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NON-PERPENDICULAR PERIODIC ORBITS IN MIRROR DIRECTIONS OF RIGHT TRIANGLES

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ABSTRACT. Let T be a right triangle with an acute angle a. A direction is a *mirror direction* of T if it is the axis of a reflection in the group generated by the reflections in the sides of T — for instance, any direction parallel or perpendicular to a side. A billiard path that meets a side at a right angle retraces itself, its directions are mirror directions, and it is periodic as soon as it meets a side at a right angle a second time. We ask whether every periodic billiard path in a mirror direction is of this kind. When $\cos 2a$ is transcendental it is: every such path meets a side at a right angle, and in the directions parallel to the hypotenuse there is no periodic path at all. The proof is a holonomy computation on Hooper's chain of rhombi, and it bounds the degree of $\cos 2a$ for any exception. When $\cos 2a$ is algebraic and a/π irrational, exceptions can occur: we construct non-perpendicular periodic cylinders in mirror directions explicitly — among them an infinite family indexed by the solutions of $u^2 - 2v^2 = -1$ — and show that the triangles carrying them form a countable dense set in each of the four classes of mirror directions. Corollaries include the absence of odd-period orbits in right triangles and the existence of one-way periodic paths in every right triangle (answering two questions of Holt), a description of the odd orbits of isosceles triangles, and the statement that Ruijgrok's third conjecture, read for right triangles, fails on a countable dense set of triangles, all with $\cos 2a$ algebraic, and holds whenever $\cos 2a$ is transcendental.

1. INTRODUCTION

1.1. **The question.** A billiard path in a polygon that meets a side at a right angle retraces itself, and it is periodic as soon as it meets a side at a right angle a second time. In a right triangle T these *perpendicular* orbits are plentiful: Cipra, Hanson and Kolan showed that almost every orbit leaving a leg perpendicularly returns to it [CHK95], Troubetzkoy that for an open set of irrational right triangles every non-singular perpendicular orbit but one is periodic, and that so is every non-singular orbit of their invariant surface but one [Tro05, Thm 3]; Hooper showed that no periodic path of a right triangle is stable [Hoo07]. The directions in which perpendicular orbits travel are the *mirror directions*: the axes of the reflections in the group G generated by the reflections in the sides of T. For a right triangle whose acute angle a is an irrational multiple of π, they fall into four G-orbits, the **mirror classes**: the directions of the two legs (each perpendicular to the other leg), of the hypotenuse, and of its normal.

This paper asks whether a periodic path in a mirror direction must be perpendicular. The question goes back to Ruijgrok's numerical study of triangular billiards, whose third conjecture asserts that an irrational triangle has no periodic orbit of this kind that is not perpendicular [Rui91]; for right triangles, his *half-integer* orbits are exactly the paths in mirror directions (the proof of Corollary 6.8).

Perpendicular orbits are periodic for a reason of symmetry: the reflection in the side a path meets at a right angle carries the path to itself, reversed. The mirror directions are exactly the directions fixed by a reflection in G, so they are where this mechanism can act, and the question is whether it is the only one there. Theorem A below says that it is when $\cos 2a$ is transcendental; Theorem B, that it is not on a countable dense set of triangles. For context: in a rational polygon periodic orbits are dense in the phase space [BGKT98]; for a dense G_δ of polygons, necessarily irrational ones, the billiard flow is ergodic [KMS86]; see [MT02] for a survey. Whether every triangle has a periodic billiard path was long open: Schwartz proved it when all angles are at most $100°$ [Sch09], Tokarsky, Garber, Marinov and Moore up to $112.3°$ [TGMM18], and Forni has recently announced a proof for every polygon [For26]; in a right triangle perpendicular orbits settle it [Sch09, §1.1]; the structure of the periodic paths in mirror directions is the natural next question.

1.2. **Results.** Write $T = Tri(a)$ for the right triangle with an acute angle a (§2.1). The answer depends on the arithmetic of $\cos 2a$.

Theorem A (Theorem 3.1)**.** *If $\cos 2a$ is transcendental, every periodic billiard path of T in a mirror direction meets a side at a right angle; in particular, since a path with a segment parallel to the hypotenuse never meets a side at a right angle, there is no periodic path parallel to the hypotenuse.*

The proof is a holonomy computation on Hooper's chain of rhombi: a non-perpendicular periodic path stays on one side of the *fold* of its direction, and there its closing condition — that its translation vector have no component normal to its direction — says that a nonzero integer polynomial vanishes at $\cos 2a$. So any exception has $\cos 2a$ algebraic, of degree bounded by how far the path reaches along the chain (Theorem 3.4); the bound is attained in the leg classes. Rational triangles have no exceptions either, by a different argument (Theorem 4.4, from [Neck, Corollary 12.1]).

Theorem B (§5, Theorem 6.1 and §6.4)**.** *Non-perpendicular periodic cylinders in mirror directions exist. In each of the four mirror classes the set of $x = 2a/\pi$ with $a/\pi \notin \mathbb{Q}$ at which $Tri(a)$ carries one is countable and dense in $[0, 1]$, consisting of points with $\cos 2a$ algebraic.*

We give explicit examples (the right triangle with legs $1 : \sqrt{7}$ carries one of period 16). The density comes from a crossing argument: a cylinder that persists over an interval of angles crosses the directions of every mirror

class densely (§6); five explicit corner words suffice to cover all angles, with tiles bounded by algebraic numbers of degree at most three in $\tan^2 a$, and a theorem of Ihara, Serre and Tate on torsion points of curves [Lan65] shows that each word crosses the mirror directions at only finitely many rational angles. The cylinders that cross only two doors of the chain are described completely (§7): they come in ladders converging to rational angles, a ladder is infinite if its limit has $|\cos 2a^*| > \frac{1}{2}$ and finite if $0 < |\cos 2a^*| < \frac{1}{2}$, and the two ladders at $a = \pi/8$ together yield an infinite family with rational $\cos 2a$, indexed by the solutions of the negative Pell equation $u^2 - 2v^2 = -1$.

So the picture is a trichotomy:

a	non-perpendicular periodic paths in mirror directions
$a/\pi \in \mathbb{Q}$	none (Theorem 4.4, [Neck, Corollary 12.1])
$\cos 2a$ transcendental	none (Theorem A)
$\cos 2a$ algebraic, $a/\pi \notin \mathbb{Q}$	at a countable dense set of triangles in each class (Theorem B); whether at all of them is open (§8)

Corollaries. No right triangle has a periodic path of odd period (Corollary 4.1), which answers a question of Holt [Hol93]. Theorem A transfers to isosceles triangles class by class (Corollary 4.2), and the odd orbits of an isosceles triangle are described: each primitive one bounces once at the midpoint of the base and crosses the axis at a right angle exactly once (Proposition 4.3). Ruijgrok's third conjecture, read for right triangles, fails on a countable dense set of triangles, all with $\cos 2a$ algebraic, and holds whenever $\cos 2a$ is transcendental (Corollary 6.8). Every right triangle carries a periodic cylinder, in some direction, none of whose paths meets a side at a right angle (Corollary 6.7); perpendicularity is forced only in mirror directions.

1.3. **Relation to earlier work.** The chain of rhombi is in [CHK95], and as a translation surface it is Hooper's $MT(T)$ [Hoo07, §2]; Lemma 3.2 is the elementary closing condition for a path developed in it; his description of the core of a periodic cylinder [Hoo07, Thm 3] is used throughout. What is new in §3 is the fold (Lemma 3.3) and the degree bound it yields (Theorem 3.4). Troubetzkoy proved density of periodic orbits in the phase space of irrational right triangles, building them from perpendicular directions, and observed that the direction of the hypotenuse splits the invariant surface in two, which is why his theorem on escape orbits [Tro05b, Thm 5] excludes it; Theorem A shows that for transcendental $\cos 2a$ this direction has no periodic orbit at all, while Gutkin and Troubetzkoy's strong recurrence [GT96] shows that it is nevertheless recurrent. Periodic *mirror trajectories* in right triangles were introduced by Vorobets, Galperin and Stëpin [GSV92, Conclusion, Fig. 50] and analysed by Galperin and Zvonkine [GZ03, §3], indexed by a *type* $n \geq 1$ — the trajectory starts parallel to the line through the centre of the first

rhombus of their corridor and that of its n-th upper rhombus, and has period $10n-4$ [GZ03, Prop. 3]; every type exists exactly when the smaller angle is at most $\pi/6$ [GZ03, Thm 2]. Those of type 2, of period 16, are the cylinders of two of the five words of §6.3, and the two-door cylinders of §7 live in the same corridor (§7). Our contribution there is that these cylinders are not perpendicular and that, crossed with the mirror classes, they produce exceptional triangles. As far as we can tell, neither Theorem A nor any non-perpendicular periodic cylinder in a mirror direction of a right triangle has appeared in print, and Theorem B contradicts a claim that has: [Tro05, abstract, (iii)] (Remark 6.10). Vorobets, Galperin and Stëpin show that in a polygon whose angles are rationally independent the rotational part of the holonomy of a periodic orbit forces it to be stable [GSV92, §6]; a right triangle lies outside that class, its acute angles summing to $\pi/2$, and Theorem A works inside this one-parameter family with the translational part of the holonomy, which there is a polynomial in $\cos 2a$ that the fold forces to be nonzero. The structure of the exceptional cylinders beyond the two-door ones of §7, and their number, will be treated elsewhere.

1.4. **Organisation.** §2 sets up the mirror classes and the chain of rhombi. §3 proves Theorem A. §4 collects corollaries: odd periods, isosceles triangles, and the rational case. §5 constructs exceptional cylinders, §6 proves the density, and §7 describes the two-door cylinders completely. §8 lists open questions. The exact certificates used in §5 and §6 are deposited with the paper.

2. Setup

2.1. **The triangle, its group, and the mirror directions.** Throughout, $T = Tri(a)$ is the right triangle with vertices $O = (0,0)$, $A = (1,0)$, $B = (0, \tan a)$, $0 < a < \pi/2$, so that the right angle is at O and the angle a at A. We write $x \coloneqq 2a/\pi \in (0,1)$, $y \coloneqq \cos 2a$, H for the hypotenuse AB, $h \coloneqq \sin a$ for the distance from O to H, and $e(\varphi) \coloneqq (\cos\varphi, \sin\varphi)$. Directions are taken modulo π unless a sign is needed; ρ_φ is the rotation by φ.

Let G be the group generated by the reflections $\sigma_O A$, $\sigma_O B$, σ_H in the lines of the three sides. Since $\sigma_H \sigma_O A = \rho_{\pm 2a}$ and $\sigma_O A \sigma_O B = \rho_\pi$, the rotations in G are the ρ_φ with $\varphi \in 2a\mathbb{Z} + \pi\mathbb{Z}$, and every element of G is either such a rotation or $\rho_\varphi \sigma_O A$, the reflection in the line of angle $\varphi/2$. The axes of the reflections in G are therefore the directions in $a\mathbb{Z} + (\pi/2)\mathbb{Z}$. (The elements of G are affine maps; when only their action on directions matters we use the same letter for an element and its linear part, as we just did.)

Definition 2.1. A direction θ is a **mirror direction** of T if it is the axis of a reflection in G, i.e. $\theta \in a\mathbb{Z} + (\pi/2)\mathbb{Z} \pmod{\pi}$. The four directions of the sides and of their normals are mirror directions, and so is every image of one of them under G.

G acts on directions through $\varphi \mapsto \varphi + 2an$ and $\varphi \mapsto -\varphi$. So when $a/\pi \notin \mathbb{Q}$ the mirror directions fall into exactly four G-orbits, the **mirror classes**:

class	directions (mod π)	contains
$\perp OB$	$2aj$, $j \in \mathbb{Z}$	the direction of OA
$\perp OA$	$\pi/2 + 2aj$	the direction of OB
$\parallel H$	$(2j-1)a$	the direction of H
$\perp H$	$\pi/2 + (2j-1)a$	the normal to H

(The four orbits are distinct because $a/\pi \notin \mathbb{Q}$; their union is $a\mathbb{Z}+(\pi/2)\mathbb{Z}$.) We call $\perp OB$ and $\perp OA$ the **leg classes**; they are exchanged by swapping the roles of the two legs, i.e. by $a \mapsto \pi/2 - a$.

The direction of a billiard path changes at each bounce by an element of G, so the class of its direction is constant along the path, and we speak of a **path in the class cls**. A billiard path is **perpendicular** if it meets a side at a right angle (at a point that is not a vertex); it then retraces itself, every direction it takes is a mirror direction, and if it is periodic it meets a side at a right angle exactly twice per period. A periodic path in a mirror direction that never meets a side at a right angle is **non-perpendicular**. Periodic paths come in **cylinders** — maximal open families of parallel periodic paths with the same sequence of bounces — and all paths of a cylinder are perpendicular or none is, since they meet the sides in the same order and at the same angles. The **width** of a cylinder is the distance between its two boundary lines, measured perpendicular to its direction.

Definition 2.2. For a class cls, the **exceptional set** $E_{\mathrm{cls}} \subset (0,1)$ is the set of $x = 2a/\pi$ such that $a/\pi \notin \mathbb{Q}$ and $Tri(a)$ carries a cylinder of non-perpendicular periodic paths in the class cls.

2.2. **The unfolding as a chain of rhombi.** We use the chain of rhombi ([CHK95]; as a translation surface, Hooper's $MT(T)$ [Hoo07, §2]) as the unfolding of an irrational right triangle; we re-derive what we need.

Rhombi and levels. Reflecting T in its two legs gives the rhombus R with centre O and vertices $\pm A, \pm B$. Its diagonals are copies of the legs, and its four edges are copies of H, each at distance h from O. R is the union of the copies kT, $k \in K := \{1, \sigma_O A, \sigma_O B, \rho_\pi\}$.

