sharp \circ \varphi_{b^\sharp})/N^\sharp$ differs by $O(\delta^2)$ in $C^m(U_2)$ (take k large) from the map given by the data $(Q_\delta \circ \psi_\delta, \hat{a}, \hat{b})$, which is $e^{-it_\delta}\widetilde{W}_\delta(e^{it_\delta}w)$ by Lemma 9.5; so $e^{it_\delta}W^\sharp(e^{-it_\delta}w) = \widetilde{W}_\delta + O(\delta^2)$ in $C^m(U_2)$. So $(1+\delta\upsilon_\delta'(0))\,e^{it_\delta}W^\sharp(e^{-it_\delta}w) = W + \delta\upsilon_\delta^\sharp$ with $\upsilon_\delta^\sharp = \upsilon_\delta + O(\delta)$ in $C^m(U_2)$: $\upsilon_\delta^\sharp$ is bounded, $\upsilon_\delta^\sharp(0) = 0$, $\varpi_j(\upsilon_\delta^\sharp) = 2 + O(\delta)$ and $\varpi_l(\upsilon_\delta^\sharp) = 1 + O(\delta)$. By Lemma 9.6 on U_2, $W + \delta\upsilon_\delta^\sharp$ is univalent near the closed disk for small δ, and so are $W^\sharp$ and $H^\sharp$. With Step 3, $(Q^\sharp, a^\sharp, b^\sharp)$ gives a pair in $\mathcal{L}_n$.

Step 5: gain and nearness. By Lemma 9.5, the zero $\varphi_{\hat{b}}^{-1}(\hat{a})$ of $B_\delta(e^{it_\delta}w)$ is $e^{-it_\delta}(1-\delta)\alpha$. As $a^\sharp = \hat{a} + O(\delta^2)$ and $b^\sharp = \hat{b}$, the zero of the factor $B^\sharp$ of the pair of $(Q^\sharp, a^\sharp, b^\sharp)$ has modulus $(1-\delta)|\alpha| + O(\delta^2)$, and $B(0) = |\alpha|$; so $1/B^\sharp(0)$ is at least $(1-\delta)^{-1}/B(0) - C\delta^2$, which exceeds $1/B(0)$ for small δ. As $s_\delta \to 0$, $Q^\sharp \to Q$, $a^\sharp \to a$ and $b^\sharp \to b$. The last statement of the theorem follows as in Theorem 9.1: if F' has no zero on $\mathbb{T}$, then $p = e = 0$ and $n_H = 2 + 2(q+r)$. □

So a pair in $\mathcal{L}_n$ with $\deg Q = n$, no zero of F' on $\mathbb{T}$ and $q + r = n/2$ has $n_H = n+2$, and Theorem 9.7 improves it directly. For odd n the bound $q + r \ge (n+1)/2$ is that of Theorem 9.1;

for even n it is larger by one. If the maximisers of Corollary 9.2 have $\deg Q_n = n$, then $q_n + r_n \geq (n+1-\mathcal{X})/2$. In $\mathcal{G}_n$, Lemma 9.5 and Step 2 apply to H, but Step 3 would have to keep the boundary zeros of $(HB_a)'$, which is not a polynomial; whether Theorem 9.7 has a form for $\mathcal{G}_n$ is open. By Lemma 9.5 the construction of [23, Lemma 6.2] is invariant under the automorphisms of the disk, while the condition $\deg Q \leq n$ is not. The mechanism of Lemma 9.5 is that of the proof of [23, Lemma 6.2], where WB is a constant multiple of $Q \circ \varphi$.

Remark 9.8. Let (W, B) be a local maximiser of the gain over $\mathcal{L}_n$ with data (Q, a, b) and $\deg Q = n$. By Theorem 9.7, $n_H \geq n+3$, that is, $\sum_{i=1}^{p} m_i \geq n+1-p-(2q+2r-e)$. The zeros of F' on $\mathbb{T}$ correspond to those of Q' on $\mathbb{T}$, with the same orders, and $\deg Q' = n-1$. So Q' has at most

$$n-1-\sum_{i=1}^{p} m_i \;\leq\; p-2+2q+2r-e$$

zeros off the circle, counted with multiplicity. In particular, if $q = r = 0$ (so $e = 0$), then $p \geq 2$, and all zeros of Q' but at most $p-2$ lie on the circle. Pairs IX and XI have $p = 1$ and $q = r = 0$, so they are not local maximisers over $\mathcal{L}_n$ with $\deg Q = n$, as Theorem 9.1 also shows.

Releasing boundary zeros. In $\mathcal{L}_n$ the gain $\sqrt{J(a,b)}$ depends on a and b only. Keeping them fixed, the zeros of Q' on $\mathbb{T}$ can be moved off the circle without changing the gain; we use the following openness lemma.

Lemma 9.9. *Let H be injective on $\{|z| < r_0\}$ with $r_0 > 1$, let $1 < r < r_0$ and $\overline{D}_r = \{|z| \leq r\}$. If $G_k \to H$ in $C^1(\overline{D}_r)$, then G_k is injective on $\overline{D}_r$ for all large k.*

Proof. $H' \neq 0$ on $\overline{D}_r$, as H is injective. If the claim fails, then along a subsequence, $G_k(z_k) = G_k(z_k')$ with $z_k \neq z_k'$ in $\overline{D}_r$, and we may assume $z_k \to z$ and $z_k' \to z'$ in $\overline{D}_r$. If $z \neq z'$, then $H(z) = H(z')$, against injectivity. If $z = z'$, then

$$0 = \frac{G_k(z_k) - G_k(z_k')}{z_k - z_k'} = \int_0^1 G_k'\big(z_k' + s(z_k - z_k')\big)\, ds \;\longrightarrow\; H'(z) \neq 0,$$

since the segment lies in the convex set $\overline{D}_r$, $G_k' \to H'$ uniformly there and H' is uniformly continuous on $\overline{D}_r$. □

Proof of Proposition 7.36. The last sentence is Proposition 7.35(i) with $k_\zeta = 5$. We work in the variable $z = \varphi(w)$ of H, as in Section 8; by Remark 7.33 it suffices to find H with the stated properties at $z = 1$.

The start. Let a, b and Q be the exact data of pair IX (Theorem 8.9), and $H_0 = (1-\bar{a}z)Q/(z-a)$. By Theorem 8.9 and Lemma 8.6, H_0 is univalent on a disk $\{|z| < r\}$ with $r > 1$, and by (C) of Lemma 8.6 (checked in Theorem 8.9), $Q' = (z-1)^5 R$ with $R(1) \neq 0$. Put $G_0(t) = g_0(e^{it})$ with $g_0 = zH_0'/H_0$. On the circle $zB_a'/B_a = P_a$, so $g_0 + P_a = zQ'/Q$, and as $(e^{it}-1)^5 = (it)^5(1+O(t))$,

$$G_0(t) + P_a(e^{it}) = c\,t^5 + O(t^6), \qquad c = \frac{iR(1)}{Q(1)}.$$

In exact rational arithmetic over $\mathbb{Q}(i)$, $\operatorname{Re}(R(1)/Q(1)) > 0$, so $\operatorname{Im} c > 0$ (numerically $c \approx 54.084 + 12.880\,i$). As P_a is real on the circle, $u_0' = -\operatorname{Im} G_0 = -\operatorname{Im}(c)\,t^5 + O(t^6)$: the function $u_0 = \log|H_0|$ has a strict local maximum at $z = 1$ with $k_\zeta = 5$. Also $\operatorname{Re} G_0(0) = -P_a(1) < 0$, so $f_0 = -\operatorname{Re} G_0$ is positive near 0; and $f_0 - P_a(e^{it}) = O(t^5)$, so $\Upsilon(f_0)(0) = \Upsilon(P_a)(1) = 0$ by Lemma 7.32(a).

The family. For $y = (y_0, \dots, y_5) \in \mathbb{C}^6$ put $H_y = H_0 + (z-b)\sum_{l=0}^{5} y_l (z-1)^l$. Then $H_y(b) = 0$, $\deg H_y = 14$, and $H_y'(b) \neq 0$ for small y. Let $G_y(t) = e^{it}H_y'(e^{it})/H_y(e^{it})$,

$$\operatorname{jet}(y) = \big(\operatorname{Im} G_y^{(j)}(0)\big)_{0 \leq j \leq 4} \in \mathbb{R}^5, \qquad \Upsilon_y = \Upsilon(-\operatorname{Re} G_y)(0).$$

Both are real-analytic near $y = 0$, $\operatorname{jet}(0) = 0$ since $u_0' = O(t^5)$, and $\Upsilon_0 = 0$.

The derivative is onto. The first-order change of $g = zH'/H$ is $\delta g = z\,(\delta H/H_0)'$, with $\delta H = (z-b)\sum_l \delta y_l (z-1)^l$. As $b \neq 1$, the derivatives of orders $0,\dots,5$ of δH at $z=1$ are arbitrary; as $H_0(1) \neq 0$, so are those of $\delta H/H_0$ (a triangular change); hence the derivatives of orders $0,\dots,4$ of δg at $z=1$ are arbitrary (again a triangular change), and so are the t-derivatives of orders $0,\dots,4$ of δG at 0, since $d/dt = iz\,d/dz$ at $z = e^{it}$. So $d\,\mathrm{jet}(0)$ maps onto $\mathbb{R}^5$, and there is a first-order change δy with $\delta G^{(j)}(0) = 0$ for $j \neq 2$ and $\delta G''(0) = \rho$ real and nonzero. It lies in the kernel of $d\,\mathrm{jet}(0)$, and along it only $f''(0)$ changes, so $\delta\Upsilon = -\rho/f_0(0)$, of either sign.

