hexagon:2611.00300v1

Integral torus partition functions just above $c = 1$ have no fixed spectral gap

Tzu-Chen Huang

November 2026

Abstract

We consider modular-invariant torus partition functions of central charge $c > 1$ that have a unique vacuum, an expansion in non-degenerate Virasoro characters with positive integer multiplicities, integer spins, no nonvacuum state at the unitarity bound, and parity symmetry. We prove that for every $g > 0$ there is $\epsilon(g) > 0$ such that no such partition function with $1 < c < 1 + \epsilon(g)$ has all nonvacuum primary dimensions at least g. Thus the spectral gap closes as $c \to 1^+$ in this class, as suggested by the numerical bootstrap study of Benjamin, Langhoff, Li and Thaler. The proof is by contradiction and compactness. Using the crystalline-measure reduction of Eberhardt and Levine, we show that every limit at $c = 1$ is an equal-weight average of two compact-boson partition functions, at least one at a half-integer radius. Positive Gaussian test functions adapted to the two circle lattices then force every state of a nearby partition function, through a logarithmically growing energy window, to lie within a power of $c - 1$ of a lattice point. On the other hand, the central-charge shift of the vacuum contributes a second-order term at zero momentum that no distribution supported on circle lattices and compatible with modular invariance can carry; a finite family of Gaussian tests detects this. Limits with two half-integer radii, where lattice points collide, require a separate analysis. The threshold $\epsilon(g)$ is not effective; we describe the additional estimates that would make it so.

Contents

1 Introduction **2**
1.1 The question . . . 2
1.2 The result . . . 3
1.3 Outline of the proof . . . 3
1.4 Remarks . . . 4

2 Setting and conventions **5**
2.1 The admissible class . . . 5
2.2 Momentum variables and the spectral measures . . . 6
2.3 The Lorentzian Fourier transform and circle lattices . . . 6
2.4 Analytic evaluations and the Gaussian identity . . . 7
2.5 Derivatives in the squared momenta . . . 9
2.6 Thermal bounds . . . 10
2.7 Compactness . . . 10

3 Classification of the limits at $c = 1$ **12**
3.1 The crystalline reduction . . . 12
3.2 The mixture weights . . . 13
3.3 The classification . . . 13
3.4 A first consequence: emergent near-currents . . . 15

4 Rigidity of lattice-supported modular distributions and the localization obstruction **15**
4.1 Distributions with discrete support . . . 15
4.2 Polynomial comb representations . . . 18
4.3 Finite unions of circle lattices . . . 21
4.4 A finite certificate for the origin coefficients . . . 25
4.5 One moving radius . . . 30
4.6 The localization obstruction . . . 34

5 Positive kernels and the reduction to half-integer radii **39**
5.1 Conventions . . . 39
5.2 Trigonometric kernels and their Fourier transforms . . . 40
5.3 Central-shift and tail estimates . . . 42
5.4 The peak budget . . . 43
5.5 Local zeros and the three-projection lemma . . . 46
5.6 Two kernel families and their finite hypotheses . . . 49
5.7 The case $R \notin \{1, 2\}$ . . . 52
5.8 The case $R \in \{1, 2\}$: the scalar row . . . 54
5.9 Choice of the integer data . . . 59
5.10 The four-scalar anchor . . . 62

6 Limits with two half-integer radii and the proof of Theorem A **65**
6.1 Conventions . . . 65
6.2 A quantitative projection onto the invariant distributions . . . 68
6.3 The reconstruction operator . . . 71
6.4 The Taylor array of the spectrum and the normal form . . . 73
6.5 Pairing, pinning and the comparison measure . . . 78
6.6 Kernel data at a pair of half-integers and their collisions . . . 82
6.7 The resolved peak budget . . . 87
6.8 Exclusion of resolved sequences . . . 90
6.9 Proof of Theorem A . . . 95

7 Remarks on effectivity **97**
7.1 Observables and certified local exclusions . . . 97
7.2 A computable modulus for the approach to the boundary . . . 99
7.3 Gluing certified local exclusions . . . 101
7.4 A smallness cutoff . . . 102
7.5 Computable origin certificates . . . 103
7.6 What remains . . . 107

Statement on the use of AI **109**

1 Introduction

1.1 The question

The torus partition function of a compact unitary two-dimensional conformal field theory is invariant under the modular group and expands in characters of the Virasoro algebra with nonnegative integer multiplicities. The modular bootstrap studies the constraints that these two properties place on the spectrum; the most basic is an upper bound on the dimension Δ_{gap} of the lightest nonvacuum primary [3, 4, 5, 6]. Most such bounds use only positivity

of the multiplicities, and therefore hold for a larger class of "partition functions" with real nonnegative multiplicities. Integrality is harder to exploit [7], but it can be decisive.

A striking instance was found recently by Benjamin, Langhoff, Li and Thaler [1]. For central charge c slightly above one, and for parity-symmetric spectra without conserved currents beyond the Virasoro algebra, the numerical bound from positivity alone is $\Delta_{\mathrm{gap}} \le c/6 + 1/3$ [5], close to $\frac{1}{2}$. Requiring in addition that the number of primaries in certain windows be an integer, they obtained numerical bounds that are smaller by an order of magnitude at $c - 1 = 3 \times 10^{-3}$, and suggested that the gap might vanish as $c \to 1^+$. No compact unitary theory with c slightly above one is known, and whether one exists is open. At $c = 1$, by contrast, the compact boson has a positive gap at every radius. The purpose of this paper is to prove that, in the class studied in [1], a positive gap cannot persist as $c \to 1^+$.

1.2 The result

Write $c = 1 + 12a$. We consider functions

$$Z(\tau) = |\chi_{\mathrm{vac}}(\tau)|^2 + \sum_j \chi_{h_j}(\tau)\,\overline{\chi_{\bar h_j}(\tau)}, \qquad \chi_h = \frac{q^{h-a/2}}{\eta}, \quad \chi_{\mathrm{vac}} = \frac{q^{-a/2}(1-q)}{\eta},$$

on the upper half-plane, where each pair of weights $(h_j, \bar h_j)$ is repeated according to its multiplicity. We call Z *admissible of gap* g if it is invariant under $\tau \mapsto -1/\tau$ and $\tau \mapsto \tau + 1$, the sum converges for all τ, the spins $h_j - \bar h_j$ are integers, the spectrum is invariant under $(h, \bar h) \mapsto (\bar h, h)$, and

$$h_j > 0, \qquad \bar h_j > 0, \qquad \Delta_j = h_j + \bar h_j \ge g$$

for every j; see Definition 2.1. The first two inequalities say that the vacuum is the only state at the unitarity bound $\Delta = |s|$. No lower bound on the twist $\Delta_j - |s_j|$ is assumed, so currents are allowed to emerge in a limit.

Theorem A. *For every $g > 0$ there exists $\epsilon(g) > 0$ such that no admissible partition function of gap g has central charge in $(1, 1+\epsilon(g))$. Equivalently, if Z_n are admissible at central charges $c_n \downarrow 1$, then their gaps tend to zero.*

The statement concerns partition functions and makes no reference to correlation functions. It does not assert that admissible partition functions with c close to one exist. The threshold $\epsilon(g)$ is obtained by contradiction from a compactness argument, and the proof gives no numerical value for it.

1.3 Outline of the proof

Suppose that admissible partition functions of gap g exist along a sequence $a \downarrow 0$. The argument has five steps.

Limits (Sections 2 and 3). Modular invariance and positivity bound the number of states below any energy uniformly in a, so a subsequence converges to a modular-invariant function at $c = 1$ which is still a nonnegative integer combination of the characters $q^h\bar q^{\bar h}/|\eta|^2$ plus the vacuum term $|1-q|^2/|\eta|^2$. This is a stronger positivity property than positivity in the irreducible characters at $c = 1$. The momenta of such a function form a self-dual crystalline measure in the plane, and the theorem of Eberhardt and Levine [2] reduces it to a finite combination of compact-boson partition functions Z_r. Examining the coefficients of a few monomials, we find that the limit is

$$\tfrac{1}{2}(Z_R + Z_t), \qquad 1 \le R, t \le (2g)^{-1/2},$$

with at least one of $2R$, $2t$ an integer (Theorem 3.5). The half-integrality comes from the scalar of dimension two that the vacuum character subtracts.

An obstruction at zero momentum (Section 4). In the variables $h = P^2$, $\bar{h} = Q^2$ a character is a Gaussian in the momentum $\sqrt{h - a/2}$, and modular invariance becomes invariance under a Fourier transform. At $a = 0$ the momenta of $\frac{1}{2}(Z_R + Z_t)$ lie on two circle lattices. At $a > 0$ the vacuum sits at $P^2 = Q^2 = -a/2$; expanded about the origin, this displacement is the distribution $-\frac{a}{4}(\partial_P^2 + \partial_Q^2)\delta_0$. We show that a distribution supported on a finite union of circle lattices and invariant under the Fourier transform and under the spin phase is a polynomial in the boost generator applied to the lattice sums (Theorems 4.9 and 4.11); such a distribution has no derivative terms at the origin. A linear-algebra argument, which is possible because the rigidity statement holds without any growth condition, converts this into a finite family of Gaussian test functions that detect an origin derivative (Theorem 4.19). The consequence is Theorem 4.23: it is impossible that every state with $\Delta \le u_* \log(1/a)$, for some $u_* > 1/(2\pi)$, lies within a^p in dimension, with the same spin, of a point of a fixed finite union of circle lattices together with one circle lattice of variable radius. The logarithmic window is what makes the tail of the spectrum negligible against a.

Localization by positive kernels (Section 5). To produce such a localization we use functions $F \ge 0$ that are Gaussians times trigonometric polynomials vanishing on the two limiting lattices and at the vacuum; a functional of this type for the self-dual boson, built from a triangular function and its Fourier transform, appears in [8, Section 3.2]. The Fourier transform of F is a finite sum of narrow Gaussian peaks, and the exact identity $\mathcal{L}_a(F) = \mathcal{L}_a(\mathbb{S}F)$ between the spectral sums bounds $\sum_j F(X_j)$ by $O(a \log^2(1/a))$ once the negative peaks are matched by states whose positions are known. For a limit $\frac{1}{2}(Z_R + Z_t)$ in which t is not a half-integer, the trigonometric polynomial can be chosen, depending on the arithmetic of t, so that this is the case, and one obtains the forbidden localization. Hence both radii of a limit are half-integers (Theorem 5.21).

The scale of the departure (Sections 5 and 6). The finitely many remaining limits lie on a common rational grid. Four scalar states at one fixed energy now play the role of the anchors, and their displacement δ_a controls the distance of the whole spectrum from the grid. By the obstruction, δ_a cannot be bounded by a power of a (Theorem 5.25). A quantitative form of the rigidity theorem then shows that, on each fixed finite set of states, the spectrum agrees to a high power of δ_a with that of $\frac{1}{2}(Z_R + Z_{t_a})$ for a half-integer R and a radius t_a with $|t_a - T|$ comparable to δ_a, where T is the half-integer limit of t_a (Theorem 6.13 and (6.82)).

Resolving the collision (Section 6). Since t_a is not a half-integer, the kernels of the third step apply at the radius t_a, but their peaks and zeros now collide as $t_a \to T$. Using Gaussians narrow enough to resolve the collisions costs only powers of $|t_a - T|^{-1}$, which are smaller than any negative power of a by the fourth step. The result is again a localization within a power of a of fixed lattices and the moving lattice Λ_{t_a}, contradicting Theorem 4.23.

Section 7 discusses what is needed to make $\epsilon(g)$ effective. All test functions used in the proof are finite or compact superpositions of Gaussians, for which the modular identity is exact; the argument does not use a finite number of derivatives of the modular crossing equation at $\tau = i$, and no formal power series in $c - 1$ is summed.

1.4 Remarks

The hypotheses are those of a torus partition function, and the limits that have to be excluded reflect this. The average of two compact-boson partition functions at different radii is not the partition function of a conformal field theory, but it satisfies every condition that survives the limit, and we know of no argument that discards it using the torus data alone. These averages

are responsible for the moving radius in the second step and for most of the work in the last two.

Parity symmetry is included in the definition because it is part of the setting of [1]. No step of the proof uses it: the classification of Section 3 does not assume it, and states of opposite spin are treated separately throughout (Remark 6.3). The assumption that no nonvacuum state has $h = 0$ or $\bar{h} = 0$ is essential to the method: it makes the long multiplicities at $c > 1$ nonnegative term by term, and this positivity is inherited by the limit.

Notation

Section 2 fixes the conventions. Throughout, $a = (c-1)/12$, $L = \log(1/a)$, $X = (P, Q)$ are momenta and $(u, v) = (P+Q, P-Q)$ light-cone coordinates, Λ_r is the circle lattice of radius r, and a radius is called a half-integer if twice it is an integer.

2 Setting and conventions

Throughout, $\tau = x + iy$ lies in the upper half-plane $\mathbb{H}$, $q = e^{2\pi i\tau}$, $\bar{q} = e^{-2\pi i\bar{\tau}}$, and η is the Dedekind eta function. We write

$$c = 1 + 12a, \qquad a \geq 0,$$

so that $a > 0$ is the regime of interest and $a = 0$ is the boundary $c = 1$. For small $a > 0$ we put $L = \log(1/a)$. Constants called C, c, κ may change from line to line; their dependence is indicated when it matters.

2.1 The admissible class

For $c > 1$ the non-degenerate Virasoro characters and the vacuum character are

$$\chi_h(\tau) = \frac{q^{h-a/2}}{\eta(\tau)} \quad (h > 0), \qquad \chi_{\mathrm{vac}}(\tau) = \frac{q^{-a/2}(1-q)}{\eta(\tau)}.$$

Definition 2.1 (Admissible partition functions)**.** Let $0 < g \leq \frac{1}{2}$ and $a > 0$. An *admissible partition function of gap g at $c = 1 + 12a$* is a function on $\mathbb{H}$ of the form

$$Z(\tau) = |\chi_{\mathrm{vac}}(\tau)|^2 + \sum_{j \in \mathcal{J}} \chi_{h_j}(\tau)\, \overline{\chi_{\bar{h}_j}(\tau)}, \tag{2.1}$$

where $\mathcal{J}$ is a countable index set and the following hold.

(i) *No currents, integral spin, gap.* For every j, $h_j > 0$, $\bar{h}_j > 0$, the spin $s_j = h_j - \bar{h}_j$ is an integer, and the dimension $\Delta_j = h_j + \bar{h}_j$ satisfies $\Delta_j \geq g$.

(ii) *Convergence.* $\sum_j e^{-2\pi y \Delta_j} < \infty$ for every $y > 0$.

(iii) *Modular invariance.* $Z(-1/\tau) = Z(\tau) = Z(\tau + 1)$ for all $\tau \in \mathbb{H}$.

(iv) *Parity.* The family $(h_j, \bar{h}_j)_{j \in \mathcal{J}}$ is invariant, with multiplicity, under $(h, \bar{h}) \mapsto (\bar{h}, h)$.

We call the indices j the (nonvacuum) *states*, and $(\Delta_j, s_j)_{j \in \mathcal{J}}$ the *spectrum*.

An index is repeated according to multiplicity, so the multiplicity of each weight $(h, \bar{h})$ is a positive integer; by (ii) it is finite, and only finitely many states have dimension below any given bound. The vacuum term is the unique shortened one and has coefficient one. The inequalities $h_j, \bar{h}_j > 0$ say that $\Delta_j > |s_j|$: there is no nonvacuum state with $h = 0$ or $\bar{h} = 0$. Nothing is assumed about the size of $\Delta_j - |s_j|$. The hypotheses concern the torus partition function only; no operator product data enter.

Remark 2.2. The restriction $g \le \frac12$ is harmless: a partition function whose states all have $\Delta_j \ge g > \frac12$ is admissible of gap $\frac12$.

Put

$$H_a(\tau) = (1-q)(1-\bar q) + \sum_{j\in\mathcal J} q^{h_j}\bar q^{\bar h_j}, \qquad \widehat Z_a(\tau) = \sqrt{y}\,|\eta(\tau)|^2 Z(\tau) = \sqrt{y}\,e^{2\pi a y} H_a(\tau). \tag{2.2}$$

Since $\sqrt{y}\,|\eta|^2$ is invariant under the modular group, $\widehat Z_a(-1/\tau) = \widehat Z_a(\tau) = \widehat Z_a(\tau+1)$. At $a=0$ the same formulas are used for the limiting objects of Theorem 2.9, where $h_j, \bar h_j \ge 0$ is allowed.

2.2 Momentum variables and the spectral measures

We use real *momentum* coordinates $X = (P,Q) \in \mathbb R^2$, related to the weights by $h = P^2$, $\bar h = Q^2$, and the *light-cone* coordinates

$$u = P+Q, \qquad v = P-Q, \qquad \Delta = P^2+Q^2 = \tfrac12(u^2+v^2), \qquad s = P^2-Q^2 = uv.$$

A function or distribution on $\mathbb R^2$ is called *even* if it is invariant under each of the reflections $P\mapsto -P$ and $Q\mapsto -Q$ separately. For $h \ge 0$ let A_h be the even probability measure on $\mathbb R$ carried by $\{\pm\sqrt h\}$,

$$A_h = \tfrac12(\delta_{\sqrt h} + \delta_{-\sqrt h}) \quad (h>0), \qquad A_0 = \delta_0.$$

A state of weights $(h,\bar h)$ is represented by $A_h \otimes A_{\bar h}$. For an admissible partition function we define

$$\rho_a = \sum_{j\in\mathcal J} A_{h_j}\otimes A_{\bar h_j}, \qquad \nu_{\mathrm{vac}} = \sum_{k,l\in\{0,1\}} (-1)^{k+l} A_k \otimes A_l, \qquad \mu_a = \nu_{\mathrm{vac}} + \rho_a. \tag{2.3}$$

Thus ρ_a is a positive even measure with $|X|^2 = \Delta$ on its support, each state carries total mass one, and atoms of ρ_a have masses in $\frac14\mathbb Z_{>0}$. We call ρ_a the *long measure* and μ_a the *Fock measure* of the partition function; both are built from the physical weights $(h_j,\bar h_j)$ and do not involve a otherwise. The *counting function* and the thermal sum are

$$N_a(E) = \rho_a(\{|X|^2 \le E\}) = \#\{j : \Delta_j \le E\}, \qquad P_a(y) = \int e^{-2\pi y|X|^2}\,\mathrm d\rho_a(X) = \sum_j e^{-2\pi y\Delta_j}.$$

With the Gaussians

$$G_\tau(P,Q) = \sqrt{y}\,e^{2\pi i\tau P^2 - 2\pi i\bar\tau Q^2}, \tag{2.4}$$

equation (2.2) reads $\widehat Z_a(\tau) = e^{2\pi a y}\langle \mu_a, G_\tau\rangle$.

2.3 The Lorentzian Fourier transform and circle lattices

Let

$$B(X,X') = 2(PP' - QQ') = uv' + vu'$$

be the Lorentzian pairing, and define for Schwartz functions

$$(\mathbb S f)(X) = 2\int_{\mathbb R^2} f(X')\,e^{2\pi i B(X,X')}\,\mathrm dP'\,\mathrm dQ' = \int_{\mathbb R^2} f(X')\,e^{2\pi i(uv'+vu')}\,\mathrm du'\,\mathrm dv'. \tag{2.5}$$

In light-cone coordinates $\mathbb S$ is the unitary Fourier transform followed by the exchange of the two variables; it preserves evenness, and $\mathbb S^2 f(X) = f(-X)$. It extends to tempered distributions by duality. The *phase operator* is multiplication by a unimodular function,

$$\mathcal C T = e^{2\pi i(P^2-Q^2)}\,T,$$

and the *boost generator* is the vector field

$$\mathcal{K} = P\partial_Q + Q\partial_P = u\partial_u - v\partial_v. \tag{2.6}$$

A direct computation gives

$$\mathbb{S}G_\tau = G_{-1/\tau}, \qquad \mathcal{C}G_\tau = G_{\tau+1}, \tag{2.7}$$

and, for $b > 0$, $z \in \mathbb{C}^2$ and $J = \mathrm{diag}(1, -1)$,

$$\phi_{b,z}(X) = e^{-b|X|^2 + z\cdot X}, \qquad \mathbb{S}\phi_{b,z}(X) = \frac{2\pi}{b}\exp\Big(\frac{(z + 4\pi i JX)\cdot(z + 4\pi i JX)}{4b}\Big). \tag{2.8}$$

Here and below, in formulas for entire functions $|X|^2$ denotes the polynomial $P^2 + Q^2$ and $z \cdot X$ the bilinear product, without complex conjugation. In particular $\mathbb{S}\phi_{b,z}$ is a constant times $\phi_{4\pi^2/b,\, 2\pi i Jz/b}$.

For $r > 0$ the *circle lattice* of radius r is

$$\Lambda_r = \Big\{(P,Q) = \Big(\frac{mr + n/r}{2}, \frac{mr - n/r}{2}\Big) : m, n \in \mathbb{Z}\Big\} = \{(u,v) = (mr,\, n/r) : m, n \in \mathbb{Z}\}. \tag{2.9}$$

The reflection $Q \mapsto -Q$ exchanges u and v and maps Λ_r onto $\Lambda_{1/r}$. On Λ_r one has $B(\lambda, \lambda') = mn' + nm'$ and $s = mn$, so Λ_r is an even unimodular lattice for B. Put

$$\begin{aligned} e_r &= \sum_{\lambda\in\Lambda_r} \delta_\lambda, \qquad \rho_r^{\mathrm{F}} = \tfrac{1}{2}(e_r + e_{1/r}), \\ \Theta_r(\tau) &= \sum_{m,n\in\mathbb{Z}} q^{(mr+n/r)^2/4}\, \bar q^{(mr-n/r)^2/4}, \qquad Z_r = \frac{\Theta_r}{|\eta|^2}. \end{aligned} \tag{2.10}$$

Poisson summation and integrality of the spin give

$$\mathbb{S}e_r = e_r, \qquad \mathcal{C}e_r = e_r, \tag{2.11}$$

and ρ_r^{F} is the even measure with $\langle \rho_r^{\mathrm{F}}, G_\tau\rangle = \sqrt{y}\,\Theta_r(\tau)$. The function $Z_r = Z_{1/r}$ is the partition function of the compact boson at radius r, with $r = 1$ the self-dual radius; $\rho_r^{\mathrm{F}} = \rho_{1/r}^{\mathrm{F}}$ is its Fock measure. A point of Λ_r with charges (m, n) has dimension $\frac{1}{2}(m^2r^2 + n^2/r^2)$ and spin mn.

We call a radius r a *half-integer* if $2r \in \mathbb{Z}$; integers are included. For radii $R, t \geq 1$ we write

$$\sigma(R,t) = \tfrac{1}{2}\rho_R^{\mathrm{F}} + \tfrac{1}{2}\rho_t^{\mathrm{F}} - \nu_{\mathrm{vac}} \tag{2.12}$$

for the long measure of the average $\frac{1}{2}(Z_R + Z_t)$, and $\mathcal{B}_g$ for the set of all $\sigma(R,t)$ with $1 \leq R, t \leq (2g)^{-1/2}$ and at least one of R, t a half-integer. Theorem 3.5 shows that these are positive measures and are exactly the possible limits.

2.4 Analytic evaluations and the Gaussian identity

At $a > 0$ the character χ_h is a Gaussian in the momentum $\sqrt{h - a/2}$, which is imaginary for the vacuum and possibly for states with $h < a/2$. We therefore evaluate entire test functions at shifted squared momenta instead of introducing a measure on real momentum space.

For an entire function f on $\mathbb{C}^2$ let $f^{\mathrm{ev}}(P,Q) = \frac{1}{4}\sum_{\pm,\pm} f(\pm P, \pm Q)$ be its even part. It is an entire function of (P^2, Q^2): there is a unique entire $f^\natural$ on $\mathbb{C}^2$ with

$$f^{\mathrm{ev}}(P,Q) = f^\natural(P^2, Q^2). \tag{2.13}$$

For $h, \bar{h} \geq 0$, $f^\natural(h,\bar{h}) = \langle A_h \otimes A_{\bar{h}}, f\rangle$. Given an admissible partition function at $c = 1 + 12a$, define

$$\mathcal{L}_a(f) = \sum_{k,l\in\{0,1\}} (-1)^{k+l} f^\natural(k - \tfrac{a}{2},\, l - \tfrac{a}{2}) + \sum_{j\in\mathcal{J}} f^\natural(h_j - \tfrac{a}{2},\, \bar{h}_j - \tfrac{a}{2}) \tag{2.14}$$

whenever the sum over $\mathcal{J}$ converges absolutely. The first sum is the *shifted vacuum*; the terms of the second are the *shifted* (or analytic) evaluations of the states, as opposed to their *physical* evaluations $f^\natural(h_j, \bar{h}_j)$, whose sum with the unshifted vacuum terms is $\langle \mu_a, f\rangle$. By construction

$$\mathcal{L}_a(G_\tau) = \widehat{Z}_a(\tau), \qquad \tau \in \mathbb{H}. \tag{2.15}$$

The evaluation points of a state are $(\pm P, \pm Q)$ with $P^2 = h - \frac{a}{2}$, $Q^2 = \bar{h} - \frac{a}{2}$. Hence $P^2 + Q^2 = \Delta - a$ is real, $P^2 - Q^2 = s$ is the spin, at most one of P, Q is imaginary when $a < g$, and

$$|\operatorname{Im} P|,\ |\operatorname{Im} Q| \leq \sqrt{a/2}, \qquad |\operatorname{Re} X|^2 \geq \Delta - a. \tag{2.16}$$

Definition 2.3 (Gaussian mixtures)**.** For $b > 0$ let $\mathcal{G}_b$ be the space of finite sums of functions

$$f(X) = \int_\Omega p(X,\omega)\, e^{-b|X|^2 + z(\omega)\cdot X}\, \mathrm{d}m(\omega),$$

where Ω is a compact metric space, m a finite complex Borel measure on it, $z : \Omega \to \mathbb{C}^2$ is continuous, and p is a polynomial in X whose coefficients are continuous functions of ω.

Every $f \in \mathcal{G}_b$ is entire, and so are all its derivatives, with $\partial^\alpha f \in \mathcal{G}_b$. Products of an element of $\mathcal{G}_b$ with polynomials, with exponentials $e^{w\cdot X}$, $w \in \mathbb{C}^2$, and with functions such as $\operatorname{sinc}(\ell(X)) = \frac{1}{2}\int_{-1}^{1} e^{i\sigma\ell(X)}\,\mathrm{d}\sigma$ for a real linear form ℓ, remain in $\mathcal{G}_b$; so do real translates. By (2.8), $\mathbb{S}$ maps $\mathcal{G}_b$ onto $\mathcal{G}_{4\pi^2/b}$. For every $0 < b' < b$, every multi-index α and every $\varrho < \infty$ there is C with

$$|\partial^\alpha f(X)| \leq C\, e^{-b'|\operatorname{Re} X|^2} \qquad \text{for } |\operatorname{Im} X| \leq \varrho. \tag{2.17}$$

Lemma 2.4 (The exact Gaussian identity)**.** *Let Z be admissible at $c = 1 + 12a$; or let $a = 0$ and let $Z_0 = H_0/|\eta|^2$ be modular invariant with H_0 of the form (2.2), $h_j, \bar{h}_j \geq 0$, convergent at every $y > 0$. For every $b > 0$ and every $f \in \mathcal{G}_b$ the sums defining $\mathcal{L}_a(f)$ and $\mathcal{L}_a(\mathbb{S}f)$ converge absolutely, and*

$$\mathcal{L}_a(f) = \mathcal{L}_a(\mathbb{S}f). \tag{2.18}$$

Proof. By (2.16) and (2.17), each term of the sum over states is bounded by $Ce^{-b'(\Delta_j - a)}$, which is summable by hypothesis (ii) of Definition 2.1. The same bound holds uniformly for $\phi_{b,z}$ with (b,z) in a compact subset of $\{\operatorname{Re} b > 0\} \times \mathbb{C}^2$, and for its derivatives in the parameters. Consequently $F(b,z) = \mathcal{L}_a(\phi_{b,z}) - \mathcal{L}_a(\mathbb{S}\phi_{b,z})$ is holomorphic on $\{\operatorname{Re} b > 0\} \times \mathbb{C}^2$, and one may differentiate under $\mathcal{L}_a$.

First let $b = 2\pi$. By (2.7), (2.15) and modular invariance, $\mathcal{L}_a(G_\tau) = \mathcal{L}_a(\mathbb{S}G_\tau)$ for all τ. The two sides are $\widehat{Z}_a(\tau)$ and $\widehat{Z}_a(-1/\tau)$, whose defining series may be differentiated termwise in x and y, by hypothesis (ii) and $|s_j| \leq \Delta_j$; and $\mathbb{S}$ commutes with ∂_x, ∂_y applied to G_τ. Differentiating in x and y at $\tau = i$ produces, on the left, $\mathcal{L}_a$ of polynomials in $P^2 - Q^2$ and $P^2 + Q^2$ times $e^{-2\pi|X|^2}$; the factor $\sqrt{y}$ contributes only terms of lower degree. By induction on the degree, the identity $\mathcal{L}_a(\psi) = \mathcal{L}_a(\mathbb{S}\psi)$ holds for every $\psi = P^{2i}Q^{2k}e^{-2\pi|X|^2}$. It holds trivially for monomials that are odd in P or in Q, because $\mathcal{L}_a$ annihilates functions with vanishing even part and $\mathbb{S}$ preserves the parity in each variable. These functions are the Taylor coefficients of $z \mapsto \phi_{2\pi,z}$ at $z = 0$, so the entire function $F(2\pi,\cdot)$ vanishes identically.

Next, $\partial_b \phi_{b,z} = -(\partial_{z_1}^2 + \partial_{z_2}^2)\phi_{b,z}$, and the same relation holds after applying $\mathbb{S}$. Therefore every b-derivative of F at $b = 2\pi$ is a z-derivative of $F(2\pi,\cdot)$ and vanishes, and $F \equiv 0$ by the identity theorem. Differentiating in z and integrating over Ω, which is justified by the uniform bounds, gives (2.18) for $f \in \mathcal{G}_b$. □

The lemma never takes a Fourier transform of a distribution supported at complex points. It applies to test functions that are not even: both sides see only the even part.

2.5 Derivatives in the squared momenta

Taylor expansions of an evaluation $f^\natural(h,\bar h)$ in the weights are controlled by the following formula, which is regular at $h=0$.

Lemma 2.5. *Let ϕ be an entire function of one variable and let $\varphi(h)=\frac12(\phi(\sqrt h)+\phi(-\sqrt h))$, an entire function of h. For every integer $k\ge1$ and every $h\in\mathbb C$,*

$$\varphi^{(k)}(h)=\frac{1}{4^k(k-1)!}\int_{-1}^{1}(1-t^2)^{k-1}\,\phi^{(2k)}(t\sqrt h)\,\mathrm dt. \tag{2.19}$$

For $h>0$, $\varphi^{(k)}(h)$ is also a linear combination of $h^{-(2k-l)/2}(\phi^{(l)}(\sqrt h)+(-1)^l\phi^{(l)}(-\sqrt h))$, $1\le l\le k$, with coefficients depending only on k and l.

Proof. For $k=1$ and $h>0$ the right side of (2.19) is $\frac{1}{4\sqrt h}(\phi'(\sqrt h)-\phi'(-\sqrt h))=\varphi'(h)$. If the formula holds for k, differentiate it in h, use $\partial_h\phi^{(2k)}(t\sqrt h)=\frac{t}{2\sqrt h}\,\phi^{(2k+1)}(t\sqrt h)$ and $t(1-t^2)^{k-1}=-\frac{1}{2k}\frac{d}{dt}(1-t^2)^k$, and integrate by parts once in t; the boundary terms vanish and one obtains the formula for $k+1$, the constant changing by the factor $1/(4k)$. Both sides are entire in h (the integrand is even in t), so the identity extends from $h>0$ to $\mathbb C$. The second statement follows by differentiating $\phi(\pm\sqrt h)$ directly. □

Lemma 2.6 (Uniform bounds for weight derivatives)**.** *Let f be entire on $\mathbb C^2$, let $k\ge0$ be an integer, and suppose that for some $M<\infty$ and $\beta>0$*

$$|\partial^\alpha f(X)|\le M\,e^{-\beta|\operatorname{Re}X|^2}\qquad\text{for }|\alpha|\le 2k,\quad|\operatorname{Im}P|,|\operatorname{Im}Q|\le1.$$

Then for all real $h,\bar h\ge-1$ and all $i+l\le k$,

$$|\partial_h^i\partial_{\bar h}^l f^\natural(h,\bar h)|\le C_k\,M\,e^{4\beta}\,e^{-\beta(h+\bar h)}, \tag{2.20}$$

where C_k depends only on k.

Proof. Treat the two variables separately. In a variable whose weight is at most 2, use (2.19) when a derivative falls on it and the definition otherwise: the function is evaluated at momenta $t\sqrt h$, $|t|\le1$, whose imaginary parts are at most one and whose real parts are bounded, and $1\le e^{2\beta}e^{-\beta h}$. In a variable whose weight exceeds 2, use the endpoint form of Lemma 2.5: the momentum is $\pm\sqrt h$, real, the coefficients $h^{-(2i-l)/2}$ are bounded, and the Gaussian factor is $e^{-\beta h}$. The integral formula in one variable commutes with either operation in the other. □

We shall use Lemma 2.6 in two ways. First, moving a state of weights $(h,\bar h)$ along its fixed-spin curve changes both weights at the same rate, and for $\Phi(d)=f^\natural(h+\frac d2,\bar h+\frac d2)$ one has $|\Phi^{(k)}(d)|\le C_kMe^{4\beta}e^{-\beta(\Delta+d)}$. Secondly, taking $k=1$ and integrating over the central shift gives the following.

Corollary 2.7 (Central-shift replacement)**.** *Under the hypotheses of Lemma 2.6 with $k=1$, for $0\le a\le1$ and all $h,\bar h\ge0$,*

$$|f^\natural(h-\tfrac a2,\bar h-\tfrac a2)-f^\natural(h,\bar h)|\le C\,a\,M\,e^{5\beta}\,e^{-\beta(h+\bar h)}.$$

2.6 Thermal bounds

Lemma 2.8 (Thermal cap, counting bound, tails)**.** *Let $0 < g \le \frac{1}{2}$, put $C_g = 2e^{\pi/2}g^{-2}$ and $A_g = e^{2\pi}(1 + C_g)$, and let Z be admissible of gap g at $c = 1 + 12a$ with $0 < a \le g^2/4$. Then*

$$P_a(g^2) \le C_g, \tag{2.21}$$

$$P_a(y) \le (1 + C_g)\, y^{-1} e^{2\pi a/y} \qquad (0 < y \le 1), \tag{2.22}$$

$$N_a(E) \le A_g\, E\, e^{2\pi a E} \qquad (E \ge 1), \tag{2.23}$$

and, for $\beta > 2\pi a$ and $E_0 \ge 1$,

$$\sum_{\Delta_j > E_0} e^{-\beta\Delta_j} \le A_g\, \beta\Big(\frac{E_0}{\beta - 2\pi a} + \frac{1}{(\beta - 2\pi a)^2}\Big) e^{-(\beta - 2\pi a)E_0}. \tag{2.24}$$

The same bounds hold with $a = 0$ for every modular-invariant function $Z_0 = H_0/|\eta|^2$ in which H_0 has the form (2.2) with $h_j, \bar h_j \ge 0$, integer spins, $\Delta_j \ge g$, and convergence at every $y > 0$. This includes the limits of Theorem 2.9 and, by Theorem 3.5, every element of $\mathcal{B}_g$.

Proof. Let $V(y) = (1 - e^{-2\pi y})^2$. On the imaginary axis (2.2) is $\widehat{Z}_a(iy) = \sqrt{y}\, e^{2\pi a y}(V(y) + P_a(y))$, and invariance under $\tau \mapsto -1/\tau$ gives

$$V(1/y) + P_a(1/y) = y\, e^{2\pi a(y - 1/y)}(V(y) + P_a(y)), \qquad y > 0. \tag{2.25}$$

Take $y = Y = g^{-2} \ge 4$ and put $C_Y = Y e^{2\pi a(Y - 1/Y)}$. Then

$$P_a(1/Y) - C_Y P_a(Y) = C_Y V(Y) - V(1/Y) \le C_Y,$$

and the contribution of a state to the left side is $e^{-2\pi\Delta/Y}\big(1 - Y e^{-2\pi(Y - 1/Y)(\Delta - a)}\big)$. Since $\Delta \ge g$, $a \le g/2$ and $Y - 1/Y \ge 3/(4g^2)$, the subtracted term is at most $g^{-2}e^{-3\pi/(4g)} \le z^2 e^{-2z}$ with $z = 1/g \ge 2$, hence at most $4e^{-4} < \frac{1}{2}$. Therefore $\frac{1}{2}P_a(g^2) \le C_Y \le g^{-2}e^{\pi/2}$, which is (2.21).

For $0 < y \le 1$, (2.25) with y replaced by $1/y$, monotonicity of P_a, $0 \le V \le 1$ and $P_a(1) \le P_a(g^2)$ give (2.22). Then $N_a(E) \le e^{2\pi y E} P_a(y)$ with $y = 1/E$ gives (2.23). Finally, integrating by parts against $e^{-\beta E}$ and discarding the nonpositive boundary term at E_0,

$$\sum_{\Delta_j > E_0} e^{-\beta\Delta_j} \le \beta \int_{E_0}^{\infty} N_a(E) e^{-\beta E}\, \mathrm{d}E \le A_g \beta \int_{E_0}^{\infty} E\, e^{-(\beta - 2\pi a)E}\, \mathrm{d}E,$$

which is (2.24). For $a = 0$ the argument is unchanged. □

In particular, for $0 < \beta \le 1$ and $a \le \min\{g^2/4, \beta/(4\pi)\}$,

$$\sum_j e^{-\beta\Delta_j} \le C'_g\, \beta^{-1}, \tag{2.26}$$

by (2.22) with $y = \beta/(2\pi)$, and $N_a(ML) \le A_g ML e^{2\pi a ML} = O_M(L)$ for every fixed M: a logarithmic energy window contains $O(L)$ states.

2.7 Compactness

Theorem 2.9 (Compactness)**.** *Fix $0 < g \le \frac{1}{2}$. Let Z_n be admissible of gap g at $c_n = 1 + 12a_n$ with $a_n \downarrow 0$, with states $(h_j^{(n)}, \bar h_j^{(n)})_{j \in \mathcal{J}_n}$. After passing to a subsequence there is a countable family $(h_j, \bar h_j)_{j \in \mathcal{J}_0}$ with*

$$h_j, \bar h_j \ge 0, \qquad h_j - \bar h_j \in \mathbb{Z}, \qquad h_j + \bar h_j \ge g, \qquad \sum_{j \in \mathcal{J}_0} e^{-2\pi y(h_j + \bar h_j)} < \infty \quad (y > 0),$$

such that:

(a) *the long measures* ρ_{a_n} *converge vaguely on* $\mathbb{R}^2$ *to* $\rho_0 = \sum_{j\in\mathcal{J}_0} A_{h_j} \otimes A_{\bar{h}_j}$;

(b) $H_{a_n} \to H_0 = (1-q)(1-\bar{q}) + \sum_{j\in\mathcal{J}_0} q^{h_j}\bar{q}^{\bar{h}_j}$ *locally uniformly on* $\mathbb{H}$, *together with all derivatives in* x *and* y;

(c) $Z_0 = H_0/|\eta|^2$ *satisfies* $Z_0(-1/\tau) = Z_0(\tau) = Z_0(\tau+1)$, *and the family* $\mathcal{J}_0$ *is parity invariant.*

The limit may contain states with $h_j = 0$ or $\bar{h}_j = 0$; these are terms $q^h\bar{q}^{\bar{h}}/|\eta|^2$ and not shortened characters. The vacuum term of H_0 is still $(1-q)(1-\bar{q})$ with coefficient one.

Proof. By (2.21), the number of states of Z_n with $\Delta \le E$ is at most $C_g e^{2\pi g^2 E}$, uniformly in n once $a_n \le g^2/4$, and their spins are integers with $|s| \le E$. A diagonal argument over $E \in \mathbb{Z}_{\ge 0}$ yields a subsequence along which, for every E that is not a limiting dimension, the finitely many states below E converge together with their spins. The limits form a locally finite family $\mathcal{J}_0$ in which coinciding limits are repeated; the closed conditions $\Delta \ge g$ and $h, \bar{h} \ge 0$ persist, spins stay integral, and parity invariance is preserved. This is (a).

For (b) we need uniform tails. By (2.21) and (2.22), $P_{a_n}(y)$ is bounded uniformly in n for each fixed $y > 0$. Hence on a compact subset of $\mathbb{H}$ with $y \ge y_0$, the contribution to H_{a_n} of states with $\Delta > E$ is at most $P_{a_n}(y_0/2)\, e^{-\pi y_0 E}$, uniformly in n; a derivative of order k in (x, y) costs a factor at most $(2\pi\Delta)^k$ because $|s| \le \Delta$, and is absorbed by replacing $y_0/2$ with $y_0/4$. Together with (a) this gives (b), and the convergence of the sum for $\mathcal{J}_0$ at every $y > 0$ follows from Fatou's lemma. The modular identities for $\sqrt{y}\, e^{2\pi a_n y} H_{a_n}$ pass to the limit, which is (c). Since $\Delta \ge g$ on $\mathcal{J}_0$, no state of the limit has weights $(0,0)$, so the coefficient of $q^0\bar{q}^0$ in H_0 is one. □

Definition 2.10. In the situation of Theorem 2.9 we say that the sequence *converges to* ρ_0 if (a) holds for the sequence itself. An *admissible sequence of gap* g is a sequence Z_n as in that theorem; we often index its objects by $a = a_n$ rather than by n, writing ρ_a, μ_a, $\mathcal{L}_a$.

We shall use vague convergence in the following form. Atoms of all the measures ρ_a have masses in $\frac{1}{4}\mathbb{Z}_{>0}$, and each state has total mass one. A *cluster* of ρ_0 is the set of its states with a given dimension and signed spin.

Corollary 2.11 (Stabilization of clusters)**.** *Let an admissible sequence converge to* ρ_0, *and let* $U \subset \mathbb{R}^2$ *be a bounded open set, invariant under the two reflections, whose boundary does not meet* $\operatorname{supp}\rho_0$. *For all sufficiently small* a *in the sequence, the number of states of* ρ_a *with support in* U *equals that of* ρ_0, *no state of* ρ_a *meets* ∂U, *and the dimensions of the states in* U *converge, with multiplicity and with their spins, to those of* ρ_0 *in* U.

Proof. Choose $E > \sup_U |X|^2$ that is not a dimension of ρ_0. By vague convergence and integrality of the masses, for small a the states of ρ_a with $\Delta < E$ are in bijection with those of ρ_0, with equal spins and converging weights. Since U is invariant under the two reflections, the support of a state lies in U or is disjoint from it. A state of ρ_0 has its support in the open set U or in the open set $\mathbb{R}^2 \setminus \overline{U}$, and so, for small a, has the corresponding state of ρ_a. States with $\Delta \ge E$ do not meet $\overline{U}$. □

Statements that hold "for all sufficiently small admissible a" are always obtained from statements about sequences by contradiction, and the resulting thresholds are not effective.

3 Classification of the limits at $c = 1$

Theorem 2.9 produces limits of the following kind. Fix $0 < g \le \frac{1}{2}$. A *boundary partition function of gap* g is a function $Z_0 = H_0/|\eta|^2$ on $\mathbb{H}$ with $Z_0(-1/\tau) = Z_0(\tau) = Z_0(\tau+1)$ and

$$H_0 = (1-q)(1-\bar{q}) + \sum_{j\in\mathcal{J}_0} q^{h_j}\bar{q}^{\bar{h}_j}, \qquad h_j, \bar{h}_j \ge 0, \quad h_j - \bar{h}_j \in \mathbb{Z}, \quad h_j + \bar{h}_j \ge g, \tag{3.1}$$

where $\mathcal{J}_0$ is countable and the sum converges absolutely on $\mathbb{H}$. Its long measure is $\rho_0 = \sum_{j\in\mathcal{J}_0} A_{h_j} \otimes A_{\bar{h}_j}$ and its Fock measure is $\mu_0 = \nu_{\mathrm{vac}} + \rho_0$. Parity invariance is not assumed in this section. Collecting equal exponents, $H_0 = \sum d_{h,\bar{h}}\, q^h \bar{q}^{\bar{h}}$ with integers $d_{h,\bar{h}}$, the *Fock coefficients*; this expansion is unique. They are nonnegative except possibly at $(h,\bar{h}) = (1,0)$ and $(0,1)$, where the vacuum contributes -1, and $d_{0,0} = 1$.

3.1 The crystalline reduction

The classification of all modular-invariant partition functions at $c = 1$ with discrete spectrum, under axioms that include positivity in the irreducible $c = 1$ characters, is due to Eberhardt and Levine [2]. We use the harmonic-analytic part of their argument, in the following form.

Proposition 3.1 ([2, Theorem 3.1 and Sections 3.3–3.5])**.** *Let μ be an even, real, purely atomic measure on $\mathbb{R}^2$ whose atoms have masses in $\frac{1}{4}\mathbb{Z}$. Suppose that*

(a) $|\mu|(\{|X| \le \varrho\}) \le C\varrho^2$ *for all* $\varrho \ge 1$*;*

(b) $\mathbb{S}\mu = \mu$ *as tempered distributions;*

(c) $P^2 - Q^2 \in \mathbb{Z}$ *on* $\operatorname{supp}\mu$*.*

Then there are a finite set $\mathcal{R} \subset [1,\infty)$ and complex numbers c_r with $\mu = \sum_{r\in\mathcal{R}} c_r\, \rho_r^{\mathrm{F}}$.

Remark 3.2. We record how this statement is extracted from [2], since it is not displayed there in this form. Theorem 3.1 of [2], applied to 4μ, requires a tempered atomic measure with integer masses whose transform $\mathbb{S}\mu$ is a measure of upper bounded density, $|\mathbb{S}\mu|(\{|X| \le \varrho\}) \le C\varrho^2$ for $\varrho \ge 1$; hypothesis (a) gives temperedness and, by (b), this bound. The subsequent reduction [2, (127)–(150)] of the resulting finite union of lattice cosets to combinations of circle (Narain) combs uses, in addition, only spin quantization (c), evenness, and (b). The authors note that positivity enters their argument up to that point only through the density bound, and that uniqueness of the vacuum is not used. Their proof of Theorem 3.1 relies on Cohen's idempotent theorem [9] and related results on Fourier quasicrystals, which we do not reprove. Alternatively, a function of the form (3.1) satisfies the full list of axioms of [2], so their final classification applies to it directly; the additional solutions found there have a negative Fock coefficient at $(h,\bar{h}) = (p^2, 0)$ for large primes p and are excluded by (3.1). Both routes lead to Lemma 3.4 below.

Lemma 3.3. *The Fock measure μ_0 of a boundary partition function satisfies the hypotheses of Proposition 3.1.*

Proof. The measure is even and atomic, and its atoms have masses in $\frac{1}{4}\mathbb{Z}$ by construction; (c) holds because the spins are integers and the points of the four vacuum terms have spin 0 or ± 1.

For (a), let $P_0(y) = \sum_{j\in\mathcal{J}_0} e^{-2\pi y(h_j+\bar{h}_j)}$. Since $|\nu_{\mathrm{vac}}|$ consists of the four vacuum terms with positive signs,

$$\int e^{-2\pi y|X|^2}\, \mathrm{d}|\mu_0|(X) \le P_0(y) + (1+e^{-2\pi y})^2 = H_0(iy) + 4e^{-2\pi y}.$$

Modular invariance of $\sqrt{y}\,H_0$ gives $H_0(iy) = y^{-1}H_0(i/y) \le y^{-1}(1 + P_0(1))$ for $0 < y \le 1$. Taking $y = \varrho^{-2}$ and using $e^{-2\pi y|X|^2} \ge e^{-2\pi}$ on $\{|X| \le \varrho\}$,

$$|\mu_0|(\{|X| \le \varrho\}) \le e^{2\pi}\big((1 + P_0(1))\varrho^2 + 4\big).$$

In particular μ_0 is a tempered distribution. For (b), note that $\mathcal{L}_0(f) = \langle \mu_0, f\rangle$, so Lemma 2.4 gives $\langle \mu_0, f\rangle = \langle \mu_0, \mathbb{S}f\rangle$ for all $f \in \mathcal{G}_{2\pi}$. This class contains the Hermite functions, which span a dense subspace of the Schwartz space, and both sides are continuous there. Hence $\mathbb{S}\mu_0 = \mu_0$. □

The proof of (a) uses positivity only through the observation that the total variation of μ_0 exceeds its thermal trace by at most four vacuum units; the general density estimate of [2, Section 2] is not needed.

3.2 The mixture weights

By Proposition 3.1 and Lemma 3.3, $\mu_0 = \sum_{r\in\mathcal{R}} c_r \rho_r^{\mathrm{F}}$ for a finite set $\mathcal{R} \subset [1,\infty)$ of distinct radii, and pairing with G_τ gives

$$H_0 = \sum_{r\in\mathcal{R}} c_r\, \Theta_r. \tag{3.2}$$

Recall from (2.10) that the monomials of Θ_t are indexed by the points $(u, v) = (mt, n/t)$ of Λ_t, and that two points give the same monomial $q^h\bar{q}^{\bar{h}}$ exactly when they are related by $(u, v) \mapsto \pm(u, v)$ or $\pm(v, u)$.

