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SINGULAR PERPENDICULAR ORBITS AND COMPLETE PERIODICITY IN RATIONAL TRIANGLES

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Abstract. Let Δ be a triangle whose angles are rational multiples of π. A billiard path leaving a side σ at a right angle either meets a vertex first — its starting point is then *singular* — or meets a side at a right angle again: σ itself, at a new point, or another side, where it **retraces** itself. The singular points cut σ into finitely many intervals, and each interval retraces or is carried onto another by the return map, reversing orientation (a *mirror pair*) or preserving it (a *translation pair*).

We count these intervals exactly. The singular points of the sides perpendicular to a direction class are in bijection with the *slots* — a vertex with a direction of the class pointing inward — whose path meets a side at a right angle before any vertex, so the number of intervals is $v + S - 2p - s'$: S slots, v perpendicular sides, p pairs of mutually reverse vertex-to-vertex paths meeting no side at a right angle, and s' slots whose path never reaches a vertex. This bounds the retracing bands and translation pairs, sharply on isosceles triangles, and makes $s' = 0$ equivalent to complete periodicity of the class. Our main theorem is that $p = 0$ **forces complete periodicity**: a minimal component needs a swapped pair. The converse fails. In every isosceles triangle (a, a, b) with N even and $a \geq 3$ the class of the base direction has $p \geq 1$, and it is completely periodic, by a companion paper (computer-assisted in part), whenever $b \geq 4$ and N has at most one odd prime factor. At odd N, two infinite families of scalene triangles have $p \geq 1$ and are completely periodic in the checked range.

1. Introduction

1.1. **The question.** A billiard path in a polygon that meets a side at a right angle reverses its direction there and retraces itself. If it meets a side at a right angle twice it is periodic, and Boshernitzan [Bos92, Proposition 1.8] and, in the same year, Vorobets, Gal'perin and Stepin [VGS92, §5, proof of Theorem 2] proved that in a rational polygon every regular path leaving a side at a right angle is periodic. So the beam of paths perpendicular to a side σ is a union of periodic bands, interrupted at finitely many singular launch points [BGKT98, proof of Theorem 4], those whose path reaches a vertex first.

This paper is about the structure and the number of those bands in a rational triangle, and about what that number says about the dynamics of the whole direction class. For right triangles the structure is the subject of the companion papers [Orph] and [Neck]: the return map pairs the branches

of a leg by mirror maps [Orph, Theorem 2.2, Lemma 4.9], except for at most one branch that retraces (the *orphan*, [Orph, Theorem A]), and the number of bands is governed by a covering identity whose defect certifies complete periodicity [Neck, Corollary 7.8, Theorem 7.6]. Off the right angle, much of this breaks: bands are also paired by translations, as Ruijgrok observed numerically in a generic triangle [Rui91, §3]; retracing bands may be many per side; and when the denominator of the angles is odd every side may carry them. What survives is a counting identity, and its proof here is elementary: a bijection, with no surface and no Euler characteristic.

1.2. **Results.** Let Δ have angles $\pi a_0/N$, $\pi a_1/N$, $\pi a_2/N$ with $\gcd(a_0, a_1, a_2) = 1$, so $N = a_0 + a_1 + a_2$. The directions perpendicular to the sides fall into *classes* under the group generated by the reflections in the sides (one class when N is odd, two when N is even; §2). For a class C, a side is *vertical* if its perpendicular lies in C and *horizontal* if its own direction does; v and h count them. A *slot* is a vertex V together with a direction $u \in C$ strictly inside the angle at V; its *prong* is the billiard path leaving V along u, followed to its first vertex. Write n_σ for the number of intervals into which the singular points cut σ.

Theorem 1.1 (slot identity; Theorem 4.2)**.** *For every class C,*

$$\sum_{\sigma \text{ vertical}} n_\sigma = v + S - 2p - s',$$

where S is the number of slots, p the number of pairs $\{\gamma, \gamma^{-1}\}$ of prongs that reach a vertex without meeting any side at a right angle, and s' the number of prongs that never reach a vertex. $S = N$ when N is odd, and $S = N/2 - h$ when N is even.

The proof is a bijection: a singular point, read backwards, is a prong. At a right triangle the identity is [Neck, Corollary 7.8], proved there by an Euler-characteristic count on the separatrix diagram, which extends the completely periodic case [Neck, Theorem 3.4]; the bijection replaces that count. A weaker form of its inequality half — at most $n + m + p - 3$ perpendicular paths from a side into a vertex, in a triangle with angles nx, mx, px where $\pi/2 = qx$ for an integer q — is stated among known results by Tokarsky, Garber, Marinov and Moore [TGMM18, §11, Fact II and the conclusion after it]. It equals our slot total only at even N, summed over both classes, and reads $2N - 3$ against our N at odd N (Remark 4.4(a)). What is new is the bijection, the exact defect and its split by class, which are what the rest of the paper consumes.

Theorem 1.2 (band bound; Theorem 5.2)**.** *Let N be even. On the class of the two sides meeting at the vertex whose numerator is even, the number of retracing bands plus the number of translation pairs is $\lceil N/4 \rceil - M - p - s'/2 \leq \lceil N/4 \rceil$, M the number of mirror pairs. On the legs of an isosceles triangle $M = 0$ and equality holds exactly when $p = s' = 0$ (Theorem 6.2).*

A bound by $g'+1$, g' the genus of the quotient surface, holds for right and isosceles triangles but not in general (Remark 5.3): the slack is the number of mirror pairs, not the topology.

Theorem 1.3 (Theorem 7.8 and Proposition 7.6)**.** *The class C is completely periodic if and only if every prong reaches a vertex ($s' = 0$), and* ***it is completely periodic whenever*** *$p = 0$. Hence $\sum_{\sigma\ vertical} n_\sigma = v + S$ implies complete periodicity (Theorem 1.1 and the first statement); at odd N, $\sum_\sigma n_\sigma = N + 3$ implies Boshernitzan's Conjecture 2.2(b) for Δ placed with a side horizontal (Corollary 7.9).*

That $p = 0$ might explain Conjecture 2.2(b) — every saddle connection of the direction crosses a side perpendicular to it — was raised by Delecroix [Del22] as a question to test; Theorem 1.3 proves it sufficient. It is not necessary:

Theorem 1.4 (Propositions 8.1 and 8.2)**.** *Let $b \geq 4$ with $3 \nmid b$. The triangle $(3, b, 2b)$ if b is even, and $(3, b, 2b+3)$ if b is odd, has $p \geq 1$, and $p \geq \lfloor b/6 \rfloor$ when $b \equiv 1, 2 \pmod 6$.*

All members with $N \leq 303$ are completely periodic (§9). At even N the converse fails with both halves proved, the second through a companion paper. In every isosceles triangle (a, a, b) with N even and $a \geq 3$, the class of the base direction has a swapped pair, of two bounces if $b < a$ and of four if $a < b < 2a$. It is completely periodic whenever $b \geq 4$ and N has at most one odd prime factor (Proposition 8.6, with [Meth]; computer-assisted in part).

1.3. **Organisation.** §2 fixes the setting. §3 recalls the return map, which behaves in any rational polygon as in a right triangle except for translation pairs, and says which sides a retracing band can join. §4 proves the slot identity, §5 the band bound, §6 the isosceles case. §7 builds the surface and proves Theorem 1.3. §8 constructs the families, states a conjecture, reduces isosceles triangles at odd N to right triangles, and shows that at even N the converse of Theorem 7.8 fails. §9 records the computations; §10 lists open problems.

2. Setting

Triangles, directions, the group. Fix Δ as in §1.2, with vertices V_0, V_1, V_2 and the angle $\pi a_i/N$ at V_i. Place Δ so that its side directions are $\theta_0 + \pi m_\sigma/N$ modulo π, with integers m_σ; going around the triangle, adjacent sides differ by the numerator at their common vertex, so up to sign the three indices are 0, a_0, $a_0 + a_1$. Let D_N be the group generated by the linear parts of the reflections in the sides; it is the dihedral group of order $2N$, its rotations being the multiples of $2\pi/N$. We index *oriented* directions in units of π/N, modulo $2N$, measured from θ_0: a rotation in D_N adds an even integer, and the reflection in side j sends $i \mapsto 2m_j - i$.

Classes. The direction perpendicular to σ and pointing into Δ has index $m_\sigma + N/2$, for a suitable representative of m_σ modulo $2N$ — a half-integer when N is odd. A **class** is a D_N-orbit of such a direction. When N is odd, write a half-integer as $k + \frac{1}{2}$: rotations preserve the parity of k and the reflection $i \mapsto 2m_j - i$ sends $k \mapsto 2m_j - k - 1$, changing it, so one class contains all $2N$ half-integers: there is **one** class, every side is vertical, and no side is horizontal. When N is even the indices are integers and D_N preserves their parity, so there are **two** classes, the integers of each parity. In either case a class C satisfies $C = -C$. A side is **vertical** for C if its perpendicular lies in C, **horizontal** if its own direction does; at even N every side is vertical for exactly one class and horizontal for exactly one — the same class when $4|N$, since then $m_\sigma + N/2 \equiv m_\sigma \pmod 2$, and the other class when $N \equiv 2 \pmod 4$.

Beams and branches. For a point s interior to σ let γ_s be the billiard path leaving s perpendicular to σ. It is **singular** if it reaches a vertex before it meets any side at a right angle. A non-singular path does meet a side at a right angle: if it reached no vertex at all it would be a regular orbit perpendicular to a side, hence periodic [Bos92, Proposition 1.8], [VGS92, §5], and a periodic path returning to s with its initial velocity arrives there perpendicularly. The first such meeting is either a **return** to σ at a point $J(s)$, or a **head-on** collision with another side, after which γ_s retraces itself to s; in the second case put $J(s) \coloneqq s$. A **branch** of σ is a component of the set of non-singular points of σ. The singular set is finite (Corollary 4.3), and n_σ denotes the number of branches, one more than the number of singular points.

Slots and prongs. A **slot** of C is a pair (V, u), V a vertex and $u \in C$ a direction strictly inside the angle at V. Its **prong** is the billiard path leaving V in direction u, followed until it first reaches a vertex (or forever, if it never does). Write S for the number of slots, s' for the number of prongs that never reach a vertex, and p for the number of unordered pairs of distinct prongs that are each other's reverse, reach a vertex, and meet no side at a right angle on the way (step (iii) of the proof of Theorem 4.2 shows these do come in pairs). We call such a pair a **swapped pair**: each prong ends at the other's slot.

