THE BENSON–ETINGOF–OSTRIK CONJECTURE

Abstract. We prove the conjecture of Benson–Etingof–Ostrik (2023): every symmetric tensor category of moderate growth over an algebraically closed field of positive characteristic admits a symmetric tensor functor to a higher Verlinde category, possibly of infinite height. Using maximal nilpotence and geometric reductivity, we reduce the problem to proving that finitely generated maximally nilpotent tensor categories are finite. The main ingredients are the construction of generic points using alternating norms and a Frobenius reduction of the fundamental coend to an ordinary Hopf subalgebra. The results of Coulembier–Flake–Newton–Ostrik supply maximal nilpotence of the Verlinde targets, the top alternating norm line, and the reduction to finitely generated sources.

hexagon:2611.00301v2

Part 1. All positive characteristics

1. Main theorem and preliminaries

We follow [EGNO15] for the basic conventions on tensor categories, duality, ind-completions, Deligne products, and extension groups. All tensor categories are symmetric, with $\mathrm{End}(\mathbf{1}) = k$; tensor functors are exact, symmetric, and linear over the indicated ground field. Unless a different hypothesis is stated, the ground field k is algebraically closed of characteristic $p > 0$.

The main theorem resolves the conjecture of Benson, Etingof, and Ostrik [BEO23, Conjecture 1.4] in full generality.

Theorem 1.1 (Benson–Etingof–Ostrik conjecture)**.** *Let k be an algebraically closed field of characteristic $p > 0$. For every symmetric tensor category $\mathcal{C}$ over k of moderate growth, there exists an exact faithful k-linear symmetric monoidal functor*

$$F : \mathcal{C} \longrightarrow \mathrm{Ver}_{p^\infty,k}\,.$$

If $\mathcal{C}$ is finitely generated, F factors through $\mathrm{Ver}_{p^r,k}$ for some finite $r \geq 1$; in particular, $\mathcal{C}$ fibres over a finite tensor category.

Here $\mathrm{Ver}_{p^r,k}$ is the finite higher Verlinde category of height r [BEO23]. We omit k when it is fixed, and put $\mathrm{Ver}_{p^\infty,k} = \bigcup_{r\geq 1} \mathrm{Ver}_{p^r,k}$ under the standard tensor embeddings. A fibre functor with target $\mathcal{D}$ means a tensor functor to $\mathcal{D}$.

Theorem 6.4 proves the intermediate assertion that finitely generated maximally nilpotent (MN) categories are finite. Moderate growth is automatic for MN categories by Lemma 4.3(4). The finiteness proof uses the ordinary-coend results established in Parts 2 and 3. These results depend only on the preceding common lemmas. We then prove the main theorem for finitely generated sources and apply [CFNO26, Proposition 4.4] to pass to arbitrary sources. Proposition 7.1 deduces the embedding theorem for arbitrary MN categories. Section 9 also proves that every tensor subcategory of an MN category in odd characteristic is Serre; this is a consequence of the finiteness theorem, not an input to its proof.

Notation specific to this paper. A tensor subcategory is full and closed under subquotients, finite sums, tensor products, and duals; it need not be closed under extensions. Write $\langle G\rangle$ for the tensor subcategory generated by G. Thus *finitely generated* means $\mathcal{C} = \langle G\rangle$, whereas *finite* has the usual meaning of [EGNO15, Section 1.8]: equivalence as an abelian category to finite-dimensional

Date: 13 November 2026.

modules over a finite-dimensional algebra. Write $\ell(X)$ for composition length. Moderate growth means that for every X some $b_X > 0$ satisfies $\ell(X^{\otimes n}) \le b_X^n$ for all $n \ge 1$.

We use $\mathbf{1}$, X^*, ev_X, coev_X, and $c_{X,Y}$ for the unit, dual, evaluation, coevaluation, and symmetry. An invertible object is also called a *line*; it is *even* if its self-symmetry is the identity. The categories of vector spaces and super vector spaces are denoted by Vec_k and sVec_k. An *ordinary* ind-object is a direct sum of copies of $\mathbf{1}$. A *finite object* is an object of $\mathcal{C}$, as opposed to an arbitrary object of $\operatorname{Ind}\mathcal{C}$.

Algebras are commutative and unital ind-algebras unless stated otherwise. We write $\operatorname{Sym}^n X = (X^{\otimes n})_{S_n}$ for ordinary symmetric powers and $\operatorname{Sym} X = \bigoplus_{n\ge 0} \operatorname{Sym}^n X$. An algebra is finitely generated if it is a quotient of $\operatorname{Sym} X$ for a finite X; an A-module is finitely generated if it is a quotient of $A \otimes X$ for such an X. We write $\operatorname{Mod}_A(\operatorname{Ind}\mathcal{C})$ for all A-modules in $\operatorname{Ind}\mathcal{C}$, with relative tensor product $\otimes_A$.

Alternating norms. For the action of S_n through the symmetry, put

$$a_n^X = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\sigma, \qquad A_n(X) = \operatorname{im} a_n^X, \qquad \operatorname{ad}(X) = \sup\{n \ge 0 : A_n(X) \ne 0\}.$$

We call a_n^X the antisymmetrizer or alternating norm, and $\operatorname{ad}(X)$ its top nonzero degree. We use $A_0(X) = \mathbf{1}$ and $\operatorname{ad}(0) = 0$. There is no factorial division; in characteristic two all the signs are 1. In positive characteristic, over any field, moderate growth is equivalent to finiteness of these degrees, and

$$\ell(X) \le \operatorname{ad}(X), \qquad \operatorname{ad}(X \otimes Y) \le \operatorname{ad}(X)\operatorname{ad}(Y)$$

[CEO23, Proposition 4.7(4), Lemma 4.8, and Corollary 4.11]. The top norm line theorem of [CFNO26, Proposition 5.1] is recorded in Lemma 5.1.

Existence and descent. A category is *incompressible* if every outgoing tensor functor embeds it as a tensor subcategory. A tensor functor is *surjective* if every target object is a subquotient of an image, as in [CEO24, Definition 3.1.4]. We use the following established results.

(1) Every category of moderate growth over an algebraically closed field admits a surjective tensor functor to an incompressible category of moderate growth [CEO24, Theorem 5.2.1]. Its target is finitely generated if the source is.
(2) Every finite tensor category over k admits a tensor functor to a finite higher Verlinde category [CF26, Theorem B], the finite Benson–Etingof–Ostrik theorem. The Verlinde categories are defined over the prime field [BEO23, Section 4.9]; their scalar extension is made explicit in the proof of Theorem 1.1.
(3) If $\mathcal{V}$ is MN and geometrically reductive, then an exact k-linear symmetric monoidal functor $\mathcal{C} \to \operatorname{Mod}_R(\operatorname{Ind}\mathcal{V})$, for a nonzero commutative ind-algebra R, implies the existence of a tensor functor $\mathcal{C} \to \mathcal{V}$ [CS26, Appendix B, Theorem B.1].
(4) To prove that every moderate-growth tensor category over k admits a tensor functor to $\mathrm{Ver}_{p^\infty,k}$, it suffices to treat finitely generated categories [CFNO26, Proposition 4.4].

MN is defined in Section 4. Geometric reductivity (GR) means that every epimorphism $X \twoheadrightarrow \mathbf{1}$ has a split positive symmetric power. We prove MN implies GR in Section 5. The Frobenius reduction used for finiteness is stated at its point of use in Section 6.

We also use Gabriel localization: quotienting a Grothendieck category by a Serre subcategory closed under coproducts gives an exact quotient functor with a fully faithful right adjoint [Gar99, Section 1.2]. The next two sections establish the tensor structure and finiteness properties of our particular quotient.

2. A highest-norm identity and flatness

This section and Section 3 allow an arbitrary ground field of positive characteristic, except where stated otherwise.

Lemma 2.1 (Parametrized inner inverse)**.** *Let $\mathcal{M}$ be an additive symmetric monoidal category with additive tensor product. Let P, Q be dualizable, and $0 \neq f : P \to Q$. Suppose either $a_n^P = 0$ for all sufficiently large n, or $a_n^Q = 0$ for all sufficiently large n. There are a dualizable object T, a nonzero map $t : T \to \mathbf{1}$, and $g : T \otimes Q \to P$ such that*

$$fg(\mathrm{id}_T \otimes f) = f(t \otimes \mathrm{id}_P). \tag{2.1}$$

If $\mathcal{M}$ is abelian with simple unit, t is epic.