Lemma 3.4. *In (3.2), $c_r \in \frac{1}{2}\mathbb{Z}_{\ge 0}$ for every $r \in \mathcal{R}$, and $\sum_{r\in\mathcal{R}} c_r = 1$.*

Proof. Fix $r \in \mathcal{R}$ with $r > 1$ and choose a prime p larger than the squares of all radii in $\mathcal{R}$. Consider the point $(u, v) = (r, p/r) \in \Lambda_r$, of spin p, and let $(h, \bar{h})$ be its weights. A point $(mt, n/t)$ of Λ_t, $t > 0$, has the same weights if and only if it is one of $\pm(r, p/r)$, $\pm(p/r, r)$. In the first case $mn = p$ and $t = r/m$, so $t \in \{r, r/p\}$; in the second $mn = p$ and $t = p/(rm)$, so $t \in \{p/r, 1/r\}$. For $t \in \mathcal{R}$, $t \ne r$, all four values are impossible: $r/p < 1$, $1/r < 1$, and $p/r = t$ would give $p = rt < p$. For $t = r$ the first case contributes the two points $\pm(r, p/r)$, and the second would require $r^2 = p/m \in \{1, p\}$, which is excluded. Hence the coefficient of $q^h\bar{q}^{\bar{h}}$ in (3.2) is exactly $2c_r$. Since the spin p exceeds one, no vacuum monomial contributes at these weights, so $2c_r = d_{h,\bar{h}} \in \mathbb{Z}_{\ge 0}$.

If $1 \in \mathcal{R}$, choose a prime p larger than every radius in $\mathcal{R}$ and consider $(u, v) = (p, p) \in \Lambda_1$, of weights $(p^2, 0)$. A point $(mt, n/t)$ with these weights is $\pm(p, p)$, so $mn = p^2$ and $t = p/m \in \{p, 1, 1/p\}$; only $t = 1$ lies in $\mathcal{R}$, and it contributes two points. Thus $2c_1 = d_{p^2,0} \in \mathbb{Z}_{\ge 0}$, because $(p^2, 0) \ne (1, 0)$.

Finally the coefficient of $q^0\bar{q}^0$ is one in H_0 and in every Θ_r, so $\sum_r c_r = 1$. □

3.3 The classification

For $r \ge 1$ put

$$m_r = \mathbf{1}_{\{2r\in\mathbb{Z}\}} + \mathbf{1}_{\{r\in\{1,2\}\}} \in \{0, 1, 2\}. \tag{3.3}$$

Theorem 3.5 (Classification of the limits)**.** *Let $0 < g \le \frac{1}{2}$.*

(i) Every boundary partition function of gap g, and in particular every limit in Theorem 2.9, has the form

$$Z_0 = \tfrac{1}{2}(Z_R + Z_t), \qquad 1 \le R, t \le (2g)^{-1/2}, \qquad 2R \in \mathbb{Z} \ \text{ or } \ 2t \in \mathbb{Z}, \tag{3.4}$$

where $R = t$ is allowed. Equivalently $\rho_0 = \sigma(R, t) \in \mathcal{B}_g$.

(ii) Conversely, for every pair (R,t) as in (3.4) the measure $\sigma(R,t)$ is positive, each of its weights $(h,\bar{h})$ has a nonnegative integer multiplicity, all its states have integer spin and dimension at least $1/(2\max(R,t)^2) \geq g$, and $\frac{1}{2}(Z_R + Z_t)$ is modular invariant.

Proof. (i) By Lemma 3.4 either one radius occurs with weight one or two radii occur with weight $\frac{1}{2}$, so $H_0 = \frac{1}{2}(\Theta_R + \Theta_t)$ with $R, t \geq 1$, possibly equal. The long part of H_0 is

$$\tfrac{1}{2}(\Theta_R + \Theta_t) - (1-q)(1-\bar{q}) = \tfrac{1}{2}(\Theta_R + \Theta_t) - 1 + q + \bar{q} - q\bar{q}, \tag{3.5}$$

and all its coefficients must be nonnegative. Only the coefficient at $(h,\bar{h}) = (1,1)$ can fail to be. The points of Λ_r with weights $(1,1)$ are $(u,v) = (\pm 2, 0)$ and $(0, \pm 2)$, which lie in Λ_r when $2/r \in \mathbb{Z}$, respectively $2r \in \mathbb{Z}$. For $r \geq 1$ the first condition holds only for $r \in \{1,2\}$, so the coefficient of $q\bar{q}$ in Θ_r is $2m_r$, and the requirement is

$$m_R + m_t \geq 1,$$

that is, $2R \in \mathbb{Z}$ or $2t \in \mathbb{Z}$. The lowest nonzero dimension in Θ_r, $r \geq 1$, is $1/(2r^2)$, attained at $(u,v) = (0, \pm 1/r)$. The corresponding scalar monomial has weights different from $(1,1)$, so it survives in (3.5), and the gap condition gives $\max(R,t) \leq (2g)^{-1/2}$.

(ii) Every coefficient of Θ_r other than the constant term is even, because $\lambda \mapsto -\lambda$ acts freely on $\Lambda_r \setminus \{0\}$ and preserves the weights. Hence $\frac{1}{2}(\Theta_R + \Theta_t)$ has integer coefficients and constant term one. In (3.5) the constant term cancels, the coefficients at $(1,0)$ and $(0,1)$ are increased by one, and the coefficient at $(1,1)$ is $m_R + m_t - 1 \geq 0$. Thus all coefficients are nonnegative integers. The remaining assertions follow from the properties of Θ_r already used. □

Remark 3.6. The average of two different circle partition functions, for example $\frac{1}{2}(Z_1 + Z_{\sqrt{2}})$ with gap $\frac{1}{4}$, is not the partition function of a conformal field theory: it is excluded in [2, Section 4.1] by an argument that uses current Ward identities and nondegeneracy of two-point functions. Neither is available for a limit of partition functions, so these averages must be retained here.

We collect the elementary properties of the measures $\sigma(R,t)$ that are used repeatedly.

Lemma 3.7. *Let $\sigma(R,t) \in \mathcal{B}_g$ and $\mu_0 = \frac{1}{2}\rho_R^{\mathrm{F}} + \frac{1}{2}\rho_t^{\mathrm{F}}$.*

(i) $\mathbb{S}\mu_0 = \mu_0$, $\mathcal{C}\mu_0 = \mu_0$, and $\operatorname{supp}\mu_0 \subset \Lambda_R \cup \Lambda_{1/R} \cup \Lambda_t \cup \Lambda_{1/t}$.

(ii) ν_{vac} is supported on the nine points with $P, Q \in \{0, \pm 1\}$, which lie in Λ_1; in light-cone coordinates they are $(0,0)$, $(\pm 1, \pm 1)$, $(\pm 2, 0)$ and $(0, \pm 2)$.

(iii) Every cluster of $\sigma(R,t)$ consists of at most four states.

(iv) The scalar cluster of $\sigma(R,t)$ at dimension two consists of $m_R + m_t - 1$ states, with m_r as in (3.3); the Fock measure μ_0 has $m_R + m_t$ unit states there.

(v) No state of $\sigma(R,t)$ has dimension below $1/(2\max(R,t)^2)$, and for every E the number of its states of dimension at most E is at most $A_g \max(E,1)$.

Proof. (i) follows from (2.11) and (ii) from (2.3). For (iii), the coefficient of a monomial in Θ_r is the number of points of Λ_r among the at most four points $\pm(u,v)$, $\pm(v,u)$, hence at most four, and so is the coefficient in the average; the vacuum subtraction increases a coefficient only at $(1,0)$ and $(0,1)$, where the coefficient of Θ_r is $2 \cdot \mathbf{1}_{\{r=1\}}$ and the number of states is at most three. (iv) was shown in the proof of Theorem 3.5, and (v) there and in Lemma 2.8. □

3.4 A first consequence: emergent near-currents

Although it is not needed in the sequel, the classification already shows that an admissible partition function with c close to one must contain spin-one states close to the unitarity bound. For $r \geq 1$ put $D_r = \frac{1}{2}(r^2 + r^{-2})$ and let $B_g = g + \frac{1}{4g} = D_{(2g)^{-1/2}}$.

Corollary 3.8. *Fix $0 < g \leq \frac{1}{2}$, $b > B_g$ and $\delta > 0$. For all sufficiently small $a > 0$, every admissible partition function of gap g at $c = 1 + 12a$ has exactly three pairs of states of spin ± 1 with $\Delta \leq b$, and at least one of them has $1 < \Delta < 1 + \delta$.*

Proof. In Θ_r the monomials of spin $+1$ come from the two points $(u, v) = \pm(r, 1/r)$ of Λ_r, at dimension D_r; with weight $\frac{1}{2}$ they contribute one state to $\sigma(R, t)$, and the term q of (3.5) contributes one state at dimension one. Spin -1 is symmetric. Hence the spin-one states of $\sigma(R, t) \in \mathcal{B}_g$ have dimensions 1, D_R, D_t, all in $[1, B_g]$. If the statement failed along a sequence $a_n \downarrow 0$, Theorems 2.9 and 3.5 and Corollary 2.11, applied to $U = \{|X|^2 < b'\}$ and $U = \{|X|^2 < 1 + \delta'\}$ with $b' \geq b$ and $0 < \delta' \leq \delta$ chosen so that b' and $1 + \delta'$ are not dimensions of ρ_0, would give a contradiction. The inequality $\Delta > 1$ is part of Definition 2.1. □

4 Rigidity of lattice-supported modular distributions and the localization obstruction

Expanding the spectrum of an admissible partition function to finite order about points of circle lattices produces a distribution on momentum space with discrete support. This section determines the distributions of this kind that are invariant under $\mathbb{S}$ and $\mathcal{C}$, and derives a quantitative consequence. The results are of three kinds.

(a) A tempered distribution that is supported on a finite union of circle lattices and invariant under $\mathbb{S}$ and $\mathcal{C}$ is a finite sum of polynomials in the boost generator $\mathcal{K}$ applied to circle combs (Theorems 4.7, 4.9 and 4.11). In particular its local expansion at the origin is a multiple of δ_0, and in the even sector it is a combination of radius derivatives of the circle Fock measures ρ_r^{F}.

(b) The statement about the origin has a finite quantitative form: finitely many Gaussian-mixture test functions bound every derivative coefficient at the origin of a $\mathcal{C}$-invariant distribution by its $\mathbb{S}$-defect (Theorem 4.19), with at most a finite-power loss when one radius moves (Proposition 4.20).

(c) The shifted vacuum of an admissible partition function contributes $-\frac{a}{4}(\partial_P^2 + \partial_Q^2)\delta_0$ at the origin. Consequently the low spectrum cannot be localized, with a power-law error in a logarithmic window, on a fixed finite union of circle lattices together with one moving pair (Theorems 4.22 and 4.23).

4.1 Distributions with discrete support

In this section T denotes a distribution on $\mathbb{R}^2$. Multi-indices refer to the momentum coordinates, $\partial^\alpha = \partial_P^{\alpha_1} \partial_Q^{\alpha_2}$, and $\mathcal{A}_q = \{\alpha \in \mathbb{Z}^2_{\geq 0} : |\alpha| \leq q\}$.

Local coefficients. Let $S \subset \mathbb{R}^2$ be closed and discrete. A distribution T with $\operatorname{supp} T \subset S$ has a unique expansion

$$T = \sum_{x \in S} \sum_{\alpha} c_\alpha(x)\, \partial^\alpha \delta_x, \tag{4.1}$$

in which, for each x, only finitely many $c_\alpha(x)$ are nonzero; conversely every such array of complex numbers defines a distribution, the sum being locally finite. We say that T has *local*

order at most q if $c_\alpha(x) = 0$ whenever $|\alpha| > q$. The numbers $c_\alpha(0)$ with $|\alpha| \geq 1$ are the *positive-order origin coefficients* of T. A tempered distribution has finite order, so a tempered distribution with support in S has local order at most q for some $q < \infty$. A function *vanishes to order at least $q+1$* at a point if all its derivatives of order at most q vanish there.

Lemma 4.1. *Let T be a tempered distribution with support in a closed discrete set S and of local order at most q, with coefficients as in (4.1).*

(i) Let $F_0 \subset S$ be finite and let ψ be a Schwartz function that vanishes to order at least $q+1$ at every point of $S \setminus F_0$. Then

$$\langle T, \psi \rangle = \sum_{x \in F_0} \sum_{|\alpha| \leq q} (-1)^{|\alpha|} c_\alpha(x)\, \partial^\alpha \psi(x).$$

(ii) Let ω be a smooth function, all of whose derivatives are polynomially bounded, that vanishes to order at least $q+1$ at every point of S. Then $\omega T = 0$.

Proof. (i) Let $\chi \in C_c^\infty(\mathbb{R}^2)$ equal one near the origin and put $\chi_\varrho(X) = \chi(X/\varrho)$. Then $\chi_\varrho \psi \to \psi$ in the Schwartz space as $\varrho \to \infty$, so $\langle T, \psi \rangle = \lim_\varrho \langle T, \chi_\varrho \psi \rangle$. The support of $\chi_\varrho \psi$ meets finitely many points of S, and by (4.1)

$$\langle T, \chi_\varrho \psi \rangle = \sum_{x \in S} \sum_{|\alpha| \leq q} (-1)^{|\alpha|} c_\alpha(x)\, \partial^\alpha (\chi_\varrho \psi)(x).$$

By the Leibniz rule the derivatives of order at most q of $\chi_\varrho \psi$ vanish at every $x \in S \setminus F_0$, and for large ϱ they agree with those of ψ at the points of F_0.

(ii) For a Schwartz function f, the function ωf is a Schwartz function vanishing to order at least $q+1$ on S, so $\langle \omega T, f \rangle = \langle T, \omega f \rangle = 0$ by (i) with $F_0 = \emptyset$. □

Translations, characters, reflections. For $w \in \mathbb{R}^2$ let $(\tau_w f)(X) = f(X + w)$ and $\chi_w(X) = e^{2\pi i B(w,X)}$; on distributions, $\langle \tau_w T, f \rangle = \langle T, \tau_{-w} f \rangle$, so that $\tau_w \delta_x = \delta_{x-w}$. From (2.5), $\mathbb{S}(\chi_w f) = \tau_w \mathbb{S} f$ and $\mathbb{S}(\tau_{-w} f) = \chi_w \mathbb{S} f$ for Schwartz functions, and by duality $\mathbb{S}(\chi_w T) = \tau_w \mathbb{S} T$ for tempered T. Hence, for vectors w_i and integers $m_i \geq 0$,

$$\prod_i (\tau_{w_i} - I)^{m_i} = \mathbb{S} \circ \Big(\prod_i (\chi_{w_i} - 1)^{m_i} \Big) \circ \mathbb{S}^{-1} \tag{4.2}$$

on tempered distributions, the middle factor acting by multiplication. Let $\check{T}$ denote the image of T under $X \mapsto -X$. Since $\mathbb{S}^2 f = \check{f}$, the distribution $\mathbb{S}^{-1} T$ is the reflection of $\mathbb{S}T$; it has the same local orders as $\mathbb{S}T$, and its support is the reflection of that of $\mathbb{S}T$. If T is even, then $\mathbb{S}^{-1} T = \mathbb{S}T$, and $\langle T, f \rangle = \langle T, f^{\mathrm{ev}} \rangle$ for every test function f.

Lattices, grids and combs. For a full-rank lattice $\Gamma \subset \mathbb{R}^2$ the dual lattice with respect to the Lorentzian pairing is

$$\Gamma^\vee = \{ w \in \mathbb{R}^2 : B(w, \lambda) \in \mathbb{Z} \text{ for all } \lambda \in \Gamma \}, \qquad (\Gamma^\vee)^\vee = \Gamma.$$

If $w \in \Gamma^\vee$ then $\chi_w - 1$ vanishes on Γ. We call Γ a *grid* if $\Gamma^\vee \subset \Gamma$. The quotient $\Gamma / \Gamma^\vee$ is then finite, and for $\gamma \in \Gamma / \Gamma^\vee$ we define the *coset comb* and the space of comb measures

$$e_\gamma = \sum_{\lambda \in \gamma + \Gamma^\vee} \delta_\lambda, \qquad \mathcal{E}(\Gamma) = \operatorname{span}\{ e_\gamma : \gamma \in \Gamma / \Gamma^\vee \}.$$

Two families of grids occur. The *rational momentum grid* $(2k)^{-1}\mathbb{Z}^2$, in the coordinates (P,Q) and for an integer $k \geq 1$, has dual $k\mathbb{Z}^2$. The *hyperbolic grid*

$$d^{-1}\Lambda_r = \Big\{(u,v) = \Big(\frac{mr}{d}, \frac{n}{dr}\Big) : m,n \in \mathbb{Z}\Big\}, \qquad d \geq 1 \text{ an integer},\ r > 0,$$

has dual $d\Lambda_r$, which consists of the points with $m,n \in d^2\mathbb{Z}$; its spin function is $s = mn/d^2$. For $d = 1$ the grid is the self-dual lattice Λ_r and its only comb is e_r.

Poisson summation takes the following form. Let $(\mathsf{b}_1, \mathsf{b}_2)$ be a basis of $\Gamma^\vee$ and $(\mathsf{b}_1^*, \mathsf{b}_2^*)$ the basis of Γ with $B(\mathsf{b}_i, \mathsf{b}_k^*) = \mathbf{1}_{i=k}$. For a Schwartz function f and $\gamma \in \Gamma$ put $\phi(x) = f(x_1\mathsf{b}_1^* + x_2\mathsf{b}_2^*)\, e^{2\pi i B(\gamma, x_1\mathsf{b}_1^* + x_2\mathsf{b}_2^*)}$, $x \in \mathbb{R}^2$. The substitution $X' = x_1\mathsf{b}_1^* + x_2\mathsf{b}_2^*$ in (2.5) gives $\mathbb{S}f(\gamma + n_1\mathsf{b}_1 + n_2\mathsf{b}_2) = c_\Gamma \int \phi(x) e^{2\pi i n\cdot x}\, \mathrm{d}x$, where c_Γ is the absolute value of the determinant of $(\mathsf{b}_1^*, \mathsf{b}_2^*)$ in light-cone coordinates, and the classical Poisson formula $\sum_{n\in\mathbb{Z}^2} \int \phi(x) e^{2\pi i n\cdot x}\, \mathrm{d}x = \sum_{n\in\mathbb{Z}^2} \phi(n)$ yields $\langle \mathbb{S}e_\gamma, f\rangle = c_\Gamma \sum_{\mu\in\Gamma} e^{2\pi i B(\gamma,\mu)} f(\mu)$. The phase depends only on the classes of γ and μ modulo $\Gamma^\vee$, so

$$\mathbb{S}e_\gamma = c_\Gamma \sum_{\delta \in \Gamma/\Gamma^\vee} e^{2\pi i B(\gamma,\delta)}\, e_\delta, \qquad c_\Gamma = [\Gamma : \Gamma^\vee]^{-1/2}. \tag{4.3}$$

The value of c_Γ follows because the matrix of B in light-cone coordinates has determinant -1, so that the covolumes of Γ and $\Gamma^\vee$ are reciprocal. For $\Gamma = d^{-1}\Lambda_r$ one has $c_\Gamma = d^{-2}$, and (4.3) with $d = 1$ is the first identity in (2.11). In particular $\mathcal{E}(\Gamma)$ is mapped into itself by $\mathbb{S}$, and by $\mathbb{S}^{-1}$ because $\check{e}_\gamma = e_{-\gamma}$.

Polynomial differential operators. Let $\mathcal{W}$ be the algebra of linear differential operators on $\mathbb{R}^2$ with polynomial coefficients. It acts on distributions and preserves temperedness. Every $\mathsf{A} \in \mathcal{W}$ has a unique *normal form* $\mathsf{A} = \sum_\alpha \partial^\alpha \circ p_\alpha$ with polynomials p_α, almost all zero, and for a closed discrete set S

$$\mathsf{A} \sum_{\lambda\in S} \delta_\lambda = \sum_\alpha \sum_{\lambda\in S} p_\alpha(\lambda)\, \partial^\alpha \delta_\lambda. \tag{4.4}$$

We use the generators

$$\mathsf{x}_1 = P, \qquad \mathsf{x}_2 = Q, \qquad \mathsf{y}_1 = \frac{\partial_P}{4\pi i}, \qquad \mathsf{y}_2 = -\frac{\partial_Q}{4\pi i}, \tag{4.5}$$

whose only nonzero commutators are

$$[\mathsf{y}_1, \mathsf{x}_1] = \frac{1}{4\pi i}, \qquad [\mathsf{y}_2, \mathsf{x}_2] = -\frac{1}{4\pi i}. \tag{4.6}$$

For $\mathsf{A} \in \mathcal{W}$ and an integer ν put $\sigma(\mathsf{A}) = \mathbb{S}\mathsf{A}\mathbb{S}^{-1}$ and $\kappa_\nu(\mathsf{A}) = \mathcal{C}^\nu \mathsf{A} \mathcal{C}^{-\nu}$. Differentiating (2.5) under the integral sign and integrating by parts gives, for Schwartz functions, $\mathbb{S}(Pf) = (4\pi i)^{-1}\partial_P \mathbb{S}f$, $\mathbb{S}(\partial_P f) = -4\pi i P\, \mathbb{S}f$, $\mathbb{S}(Qf) = -(4\pi i)^{-1}\partial_Q \mathbb{S}f$ and $\mathbb{S}(\partial_Q f) = 4\pi i Q\, \mathbb{S}f$; these identities extend to tempered distributions by duality. Together with $\mathcal{C}^\nu \partial_P \mathcal{C}^{-\nu} = \partial_P - 4\pi i \nu P$ and $\mathcal{C}^\nu \partial_Q \mathcal{C}^{-\nu} = \partial_Q + 4\pi i \nu Q$ they show that σ and κ_ν are automorphisms of $\mathcal{W}$, given on the generators by

$$\sigma : (\mathsf{x}_j, \mathsf{y}_j) \mapsto (\mathsf{y}_j, -\mathsf{x}_j), \qquad \kappa_\nu : (\mathsf{x}_j, \mathsf{y}_j) \mapsto (\mathsf{x}_j,\ \mathsf{y}_j - \nu \mathsf{x}_j), \qquad j = 1, 2. \tag{4.7}$$

The boost generator (2.6) is $\mathcal{K} = -4\pi i\, (\mathsf{x}_1\mathsf{y}_2 - \mathsf{x}_2\mathsf{y}_1)$. Since $\mathsf{x}_1, \mathsf{y}_1$ commute with $\mathsf{x}_2, \mathsf{y}_2$, the element $\mathsf{x}_1\mathsf{y}_2 - \mathsf{x}_2\mathsf{y}_1$ is fixed by both substitutions in (4.7). Therefore

$$\mathbb{S}\mathcal{K} = \mathcal{K}\mathbb{S}, \qquad \mathcal{C}\mathcal{K} = \mathcal{K}\mathcal{C}, \qquad \mathcal{K}\delta_0 = 0, \tag{4.8}$$

the last identity because $P\partial_Q \delta_0 = \partial_Q(P\delta_0) = 0$ and likewise for $Q\partial_P \delta_0$. Thus $p(\mathcal{K})\delta_0 = p(0)\delta_0$ for every polynomial p. Away from the origin the opposite holds. For $x = (P_x, Q_x)$ the identity

$f\,\partial^\alpha\delta_x = \sum_{\beta\le\alpha}(-1)^{|\alpha-\beta|}\binom{\alpha}{\beta}(\partial^{\alpha-\beta}f)(x)\,\partial^\beta\delta_x$ shows that $\mathcal{K}\partial^\alpha\delta_x = (Q_x\partial_P + P_x\partial_Q)\partial^\alpha\delta_x$ plus terms of order at most $|\alpha|$, and by induction

$$p(\mathcal{K})\delta_x = \hat{p}\,(Q_x\partial_P + P_x\partial_Q)^j\delta_x + (\text{terms of order} < j) \tag{4.9}$$

for a polynomial p of degree j with leading coefficient $\hat{p}$. If $x \neq 0$ this distribution is nonzero and has local order exactly j. In particular the powers of $\mathcal{K}$ are linearly independent in $\mathcal{W}$.

4.2 Polynomial comb representations

Lemma 4.2 (Two difference equations modulo finite support)**.** *Let $c : \mathbb{Z}^2 \to \mathbb{C}$ be an arbitrary function and $m \ge 1$ an integer. Let $\mathsf{D}_1, \mathsf{D}_2$ be the forward differences in the two coordinate directions, $(\mathsf{D}_1 c)(n_1, n_2) = c(n_1+1, n_2) - c(n_1, n_2)$ and similarly for D_2. If $\mathsf{D}_1^m c$ and $\mathsf{D}_2^m c$ have finite support, then $c = p + c_0$, where c_0 has finite support and p is the restriction to $\mathbb{Z}^2$ of a polynomial of degree at most $m-1$ in each variable separately. No growth condition on c is assumed.*

Proof. Identify a finitely supported function f on $\mathbb{Z}^2$ with the Laurent polynomial $\sum_n f(n) z_1^{n_1} z_2^{n_2}$. Under this identification D_i becomes multiplication by $z_i^{-1} - 1$. Put $f_i = \mathsf{D}_i^m c$ and $\pi_i = (z_i^{-1} - 1)^m$. Since D_1 and D_2 commute, $\mathsf{D}_2^m f_1 = \mathsf{D}_1^m f_2$, that is, $\pi_2 f_1 = \pi_1 f_2$ in the ring $\mathbb{C}[z_1^{\pm1}, z_2^{\pm1}]$. This ring is a unique factorization domain in which $1 - z_1$ and $1 - z_2$ are non-associate primes, so π_1 divides f_1: there is a Laurent polynomial c_0 with $f_1 = \pi_1 c_0$. Cancelling π_1 in $\pi_2\pi_1 c_0 = \pi_1 f_2$ gives $f_2 = \pi_2 c_0$. Regarding c_0 as a finitely supported function, we obtain $\mathsf{D}_1^m(c - c_0) = \mathsf{D}_2^m(c - c_0) = 0$.

Put $p = c - c_0$. For fixed n_2, the function $n_1 \mapsto p(n_1, n_2)$ on $\mathbb{Z}$ has vanishing m-th difference. Such a function is determined, in both directions, by its values at $0, \dots, m-1$, and the Newton polynomial $\sum_{k<m}\binom{n_1}{k}(\mathsf{D}_1^k p)(0, n_2)$ has the same values and also vanishing m-th difference; hence $p(n_1, n_2) = \sum_{k<m}\binom{n_1}{k}(\mathsf{D}_1^k p)(0, n_2)$ for all $n_1 \in \mathbb{Z}$. Each coefficient $n_2 \mapsto (\mathsf{D}_1^k p)(0, n_2)$ has vanishing m-th difference in n_2 and is, by the same argument, a polynomial of degree at most $m-1$ in n_2. □

The corresponding statement in one variable is false: the indicator function of $\mathbb{Z}_{\ge 0}$ has a finitely supported first difference. Two independent directions are essential.

Lemma 4.3 (Differences along one class)**.** *Let $(\Gamma_i)_{i\in\mathsf{I}}$ be finitely many full-rank lattices in $\mathbb{R}^2$ with $\Gamma_i \cap \Gamma_{i'} = \{0\}$ for $i \neq i'$, let $S = \bigcup_i \Gamma_i$, let $m \ge 1$ be an integer and $j \in \mathsf{I}$.*

(i) For any subgroups $G_i \subset \Gamma_i$ of finite index, $i \neq j$, there are vectors $w_i \in G_i$ such that

$$\sum_{i\neq j} k_i w_i \notin \Gamma_j \qquad \text{whenever } 0 \le k_i \le m \text{ and not all } k_i \text{ vanish.} \tag{4.10}$$

In particular $w_i \neq 0$.

(ii) Let $w_i \in \Gamma_i$, $i \neq j$, satisfy (4.10), let $w \in \Gamma_j$, and put $\mathsf{D} = (\tau_w - I)^m \prod_{i\neq j}(\tau_{w_i} - I)^m$. There is a finite set $F_0 \subset \Gamma_j$ with the following property. If T is any distribution of the form (4.1) on S, with arbitrary coefficients, and $\mathsf{D}T = 0$, then for every α

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}c_\alpha(y + kw) = 0 \qquad \text{for all } y \in \Gamma_j \setminus F_0. \tag{4.11}$$

If $\mathsf{I} = \{j\}$, then $\mathsf{D} = (\tau_w - I)^m$ and $F_0 = \emptyset$.

Proof. (i) We choose the vectors one at a time. Suppose that w_i has been chosen for i in a subset $\mathsf{I}' \subset \mathsf{I} \setminus \{j\}$, in such a way that (4.10) holds for all tuples with $k_i = 0$ for $i \notin \mathsf{I}'$, and let $i_0 \in \mathsf{I} \setminus (\mathsf{I}' \cup \{j\})$. The additional conditions for $\mathsf{I}' \cup \{i_0\}$ are those with $k_{i_0} \geq 1$ and $k_i = 0$ for $i \notin \mathsf{I}' \cup \{i_0\}$. Each of these finitely many conditions is violated by at most one vector of G_{i_0}: if w and w' both violate it, then $k_{i_0}(w - w') \in \Gamma_j \cap \Gamma_{i_0} = \{0\}$. Since G_{i_0} is infinite, an admissible w_{i_0} exists. The condition with $k_i = 1$ and all other coefficients zero says $w_i \notin \Gamma_j$, so $w_i \neq 0$.

(ii) Extend c_α by zero outside S. Expanding the product, $\mathsf{D} = \sum_{\mathbf{k}} \varkappa_{\mathbf{k}} \tau_{\mathsf{w}_{\mathbf{k}}}$, where $\mathbf{k} = (k_i)_{i \in \mathsf{I}}$ runs over $\{0, \dots, m\}^{\mathsf{I}}$, $\varkappa_{\mathbf{k}} = \prod_i (-1)^{m-k_i} \binom{m}{k_i}$ and $\mathsf{w}_{\mathbf{k}} = k_j w + \sum_{i \neq j} k_i w_i$. Since $\tau_{w'} \delta_x = \delta_{x - w'}$, the coefficient of $\partial^\alpha \delta_y$ in $\mathsf{D}T$ is $\sum_{\mathbf{k}} \varkappa_{\mathbf{k}} \, c_\alpha(y + \mathsf{w}_{\mathbf{k}})$. Let $y \in \Gamma_j$. If $k_i = 0$ for all $i \neq j$, then $\mathsf{w}_{\mathbf{k}} = k_j w$, and these terms add up to $(-1)^{m(|\mathsf{I}|-1)}$ times the left side of (4.11). For every other $\mathbf{k}$ we have $y + \mathsf{w}_{\mathbf{k}} \notin \Gamma_j$ by (4.10), so $c_\alpha(y + \mathsf{w}_{\mathbf{k}})$ can be nonzero only if $y + \mathsf{w}_{\mathbf{k}} \in \Gamma_i$ for some $i \neq j$. For fixed $\mathbf{k}$ and i at most one $y \in \Gamma_j$ has this property, because the difference of two such points lies in $\Gamma_j \cap \Gamma_i$. Let F_0 be the finite set of all $y \in \Gamma_j$ obtained in this way. For $y \in \Gamma_j \setminus F_0$ the vanishing of the coefficient of $\partial^\alpha \delta_y$ in $\mathsf{D}T$ is (4.11). □

Lemma 4.4 (Polynomial combs for transverse grids)**.** *Let $(\Gamma_i)_{i \in \mathsf{I}}$ be finitely many grids in $\mathbb{R}^2$ with $\Gamma_i \cap \Gamma_{i'} = \{0\}$ for $i \neq i'$, and let $e_{i,\gamma}$, $\gamma \in \Gamma_i / \Gamma_i^\vee$, be the coset combs of Γ_i.*

(i) Faithfulness. *If operators $\mathsf{A}_{i,\gamma} \in \mathcal{W}$ satisfy $\sum_{i,\gamma} \mathsf{A}_{i,\gamma} e_{i,\gamma} = 0$, then $\mathsf{A}_{i,\gamma} = 0$ for all i and γ.*

(ii) Representation. *If T is a tempered distribution such that T and $\mathbb{S}T$ are supported in $\bigcup_i \Gamma_i$, then there are operators $\mathsf{A}_{i,\gamma} \in \mathcal{W}$, unique by the first part, with*

$$T = \sum_{i \in \mathsf{I}} \sum_{\gamma \in \Gamma_i / \Gamma_i^\vee} \mathsf{A}_{i,\gamma} \, e_{i,\gamma}. \tag{4.12}$$

Proof. (i) Write $\mathsf{A}_{i,\gamma} = \sum_\alpha \partial^\alpha \circ p_{i,\gamma,\alpha}$ in normal form. A point $\lambda \neq 0$ of $\bigcup_i \Gamma_i$ lies in exactly one Γ_i and in exactly one coset $\gamma + \Gamma_i^\vee$, and by (4.4) the coefficient of $\partial^\alpha \delta_\lambda$ in the sum is $p_{i,\gamma,\alpha}(\lambda)$. Hence each $p_{i,\gamma,\alpha}$ vanishes on the coset $\gamma + \Gamma_i^\vee$ with at most the origin removed. In lattice coordinates (n_1, n_2) on this coset, the polynomial vanishes identically on every line $n_2 = \text{const}$ that does not contain the removed point, hence on infinitely many parallel lines, hence identically.

(ii) Let $S = \bigcup_i \Gamma_i$ and let m be a common bound for the local orders of T and $\mathbb{S}T$; write T as in (4.1). Fix $j \in \mathsf{I}$ and a basis $(\mathsf{b}_1, \mathsf{b}_2)$ of $\Gamma_j^\vee$. For $\iota \in \{1, 2\}$ choose vectors $w_i^{(\iota)} \in \Gamma_i^\vee$, $i \neq j$, as in Lemma 4.3(i) with $m + 1$ in place of m, and put

$$\mathsf{D}^{(\iota)} = (\tau_{\mathsf{b}_\iota} - I)^{m+1} \prod_{i \neq j} (\tau_{w_i^{(\iota)}} - I)^{m+1}, \qquad \omega^{(\iota)} = (\chi_{\mathsf{b}_\iota} - 1)^{m+1} \prod_{i \neq j} (\chi_{w_i^{(\iota)}} - 1)^{m+1}.$$

Every point of S lies in some Γ_i, on which the i-th factor of $\omega^{(\iota)}$ vanishes to order $m+1$, because the vector of that factor lies in $\Gamma_i^\vee$. The distribution $\mathbb{S}^{-1}T$ is supported in S and has local order at most m, so $\omega^{(\iota)} \mathbb{S}^{-1} T = 0$ by Lemma 4.1(ii), and $\mathsf{D}^{(\iota)} T = 0$ by (4.2). Lemma 4.3(ii) shows that, for every α and every coset $\gamma + \Gamma_j^\vee$, the function $(n_1, n_2) \mapsto c_\alpha(\gamma + n_1 \mathsf{b}_1 + n_2 \mathsf{b}_2)$ on $\mathbb{Z}^2$ has finitely supported differences of order $m + 1$ in both directions. By Lemma 4.2 it differs from a polynomial in (n_1, n_2) at finitely many points only. Thus there are polynomials $p_{j,\gamma,\alpha}$ on $\mathbb{R}^2$, zero for $|\alpha| > m$, with $c_\alpha(\lambda) = p_{j,\gamma,\alpha}(\lambda)$ for all but finitely many $\lambda \in \gamma + \Gamma_j^\vee$.

Carry this out for every j, put $\mathsf{A}_{j,\gamma} = \sum_\alpha \partial^\alpha \circ p_{j,\gamma,\alpha}$ and $F = T - \sum_{j,\gamma} \mathsf{A}_{j,\gamma} e_{j,\gamma}$. By (4.4) and the previous paragraph, F has finite support. Each $\mathsf{A}_{j,\gamma} e_{j,\gamma}$ is tempered, and $\mathbb{S}(\mathsf{A}_{j,\gamma} e_{j,\gamma}) = \sigma(\mathsf{A}_{j,\gamma}) \, \mathbb{S} e_{j,\gamma}$ is supported in Γ_j by (4.3). Hence $\mathbb{S}F$ is supported in the closed discrete set S. But F is a finite combination of derivatives of point masses, so $\mathbb{S}F$ is a real-analytic function, a finite sum of polynomials times characters. A continuous function supported in a discrete set vanishes; hence $\mathbb{S}F = 0$ and $F = 0$. □

Lemma 4.5 (Operators invariant under two shears)**.** *Let* $\mathsf{A} \in \mathcal{W}$ *and let* ν *be a nonzero integer. If* A *is invariant under conjugation by* $\mathcal{C}^\nu$ *and by* $\mathbb{S}\mathcal{C}^\nu\mathbb{S}^{-1}$*, that is,*

$$\kappa_\nu(\mathsf{A}) = \mathsf{A} \qquad \text{and} \qquad \sigma\kappa_\nu\sigma^{-1}(\mathsf{A}) = \mathsf{A},$$

then $\mathsf{A} = p(\mathcal{K})$ *for a polynomial* p*.*

Proof. For a matrix $\mathsf{g} = \left(\begin{smallmatrix} \mathsf{g}_{11} & \mathsf{g}_{12} \\ \mathsf{g}_{21} & \mathsf{g}_{22} \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$ the substitution

$$\Phi_\mathsf{g} : \ (\mathsf{x}_j, \mathsf{y}_j) \mapsto (\mathsf{g}_{11}\mathsf{x}_j + \mathsf{g}_{12}\mathsf{y}_j, \ \mathsf{g}_{21}\mathsf{x}_j + \mathsf{g}_{22}\mathsf{y}_j), \qquad j = 1, 2,$$

preserves the relations (4.6), because $[\mathsf{g}_{21}\mathsf{x}_j + \mathsf{g}_{22}\mathsf{y}_j, \ \mathsf{g}_{11}\mathsf{x}_j + \mathsf{g}_{12}\mathsf{y}_j] = \det \mathsf{g}\,[\mathsf{y}_j, \mathsf{x}_j]$ and the generators with different indices commute. Hence Φ_g is an automorphism of $\mathcal{W}$, and $\Phi_\mathsf{g}\Phi_{\mathsf{g}'} = \Phi_{\mathsf{g}'\mathsf{g}}$. By (4.7),

$$\kappa_\nu = \Phi_\mathsf{g} \text{ with } \mathsf{g} = \begin{pmatrix} 1 & 0 \\ -\nu & 1 \end{pmatrix}, \qquad \sigma\kappa_\nu\sigma^{-1} = \Phi_\mathsf{g} \text{ with } \mathsf{g} = \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix};$$

for the second, σ^{-1} maps $(\mathsf{x}_j, \mathsf{y}_j)$ to $(-\mathsf{y}_j, \mathsf{x}_j)$, so $\sigma\kappa_\nu\sigma^{-1}(\mathsf{x}_j) = \sigma(-\mathsf{y}_j + \nu\mathsf{x}_j) = \mathsf{x}_j + \nu\mathsf{y}_j$ and $\sigma\kappa_\nu\sigma^{-1}(\mathsf{y}_j) = \sigma(\mathsf{x}_j) = \mathsf{y}_j$.

Let $\mathcal{W}_n \subset \mathcal{W}$ be the finite-dimensional span of all products of at most n generators; every Φ_g preserves $\mathcal{W}_n$. Choose n with $\mathsf{A} \in \mathcal{W}_n$. For $z \in \mathbb{C}$ let $\alpha_z = \Phi_\mathsf{g}$ with $\mathsf{g} = \left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right)$. Then $\alpha_z\alpha_{z'} = \alpha_{z+z'}$, and $z \mapsto \alpha_z(\mathsf{A})$ is a polynomial map from $\mathbb{C}$ to $\mathcal{W}_n$. By hypothesis $\alpha_{-\nu}(\mathsf{A}) = \mathsf{A}$, hence $\alpha_{k\nu}(\mathsf{A}) = \mathsf{A}$ for all $k \in \mathbb{Z}$, and the polynomial $\alpha_z(\mathsf{A}) - \mathsf{A}$ has infinitely many zeros. Thus $\alpha_z(\mathsf{A}) = \mathsf{A}$ for all $z \in \mathbb{C}$. Likewise, with $\beta_z = \Phi_\mathsf{g}$ for $\mathsf{g} = \left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right)$, the hypothesis $\beta_\nu(\mathsf{A}) = \mathsf{A}$ and the polynomial dependence on z give $\beta_z(\mathsf{A}) = \mathsf{A}$ for all $z \in \mathbb{C}$. The set of g with $\Phi_\mathsf{g}(\mathsf{A}) = \mathsf{A}$ is a subgroup of $\mathrm{SL}_2(\mathbb{C})$ that contains all elementary matrices, which generate $\mathrm{SL}_2(\mathbb{C})$. Therefore $\Phi_\mathsf{g}(\mathsf{A}) = \mathsf{A}$ for every $\mathsf{g} \in \mathrm{SL}_2(\mathbb{C})$.

We conclude by induction on n; the case $n = 0$ is trivial since $\mathcal{W}_0 = \mathbb{C}$. By (4.6) the associated graded algebra of the filtration $(\mathcal{W}_n)$ is the polynomial ring $\mathbb{C}[\mathsf{x}_1, \mathsf{y}_1, \mathsf{x}_2, \mathsf{y}_2]$, and Φ_g acts on it by the same linear substitution. Let $\mathsf{a}^{\mathrm{top}}$ be the class of A in $\mathcal{W}_n/\mathcal{W}_{n-1}$, a homogeneous polynomial of degree n in the two vectors $w_1 = (\mathsf{x}_1, \mathsf{y}_1)$ and $w_2 = (\mathsf{x}_2, \mathsf{y}_2)$ with $\mathsf{a}^{\mathrm{top}}(\mathsf{g}w_1, \mathsf{g}w_2) = \mathsf{a}^{\mathrm{top}}(w_1, w_2)$ for all $\mathsf{g} \in \mathrm{SL}_2(\mathbb{C})$. If the determinant $\mathsf{k} = \det(w_1, w_2) = \mathsf{x}_1\mathsf{y}_2 - \mathsf{x}_2\mathsf{y}_1$ is nonzero, the matrix g with columns w_1/k and w_2 has determinant one and maps $(\mathsf{k}, 0)$ to w_1 and $(0, 1)$ to w_2; hence $\mathsf{a}^{\mathrm{top}}(w_1, w_2) = \mathsf{a}^{\mathrm{top}}((\mathsf{k}, 0), (0, 1))$ is a polynomial in k on the dense set $\{\mathsf{k} \neq 0\}$, and therefore everywhere. By homogeneity $\mathsf{a}^{\mathrm{top}} = \vartheta\,(\mathsf{x}_1\mathsf{y}_2 - \mathsf{x}_2\mathsf{y}_1)^{n/2}$ for a constant ϑ, with $\vartheta = 0$ if n is odd. The element $\mathsf{x}_1\mathsf{y}_2 - \mathsf{x}_2\mathsf{y}_1 = -\mathcal{K}/(4\pi i)$ of $\mathcal{W}$ is fixed by every Φ_g, since $\mathsf{x}_1, \mathsf{y}_1$ commute with $\mathsf{x}_2, \mathsf{y}_2$ and $\det \mathsf{g} = 1$. Consequently $\mathsf{A} - \vartheta(-\mathcal{K}/(4\pi i))^{n/2}$ lies in $\mathcal{W}_{n-1}$ and is fixed by every Φ_g. By the induction hypothesis it is a polynomial in $\mathcal{K}$, and so is A. □

Lemma 4.6 (Combs on one grid)**.** *Let* Γ *be a grid.*

(i) If T *is a tempered distribution such that* T *and* $\mathbb{S}T$ *are supported in* Γ*, then* $T = \sum_{\gamma \in \Gamma/\Gamma^\vee} \mathsf{A}_\gamma e_\gamma$ *with uniquely determined operators* $\mathsf{A}_\gamma \in \mathcal{W}$*.*

(ii) There is an integer $\nu \geq 1$ *such that* $\nu s(\lambda) \in \mathbb{Z}$ *for all* $\lambda \in \Gamma$*, where* $s = P^2 - Q^2$*; one may take* $\nu = 4k^2$ *for* $\Gamma = (2k)^{-1}\mathbb{Z}^2$ *and* $\nu = d^2$ *for* $\Gamma = d^{-1}\Lambda_r$*. For every such* ν*, both* $\mathcal{C}^\nu$ *and* $\mathbb{S}\mathcal{C}^\nu\mathbb{S}^{-1}$ *fix every element of* $\mathcal{E}(\Gamma)$*.*

Proof. Part (i) is Lemma 4.4 for a single grid. For (ii), let n_0 be the exponent of the finite group $\Gamma/\Gamma^\vee$. For $\lambda \in \Gamma$ we have $n_0\lambda \in \Gamma^\vee$, so $2n_0 s(\lambda) = B(n_0\lambda, \lambda) \in \mathbb{Z}$, and $\nu = 2n_0$ has the required property; the two special values are read off from $s = P^2 - Q^2$ and $s = mn/d^2$. If νs is integral on Γ, then $\mathcal{C}^\nu$ fixes every measure carried by Γ, in particular every element of $\mathcal{E}(\Gamma)$. Since $\mathbb{S}^{-1}$ maps $\mathcal{E}(\Gamma)$ into itself by (4.3), $\mathbb{S}\mathcal{C}^\nu\mathbb{S}^{-1}e = \mathbb{S}\mathbb{S}^{-1}e = e$ for $e \in \mathcal{E}(\Gamma)$. □

Theorem 4.7 (Rigidity on a grid)**.** *Let Γ be a grid, for instance a rational momentum grid $(2k)^{-1}\mathbb{Z}^2$ or a hyperbolic grid $d^{-1}\Lambda_r$, and let T be a tempered distribution with* $\operatorname{supp} T \subset \Gamma$ *and*

$$\mathbb{S}T = T, \qquad \mathcal{C}T = T.$$

Then there are unique polynomials p_γ, $\gamma \in \Gamma/\Gamma^\vee$, such that

$$T = \sum_{\gamma \in \Gamma/\Gamma^\vee} p_\gamma(\mathcal{K})\, e_\gamma. \tag{4.13}$$

In particular the local expansion of T at the origin is $p_0(0)\,\delta_0$: all positive-order origin coefficients of T vanish.

Proof. By Lemma 4.6(i), $T = \sum_\gamma \mathsf{A}_\gamma e_\gamma$. Let ν be as in Lemma 4.6(ii). Since $\mathcal{C}^\nu T = T$ and $\mathcal{C}^\nu e_\gamma = e_\gamma$,

$$\sum_\gamma \mathsf{A}_\gamma e_\gamma = \mathcal{C}^\nu T = \sum_\gamma \kappa_\nu(\mathsf{A}_\gamma)\, \mathcal{C}^\nu e_\gamma = \sum_\gamma \kappa_\nu(\mathsf{A}_\gamma)\, e_\gamma,$$

and the uniqueness in Lemma 4.6(i) gives $\kappa_\nu(\mathsf{A}_\gamma) = \mathsf{A}_\gamma$. The operator $\mathbb{S}\mathcal{C}^\nu\mathbb{S}^{-1}$ also fixes T and every e_γ, and the same computation gives $\sigma\kappa_\nu\sigma^{-1}(\mathsf{A}_\gamma) = \mathsf{A}_\gamma$. By Lemma 4.5, $\mathsf{A}_\gamma = p_\gamma(\mathcal{K})$. The polynomials are unique by the uniqueness of the A_γ and the linear independence of the powers of $\mathcal{K}$. Only the coset $\gamma = 0$ contains the origin, near which $e_0 = \delta_0$, and $p_0(\mathcal{K})\delta_0 = p_0(0)\delta_0$ by (4.8). □

Neither positivity nor integrality of the coefficients of T, nor any bound on its local order, enters Theorem 4.7.

4.3 Finite unions of circle lattices

Two radii $r, r' > 0$ are *commensurable* if $r/r' \in \mathbb{Q}$. If they are not, then $\Lambda_r \cap \Lambda_{r'} = \{0\}$, and the same holds for $d^{-1}\Lambda_r$ and $d'^{-1}\Lambda_{r'}$: at a common point the light-cone coordinate u lies in $\mathbb{Q}r \cap \mathbb{Q}r' = \{0\}$ and v in $\mathbb{Q}r^{-1} \cap \mathbb{Q}r'^{-1} = \{0\}$.

Definition 4.8. A *transverse family of hyperbolic grids* is a finite family $\Gamma_i = d_i^{-1}\Lambda_{r_i}$, $i \in \mathsf{I}$, with integers $d_i \geq 1$ and pairwise incommensurable radii $r_i > 0$. Its *support* is $S = \bigcup_i \Gamma_i$, its *phase exponent* is $\nu = \operatorname{lcm}\{d_i^2 : i \in \mathsf{I}\}$, and its combs are the coset combs $e_{i,\gamma}$, $\gamma \in \Gamma_i/\Gamma_i^\vee$. The family is *adapted* to a finite set $\mathcal{R} \subset (0,\infty)$ if $\bigcup_{r\in\mathcal{R}} \Lambda_r \subset S$.

For such a family,

$$\Gamma_i^\vee = d_i\Lambda_{r_i} \subset \Gamma_i, \qquad \Gamma_i \cap \Gamma_{i'} = \{0\} \quad (i \neq i'), \qquad \nu s(\lambda) \in \mathbb{Z} \quad (\lambda \in S), \tag{4.14}$$

so that Lemmas 4.3 and 4.4 apply to it, and $\mathcal{C}^\nu$ fixes every comb $e_{i,\gamma}$. Every finite set $\mathcal{R}$ has an adapted family: choose one representative r_i in each commensurability class of $\mathcal{R}$; if $r = (k/k')r_i$ with positive integers k, k', then $\Lambda_r \subset (kk')^{-1}\Lambda_{r_i}$, and d_i may be taken to be a common multiple of the products kk' occurring in the class.