The point. By the implicit function theorem, $\mathrm{jet}^{-1}(0)$ is, near $y=0$, a real-analytic submanifold of $\mathbb{C}^6 = \mathbb{R}^{12}$ through 0 with tangent space $\ker d\,\mathrm{jet}(0)$. Since $\Upsilon_0 = 0$ and the derivative of $y \mapsto \Upsilon_y$ along this submanifold is nonzero at 0, there are $y \in \mathrm{jet}^{-1}(0)$ arbitrarily close to 0 with $\Upsilon_y < 0$. Fix such a y, close enough to 0. By Lemma 9.9, H_y is injective on $\{|z| \le r'\}$ for a fixed $r' \in (1,r)$. As $\mathrm{jet}(y) = 0$, $u_y' = -\operatorname{Im} G_y = O(t^5)$, and $\operatorname{Im} G_y^{(5)}(0)$ has the sign of $\operatorname{Im} G_0^{(5)}(0) = 5!\operatorname{Im} c > 0$; so $u_y = \log|H_y|$ has a strict local maximum at $z=1$ with $k_\zeta = 5$. Also $\operatorname{Re} G_y(0) < 0$ and $\Upsilon(-\operatorname{Re} G_y)(0) = \Upsilon_y < 0$. Finally $W = (H_y \circ \varphi)/N_b$ with $N_b = (1-|b|^2)H_y'(b)$ is in $\mathcal{S}$ and univalent on a neighbourhood of the closed disk, as in the proof of Lemma 8.5, and by Remark 7.33 the point $\zeta = \varphi^{-1}(1)$ has the stated properties. □

Proposition 9.10. *Let (W,B) be a pair in $\mathcal{L}_n$ with data (Q,a,b) such that H is univalent on a disk $\{|z| < r\}$ with $r > 1$. Let $\xi_1,\dots,\xi_p$ be the zeros of Q' on $\mathbb{T}$, of orders $m_1,\dots,m_p$, put $M = \sum_i m_i$ and $\Pi = (z-\xi_1)^{m_1}\cdots(z-\xi_p)^{m_p}$, and write $Q' = \Pi R$, so that R has no zero on the closed disk. Suppose that*

$$(\mathrm{R}_\mathcal{L}) \qquad \int_a^b \Pi\tilde{R}\,dt \neq 0 \quad \text{for some polynomial } \tilde{R} \text{ of degree at most } n-1-M.$$

Let $\hat{\xi} = (\hat{\xi}_1,\dots,\hat{\xi}_M)$ be close to the list ξ in which each ξ_i appears m_i times, with $|\hat{\xi}_j| \ge 1$ for all j. Then there is a polynomial $Q_{\hat{\xi}}$ of degree at most n with $Q_{\hat{\xi}}(a) = Q_{\hat{\xi}}(b) = 0$ and $Q_{\hat{\xi}}' = \prod_j (z-\hat{\xi}_j)\,R_{\hat{\xi}}$, where $R_{\hat{\xi}}$ has no zero on the closed disk, such that $Q_{\hat{\xi}} \to Q$ coefficientwise as $\hat{\xi} \to \xi$, and, for $\hat{\xi}$ close enough to ξ, $(Q_{\hat{\xi}},a,b)$ gives a pair $(W_{\hat{\xi}},B)$ in $\mathcal{L}_n$, with the same B and so the same gain $\sqrt{J(a,b)}$, whose $H_{\hat{\xi}}$ is univalent on $\{|z| < r'\}$ for a given $r' \in (1,r)$. If all $|\hat{\xi}_j| > 1$, then $(W_{\hat{\xi}},B)$ is a Blaschke model. If $p=1$ and $m_1 \le n-2$, then $(R_\mathcal{L})$ holds.

Proof. A polynomial P of degree at most $n-1$ is the derivative of a polynomial of degree at most n that vanishes at a and b if and only if $\int_a^b P = 0$, and that polynomial is then $\int_a^z P$. Put $\Pi_{\hat{\xi}} = \prod_j (z-\hat{\xi}_j)$ and, with $\tilde{R}$ from $(\mathrm{R}_\mathcal{L})$,

$$\hat{y}(\hat{\xi}) = -\frac{\int_a^b \Pi_{\hat{\xi}} R\,dt}{\int_a^b \Pi_{\hat{\xi}}\tilde{R}\,dt}, \qquad Q_{\hat{\xi}}(z) = \int_a^z \Pi_{\hat{\xi}}\big(R + \hat{y}(\hat{\xi})\tilde{R}\big)\,dt.$$

The denominator is continuous in $\hat{\xi}$ and not 0 at $\hat{\xi} = \xi$, and $\int_a^b \Pi R = \int_a^b Q' = 0$; so $\hat{y}$ is defined and continuous near ξ, with $\hat{y}(\xi) = 0$. Then $\deg Q_{\hat{\xi}} \le n$, $Q_{\hat{\xi}}(a) = Q_{\hat{\xi}}(b) = 0$, $Q_{\hat{\xi}}' = \Pi_{\hat{\xi}} R_{\hat{\xi}}$ with $R_{\hat{\xi}} = R + \hat{y}(\hat{\xi})\tilde{R}$, and $Q_{\hat{\xi}} \to Q$ coefficientwise. So $R_{\hat{\xi}} \to R$ uniformly on the closed disk, where $\min|R| > 0$, and $R_{\hat{\xi}}$ has no zero there for $\hat{\xi}$ close. The polynomial $H_{\hat{\xi}} = (1-\bar{a}z)Q_{\hat{\xi}}/(z-a)$ has $H_{\hat{\xi}}(b) = 0$ and tends to H coefficientwise, so in C^1 on $\{|z| \le r'\}$, and by Lemma 9.9 it is univalent on $\{|z| < r'\}$ for $\hat{\xi}$ close. As $|\hat{\xi}_j| \ge 1$, $Q_{\hat{\xi}}' \neq 0$ in $\mathbb{D}$, so $(Q_{\hat{\xi}},a,b)$ gives a pair in $\mathcal{L}_n$; its factor B, the multiple of $B_a \circ \varphi$ by the constant of modulus 1 with $B(0) > 0$, depends on a and b only. If all $|\hat{\xi}_j| > 1$, $Q_{\hat{\xi}}'$ has no zero on the closed disk, and Lemma 8.5 (with $k=1$) applies. For $p=1$ and $\Pi = (z-\xi)^m$, take $\tilde{R} = 1$ and $\tilde{R} = z-\xi$, of degree at most $n-1-m$ when $m \le n-2$. If both

integrals vanish, then $(b-\xi)^{m+1} = (a-\xi)^{m+1}$ and $(b-\xi)^{m+2} = (a-\xi)^{m+2}$; as $a-\xi$ and $b-\xi$ are not 0, dividing gives $b = a$, which is excluded. □

Corollary 9.11. *Let (a, b, Q) be the data of one of pairs VI, VII, VIII, IX, XI, XII, XVI and XVII, of degree $n = 18, 17, 16, 14, 12, 12, 6$ and 6. Then there are Blaschke models with one zero and gain exactly $\sqrt{J(a,b)}$, given by polynomials $Q_{\hat\xi}$ of degree n with $Q_{\hat\xi}(a) = Q_{\hat\xi}(b) = 0$.*

Proof. By Theorems 8.7 and 8.9, facts (R), (S) and (C) of Lemma 8.6 hold for these pairs, with $\Pi = (z-1)^m$, where $m = 3$ for pairs XVI and XVII and $m = 5$ for the others. So H is univalent on a disk $\{|z| < r\}$ with $r > 1$ (Lemma 8.6), and $Q' = (z-1)^m R$ with R zero-free on the closed disk: $p = 1$ and $m_1 = m \le n-2$. Proposition 9.10 with all $|\hat\xi_j| > 1$ gives the Blaschke models; their degree is that of Q, since $Q_{\hat\xi} \to Q$. □

Remark 9.12. (a) *Order of the zero.* Among the pairs in $\mathcal{L}_n$ with H univalent past the circle and one zero of Q' on $\mathbb{T}$, of order $m \le n-2$, every gain reached with order m is reached, at the same (a, b) and degree, with every lower order and by a Blaschke model. So in this class a zero of order 7 (which needs $n \ge 9$) gives no larger supremum than order 5. The statement does not cover pairs with boundary critical points or contact pairs. (b) *Same gain.* Each pair of Proposition 9.10 is a limit of Blaschke models in $\mathcal{L}_n$ with exactly its gain, whereas [23, Lemma 2.1] gives regular pairs with the smaller gain $t/B(0)$, $t < 1$. (c) *Local maximisers.* If a local maximiser over $\mathcal{L}_n$ had H univalent past the circle and ($\mathrm{R}_{\mathcal{L}}$), the Blaschke models of (b), arbitrarily close to it, have $n_H = 2 \le n+1$, so Theorem 9.1(L) would give pairs of larger gain arbitrarily close to it. So a local maximiser over $\mathcal{L}_n$ has $q + r \ge 1$, or ($\mathrm{R}_{\mathcal{L}}$) fails; for $p = 1$ the latter means $m_1 = n-1$. This complements Theorem 9.1, which needs neither $q = r = 0$ nor ($\mathrm{R}_{\mathcal{L}}$) and, when $q = r = 0$, excludes $M \le n-1-p$. (d) *Sharpness.* If $M = n-1$, then R is a constant, so every $\tilde R$ of degree 0 is a multiple of R, and $\int_a^b \Pi R = \int_a^b Q' = 0$; so ($\mathrm{R}_{\mathcal{L}}$) needs $M \le n-2$. It fails, for instance, for the pair of [23, Thm. 6.1], where $Q' = 4(z-1)^3$ and $n = 4$.

The same argument works with several zeros, for the pairs of Lemma 8.4; now k corrections are needed.

Proposition 9.13. *Let $k \ge 1$, and let Q, $b, a_1, \dots, a_k$ and H be as in Lemma 8.4, with $\deg Q \le n$ and with H univalent on a disk $\{|z| < r\}$ with $r > 1$. Let $\xi_1, \dots, \xi_p$ be the zeros of Q' on $\mathbb{T}$, of orders $m_1, \dots, m_p$, put $M = \sum_i m_i$ and $\Pi = (z-\xi_1)^{m_1} \cdots (z-\xi_p)^{m_p}$, and write $Q' = \Pi R$, so that R has no zero on the closed disk. Suppose that*

($\mathrm{R}^k_{\mathcal{L}}$) $\det\Big[\int_b^{a_i} \Pi \tilde R_j \, dt\Big]_{i,j=1}^k \ne 0$ *for some polynomials $\tilde R_1, \dots, \tilde R_k$ of degree at most $n-1-M$.*

Let $\hat\xi = (\hat\xi_1, \dots, \hat\xi_M)$ be close to the list ξ in which each ξ_i appears m_i times, with $|\hat\xi_j| \ge 1$ for all j. Then there is a polynomial $Q_{\hat\xi}$ of degree at most n with $Q_{\hat\xi}(b) = Q_{\hat\xi}(a_1) = \cdots = Q_{\hat\xi}(a_k) = 0$ and $Q'_{\hat\xi} = \prod_j (z - \hat\xi_j)\, R_{\hat\xi}$, where $R_{\hat\xi}$ has no zero on the closed disk, such that $Q_{\hat\xi} \to Q$ coefficientwise as $\hat\xi \to \xi$, and, for $\hat\xi$ close enough to ξ, Lemma 8.4 applied to $(Q_{\hat\xi}, b, a_1, \dots, a_k)$ gives an admissible pair $(W_{\hat\xi}, B)$, with the same B and so the same gain $\prod_j J(a_j, b)^{1/2}$, whose $H_{\hat\xi}$ is univalent on $\{|z| < r'\}$ for a given $r' \in (1, r)$. If all $|\hat\xi_j| > 1$, then $(W_{\hat\xi}, B)$ is a Blaschke model.