Proof. Choose the largest $n \geq 1$ for which

$$F_n = a_n^Q f^{\otimes n} = f^{\otimes n} a_n^P \neq 0.$$

Put $T = (Q^{\otimes n})^* \otimes P^{\otimes n}$, and define

$$t = \mathrm{ev}_{Q^{\otimes n}}(\mathrm{id} \otimes F_n) : T \longrightarrow \mathbf{1}.$$

The duality bijection $\mathrm{Hom}(P^{\otimes n}, Q^{\otimes n}) \simeq \mathrm{Hom}(T, \mathbf{1})$ makes $t \neq 0$. This is a matrix-coefficient map, not a trace.

The coset decomposition of S_{n+1} gives the following integral identity, with object superscripts suppressed and $(i, n+1)$ denoting the transposition of positions i and $n+1$:

$$a_{n+1} = (a_n \otimes \mathrm{id}) \left(\mathrm{id} - \sum_{i=1}^{n} (i, n+1) \right). \tag{2.2}$$

Apply it to $f^{\otimes(n+1)}$ and evaluate its first n outputs against $(Q^{\otimes n})^*$. Its left side is zero. The identity term on the right is $f(t \otimes \mathrm{id}_P)$.

Here is an explicit factorization of the other terms. Let

$$b_i : P^{\otimes n} \otimes Q \longrightarrow Q^{\otimes n} \otimes P$$

interchange the i-th parameter and the last input, apply f to the other $n-1$ parameter factors, and leave the last output factor unchanged. Define

$$g_i = (\mathrm{ev}_{Q^{\otimes n}} \otimes \mathrm{id}_P)(\mathrm{id} \otimes a_n^Q \otimes \mathrm{id}_P)(\mathrm{id} \otimes b_i).$$

Naturality of the symmetry gives the explicit identification

$$(\mathrm{ev}_{Q^{\otimes n}} \otimes \mathrm{id}_Q)\big(\mathrm{id} \otimes [(a_n^Q \otimes \mathrm{id}_Q) f^{\otimes(n+1)} (i, n+1)_P]\big) = f g_i (\mathrm{id}_T \otimes f).$$

Thus $g = \sum_i g_i$ gives (2.1). The construction uses neither scalar points nor factorial division; all signs come from the symmetry. If the unit is simple, every nonzero map to it is epic. □

In an abelian monoidal category with right exact tensor product, an object M is *flat* if $M \otimes -$ is exact. A monomorphism is *pure* if it stays monic after tensoring with any object.

Lemma 2.2 (Purity of an image)**.** *Suppose $\mathcal{M}$ is abelian symmetric monoidal, tensor product is right exact, and the unit is simple. Let $f : P \to Q$ satisfy Lemma 2.1. Then $\mathrm{im}(f) \hookrightarrow Q$ remains monic after tensoring with any object. If Q is flat,* $\mathrm{coker}\, f$ *is flat.*

Proof. Write $f = j\pi$, where $\pi : P \twoheadrightarrow I = \mathrm{im}\, f$ and $j : I \hookrightarrow Q$. Set $r = \pi g$. Cancel j in (2.1), and descend along the epimorphism $\mathrm{id}_T \otimes \pi$. This gives

$$r(\mathrm{id}_T \otimes j) = t \otimes \mathrm{id}_I\,. \tag{2.3}$$

For any M, let $J = \ker(j \otimes \mathrm{id}_M)$. Restrict (2.3), tensored with M, to $T \otimes J$. It says that the composite

$$T \otimes J \xrightarrow{t \otimes \mathrm{id}_J} J \hookrightarrow I \otimes M$$

is zero. The first arrow is epic by right exactness, so $J = 0$. This proves purity without assuming that M is flat.

For completeness, let $M' \hookrightarrow M$ have cokernel M''. Purity gives exact rows

$$0 \longrightarrow I \otimes M' \longrightarrow Q \otimes M' \longrightarrow (\operatorname{coker} f) \otimes M' \longrightarrow 0$$

and the analogous row for M. Flatness of Q, together with the monomorphisms from I, makes both left vertical arrows monic. The snake lemma identifies the possible kernel on the right with the kernel of $I \otimes M'' \to Q \otimes M''$. This is zero by purity. Hence $\operatorname{coker} f$ is flat. □

Remark 2.3. For an ind-algebra A in a tensor category, call $A \otimes X$, with X finite, *finite free*. Such objects need not be projective. Nevertheless every A-module is a filtered colimit of cokernels of maps between finite free modules.

Indeed, take a free cover $P = \bigoplus_\alpha (A \otimes X_\alpha) \twoheadrightarrow M$. For each finite subsum P_J, its relation module $\ker(P_J \to M)$ is the union of images of finite free modules. Take the cokernels of all finite sets of these relations in all such P_J. These form a filtered system with colimit M. Each is a cokernel of a map $A \otimes X \to A \otimes Y$. The relevant compactness is just

$$\operatorname{Hom}_A(A \otimes X, -) = \operatorname{Hom}_{\operatorname{Ind}\mathcal{C}}(X, -),$$

which commutes with filtered colimits. It does not assert projectivity; on the right, the module's A-action is forgotten.

3. The generic category of a domain

For subobjects $E, F \subset A$, their product is the image of $E \otimes F \to A$; similarly, E^n denotes the image of $E^{\otimes n} \to A$ under multiplication. An ideal of A is an A-submodule of its regular module. A nonzero algebra is *simple* if its only ideals are zero and itself. The ideal generated by E is $AE = \operatorname{im}(A \otimes E \to A)$. It is *finitely generated* when E can be chosen finite. For an ideal I and an A-module M, write $IM = \operatorname{im}(I \otimes_A M \to M)$; I annihilates M if $IM = 0$. We call $A \neq 0$ a *domain* if the product of any two nonzero ideals is nonzero. This definition is internal to the tensor category.

Lemma 3.1 (Torsion localization)**.** *Let A be a domain in $\operatorname{Ind}\mathcal{C}$. Let $\mathcal{T}_A$ consist of the modules that are unions of submodules annihilated by nonzero finitely generated ideals of A. These are called* torsion *modules;* torsion free *means having no nonzero torsion submodule. Then $\mathcal{T}_A$ is a localizing Serre tensor ideal, where tensor ideal means closed under tensoring with arbitrary A-modules. The Gabriel quotient*

$$\mathcal{Q}_A = \operatorname{Mod}_A(\operatorname{Ind}\mathcal{C})/\mathcal{T}_A, \qquad q : \operatorname{Mod}_A(\operatorname{Ind}\mathcal{C}) \longrightarrow \mathcal{Q}_A$$

is a nonzero Grothendieck category with a right exact symmetric tensor product that is cocontinuous, meaning colimit-preserving, in each variable. The functor q is exact and strong symmetric monoidal, and qA is its simple tensor unit. The tensor product on $\mathcal{Q}_A$ is induced by $\otimes_A$; we write it simply as $\otimes$.

Proof. Products of nonzero finitely generated ideals are again nonzero and finitely generated. A finite sum of annihilated submodules is therefore annihilated by one such ideal. Closure under subobjects, quotients, and sums follows.

For extension closure, take an extension with torsion endpoints and a finitely generated submodule E of its middle term. Its image in the quotient is killed by a nonzero finitely generated ideal I. Hence IE lies in the torsion submodule. The module IE is finitely generated: tensor finite free covers of I and E, and use right exactness. Thus $JIE = 0$ for another nonzero finitely generated ideal J. Since every module is the union of its finitely generated submodules, extension closure follows. Tensoring an I-annihilated module with any module preserves annihilation; colimits give the tensor-ideal assertion.

Tensor descent requires checking kernels, not just cokernels. Suppose $j : M' \hookrightarrow M$ has cokernel killed by a nonzero ideal I. The action $I \otimes_A M \to M$ factors through M'. Its restriction to $I \otimes_A M'$ agrees with the action on M': compose with j and cancel that monomorphism *before* tensoring further. For an arbitrary module N, the action on $M' \otimes_A N$ therefore factors as

$$I \otimes_A M' \otimes_A N \longrightarrow I \otimes_A M \otimes_A N \longrightarrow M' \otimes_A N.$$

It kills $\ker(j \otimes_A \mathrm{id}_N)$. The cokernel after tensoring is killed by I as well. If M/M' is only locally torsion, write it as a filtered union of annihilated submodules and take their inverse images in M. Exactness of filtered colimits expresses the kernel after tensoring as a union of the just-considered kernels. For an epimorphism with torsion kernel, right exactness suffices. Thus tensoring preserves every morphism with torsion kernel and cokernel, and descends to the Gabriel quotient.