Theorem 4.9 (No origin jets on a finite union of circle lattices)**.** *Let $(\Gamma_i)_{i\in\mathsf{I}}$ be a transverse family of hyperbolic grids with support S; for instance a family adapted to a finite set of radii $\mathcal{R}$, in which case $S \supset \bigcup_{r\in\mathcal{R}} \Lambda_r$. Let T be a tempered distribution with* $\operatorname{supp} T \subset S$ *and*

$$\mathbb{S}T = T, \qquad \mathcal{C}T = T.$$

Then there are unique polynomials $p_{i,\gamma}$ with

$$T = \sum_{i\in\mathsf{I}} \sum_{\gamma \in \Gamma_i/\Gamma_i^\vee} p_{i,\gamma}(\mathcal{K})\, e_{i,\gamma}. \tag{4.15}$$

In particular, a tempered distribution that is invariant under $\mathbb{S}$ and $\mathcal{C}$ and supported on a finite union of circle lattices has a local expansion at the origin equal to a multiple of δ_0: all its positive-order origin coefficients vanish.

Proof. By (4.14) and Lemma 4.4(ii), $T = \sum_{i,\gamma} \mathsf{A}_{i,\gamma} e_{i,\gamma}$. Let ν be the phase exponent. By Lemma 4.6(ii), applied to each Γ_i with this common ν, the operators $\mathcal{C}^\nu$ and $\mathbb{S}\mathcal{C}^\nu\mathbb{S}^{-1}$ fix every $e_{i,\gamma}$; they also fix T. As in the proof of Theorem 4.7,

$$\sum_{i,\gamma} \mathsf{A}_{i,\gamma} e_{i,\gamma} = \sum_{i,\gamma} \kappa_\nu(\mathsf{A}_{i,\gamma})\, e_{i,\gamma} = \sum_{i,\gamma} \sigma\kappa_\nu\sigma^{-1}(\mathsf{A}_{i,\gamma})\, e_{i,\gamma},$$

and the faithfulness statement of Lemma 4.4(i) shows that each $\mathsf{A}_{i,\gamma}$ is fixed by κ_ν and by $\sigma\kappa_\nu\sigma^{-1}$. Lemma 4.5 gives $\mathsf{A}_{i,\gamma} = p_{i,\gamma}(\mathcal{K})$. The polynomials are unique by Lemma 4.4(i) and the linear independence of the powers of $\mathcal{K}$. Near the origin only the combs $e_{i,0}$ contribute, each equal to δ_0 there, and $p_{i,0}(\mathcal{K})\delta_0 = p_{i,0}(0)\delta_0$ by (4.8). □

The finite-dimensional problem that remains is to find the invariant elements of $\mathcal{E}(\Gamma)$ for a hyperbolic grid. The next lemma is the special case, for such a grid, of the finite Fourier analysis in [2, Section 3.5]; we include the proof because the coefficients are arbitrary complex numbers here.

Lemma 4.10 (Invariant comb measures on a hyperbolic grid)**.** *Let $\Gamma = d^{-1}\Lambda_r$ with an integer $d \geq 1$ and $r > 0$. The elements of $\mathcal{E}(\Gamma)$ that are invariant under both $\mathbb{S}$ and $\mathcal{C}$ are exactly the complex linear combinations of the circle combs*

$$e_{rk/d}, \qquad k \text{ a positive divisor of } d^2,$$

and these combs are linearly independent.

Proof. Index the cosets of $\Gamma^\vee$ by $(m,n) \in (\mathbb{Z}/d^2\mathbb{Z})^2$, the coset $e_{m,n}$ containing the point $(u,v) = (mr/d,\, n/(dr))$. On Γ the spin is mn/d^2 and the pairing is $B = (mn' + nm')/d^2$. By (4.3) and $c_\Gamma = d^{-2}$,

$$\mathbb{S}e_{m,n} = \frac{1}{d^2} \sum_{m',n'} e^{2\pi i(mn'+nm')/d^2}\, e_{m',n'}, \qquad \mathcal{C}e_{m,n} = e^{2\pi imn/d^2}\, e_{m,n}.$$

A measure $\sum \phi(m,n) e_{m,n}$ is therefore invariant if and only if

$$\phi(m,n) = \frac{1}{d^2} \sum_{m',n'} \phi(m',n')\, e^{2\pi i(m'n+n'm)/d^2}, \qquad (e^{2\pi imn/d^2} - 1)\phi(m,n) = 0. \tag{4.16}$$

Let $\widehat{\phi}(m,n) = \sum_{n'} \phi(m,n') e^{2\pi inn'/d^2}$ be the Fourier transform in the second variable, which determines ϕ. The second condition in (4.16) holds if and only if $\widehat{\phi}(m,n+m) = \widehat{\phi}(m,n)$ for all m,n. The right side of the first condition equals $d^{-2}\sum_{m'} e^{2\pi im'n/d^2}\widehat{\phi}(m',m)$, whose transform in n at the point (m,n) is $\widehat{\phi}(-n,m)$; so the first condition holds if and only if $\widehat{\phi}(m,n) = \widehat{\phi}(-n,m)$ for all m,n. The maps $(m,n) \mapsto (m,n+m)$ and $(m,n) \mapsto (-n,m)$ are right multiplication of the row vector (m,n) by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, which generate $\mathrm{SL}_2(\mathbb{Z})$. Hence ϕ is invariant if and only if $\widehat{\phi}$ is constant on the orbits of $\mathrm{SL}_2(\mathbb{Z})$ on $(\mathbb{Z}/d^2\mathbb{Z})^2$.

These orbits are the level sets of $k = \gcd(m,n,d^2)$. Indeed, the ideal of $\mathbb{Z}$ generated by m,n,d^2 is unchanged by the action. Conversely, let $\gcd(m,n,d^2) = k$ and write $(m,n) = k(m_1,n_1)$ with integers satisfying $\gcd(m_1,n_1,d^2/k) = 1$. We may take $1 \leq n_1 \leq d^2/k$. By the Chinese remainder theorem there is an integer k' such that $m_2 = m_1 + k'd^2/k$ is not divisible by any prime that divides n_1 but not d^2/k; the primes dividing both n_1 and d^2/k do not divide m_1, hence not m_2. Thus $\gcd(m_2,n_1) = 1$, and there is a matrix in $\mathrm{SL}_2(\mathbb{Z})$ with first

row (m_2, n_1). It maps $(k, 0)$ to $(km_2, kn_1) \equiv (m, n)$ modulo d^2, so (m, n) lies in the orbit of $(k, 0)$.

Consequently the invariant $\widehat{\phi}$ are the functions of $\gcd(m, n, d^2)$. A basis of this space is given by the functions $\mathbf{1}_{k|m}\mathbf{1}_{k|n}$, $k \mid d^2$, since these are obtained from the indicators of the level sets by a triangular change of basis with respect to divisibility. Inverting the transform,

$$\widehat{\phi} = \mathbf{1}_{k|m}\mathbf{1}_{k|n} \quad \Longleftrightarrow \quad \phi(m,n) = \frac{1}{d^2}\sum_{n' \colon k|n'} \mathbf{1}_{k|m}\, e^{-2\pi i n n'/d^2} = \frac{1}{k}\,\mathbf{1}_{k|m}\,\mathbf{1}_{(d^2/k)|n}.$$

The corresponding measure is k^{-1} times the comb of the lattice of points $(u, v) = (m_1 kr/d,\ n_1 d/(kr))$, $m_1, n_1 \in \mathbb{Z}$, which is $\Lambda_{rk/d}$. □

Theorem 4.11 (Classification of the invariant distributions)**.** *Let $\mathcal{R} \subset (0, \infty)$ be finite, let $F \subset \Lambda_1$ be finite, and let T be a tempered distribution with*

$$\mathbb{S}T = T, \qquad \mathcal{C}T = T, \qquad \operatorname{supp} T \subset \bigcup_{r\in\mathcal{R}} \Lambda_r \cup F.$$

(i) There are polynomials p_r, $r \in \mathcal{R}$, such that

$$T = \sum_{r\in\mathcal{R}} p_r(\mathcal{K})\, e_r. \tag{4.17}$$

(ii) The polynomials p_r are uniquely determined by T, and $\deg p_r \le q$ if T has local order at most q. Conversely, every distribution of the form (4.17) is tempered and invariant under $\mathbb{S}$ and $\mathcal{C}$.

(iii) Suppose that $\mathcal{R}$ is closed under $r \mapsto 1/r$. Then T is even if and only if $p_{1/r}(z) = p_r(-z)$ for all $r \in \mathcal{R}$, and if and only if T is a finite linear combination of the radius derivatives

$$(r\partial_r)^j \rho_r^{\mathrm{F}} = \frac{(-1)^j}{2}\,\mathcal{K}^j(e_r + (-1)^j e_{1/r}), \qquad r \in \mathcal{R},\ r \ge 1,\ j \ge 0. \tag{4.18}$$

Here $(r\partial_r)^j\rho_r^{\mathrm{F}}$ denotes the tempered distribution $f \mapsto (t\partial_t)^j \langle \rho_t^{\mathrm{F}}, f\rangle\big|_{t=r}$. For $r = 1$ the derivatives of odd order vanish. The coefficients of the combination are unique if only even j are used at $r = 1$, and only orders $j \le q$ occur if T has local order at most q.

Proof. Private points. We first record a fact about circle lattices. Let $\mathcal{R}' \subset (0, \infty)$ be finite, $r \in \mathcal{R}'$, and let ϖ be a prime with $\varpi > r/r'$ for all $r' \in \mathcal{R}'$. The point x_ϖ with light-cone coordinates $(u, v) = (r, \varpi/r)$ lies in Λ_r. If it lies in $\Lambda_{r'}$, say $(r, \varpi/r) = (m'r', n'/r')$, then m', n' are positive integers with $m'n' = \varpi$, so $m' = 1$ and $r' = r$, or $m' = \varpi$ and $r' = r/\varpi$. The second case is excluded for $r' \in \mathcal{R}'$ by the choice of ϖ. Thus x_ϖ belongs to $\Lambda_{r'}$ for no $r' \in \mathcal{R}' \setminus \{r\}$; we call it a *private point* of r relative to $\mathcal{R}'$. There are infinitely many of them, and they are different from the origin.

Uniqueness and degrees. Suppose (4.17) holds, and let x be a private point of r relative to $\mathcal{R}$. Near x we have $T = p_r(\mathcal{K})\delta_x$. By (4.9) this determines p_r, and its local order is $\deg p_r$. This proves the first two assertions of (ii). The converse follows from (2.11) and (4.8).

Existence. Let $(\Gamma_i)_{i\in\mathsf{I}}$, with $\Gamma_i = d_i^{-1}\Lambda_{r_i}$, be a transverse family adapted to $\mathcal{R} \cup \{1\}$. Its support contains $\operatorname{supp} T$, and Theorem 4.9 gives $T = \sum_{i,\gamma} p_{i,\gamma}(\mathcal{K})e_{i,\gamma}$. Collecting powers of $\mathcal{K}$, write

$$T = \sum_{j\ge 0} \mathcal{K}^j \eta_j, \qquad \eta_j = \sum_i \eta_{j,i}, \quad \eta_{j,i} \in \mathcal{E}(\Gamma_i),$$

with only finitely many $\eta_j \neq 0$. Each $\mathcal{E}(\Gamma_i)$ is mapped into itself by $\mathbb{S}$, by (4.3), and by $\mathcal{C}$, which multiplies $e_{i,\gamma}$ by the constant $e^{2\pi i s(\gamma)}$. Since $\mathcal{K}$ commutes with $\mathbb{S}$ and $\mathcal{C}$,

$$0 = \mathbb{S}T - T = \sum_j \mathcal{K}^j(\mathbb{S}\eta_j - \eta_j), \qquad 0 = \mathcal{C}T - T = \sum_j \mathcal{K}^j(\mathcal{C}\eta_j - \eta_j).$$

Expanding $\mathbb{S}\eta_{j,i}-\eta_{j,i}$ in the combs $e_{i,\gamma}$, the first identity has the form $\sum_{i,\gamma}(\sum_j \vartheta_{i,\gamma,j}\mathcal{K}^j)e_{i,\gamma}=0$ with complex numbers $\vartheta_{i,\gamma,j}$. By Lemma 4.4(i) each operator $\sum_j \vartheta_{i,\gamma,j}\mathcal{K}^j$ vanishes, and by the linear independence of the powers of $\mathcal{K}$ all $\vartheta_{i,\gamma,j}$ vanish. Hence $\mathbb{S}\eta_{j,i}=\eta_{j,i}$, and in the same way $\mathcal{C}\eta_{j,i}=\eta_{j,i}$. By Lemma 4.10, $\eta_{j,i}$ is a combination of circle combs $e_{r'}$ with radii $r'=r_i k/d_i$, $k\mid d_i^2$. Therefore

$$T=\sum_{r'\in\mathcal{R}'} p_{r'}(\mathcal{K})\,e_{r'}$$

for a finite set $\mathcal{R}'\supset\mathcal{R}\cup\{1\}$ and polynomials $p_{r'}$. Let $r'\in\mathcal{R}'\setminus\mathcal{R}$ and suppose $p_{r'}\neq 0$. Choose a private point x of r' relative to $\mathcal{R}'$ that does not lie in the finite set F. Near x we have $T=p_{r'}(\mathcal{K})\delta_x\neq 0$ by (4.9), so $x\in\operatorname{supp}T$; but $x\notin F$, and $x\notin\Lambda_r$ for $r\in\mathcal{R}$ because $\mathcal{R}\subset\mathcal{R}'\setminus\{r'\}$. This contradicts the hypothesis on the support, so $p_{r'}=0$ for $r'\notin\mathcal{R}$, which proves (i).

The even sector. Let $\mathfrak{r}$ be the reflection $Q\mapsto -Q$. It maps e_r to $e_{1/r}$ and conjugates $\mathcal{K}=P\partial_Q+Q\partial_P$ to $-\mathcal{K}$, so $\mathfrak{r}_*(p(\mathcal{K})e_r)=p(-\mathcal{K})e_{1/r}$. The reflection $X\mapsto -X$ fixes every e_r and commutes with $\mathcal{K}$, hence fixes T. Thus T is even if and only if $\mathfrak{r}_*T=T$, which by the uniqueness in (ii) means $p_{1/r}(z)=p_r(-z)$ for all $r\in\mathcal{R}$. In that case p_1 is an even polynomial and, writing $p_r(z)=\sum_j p_{r,j}z^j$,

$$T=\sum_{r\in\mathcal{R},\,r>1}\ \sum_j p_{r,j}\,\mathcal{K}^j\,(e_r+(-1)^j e_{1/r})+\sum_{j\ \text{even}} p_{1,j}\,\mathcal{K}^j e_1,$$

the last sum being present only if $1\in\mathcal{R}$. It remains to prove (4.18); the uniqueness and degree statements then follow from (ii), and each radius derivative is even because ρ_r^{F} is. For a Schwartz function f, $\langle e_r,f\rangle=\sum_{m,n}f(mr,n/r)$ in light-cone coordinates. Since $r\partial_r\big[f(mr,n/r)\big]=\big((u\partial_u-v\partial_v)f\big)(mr,n/r)$ and $\mathcal{K}f$ is again a Schwartz function, the series may be differentiated termwise any number of times, locally uniformly in r, and

$$(r\partial_r)^j\langle e_r,f\rangle=\langle e_r,\mathcal{K}^j f\rangle=\langle(-\mathcal{K})^j e_r,f\rangle,$$

because the transpose of $\mathcal{K}$ is $-\mathcal{K}$. Replacing r by $1/r$ changes the sign of $r\partial_r$, so $(r\partial_r)^j e_{1/r}=\mathcal{K}^j e_{1/r}$. Adding the two and dividing by two gives (4.18), in view of $\rho_r^{\mathrm{F}}=\frac12(e_r+e_{1/r})$. □

Theorem 4.11 applies in particular to distributions supported on a rational grid. The link is the following elementary fact, which is also used in Section 4.6.

Lemma 4.12 (Integral-spin points of a rational light-cone grid)**.** *Let $d\geq 1$ be an integer. Every point with light-cone coordinates $(u,v)=(m/d,n/d)$, $m,n\in\mathbb{Z}$, and integral spin $uv\in\mathbb{Z}$ lies in*

$$\bigcup_{k\mid d^2}\Lambda_{k/d},$$

the union over the positive divisors k of d^2. The set $\mathcal{R}_d=\{k/d : k\mid d^2\}$ of these radii contains 1 and is closed under $r\mapsto 1/r$, and $\Lambda_r\subset d^{-1}\Lambda_1$ for $r\in\mathcal{R}_d$.

Proof. Let $k=d^2/\gcd(n,d^2)$, with the convention $\gcd(0,d^2)=d^2$. Then d^2 divides nk. Moreover $d^2\mid mn$ implies that k divides $m\cdot n/\gcd(n,d^2)$, and k is coprime to $n/\gcd(n,d^2)$, so $k\mid m$. Hence

$$u=\frac{m}{k}\cdot\frac{k}{d},\qquad v=\frac{nk}{d^2}\Big/\frac{k}{d}$$

with integers m/k and nk/d^2, that is, $(u,v)\in\Lambda_{k/d}$. The radius k/d equals 1 for $k=d$, and $(k/d)^{-1}=(d^2/k)/d$. Finally a point $(m_1k/d,\,n_1d/k)$ of $\Lambda_{k/d}$ has $v=n_1(d^2/k)/d\in d^{-1}\mathbb{Z}$. □

Corollary 4.13 (Invariant distributions on a rational grid)**.** *Let $d \geq 1$ and $q \geq 0$ be integers, and let $\mathcal{R}_d$ be as in Lemma 4.12. Let T be a tempered distribution of local order at most q, supported in the light-cone grid $d^{-1}\Lambda_1 = \{(u,v) \in d^{-1}\mathbb{Z}^2\}$, with $\mathbb{S}T = T$ and $\mathcal{C}T = T$. Then*

$$T = \sum_{r \in \mathcal{R}_d} p_r(\mathcal{K})\, e_r$$

with unique polynomials p_r of degree at most q. If in addition T is even, then T is a linear combination, with unique coefficients, of the distributions

$$(r\partial_r)^j \rho_r^{\mathrm{F}}, \qquad r \in \mathcal{R}_d,\ r \geq 1,\ 0 \leq j \leq q, \quad j \text{ even if } r = 1. \tag{4.19}$$

Conversely, each distribution in (4.19) is tempered, even, invariant under $\mathbb{S}$ and $\mathcal{C}$, supported in $d^{-1}\Lambda_1$, and of local order at most q. Hence the space of even tempered distributions of local order at most q that are supported in $d^{-1}\Lambda_1$ and invariant under $\mathbb{S}$ and $\mathcal{C}$ has the basis (4.19). The same description holds, with $d = 2k$, for the invariant distributions supported in a rational momentum grid $(2k)^{-1}\mathbb{Z}^2$: they are the elements of this space whose support lies in that grid.

Proof. The smooth function $e^{2\pi i s} - 1$ annihilates T, so T vanishes near every point at which this function is nonzero; that is, the spin is integral on $\operatorname{supp} T$. By Lemma 4.12, $\operatorname{supp} T \subset \bigcup_{r \in \mathcal{R}_d} \Lambda_r$, and Theorem 4.11 applies with $\mathcal{R} = \mathcal{R}_d$ and $F = \emptyset$. The converse statements follow from (4.18), Theorem 4.11(ii) and the inclusion $\Lambda_r \subset d^{-1}\Lambda_1$. For the last assertion note that $(2k)^{-1}\mathbb{Z}^2$ is contained in the light-cone grid $(2k)^{-1}\Lambda_1$, because $u = P+Q$ and $v = P-Q$. □

4.4 A finite certificate for the origin coefficients

Theorem 4.9 shows that $\mathbb{S}$- and $\mathcal{C}$-invariance forbid positive-order origin coefficients. For distributions of bounded local order, finitely many test functions detect this obstruction, as we show in this subsection. The first step replaces Lemma 4.5 by a statement that uses the phase operator only.

Lemma 4.14 (Operators commuting with the spin)**.** *Let $\mathsf{A} \in \mathcal{W}$ commute with multiplication by the spin function $s = P^2 - Q^2 = uv$. Then*

$$\mathsf{A} = \sum_{j=0}^{m} \mathsf{q}_j\, \mathcal{K}^j$$

with polynomials $\mathsf{q}_j = \mathsf{q}_j(P,Q)$, and consequently $\mathsf{A}\delta_0 = \mathsf{q}_0(0)\,\delta_0$. The same conclusions hold if $\kappa_\nu(\mathsf{A}) = \mathsf{A}$ for some nonzero integer ν.

Proof. We use light-cone coordinates and induct on the differential order m of A; operators of order zero are polynomials. Let $m \geq 1$, write $\mathsf{A} = \sum_{i+k \leq m} \mathsf{a}_{ik}(u,v)\,\partial_u^i \partial_v^k$, and let $\mathsf{a}^{\mathrm{top}} = \sum_{i+k=m} \mathsf{a}_{ik}\,\xi_u^i \xi_v^k$ be its principal symbol. Since $[\partial_u^i \partial_v^k, uv] = iv\,\partial_u^{i-1}\partial_v^k + ku\,\partial_u^i \partial_v^{k-1} + ik\,\partial_u^{i-1}\partial_v^{k-1}$, the commutator $[\mathsf{A}, s]$ has order at most $m-1$, and its part of order $m-1$ has symbol $(v\partial_{\xi_u} + u\partial_{\xi_v})\mathsf{a}^{\mathrm{top}}$. Hence

$$(v\partial_{\xi_u} + u\partial_{\xi_v})\mathsf{a}^{\mathrm{top}} = 0.$$

Fix (u,v) with $uv \neq 0$. In the linear coordinates $\ell = u\xi_u - v\xi_v$ and $\ell' = \xi_u/v$ on the ξ-plane the derivation $v\partial_{\xi_u} + u\partial_{\xi_v}$ is $\partial_{\ell'}$. So $\mathsf{a}^{\mathrm{top}}$, a homogeneous polynomial of degree m in ξ, depends on ℓ only: $\mathsf{a}^{\mathrm{top}} = \mathsf{q}\,(u\xi_u - v\xi_v)^m$ with $\mathsf{q} = \mathsf{q}(u,v)$. Comparing the coefficients of ξ_u^m and ξ_v^m gives $\mathsf{a}_{m0} = \mathsf{q}\,u^m$ and $\mathsf{a}_{0m} = \mathsf{q}\,(-v)^m$, hence the polynomial identity $(-v)^m \mathsf{a}_{m0} = u^m \mathsf{a}_{0m}$. As u^m and v^m are coprime, u^m divides a_{m0}, so $\mathsf{q} = \mathsf{a}_{m0}/u^m$ is a polynomial, and $\mathsf{a}^{\mathrm{top}} = \mathsf{q}\,(u\xi_u - v\xi_v)^m$ holds identically.

The operator $\mathsf{q}\,\mathcal{K}^m$, with $\mathcal{K} = u\partial_u - v\partial_v$, has principal symbol $\mathsf{a}^{\mathrm{top}}$, and it commutes with s because $[\mathcal{K}, s] = \mathcal{K}(uv) = 0$. Thus $\mathsf{A} - \mathsf{q}\,\mathcal{K}^m$ commutes with s and has order less than m, and the induction hypothesis applies. The statement about δ_0 follows from (4.8).

Finally let $\kappa_\nu(\mathsf{A}) = \mathsf{A}$ with $\nu \neq 0$. For $z \in \mathbb{C}$,

$$e^{2\pi i z s}\,\mathsf{A}\,e^{-2\pi i z s} = \sum_{k=0}^{m} \frac{(2\pi i z)^k}{k!}\,\mathrm{ad}_s^k(\mathsf{A}), \qquad \mathrm{ad}_s(\mathsf{A}) = [s, \mathsf{A}],$$

the sum being finite because ad_s lowers the order. This polynomial in z takes the value A at all integer multiples of ν, hence is constant, and the coefficient of z gives $[s, \mathsf{A}] = 0$. □

Lemma 4.15 (Formal rigidity without growth conditions)**.** *Let $(\Gamma_i)_{i\in\mathsf{I}}$ be a transverse family of hyperbolic grids with support S and phase exponent ν, and let $q \geq 1$. There is a finite list $\mathsf{D}_1, \dots, \mathsf{D}_{n_0}$ of translation operators*

$$\mathsf{D}_l = \prod_{i\in\mathsf{I}} (\tau_{w_{i,l}} - I)^{m_l}, \qquad w_{i,l} \in \Gamma_i^\vee \setminus \{0\}, \qquad m_l \in \{q+1,\, 2q+1\},$$

with the following property. Let $(c_\alpha(x))_{x\in S,\,\alpha\in\mathcal{A}_q}$ be an arbitrary array of complex numbers, subject to no growth condition, and let

$$T = \sum_{x\in S} \sum_{|\alpha|\leq q} c_\alpha(x)\,\partial^\alpha \delta_x.$$

If $\mathsf{D}_l T = 0$ for all l and $\mathcal{C}^\nu T = T$, then $c_\alpha(0) = 0$ for all α with $1 \leq |\alpha| \leq q$.

Proof. The list. For each $j \in \mathsf{I}$ fix a basis $(\mathsf{b}_{j,1}, \mathsf{b}_{j,2})$ of $\Gamma_j^\vee$. For each j and $\iota \in \{1, 2\}$ the list contains the operator

$$\mathsf{D}_{j,\iota} = (\tau_{\mathsf{b}_{j,\iota}} - I)^{q+1} \prod_{i\neq j} (\tau_{w_i} - I)^{q+1},$$

where the vectors $w_i = w_i^{(j,\iota)} \in \Gamma_i^\vee$ are chosen by Lemma 4.3(i) with $m = q+1$ and $G_i = \Gamma_i^\vee$. In addition the list contains one operator $\mathsf{D}_* = \prod_{i\in\mathsf{I}} (\tau_{w_i^*} - I)^{2q+1}$ with arbitrary nonzero $w_i^* \in \Gamma_i^\vee$.

Polynomial structure. Let T be as in the statement. By Lemma 4.3(ii), applied to $\mathsf{D}_{j,1}$ and $\mathsf{D}_{j,2}$, for every j, every coset $\gamma + \Gamma_j^\vee$ and every α the function $(n_1, n_2) \mapsto c_\alpha(\gamma + n_1\mathsf{b}_{j,1} + n_2\mathsf{b}_{j,2})$ has finitely supported differences of order $q+1$ in both directions. Lemma 4.2, which requires no growth condition, shows that it agrees outside a finite set with a polynomial of degree at most q in each of n_1, n_2. As in the proof of Lemma 4.4(ii) we obtain

$$T = T_0 + F, \qquad T_0 = \sum_{i,\gamma} \mathsf{A}_{i,\gamma} e_{i,\gamma}, \qquad \mathsf{A}_{i,\gamma} = \sum_{|\alpha|\leq q} \partial^\alpha \circ p_{i,\gamma,\alpha},$$

where F has finite support and the polynomials $p_{i,\gamma,\alpha}$ have total degree at most $2q$.

The finite part vanishes. For $w \in \Gamma_i^\vee$ translation by w preserves $e_{i,\gamma}$, so $(\tau_w - I)^{2q+1}(\partial^\alpha(p\,e_{i,\gamma})) = \partial^\alpha(((\tau_w - I)^{2q+1}p)\,e_{i,\gamma}) = 0$ for every polynomial p of degree at most $2q$. Since translations commute, $\mathsf{D}_* T_0 = 0$, and therefore $\mathsf{D}_* F = \mathsf{D}_* T - \mathsf{D}_* T_0 = 0$. The distribution F is tempered, and by (4.2) $\omega_*\,\mathbb{S}^{-1}F = 0$ with $\omega_* = \prod_i (\chi_{w_i^*} - 1)^{2q+1}$. Since F has finite support, $\mathbb{S}^{-1}F$ is a real-analytic function, and ω_* vanishes only on the locally finite family of lines $B(w_i^*, X) \in \mathbb{Z}$. Hence $\mathbb{S}^{-1}F$ vanishes on a dense open set, so $\mathbb{S}^{-1}F = 0$ and $F = 0$.

Conclusion. Thus $T = T_0$ is a polynomial comb. Since $\mathcal{C}^\nu$ fixes T and every $e_{i,\gamma}$, $\sum_{i,\gamma} \mathsf{A}_{i,\gamma} e_{i,\gamma} = \sum_{i,\gamma} \kappa_\nu(\mathsf{A}_{i,\gamma}) e_{i,\gamma}$, and Lemma 4.4(i) gives $\kappa_\nu(\mathsf{A}_{i,\gamma}) = \mathsf{A}_{i,\gamma}$. Near the origin $T = \sum_i \mathsf{A}_{i,0}\delta_0$, and by Lemma 4.14 each term is a multiple of δ_0. □

Lemma 4.16 (A finite consequence of finitely supported equations)**.** *Let $\mathcal{I}$ be a set. Call an element r of the space $\mathbb{C}^{(\mathcal{I})}$ of finitely supported functions on $\mathcal{I}$ a* row*, and let it act on arbitrary arrays $c \in \mathbb{C}^{\mathcal{I}}$ by $\mathsf{r}(c) = \sum_{\imath \in \mathcal{I}} \mathsf{r}_\imath c_\imath$. Let R be a family of rows and λ a row. If $\lambda(c) = 0$ for every $c \in \mathbb{C}^{\mathcal{I}}$ such that $\mathsf{r}(c) = 0$ for all $\mathsf{r} \in \mathsf{R}$, then λ is a finite linear combination of elements of R.*

Proof. Let $U \subset \mathbb{C}^{(\mathcal{I})}$ be the linear span of R. If $\lambda \notin U$, there is a linear functional Λ on $\mathbb{C}^{(\mathcal{I})}$ that vanishes on U and satisfies $\Lambda(\lambda) = 1$. Let $c_\imath$ be the value of Λ on the indicator function of $\{\imath\}$. Then $\mathsf{r}(c) = \Lambda(\mathsf{r})$ for every row r, so c satisfies all equations of R while $\lambda(c) = 1$, a contradiction. Hence $\lambda \in U$. □

The array c produced in this proof satisfies no growth condition. This is the reason for stating Lemma 4.15 for arbitrary arrays: a rigidity statement for tempered arrays only would not yield a finite combination of rows.

Uniform families of Gaussian mixtures. To control the test functions, also in families, we fix the following notation. Let $n \geq 0$ be an integer; for $n = 0$ the cube $[-1,1]^n$ is a point of unit mass. For a continuous map $z : [-1,1]^n \to \mathbb{C}^2$ and a polynomial $p(X,\xi) = \sum_{|\gamma| \leq n'} p_\gamma(\xi) X^\gamma$ whose coefficients are continuous functions of $\xi \in [-1,1]^n$, put

$$\mathfrak{m}[p,z](X) = \int_{[-1,1]^n} p(X,\xi)\, e^{-2\pi|X|^2 + z(\xi)\cdot X}\, \mathrm{d}\xi,$$
$$\|p\| = \max_\gamma \sup_\xi |p_\gamma(\xi)|, \qquad \|z\| = \sup_\xi |z(\xi)|.$$

Lemma 4.17. *Let $n, n' \geq 0$ be integers and $\mathsf{z}_0 < \infty$, and let p, z be as above, with p of degree at most n' in X and $\|z\| \leq \mathsf{z}_0$.*

(i) $\mathfrak{m}[p,z] \in \mathcal{G}_{2\pi}$*, and* $e^{w\cdot X}\mathfrak{m}[p,z] = \mathfrak{m}[p, z+w]$ *for* $w \in \mathbb{C}^2$*.*

(ii) $\mathbb{S}\,\mathfrak{m}[p,z] = \mathfrak{m}[\tilde{p}, iJz]$*, where* $J = \operatorname{diag}(1,-1)$ *and $\tilde{p}$ is a polynomial of degree at most n' in X with $\|\tilde{p}\| \leq C\|p\|$, $C = C(n', \mathsf{z}_0)$.*

(iii) For every integer $k_0 \geq 0$, every $\varrho < \infty$ and every $0 < b < 2\pi$ there is $C = C(n, n', \mathsf{z}_0, k_0, \varrho, b)$ with

$$|\partial^\alpha \mathfrak{m}[p,z](X)| \leq C\,\|p\|\, e^{-b|\operatorname{Re} X|^2} \qquad \text{for } |\alpha| \leq k_0,\ |\operatorname{Im} X| \leq \varrho.$$

Proof. (i) is immediate from Definition 2.3. For (ii), (2.8) with $b = 2\pi$ reads $\mathbb{S}\phi_{2\pi,z} = e^{z\cdot z/(8\pi)}\phi_{2\pi,iJz}$. We have $X^\gamma \phi_{2\pi,z} = \partial_z^\gamma \phi_{2\pi,z}$, and $\mathbb{S}$ commutes with differentiation in the parameter z, by differentiation under the integral (2.5). Hence

$$\mathbb{S}\big(X^\gamma \phi_{2\pi,z}\big) = \partial_z^\gamma \big[e^{z\cdot z/(8\pi)}\phi_{2\pi,iJz}\big] = \mathsf{H}_\gamma(X,z)\, e^{z\cdot z/(8\pi)}\, \phi_{2\pi,iJz},$$

where H_γ is a polynomial in (X,z) of degree at most $|\gamma|$ with universal coefficients. Integrating over $[-1,1]^n$, which is permitted by Fubini's theorem, gives (ii) with $\tilde{p}(X,\xi) = \sum_\gamma p_\gamma(\xi)\, \mathsf{H}_\gamma(X, z(\xi))\, e^{z(\xi)\cdot z(\xi)/(8\pi)}$. For (iii), $\partial^\alpha\big(X^\gamma e^{-2\pi|X|^2 + z\cdot X}\big)$ is a polynomial in (X,z) of degree at most $n' + k_0$ in X times $e^{-2\pi|X|^2 + z\cdot X}$, and for $|\operatorname{Im} X| \leq \varrho$ and $|z| \leq \mathsf{z}_0$

$$\big|e^{-2\pi|X|^2 + z\cdot X}\big| \leq e^{2\pi\varrho^2 + \mathsf{z}_0\varrho}\, e^{-2\pi|\operatorname{Re} X|^2 + \mathsf{z}_0|\operatorname{Re} X|}.$$

The factor $e^{-(2\pi - b)|\operatorname{Re} X|^2}$ absorbs the polynomial and $e^{\mathsf{z}_0|\operatorname{Re} X|}$. □

Lemma 4.18 (Cardinal test functions). *Let $\mathfrak{c}_1, \dots, \mathfrak{c}_{n_1}$ be cosets of lattices in $\mathbb{R}^2$, given as*

$$\mathfrak{c}_i = \{X \in \mathbb{R}^2 : \ell_{i,1}(X - x_i) \in \mathbb{Z},\ \ell_{i,2}(X - x_i) \in \mathbb{Z}\}$$

with points $x_i \in \mathbb{R}^2$ and linearly independent real linear forms $\ell_{i,1}, \ell_{i,2}$. Let $y \in \mathbb{R}^2$, let $q \geq 0$ be an integer and $\beta \in \mathcal{A}_q$. For each i with $y \notin \mathfrak{c}_i$ choose $\ell_i^\star \in \{\ell_{i,1}, \ell_{i,2}\}$ with $\ell_i^\star(y - x_i) \notin \mathbb{Z}$. Put

$$\psi_0(X) = e^{-2\pi|X-y|^2} \prod_{i:\, y \in \mathfrak{c}_i} \Big[\operatorname{sinc}\big(\pi \ell_{i,1}(X-y)\big) \operatorname{sinc}\big(\pi \ell_{i,2}(X-y)\big)\Big]^{q+1} \times \prod_{i:\, y \notin \mathfrak{c}_i} \left[\frac{e^{2\pi i \ell_i^\star(X-x_i)} - 1}{e^{2\pi i \ell_i^\star(y-x_i)} - 1}\right]^{q+1},$$

and $\psi = \mathtt{t}\,\psi_0$, where $\mathtt{t}$ is the Taylor polynomial of degree q at y of the function $(-1)^{|\beta|}(X-y)^\beta/(\beta!\,\psi_0(X))$, which is analytic near y because $\psi_0(y) = 1$.

(i) ψ vanishes to order at least $q+1$ at every point of $\bigcup_i \mathfrak{c}_i$ other than y, and $\partial^\alpha \psi(y) = (-1)^{|\beta|}\mathbf{1}_{\alpha=\beta}$ for $|\alpha| \leq q$. Consequently, for every tempered distribution U supported in $\bigcup_i \mathfrak{c}_i$ and of local order at most q, $\langle U, \psi\rangle$ is the coefficient of $\partial^\beta \delta_y$ in U.

(ii) ψ is a finite sum of functions $\mathfrak{m}[p,z]$ as in Lemma 4.17; in particular $\psi \in \mathcal{G}_{2\pi}$. If $|y|$, $|x_i|$ and the norms of the forms $\ell_{i,1}, \ell_{i,2}$ are bounded by $\mathtt{z}_1$, then the number of terms, the integers n, n' and a bound for $\|z\|$ can be chosen in terms of q, n_1 and $\mathtt{z}_1$ only, and

$$\|p\| \leq C(q, n_1, \mathtt{z}_1)\, \mathfrak{d}^{-(q+1)^2}, \qquad \mathfrak{d} = \prod_{i:\, y \notin \mathfrak{c}_i} |e^{2\pi i \ell_i^\star(y-x_i)} - 1|.$$

Proof. (i) Let $x \in \mathfrak{c}_i$, $x \neq y$. If $y \in \mathfrak{c}_i$, then $\ell_{i,1}(x-y)$ and $\ell_{i,2}(x-y)$ are integers, not both zero, so one of the two sinc factors with index i vanishes at x and its $(q+1)$-st power vanishes to order $q+1$. If $y \notin \mathfrak{c}_i$, the numerator of the exponential factor with index i vanishes on $\mathfrak{c}_i$. All factors are entire, so ψ_0 and ψ vanish to order at least $q+1$ at x. At y every factor equals one, and by the choice of $\mathtt{t}$, $\psi(X) = (-1)^{|\beta|}(X-y)^\beta/\beta! + O(|X-y|^{q+1})$. The last assertion follows from Lemma 4.1(i) with $F_0 = \{y\}$.

(ii) Write $\ell(X) = \vec{\ell}\cdot X$ for a linear form ℓ, and $\psi_0 = \mathfrak{d}_0^{-1}\psi_0^\circ$, where $\mathfrak{d}_0 = \prod_{i:\, y \notin \mathfrak{c}_i}(e^{2\pi i \ell_i^\star(y-x_i)} - 1)^{q+1}$ and ψ_0° is ψ_0 with the denominators removed; thus $|\mathfrak{d}_0| = \mathfrak{d}^{q+1}$. We have

$$e^{-2\pi|X-y|^2} = e^{-2\pi y\cdot y}\, e^{-2\pi|X|^2 + 4\pi y\cdot X},$$

$$\operatorname{sinc}(\pi\ell(X-y)) = \frac{1}{2}\int_{-1}^{1} e^{-i\pi\xi\ell(y)}\, e^{i\pi\xi\vec{\ell}\cdot X}\, \mathrm{d}\xi,$$

$$(e^{2\pi i\ell(X-x)} - 1)^{q+1} = \sum_{k=0}^{q+1} (-1)^{q+1-k} \binom{q+1}{k} e^{-2\pi i k\ell(x)}\, e^{2\pi i k\vec{\ell}\cdot X}.$$

Multiplying out, ψ_0° is a sum of at most $(q+2)^{n_1}$ functions $\mathfrak{m}[p,z]$ with $n \leq 2(q+1)n_1$, with polynomials p of degree zero in X and $\|p\| \leq 2^{(q+1)n_1}$, and with $\|z\|$ bounded in terms of $q, n_1, \mathtt{z}_1$. By Lemma 4.17(iii) the derivatives of ψ_0° of order at most q at y are bounded in terms of the same quantities. Since $\psi_0^\circ(y) = \mathfrak{d}_0$ and $|\mathfrak{d}_0| \leq 2^{(q+1)n_1}$, the Taylor coefficients at y, up to order q, of

$$\frac{\mathfrak{d}_0}{\psi_0^\circ} = \sum_{k\geq 0}\Big(-\frac{\psi_0^\circ - \mathfrak{d}_0}{\mathfrak{d}_0}\Big)^k$$

are bounded by a constant times $|\mathfrak{d}_0|^{-q}$; only the terms with $k \leq q$ contribute to them, because $\psi_0^\circ - \mathfrak{d}_0$ vanishes at y. Hence the coefficients of $\mathtt{t}$, in powers of $X-y$ and therefore also in powers of X, are bounded by $C|\mathfrak{d}_0|^{-q}$, and $\psi = (\mathfrak{d}_0^{-1}\mathtt{t})\,\psi_0^\circ$ has the asserted form with $n' = q$ and $\|p\| \leq C|\mathfrak{d}_0|^{-q-1} = C\mathfrak{d}^{-(q+1)^2}$. □

Rows. Fix a transverse family $(\Gamma_i)_{i\in\mathsf{I}}$ of hyperbolic grids with support S and phase exponent ν, an integer $q \geq 1$, and the operators $\mathsf{D}_1, \dots, \mathsf{D}_{n_0}$ of Lemma 4.15. Expanding the products,

$$\mathsf{D}_l = \sum_{\mathbf{k}} \varkappa_{l,\mathbf{k}}\, \tau_{\mathsf{w}_{l,\mathbf{k}}}, \qquad \omega_l := \prod_{i\in\mathsf{I}} (\chi_{w_{i,l}} - 1)^{m_l} = \sum_{\mathbf{k}} \varkappa_{l,\mathbf{k}}\, \chi_{\mathsf{w}_{l,\mathbf{k}}}, \tag{4.20}$$

where $\mathbf{k} = (k_i)_{i\in\mathsf{I}}$ runs over $\{0, \dots, m_l\}^{\mathsf{I}}$, $\varkappa_{l,\mathbf{k}} = \prod_i (-1)^{m_l - k_i} \binom{m_l}{k_i}$ and $\mathsf{w}_{l,\mathbf{k}} = \sum_i k_i w_{i,l}$. By (4.2), $\mathsf{D}_l = \mathbb{S}\,\omega_l\,\mathbb{S}^{-1}$. For an array $c \in \mathbb{C}^{S\times\mathcal{A}_q}$ let $T_c = \sum_{x\in S} \sum_{|\alpha|\leq q} c_\alpha(x) \partial^\alpha \delta_x$, and put $c_\alpha(x) = 0$ for $x \notin S$. The coefficient of $\partial^\beta \delta_y$ in $\mathsf{D}_l T_c$ is the *translation row*

$$\mathsf{r}_{l,y,\beta}(c) = \sum_{\mathbf{k}} \varkappa_{l,\mathbf{k}}\, c_\beta(y + \mathsf{w}_{l,\mathbf{k}}), \qquad 1 \leq l \leq n_0,\ y \in \mathbb{R}^2,\ \beta \in \mathcal{A}_q. \tag{4.21}$$

Write $\partial^\gamma e^{2\pi i\nu s} = \mathsf{h}_\gamma\, e^{2\pi i\nu s}$ with polynomials $\mathsf{h}_\gamma \in \mathbb{Q}(\pi i)[P, Q]$, $\mathsf{h}_0 = 1$. Since $e^{2\pi i\nu s} = 1$ on S, the coefficient of $\partial^\beta \delta_x$ in $\mathcal{C}^\nu T_c - T_c$ is the *phase row*

$$\mathsf{p}_{x,\beta}(c) = \sum_{\substack{\alpha\in\mathcal{A}_q \\ \alpha\geq\beta,\ \alpha\neq\beta}} (-1)^{|\alpha-\beta|} \binom{\alpha}{\beta} \mathsf{h}_{\alpha-\beta}(x)\, c_\alpha(x), \qquad x \in S,\ \beta \in \mathcal{A}_q. \tag{4.22}$$

Both kinds of rows are finitely supported, and an array c satisfies $\mathsf{D}_l T_c = 0$ for all l and $\mathcal{C}^\nu T_c = T_c$ if and only if all translation rows and all phase rows vanish on c.

Theorem 4.19 (A finite Gaussian-mixture certificate)**.** *Let Σ be a finite union of circle lattices and let $q \geq 1$ be an integer. There are finitely many functions $f_1, \dots, f_n$ on $\mathbb{C}^2$ and a constant $C_{\Sigma,q}$ with the following properties.*

(i) For every l, both f_l and $\mathbb{S}f_l$ belong to the class $\mathcal{G}_{2\pi}$ of Definition 2.3. In particular they are entire, their restrictions to $\mathbb{R}^2$ are Schwartz functions, they satisfy the bounds (2.17) for every $b' < 2\pi$, and Lemma 2.4 applies to each f_l.

(ii) Every even tempered distribution T with $\operatorname{supp} T \subset \Sigma$, of local order at most q and with $\mathcal{C}T = T$, satisfies

$$\max_{1\leq|\alpha|\leq q} |c_\alpha(0)| \leq C_{\Sigma,q} \max_{1\leq l\leq n} \bigl|\langle (\mathbb{S} - I)T, f_l\rangle\bigr|, \tag{4.23}$$

where $c_\alpha(0)$ are the origin coefficients of T in (4.1).

The functions f_l and the constant depend only on Σ and q.

Proof. Let $\Sigma = \bigcup_{r\in\mathcal{R}} \Lambda_r$, let $(\Gamma_i)_{i\in\mathsf{I}}$ be a transverse family adapted to $\mathcal{R}$, and use the notation of the preceding paragraph.

Row identities. Let R be the family of all rows (4.21) and (4.22) on $\mathcal{I} = S \times \mathcal{A}_q$. By Lemma 4.15, every array on which all rows of R vanish satisfies $c_\alpha(0) = 0$ for $1 \leq |\alpha| \leq q$. By Lemma 4.16, for each such α there are finitely many translation rows $\mathsf{r}_\mu = \mathsf{r}_{l_\mu,y_\mu,\beta_\mu}$, $\mu \in \mathsf{M}_\alpha$, finitely many phase rows $\mathsf{p}_{\mu'}$, $\mu' \in \mathsf{M}'_\alpha$, and complex numbers $\vartheta_\mu, \vartheta'_{\mu'}$ with

$$c_\alpha(0) = \sum_{\mu\in\mathsf{M}_\alpha} \vartheta_\mu\, \mathsf{r}_\mu(c) + \sum_{\mu'\in\mathsf{M}'_\alpha} \vartheta'_{\mu'}\, \mathsf{p}_{\mu'}(c) \qquad \text{for all } c \in \mathbb{C}^{S\times\mathcal{A}_q}. \tag{4.24}$$

Extraction. Fix $\mu \in \mathsf{M}_\alpha$ and write $(l, y, \beta) = (l_\mu, y_\mu, \beta_\mu)$. For every array c the distribution $\mathsf{D}_l T_c$ is supported in the union of the cosets $\Gamma_i - \mathsf{w}_{l,\mathbf{k}}$, $i \in \mathsf{I}$, $\mathbf{k} \in \{0, \dots, m_l\}^{\mathsf{I}}$, and has local order at most q. Each of these cosets has the form required in Lemma 4.18, with the point $-\mathsf{w}_{l,\mathbf{k}}$ and the coordinate forms $d_i u/r_i$, $d_i r_i v$ of Γ_i. Let ψ_μ be the function ψ of that lemma for these cosets, the point y, the order q and the index β. If T_c is tempered, then

$$\langle \mathsf{D}_l T_c, \psi_\mu\rangle = \mathsf{r}_\mu(c) \tag{4.25}$$

by Lemma 4.18(i) and (4.21).

The tests. Put

$$f_\mu = \omega_{l_\mu}\,\mathbb{S}\psi_\mu, \qquad \mu \in \mathsf{M}_\alpha,\ 1 \le |\alpha| \le q.$$

By Lemma 4.18(ii), ψ_μ is a finite sum of functions $\mathfrak{m}[p,z]$. By Lemma 4.17(ii) so is $\mathbb{S}\psi_\mu$; by (4.20) and Lemma 4.17(i) so is f_μ, the multiplier ω_l being a finite sum of exponentials $\chi_{\mathsf{w}}(X) = e^{2\pi i B(\mathsf{w},X)}$; and by Lemma 4.17(ii) again so is $\mathbb{S}f_\mu$. This proves (i).

The estimate. Let T be as in (ii) and let c be its coefficient array, extended by zero to $S \supset \Sigma$, so that $T = T_c$. Since $\mathcal{C}T = T$ we have $\mathcal{C}^\nu T = T$, and all phase rows vanish on c. Every point of S lies in some Γ_i, on which the factor $(\chi_{w_{i,l}} - 1)^{m_l}$ of ω_l vanishes to order $m_l \ge q+1$, because $w_{i,l} \in \Gamma_i^\vee$. Hence $\omega_l T = 0$ by Lemma 4.1(ii). As T is even, $\mathbb{S}^{-1}T = \mathbb{S}T$, and therefore

$$\mathsf{D}_l T = \mathbb{S}\,\omega_l\,\mathbb{S}^{-1}T = \mathbb{S}\,\omega_l\,\mathbb{S}T = \mathbb{S}\,\omega_l\,(\mathbb{S} - I)T.$$

Pairing with ψ_μ and using (4.25),

$$\mathsf{r}_\mu(c) = \langle \mathsf{D}_{l_\mu} T, \psi_\mu\rangle = \langle \omega_{l_\mu}(\mathbb{S}-I)T, \mathbb{S}\psi_\mu\rangle = \langle (\mathbb{S}-I)T, f_\mu\rangle.$$

Consequently (4.24) gives $c_\alpha(0) = \sum_{\mu\in\mathsf{M}_\alpha} \vartheta_\mu \langle (\mathbb{S}-I)T, f_\mu\rangle$, and (4.23) follows with $C_{\Sigma,q} = \max_\alpha \sum_{\mu\in\mathsf{M}_\alpha} |\vartheta_\mu|$. □

The tests f_l are not even. In (4.23) the distributions T and $\mathbb{S}T$ are even, so the pairings involve only the even parts of f_l and $\mathbb{S}f_l$, that is, the functions $f_l^\natural$ and $(\mathbb{S}f_l)^\natural$ of (2.13). The analytic evaluation $\mathcal{L}_a$ and Lemma 2.4 likewise involve only these even parts. The constant $C_{\Sigma,q}$ is not explicit, since the row identities (4.24) are obtained from Lemma 4.16.