Proof. A polynomial P of degree at most $n-1$ is the derivative of a polynomial of degree at most n that vanishes at $b, a_1, \dots, a_k$ if and only if $\int_b^{a_i} P = 0$ for $i = 1, \dots, k$, and that polynomial is then $\int_b^z P$. Put $\Pi_{\hat\xi} = \prod_j (z - \hat\xi_j)$; let $\hat G(\hat\xi)$ be the $k \times k$ matrix with entries $\hat G_{ij}(\hat\xi) = \int_b^{a_i} \Pi_{\hat\xi} \tilde R_j \, dt$, with $\tilde R_j$ from ($\mathrm{R}^k_{\mathcal{L}}$), and $\hat g(\hat\xi)$ the vector with entries $\hat g_i(\hat\xi) = \int_b^{a_i} \Pi_{\hat\xi} R \, dt$. Both are continuous in $\hat\xi$, $\det \hat G(\xi) \ne 0$ by ($\mathrm{R}^k_{\mathcal{L}}$), and $\hat g_i(\xi) = \int_b^{a_i} Q' = Q(a_i) - Q(b) = 0$. So for $\hat\xi$ near ξ the vector

$\hat y(\hat\xi) = -\hat G(\hat\xi)^{-1}\hat g(\hat\xi)$ is defined and continuous, with $\hat y(\xi) = 0$. Put

$$R_{\hat\xi} = R + \sum_{j=1}^{k} \hat y_j(\hat\xi)\,\tilde R_j, \qquad Q_{\hat\xi}(z) = \int_b^z \Pi_{\hat\xi} R_{\hat\xi}\,dt.$$

Then $\deg Q_{\hat\xi} \le n$, since $\deg \Pi_{\hat\xi} R = \deg Q' \le n-1$ and $\deg \Pi_{\hat\xi}\tilde R_j \le n-1$; $Q_{\hat\xi}(b) = 0$; and $Q_{\hat\xi}(a_i) = \hat g_i(\hat\xi) + (\hat G(\hat\xi)\hat y(\hat\xi))_i = 0$ for each i. Also $Q'_{\hat\xi} = \Pi_{\hat\xi} R_{\hat\xi}$ and $Q_{\hat\xi} \to Q$ coefficientwise. So $R_{\hat\xi} \to R$ uniformly on the closed disk, where $\min|R| > 0$, and $R_{\hat\xi}$ has no zero there for $\hat\xi$ close. The polynomial $H_{\hat\xi} = Q_{\hat\xi}\prod_j (1-\bar a_j z)/(z-a_j)$ has $H_{\hat\xi}(b) = 0$; the division of $Q_{\hat\xi}$ by $\prod_j (z-a_j)$ is exact, and its quotient depends continuously on $Q_{\hat\xi}$, so $H_{\hat\xi} \to H$ coefficientwise, hence in C^1 on $\{|z| \le r'\}$, and by Lemma 9.9 $H_{\hat\xi}$ is univalent on $\{|z| < r'\}$ for $\hat\xi$ close. As $|\hat\xi_j| \ge 1$, $Q'_{\hat\xi} \ne 0$ in $\mathbb{D}$, so Lemma 8.4 applies to $(Q_{\hat\xi}, b, a_1, \dots, a_k)$; its factor B depends on $b, a_1, \dots, a_k$ only. If all $|\hat\xi_j| > 1$, $Q'_{\hat\xi}$ has no zero on the closed disk, and Lemma 8.5 applies. □

For $k = 1$ the condition $(\mathrm{R}^k_{\mathcal L})$ is $(\mathrm{R}_{\mathcal L})$, up to sign. For $k \ge 2$ the determinant is computed directly, as for pair I below.

Corollary 9.14. *Let b, a_1, a_2 and Q be the data of pair I, and let $J_{\mathrm I}$ be as in Theorem 8.8. Then there are Blaschke models with two zeros and gain exactly $\sqrt{J_{\mathrm I}}$, given by polynomials $Q_{\hat\xi}$ of degree 18 with $Q_{\hat\xi}(b) = Q_{\hat\xi}(a_1) = Q_{\hat\xi}(a_2) = 0$.*

Proof. By the proof of Theorem 8.8, facts (R), (S) and (C) of Lemma 8.6 hold, with $\Pi = (z-1)^5$. So H is univalent on a disk $\{|z| < r\}$ with $r > 1$ (Lemma 8.6), and $Q' = (z-1)^5 R$ with R zero-free on the closed disk: $p = 1$, $\xi_1 = 1$, $M = m_1 = 5$ and $n = 18$. Take $\tilde R_1 = 1$ and $\tilde R_2 = z - 1$, of degree at most $n - 1 - M = 12$. Then

$$\int_b^{a_i} (t-1)^5 \tilde R_j(t)\,dt = \frac{(a_i-1)^{5+j} - (b-1)^{5+j}}{5+j} \qquad (i, j = 1, 2),$$

and the determinant of this 2×2 matrix, computed in exact arithmetic over $\mathbb{Q}(i)$, is $-0.7186277\ldots + 0.6643654\ldots\, i$. It is not 0, so $(\mathrm{R}^k_{\mathcal L})$ holds with $k = 2$. Proposition 9.13 with all $|\hat\xi_j| > 1$ gives the Blaschke models; their degree is that of Q, since $Q_{\hat\xi} \to Q$. □

It is open whether the supremum of the gain over $\mathcal{L}_n$ is attained, and how its maximisers behave as $n \to \infty$. By Theorem 9.1 and Corollary 9.2, if local maximisers exist for all n and $\sum_i (m_i + 1)$ stays bounded, their numbers $q_n + r_n$ of defects grow at least linearly in n, while by Lemmas 7.12 and 7.17 a single W meromorphic past the circle has finitely many defects unless $W(\mathbb{D})$ is a slit plane.

10. Adding zeros near the circle

This section is about adding zeros of B near the circle while W is kept. We show that a critical point of WB on the circle can block this (Proposition 10.1); for $W(w) \not\equiv w$ meromorphic past the circle we count exactly the zeros of $(WB)'$ in the disk when the new zeros lie on fixed radii (Propositions 10.2 and 10.4); and we show that in the polynomial family of [23, Lemma 6.2] the number of zeros of B is locally constant (Proposition 10.3).

We first recall the local effect of one new zero of B near the circle, with W kept. Let (W, B) be a Blaschke model, $F = WB$, ξ_0 a point of the circle and $a = (1-s)\xi_0$. Put $\chi = wF'/F$, and on the circle $u = \log|W|$, with derivatives in θ. As $|B| = 1$ on the circle, $\frac{d}{d\theta}\log F(e^{i\theta}) = i\chi$, so $\chi = \frac{d}{d\theta}\arg F - iu'$ there; in particular $\chi(\xi_0)$ is real if $u'(\xi_0) = 0$. By [23, Lemma 8.1, Rem. 8.3, Prop. 8.8], if $u' = 0$, $u'' > 0$ and $\chi < 0$ at ξ_0, then (W, BB_a) is a Blaschke model for all small $s > 0$,

and its gain is larger by the factor $1/(1-s)$. By Proposition 10.2(i) below, with $M=1$, $\zeta_1=\xi_0$ and $r=1-s$, the derivative $(WBB_a)'$ has a zero in $\mathbb{D}$ for all small s if $u'(\xi_0)\neq 0$, or if $u'(\xi_0)=0$ and either $\chi(\xi_0)>0$ or $u''(\xi_0)<0$.

A critical point of F on the circle can block this route. For $a\in\mathbb{D}\setminus\{0\}$ put $\widetilde{B}_a=-\frac{|a|}{a}B_a$, with B_a as in Section 2, so that $\widetilde{B}_a(0)=|a|$.

Proposition 10.1. *Let (W,B) be admissible, and let ζ_0 be a point of the circle such that $F=WB$ extends holomorphically to a neighbourhood of ζ_0, with $F'(w)=C(w-\zeta_0)^m(1+O(w-\zeta_0))$, $m\geq 1$, $C\neq 0$. Then $F(\zeta_0)\neq 0$; put $c_{\zeta_0}=(i\zeta_0)^m\zeta_0 C/F(\zeta_0)$, so that $\chi(\zeta_0 e^{it})=c_{\zeta_0}t^m+O(t^{m+1})$ for $\chi=wF'/F$, as in Section 7.7 (for $m=1$, $\operatorname{Im}c_{\zeta_0}=-u''(\theta_0)$ with $\zeta_0=e^{i\theta_0}$ and $u=\log|W|$ on the circle).*

(a) *For all small $s>0$ the pair $(W,B\widetilde{B}_a)$ with $a=(1-s)\zeta_0$ is not admissible.*

(b) *Let $\eta_0>0$, and let $m\geq 3$, or $m=2$ and $c_{\zeta_0}\notin(-\infty,0)$, or $m=1$ and $\operatorname{Im}c_{\zeta_0}>0$. There is $s_0>0$ such that $(W,B\widetilde{B}_a)$ with $a=(1-s)\xi_0$ is not admissible whenever $|\xi_0|=1$, $|\xi_0-\zeta_0|\geq\eta_0$ and $0<s<s_0$.*

Proof. Since $B\neq 0$ near the circle, $W=F/B$ extends holomorphically near ζ_0. If $W(\zeta_0)=0$, the points $w_n=(1-1/n)\zeta_0$ satisfy $W(w_n)\to 0=W(0)$, an interior point of $W(\mathbb{D})$, so $w_n\to W^{-1}(0)=0$ by continuity of W^{-1}, which is absurd; hence $F(\zeta_0)\neq 0$. On $\mathbb{D}\setminus\{a\}$ we have $(F\widetilde{B}_a)'=\widetilde{B}_a\,G_a$ with $G_a=F'+\Gamma_a$ and $\Gamma_a(w)=(1-|a|^2)F(w)/((w-a)(1-\bar{a}w))$, so a zero of G_a in $\mathbb{D}\setminus\{a\}$ is a zero of $(F\widetilde{B}_a)'$ in $\mathbb{D}$.