The usual localization theorem gives the Grothendieck category, the exact quotient functor, and its fully faithful right adjoint q_*. Tensor preserves colimits in the quotient: lift a diagram by q_*, compute its colimit before applying q, and use that both q and the original relative tensor product preserve colimits. The symmetry and associativity constraints descend.

There is no nonzero torsion ideal in A: a nonzero finite subobject of such an ideal would generate a nonzero ideal whose product with another nonzero ideal is zero. Consequently $qA \neq 0$. For every nonzero ideal I, choose a nonzero finitely generated ideal inside I; it kills A/I. Thus $qI \simeq qA$. Every subobject $U \subset qA$ is the image of an ideal of A. Explicitly, pull back $q_*U \subset q_*qA$ along $A \to q_*qA$, and apply the exact functor q. Hence qA is simple. □

Theorem 3.2 (The generic tensor category)**.** *Suppose every object of $\mathcal{C}$ has finite alternating norm degree. For a domain A, tensor product in $\mathcal{Q}_A$ is exact. Every object of $\mathcal{Q}_A$ is a union of finite-length subobjects. The full finite-length subcategory $\mathcal{D}_A$ is a tensor category over the field*

$$K_A = \mathrm{End}_{\mathcal{Q}_A}(qA).$$

Scalars on the nonzero unit give an embedding $k \hookrightarrow K_A$. The functor

$$F_A : \mathcal{C} \longrightarrow \mathcal{D}_A, \qquad X \longmapsto P_X = q(A \otimes X)$$

is k-linear, exact, faithful, and strong symmetric monoidal, with

$$\ell(P_X) \leq \mathrm{ad}(X). \tag{3.1}$$

Every object of $\mathcal{D}_A$ is a quotient of some P_X. If $\mathcal{C}$ has moderate growth in positive characteristic, then so does $\mathcal{D}_A$. The field K_A need not equal k; equality is proved under the additional hypotheses of Lemma 3.5.

Proof. The P_X are dualizable, with dual P_{X^*}, and generate $\mathcal{Q}_A$ under sums and quotients. Tensoring by a dualizable object has both adjoints, so the P_X are flat. They inherit vanishing of high alternating norms. For every nonzero map $f : P_X \to P_Y$, Lemmas 2.1 and 2.2 show that $\operatorname{coker} f$ is flat; this is also true when $f = 0$. Apply q to the filtered presentations in Remark 2.3. Every object of $\mathcal{Q}_A$ is a filtered colimit of these flat cokernels. Filtered colimits are exact and commute with tensor product, so every object is flat. Tensor product is therefore exact.

Tensor products of nonzero objects are nonzero. A nonzero map $P_X \to M$ transposes to a nonzero map $qA \to M \otimes P_{X^*}$, which is monic because qA is simple. Tensoring it with $N \neq 0$ proves $M \otimes N \neq 0$.

We adapt the filtration argument of [CEO23, Lemma 4.9] to this setting, where finite length is not yet known. Suppose an object P has a strict finite filtration with d nonzero factors $Q_1, \dots, Q_d$. Exact tensor product gives the usual tensor filtration, whose associated graded is $(\bigoplus_i Q_i)^{\otimes d}$, compatibly with permutations. On the ordered summand $Q_1 \otimes \cdots \otimes Q_d$, the composite of the alternating norm and projection back to that summand is the identity: every other permutation lands in a different ordered summand. This summand is nonzero. Thus $a_d^P \neq 0$. Applied to P_X, this proves (3.1), without assuming finite length in advance.

Images of finite sums of the P_X exhaust every object. They have finite length, proving local finite length. A finite-length object M is a quotient of one P_Y; its kernel is again finite-length and is covered by a P_X. Hence

$$P_X \longrightarrow P_Y \longrightarrow M \longrightarrow 0.$$

The category is closed monoidal: cocontinuous tensor functors on a Grothendieck category have right adjoints. The internal Hom $\underline{\mathrm{Hom}}(M,N)$ is characterized by

$$\mathrm{Hom}(U, \underline{\mathrm{Hom}}(M,N)) \simeq \mathrm{Hom}(U \otimes M, N).$$

Set $M^\vee = \underline{\mathrm{Hom}}(M, qA) = \ker(P_{Y^*} \to P_{X^*})$, a finite-length object. For $P = P_X, P_Y$, dualizability makes the canonical map

$$\underline{\mathrm{Hom}}(P, qA) \otimes N \longrightarrow \underline{\mathrm{Hom}}(P, N)$$

an isomorphism. Left exactness of internal Hom in its first variable and exactness of $- \otimes N$ identify the corresponding map for M with the isomorphism

$$M^\vee \otimes N \xrightarrow{\sim} \ker(P_{Y^*} \otimes N \to P_{X^*} \otimes N) = \underline{\mathrm{Hom}}(M, N).$$

Invertibility of this canonical map for every N is equivalent to dualizability of M: its adjunction unit and counit give coevaluation and evaluation, with the duality identities. Thus M is dualizable with dual $M^\vee$. Tensor products of finite-length objects are finite-length, since they are quotients of objects $P_X \otimes P_Y = P_{X \otimes Y}$.

The endomorphism ring of the simple unit is a division ring, and is commutative by the monoidal unit axioms. It is the field K_A. Independent maps $qA \to V$ give a monomorphism from the corresponding finite sum of units. Consequently

$$\dim_{K_A} \mathrm{Hom}(qA, V) \le \ell(V), \qquad \dim_{K_A} \mathrm{Hom}(M, N) \le \ell(M^\vee \otimes N).$$

This proves Hom-finiteness. Essential smallness follows because the P_X form a set and, up to isomorphism, the quotients of each object in a Grothendieck category form a set. These establish all the tensor-category axioms for $\mathcal{D}_A$.

For M a quotient of P_X, the norm bounds give

$$\ell(M^{\otimes n}) \le \ell(P_{X^{\otimes n}}) \le \mathrm{ad}(X^{\otimes n}) \le \mathrm{ad}(X)^n.$$

This proves moderate growth. Finally F_A is exact because free modules and q are exact. For $X \neq 0$, coevaluation is monic $\mathbf{1} \hookrightarrow X \otimes X^*$; its image under F_A prevents $F_A(X)$ from vanishing. An exact functor that detects nonzero objects is faithful. In particular it preserves alternating norm degree exactly. □

Corollary 3.3 (Simple commutative algebras)**.** *Let $\mathcal{C}$ have moderate growth over a field of positive characteristic. Every simple commutative ind-algebra A is* absolutely flat*: tensor product $\otimes_A$ is exact on* $\mathrm{Mod}_A(\mathrm{Ind}\,\mathcal{C})$*. Moreover,*

$$\ell_A(A \otimes X) \le \mathrm{ad}(X) \qquad (X \in \mathcal{C}),$$

where ℓ_A denotes length in the module category. Thus A is artinian *in the terminology of* [CS26]*: every finite free A-module has finite length.*

Proof. A simple algebra is a domain, and its only nonzero ideal is A. Consequently $\mathcal{T}_A = 0$, so Theorem 3.2 applies directly to $\mathcal{Q}_A = \mathrm{Mod}_A(\mathrm{Ind}\,\mathcal{C})$. □

This answers [CS26, Questions 3.27–3.28] for moderate-growth categories in positive characteristic.

Corollary 3.4 (Scalar extension)**.** *Let $\mathcal{C}$ have moderate growth in positive characteristic over a field k, and let L/k be a field extension. The finite-length L-modules in* $\mathrm{Ind}\,\mathcal{C}$ *form a moderate-growth*

tensor category $\mathcal{C}_L$ over L. The functor $X \mapsto L \otimes_k X$ is exact, faithful, strong symmetric monoidal, and surjective. It preserves finite tensor generation. For $X, Y \in \mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{C}_L}(L \otimes X, L \otimes Y) = L \otimes_k \mathrm{Hom}_{\mathcal{C}}(X, Y). \tag{3.2}$$

Proof. Regard L as an ordinary commutative ind-algebra. Every subobject of a sum of units is again a sum of units, so its internal ideals are precisely its ordinary ideals. Thus L is simple and its torsion subcategory is zero. Apply Theorem 3.2 with $A = L$. Since $\mathrm{End}_L(L) = \mathrm{Hom}(\mathbf{1}, L) = L$, it gives the tensor category $\mathcal{C}_L$ over L, its moderate growth, and all the stated properties of free extension. In particular, every finite-length L-module is a quotient of some $L \otimes X$; this gives surjectivity and preservation of finite generation. Finally, free-module adjunction and compactness of X give (3.2). □

Lemma 3.5 (A generic point over the original field)**.** *Let k be uncountable and algebraically closed of positive characteristic, and let $\mathcal{C}$ have moderate growth. Let B be a finitely generated commutative ind-algebra, and let $E \subset B$ be finite with $E^n \neq 0$ for all $n \geq 1$. There are a moderate-growth tensor category $\mathcal{D}$ over k, a tensor functor $F : \mathcal{C} \to \mathcal{D}$, and a unital algebra morphism*

$$\epsilon : F(B) \longrightarrow \mathbf{1}$$

whose restriction to $F(E)$ is nonzero. Here F on ind-objects means its colimit extension.