4.5 One moving radius

The constant in Theorem 4.19 depends on the union of lattices. When one radius varies, it can be controlled up to a finite power of the distance to a limiting radius.

Proposition 4.20 (A certificate for one moving radius)**.** *Let $\mathcal{R}_0 \subset (0,\infty)$ be finite, let $\Sigma_0 = \bigcup_{r\in\mathcal{R}_0} \Lambda_r$, and for $t > 0$ put*

$$\Sigma(t) = \Sigma_0 \cup \Lambda_t \cup \Lambda_{1/t}.$$

Fix $t_0 > 0$ and an integer $q \ge 1$. There exist $\varepsilon_0 \in (0, t_0)$, integers $n \ge 1$ and $K \ge 0$, and, for every t with $0 < |t - t_0| < \varepsilon_0$, functions $f_{1,t}, \dots, f_{n,t}$ on $\mathbb{C}^2$ such that the following hold.

(i) $f_{l,t} \in \mathcal{G}_{2\pi}$ and $\mathbb{S}f_{l,t} \in \mathcal{G}_{2\pi}$ for all l and t.

(ii) For every t with $0 < |t-t_0| < \varepsilon_0$, every even tempered distribution T with $\operatorname{supp} T \subset \Sigma(t)$, of local order at most q and with $\mathcal{C}T = T$, satisfies

$$\max_{1\le|\alpha|\le q} |c_\alpha(0)| \le \sum_{l=1}^{n} \bigl|\langle (\mathbb{S}-I)T, f_{l,t}\rangle\bigr|. \tag{4.26}$$

(iii) For every integer $k_0 \ge 0$, every $\varrho < \infty$ and every $0 < b < 2\pi$ there is a constant C such that, for all l, all $|\alpha| \le k_0$, all $X \in \mathbb{C}^2$ with $|\operatorname{Im} X| \le \varrho$ and all t with $0 < |t - t_0| < \varepsilon_0$,

$$|\partial^\alpha f_{l,t}(X)| + |\partial^\alpha \mathbb{S}f_{l,t}(X)| \le C\,|t-t_0|^{-K}\, e^{-b|\operatorname{Re} X|^2}. \tag{4.27}$$

The numbers ε_0, n, K and the functions $f_{l,t}$ can be chosen depending only on $\mathcal{R}_0$, t_0 and q; the constant C depends in addition on k_0, ϱ and b.

The bound (4.27) may degenerate as $t \to t_0$; only its rate is controlled. In the applications the order of the choices matters: the local order q, and with it K, is fixed before any quantity that is to be balanced against the loss $|t-t_0|^{-K}$.

Proof. Step 1: the family of grids. Let $(\Gamma_i)_{i\in\mathsf{I}}$, $\Gamma_i = d_i^{-1}\Lambda_{r_i}$, be a transverse family adapted to $\mathcal{R}_0$, with phase exponent ν. Put $\mathsf{I}^{\pm} = \mathsf{I} \sqcup \{+,-\}$, $d_{\pm} = 1$, and for $t > 0$

$$\Gamma_+(t) = \Lambda_t, \qquad \Gamma_-(t) = \Lambda_{1/t}, \qquad \Gamma_i(t) = \Gamma_i \ \ (i \in \mathsf{I}), \qquad S(t) = \bigcup_{i\in\mathsf{I}^{\pm}} \Gamma_i(t) \supset \Sigma(t).$$

For each $i \in \mathsf{I}^{\pm}$ let $\ell^t_{i,1}, \ell^t_{i,2}$ be the linear forms

$$\Big(\frac{d_i u}{r_i},\, d_i r_i v\Big) \ \ (i \in \mathsf{I}), \qquad \Big(\frac{u}{t},\, tv\Big) \ \ (i = +), \qquad \Big(tu,\, \frac{v}{t}\Big) \ \ (i = -),$$

so that $\Gamma_i(t)$ is the set where both forms are integral, and let $x_{i,m,n}(t)$ be the point of $\Gamma_i(t)$ at which they take the values $(m,n) \in \mathbb{Z}^2$. The dual lattice $\Gamma_i(t)^{\vee}$ consists of the points $x_{i,m,n}(t)$ with $m, n \in d_i^2\mathbb{Z}$, and $\nu s \in \mathbb{Z}$ on $S(t)$, for every $t > 0$.

Let $\mathbb{F} = \mathbb{Q}(\pi i,\, r_i : i \in \mathsf{I}) \subset \mathbb{C}$, a countable field. A *Laurent vector* is a map $y : (0,\infty) \to \mathbb{R}^2$ whose coordinates are given by elements of $\mathbb{F}[t, t^{-1}]$ (real-valued for $t > 0$). The maps $t \mapsto x_{i,m,n}(t)$ are Laurent vectors, Laurent vectors form a group under addition, and $t \mapsto \ell^t_{i,\iota}(y(t))$ is given by an element of $\mathbb{F}[t,t^{-1}]$ for every Laurent vector y and $\iota \in \{1,2\}$. Fix a number $t_* > 0$ that is transcendental over $\mathbb{F}$. Evaluation at t_* is an injective ring homomorphism $\mathbb{F}[t,t^{-1}] \to \mathbb{C}$ and extends to an embedding of the field $\mathbb{F}(t)$ of rational functions. Hence:

> two elements of $\mathbb{F}[t,t^{-1}]$, or two Laurent vectors, that agree at t_* agree identically. (4.28)

The radii r_i $(i \in \mathsf{I})$, t_* and $1/t_*$ are pairwise incommensurable. Thus $(\Gamma_i(t_*))_{i\in\mathsf{I}^{\pm}}$ is a transverse family of hyperbolic grids with phase exponent ν, to which Lemma 4.15 and the constructions of Section 4.4 apply. For other values of t the lattices $\Gamma_i(t)$ may be commensurable and may share nonzero points.

Step 2: labels. Let $\Xi = \{0\} \sqcup (\mathsf{I}^{\pm} \times (\mathbb{Z}^2 \setminus \{0\}))$ and put $x_0(t) = 0$, $x_\lambda(t) = x_{i,m,n}(t)$ for $\lambda = (i,m,n)$. The label 0 is regarded as the label $(i,0,0)$ of every class i. For each t the map $\lambda \mapsto x_\lambda(t)$ maps Ξ onto $S(t)$ and is injective on the labels of any one class; at $t = t_*$ it is a bijection. A *label array* is an element $\hat c \in \mathbb{C}^{\Xi\times\mathcal{A}_q}$, and we put

$$T^t[\hat c] = \sum_{\lambda\in\Xi} \sum_{|\alpha|\le q} \hat c_\alpha(\lambda)\, \partial^\alpha \delta_{x_\lambda(t)},$$

a locally finite sum. Conversely, let T be a distribution supported in $S(t)$ of local order at most q, with coefficients $c_\alpha(x)$. For each $x \in S(t)$ choose one label $\lambda(x)$ with $x_{\lambda(x)}(t) = x$, and define

$$\hat c_\alpha(\lambda(x)) = c_\alpha(x) \quad (x \in S(t)), \qquad \hat c_\alpha(\lambda) = 0 \quad \text{if } \lambda \ne \lambda(x_\lambda(t)). \tag{4.29}$$

Then $T^t[\hat c] = T$ and $\hat c_\alpha(0) = c_\alpha(0)$. When several labels are mapped to the same point, the coefficients of T at that point are thus attributed to one of them and the others receive zero.

Step 3: rows depending on the parameter. Apply Lemma 4.15 to the family $(\Gamma_i(t_*))_{i\in\mathsf{I}^{\pm}}$ and the order q. Its translation vectors are $w_{i,l}(t_*)$ for Laurent vectors $w_{i,l} = x_{i,m_{i,l},n_{i,l}}$ with $(m_{i,l},n_{i,l}) \in d_i^2\mathbb{Z}^2 \setminus \{0\}$. For every $t > 0$ we have $w_{i,l}(t) \in \Gamma_i(t)^{\vee} \setminus \{0\}$, and we define $\mathsf{D}_l(t)$, $\omega_l(t)$, $\varkappa_{l,\mathbf{k}}$ and the Laurent vectors $\mathsf{w}_{l,\mathbf{k}}$ as in (4.20). For a Laurent vector y, $\beta \in \mathcal{A}_q$ and $t > 0$ define the functional on label arrays

$$\mathsf{r}^t_{l,y,\beta}(\hat c) = \sum_{\mathbf{k}} \varkappa_{l,\mathbf{k}} \sum_{\substack{\lambda\in\Xi \\ x_\lambda(t) = y(t) + \mathsf{w}_{l,\mathbf{k}}(t)}} \hat c_\beta(\lambda).$$

The inner sum has at most one term from each class. Since $\tau_w \delta_x = \delta_{x-w}$,

$$\mathsf{r}^t_{l,y,\beta}(\hat{c}) = \text{the coefficient of } \partial^\beta \delta_{y(t)} \text{ in } \mathsf{D}_l(t)\, T^t[\hat{c}] \tag{4.30}$$

for every $t > 0$ and every label array, whether or not labels coincide at t. For $\lambda \in \Xi$ and $\beta \in \mathcal{A}_q$ define

$$\mathsf{p}^t_{\lambda,\beta}(\hat{c}) = \sum_{\substack{\alpha \in \mathcal{A}_q \\ \alpha \geq \beta,\ \alpha \neq \beta}} (-1)^{|\alpha-\beta|} \binom{\alpha}{\beta} \mathsf{h}_{\alpha-\beta}(x_\lambda(t))\, \hat{c}_\alpha(\lambda),$$

with h_γ as in (4.22). Its coefficients are elements of $\mathbb{F}[t, t^{-1}]$. If T is supported in $S(t)$, has local order at most q and satisfies $\mathcal{C}^\nu T = T$, and $\hat{c}$ is a label array of T as in (4.29), then

$$\mathsf{p}^t_{\lambda,\beta}(\hat{c}) = 0 \qquad \text{for all } \lambda \in \Xi,\ \beta \in \mathcal{A}_q. \tag{4.31}$$

Indeed, for $\lambda = \lambda(x)$ the left side is the coefficient of $\partial^\beta \delta_x$ in $\mathcal{C}^\nu T - T$, because $e^{2\pi i \nu s(x)} = 1$; for all other labels $\hat{c}_\alpha(\lambda) = 0$.

Finally let the *generic row* be

$$\mathsf{r}^{\mathrm{gen}}_{l,y,\beta}(\hat{c}) = \sum_{\mathbf{k}} \varkappa_{l,\mathbf{k}} \sum_{\substack{\lambda \in \Xi \\ x_\lambda \equiv y + \mathsf{w}_{l,\mathbf{k}}}} \hat{c}_\beta(\lambda),$$

where $\equiv$ denotes equality of Laurent vectors; the inner sum has at most one term. Its coefficients are integers that do not depend on t, and $\mathsf{r}^{t_*}_{l,y,\beta} = \mathsf{r}^{\mathrm{gen}}_{l,y,\beta}$ by (4.28).

Step 4: the row identity at t_*. At $t = t_*$ label arrays are the same as arrays on $S(t_*) \times \mathcal{A}_q$, the functionals $\mathsf{p}^{t_*}_{\lambda,\beta}$ are the phase rows (4.22), and the translation row (4.21) at a point of $\mathbb{R}^2$ is either zero or equal to $\mathsf{r}^{t_*}_{l,y,\beta}$ for a Laurent vector y whose value at t_* is that point: if the row is nonzero, the point is $x_\lambda(t_*) - \mathsf{w}_{l,\mathbf{k}}(t_*)$ for some λ and $\mathbf{k}$. Exactly as for (4.24), Lemmas 4.15 and 4.16 give, for each α with $1 \leq |\alpha| \leq q$, finitely many triples $\mu = (l_\mu, y_\mu, \beta_\mu)$, $\mu \in \mathsf{M}_\alpha$, with Laurent vectors y_μ, finitely many pairs $\mu' = (\lambda_{\mu'}, \beta_{\mu'})$, $\mu' \in \mathsf{M}'_\alpha$, and complex numbers ϑ^*_μ, $\vartheta^*_{\mu'}$ such that for all label arrays

$$\hat{c}_\alpha(0) = \sum_{\mu \in \mathsf{M}_\alpha} \vartheta^*_\mu\, \mathsf{r}^{\mathrm{gen}}_\mu(\hat{c}) + \sum_{\mu' \in \mathsf{M}'_\alpha} \vartheta^*_{\mu'}\, \mathsf{p}^{t_*}_{\mu'}(\hat{c}),$$

where $\mathsf{r}^{\mathrm{gen}}_\mu = \mathsf{r}^{\mathrm{gen}}_{l_\mu, y_\mu, \beta_\mu}$ and $\mathsf{p}^t_{\mu'} = \mathsf{p}^t_{\lambda_{\mu'}, \beta_{\mu'}}$.

Step 5: the row identity for all parameters. All the rows in the last identity, and the coordinate functional $\hat{c} \mapsto \hat{c}_\alpha(0)$, are supported in one finite subset of $\Xi \times \mathcal{A}_q$ that does not depend on t. Restricted to these columns, the rows $\mathsf{r}^{\mathrm{gen}}_\mu$ and $\mathsf{p}^t_{\mu'}$ form a matrix with entries in $\mathbb{F}[t, t^{-1}]$. The identity says that, after evaluation at t_*, a coordinate vector lies in the row space of this matrix. Evaluation at t_* is an embedding of the field $\mathbb{F}(t)$ into $\mathbb{C}$, and the solvability of a linear system is not affected by an extension of the field of scalars. Hence there are rational functions $\vartheta_\mu, \vartheta_{\mu'} \in \mathbb{F}(t)$ with

$$\hat{c}_\alpha(0) = \sum_{\mu \in \mathsf{M}_\alpha} \vartheta_\mu(t)\, \mathsf{r}^{\mathrm{gen}}_\mu(\hat{c}) + \sum_{\mu' \in \mathsf{M}'_\alpha} \vartheta_{\mu'}(t)\, \mathsf{p}^t_{\mu'}(\hat{c}) \tag{4.32}$$

as an identity over $\mathbb{F}(t)$, and therefore for all label arrays and all real $t > 0$ outside the finite set E' of poles of the functions $\vartheta_\mu, \vartheta_{\mu'}$, $\mu \in \mathsf{M}_\alpha$, $\mu' \in \mathsf{M}'_\alpha$, $1 \leq |\alpha| \leq q$.

Step 6: incidences and denominators. Fix $\mu \in \mathsf{M}_\alpha$ and write $(l, y, \beta) = (l_\mu, y_\mu, \beta_\mu)$. For $i \in \mathsf{I}^\pm$, $\mathbf{k} \in \{0, \dots, m_l\}^{\mathsf{I}^\pm}$ and $\iota \in \{1, 2\}$ let

$$\mathsf{v}_{i,\mathbf{k},\iota}(t) = \ell^t_{i,\iota}\big(y(t) + \mathsf{w}_{l,\mathbf{k}}(t)\big) \in \mathbb{F}[t, t^{-1}],$$

so that $y(t)$ lies in the coset $\Gamma_i(t) - \mathsf{w}_{l,\mathbf{k}}(t)$ if and only if $\mathsf{v}_{i,\mathbf{k},1}(t)$ and $\mathsf{v}_{i,\mathbf{k},2}(t)$ are integers. Call the pair $(i,\mathbf{k})$ of *type one* if both functions are constant with integer values, and of *type two* otherwise. For a pair of type two choose $\iota = \iota(i,\mathbf{k})$ such that $\mathsf{v}^\star_{i,\mathbf{k}} := \mathsf{v}_{i,\mathbf{k},\iota}$ is not an integer constant. By (4.28), $\mathsf{v}^\star_{i,\mathbf{k}}(t_*) \notin \mathbb{Z}$. Consequently the function

$$\mathfrak{d}_\mu(t) = \prod_{(i,\mathbf{k}) \text{ of type two}} (e^{2\pi i \mathsf{v}^\star_{i,\mathbf{k}}(t)} - 1)$$

is real-analytic on $(0,\infty)$ and does not vanish at t_*. Its zeros, and the zeros of the finitely many functions $\mathfrak{d}_\mu$ together, form a closed discrete subset E'' of $(0,\infty)$. We claim that

$$\mathsf{r}^t_\mu = \mathsf{r}^{\mathrm{gen}}_\mu \qquad \text{for all } t \in (0,\infty) \setminus \mathsf{E}''. \tag{4.33}$$

Every term of $\mathsf{r}^{\mathrm{gen}}_\mu$ occurs in r^t_μ, and the two functionals differ only by terms $\hat{c}_\beta(\lambda)$ with $x_\lambda(t) = y(t) + \mathsf{w}_{l,\mathbf{k}}(t)$ for some $\mathbf{k}$. Let λ and $\mathbf{k}$ satisfy this equation at some $t \notin \mathsf{E}''$, and let i be a class of λ. Then $y(t)$ lies in $\Gamma_i(t) - \mathsf{w}_{l,\mathbf{k}}(t)$, so both $\mathsf{v}_{i,\mathbf{k},\iota}(t)$ are integers. Since $t \notin \mathsf{E}''$, the pair $(i,\mathbf{k})$ is of type one: $\mathsf{v}_{i,\mathbf{k},1} \equiv m$ and $\mathsf{v}_{i,\mathbf{k},2} \equiv n$ with integers m, n. Hence $y + \mathsf{w}_{l,\mathbf{k}} \equiv x_{i,m,n}$ as Laurent vectors, and $x_\lambda(t) = x_{i,m,n}(t)$. As the labels of the class i are mapped injectively, $\lambda = (i,m,n)$, and the term $\hat{c}_\beta(\lambda)$ occurs in $\mathsf{r}^{\mathrm{gen}}_\mu$ as well. This proves (4.33).

Step 7: the tests. Choose $\varepsilon_0 \in (0, t_0)$ so small that the set $\{t : 0 < |t - t_0| \le \varepsilon_0\}$ contains no point of $\mathsf{E}' \cup \mathsf{E}''$; this is possible because both sets are closed and discrete in $(0,\infty)$. For $0 < |t - t_0| < \varepsilon_0$ and $\mu \in \mathsf{M}_\alpha$, $1 \le |\alpha| \le q$, let $\psi_{\mu,t}$ be the function ψ of Lemma 4.18 for the cosets $\Gamma_i(t) - \mathsf{w}_{l_\mu,\mathbf{k}}(t)$, written with the forms $\ell^t_{i,1}, \ell^t_{i,2}$ and the points $-\mathsf{w}_{l_\mu,\mathbf{k}}(t)$, for the point $y_\mu(t)$, the order q, the index β_μ, and the choices $\ell^\star = \ell^t_{i,\iota(i,\mathbf{k})}$ of Step 6. These choices are legitimate: for a pair of type one, $y_\mu(t)$ lies in the coset for every t; for a pair of type two and $t \notin \mathsf{E}''$, the chosen form is not integral at $y_\mu(t) + \mathsf{w}_{l_\mu,\mathbf{k}}(t)$, so $y_\mu(t)$ does not lie in the coset. The quantity $\mathfrak{d}$ of Lemma 4.18(ii) is $|\mathfrak{d}_\mu(t)|$. Define

$$f_{\mu,t} = \vartheta_\mu(t)\, \omega_{l_\mu}(t)\, \mathbb{S}\psi_{\mu,t}.$$

The functions $f_{l,t}$ of the proposition are the $f_{\mu,t}$, $\mu \in \mathsf{M}_\alpha$, $1 \le |\alpha| \le q$, in some enumeration. Assertion (i) follows as in the proof of Theorem 4.19, from Lemmas 4.18(ii) and 4.17.

Step 8: proof of (ii). Let $0 < |t - t_0| < \varepsilon_0$ and let T be as in (ii). Then $\operatorname{supp} T \subset \Sigma(t) \subset S(t)$. Every point of $S(t)$ lies in some $\Gamma_i(t)$, $i \in \mathsf{I}^\pm$, and the factor $(\chi_{w_{i,l}(t)} - 1)^{m_l}$ of $\omega_l(t)$ vanishes on $\Gamma_i(t)$ to order $m_l \ge q+1$, because $w_{i,l}(t) \in \Gamma_i(t)^\vee$. This holds for every t, also when the lattices $\Gamma_i(t)$ are not transverse, since it uses only a class to which the point belongs and the factor of that class. Hence $\omega_l(t)T = 0$ by Lemma 4.1(ii), and as in the proof of Theorem 4.19, using that T is even,

$$\mathsf{D}_l(t)T = \mathbb{S}\,\omega_l(t)\,(\mathbb{S} - I)T.$$

Let $\hat{c}$ be a label array of T as in (4.29). By (4.33), (4.30) and Lemma 4.18(i),

$$\mathsf{r}^{\mathrm{gen}}_\mu(\hat{c}) = \mathsf{r}^t_\mu(\hat{c}) = \langle \mathsf{D}_{l_\mu}(t)T, \psi_{\mu,t}\rangle = \langle (\mathbb{S} - I)T,\, \omega_{l_\mu}(t)\,\mathbb{S}\psi_{\mu,t}\rangle.$$

Since $\mathcal{C}^\nu T = T$, the phase rows vanish on $\hat{c}$ by (4.31). Inserting this into (4.32), which is valid because $t \notin \mathsf{E}'$, and using $\hat{c}_\alpha(0) = c_\alpha(0)$,

$$c_\alpha(0) = \sum_{\mu \in \mathsf{M}_\alpha} \langle (\mathbb{S} - I)T, f_{\mu,t}\rangle,$$

which implies (4.26).

Step 9: proof of (iii). For $|t - t_0| \le \varepsilon_0$ the points $y_\mu(t)$, the vectors $\mathsf{w}_{l,\mathbf{k}}(t)$ and the forms $\ell^t_{i,\iota}$ are bounded, since t stays in a compact subset of $(0,\infty)$. By Lemma 4.18(ii), each $\psi_{\mu,t}$

is a sum of a bounded number of functions $\mathfrak{m}[p,z]$ with fixed n and n', with $\|z\|$ bounded uniformly in t, and with $\|p\| \le C|\mathfrak{d}_\mu(t)|^{-(q+1)^2}$. By Lemma 4.17(i) and (ii) the same holds for $\mathbb{S}\psi_{\mu,t}$, for $f_{\mu,t} = \vartheta_\mu(t)\omega_{l_\mu}(t)\mathbb{S}\psi_{\mu,t}$ and for $\mathbb{S}f_{\mu,t}$, with

$$\|p\| \le C\,|\vartheta_\mu(t)|\,|\mathfrak{d}_\mu(t)|^{-(q+1)^2};$$

here we use that $\omega_l(t)$ is a sum of exponentials $e^{2\pi i B(\mathsf{w}_{l,\mathbf{k}}(t),X)}$ with fixed integer coefficients and bounded $\mathsf{w}_{l,\mathbf{k}}(t)$. The function $\mathfrak{d}_\mu$ is real-analytic near t_0 and not identically zero, so it vanishes at t_0 to some finite order $k_\mu \ge 0$. Since it has no other zero in $[t_0-\varepsilon_0, t_0+\varepsilon_0]$, the quotient $\mathfrak{d}_\mu(t)/(t-t_0)^{k_\mu}$ is continuous and nonvanishing on this interval, and $|\mathfrak{d}_\mu(t)| \ge C^{-1}|t-t_0|^{k_\mu}$ there. Similarly the rational function ϑ_μ has at most a pole of finite order k'_μ at t_0 and no other pole in the interval, so $|\vartheta_\mu(t)| \le C|t-t_0|^{-k'_\mu}$. With $K = \max_\mu((q+1)^2k_\mu + k'_\mu)$ we obtain $\|p\| \le C|t-t_0|^{-K}$, and Lemma 4.17(iii) gives (4.27). The exponent K is determined by the chosen row identities and the denominators, which were fixed in terms of $\mathcal{R}_0$, t_0 and q alone; the numbers k_0, ϱ and b affect only C. □

4.6 The localization obstruction

In this subsection the certificates are applied to the spectrum of an admissible partition function. The distribution to which they are applied is a finite Taylor expansion of the analytic evaluation (2.14) about real lattice points.

Weight derivatives of the state measures. For $h > 0$, an integer $k \ge 0$ and a smooth function ϕ on $\mathbb{R}$, the function $h \mapsto \langle A_h, \phi\rangle = \frac{1}{2}(\phi(\sqrt{h}) + \phi(-\sqrt{h}))$ is smooth on $(0,\infty)$, and by the chain rule its k-th derivative is a linear combination of the numbers $\phi^{(l)}(\pm\sqrt{h})$, $l \le k$. This defines a distribution $A_h^{(k)}$ on $\mathbb{R}$ which is even, supported in $\{\pm\sqrt{h}\}$ and of order at most k. For $h = 0$ we put

$$A_0^{(k)} = \frac{k!}{(2k)!}\,\delta_0^{(2k)}.$$

If ϕ is entire, then $\frac{1}{2}(\phi(\sqrt{h}) + \phi(-\sqrt{h})) = \sum_k \phi^{(2k)}(0)h^k/(2k)!$ is an entire function of h, and $\langle A_h^{(k)}, \phi\rangle$ is its k-th derivative for every $h \ge 0$. Applying this in each variable, for every entire function f on $\mathbb{C}^2$ and all $h, \bar{h} \ge 0$,

$$\langle A_h^{(i)} \otimes A_{\bar{h}}^{(l)}, f\rangle = \partial_h^i \partial_{\bar{h}}^l\, f^\natural(h,\bar{h}). \tag{4.34}$$

For a real point $X = (P,Q)$ and $k \ge 0$ let

$$\Upsilon_X^{(k)} = 2^{-k} \sum_{i+l=k} \binom{k}{i} A_{P^2}^{(i)} \otimes A_{Q^2}^{(l)}. \tag{4.35}$$

By (4.34), for entire f,

$$\langle \Upsilon_X^{(k)}, f\rangle = \frac{d^k}{d\zeta^k}\Big|_{\zeta=0} f^\natural(P^2 + \tfrac{\zeta}{2},\, Q^2 + \tfrac{\zeta}{2}). \tag{4.36}$$

Thus $\Upsilon_X^{(k)}$ is the k-th derivative of the state measure $A_h \otimes A_{\bar{h}}$ along the curve of constant spin through $(h,\bar{h}) = (P^2,Q^2)$, parametrized by the dimension; the variable ζ is the displacement called d in the remark following Lemma 2.6. It is an even distribution supported in $\{(\pm P, \pm Q)\}$ and of local order at most $2k$.

Lemma 4.21 (The Taylor distribution)**.** *Let Z be an admissible partition function of gap g at $c = 1 + 12a$ with $0 < a \le 1$. Let $N \ge 1$ be an integer, $E_* > 0$ and $0 \le \varepsilon < g$. Let $\Sigma \subset \mathbb{R}^2$ be invariant under the reflections $P \mapsto -P$ and $Q \mapsto -Q$ and contain the nine points with $P, Q \in \{0, \pm 1\}$. Suppose that to every state j with $\Delta_j \le E_*$ there is assigned a real point $X_j = (P_j, Q_j) \in \Sigma$ with*

$$P_j^2 - Q_j^2 = s_j, \qquad |\Delta_j - |X_j|^2| \le \varepsilon.$$

Put $\zeta_j = \Delta_j - |X_j|^2 - a$ and define

$$T_a = \sum_{k,l \in \{0,1\}} (-1)^{k+l} \sum_{n=0}^{N} \frac{(-a)^n}{n!}\, \Upsilon^{(n)}_{(\sqrt{k},\sqrt{l})} + \sum_{j:\, \Delta_j \le E_*} \sum_{n=0}^{N} \frac{\zeta_j^n}{n!}\, \Upsilon^{(n)}_{X_j}. \tag{4.37}$$

(i) T_a is an even distribution with finite support contained in Σ, of local order at most $2N$, and $\mathcal{C}T_a = T_a$.

(ii) The local expansion of T_a at the origin is $\delta_0 - \frac{a}{4}(\partial_P^2 + \partial_Q^2)\delta_0$ plus terms of order at least four. In particular the coefficient of $\partial_P^2 \delta_0$ in T_a is $-a/4$, and no state contributes at the origin.

(iii) Let $f \in \mathcal{G}_{b_0}$ for some $b_0 > 0$, let $b > 0$ and $M < \infty$, and suppose that

$$|\partial^\alpha F(X)| \le M\, e^{-b|\operatorname{Re} X|^2} \qquad \text{for } F \in \{f, \mathbb{S}f\},\ |\alpha| \le 2N+2,\ |\operatorname{Im} P|, |\operatorname{Im} Q| \le 1. \tag{4.38}$$

Then, with a constant C_N depending only on N,

$$\begin{aligned} |\langle (\mathbb{S} - I)T_a, f\rangle| \le C_N\, e^{5b} M \Big[&\#\{j : \Delta_j \le E_*\}\, (\varepsilon + a)^{N+1} + a^{N+1} \\ &+ \sum_{j:\, \Delta_j > E_*} e^{-b\Delta_j} \Big]. \end{aligned} \tag{4.39}$$

Proof. (i) Finitely many states have $\Delta_j \le E_*$, so the sum is finite. Each $\Upsilon^{(n)}_X$ with $n \le N$ is even, of local order at most $2N$, and supported in $\{(\pm P, \pm Q)\}$, which is contained in Σ for $X = X_j$ and for the vacuum points. For the phase, let $X = (P, Q)$ be real with $P^2 - Q^2 \in \mathbb{Z}$ and let f be entire. The function $e^{2\pi i(P^2 - Q^2)} f$ has the even evaluation $e^{2\pi i(h-\bar h)} f^\natural(h, \bar h)$, and along the curve $(h, \bar h) = (P^2 + \frac{\zeta}{2}, Q^2 + \frac{\zeta}{2})$ the factor $e^{2\pi i(h-\bar h)}$ is identically one. By (4.36), $\langle \mathcal{C}\Upsilon^{(n)}_X, f\rangle = \langle \Upsilon^{(n)}_X, f\rangle$ for all entire f. Both sides are distributions of finite order with support in the same finite set, and polynomials realize arbitrary jets at finitely many points; hence $\mathcal{C}\Upsilon^{(n)}_X = \Upsilon^{(n)}_X$. This applies to X_j, because $P_j^2 - Q_j^2 = s_j \in \mathbb{Z}$, and to the vacuum points.

(ii) For a state, $|X_j|^2 \ge \Delta_j - \varepsilon \ge g - \varepsilon > 0$, so none of the points $(\pm P_j, \pm Q_j)$ is the origin. The vacuum terms with $(k, l) \ne (0, 0)$ are supported away from the origin. For $(k, l) = (0, 0)$, by the definition of $A_0^{(i)}$,

$$\Upsilon^{(n)}_{(0,0)} = 2^{-n} \sum_{i+l=n} \binom{n}{i} \frac{i!\, l!}{(2i)!\, (2l)!}\, \partial_P^{2i} \partial_Q^{2l} \delta_0,$$

which is δ_0 for $n = 0$, equals $\frac{1}{4}(\partial_P^2 + \partial_Q^2)\delta_0$ for $n = 1$, and has order $2n \ge 4$ for $n \ge 2$. The term $n = 1$ carries the factor $-a$.

(iii) Let $F \in \{f, \mathbb{S}f\}$; recall that $\mathbb{S}f \in \mathcal{G}_{4\pi^2/b_0}$. For a state with $\Delta_j \le E_*$ put $\Phi_j(\zeta) = F^\natural(P_j^2 + \frac{\zeta}{2}, Q_j^2 + \frac{\zeta}{2})$, and for $k, l \in \{0, 1\}$ put $\Psi_{k,l}(\zeta) = F^\natural(k + \frac{\zeta}{2}, l + \frac{\zeta}{2})$. These are entire functions of ζ. Since $P_j^2 = \frac{1}{2}(|X_j|^2 + s_j)$ and $h_j = \frac{1}{2}(\Delta_j + s_j)$, and similarly for Q_j^2 and $\bar h_j$,

$$\Phi_j(\zeta_j) = F^\natural(h_j - \tfrac{a}{2}, \bar h_j - \tfrac{a}{2}), \qquad \Psi_{k,l}(-a) = F^\natural(k - \tfrac{a}{2}, l - \tfrac{a}{2}),$$

which are the shifted evaluations in (2.14). By (4.36) and (4.37), $\langle T_a, F\rangle$ is the sum of the Taylor polynomials of degree N at $\zeta = 0$ of these functions, evaluated at ζ_j and at $-a$ respectively. Therefore

$$\begin{aligned}\mathcal{L}_a(F) - \langle T_a, F\rangle = &\sum_{k,l\in\{0,1\}} (-1)^{k+l}\,\mathrm{Rem}_N[\Psi_{k,l}](-a) + \sum_{j:\,\Delta_j\le E_*} \mathrm{Rem}_N[\Phi_j](\zeta_j)\\ &+ \sum_{j:\,\Delta_j>E_*} F^\natural(h_j - \tfrac{a}{2}, \bar h_j - \tfrac{a}{2}),\end{aligned} \tag{4.40}$$

where $\mathrm{Rem}_N[\Phi](\zeta)$ denotes the Taylor remainder of order N, and where the sums defining $\mathcal{L}_a(F)$ converge absolutely by Lemma 2.4. For real ζ,

$$|\mathrm{Rem}_N[\Phi](\zeta)| \le \frac{|\zeta|^{N+1}}{(N+1)!} \max_{\zeta' \text{ between } 0 \text{ and } \zeta} |\Phi^{(N+1)}(\zeta')|,$$

and $\Phi_j^{(N+1)}(\zeta')$ is the average, with weights $2^{-N-1}\binom{N+1}{i}$, of the derivatives $\partial_h^i \partial_{\bar h}^{N+1-i} F^\natural$ at the point $(P_j^2 + \frac{\zeta'}{2}, Q_j^2 + \frac{\zeta'}{2})$. For ζ' between 0 and ζ_j, the first coordinate of this point lies between $P_j^2 \ge 0$ and $h_j - \frac{a}{2} \ge -\frac{a}{2}$, the second between $Q_j^2 \ge 0$ and $\bar h_j - \frac{a}{2} \ge -\frac{a}{2}$, and their sum $|X_j|^2 + \zeta'$ is at least $-a$. Thus both arguments are real and at least -1, whatever their size, and Lemma 2.6 applies with $k = N+1$ and $\beta = b$, by (4.38). That lemma is uniform over all real arguments $h, \bar h \ge -1$. Hence no separate treatment is needed for paths along which a squared momentum is small, vanishes, becomes slightly negative, or passes from the range in which the proof of that lemma uses the integral formula (2.19) to the range in which it uses the endpoint formula. We obtain $|\Phi_j^{(N+1)}(\zeta')| \le C_{N+1} M e^{4b} e^b$ and, since $|\zeta_j| \le \varepsilon + a$,

$$|\mathrm{Rem}_N[\Phi_j](\zeta_j)| \le C_{N+1} M e^{5b} (\varepsilon + a)^{N+1}.$$

The same argument for $\Psi_{k,l}$ on $[-a, 0]$, where the arguments are at least $-\frac{a}{2}$ and their sum is at least $-a$, gives $|\mathrm{Rem}_N[\Psi_{k,l}](-a)| \le C_{N+1} M e^{5b} a^{N+1}$. For the states with $\Delta_j > E_*$, Lemma 2.6 without derivatives gives

$$|F^\natural(h_j - \tfrac{a}{2}, \bar h_j - \tfrac{a}{2})| \le C_{N+1} M e^{4b} e^{-b(\Delta_j - a)} \le C_{N+1} M e^{5b} e^{-b\Delta_j}.$$

Hence $|\mathcal{L}_a(F) - \langle T_a, F\rangle|$ is bounded by $4 C_{N+1} e^{5b} M$ times the bracket in (4.39). Finally, by Lemma 2.4, $\mathcal{L}_a(\mathbb{S}f) = \mathcal{L}_a(f)$, so

$$\langle (\mathbb{S} - I) T_a, f\rangle = \langle T_a, \mathbb{S}f\rangle - \langle T_a, f\rangle = \big[\mathcal{L}_a(f) - \langle T_a, f\rangle\big] - \big[\mathcal{L}_a(\mathbb{S}f) - \langle T_a, \mathbb{S}f\rangle\big],$$

and (4.39) follows with $C_N = 8 C_{N+1}$. □

The individual coefficients of T_a in (4.1) need not be bounded: they can be large when a nonzero squared momentum P_j^2 or Q_j^2 is small. The proof does not use them; Lemma 2.6 controls their combined action on entire test functions.

Theorem 4.22 (No localization on a fixed union of circle lattices)**.** *Let $0 < g \le \frac{1}{2}$. Let $\mathcal{R}_0 \subset (0, \infty)$ be a finite set of radii with $1 \in \mathcal{R}_0$ and $1/r \in \mathcal{R}_0$ for all $r \in \mathcal{R}_0$, so that $\Sigma_0 = \bigcup_{r\in\mathcal{R}_0} \Lambda_r$ contains Λ_1 and is invariant under the reflection exchanging u and v. Let $p > 0$, $C < \infty$, $A < \infty$ and $u_* > 1/(2\pi)$. There is no admissible sequence of gap g, with parameters $a \to 0$ and $L = \log(1/a)$, such that for every a in the sequence every state j with $\Delta_j \le u_* L$ has a real point $X_j = (P_j, Q_j) \in \Sigma_0$ with*

$$P_j^2 - Q_j^2 = s_j \qquad \text{and} \qquad |\Delta_j - |X_j|^2| \le C\, a^p L^A. \tag{4.41}$$

The assignment $j \mapsto X_j$ is not required to be injective, and no relation between the number of states assigned to a point and any multiplicity is assumed.

Proof. Suppose that such a sequence exists. Fix b with $1/u_* < b < 2\pi$ and an integer $N \geq 1$ with $(N+1)\min(p,1) > 1$, and put $q = 2N$. Let $f_1, \dots, f_n$ and $C_{\Sigma_0,q}$ be given by Theorem 4.19 for $\Sigma = \Sigma_0$. Since f_l and $\mathbb{S}f_l$ lie in $\mathcal{G}_{2\pi}$, the bound (2.17) with $b' = b$ provides a constant M such that (4.38) holds for every $f = f_l$.

Let a be an element of the sequence, so small that $a \leq g^2/4$, $2\pi a < b$, $u_*L \geq 1$ and $\varepsilon_a := Ca^pL^A < g$. The set Σ_0 is invariant under $Q \mapsto -Q$, which exchanges u and v, and under $X \mapsto -X$, hence under both reflections, and it contains the points with $P, Q \in \{0, \pm 1\}$, which lie in Λ_1. Let T_a be the distribution (4.37) for $\Sigma = \Sigma_0$, $E_* = u_*L$, $\varepsilon = \varepsilon_a$ and the points X_j of the hypothesis. By Lemma 4.21(i), T_a is an even tempered distribution supported in Σ_0, of local order at most q, with $\mathcal{C}T_a = T_a$. By Lemma 4.21(ii) and (4.23),

$$\frac{a}{4} \leq C_{\Sigma_0,q} \max_l |\langle (\mathbb{S} - I)T_a, f_l \rangle|.$$

We bound the right side by (4.39). By (2.23), the number of states with $\Delta_j \leq u_*L$ is at most $A_g u_* L\, e^{2\pi a u_* L} \leq 2A_g u_* L$ for small a. With $A_+ = \max(A, 0)$ we have $\varepsilon_a + a \leq (C+1)a^{\min(p,1)}L^{A_+}$. By (2.24) with $\beta = b$ and $E_0 = u_*L$,

$$\sum_{\Delta_j > u_*L} e^{-b\Delta_j} \leq A_g b\Big(\frac{u_*L}{b - 2\pi a} + \frac{1}{(b-2\pi a)^2}\Big) e^{-(b-2\pi a)u_*L} = O\big(L\, a^{bu_*}\big),$$

because $e^{2\pi a u_* L} \to 1$. Altogether

$$\max_l |\langle (\mathbb{S} - I)T_a, f_l \rangle| \leq C' \Big[L^{1+(N+1)A_+}\, a^{(N+1)\min(p,1)} + a^{N+1} + L\, a^{bu_*} \Big] \tag{4.42}$$

with C' independent of a. Since $(N+1)\min(p,1) > 1$, $N+1 > 1$ and $bu_* > 1$, the right side is $o(a)$. This contradicts the lower bound $a/4$ for small a. □

Theorem 4.23 (No localization with one moving radius)**.** *Let $0 < g \leq \frac{1}{2}$. Let $\mathcal{R}_0 \subset (0, \infty)$ be a finite set of radii with $1 \in \mathcal{R}_0$ and $1/r \in \mathcal{R}_0$ for all $r \in \mathcal{R}_0$, and let $\Sigma_0 = \bigcup_{r \in \mathcal{R}_0} \Lambda_r$. Let $I \subset (0, \infty)$ be a compact interval, and let $p > 0$, $C < \infty$, $A < \infty$ and $u_* > 1/(2\pi)$. There is no admissible sequence of gap g, with parameters $a \to 0$ and $L = \log(1/a)$, together with radii $t_a \in I$, such that for every a in the sequence every state j with $\Delta_j \leq u_*L$ has a real point*

$$X_j = (P_j, Q_j) \in \Sigma_0 \cup \Lambda_{t_a} \cup \Lambda_{1/t_a}$$

with

$$P_j^2 - Q_j^2 = s_j \qquad \text{and} \qquad |\Delta_j - |X_j|^2| \leq C\, a^p L^A.$$

The assignment $j \mapsto X_j$ is not required to be injective, no relation between the number of states assigned to a point and any multiplicity is assumed, and $\Sigma_0 \cup \Lambda_{t_a} \cup \Lambda_{1/t_a}$ is not required to be the support of an element of $\mathcal{B}_g$.

Proof. Suppose that such a sequence exists. Passing to a subsequence we may assume $t_a \to t_0 \in I$. We use the notation $\Sigma(t)$ of Proposition 4.20.

Choice of parameters. Fix b with $1/u_* < b < 2\pi$ and an integer $N \geq 1$ with $(N+1)\min(p,1) > 1$, and put $q = 2N$. Apply Proposition 4.20 to $\mathcal{R}_0$, t_0 and q; this provides ε_0, n, the exponent K and the tests $f_{l,t}$. Then apply part (iii) of the proposition with $k_0 = 2N + 2$, the strip $|\operatorname{Im} X| \leq \sqrt{2}$ and the chosen b; this provides the constant C_1 in (4.27). Finally fix $\eta > 0$ so small that

$$K\eta < \tfrac{1}{2} \min\big\{(N+1)\min(p,1) - 1,\ bu_* - 1\big\}. \tag{4.43}$$

Thus N, q, the number of derivatives and K are fixed before η. Either $|t_a - t_0| \geq a^\eta$ for infinitely many a in the sequence, or $|t_a - t_0| < a^\eta$ for all sufficiently small a.

First case: $|t_a - t_0| \ge a^\eta$ along a subsequence. Restrict to this subsequence and to a so small that $|t_a - t_0| < \varepsilon_0$, $a \le g^2/4$, $2\pi a < b$, $u_* L \ge 1$ and $\varepsilon_a := Ca^p L^A < g$. The set $\Sigma(t_a)$ is invariant under both reflections and contains Λ_1. Let T_a be the distribution (4.37) for $\Sigma = \Sigma(t_a)$, $E_* = u_* L$, $\varepsilon = \varepsilon_a$ and the given points X_j. By Lemma 4.21(i) and (ii) and by (4.26) with $t = t_a$,

$$\frac{a}{4} \le \sum_{l=1}^{n} |\langle (\mathbb{S} - I) T_a, f_{l,t_a} \rangle|.$$

By (4.27), each $f = f_{l,t_a}$ satisfies (4.38) with $M = C_1 |t_a - t_0|^{-K} \le C_1 a^{-K\eta}$; here $|\operatorname{Im} P|, |\operatorname{Im} Q| \le 1$ implies $|\operatorname{Im} X| \le \sqrt{2}$. The bracket in (4.39) is estimated exactly as in the proof of (4.42), which used only (2.23), (2.24) and the bound on ε_a. Hence

$$\frac{a}{4} \le C'\, a^{-K\eta} \Big[L^{1+(N+1)A_+}\, a^{(N+1)\min(p,1)} + a^{N+1} + L\, a^{bu_*} \Big], \qquad A_+ = \max(A, 0),$$

with C' independent of a. By (4.43) and $N + 1 \ge (N+1)\min(p,1)$, each of the exponents $(N+1)\min(p,1) - K\eta$, $N + 1 - K\eta$ and $bu_* - K\eta$ is greater than one, so the right side is $o(a)$, a contradiction.

Second case: $|t_a - t_0| < a^\eta$ for all small a. We move the points on the variable lattices to the limiting radius. Let j be a state with $\Delta_j \le u_* L$. If $X_j \in \Sigma_0$ put $X_j' = X_j$. If $X_j \notin \Sigma_0$ and $X_j \in \Lambda_{t_a}$, with light-cone coordinates $(mt_a, n/t_a)$, let $X_j' \in \Lambda_{t_0}$ be the point $(mt_0, n/t_0)$; otherwise $X_j \in \Lambda_{1/t_a}$ has coordinates $(m/t_a, nt_a)$, and we let $X_j' \in \Lambda_{1/t_0}$ be the point $(m/t_0, nt_0)$. In each case X_j' has the same spin $mn = s_j$ as X_j, and $(m,n) \ne (0,0)$ in the last two cases. The dimension of the point with charges (m,n) at radius t is $E(t) = \frac{1}{2}(m^2 t^2 + n^2/t^2)$, and $|\partial_t E(t)| = |m^2 t - n^2/t^3| \le (2/t)\, E(t)$. Integrating $\partial_t \log E$ over the interval between t_a and t_0, which is contained in I, gives

$$\big| |X_j'|^2 - |X_j|^2 \big| \le |X_j|^2 (e^{2|t_a - t_0|/\min I} - 1) \le C_2\, |X_j|^2\, a^\eta \le C_2\, (u_* L + 1)\, a^\eta$$

for small a, where we used $|X_j|^2 \le \Delta_j + Ca^p L^A \le u_* L + 1$; the same bound holds for the points of $\Lambda_{1/t}$, with m and n exchanged. Therefore every state with $\Delta_j \le u_* L$ has a real point $X_j' = (P_j', Q_j')$ with

$$X_j' \in \Sigma_0' := \Sigma_0 \cup \Lambda_{t_0} \cup \Lambda_{1/t_0}, \qquad P_j'^2 - Q_j'^2 = s_j, \qquad \big| \Delta_j - |X_j'|^2 \big| \le C_3\, a^{\min(p,\eta)} L^{\max(A,1)}.$$

The set $\mathcal{R}_0 \cup \{t_0, 1/t_0\}$ is finite, contains 1 and is closed under inversion. The elements of the sequence with a below a suitable threshold form an admissible sequence of gap g, and for it the last display contradicts Theorem 4.22, applied with $\mathcal{R}_0 \cup \{t_0, 1/t_0\}$, the exponent $\min(p, \eta) > 0$, the constants C_3 and $\max(A, 1)$, and the same g and u_*. □

By Lemma 4.12, the hypothesis on $\mathcal{R}_0$ in Theorems 4.22 and 4.23 is satisfied by the set $\mathcal{R}_d$ of radii that carries the integral-spin points of the light-cone grid $d^{-1}\Lambda_1$. The two theorems are equivalent to a statement about individual partition functions.

Corollary 4.24 (Uniform form)**.** *Let g, $\mathcal{R}_0$, Σ_0, I, p, C, A and u_* be as in Theorem 4.23. There is $a_* > 0$ with the following property. If Z is an admissible partition function of gap g at $c = 1 + 12a$ with $0 < a < a_*$, $L = \log(1/a)$, and $t \in I$, then there is a state j with $\Delta_j \le u_* L$ for which no real point $X_j = (P_j, Q_j) \in \Sigma_0 \cup \Lambda_t \cup \Lambda_{1/t}$ satisfies*

$$P_j^2 - Q_j^2 = s_j \qquad \text{and} \qquad \big| \Delta_j - |X_j|^2 \big| \le C\, a^p L^A.$$

The same holds, with a threshold depending only on g, $\mathcal{R}_0$, p, C, A and u_*, when the lattices Λ_t and $\Lambda_{1/t}$ are omitted.

Proof. If no such a_* existed, there would be admissible partition functions Z_n of gap g at $c_n = 1 + 12a_n$, with a_n strictly decreasing to zero, and radii $t_n \in I$, for which every state with $\Delta_j \le u_* \log(1/a_n)$ has a point as described. This is an admissible sequence excluded by Theorem 4.23. The statement without the variable lattices follows in the same way from Theorem 4.22. Conversely, the corollary implies the two theorems, because an admissible sequence contains partition functions with $a < a_*$. □

5 Positive kernels and the reduction to half-integer radii

By Theorem 3.5, every limit of an admissible sequence of gap g is a measure $\sigma(R, t_0)$ in which at least one radius, which we call R, is a half-integer. In this section we prove that the other radius is a half-integer as well (Theorem 5.21), and that near the remaining limits four scalar states at one fixed energy must move more slowly than every power of a (Theorem 5.25). Both results are obtained from Gaussian kernels that are nonnegative on the physical spectrum and whose Lorentzian Fourier transforms are finite sums of narrow Gaussians. The exact Gaussian identity of Lemma 2.4 turns information on finitely many low states into a bound for the sum of the kernel over all states; the bound localizes every state of a logarithmic energy window near finitely many circle lattices, with its exact spin, and this is excluded by Theorems 4.22 and 4.23.