Unfold a billiard path: each time it meets a side, reflect the current copy of T in that side, so that the path becomes a straight line through copies gT, $g \in G$. Group the copies into rhombi $gR = \cup_{k\in K} gkT$, keeping the labels gk. Two labelled rhombi g_1R, g_2R coincide up to translation iff $g_1K = g_2K$. Every coset gK contains a rotation ρ_φ, and $\rho_\varphi K$ determines φ modulo π (because $\rho_\varphi \sigma_O B = \rho_{\varphi+\pi}\sigma_O A$). Since $\varphi \in 2a\mathbb{Z} + \pi\mathbb{Z}$, we may take $\varphi = 2an$; and if $a/\pi \notin \mathbb{Q}$, then $2an \equiv 2an' \pmod{\pi}$ forces $n = n'$. Hence:

Lemma 2.3. *If $a/\pi \notin \mathbb{Q}$, the labelled rhombi of the unfolding are indexed, up to translation, by the **level** $n \in \mathbb{Z}$: the rhombus of level n is $R_n := \rho_{2an}R$, with its labels, and distinct levels give distinct rhombi.* □

Meeting a leg means crossing a diagonal inside a rhombus and does not change the rhombus.

Doors. The edges of R_n have directions $\pm a + 2an$. Reflecting R_n in an edge of direction $a+2an$ gives the rhombus with edge directions $a+2an$ and $3a+2an$, which is R_{n+1}; reflecting it in an edge of direction $-a+2an$ gives R_{n-1}. So the pair of parallel edges of R_n of direction $a+2an$ — equivalently, of R_{n+1} of direction $-a+2a(n+1)$ — is a **door** between the levels n and $n+1$. We index it by $k = n + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2}$. The two edges of door k have the outward normals $\pm e(\psi_k)$, with

$$\psi_k \coloneqq \pi/2 + 2ak,$$

and crossing either of them, from level n, leads to level $n+1$. The unfolding of T is thus a bi-infinite chain of rhombi, consecutive levels joined through doors (Figure 1) — an infinite translation surface which, with its rhombus centres filled in as regular points, is homeomorphic, like the invariant surface of every irrational polygon, to the Loch Ness monster [Val09]; removing the centres gives Hooper's $MT(T)$ (Proposition 5.3).

All directions are measured in the frame of level 0, where $R_0 = R$. A straight line of direction θ there is the billiard direction $g^{-1}\theta$ in the copy gT; since each class is G-invariant, "the line is in the class cls" does not depend on the copy.

Periodicity. Fix a direction θ. Suppose a line of direction θ, after finitely many crossings, is again in a rhombus of the level it started in, at the same perpendicular offset from that rhombus's centre. By Lemma 2.3 the two rhombi are translates of each other, labels included, so the billiard state — a point of T and a direction — recurs: the line is a periodic billiard path. Conversely, a periodic billiard path unfolds to such a line after one or two periods (two when the copy reached after one period is the image of the first under the reflection with axis θ; see §3.1). The centre of a rhombus is the right-angle vertex of its four copies, where the total angle is 2π, so it is a regular point of the unfolding and a line through a centre simply continues straight; we use this whenever a developed line passes through a centre.

3. The dichotomy

Theorem 3.1. *Let T be a right triangle with acute angle a such that $y = \cos 2a$ is transcendental. Then every periodic billiard path of T in a mirror direction meets a side of T at a right angle. In particular T has no periodic billiard path in the class $\parallel H$, whose directions never meet a side at a right angle.*

More precisely, the proof gives a degree bound valid whenever $a/\pi \notin \mathbb{Q}$ (Theorem 3.4).

Since y transcendental implies $a/\pi \notin \mathbb{Q}$, Lemma 2.3 applies, and we work in the chain of rhombi throughout.

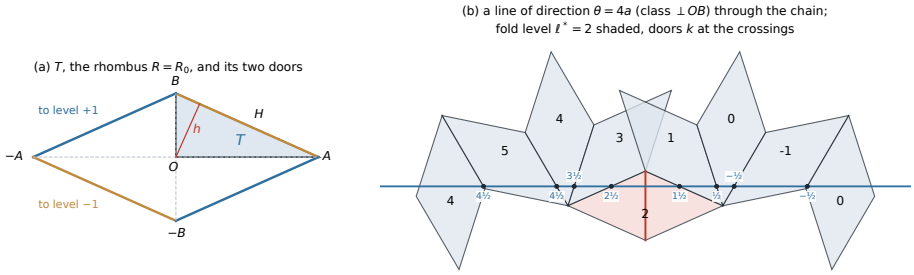


FIGURE 1. (a) The triangle T, the rhombus $R = R_0$ formed by its four copies around O, and the distance h from O to H. The edges of direction a form the door to level 1, those of direction $-a$ the door to level -1. (b) A line of direction $\theta = 4a$ (class $\perp OB$; $\theta = 2aj$ with $j = 2$ in the notation of §3.2) developed through the chain, drawn horizontally, at $a = 0.42$: the rhombus of level $\ell^* = 2$ (shaded) and the four rhombi the line crosses on each side of it, of levels $3, 4, 5, 4$ and $1, 0, -1, 0$ (a turn sends the line back into a level it has left). Rhombi are labelled by level and crossings by the door k. Level ℓ^* is the fold of θ (§3.2): there the line crosses the copy of OB (red) at a right angle, and the reflection in that diagonal maps the picture to itself and acts on levels as $r \mapsto 4 - r$. At this offset the line turns — leaves a rhombus through the door it entered by (Lemma 7.2) — in levels 5 and -1, which that reflection exchanges. Rhombi far from the line may overlap in the plane: the development is an immersion, not an embedding.

3.1. **The holonomy of a periodic path.** Let γ be a periodic billiard path of direction θ, and develop one period of it as a straight segment through the chain, from a point p in a copy T to the point $p+V$ in a copy gT, where the billiard state recurs: $g^{-1}\theta = \theta$ as an oriented direction. So g fixes the vector $e(\theta)$: either $g = 1$, or g is the reflection in the line $\mathbb{R}e(\theta)$, which lies in G exactly when θ is a mirror direction.

Lemma 3.2 (holonomy)**.** *If $g = 1$, then*

$$\sum_k c_k \sin(\psi_k - \theta) = 0,$$

where $c_k \in \mathbb{Z}$ is the signed number of crossings of door k defined in the proof (so $c_k = 0$ for every door not crossed), and not all c_k vanish.

Proof. With $g = 1$ the last copy is the translate of the first by V, and so are the rhombi containing them. Each crossing of a door edge replaces the current rhombus by its reflection in that edge, moving its centre by $2h\nu$, where $\nu = \pm e(\psi_k)$ is the edge's outward normal; crossing a diagonal does

not move the rhombus. Hence $V = 2h\sum_k c_k e(\psi_k)$, where $c_k \in \mathbb{Z}$ counts the crossings of door k through an edge of outward normal $+e(\psi_k)$ minus those through an edge of outward normal $-e(\psi_k)$. On the other hand $V = \ell e(\theta)$, where $\ell > 0$ is the length of the period. The component of V normal to θ gives the identity, and $V \neq 0$ forces some $c_k \neq 0$. □

3.2. **The fold.** Fix a class and a direction θ in it. We locate the places where a line of direction θ can meet a side at a right angle, using that $a/\pi \notin \mathbb{Q}$. Put $\chi_k \coloneqq \psi_k - \theta \pmod{\pi}$, the angle of door k's normal measured from the direction; where a sine of $\psi_k - \theta$ is taken (Lemmas 3.2 and 3.3), θ is the representative written in the class table of §2.1. A crossing of a door edge is a right-angle hit on H iff $\chi_k \equiv 0 \pmod{\pi}$. The diagonals of level n bisect the angles between its edges, so their normals are $(\chi_{n-\frac{1}{2}} + \chi_{n+\frac{1}{2}})/2 = \pi/2 + 2an - \theta$ and that plus $\pi/2$, and a crossing of a diagonal is a right-angle hit on a leg iff $2an - \theta \in (\pi/2)\mathbb{Z}$. Write θ as in the class table, with index j:

- **leg class** $\perp OB$, $\theta = 2aj$. A door is never hit at a right angle (that would need $2a(k-j) \in \pi/2 + \pi\mathbb{Z}$ with $k - j \in \mathbb{Z} + \frac{1}{2}$), and a diagonal is hit at a right angle only at the level $\ell^* \coloneqq j$. In level ℓ^* the line is perpendicular to the diagonal that is a copy of OB, and every line through the rhombus crosses that diagonal (the rhombus projects onto the diagonal's line exactly as the diagonal does). The class $\perp OA$ is the same with the legs exchanged.
- $\perp H$, $\theta = \pi/2 + (2j-1)a$. A diagonal is never hit at a right angle ($2an - \theta$ is $\pi/2$ plus an odd multiple of a), and a door is hit at a right angle exactly when the line crosses the door $k^* \coloneqq j - \frac{1}{2}$.
- $\parallel H$, $\theta = (2j-1)a$. Nothing is ever hit at a right angle: $\chi_k = \pi/2 + 2a(k - k^*)$ with $k^* \coloneqq j - \frac{1}{2}$ and $k - k^* \in \mathbb{Z} \setminus \{0\}$ for every door the line can cross, and $2an - \theta$ is an odd multiple of a. The door k^* itself is parallel to θ and is never crossed.

We call ℓ^* (a level) or k^* (a door) the **fold** of θ. The reflection of G with axis θ acts on the chain as $r \mapsto 2\ell^* - r$ resp. $r \mapsto 2k^* - r$ on levels, so it exchanges the two sides of the fold, and the two half-chains are joined only through it. Consequently:

Lemma 3.3 (one side)**.** *A non-perpendicular periodic path in a mirror direction stays, when developed, on one side of the fold of its direction, and one period of it develops with $g = 1$. On that side, setting $m \coloneqq k - \ell^*$ resp. $m \coloneqq k - k^*$ for the doors k it crosses,*

class	$\sin(\psi_k - \theta)$	*doors crossed*
$\perp OB$	$\cos(2am) = \cos a \cdot V_{\lvert m\rvert-\frac{1}{2}}(y)$	$\lvert m\rvert \geq 3/2$
$\perp OA$	$\sin(2am) = \pm(-1)^{\lvert m\rvert-\frac{1}{2}} \sin a \cdot V_{\lvert m\rvert-\frac{1}{2}}(-y)$	$\lvert m\rvert \geq 3/2$
$\perp H$	$\sin(2am) = \pm \sin 2a \cdot U_{\lvert m\rvert-1}(y)$	$\lvert m\rvert \geq 2$
$\parallel H$	$\cos(2am) = T_{\lvert m\rvert}(y)$	$\lvert m\rvert \geq 1$

where T_N, U_N, V_N are the Chebyshev polynomials of the first, second and third kind, $V_{-1} = V_0 = 1$, $V_N = 2yV_{N-1} - V_{N-2}$, so that $\cos((2N+1)a) = \cos a \cdot V_N(\cos 2a)$; each of T_N, U_N, V_N has degree N and leading coefficient a power of 2, and the signs $\pm$ in the $\perp OA$ and $\perp H$ rows are fixed on each side of the fold.

Proof. In a leg class, a path that enters level ℓ^* is perpendicular, so a non-perpendicular path avoids level ℓ^* and with it the doors $\ell^* \pm \frac{1}{2}$. In $\perp H$ it never crosses k^*; it also avoids the two levels $k^* \pm \frac{1}{2}$ adjacent to the fold, because a line of direction θ through either of them meets one of the two edges of door k^*, at a right angle (the two parallel edges' projections onto their common direction cover the rhombus's). In $\parallel H$ the door k^* is parallel to the flow. In each case the path stays on one side of the fold. The starting copy T lies in level 0; the reflection with axis θ would put the endpoint copy in level $2\ell^*$ (resp. $2k^*$), across the fold, whereas the endpoint copy lies on the same side as T; so $g = 1$. The table is $\psi_k - \theta$ evaluated from the class table, with the Chebyshev identities. □

3.3. **Proof of the theorem.**

Theorem 3.4. *Let $a/\pi \notin \mathbb{Q}$, and let γ be a non-perpendicular periodic billiard path of $Tri(a)$ in a mirror direction, and let D be the largest $|m|$ among the doors its development crosses. Then $y = \cos 2a$ is algebraic, of degree at most $D - \frac{1}{2}$ (leg classes), $D - 1$ ($\perp H$), or D ($\parallel H$).*

Proof. By Lemma 3.3 one period develops with $g = 1$, so Lemma 3.2 gives $\sum_k c_k \sin(\psi_k - \theta) = 0$ with integers c_k, not all zero. On one side of the fold each value of $|m|$ names exactly one door, so by Lemma 3.3 the left side is a fixed nonzero factor ($\cos a$, $\pm\sin a$, $\pm\sin 2a$ or 1) times $P(y) := \sum_k c_k F_{|m_k|}(y)$, where $F_{|m|}$ is $V_{|m|-\frac{1}{2}}(y)$, $(-1)^{|m|-\frac{1}{2}} V_{|m|-\frac{1}{2}}(-y)$, $U_{|m|-1}(y)$ or $T_{|m|}(y)$. These polynomials have distinct degrees for distinct $|m|$, so P is a nonzero integer polynomial of degree at most the stated bound, and $P(y) = 0$. □

The bound is attained in the leg classes: the irrational members of the family of Corollary 7.13 cross the doors $|m| = 3/2, 5/2$ of a leg class and have $\cos 2a$ quadratic.

Proof of Theorem 3.1. If y is transcendental then $a/\pi \notin \mathbb{Q}$, and Theorem 3.4 excludes every non-perpendicular periodic path in a mirror direction. In the class $\parallel H$ no path is perpendicular (§3.2), so there is no periodic path at all. □

Remark 3.5. For the leg classes Theorem 3.1 can also be proved through an interval exchange on the leg (the fan); we do not give that proof here. The proof above needs only Hooper's chain and the closing of the developed path.