(b) Let $|\xi_0-\zeta_0|\geq\eta_0$ and $s\leq\eta_0/8$. On a closed disk U' about ζ_0 of radius $r_1\leq\eta_0/4$ on which F is holomorphic, $|(w-a)(1-\bar{a}w)|\geq(\eta_0/2)^2$, so $\Gamma_a(\zeta_0+\hat{h})=\lambda F(\zeta_0)/\zeta_0+O(s|\hat{h}|)$, where $\lambda=(1-|a|^2)/|\zeta_0-a|^2\in[s/4,\,2s/(\eta_0-s)^2]$, and the constant depends only on F and η_0. Put $\rho=\lambda^{1/m}\rho_1$ with $\rho_1^m=-F(\zeta_0)/(\zeta_0 C)$. Then $G_a(\zeta_0+\rho t)=(\lambda F(\zeta_0)/\zeta_0)(1-t^m+E(t))$ with $E(t)=O(|\rho|)$ uniformly on $|t|\leq 2$. Choose j and $q>0$ such that $\operatorname{Re}(\bar{\zeta}_0\rho e^{2\pi ij/m})\leq -q|\rho|$. This is possible: for $m\geq 3$ with $q=1/2$, because some m-th root of unity lies within π/m of any direction; for $m=2$ unless $\bar{\zeta}_0\rho_1$ is purely imaginary, which happens exactly when $c_{\zeta_0}<0$; for $m=1$ because $\operatorname{Re}(\bar{\zeta}_0\rho)=-\lambda\operatorname{Im}c_{\zeta_0}/|c_{\zeta_0}|^2$. The direction of ρ depends neither on ξ_0 nor on s, so j and q do not either. Fix $r_0>0$ smaller than half the distance between neighbouring m-th roots of unity and than $q/2$. By Rouché's theorem, for $|\rho|$ small there is a zero t' with $|t'-e^{2\pi ij/m}|<r_0$. Since $|\zeta_0+\rho t'|^2=1+2\operatorname{Re}(\bar{\zeta}_0\rho t')+|\rho t'|^2$ and $r_0<q/2$, the point $\zeta_0+\rho t'$ lies in $\mathbb{D}$ when $|\rho|$ is small; it is then far from a, which lies near ξ_0. All constants depend only on F, η_0 and m.

(a) Now $a=(1-s)\zeta_0$. With $w=\zeta_0+\hat{h}$ one has exactly $(w-a)(1-\bar{a}w)=-(1-s)\bar{\zeta}_0\hat{h}^2+s^2\hat{h}+s^2\zeta_0$, so $N_a:=(w-a)(1-\bar{a}w)\,G_a=-C\bar{\zeta}_0\hat{h}^{m+2}(1+O(\hat{h}))+2sF(\zeta_0)+O(s|\hat{h}|+s^2)$. With $\hat{h}=\rho t$ and $\rho^{m+2}=2s\zeta_0F(\zeta_0)/C$ this is $2sF(\zeta_0)(1-t^{m+2}+o(1))$ on $|t|\leq 2$. As in (b), since $m+2\geq 3$, some $(m+2)$-th root of unity ω has $\operatorname{Re}(\bar{\zeta}_0\rho\,\omega)\leq-|\rho|/2$, and by Rouché's theorem N_a has a zero $\zeta_0+\rho t'$ with t' near ω; as in (b) it lies in $\mathbb{D}$, at distance of order $s^{1/(m+2)}$ from ζ_0, hence not at a or $1/\bar{a}$. □

The splitting of a multiple zero under a small perturbation is standard; the point of Proposition 10.1 is that the perturbation has a direction that depends only on F near ζ_0. Case (b) is not covered by [23, Lemma 8.1, Prop. 8.8], which treat regular pairs. For every boundary pair of Table 2, $(WB)'$ has a zero of order 5 (pairs I–XII) or 3 (pairs XVI and XVII) on the circle (Section 8), so a new zero near any fixed boundary point, or on the radius through that zero, destroys admissibility when W is kept. The proposition does not cover the case that ξ_0 tends to ζ_0 together with s, the route of [23, Lemma 2.1], which first replaces W, or joint variations of W and B.

For zeros added on fixed radii, Theorem 7.26 gives an exact count. Let $W(w)\not\equiv w$, P, u and B (of degree d) be as in Section 7.7, with $F=WB$, $\chi=wF'/F$, the sets Z_χ and the indices ι_ζ there, and let E be the set of zeros of u' on $\mathbb{T}\setminus P$, a finite set. A point $x\in E$ is *χ-negative* if $\chi(x)<0$, with *χ-depth* $\eta(x)=-\chi(x)$, and *χ-positive* if $\chi(x)>0$; the other points of E are the zeros of F'

on $\mathbb{T}\setminus P$, and at each of them $\iota\ge 0$ by the definition of the index. A *χ-negative maximum* is a χ-negative point at which u has a strict local maximum, and similarly for a *χ-negative minimum*. For distinct points $\zeta_1,\dots,\zeta_M$ of $\mathbb{T}\setminus P$ and $0<r<1$ put $C_r=\prod_j \widetilde{B}_{r\zeta_j}$, so that $C_r(0)=r^M$, $F_r=FC_r$ and $\sigma_r = wC_r'/C_r$. On the circle σ_r is real and positive; since $|x-r\zeta|^2=(1-r)^2+r|x-\zeta|^2$ for $|x|=|\zeta|=1$,

$$\sigma_r(x)=\sum_{j=1}^{M}\frac{1-r^2}{(1-r)^2+r|x-\zeta_j|^2}\ \ge\ M\,\frac{1-r}{1+r},\qquad \sigma_r(\zeta_j)\ge\frac{1+r}{1-r}, \tag{2}$$

and $\sigma_r(x)\to 0$ as $r\to 1$ if x is none of the ζ_j. So $\chi_r := wF_r'/F_r = \chi+\sigma_r$ has the imaginary part $-u'$ on the circle, and u does not depend on r.

Proposition 10.2. *With this notation:*

(i) *For all $r<1$ close enough to 1, F_r' has no zero on $\mathbb{T}\setminus P$, and*

$$N_{\mathbb{D}}(F_r') = N_{\mathbb{D}}(F') + \sum_{j=1}^{M}\bigl(1+\iota_j^*\bigr) + \sum_{\zeta}\iota_\zeta,$$

where $\iota_j^*=\iota_{\zeta_j}$ if $\zeta_j\in Z_\chi$ and $\iota_j^*=0$ otherwise, and the last sum is over the zeros ζ of F' on $\mathbb{T}\setminus P$ that are none of the ζ_j. Every term is non-negative. Hence (W,BC_r) is admissible for all r close to 1 if and only if (W,B) is admissible, every ζ_j is a χ-negative minimum, and every zero of F' on $\mathbb{T}\setminus P$ has index 0; its gain is then $r^{-M}/B(0)$.

(ii) *For every $r\in(0,1)$ such that $\chi(x)+\sigma_r(x)\ne 0$ for all $x\in E$,*

$$N_{\mathbb{D}}(F_r') = d+M-\sum\bigl\{\varepsilon_x:\ x\in E,\ \chi(x)+\sigma_r(x)<0\bigr\}. \tag{3}$$

(iii) *Let $M\ge 1$, suppose that the conditions of (i) hold, and let r^* be the largest $r\in(0,1)$ at which $\chi(x)+\sigma_r(x)=0$ for a χ-negative maximum x or for $x=\zeta_j$. Then r^* exists, and (W,BC_r) is admissible for every $r\in(r^*,1)$. If exactly one point of E has $\chi+\sigma_{r^*}=0$, with a change of sign, then $N_{\mathbb{D}}(F_r')=1$ for $r<r^*$ close to r^*.*

(iv) *In the setting of (iii), at least $d+M$ χ-negative maxima are none of the ζ_j, and $1-r^*<2\eta(x)/M$ for each of them with $2\eta(x)/M<1$. If $M=1$ and $\eta(\zeta_1)\le 1$, then $1-r^*$ is the least root in $(0,1)$, over the χ-negative maxima x, of*

$$f_x(Y)=(1+\eta)Y^2-(2+\eta D^2)Y+\eta D^2,\qquad \eta=\eta(x),\ D=|x-\zeta_1|;$$

here $f_x(1-r)=\bigl((1-r)^2+rD^2\bigr)\bigl(\eta(x)-\sigma_r(x)\bigr)$, which has the sign of $\eta(x)-\sigma_r(x)$. If $\eta(x)<1$, then f_x has exactly one root in $(0,1)$, and it is $\eta D^2(1+O(\eta))/2$.

Proof. (i) By Theorem 7.26(i), which holds for every finite Blaschke product, $N_{\mathbb{D}}(F')=d-\sum_{Z_\chi}\iota$ and $N_{\mathbb{D}}(F_r')=d+M-\sum_{Z_{\chi_r}}\iota$. As E is finite, by (2) we may take r so close to 1 that $\sigma_r(x)<\eta(x)$ at every χ-negative point x that is none of the ζ_j, and $\sigma_r(\zeta_j)>|\chi(\zeta_j)|$ for every j. Outside E, $\operatorname{Im}\chi_r=-u'\ne 0$. At a point $x\in E$ that is none of the ζ_j: if $\chi(x)\ge 0$, then $\chi_r(x)\ge\sigma_r(x)>0$; if $\chi(x)<0$, then $\chi_r(x)<0$, so $x\in Z_{\chi_r}$ with $m_x=0$ and index ε_x, its index for χ as well. At the ζ_j in E, $\chi_r>0$. So χ_r has no zero on $\mathbb{T}\setminus P$, nor has F_r', and Z_{χ_r} is the set of χ-negative points other than the ζ_j, with the same indices. Subtracting, $N_{\mathbb{D}}(F_r')-N_{\mathbb{D}}(F')$ is M plus the sum of ι over the points of Z_χ outside Z_{χ_r}, that is, over the ζ_j in Z_χ and over the zeros of F' on the circle that are none of the ζ_j: this is the formula. The index is at least 0 at a zero of F' and is $\varepsilon\ge -1$ at a χ-negative point, so every term is non-negative, and $N_{\mathbb{D}}(F_r')=0$ if and only if $N_{\mathbb{D}}(F')=0$ and every term vanishes. The term $1+\iota_j^*$ vanishes if and only if $\iota_j^*=-1$, that is, ζ_j is a χ-negative point with $\varepsilon=-1$; then no ζ_j is a zero of F', and the last sum vanishes if and only if all zeros of F' on the circle have index 0. Finally W is unchanged and $BC_r(0)=r^MB(0)>0$.

(ii) If $\chi+\sigma_r\neq 0$ on E, then χ_r has no zero on $\mathbb{T}\setminus P$, the points of Z_{χ_r} have $m=0$ and index ε, and Theorem 7.26(i) gives (3).