5.1 Conventions

We write $[u, v]$ for the point of $\mathbb{R}^2$ whose light-cone coordinates are u and v, that is, the point $(P, Q) = (\frac{u+v}{2}, \frac{u-v}{2})$. All vectors entering kernels are given in this form. Recall that $|X|^2 = P^2 + Q^2 = \frac{1}{2}(u^2 + v^2)$, that the spin of a real point is $uv = P^2 - Q^2$, and that $B(X, Y) = uv_Y + vu_Y = 2(PP_Y - QQ_Y)$ for $Y = [u_Y, v_Y] = (P_Y, Q_Y)$. For such Y put

$$Y^\sharp = [v_Y, u_Y] = (P_Y, -Q_Y), \qquad B(X, Y^\sharp) = uu_Y + vv_Y = 2(PP_Y + QQ_Y).$$

The map $Y \mapsto Y^\sharp$ is linear and maps Λ_r onto $\Lambda_{1/r}$. A real point is *off both axes* if $PQ \neq 0$, equivalently $u^2 \neq v^2$. The open disc of radius ϱ about $x \in \mathbb{R}^2$, for the Euclidean norm of (P, Q), is denoted $D(x, \varrho)$. The *vacuum support* is

$$V = \operatorname{supp} \nu_{\mathrm{vac}} = \{(P, Q) : P, Q \in \{0, \pm 1\}\} = \{[0, 0],\ \pm[1, 1],\ \pm[1, -1],\ \pm[2, 0],\ \pm[0, 2]\} \subset \Lambda_1,$$

and by (2.3)

$$\nu_{\mathrm{vac}}(\{(0, 0)\}) = 1, \qquad \nu_{\mathrm{vac}}(\{(\pm 1, 0)\}) = \nu_{\mathrm{vac}}(\{(0, \pm 1)\}) = -\tfrac{1}{2}, \qquad \nu_{\mathrm{vac}}(\{(\pm 1, \pm 1)\}) = \tfrac{1}{4}. \tag{5.1}$$

Let Z be admissible at $c = 1 + 12a$. For a state j and $\sigma \in \{\pm 1\}^2$ we write

$$X_j^\sigma = (\sigma_1 \sqrt{h_j},\, \sigma_2 \sqrt{\bar{h}_j}), \qquad X_{j,a}^\sigma = (\sigma_1 \sqrt{h_j - \tfrac{a}{2}},\, \sigma_2 \sqrt{\bar{h}_j - \tfrac{a}{2}})$$

for its *physical points* and its *shifted points*, with one determination of each square root fixed. Thus $\rho_a = \sum_j \frac{1}{4} \sum_\sigma \delta_{X_j^\sigma}$, and for every entire f

$$f^\natural(h_j, \bar{h}_j) = \tfrac{1}{4} \sum_\sigma f(X_j^\sigma), \qquad f^\natural(h_j - \tfrac{a}{2}, \bar{h}_j - \tfrac{a}{2}) = \tfrac{1}{4} \sum_\sigma f(X_{j,a}^\sigma). \tag{5.2}$$

Each coordinate of $X_{j,a}^\sigma$ is real or purely imaginary, and

$$\left|\operatorname{Re} X_{j,a}^\sigma - X_j^\sigma\right| \le \sqrt{a}, \qquad \left|\operatorname{Im} X_{j,a}^\sigma\right|^2 \le a, \tag{5.3}$$

because $|\sqrt{h-a/2}-\sqrt{h}| \le \sqrt{a/2}$ for $h \ge a/2$, while for $0 \le h < a/2$ the shifted coordinate is imaginary of modulus at most $\sqrt{a/2}$ and $\sqrt{h} < \sqrt{a/2}$. The same notation and the same bounds are used for the four vacuum weights $(k,l) \in \{0,1\}^2$, whose physical points form V. For a function f that is not even, the physical pairing is by definition the pairing with the even measure μ_a,

$$\langle \mu_a, f\rangle = \sum_{k,l\in\{0,1\}} (-1)^{k+l} f^\natural(k,l) + \sum_j f^\natural(h_j, \bar h_j), \tag{5.4}$$

so that every state enters through all four of its physical points, each with weight $\frac14$. The analytic evaluation $\mathcal{L}_a(f)$ of (2.14) is the same expression with all weights shifted by $-\frac{a}{2}$.

5.2 Trigonometric kernels and their Fourier transforms

A *trigonometric polynomial* is a function

$$\Phi(X) = \sum_{i\in\Xi} c_i\, e^{2\pi i B(X,\xi_i)}, \tag{5.5}$$

where Ξ is a finite index set, $\xi_i \in \mathbb{R}^2$ and $c_i \in \mathbb{R}$. We always assume that the family $(\xi_i, c_i)_{i\in\Xi}$ is invariant under $\xi \mapsto -\xi$, so that Φ is real on $\mathbb{R}^2$. The points ξ_i are called the *centres* of Φ; they need not be distinct, and each index is treated separately. For $\epsilon > 0$ and $\xi \in \mathbb{R}^2$ put

$$\Phi_\epsilon(X) = e^{-2\pi\epsilon^2|X|^2}\,\Phi(X) = e^{-\pi\epsilon^2(u^2+v^2)}\,\Phi(X), \qquad \gamma_{\epsilon,\xi}(X) = \epsilon^{-2} e^{-2\pi|X-\xi|^2/\epsilon^2}. \tag{5.6}$$

We call Φ_ϵ a *kernel*; its first factor is the broad Gaussian and $\gamma_{\epsilon,\xi}$ is a narrow Gaussian. As in Section 2, $|X-\xi|^2$ denotes the polynomial $(P-P_\xi)^2 + (Q-Q_\xi)^2$ when X is complex, so that

$$|\gamma_{\epsilon,\xi}(X)| = \epsilon^{-2}\exp\Big(-\frac{2\pi}{\epsilon^2}\big(|\operatorname{Re} X - \xi|^2 - |\operatorname{Im} X|^2\big)\Big). \tag{5.7}$$

The building blocks are, for real Y and an integer $D \ge 1$,

$$S_Y(X) = \sin^2(\pi B(X,Y)), \qquad \Omega_D(X) = \sin^2(\pi D u) + \sin^2(\pi D v) = S_{[0,D]}(X) + S_{[D,0]}(X), \tag{5.8}$$

and we write

$$C = [2,0], \qquad C^\sharp = [0,2], \qquad \Omega = \Omega_2 = S_C + S_{C^\sharp} = \sin^2(2\pi u) + \sin^2(2\pi v).$$

A *product kernel* is built from a product of four sine-square factors $S_{Y_1}S_{Y_1^\sharp}S_{Y_2}S_{Y_2^\sharp}$, and a *sum kernel* from $\Omega_D S_Y S_{Y^\sharp}$ or from Ω_D alone. If $Y \in \Lambda_r$, then $B(\cdot, Y)$ is integral on Λ_r and $B(\cdot, Y^\sharp)$ is integral on $\Lambda_{1/r}$; hence

$$S_Y S_{Y^\sharp} = 0 \ \text{ on } \Lambda_r \cup \Lambda_{1/r} \quad (Y \in \Lambda_r), \qquad \Omega_D = 0 \ \text{ on } \{[u,v] : Du, Dv \in \mathbb{Z}\}. \tag{5.9}$$

Lemma 5.1 (Fourier expansion of kernels)**.** *Let Φ be a trigonometric polynomial as in (5.5) and let $\epsilon > 0$. Then $\Phi_\epsilon \in \mathcal{G}_{2\pi\epsilon^2}$,*

$$\mathbb{S}\Phi_\epsilon = \sum_{i\in\Xi} c_i\,\gamma_{\epsilon,\xi_i}, \qquad \textit{that is,} \qquad \mathbb{S}\Phi_\epsilon(X) = \epsilon^{-2}\sum_{i\in\Xi} c_i \exp\Big(-\frac{\pi}{\epsilon^2}\big((u-u_{\xi_i})^2 + (v-v_{\xi_i})^2\big)\Big), \tag{5.10}$$

and for every admissible partition function at $c = 1 + 12a$

$$\mathcal{L}_a(\Phi_\epsilon) = \sum_{i\in\Xi} c_i\,\mathcal{L}_a(\gamma_{\epsilon,\xi_i}), \tag{5.11}$$

all evaluations being absolutely convergent. The centres and coefficients of the building blocks are as follows.

(i) *For* $\Phi = S_{Y_1}S_{Y_2}S_{Y_3}S_{Y_4}$*:* $\Xi = \{-1,0,1\}^4$*,* $\xi_\eta = \sum_{i=1}^4 \eta_i Y_i$ *and*

$$c_\eta = \frac{1}{16}\Big(-\frac{1}{2}\Big)^{|\eta|}, \qquad |\eta| = \#\{i : \eta_i \neq 0\}.$$

The coefficient is negative exactly at the centres with $|\eta| \in \{1,3\}$*.*

(ii) *For* $\Phi = \Omega_D S_Y S_{Y^\sharp}$*: the* 45 *centres are* $\zeta + \eta_1 Y + \eta_2 Y^\sharp$ *with* $\zeta \in \{0, \pm[D,0], \pm[0,D]\}$ *and* $\eta \in \{-1,0,1\}^2$*, and*

	$\eta = 0$	$\lvert\eta\rvert = 1$	$\lvert\eta\rvert = 2$
$\zeta = 0$	$\frac{1}{4}$	$-\frac{1}{8}$	$\frac{1}{16}$
$\zeta \neq 0$	$-\frac{1}{16}$	$\frac{1}{32}$	$-\frac{1}{64}$

are the coefficients. There are 1, 4, 4 *centres in the first row and* 4, 16, 16 *in the second.*

(iii) *For* $\Phi = \Omega_D$*: the centres are* 0 *with coefficient* 1 *and* $\pm[D,0]$*,* $\pm[0,D]$ *with coefficient* $-\frac{1}{4}$*.*

Proof. In momentum coordinates $e^{2\pi i B(X,\xi)} = e^{z\cdot X}$ with $z = 4\pi i(P_\xi, -Q_\xi)$, so Φ_ϵ is a finite sum of functions $\phi_{b,z}$ with $b = 2\pi\epsilon^2$ and belongs to $\mathcal{G}_b$. By (2.8), with $J = \mathrm{diag}(1,-1)$ and $z = 4\pi i J\xi$,

$$\mathbb{S}\big(e^{-2\pi\epsilon^2|X|^2} e^{2\pi i B(X,\xi)}\big) = \frac{1}{\epsilon^2}\exp\Big(\frac{(4\pi i)^2\, J(\xi+X)\cdot J(\xi+X)}{8\pi\epsilon^2}\Big) = \gamma_{\epsilon,-\xi}(X).$$

Summing over i and using the invariance of the family under $\xi \mapsto -\xi$ gives (5.10); the second form is the same formula in light-cone coordinates, since $2|X-\xi|^2 = (u-u_\xi)^2 + (v-v_\xi)^2$. Lemma 2.4 gives $\mathcal{L}_a(\Phi_\epsilon) = \mathcal{L}_a(\mathbb{S}\Phi_\epsilon)$ with absolute convergence, and each $\gamma_{\epsilon,\xi}$ lies in $\mathcal{G}_{2\pi/\epsilon^2}$, which gives (5.11).

The tables follow from

$$S_Y(X) = \tfrac{1}{2} - \tfrac{1}{4}e^{2\pi i B(X,Y)} - \tfrac{1}{4}e^{-2\pi i B(X,Y)}, \qquad \Omega_D(X) = 1 - \tfrac{1}{4}\sum_{\zeta\in\{\pm[D,0],\pm[0,D]\}} e^{2\pi i B(X,\zeta)},$$

the second because $B(X,[0,D]) = Du$ and $B(X,[D,0]) = Dv$. Multiplying four factors of the first kind gives the coefficient $(\frac{1}{2})^{4-|\eta|}(-\frac{1}{4})^{|\eta|}$ of (i). In (ii) the coefficient is the product of 1 or $-\frac{1}{4}$, according as $\zeta = 0$ or not, with $(\frac{1}{2})^{2-|\eta|}(-\frac{1}{4})^{|\eta|} \in \{\frac{1}{4}, -\frac{1}{8}, \frac{1}{16}\}$. In each case the family is invariant under $\xi \mapsto -\xi$. □

Lemma 5.2 (Evenness and positivity)**.** *Let* $Y \in \mathbb{R}^2$ *and let* $D \geq 1$ *be an integer.*

(i) *As entire functions on* $\mathbb{C}^2$*,*

$$\begin{aligned} S_Y(X)S_{Y^\sharp}(X) &= \tfrac{1}{4}\big(\cos(4\pi Q_Y Q) - \cos(4\pi P_Y P)\big)^2, \\ \Omega_D(X) &= 1 - \cos(2\pi DP)\cos(2\pi DQ). \end{aligned} \tag{5.12}$$

In particular $S_Y S_{Y^\sharp}$ *and* Ω_D *are even.*

(ii) *If* Φ *is a product of functions* $S_Y S_{Y^\sharp}$*, then* $\Phi_\epsilon^\natural(h,\bar h) \geq 0$ *for all real* $h, \bar h$ *of either sign. Consequently, for an admissible partition function at* $c = 1 + 12a$*, the physical and the shifted evaluations of a product kernel at every state are nonnegative, also when one shifted momentum is imaginary.*

(iii) *If* Φ *is* Ω_D *times a product of functions* $S_Y S_{Y^\sharp}$*, then* $\Phi_\epsilon \geq 0$ *on* $\mathbb{R}^2$*, so the physical evaluations* $\Phi_\epsilon^\natural(h_j,\bar h_j) = \Phi_\epsilon(X_j^\sigma)$ *are nonnegative. The shifted evaluation of* Ω_D *at a point with* P *real and* $Q = i\tau$ *imaginary equals* $1 - \cos(2\pi DP)\cosh(2\pi D\tau)$ *and may be negative.*

Proof. With $\alpha = 2\pi P_Y P$ and $\beta = 2\pi Q_Y Q$ one has $\pi B(X,Y) = \alpha - \beta$ and $\pi B(X,Y^\sharp) = \alpha + \beta$, and $\sin(\alpha-\beta)\sin(\alpha+\beta) = \frac{1}{2}(\cos 2\beta - \cos 2\alpha)$. Similarly $\sin^2(\pi Du) + \sin^2(\pi Dv) = 1 - \frac{1}{2}(\cos(2\pi Du) + \cos(2\pi Dv))$ and $u = P + Q$, $v = P - Q$. This is (i). The function $x \mapsto \cos(4\pi P_Y\sqrt{x})$ is entire with real Taylor coefficients, hence real for real x of either sign. Therefore $(S_Y S_{Y^\sharp})^\natural(h,\bar h)$ is the square of a real number for real $h, \bar h$; the broad Gaussian contributes the positive factor $e^{-2\pi\epsilon^2(h+\bar h)}$, and (ii) follows. Part (iii) is immediate from (5.8) and (5.12). □

Only physical evaluations enter the budgets below; part (ii) is recorded because it shows that product kernels give positive sum rules directly on the analytic evaluations.

5.3 Central-shift and tail estimates

Lemma 5.3 (Central shift for kernels)**.** *Let $0 < g \le \frac{1}{2}$, and let Φ be a trigonometric polynomial with $|\xi_i| \le \varrho_*$ for all i and $\sum_i |c_i| \le c_*$. There is $C = C(g,\varrho_*,c_*)$ such that, for $0 < \epsilon \le \frac{1}{2}$, every admissible partition function of gap g at $c = 1 + 12a$ with $0 < a \le \min\{g^2/4, \epsilon^2/4\}$ satisfies*

$$|\Phi_\epsilon^\natural(h - \tfrac{a}{2}, \bar h - \tfrac{a}{2}) - \Phi_\epsilon^\natural(h,\bar h)| \le C\,a\,e^{-\pi\epsilon^2(h+\bar h)} \qquad (h,\bar h \ge 0), \tag{5.13}$$

$$\sum_j |\Phi_\epsilon^\natural(h_j - \tfrac{a}{2}, \bar h_j - \tfrac{a}{2}) - \Phi_\epsilon^\natural(h_j,\bar h_j)| \le C\,a\,\epsilon^{-2}, \tag{5.14}$$

$$\Big|\sum_{k,l\in\{0,1\}} (-1)^{k+l}\Phi_\epsilon^\natural(k - \tfrac{a}{2}, l - \tfrac{a}{2}) - \langle \nu_{\mathrm{vac}}, \Phi_\epsilon\rangle\Big| \le C\,a. \tag{5.15}$$

In particular, if Φ vanishes on V, the shifted vacuum value of Φ_ϵ is at most Ca in absolute value.

Proof. Let $|\operatorname{Im} P|, |\operatorname{Im} Q| \le 1$ and $|\alpha| \le 2$. Since $|\operatorname{Im} B(X,\xi)| \le 2|\operatorname{Im} X||\xi|$, each exponential $e^{2\pi i B(X,\xi_i)}$ and its derivatives of order at most two are bounded by $(1 + 4\pi\varrho_*)^2 e^{4\sqrt{2}\pi\varrho_*}$. The broad Gaussian has modulus $e^{-2\pi\epsilon^2(|\operatorname{Re} X|^2 - |\operatorname{Im} X|^2)} \le e^{4\pi} e^{-2\pi\epsilon^2|\operatorname{Re} X|^2}$, and its derivatives of order at most two are this function times polynomials of degree at most two in $\epsilon^2 P$, $\epsilon^2 Q$ and ϵ^2, hence bounded by a constant times $(1 + \epsilon^2|\operatorname{Re} X|)^2 e^{-2\pi\epsilon^2|\operatorname{Re} X|^2} \le C' e^{-\pi\epsilon^2|\operatorname{Re} X|^2}$. Therefore

$$|\partial^\alpha \Phi_\epsilon(X)| \le M\, e^{-\pi\epsilon^2|\operatorname{Re} X|^2}, \qquad M = C''(\varrho_*)\, c_*, \tag{5.16}$$

on this strip. Corollary 2.7 with $\beta = \pi\epsilon^2 \le 1$ gives (5.13), and applying it to the four vacuum weights gives (5.15). Since $a \le \beta/(4\pi)$, summing (5.13) over the states with (2.26) gives (5.14). □

Narrow test functions are not uniformly bounded on a strip of fixed width. For them we use the following variant of Corollary 2.7, in which only the strip actually visited by the central shift enters.

Lemma 5.4 (Central shift on the thin strip)**.** *Let $0 < a \le 1$, let f be entire on $\mathbb{C}^2$, and suppose that for some $M < \infty$ and $\beta > 0$*

$$|\partial^\alpha f(X)| \le M\, e^{-\beta|\operatorname{Re} X|^2} \qquad \text{for } |\alpha| \le 2, \quad |\operatorname{Im} P|, |\operatorname{Im} Q| \le \sqrt{a/2}.$$

Then for all $h, \bar h \ge 0$

$$|f^\natural(h - \tfrac{a}{2}, \bar h - \tfrac{a}{2}) - f^\natural(h,\bar h)| \le a\, M\, e^{3\beta}\, e^{-\beta(h+\bar h)}.$$

Proof. It suffices to show that $|\partial_h f^\natural(h',\bar h')| \le \frac12 M e^{2\beta} e^{-\beta(h'+\bar h')}$ for real $h',\bar h' \ge -\frac{a}{2}$, and the same for $\partial_{\bar h}$; integrating $\frac{d}{ds} f^\natural(h-\frac{s}{2},\bar h-\frac{s}{2})$ over $0\le s\le a$ then gives the claim, with $e^{\beta a}\le e^\beta$. Fix $\bar h'$ and a sign of $\sqrt{\bar h'}$, and let ϕ be the function $P\mapsto f(P,\pm\sqrt{\bar h'})$. If $h'\le 2$, Lemma 2.5 with $k=1$ expresses the derivative as $\frac14\int_{-1}^1 \phi''(t\sqrt{h'})\,dt$; the points $(t\sqrt{h'},\pm\sqrt{\bar h'})$ lie in the thin strip and the square of their real part is at least $\max\{\bar h',0\}\ge h'+\bar h'-2$. If $h'>2$, the derivative is $(\phi'(\sqrt{h'})-\phi'(-\sqrt{h'}))/(4\sqrt{h'})$, the points are in the thin strip, and the square of their real part is at least $h'+\bar h'$. Averaging over the sign of $\sqrt{\bar h'}$ gives the bound in both cases. □

Lemma 5.5 (Tails of narrow Gaussians)**.** *Let $0<g\le\frac12$ and $\varrho_*\ge 1$. There is $C_1=C_1(g,\varrho_*)$ such that for every admissible partition function of gap g at $c=1+12a$ with $0<a\le g^2/4$, every $x\in\mathbb{R}^2$ with $|x|\le\varrho_*$, every $0<\varrho\le\frac14$ and every $0<\epsilon\le\frac12$,*

$$\sum_j \frac14 \sum_{\sigma:\,|X_j^\sigma - x|\ge\varrho} \exp\Big(-\frac{\pi|X_j^\sigma-x|^2}{2\epsilon^2}\Big) \le C_1\, e^{-\pi\varrho^2/(2\epsilon^2)}. \tag{5.17}$$

If moreover $a\le\varrho^2/4$, then for every state and every σ with $|X_j^\sigma - x|\ge\varrho$,

$$|\gamma_{\epsilon,x}(X_{j,a}^\sigma)| \le \epsilon^{-2} e^{2\pi a/\epsilon^2} \exp\Big(-\frac{\pi|X_j^\sigma-x|^2}{2\epsilon^2}\Big), \tag{5.18}$$

and the same holds for the shifted points of the four vacuum weights.

Proof. Put $E_1 = 4\varrho_*^2 \ge 4$. By (2.23) there are at most $A_g E_1 e^{\pi E_1/8}$ states with $\Delta_j\le E_1$, and each of their terms is at most $e^{-\pi\varrho^2/(2\epsilon^2)}$. If $\Delta_j > E_1$, then $|X_j^\sigma| = \sqrt{\Delta_j} > 2|x|$, so $|X_j^\sigma - x| \ge \frac12|X_j^\sigma|$ and the term is at most $e^{-\beta\Delta_j}$ with $\beta=\pi/(8\epsilon^2)\ge\pi/2$. Since $2\pi a\le\pi/8\le\beta/2$, (2.24) bounds the sum of these terms by

$$A_g\beta\Big(\frac{2E_1}{\beta}+\frac{4}{\beta^2}\Big)e^{-\beta E_1/2} \le A_g(2E_1+\tfrac{8}{\pi})e^{-\pi/(4\epsilon^2)} \le A_g(2E_1+\tfrac{8}{\pi})e^{-\pi\varrho^2/(2\epsilon^2)}.$$

This proves (5.17). If $a\le\varrho^2/4$ and $|X_j^\sigma - x|\ge\varrho$, then by (5.3) $|\operatorname{Re} X_{j,a}^\sigma - x| \ge |X_j^\sigma - x| - \sqrt{a} \ge \frac12|X_j^\sigma - x|$ and $|\operatorname{Im} X_{j,a}^\sigma|^2 \le a$, and (5.18) follows from (5.7). □

5.4 The peak budget

By (5.11), the analytic evaluation of a kernel is a signed sum of narrow Gaussian peaks. The next lemma bounds this sum from above. Peaks with positive coefficient need only an upper bound for the number of states near the centre; peaks with negative coefficient need a lower bound, which is available when enough states are known to sit close to the centre, away from the momentum axes. The leading terms cancel by a relation among the limiting masses. The lemma is stated for a single partition function; all its hypotheses are explicit, and the limiting measure enters only through the numbers m_i.

Lemma 5.6 (Peak budget)**.** *Let $0<g\le\frac12$, $\varrho_*\ge1$, $c_*,m_*<\infty$ and $0<\varrho\le\min\{\frac14,\frac12\sqrt{g}\}$. Let Φ be a trigonometric polynomial (5.5) with $|\xi_i|\le\varrho_*$ and $\sum_i|c_i|\le c_*$, and let $(m_i)_{i\in\Xi}$ be real numbers with $0\le m_i\le m_*$ and*

$$\sum_{i\in\Xi} c_i\, m_i = 0. \tag{5.19}$$

Let Z be admissible of gap g at $c=1+12a$, let $\varepsilon\ge0$, and assume the following.

(B0) Φ vanishes on the vacuum support V.

(B1) Origin. *If $\xi_i = 0$, then $c_i > 0$ and $m_i = 1$.*

(B2) Positive centres. *If $c_i > 0$ and $\xi_i \neq 0$, then* $\operatorname{dist}(\xi_i, V) \geq \varrho$ *and* $\rho_a(D(\xi_i, \varrho)) \leq m_i$.

(B3) Negative centres. *If $c_i < 0$, then one of the following holds.*

(a) Anchored centre. *$|P_{\xi_i}| \geq 2\varrho$ and $|Q_{\xi_i}| \geq 2\varrho$; the disc $D(\xi_i, \varrho)$ contains no point of V other than possibly ξ_i itself, and in that case $\xi_i \in \{(\pm 1, \pm 1)\}$; and, with $w_i = \nu_{\mathrm{vac}}(\{\xi_i\}) \in \{0, \frac{1}{4}\}$, there are $n_i \geq 0$ distinct states j with*

$$s_j = P_{\xi_i}^2 - Q_{\xi_i}^2, \qquad |\Delta_j - |\xi_i|^2| \leq \varepsilon, \qquad \text{and} \qquad w_i + \tfrac{1}{4} n_i \geq m_i.$$

(b) Empty centre. *$m_i = 0$,* $\operatorname{dist}(\xi_i, V) \geq \varrho$ *and* $\rho_a(D(\xi_i, \varrho)) = 0$.

Then there is $C = C(g, \varrho_*, c_*, m_*)$, independent of ϱ, Z, a, ϵ and ε, such that

$$\sum_j \Phi_\epsilon^\natural(h_j, \bar{h}_j) \leq \frac{C}{\varrho^2}\Big((a + \varepsilon^2)\epsilon^{-4} + e^{-\pi\varrho^2/(4\epsilon^2)}\Big) \tag{5.20}$$

whenever

$$0 < \epsilon \leq \tfrac{1}{2}, \qquad 0 < a \leq \min\{\tfrac{1}{4}g^2,\ \tfrac{1}{4}\epsilon^2,\ \tfrac{1}{4}\varrho^2\}, \qquad \varepsilon \leq \varrho^2. \tag{5.21}$$

In particular, for fixed ϱ the right side of (5.20) is at most $C'\big((a + \varepsilon^2)\epsilon^{-4} + e^{-\kappa/\epsilon^2}\big)$ with $\kappa = \pi\varrho^2/4$ and $C' = C\varrho^{-2}$.

No lower bound on the mass near a positive centre is assumed, positive centres are allowed to carry limiting mass, and the states near a positive centre need not be close to it on the scale ϵ. No bound of this kind could hold at a positive centre placed at one of the negative atoms $(\pm 1, 0)$, $(0, \pm 1)$ of ν_{vac}, which is why (B2) and (B3) keep all centres other than the origin and $(\pm 1, \pm 1)$ away from V.

Proof. Write $\gamma_i = \gamma_{\epsilon,\xi_i}$ and $\mathsf{N} = \epsilon^{-2} e^{-\pi\varrho^2/(2\epsilon^2)}$. Since Φ is real on $\mathbb{R}^2$, the entire function $\Phi_\epsilon^\natural$ is real at real arguments and $\mathcal{L}_a(\Phi_\epsilon)$ is real, so by (5.11)

$$\mathcal{L}_a(\Phi_\epsilon) = \sum_{i \in \Xi} c_i \operatorname{Re} \mathcal{L}_a(\gamma_i). \tag{5.22}$$

Fix i and write $\xi = \xi_i$. By (5.2), $\mathcal{L}_a(\gamma_i)$ is a sum over the pairs (j, σ) and over the shifted points of the four vacuum weights. Call a pair, or a vacuum point, *far* if the corresponding physical point is at distance at least ϱ from ξ, and *near* otherwise. By (5.18), (5.17) and $e^{2\pi a/\epsilon^2} \leq e^{\pi/2}$, the far terms satisfy

$$\sum_{(j,\sigma)\ \text{far}} \tfrac{1}{4}|\gamma_i(X_{j,a}^\sigma)| + \sum_{\text{far vacuum points}} \tfrac{1}{4}|\gamma_i| \leq e^{\pi/2}(C_1 + 4)\,\mathsf{N} =: \mathsf{F}. \tag{5.23}$$

We also use $e^x \leq 1 + e^{\pi/2}x$ for $0 \leq x \leq \frac{\pi}{2}$, with $x = 2\pi a/\epsilon^2$.

The origin. Let $\xi = 0$. Every state is far, because $|X_j^\sigma| = \sqrt{\Delta_j} \geq \sqrt{g} \geq 2\varrho$, and so are the vacuum weights other than $(0, 0)$. The shifted point of the weight $(0, 0)$ is $(i\sqrt{a/2}, i\sqrt{a/2})$ up to signs, where γ_i equals $\epsilon^{-2} e^{2\pi a/\epsilon^2}$. Hence

$$\operatorname{Re} \mathcal{L}_a(\gamma_i) \leq \epsilon^{-2} + 2\pi e^{\pi/2} a\, \epsilon^{-4} + \mathsf{F} = m_i \epsilon^{-2} + 2\pi e^{\pi/2} a\, \epsilon^{-4} + \mathsf{F}.$$

Positive centres. Let $c_i > 0$ and $\xi \neq 0$. All vacuum points are far. At a near pair, (5.7) and (5.3) give $|\gamma_i(X_{j,a}^\sigma)| \leq \epsilon^{-2} e^{2\pi a/\epsilon^2}$, and the near pairs have total weight $\sum \frac{1}{4} = \rho_a(D(\xi, \varrho)) \leq m_i$. Hence

$$\operatorname{Re} \mathcal{L}_a(\gamma_i) \leq m_i \epsilon^{-2} + 2\pi e^{\pi/2} m_* a\, \epsilon^{-4} + \mathsf{F}.$$

Empty centres. In case (B3)(b) every pair and every vacuum point is far, so $|\mathcal{L}_a(\gamma_i)| \le \mathsf{F}$, and $m_i = 0$.

Anchored centres. Assume (B3)(a). If a state j has a near physical point, then $\sqrt{h_j} \ge |P_\xi| - \varrho \ge \varrho$ and likewise $\sqrt{\bar h_j} \ge \varrho$, so $h_j - \frac{a}{2}$ and $\bar h_j - \frac{a}{2}$ are positive, all four shifted points of j are real, and γ_i is positive at each of them. If j has no near point, all its pairs are far. Consequently the real part of the sum over the states is at least the sum of $\frac14\gamma_i(X^\sigma_{j,a})$ over any chosen set of pairs with near physical points, minus the far terms. Let j be one of the n_i states of (B3)(a), and let σ be the sign pattern of ξ. Since $s_j = P_\xi^2 - Q_\xi^2$, we have $|h_j - P_\xi^2| = |\bar h_j - Q_\xi^2| = \frac12|\Delta_j - |\xi|^2| \le \frac12\varepsilon$. As $P_\xi^2 \ge 4\varrho^2$, $\varepsilon \le \varrho^2$ and $a \le \varrho^2$, the number $h_j - \frac{a}{2}$ is positive, and

$$\Big|\sqrt{h_j - \tfrac{a}{2}} - |P_\xi|\Big| \le \frac{|h_j - \frac{a}{2} - P_\xi^2|}{|P_\xi|} \le \frac{\varepsilon + a}{4\varrho},$$

and the same for the second coordinate. The same computation with $a = 0$ shows that X^σ_j is near. Therefore $|X^\sigma_{j,a} - \xi|^2 \le (\varepsilon + a)^2/(8\varrho^2)$ and

$$\gamma_i(X^\sigma_{j,a}) \ge \epsilon^{-2}\max\{0, 1 - x\}, \qquad x = \frac{\pi(\varepsilon + a)^2}{4\varrho^2\epsilon^2}.$$

If $\xi \in \{(\pm1, \pm1)\}$, the vacuum weight $(1, 1)$ enters $\mathcal{L}_a$ with sign $+1$; its four shifted points $\sqrt{1 - a/2}\,(\pm1, \pm1)$ are real, the one with the signs of ξ is at squared distance $2(1 - \sqrt{1 - a/2})^2 \le a^2/2$ from ξ, and γ_i is at least $\epsilon^{-2}(1 - \pi a^2/\epsilon^2) \ge \epsilon^{-2}(1 - x)$ there. The physical points of the other vacuum weights are at distance at least one from ξ and are far. If $\xi \notin V$, all vacuum points are far. Altogether

$$\operatorname{Re}\mathcal{L}_a(\gamma_i) \ge (w_i + \tfrac14 n_i)\epsilon^{-2}\max\{0, 1 - x\} - \mathsf{F} \ge m_i\epsilon^{-2} - \frac{\pi m_*(\varepsilon^2 + a)}{2\varrho^2\epsilon^4} - \mathsf{F},$$

where we used $w_i + \frac14 n_i \ge m_i \ge 0$ and $(\varepsilon + a)^2 \le 2\varepsilon^2 + 2a$.

Conclusion. Multiplying each of the four bounds by c_i, with the appropriate sign, and inserting them into (5.22) we obtain, by (5.19),

$$\mathcal{L}_a(\Phi_\epsilon) \le \epsilon^{-2}\sum_i c_i m_i + c_*\Big(2\pi e^{\pi/2} m_* \frac{a}{\epsilon^4} + \frac{\pi m_*(\varepsilon^2 + a)}{2\varrho^2\epsilon^4} + \mathsf{F}\Big) \le \frac{C}{\varrho^2}\Big(\frac{a + \varepsilon^2}{\epsilon^4} + e^{-\pi\varrho^2/(4\epsilon^2)}\Big),$$

because $\mathsf{N} \le 4(\pi e\varrho^2)^{-1}e^{-\pi\varrho^2/(4\epsilon^2)}$. Finally, by (B0) and Lemma 5.3,

$$\sum_j \Phi^{\natural}_\epsilon(h_j, \bar h_j) \le \mathcal{L}_a(\Phi_\epsilon) + Ca + Ca\epsilon^{-2},$$

and the last two terms are bounded by $Ca\epsilon^{-4}\varrho^{-2}$. □

Along a convergent sequence, all hypotheses of Lemma 5.6 except the displacement bound in (B3)(a) follow from vague convergence.

Lemma 5.7 (The hypotheses along a convergent sequence)**.** *Let an admissible sequence of gap g converge to ρ_0, and put $\mu_0 = \nu_{\mathrm{vac}} + \rho_0$. For each a in the sequence let $\Phi^{(a)} = \sum_{i\in\Xi} c_i e^{2\pi i B(\cdot, \xi_i(a))}$ be a trigonometric polynomial with a fixed index set and fixed coefficients, such that $\xi_i(a) \to \xi_i^0$ as $a \to 0$ for every i, and $\xi_i(a) = \xi_i^0$ for all a whenever $\xi_i^0 \in V$. Let $\Phi^{(0)}$ be the trigonometric polynomial with centres ξ_i^0, assume that $\Phi^{(0)}$ vanishes on $\operatorname{supp}\mu_0$, and put $m_i = \mu_0(\{\xi_i^0\})$.*

(i) The masses satisfy (5.19): $\sum_i c_i\,\mu_0(\{\xi_i^0\}) = 0$.

(ii) There is $\varrho_0 > 0$ such that for every $0 < \varrho \le \varrho_0$ and all sufficiently small a in the sequence: condition (B2) *holds for $\Phi^{(a)}$ at every index with $\xi_i^0 \notin V$, and condition* (B3)(b) *holds at every index with $\xi_i^0 \notin V \cup \operatorname{supp}\rho_0$.*

(iii) Let ξ_i^0 be off both axes, with $\xi_i^0 \notin V \setminus \{(\pm1,\pm1)\}$, and put $n_i = 4\rho_0(\{\xi_i^0\})$. For $0 < \varrho \le \varrho_0$ and all sufficiently small a, exactly n_i states of ρ_a have a physical point in $D(\xi_i^0, \varrho)$; they have the spin $s(\xi_i^0)$ of ξ_i^0, and their dimensions converge to $|\xi_i^0|^2$. If $s(\xi_i(a)) = s(\xi_i^0)$ and these dimensions are within ε of $|\xi_i(a)|^2$, then (B3)(a) *holds for $\Phi^{(a)}$ at the index i with these n_i states, and $w_i + \frac{1}{4}n_i = m_i$.*

The thresholds in a depend on the sequence and are not effective.

Proof. (i) For $a = 0$ the evaluation $\mathcal{L}_0(f)$ is the pairing $\langle \mu_0, f\rangle$, and Lemma 2.4 (in the case $a = 0$) together with (5.10) gives $\langle \mu_0, \Phi_\epsilon^{(0)}\rangle = \sum_i c_i \langle \mu_0, \gamma_{\epsilon,\xi_i^0}\rangle$. Since $\Phi^{(0)}$ vanishes on $\operatorname{supp}\mu_0$,

$$0 = \epsilon^2 \langle \mu_0, \Phi_\epsilon^{(0)}\rangle = \sum_i c_i \int e^{-2\pi|X-\xi_i^0|^2/\epsilon^2}\, \mathrm{d}\mu_0(X) \qquad (0 < \epsilon \le 1).$$

The integrands are dominated by $e^{-2\pi|X-\xi_i^0|^2}$, which is integrable for $|\nu_{\mathrm{vac}}| + \rho_0$ by (2.23) with $a = 0$, and converge pointwise to the indicator of $\{\xi_i^0\}$ as $\epsilon \to 0$. Dominated convergence gives (i).

(ii) The set $V \cup \operatorname{supp}\rho_0$ is locally finite. Choose $\varrho_0 \le \min\{\frac{1}{4}, \frac{1}{2}\sqrt{g}\}$ so small that, for every i, the closed disc of radius $2\varrho_0$ about ξ_i^0 contains no point of $V \cup \operatorname{supp}\rho_0$ other than possibly ξ_i^0, and $|P_{\xi_i^0}|, |Q_{\xi_i^0}| \ge 3\varrho_0$ whenever ξ_i^0 is off both axes. Let $0 < \varrho \le \varrho_0$ and let a be so small that $|\xi_i(a) - \xi_i^0| < \frac{1}{2}\varrho$ for all i. Then $D(\xi_i(a), \varrho) \subset D' = D(\xi_i^0, \frac{3}{2}\varrho)$. Since $\rho_0(\partial D') = 0$, vague convergence gives $\rho_a(D') \to \rho_0(D') = \rho_0(\{\xi_i^0\})$, and since all these numbers lie in $\frac{1}{4}\mathbb{Z}$, equality holds for small a. If $\xi_i^0 \notin V$, then $\rho_0(\{\xi_i^0\}) = m_i$ and $\operatorname{dist}(\xi_i(a), V) \ge 2\varrho_0 - \frac{1}{2}\varrho \ge \varrho$, which is (B2); if moreover $\xi_i^0 \notin \operatorname{supp}\rho_0$, then $m_i = 0$ and $\rho_a(D(\xi_i(a), \varrho)) = 0$, which is (B3)(b).

(iii) Let U' be the union of the four discs of radius $\frac{3}{2}\varrho$ about the points $(\pm P_{\xi_i^0}, \pm Q_{\xi_i^0})$. It is bounded, open and invariant under the two reflections, and its boundary does not meet $\operatorname{supp}\rho_0$ because ρ_0 is even. A state of ρ_0 supported in U' has weights $(P_{\xi_i^0}^2, Q_{\xi_i^0}^2)$, both positive, and gives mass $\frac{1}{4}$ to ξ_i^0; so there are n_i such states. By Corollary 2.11, for small a exactly n_i states of ρ_a are supported in U', their spins (which are integers and converge) equal $s(\xi_i^0)$, and their dimensions converge to $|\xi_i^0|^2$. A state of ρ_a with a physical point in $D(\xi_i^0, \varrho)$ is supported in U'; conversely, by the convergence of dimensions and spins, each of the n_i states supported in U' has, for small a, a physical point in $D(\xi_i^0, \varrho)$. For small a the geometric requirements of (B3)(a) hold at $\xi_i(a)$ by the choice of ϱ_0, and $m_i = \nu_{\mathrm{vac}}(\{\xi_i^0\}) + \rho_0(\{\xi_i^0\}) = w_i + \frac{1}{4}n_i$, because $\xi_i(a) = \xi_i^0$ when $\xi_i^0 \in V$. □

5.5 Local zeros and the three-projection lemma

A budget (5.20) bounds the value of the trigonometric polynomial at every state of a logarithmic window, since all terms are nonnegative. Two deterministic lemmas convert such a bound into information on the state. The first concerns a state close to a known zero of the kernel; the second concerns an arbitrary state and several kernels.

Let $X_0 = [u_0, v_0] = (P_0, Q_0)$ be a real point off both axes. A state with the spin u_0v_0 of X_0 and with dimension $|X_0|^2 + \delta$ has weights $(P_0^2 + \frac{\delta}{2}, Q_0^2 + \frac{\delta}{2})$; for $|\delta| < 2\min\{P_0^2, Q_0^2\}$ we denote by

$$X(\delta) = \Big(\operatorname{sgn}(P_0)\sqrt{P_0^2 + \tfrac{\delta}{2}},\ \operatorname{sgn}(Q_0)\sqrt{Q_0^2 + \tfrac{\delta}{2}}\Big) \tag{5.24}$$

its physical point with the signs of X_0. This is the *fixed-spin curve* through X_0, parametrized by the dimension. For $Y=[u_Y,v_Y]$ put

$$\lambda(X_0,Y)=\frac{|u_0v_Y-v_0u_Y|}{|u_0^2-v_0^2|}. \tag{5.25}$$

Lemma 5.8 (Local zeros along a fixed-spin curve)**.** *Let $\varrho_*\ge 1$ and $\varpi>0$, let $X_0=(P_0,Q_0)$ be a real point with $|P_0|,|Q_0|\ge\varpi$ and $|X_0|\le\varrho_*$, and let $|\delta|\le\varpi^2$. Let $Y\in\mathbb{R}^2$ with $|Y|\le\varrho_*$.*

(i) $|S_Y(X(\delta))-S_Y(X_0)|\le\pi\varrho_*|\delta|/\varpi$.

(ii) If $B(X_0,Y)\in\mathbb{Z}$ and $\lambda=\lambda(X_0,Y)>0$, then $S_Y(X(\delta))\ge\lambda^2\delta^2$ whenever $|\delta|\le\min\{2\lambda\varpi^3/\varrho_*,\ 1/(3\lambda)\}$.

(iii) Let Φ be a function on $\mathbb{R}^2$ with $\Phi\ge\prod_{i=1}^{k}S_{Y_i}\prod_{i=1}^{k'}S_{Y_i'}$, where $k\ge1$, all $|Y_i|,|Y_i'|\le\varrho_*$, and, for some $\lambda>0$ and $\varkappa>0$,

$$B(X_0,Y_i)\in\mathbb{Z},\quad \lambda(X_0,Y_i)\ge\lambda\quad(1\le i\le k),\qquad S_{Y_i'}(X_0)\ge\varkappa\quad(1\le i\le k').$$

Put $\delta_1=\min\{\varpi^2,\ 2\lambda\varpi^3/\varrho_*,\ \varpi/(3\varrho_*),\ \varkappa\varpi/(2\pi\varrho_*)\}$. Then for $|\delta|\le\delta_1$

$$\Phi(X(\delta))\ge\Big(\frac{\varkappa}{2}\Big)^{k'}\lambda^{2k}\delta^{2k};\qquad\text{so}\qquad |\delta|\le\frac{1}{\lambda}\Big(\Big(\frac{2}{\varkappa}\Big)^{k'}\Gamma\Big)^{1/(2k)}\quad\text{if}\ \ \Phi(X(\delta))\le\Gamma.$$

Proof. Write $X(\delta)=(P,Q)$. For $|\delta|\le\varpi^2$ we have $|P|,|Q|\ge\varpi/\sqrt{2}$, and $dP/d\delta=1/(4P)$, $d^2P/d\delta^2=-1/(16P^3)$, and likewise for Q. The phase $\beta(\delta)=B(X(\delta),Y)=2(PP_Y-QQ_Y)$ therefore satisfies

$$\beta'=\frac12\Big(\frac{P_Y}{P}-\frac{Q_Y}{Q}\Big),\qquad |\beta'|\le\frac{\varrho_*}{\varpi},\qquad |\beta''|\le\frac{|P_Y|+|Q_Y|}{8(\varpi/\sqrt2)^3}\le\frac{\varrho_*}{2\varpi^3},$$

and $\beta'(0)=(P_YQ_0-Q_YP_0)/(2P_0Q_0)=(u_0v_Y-v_0u_Y)/(u_0^2-v_0^2)$, so that $|\beta'(0)|=\lambda(X_0,Y)\le\varrho_*/\varpi$. Since $x\mapsto\sin^2(\pi x)$ has derivative bounded by π, (i) follows. In (ii), $|\beta(\delta)-\beta(0)-\beta'(0)\delta|\le\varrho_*\delta^2/(4\varpi^3)\le\frac12\lambda|\delta|$, so the distance from $\beta(\delta)$ to the integer $\beta(0)$ lies between $\frac12\lambda|\delta|$ and $\frac32\lambda|\delta|\le\frac12$, and $\sin^2(\pi x)\ge4x^2$ for $|x|\le\frac12$ gives the claim. In (iii), each factor S_{Y_i} is at least $\lambda(X_0,Y_i)^2\delta^2\ge\lambda^2\delta^2$ by (ii), whose conditions hold because $2\lambda(X_0,Y_i)\varpi^3/\varrho_*\ge2\lambda\varpi^3/\varrho_*$ and $1/(3\lambda(X_0,Y_i))\ge\varpi/(3\varrho_*)$; each factor $S_{Y_i'}$ is at least $\frac12\varkappa$ by (i). □

We use the elementary inequality

$$\sin^2(\pi x)\ge4\,\mathrm{dist}(x,\mathbb{Z})^2\qquad(x\in\mathbb{R}). \tag{5.26}$$

Lemma 5.9 (Three projections)**.** *Let $\ell\ge1$ and $p\ge1$ be integers with $\gcd(p,2\ell)=1$, put $p_k=p+k\ell$, let $0<t_-\le t\le t_+$ and $\tau_*=\max\{t_+,1/t_-\}$, and let*

$$B_k(t)=[t,\,p_k/t]\in\Lambda_t,\qquad k=0,1,2.$$

Let $U,V_0,\Delta V\ge1$ be integers, $V_j=V_0+j\Delta V$, $A_j=[U,V_j]$ for $j=0,1,2$, and $D=2U\Delta V$. Let $X=[u,v]$ be a real point with $uv=s\in\mathbb{Z}$ and $|u|,|v|\le\Upsilon$, where $\Upsilon\ge1$, and let $0<\zeta\le1$.

(a) Free vectors. *Suppose that for every $k\in\{0,1,2\}$*

$$\mathrm{dist}\big(B(X,B_k(t)),\mathbb{Z}\big)\le\zeta\quad\text{or}\quad\mathrm{dist}\big(B(X,B_k(t)^\sharp),\mathbb{Z}\big)\le\zeta, \tag{5.27}$$

and that $(11+6p)\,\tau_*\Upsilon\zeta<1/(4\ell^2)$. Then there is $\lambda=[u_\lambda,v_\lambda]\in\Lambda_t\cup\Lambda_{1/t}$ with

$$u_\lambda v_\lambda=s,\qquad |u-u_\lambda|,\ |v-v_\lambda|\le(3+2p)\tau_*\zeta,\qquad \big||X|^2-|\lambda|^2\big|\le3(3+2p)\tau_*\zeta\Upsilon.$$

(b) Fixed vectors. *Put* $C_A = 3 + 2V_2$. *Suppose that for every* $j \in \{0,1,2\}$

$$\operatorname{dist}(B(X,A_j),\mathbb{Z}) \le \zeta \quad \text{or} \quad \operatorname{dist}(B(X,A_j^\sharp),\mathbb{Z}) \le \zeta, \tag{5.28}$$

and that $3C_A\zeta\Upsilon < D^{-2}$. *Then there are integers* m,n *with*

$$\frac{mn}{D^2} = s, \qquad \Big|u - \frac{m}{D}\Big|,\ \Big|v - \frac{n}{D}\Big| \le C_A\zeta, \qquad \Big||X|^2 - \frac{m^2+n^2}{2D^2}\Big| \le 3C_A\zeta\Upsilon.$$

(c) Product kernels. *If* $0 \le \Gamma \le 1$ *and*

$$S_{A_j}(X)S_{A_j^\sharp}(X)\,S_{B_k(t)}(X)S_{B_k(t)^\sharp}(X) \le \Gamma \qquad \text{for all } 0 \le j,k \le 2,$$

then (5.27) *holds for all* k *or* (5.28) *holds for all* j, *with* $\zeta = \frac12\Gamma^{1/8}$.

(d) Sum kernels. *If* $0 \le \Gamma \le 1$ *and* $\Omega(X)\,S_{B_k(t)}(X)S_{B_k(t)^\sharp}(X) \le \Gamma$ *for* $k = 0,1,2$, *put* $\zeta = \frac12\Gamma^{1/6}$. *Then* (5.27) *holds for all* k, *or there are integers* m,n *with* $|u - \frac{m}{2}|, |v - \frac{n}{2}| \le \frac12\zeta$. *In the second case, if* $6\zeta\Upsilon < 1$, *then* $\frac{mn}{4} = s$ *and* $\big||X|^2 - \frac18(m^2+n^2)\big| \le \frac32\zeta\Upsilon$.