Remark 3.6 (the door sign law). A line of direction θ leaves a rhombus only through an edge whose outward normal ν has $\nu \cdot e(\theta) > 0$. So every crossing of door k, in either direction, contributes the same sign $\sigma_k = \operatorname{sgn}\cos(\psi_k - \theta)$ to Lemma 3.2, and $c_k = \sigma_k n_k$, where n_k is the number of crossings. For a non-perpendicular periodic path the crossed doors form a contiguous run on one side of the fold, and Theorem 3.4's relation becomes a relation $\sum_m n_m t_m(y) = 0$ with positive even integers n_m (even because one period returns to its level, so each door is crossed as often in each direction) and polynomials $t_m = \pm F_m$ of signs fixed by the direction. This positivity is a genuine constraint on the door counts. We do not know whether it ever excludes an algebraic y: the signs of the values $t_m(y)$ change every $\approx \pi/(4a)$ doors along the run, and at an algebraic y of degree d the polynomials span a space of dimension at most d, so along a long run positive relations are easy to satisfy. For a perpendicular path the counts are symmetric about the fold, $n_{+m} = n_{-m}$, and the relation is vacuous.

Remark 3.7. The class $\parallel H$ is special for a structural reason: its fold is a door parallel to the flow, so every line of such a direction is confined to a half-chain. Troubetzkoy observed this splitting [Tro05b, p. 5], which is why his theorem on escape orbits [Tro05b, Thm 5] excludes this direction. Theorem 3.1 shows that, for transcendental $\cos 2a$, the confinement is never closed up by a periodic orbit. What the flow does there instead is discussed in §8.

4. Corollaries

4.1. **Odd periods.**

Corollary 4.1. *No right triangle, rational or irrational, has a non-singular periodic billiard path (one avoiding all three vertices) of odd period.*

Proof. Let γ be a periodic path of odd period p in a triangle T, and unfold one period along its orbit type. The composition F of the p reflections is orientation-reversing, fixes the direction v of γ, and moves the starting point z to $z + Lv$, $L > 0$. So F is a glide reflection: the reflection in a line ℓ parallel to v followed by a translation cv along it. Its displacement $F(z) - z$ is cv plus a vector normal to v of length twice the distance from z to ℓ; as $F(z) - z = Lv$, that distance is 0, so the unfolded path is the axis ℓ and $c = L$. For a triangle T' near T the same orbit type gives a glide reflection near F, with axis near ℓ and positive glide; since the original axis crosses each edge at positive distance from the vertices, the new axis crosses the same edges in the same order. So γ is *stable*: every triangle near T has a periodic path of the same orbit type. By Hooper's theorem [Hoo07, Thm 1] no periodic path of a right triangle is stable (Hooper reduces the rational case to the irrational one, [Hoo07, §2]). □

Holt showed that a right triangle has no periodic path of period 3, 5 or 7 and asked whether it has one of any odd period [Hol93, §4.3 and p. 89];

Corollary 4.1 answers this. Both ingredients are known: that odd orbits are stable is classical ([GSV92, §8A] proves it for *isolated* orbits, by an argument that uses only the oddness of the period, and [HH00, Thm 1] states it for odd orbits), and the instability is Hooper's theorem (for acute angle $\pi/(2n)$ it is due to Vorobets [Vor97, Thm 4.1]). Hooper's thesis contains both halves — odd periods are stable [Hoo06, Lemma 5.2], and right triangles have no stable periodic paths [Hoo06, Ch. 4] — without joining them. We have not found the combination in the literature; §4.3 uses it.

4.2. **Isosceles triangles.** The relation between periodic paths of an isosceles triangle and of the right triangle obtained by folding it is studied by Becker [Bec13]; we need only the following. Let $I(a)$ be the isosceles triangle with base $A'A$, $A' = (-1, 0)$, $A = (1, 0)$, apex $B = (0, \tan a)$ and base angle a; let O be the midpoint of the base, $\Delta = OB$ the axis, and σ the reflection in Δ. Folding $I(a)$ along Δ gives $Tri(a)$. The linear group of $I(a)$ is $\theta \mapsto \pm\theta + 2ka$, of index two in that of $Tri(a)$ when $a/\pi \notin \mathbb{Q}$. We call *mirror directions* of $I(a)$ the directions fixed mod π by a reflection of this group (its axes and their normals); they are exactly the mirror directions of $Tri(a)$, and its four mirror classes are, as sets of directions mod π, the four classes of $Tri(a)$:

$I(a)$	$Tri(a)$
parallel to the base, $\{2ka\}$	$\perp OB$
perpendicular to the base, $\{\pi/2 + 2ka\}$	$\perp OA$
perpendicular to a leg, $\{\pi/2 + (2j-1)a\}$	$\perp H$
parallel to a leg, $\{(2j-1)a\}$	$\parallel H$

Corollary 4.2. *Let* $\cos 2a$ *be transcendental. Every periodic billiard path of* $I(a)$ *in the class perpendicular to the base (resp. to a leg) meets the base (resp. a leg) at a right angle; the class parallel to a leg carries no periodic path; and every periodic path in the class parallel to the base crosses the axis* Δ *at a right angle, so that* σ *maps it to itself reversed, while never meeting a side at a right angle.*

Proof. A periodic path of $I(a)$ folds to a closed path of $Tri(a)$. If it passes through O, the folded path runs into the right-angle vertex of $Tri(a)$, which in the chain of rhombi is a rhombus centre, a regular point; a line through a centre crosses no door and changes neither the level nor the direction mod π. So the folded path develops as a closed line in the chain (after one or two periods). §3 uses nothing about a path except this — §3.1 develops one period as a straight segment, and §3.2 and Lemma 3.3 locate where a straight line of direction θ meets a side at a right angle — so it applies to the folded path: it meets the fold of its direction. The hit is at an interior point of a side. The path avoids the vertices of $I(a)$, among them A and B; and a line crossing a diagonal at a right angle at the centre O runs along the other diagonal, that is, along the base or up the axis into the vertex B.

By the table, the fold is a right-angle crossing of OA (the base) in the class perpendicular to the base, of H (a leg) in the class perpendicular to a leg, and of OB (the axis Δ, interior to $I(a)$) in the class parallel to the base; in the class parallel to a leg there is none. In the class parallel to the base the directions $2ka$ are never normal to the base or to a leg when $a/\pi \notin \mathbb{Q}$. □

The base-parallel clause should be compared with Gutkin and Troubetzkoy's theorem that in an isosceles triangle almost every orbit parallel to the base is periodic [GT96, Cor. 4.4]; Corollary 4.2 says what those orbits look like when $\cos 2a$ is transcendental.

4.3. **Odd orbits of isosceles triangles.** Corollary 4.1 determines the odd orbits of every isosceles triangle. A periodic path of period p is *primitive* if p is its minimal period; a path of odd period is an odd iterate of a primitive one, since its minimal period divides p.

Proposition 4.3. *Let γ be a periodic billiard path of odd period p in an isosceles triangle $I(a)$, rational or not. Then:*

1. *γ bounces at the midpoint O of the base;*
2. *$\sigma\gamma$ is γ reversed;*
3. *if γ is primitive, then in each period it bounces exactly once at O and crosses Δ at a right angle exactly once; its other crossings of Δ, if any, are oblique and are exchanged in pairs by σ composed with time reversal;*
4. *conversely, a path leaving O that meets Δ at a right angle after b bounces, without meeting a vertex or returning to O first, closes up with its σ-image into a primitive periodic path of period $2b+1$;*
5. *every odd orbit lies in the class parallel to the base, and up to reversal the primitive odd orbits are indexed by the directions $2ka \pmod{2\pi}$ with $\sin 2ka > 0$ for which the path from O in that direction meets Δ perpendicularly without meeting a vertex or returning to O first — at most one orbit per direction. In particular, for a base angle $a = m\pi/n$ in lowest terms there are at most $(n-1)/2$ primitive odd orbits if n is odd and $\lfloor (n-2)/4 \rfloor$ if n is even.*

Proof. (1) Otherwise the folded path — the image of γ under folding $I(a)$ along Δ — is a non-singular closed path of $Tri(a)$, since the other vertices of $Tri(a)$ are vertices of $I(a)$. It closes after $p+c$ bounces, where c, the number of crossings of Δ per period, is even because Δ separates $I(a)$; so its minimal period divides the odd number $p+c$, contradicting Corollary 4.1. (2) Both γ and $\sigma\gamma$ reversed pass through O with the same outgoing direction, since σ composed with the reflection in the base is -1. (3) σ composed with time reversal is an orientation-reversing involution of the orbit circle, so it has exactly two fixed points, each a right-angle crossing of Δ or a bounce at $\Delta \cap \partial I(a) = \{O, B\}$, and B is a vertex. Bounces off the fixed points pair up, so p is congruent mod 2 to the number of fixed points that are bounces at O. Every bounce at O and every right-angle crossing of Δ is a fixed point

of this involution (each gives a reversal symmetry with the same σ, and the reversal time is unique because γ is primitive), so there are no others; the remaining crossings of Δ are not fixed, so the involution pairs them. (4) Reflect the path in Δ and reverse it; the two halves join at the right-angle crossing and at O. Neither half returns to O, so the closed path bounces at O once per period and is primitive. (5) By (3), applied to the primitive root of γ, the path crosses Δ at a right angle, that is, horizontally; the directions there and at O differ by an element of the linear group of $I(a)$, so that group carries the outgoing direction at O to the horizontal in one of its two orientations, whose orbits are $\{2ka\}$ and $\{\pi + 2ka\}$ mod 2π; reversal $\varphi \mapsto \pi - \varphi$ exchanges the two, so up to reversal the direction lies in $\{2ka\}$. A primitive orbit passes O once per period by (3), so it is determined by its direction there, and by (4) every direction of the statement gives one. For $a = m\pi/n$ these directions are the multiples of $2\pi/n$ in the upper half-plane, less the vertical (which runs into B); when n is even the two orbits coincide, and reversal pairs these directions among themselves. □

Parts of Proposition 4.3 are in the literature. At irrational angles, (1) follows from Hooper's [Hoo07, Cor. 6]: in an isosceles triangle with $a/\pi \notin \mathbb{Q}$, every orbit type of a periodic path is also that of a periodic path through the midpoint of the base. An odd orbit is the only periodic path of its orbit type, since one period unfolds to a glide reflection, which preserves only its axis; so the odd orbit itself passes through O. (Hooper's "twice in its period" counts the period of the unfolded closed geodesic, which is twice the odd period, in agreement with (3).) Becker's density argument [Bec14, Cor. 5.11] gives (1) and (2) at rational angles as well. The converse (4) is, seen from O, Hooper and Schwartz's construction of odd orbits in obtuse isosceles triangles [HS09, Prop. 3.4]. The right-angle crossing in (3), the indexing in (5) and the bound appear to be new; the bound's value for the 120°-isosceles triangle ($n = 6$, at most one orbit, none attained) is in [BGHU19, Cor. 2]. For the equilateral triangle ($n = 3$) the bound is 1, so every odd orbit is an iterate of Fagnano's orbit — Baxter and Umble's classification [BU08, Cor. 2], here obtained from Hooper's instability theorem instead of a tiling count.

4.4. **Rational right triangles.** When $a/\pi \in \mathbb{Q}$ the group G is finite, Definition 2.1 still makes sense, and the first conclusion of Theorem 3.1 holds without any condition:

Theorem 4.4 ([Neck, Corollary 12.1])**.** *Every periodic billiard path of a right triangle with $a/\pi \in \mathbb{Q}$ whose direction is a mirror direction meets a side at a right angle.*

The proof is a corollary of the structure of the flow on the necklace surface of a rational right triangle and is given in [Neck, §12]. A direction is *completely periodic* if the flow in it decomposes the invariant surface into periodic cylinders. When an acute angle is π/n the triangle has the lattice

property, and by Veech's dichotomy [Vee89, Proposition (2.10)] every direction in which some billiard path is periodic is completely periodic; so in a mirror direction that carries a periodic path, Theorem 4.4 says that every cylinder of the decomposition is perpendicular. For these triangles Vorobets showed that every periodic path is symmetric under an affine reflection of the invariant surface [Vor97, Prop. 2.2]; Theorem 4.4 asks for symmetry under the reflection in a side. Among rational right triangles the lattice property holds only in that case [KS00, Thm 2]. Theorem 4.4 assumes neither property: it applies to every periodic cylinder in a mirror direction, whatever the rest of the flow in that direction does — and that need not be periodic: in the right triangle with angles $7\pi/30$, $8\pi/30$, which is not a lattice triangle, a direction perpendicular to a side has minimal components [DR22]. Together with Theorem 3.1 this leaves only the triangles with $\cos 2a$ algebraic and $a/\pi \notin \mathbb{Q}$ as possible carriers of non-perpendicular periodic paths in mirror directions. The next two sections show that such paths exist, and that the triangles carrying them are dense.

5. The exceptions are real

In this section $a/\pi \notin \mathbb{Q}$ and $\cos 2a$ is algebraic. All cylinders below are certified by exact arithmetic. The certificates scale T by $2\cos a$, so that in complex coordinates $O = 0$, $A = 2\cos a$ and $B = 2i\sin a = 2e^{ia} - 2\cos a$, and they compute in the number field $\mathbb{Q}(2\cos a)$ and its extension by e^{ia}: the word of the cylinder unfolds to a map whose linear part is exactly the identity and whose translation is exactly parallel to the direction, and the unfolded segment crosses every edge of its corridor (the chain of copies of T it unfolds through, §6.1) strictly between the edge's endpoints (exact sign computations). The certificates are deposited with the paper (see *Code availability* at the end).

Non-perpendicularity. For a cylinder in a mirror direction, being non-perpendicular is a finite check: by §3.2 a path in a leg class can meet a side at a right angle only in the level ℓ^*, a path in $\perp H$ only by crossing the door k^*, and a path in $\parallel H$ never. So it suffices that the word, developed from the cylinder's starting copy, never visits the fold. Lemmas 6.4 and 7.8 settle this for the corner-word cylinders of §6 and the two-door cylinders of §7.