Proof. Choose an ideal P maximal among those containing none of the E^n. A chain union still avoids them, since each E^n is finite, so Zorn's lemma applies. If ideals I, J are not contained in P, maximality gives $E^a \subset P + I$ and $E^b \subset P + J$ for some a, b. Then $E^{a+b} \subset P + IJ$, so $IJ \not\subset P$. Therefore $A = B/P$ is a domain.

Write $\overline{E}$ for the image of E in A. Every nonzero ideal of A contains some $\overline{E}^n$, again by maximality. Thus the nonzero ideals

$$J_n = A\overline{E}^n$$

are a decreasing cofinal family among all nonzero ideals. The module A is torsion free. The morphism formula for a Serre quotient therefore gives

$$K_A = \varinjlim_{0 \neq I \subset A} \mathrm{Hom}_A(I, A) = \varinjlim_{n} \mathrm{Hom}_A(J_n, A).$$

The transition maps are restriction along smaller ideals. Restriction to generators gives an injection

$$\mathrm{Hom}_A(J_n, A) \hookrightarrow \mathrm{Hom}_{\mathrm{Ind}\,\mathcal{C}}(\overline{E}^n, A).$$

Because B, and hence A, is finitely generated, it is a countable union of finite objects. Each of these Hom spaces is consequently countable-dimensional, and so is K_A over k. If $t \in K_A$ were transcendental, the elements $(t - c)^{-1}$, $c \in k$, would be linearly independent: clear denominators in a finite relation and evaluate at each distinct c. This contradicts countable dimension. Thus K_A/k is algebraic and $K_A = k$.

Use $\mathcal{D} = \mathcal{D}_A$ and $F = F_A$ from Theorem 3.2. The map $A \otimes B \to A$, induced by multiplication and $B \to A$, gives ϵ after applying q. Its image on $A \otimes E$ is J_1. Since $qJ_1 \simeq qA$, the restriction to $F(E)$ is even epic. □

4. Maximal nilpotence from generic points

Throughout this section the ground field is algebraically closed. For $X \in \mathcal{C}$, let

$$\widetilde{X} = \ker(X \longrightarrow \mathrm{Hom}(X, \mathbf{1})^* \otimes \mathbf{1}).$$

The displayed map is the maximal quotient that is a sum of units; it is epic. A *nilpotent subobject* $E \subset B$ means that the multiplication map $E^{\otimes n} \to B$ vanishes for some $n \geq 1$.

Definition 4.1. A category is *maximally nilpotent*, abbreviated MN, if $\widetilde{X}$, in degree one of $\mathrm{Sym}(X)$, is nilpotent for every X. Equivalently the image of $\mathrm{Sym}(\widetilde{X}) \to \mathrm{Sym}(X)$ is finite. This is the maximal-unit-quotient formulation of MN [CEO24, Theorem 7.2.2].

For finite tensor categories, maximal nilpotence is equivalent to incompressibility [CFNO26, Proposition 4.3]. The next proposition proves the implication from incompressibility over an uncountable field without assuming finiteness.

Proposition 4.2. *Over an uncountable algebraically closed field of positive characteristic, every incompressible tensor category of moderate growth is MN.*

Proof. If $\widetilde{X}$ were nonnilpotent in $B = \mathrm{Sym}(X)$, Lemma 3.5 would give F and an algebra map $F(B) \to \mathbf{1}$ nonzero on $F(\widetilde{X})$. Its degree-one restriction is a map $F(X) \to \mathbf{1}$. Incompressibility makes F full, so this map is $F(v)$ for some $v : X \to \mathbf{1}$. Every such v vanishes on $\widetilde{X}$, a contradiction. □

Lemma 4.3 (Consequences of maximal nilpotence)**.** *In an MN category:*

(1) *If* $\mathrm{Hom}(X, \mathbf{1}) = 0$, *then* $\mathrm{Sym}(X)$ *is finite.*
(2) *Every tensor functor over the ground field out of the category is full.*
(3) *Every even invertible object is the unit.*
(4) *The category has moderate growth.*

MN passes to tensor subcategories.

Proof. For (1), $\widetilde{X} = X$. Part (2) is [CEO24, Theorem 7.2.3(1)]. For (3), an invertible object is simple. If it is even, every ordinary symmetric power is its nonzero tensor power. A nonunit even invertible would contradict (1). Part (4) is [CS26, Lemma 5.4]: its hypothesis MN(i), finiteness of $\mathrm{Sym}\, L$ for every nonunit simple L, follows from (1). Neither cited result assumes GR or finiteness of the category. Inheritance is immediate from fullness of a tensor-subcategory inclusion and the definition of $\widetilde{X}$. □

Theorem 4.4 ([CFNO26, Theorem A, Proposition 4.1])**.** *Over any algebraically closed field of positive characteristic,* Ver_{p^r} *is MN for every* $1 \le r \le \infty$.

5. Nilpotent kernels and geometric reductivity

The norm images, rather than ordinary exterior powers, provide the line needed in the determinant argument.

Lemma 5.1 (The top alternating norm is an even line)**.** *Let* $X \neq 0$ *have finite alternating norm degree* $d = \mathrm{ad}(X)$. *Then* $A_d(X)$ *is invertible and its self-symmetry is the identity.*

Proof. Finite norm degree is preserved by duals and subquotients, and by finite sums and tensor products [CEO23, Lemmas 4.8–4.9]. Thus the tensor subcategory $\langle X \rangle$ has moderate growth [CEO23, Corollary 4.11]. Apply [CFNO26, Proposition 5.1] to this subcategory. Its alternating powers are defined by the same integral norms a_i^X; its top nonzero index is therefore $d = \mathrm{ad}(X)$. The result gives both invertibility and trivial self-symmetry. □

Lemma 5.2 (Divided derivatives across a nilpotent kernel)**.** *Let* $\alpha : X \twoheadrightarrow \mathbf{1}$ *have kernel* K, *and put* $B = \mathrm{Sym}\, X$. *Suppose that the multiplication map* $K^{\otimes N} \to B$ *vanishes. If* $q = p^e \ge N$, *with* $e \ge 0$, *there is a canonical isomorphism*

$$D_q : B_{2q-1} \xrightarrow{\sim} B_{q-1}, \qquad B_n = \mathrm{Sym}^n X.$$

Here D_r *is the coefficient of* z^r *in translation by* α, *that is, in the algebra map*

$$\tau : B \longrightarrow B \otimes k[z], \qquad x \longmapsto x + \alpha(x)z.$$

The formula for generators denotes the morphism $X \to B_1 \oplus B_0 z$, so does not assume vector-space points of X.

Proof. Let $J = (K) \subset B$. Then $J^N = 0$ and $B/J = \mathrm{Sym}(X/K) = k[t]$, with t of degree one. Translation fixes K, so every coefficient D_r preserves J^j. Write $M_j = J^j/J^{j+1}$, and $R_j = (M_j)_j$. The graded $k[t]$-module M_j is generated in degree j by the image of $K^{\otimes j}$. Hence multiplication gives epimorphisms

$$u_{j,n} : R_j \longrightarrow (M_j)_n, \qquad u_{j,n}(v) = t^{n-j}v \quad (n \geq j).$$

These are isomorphisms: $D_r(R_j) = 0$ for $r > 0$, and translation on the quotient is $t \mapsto t + z$; taking the coefficient of z^{n-j} gives $D_{n-j}u_{j,n} = \mathrm{id}_{R_j}$. These calculations use the induced $k[t]$-action on M_j, not a choice of a lift $\mathbf{1} \to X$. More generally,

$$D_r u_{j,n} = \binom{n-j}{r} u_{j,n-r} \quad \text{when } n - r \geq j. \tag{5.1}$$

For $0 \leq j < N \leq q$, both $q-1$ and $2q-1$ are at least j, and

$$\binom{2q-1-j}{q} = 1 \quad \text{in } k. \tag{5.2}$$

Indeed, the coefficient of u^q in

$$(1+u)^{2q-1-j} = (1+u^q)(1+u)^{q-1-j}$$

is 1, since $0 \leq q-1-j < q$. Thus D_q induces an isomorphism on every layer of the finite J-filtrations of B_{2q-1} and B_{q-1}. It is an isomorphism. □