(iii) Let O be the set of χ-negative points other than the ζ_j. By (i) and (3) for r near 1, $\sum_{x\in O}\varepsilon_x = d+M$; so at least $d+M$ points of O are χ-negative maxima. The set of r in the definition of r^* is closed in $(0,1)$ and, by (2), avoids a neighbourhood of 1. Let $\tilde r<1$ be its largest element, or $\tilde r=0$ if it is empty. As r decreases from 1 to $\tilde r$, the points with $\chi\geq 0$ stay χ_r-positive, as $\chi+\sigma_r\geq\sigma_r>0$ there; by the choice of $\tilde r$, the ζ_j stay χ_r-positive and the χ-negative maxima in O stay χ_r-negative; so only χ-negative minima and points with $\varepsilon=0$ can leave the set of χ_r-negative points. No χ-negative minimum leaves: otherwise let $r_y\in(\tilde r,1)$ be the largest r at which a χ-negative minimum $y\in O$ has $\chi(y)+\sigma_r(y)=0$. At r_y the χ_r-negative points are those of O less y and less some points with $\varepsilon\leq 0$, so their ε add up to at least $d+M+1$, and the zeros of F'_{r_y} on the circle have index at least 0; Theorem 7.26(i) gives $N_{\mathbb{D}}(F'_{r_y})\leq d+M-(d+M+1)<0$, which is absurd. So on $(\tilde r,1)$ the ε of the χ_r-negative points add up to $d+M$: $N_{\mathbb{D}}(F'_r)=0$ where $\chi+\sigma_r\neq 0$ on E, by (3), and at the other r the zeros of χ_r on E have $\varepsilon=0$, so Theorem 7.26(i) gives $N_{\mathbb{D}}(F'_r)=-\sum\iota\leq 0$, hence 0 again. So (W,BC_r) is admissible on $(\tilde r,1)$. As $r\to 0$, C_r tends locally uniformly to a unimodular constant times w^M, and $(w^MF)'$ has a zero of order M at 0, since $F(0)=0\neq F'(0)$; by Hurwitz's theorem $N_{\mathbb{D}}(F'_r)\geq M\geq 1$ for small r. So $\tilde r>0$, and $r^*=\tilde r$ exists. If exactly one point x has $\chi+\sigma_{r^*}=0$ at r^*, with a change of sign, then just below r^* the set of χ_r-negative points loses the χ-negative maximum x or gains the point $x=\zeta_j$, of $\varepsilon=-1$, and (3) gives $N_{\mathbb{D}}=1$.

(iv) The proof of (iii) shows that at least $d+M$ points of O are χ-negative maxima. If $x\in O$ is a χ-negative maximum with $2\eta(x)/M<1$, then at $1-r=2\eta(x)/M$, by (2), $\sigma_r(x)\geq M(1-r)/(1+r)>M(1-r)/2=\eta(x)$, while $\sigma_r(x)<\eta(x)$ for r near 1; so $\chi(x)+\sigma_r(x)=0$ for some r with $1-r<2\eta(x)/M$, and $r^*>1-2\eta(x)/M$. For $M=1$ put $Y=1-r$; then $1-r^2=Y(2-Y)$ and $(1-r)^2+rD^2=Y^2+(1-Y)D^2$, so $f_x(Y)=\big(Y^2+(1-Y)D^2\big)\big(\eta(x)-\sigma_r(x)\big)$ has the sign of $\eta(x)-\sigma_r(x)$. As $\sigma_r(\zeta_1)\geq(1+r)/(1-r)>1\geq\eta(\zeta_1)$, the point ζ_1 gives no r in the definition of r^*, and $1-r^*$ is the least root in $(0,1)$ of the f_x. If $\eta<1$, then $f_x(0)=\eta D^2>0>\eta-1=f_x(1)$, and f_x is a quadratic with positive leading coefficient, so it has exactly one root in $(0,1)$; the expansion of the smaller root is $\eta D^2(1+O(\eta))/2$. □

An example: one added zero. Let (W,B) be admissible with $\deg B=d$, let F' have no zero on $\mathbb{T}\setminus P$, and add one zero ($M=1$) on the radius of a χ-negative minimum ζ_1 with $\eta(\zeta_1)\leq 1$. In (i), $\iota_1^*=\iota_{\zeta_1}=\varepsilon_{\zeta_1}=-1$ and the last sum is empty, so $N_{\mathbb{D}}(F'_r)=0+(1-1)=0$: for r close to 1 the pair (W,BC_r) is admissible, with gain $r^{-1}/B(0)$. (At a χ-negative maximum the term would be $1+1=2$, and F'_r would have two zeros in the disk.) By the proof of (iii), the ε_x of the χ-negative points $x\neq\zeta_1$ add up to $d+1$, so at least $d+1$ of them are χ-negative maxima. At such a point $\chi(x)+\sigma_r(x)=0$ means $\sigma_r(x)=\eta(x)$, and ζ_1 gives no such r, as $\sigma_r(\zeta_1)>1\geq\eta(\zeta_1)$. So r^* is the largest $r<1$ at which $\sigma_r(x)=\eta(x)$ for one of these maxima, and by (iv) $1-r^*$ is the least root in $(0,1)$ of their quadratics f_x.

Proposition 10.2(i) is the exact form of Proposition 10.1 for zeros added on fixed radii: with W kept, a zero ζ_0 of F' on the circle that is none of the ζ_j blocks the construction if and only if $\iota_{\zeta_0}>0$, and it adds exactly ι_{ζ_0} zeros of $(WBC_r)'$ in the disk; if ζ_0 is one of the ζ_j, its term in (i) is $1+\iota_{\zeta_0}\geq 1$, so it always blocks, as in Proposition 10.1(a). Among the cases left open there, $m=1$ with $\operatorname{Im} c_{\zeta_0}<0$ (a minimum of u) has index 0, and $m=2$ with $c_{\zeta_0}<0$ has index $1+\varepsilon_{\zeta_0}$, so it blocks unless u has a minimum at ζ_0. Proposition 10.1(b) is uniform in the added point, which (i) does not give. For the two pairs of Remark 7.30, whose zero of order 5 has index 3 ($m=5$ by (C) of Lemma 8.6; u has a strict maximum with $k_\zeta=5$ by Section 7.8), every choice of zeros added on fixed radii near the circle, with W kept, leaves at least 3 zeros of $(WBC_r)'$ in the disk. For regular pairs the "if" of (i) with $M=1$ at a nondegenerate minimum ($u''(\zeta_1)>0$) is [23, Prop. 8.8]; (i) allows minima of any odd order. The lower bound in (2) is the mechanism of [23, Thm. 8.7(a)].

By (iv), a pair whose χ-negative maxima are shallow has little room: for one added zero, $1-r^*$ is about the least of the numbers $\eta(x)D^2/2$.

Let $\mathcal{Q}_n$ be the monic polynomials of degree n, with the norm $\|Q\|$ given by the largest modulus of a coefficient. For $Q\in\mathcal{Q}_n$ and a zero $b\in\mathbb{D}$ of Q let $A(Q,b)$ be the multiset of zeros of Q in $\mathbb{D}$ with one copy of b removed, and $H_{Q,b}=Q\prod_{a\in A(Q,b)}(1-\bar a z)/(z-a)$, a polynomial with $H_{Q,b}(b)=0$. Let U_n be the set of such (Q,b) with $H_{Q,b}$ univalent in $\mathbb{D}$. The pairs of [23, Lemma 6.2], and their analogues with k zeros of B, are built from points of U_n with $|A(Q,b)|=k$. In this family the number of zeros of B cannot change along a continuous path in U_n (degree n fixed).

Proposition 10.3. (a) *For $(Q,b)\in U_n$, $\min_{|z|=1}|Q(z)|\ge(1-|b|^2)|H'_{Q,b}(b)|/4$, and this is at least $(1-|b|^2)|Q'(b)|/4$ if b is a simple zero of Q.*

(b) *$|A(Q,b)|$ is locally constant on U_n, and U_n and each of the sets $\{|A(Q,b)|=k\}$ are closed in $\mathcal{Q}_n\times\mathbb{D}$.*

(c) *Let $b_0\in\mathbb{D}$ be a simple zero of $Q_0\in\mathcal{Q}_n$, $m_0=(1-|b_0|)|Q_0'(b_0)|$ and $\|Q_0\|_1$ the sum of the moduli of the coefficients of Q_0. If $0<\delta_1\le(1-|b_0|)/2$, $n(n+1)\delta/2+2n(n+1)\|Q_0\|_1\delta_1\le m_0/2$ and $(n+1)\delta<(1-|b_0|)m_0/16$, then no $(Q,b)\in U_{n+1}$ has $|b-b_0|<\delta_1$ and $\|Q-Q_0\,(z-\xi)\|<\delta$ for some $|\xi|=1$.*

Proof. A univalent H that is continuous on the closed disk takes no value of $H(\mathbb{D})$ on the circle (if $H(\xi)=H(z_0)$ with $|\xi|=1$, then $H((1-1/m)\xi)\to H(z_0)\in H(\mathbb{D})$, so $(1-1/m)\xi\to z_0$ by continuity of H^{-1}, although $(1-1/m)\xi\to\xi\ne z_0$). (a) Each factor $(1-\bar a z)/(z-a)$ has modulus 1 on the circle, so $|H_{Q,b}|=|Q|$ there. With $\varphi(w)=(w+b)/(1+\bar b w)$, Koebe's one-quarter theorem [16, Ch. 2] applied to $H_{Q,b}\circ\varphi$ shows that $H_{Q,b}(\mathbb{D})$ contains the disk of radius $(1-|b|^2)|H'_{Q,b}(b)|/4$ about 0, which the values on the circle avoid. If b is simple, $H'_{Q,b}(b)=Q'(b)\prod_a(1-\bar a b)/(b-a)$ and $|1-\bar a b|^2-|b-a|^2=(1-|a|^2)(1-|b|^2)>0$. (b) By (a), Q has no zero on the circle, so Rouché's theorem keeps the number of zeros in $\mathbb{D}$ fixed on the ball $\|\widetilde Q-Q\|<\min_{|z|=1}|Q|/n$. For closedness, let $(Q_m,b_m)\in U_n$ converge to $(Q_\infty,b_\infty)\in\mathcal{Q}_n\times\mathbb{D}$, pass to a subsequence along which $A(Q_m,b_m)=\{a_{1,m},\dots,a_{k,m}\}$ (with k fixed) converges to $\{\alpha_1,\dots,\alpha_k\}$, and write $Q_m=\prod_j(z-a_{j,m})\,P_m$. Monic division is continuous and the remainders vanish, so $P_m\to P_\infty=Q_\infty/\prod_j(z-\alpha_j)$, a monic polynomial of degree $n-k\ge1$. Then $H_{Q_m,b_m}=P_m\prod_j(1-\bar a_{j,m}z)\to H_\infty=P_\infty\prod_j(1-\bar\alpha_j z)$ uniformly on the closed disk; H_∞ is not constant (P_∞ is monic of degree at least 1 and the product has constant term 1), so it is univalent by Hurwitz's theorem (if H_∞ took some value at two points, Rouché's theorem on two small disjoint circles would make H_{Q_m,b_m} take it twice), and $H_\infty(b_\infty)=0$. Hence H_∞ has no zero on the circle. So no α_j lies on the circle, as $H_\infty(\alpha_j)$ would vanish through the factor $1-|\alpha_j|^2$, and P_∞ has no zero there; P_∞ has exactly one zero in $\mathbb{D}$, namely b_∞, so $A(Q_\infty,b_\infty)=\{\alpha_j\}$ and $(Q_\infty,b_\infty)\in U_n$. (c) Put $Q_\xi=Q_0\,(z-\xi)$; then $|Q'_\xi(b_0)|\ge m_0$, $|Q''_\xi|\le 2n(n+1)\|Q_0\|_1$, $|Q-Q_\xi|<(n+1)\delta$ and $|Q'-Q'_\xi|<n(n+1)\delta/2$ on the closed disk. So $|Q'(b)|\ge m_0/2$, b is a simple zero of Q, $1-|b|^2\ge(1-|b_0|)/2$, and (a) gives $\min_{|z|=1}|Q|\ge(1-|b_0|)m_0/16>(n+1)\delta>|Q(\xi)-Q_\xi(\xi)|=|Q(\xi)|$, which is absurd. □

So a pair of degree $n+1$ with two zeros cannot be reached from a one-zero pair of degree n by adding a zero of Q near the circle while b stays near its old place, and pairs with k zeros of fixed degree form a union of components of U_n. This is in line with [23, Rem. 8.11], where a zero of B may escape to the circle in a limit: inside the polynomial family this cannot happen while b stays in a compact subset of $\mathbb{D}$. The restriction to b in a compact subset of $\mathbb{D}$ is needed: $(z^n(z-b),b)\in U_{n+1}$ for every $b\in\mathbb{D}$, with $H=z-b$.