5.1. **Two examples.**

Example 5.1 (the $1 : \sqrt{7}$ triangle, a leg class)**.** Let $\cos 2a = 3/4$, so that $\tan a = 1/\sqrt{7}$ and the legs are in the ratio $1 : \sqrt{7}$; $a/\pi \notin \mathbb{Q}$ by Niven's theorem [Niv56]. The path leaving the short leg OB in the direction $2a$ (a direction of the class $\perp OB$) and following the word

$$HAHABHABHBAHAHAB$$

(H the hypotenuse, $A = OA$, $B = OB$) is periodic, of period 16. Its word unfolds to the translation by $4 \cdot (\cos 2a, \sin 2a)$ (with $OA = 1$), exactly parallel to the direction, and the launch interval on OB is $(0, \sqrt{7}/42)$, so the

cylinder has perpendicular width $\sqrt{7}/56$. Its smallest angles to the normals of the sides over a period are $\arcsin(1/8)$ (to OA), $0.236\ldots$ rad (to H) and $2a$ (to OB): no path of the cylinder meets a side at a right angle. So the $1:\sqrt{7}$ triangle carries a non-perpendicular periodic cylinder in the class $\perp OB$ (Figure 2). The word is $W\cdot AB\cdot W^R\cdot AB$ with $W=HAHABH$: the cylinder's core (Proposition 5.3) is the corner path (§6.1) of the word $HAHABH$ of Proposition 6.6, and the cylinder is the first member of the family of Corollary 7.13.

Example 5.2 (the $\sqrt{3}:\sqrt{5}$ triangle, the class $\perp H$)**.** Let $\cos 2a = 1/4$, legs $\sqrt{3}:\sqrt{5}$. The word

$$HABHBAHAHBHAHABHAHBHAHAB$$

is a non-perpendicular periodic cylinder of period 24 in the class $\perp H$, in the direction $\pi/2-11a$ (index $j=-5$), certified over $\mathbb{Q}(2\cos a)=\mathbb{Q}(\sqrt{10})$, with launch interval $(7\sqrt{15}/129, 5\sqrt{15}/43)$ on OB (in the coordinates of §2.1) and smallest angle to a normal 0.0990 rad. Up to cyclic order its word is $W\cdot AB\cdot W^R\cdot AB$ with $W=HAHBHAHABH$, so its core is the corner path of that word.

5.2. **Rational values of** $\cos 2a$**.** By Niven's theorem every rational $\cos 2a\notin\{0,\pm\frac{1}{2}\}$ gives an irrational right triangle. Table 1 lists what a systematic search finds at the rational values $y=p/q$ with $q\le 20$. A triangle is the pair $\{y,-y\}$ (exchanging the legs), so we list $y>0$, where OB is the shorter leg.

Table 1. Non-perpendicular periodic cylinders at rational $\cos 2a = y$*,* $q\le 20$*, by class; entries are periods (numbers of bounces); in each non-empty cell the search found a single cylinder (a cylinder* C *and its reverse* C^R*, Proposition 5.3, counted once). Every entry is certified exactly; blank cells are cells where the search found none, which is not a proof of absence. The search ran in two passes: every reduced* p/q *with* $q\le 20$*, the directions* $|j|\le 10$ *of each class,* 60 *launch points per direction and at most* 2000 *bounces; then a ten times deeper pass at* $q\le 12$ *(*$|j|\le 20$*,* 120 *launch points, at most* 20000 *bounces), which added the entries at* $2/3$ *and* $9/10$*.*

y	legs	$\perp OB$	$\perp OA$	$\perp H$	$\parallel H$
$1/4$	$\sqrt{3}:\sqrt{5}$			24	
$1/3$	$1:\sqrt{2}$				52
$5/8$	$\sqrt{3}:\sqrt{13}$			356	
$2/3$	$1:\sqrt{5}$			3600	
$7/10$	$\sqrt{3}:\sqrt{17}$	106			210
$3/4$	$1:\sqrt{7}$	16			30
$5/6$	$1:\sqrt{11}$	84			164
$7/8$	$1:\sqrt{15}$	338			670
$9/10$	$1:\sqrt{19}$	7356			14252

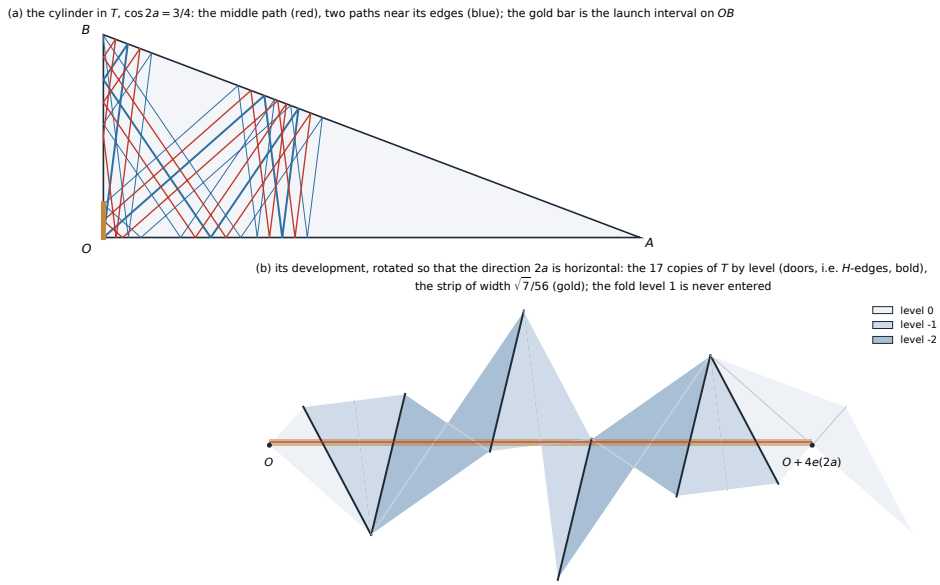


FIGURE 2. The cylinder of Example 5.1 ($\cos 2a = 3/4$). (a) Folded into T: the path launched from the middle of the launch interval $(0, \sqrt{7}/42)$ on OB (the gold bar), and two paths near the cylinder's edges (blue). (b) Its development, rotated so that the direction $2a$ is horizontal: the 17 copies of T crossed by one period, shaded by level, with the six crossings of H (the doors) bold. The strip of perpendicular width $\sqrt{7}/56$, on one side of the core through O (Proposition 5.3), runs from O to $O + 4e(2a)$, where the last copy is the first one translated. Every copy lies in the levels 0, -1, -2, on one side of the fold level $\ell^* = 1$, as Lemma 3.3 requires.

The last four rows are the triangles with legs $1 : \sqrt{(4m-1)}$, $y = (2m-1)/(2m)$, $m = 2, \ldots, 5$, which carry cylinders in both $\perp OB$ and $\parallel H$. Whether this persists for every m we do not know. At no rational value with $q \leq 20$ did the search find a cylinder in the class perpendicular to the longer leg.

5.3. **Hooper's core.** Every cylinder above has a generalized diagonal (a billiard path from a vertex to a vertex) through the right-angle vertex on its boundary, and this is general.

Proposition 5.3 (Hooper)**.** *Let T be a right triangle with $a/\pi \notin \mathbb{Q}$, and let C be a cylinder of periodic billiard paths of T. Then one boundary line of C is a generalized diagonal $\delta : O \to \ldots \to O$ of some k bounces, traversed there and back — the* ***core*** *of C; across δ lies the cylinder C^R of the paths of C traversed in reverse; and the paths of C have period $2k + 4$.*

Proof. Develop C in Hooper's translation surface $MT(T)$ — the chain of rhombi of §2.2 with the rhombus centres removed — where a path γ of C lifts to a simple closed geodesic $\tilde{\gamma}$ [Hoo07, §2]. Let i be Hooper's involution, the rotation by π about every rhombus centre. Hooper's Theorem 3 [Hoo07] states: *the two closed curves $\tilde{\gamma}$ and $i(\tilde{\gamma})$ cut $MT(T)$ into two pieces, one of which is a twice-punctured Euclidean cylinder bounded by these two curves.* The curves are disjoint [Hoo07, Prop. 4], and the punctures are rhombus centres, i.e. lifts of O. Since i is an involutive isometry exchanging $\tilde{\gamma}$ and $i(\tilde{\gamma})$, it permutes the two pieces; only one of them has finite area (the other contains both ends of the chain), so i preserves the cylinder and exchanges its boundary curves. So i acts on it as the rotation by π about each of the two punctures, and the closed geodesic through them, parallel to $\tilde{\gamma}$, is i-invariant with its orientation reversed. Folded into T it is a generalized diagonal δ from O to O, of k bounces say, run there and back. The paths between δ and $\tilde{\gamma}$ fold to paths with the bounce sequence of γ, those between δ and $i(\tilde{\gamma})$ to the same paths reversed. A path close to δ follows its k bounces, bounces once in each leg near O, follows the k bounces back, and bounces once in each leg near the other end, so its period is $2k + 4$. □

If the paths of C meet a side at a right angle, each is its own reverse, so $C^R = C$: the paths at distance ε on the two sides of δ are one and the same path.

That the middle of a perpendicular beam runs into the right-angle vertex was noticed in computations by Ruijgrok [Rui91] and by Cipra, Hanson and Kolan [CHK95], and proved for perpendicular beams by Troubetzkoy [Tro05, Thm 1], when the smaller acute angle lies in $(\pi/6, \pi/4)$, before Hooper's Theorem 3 [Hoo07].

In particular every exceptional cylinder of a mirror class is located by a generalized diagonal from O to O in that class — the starting point of the searches behind Table 1. Those searches find many more: for instance the triangle with $\cos 2a = (1 + \sqrt{41})/8$ carries non-perpendicular periodic cylinders in all four classes (here, as for the y_n below, $a/\pi \notin \mathbb{Q}$ because a non-perpendicular periodic path in a mirror direction cannot exist at rational a, by Theorem 4.4; the certificates decide non-perpendicularity exactly, edge by edge). Beyond the two-door case of §7, in our searches they come in towers and ladders, accumulating at special points; their structure and their census will be treated elsewhere. One family is worth recording here, because it accumulates at an irrational angle, whereas the ladders of §7 accumulate at rational ones: for $n \geq 1$ let y_n be the positive root of the cubic $n(8y^3 + 8y^2 - 6y - 4) + (4y^2 - 2y - 1)$, so that the y_n decrease to $y_c := (\sqrt{17} - 1)/4$. For $1 \leq n \leq 8$, $Tri(\frac{1}{2} \arccos y_n)$ carries a non-perpendicular periodic cylinder in the class $\perp OB$ of period $30n + 6$, certified exactly (see *Code availability*), and at y_c itself (where $a/\pi \notin \mathbb{Q}$, since the conjugate $-(\sqrt{17} + 1)/4$ of y_c is below -1) the family's building block closes up into a non-perpendicular

periodic cylinder of period 30. That the family continues for every n will be shown elsewhere.

6. Density

Theorem 6.1. *For each of the four mirror classes, the exceptional set E_{cls} is dense in $[0,1]$.*

The mechanism is simple. A periodic cylinder that persists over an interval of angles has a direction that varies continuously with a, while the directions of a class sweep through all of $[0,\pi/2]$ faster and faster as their index grows; so the cylinder's direction crosses the class's directions densely, and at each crossing it is a periodic cylinder in a mirror direction. The work is to produce persistent cylinders that are not perpendicular, and enough of them to cover all angles.

6.1. **Corner words.** We use complex coordinates: $O=0$, $A=1$, $B=i\tan a$. Let $\sigma_A z=\bar z$ and $\sigma_B z=-\bar z$ be the reflections in the legs ($\sigma_O A$, $\sigma_O B$ of §2.1) and $\sigma_H z=u\bar z+1-u$, $u:=e^{-2ia}$, the reflection in the hypotenuse. A **corner word** is a word $W=w_1\cdots w_k$ in the letters A, B, H (the sides). It unfolds as $g_i:=\sigma_{w_1}\circ\cdots\circ\sigma_{w_i}$, with crossed edges $e_i:=g_{i-1}(w_i)$ and endpoint $O_{\mathrm{end}}(a):=g_k(O)$. The copies $T_i:=g_i(T)$ ($T_0=T$) form the **corridor** of W, consecutive copies sharing the edge e_i. We say that W is **realised** at a if the segment from O to $O_{\mathrm{end}}(a)$ crosses the edges $e_1,\dots,e_k$ in this order, each at an interior point. Folded back into T, the segment is then a billiard path from the corner O to the corner O following W, the **corner path** of W, in the direction $\varphi_W(a):=\arg O_{\mathrm{end}}(a)$. The **tile** R_W is the set of $a\in(0,\pi/2)$ at which W is realised; it is open.