Terminology. A branch on which J is the identity is a **retracing branch**: each of its paths meets a second side at a right angle and comes back. (Every path of a beam is a *perpendicular trajectory* in the sense of [VGS92]; the retracing ones are perpendicular to two different sides, like the orbits of [VGS92, §4A, Assertion 1].) Retracing branches come in pairs on two sides, joined by a strip of retracing paths — a **retracing band** (a *band of double normals* in [Orph, Theorem 5.3]). In right triangles the retracing branch of a leg is unique when it exists, and [Orph] calls it the orphan; we keep that name for right triangles only.

3. The return map in a rational polygon

The structure of the return map is proved in [Orph, §2] for a leg of a right triangle, but the proofs use only the reflection law, time reversal and the development of the beam, and apply verbatim to any side of any rational polygon. For a polygon, N is the least common denominator of the angles in units of π, D_N the group generated by the linear parts of the reflections in the sides, and the indices m_σ, singular points, branches and J are defined as in §2.

Theorem 3.1 (the return map; [Orph, Lemma 2.1, Theorem 2.2], [Neck, proof of Lemma 5.2])**.** *Let σ be a side of a rational polygon.*

1. *(Retrace criterion.) For s interior to a branch, $J(s) = s$ if and only if γ_s meets a side head-on at an interior time; that collision is then unique, occurs at half the return time on a side other than σ, and the sequence of sides struck is a palindrome about it. The condition holds at every point of a branch or at none.*
2. *(Pairing.) J is an involution, affine with slope ± 1 on each branch, and on each branch either $J = \mathrm{id}$ or J carries it isometrically onto a **distinct** branch.*
3. *(The two kinds of pair.) On a pair, the linear part of the developing isometry is $-\mathrm{id}$ — then J has slope -1, a **mirror pair** — or the reflection r_σ fixing the direction of σ — then J has slope $+1$, a **translation pair**.*

Proof. First, the sequence of sides struck before the first right-angle meeting, and with it the developing isometry, is constant on a branch. For a non-singular s the path up to that meeting is a finite polyline avoiding the vertices, so for s' near s the path strikes the same sides in the same order; whether a meeting is at a right angle depends only on the direction, which the sequence fixes, so the same meeting is still the first right-angle one. The sequence is therefore locally constant on the non-singular points, hence constant on each branch. (1) and (2) are [Orph, Lemma 2.1, Theorem 2.2]: the first rests on uniqueness of billiard orbits and time reversal, the second on the development (the launch copy of σ and the return copy are parallel, so $J(s) = \varepsilon s + d$ with ε, d constant on the branch), time reversal ($J^2 = \mathrm{id}$, and J maps branches to branches by maximality); if $J(B) = B$, then $\varepsilon = +1$ makes J a translation of an interval onto itself, so $d = 0$, and $\varepsilon = -1$ is excluded by (1), J fixing the midpoint. For (3), as in the proof of [Neck, Lemma 5.2]: the developed path keeps its direction while the folded path arrives at σ moving opposite to its launch velocity, so the linear part g of the developing isometry negates the normal of σ; being orthogonal and preserving the line of σ, it is $-\mathrm{id}$ or r_σ, and it acts on σ's direction by $\varepsilon = -1$, resp. $+1$. □

Translation pairs are not exotic.

Example 3.2. In the triangle with angles $(\pi/6, \pi/6, 2\pi/3)$ and base $[0,1]$, a path launched upward from $s \in (0, \frac{1}{2})$ reflects off the left leg into direction $-30°$, off the base at $2s$ into direction $30°$, and off the right leg straight down: $J(s) = s + \frac{1}{2}$.

More generally, on the base of every rational isosceles triangle with N even all branches but at most two lie in translation pairs (Theorem 6.2; for odd N see Remark 6.3). In a right triangle there are none, on any side ([Orph, Lemma 4.9] and Lemma 6.1); that is where the right angle enters [Orph], and it is the main thing that fails here.

Which sides can a retracing band join? A retracing path turns around on a side τ whose developed copy is perpendicular to the beam, that is, parallel to σ.

Proposition 3.3 (side congruence)**.** *In a rational polygon, the developed copies of a side τ in the beam of σ have directions in $\theta_\tau + (2\pi/N)\mathbb{Z}$ modulo π, and this set contains the direction of σ if and only if $(m_\tau - m_\sigma) + 2k \equiv 0 \pmod N$ for some k — that is, if and only if N is odd or $m_\tau \equiv m_\sigma \pmod 2$. In a triangle with N even exactly one numerator is even, and a retracing band can only join the two sides meeting at that vertex. When N is odd every pair of sides is admissible (Figure 1(a)).*

Proof. Reflection in side j sends a line direction ψ to $2\theta_j - \psi$, and $2\theta_j - \theta_\tau = \theta_\tau + 2(\theta_j - \theta_\tau)$, so the orbit of θ_τ is $\theta_\tau + (2\pi/N)\mathbb{Z}$. In a triangle the indices are $0, a_0, a_0 + a_1$ up to sign; $\Sigma a_i = N$ even with $\gcd = 1$ forces two odd numerators and one even, so exactly the two sides at the even vertex have equal parity. □

For right triangles this is the side table of [Orph, Theorem 5.3]. The informal version for triangles is in Delecroix's notes on the conjecture [Del22].

Proposition 3.4 (no mirror pairs at odd N)**.** *If N is odd then $-\mathrm{id} \notin D_N$, so by Theorem 3.1(3) every branch of every side is retracing or lies in a translation pair.*

Example 3.5 (the equilateral triangle)**.** With $A = (0,0)$, $B = (1,0)$, a path launched up from $x < \frac{1}{2}$ reflects off AC into direction $30°$, perpendicular to BC; from $x > \frac{1}{2}$ off BC into $150°$, perpendicular to AC. So each side consists of two retracing branches, and every side carries a retracing band to each of the others — impossible in a right triangle, where at most two sides carry a retracing branch [Orph, Theorem 5.3].

4. The slot identity

Lemma 4.1 (slot count)**.** *$S = N$ when N is odd, and $S = N/2 - h$ when N is even. In both cases $C = -C$, and no segment of a perpendicular launch or of a prong runs along a side.*

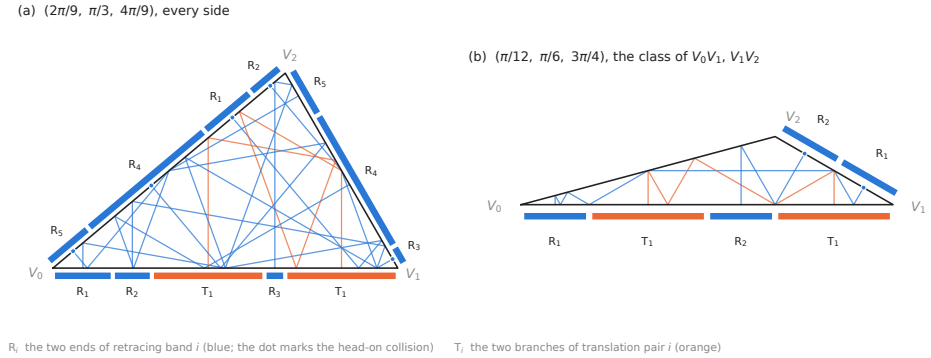


FIGURE 1. Branch partitions, coloured by kind: retracing branches blue (R), branches of translation pairs orange (T), the two branches of each band or pair carrying the same number; the thin lines are the paths from the branch midpoints, one per band or pair, and a dot marks a head-on collision. (a) The triangle $(2\pi/9, \pi/3, 4\pi/9)$, $N = 9$: every side is vertical, all three carry retracing branches, and $\sum_\sigma n_\sigma = 5 + 3 + 4 = N + 3$ (sides V_0V_1, V_1V_2, V_2V_0) (Theorem 4.2, $p = s' = 0$). (b) Example 5.4: the class of the two sides at the angle $\pi/6$, two retracing bands and one translation pair; V_0V_2 is not vertical in this class (its perpendicular has odd index), so it carries no branches.

Proof. The angle at a vertex is the open arc between its two side directions, of indices m and $m + a_i$. When N is odd it holds a_i half-integers, and C is all of them: $S = \Sigma a_i = N$. When N is even and C is the integers of parity ε, the arc holds $(a_i - b_i)/2$ of them, b_i the number of the vertex's two sides whose index has parity ε; summing, $S = (N - 2h)/2$, each horizontal side being counted at both ends. Adding N preserves half-integers, and preserves parity at even N, so $C = -C$. A launch leaves the interior of σ transversally; a reflection at an interior point of a side leaves it transversally; and a segment from an interior point of τ in the direction of another side lies on that side only if it starts at their common vertex. □

So a segment ending at a vertex V starts at an interior point of a side, and its reversed direction lies strictly inside the angle at V; and the first segment of a prong ends at an interior point of the opposite side.

Theorem 4.2 (slot identity)**.** *For every rational triangle and every class C,*

$$\sum_{\sigma \text{ vertical}} n_\sigma = v + S - 2p - s'.$$

In particular $\sum_\sigma n_\sigma = N + 3 - 2p - s'$ when N is odd, and $\sum_{\sigma \text{ vertical}} n_\sigma = N/2 - h + v - 2p - s'$ when N is even.

Proof (Figure 2). (i) *Singular points are prongs.* Let x be a singular point of a vertical side σ: γ_x reaches a vertex V before any perpendicular collision, arriving along $-u$, u strictly inside the angle at V (Lemma 4.1). The directions of γ_x are the D_N-images of σ's normal, so $u \in C$, and the reversed path is the prong of the slot (V, u): it meets σ at x at a right angle, having met no vertex and no side at a right angle before.

(ii) *Prongs that meet a side at a right angle are singular points.* Conversely let the prong of (V, u) meet a side τ at a right angle, at an interior point x, before reaching any vertex, and take the first such meeting. Then τ's normal is in C, so τ is vertical, and the prong read backwards from x is τ's perpendicular launch at x, reaching V with no perpendicular collision first: x is singular for τ. The two maps are inverse, a slot's first perpendicular collision being unique. In particular the singular set of a vertical side is finite, and $\sum_{\sigma \text{ vertical}}(n_\sigma - 1) = f$, the number of slots whose prong meets a side at a right angle before any vertex — side by side, the side of the collision being the side of the singular point.