Proof. (a) Call k of the first kind if the first alternative in (5.27) holds, and of the second kind otherwise. Two of the three indices, $k_1 < k_2$, are of the same kind. If they are of the first kind put $(\hat u,\hat v) = (u/t, tv)$, otherwise $(\hat u,\hat v) = (v/t, tu)$. In both cases $\hat u\hat v = s$, $|\hat u|, |\hat v| \le \tau_*\Upsilon$, and, since $B(X,B_k(t)) = p_k u/t + tv$ and $B(X,B_k(t)^\sharp) = tu + p_k v/t$,

$$p_{k_i}\hat u + \hat v = n_i + e_i, \qquad n_i \in \mathbb{Z}, \quad |e_i| \le \zeta \qquad (i = 1,2).$$

Let $D' = (k_2 - k_1)\ell \in \{\ell, 2\ell\}$, $m' = n_2 - n_1$ and $n' = D'n_1 - p_{k_1}m'$. Subtracting the two relations and substituting back,

$$\hat u = \frac{m'}{D'} + e_{\hat u}, \quad |e_{\hat u}| \le \frac{2\zeta}{D'} \le 2\zeta; \qquad \hat v = \frac{n'}{D'} + e_{\hat v}, \quad |e_{\hat v}| \le \zeta + \frac{2p_{k_1}\zeta}{D'} \le (3+2p)\zeta,$$

because $p_{k_1}/D' \le p+1$: if $k_1 = 1$ then $k_2 = 2$ and $D' = \ell$. Hence

$$\Big|s - \frac{m'n'}{D'^2}\Big| = |e_{\hat u}\hat v + (\hat u - e_{\hat u})e_{\hat v}| \le (5+2p)\tau_*\Upsilon\zeta + 2(3+2p)\zeta^2 \le (11+6p)\tau_*\Upsilon\zeta < \frac{1}{D'^2}.$$

Both s and $m'n'/D'^2$ lie in $D'^{-2}\mathbb{Z}$, so $m'n' = sD'^2$ and $D'^2 \mid m'n'$. We claim that $D' \mid m'$. The integer p_{k_1} is coprime to D': for $k_1 = 0$ because $\gcd(p, 2\ell) = 1$, and for $k_1 = 1$ because $D' = \ell$ and $\gcd(p+\ell, \ell) = 1$. Suppose that a prime has exponent $e \ge 1$ in D' and exponent $r < e$ in m'. It does not divide p_{k_1}, so its exponent in $n' = D'n_1 - p_{k_1}m'$ is exactly r, and its exponent in $m'n'$ is $2r < 2e$, contradicting $D'^2 \mid m'n'$. Hence $D' \mid m'$, and then $D' \mid n'$. With $m = m'/D'$ and $n = n'/D'$ we get $mn = s$, $|\hat u - m| \le 2\zeta$ and $|\hat v - n| \le (3+2p)\zeta$. In the first kind, $\lambda = [mt, n/t] \in \Lambda_t$ satisfies $|u - mt| \le 2\tau_*\zeta$ and $|v - n/t| \le (3+2p)\tau_*\zeta$; in the second kind, $\lambda = [n/t, mt] \in \Lambda_{1/t}$ satisfies the same bounds with u and v exchanged. In both cases $u_\lambda v_\lambda = mn = s$. Finally, with $c = (3+2p)\tau_*\zeta \le 1 \le \Upsilon$,

$$\big||X|^2 - |\lambda|^2\big| \le \tfrac12|u - u_\lambda|(2|u| + |u - u_\lambda|) + \tfrac12|v - v_\lambda|(2|v| + |v - v_\lambda|) \le c(2\Upsilon + c) \le 3c\Upsilon.$$

(b) Since $B(X,A_j) = V_j u + Uv$ and $B(X,A_j^\sharp) = Uu + V_j v$, two indices $j_1 < j_2$ satisfy, with $(x_1,x_2) = (u,v)$ or $(x_1,x_2) = (v,u)$,

$$V_{j_i}x_1 + Ux_2 = n_i + e_i, \qquad n_i \in \mathbb{Z}, \quad |e_i| \le \zeta \qquad (i = 1,2).$$

Let $D'' = (j_2 - j_1)\Delta V \in \{\Delta V, 2\Delta V\}$. Subtracting, $x_1 = (n_2 - n_1)/D'' + e'$ with $|e'| \le 2\zeta$, and then

$$x_2 = \frac{D''n_1 - V_{j_1}(n_2 - n_1)}{D''U} + e'', \qquad |e''| \le \frac{\zeta + 2V_{j_1}\zeta}{U} \le (1 + 2V_2)\zeta.$$

Both D'' and $D''U$ divide $D = 2U\Delta V$, so x_1 and x_2, hence u and v, are within $C_A\zeta$ of $D^{-1}\mathbb{Z}$; let m/D and n/D be the corresponding grid values. Then $|s - mn/D^2| \le C_A\zeta(2\Upsilon + C_A\zeta) \le 3C_A\zeta\Upsilon < D^{-2}$, and both numbers lie in $D^{-2}\mathbb{Z}$, so they are equal. The bound for $|X|^2$ is obtained as in (a).

(c) If $S_{A_j}S_{A_j^\sharp}(X) > \Gamma^{1/2}$ for some j, then $S_{B_k}S_{B_k^\sharp}(X) \le \Gamma^{1/2}$ for every k; otherwise $S_{A_j}S_{A_j^\sharp}(X) \le \Gamma^{1/2}$ for every j. If a product of two numbers of $[0,1]$ is at most $\Gamma^{1/2}$, one of them is at most $\Gamma^{1/4}$, and by (5.26) the corresponding phase is within $\frac{1}{2}\Gamma^{1/8}$ of an integer.

(d) If $\Omega(X) > \Gamma^{1/3}$, then $S_{B_k}S_{B_k^\sharp}(X) \le \Gamma^{2/3}$ for every k, one factor is at most $\Gamma^{1/3}$, and its phase is within $\frac{1}{2}\Gamma^{1/6}$ of an integer. Otherwise $\sin^2(2\pi u)$ and $\sin^2(2\pi v)$ are at most $\Gamma^{1/3}$, so $2u$ and $2v$ are within ζ of integers m, n. Then $|s - \frac{mn}{4}| \le \frac{1}{2}\zeta(2\Upsilon + \frac{1}{2}\zeta) \le \frac{3}{2}\zeta\Upsilon < \frac{1}{4}$, and both numbers lie in $\frac{1}{4}\mathbb{Z}$; the bound for $|X|^2$ follows as in (a). □

In (b) and (d) the associated point $[m/D, n/D]$, respectively $[m/2, n/2]$, has integral spin, so by Lemma 4.12 it lies in the finite union of circle lattices

$$\Sigma_D = \bigcup_{k \mid D^2} \Lambda_{k/D}, \tag{5.29}$$

respectively in $\Sigma_2 = \Lambda_{1/2} \cup \Lambda_1 \cup \Lambda_2$. The set Σ_D contains Λ_1 and is invariant under the reflection $[u, v] \mapsto [v, u]$, because $k \mapsto D^2/k$ permutes the divisors of D^2.

5.6 Two kernel families and their finite hypotheses

Throughout, $I \subset (0, \infty)$ is a compact interval, $\ell \ge 1$ is an integer, and p is an integer with

$$\gcd(p, 2\ell) = 1, \qquad p > \sup_{t \in I} t^2 + 2. \tag{5.30}$$

For an integer $n \ge 1$ and $t > 0$ put

$$Y_n(t) = [t,\, n/t] \in \Lambda_t, \qquad Y_n(t)^\sharp = [n/t,\, t] \in \Lambda_{1/t},$$
$$p_k = p + k\ell, \qquad B_k(t) = Y_{p_k}(t) \quad (k = 0, 1, 2).$$

The point $Y_n(t)$ has spin n and squared norm $E_n(t) = \frac{1}{2}(t^2 + n^2/t^2)$. For $t \in I$ the points $B_k(t)$ are off both axes, because $p_k > t^2$, and lie outside V, because $p_k/t > t + 2/t > 2$; and E_p is strictly decreasing on I, because $E_p'(t) = t - p^2/t^3 < 0$. For a half-integer R we put

$$\Sigma(t) = V \cup \Lambda_R \cup \Lambda_{1/R} \cup \Lambda_t \cup \Lambda_{1/t}. \tag{5.31}$$

If $\rho_0 = \sigma(R, t_0)$ and $\mu_0 = \nu_{\mathrm{vac}} + \rho_0 = \frac{1}{2}\rho_R^{\mathrm{F}} + \frac{1}{2}\rho_{t_0}^{\mathrm{F}}$, then by (2.10)

$$\mu_0(\{x\}) = \tfrac{1}{4}(e_R + e_{1/R} + e_{t_0} + e_{1/t_0})(\{x\}), \qquad \operatorname{supp}\mu_0 \subset \Sigma(t_0), \qquad \operatorname{supp}\rho_0 \subset \Sigma(t_0). \tag{5.32}$$

The product family

Product data consist of I, ℓ, p as above, a half-integer $R = K/2$ with $K \ge 3$ and $K \ne 4$, and integers $U, V_0, \Delta V \ge 1$ such that, with $V_j = V_0 + j\Delta V$ and $A_j = [U, V_j]$ for $j = 0, 1, 2$,

$$A_j \in \Lambda_R, \qquad U \ge 3, \qquad V_j \ge 3, \qquad |U - V_j| \ge 3. \tag{5.33}$$

Thus $A_j \in \Lambda_R \cap \Lambda_1$ is off both axes, and $A_j^\sharp \in \Lambda_{1/R} \cap \Lambda_1$. The ten trigonometric polynomials of the family are

$$\Phi_t^* = S_C S_{C^\sharp}\, S_{B_0(t)} S_{B_0(t)^\sharp}, \qquad \Phi_t^{j,k} = S_{A_j} S_{A_j^\sharp}\, S_{B_k(t)} S_{B_k(t)^\sharp} \quad (0 \le j,k \le 2). \tag{5.34}$$

Since $C^\sharp = [0,2] = [0\cdot R,\ K/R] \in \Lambda_R$ and $C, A_j \in \Lambda_1$, (5.9) shows that each of them vanishes on $\Sigma(t)$. By Lemma 5.1(i) their centres are

$$\xi_\eta(t) = \eta_1 Y + \eta_2 Y^\sharp + \eta_3 B_k(t) + \eta_4 B_k(t)^\sharp, \qquad \eta \in \{-1,0,1\}^4, \tag{5.35}$$

with $Y = C$, $k = 0$ for Φ_t^* and $Y = A_j$ for $\Phi_t^{j,k}$. The *product hypotheses* at $t_0 \in I$ are:

(H1) Neither $2t_0$ nor $2/t_0$ is an integer, and $B_k(t_0) \notin \Lambda_{1/t_0} \cup \Lambda_R \cup \Lambda_{1/R}$ for $k = 0,1,2$.

(H2) For each of the ten polynomials, every centre $\xi_\eta(t_0)$ with $|\eta| = 3$ lies outside $\Sigma(t_0)$.

(H3) For each of the ten polynomials, every centre $\xi_\eta(t_0)$ with $|\eta| \in \{2,4\}$ lies outside V.

(H4) $S_{B_0(t_0)}(A_j)\, S_{B_0(t_0)^\sharp}(A_j) \neq 0$ for $j = 0,1,2$, and $S_C\, S_{C^\sharp}\, S_{B_0(t_0)^\sharp}$ does not vanish at $B_k(t_0)$ for $k = 1,2$.

The centres with $|\eta| = 1$ are $\pm C, \pm C^\sharp \in V$ for Φ_t^*, and $\pm A_j$, $\pm A_j^\sharp$, $\pm B_k(t)$, $\pm B_k(t)^\sharp$, which lie outside V by (5.33) and the remark above. No condition is imposed on coincidences among the centres.

The sum family

Sum data consist of I, ℓ, p as above, a radius $R \in \{1,2\}$, and an integer w with $1 \le w < p$. We write

$$W(t) = Y_w(t) = [t,\ w/t], \qquad \mathcal{N} = \{p_0, p_1, p_2, w\}, \qquad \Phi_t^{(n)} = \Omega\, S_{Y_n(t)} S_{Y_n(t)^\sharp} \quad (n \in \mathcal{N}).$$

Since $\Lambda_1 \cup \Lambda_2 \cup \Lambda_{1/2} \subset \frac{1}{2}\mathbb{Z}^2$ in light-cone coordinates, (5.9) shows that $\Phi_t^{(n)}$ vanishes on $\Sigma(t)$. By Lemma 5.1(ii) its centres are $\zeta + \eta_1 Y_n(t) + \eta_2 Y_n(t)^\sharp$ with $\zeta \in \{0, \pm C, \pm C^\sharp\}$ and $\eta \in \{-1,0,1\}^2$. We also need the 27 points

$$x_{l,k,j}(t) = [2,2] - lC^\sharp - kW(t) - jW(t)^\sharp = \big[\,2 - kt - jw/t,\ 2 - 2l - kw/t - jt\,\big], \qquad 0 \le l,k,j \le 2. \tag{5.36}$$

The *sum hypotheses* at $t_0 \in I$ are:

(S1) Neither $2t_0$ nor $2/t_0$ is an integer, and $Y_n(t_0) \notin \Lambda_{1/t_0}$ for $n \in \mathcal{N}$.

(S2) For $n \in \mathcal{N}$, $\zeta \in \{\pm C, \pm C^\sharp\}$ and $\eta \in \{\pm 1\}^2$, the centre $\zeta + \eta_1 Y_n(t_0) + \eta_2 Y_n(t_0)^\sharp$ lies outside $\Sigma(t_0)$.

(S3) For $n \in \mathcal{N}$, $\zeta \in \{\pm C, \pm C^\sharp\}$ and $\eta \in \{\pm 1\}^2$, the centres $\eta_1 Y_n(t_0) + \eta_2 Y_n(t_0)^\sharp$, $\zeta + \eta_1 Y_n(t_0)$ and $\zeta + \eta_1 Y_n(t_0)^\sharp$ lie outside V.

(S4) $S_{B_0(t_0)^\sharp}(Y_n(t_0)) \neq 0$ for $n \in \{p_1, p_2, w\}$, and neither $2(t_0 + p/t_0)$ nor $2(t_0 - p/t_0)$ is an integer.

(S5) $x_{l,k,j}(t_0) \notin \Sigma(t_0)$ whenever $(k,j) \neq (0,0)$.

Under (S1) each $Y_n(t_0)$, $n \in \mathcal{N}$, is off both axes (if $t_0^2 = n$ then $Y_n(t_0) = [n/t_0, t_0] \in \Lambda_{1/t_0}$), lies outside $\frac{1}{2}\mathbb{Z}^2 \supset V \cup \Lambda_R \cup \Lambda_{1/R}$, and satisfies $\Omega(Y_n(t_0)) \ge \sin^2(2\pi t_0) > 0$.

Lemma 5.10 (The exceptional set is finite)**.** *For given product data, the set of $t_0 \in I$ at which one of* (H1)–(H4) *fails is finite. For given sum data, the set of $t_0 \in I$ at which one of* (S1)–(S5) *fails is finite.*

Proof. All points considered below have the form

$$\xi(t) = \big[a_1 + b_1 t + b_2 n/t,\ a_2 + b_2 t + b_1 n/t\big], \qquad a_1, a_2, b_1, b_2 \in \mathbb{Z}, \quad n \geq 1, \tag{5.37}$$

with data from a finite list: for (5.35), $[a_1, a_2] = \eta_1 Y + \eta_2 Y^\sharp$, $(b_1, b_2) = (\eta_3, \eta_4)$ and $n = p_k$; for the centres of $\Phi_t^{(n)}$, $[a_1, a_2] = \zeta$ and $(b_1, b_2) = \eta$; for (5.36), $[a_1, a_2] = [2, 2-2l]$, $(b_1, b_2) = (-k, -j)$ and $n = w$. For $t \in I$ these points stay in a fixed ball, and a point of Λ_t or $\Lambda_{1/t}$ in that ball has charges in a fixed finite set, as has a point of $V \cup \Lambda_R \cup \Lambda_{1/R}$. Thus $\xi(t) \in \Sigma(t)$ for some $t \in I$ means that one of finitely many systems

$$\xi(t) = [mt,\ n'/t], \qquad \xi(t) = [m/t,\ n't], \qquad \xi(t) = [x_1, x_2]$$

holds, with integers m, n' and a fixed point $[x_1, x_2]$. After multiplication by t, each coordinate equation is a polynomial equation of degree at most two in t, so a system has finitely many solutions unless both of its polynomials vanish identically. Comparing coefficients, and using $n \neq 0$, this happens only as follows:

$$\begin{aligned} &\xi(t) \in \Lambda_t \text{ identically} &&\Longrightarrow\quad a_1 = a_2 = 0,\ b_2 = 0;\\ &\xi(t) \in \Lambda_{1/t} \text{ identically} &&\Longrightarrow\quad a_1 = a_2 = 0,\ b_1 = 0;\\ &\xi(t) = [x_1, x_2] \text{ identically} &&\Longrightarrow\quad b_1 = b_2 = 0. \end{aligned} \tag{5.38}$$

Indeed, the first system reads $a_1 t + (b_1 - m)t^2 + b_2 n = 0$ and $a_2 t + b_2 t^2 + b_1 n - n' = 0$, the second reads $a_1 t + b_1 t^2 + b_2 n - m = 0$ and $a_2 t + (b_2 - n')t^2 + b_1 n = 0$, and the third has first equation $(a_1 - x_1)t + b_1 t^2 + b_2 n = 0$.

Product data. For $Y = [u_Y, v_Y] \in \{C, A_0, A_1, A_2\}$ we have $u_Y^2 \neq v_Y^2$, so $\eta_1 Y + \eta_2 Y^\sharp = [\eta_1 u_Y + \eta_2 v_Y, \eta_1 v_Y + \eta_2 u_Y]$ vanishes only for $\eta_1 = \eta_2 = 0$. If $|\eta| = 3$, none of the three cases of (5.38) can occur, since they would force $|\eta| \leq 1$, $|\eta| \leq 1$ and $|\eta| \leq 2$ respectively; hence (H2) fails only on a finite set. If $|\eta| \in \{2, 4\}$ and $(\eta_3, \eta_4) \neq (0, 0)$, the third case cannot occur, so $\xi_\eta(t) \in V$ for finitely many t only; if $\eta_3 = \eta_4 = 0$ the centre is one of $\pm[u_Y + v_Y, u_Y + v_Y]$, $\pm[u_Y - v_Y, v_Y - u_Y]$, which is never in V: for $Y = C$ these are $\pm[2, 2]$, $\pm[2, -2]$, and for $Y = A_j$ both coordinates have modulus at least 3 by (5.33). Hence (H3) fails only on a finite set. In (H1), $2t_0 \in \mathbb{Z}$ or $2/t_0 \in \mathbb{Z}$ holds at finitely many points of I; $B_k(t) \in \Lambda_{1/t}$ requires $t^2 = m$ with a bounded integer m; and $B_k(t) \in \Lambda_R \cup \Lambda_{1/R}$ requires t to be the first coordinate of one of finitely many fixed points. In (H4) the phases are

$$B(A_j, B_0(t)) = \frac{Up}{t} + V_j t, \qquad B(A_j, B_0(t)^\sharp) = Ut + \frac{V_j p}{t},$$
$$B(B_k(t), C) = \frac{2p_k}{t}, \qquad B(B_k(t), C^\sharp) = 2t,$$

and $B(B_k(t), B_0(t)^\sharp) = t^2 + pp_k/t^2$. Each is a nonconstant rational function of t, bounded on I, and therefore integer-valued at finitely many points of I.

Sum data. In (S2) we have $[a_1, a_2] = \zeta \neq 0$ and $b_1 b_2 \neq 0$, and in (S3) we have $(b_1, b_2) \neq (0, 0)$ and the target is a fixed point; no case of (5.38) occurs. In (S5), $a_1 = 2 \neq 0$ excludes the first two cases and $(k, j) \neq (0, 0)$ the third. (S1) is treated like (H1), and the phases in (S4) are $B(Y_n(t), B_0(t)^\sharp) = t^2 + pn/t^2$ and $2(t \pm p/t)$, again nonconstant rational functions. □

5.7 The case $R \notin \{1, 2\}$

Proposition 5.11 (Localization and exclusion for product data)**.** *Let $0 < g \le \frac{1}{2}$, let product data be given, and let t_0 be an interior point of I at which* (H1)–(H4) *hold. Then no admissible sequence of gap g converges to $\sigma(R, t_0)$.*

Proof. Suppose that an admissible sequence of gap g converges to $\rho_0 = \sigma(R, t_0)$, and let $\mu_0 = \nu_{\mathrm{vac}} + \rho_0$. All constants below depend on the data, on g and on t_0; statements hold for all sufficiently small a in the sequence.

Step 0: limiting masses. By (5.32), a point $[2, 0] = (1, 1)$ lies in Λ_r if and only if $2/r \in \mathbb{Z}$, and in $\Lambda_{1/r}$ if and only if $2r \in \mathbb{Z}$. Since $2R = K \in \mathbb{Z}$, $2/R = 4/K \notin \mathbb{Z}$, and neither $2t_0$ nor $2/t_0$ is an integer, we get $\mu_0(\{C\}) = \frac{1}{4} = \nu_{\mathrm{vac}}(\{C\})$, and the same at $-C$, $\pm C^\sharp$. By (H1), $\mu_0(\{B_k(t_0)\}) = \frac{1}{4}$. By (H4), $A_j \notin \Lambda_{t_0} \cup \Lambda_{1/t_0}$: otherwise $B(A_j, B_0(t_0))$ or $B(A_j, B_0(t_0)^\sharp)$ would be an integer. Hence $\mu_0(\{A_j\}) = \frac{1}{4} n_j^0$ with $n_j^0 \in \{1, 2\}$, according as $A_j \notin \Lambda_{1/R}$ or $A_j \in \Lambda_{1/R}$. These points are outside V, except $\pm C, \pm C^\sharp$, so the same values hold for ρ_0 at $B_k(t_0)$ and A_j, and $\rho_0(\{\pm C\}) = \rho_0(\{\pm C^\sharp\}) = 0$.

Step 1: the moving radius. By Lemma 5.7(iii), applied to the constant family $\Phi^*_{t_0}$ and its centre $B_0(t_0)$, exactly one state of ρ_a has a physical point near $B_0(t_0)$; it has spin p, and its dimension $\Delta_{0,a}$ converges to $E_p(t_0)$. Since E_p is strictly decreasing on I and t_0 is interior, there is a unique $t_a \in I$ with

$$E_p(t_a) = \Delta_{0,a}, \qquad t_a \to t_0. \tag{5.39}$$

In the same way, using the constant families $\Phi^{j,k}_{t_0}$ and their centres $B_k(t_0)$ and A_j: for $k = 1, 2$ exactly one state has a physical point near $B_k(t_0)$; it has spin p_k, and we write its dimension as $E_{p_k}(t_a) + \delta_{k,a}$, where $\delta_{k,a} \to 0$. Exactly n_j^0 states have a physical point near A_j; they have spin UV_j, and their dimensions $|A_j|^2 + \delta$ satisfy $\delta \to 0$.

Step 2: the budget of Φ^*. We apply Lemmas 5.7 and 5.6 to $\Phi^{(a)} = \Phi^*_{t_a}$, whose centres $\xi_\eta(t_a)$ converge to $\xi_\eta(t_0)$, with $m_\eta = \mu_0(\{\xi_\eta(t_0)\})$. By (H2), (H3) and the remark after (H4), the only centres with $\xi_\eta(t_0) \in V$ are the origin, for $\eta = 0$, and $\pm C$, $\pm C^\sharp$; these do not depend on a. The polynomial $\Phi^*_{t_0}$ vanishes on $\operatorname{supp} \mu_0$, so (5.19) holds. Condition (B0) holds, and (B1) holds because the coefficient at $\eta = 0$ is $\frac{1}{16}$ and $\mu_0(\{0\}) = 1$. The centres with $|\eta| \in \{2, 4\}$ have positive coefficients and satisfy (B2) by (H3). The centres with $|\eta| = 3$ satisfy (B3)(b) by (H2) and (5.32). The remaining centres have $|\eta| = 1$ and negative coefficients. At $\pm C$, $\pm C^\sharp$ condition (B3)(a) holds with $n_i = 0$ and $w_i = \frac{1}{4} = m_i$. At $\pm B_0(t_a)$ and $\pm B_0(t_a)^\sharp$, which are the four physical points of a state of spin p and dimension $E_p(t_a)$, condition (B3)(a) holds with $n_i = 1$, the state of Step 1, and with $\varepsilon = 0$, by (5.39). Fix $0 < \varrho \le \frac{1}{2}\sqrt{g}$ as in Lemma 5.7, for $\Phi^*_{t_a}$ and for the nine families $\Phi^{j,k}_{t_a}$, put $\kappa = \pi\varrho^2/4$ and choose

$$\epsilon^2 = \kappa / L, \qquad \text{so that} \qquad e^{-\kappa/\epsilon^2} = a. \tag{5.40}$$

The range (5.21) holds for small a, and (5.20) gives

$$\sum_j \Phi^*_{t_a,\epsilon}(X_j^\sigma) \le C_0\, a\, L^2. \tag{5.41}$$

Here the kernel is even, so its value at a state does not depend on σ, and all terms are nonnegative by Lemma 5.2. For a state with $\Delta_j \le E_*$, where E_* is any fixed bound, the broad Gaussian is at least $\frac{1}{2}$ for small a, and therefore

$$\Phi^*_{t_a}(X_j^\sigma) \le 2C_0\, a\, L^2 \qquad (\Delta_j \le E_*). \tag{5.42}$$

Step 3: anchoring. Consider one of the n_j^0 states near A_j, with dimension $|A_j|^2 + \delta$. Its physical point with the signs of A_j is $X(\delta)$ in the notation (5.24), with $X_0 = A_j$. We apply

Lemma 5.8(iii) to $\Phi = \Phi^*_{t_a}$ with the two vanishing factors S_C, $S_{C^\sharp}$ and the two nonvanishing factors $S_{B_0(t_a)}$, $S_{B_0(t_a)^\sharp}$. Indeed $B(A_j, C) = 2V_j$ and $B(A_j, C^\sharp) = 2U$ are integers, and by (5.25)

$$\lambda(A_j, C) = \frac{2V_j}{|U^2 - V_j^2|} > 0, \qquad \lambda(A_j, C^\sharp) = \frac{2U}{|U^2 - V_j^2|} > 0,$$

while $S_{B_0(t_a)}(A_j)$ and $S_{B_0(t_a)^\sharp}(A_j)$ converge to the positive numbers of (H4). Since $\delta \to 0$, the lemma applies for small a, and with (5.42) and $k = 2$ it gives

$$|\delta| \le C\,(aL^2)^{1/4} = C\,a^{1/4}L^{1/2} \qquad \text{for every state near } A_j. \tag{5.43}$$

Next let $k \in \{1, 2\}$ and consider the state near $B_k(t_0)$. Take $X_0 = B_k(t_a)$, which stays uniformly off both axes, the vanishing factor $S_{B_0(t_a)}$ and the nonvanishing factors S_C, $S_{C^\sharp}$, $S_{B_0(t_a)^\sharp}$. Here $B(B_k(t), B_0(t)) = p + p_k$ is an integer, and

$$\lambda(B_k(t), B_0(t)) = \frac{|t \cdot p/t - (p_k/t) \cdot t|}{|t^2 - p_k^2/t^2|} = \frac{k\ell}{p_k^2/t^2 - t^2}$$

is bounded below on I; the three nonvanishing factors at $B_k(t_a)$ converge to the positive numbers of (H4). Lemma 5.8(iii) with $k = 1$ gives

$$|\delta_{k,a}| \le C\,(aL^2)^{1/2} = C\,a^{1/2}L \qquad (k = 1, 2), \tag{5.44}$$

and $\delta_{0,a} = 0$ by (5.39).

Step 4: the nine budgets. Fix j, k and apply Lemmas 5.7 and 5.6 to $\Phi^{(a)} = \Phi^{j,k}_{t_a}$. The only centre with $\xi_\eta(t_0) \in V$ is the origin. Conditions (B0)–(B2) and (B3)(b) for $|\eta| = 3$ hold as in Step 2. The centres with $|\eta| = 1$ are $\pm A_j$, $\pm A_j^\sharp$, which are the physical points of a state of spin UV_j and dimension $|A_j|^2$, and $\pm B_k(t_a)$, $\pm B_k(t_a)^\sharp$. By Step 1 and Lemma 5.7(iii), condition (B3)(a) holds at the former with the n_j^0 states near A_j and at the latter with the single state near $B_k(t_0)$, and by (5.43) and (5.44) with

$$\varepsilon = \varepsilon_a = C\,a^{1/4}L^{1/2}.$$

With the width (5.40), (5.20) gives

$$\sum_j \Phi^{j,k}_{t_a,\epsilon}(X_j^\sigma) \le C\big((a + a^{1/2}L)L^2 + a\big) \le C\,a^{1/2}L^3, \qquad 0 \le j, k \le 2. \tag{5.45}$$

Step 5: localization. Let j be a state with $\Delta_j \le L$, and let $X = [u, v]$ be one of its physical points. The broad Gaussian at X is at least $e^{-2\pi\kappa}$, all terms of (5.45) are nonnegative, and so

$$\Phi^{j',k}_{t_a}(X) \le \Gamma_a := C\,a^{1/2}L^3 \qquad \text{for all } 0 \le j', k \le 2.$$

Moreover $uv = s_j \in \mathbb{Z}$ and $|u|, |v| \le \sqrt{2\Delta_j} \le \Upsilon := \sqrt{2L}$. We apply Lemma 5.9(c) with $t = t_a$, $t_\pm$ the endpoints of I, and

$$\zeta = \tfrac{1}{2}\Gamma_a^{1/8} = C\,a^{1/16}L^{3/8}.$$

Since $\zeta\Upsilon \to 0$, the smallness conditions of parts (a) and (b) of that lemma hold for small a. Consequently there is a real point X_j', equal either to a point of $\Lambda_{t_a} \cup \Lambda_{1/t_a}$ or to a point $[m/D, n/D]$ with $D = 2U\Delta V$, whose spin is exactly s_j, and

$$|\Delta_j - |X_j'|^2| \le C\,\zeta\,\Upsilon \le C\,a^{1/16}L^{7/8}. \tag{5.46}$$

In the second case $X_j' \in \Sigma_D$ by (5.29).

Step 6: contradiction. The set Σ_D is a fixed finite union of circle lattices, invariant under $[u, v] \mapsto [v, u]$ and containing Λ_1; the radii t_a lie in the compact interval I; and by Step 5 every state with $\Delta_j \le u_* L$, $u_* = 1 > 1/(2\pi)$, has a real point $X_j' \in \Sigma_D \cup \Lambda_{t_a} \cup \Lambda_{1/t_a}$ whose spin is exactly s_j and which satisfies $|\Delta_j - |X_j'|^2| \le Ca^{1/16}L^{7/8}$, for all sufficiently small a in the sequence. Discarding finitely many terms leaves an admissible sequence of gap g, and for it this contradicts Theorem 4.23. □

5.8 The case $R \in \{1,2\}$: the scalar row

For $R \in \{1,2\}$ the limit $\sigma(R,t_0)$ has a long scalar state of dimension two, at the points $(\pm 1,\pm 1)$ where ν_{vac} has positive atoms. Its displacement enters every budget through the centres $\pm C$, $\pm C^\sharp$, and must be bounded first. This is done with one explicit combination of translates of a narrow test function, whose Fourier transform is dominated by a sum kernel.

For $x \in \mathbb{R}^2$ and $\epsilon > 0$ put

$$\begin{aligned} \psi_{\epsilon,x}(X) &= (P - P_x)\, e^{-2\pi|X-x|^2/\epsilon^2}, \qquad \psi_\epsilon = \psi_{\epsilon,[2,2]}, \\ \text{that is,}\quad \psi_\epsilon(P,Q) &= (P-2)\, e^{-2\pi((P-2)^2+Q^2)/\epsilon^2}. \end{aligned} \tag{5.47}$$

These functions are not even. For sum data and $t \in I$ we define the *row* as the following third-order iterated second difference, acting on functions:

$$(\mathsf{D}_t f)(X) = \sum_{l,k,j=0}^{2} b_l b_k b_j\, f(X + lC^\sharp + kW(t) + jW(t)^\sharp), \qquad (b_0,b_1,b_2) = (1,-2,1). \tag{5.48}$$

Pairing $\mathsf{D}_t f$ with μ_a is the same as pairing f with the measure obtained from μ_a by the operator $(\tau_{C^\sharp}-1)^2(\tau_{W(t)}-1)^2(\tau_{W(t)^\sharp}-1)^2$, where $\tau_Y \delta_X = \delta_{X+Y}$; we shall only use the function side.

Lemma 5.12 (Preimages of the row)**.** *Let $t > 0$, let $w \geq 1$ be an integer and $0 < \epsilon \leq 1$.*

(i) With the points $x_{l,k,j}(t)$ of (5.36),

$$\mathsf{D}_t \psi_\epsilon = \sum_{l,k,j=0}^{2} b_l b_k b_j\, \psi_{\epsilon, x_{l,k,j}(t)}.$$

The three points with $k = j = 0$ do not depend on t: they are $x_{0,0,0} = [2,2] = (2,0)$, $x_{1,0,0} = [2,0] = (1,1)$ and $x_{2,0,0} = [2,-2] = (0,2)$, with coefficients $1,-2,1$.

(ii) For real X and x, $|\psi_{\epsilon,x}(X)| \leq e^{-\pi|X-x|^2/(2\epsilon^2)}$.

(iii) The middle preimage. *$\psi_{\epsilon,(1,1)}$ vanishes at $(1,1)$. If $|z| \leq 1$ and $h = 1 + \frac{z}{2}$, then*

$$\Big|\tfrac{1}{4}\psi_{\epsilon,(1,1)}(\sqrt{h},\sqrt{h}) - \frac{z}{16}\Big| \leq \frac{z^2}{32} + \frac{|z|^3}{2\epsilon^2}.$$

(iv) The axis preimages. *If $0 \leq \delta \leq 1$, $h = 4 + \frac{\delta}{2}$ and $\bar h = \frac{\delta}{2}$, then $|\psi_{\epsilon,(2,0)}(\sqrt{h}, \pm\sqrt{\bar h})| \leq \frac{\delta}{8}$. If $h = \frac{\delta}{2}$ and $\bar h = 4 + \frac{\delta}{2}$, then $\psi_{\epsilon,(0,2)}(\sqrt{h},\sqrt{\bar h}) + \psi_{\epsilon,(0,2)}(-\sqrt{h},\sqrt{\bar h}) = 0$.*

Proof. (i) For $y \in \mathbb{R}^2$, $\psi_\epsilon(X+y) = (P + P_y - 2)e^{-2\pi|X+y-(2,0)|^2/\epsilon^2} = \psi_{\epsilon,(2,0)-y}(X)$, and $(2,0) - y = [2,2] - y$ equals $x_{l,k,j}(t)$ for $y = lC^\sharp + kW(t) + jW(t)^\sharp$. (ii) With $\nu = |X - x|$ we have $|\psi_{\epsilon,x}(X)| \leq \nu e^{-\pi\nu^2/\epsilon^2} e^{-\pi\nu^2/\epsilon^2}$ and $\nu e^{-\pi\nu^2/\epsilon^2} \leq \epsilon(2\pi e)^{-1/2} \leq 1$. (iii) Let $q = \sqrt{1 + z/2} - 1$. Then $|q| \leq \frac{1}{2}|z|$ and $q - \frac{z}{4} = -\frac{1}{2}q^2$, so $|q - \frac{z}{4}| \leq \frac{1}{8}z^2$. The value in question is $\frac{1}{4} q\, e^{-4\pi q^2/\epsilon^2}$, and $|q|(1 - e^{-4\pi q^2/\epsilon^2}) \leq 4\pi|q|^3/\epsilon^2 \leq \pi|z|^3/(2\epsilon^2)$. (iv) In the first case $0 \leq \sqrt{h} - 2 = \frac{\delta}{2}/(\sqrt{h}+2) \leq \frac{\delta}{8}$ and the Gaussian is at most one. In the second case $\psi_{\epsilon,(0,2)}(P,Q) = P e^{-2\pi(P^2+(Q-2)^2)/\epsilon^2}$ is odd in P. □

Lemma 5.13 (The multiplier of the row)**.** *Let $0 < g \leq \frac{1}{2}$, let $I \subset (0,\infty)$ be a compact interval, $w \geq 1$ an integer, $t \in I$, $W(t) = [t, w/t]$ and $0 < \epsilon \leq \frac{1}{2}$.*

(i) *For* $x \in \mathbb{R}^2$,

$$\mathbb{S}\psi_{\epsilon,x}(X) = i\epsilon^4 P\, e^{-2\pi\epsilon^2|X|^2} e^{2\pi i B(X,x)}; \qquad \textit{in particular} \quad \mathbb{S}\psi_\epsilon(X) = i\epsilon^4 P\, e^{-2\pi\epsilon^2|X|^2} e^{8\pi i P}.$$

(ii) $\mathbb{S}(\mathsf{D}_t\psi_\epsilon) = \omega_t \cdot \mathbb{S}\psi_\epsilon$ *with the multiplier*

$$\omega_t(X) = \prod_{Y \in \{C^\sharp, W(t), W(t)^\sharp\}} \left(e^{-2\pi i B(X,Y)} - 1\right)^2,$$

which vanishes on $\{[u,v] : 2u \in \mathbb{Z}\} \supset V$ *and satisfies, on* $\mathbb{R}^2$,

$$\begin{aligned} |\omega_t| &= 64\, S_{C^\sharp} S_{W(t)} S_{W(t)^\sharp} \le 64\, \Omega\, S_{W(t)} S_{W(t)^\sharp}, \\ |\mathbb{S}(\mathsf{D}_t\psi_\epsilon)(X)| &\le 16\, \epsilon^3\, e^{-\pi\epsilon^2|X|^2}\, \Omega(X) S_{W(t)}(X) S_{W(t)^\sharp}(X). \end{aligned} \tag{5.49}$$

(iii) *There is* $C = C(g, I, w)$ *such that for every admissible partition function of gap* g *at* $c = 1 + 12a$ *with* $0 < a \le \min\{g^2/4, \epsilon^2/4\}$,

$$\left|\mathcal{L}_a(\mathsf{D}_t\psi_\epsilon) - \langle \mu_a, \mathsf{D}_t\psi_\epsilon\rangle\right| \le Ca\epsilon^{-1}, \qquad \left|\mathcal{L}_a(\mathbb{S}\mathsf{D}_t\psi_\epsilon) - \langle \mu_a, \mathbb{S}\mathsf{D}_t\psi_\epsilon\rangle\right| \le Ca\epsilon, \tag{5.50}$$

and consequently, with $\Phi_t^{(w)} = \Omega S_{W(t)} S_{W(t)^\sharp}$ *and the kernel* $\Phi^{(w)}_{t,\epsilon/\sqrt{2}}$ *of width* $\epsilon/\sqrt{2}$,

$$\left|\langle \mu_a, \mathsf{D}_t\psi_\epsilon\rangle\right| \le 16\, \epsilon^3 \sum_j \Phi^{(w)}_{t,\epsilon/\sqrt{2}}(X_j^\sigma) + 2Ca\epsilon^{-1}. \tag{5.51}$$

Here the pairings with μ_a *are the physical pairings* (5.4)*: the row is evaluated on the unshifted weights, and the whole effect of the central shift is contained in* (5.50).

Proof. (i) By the computation in the proof of Lemma 5.1, $\gamma_{\epsilon,x} = \mathbb{S}f_{-x}$ with $f_\xi(X) = e^{-2\pi\epsilon^2|X|^2} e^{2\pi i B(X,\xi)}$, and $\mathbb{S}^2 f(X) = f(-X)$; hence $\mathbb{S}\gamma_{\epsilon,x}(X) = f_{-x}(-X) = e^{-2\pi\epsilon^2|X|^2} e^{2\pi i B(X,x)}$. Since $\psi_{\epsilon,x} = \frac{\epsilon^4}{4\pi}\partial_{P_x}\gamma_{\epsilon,x}$ and the difference quotients in P_x converge in the Schwartz space, on which $\mathbb{S}$ is continuous, we get $\mathbb{S}\psi_{\epsilon,x} = \frac{\epsilon^4}{4\pi}\partial_{P_x}\left(e^{-2\pi\epsilon^2|X|^2} e^{2\pi i B(X,x)}\right)$, and $\partial_{P_x} B(X,x) = 2P$. For $x = (2,0)$, $B(X,x) = 4P$.

(ii) A change of variables in (2.5) gives $\mathbb{S}(f(\cdot + Y)) = e^{-2\pi i B(\cdot,Y)}\,\mathbb{S}f$. Hence $\mathbb{S}\mathsf{D}_t f$ is $\mathbb{S}f$ times the product over $Y \in \{C^\sharp, W(t), W(t)^\sharp\}$ of $\sum_l b_l e^{-2\pi i l B(X,Y)} = (e^{-2\pi i B(X,Y)} - 1)^2$. On $\mathbb{R}^2$, $|e^{-2\pi i\theta} - 1|^2 = 4\sin^2(\pi\theta)$, and $S_{C^\sharp} = \sin^2(2\pi u) \le \Omega$. Finally $|\mathbb{S}\psi_\epsilon(X)| = \epsilon^4 |P| e^{-2\pi\epsilon^2|X|^2} \le (2\pi e)^{-1/2}\epsilon^3 e^{-\pi\epsilon^2|X|^2}$ and $64(2\pi e)^{-1/2} \le 16$.

(iii) By (i) of Lemma 5.12, $\mathsf{D}_t\psi_\epsilon$ is a combination, with coefficients of total modulus 64, of functions $\psi_{\epsilon,x}$ with $|x| \le \varrho_1$, where ϱ_1 depends on I and w. Write $\psi_{\epsilon,x}(X) = \epsilon\,\varphi((X-x)/\epsilon)$ with $\varphi(Y) = Y_1 e^{-2\pi Y\cdot Y}$. For $|\alpha| \le 2$ we have $|\partial^\alpha\varphi(Y)| \le c(1+|Y|)^3 e^{-2\pi(|\operatorname{Re} Y|^2 - |\operatorname{Im} Y|^2)}$. If $|\operatorname{Im} P|, |\operatorname{Im} Q| \le \sqrt{a/2}$, then $|\operatorname{Im}(X-x)/\epsilon|^2 \le a/\epsilon^2 \le \frac14$, and therefore

$$|\partial^\alpha\psi_{\epsilon,x}(X)| \le c'\epsilon^{-1} e^{-\pi|\operatorname{Re} X - x|^2/\epsilon^2} \le c' e^{\pi\varrho_1^2}\,\epsilon^{-1}\, e^{-|\operatorname{Re} X|^2},$$

using $\epsilon \le 1$ and $|X-x|^2 \ge \frac12|X|^2 - |x|^2$. Lemma 5.4 with $\beta = 1$ and $M = C\epsilon^{-1}$ bounds the change of each evaluation under the central shift by $Ca\epsilon^{-1}e^{-(h+\bar h)}$; summing over the states with (2.26), and over the four vacuum weights, gives the first bound of (5.50). For the second, by (i) and (ii) the function $\mathbb{S}\mathsf{D}_t\psi_\epsilon$ is $i\epsilon^4 P\, e^{-2\pi\epsilon^2|X|^2}$ times a combination, with coefficients of total modulus 64, of exponentials $e^{2\pi i B(X,\xi)}$ with $|\xi| \le \varrho_1$. Arguing as for (5.16), on the strip $|\operatorname{Im} P|, |\operatorname{Im} Q| \le 1$ its derivatives of order at most two are bounded by a constant times

$$\epsilon^4(1 + |\operatorname{Re} X|)(1 + \epsilon^2|\operatorname{Re} X|)^2 e^{-2\pi\epsilon^2|\operatorname{Re} X|^2} \le C\epsilon^3 e^{-\pi\epsilon^2|\operatorname{Re} X|^2},$$

and Corollary 2.7 with $\beta = \pi\epsilon^2$, followed by (2.26), gives the bound $Ca\epsilon$.

The function $\mathsf{D}_t\psi_\epsilon$ belongs to $\mathcal{G}_{2\pi/\epsilon^2}$, so $\mathcal{L}_a(\mathsf{D}_t\psi_\epsilon) = \mathcal{L}_a(\mathbb{S}\mathsf{D}_t\psi_\epsilon)$ by Lemma 2.4. In $\langle\mu_a, \mathbb{S}\mathsf{D}_t\psi_\epsilon\rangle$ the vacuum terms vanish because $\omega_t = 0$ on V, and by (5.2) and (5.49) the term of a state is at most $16\epsilon^3 e^{-\pi\epsilon^2\Delta_j}\Phi_t^{(w)}(X_j^\sigma)$ in modulus, the function $\Phi_t^{(w)}$ being even. Combining this with (5.50) gives (5.51). □

Proposition 5.14 (The scalar near dimension two)**.** *Let $0 < g \le \frac12$, let sum data be given, and let t_0 be an interior point of I at which* (S1)–(S5) *hold. Suppose that an admissible sequence of gap g converges to $\sigma(R, t_0)$. Then there is $\varrho > 0$ such that, for all sufficiently small a in the sequence, exactly one state has a physical point in $D((1,1),\varrho)$; it has spin zero, and its dimension $2 + z_a$ satisfies $z_a \to 0$ and*

$$|z_a| \le C\,a\,L^{5/2}; \qquad \textit{in particular} \quad |z_a| < a^{1/3} \tag{5.52}$$

for all sufficiently small a in the sequence. The constant C depends on g, the data and t_0.

Proof. Let $\rho_0 = \sigma(R, t_0)$, $\mu_0 = \nu_{\mathrm{vac}} + \rho_0$, and $\varsigma_a = |z_a|$.

Step 0: the low clusters. Since $R \in \{1, 2\}$, both $2R$ and $2/R$ are integers, so $C \in \Lambda_R \cap \Lambda_{1/R}$, while $C \notin \Lambda_{t_0} \cup \Lambda_{1/t_0}$ by (S1). By (5.32), $\mu_0(\{C\}) = \frac12$ and $\rho_0(\{C\}) = \frac14$, and the same holds at $-C$, $\pm C^\sharp$. Lemma 5.7(iii), applied to $\Phi_{t_0}^{(p)}$ and its centre $C = (1,1)$, gives the first assertion, with spin zero and $z_a \to 0$. By (S1) and the remark after (S5), $\mu_0(\{Y_n(t_0)\}) = \rho_0(\{Y_n(t_0)\}) = \frac14$ for $n \in \mathcal{N}$. As in Step 1 of the proof of Proposition 5.11, exactly one state has a physical point near $B_0(t_0)$, and its dimension $\Delta_{0,a}$ defines $t_a \in I$ by (5.39), with $t_a \to t_0$; for $n \in \{p_1, p_2, w\}$ exactly one state has a physical point near $Y_n(t_0)$, it has spin n, and its dimension is $E_n(t_a) + \delta_{n,a}$ with $\delta_{n,a} \to 0$. Finally, the point $[2,2] = (2,0)$ lies in Λ_r, and in $\Lambda_{1/r}$, if and only if $2r$ and $2/r$ are both integers. Hence $\rho_0(\{(\pm2,0)\}) = \mu_0(\{(\pm2,0)\}) = \frac12$, which is the measure of one state of weights $(4,0)$, and likewise at $(0,\pm2)$. By Corollary 2.11, exactly one state of ρ_a has physical points near $(\pm2,0)$; it has spin 4 and dimension $4 + \delta_a^+$ with $\delta_a^+ \to 0$. Exactly one state has physical points near $(0,\pm2)$; it has spin -4 and dimension $4 + \delta_a^-$ with $\delta_a^- \to 0$. Here $\delta_a^\pm > 0$ because the weights of a state are positive.

Step 1: the scale. Fix $0 < \varrho \le \frac12\sqrt{g}$ as in Lemma 5.7 for the four families $\Phi_{t_a}^{(n)}$, $n \in \mathcal{N}$, and so small that the closed discs of radius 2ϱ about the points $x_{l,k,j}(t_0)$ contain no point of $V \cup \operatorname{supp}\rho_0$ other than possibly their centres; by (S5) the centre itself is not in $V \cup \operatorname{supp}\rho_0$ when $(k,j) \ne (0,0)$. Put

$$b_a = \varsigma_a + a, \qquad H = \log(1/b_a) \le L, \qquad \beta_a = a + \varsigma_a^2, \qquad \epsilon^2 = \frac{\pi\varrho^2}{16H}. \tag{5.53}$$

Then $b_a \to 0$, $e^{-\pi\varrho^2/(4\epsilon^2)} = b_a^4 \le 8\beta_a$, and $aH \le b_a \log(1/b_a) \to 0$, so that the range (5.21) holds for the widths ϵ and $\epsilon/\sqrt{2}$ when a is small.

Step 2: the budget of $\Phi^{(p)}$. We verify the hypotheses of Lemma 5.6 for $\Phi^{(a)} = \Phi_{t_a}^{(p)}$, using Lemma 5.7 with $m_i = \mu_0(\{\xi_i^0\})$ and the table of Lemma 5.1(ii). The centres lying in V at t_0 are the origin and $\pm C, \pm C^\sharp$, by (S2), (S3) and the remark after (S5); they are fixed. The origin has coefficient $\frac14$ and $\mu_0(\{0\}) = 1$. The centres $\pm B_0 \pm B_0^\sharp$, $\zeta \pm B_0$ and $\zeta \pm B_0^\sharp$ with $\zeta \ne 0$ have positive coefficients and satisfy (B2) by (S3). The centres $\zeta \pm B_0 \pm B_0^\sharp$ with $\zeta \ne 0$ satisfy (B3)(b) by (S2). At $\pm C$, $\pm C^\sharp$ condition (B3)(a) holds with $w_i = \frac14$, $n_i = 1$, the scalar state of Step 0, whose spin is zero and whose dimension differs from $|C|^2 = 2$ by ς_a; here $m_i = \frac12$. At $\pm B_0(t_a)$, $\pm B_0(t_a)^\sharp$ it holds with $n_i = 1$ and no displacement. With $\varepsilon = \varsigma_a$, (5.20) gives

$$\sum_j \Phi_{t_a,\epsilon}^{(p)}(X_j^\sigma) \le C((a + \varsigma_a^2)H^2 + b_a^4) \le C\beta_a H^2, \tag{5.54}$$

and, as in (5.42), $\Phi_{t_a}^{(p)}(X_j^\sigma) \le C\beta_a H^2$ for every state with $\Delta_j \le E_*$.