Lemma 6.2 (corridor criterion)**.** *Let W start and end with H and contain no doubled letter. Then $a\in R_W$ iff for every i the two endpoints P, Q of e_i lie strictly on opposite sides of the line through O and O_{end}, i.e. $f_P f_Q<0$ with $f_P:=\Im(\mathit{conj}(O_{\mathrm{end}})\cdot P)$.*

Proof. "Only if" is clear. Conversely, let $L(s):=s\cdot O_{\mathrm{end}}$. Since W has no doubled letter, $e_i\neq e_{i+1}$; these are two distinct sides of T_i, and together they contain its three vertices. The sign conditions put every vertex of $T_1,\dots,T_{k-1}$ off L, so L meets each e_i at a single interior point $L(s_i)$. A line missing the vertices of a triangle and crossing two of its sides meets it in the chord between them, so $L\cap T_i=[L(s_i),L(s_{i+1})]$. Since $w_1=H$, $L\cap T_0=[O,L(s_1)]$, and since $w_k=H$, $L\cap T_k=[L(s_k),O_{\mathrm{end}}]$, where O_{end} is the vertex of T_k opposite e_k. So L enters T_i at $L(s_i)$ and leaves it at $L(s_{i+1})$; since T_{i-1} and T_i lie on opposite sides of e_i, which L crosses only at $L(s_i)$, the chords $L\cap T_{i-1}$ and $L\cap T_i$ lie on opposite sides of $L(s_i)$ along L. So the sequence $0,s_1,\dots,s_k,1$ of chord endpoints is strictly monotone, and since it begins at 0 and ends at 1 it increases: $0<s_1<\cdots<s_k<1$. □

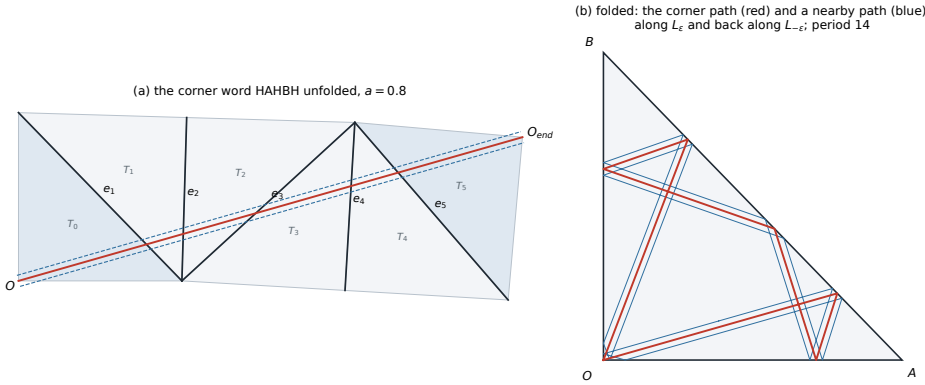


FIGURE 3. The corner word $W = HAHBH$ at $a = 0.8$, inside its tile (Proposition 6.6). (a) The copies $T_0, \dots, T_5$ of the unfolding, the crossed edges $e_1, \dots, e_5$, the segment from O to O_{end} (red), and two parallel lines $L_{\pm\varepsilon}$ (dashed). (b) Folded into T: the corner path from O to O (red) and a nearby path of the cylinder of Lemma 6.3 (blue), periodic of period $2k+4 = 14$ with word $HAHBH \cdot BA \cdot HBHAH \cdot AB$. The blue path runs along L_ε, turns at the corner and returns along $L_{-\varepsilon}$, so it appears on both sides of the corner path.

Lemma 6.3 (a realised corner word is a cylinder)**.** *Let W be realised at a. Every billiard path parallel to the corner path of W and close enough to it is periodic, with period $2k + 4$.*

Proof. Let L_ε be the line parallel to the corner path at signed distance ε. For $|\varepsilon|$ small, L_ε crosses the same edges in the same order, by the strict inequalities, and arrives at the corner O_{end}, where the copy T_k has its right angle, from inside that angle. A billiard path entering a right-angled corner close to the vertex reflects in both legs and leaves antiparallel; the two reflections compose to the point reflection in the vertex, which carries the line at distance ε to the line at distance $-\varepsilon$. So after the corner the path is $L_{-\varepsilon}$ traversed backwards: it retraces the corridor of W on the other side of the corner path, reaches the corner O, and the same reflection returns it to L_ε in its original direction. The period consists of k bounces each way and two at each corner. □

The corner path itself is a generalized diagonal from O to O, the core of the cylinders on either side of it, each the other reversed (compare Proposition 5.3). The argument uses no irrationality of a. Figure 3 shows the construction for $W = HAHBH$.

Lemma 6.4 (non-perpendicularity)**.** *Let W be realised at a. The paths of the cylinder of Lemma 6.3 meet a side at a right angle iff W is a palindrome of odd length.*

Proof. A path of the cylinder hits the edges e_i at the angles of the corner path, and it hits the legs near the two corners. Near a corner the path's direction lies strictly inside the right angle, so it is perpendicular to neither leg. If the corner path hits some e_i at a right angle, it retraces itself back to O, so $W = vsv^R$ has odd length. Conversely, if $W = W^R$ has odd length, the reversed corner path leaves O following $W^R = W$; a corner path following W has direction $\arg O_{\text{end}}$, so the two paths coincide, the path is symmetric about its middle bounce, and that bounce is at a right angle. □

6.2. **Crossings.** Fix a class cls. Its directions are $\theta_j(a) = r_j a + c \pmod{\pi}$ with $j \in \mathbb{Z}$, slope $r_j \in \{2j, 2j-1\}$ and $c \in \{0, \pi/2\}$ (§2.1), and each class is closed under $\varphi \mapsto -\varphi$. Let $\gamma_j(a) \coloneqq |\theta_j(a)|$, where θ_j is reduced mod π into $(-\pi/2, \pi/2]$. Then γ_j is a triangle wave with values in $[0, \pi/2]$, monotone on pieces of length $\pi/(2|r_j|)$, each running from 0 to $\pi/2$ or back.

Recall $u = e^{-2ia}$ from §6.1. The reflections σ_A, σ_B, σ_H have coefficients in $\mathbb{Z}[u]$, and on $|u| = 1$ complex conjugation acts on such coefficients by $u \mapsto u^{-1}$; so for a corner word W the endpoint is $O_{\text{end}}(a) = F_W(u)$ for a Laurent polynomial $F_W \in \mathbb{Z}[u, u^{-1}]$, and for real a

$$e^{2i\varphi_W(a)} = F_W(u)/F_W(u^{-1}) =: \Phi_W(u).$$

We call W **non-degenerate** if Φ_W is not of the form ζu^n with ζ a constant, i.e. if φ_W is not an affine function of a.

Lemma 6.5. *Let W be non-degenerate, realised on an open interval J, and not an odd palindrome. Then the set of $a \in J$ with $a/\pi \notin \mathbb{Q}$ and $\varphi_W(a) = \gamma_j(a)$ for some j is dense in J, and $2a/\pi \in E_{\text{cls}}$ for each such a.*

Proof. Let $I \subset J$ be open. For $|r_j|$ large some monotone piece $[\alpha, \beta]$ of γ_j lies in I. The function $\varphi_W - \gamma_j$ is continuous on $[\alpha, \beta]$ ($O_{\text{end}} \neq 0$ there) and has opposite signs at the ends, since $\gamma_j \in \{0, \pi/2\}$ there and $\varphi_W \in (0, \pi/2)$. So it vanishes at some $a \in (\alpha, \beta)$, and the same holds on every open subinterval of I: the crossings are dense in J. At a crossing $\varphi_W(a) \equiv \pm\theta_j(a) \pmod{\pi}$, so by Lemma 6.3 $Tri(a)$ carries a periodic cylinder in a direction of cls, and by Lemma 6.4 none of its paths meets a side at a right angle.

It remains to see that only finitely many crossings, over all j and all classes together, have $a/\pi \in \mathbb{Q}$. At such a crossing u is a root of unity, and so is $\Phi_W(u) = e^{\pm 2i\theta_j(a)}$, because $\theta_j(a) \in a\mathbb{Z} + (\pi/2)\mathbb{Z} \subset \pi\mathbb{Q}$. So $(u, \Phi_W(u))$ is a torsion point of the algebraic group $\mathbb{G}_m^2$ lying on the curve Γ_W, the Zariski closure in $\mathbb{G}_m^2$ of the graph $\{(u, \Phi_W(u))\}$ (u not a zero or pole of Φ_W), which is irreducible and does not depend on j or on the class. By a theorem of Ihara, Serre and Tate [Lan65], an irreducible curve in $\mathbb{G}_m^2$ containing infinitely many torsion points is the translate of a one-dimensional subtorus by a torsion point, i.e. the zero set of some $u^\alpha v^\beta - \zeta$ with integers $(\alpha, \beta) \neq (0, 0)$ and ζ a root of unity. On Γ_W this reads $\Phi_W(u)^\beta = \zeta u^{-\alpha}$, with $\beta \neq 0$ since u is not constant on Γ_W; so Φ_W has its zeros and poles only at 0 and ∞, and is a constant times a power of u: $\Phi_W(u) = \zeta' u^n$, which non-degeneracy

excludes. So Γ_W has only finitely many torsion points, and removing the corresponding angles from the dense set of crossings leaves a dense set. □

6.3. **Five tiles cover all angles.** Write $s \coloneqq \tan^2 a$. For a word W and a vertex image P, f_P is a rational function of $\tan a$ whose denominator is a power of $1+s$; so the sign conditions of Lemma 6.2 are sign conditions on explicit polynomials. Exchanging the legs maps $Tri(a)$ to a triangle similar to $Tri(\pi/2-a)$, so $R_{W'} = \pi/2 - R_W$, where W' is W with $A \leftrightarrow B$.

Proposition 6.6. *The tiles of the words*

W	R_W	*endpoints* ($s = \tan^2 a$)
$HAHABH$	$(0, a_1)$	$5s = 3$, *i.e.* $\cos 2a_1 = 1/4$ ($a_1 = 0.65905\ldots$)
$HAHBHAHABH$	(a_2, a_3)	*smaller positive root of* $s^3 - 5s^2 - 93s + 41$ ($a_2 = 0.58131\ldots$), *smaller positive root of* $s^3 - 77s^2 + 51s + 1$ ($a_3 = 0.69222\ldots$)
$HAHBH$	(a_4, a_5)	$s^2 + 10s - 7 = 0$ ($a_4 = 0.68108\ldots$), $7s^2 - 10s - 1 = 0$ ($a_5 = 0.88970\ldots$); *i.e.* $\lvert\cos 2a\rvert < (\sqrt{2}-1)/2$

together with the leg swaps $HBHAHBHBAH$ and $HBHBAH$ of the first two, cover $(0, \pi/2)$*:*

$$(0,\pi/2) = (0,a_1) \cup (a_2,a_3) \cup (a_4,a_5) \cup (\pi/2-a_3, \pi/2-a_2) \cup (\pi/2-a_1, \pi/2).$$

None of the five words is a palindrome of odd length, and all five are non-degenerate.

Proof. Each word starts and ends with H and has no doubled letter, so Lemma 6.2 applies (Figure 4 shows the cover); Table 2 lists, for each crossed edge, the sign-carrying factors of f_P and f_Q (computed exactly by the deposited script; see *Code availability*). Reading off the signs gives the tiles; for $HAHABH$, for example, all six pairs have opposite signs iff $5s < 3$. The overlaps are $a_2 < a_1$, $a_4 < a_3$, and their mirror images, which are inequalities between explicit algebraic numbers. $HAHABH$ and $HAHBHAHABH$ have even length and $HAHBH \neq HBHAH$. A corner path read backwards is a corner path of the reversed word, so $R_{W^R} = R_W$; as $HAHBH$ is the leg swap of its reverse $HBHAH$, its tile satisfies $R = \pi/2 - R$, that is $a_4 + a_5 = \pi/2$, which is why it needs no separate leg swap in the cover. Finally, composing the reflections of §6.1 gives $F_W(u) = 1 + u - 2u^2$, $1 - 2u^2 - u^3 + 2u^4$ and $1 - 2u^2 + u^3$ for the three words of the table, and $1 - 2u + 3u^3 - 2u^4$ and $1 - 3u + 2u^2$ for the two leg swaps; none satisfies $F_W(u) = \zeta u^m F_W(u^{-1})$, so no Φ_W is a monomial. □

Table 2. For each crossed edge e_i (named by its letter), the sign-carrying factors of f_P and f_Q, as polynomials in $s = \tan^2 a$ (positive factors $\tan^j a$, powers of $1+s$ and positive constants dropped). W is realised iff every pair

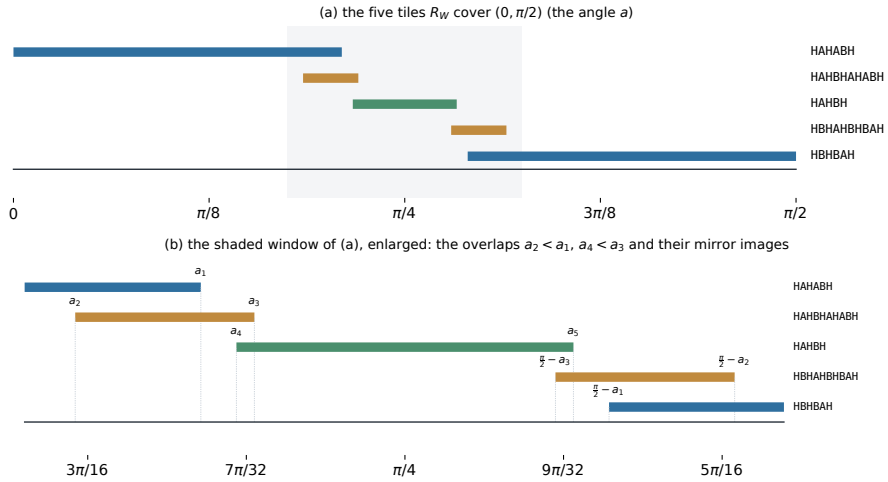


FIGURE 4. The five tiles of Proposition 6.6 on $0 < a < \pi/2$, each labelled by its word. (a) The whole interval. (b) The shaded window of (a), enlarged, where the five tiles overlap: $a_2 < a_1$, $a_4 < a_3$, and their mirror images under $a \mapsto \pi/2 - a$.

has opposite signs. A repeated pair is two distinct edges with a common endpoint.

W	edges $e_1, \dots, e_k$
$HAHABH$	$(5s-3, 7-s)$, $(1, 5s-3)$, $(5s-3, s+9)$, $(1, 5s-3)$, $(1, -1)$, $(1, -1)$
$HAHBH$	$(7s^2-10s-1, -(s^2-14s+1))$, $(3s-1, 7s^2-10s-1)$, $(7s^2-10s-1, s^2+10s-7)$, $(s-3, s^2+10s-7)$, $(s^2-14s+1, s^2+10s-7)$
$HAHBHAHABH$	$(-7s^3+67s^2-53s+1, (s-3)(s^2-26s+5))$, $((7-s)(3s-1), -7s^3+67s^2-53s+1)$, $(-7s^3+67s^2-53s+1, -s^3+5s^2+93s-41)$, $(-s^2+18s-13, -s^3+5s^2+93s-41)$, $(-s^3+77s^2-51s-1, -s^3+5s^2+93s-41)$, $(3s-1, -s^3+77s^2-51s-1)$, $(-s^3+77s^2-51s-1, s^3-5s^2+99s-23)$, $(3s-1, -s^3+77s^2-51s-1)$, $(3s-1, -(s-3)^2)$, $(s^2+18s+1, -(s-3)^2)$

6.4. **Proof of Theorem 6.1, and two consequences.**

Proof of Theorem 6.1. By Proposition 6.6 every $a_0 \in (0, \pi/2)$ lies in the tile J of a non-degenerate word that is not an odd palindrome, and by Lemma 6.5 E_{cls} is dense in $2J/\pi$. So $cl(E_{\text{cls}}) \supseteq (0, 1)$. □

Theorem 4.4 is not used here: the rational crossings are excluded by Lemma 6.5 itself. The explicit ladders of §7 give the density again on the outer thirds, with explicit cylinders and widths, for every class (Proposition 7.12).