(iii) *The rest pair up.* A prong that meets a side at a right angle reverses there and retraces back to V, by time reversal as in Theorem 3.1(1); every other prong either never reaches a vertex (s' of them) or reaches a vertex W with no perpendicular collision. In the last case the reversed path is the prong of a slot (W, u') of the same kind, as in (i), and $(W, u') \neq (V, u)$: equality would make the path its own reverse, $\gamma(L - t) = \gamma(t)$, forcing a velocity reversal at $t = L/2$, which is a perpendicular collision at a point that is not a vertex. So these slots fall into p pairs, $S = f + 2p + s'$, and the identity is (ii) rearranged. The values of S are Lemma 4.1. □

Corollary 4.3. *The singular set of every side is finite (every side is vertical for some class), and* $\sum_{\sigma \text{ vertical}} n_\sigma \leq v + S$, *with equality if and only if* $p = s' = 0$. *At odd* N, $\sum_\sigma n_\sigma \leq N + 3$.

Remark 4.4. (a) Theorem 4.2 sharpens the injection *singular point* $\mapsto$ *(first vertex, direction)* behind the finiteness theorems of [BGKT98] and [Orph, Proposition 3.4] into a bijection onto an exactly counted set. Summed over both classes at even N, the slot total is $N - 3$, which is Tokarsky et al.'s bound $n + m + p - 3$ [TGMM18, §11, Fact II] in our units; at odd N their bound reads $2N - 3$, against $S = N$ here. (b) At a right triangle ($N = 2Q$) Theorem 4.2 is the covering identity of [Neck, Corollary 7.8], whose defect $D = 2p + (z - X)$ (in [Neck]'s notation) is defined on a surface; Lemma 7.5 below identifies its p with the billiard p, so $z - X = s'$. Here no complete periodicity, no surface and no Euler characteristic are used.

5. Bands and the band bound

Let R be the number of retracing bands of C — pairs of retracing branches on two vertical sides joined by retracing paths — M the number of mirror pairs, and T the number of translation pairs, summed over the vertical sides.

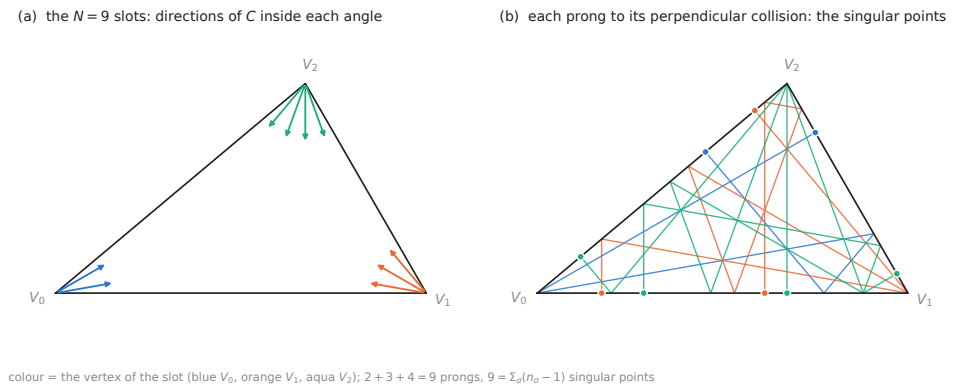


FIGURE 2. The bijection of Theorem 4.2 in the triangle of Figure 1(a). (a) The $S = N = 9$ slots, a_i at V_i. (b) Every prong meets a side at a right angle before any vertex; the meeting points (dots, coloured by the slot's vertex) are the $9 = \sum_\sigma (n_\sigma - 1)$ singular points, the cuts between the branches of Figure 1(a).

Proposition 5.1 (band count)**.**

$$\sum_{\sigma \text{ vertical}} n_\sigma = 2(R + M + T).$$

Proof. By Theorem 3.1 every branch is retracing or one of a pair on its own side. A retracing branch B of σ turns head-on at a point x of a side $\tau \neq \sigma$, which is therefore vertical; read from x, the same segment is τ's perpendicular launch, first meeting a side at a right angle at a point of σ, so x lies in a retracing branch $\varphi(B)$ of τ. The map $s \mapsto x(s)$ is continuous and injective on B with non-singular values, so $x(B) \subseteq \varphi(B)$; the same map read from τ sends $\varphi(B)$ into the branch of σ containing B. So $x(B) = \varphi(B)$ and $\varphi \circ \varphi = \mathrm{id}$, with no fixed points since $\tau \neq \sigma$: retracing branches come in pairs, one pair per band. □

Theorem 5.2 (band bound)**.** *Let N be even. On the class of the two sides meeting at the vertex with even numerator,*

$$R + T = \lceil N/4 \rceil - M - p - s'/2 \leq \lceil N/4 \rceil,$$

with equality if and only if $M = p = s' = 0$. On the other class $R = 0$ and $M + T = \lfloor N/4 \rfloor - p - s'/2$. When N is odd, $M = 0$ and $R + T = (N + 3)/2 - p - s'/2$.

Proof. Theorem 4.2 and Proposition 5.1 give $R+M+T = (v+S-2p-s')/2$. On the first class the two sides at the even vertex have indices of equal parity and the third side the other parity, so $v = 2$; and $h = 2$ if $4|N$ (then $m + N/2 \equiv m$), $h = 1$ if $N \equiv 2 \pmod 4$. So $(v + N/2 - h)/2$ is $N/4$, resp. $(N+2)/4$: $\lceil N/4 \rceil$ in both cases. On the second class $v = 1$ and $h = 1$, resp.

2, giving $\lfloor N/4 \rfloor$; with one vertical side there is no retracing band. At odd N, $v = 3$, $S = N$, and $M = 0$ by Proposition 3.4. □

Remark 5.3 (the genus is not the variable). At even N the unfolded surface S_Δ of §7 carries the rotation τ by π, and the quotient S_Δ/τ has genus g' with $2 - 2g' = -N/2 + \sum_i \gcd(a_i, N/2)$; $g' = 0$ exactly for right triangles. (Put $Q := N/2$. The $d_i = \gcd(a_i, Q)$ are pairwise coprime, since a common divisor of a_i, a_j, Q divides $a_k = 2Q - a_i - a_j$; so $d_0 d_1 d_2 | Q$. For positive integers $x+y+z \leq$ xyz $+2$, with equality iff two of them are 1. Hence $\Sigma d_i \leq Q+2$, with equality iff some $d_i = Q$, that is $a_i = Q$, a right angle.) For right and isosceles triangles $R+T \leq g'+1$ on each class. At a right triangle ($R+T \leq 1$) this is [Orph, Theorem 5.3] (at most one retracing branch per side), with the absence of translation pairs ([Orph, Lemma 4.9] and Lemma 6.1). For an isosceles triangle $g' = (Q - \gcd(2, Q))/2$, so $g' + 1 = \lceil N/4 \rceil$, and the bound is Theorem 5.2. It fails in general.

Example 5.4. Let Δ have angles $\pi/12, \pi/6, 3\pi/4$ at V_0, V_1, V_2 ($N = 12$, $g' = 1$), with $V_0V_1 = [0, 1]$, so that V_2 lies over $x_2 = (\sqrt{3}+1)/4$. On the class of V_0V_1 and V_1V_2, the two sides at the angle $\pi/6$, $R+T = 3 = \lceil N/4 \rceil > g'+1$.

Proof. Directions are in degrees; the perpendiculars to V_0V_1, V_1V_2, V_2V_0 are 90, 60, 105 modulo 180. (i) For $s \in (\frac{1}{2}, x_2)$ the path launched up from s meets V_2V_0 and leaves in direction -60, meets the base at $2s/\sqrt{3}$ and leaves at 60, and meets V_1V_2 head-on: the ray at 60 from a base point y passes right of V_2 exactly when $y > 1/\sqrt{3}$. (ii) For $s \in (0, (\sqrt{3}-1)/4)$ the path leaves its reflections in the sides $2, 0, 2, 0, 2, 1, 0$ in directions $-60, 60, -30, 30, 0, -60, 60$, and then meets V_1V_2 head-on. (iii) For $s \in (x_2, 1)$ the path leaves its reflections in $1, 0, 2, 0, 2$ in directions $210, 150, 240, 120, 270$ and lands perpendicularly on the base at $s - \frac{1}{2}$, with the reversed sequence of sides from the landing point: a translation pair with $J(s) = s + \frac{1}{2}$ on $((\sqrt{3}-1)/4, \frac{1}{2})$. The retracing intervals of (i) and (ii) are separated by that non-retracing interval, so they lie in different branches (Theorem 3.1(1)) and, by Proposition 5.1's pairing, in two different retracing bands; with (iii), $R + T \geq 3$. Theorem 5.2 gives $R + T \leq \lceil 12/4 \rceil = 3$. □

So $M = p = s' = 0$ on this class, and the four intervals of (i)–(iii) are all the branches of V_0V_1. The two halves of V_1V_2 are the far ends of the bands. Theorem 5.2 shows where the slack goes: at a right triangle $R+T \leq 1$, and the mirror pairs, with $p + s'/2$, absorb the rest of $\lceil N/4 \rceil$.

6. Isosceles triangles: the bound is sharp

Let $I = (a, a, b)$ with $N = 2a + b$ even, so b is even and a odd; the apex is the even vertex and the legs ℓ, ℓ' form the class of Theorem 5.2. Halving I along its axis gives the right triangle T with $2\alpha/\pi = P/Q := (b/2)/(N/2)$: apex O, base vertex A, foot R, hypotenuse $H = \ell$, leg L_1 the half base, leg L_2 the axis. Let n_H, $n(P/Q)$ be the numbers of branches of the beams of T

perpendicular to H and to L_1, and $\nu_H, \nu \in \{0,1\}$ their numbers of retracing branches [Orph, Theorem 5.3].

The beam of T perpendicular to its hypotenuse behaves like the leg beams of [Orph].