Lemma 5.3 (A norm determinant gives a section)**.** *In a moderate-growth category, a morphism α satisfying Lemma 5.2 has a split symmetric power. More precisely, if $q = p^e \geq N$ and $d = \mathrm{ad}(\mathrm{Sym}^{q-1} X)$, there is a morphism*

$$s : \mathbf{1} \longrightarrow \mathrm{Sym}^{dq} X, \qquad \mathrm{Sym}^{dq}(\alpha)s = \mathrm{id}_{\mathbf{1}}\,.$$

Proof. Set $Y = B_{q-1}$ and $Z = B_{2q-1}$. The symmetric algebra has its standard Hopf coproduct, primitive on X. Its homogeneous component $\Delta_{q-1,q} : Z \to Y \otimes B_q$ satisfies

$$(\mathrm{id}_Y \otimes \mathrm{Sym}^q \alpha)\Delta_{q-1,q} = D_q.$$

Consequently

$$f = \Delta_{q-1,q} D_q^{-1} : Y \longrightarrow Y \otimes B_q \quad \text{satisfies} \quad (\mathrm{id}_Y \otimes \mathrm{Sym}^q \alpha) f = \mathrm{id}_Y\,.$$

The map $\mathrm{Sym}^{q-1} \alpha$ is epic, so $Y \neq 0$. Its top norm $L = A_d(Y)$ is invertible by Lemma 5.1.

Tensor f with itself d times, move its coefficient factors to the right, and multiply them. This gives

$$f^{[d]} : Y^{\otimes d} \longrightarrow Y^{\otimes d} \otimes B_{dq}.$$

Commutativity of B makes this map S_d-equivariant, with S_d acting on the Y-factors. In particular

$$f^{[d]} a_d^Y = (a_d^Y \otimes \mathrm{id}) f^{[d]}.$$

Exact tensor product therefore induces a map $L \to L \otimes B_{dq}$. Since L is invertible, it is $\mathrm{id}_L \otimes s$ for a unique $s : \mathbf{1} \to B_{dq}$. Evaluating the coefficients by $\mathrm{Sym}^{dq} \alpha$ turns $f^{[d]}$ into $\mathrm{id}_{Y^{\otimes d}}$, and hence the induced map into id_L. Cancelling L proves $\mathrm{Sym}^{dq}(\alpha)s = \mathrm{id}_{\mathbf{1}}$. □

Theorem 5.4 (MN implies GR)**.** *Over an algebraically closed field of positive characteristic, every MN tensor category is geometrically reductive: for each epimorphism $\alpha : X \twoheadrightarrow \mathbf{1}$, some $\mathrm{Sym}^n \alpha$ splits.*

Proof. Moderate growth follows from Lemma 4.3(4). Choose a subobject $X' \subset X$ of minimal length for which $\alpha' = \alpha|_{X'}$ is epic. No proper subobject of X' maps onto $\mathbf{1}$. Thus every proper subobject is contained in $K' = \ker \alpha'$. In particular K' is the unique maximal subobject, and the head of X' is the simple object $\mathbf{1}$. It follows that $\mathrm{Hom}(X', \mathbf{1}) = k\alpha'$ and $\widetilde{X'} = K'$. MN makes K' nilpotent in $\mathrm{Sym}\, X'$. Lemma 5.3 gives a section of a symmetric power of α'. Composing it with $\mathrm{Sym}^n X' \to \mathrm{Sym}^n X$ gives the required section. □

6. Fundamental coends and finiteness

For a tensor category $\mathcal{A}$, its fundamental coend is the commutative Hopf ind-algebra

$$H_{\mathcal{A}} = \int^{X \in \mathcal{A}} X^* \otimes X.$$

Its coefficient maps are $c_X : X^* \otimes X \to H_{\mathcal{A}}$. Dinaturality means that for $f : X \to Y$ the two maps $Y^* \otimes X \to H_{\mathcal{A}}$, obtained using f or f^*, agree. The coend notation $\int^X$ expresses universality: maps $H_{\mathcal{A}} \to M$ correspond to families of maps $X^* \otimes X \to M$ satisfying these same identities. Multiplication is induced by tensor products, the unit by $X = \mathbf{1}$, and the antipode by duality. Equivalently, there are universal coactions $\rho_X : X \to X \otimes H_{\mathcal{A}}$, natural in X and compatible with tensor products.

Lemma 6.1 (Coefficient generation and finiteness)**.** *If $\mathcal{A} = \langle G \rangle$, its coend is generated as an algebra by the coefficient images of G and G^*. For every simple S, the map $c_S : S^* \otimes S \to H_{\mathcal{A}}$ is monic. Moreover $H_{\mathcal{A}}$ is a finite object if and only if $\mathcal{A}$ is a finite tensor category.*

Proof. Coefficients of tensor products are products of coefficients; coefficients of direct sums are sums of coefficient images. For a monomorphism $U \hookrightarrow X$, dinaturality and the epimorphism $X^* \twoheadrightarrow U^*$ show that the coefficient image of U is contained in that of X. The same holds for a quotient, using the original epimorphism. This proves the generation assertion.

For simple coefficients, we recall the external-coend argument in [CEO23, proof of Lemma 8.2]; the same coend will prove the finiteness criterion. Consider

$$K = \int^{X \in \mathcal{A}} X^* \boxtimes X \quad \text{in } \mathrm{Ind}(\mathcal{A} \boxtimes \mathcal{A}).$$

The Deligne product exists, and biexact functors induce exact functors from it [EGNO15, Proposition 1.11.2]. Tensor multiplication $m : \mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}$ is therefore exact. Over our algebraically closed field, every simple in the Deligne product is $S \boxtimes T$ for simples S, T, and $S \otimes T \neq 0$. Thus m kills no nonzero object and is faithful. Its ind-extension is exact and preserves colimits. Thus $m(K) = H_{\mathcal{A}}$. The map $S^* \boxtimes S \to K$ is nonzero: after applying m and the coend counit its composite is $\mathrm{ev}_S \neq 0$. Its source is simple, so it is monic. Applying m proves the assertion about c_S.

Now suppose $H_{\mathcal{A}}$ is finite. Every finite subobject $K' \subset K$ satisfies

$$\ell(K') \leq \ell(mK') \leq \ell(H_{\mathcal{A}}).$$

A filtered union of finite-length subobjects with a common length bound stabilizes. Hence K is finite.

Choose a faithful exact linear functor $\omega : \mathcal{A} \to \mathrm{Vec}_k$. Such a functor exists by [EGNO15, Theorem 1.9.15]; no monoidal structure on ω is asserted. Put $\eta(X) = \omega(X^*)^*$. The functor $\eta \boxtimes \omega$ sends $U \boxtimes V$ to $\eta(U) \otimes_k \omega(V)$. Applying its ind-extension to K gives the reconstruction coalgebra

$$C_\omega = \int^{X \in \mathcal{A}} \omega(X)^* \otimes \omega(X).$$

It is finite-dimensional because K is finite. Coalgebra reconstruction [EGNO15, Theorem 1.10.1] identifies $\mathcal{A}$ with finite-dimensional C_ω-comodules, equivalently with finite-dimensional modules over the ordinary, possibly noncommutative algebra C_ω^*. This proves finiteness.

Conversely, if $\mathcal{A}$ is finite, choose a projective generator P. Every object is a quotient of a finite sum of copies of P. The coefficient argument at the start of the proof therefore makes $P^* \otimes P \to H_{\mathcal{A}}$ epic, so the coend is finite. □

Corollary 6.2 (An ordinary coend forces pointedness)**.** *If $H_{\mathcal{A}}$ is ordinary, every simple of $\mathcal{A}$ is invertible.*

Proof. For simple S, the preceding lemma makes $S^* \otimes S$ a subobject of $H_{\mathcal{A}}$, hence a finite sum of units. Schur's lemma gives

$$\dim_k \operatorname{Hom}(\mathbf{1}, S^* \otimes S) = \dim_k \operatorname{End}(S) = 1,$$

so $S^* \otimes S \simeq \mathbf{1}$. □

Lemma 6.3 (A Hopf subalgebra of the fundamental coend)**.** *Let $K \subset H_{\mathcal{C}}$ be a Hopf ind-subalgebra. There is a tensor subcategory $\mathcal{D} \subset \mathcal{C}$ whose coend, viewed in* $\operatorname{Ind}\mathcal{C}$*, is K. It consists of the objects whose canonical coaction factors through $X \otimes K \subset X \otimes H_{\mathcal{C}}$.*

Proof. Write $G = \pi(\mathcal{C})$ for the fundamental group, so $\mathcal{O}(G) = H_{\mathcal{C}}$, and $Q = \operatorname{Spec} K$. The inclusion defines a map $q : G \to Q$. The canonical G-action on its own coordinate algebra is the adjoint action [CEO24, Sections 4.1.1–4.1.3]. Restricting to K, and using its Hopf structure, shows that the adjoint action on Q, pulled back along q, agrees with the canonical action on $\mathcal{O}(Q)$. Thus (Q, q) is a $\mathcal{C}$-group in the sense of [CEO24, Section 4.1].