Corollary 6.7. *Every right triangle carries a periodic billiard cylinder none of whose paths meets a side at a right angle.*

Proof. By Proposition 6.6 every a lies in the tile of a word that is not an odd palindrome; apply Lemmas 6.3 and 6.4. □

So perpendicularity is a phenomenon of mirror directions only. Off the near-isosceles band $[a_1, \pi/2 - a_1]$, Corollary 6.7 is in substance due to Vorobets, Galperin and Stëpin and to Galperin and Zvonkine: the mirror trajectories of type 2 [GSV92, Conclusion], [GZ03, §3] exist exactly when the smaller angle is below a_1 [GZ03, Thm 2 and Remark 3], and they are the cylinders of $HAHABH$ (or of its leg swap). That these cylinders are non-perpendicular, Lemma 6.4, is not remarked there. For $a/\pi \in \mathbb{Q}$ the existence of periodic cylinders is Masur's theorem [Mas86]; what Corollary 6.7 adds there is that one can be chosen non-perpendicular. Corollary 6.7 also answers Holt's question *"Are there one-way periodic paths through right triangles?"* [Hol93, p. 89] — paths that never retrace — in every right triangle; off the band $[a_1, \pi/2 - a_1]$ the type-2 mirror trajectories [GSV92], [GZ03] already give them.

The size of the exceptional set. By Theorem 3.4, E_{cls} consists of points with $\cos 2a$ algebraic, so it is countable. More precisely, a given word in a given direction θ_j is an exceptional cylinder of only finitely many triangles: the door counts c_k relative to the fold are determined by the word and j, so $\cos 2a$ is a root of one fixed nonzero polynomial. Thus E_{cls} is a countable dense set, and no exceptional cylinder persists over an interval of angles; on the other hand one triangle can carry several (§8).

Ruijgrok's third conjecture. Ruijgrok calls a billiard path *half-integer* if each of its collision angles is of the form $\theta = q_1\alpha_1 + q_2\alpha_2 + q_3\alpha_3 + \pi/2$ with integers q_ℓ (α_ℓ the angles of the triangle), and *weak-orthogonal* if moreover no collision is perpendicular; his third conjecture asserts that an irrational triangle has no weak-orthogonal periodic orbit [Rui91, p. 958]. Ruijgrok does not define *irrational triangle*, but he applies it to right triangles ("every irrational rectangular triangle", [Rui91, p. 957]) and treats them separately in his §2; for a right triangle it means $a/\pi \notin \mathbb{Q}$.

Corollary 6.8 (Ruijgrok's third conjecture for right triangles)**.** *For irrational right triangles, the conjecture fails on the countable dense set* $E := \cup_{\mathrm{cls}} E_{\mathrm{cls}}$ *of triangles, all of which have* $\cos 2a$ *algebraic, and holds whenever* $\cos 2a$ *is transcendental.*

Proof. In a right triangle the half-integer paths are exactly the paths in mirror directions (in any convention for the collision angle, since $\pi/2$ lies in the shift group), and the weak-orthogonal ones are the non-perpendicular

ones. So the corollary is Theorems 3.4 and 6.1 for the first clause and Theorem 3.1 for the second. □

Corollary 7.13 gives an explicit infinite sequence of counterexamples with rational $\cos 2a$.

Ruijgrok's fourth conjecture, that in an irrational triangle a periodic orbit is either integer or bi-orthogonal [Rui91, p. 958], fails for every irrational right triangle outside a countable set. In a right triangle $\pi/2$ is an angle, so his integer orbits are the paths in mirror directions, and the cylinder of Corollary 6.7 lies in a non-mirror direction except at the countably many points of E.

Corollary 6.9 (isosceles triangles)**.** *For each of the four mirror classes of the isosceles triangles $I(a)$ (§4.2), the base angles at which $I(a)$ carries an exceptional periodic cylinder in that class are dense in $(0, \pi/2)$.*

Here exceptional means, against Corollary 4.2: in the classes perpendicular to the base or to a leg, a cylinder whose paths do not meet the base, resp. a leg, at a right angle; parallel to a leg, any periodic cylinder; parallel to the base, a cylinder whose paths do not cross the axis at a right angle.

Proof. Let $2a/\pi \in E_{\text{cls}}$ for the class cls of $Tri(a)$ matching the isosceles class (§4.2), and take a path γ of the exceptional cylinder. Lifting it to $I(a)$ — reflecting it across OB at each hit — gives a billiard path of $I(a) = Tri(a) \cup \sigma Tri(a)$; its hits on the base and the legs are the hits of γ on OA and H, at the same angles, and its crossings of Δ are the hits of γ on OB. It closes after one or two periods of γ, and the cylinder lifts. Its direction lies in the isosceles class, because each class is closed under $\theta \mapsto -\theta$. No path of the original cylinder meets a side at a right angle, so the lift is exceptional in each of the four senses. Theorem 6.1 gives the density. □

Remark 6.10 (a claim of Troubetzkoy). Let $\pi/6 < a < \pi/4$, i.e. $x \in (1/3, 1/2)$, so that a is the smaller acute angle and OA the leg adjacent to it. For irrational right triangles in this range Troubetzkoy states that the periodic orbits perpendicular to that leg fill the invariant surface of their direction [Tro05, abstract, (iii)], and the proof of [Tro05, Thm 3] concludes that the perpendicular periodic cylinders exhaust that surface; the proof of [Tro05b, Thm 6] repeats the step. In our terms the surface is the class $\perp OA$. By Theorem 6.1, $E_{\perp} OA$ is dense in $(1/3, 1/2)$, and at each of its points the class $\perp OA$ contains a cylinder of non-perpendicular periodic paths — an open set disjoint from the perpendicular cylinders. An explicit instance, certified exactly (see Code availability): at $\cos 2a = (\sqrt{7} - 1)/4$ ($a \approx 32.85°$), a cylinder of period 944 in the class $\perp OA$. The theorems of [Tro05] and [Tro05b] as stated — every non-singular orbit of that surface but the escape orbit is periodic — are not affected: the paths of these cylinders are periodic, though not perpendicular.

7. Two-door cylinders

The simplest exceptional cylinders cross only two doors of the chain. A one-dimensional model of how a line moves through the chain describes them completely: each is determined by one integer N (Theorem 7.4), they come in ladders converging to rational angles, a ladder is infinite if its limit has $|\cos 2a^*| > \frac{1}{2}$ and finite if $0 < |\cos 2a^*| < \frac{1}{2}$ (Theorem 7.9), and the two ladders at $a = \pi/8$ are given by two quadratics (Corollary 7.13); their rational members are indexed by the negative Pell equation.

These cylinders are the mirror trajectories of Vorobets, Galperin and Stëpin [GSV92, Conclusion, Fig. 50] and Galperin and Zvonkine [GZ03, §3]: three consecutive levels joined by their two doors form the corridor of rhombi of [GZ03, §3], and the cylinder of Theorem 7.4 with parameter N has the period $10(N+1)-4$ of their type $N+1$. The threshold of Theorem 7.9 is theirs as well: by [GZ03, Thm 2], mirror trajectories of every type exist when the smaller angle is at most $\pi/6$, and of only finitely many types when it exceeds $\pi/6$. What is new here is the crossing with the mirror classes (equation (N) of §7.3), the ladders converging to rational angles, the non-perpendicularity of the cylinders (Lemma 7.8) and the family of §7.4.

7.1. **The offset model.** Rotate the picture so that θ points along $+e_x$. Door k then has normals $\pm e(\chi_k)$ with

$\chi_k = \psi_k - \theta = \pi/2 + 2ak - \theta$ (mod π, as in §3.2),

and the rhombus of level n is $\{p : |(p-c)\cdot e(\chi_{n-\frac{1}{2}})| \le h, |(p-c)\cdot e(\chi_{n+\frac{1}{2}})| \le h\}$, c its centre. Along a horizontal line, let s be the line's y-coordinate minus c_y: its signed offset from the centre of the current rhombus.

Lemma 7.1 (moves)**.** *Crossing door k, in either direction, changes s by $\mu_k := -2h \cdot \operatorname{sgn}(\cos\chi_k) \cdot \sin\chi_k$.*

Proof. The line leaves through the edge whose outward normal ν has $\nu\cdot e_x > 0$, i.e. $\nu = \operatorname{sgn}(\cos\chi_k)e(\chi_k)$. The next rhombus is the reflection of the current one in that edge, so its centre is $c + 2h\nu$, and s drops by $2h\nu_y$. □

Lemma 7.2 (turning and passing)**.** *Let a level have door normals χ_1, χ_2 with $|\cos\chi_1| > |\cos\chi_2| > 0$, and put $\tau := h(|\cos\chi_1| - |\cos\chi_2|)/\sin 2a$. A line entering the level through door 1 leaves through door 1 (it **turns**) if $|s| < \tau$, and through door 2 (it **passes**) if $|s| > \tau$. A line entering through door 2 always passes.*

Door 1 is the level's **turning door**, and τ its **turning strip**.

Proof. The rhombus is the intersection of two slabs, slab i bounded by the two edges of door i. Replacing χ_i by $\chi_i + \pi$ changes neither slab, so take $\cos\chi_i > 0$. The horizontal line at height s meets slab i in the interval $x \in [-(h + s\sin\chi_i)/\cos\chi_i, (h - s\sin\chi_i)/\cos\chi_i]$, of length $2h/\cos\chi_i$ and midpoint $-s\tan\chi_i$. The line enters through an edge of the door whose interval has the larger left endpoint and leaves through an edge of the door

whose interval has the smaller right endpoint, so it enters and leaves through the same door iff that door's interval lies inside the other's. That door is then the one with the shorter interval, door 1, and the condition is $|s| \cdot |\tan\chi_1 - \tan\chi_2| < h(1/\cos\chi_2 - 1/\cos\chi_1)$, i.e. $|s| \cdot |\sin(\chi_1 - \chi_2)| < h(\cos\chi_1 - \cos\chi_2)$, where $|\sin(\chi_1 - \chi_2)| = \sin 2a$ since $\chi_1 - \chi_2 \equiv \pm 2a$. The longer interval never lies inside the shorter one. □

Both lemmas depend on χ only modulo π, and both are invariant under $(\chi, s) \mapsto (-\chi, -s)$, which leaves τ unchanged and replaces μ by $-\mu$.

7.2. **The two-door law.** Let $a/\pi \notin \mathbb{Q}$, and let C be a non-perpendicular periodic cylinder in a mirror class whose core crosses exactly two doors. By Lemma 3.3 they lie on one side of the fold, and they are consecutive, since a line can move between levels only through the doors in between. Let M be the level between them. Call d_X the turning door of M and d_I its other door, and X, I the levels beyond d_X and d_I; let μ_X, μ_I be the moves of Lemma 7.1 and τ_M, τ_X, τ_I the turning strips of Lemma 7.2. Since the core enters X and I and leaves them through the door it came in by, X turns through d_X and I through d_I. By Proposition 5.3 the core passes through two rhombus centres; the half of it between them is a corner path, crossing d_I and d_X some n_I, n_X times. Put $S \coloneqq n_I + n_X$.

Lemma 7.3 (ratio)**.** *Let $\varepsilon \coloneqq \mu_I + \mu_X$ be the drift of a pass through M. Then $|\varepsilon|/\tau_M = 2(1 + |y|) > 2$.*

Proof. Closure forces $\mu_I \mu_X < 0$, so exactly one multiple of $\pi/2$ lies strictly between χ_{d_I} and χ_{d_X}, whose difference is $\pm 2a$. Suppose it is $\equiv 0 \pmod{\pi}$, and write the two values as ξ_1 and $-\xi_2$ with $\xi_1, \xi_2 \in (0, \pi/2)$ and $\xi_1 + \xi_2 = 2a$. Then $|\varepsilon|/\tau_M = 2\sin 2a \cdot |\sin\xi_1 - \sin\xi_2| / |\cos\xi_1 - \cos\xi_2| = 2\sin 2a \cdot \cot a = 4\cos^2 a$. The level beyond the door with value ξ_1 turns through it, and its other door has value $\xi_1 + 2a = 2\xi_1 + \xi_2$; so $\cos\xi_1 > |\cos(2\xi_1 + \xi_2)|$, i.e. $3\xi_1 + \xi_2 < \pi$. Likewise $\xi_1 + 3\xi_2 < \pi$ on the other side. Adding, $\xi_1 + \xi_2 < \pi/2$, so $a < \pi/4$ and $4\cos^2 a = 2(1 + |y|)$. If the multiple is $\equiv \pi/2 \pmod{\pi}$, write the two values as $\pi/2 + \xi_1$ and $\pi/2 - \xi_2$ with $\xi_1, \xi_2 \in (0, \pi/2)$ and $\xi_1 + \xi_2 = 2a$. Then $|\varepsilon|/\tau_M = 2\sin 2a \cdot |\cos\xi_1 - \cos\xi_2| / |\sin\xi_1 - \sin\xi_2| = 2\sin 2a \cdot \tan a = 4\sin^2 a$. The level beyond the door with value $\pi/2 + \xi_1$ turns through it, and its other door has value $\pi/2 + 2\xi_1 + \xi_2$; so $\sin\xi_1 > |\sin(2\xi_1 + \xi_2)|$, i.e. $3\xi_1 + \xi_2 > \pi$. Likewise $\xi_1 + 3\xi_2 > \pi$ on the other side. Adding, $\xi_1 + \xi_2 > \pi/2$, so $a > \pi/4$ and $4\sin^2 a = 2(1 + |y|)$. □