For pairs of the type of [23, Lemma 8.1], with all zeros of B near the circle, there is an exact count of the zeros of $(WB)'$. Let W, P, u, g and Z_- be as in Section 7.6, with $W(w)\not\equiv w$, so that u is constant on no arc; let Crit be the set of zeros of W' on $\mathbb{T}\setminus P$. By Lemma 7.12(b), which holds on $\mathbb{T}\setminus P$ (proof of Lemma 7.17), these zeros are simple, so g has a simple zero at each $c\in$ Crit. A zero of u' is a *maximum*, a *minimum* or an *inflection* of u if u has a strict local maximum there,

a strict local minimum, or neither. A point of Z_- that is a minimum of u is a *downward crossing*, and one that is a maximum is an *upward crossing*. For distinct $\zeta_1, \dots, \zeta_M \in \mathbb{T} \setminus (P \cup \mathrm{Crit})$ and $0 < r < 1$ let C_r be the Blaschke product with zeros $r\zeta_j$ and $C_r(0) = r^M$.

Proposition 10.4. *Suppose that no point of* Crit *is an inflection of u. Then for all $r < 1$ close enough to 1, $(WC_r)'$ has no zero on $\mathbb{T} \setminus P$, and the number of its zeros in $\mathbb{D}$ is*

$$\#\{c \in \mathrm{Crit} : \ c \text{ is a maximum of } u\} + \sum_{j=1}^{M} \hat{\nu}_j, \qquad \hat{\nu}_j = \begin{cases} 0 & \text{if } \zeta_j \text{ is a downward crossing,} \\ 2 & \text{if } \zeta_j \text{ is an upward crossing,} \\ 1 & \text{otherwise.} \end{cases}$$

In particular (W, C_r) is admissible for all r close to 1 if and only if no point of Crit *is a maximum of u and every ζ_j is a downward crossing.*

Proof. Let $F_r = WC_r$, $\sigma_r = wC_r'/C_r$, real and positive on $\mathbb{T}$, and $\chi_r = g + \sigma_r = wF_r'/F_r$. In the closed disk minus P, F_r' and χ_r have the same zeros, since F_r vanishes only at 0 and at the $r\zeta_j$, where $F_r' \neq 0$. Put $s = 1 - r$. Fix $\rho_0 > 0$ such that the closed disks Δ_x of radius ρ_0 about the points x of $P \cup \mathrm{Crit} \cup \{\zeta_j\}$ are disjoint and lie in U, g is injective on each Δ_c, $c \in \mathrm{Crit}$, and on $\mathbb{T} \cap \Delta_x$ the function u' vanishes at most at x for $x \in \mathrm{Crit} \cup \{\zeta_j\}$; moreover $\operatorname{Im} g \neq 0$ on $\mathbb{T} \cap \Delta_j$ if $g(\zeta_j) \notin \mathbb{R}$, and $\operatorname{Re} g > 0$ there if $g(\zeta_j) > 0$. On the closed disk outside Δ_j, $|w - r\zeta_j| \geq \rho_0/2$ and $|1 - r\bar{\zeta}_j w| \geq |w - r\zeta_j|$, so $|\sigma_r| \leq Cs$ there.

Away from the disks. On the closed disk minus the open disks Δ_x, g is continuous and zero-free, so $|g|$ is bounded below, and $\chi_r \neq 0$ for small s. *Poles.* On Δ_p minus p, $g = -mp/(w-p) + O(1)$, so $|g| \geq 1$ for ρ_0 small, and $\chi_r \neq 0$.

Critical points. Let $c \in \mathrm{Crit}$. On $\partial\Delta_c$, $|\sigma_r| \leq Cs < \min|g|$, so by Rouché's theorem χ_r has exactly one zero w_r in Δ_c. On the arc $A = \mathbb{T} \cap \Delta_c$, $\operatorname{Im} \chi_r = -u'$ vanishes only at c, where $\chi_r(c) = \sigma_r(c) > 0$; so $w_r \notin \mathbb{T}$, and $w_r \in \mathbb{D}$ if and only if χ_r winds once around 0 along the boundary of $\Delta_c \cap \mathbb{D}$, traversed as A followed by the inner arc. On the inner arc $\chi_r = g + O(s)$, and g is conformal at c with $g(c) = 0$, so there the argument of χ_r increases by $\pi + O(\rho_0) + o(1)$. If c is a maximum of u, then $u' > 0$ before c and $u' < 0$ after it, so on A the curve χ_r runs from the lower half-plane to the upper one through a positive point; its argument increases by an amount in $(0, 2\pi)$ which, as the end points are $\mp g'(c) ic\rho_0 + O(\rho_0^2) + o(1)$, is $\pi + O(\rho_0) + o(1)$. The total is 2π, and $w_r \in \mathbb{D}$. If c is a minimum, the half-planes are exchanged, the change on A is $-\pi + O(\rho_0) + o(1)$, the total is 0, and w_r lies outside the closed disk.

The zeros of C_r. In $\Delta_j \cap \mathbb{D}$, χ_r has exactly one pole, at $r\zeta_j$, as $1/(r\bar{\zeta}_j)$ lies outside. On $A_j = \mathbb{T} \cap \Delta_j$, χ_r has no zero: if $g(\zeta_j) \notin \mathbb{R}$, then $\operatorname{Im} \chi_r = \operatorname{Im} g \neq 0$; if $g(\zeta_j) > 0$, then $\operatorname{Re} \chi_r > \operatorname{Re} g > 0$; and if $g(\zeta_j) < 0$, then $\operatorname{Im} \chi_r = -u'$ vanishes only at ζ_j, where $\chi_r = g(\zeta_j) + (1+r)/(1-r) + O(s) > 0$. On the inner arc $\chi_r = g + O(s)$, and g has no zero or pole in $\Delta_j \cap \overline{\mathbb{D}}$, so the number of zeros of χ_r in $\Delta_j \cap \mathbb{D}$ is

$$1 + \frac{1}{2\pi}\big(\Delta_{A_j} \arg \chi_r - \Delta_{A_j} \arg g\big) + o(1),$$

where both curves have the same end points up to $o(1)$; we show that it is $\hat{\nu}_j$. If $g(\zeta_j) \notin \mathbb{R}$ or $g(\zeta_j) > 0$, both curves stay in one open half-plane, and the number is 1. At a downward crossing $\operatorname{Im} g = \operatorname{Im} \chi_r$ passes from positive to negative; g crosses $\mathbb{R}$ once, at $g(\zeta_j) < 0$, so its argument increases by an amount in $(0, 2\pi)$, while χ_r crosses once at a positive point, so its argument changes by an amount in $(-2\pi, 0)$; the difference is -2π and the number is 0. At an upward crossing the difference is 2π, and the number is 2. At an inflection with $g(\zeta_j) < 0$, $\operatorname{Im} \chi_r = \operatorname{Im} g$ keeps one sign on both sides of ζ_j, so both curves stay in one closed half-plane minus 0, where a continuous argument gives both changes as the same difference of end values, and the number is 1. Summing over the disks proves the count, and the last statement follows. □

If $c \in \mathrm{Crit}$ is the end of a slit, that is, a punctured arc of $\mathbb{T}$ about c lies in Cont, then the partner map β of Lemma 7.12(e) fixes c and reverses the circle near c, and $u \circ \beta = u$. So $u - u(c)$ has one sign on both sides of c: c is never an inflection, and it is a maximum of u exactly when the distance to 0 decreases as one leaves the end along the slit. The count of the $\hat{\nu}_j$ is the local form of the proof of [23, Lemma 8.1], and for Blaschke models it is Proposition 10.2(i) with $B \equiv 1$; at a point of Crit with $u'' \neq 0$, the maximum case is Proposition 10.1(b) with $m = 1$ and B constant.

Corollary 10.5. *Let W be as above, and let it map $\mathbb{D}$ onto a slit plane. There are $M \geq 1$ and points $\zeta_1, \dots, \zeta_M$ for which (W, C_r) is admissible for all $r < 1$ close to 1 if and only if the distance to 0 increases as one leaves each end of a slit and some slit has an interior strict local minimum of this distance.*

Proof. Each point of a slit is the image of two points of the circle, one from each edge. At an interior strict local minimum of the distance to 0 along the slit, the slit is tangent to a circle about 0 and $\arg W$ moves in opposite directions on the two edges: one edge gives a downward crossing, the other a point with $g > 0$. An interior strict local maximum gives an upward crossing and a point with $g > 0$. With the paragraph before the corollary, which treats the ends of the slits, the corollary follows from the last statement of Proposition 10.4. □

If, for instance, the slit is a single arc with one finite end, then the distance to 0 must increase from the end, decrease to an interior minimum and increase again to ∞: the slit turns back toward 0, a *hook*. Straight slits never qualify (Corollary 7.24 shows more), and by Proposition 7.20 no rational map onto a slit plane has a hook. The gains r^{-M} of these pairs tend to 1 as $r \to 1$.