Step 3: the state near W. Its physical point with the signs of $W(t_a)$ is $X(\delta_{w,a})$ for $X_0 = W(t_a)$. We apply Lemma 5.8(iii) to $\Phi^{(p)}_{t_a} \geq S_{C^\sharp} S_{B_0(t_a)} S_{B_0(t_a)^\sharp}$, with the vanishing factor $S_{B_0(t_a)}$ and the nonvanishing factors $S_{C^\sharp}$ and $S_{B_0(t_a)^\sharp}$. Indeed $B(W(t), B_0(t)) = p + w \in \mathbb{Z}$,

$$\lambda(W(t), B_0(t)) = \frac{p - w}{|t^2 - w^2/t^2|}$$

is bounded below near t_0 because $w < p$, $S_{C^\sharp}(W(t_a)) = \sin^2(2\pi t_a) \to \sin^2(2\pi t_0) > 0$, and $S_{B_0(t_a)^\sharp}(W(t_a))$ converges to the positive number of (S4). With $k = 1$ the lemma gives

$$|\delta_{w,a}| \leq C\,(\beta_a H^2)^{1/2} = C\beta_a^{1/2} H. \tag{5.55}$$

Step 4: the axis states. Let $X = [u, v]$, with $u, v > 0$, be a physical point of the state near $(\pm 2, 0)$. Then $uv = 4$, $u, v \to 2$, and

$$\delta_a^+ = \tfrac{1}{2}(u^2 + v^2) - uv = \tfrac{1}{2}(u - v)^2 \leq (u - 2)^2 + (v - 2)^2 \leq \tfrac{1}{16}\,\Omega(X),$$

by (5.26), since $\operatorname{dist}(2u, \mathbb{Z}) = 2|u - 2|$ and $\operatorname{dist}(2v, \mathbb{Z}) = 2|v - 2|$ for small a. The phases $B([2, 2], B_0(t))$ and $B([2, 2], B_0(t)^\sharp)$ both equal $2(t + p/t)$, so by (S4) the product $S_{B_0(t_a)} S_{B_0(t_a)^\sharp}$ at X is bounded below for small a. With Step 2 this gives $\delta_a^+ \leq C\beta_a H^2$. For the state near $(0, \pm 2)$ take $X = [u, v]$ with $u > 0 > v$; then $uv = -4$, $\delta_a^- = \tfrac{1}{2}(u + v)^2 \leq (u - 2)^2 + (v + 2)^2 \leq \tfrac{1}{16}\Omega(X)$, the phases at $[2, -2]$ are $\pm 2(t - p/t)$, and (S4) gives in the same way

$$0 < \delta_a^\pm \leq C\beta_a H^2. \tag{5.56}$$

Step 5: the budget of $\Phi^{(w)}$ at width $\epsilon/\sqrt{2}$. The hypotheses of Lemma 5.6 for $\Phi^{(w)}_{t_a}$ are verified as in Step 2, with W in place of B_0. The only difference is that the state at the centres $\pm W(t_a)$, $\pm W(t_a)^\sharp$ is displaced by $\delta_{w,a}$. By (5.55) we may take $\varepsilon = C\beta_a^{1/2} H$, which also bounds ς_a and tends to zero. Applying (5.20) with the width $\epsilon/\sqrt{2}$,

$$\sum_j \Phi^{(w)}_{t_a, \epsilon/\sqrt{2}}(X_j^\sigma) \leq C\big((a + \beta_a H^2) H^2 + b_a^8\big) \leq C\beta_a H^4. \tag{5.57}$$

Step 6: the row. By (5.51) with $t = t_a$, (5.57) and $\epsilon^2 = \pi\varrho^2/(16H)$,

$$\big|\langle \mu_a, \mathsf{D}_{t_a} \psi_\epsilon \rangle\big| \leq C\epsilon^3 \beta_a H^4 + Ca\epsilon^{-1} \leq C\beta_a H^{5/2}. \tag{5.58}$$

We now evaluate the left side by Lemma 5.12(i), one preimage $x = x_{l,k,j}(t_a)$ at a time. Recall that $\langle \mu_a, \psi_{\epsilon,x} \rangle$ is a sum over the physical points of all states, with weights $\frac{1}{4}$, and over the points of V with the weights (5.1). By Lemma 5.12(ii) and (5.17), the points at distance at least ϱ from x contribute at most $(C_1 + 4) e^{-\pi\varrho^2/(2\epsilon^2)} = (C_1 + 4) b_a^8$ in total. It remains to find the points within ϱ of x.

If $(k, j) \neq (0, 0)$, there are none for small a: the disc $D(x_{l,k,j}(t_a), \varrho)$ is contained in $D' = D(x_{l,k,j}(t_0), \frac{3}{2}\varrho)$, which contains no point of V, and $\rho_a(D') \to \rho_0(D') = 0$ by vague convergence, hence $\rho_a(D') = 0$.

For $x_{1,0,0} = (1, 1)$ the points within ϱ are the vacuum atom at $(1, 1)$, where $\psi_{\epsilon,(1,1)}$ vanishes, and the point $(\sqrt{h}, \sqrt{h})$, $h = 1 + \frac{1}{2} z_a$, of the scalar state of Step 0, which has weight $\frac{1}{4}$. By Lemma 5.12(iii),

$$\langle \mu_a, \psi_{\epsilon,(1,1)} \rangle = \frac{z_a}{16} + O(\varsigma_a^2 + \varsigma_a^3 H + b_a^8).$$

For $x_{0,0,0} = (2, 0)$ the points within ϱ are the two points $(\sqrt{h}, \pm\sqrt{h})$ of the state of dimension $4 + \delta_a^+$, each of weight $\frac{1}{4}$; by Lemma 5.12(iv) and (5.56) they contribute at most $\frac{1}{16}\delta_a^+ \leq C\beta_a H^2$ in modulus. For $x_{2,0,0} = (0, 2)$ the points within ϱ are the two points $(\pm\sqrt{h}, \sqrt{h})$ of the state

of dimension $4+\delta_a^-$, and their contributions cancel by Lemma 5.12(iv). Since the coefficient of the middle preimage is $b_1=-2$,

$$\langle \mu_a, \mathsf{D}_{t_a}\psi_\epsilon\rangle = -\frac{z_a}{8} + O(\beta_a H^2). \tag{5.59}$$

Conclusion. By (5.58) and (5.59), $\frac{1}{8}\varsigma_a \le C(a+\varsigma_a^2)H^{5/2}$. Since $\varsigma_a H^{5/2} \le b_a \log^{5/2}(1/b_a) \to 0$, the term $C\varsigma_a^2 H^{5/2}$ is at most $\frac{1}{16}\varsigma_a$ for small a, and then $\varsigma_a \le 16CaH^{5/2} \le 16CaL^{5/2}$. □

Proposition 5.15 (Localization and exclusion for sum data)**.** *Let $0<g\le\frac12$, let sum data be given, and let t_0 be an interior point of I at which* (S1)–(S5) *hold. Then no admissible sequence of gap g converges to $\sigma(R,t_0)$.*

Proof. Suppose that such a sequence exists, and keep the notation of the proof of Proposition 5.14. We use only the bound $\varsigma_a < a^{1/3}$ of (5.52). Choose the width as in (5.40), $\epsilon^2=\kappa/L$ with $\kappa=\pi\varrho^2/4$.

Step 1. The hypotheses of Lemma 5.6 for $\Phi_{t_a}^{(p)}$ were verified in Step 2 of the preceding proof, with $\varepsilon=\varsigma_a<a^{1/3}$. Hence

$$\sum_j \Phi_{t_a,\epsilon}^{(p)}(X_j^\sigma) \le C((a+a^{2/3})L^2+a) \le C\,a^{2/3}L^2,$$

and $\Phi_{t_a}^{(p)}(X_j^\sigma)\le Ca^{2/3}L^2$ for states with $\Delta_j\le E_*$.

Step 2. Let $k\in\{1,2\}$. For the state near $B_k(t_0)$ we apply Lemma 5.8(iii) at $X_0=B_k(t_a)$ to $\Phi_{t_a}^{(p)} \ge S_{C^\sharp}S_{B_0(t_a)}S_{B_0(t_a)^\sharp}$. As in Step 3 of the proof of Proposition 5.11, the factor $S_{B_0(t_a)}$ vanishes at $B_k(t_a)$ with rate $k\ell/(p_k^2/t_a^2-t_a^2)$; the factors $S_{C^\sharp}$ and $S_{B_0(t_a)^\sharp}$ at $B_k(t_a)$ converge to $\sin^2(2\pi t_0)>0$ and to the positive number of (S4). Therefore

$$|\delta_{p_k,a}| \le C\,(a^{2/3}L^2)^{1/2} = C\,a^{1/3}L \qquad (k=1,2),$$

and $\delta_{p_0,a}=0$.

Step 3. For $k=0,1,2$ the hypotheses of Lemma 5.6 hold for $\Phi_{t_a}^{(p_k)}$, as in Step 2 of the preceding proof, with $\varepsilon=Ca^{1/3}L$, which bounds both ς_a and $|\delta_{p_k,a}|$. Hence

$$\sum_j \Phi_{t_a,\epsilon}^{(p_k)}(X_j^\sigma) \le C((a+a^{2/3}L^2)L^2+a) \le C\,a^{2/3}L^4, \qquad k=0,1,2.$$

Step 4. Let j be a state with $\Delta_j\le L$ and $X=[u,v]$ one of its physical points. As in Step 5 of the proof of Proposition 5.11,

$$\Omega(X)\,S_{B_k(t_a)}(X)S_{B_k(t_a)^\sharp}(X) \le \Gamma_a := C\,a^{2/3}L^4 \qquad (k=0,1,2),$$

$uv=s_j\in\mathbb{Z}$ and $|u|,|v|\le\Upsilon=\sqrt{2L}$. Lemma 5.9(d) and (a), with $\zeta=\frac12\Gamma_a^{1/6}=Ca^{1/9}L^{2/3}$, give a real point X_j', either in $\Lambda_{t_a}\cup\Lambda_{1/t_a}$ or of the form $[m/2,n/2]$, whose spin is exactly s_j and which satisfies

$$\bigl|\Delta_j - |X_j'|^2\bigr| \le C\,\zeta\,\Upsilon \le C\,a^{1/9}L^{7/6}. \tag{5.60}$$

In the second case $X_j'\in\Sigma_2=\Lambda_{1/2}\cup\Lambda_1\cup\Lambda_2$ by (5.29).

Step 5. As in Step 6 of the proof of Proposition 5.11, after discarding finitely many terms of the sequence this contradicts Theorem 4.23, applied with the fixed union Σ_2, the interval I, $u_*=1$, and the error bound (5.60). □

Remark 5.16. If the full bound $|z_a|\le CaL^{5/2}$ of (5.52) is used in Step 1, the same steps give budgets $O(aL^2)$ and $O(aL^4)$ and the localization error $O(a^{1/6}L^{7/6})$ in (5.60). Any positive power of a suffices for Theorem 4.23.

5.9 Choice of the integer data

By Lemma 5.10, fixed data leave finitely many exceptional radii. We now show that for every radius $t_0 > 1$ which is not a half-integer the data can be chosen, depending on t_0, so that the finite hypotheses hold at t_0. All points involved have the form (5.37),

$$\xi = \big[a_1 + b_1 t_0 + b_2 n/t_0,\ a_2 + b_2 t_0 + b_1 n/t_0\big], \qquad a_1, a_2, b_1, b_2 \in \mathbb{Z}, \quad n \geq 1,$$

and we recall that $\Lambda_{t_0} = \{[mt_0, n'/t_0]\}$ and $\Lambda_{1/t_0} = \{[m/t_0, n't_0]\}$ with $m, n' \in \mathbb{Z}$.

Lemma 5.17 (Incidences at an irrational radius)**.** *Let $t_0 > 0$ be irrational and let ξ be as above.*

(i) If $(b_1, b_2) \neq (0,0)$ and $\xi \in \mathbb{Q}^2$, then $b_2 = \mathsf{s} b_1$ with a sign $\mathsf{s} \in \{\pm 1\}$, and $t_0 + \mathsf{s} n/t_0 \in \mathbb{Q}$. For each sign there is at most one integer $n \geq 1$ with $t_0 + \mathsf{s} n/t_0 \in \mathbb{Q}$; if $t_0^2 \in \mathbb{Q}$, this happens only for $\mathsf{s} = -1$ and $n = t_0^2$.

(ii) If $(a_1, a_2) \neq (0,0)$ and $\xi \in \Lambda_{t_0} \cup \Lambda_{1/t_0}$, then $t_0^2 \notin \mathbb{Q}$, $a_1 a_2 \neq 0$, and t_0 satisfies a relation $t_0^2 = e_1 t_0 + e_0$ with $e_0, e_1 \in \mathbb{Q} \setminus \{0\}$. Moreover $b_2 \neq 0$ and $n = a_1 e_0/(e_1 b_2)$ if $\xi \in \Lambda_{t_0}$, and $b_1 \neq 0$ and $n = a_2 e_0/(e_1 b_1)$ if $\xi \in \Lambda_{1/t_0}$. In particular, for given t_0, a_1, a_2, b_1, b_2 there are at most two such integers n.

Proof. (i) Both $b_1 t_0 + b_2 n/t_0$ and $b_2 t_0 + b_1 n/t_0$ are rational, hence so is $(b_1^2 - b_2^2)t_0$, and $b_1^2 = b_2^2 \neq 0$. Then $b_1(t_0 + \mathsf{s} n/t_0) \in \mathbb{Q}$. If this holds for $n \neq n'$ with the same sign, then $(n - n')/t_0 \in \mathbb{Q}$, which is impossible. If $t_0^2 \in \mathbb{Q}$, then $(t_0^2 + \mathsf{s} n)/t_0 \in \mathbb{Q}$ forces $t_0^2 + \mathsf{s} n = 0$.

(ii) Since $\xi \mapsto \xi^\sharp$ maps Λ_{1/t_0} onto Λ_{t_0} and exchanges (a_1, b_1) with (a_2, b_2), it suffices to treat $\xi \in \Lambda_{t_0}$. Multiplying the two coordinate equations $\xi = [mt_0, n'/t_0]$ by t_0 gives

$$a_1 t_0 + (b_1 - m)t_0^2 + b_2 n = 0, \qquad a_2 t_0 + b_2 t_0^2 + b_1 n - n' = 0. \tag{5.61}$$

If $t_0^2 \in \mathbb{Q}$, these give $a_1 t_0 \in \mathbb{Q}$ and $a_2 t_0 \in \mathbb{Q}$, so $a_1 = a_2 = 0$, contrary to the hypothesis. If $1, t_0, t_0^2$ are linearly independent over $\mathbb{Q}$, the first equation gives $a_1 = 0$ and $b_2 n = 0$, so $b_2 = 0$, and the second gives $a_2 = 0$; again a contradiction. Hence $t_0^2 = e_1 t_0 + e_0$ with unique rational e_0, e_1, and $e_1 \neq 0$. Substituting and comparing the coefficients of 1 and t_0,

$$a_1 = -(b_1 - m)e_1, \qquad b_2 n = -(b_1 - m)e_0, \qquad a_2 = -b_2 e_1.$$

If $b_2 = 0$, then $a_2 = 0$, hence $a_1 \neq 0$, $b_1 \neq m$ and $e_0 = 0$, which would make $t_0 = e_1$ rational. So $b_2 \neq 0$, $a_2 \neq 0$, and $n = a_1 e_0/(e_1 b_2)$; since $n \neq 0$, also $a_1 \neq 0$ and $e_0 \neq 0$. □

Lemma 5.18 (Data for an irrational radius)**.** *Let $I \subset (0, \infty)$ be a compact interval and $t_0 \in I$ irrational. Put $\ell = t_0^2$ if t_0^2 is an integer and $\ell = 1$ otherwise.*

(a) Let $R = K/2$ with $K \geq 3$, $K \neq 4$, and put $U = 2K$, $V_j = 2(2K + 3 + j)$ for $j = 0, 1, 2$. These are product data for every p satisfying (5.30), and for all but finitely many such p the hypotheses (H1)–(H4) hold at t_0.

(b) Let $R \in \{1, 2\}$. There is an integer $w \geq 1$ such that, for all but finitely many $p > w$ satisfying (5.30), the hypotheses (S1)–(S5) hold at t_0.

Proof. The integers p with $\gcd(p, 2\ell) = 1$ contain the progression $1 + 2\ell\mathbb{Z}_{\geq 0}$, so infinitely many p satisfy (5.30). If $t_0^2 = \ell$ is an integer, then $\ell \geq 2$ and $\gcd(p_k, \ell) = \gcd(p, \ell) = 1$ for $k = 0, 1, 2$. Below, "excluding p" means discarding finitely many values of p. By Lemma 5.17(i) we exclude the at most six p for which

$$t_0 + \mathsf{s} p_k/t_0 \in \mathbb{Q} \qquad \text{for some } \mathsf{s} \in \{\pm 1\},\ k \in \{0, 1, 2\}. \tag{5.62}$$

Three facts are used in both parts. First, neither $2t_0$ nor $2/t_0$ is an integer. Secondly, $Y_n(t_0) = [t_0, n/t_0] \in \Lambda_{1/t_0}$ means $t_0^2 = m$ and $n = n't_0^2$ with integers m, n', so it requires t_0^2 to be an integer dividing n; this fails for $n = p_k$. Thirdly, the phase $t_0^2 + pp_k/t_0^2$ of $S_{B_0(t_0)^\sharp}$ at $B_k(t_0)$ is, for each k, an integer for at most one p: if $t_0^2 \notin \mathbb{Q}$ and two values $p < p'$ gave integers, then, with $p'_k = p' + k\ell$, the number $(p'p'_k - pp_k)/t_0^2$ would be a nonzero integer and t_0^2 rational; if $t_0^2 = A/B$ in lowest terms with $B > 1$, integrality of $A/B + pp_kB/A$ gives $B \mid A^2$ after multiplication by AB, which is impossible; and if $t_0^2 = \ell$ is an integer, integrality gives $\ell \mid pp_k$, contradicting $\gcd(pp_k, \ell) = 1$.

(a) The conditions (5.33) hold: $A_j = [4R,\, K(2K+3+j)/R] \in \Lambda_R$, $U = 2K \geq 6$, and $V_j - U = 2K + 6 + 2j \geq 3$.

(H1): $B_k(t_0)$ has the irrational coordinate t_0, so it is not in $\Lambda_R \cup \Lambda_{1/R} \subset \mathbb{Q}^2$; the rest was shown above.

(H2): Let $|\eta| = 3$. In the notation (5.37), $[a_1, a_2] = \eta_1 Y + \eta_2 Y^\sharp \neq 0$ and $(b_1, b_2) = (\eta_3, \eta_4) \neq (0,0)$. By Lemma 5.17(i) and the exclusion (5.62), $\xi_\eta(t_0) \notin \mathbb{Q}^2 \supset V \cup \Lambda_R \cup \Lambda_{1/R}$. By Lemma 5.17(ii), $\xi_\eta(t_0) \in \Lambda_{t_0} \cup \Lambda_{1/t_0}$ is possible only if p_k is one of at most two numbers determined by t_0 and (a_1, a_2, b_1, b_2). The latter range over a finite set which does not depend on p, so we exclude finitely many p. In particular, if $t_0^2 \in \mathbb{Q}$ no such incidence occurs: for the fixed vector C or $C^\sharp$, for instance, the two equations (5.61) force $a_1 t_0 \in \mathbb{Q}$ and $a_2 t_0 \in \mathbb{Q}$, while one of a_1, a_2 is ± 2.

(H3): Let $|\eta| \in \{2, 4\}$. If $(\eta_3, \eta_4) = (0,0)$, the centre is $\pm Y \pm Y^\sharp$ and is not in V, as in the proof of Lemma 5.10. Otherwise $\xi_\eta(t_0) \notin \mathbb{Q}^2 \supset V$ by Lemma 5.17(i) and (5.62).

(H4): The phases at A_j are $Up/t_0 + V_j t_0$ and $Ut_0 + V_j p/t_0$. If one of them were an integer for two values $p \neq p'$, then $U(p-p')/t_0$ or $V_j(p-p')/t_0$ would be an integer and t_0 rational; so we exclude at most six p. At $B_k(t_0)$ the phases of S_C and $S_{C^\sharp}$ are $2p_k/t_0$ and $2t_0$, which are irrational, and the phase of $S_{B_0(t_0)^\sharp}$ was treated above.

(b) We first choose w. Let $\mathfrak{W}$ be the finite set of integers n given by Lemma 5.17(ii) for $[a_1, a_2] = [2, 2-2l]$ and $(b_1, b_2) = (-k, -j)$ with $0 \leq l, k, j \leq 2$, $(k, j) \neq (0,0)$. We require

$$t_0 + w/t_0 \notin \mathbb{Q}, \qquad t_0 - w/t_0 \notin \mathbb{Q}, \qquad w \notin \mathfrak{W}, \qquad \gcd(w, t_0^2) = 1 \ \text{ if } t_0^2 \in \mathbb{Z}. \tag{5.63}$$

By Lemma 5.17(i) the first two conditions exclude at most two integers, and none other than $w = t_0^2$ when $t_0^2 \in \mathbb{Q}$; so an integer $w \geq 1$ with (5.63) exists, for instance in the progression $1 + t_0^2 \mathbb{Z}_{\geq 0}$ when t_0^2 is an integer. Fix it, and let $p > w$.

(S1): shown above for $n = p_k$; for $n = w$, t_0^2 would be an integer dividing w, contrary to (5.63).

(S2): Here $[a_1, a_2] = \zeta \in \{\pm[2,0], \pm[0,2]\}$, so $a_1 a_2 = 0$ and $(a_1, a_2) \neq 0$, and Lemma 5.17(ii) shows that the centre is not in $\Lambda_{t_0} \cup \Lambda_{1/t_0}$. It is not in $\mathbb{Q}^2 \supset V \cup \Lambda_R \cup \Lambda_{1/R}$ by Lemma 5.17(i), using (5.63) for $n = w$ and (5.62) for $n = p_k$.

(S3): The centres have $(b_1, b_2) \neq (0,0)$ and are not in $\mathbb{Q}^2$, for the same reason.

(S4): For $n = p_1, p_2$ the phase $t_0^2 + pn/t_0^2$ was treated above. For $n = w$ the same three cases apply: if $t_0^2 \notin \mathbb{Q}$, at most one p gives an integer, since otherwise $(p'-p)w/t_0^2$ would be a nonzero integer; if $t_0^2 = A/B$ with $B > 1$, integrality gives $B \mid A^2$; if $t_0^2 = \ell$, it gives $\ell \mid pw$, contradicting $\gcd(pw, \ell) = 1$. Finally $2(t_0 \pm p/t_0) \in \mathbb{Z}$ holds for at most one p for each sign, because two such values would make $2(p-p')/t_0$ an integer.

(S5): Let $(k, j) \neq (0,0)$. If $x_{l,k,j}(t_0) \in \mathbb{Q}^2$, then by Lemma 5.17(i) $j = \mathsf{s}k$, which for $k, j \geq 0$ means $j = k > 0$ and $\mathsf{s} = 1$, and $t_0 + w/t_0 \in \mathbb{Q}$, contrary to (5.63). Since $a_1 = 2 \neq 0$, Lemma 5.17(ii) shows that $x_{l,k,j}(t_0) \in \Lambda_{t_0} \cup \Lambda_{1/t_0}$ would force $w \in \mathfrak{W}$. □

For rational radii the incidences are decided by denominators. Let $\alpha, \beta \geq 1$ be coprime integers. For integers c_1, c_2,

$$\alpha\Big(c_1\frac{\alpha}{\beta} + c_2\frac{\beta}{\alpha}\Big) \in \mathbb{Z} \iff \beta \mid c_1, \qquad \beta\Big(c_1\frac{\alpha}{\beta} + c_2\frac{\beta}{\alpha}\Big) \in \mathbb{Z} \iff \alpha \mid c_2, \tag{5.64}$$

because the two numbers are $c_1\alpha^2/\beta + c_2\beta$ and $c_1\alpha + c_2\beta^2/\alpha$.

Lemma 5.19 (Incidences at a rational radius). *Let $t_0 = \alpha/\beta$ in lowest terms with $\beta \geq 3$ and $\alpha \geq 4$, let $n \geq 1$ be an integer coprime to α, and let ξ be as in (5.37) with $t = t_0$ and $|b_1|, |b_2| \leq 2$.*

(i) If $|b_1|, |b_2| \leq 1$ and one of $2\xi_1$, $2\xi_2$ is an integer, then $b_1 = b_2 = 0$. Here $\xi = [\xi_1, \xi_2]$.

(ii) If $2\xi_1$ and $2\xi_2$ are both integers, then $b_1 = b_2 = 0$.

(iii) If $\xi \in \Lambda_{t_0}$, then $b_2 = 0$ and $\alpha \mid a_1$. If $\xi \in \Lambda_{1/t_0}$, then $b_1 = 0$ and $\beta \mid a_1$, $\alpha \mid a_2$.

Proof. We have $\xi_1 = a_1 + b_1\alpha/\beta + b_2 n\beta/\alpha$ and $\xi_2 = a_2 + b_2\alpha/\beta + b_1 n\beta/\alpha$, and $\alpha \mid cn$ is equivalent to $\alpha \mid c$. By (5.64), if $2\xi_1 \in \mathbb{Z}$ then $\beta \mid 2b_1$ and $\alpha \mid 2b_2$, and if $2\xi_2 \in \mathbb{Z}$ then $\beta \mid 2b_2$ and $\alpha \mid 2b_1$. For $|b_i| \leq 1$ we have $|2b_i| \leq 2 < \beta, \alpha$, which gives (i). In (ii), $\alpha\beta$ divides $2b_1$ and $2b_2$, and $|2b_i| \leq 4 < 12 \leq \alpha\beta$. For (iii), let $\xi = [m\alpha/\beta, n'\beta/\alpha] \in \Lambda_{t_0}$. The first coordinate gives $(b_1 - m)\alpha/\beta + b_2 n\beta/\alpha = -a_1 \in \mathbb{Z}$, so $\alpha \mid b_2$ and $\beta \mid (b_1 - m)$ by (5.64); hence $b_2 = 0$, because $|b_2| \leq 2 < \alpha$, and then $a_1 = (m - b_1)\alpha/\beta$ is a multiple of α. If $\xi = [m\beta/\alpha, n'\alpha/\beta] \in \Lambda_{1/t_0}$, the first coordinate gives $b_1\alpha/\beta + (b_2 n - m)\beta/\alpha = -a_1$, so $\beta \mid b_1$, hence $b_1 = 0$, and $\alpha \mid (b_2 n - m)$, hence $\beta \mid a_1$; the second coordinate gives $(b_2 - n')\alpha/\beta = -a_2$ with $\beta \mid (b_2 - n')$, hence $\alpha \mid a_2$. □

Lemma 5.20 (Data for a rational radius). *Let $t_0 = \alpha/\beta > 1$ in lowest terms with $\beta \geq 3$, let $I \subset (0, \infty)$ be a compact interval containing t_0, and put $\ell = \alpha$. Let p satisfy $p \equiv 1 \pmod{2\alpha}$ and $p > \sup_I t^2 + 2$; then (5.30) holds.*

(a) Let $R = K/2$ with $K \geq 3$, $K \neq 4$, and put

$$U = 2K\alpha\beta, \qquad V_j = 2 + 2\alpha\beta(K + 2 + j) \quad (j = 0, 1, 2).$$

These are product data, and (H1)–(H4) hold at t_0.

(b) Let $R \in \{1, 2\}$ and $w = 1$. Then (S1)–(S5) hold at t_0.

Proof. Since $\alpha > \beta \geq 3$ we have $\alpha \geq 4$, and the integers p_k and $w = 1$ are congruent to 1 modulo α, hence coprime to α; so Lemma 5.19 applies to all points considered. Neither $2t_0 = 2\alpha/\beta$ nor $2/t_0 = 2\beta/\alpha$ is an integer. By Lemma 5.19(iii) with $a_1 = a_2 = 0$ and $(b_1, b_2) = (1, 0)$, $Y_n(t_0) \notin \Lambda_{1/t_0}$ for $n \in \{p_0, p_1, p_2, 1\}$. The phase $t_0^2 + pn/t_0^2 = \alpha^2/\beta^2 + pn\beta^2/\alpha^2$ is not an integer: otherwise multiplication by $\alpha^2\beta^2$ would give $\beta^2 \mid \alpha^4$.

(a) We have $A_j = [4\alpha\beta R, K(1 + \alpha\beta(K + 2 + j))/R] \in \Lambda_R$, $U, V_j \geq 3$, and $V_j - U = 2 + 2\alpha\beta(2 + j) \geq 3$; so (5.33) holds. Note that

$$U \equiv 0, \qquad V_j \equiv 2 \pmod{\alpha\beta}. \tag{5.65}$$

(H1): By Lemma 5.19(i) with $(b_1, b_2) = (1, 0)$, neither $2t_0$ nor $2p_k/t_0$ is an integer, whereas a point of $\Lambda_R = \{[mK/2, 2n'/K]\}$ has $2\xi_1 \in \mathbb{Z}$ and a point of $\Lambda_{1/R}$ has $2\xi_2 \in \mathbb{Z}$. The rest was shown above.

(H2): Let $|\eta| = 3$, so $(b_1, b_2) = (\eta_3, \eta_4) \neq (0, 0)$. By Lemma 5.19(i), neither $2\xi_1$ nor $2\xi_2$ is an integer, so $\xi_\eta(t_0) \notin V \cup \Lambda_R \cup \Lambda_{1/R}$. If $\xi_\eta(t_0) \in \Lambda_{t_0}$, then $\eta_4 = 0$ by Lemma 5.19(iii), hence $\eta_1\eta_2 \neq 0$, and α divides $a_1 = \eta_1 u_Y + \eta_2 v_Y$. For $Y = A_j$ this number is congruent to $2\eta_2 = \pm 2$ modulo α by (5.65), and for $Y = C$ it equals $2\eta_1 = \pm 2$; both are impossible since $\alpha \geq 4$. If $\xi_\eta(t_0) \in \Lambda_{1/t_0}$, then $\eta_3 = 0$, $\eta_1\eta_2 \neq 0$, and α divides $a_2 = \eta_1 v_Y + \eta_2 u_Y$, which is congruent to $2\eta_1$ for $Y = A_j$ and equals $2\eta_2$ for $Y = C$; again impossible.

(H3): If $(\eta_3, \eta_4) \neq (0, 0)$, then $\xi_\eta(t_0) \notin V$ by Lemma 5.19(i). Otherwise the centre is $\pm Y \pm Y^\sharp \notin V$, as in the proof of Lemma 5.10.

(H4): By (5.65),

$$B(A_j, B_0(t_0)) = \frac{Up\beta}{\alpha} + \frac{V_j\alpha}{\beta} \equiv \frac{2\alpha}{\beta}, \qquad B(A_j, B_0(t_0)^\sharp) = \frac{U\alpha}{\beta} + \frac{V_jp\beta}{\alpha} \equiv \frac{2p\beta}{\alpha} \pmod{\mathbb{Z}},$$

and neither is an integer, because $\beta \nmid 2$ and $\alpha \nmid 2$. At $B_k(t_0)$ the phases of S_C and $S_{C^\sharp}$ are $2p_k\beta/\alpha$ and $2\alpha/\beta$, not integers, and the phase of $S_{B_0(t_0)^\sharp}$ was treated above.

(b) (S1) and the first part of (S4) were shown above. If $2(t_0 \pm p/t_0) = 2\alpha/\beta \pm 2p\beta/\alpha$ were an integer, (5.64) would give $\beta \mid 2$. In (S2) and (S3) the centres have $|b_1|, |b_2| \le 1$ and $(b_1, b_2) \neq (0, 0)$, so by Lemma 5.19(i) they are not in $\frac{1}{2}\mathbb{Z}^2 \supset V \cup \Lambda_R \cup \Lambda_{1/R}$; in (S2) moreover $b_1b_2 \neq 0$, which excludes $\Lambda_{t_0} \cup \Lambda_{1/t_0}$ by Lemma 5.19(iii). In (S5), $[a_1, a_2] = [2, 2-2l]$ and $(b_1, b_2) = (-k, -j) \neq (0, 0)$. By Lemma 5.19(ii) the point is not in $\frac{1}{2}\mathbb{Z}^2$. If it were in Λ_{t_0}, then $\alpha \mid a_1 = 2$; if it were in Λ_{1/t_0}, then $\beta \mid a_1 = 2$. Both are impossible. □

Theorem 5.21 (Both limiting radii are half-integers)**.** *Let $0 < g \le \frac{1}{2}$. Every limit of an admissible sequence of gap g is of the form $\sigma(R, T)$ with*

$$2R \in \mathbb{Z}, \qquad 2T \in \mathbb{Z}, \qquad 1 \le R, T \le (2g)^{-1/2}.$$

Proof. By Theorem 3.5 the limit is $\sigma(R, t_0)$ with $1 \le R, t_0 \le R_{\max} = (2g)^{-1/2}$ and at least one half-integer radius; since $\sigma(R, t_0) = \sigma(t_0, R)$ we may assume $2R \in \mathbb{Z}$. Suppose that $2t_0 \notin \mathbb{Z}$. Then $t_0 > 1$ is an interior point of $I = [\frac{1}{2}, R_{\max} + 1]$. If $R \notin \{1, 2\}$, then $R = K/2$ with $K \ge 3$ and $K \neq 4$, and Lemma 5.18(a) or Lemma 5.20(a), according as t_0 is irrational or rational (in which case its denominator is at least 3), provides product data for which (H1)–(H4) hold at t_0; Proposition 5.11 then shows that no admissible sequence of gap g converges to $\sigma(R, t_0)$. If $R \in \{1, 2\}$, parts (b) of the same lemmas provide sum data for which (S1)–(S5) hold at t_0, and Proposition 5.15 gives the same contradiction. □

5.10 The four-scalar anchor

By Theorem 5.21 only finitely many limits remain for each gap, and all of them are supported on one rational grid. Put

$$N_g = \lfloor 2(2g)^{-1/2} \rfloor, \qquad d = 2\,N_g!, \qquad E_d = \tfrac{1}{2}d^2, \qquad \eta_d = \frac{1}{8d^2}. \tag{5.66}$$

Since $g \le \frac{1}{2}$ we have $N_g \ge 2$ and $d \ge 4$. If $R = K/2$ is a half-integer with $1 \le R \le (2g)^{-1/2}$, then $2 \le K \le N_g$, so $dR = N_g!\,K$ and $d/R = 4N_g!/K$ are integers. Consequently, in light-cone coordinates,

$$\Lambda_R \cup \Lambda_{1/R} \subset d^{-1}\mathbb{Z}^2, \qquad [\pm d, 0],\ [0, \pm d] \in \Lambda_R \cap \Lambda_{1/R} \tag{5.67}$$

for every such R. The points $[\pm d, 0]$, $[0, \pm d]$ are the four points $(\pm\frac{d}{2}, \pm\frac{d}{2})$; they have spin zero and squared norm E_d.

Lemma 5.22 (The common four-scalar cluster)**.** *Let $0 < g \le \frac{1}{2}$. There is $a_1 = a_1(g) > 0$ such that every admissible partition function of gap g at $c = 1 + 12a$ with $0 < a < a_1$ has exactly four states of spin zero with dimension in $(E_d - \eta_d, E_d + \eta_d)$. Writing their dimensions as $E_d + z_{i,a}$, $1 \le i \le 4$, and*

$$\delta_a = \max_{1 \le i \le 4} |z_{i,a}|,$$

we have $\delta_a \to 0$ uniformly: for every $\eta > 0$ there is $a(\eta) > 0$ such that $\delta_a < \eta$ for every admissible partition function of gap g with $0 < a < a(\eta)$.

The quantity δ_a depends on the partition function and not only on a; we suppress this in the notation.

Proof. Let $\rho_0 = \sigma(R,T)$ with half-integers $1 \le R,T \le (2g)^{-1/2}$, and $\mu_0 = \nu_{\mathrm{vac}} + \rho_0$. By (5.67) and (5.32), μ_0 has mass $\frac14 \cdot 4 = 1$ at each of the points $(\pm\frac{d}{2}, \pm\frac{d}{2})$, and these are outside V because $\frac{d}{2} \ge 2$. Hence ρ_0 has total mass four on these points, that is, exactly four states of weights $(\frac14 d^2, \frac14 d^2)$. Every point of $\operatorname{supp}\rho_0$ of spin zero is of the form $[m/d, 0]$ or $[0, n/d]$ and has dimension in $\frac{1}{2d^2}\mathbb{Z}$; since $\eta_d < \frac{1}{2d^2}$, no state of ρ_0 of spin zero other than these four has dimension in $[E_d - \eta_d, E_d + \eta_d]$. Let

$$U_d = \{X \in \mathbb{R}^2 : \ |P^2 - Q^2| < \tfrac12, \quad E_d - \eta_d < |X|^2 < E_d + \eta_d\}.$$

This set is bounded, open and invariant under the two reflections; its boundary does not meet $\operatorname{supp}\rho_0$, because the points of $\operatorname{supp}\rho_0$ have integral spin and those of spin zero have dimension in $\frac{1}{2d^2}\mathbb{Z}$, which does not contain $E_d \pm \eta_d$; and the states of ρ_0 supported in U_d are the four states above. A state of an admissible partition function has spin zero and dimension in $(E_d - \eta_d, E_d + \eta_d)$ if and only if it is supported in U_d.

Suppose that the lemma fails. Then there are $\eta > 0$ and admissible partition functions of gap g at $c = 1 + 12a_n$, $a_n \to 0$, for each of which the number of states supported in U_d is not four, or is four with $\delta_{a_n} \ge \eta$. By Theorem 2.9 a subsequence converges to some ρ_0, which is of the above form by Theorem 5.21. By Corollary 2.11, along this subsequence the number of states supported in U_d is eventually four and their dimensions converge to E_d, a contradiction. □

The kernel adapted to the grid is built from Ω_d alone,

$$\Omega_{d,\epsilon}(X) = e^{-2\pi\epsilon^2|X|^2}\big(\sin^2(\pi d u) + \sin^2(\pi d v)\big).$$

Proposition 5.23 (The four-scalar budget)**.** *Let $0 < g \le \frac12$. There are constants C, $\kappa > 0$ and $a_0 > 0$, depending only on g, such that for every admissible partition function of gap g at $c = 1 + 12a$ with $0 < a < a_0$, and every $0 < \epsilon \le \frac12$ with $a \le \frac14\epsilon^2$,*

$$\sum_j e^{-2\pi\epsilon^2\Delta_j}\big(\sin^2(\pi d u_j) + \sin^2(\pi d v_j)\big) \le C\Big((a + \delta_a^2)\epsilon^{-4} + e^{-\kappa/\epsilon^2}\Big), \tag{5.68}$$

where $[u_j, v_j]$ is any physical point of the state j.

Proof. We apply Lemma 5.6 to $\Phi = \Omega_d$, with $\varrho = \min\{\frac14, \frac12\sqrt{g}\}$, $\varrho_* = d/\sqrt{2}$, $c_* = 2$ and $m_* = 1$. By Lemma 5.1(iii) the centres are the origin, with coefficient 1, and the four points $[\pm d, 0]$, $[0, \pm d]$, with coefficient $-\frac14$. We take $m_i = 1$ at all five centres, so that (5.19) reads $1 - 4 \cdot \frac14 = 0$. Condition (B0) holds because the points of V have integral light-cone coordinates, (B1) holds, and (B2) is empty. Let $\xi = (\pm\frac{d}{2}, \pm\frac{d}{2})$ be one of the negative centres. Its coordinates have modulus $\frac{d}{2} \ge 2 \ge 2\varrho$, the disc $D(\xi, \varrho)$ contains no point of V, and $w_i = 0$. For $a < a_1(g)$, Lemma 5.22 provides $n_i = 4$ states of spin $0 = P_\xi^2 - Q_\xi^2$ with $|\Delta_j - |\xi|^2| = |z_{i,a}| \le \delta_a$, and $w_i + \frac14 n_i = 1 = m_i$. Thus (B3)(a) holds with $\varepsilon = \delta_a$. By the uniform convergence in Lemma 5.22 there is $a_0 \le a_1$, depending only on g, such that $\delta_a \le \varrho^2$ and $a \le \min\{\frac14 g^2, \frac14\varrho^2\}$ whenever $a < a_0$. Then (5.21) holds, and (5.20) is the assertion, with $\kappa = \pi\varrho^2/4$ and a constant depending only on g and $d = d(g)$. The left side of (5.20) is the left side of (5.68) because $\Omega_{d,\epsilon}$ is even. □

The constants in Proposition 5.23 do not depend on a limiting measure: Lemma 5.6 is applied to each partition function separately, and the only input from compactness is the threshold $a_0(g)$ of Lemma 5.22.

Corollary 5.24 (Localization on the scale of the four scalars)**.** *Let $0 < g \le \frac{1}{2}$ and put*

$$b_a = \sqrt{a} + \delta_a, \qquad H_a = \log(1/b_a).$$

For every $M > 0$ there are $C_M < \infty$ and $a_M > 0$, depending only on g and M, such that for every admissible partition function of gap g with $0 < a < a_M$ and every state with $\Delta_j \le M H_a$, with physical point $[u_j, v_j]$, there are unique integers m, n with

$$\frac{mn}{d^2} = s_j, \qquad (m,n) \neq (0,0), \qquad \Big|u_j - \frac{m}{d}\Big| + \Big|v_j - \frac{n}{d}\Big| \le C_M\, b_a H_a. \tag{5.69}$$

Moreover $|\Delta_j - \frac{m^2+n^2}{2d^2}| \le C_M\, b_a H_a^{3/2}$, and if $|m| = |n|$, that is, if the associated point lies on a momentum axis, then $0 \le \Delta_j - \frac{m^2+n^2}{2d^2} \le C_M\, b_a^2 H_a^2$.

Proof. By Lemma 5.22, $b_a \to 0$ uniformly over the class as $a \to 0$. Take $\epsilon^2 = \kappa/(2H_a)$ in Proposition 5.23; this is admissible for small a because $aH_a \le b_a^2 \log(1/b_a) \to 0$. Then $e^{-\kappa/\epsilon^2} = b_a^2$ and $a + \delta_a^2 \le b_a^2$, so the right side of (5.68) is at most $C b_a^2 H_a^2$. For a state with $\Delta_j \le M H_a$ the broad Gaussian is at least $e^{-\pi\kappa M}$, and all terms are nonnegative; hence $\sin^2(\pi d u_j) + \sin^2(\pi d v_j) \le C_M b_a^2 H_a^2$, and by (5.26) there are integers m, n with $|u_j - \frac{m}{d}|, |v_j - \frac{n}{d}| \le c := C_M b_a H_a$. For small a we have $c < 1/(2d)$, so m, n are unique. Since $|u_j|, |v_j| \le \sqrt{2MH_a}$,

$$\Big|s_j - \frac{mn}{d^2}\Big| \le c(|u_j| + |v_j|) + c^2 \le C_M\, b_a H_a^{3/2},$$

which is smaller than d^{-2} for small a; as both numbers lie in $d^{-2}\mathbb{Z}$, they are equal. The same computation bounds $|\Delta_j - \frac{m^2+n^2}{2d^2}|$. If $(m,n) = (0,0)$, then $\Delta_j = \frac{1}{2}(u_j^2 + v_j^2) \le c^2 < g$, which is impossible. If $|m| = |n|$, then $|s_j| = m^2/d^2 = \frac{m^2+n^2}{2d^2}$ and

$$0 \le \Delta_j - |s_j| = \tfrac{1}{2}\big(|u_j| - |v_j|\big)^2 \le \tfrac{1}{2}\Big(\Big||u_j| - \frac{|m|}{d}\Big| + \Big||v_j| - \frac{|n|}{d}\Big|\Big)^2 \le 2c^2.$$

All thresholds depend only on g and M, because the convergence $b_a \to 0$ is uniform. □

Theorem 5.25 (The four scalars move slowly)**.** *Let $0 < g \le \frac{1}{2}$ and $p > 0$. For every $k \in \mathbb{Z}_{\ge 0}$ there is $a(p,k) > 0$ such that*

$$\delta_a \ge k\, a^p$$

for every admissible partition function of gap g at $c = 1 + 12a$ with $0 < a < a(p,k)$. In other words, $\delta_a / a^p \to \infty$ as $a \to 0$, uniformly over the admissible partition functions of gap g.

Proof. It suffices to treat $p \le \frac{1}{2}$. If the statement fails, there are $k \in \mathbb{Z}_{\ge 0}$ and an admissible sequence of gap g with $a \to 0$ and $\delta_a < k a^p$. Apply Proposition 5.23 with $\epsilon^2 = \kappa/(2L)$, so that $e^{-\kappa/\epsilon^2} = a^2$; the condition $a \le \frac{1}{4}\epsilon^2$ holds for small a. The right side of (5.68) is then at most $C(a + k^2 a^{2p})L^2 + Ca^2 \le C' a^{2p} L^2$. For a state with $\Delta_j \le L$ the broad Gaussian is at least $e^{-\pi\kappa}$, so $\sin^2(\pi d u_j) + \sin^2(\pi d v_j) \le C a^{2p} L^2$. As in the proof of Corollary 5.24, for small a there are integers m, n with

$$\Big|u_j - \frac{m}{d}\Big|,\ \Big|v_j - \frac{n}{d}\Big| \le C\, a^p L, \qquad \frac{mn}{d^2} = s_j, \qquad \Big|\Delta_j - \frac{m^2+n^2}{2d^2}\Big| \le C\, a^p L^{3/2},$$

where the exactness of the spin uses $|u_j|, |v_j| \le \sqrt{2L}$ and $a^p L^{3/2} \to 0$. The point $X_j' = [m/d, n/d]$ has integral spin, so by Lemma 4.12 it lies in the fixed finite union Σ_d of (5.29), which contains Λ_1 and is invariant under $[u,v] \mapsto [v,u]$. Thus every state with $\Delta_j \le u_* L$, $u_* = 1 > 1/(2\pi)$, has a real point $X_j' \in \Sigma_d$ with spin exactly s_j and $|\Delta_j - |X_j'|^2| \le C a^p L^{3/2}$, for all sufficiently small a in the sequence. Discarding finitely many terms leaves an admissible sequence of gap g, and for it this contradicts Theorem 4.22. □

Theorem 5.25 gives no upper bound for δ_a by a power of a; it is used below only in the form $\delta_a^{-1} \le a^{-p}$ for every fixed $p > 0$ and small a.

6 Limits with two half-integer radii and the proof of Theorem A

By Theorem 5.21, every limit of an admissible sequence of gap g is a measure $\sigma(R,T)$ in which both radii are half-integers. In this section we exclude these finitely many limits and complete the proof of Theorem A. The kernels of Section 5 cannot be used at the limiting pair (R,T) itself, because their Fourier centres and their zeros collide there. The argument has three parts.

(a) A quantitative form of the rigidity theorem on the common rational grid (Proposition 6.6) shows that the physical Fock measure μ_a agrees, on fixed compact sets and up to $a+b_a^{n+1}$ for every fixed n, times a power of $\log(1/b_a)$, with a combination of at most four circle measures reconstructed from finitely many of its own states (Lemma 6.11).

(b) Positivity, the vacuum term and a moment argument turn this combination into one comparison measure $\sigma(R,t_a)$ with an exact half-integer R and a radius $t_a \to T$, which reproduces the states of every fixed cluster up to a power of b_a (Theorem 6.13); moreover $|t_a - T|$ is comparable to the four-scalar displacement δ_a.

(c) At the radius $t_a \neq T$ the kernels of Section 5 are available again. Their centres and zeros are separated on the scale $|t_a - T|^2$ (Lemma 6.16), and Gaussians of this width give peak budgets that lose only powers of $|t_a - T|^{-1}$ (Lemma 6.21). By Theorem 5.25 such powers are smaller than every negative power of a, and the localization excluded by Theorem 4.23 follows.

6.1 Conventions

Throughout this section $0 < g \le \frac{1}{2}$ is fixed, $R_{\max} = (2g)^{-1/2}$, and d, E_d, η_d are the numbers of (5.66). Points of $\mathbb{R}^2$ are written $[u,v]$ in light-cone coordinates and (P,Q) in momentum coordinates, as in Section 5.1. For an admissible partition function of gap g at $c = 1+12a$ with $a < a_1(g)$,

$$\delta_a, \qquad b_a = \sqrt{a} + \delta_a, \qquad H_a = \log(1/b_a)$$

are the quantities of Lemma 5.22 and Corollary 5.24; thus $a \le b_a^2$, and $b_a \to 0$ uniformly as $a \to 0$.