Theorem 7.4 (the two-door law)**.** *The core of C passes through the centre of M, and the corner path from that centre to the other one turns in M nowhere else. The counts are $(n_I, n_X) = (N, N + 1)$ for some $N \geq 1$, and the period of C is $5S + 1$ bounces.*

Proof. Between consecutive visits to X, the core either turns in M ($X \to M \to X$), which changes its offset e by $2\mu_X$ and happens iff $|e + \mu_X| < \tau_M$, or makes the loop $X \to M \to I \to M \to X$, which changes e by 2ε. The

turning offsets $o_1, \dots, o_J$ of one period are distinct points of an interval Z of length $2\tau_M$ (equal offsets in the same rhombus would be the same chord), and $o_{i+1} \equiv o_i + 2\mu_X \pmod{2\varepsilon}$. Since $2\tau_M < |\varepsilon|$ (Lemma 7.3), Z embeds in $\mathbb{R}/2\varepsilon\mathbb{Z}$, so the o_i are J equally spaced points of a circle of length $2|\varepsilon|$ inside an arc of length $2\tau_M$: $(J-1)/J < \tau_M/|\varepsilon| < \frac{1}{2}$, so $J = 1$ (and $J \geq 1$, or e drifts forever). Hooper's involution — rotation by π about every centre — preserves C and its core, reverses the core and negates offsets, so it maps the unique turn to itself: the turn passes through the centre of M. From there the corner path leaves through d_X and visits $X, I, X, \dots$ at offsets $\mu_X + j\varepsilon$, $j = 0, 1, \dots$, without turning in M again. It stops at the next centre, which is not that of M: it would be met at offset 0 on an arrival in M, either from I, whose arrivals have offsets $(j+1)\varepsilon \neq 0$ (j odd), or from X, through the turning door d_X with $|0| < \tau_M$, which would be a second turn, or the first one again, and then the core would pass through only one rhombus centre, against Proposition 5.3. So the second centre is that of X or of I, and $n_X = n_I + 1$. Counting bounces — a pass through a rhombus is one leg hit and a turn is two — the corner path has $2S - 1 + n_T$ bounces, n_T the number of turns, here $n_T = (S-1)/2$; by Proposition 5.3 the period is $2(2S - 1 + n_T) + 4 = 5S + 1$. □

The same bookkeeping is an exact existence criterion.

Proposition 7.5 (existence)**.** *Let $a/\pi \notin \mathbb{Q}$, and let M be a level such that M and its two neighbours lie strictly on one side of the fold, with turning door d_X and other door d_I, such that $\mu_I \mu_X < 0$ and the levels X, I beyond them turn through d_X, d_I. A two-door cylinder with middle level M and counts $(N, N+1)$ exists iff $N\mu_I + (N+1)\mu_X = 0$, $|\mu_X| < \tau_X$, and, if $N \geq 2$, $|\mu_X + \varepsilon| < \tau_I$.*

Proof. Necessity: Theorem 7.4 forces the corner path from the centre of M, whose first visits to X and I are at offsets μ_X and $\mu_X + \varepsilon$. Sufficiency: follow the line through the centre of M out through d_X. Suppose it has reached its i-th visit to X or I (X for even i, I for odd i, starting at $i = 0$) at offset $\mu_X + i\varepsilon = (i - N)\varepsilon$, using $\mu_X = -N\varepsilon$. For $i < N$ it turns there — the offsets decrease in absolute value, so the conditions $|\mu_X| < \tau_X$ and $|\mu_X + \varepsilon| < \tau_I$ cover every visit — and it returns to M. From I it enters M through d_I and passes; from X it enters through d_X at offset $(i - N)\varepsilon + \mu_X = (i - 2N)\varepsilon$, of absolute value at least $(N+1)|\varepsilon| \geq 2|\varepsilon| > \tau_M$ (the computation of Lemma 7.3 uses only the hypotheses made here), so it passes as well. At $i = N$ the offset is 0: the line reaches the centre of X or I, a regular point, so it is a corner path, the core of a cylinder (compare Lemma 6.3). By §3.2 the cylinder is non-perpendicular: its lines stay in I, M, X, none of which is a fold level, and cross only d_I and d_X, neither of which is a fold door. □

So the members of a two-door family are the solutions of $N = \kappa(a) \coloneqq -\mu_X/\varepsilon$ in integers $N \geq 1$ at which the two strip conditions of Proposition 7.5 hold.

7.3. **Ladders.** As $N \to \infty$ the members accumulate at the **equal-weight points** $\varepsilon = 0$ (since $|\mu_X| \le 2h$, a large $N = -\mu_X/\varepsilon$ forces ε to be small, and each value of N is taken at only finitely many angles away from these points, see the proof of Theorem 7.9), where $|\sin\chi_{d_I}| = |\sin\chi_{d_X}|$, i.e. $\chi_{d_I}+\chi_{d_X} \equiv 0 \pmod\pi$. These are rational angles. At such a point either the two values χ are $\equiv \pm a^*$ (the **first type**) or they are $\equiv \pi/2 \pm a^*$ (the **second type**). For instance, for the doors at depth $|m| = m_0, m_0+1$ of the class $\perp OB$ ($m_0 \in \mathbb{Z}+\frac12$) the equal-weight points are $a^* = l\pi/(4m_0+2)$, $l = 1,\dots,2m_0$, of the first type for odd l and of the second for even l; for $\perp OA$ the parities are exchanged. By the proof of Lemma 7.3 a ladder can converge to a point of the first type only if $a^* \le \pi/4$, and to one of the second type only if $a^* \ge \pi/4$. A **ladder** is the sequence of members on one side of one equal-weight point a^* at which $0 < \eta < a$, with η as defined below (for the second type, after exchanging the legs) — the range of the first case of Lemma 7.3.

Fix a ladder of the first type with limit a^*, and let L be its middle level, so that the crossed doors are $L \pm \frac12$. Put $\varphi(a) := (\chi_{L-\frac12} + \chi_{L+\frac12})/2 = \pi/2 + 2aL - \theta(a)$, where $\theta(a)$ is the direction of the class, affine in a. Then

(C)
$$\varphi(a^*) \equiv 0 \pmod\pi, \qquad \text{and } \varphi(a) = \varphi(a^*) + K(a-a^*) \text{ with } K \in \mathbb{Z}\setminus\{0\},$$

with $K \neq 0$: for a leg class $K = 2(L-\ell^*)$ and L is not the fold level, and for the hypotenuse classes K is odd. Write $\delta := K(a-a^*)$ and $\eta := |\delta|$. Modulo π the four doors around L have $\chi_{L\mp\frac12} = \delta \mp a$ and $\chi_{L\mp 3/2} = \delta \mp 3a$. Assume

(H)
$$0 < \eta < a \text{ and } 3a + \eta < \pi/2,$$

which holds on a punctured neighbourhood of a^* when $3a^* < \pi/2$, and put $r := \sin\eta$, $p := \sin(2a-\eta)$, $q := \sin(2a+\eta)$; then $0 < r < p < q$.

Lemma 7.6 (the doors)**.** *Assume (H) and $\delta < 0$. All four χ above lie in $(-\pi/2, \pi/2)$. The turning door of $M := L$ is $L+\frac12$, so $X = L+1$ and $I = L-1$, with $\tau_M = r\tan a$, $\tau_X = p\tan a$ (turning door $L+\frac12$) and $\tau_I = q\tan a$ (turning door $L-\frac12$). In units of $\tan a$ ($\hat s := s/\tan a$), crossing $L+\frac12$ adds $\lambda_X := r-p < 0$ and crossing $L-\frac12$ adds $\lambda_I := q+r > 0$. For $\delta > 0$ the same holds with the doors $L \pm \frac12$ exchanged and $\hat s := -s/\tan a$.*

Proof. For $\delta = -\eta$ the four values are $-3a-\eta, -a-\eta, a-\eta, 3a-\eta$, all of modulus $< \pi/2$ by (H). By $\cos u - \cos v = 2\sin((u+v)/2)\sin((v-u)/2)$: for M, $\cos(a-\eta) - \cos(a+\eta) = 2\sin a\sin\eta$; for X, $\cos(a-\eta) - \cos(3a-\eta) = 2\sin(2a-\eta)\sin a$; for I, $\cos(a+\eta) - \cos(3a+\eta) = 2\sin(2a+\eta)\sin a$. Lemma 7.2 gives the turning doors and the strips, and Lemma 7.1 the moves, by $2\cos a\sin(a\mp\eta) = \sin(2a\mp\eta) \mp \sin\eta$. The case $\delta > 0$ is the image under $(\chi, s) \mapsto (-\chi, -s)$. □

Lemma 7.7 (the one-block word)**.** *Assume (C) and (H). Consider the state "in M, just entered from X, at offset $\hat s = w$", and the two loops back to it: t — turn in M, to X, turn in X, back to M; and z — pass M, to I, turn in*

I, back to M, pass M, to X, turn in X, back to M. If $N \geq 1$ is an integer with

$$N = \sin(a-\eta)/(2\cos a \cdot \sin\eta), \tag{N}$$

then the lines that follow $\Omega_N \coloneqq t \cdot z^N$ for one period are exactly those with $w \in (-r, r)$, and each returns to the state with the same w. The line $w = 0$ is the core, and the lines w and $-w$ are the same billiard paths traversed in opposite directions; so $Tri(a)$ carries a cylinder of periodic paths with itinerary Ω_N, of width $\tan a \cdot \sin\eta$.

Proof. By Lemma 7.6, t from w requires $|w| < r$ (turn in M) and $|w + \lambda_X| < p$ (turn in X), and changes w by $2\lambda_X$. The loop z from an offset v requires $|v| > r$, $|v + \lambda_I| < q$ and $|v + 2\lambda_I + \lambda_X| < p$, and changes v by $\Delta_z = 2(\lambda_X + \lambda_I) = 8r\cos^2 a > 0$. The word closes iff $2\lambda_X + N\Delta_z = 0$, which is (N). The z-blocks start at $w_j = w + 2\lambda_X + j\Delta_z$, increasing in j, with $w_N = w$. Given $|w| < r$, the conditions $|w_j| > r$ hold automatically ($w_j \leq w - \Delta_z < r - 8r\cos^2 a < -r$, as $\cos^2 a > 1/4$, indeed $a < \pi/6$ by (H)); the conditions in I reduce to $w > 2p - 2q - 3r$ and $w < 2q - 2p + 3r$; those in X to $w > 2p - 2q - 5r$, $w < r$ and $-r < w < 2p - r$. Since $0 < r < p < q$, the binding constraints are $-r < w < r$. The line $w = 0$ passes through the centre of M; Hooper's involution about that centre maps the line at offset w to the line at offset $-w$ reversed. □

In the notation of §7.2, (N) is the equation $N = \kappa(a)$, and the corner path of Ω_N has counts $(n_I, n_X) = (N, N+1)$.

Lemma 7.8 (no right angle)**.** *Under (H), no line of direction θ that stays in the levels $L-1, L, L+1$ and crosses only the doors $L \pm \frac{1}{2}$ meets a side at a right angle.*

Proof. As in §3.2: a right-angle hit on H needs a crossed door with $\chi \equiv 0$, but $0 < |\delta \mp a| < \pi/2$; a right-angle hit on a leg needs a level whose average $(\chi_{n-\frac{1}{2}} + \chi_{n+\frac{1}{2}})/2$ lies in $(\pi/2)\mathbb{Z}$, but for $n = L, L \pm 1$ these averages are δ and $\delta \pm 2a$, with $0 < |\delta| < \pi/2$ and $0 < |\delta \pm 2a| \leq \eta + 2a < \pi/2$. □

Theorem 7.9. *A two-door ladder at a^* is infinite if $|\cos 2a^*| > \frac{1}{2}$ and finite if $0 < |\cos 2a^*| < \frac{1}{2}$.*

Proof. Take the first type; the second is the same with the legs exchanged ($a \mapsto \pi/2 - a$). Suppose first $|\cos 2a^*| > \frac{1}{2}$, i.e. $a^* < \pi/6$. Then (H) holds on a punctured neighbourhood of a^*, and on each side of a^* the function $N(a) \coloneqq \sin(a-\eta)/(2\cos a \sin\eta)$ is continuous and tends to $+\infty$ as $a \to a^*$. By the intermediate value theorem every large integer N is attained at some a_N on that side, and by Lemmas 7.7 and 7.8 $Tri(a_N)$ carries a non-perpendicular cylinder with middle level L; by Lemma 7.10 below all but finitely many a_N have $a_N/\pi \notin \mathbb{Q}$, so the ladder is infinite.

Now let $\pi/6 < a^* < \pi/4$, and a near a^*. Then $3a - \eta > \pi/2$, so the outer door $L + 3/2$ of X has $\cos\chi < 0$ (for $\delta < 0$; for $\delta > 0$ exchange

the roles of X and I). The doors $L \pm \frac{1}{2}$, and with them the moves and τ_M, are as in Lemma 7.6, and $L + \frac{1}{2}$ is still the turning door of X, since $|\cos(3a-\eta)| < \cos(a-\eta)$. But by Lemma 7.2 the turning strip of X shrinks to $(p-c)\tan a$ with $c \coloneqq |\cos(3a-\eta)|/\sin a$. By Theorem 7.4 the core of a member passes through the centre of M and then turns in X at offset λ_X (in units of $\tan a$), so $|\lambda_X| = p - r < p - c$, i.e. $\sin\eta > |\cos(3a-\eta)|/\sin a$. As $a \to a^*$ the left side tends to 0 and the right to $|\cos 3a^*|/\sin a^* > 0$, so no member lies near a^*. Away from a^* the ladder has only finitely many members: at a member $a + \eta < \pi/2$ (otherwise 0 and one of $\pm\pi/2$ both lie between the two values $\delta \mp a$, against $\mu_I \mu_X < 0$), so the moves are computed as in Lemma 7.6 and (N) holds with $N = \kappa(a) = \sin(a-\eta)/(2\cos a \sin\eta)$, which is bounded there (it tends to 0 as $\eta \to a$); and for each fixed N, by the computation in the proof of Lemma 7.10, (N) is a nonzero polynomial equation in e^{ia}, with at most $2|K|+2$ roots. So the ladder is finite. □

At $|\cos 2a^*| = \frac{1}{2}$ the proof gives nothing; [GZ03, Thm 2] gives mirror trajectories of every type at $a = \pi/6$, which suggests that these ladders are infinite (§8). At $\cos 2a^* = 0$ the argument above excludes members near a^* with $a < \pi/4$ only.