Lemma 6.1 (the hypotenuse beam)**.** *Let T be a right triangle, $2\alpha/\pi = P/Q$. On every side of T, J has slope -1 on each non-retracing branch, so no branch is carried onto another by a translation. Every mirror pair of the H beam is adjacent across a height whose orbit reaches R before any other vertex, and conversely. A retracing branch of the H beam occurs only at odd Q, and it turns around on a copy of L_1 if P is odd and of L_2 if P is even.*

Proof. For L_1 this is [Orph, Lemma 4.9], and for L_2 the same lemma at $(Q-P)/Q$, the two legs exchanging roles. For H, fold by the stabiliser $V_4^H = \{\pm 1, r_P, r_{P+Q}\}$ of the direction of H in place of the V_4 of [Orph, Lemma 4.8], indices in units of $\pi/2Q$ as in its proof, where H has index P and its beam $P+Q$. That proof runs as written with $\{P, P+Q\}$ for $\{Q, 0\}$: a side is a boundary side of $D_H \coloneqq S_\alpha/V_4^H$, S_α the unfolding of T [Orph, Theorem 6.1], exactly when its copy is parallel to H or to the beam, and a side of the right parity reaches each of the two directions in a single copy. The boundary sides parallel to H are H itself, written $\bar{\Lambda}$, with ends $\bar{O}$ and $\bar{A}$, and, when Q is odd, one copy $\bar{W}$ of L_1 (if P is odd) or of L_2 (if P is even). The boundary sides parallel to the beam are one copy of H when Q is even, and when Q is odd one copy of the other leg — the copy that turns one leg parallel to H turns the other parallel to the beam — so there is one boundary image of R when Q is odd, where these two meet, and none when Q is even. With $f=2$, $b=0$ at even Q and $f=3$, $b=1$ at odd Q, the Euler count closing the proof of [Orph, Lemma 4.8] again gives $\chi(D_H) = 1$: D_H is a disc, with $\lfloor Q/2 \rfloor$ poles. The proof of [Orph, Lemma 4.9] then applies word for word, and so do steps (ii) and (iii) of [Orph, Lemma 4.10] and [Orph, Corollary 4.14], with $\bar{O}$, $\bar{A}$ in place of the ends $\bar{R}$, $\bar{A}$ of $\bar{\Lambda}$, and the boundary image of R, if there is one, in place of $\bar{R}$ and $\bar{R}'$. That gives the slope and the adjacency. A retracing leaf of the H beam ends on a vertical boundary side other than $\bar{\Lambda}$, which is $\bar{W}$; so it occurs only at odd Q, as [Orph, Theorem 5.3] requires, and turns around on $\bar{W}$. □

Theorem 6.2 (isosceles beams)**.**

1. *(Legs.) Every branch of a leg beam retraces, onto the other leg, and $n_\ell = (n_H + \nu_H)/2$.*
2. *(Base.) No branch of the base retraces. There is one mirror pair if $\nu = 1$ (possible only when Q is even), and every other branch lies in a translation pair, $(n(P/Q) - \nu)/2$ of them.*
3. *(Sharpness.) On the legs' class there are no mirror or translation pairs and $R = n_\ell$, so $R = \lceil N/4 \rceil - p - s'/2$: Theorem 5.2 holds with equality if and only if $p = s' = 0$. When $Q \geq 4$ is even this happens exactly when the class of T perpendicular to H is completely periodic.*

Proof. An orbit of I folds onto an orbit of T, recording the parity of its crossings of the axis; the vertices of I are images of O and A, and R becomes the midpoint of the base, a regular point. So each branch of I is a union of branches of T joined across heights whose orbit reaches R before any other vertex. By [Orph, Corollary 4.14] (with Lemma 6.1 for H) those heights are exactly the common endpoints of mirror pairs, and each mirror pair $B, J(B)$ of T is joined across its centre c into one branch $F = B \cup \{c\} \cup J(B)$ of I; the retracing branch of T, if any, is a branch of I by itself. Two such unions never merge: a common endpoint would be a height reaching R first, hence the centre of a mirror pair, and each branch has one partner.

The crossing parity. An orbit of T that ends head-on on L_2 crosses the axis in I and ends head-on at the mirror image of its launch point; one that ends at $J(s)$ after an even number of crossings ends at $J(s)$ in I, and after an odd number at its mirror image. On a union F the number is odd: were it even, the return map J_I of I would be the reflection of F about c, so $J_I(c) = c$ at a non-singular height of I, and Theorem 3.1(1) would give $J_I = \mathrm{id}$ on F.

(1) On the leg the unions and the retracing branch give $(n_H - \nu_H)/2 + \nu_H$ branches. On a union, an odd number of crossings means that the orbit meets the other leg head-on, so the union retraces onto the other leg. The retracing branch of the H beam occurs only at odd Q (Lemma 6.1), where $P = b/2$ is even since a is odd; it turns around on a copy of L_2, beyond which lies the other leg, so it retraces onto the other leg too. (2) On the base, the launch at the midpoint runs up the axis into the apex, so each branch lies in one half, which carries the branches of L_1; each mirror pair of T becomes one branch in each half, and since its number of crossings is odd, the composite of J with the reflection in the axis, which preserves orientation, carries it onto the other half: a translation pair — $(n(P/Q) - \nu)/2$ of them. The retracing branch of L_1, which exists only when P is odd, that is Q even, turns around on a copy of L_2 [Orph, Theorem 4.6], that is on the axis, so in I it and its mirror image form the mirror pair. (3) The leg beams have no pairs by (1), so $R = n_\ell$ and Theorem 5.2 reads as stated. At even Q, $\nu_H = 0$ (Lemma 6.1) and $\lceil N/4 \rceil = Q/2$, so equality is $n_H = Q$, which is complete periodicity of the H class of T by [Neck, Corollary 7.11]; on that class (H its only vertical side) the corollary needs no hypothesis. □

Remark 6.3. At odd Q equality needs in addition $\nu_H = 1$, the existence of a retracing branch of T's hypotenuse beam; that is, by [Orph, Theorem 5.3], the existence of the orphan at the complementary centre $(Q-P)/Q$ ([Orph, Theorem A(c)]), proved when Q is a prime power and at further centres ([Orph, Theorem 5.3], [Meth]). At odd N (where P and Q are odd) the picture changes at the retracing branches of T (the proof of Theorem 6.2): the two base branches containing T's retracing branch of L_1 and its mirror image, when it exists, retrace onto the legs instead of forming a mirror pair, and the leg branch containing T's retracing branch of H, when it exists,

retraces onto the base (it turns on a copy of L_1); every other leg branch retraces onto the other leg and every other base branch lies in a translation pair. The equilateral triangle (Example 3.5) is the smallest case. The counts above are stated for even N only.

7. Complete periodicity

7.1. **The surface and the involution.** Let S_Δ be the unfolding of Δ [MT02]: copies (Δ, g), $g \in D_N$, with (x, g) glued to (x, gr_j) for x on side j, $r_j \in D_N$ the reflection in the direction of side j; it is a translation surface, and we mark every image of a vertex. Put $X \coloneqq S_\Delta$ when N is odd. When N is even, $-1 \in D_N$ is central and $\tau(x, g) = (x, -g)$ is a fixed-point-free involution off the marked points with derivative -1; put $X \coloneqq S_\Delta/\tau$, a half-translation surface. Write Σ for the set of marked points.

Fix $\omega \in C$. The billiard paths with directions in C are the projections of the leaves of the foliation F_ω of X in direction $\pm\omega$, and C is **completely periodic** if F_ω decomposes into cylinders — equivalently, every regular billiard path with direction in C is periodic. (All directions in C are equivalent: left multiplication by $g \in D_N$ carries F_ω to $F_{g\omega}$.)

Let $r \in D_N$ be the reflection with $r\omega = -\omega$ — the reflection in the axis perpendicular to ω; it lies in D_N, being $gr_\sigma g^{-1}$ for any g with $g(m_\sigma + N/2) = \omega$. Define

$$\iota(x, g) \coloneqq (x, rg).$$

It is well defined, since left multiplication commutes with the gluing $g \mapsto gr_j$ and with τ; it is an orientation-reversing isometric involution, with derivative r, and preserves F_ω.

Lemma 7.1 (the fixed set)**.** $\mathrm{Fix}(\iota)$ *is a union of side copies and their endpoints: the copies* (σ, g) *with* $gr_\sigma g^{-1} = r$*, which are perpendicular to* ω *(**vertical arcs**), and, when* N *is even, those with* $gr_\sigma g^{-1} = -r$*, which are parallel to* ω *(**horizontal arcs**). No interior point of a copy is fixed.*

Proof. $\iota(x, g) = (x, g)$ requires $rg = g$ (or $rg = -g$ at even N), impossible, or x on a side σ with $rg = gr_\sigma$ (or, at even N, $rg = -gr_\sigma$), i.e. $gr_\sigma g^{-1} = \pm r$. The developed copy (σ, g) has direction $g(m_\sigma)$, whose reflection is $gr_\sigma g^{-1}$; it equals r iff the copy is perpendicular to ω, and $-r$ iff it is parallel. A fixed marked point is an endpoint of a fixed copy: near it ι is an orientation-reversing isometry of a cone fixing its apex, which fixes exactly two rays. □

A copy of σ is vertical exactly when σ is a vertical side for C, and X has vertical arcs: the copy (σ, g) with $g(m_\sigma + N/2) = \omega$.

7.2. **Saddle connections and reversibility.** The next two lemmas are proved in [Neck, Lemma 3.3, Lemma 7.2] for the surface of a right triangle; the proofs use only that ι is a foliation-preserving isometric involution whose fixed set is as in Lemma 7.1, and we repeat them in this generality. Lemma 7.3 is also a case of Vorobets' statement that a trajectory reversed by

an affine reflection of a flat surface is periodic or a saddle connection [Vor97, Proposition 2.1], which he derives from Stepin's argument for perpendicular orbits.