11. Open problems

(1) *Do more zeros help?* Jatar and Ng [23, Sect. 9] ask whether $\mathcal{B}_d$, the supremum of the gains over pairs with $\deg B \leq d$, is strictly increasing, and whether $\mathcal{B}_1 = K^*$. Since $K^* = \sup_d \mathcal{B}_d$ [23, Thm. 1.2], $K^* = 4$ [18, Thm. 1.1], and every $\mathcal{B}_d$ is less than 4 [23, Thm. 1.3], we have $\mathcal{B}_d < K^*$ for every d. So $\mathcal{B}_1 < K^*$, and the sequence $\mathcal{B}_d$ is not eventually constant; whether it is strictly increasing remains open. They prove $1.1018 < \mathcal{B}_1 < \sqrt{7/5}$; Theorems 8.7 and 8.15 give $1.1363375 < \mathcal{B}_1 < 1.177801$, and Theorems 8.8 and 8.19 give $1.1559837 < \mathcal{B}_2 < 100/61$. By Corollary 7.10 a pair attaining $\mathcal{B}_d$ with $d \geq 1$ is never a Blaschke model. The root transform of Section 6 keeps the gain but not (M3): by Lemma 6.1(c), it gives a Blaschke model with n zeros exactly when the one-zero pair passes the test of Proposition 2.11 with σ replaced by $n\sigma$. So symmetric pairs lead to the problems gain_n of Corollary 6.3, and whether $\sup_\Psi \mathrm{gain}_2(\Psi) > \mathcal{B}_1$ is open. For the pairs of Lemma 8.4 whose Q' has one zero on the circle, Proposition 8.11 gives a necessary condition for $k = 3$ zeros of B; whether such pairs with three zeros exist is open.

(2) *An explicit polynomial.* The proofs that $K^* > 1$ ([1], and [23, Prop. 4.2–4.3] through Runge approximation and Hilbert's lemniscate theorem) give no explicit polynomial and no bound for its degree. Adamczewski and Colbrook [2] give one: a polynomial p of degree 29 984 with $m(p) > 1.0000004$, certified by interval arithmetic. The larger lower bounds for K^*, including the one proved here, also pass from pairs to polynomials through Runge approximation and give no explicit polynomial. Find an explicit polynomial p with $m(p) > 1.1$, of stated degree, with a proof of this inequality.

(3) *Degree dependence.* $K(n)$ is nondecreasing in n [7] and tends to $K^* = 4$ [18]. How fast? The proof in [18] is not effective: for a given $A < 4$ it gives no bound on the least n with $K(n) > A$.

(4) *Slit maps and hooks.* By Corollaries 7.19 and 7.25, an extremal W that is meromorphic past the circle is a non-rational map onto a slit plane. By Corollary 10.5, such a map carries pairs (W, C_r) whose zeros $r\zeta_j$ lie on fixed radii, with r close to 1, only if some slit has an

interior strict local minimum of the distance to 0, for instance a hook; and the gains of these pairs tend to 1. Whether a map onto the plane minus a hook, meromorphic past the circle, carries a pair with gain above 1.1559837 is open; by Theorem 7.22, in the notation of Section 7.6, Z_- then has a point at which u is not a nondegenerate minimum.

Appendix A. The data of pair I

The zeros are

$$b = -\tfrac{3238273}{16777216} + \tfrac{6869081}{8388608}\,i, \qquad a_1 = -\tfrac{4425427}{16777216} - \tfrac{256269}{2097152}\,i, \qquad a_2 = \tfrac{2126359}{8388608} - \tfrac{6838355}{8388608}\,i.$$

Then

$$Q'(z) = 18\,(z-1)^5\,\hat{s}(z)\prod_{j=1}^{10}(z-\rho_j),$$

with

$$\begin{aligned}
\rho_1 &= \tfrac{524767}{524288} + \tfrac{3303}{1048576}\,i, & \rho_2 &= \tfrac{262319}{262144} + \tfrac{3785}{1048576}\,i, & \rho_3 &= \tfrac{510667}{1048576} + \tfrac{596603}{524288}\,i,\\
\rho_4 &= \tfrac{140827}{1048576} + \tfrac{1177373}{1048576}\,i, & \rho_5 &= -\tfrac{20639}{32768} + \tfrac{439225}{524288}\,i, & \rho_6 &= -\tfrac{105435}{131072} - \tfrac{652651}{1048576}\,i,\\
\rho_7 &= -\tfrac{291711}{524288} - \tfrac{113639}{131072}\,i, & \rho_8 &= -\tfrac{96673}{524288} - \tfrac{1084035}{1048576}\,i, & \rho_9 &= \tfrac{524719}{524288} - \tfrac{3195}{262144}\,i,\\
\rho_{10} &= \tfrac{262337}{262144} - \tfrac{4087}{1048576}\,i.
\end{aligned}$$

Put $Q(z) = \int_b^z Q'(t)\,dt$, so that $Q(b) = 0$ and Q is monic of degree 18. The quadratic $\hat{s}(z) = z^2 + s_1 z + s_0$ is fixed by $Q(a_1) = Q(a_2) = 0$: with $\mathcal{V}(t) = (t-1)^5\prod_j(t-\rho_j)$, (s_0, s_1) is the unique solution of the linear system

$$\int_b^{a_i} (t^2 + s_1 t + s_0)\,\mathcal{V}(t)\,dt = 0 \qquad (i = 1, 2),$$

and s_0, s_1 are Gaussian rationals whose real and imaginary parts have denominators of 451 to 458 digits; numerically the zeros of $\hat{s}$ are $1.00000688\ldots - 0.00152261\ldots\,i$ and $-0.41009053\ldots + 1.04265219\ldots\,i$. Then $H(z) = (1-\bar{a}_1 z)(1-\bar{a}_2 z)\,Q(z)/((z-a_1)(z-a_2))$ is a polynomial of degree 18, and $R(z) = 18\,\hat{s}(z)\prod_j(z-\rho_j)$.

For these data the facts (D), (C), (R) and (S) of the proof of Theorem 8.8 hold; each is verified in exact arithmetic over $\mathbb{Q}(i)$; (R) is checked as described after Lemma 8.6, with the resultant of degree at most 578 in c computed at the 581 integer points $c = 0, 1, \ldots, 580$ (as in [19, Sect. 5]). For (C) the division of Q' by $(z-1)^5$ is exact, $R(1) \neq 0$, and Lemma 7.11 is applied to R, of degree 12; for (S) it is applied to H', of degree 17.

The zeros of R, H' and the $\mathcal{D}_c$ lie outside $\overline{\mathbb{D}}$. We call the distance from the unit circle to the nearest zero of such a polynomial its *margin*. The zero of R nearest to the circle, a zero of $\hat{s}$, has modulus $1.00000804\ldots$; four of the ρ_j lie within 0.0123 of $z = 1$ as well. The zero of H' nearest to the circle has modulus $1.00001141\ldots$; since $\mathcal{D}_{\pm1}(w) = H'(\pm w)$, this gives the margin at $c = \pm1$. Here the smallest margin over $c \in [-1, 1]$ is not at $c = \pm1$: numerically the zeros of the $\mathcal{D}_c$ come closest near $c = \pm0.9231$, within about $7.028 \cdot 10^{-6}$ of the circle. Condition (S) concerns $c = \pm1$ only; the other c are covered by (R) (Figure 3). A small margin has a geometric meaning. For $-1 < c < 1$ and $c = \cos\theta$, a zero w of $\mathcal{D}_c$ gives $H(we^{i\theta}) = H(we^{-i\theta})$ with $|w| > 1$; so when $|w|$ is close to 1, two points of the boundary curve of $H(\mathbb{D})$ nearly coincide. Numerically, at the dips of Figure 3 this happens along the narrow inlet of Figure 2, whose two sides almost touch.

Appendix B. The data of pair VI

The zeros are

$$a = \tfrac{5104807}{16777216} + \tfrac{5629749}{16777216}\,i, \qquad b = \tfrac{7625713}{16777216} - \tfrac{10960539}{16777216}\,i.$$

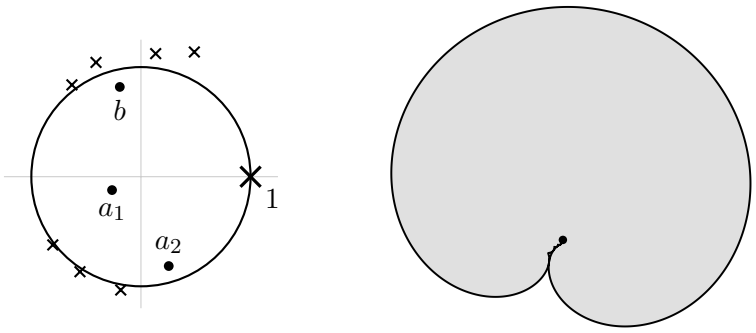


FIGURE 2. Left: the data of Theorem 8.8. Dots: the zeros b, a_1, a_2 of Q in $\mathbb{D}$. Crosses: the 17 critical points of Q, counted with multiplicity; none lies in $\mathbb{D}$. The large cross at $z = 1$ stands for the ten of them within 0.0123 of 1: the zero of order 5, one zero of $\hat{s}$ and four of the ρ_j. The small crosses are the other seven. Right: the image $H(\mathbb{D}) = N_b\, W(\mathbb{D})$, computed numerically; axes omitted. The dot is $H(b) = 0$, next to the tip of a narrow inlet whose two sides almost touch.

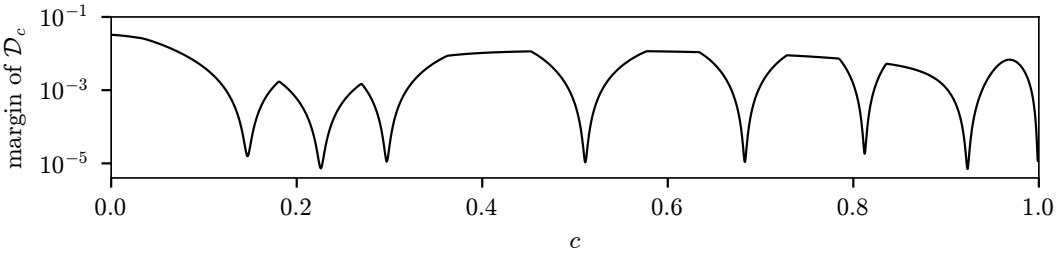


FIGURE 3. For pair I: how close the zeros of $\mathcal{D}_c$ come to the unit circle, computed numerically for $0 \le c \le 1$, on a logarithmic scale; since $\mathcal{D}_{-c}(w) = \mathcal{D}_c(-w)$, the interval $-1 \le c \le 0$ gives its mirror image. By (R) of Lemma 8.6 the margin is positive for every c. Its lowest point is near $c = 0.9231$. At each dip a zero of $\mathcal{D}_c$ is close to the circle.