Members. By Theorem 7.4, every member of a ladder of the first type with $a^* < \pi/6$ that lies in the neighbourhood of a^* where (H) holds is one of the a_N above.

Lemma 7.10. *All but finitely many a_N satisfy $a_N/\pi \notin \mathbb{Q}$.*

Proof. $\varphi(a) = c_0 + Ka$ with $c_0 \in \{0, \pi/2\}$, and (N), written as $\sin(a - |\delta|) = 2N\cos a \sin|\delta|$, expands into sines (if $c_0 = 0$) or cosines (if $c_0 = \pi/2$) of integer multiples of a, of frequency at most $|K|+1$. In $z = e^{ia}$, after multiplying by $z^{|K|+1}$ and a constant, it is an integer polynomial of degree $2|K|+2$ with leading coefficient $\pm N$ or $\pm(N+1)$, hence nonzero. So e^{ia_N} is algebraic of degree at most $2|K|+2$, a bound independent of N. If $a_N/\pi \in \mathbb{Q}$, e^{ia_N} is a root of unity whose order has Euler totient at most $2|K|+2$, and there are finitely many such orders. □

Lemma 7.11 (placing the centre)**.** *For $x^* = d/Q \in (0, 1/3)$ in lowest terms and $a^* = \pi x^*/2$, condition (C) holds for some L: in $\perp OA$ with $\theta = \pi/2$ always; in $\perp OB$ with $\theta = 0$ if Q is even; in $\parallel H$ with $\theta = -a$ if d and Q are odd; in $\perp H$ with $\theta = \pi/2 - a$ if d is even.*

Proof. In $\perp OA$ the fold is the level 0 and $\varphi = 2aL$, so $L \coloneqq Q$ gives $\varphi(a^*) = \pi d$. In $\perp OB$, $\varphi = \pi/2 + 2aL$, and we need $dL/Q \in \frac{1}{2} + \mathbb{Z}$; with $Q = 2Q'$ take $L \equiv d^{-1}Q' \pmod{2Q'}$. In $\parallel H$, $\varphi = \pi/2 + (2L+1)a$, and we need $d(2L+1) \equiv Q \pmod{2Q}$; $d^{-1}Q$ is odd modulo $2Q$, so this is solvable. In $\perp H$, $\varphi = (2L+1)a$, and $2L+1 \coloneqq Q$ works because d is even. In each case $K \neq 0$. □

Proposition 7.12 (explicit ladders)**.** *Let $x^* = d/Q \in (0, 1/3)$ in lowest terms, and let cls be $\perp OA$, or $\perp OB$ if Q is even, $\parallel H$ if d and Q are*

odd, or $\perp H$ if d is even. Then x^* is the limit of points $x_N \in E_{\mathrm{cls}}$ at which $Tri(a_N)$ carries a non-perpendicular cylinder of width $\tan a_N \cdot \sin \eta_N$, with $\eta_N = |K(a_N - a^*)| \to 0$. The same holds at $x^* = d'/Q \in (2/3, 1)$ in lowest terms for $\perp OB$, and for $\perp OA$ if Q is even, $\| H$ if d' is even, and $\perp H$ if d' and Q are odd.

Proof. Lemma 7.11 gives (C), the proof of Theorem 7.9 the members $a_N \to a^*$, Lemma 7.10 their irrationality with finitely many exceptions, and Lemmas 7.7 and 7.8 the cylinders. Exchanging the legs maps $x \mapsto 1-x$ and the classes $\perp OB \leftrightarrow \perp OA$, $\perp H \mapsto \perp H$, $\| H \mapsto \| H$, and $d/Q \mapsto (Q-d)/Q$. □

Each of these sets of rationals is dense in $[0, 1/3]$, so Proposition 7.12 reproves the density of Theorem 6.1, with explicit cylinders, for every class on both outer thirds. Theorem 7.9 is why it stops at $x = 1/3$; the middle third needs the accumulation at irrational points that Lemma 6.5 provides.

7.4. The $\sqrt{2}$ ladder and the negative Pell equation.

Corollary 7.13. *For every integer $N \geq 1$, let y be the positive root of $(4N+4)y^2 - 2y - (2N+1)$ or of $4Ny^2 + 2y - (2N+1)$. Then $a/\pi \notin \mathbb{Q}$ for $a = \frac{1}{2} \arccos y$, and $Tri(a)$ carries a non-perpendicular periodic cylinder in the class $\perp OB$, of period $10N+6$. Such a y is rational iff $u^2 - 2v^2 = -1$ for some integer v, with $u = 4N+3$, resp. $4N+1$. So every solution of the negative Pell equation $u^2 - 2v^2 = -1$ with $u > 1$ gives a right triangle with rational $\cos 2a$ carrying a non-perpendicular periodic cylinder — $y = 3/4$ (Example 5.1), $7/10$, $17/24$, … — with periods $16, 106, 596, 3486, 20296, \ldots$, i.e. $5p'p + 1$ for consecutive numerators p', p of the convergents of $\sqrt{2}$.*

Proof. Take the class $\perp OB$, $\theta = 0$ and $L = 2$, so that $\varphi(a) = \pi/2 + 4a$: (C) holds at $a^* = \pi/8$ with $K = 4$. On the side $a < \pi/8$ we have $\eta = \pi/2 - 4a$, and (N) reads $N = -\cos 5a/(2\cos a \cos 4a)$; since $\cos 5a = \cos a \cdot (4y^2 - 2y - 1)$ and $\cos 4a = 2y^2 - 1$, this is $(4N+4)y^2 - 2y - (2N+1) = 0$. On the side $a > \pi/8$, $\eta = 4a - \pi/2$, and (N) reads $N = -\cos 3a/(2\cos a \cos 4a)$, i.e. $4Ny^2 + 2y - (2N+1) = 0$.

The first quadratic is negative at $y = 1/\sqrt{2}$ and equals $(N-1)/4 \geq 0$ at $y = 3/4$, so its positive root lies in $(1/\sqrt{2}, 3/4]$, i.e. $a \in [\frac{1}{2}\arccos(3/4), \pi/8)$. There (H) reads $\pi/2 - 4a < a$, i.e. $a > \pi/10$, which holds since $\cos(\pi/5) > 3/4$, and $3a + \eta = \pi/2 - a < \pi/2$. The second quadratic is positive at $y = 1/\sqrt{2}$, and at $y = \cos(2\pi/7)$ its value is $2N\cos(4\pi/7) + 2\cos(2\pi/7) - 1 \leq 2\cos(4\pi/7) + 2\cos(2\pi/7) - 1 = 2\cos(\pi/7) - 2 < 0$, using $\cos(2\pi/7) + \cos(4\pi/7) + \cos(6\pi/7) = -\frac{1}{2}$. So its positive root lies in $(\cos(2\pi/7), 1/\sqrt{2})$, i.e. $a \in (\pi/8, \pi/7)$, where (H) reads $4a - \pi/2 < a$ ($a < \pi/6$) and $3a + \eta = 7a - \pi/2 < \pi/2$ ($a < \pi/7$). So (H) holds at every member, and Lemmas 7.7 and 7.8 give the cylinders, of period $5S + 1 = 10N + 6$ by Theorem 7.4. For the irrationality of a/π: $2y$ lies in $(1, 3/2]$, so if it is rational it is not an integer; if it is irrational its minimal polynomial is $(N+1)z^2 - z - (2N+1)$, resp. $Nz^2 + z - (2N+1)$, which is primitive and not monic except for the

second at $N = 1$, where the conjugate $(-1 - \sqrt{13})/2$ of $2y$ is below -2. In every case $2y = 2\cos 2a$ is not an algebraic integer all of whose conjugates lie in $[-2, 2]$, so $a/\pi \notin \mathbb{Q}$.

A root is rational iff the discriminant $4(8N^2 + 12N + 5)$, resp. $4(8N^2 + 4N + 1)$, is a square, i.e. iff $8N^2 + 12N + 5 = v^2$ resp. $8N^2 + 4N + 1 = v^2$, which is $u^2 - 2v^2 = -1$ with $u = 4N + 3$, resp. $u = 4N + 1$. Every solution has u odd; $u = 1$ gives $N = 0$, and every other solution has $u \geq 7$, hence $N \geq 1$. The solutions $u = 7, 41, 239, 1393, 8119$ give $N = 1, 10, 59, 348, 2029$ and the periods listed. □

The first member, $N = 1$ on the side $a < \pi/8$, is Example 5.1: there $\sin\eta = 1/8$, the smallest angle quoted in Example 5.1, and the width $\tan a \cdot \sin\eta = \sqrt{7}/56$ is the one found there.

8. Remarks and open questions

The class $\parallel H$ off algebraic points. For transcendental $\cos 2a$, a line of a direction in $\parallel H$ is confined to a half-chain (Remark 3.7) and is never periodic (Theorem 3.1). The flow in such a direction is nevertheless recurrent for almost every starting point: Gutkin and Troubetzkoy proved that right triangles are strongly recurrent, in every direction [GT96, Cor. 4.3]. What the flow does beyond that is open. On a simpler chain-like infinite surface, the infinite staircase, the invariant measures of every directional flow are classified [HHW13]. Call an orbit a forward (backward) *escape orbit* if its level tends to $+\infty$ ($-\infty$); Troubetzkoy proved that every direction not parallel to H has exactly one of each [Tro05b, Thm 5]. We expect, but have not proved, that in $\parallel H$ each half-chain carries exactly one escape orbit, and that there is no minimal component of finite area. Non-periodic trajectories that are not dense were found in polygons by Galperin [Gal83] and, by means of induced interval exchanges, in a family of isosceles triangles by Slipantschuk, Bandtlow and Just [SBJ24] (see also [ZSBJ23]); as they note, the construction gives such trajectories in right triangles too, in a direction fixed by a generalised diagonal, which is not a mirror direction. Whether the confined $\parallel H$ directions carry them is open.

Ladders at $\pi/6$. Is a two-door ladder at $a^* = \pi/6$ (or $\pi/3$) infinite? Theorem 7.9 leaves $|\cos 2a^*| = \frac{1}{2}$ open; mirror trajectories of every type exist at $a = \pi/6$ [GZ03, Thm 2], which suggests that it is.

Isosceles odd orbits. Does an isosceles triangle with irrational base angle have finitely or infinitely many odd orbits? By Proposition 4.3 the primitive odd orbits, up to reversal, are indexed by the directions $2ka$ from the base midpoint that reach the axis perpendicularly without meeting a vertex or returning to the midpoint first. The rational case, including when the bound of Proposition 4.3(5) is attained, is treated in [Neck, Corollary 12.5 and Remark 12.6].

Scalene triangles. A triangle with angles α, β, γ rationally independent has eight mirror classes. With one side along the real axis the mirror directions are $k\alpha + m\beta$ and these plus $\pi/2$; the first split by the parities of k and m into four classes of *parallel type* — three contain the direction of a side, the fourth none — and the second into the four classes of *perpendicular type*. At $\gamma = \pi/2$ they merge in pairs into the four of a right triangle. In every acute triangle Fagnano's orbit is a periodic orbit in a class of parallel type that meets no side at a right angle, so the parallel-type half of Theorem 3.1 fails for general triangles. *Is every periodic orbit of a typical triangle in a perpendicular-type mirror class perpendicular?*

Sharpness. Is there an irrational right triangle with algebraic $\cos 2a$ that carries no non-perpendicular periodic cylinder in some mirror class? More strongly: is there one that carries none in any mirror class? Of the 126 triangles with rational $\cos 2a = p/q$, $q \le 20$, our search found cylinders at only nine (Table 1), two of them only in a ten times deeper pass run at $q \le 12$; and every absence we know is a search that found nothing, not a proof. In the other direction, one triangle can carry several exceptional cylinders in the same class — up to six in our data, often nested in towers — and whether a triangle can carry infinitely many is open.

Selecting the exceptional points. Theorem 3.4 confines the exceptional set to algebraic $\cos 2a$ with explicit degree bounds; Remark 3.6 suggests that the sign constraints on door counts do little to narrow it further. We know no arithmetic characterization of the exceptional points.

Growth. Systematic searches find exceptional cylinders in abundance (§5.3). Their census, and the growth of their number counted by word length, are left for later work.

Code availability. The computations of §5 and §6 are deposited at `https://github.com/josh-boorstein/billiards`, directory `mirror_A/`, one script per statement: the exact certificate of every cylinder of §5 from its word (Examples 5.1 and 5.2, Table 1, the four cylinders at $\cos 2a = (1+\sqrt{41})/8$, and the members $n \le 8$ of the family of §5.3 with its period-30 cylinder at y_c), the cylinder of Remark 6.10, and the tiles of Proposition 6.6 with Table 2 and the five F_W. A small library, `lib/`, carries the arithmetic in $\mathbb{Q}(2\cos a)$ with certified signs. The figures are drawn by a script that checks every number in their captions. Every theorem, proposition, lemma and corollary is proved in the text. Example 5.2, Table 1, the cylinders at $\cos 2a = (1+\sqrt{41})/8$, the members of the family of §5.3 and the cylinder of Remark 6.10 are computer-certified (exact arithmetic), and for that family the text claims only the certified members; Table 2 and the five F_W were computed by the deposited script. The version used here is release v1.0, archived at Zenodo as doi:10.5281/zenodo.24037744.

Use of AI

This paper — its text, its proofs and the code behind Sections 5 and 6 — was drafted by an AI system (Claude, Anthropic) working under the author's direction, and checked by independent AI review passes and by the exact computations described in the paragraph *Code availability*. The author has read parts of it in detail and takes responsibility for its content.

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