Lemma 7.2 (trichotomy)**.** *Let e be a saddle connection of F_ω. Exactly one of the following holds: (1) e is a horizontal arc; (2) $\iota(e) = e$, e is not contained in* $\mathrm{Fix}(\iota)$*, and e meets* $\mathrm{Fix}(\iota)$ *in exactly one point, which lies on a vertical arc; (3) $\iota(e) \neq e$. Conversely, every point of a saddle connection lying in the interior of a vertical arc lies on one of type (2).*

Proof. If $e \subset \mathrm{Fix}(\iota)$ it is a leaf inside $\mathrm{Fix}(\iota)$, hence a horizontal arc, vertical arcs being transverse to F_ω. If $\iota(e) = e$ and $e \not\subset \mathrm{Fix}(\iota)$, ι restricts to an involution of the arc e other than the identity, which reverses it and fixes exactly its midpoint; that point is not marked (it is interior to e) and is not on a horizontal arc (two leaves do not cross), so by Lemma 7.1 it lies on a vertical arc. Conversely, if e passes through an interior point y of a vertical arc, $\iota(e)$ is a leaf through y in the same direction, so $\iota(e) = e$. □

Lemma 7.3 (reversibility on a vertical arc)**.** *Let κ be a vertical arc. Every leaf through an interior point of κ that returns to κ is closed; hence almost every point of κ lies on a closed leaf.*

Proof. At $x \in \kappa$ the derivative of ι fixes the tangent of κ and negates the direction ω. Let $\tilde{\kappa} = \kappa \times \{\pm\}$ be the transversal doubled by a choice of direction along the leaf, ρ the first-return map on it, and $\varsigma(x, d) = (x, \bar{d})$. Time reversal gives $\rho\varsigma\rho = \varsigma$. And ι carries the leaf segment from (x, d) to the one from $(x, \bar{d})$, fixing its endpoint on κ, so $\rho\varsigma = \varsigma\rho$. Hence $\rho^2 = \mathrm{id}$: the leaf from x returns to x in its original direction, so it is closed. Poincaré recurrence makes almost every point return. □

(For a beam this is the statement that the return map is an involution, Theorem 3.1(2), read on the surface.)

Corollary 7.4. *A saddle connection meeting the interior of a vertical arc has a cylinder on each side, so it bounds no minimal component.*

Proof. There are finitely many saddle connections in direction ω, so near a crossing point the points of the arc on either side lie in the components of the complement of the saddle connections adjacent to e (possibly the same); by Lemma 7.3 almost all lie on closed leaves. □

7.3. **The dictionary.**

Lemma 7.5 (dictionary)**.** *The saddle connections of F_ω are the horizontal arcs and the lifts of the billiard paths from a vertex to a vertex, with no vertex between, whose directions lie in C. The lifts of one such path form one ι-orbit, of size* 1 *if and only if the path meets a side at a right angle. Hence the type-(3) saddle connections of Lemma 7.2 form exactly p pairs, and the prongs that never reach a vertex lift to ι-pairs of separatrices that never close.*

Proof. A leaf meeting a side copy at an interior point crosses it transversally, or contains it and runs to its endpoints; so a saddle connection that is not a horizontal arc projects to a vertex-to-vertex billiard path with directions in $D_N\omega = C$, and every such path lifts. A trajectory leaving V in direction $u \in C$ lifts through the copies g with $gu = \pm\omega$, modulo τ at even N. When N is odd no reflection fixes u (axes have integer indices, u is a half-integer), so D_N acts freely on C and the two lifts start in the copies g_1 ($g_1u = \omega$) and rg_1 ($rg_1u = -\omega$): they are ι-images. When N is even, u is fixed by exactly one reflection s, with axis parallel to u; the solutions modulo τ are $[g_1]$ and $[g_1s]$, and $g_1sg_1^{-1}$ is the reflection with axis parallel to ω, which is $-r$; so $[g_1s] = [rg_1]$, again ι-images. An ι-invariant lift is of type (2) and meets a vertical arc at its midpoint, which is a right-angle collision of the path; conversely such a collision is a crossing of a vertical arc, and the converse in Lemma 7.2 makes the lift invariant. □

So at a right triangle the billiard p and s' of Theorem 4.2 are the surface quantities p and $s/2$ of [Neck], which answers the identification promised in Remark 4.4(b).

7.4. **Complete periodicity is a prong condition.**

Proposition 7.6. *C is completely periodic if and only if every prong reaches a vertex: $s' = 0$.*

Proof. A direction on a flat surface with marked points is completely periodic if and only if every separatrix closes into a saddle connection: if all close, each component of the complement of the saddle connections is flat with geodesic boundary and corners of angle π, hence of Euler characteristic 0, hence a cylinder; conversely in a cylinder decomposition a leaf from a marked point lies in a cylinder or on its boundary and returns to it. By Lemma 7.5 and its proof the separatrices of F_ω are the horizontal arcs, which are saddle connections, and the lifts of the prongs. □

Corollary 7.7 (the count certificate)**.** *If $\sum_{\sigma\ vertical} n_\sigma = v + S$, then C is completely periodic. On a completely periodic class $\sum_{\sigma\ vertical} n_\sigma = v + S - 2p$; at odd N, $\sum_\sigma n_\sigma = N + 3 - 2p$.*

Proof. Theorem 4.2 gives $2p + s' = 0$; apply Proposition 7.6. □

Corollary 7.7 reads complete periodicity off a count, but it is no stronger than Proposition 7.6 itself, a finite check of the prongs. What the next theorem adds is that the *pairs* are the only obstruction.

Theorem 7.8 ($p = 0$ implies complete periodicity)**.** *Every saddle connection on the boundary of a minimal component of F_ω is a horizontal arc or a lift of a prong of a swapped pair, and at least one is of the second kind. In particular, if $p = 0$, then C is completely periodic: $p = 0$ implies $s' = 0$.*

Proof. Let Ω be a minimal component. Its frontier in $X \setminus \Sigma$ is a finite union of open saddle connections. None is of type (2) of Lemma 7.2, by Corollary 7.4;

so each is a horizontal arc or of type (3). Suppose that $\partial\Omega$ contains no saddle connection of type (3), as when $p = 0$; then $\partial\Omega$ is a union of horizontal arcs (empty when N is odd). Then $\iota(\Omega)$ is a minimal component, and it lies across each arc of $\partial\Omega$ from Ω, since ι fixes the arc pointwise and exchanges its two sides. So $U \coloneqq \Omega \cup \iota(\Omega) \cup (\partial\Omega \setminus \Sigma)$ is open; it is also closed in $X \setminus \Sigma$, since the closures of Ω and $\iota(\Omega)$ in $X \setminus \Sigma$ add only $\partial\Omega \setminus \Sigma = \partial\iota(\Omega) \setminus \Sigma$. $X \setminus \Sigma$ is connected (S_Δ is, the reflections r_j generating D_N, and removing finitely many points disconnects no surface), so $U = X \setminus \Sigma$. Take a vertical arc κ (§7.1). Its interior points are not on horizontal arcs — distinct side copies meet only at marked points — so they lie in $\Omega \cup \iota(\Omega)$, and by Lemma 7.3 almost all of them lie on closed leaves. A minimal component contains no closed leaf, a contradiction. So $\partial\Omega$ contains a saddle connection of type (3), which by Lemma 7.5 is a lift of a prong of a swapped pair; when $p = 0$ there is none, and no minimal component exists. □

At a right triangle this is [Neck, Theorem 7.6], whose proof first confines a minimal component to one disc of the complement of $\mathrm{Fix}(\iota)$ [Neck, Corollary 7.4]; the symmetrisation $\Omega \cup \iota(\Omega)$ makes that step unnecessary. At right triangles the converse holds as well [Neck, Corollary 7.11] — unconditionally at prime-power Q, on the Veech locus $\min(P, Q - P) \leq 2$ and on every class with one vertical side, and on the remaining two classes under the hypothesis (OB) of [Neck, §1.5]. In general it fails: §8 constructs triangles with $p \geq 1$. At odd N those computed are completely periodic (§9); at even N, Proposition 8.6 proves complete periodicity of an infinite family of isosceles triangles with $p \geq 1$.

7.5. **Boshernitzan's Conjecture 2.2(b).** Boshernitzan [Bos92, Conjecture 2.2] considers a rational triangle with angles $(\alpha, \beta, \chi)\pi$, d the least common denominator and $d' = \mathrm{lcm}(2, d)$, placed so that one side is horizontal or vertical (more generally, at an angle to the horizontal that is a multiple of π/d'), and conjectures in part (b) that for odd d every regular orbit is periodic in each direction of $S^1(2d') \setminus S^1(d')$ — the odd multiples of $\pi/2d$. Here $d = N$.

Corollary 7.9. *Let N be odd and place Δ with a side horizontal. Then $S^1(2d') \setminus S^1(d')$ is the class C, and if $p = 0$ — in particular if $\sum_\sigma n_\sigma = N+3$ — Conjecture 2.2(b) holds for Δ.*

Proof. With a side horizontal the side directions are the multiples of π/N, the perpendiculars are $\pi/2 + k\pi/N = (N+2k)\pi/2N$, odd multiples of $\pi/2N$, and there are $2N$ oriented ones: C is the whole set. Apply Theorem 7.8 and Corollary 7.7. □

Remark 7.10 (placement). Corollary 1.7 of [Bos92], whose hypotheses Conjecture 2.2 assumes, also admits a triangle with a *vertical* side. For odd d that placement makes $S^1(2d') \setminus S^1(d')$ the set of the side directions and their images under D_N instead, where (b) read literally fails: Delecroix and

Rüth found minimal components in directions parallel to a side [DR22], and Boshernitzan's own example — the direction $5\pi/11$ in $(2/11, 3/11, 6/11)$ *with the largest side horizontal*, where he reports orbits that do not close after $2 \cdot 10^8$ reflections [Bos92, §2] — lies in the orbit of the side directions there, and after a quarter turn to the vertical placement it would lie in $S^1(2d') \setminus S^1(d')$, contradicting his computer verification of (b) for $d \le 16$. More generally Corollary 1.7 admits a side at any multiple of π/d': at an even multiple $S^1(2d') \setminus S^1(d')$ is again the perpendicular class, and Corollary 7.9 applies unchanged; at an odd multiple it is the side-direction set, as for a vertical side. So (b) is meant with a side at an even multiple of π/d' — for instance horizontal — the placement of Corollary 7.9 (and the reading adopted, without comment, in the restatement of [DR22]).

Remark 7.11. Delecroix proposed testing, in the flatsurvey project [Del22], whether — when complete periodicity holds, especially for odd d — every saddle connection of the direction crosses a side perpendicular to it, as a possible explanation of the conjecture. By Lemma 7.5 that is $p = 0$ at odd N (at even N, for the saddle connections other than the horizontal arcs), and Theorem 7.8 shows it suffices. [DR22] reports an exhaustive search up to $d = 189$ that found no counterexample to (b), and counterexamples to parts (a) and (e) of Conjecture 2.2, among them $(7, 7, 16)$ and $(7, 8, 15)$.