Apply internal Tannaka duality [CEO24, Theorem 4.2.1] to $\operatorname{Rep}_{\mathcal{C}}(Q, q)$, the representations whose restriction along q is their canonical G-action. Forgetting the representation is exact and symmetric monoidal. It is fully faithful: every morphism in $\mathcal{C}$ respects canonical $H_{\mathcal{C}}$-coactions, and the monomorphism $K \hookrightarrow H_{\mathcal{C}}$ then implies that it respects the K-coactions. Its image is closed under subquotients. For example, for $U \subset X$, exact tensor product gives

$$(U \otimes H_{\mathcal{C}}) \cap (X \otimes K) = U \otimes K \quad \text{inside } X \otimes H_{\mathcal{C}},$$

which factors the canonical coaction on U through K. Quotients follow by the same exactness. Sums, tensor products, and duals follow from the Hopf structure. The image is therefore the asserted tensor subcategory $\mathcal{D}$.

The inverse direction of [CEO24, Theorem 4.2.1] identifies the internal group of tensor automorphisms $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ with Q, where $\omega : \operatorname{Rep}_{\mathcal{C}}(Q, q) \to \mathcal{C}$ is the forgetful functor. Equivalently, its coefficient map is an isomorphism

$$\int^{X \in \operatorname{Rep}_{\mathcal{C}}(Q,q)} \omega(X)^* \otimes \omega(X) \xrightarrow{\sim} \mathcal{O}(Q) = K.$$

Full faithfulness identifies the source with $H_{\mathcal{D}}$ in $\operatorname{Ind}\mathcal{C}$, so $H_{\mathcal{D}} \simeq K$. □

Frobenius reduction. For a commutative ind-algebra H, put

$$H^{[1]} = \operatorname{im}\big((H^{\otimes p})^{S_p} \longrightarrow H\big), \qquad H^{[r+1]} = (H^{[r]})^{[1]},$$

where the arrow is multiplication and the superscript S_p denotes invariants. These are image subalgebras, not scalar twists; the displayed description is [CS25, Lemma 4.2]. For a finitely generated commutative Hopf ind-algebra H in an MN, GR category, the reduction uses the following statements.

(1) H is finitely generated as a module over $H^{[1]}$, and $H^{[1]}$ is finitely generated [CS25, Lemma 4.4(2),(3)]. Iteration makes every $H^{[r]}$ finitely generated as an algebra and H finitely generated as a module over it: compose finite free module covers at successive stages.
(2) Every $H^{[r]}$ is a Hopf subalgebra of H [CS25, Lemma 4.17(1)].
(3) For sufficiently large r, $\operatorname{Spec} H^{[r]}$ is purely even [CS25, Proposition 4.19(2)].

Here "purely even" means that the ideal generated by the joint kernel of all maps $H^{[r]} \to \mathbf{1}$ is zero. Thus every finite subobject embeds in a finite sum of units, and $H^{[r]}$ is ordinary. Conversely an ordinary algebra is purely even. No finiteness of the ambient category or of its set of simples is assumed in these results.

Theorem 6.4 (Finite generation implies finiteness for MN categories)**.** *Let k be algebraically closed of positive characteristic. Every finitely generated MN tensor category $\mathcal{C}$ is finite.*

Proof. By Lemma 4.3(4) and Theorem 5.4, the category has moderate growth and is geometrically reductive. By Lemma 6.1, its fundamental coend $H = H_{\mathcal{C}}$ is a finitely generated commutative Hopf ind-algebra. Apply the Frobenius reduction just stated. For a sufficiently large r,

$$K = H^{[r]} \subset H$$

is an ordinary Hopf subalgebra, and H is finitely generated as a K-module.

Let $\mathcal{D} \subset \mathcal{C}$ be the tensor subcategory from Lemma 6.3. Its coend is K. This coend is ordinary also as an ind-object of $\mathcal{D}$: the inclusion is full and closed under subobjects, and every finite sum of units occurring in K belongs to $\mathcal{D}$. The category $\mathcal{D}$ inherits MN. If $p > 2$, Proposition 8.1 identifies $\mathcal{D}$ with Vec or sVec. If $p = 2$, Proposition 11.2 makes $\mathcal{D}$ finite.

Lemma 6.1 makes $K = H_{\mathcal{D}}$ finite. Since H is a finitely generated K-module, it is a quotient of $K \otimes W$ for some finite $W \in \mathcal{C}$, so it too is finite. Apply the same lemma to conclude that $\mathcal{C}$ is finite. □

Corollary 6.5. *Every finitely generated MN tensor category embeds as a tensor subcategory of a finite higher Verlinde category.*

Proof. It is finite by Theorem 6.4, and hence incompressible by [CEO24, Theorem 7.2.3(2)]. Its functor to a finite higher Verlinde category supplied by [CF26, Theorem B] is therefore a tensor-subcategory embedding. □

A tensor category is *subterminal* if any two tensor functors with the same source and that target are isomorphic as symmetric monoidal functors.

Corollary 6.6 (Subterminality [CFNO26, Proposition 4.2])**.** *The categories Ver_{p^r}, for $1 \le r \le \infty$, are subterminal.*

7. Proof of the main theorem

Proof of Theorem 1.1. We first treat finitely generated sources $\mathcal{C} = \langle G \rangle$. Suppose k is uncountable. By [CEO24, Theorem 5.2.1], choose a surjective tensor functor $F : \mathcal{C} \to \mathcal{I}$ to an incompressible category of moderate growth. Surjectivity makes $\mathcal{I} = \langle F(G) \rangle$. Proposition 4.2 makes $\mathcal{I}$ MN, so Corollary 6.5 embeds it in $\mathrm{Ver}_{p^r,k}$ for some finite r. Composition proves the finitely generated assertion over k.

For a general algebraically closed k, choose an uncountable algebraically closed extension L/k. By Corollary 3.4, $\mathcal{C}_L$ has moderate growth and is generated by $L \otimes G$. The preceding paragraph gives a tensor functor $\mathcal{C}_L \to \mathrm{Ver}_{p^r,L}$ for some finite r. The construction of higher Verlinde categories over the prime field [BEO23, Section 4.9] identifies $(\mathrm{Ver}_{p^r,k})_L$ with $\mathrm{Ver}_{p^r,L}$. Thus restriction along $\mathcal{C} \to \mathcal{C}_L$ gives an exact k-linear strong symmetric monoidal functor

$$\mathcal{C} \longrightarrow \mathrm{Mod}_L(\mathrm{Ind}\,\mathrm{Ver}_{p^r,k}).$$

The target category $\mathrm{Ver}_{p^r,k}$ is MN by Theorem 4.4 and GR by [CFNO26, Proposition 4.5]. Apply the neutrality theorem [CS26, Appendix B, Theorem B.1] with $R = L$ to obtain $\mathcal{C} \to \mathrm{Ver}_{p^r,k}$.

We have proved the assertion for every finitely generated source over k. The reduction [CFNO26, Proposition 4.4] now gives a tensor functor $\mathcal{C} \to \mathrm{Ver}_{p^\infty,k}$ for every moderate-growth tensor category $\mathcal{C}$. □

Proposition 7.1 (Embedding an arbitrary MN category)**.** *Every MN tensor category over k embeds as a tensor subcategory of* $\mathrm{Ver}_{p^\infty,k}$.

Proof. Let $\mathcal{A}$ be MN. It has moderate growth by Lemma 4.3(4), so Theorem 1.1 gives a tensor functor $F : \mathcal{A} \to \mathrm{Ver}_{p^\infty,k}$. Every finitely generated tensor subcategory $\mathcal{E} \subset \mathcal{A}$ is MN and finite by Theorem 6.4, hence incompressible by [CFNO26, Proposition 4.3]. Thus $F|_{\mathcal{E}}$ is a tensor-subcategory embedding. Every pair of objects belongs to one such $\mathcal{E}$, so F is fully faithful. Every subquotient of $F(X)$ lies in the image of $F|_{\langle X\rangle}$. Hence the image of F is a tensor subcategory. □

Remark 7.2 (Uniqueness). The resulting tensor functor $\mathcal{C} \to \mathrm{Ver}_{p^\infty,k}$ is unique up to isomorphism: $\mathrm{Ver}_{p^\infty,k}$ is subterminal by Corollary 6.6. The same result gives uniqueness for any fixed finite target $\mathrm{Ver}_{p^r,k}$ whenever such a functor exists.