Then

$$Q'(z) = 18\,(z-1)^5\,(z-\rho_0)\prod_{j=1}^{11}(z-\rho_j),$$

with

$$\begin{aligned}
\rho_1 &= \tfrac{255391}{524288} + \tfrac{1143071}{1048576}\,i, & \rho_2 &= \tfrac{88671}{1048576} + \tfrac{1246607}{1048576}\,i, & \rho_3 &= -\tfrac{58043}{262144} + \tfrac{1155163}{1048576}\,i,\\
\rho_4 &= -\tfrac{150829}{262144} + \tfrac{480931}{524288}\,i, & \rho_5 &= -\tfrac{892479}{1048576} + \tfrac{334139}{524288}\,i, & \rho_6 &= -\tfrac{751281}{1048576} - \tfrac{841275}{1048576}\,i,\\
\rho_7 &= -\tfrac{477637}{1048576} - \tfrac{1061955}{1048576}\,i, & \rho_8 &= -\tfrac{90629}{1048576} - \tfrac{1208029}{1048576}\,i, & \rho_9 &= \tfrac{255145}{1048576} - \tfrac{1211027}{1048576}\,i,\\
\rho_{10} &= \tfrac{18039}{32768} - \tfrac{1032877}{1048576}\,i, & \rho_{11} &= \tfrac{949373}{1048576} - \tfrac{669111}{1048576}\,i.
\end{aligned}$$

Put $Q(z) = \int_b^z Q'(t)\,dt$, so that $Q(b) = 0$ and Q is monic of degree 18. The root ρ_0 is fixed by $Q(a) = 0$: with $\mathcal{V}(t) = (t-1)^5 \prod_j (t-\rho_j)$,

$$\rho_0 = \frac{\int_b^a t\,\mathcal{V}(t)\,dt}{\int_b^a \mathcal{V}(t)\,dt},$$

a Gaussian rational whose real and imaginary parts have denominators of 234 digits; numerically $\rho_0 = 0.85175723\ldots + 0.82891526\ldots\, i$. Then $H(z) = (1-\bar{a}z)\,Q(z)/(z-a)$ is a polynomial of degree 18, and $R(z) = 18\,(z-\rho_0)\prod_j (z-\rho_j)$.

For these data the facts (D), (C), (R) and (S) of the proof of Theorem 8.7 hold; each is verified in exact arithmetic over $\mathbb{Q}(i)$. For (C) the division of Q' by $(z-1)^5$ is exact, $R(1) \neq 0$, and Lemma 7.11 is applied to R, of degree 12; for (S) it is applied to H', of degree 17. (R) is checked as described after Lemma 8.6, with the resultant of degree at most 578 in c computed at the 581 integer points $c = 0, 1, \ldots, 580$. The margins are small. The zero of R nearest to the circle has modulus $1.06329949\ldots$. The zero of H' nearest to the circle has modulus $1.00001014\ldots$; since $\mathcal{D}_{\pm 1}(w) = H'(\pm w)$, this is the margin at $c = \pm 1$. Numerically the zeros of the $\mathcal{D}_c$ come slightly closer near $c = \pm 0.3257$, within about $9.24 \cdot 10^{-6}$. Condition (S) concerns $c = \pm 1$ only; the other c are covered by (R).

Appendix C. The computations for Theorems 8.15 and 8.19

All enclosures below are computed in ball arithmetic [24] from exact rational inputs.

Theorem 8.15. There are two ranges $[s_-, s_+]$ of $s = |a|$: $[0.845, 0.849]$ with $\rho = 827/1000$, and $[0.849, 0.84904]$ with $\rho = 8271/10000$. For each, the rectangle $[s_-, s_+] \times [0, 355/113]$ of pairs (s, θ), where $355/113 > \pi$, is covered by boxes, and on each box we certify

$$\left|\operatorname{Log}\bigl(-\sigma_s(\rho e^{i\theta})\bigr)\right|^2 - L(\rho)^2 > 0 \qquad \text{for all } (s,\theta) \text{ in the box},$$

by a ball enclosure of the left side at 200 bits. This is the hypothesis of Lemma 8.14 for every s in the range: the range $\theta \in [0, \pi]$ suffices because $\sigma_s(\bar{w}) = \overline{\sigma_s(w)}$. The enclosure of $|\operatorname{Arg}\xi|$ uses

$$|\operatorname{Arg}\xi| = \begin{cases} \arctan(|\operatorname{Im}\xi| / \operatorname{Re}\xi) & \text{if } \operatorname{Re}\xi > 0, \\ \pi - \arctan(|\operatorname{Im}\xi| / |\operatorname{Re}\xi|) & \text{if } \operatorname{Re}\xi < 0, \\ \arccos(\operatorname{Re}\xi / |\xi|) & \text{otherwise}, \end{cases}$$

so that no branch cut is crossed. Adaptive bisection needs 96282 boxes for $[0.845, 0.849]$ and 476313 boxes for $[0.849, 0.84904]$. The margin is small: at $s = 0.849$, $\rho = 0.827$ the minimum of $|\operatorname{Log}(-\sigma_s)| - L(\rho)$ on the circle is about $1.9 \cdot 10^{-4}$, and at $s = 0.84905$, $\rho = 0.8271$, $\theta = 0.2848$ the enclosure of $|\operatorname{Log}(-\sigma_s)|^2 - L(\rho)^2$ is negative.

Theorem 8.19. Here all enclosures are computed at 128 bits.

The boxes. The cover of the parameter set X_{61} of the proof has 28853 boxes, obtained by halving the widest side, starting from the $8 \times 16 \times 8$ grid of $[0,1] \times [-1,1] \times [0,1]$; a box is discarded only if an exact test shows that it misses X_{61}. Each centre is of the form $w_C = (k + li)/40$ and each radius of the form $\rho = k'/1000$; 25454 boxes use $w_C = 0$.

Condition (i) is checked in exact rational arithmetic over the whole box: the exact range of $|a - w_C|^2$ over the segment of values of a_1 and over the rectangle of values of a_2 is compared with $(\rho \pm 1/200)^2$.

Condition (ii). Write $p = (r_1, x_2, y_2)$, p_0 for the centre of the box, $\hat{t}_k$ for its half-widths, $t = p - p_0$ (so $|t_k| \le \hat{t}_k$) and $\partial_k = \partial/\partial p_k$. The interval $\theta \in [0, 710/113] \supset [0, 2\pi]$ is covered adaptively by pieces I, with midpoint θ_I and half-width η. On each piece we certify

$$\left|\operatorname{Log}\bigl(-\sigma(p,w)\bigr)\right| - L(r_I) > 0 \qquad \text{for every } p \text{ in the box and every } w = w_C + \rho e^{i\theta},\ \theta \in I,$$

where r_I is an upper bound of $|w|$ on the piece; since L is increasing, this gives (ii) on the piece. The left side is bounded below by the first of the following five certified bounds that is positive.

(a) The ball enclosure of $|\operatorname{Log}(-\sigma)|^2 - L(r_I)^2$ over the box and the piece, computed as for Theorem 8.15, with $|\operatorname{Log}\xi| \ge |\log|\xi||$ used instead where $|\xi| < e^{-L}$ or $|\xi| > e^{L}$ (this is needed where σ has a zero on the circle).

(b) The same, applied to an enclosure of σ from the mean value theorem: for fixed w,

$$\sigma(p,w)-\sigma(p_0,w)=\sum_k t_k\int_0^1 \partial_k\sigma(p_0+st,w)\,ds \;\in\; \sum_k[-\hat t_k,\hat t_k]\,E_k,$$

where E_k is the enclosure of $\partial_k\sigma$ over the box; each integral lies in E_k, since an enclosure is a rectangle and hence convex.

(c) The same with θ moving as well, from θ_I: this adds $[-\eta,\eta]$ times an enclosure of $\partial\sigma/\partial\theta = i(w-w_C)\,\partial\sigma/\partial w$.

(d) Where $\inf|\sigma|>0$ is certified on the box: let ϕ_0 be a logarithm of $-\sigma(p_0,w)$ and ϕ the continuous logarithm of $-\sigma(\cdot,w)$ along the segment from p_0 to p with $\phi(p_0)=\phi_0$. Then

$$\phi(p)=\phi_0+\sum_k t_k g_k+\epsilon,\qquad g_k=\frac{\partial_k\sigma}{\sigma}(p_0,w),\qquad |\epsilon|\le\frac12\sum_{k,l}\hat t_k\hat t_l\sup\bigl|\partial_k\partial_l\phi\bigr|,$$

where $\partial_k\partial_l\phi=\partial_k\partial_l\sigma/\sigma-\partial_k\sigma\,\partial_l\sigma/\sigma^2$. Since $|\operatorname{Log}(-\sigma(p,w))|=\min_{n\in\mathbb{Z}}|\phi(p)-2\pi i n|$, it is at least the least of the bounds

$$|\phi(p)-2\pi i n|\;\ge\;\operatorname{Re}\bigl(\bar u(\phi_0-2\pi i n)\bigr)-\sum_k\hat t_k\bigl|\operatorname{Re}(\bar u g_k)\bigr|-|\epsilon|$$

for the three n nearest to $\operatorname{Im}\phi_0/2\pi$, with a unit u close to the direction of $\phi_0-2\pi i n$, and

$$|\phi(p)-2\pi i n|\;\ge\;4\pi-|\operatorname{Im}\phi_0-2\pi n^*|-|\phi(p)-\phi_0|,\qquad |\phi(p)-\phi_0|\le\sum_k\hat t_k|g_k|+|\epsilon|,$$

for the other n, where n^* is the middle one of the three.

(e) Where $\inf|\sigma|>0$ is certified on the box:

$$\bigl|\operatorname{Log}\bigl(-\sigma(p,w)\bigr)\bigr|\;\ge\;\bigl|\operatorname{Log}\bigl(-\sigma(p_0,w)\bigr)\bigr|-\sum_k\hat t_k\,\frac{\sup|\partial_k\sigma|}{\inf|\sigma|};$$

along a segment in the box, with w fixed, a continuous logarithm of $-\sigma$ moves by at most this sum, and $|\operatorname{Log}\xi|=\min_{n\in\mathbb{Z}}|\log\xi+2\pi i n|$ is 1-Lipschitz in that logarithm.

The partial derivatives are exact: each term of σ is $w/(w-a)+\bar a w/(1-\bar a w)$, with $\partial/\partial a=w/(w-a)^2$, $\partial/\partial\bar a=w/(1-\bar a w)^2$, $\partial^2/\partial a^2=2w/(w-a)^3$, $\partial^2/\partial\bar a^2=2w^2/(1-\bar a w)^3$ and $\partial^2/\partial a\,\partial\bar a=0$, and its derivative in w is $-a/(w-a)^2+\bar a/(1-\bar a w)^2$.

Condition (iii). The zeros of $\widehat P_0$ at a rational point of the box are isolated by validated root enclosures and compared with C.

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