8. The converse fails: swapped prongs

At odd N, $p \ge 1$ is rare: among all scalene triangles with odd $N \le 51$ it occurs only when some $\gcd(a_i, N) \ge 3$ (§9). In the instances we examined, each swapped prong is a vanishing sum of roots of unity in disguise — its holonomy is parallel to the direction, so its component along the normal, a sum of roots of unity, vanishes — and after cancelling opposite pairs the sum reduces to rotated regular q-gons with $q|N$. The simplest instances come from the identity $\cos \pi/3 = \frac{1}{2}$.

Proposition 8.1. *(F1) For $b \equiv 4 \pmod 6$ let $\Delta = (3, b, 2b)$, $N = 3(b+1)$; (F2) for $b \equiv 5 \pmod 6$ let $\Delta = (3, b, 2b+3)$, $N = 3(b+2)$. In both, N is odd and V_0 has angle $\alpha_0 = \pi/(N/3)$, and Δ has a prong from V_2 to V_1 (F1), resp. from V_1 to V_2 (F2), with $2(b-1)/3$, resp. $(2b+2)/3$, reflections, all in the two sides at V_0, and no right-angle collision. Hence $p \ge 1$ and $\sum_\sigma n_\sigma \le N + 1$.*

Proof. Take circumdiameter 1, so $|V_0V_1| = \sin\alpha_2$ and $|V_0V_2| = \sin\alpha_1$, and develop the fan of copies of Δ about V_0, the copies of V_0V_1 on rays at even multiples of α_0 and those of V_0V_2 on odd multiples (Figure 3). (F1) Let $Y = V_2$ on the ray at angle α_0 and Z the image of V_1 on the ray at angle $-n\alpha_0$, $n = 2(b-1)/3$ (even). Then $\angle YV_0Z = (n+1)\alpha_0 = 2\alpha_1 + \pi/N$, and since $\alpha_1 + \pi/N = \pi/3$, the triangle V_0YZ has angles $\pi/2 - \alpha_1$ at Y and $\pi/6$ at Z: these sum to $\pi - \angle YV_0Z$, and since $\sin\theta/\sin(c-\theta)$ is increasing in θ on $(0, c)$, the angles are fixed by the law of sines, which, with $|V_0Y| = \sin\alpha_1$

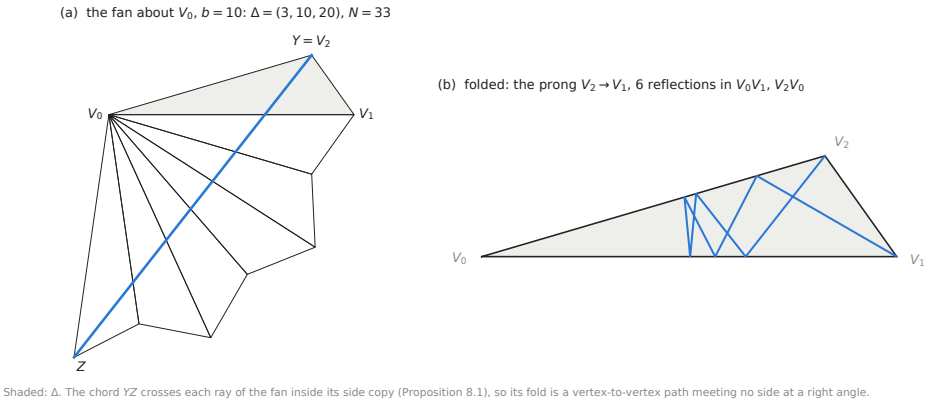


FIGURE 3. Proposition 8.1 at $b = 10$: $\Delta = (3, 10, 20)$, $N = 33$. (a) The fan of copies of Δ about V_0 (shaded: Δ) and the chord from $Y = V_2$ to the image Z of V_1 on the ray at angle $-6\alpha_0$. (b) Its fold: a prong from V_2 to V_1 with 6 reflections in the two sides at V_0, meeting no side at a right angle.

and $|V_0Z| = \sin 2\alpha_1$, reads $\sin 2\alpha_1/\cos\alpha_1 = \sin\alpha_1/\sin(\pi/6)$, true. (F2) Let $Y = V_1$ at angle 0 and Z the image of V_2 at angle $(n+1)\alpha_0 = 2\alpha_1 + 5\pi/N$, $n = (2b+2)/3$; with $x = \pi/N$ and $\alpha_1 + 2x = \pi/3$, the angles at Y and Z are $\pi/2 - \alpha_1 - \alpha_0$ and $\pi/6$, and the law of sines is $2\cos(\pi/3+x)\sin(\pi/3+x) = \sin(\pi/3-2x)$, true. In both, Y is on a ray of integer index and the angle at Y is a half-integer multiple of π/N, so the segment YZ has a direction in the class C.

It stays in the fan. The distance from V_0 to the line YZ along the ray at angle θ is convex in θ. On the rays carrying the longer side copies ($\sin\alpha_2 > \sin\alpha_1$: in (F1) $\alpha_2 = 2\alpha_1$ with $\alpha_1 < \pi/3$, in (F2) $\sin(\pi/3+x) > \sin(\pi/3-2x)$) it is therefore below the larger endpoint value, $\sin\alpha_2$; on the others its maximum is at the two extreme rays crossed, where the conditions are (F1) $\cos\alpha_1 < \cos(\alpha_1 - 2\pi/N)$ and $\cos\alpha_1 < \cos(\alpha_1 - 6\pi/N)$, the latter since $b > 3$; (F2) $\sin(\pi/3 - 2x) < \sin(\pi/3 + 4x)$, since $x < \pi/6$, and $\sin(5\pi/6 - 6x) > \frac{1}{2}$, since $N > 9$. So every crossing is interior to its side copy, the path reaches no vertex before Z, and its endpoints are distinct vertices. A prong that meets a side at a right angle retraces to its own vertex (step (iii) of the proof of Theorem 4.2), so this one meets none, and with its reverse it is a swapped pair. □

The other residues need longer paths, and there are more of them. For $m \geq 1$ let $N = 18m + 9$, $x = \pi/N$, and for $k = 0, \ldots, m-1$ let $w_{m,k}$ be the word $(02)^{2m-1-k}0 \cdot 1 \cdot (02)^{2m+2+k} \cdot 1 \cdot (02)^k 0$ in the sides $0 = V_0V_1$, $1 = V_1V_2$, $2 = V_2V_0$ (length $8m + 6 + 2k$).

Proposition 8.2. *In both $\Delta_1 = (3, 6m+2, 12m+4)$ and $\Delta_2 = (3, 6m+1, 12m+5)$, for each $k = 0, \ldots, m-1$ the billiard path leaving V_2 in direction*

$\psi_k = -5\pi/6 + (8+3k)x$ (Δ_1), resp. $-5\pi/6 + (7+3k)x$ (Δ_2), has bounce word $w_{m,k}$ and ends at V_2, reaching no vertex between and meeting no side at a right angle. Hence $p \geq m$ and $\sum_\sigma n_\sigma \leq N + 3 - 2m$.

Proof. $\alpha_0 = 3x$; in Δ_1, $\alpha_1 = \pi/3 - x$, $\alpha_2 = 2\pi/3 - 2x$; in Δ_2, $\alpha_1 = \pi/3 - 2x$, $\alpha_2 = 2\pi/3 - x$. With circumdiameter 1 the altitude from V_0 is $\eta = \sin\alpha_1 \sin\alpha_2$. A run of reflections in sides 0 and 2 develops the fan about an image of V_0 (a *star*: images of V_1 on even rays, of V_2 on odd rays, rays α_0 apart), and a reflection in side 1 reflects the star across a copy of V_1V_2, moving its centre by 2η along that copy's normal. Write $n_1 = 4m - 2 - 2k$. Star 1 has centre $O_1 = 0$ and ray j at angle $j\alpha_0$; the path starts at the image Y of V_2 on ray 1, crosses rays $0, -1, \dots, -n_1$, and leaves across the copy e_1 of side 1 between rays $-n_1$ and $-(n_1+1)$, whose normal is $\varphi_1 = -n_1\alpha_0 - (\pi/2 - \alpha_1)$. Star 2, the reflection of star 1 in e_1, has centre $O_2 = 2\eta e^{i\varphi_1}$ and ray j at angle $2\varphi_1 + \pi - j\alpha_0$; the path crosses rays $-n_1, \dots, 5 + 4k$ and leaves across e_2, between rays $5+4k$ and $6+4k$, with normal $\varphi_2 = 2\varphi_1 + \pi - ((6+4k)\alpha_0 - (\pi/2 - \alpha_1))$. Star 3 has centre $O_3 = O_2 + 2\eta e^{i\varphi_2}$ and ray j at angle $2(\varphi_2 - \varphi_1) + j\alpha_0$; the path crosses rays $6+4k, \dots, 6+6k$ and ends at the image Z of V_2 on ray $7+6k$. Since $12m \cdot x = 2\pi/3 - 6x$, every angle in this construction is a fixed integer combination of $\pi/6$, x and $y \coloneqq kx$, and (m,k) enter only through the point (x,y) of the region $0 < x \leq \pi/27$, $0 \leq y \leq Y(x) \coloneqq (\pi - 27x)/18$ ($k \leq m-1$ is $y \leq Y(x)$).