Remark 7.3 (Dependencies, ground fields, and height). The results of [CFNO26] used for the Verlinde targets, the top norm line, and the reduction to finitely generated sources are independent of Theorem 1.1. The generic-flatness argument is proved directly: [CFNO26, Proposition 4.8] assumes a tensor functor to Ver_{p^∞}, whose existence is the conclusion sought here. Likewise, the geometric reductivity of finite categories in [CFNO26, Proposition 4.5] supplies the descent target, while Theorem 5.4 is needed for MN categories before their finiteness is established.

The uncountability assumption in Lemma 3.5 ensures $K_A = k$. Scalar extension and neutrality remove this assumption for finitely generated sources, after which [CFNO26, Proposition 4.4] gives the general case. The proof gives no explicit bound on the height r in terms of a tensor generator.

Remark 7.4 (The generic quotient). A domain need not be simple: the domain $k[x, y]$ in $\mathrm{Rep}(\mathrm{SL}_2)$ has the proper stable ideal (x, y). Passing to $\mathcal{Q}_A$ makes the unit simple; this is the property used in the highest-norm argument.

Part 2. Odd characteristic

Throughout this part, k is algebraically closed and $p > 2$.

8. Ordinary coends

We use the theorem that the maximal Tannakian and super-Tannakian subcategories are Serre, together with its consequence for categories whose semisimple objects form an integral fusion category [EG21a, Appendix, Theorem 7.1 and Proposition 7.2]. Here Tannakian and super-Tannakian mean admitting a tensor functor to Vec_k and sVec_k, respectively.

Proposition 8.1. *An MN tensor category $\mathcal{U}$ with ordinary fundamental coend is equivalent to* Vec_k *or* sVec_k.

Proof. By Corollary 6.2, all simples are invertible. The self-symmetry of an invertible object is a scalar ± 1, and Lemma 4.3 makes every even invertible the unit. The product of two odd invertibles is even. Consequently there is at most one nonunit simple ε, and if it exists then $\varepsilon^{\otimes 2} \simeq \mathbf{1}$ and its self-symmetry is -1. The semisimple objects therefore form a tensor subcategory with at most two invertible simples, hence an integral fusion category. The cited theorem makes $\mathcal{U}$ super-Tannakian. Its tensor functor to sVec is full by MN and faithful by exactness. All short exact sequences split after this functor, so their splittings lift. Thus $\mathcal{U}$ is semisimple, and its fully faithful image in sVec is Vec or sVec. □

Example 8.2 (Parity of the top norm). The parity assertion concerns the norm images $A_n(X)$; evenness describes their self-symmetry, while the integer d can be odd. For instance, $\mathrm{ad}(\mathbf{1}) = 1$

and $A_1(\mathbf{1}) = \mathbf{1}$ is even. For example, let ε be the one-dimensional odd object of sVec_k. Each permutation $\sigma \in S_n$ acts on $\varepsilon^{\otimes n}$ by $\mathrm{sgn}(\sigma)$, so

$$a_n^{\varepsilon} = n!\,\mathrm{id}_{\varepsilon^{\otimes n}}, \qquad A_n(\varepsilon) = \begin{cases} \varepsilon^{\otimes n}, & 0 \le n < p, \\ 0, & n \ge p. \end{cases}$$

Thus $\mathrm{ad}(\varepsilon) = p-1$ and $A_{p-1}(\varepsilon) \simeq \mathbf{1}$ is even. At $p = 3$, $A_1(\varepsilon)$ is odd but the highest nonzero term is $A_2(\varepsilon) = \mathbf{1}$. The ordinary exterior power $\bigwedge^n X$ is the largest quotient of $X^{\otimes n}$ on which each permutation acts by its sign. For the odd line, $\bigwedge^n \varepsilon \simeq \varepsilon^{\otimes n}$ is nonzero for every n.

9. Tensor subcategories are Serre

At odd primes, every tensor subcategory of a finite higher Verlinde category is Serre [BEO23, Proposition 4.59 and Corollary 4.61]. Together with the common finiteness theorem, this gives a short proof of the corresponding statement in any MN category.

Proposition 9.1 (Tensor subcategories of MN categories are Serre)**.** *Let $\mathcal{A}$ be MN, and let $\mathcal{D} \subset \mathcal{A}$ be a tensor subcategory. Then every extension in $\mathcal{A}$ with endpoints in $\mathcal{D}$ belongs to $\mathcal{D}$.*

Proof. Let $0 \to A \to X \to B \to 0$ have $A, B \in \mathcal{D}$. The category $\mathcal{E} = \langle X \rangle \subset \mathcal{A}$ contains A, B, is finitely generated, and inherits MN. By Corollary 6.5, it embeds in a finite higher Verlinde category. The image of its tensor subcategory $\langle A \oplus B \rangle$ is Serre there and therefore contains the image of X. Full faithfulness gives $X \in \langle A \oplus B \rangle \subset \mathcal{D}$. □

Corollary 9.2. *In any MN tensor category in odd characteristic,*

$$\mathrm{Ext}^1(\mathbf{1}, \mathbf{1}) = 0.$$

Proof. Apply Proposition 9.1 to the semisimple subcategory $\mathcal{D} = \mathrm{Vec}$. □

Part 3. Characteristic two

Throughout this part, k is algebraically closed of characteristic two, except for the normalization lemma, which only requires perfection.

10. The unit extension and the Frobenius functor

The ordinary-coend case need not be semisimple in this characteristic. Let

$$\mathcal{V} = \mathrm{Ver}_4^+ = \mathrm{Rep}\big(k[d]/(d^2),\ R = 1 + d \otimes d\big),$$

where d is primitive. This is the symmetric category of vector spaces with a square-zero operator d, with symmetry $v \otimes w \mapsto w \otimes v + dw \otimes dv$ [BE19, EG21b].

Example 10.1 (A nonsplit extension of the unit). In $\mathcal{V} = \mathrm{Ver}_4^+$, let E have basis x, y, with $d(y) = x$ and $d(x) = 0$. The sequence

$$0 \longrightarrow \mathbf{1} \xrightarrow{1 \mapsto x} E \xrightarrow{\alpha} \mathbf{1} \longrightarrow 0, \qquad \alpha(x) = 0, \quad \alpha(y) = 1,$$

does not split. As a graded ordinary algebra, $\mathrm{Sym}\, E$ is $k[x,y]/(x^2)$, with d acting as a derivation. Thus y^2 is invariant and gives a section of $\mathrm{Sym}^2 \alpha$. Here $N = q = 2$, and the map $D_2 : B_3 \to B_1$ sends $y^3 \mapsto y$ and $xy^2 \mapsto x$. This illustrates why the divided derivative remains useful even though ordinary differentiation in degree two vanishes on squares.

In characteristic two the Frobenius functor has the description

$$\mathrm{Fr}(X) = \ker(1 + c_{X,X}) / \mathrm{im}(1 + c_{X,X}).$$

It is additive and symmetric monoidal, with target the Frobenius scalar twist of the category [CEO23, Section 3]; scalar twisting does not affect composition length. A category is *Frobenius*

exact when this functor is exact. A Frobenius-exact category of moderate growth admits a tensor functor to $\mathrm{Ver}_2 = \mathrm{Vec}$ [CEO23, Theorem 1.1].

An exact faithful quasi-tensor functor is equipped with natural tensor isomorphisms and an isomorphism between the image of the unit and the target unit, without imposing associativity coherence. Such a functor to a semisimple category extends from a tensor subcategory containing all simples [EG21a, Appendix, Proposition 7.5]. This extension theorem applies without a finiteness hypothesis on the source.

11. MN categories with an ordinary coend

The following lemma extracts from [EG21b, Proposition 2.5] a normalization argument for finite-dimensional commutative local algebras.

Lemma 11.1 (Normalizing a triangular element)**.** *Let k be a perfect field of characteristic two, and let B be a finite-dimensional commutative local k-algebra with maximal ideal $\mathfrak{m}$ and residue field k. Suppose*

$$R \in 1 + \mathfrak{m} \otimes \mathfrak{m}, \qquad R_{21}R = 1.$$

There are $Q \in 1 + \mathfrak{m} \otimes \mathfrak{m}$ and $d \in \mathfrak{m}$, with $d^2 = 0$, such that

$$Q_{21}^{-1} R Q = 1 + d \otimes d.$$

The subscript 21 interchanges the two tensor factors.

Proof. Filter B by powers of $\mathfrak{m}$, and $B \otimes B$ by total degree:

$$\mathcal{F}^s = \sum_{i+j \geq s} \mathfrak{m}^i \otimes \mathfrak{m}^j \qquad (i, j \geq 0).$$

Write $\deg b$ for the largest j with $b \in \mathfrak{m}^j$, and set $\deg 0 = \infty$. Start with $d = 0$ and $n = 2$. Inductively suppose $R = 1 + d \otimes d$ modulo terms of total degree at least n, with $d = 0$ or $2 \deg d < n$. Let e be the degree-n error. Triangularity gives $\deg(d^2) > n/2$ and $e = e_{21}$. Indeed, $d^2 \otimes d^2$ must vanish below degree n; in degree n a possible leading term $w \otimes w$ would equal $e + e_{21}$. The latter has zero diagonal in a homogeneous basis, forcing $w = 0$, and then $e = e_{21}$.

Since k is perfect, write

$$e = v \otimes v + h + h_{21},$$

where v lies in the degree-$n/2$ piece of $\operatorname{gr} B$, and $v = 0$ for odd n. Take square roots of the diagonal entries to obtain v, and use the remaining off-diagonal entries for h. Both tensor factors of the error have positive degree, so choose lifts $v \in \mathfrak{m}$ and $h \in \mathfrak{m} \otimes \mathfrak{m}$. Twisting by $1 + h$ leaves the degree-n error $v \otimes v$. If $d = 0$, replace it by v. Otherwise twist further by $1 + d \otimes v$ and replace d by $d + v$. More explicitly, modulo $\mathcal{F}^{n+1}$,

$$\begin{aligned}(1 + v \otimes d)^{-1}(1 + d \otimes d + v \otimes v)(1 + d \otimes v)\\ \equiv 1 + (d + v) \otimes (d + v).\end{aligned}$$

Indeed, $\deg(d^2) > n/2$ and $\deg v \geq n/2 > \deg d$ make all omitted products have degree greater than n. For example, $(d \otimes d)(d \otimes v) = d^2 \otimes dv$ has degree greater than $n + \deg d$, and $(d \otimes v)(v \otimes d)$ has degree at least $n + 2 \deg d$. The induction terminates because $\mathfrak{m}$ is nilpotent; the product of its twists gives Q. Finally $(1 + d \otimes d)^2 = 1$ gives $d^2 \otimes d^2 = 0$, hence $d^2 = 0$. □

Proposition 11.2. *Let k be algebraically closed of characteristic two. An MN tensor category $\mathcal{U}$ with ordinary fundamental coend is finite. Its underlying abelian category is equivalent to Vec_k or to finite-dimensional modules over $k[d]/(d^2)$.*

Proof. Corollary 6.2 and Lemma 4.3 show that the only simple of $\mathcal{U}$ is $\mathbf{1}$: every simple is invertible, and its self-symmetry is a scalar whose square is 1, hence is 1 in characteristic two. Thus length is multiplicative under tensor products, $\mathcal{U}$ has moderate growth, and its Frobenius–Perron dimension is composition length. By [EO21, Proposition 7.4],

$$\mathcal{U}_{\mathrm{ex}} = \{X \in \mathcal{U} : \ell(\mathrm{Fr}\, X) = \ell(X)\}$$

is a Frobenius-exact tensor subcategory. The cited proposition requires finitely many simples, not a finite category. It admits a tensor functor to Vec by [CEO23, Theorem 1.1]. It inherits MN, so this functor is full. Consequently $\mathcal{U}_{\mathrm{ex}} = \mathrm{Vec}$ inside $\mathcal{U}$.

Commutative reconstruction. Extend the identity on $\mathrm{Vec} \subset \mathcal{U}$ to an exact faithful quasi-tensor functor $F : \mathcal{U} \to \mathrm{Vec}$, using [EG21a, Appendix, Proposition 7.5]. Exactness gives $\dim F(X) = \ell(X)$. Coalgebra reconstruction identifies $\mathcal{U}$ with the finite-dimensional continuous modules over the initially possibly noncommutative algebra

$$A = \mathrm{Nat}(F, F), \qquad J = \ker(A \longrightarrow \mathrm{End}(F(\mathbf{1})) = k).$$

Use finite collections of objects containing $\mathbf{1}$ to define the action topology. Then A is an inverse limit of finite-dimensional algebras. Elements of J lower composition filtrations, so these finite action quotients are local with residue field k. Under reconstruction, F is the forgetful functor.

Write $\gamma_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ for its tensor isomorphisms and identify $F(\mathbf{1}) = k$. The unit constraints identify $\gamma_{\mathbf{1},-}$ and $\gamma_{-,\mathbf{1}}$ with units $a, b \in A$ having the same value $\lambda \in k^\times$ on $\mathbf{1}$. Replacing $\gamma_{X,Y}$ by

$$\lambda\, \gamma_{X,Y}(b_X^{-1} \otimes a_Y^{-1})$$

makes both unit isomorphisms the identity. We use this normalized quasi-tensor structure from now on.

In fact A is commutative. The pairing

$$F(X^*) \otimes F(X) \xrightarrow{\gamma_{X^*,X}} F(X^* \otimes X) \xrightarrow{F(\mathrm{ev}_X)} k$$

defines a natural map $\delta_X : F(X^*) \to F(X)^*$. It is an isomorphism on $\mathbf{1}$; both sides are exact contravariant functors, so induction on length makes it an isomorphism for every X. This argument does not require associativity coherence for γ. Thus, with $H = H_{\mathcal{U}}$,

$$F(H) \simeq \int^{X \in \mathcal{U}} F(X)^* \otimes F(X).$$

These vector-space identifications carry the natural map

$$\mathrm{Nat}(\mathrm{id}_{\mathcal{U}}, \mathrm{id}_{\mathcal{U}}) \longrightarrow A, \qquad u \longmapsto F(u),$$

to $\mathrm{Hom}(H, \mathbf{1}) \to \mathrm{Hom}_k(F(H), k)$: the functional corresponding to u restricts on $X^* \otimes X$ to $\mathrm{ev}_X(\mathrm{id}_{X^*} \otimes u_X)$, and naturality of γ identifies its image with the functional for $F(u_X)$. The map is bijective because H is ordinary. The map preserves composition, and natural endomorphisms of the identity commute. Hence A is commutative.

Bounding each finite quotient. Let B be a finite action quotient of A, with maximal ideal $\mathfrak{m}$. By reconstruction, its finite-dimensional modules are objects of $\mathcal{U}$, via restriction of scalars. On pairs of B-modules, transport the symmetry through the quasi-tensor isomorphisms and remove the ordinary flip. This is a natural endomorphism of the ordinary tensor product of the two underlying vector spaces. Evaluating on regular modules identifies it with an element $R_B \in B \otimes B$. The symmetry and unit axioms give

$$R_B \in 1 + \mathfrak{m} \otimes \mathfrak{m}, \qquad (R_B)_{21} R_B = 1.$$

Apply Lemma 11.1 to obtain Q_B, d_B, and let $X \in \mathcal{U}$ correspond to the cyclic module $B/d_B B$. On $F(X) \otimes F(X)$, the operator $Q_{B,21}^{-1} R_B Q_B$ is the identity. Equivalently the transported symmetry

τR_B is conjugate to the ordinary flip τ, since

$$\tau Q_{B,21}^{-1} R_B Q_B = Q_B^{-1}(\tau R_B) Q_B.$$

The cohomology of $1+\tau$ on $V \otimes V$ has basis $[v_i \otimes v_i]$, for any basis (v_i) of V. Exactness of F and the Frobenius formula therefore give $\ell(\operatorname{Fr} X) = \dim F(X) = \ell(X)$. Hence $X \in \mathcal{U}_{\mathrm{ex}} = \mathrm{Vec}$.

Thus $B/d_B B$ is a nonzero cyclic trivial B-module, so is k. Every $b \in B$ is therefore $\lambda + d_B b'$. Since $d_B^2 = 0$, this gives $d_B B = k d_B$ and $\dim_k B \leq 2$. Their duals form a directed union equal to the reconstruction coalgebra, so it too has dimension at most two, as does A. The local algebra A is consequently k or $k[d]/(d^2)$, and reconstruction proves the assertion.

Only the action of R_B on the displayed module was used. No tensor structure on B-modules or compatibility between the choices of Q_B for different quotients is needed. □

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