Direction. $Z - Y = \sin\alpha_1 \cdot [2\sin\alpha_2(e^{i\varphi_1} + e^{i\varphi_2}) + e^{i\theta_Z} - e^{i\alpha_0}]$, $\theta_Z \coloneqq \arg(Z - O_3)$, and the condition is $\Im((Z-Y)e^{-i\psi_k}) = 0$. Put $\beta \coloneqq 3x + 3y$. In Δ_1, $\varphi_1 - \psi_k = \beta$; $\varphi_2 = -\pi/2 + 5x$ does not depend on k, and $\varphi_2 - \psi_k = \pi/3 - \beta$; $\theta_Z - \psi_k = 3\pi/2 + x + 3y$; and $\alpha_0 - \psi_k = 5\pi/6 - 5x - 3y$. So the condition reads

$$2\sin\alpha_2[\sin\beta + \sin(\pi/3 - \beta)] - \cos(x + 3y) - \sin(5\pi/6 - 5x - 3y) = 0.$$

Here $\sin\beta + \sin(\pi/3 - \beta) = 2\sin(\pi/6)\cos(\beta - \pi/6) = \cos(\beta - \pi/6)$ — the identity $\cos\pi/3 = \frac{1}{2}$ of Proposition 8.1 — and $2\sin\alpha_2\cos(\beta - \pi/6) = \sin(\alpha_2 + \beta - \pi/6) + \sin(\alpha_2 - \beta + \pi/6)$, which at $\alpha_2 = 2\pi/3 - 2x$ is $\cos(x + 3y) + \sin(5\pi/6 - 5x - 3y)$: the condition holds, for every (x,y). Δ_2 is the same with $\alpha_2 = 2\pi/3 - x$, $\varphi_2 = -\pi/2 + 4x$, and $x + 3y$, $5x$ replaced by $2x + 3y$, $4x$. The direction ψ_k is a half-integer multiple of x, so it lies in the class C, and it points into the angle at V_2: $\psi_k - (\alpha_0 - \pi) = \pi/6 + 5x + 3y$ (Δ_1), $\pi/6 + 4x + 3y$ (Δ_2), both in $(0, \pi/3 + x/2] \subset (0, \alpha_2)$.

Containment. Put $\mathrm{side}(z) \coloneqq \Im((z-Y)e^{-i\psi_k})$; the line crosses a segment OP iff $\mathrm{side}(O)\,\mathrm{side}(P) < 0$. In each star it suffices to check the extreme crossed rays of each vertex type — the vertices of one type lie on a circle about the centre, the centre is on the near side, and the extreme rays of one type span less than π (at most $2\pi/3$, $\pi - 9x$ and $\pi/3$ in stars 1, 2, 3), so every ray between them is crossed — together with the exit copies e_1, e_2: fourteen conditions, two of which (star 3's rays $7+4k$ and $5+6k$) arise only when $k \geq 1$ (at $k = 0$ star 3's rays $6+4k$ and $6+6k$ coincide). A line

meets each triangle copy's boundary twice, so consecutive crossings lie in a common copy and the path is the straight segment through the claimed copies, interior at every crossing.

Each side function involved is exactly of the form $a(x)\cos 3y + b(x)\sin 3y$: the vertices carry the phases 1 or e^{6iy}, and ψ_k carries e^{3iy}. On an interval of y shorter than $\pi/3$ such a function, if not identically zero, has at most one zero, and that zero is simple; and $3Y(x) = \pi/6 - 9x/2 < \pi$. So a side function has constant sign on $0 \le y \le Y(x)$ as soon as its two end values $f(x,0)$ and $f(x,Y(x))$ are nonzero and of the same sign; for the two conditions with $k \ge 1$, which force $m \ge 2$ and so $x \le \pi/45$, the ends are $y = x$ and $y = Y(x)$. The end values are trigonometric polynomials in $u = x/2$ with coefficients in $\mathbb{Q}(\sqrt{3})$; with $t = \tan(u/2)$ each is $(A(t) + \sqrt{3}B(t))/(1+t^2)^d$ with $A, B \in \mathbb{Q}[t]$, and the range is $t \in (0, \tan(\pi/108)] \subset (0, 3/100]$ (resp. $(0, \tan(\pi/180)] \subset (0, 7/400]$). An exact real-root count (Sturm) shows that the norm $A^2 - 3B^2 \in \mathbb{Q}[t]$, of degree at most 92, has, once its factor t^j is removed, no root there for every end value but one. The exception is $\mathrm{side}(O_1) = -\sin\alpha_1 \sin(5\pi/6 - 5x)$ for Δ_1 at $y = 0$, which is visibly nonzero; its norm vanishes at the conjugate point $x = \pi/30$. Read exactly at $t = 1/100$, the signs satisfy all fourteen conditions, in both families. This step is computer-assisted but exact: no floating-point value enters.

Finally $w_{m,k}$ is not a palindrome (its outer runs have lengths $4m-1-2k > 2k+1$), so the path meets no side at a right angle (a prong that does retraces, by step (iii) of the proof of Theorem 4.2, and its word is then a palindrome by time reversal, as in Theorem 3.1(1)). Its two ends are V_2 at different slots — equal slots would make the path its own reverse, with a palindromic word — so with its reverse it is a swapped pair. The reverse has first run of length $2k+1 \le 2m-1$, while every $w_{m,l}$, $l \le m-1$, has first run $4m-1-2l \ge 2m+1$; so no path of the statement is the reverse of another, and the m pairs are distinct. (The reverse is the path in direction ψ_{2m-1-k}, the formula for ψ_k read at $2m-1-k$.) □

Propositions 8.1 and 8.2 together give Theorem 1.4: $b \equiv 4, 5 \pmod 6$ are Proposition 8.1 and $b \equiv 2, 1 \pmod 6$, $b \ge 7$, Proposition 8.2, with $m = \lfloor b/6 \rfloor$. (At $b = 1, 2$, $N = 9$, $p = 0$.)

Remark 8.3. On the residues of Proposition 8.2 the exact count is $p = m$ in all computed cases (§9), so there the $2m$ swapped prongs are exactly those of Proposition 8.2 and their reverses; we have not proved that no other prong is swapped. So on these triangles the defect $2p$ of Corollary 7.7 grows linearly with b, while the class stays completely periodic wherever it has been checked.

Conjecture 8.4. *If N is odd and $p \ge 1$, then $\gcd(a_i, N) \ge 3$ for some i.*

By Theorem 7.8 and Corollary 7.9, Conjecture 8.4 implies Conjecture 2.2(b) for every triangle with N prime, placed with a side horizontal. It is not a criterion: of the 626 scalene triangles with odd $N \le 51$ and

some $\gcd(a_i, N) \geq 3$, only 18 have $p \geq 1$ (§9). The evidence is for scalene triangles. For an isosceles triangle (a, a, b) with N odd every $\gcd(a_i, N)$ is 1 ($\gcd(a, N) = \gcd(a, b)$, $\gcd(b, N) = \gcd(b, 2a)$), so the conjecture predicts $p = 0$; the next remark shows that this is a statement about right triangles.

Remark 8.5 (isosceles triangles at odd N). Let $I = (a, a, b)$, N odd, and let T be its right half, with $2\alpha/\pi = b/N$ at the apex O, base vertex A and right angle at R. The reflection in the axis is not in D_N (the axis has index $N/2$) but is in T's group D_{2N}, so the unfolding of I is the unfolding of T, with the images of R unmarked; the class of I is the class of T in which both T's hypotenuse and its half base are vertical, and complete periodicity is the same on both. Write p_T^{OA} for the number of swapped pairs of that class of T joining images of O and A, and p_T^{OR} for those joining an image of O or A to an image of R. Folding paths of I onto T shows $p_I = 2p_T^{OA} + p_T^{OR}$: a swapped pair of T between images of O and A lifts to two swapped pairs of I, a swapped pair from O or A into R lifts to one — the paths of I through the midpoint of the base, each symmetric to its own reverse under the axis reflection — and pairs from R to R lift to closed paths. Hence, by Theorem 7.8 and [Neck, Corollary 7.11], $p_I = 0$ if and only if the class is completely periodic — unconditionally when N is a prime power or $\min(b, 2a) \leq 2$, and otherwise under the hypothesis (OB) of [Neck, §1.5] at b/N. So for isosceles triangles Conjecture 8.4 implies Conjecture 2.2(b), and conversely wherever (OB) holds: there both are complete periodicity of one class of right triangles at odd denominator. It holds for every isosceles triangle with odd $N \leq 39$ (§9), and it is proved in [Meth, Corollary 8.6] whenever N has at most one odd prime factor (with $b \geq 3$; computer-assisted in part) and on further families of composite N, through this reduction.

At even N the isosceles picture is different, and there the converse of Theorem 7.8 fails with both halves proved. Let C_0 be the class of $I = (a, a, b)$ containing the direction of the base: in the indexing of §2, with the base at index 0, the even integers. The legs have the odd indices $\pm a$, so the legs are vertical in C_0 when $N/2$ is odd and the base is vertical in C_0 when $N/2$ is even.

Proposition 8.6 (isosceles triangles at even N)**.** *Let $I = (a, a, b)$ with $N = 2a + b$ even and $a \geq 3$. Then C_0 has $p \geq 1$. If moreover $b = 2$, or $b \geq 4$ and either N has at most one odd prime factor or $a \not\equiv \pm 2 \pmod q$ for every odd prime $q|N$, then both classes of I are completely periodic.*

Proof. Put the base AA' on the horizontal axis, $A = (-1, 0)$, $A' = (1, 0)$, so the apex is $(0, \tan\beta)$ with base angle $\beta = \pi a/N$; let $\theta := \pi - 2\beta = \pi b/N$, the apex angle. First let $b < 2a$, that is $\theta < \pi/2$; as N is even, $b \neq a$. Suppose first $b < a$, that is $\theta < \beta$. Then the path leaving A at angle θ above the base points into the angle at A. It meets the leg A'–apex at $x = (\tan\beta - \tan\theta)/(\tan\beta + \tan\theta) \in (0, 1)$, an interior point, at angle β to that leg. The reflected direction is horizontal, so the path crosses the axis

at a right angle, at a height below the apex, and meets the other leg at the mirror point, again at angle β. By the symmetry in the axis it then runs to A'. Its direction has index b, which is even, so it is the prong of a slot of C_0. Both angles of incidence are $\beta < \pi/2$ (the axis is not a side), so it meets no side at a right angle. With its reverse, which ends at A, it is a pair counted by p.

Now suppose $a < b < 2a$, that is $\beta < \theta < \pi/2$, and start at the apex instead. The path leaving the apex at angle θ below the horizontal is steeper than the leg A'–apex, so it points into the angle at the apex. It meets the base at $x_0 = \tan\beta/\tan\theta \in (0,1)$, at angle θ, leaves it at angle θ above the base, and meets the leg A'–apex at $x = 2\tan\beta/(\tan\beta + \tan\theta) \in (x_0, 1)$, at angle $\pi - \beta - \theta = \beta$. As before the reflected direction is horizontal, and by the symmetry in the axis the path meets the base at $-x_0$ and returns to the apex, after four bounces (base, leg, leg, base) at angles θ, β, β, θ, all below $\pi/2$. Its direction has index $2N - b$, which is even, and its reverse leaves the apex with index $N + b$, a different slot since $b \neq 2a$; so again C_0 has a pair counted by p.

Now let $b > 2a$ ($b = 2a$ would force $a = 1$). Let T be the half of I to the right of the axis, with its right angle at the midpoint F of the base. It is the right triangle of [Neck] at the centre $P/Q = a/(N/2)$ (in lowest terms, as $\gcd(a, N/2) = \gcd(a, b/2) = 1$): [Neck]'s vertex O, of angle $\pi a/N$, is A', its right angle R is F, its third vertex is the apex, and its leg L_1 is the axis, of length ℓ. (These names differ from those of Remark 8.5.) The standing hypothesis of [Neck, §9], $P \geq 3$ and $Q - P \geq 3$, holds, since $Q - P = b/2 > a \geq 3$. The beam of the axis in T (§2) is the beam of horizontal paths of I leaving the axis to the right; its directions lie in C_0, a meeting with the axis is a crossing of it in I, and its branches are those of [Neck] (by [Neck, Lemma ($\star$)] of §5, a branch boundary of [Neck] reaches its vertex before the first right-angle meeting). If a singular point of the axis has a path that first reaches A' or the apex, V say, then that path and its mirror image in the axis form a path of I from the mirror image of V to V that crosses the axis at a right angle and meets no side of I at a right angle. Its two ends are prongs of C_0, each the other's reverse, and distinct (at the apex they are mirror directions, equal only for the axis itself), so they form a swapped pair.

By [Neck, Lemma 8.1 (1)], with $m \coloneqq \lceil 2Q/P \rceil - 1$, the path from a certain interior point y of the axis first reaches the apex when m is odd, which gives a pair as just shown, and reaches F when m is even. In the latter case suppose that no singular point has a path first reaching A' or the apex. A path that first reaches F is sent back along itself by the two legs of T there and returns to its starting point, so no path of the beam meets an image of A' or of the apex before its first right-angle meeting, and the whole axis is one crossing interval in the sense of [Neck, §9]. By [Neck, Proposition 9.1] it carries at most two branches, and if two, a mirror pair with the singular point between them. The point y is singular, so the branches are $(0, y)$ and

(y, ℓ), measured from F, with $J(s) = 2y - s$ [Neck, proof of Proposition 5.3 (ii)]; as J maps one onto the other, $y = \ell/2$ and $J(s) = \ell - s$. On $(0, \ell/2)$ the sides struck and the developing isometry are constant (proof of Theorem 3.1), so in one development the paths from the points s are parallel segments, from s to the point $\ell - s$ of one copy of the axis. As $s \to 0$ they converge to the segment that runs from F along the base to A' and on to the image of the apex at the end of that copy. Let σ be its part from A' to the first image of A' or of the apex on it (images of F are regular points of the development; in I they are the midpoint of the base, met at an angle other than $\pi/2$, since σ does not run along the axis). σ leaves A' strictly inside the angle there, as $\pi/(\pi a/N) = 2Q/P$ is not an integer. It meets no side at a right angle: the paths from small s run parallel to it and strike the same copies of the sides, so they would meet that side at a right angle as well, before their return at $\ell - s$. Unfolded across the axis, σ is a path of I in C_0 from A' to a vertex, meeting no side at a right angle, and it is not its own reverse, which would need a right angle or a vertex at its midpoint. With its reverse it is a swapped pair.

At $b = 2$ (and at $b = 4$) the right half has an angle π/N (resp. $2\pi/N$), so it is a lattice triangle [Vee89], [KS00, Theorem 2]; every direction with a saddle connection is completely periodic, and each class of I contains a horizontal arc, so both classes are completely periodic with no appeal to [Meth] (the unfolding of I covers that of its right half; cf. [CHM08, §1], whose billiard example is the triangle $(3, 3, 4)$). Under the remaining hypotheses (with $a \geq 3$ and $b \geq 4$), [Meth, Corollary 8.6] (parts 1 and 2) gives Conjecture 2.2(e) for I: every direction in $S^1(2d')$, the multiples of π/N measured from the base, is completely periodic, and these are the directions of both classes of I. Its proof makes both perpendicular classes of the right half completely periodic ([Meth, Corollary 8.1], parts 1 and 2; part 1 is computer-assisted in part) and transfers them to I by [Meth, Lemma 8.4(ii)]. □

So the converse of Theorem 7.8 already fails at $(3, 3, 2)$. The first triangle of Proposition 8.6 whose right half is off the lattice locus $\min(P, Q - P) \leq 2$ [KS00] is the obtuse $(3, 3, 8)$ (half 3/7), which needs [Meth, Corollary 8.1, part 1]; the first acute ones are $(5, 5, 6)$ and, with $b < a$, $(7, 7, 6)$.

Neither acute pair owes anything to arithmetic. Each exists because C_0 contains the base direction, so a path can be symmetric under the axis with a horizontal middle segment. That is also why the count $p_I = 2p_T^{OA} + p_T^{OR}$ of Remark 8.5 has no analogue on C_0. The middle segment crosses the axis at a right angle, so each path folds onto a path of T that retraces from L_2, which p_T does not count. The swapped pairs of Propositions 8.1 and 8.2, at odd N, are not of this kind. When $N/2$ is odd, C_0 is the legs' class of Theorem 6.2, so there the band bound of Theorem 5.2 is strict ($R < \lceil N/4 \rceil$) whenever $a \geq 3$, also where the class is completely periodic. Theorem 6.2(3) identifies equality with complete periodicity only for even $N/2$.

9. Computations

Everything above is proved by hand except the containment step of Proposition 8.2 (an exact root count) and, through [Meth, Corollary 8.1, part 1], the case of Proposition 8.6 in which N has one odd prime factor and $a \equiv \pm 2$ modulo it; the computations below are checks, not inputs. Branches and prongs were computed exactly, in the cyclotomic field of the triangle, from the interval exchange of the unfolded surface: an exact classifier of the branches of every side (retracing, mirror, translation) and an exact tracer of every prong.

- *The slot identity and the band bound* hold on every class of every scalene triangle with odd $N \le 21$ or even $N \le 34$ (820 classes), the classifier's branch counts and the tracer's prong census agreeing side by side. The identity is not vacuous: on the non-completely-periodic class of $(7, 8, 15)$ (found in [DR22]) the tracer finds $s' = 6$, $p = 1$, and $\Sigma n_\sigma = 15 - 2 + 1 - 2 - 6 = 6$, as observed.
- *Odd N.* Among the 1578 scalene triangles with odd $N \le 51$, $p \ge 1$ occurs on 18, every one among the 626 with some $\gcd(a_i, N) \ge 3$, and every one completely periodic ($s' = 0$); with $N \le 21$ they are $(2, 5, 8)$, $(3, 4, 8)$, $(3, 5, 13)$ and $(4, 6, 11)$, where $\sum_\sigma n_\sigma = N + 1$. At the prime values $N = 23, 29, \ldots, 47$, $p = 0$ on every triangle.
- *The families of Theorem 1.4* with $N \le 303$: every member is completely periodic, and p is 1 on the residues of Proposition 8.1 and $(b-2)/6$, resp. $(b-1)/6$, on those of Proposition 8.2 — its lower bound, attained.
- *Isosceles triangles* with even $N \le 30$: the formulae of Theorem 6.2(1)–(2) hold. With odd $N \le 39$ (158 triangles): the class of Remark 8.5 is completely periodic with $p = 0$ on all of them — 157 read off an exact certificate for the right half, and $(17, 17, 3)$, whose half needs separatrices of length above 10^9, by an exact decomposition of the direction; the exact decomposition of each triangle's own directions agrees on all 158. With even $N \le 60$ and $b < 2a$ (91 triangles, 31 with $a < b < 2a$), both classes are completely periodic by an exact decomposition, including the members outside Proposition 8.6's hypotheses. On the same instrument the minimal components are found in a perpendicular class of $(7, 7, 16)$ ($b > a$; found in [DR22]) and, for $(4, 4, 7)$, in the side directions, outside its perpendicular class.

Code availability. The root count of Proposition 8.2 and the computations listed above are deposited at `https://github.com/josh-boorstein/billiards`, whose manifest maps each one to the statement it supports; this note's code is in the directories `lib/` and `polygon/`. The version used here is release v1.1, archived at Zenodo as doi:10.5281/zenodo.24093456.

10. Open problems

1. **Conjecture 2.2(b).** A counterexample would need a minimal component bounded by swapped pairs alone (Theorem 7.8). Conjecture 8.4 would settle prime N; for isosceles triangles it is equivalent to Conjecture 2.2(b) wherever (OB) holds (Remark 8.5).
2. **When is $p \geq 1$?** Beyond Theorem 1.4 the known swapped prongs at odd N come from pentagon identities as well as the triangle identity of §8; a family built on $\cos(2\pi/5)$ should exist. At even N, Proposition 8.6 gives one swapped pair on C_0; in each of the 22 isosceles triangles whose prongs we traced completely, C_0 has exactly $(n-1)/2$ of them, n the branch count of [Neck] at $a/(N/2)$. Is that an identity?
3. **Exact counts.** Theorem 4.2 reduces $\sum_{\sigma \text{ vertical}} n_\sigma$ to p and s'; the split among the vertical sides is not determined. At even N, in non-right triangles, one of the two sides at the even vertex frequently carries only retracing branches; when, and why, is open.
4. **Polygons.** Theorem 3.1, the first sentence of Proposition 3.3 and Proposition 3.4 hold in any rational polygon, and the proof of Proposition 5.1 does not use that Δ is a triangle; we have not checked the slot identity or Theorem 7.8 beyond triangles.
5. **Irrational triangles.** With no rotation by π in the group there are no mirror pairs, and the return map pairs branches only by translation (as in Ruijgrok's gapped twins [Rui91, §4]). Is there a structure theorem?

Use of AI

This paper — its text, its proofs and the code it deposits (Section 9, *Code availability*) — was drafted by an AI system (Claude, Anthropic) working under the author's direction, and checked by independent AI review passes and by the exact computations described in Section 9. The author has read parts of it in detail and takes responsibility for its content.

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