The grid. We use the light-cone grid of denominator d, which is the hyperbolic grid $d^{-1}\Lambda_1$ of Section 4.1,

$$\Gamma = d^{-1}\Lambda_1 = \{[m/d,\, n/d] : m,n \in \mathbb{Z}\}, \qquad \Gamma^\vee = d\Lambda_1 = \{[md,\, nd] : m,n \in \mathbb{Z}\},$$
$$\mathsf{b}_1 = [d,0], \qquad \mathsf{b}_2 = [0,d]. \tag{6.1}$$

The vectors $\mathsf{b}_1, \mathsf{b}_2$ form a basis of $\Gamma^\vee$. In momentum coordinates $[m/d, n/d] = (\frac{m+n}{2d}, \frac{m-n}{2d})$, so $\Gamma \subset (2d)^{-1}\mathbb{Z}^2$, and a nonzero momentum coordinate of a point of Γ has modulus at least $(2d)^{-1}$. The reflections $P \mapsto -P$ and $Q \mapsto -Q$ are the maps $[u,v] \mapsto [-v,-u]$ and $[u,v] \mapsto [v,u]$; they preserve Γ and $\Gamma^\vee$. This is the grid of Corollary 5.24: the point associated there with a state is $[m/d, n/d] \in \Gamma$. Let

$$\mathcal{R}_d = \{k/d : \ k \text{ a positive divisor of } d^2\}, \qquad \mathcal{R}_d^+ = \{r \in \mathcal{R}_d : r \ge 1\}, \qquad \Sigma_d = \bigcup_{r \in \mathcal{R}_d} \Lambda_r \subset \Gamma$$

be the radii and the union of circle lattices of Lemma 4.12 and (5.29); we have $\mathcal{R}_d \subset [1/d, d]$. Every half-integer r with $1 \le r \le R_{\max}$ belongs to $\mathcal{R}_d^+$: if $r = K/2$, then $2 \le K \le N_g$ and $r = k/d$ with $k = KN_g!$, which divides $d^2 = 4(N_g!)^2$ because K divides $N_g!$.

States and windows. A *state* is a pair $(\Delta, s) \in \mathbb{R} \times \mathbb{Z}$ with $\Delta \geq |s|$; its weights are $h = \frac{1}{2}(\Delta + s)$ and $\bar{h} = \frac{1}{2}(\Delta - s)$, its *state measure* is $A_h \otimes A_{\bar{h}}$, and its physical points are $(\pm\sqrt{h}, \pm\sqrt{\bar{h}})$. A *family of states* is a collection of states, with repetitions allowed, having finitely many members below every dimension; its measure is the sum of the state measures. Thus ρ_a is the measure of the family $(\Delta_j, s_j)_{j\in\mathcal{J}}$. For $(E_0, s_0) \in \mathbb{R} \times \mathbb{Z}$ and $\eta > 0$ the *window*

$$\mathcal{W}_\eta(E_0, s_0) = \{X \in \mathbb{R}^2 : \ |P^2 - Q^2 - s_0| < \tfrac{1}{2}, \ \ ||X|^2 - E_0| < \eta\} \tag{6.2}$$

is a bounded open set, invariant under the two reflections. A state is supported in it if and only if its spin is s_0 and its dimension lies in $(E_0 - \eta, E_0 + \eta)$. For a family of states $\mathfrak{S}$ we call the window $\mathcal{W}_\eta(E_0, s_0)$ *isolated* if no state of $\mathfrak{S}$ with spin s_0 has dimension in $[E_0 - 3\eta, E_0 + 3\eta] \setminus \{E_0\}$. Since a family has finitely many states below every dimension, every pair (E_0, s_0) has isolated windows, and finitely many pairs have a common η.

Weight tests. A *weight test* is a function on $\mathbb{R}^2$ of the form

$$\phi(P, Q) = \Theta(P^2, Q^2), \qquad \Theta \in C_c^\infty(\mathbb{R}^2). \tag{6.3}$$

It is smooth, compactly supported and even, and $\langle A_h \otimes A_{\bar{h}}, \phi\rangle = \Theta(h, \bar{h})$ for $h, \bar{h} \geq 0$. All compactly supported test functions used below are of this form. We fix once and for all two cutoffs

$$\vartheta^{\mathrm{sp}} \in C_c^\infty\big((-\tfrac{1}{2}, \tfrac{1}{2})\big), \quad \vartheta^{\mathrm{sp}} = 1 \text{ on } [-\tfrac{1}{4}, \tfrac{1}{4}]; \qquad \vartheta \in C_c^\infty\big((-2, 2)\big), \quad \vartheta = 1 \text{ on } [-1, 1].$$

For $(E_0, s_0) \in \mathbb{R} \times \mathbb{Z}$, $\eta > 0$ and a smooth function φ on a neighbourhood of $[E_0 - 2\eta, E_0 + 2\eta]$, the function

$$\phi(X) = \vartheta^{\mathrm{sp}}\big(P^2 - Q^2 - s_0\big)\, \vartheta\Big(\frac{|X|^2 - E_0}{\eta}\Big)\, \varphi(|X|^2) \tag{6.4}$$

is a weight test, with $\Theta(h, \bar{h}) = \vartheta^{\mathrm{sp}}(h - \bar{h} - s_0)\, \vartheta((h + \bar{h} - E_0)/\eta)\, \varphi(h + \bar{h})$. For a state (Δ, s),

$$\big\langle A_h \otimes A_{\bar{h}}, \phi\big\rangle = \mathbf{1}_{\{s = s_0\}}\, \vartheta\Big(\frac{\Delta - E_0}{\eta}\Big)\, \varphi(\Delta), \tag{6.5}$$

because the spin is an integer. In particular the pairing equals $\varphi(\Delta)$ if the state is supported in $\mathcal{W}_\eta(E_0, s_0)$ and vanishes if its spin differs from s_0 or $|\Delta - E_0| \geq 2\eta$.

The states of the comparison measures. For $t > 0$ and $(m, n) \in \mathbb{Z}^2$ let

$$E_{m,n}(t) = \tfrac{1}{2}(m^2 t^2 + n^2/t^2)$$

be the squared norm of the point $[mt, n/t] \in \Lambda_t$, whose spin is mn. A *class* is a set $\{(m, n), (-m, -n)\}$ with $(m, n) \neq (0, 0)$.

Lemma 6.1 (States of $\sigma(R, t)$)**.** *Let $t > 0$.*

(i) $\frac{1}{2}\rho_t^{\mathrm{F}} - \frac{1}{2}\delta_0$ *is the measure of the family that contains, for each class $\{\pm(m, n)\}$, one state of dimension $E_{m,n}(t)$ and spin mn.*

(ii) Let $R \geq 1$ be a half-integer. Then $\sigma(R, t)$ is the measure of the family $\mathfrak{S}(R, t)$ consisting of the following states. Fixed states: *one state $(E_{m,n}(R), mn)$ for each class other than the class of $(0, 2R)$, and the two states $(1, 1)$ and $(1, -1)$.* Moving states: *one state $(E_{m,n}(t), mn)$ for each class. In particular $\sigma(R, t)$ is a positive measure, its states have integer spins, and $\sigma(R, t)(\{x\})$ is $\frac{1}{4}$ times the number of pairs consisting of a state of $\mathfrak{S}(R, t)$ and one of its four sign patterns whose physical point is x.*

(iii) Let $I \subset (0,\infty)$ be a compact interval and $E < \infty$. Only finitely many classes satisfy $E_{m,n}(t) \le E$ for some $t \in I$, and there is $C = C(I,E)$ such that $|E_{m,n}(t) - E_{m,n}(t')| \le C|t-t'|$ for these classes and all $t, t' \in I$.

Proof. (i) By (2.10), and since $[n/t, mt]$ runs over $\Lambda_{1/t}$ when (m,n) runs over $\mathbb{Z}^2$,

$$\tfrac{1}{2}\rho_t^{\mathrm{F}} = \tfrac{1}{4}\sum_{(m,n)\in\mathbb{Z}^2}(\delta_{[mt,n/t]}+\delta_{[n/t,mt]}) = \tfrac{1}{2}\delta_0 + \sum_{\{\pm(m,n)\}}\tfrac{1}{4}(\delta_{[mt,n/t]}+\delta_{-[mt,n/t]}+\delta_{[n/t,mt]}+\delta_{-[n/t,mt]}).$$

If $[u,v] = (P,Q)$, the four points $\pm[u,v]$, $\pm[v,u]$ are the points $(\pm P, \pm Q)$ with all combinations of signs. Hence the term of a class is the state measure of the state with the weights of $[mt, n/t]$, that is, with dimension $E_{m,n}(t)$ and spin mn.

(ii) By (i) and (2.12), $\frac{1}{2}\rho_R^{\mathrm{F}} + \frac{1}{2}\rho_t^{\mathrm{F}}$ is δ_0 plus the measure of the family of all states $(E_{m,n}(R), mn)$ and $(E_{m,n}(t), mn)$. By (2.3), ν_{vac} is δ_0 minus the state measures of $(1,1)$ and $(1,-1)$ plus the state measure of $(2,0)$. Since $2R$ is an integer, the class of $(0,2R)$ contributes the state $(E_{0,2R}(R), 0) = (2,0)$, which cancels the last term. The remaining assertions are immediate.

(iii) If $E_{m,n}(t) \le E$ with $t \in I$, then $|m| \le \sqrt{2E}/\min I$ and $|n| \le \sqrt{2E}\max I$. For such classes $\partial_t E_{m,n}(t) = m^2 t - n^2/t^3$ is bounded on I. □

Lemma 6.2 (Windows)**.** *Let $R, T \ge 1$ be half-integers, let $(E_0, s_0) \in \mathbb{R}\times\mathbb{Z}$, and let $\mathcal{W}_\eta(E_0,s_0)$ be isolated for $\mathfrak{S}(R,T)$. Let N_0 be the number of states of $\mathfrak{S}(R,T)$ with dimension E_0 and spin s_0.*

(i) There is $\varepsilon_1 > 0$ such that for $|t-T| < \varepsilon_1$ the states of $\mathfrak{S}(R,t)$ with spin s_0 and dimension in $(E_0-2\eta, E_0+2\eta)$ are the fixed states equal to (E_0,s_0) and the moving states of the classes with $mn = s_0$ and $E_{m,n}(T) = E_0$. Their number is N_0, and their dimensions lie in $(E_0-\eta, E_0+\eta)$.

(ii) If an admissible sequence of gap g converges to $\sigma(R,T)$, then for all sufficiently small a in the sequence exactly N_0 states of ρ_a have spin s_0 and dimension in $(E_0-2\eta, E_0+2\eta)$; their dimensions lie in $(E_0-\eta, E_0+\eta)$ and converge to E_0.

Proof. (i) By Lemma 6.1(ii) the family $\mathfrak{S}(R,T)$ consists of the fixed states and of the moving states at $t = T$, so N_0 is the number of states described in (i). The fixed states do not depend on t; those with spin s_0 and dimension in $(E_0-2\eta, E_0+2\eta)$ are equal to (E_0,s_0), by isolation. By Lemma 6.1(iii), finitely many classes have $mn = s_0$ and $E_{m,n}(t) \le E_0 + 2\eta$ for some $t \in [\frac{1}{2}T, 2T]$, and for a suitable $\varepsilon_1 \le \frac{1}{2}T$ they satisfy $|E_{m,n}(t) - E_{m,n}(T)| < \eta$ when $|t-T| < \varepsilon_1$. For such a class either $E_{m,n}(T) = E_0$, or $|E_{m,n}(T) - E_0| > 3\eta$ by isolation, and then $|E_{m,n}(t) - E_0| > 2\eta$.

(ii) The windows $\mathcal{W}_\eta(E_0,s_0)$ and $\mathcal{W}_{2\eta}(E_0,s_0)$ are bounded, open and invariant under the reflections. A boundary point has spin $s_0 \pm \frac{1}{2}$, or it has integer spin s_0 and squared norm $E_0 \pm \eta$, respectively $E_0 \pm 2\eta$; by isolation and the integrality of the spins of $\sigma(R,T)$, the boundaries do not meet $\operatorname{supp}\sigma(R,T)$. Both windows contain the same N_0 states of $\sigma(R,T)$, and Corollary 2.11 gives the claim. □

Remark 6.3. The parity hypothesis (iv) of Definition 2.1 is not used in this section. The only symmetries used are the two reflections $P \mapsto -P$ and $Q \mapsto -Q$, under which ρ_a and μ_a are invariant by construction. In particular the spaces of Section 6.2 are defined by evenness alone, and clusters of opposite spins are treated independently.

6.2 A quantitative projection onto the invariant distributions

Corollary 4.13 describes the even tempered distributions supported in Γ that are invariant under $\mathbb{S}$ and $\mathcal{C}$. We need a stable version: a distribution on Γ that is exactly invariant under $\mathcal{C}$ and almost invariant under $\mathbb{S}$, as measured by finitely many tests per unit area in a window of logarithmic size, is close to an invariant one on every fixed compact set. In this subsection $d \geq 1$ is an arbitrary integer and Γ, b_1, b_2 are given by (6.1); T and U denote distributions.

Polynomial combs. Let $q \geq 1$ be an integer. Let W_q be the space of distributions

$$\mathsf{U} = \sum_{\mathfrak{c} \in \Gamma/\Gamma^\vee} \sum_{|\alpha| \leq q} \sum_{z \in \mathfrak{c}} p_{\mathfrak{c},\alpha}(z)\, \partial^\alpha \delta_z \tag{6.6}$$

that are even, where each $p_{\mathfrak{c},\alpha}$ is a polynomial on $\mathbb{R}^2$ of degree at most q in each of the light-cone coordinates u and v. Here $\partial^\alpha = \partial_P^{\alpha_1} \partial_Q^{\alpha_2}$ as in Section 4. A polynomial is determined by its values on a coset of $\Gamma^\vee$, so the polynomials in (6.6) are determined by U, and W_q is a finite-dimensional space of tempered distributions of local order at most q supported in Γ. We write $c_\alpha^{\mathsf{U}}(z) = p_{\mathfrak{c},\alpha}(z)$, $z \in \mathfrak{c}$, for the coefficients of U in (4.1), fix a norm $\|\cdot\|$ on W_q, and put

$$\mathsf{V}_q = \{\mathsf{U} \in \mathsf{W}_q : \ \mathbb{S}\mathsf{U} = \mathsf{U},\ \mathcal{C}\mathsf{U} = \mathsf{U}\}.$$

Neither $\mathbb{S}$ nor $\mathcal{C}$ maps W_q into itself.

Lemma 6.4 (The invariant subspace)**.** *The space V_q is the space of all even tempered distributions of local order at most q that are supported in Γ and invariant under $\mathbb{S}$ and $\mathcal{C}$. It has the basis (4.19),*

$$(r\partial_r)^j \rho_r^{\mathrm{F}}, \qquad r \in \mathcal{R}_d^+, \quad 0 \leq j \leq q, \quad j \text{ even if } r = 1.$$

Proof. Every element of V_q is an even tempered distribution of local order at most q, supported in $\Gamma = d^{-1}\Lambda_1$ and invariant, so by Corollary 4.13 it is a combination of the displayed distributions, which are linearly independent. It remains to show that each of them belongs to W_q; it is even and invariant by the same corollary. By (4.18) it suffices to show that $\mathcal{K}^j e_r$ has the form (6.6) for $r \in \mathcal{R}_d$ and $j \leq q$. The lattice Λ_r is contained in Γ by Lemma 4.12, and it contains $\Gamma^\vee$: for $r = k/d$ with $k \mid d^2$ we have $[d, 0] = [(d^2/k)\, r, 0]$ and $[0, d] = [0, k/r]$. Hence Λ_r is a union of cosets of $\Gamma^\vee$. As operators on distributions, $\mathcal{K} = u\partial_u - v\partial_v = \partial_u \circ u - \partial_v \circ v$, and by induction on j, using $u \circ \partial_u^i = \partial_u^i \circ u - i\, \partial_u^{i-1}$ and the same relation for v, the operator $\mathcal{K}^j$ is a combination of operators $\partial_u^i \partial_v^l \circ u^{i'} v^{l'}$ with $i + l \leq j$, $i' \leq j$ and $l' \leq j$. Since ∂_u and ∂_v are combinations of ∂_P and ∂_Q, (4.4) shows that $\mathcal{K}^j e_r$ is of the form (6.6) with polynomials of degree at most $j \leq q$ in each of u and v. □

Thus V_q does not depend on the choice of the comb space beyond its order: it consists of radius-derivative jets, of order at most q, of the circle measures of *all* radii in $\mathcal{R}_d^+$, the self-dual radius included.

Lemma 6.5 (A finite-dimensional projection)**.** *Let $q \geq 1$. There are a finite family $\mathcal{F}_q$ of functions of the form $X^\gamma e^{-2\pi|X|^2}$, a linear projection Π_q of W_q onto V_q, and a constant C, all depending only on d, q and the norm, such that for every $\mathsf{U} \in \mathsf{W}_q$*

$$\|\mathsf{U} - \Pi_q \mathsf{U}\| \leq C \max_{f \in \mathcal{F}_q} \Big(\big|\langle (\mathbb{S} - I)\mathsf{U}, f\rangle\big| + \big|\langle (\mathcal{C} - I)\mathsf{U}, f\rangle\big| \Big). \tag{6.7}$$

Proof. Let $\mathcal{D}$ be the linear span of the functions $X^\gamma e^{-2\pi|X|^2}$, $\gamma \in \mathbb{Z}^2_{\geq 0}$. It is the span of the Hermite functions and is dense in the Schwartz space. We claim that for every finite-dimensional

subspace Y of the space of tempered distributions there are finitely many monomial functions $f_1, \dots, f_k \in \mathcal{D}$ such that $\mathsf{U} \mapsto (\langle \mathsf{U}, f_i \rangle)_{i \le k}$ is injective on Y. Indeed, the image of the linear map from $\mathcal{D}$ to the dual space Y^*, $f \mapsto \langle \cdot, f \rangle$, is a subspace of Y^*. If it were a proper subspace, some nonzero $\mathsf{U} \in \mathsf{Y}$ would satisfy $\langle \mathsf{U}, f \rangle = 0$ for all $f \in \mathcal{D}$, and hence, by density and continuity, $\mathsf{U} = 0$. So the map is onto, and finitely many of the spanning monomial functions are mapped onto a spanning set of Y^*.

Apply the claim to the finite-dimensional spaces $(\mathbb{S} - I)\mathsf{W}_q$ and $(\mathcal{C} - I)\mathsf{W}_q$, and let $\mathcal{F}_q$ be the union of the two families obtained. The linear map

$$\mathfrak{D} : \mathsf{W}_q \to \mathbb{C}^{\mathcal{F}_q} \times \mathbb{C}^{\mathcal{F}_q}, \qquad \mathfrak{D}\mathsf{U} = \Big(\langle (\mathbb{S} - I)\mathsf{U}, f \rangle,\ \langle (\mathcal{C} - I)\mathsf{U}, f \rangle \Big)_{f \in \mathcal{F}_q}$$

then has kernel exactly V_q. Choose a complement $\mathsf{W}_q = \mathsf{V}_q \oplus \mathsf{Y}_q$ and let Π_q be the projection along Y_q. The map $\mathfrak{D}$ is injective on the finite-dimensional space Y_q, so $\|\mathsf{Y}\| \le C \|\mathfrak{D}\mathsf{Y}\|_\infty$ for $\mathsf{Y} \in \mathsf{Y}_q$. Applying this to $\mathsf{U} - \Pi_q \mathsf{U} \in \mathsf{Y}_q$ and using $\mathfrak{D}\Pi_q \mathsf{U} = 0$ gives (6.7). □

The lemma bounds the distance to the whole invariant subspace, and not only a linear functional that vanishes on it. The constant is not explicit.

Probes. For $y \in \Gamma$ and $\beta \in \mathcal{A}_q$ let $\psi_{y,\beta}$ be the function ψ of Lemma 4.18 for the single coset $\mathfrak{c}_1 = \Gamma$, written with $x_1 = 0$ and the forms $\ell_{1,1}(X) = du$, $\ell_{1,2}(X) = dv$, for the point y, the order q and the index β. Thus $\psi_{y,\beta} = \mathsf{t}\,\psi_0$ with

$$\psi_0(X) = e^{-2\pi|X-y|^2} \Big[\operatorname{sinc}\big(\pi d(u - u_y)\big) \operatorname{sinc}\big(\pi d(v - v_y)\big) \Big]^{q+1}$$

and t the Taylor polynomial of degree q at y of $(-1)^{|\beta|}(X-y)^\beta / (\beta!\,\psi_0(X))$. This construction depends on X and y only through $X - y$, so

$$\psi_{y,\beta}(X) = \psi_{0,\beta}(X - y), \qquad y \in \Gamma. \tag{6.8}$$

By Lemma 4.18, $\psi_{0,\beta}$ is a finite sum of functions $\mathfrak{m}[p, z]$ as in Lemma 4.17, and for every tempered distribution U supported in Γ and of local order at most q,

$$\langle \mathsf{U}, \psi_{y,\beta} \rangle = \text{the coefficient of } \partial^\beta \delta_y \text{ in } \mathsf{U}. \tag{6.9}$$

With the characters $\chi_w(X) = e^{2\pi i B(w,X)}$ of Section 4.1 put

$$\begin{aligned} \omega_\iota &= (\chi_{\mathsf{b}_\iota} - 1)^{q+1} \quad (\iota = 1, 2), \\ \mathcal{P}_q(\varrho) &= \{ \omega_\iota\, \mathbb{S}\psi_{y,\beta} :\ \iota \in \{1,2\},\ \beta \in \mathcal{A}_q,\ y \in \Gamma,\ |u_y|, |v_y| \le \varrho \}. \end{aligned} \tag{6.10}$$

Since $\chi_{\mathsf{b}_1}(X) = e^{2\pi i dv}$ and $\chi_{\mathsf{b}_2}(X) = e^{2\pi i du}$, the function ω_ι vanishes to order $q+1$ at every point of Γ.

Proposition 6.6 (Quantitative grid projection)**.** *Let $d, q \ge 1$ be integers, let $A_0 \ge 0$ and $A_1 > 0$, and put $D_0 = 4A_1/\pi$. Let $\mathcal{F}_q$ and Π_q be as in Lemma 6.5. For every compact set $K \subset \mathbb{R}^2$ there is a constant C_K, depending only on d, q, A_0, A_1 and K, with the following property. Let $0 < b < e^{-1}$, $H = \log(1/b)$, $C_0 < \infty$ and $\varepsilon \ge 0$, and let*

$$\mathsf{T} = \sum_{z \in \Gamma} \sum_{|\alpha| \le q} c_\alpha(z)\, \partial^\alpha \delta_z$$

be a distribution such that

(i) $|c_\alpha(z)| \le C_0 (1 + |z|)^{A_0}$ for all $z \in \Gamma$ and $|\alpha| \le q$;

(ii) T *is even and* $\mathcal{C}\mathsf{T} = \mathsf{T}$*;*

(iii) $|\langle(\mathbb{S} - I)\mathsf{T}, f\rangle| \le \varepsilon$ *for every* $f \in \mathcal{F}_q$ *and every* $f \in \mathcal{P}_q(\sqrt{D_0 H} + (q+3)d)$*.*

Then there is $\mathsf{V} \in \mathsf{V}_q$ *such that*

$$\max_{z \in \Gamma \cap K} \max_{|\alpha| \le q} |c_\alpha(z) - c_\alpha^{\mathsf{V}}(z)| \le C_K \Big(\varepsilon \, (1+H)^{q+1} + C_0 \, b^{A_1} \Big). \tag{6.11}$$

The maximum includes the coefficients of order zero.

No reference measure is subtracted from T, the coefficients of T need not be small anywhere, and the number D_0, which fixes the window of the probes, depends only on A_1.

Proof. Constants called C depend only on d, q, A_0 and A_1. By (i), T is a tempered distribution of local order at most q supported in Γ.

Step 1: differences. Let $\mathsf{D}_\iota = (\tau_{\mathsf{b}_\iota} - I)^{q+1}$. Since $\tau_w \delta_x = \delta_{x-w}$, the distribution $\mathsf{D}_\iota \mathsf{T}$ is supported in Γ, has local order at most q, and its coefficient at $\partial^\beta \delta_y$ is the forward difference

$$(\Delta_\iota^{q+1} c_\beta)(y) = \sum_{k=0}^{q+1} (-1)^{q+1-k} \binom{q+1}{k} c_\beta(y + k\mathsf{b}_\iota). \tag{6.12}$$

By (4.2), $\mathsf{D}_\iota = \mathbb{S}\,\omega_\iota\,\mathbb{S}^{-1}$. As T is even, $\mathbb{S}^{-1}\mathsf{T} = \mathbb{S}\mathsf{T}$, and $\omega_\iota \mathsf{T} = 0$ by Lemma 4.1(ii). Hence, by (6.9),

$$(\Delta_\iota^{q+1} c_\beta)(y) = \langle \mathsf{D}_\iota \mathsf{T}, \psi_{y,\beta} \rangle = \langle \omega_\iota \, \mathbb{S}\mathsf{T}, \mathbb{S}\psi_{y,\beta} \rangle = \big\langle (\mathbb{S} - I)\mathsf{T}, \, \omega_\iota \, \mathbb{S}\psi_{y,\beta} \big\rangle,$$

and by (iii)

$$|(\Delta_\iota^{q+1} c_\beta)(y)| \le \varepsilon \qquad \text{for } y \in \Gamma, \ |u_y|, |v_y| \le \sqrt{D_0 H} + (q+3)d, \ \iota \in \{1,2\}, \ |\beta| \le q. \tag{6.13}$$

Step 2: interpolation in one variable. Let $N' \ge q$ be an integer, let c be a function on $\{-N', \dots, N'\}$ whose forward differences satisfy $|\Delta^{q+1} c(i)| \le \varepsilon$ for $-N' \le i \le N' - q - 1$, and let p be the polynomial of degree at most q with $\mathsf{p}(i) = c(i)$ for $0 \le i \le q$. Then

$$|c(i) - \mathsf{p}(i)| \le \varepsilon \, (q + |i|)^{q+1} \qquad (|i| \le N'). \tag{6.14}$$

To see this, let $e = c - \mathsf{p}$, so that $e(0) = \cdots = e(q) = 0$ and $\Delta^{q+1} e = \Delta^{q+1} c$. We show $|e(i)| \le \varepsilon \binom{i}{q+1}$ for $0 \le i \le N'$ by induction on $q \ge 0$. For $q = 0$, $e(i) = \sum_{l<i} \Delta e(l)$ and $e(0) = 0$. For $q \ge 1$ the function Δe vanishes at $0, \dots, q-1$ and its differences of order q are bounded by ε, so $|\Delta e(l)| \le \varepsilon \binom{l}{q}$ by the induction hypothesis, and $|e(i)| \le \varepsilon \sum_{l<i} \binom{l}{q} = \varepsilon \binom{i}{q+1}$. Only the differences $\Delta^{q+1} c(l)$ with $0 \le l \le i - q - 1$ are used. For negative i apply this to $\check{e}(i) = e(q - i)$, which vanishes at $0, \dots, q$ and satisfies $\Delta^{q+1} \check{e}(i) = (-1)^{q+1} \Delta^{q+1} e(-i-1)$; this gives $|e(i)| \le \varepsilon \binom{q-i}{q+1}$ for $-N' \le i < 0$ and uses the differences at $i, \dots, -1$. Finally $\binom{q+|i|}{q+1} \le (q + |i|)^{q+1}$.

Step 3: interpolation on a coset. Put

$$N' = \lceil \sqrt{D_0 H}/d \rceil + q + 1, \qquad \mathsf{Q}_H = \{[u,v] : |u|, |v| \le \sqrt{D_0 H}\}.$$

Fix a coset $\mathfrak{c}$ of $\Gamma^\vee$ in Γ, let $z_\mathfrak{c}$ be its point with $u, v \in [0, d)$, fix α, and let $c(n_1, n_2) = c_\alpha(z_\mathfrak{c} + n_1 \mathsf{b}_1 + n_2 \mathsf{b}_2)$. The point $z_\mathfrak{c} + n_1 \mathsf{b}_1 + n_2 \mathsf{b}_2$ has light-cone coordinates $u_{z_\mathfrak{c}} + n_1 d$ and $v_{z_\mathfrak{c}} + n_2 d$. If it lies in Q_H, then $|n_1|, |n_2| \le N'$; and if $|n_1|, |n_2| \le N'$, its coordinates are bounded by $d(N'+1) \le \sqrt{D_0 H} + (q+3)d$. By (6.13), the differences of order $q+1$ of c in each of the two variables are therefore bounded by ε whenever their stencil lies in $\{-N', \dots, N'\}^2$. Let $p(n_1, n_2)$ be the polynomial of degree at most q in each variable with $p = c$ on $\{0, \dots, q\}^2$, and let $\ell_0, \dots, \ell_q$ be the Lagrange polynomials of the nodes $0, \dots, q$; they satisfy $|\ell_k(i)| \le (q + |i|)^q$

for integers i, because the denominators $\prod_{l\neq k}|k-l|$ are at least one. For $0\le k\le q$ the function $p(\cdot,k)$ is the interpolating polynomial of $c(\cdot,k)$, so (6.14) gives $|c(n_1,k)-p(n_1,k)|\le \varepsilon(q+|n_1|)^{q+1}$. For fixed n_1 the interpolating polynomial of $c(n_1,\cdot)$ is $\sum_k \ell_k(n_2)c(n_1,k)$, and $p(n_1,n_2)=\sum_k \ell_k(n_2)p(n_1,k)$. Hence, for $|n_1|,|n_2|\le N'$,

$$\begin{aligned}|c(n_1,n_2)-p(n_1,n_2)| &\le \varepsilon(q+|n_2|)^{q+1}+\sum_{k=0}^{q}|\ell_k(n_2)|\,\varepsilon\,(q+|n_1|)^{q+1}\\ &\le (q+2)\,(q+N')^{2q+1}\,\varepsilon \le C\,(1+H)^{q+1}\varepsilon.\end{aligned}\tag{6.15}$$

Moreover, by (i) the $(q+1)^2$ initial values are bounded by CC_0, so

$$|p(n_1,n_2)|\le \sum_{i,k}|\ell_i(n_1)||\ell_k(n_2)||c(i,k)|\le C\,C_0\,(q+|n_1|)^q(q+|n_2|)^q.\tag{6.16}$$

Let $p_{\mathfrak{c},\alpha}$ be the polynomial on $\mathbb{R}^2$ with $p_{\mathfrak{c},\alpha}(z_{\mathfrak{c}}+n_1\mathsf{b}_1+n_2\mathsf{b}_2)=p(n_1,n_2)$; it has degree at most q in each of u and v.

Step 4: a comb close to T. Let U° be the distribution (6.6) formed with the polynomials $p_{\mathfrak{c},\alpha}$ of Step 3, and let $\widetilde{\mathsf{U}}=\frac14\sum_{\mathsf{g}}\mathsf{g}_*\mathsf{U}^\circ$ be its average over the group of the four maps $(P,Q)\mapsto(\pm P,\pm Q)$. Each of these maps preserves Γ and $\Gamma^\vee$, maps $\partial^\alpha\delta_z$ to $\pm\partial^\alpha\delta_{\mathsf{g}z}$, and permutes the light-cone coordinates up to signs; hence $\mathsf{g}_*\mathsf{U}^\circ$ is again of the form (6.6), and $\widetilde{\mathsf{U}}\in\mathsf{W}_q$. Since T is even, $\mathsf{T}-\widetilde{\mathsf{U}}=\frac14\sum_{\mathsf{g}}\mathsf{g}_*(\mathsf{T}-\mathsf{U}^\circ)$, and the square Q_H is invariant under the four maps. Writing $\tilde c_\alpha$ for the coefficients of $\widetilde{\mathsf{U}}$, we obtain from (6.15) and (6.16)

$$|c_\alpha(z)-\tilde c_\alpha(z)|\le C(1+H)^{q+1}\varepsilon\quad(z\in\Gamma\cap\mathsf{Q}_H),\qquad |\tilde c_\alpha(z)|\le C\,C_0\,(1+|z|)^{2q}\quad(z\in\Gamma).\tag{6.17}$$

Step 5: the defects of $\widetilde{\mathsf{U}}$. Let G be one of the functions f, $\mathbb{S}f$, $(e^{2\pi i(P^2-Q^2)}-1)f$ with $f\in\mathcal{F}_q$. Each of them is a polynomial times $e^{-2\pi|X|^2}$, times a function all of whose derivatives are polynomially bounded on $\mathbb{R}^2$; for $\mathbb{S}f$ this follows from Lemma 4.17(ii) with $z=0$. Hence $|\partial^\alpha G(X)|\le Ce^{-\pi|X|^2}$ for real X and $|\alpha|\le q$. The pairing of the tempered distribution $\mathsf{T}-\widetilde{\mathsf{U}}$ with G is the absolutely convergent sum of the terms $(-1)^{|\alpha|}(c_\alpha-\tilde c_\alpha)(z)\,\partial^\alpha G(z)$. A point $z\in\Gamma\setminus\mathsf{Q}_H$ has $|z|^2=\frac12(u_z^2+v_z^2)>\frac12 D_0H$, hence $e^{-\pi|z|^2}\le e^{-\pi D_0H/4}e^{-\pi|z|^2/2}=b^{A_1}e^{-\pi|z|^2/2}$. By (i) and (6.17),

$$\begin{aligned}|\langle\mathsf{T}-\widetilde{\mathsf{U}},G\rangle| &\le C(1+H)^{q+1}\varepsilon\sum_{z\in\Gamma}e^{-\pi|z|^2}+C\,C_0\,b^{A_1}\sum_{z\in\Gamma}(1+|z|)^{A_0+2q}e^{-\pi|z|^2/2}\\ &\le C\big((1+H)^{q+1}\varepsilon+C_0\,b^{A_1}\big)=:\varepsilon'.\end{aligned}\tag{6.18}$$

For $f\in\mathcal{F}_q$ we have $\langle(\mathbb{S}-I)\widetilde{\mathsf{U}},f\rangle=\langle(\mathbb{S}-I)\mathsf{T},f\rangle-\langle\mathsf{T}-\widetilde{\mathsf{U}},\mathbb{S}f-f\rangle$ and, by (ii), $\langle(\mathcal{C}-I)\widetilde{\mathsf{U}},f\rangle=-\langle\mathsf{T}-\widetilde{\mathsf{U}},(e^{2\pi i(P^2-Q^2)}-1)f\rangle$. By (iii) and (6.18) both are at most $\varepsilon+2\varepsilon'$ in modulus.

Step 6: projection. Put $\mathsf{V}=\Pi_q\widetilde{\mathsf{U}}\in\mathsf{V}_q$. By (6.7), $\|\widetilde{\mathsf{U}}-\mathsf{V}\|\le C(\varepsilon+\varepsilon')$. For fixed $z\in\Gamma$ and α the coefficient $\mathsf{U}\mapsto c_\alpha^{\mathsf{U}}(z)$ is a linear functional on the finite-dimensional space W_q, so $|\tilde c_\alpha(z)-c_\alpha^{\mathsf{V}}(z)|\le C'_K(\varepsilon+\varepsilon')$ for the finitely many $z\in\Gamma\cap K$. If $K\subset\mathsf{Q}_H$, the first bound of (6.17) gives (6.11). If $K\not\subset\mathsf{Q}_H$, then b is bounded below by a positive number b_K depending on K and D_0, and (i) and the second bound of (6.17) give $|c_\alpha(z)-\tilde c_\alpha(z)|\le C''_KC_0\le C''_Kb_K^{-A_1}C_0\,b^{A_1}$ on K. □

6.3 The reconstruction operator

We now construct a linear operator, fixed once g is given, which recovers every element of every space V_q from finitely many of its local coefficients, and which sends the Fock measure of a partition function to a positive combination of at most four circle measures. The grid is again (6.1) with the integer d of (5.66).

Fix a prime $\mathfrak{p} > d^2$ and put

$$\mathsf{E}(t) = E_{1,\mathfrak{p}}(t) = \tfrac{1}{2}(t^2 + \mathfrak{p}^2/t^2), \qquad \mathsf{E}(1/t) = E_{\mathfrak{p},1}(t).$$

Since $\mathsf{E}'(t) = t - \mathfrak{p}^2/t^3 < 0$ for $0 < t < \sqrt{\mathfrak{p}}$, the function E maps $(0, \sqrt{\mathfrak{p}})$ diffeomorphically onto $(\mathfrak{p}, \infty)$; we write $\mathsf{E}^{-1} : (\mathfrak{p}, \infty) \to (0, \sqrt{\mathfrak{p}})$ for the inverse of this restriction. The classes with $mn = \mathfrak{p}$ are those of $(1, \mathfrak{p})$ and $(\mathfrak{p}, 1)$. By Lemma 6.1(i), for every $t > 0$,

> the family of $\frac{1}{2}\rho_t^{\mathrm{F}} - \frac{1}{2}\delta_0$ has exactly two states of spin $\mathfrak{p}$; their dimensions are $\mathsf{E}(t)$ and $\mathsf{E}(1/t)$. (6.19)

The radii of $\mathcal{R}_d$ lie in $[1/d, d] \subset (\mathfrak{p}^{-1/2}, \mathfrak{p}^{1/2})$, so the numbers $\mathsf{E}(r)$, $r \in \mathcal{R}_d$, are distinct and greater than $\mathfrak{p}$; let $g_0 > 0$ be the smallest distance between two of them. Choose $\eta_* \in (0, 1]$ such that

(a) $8\eta_* \le g_0$ and $\mathsf{E}(d) - 2\eta_* > \mathfrak{p}$;

(b) for every $r \in \mathcal{R}_d^+$ with $r > 1$ and every E with $|E - \mathsf{E}(r)| \le 2\eta_*$ we have $\bigl|\mathsf{E}(1/\mathsf{E}^{-1}(E)) - \mathsf{E}(1/r)\bigr| \le \frac{1}{2}g_0$.

Condition (b) holds for small η_* by continuity. For $r \in \mathcal{R}_d^+$ put

$$\vartheta_r(E) = \vartheta\Bigl(\frac{E - \mathsf{E}(r)}{\eta_*}\Bigr), \qquad \mathsf{c}_r = \begin{cases} \frac{1}{2}, & r > 1, \\ \frac{1}{4}, & r = 1. \end{cases}$$

By (a) the supports of the functions ϑ_r are pairwise disjoint and contained in $(\mathfrak{p}, \infty)$. For a weight test ϕ let $\mathsf{f}_\phi(t) = \langle \rho_t^{\mathrm{F}}, \phi\rangle$, $t > 0$, and define

$$(P_*'\phi)(X) = \vartheta^{\mathrm{sp}}(P^2 - Q^2 - \mathfrak{p})\, G_\phi(|X|^2), \qquad G_\phi(E) = \sum_{r \in \mathcal{R}_d^+} \mathsf{c}_r\, \vartheta_r(E)\, \mathsf{f}_\phi(\mathsf{E}^{-1}(E)), \tag{6.20}$$

where the term with index r is zero outside the support of ϑ_r. For a distribution U on $\mathbb{R}^2$ we define $P_*\mathsf{U}$ on weight tests by $\langle P_*\mathsf{U}, \phi\rangle = \langle \mathsf{U}, P_*'\phi\rangle$. The clusters of spin $\mathfrak{p}$ and dimension $\mathsf{E}(r)$ are the *private clusters*; there is one for every radius of $\mathcal{R}_d^+$, whether or not it occurs in a limit.

Lemma 6.7 (The reconstruction operator)**.** *The map P_*' is linear, depends only on g, and has the following properties.*

(i) $P_*'\phi$ is a weight test for every weight test ϕ, and for every state (Δ, s) with weights $(h, \bar{h})$

$$\langle A_h \otimes A_{\bar{h}}, P_*'\phi\rangle = \mathbf{1}_{\{s=\mathfrak{p}\}}\, G_\phi(\Delta). \tag{6.21}$$

(ii) $P_*\nu_{\mathrm{vac}} = 0$.

(iii) For every $r \in \mathcal{R}_d^+$ there is an open interval $I_r \ni r$ such that $P_*\rho_t^{\mathrm{F}} = \rho_t^{\mathrm{F}}$ for $t \in I_r$, that is, $\langle \rho_t^{\mathrm{F}}, P_*'\phi\rangle = \langle \rho_t^{\mathrm{F}}, \phi\rangle$ for every weight test ϕ.

(iv) $P_*\mathsf{V} = \mathsf{V}$ for every $q \ge 1$ and every $\mathsf{V} \in \mathsf{V}_q$.

(v) Let an admissible sequence of gap g converge to $\sigma(R_1, R_2)$ with half-integers R_1, R_2, and for $r \in \mathcal{R}_d^+$ put

$$n_r = \#\{i \in \{1, 2\} : R_i = r\} \cdot (1 + \mathbf{1}_{\{r=1\}}).$$

For all sufficiently small a in the sequence and every $r \in \mathcal{R}_d^+$, exactly n_r states of ρ_a have spin $\mathfrak{p}$ and dimension in $(\mathsf{E}(r) - 2\eta_*, \mathsf{E}(r) + 2\eta_*)$; their dimensions $\Delta_{r,1} \le \cdots \le \Delta_{r,n_r}$ lie in $(\mathsf{E}(r) - \eta_*, \mathsf{E}(r) + \eta_*)$, the radii $t_{r,i} = \mathsf{E}^{-1}(\Delta_{r,i})$ converge to r, and

$$P_*\mu_a = \sum_{r \in \mathcal{R}_d^+} \mathsf{c}_r \sum_{i=1}^{n_r} \rho_{t_{r,i}}^{\mathrm{F}}, \qquad \sum_r n_r \le 4, \qquad \sum_r \mathsf{c}_r n_r = 1. \tag{6.22}$$

In (v) only the radii $r \in \{R_1, R_2\}$ have $n_r \neq 0$. The private clusters of all other radii of $\mathcal{R}_d^+$ are needed for (iv), and they are empty for small a.

Proof. (i) We have $\mathsf{f}_\phi(t) = \frac{1}{2}\sum_{m,n}(\phi([mt, n/t]) + \phi([m/t, nt]))$. Since ϕ has compact support, for t in a compact subset of $(0,\infty)$ only finitely many terms are different from zero, so f_ϕ is smooth. Hence G_ϕ is smooth and compactly supported on $\mathbb{R}$, and $P_*'\phi$ is the weight test with $\Theta(h,\bar h) = \vartheta^{\mathrm{sp}}(h - \bar h - \mathfrak{p})\, G_\phi(h + \bar h)$. Formula (6.21) follows as (6.5).

(ii) By (2.3) the measure ν_{vac} is a combination of the state measures of $(0,0)$, $(1,\pm1)$ and $(2,0)$, none of which has spin $\mathfrak{p}$.

(iii) By Lemma 6.1(i), (6.19) and (6.21), and since $(P_*'\phi)(0) = 0$,

$$\langle \rho_t^{\mathrm{F}}, P_*'\phi\rangle = 2\, G_\phi(\mathsf{E}(t)) + 2\, G_\phi(\mathsf{E}(1/t)), \qquad t > 0. \tag{6.23}$$

Let $r > 1$ and $I_r = \{t \in (0, \sqrt{\mathfrak{p}}) : |\mathsf{E}(t) - \mathsf{E}(r)| < \eta_*\}$. For $t \in I_r$ we have $\vartheta_r(\mathsf{E}(t)) = 1$, $\vartheta_{r'}(\mathsf{E}(t)) = 0$ for $r' \neq r$, and $\mathsf{E}^{-1}(\mathsf{E}(t)) = t$, so $G_\phi(\mathsf{E}(t)) = \frac{1}{2}\mathsf{f}_\phi(t)$. By (b), $\mathsf{E}(1/t)$ is within $\frac{1}{2}g_0$ of $\mathsf{E}(1/r)$. As $1/r \in \mathcal{R}_d \setminus \mathcal{R}_d^+$, the number $\mathsf{E}(1/r)$ is at distance at least g_0 from every $\mathsf{E}(r')$, $r' \in \mathcal{R}_d^+$, so $\mathsf{E}(1/t)$ is at distance at least $\frac{1}{2}g_0 \geq 4\eta_*$ from them and $G_\phi(\mathsf{E}(1/t)) = 0$. Thus (6.23) equals $\mathsf{f}_\phi(t)$. For $r = 1$ let I_1 be the set of $t \in (\mathfrak{p}^{-1/2}, \mathfrak{p}^{1/2})$ with $|\mathsf{E}(t) - \mathsf{E}(1)| < \eta_*$ and $|\mathsf{E}(1/t) - \mathsf{E}(1)| < \eta_*$; it is an open interval containing 1. For $t \in I_1$ both $\mathsf{E}(t)$ and $\mathsf{E}(1/t)$ lie where $\vartheta_1 = 1$ and all other cutoffs vanish, and $\mathsf{E}^{-1}(\mathsf{E}(t)) = t$, $\mathsf{E}^{-1}(\mathsf{E}(1/t)) = 1/t$. Hence (6.23) equals $\frac{1}{2}\mathsf{f}_\phi(t) + \frac{1}{2}\mathsf{f}_\phi(1/t) = \mathsf{f}_\phi(t)$, because $\rho_{1/t}^{\mathrm{F}} = \rho_t^{\mathrm{F}}$.

(iv) By Lemma 6.4 it suffices to treat $\mathsf{V} = (r\partial_r)^j \rho_r^{\mathrm{F}}$ with $r \in \mathcal{R}_d^+$. By Theorem 4.11(iii), V is the distribution $f \mapsto (t\partial_t)^j \langle \rho_t^{\mathrm{F}}, f\rangle|_{t=r}$ on Schwartz functions, so we have for every weight test ϕ

$$\langle \mathsf{V}, P_*'\phi\rangle = (t\partial_t)^j \langle \rho_t^{\mathrm{F}}, P_*'\phi\rangle\big|_{t=r} = (t\partial_t)^j \langle \rho_t^{\mathrm{F}}, \phi\rangle\big|_{t=r} = \langle \mathsf{V}, \phi\rangle,$$

since the two functions of t coincide on the open interval I_r by (iii).

(v) By Lemma 6.1(ii) and (6.19), the states of spin $\mathfrak{p}$ of $\sigma(R_1, R_2)$ have the dimensions $\mathsf{E}(R_i)$ and $\mathsf{E}(1/R_i)$, $i = 1, 2$. Since $1 \leq R_i \leq R_{\max}$ by Theorem 3.5, all four radii belong to $\mathcal{R}_d$. By (a), such a state has dimension within $3\eta_*$ of $\mathsf{E}(r)$, $r \in \mathcal{R}_d^+$, only if its radius equals r, in which case the dimension is $\mathsf{E}(r)$; this happens for $R_i = r$, and for $1/R_i$ when $R_i = r = 1$. Hence $\mathcal{W}_{\eta_*}(\mathsf{E}(r), \mathfrak{p})$ is isolated for $\mathfrak{S}(R_1, R_2)$ and contains n_r of its states, and Lemma 6.2(ii) gives the statements about the states of ρ_a; the convergence $t_{r,i} \to r$ follows from the continuity of E^{-1}. By (ii) and (6.21),

$$\langle \mu_a, P_*'\phi\rangle = \sum_{j:\, s_j = \mathfrak{p}} G_\phi(\Delta_j) = \sum_{r \in \mathcal{R}_d^+} \mathsf{c}_r \sum_{i=1}^{n_r} \mathsf{f}_\phi(\mathsf{E}^{-1}(\Delta_{r,i})),$$

because a state of spin $\mathfrak{p}$ with $\vartheta_r(\Delta_j) \neq 0$ is one of the n_r states above, for which $\vartheta_r(\Delta_j) = 1$. Finally $\sum_r n_r = \sum_i (1 + \mathbf{1}_{\{R_i = 1\}}) \leq 4$ and $\sum_r \mathsf{c}_r n_r = \sum_i \mathsf{c}_{R_i}(1 + \mathbf{1}_{\{R_i=1\}}) = 1$. □

6.4 The Taylor array of the spectrum and the normal form

We expand the spectrum of an admissible partition function to a finite order about the grid points supplied by Corollary 5.24. The resulting distribution satisfies the hypotheses of Proposition 6.6. The estimates must be uniform for probes centred anywhere in a window of radius of order $\sqrt{H_a}$, and for their Fourier transforms, which oscillate with frequencies of that order. A Gaussian envelope centred at the origin, as in Lemma 2.6, would lose a factor exponential in H_a for such functions. We therefore use the following pointwise form of that lemma.

For real x let $\mathsf{J}(x) \subset \mathbb{C}$ be the set $\{\tau\sqrt{x} : -1 \leq \tau \leq 1\}$ if $x \leq 2$ and the set $\{\pm\sqrt{x}\}$ if $x > 2$, where $\sqrt{x} = i\sqrt{|x|}$ for $x < 0$.

Lemma 6.8 (Pointwise bounds for weight derivatives)**.** *Let F be an entire function on $\mathbb{C}^2$ and $k \ge 0$ an integer. For all real $h, \bar h$ and all $i + l \le k$,*

$$|\partial_h^i \partial_{\bar h}^l F^\natural(h, \bar h)| \le C_k \max\{|\partial^\alpha F(P, Q)| : \ |\alpha| \le 2k, \ P \in \mathsf{J}(h), \ Q \in \mathsf{J}(\bar h)\},$$

where C_k depends only on k.

Proof. First let ϕ be an entire function of one variable, $\varphi(h) = \frac12(\phi(\sqrt h) + \phi(-\sqrt h))$ and $i \ge 0$. We claim that $|\varphi^{(i)}(h)| \le C_i \max\{|\phi^{(l)}(P)| : l \le 2i, \ P \in \mathsf{J}(h)\}$ for real h. For $i = 0$ this is clear, since $\pm\sqrt h \in \mathsf{J}(h)$. For $i \ge 1$ and $h \le 2$ it follows from (2.19), in which the points $t\sqrt h$ lie in $\mathsf{J}(h)$ and the weight $(1-t^2)^{i-1}/(4^i(i-1)!)$ has integral at most one. For $i \ge 1$ and $h > 2$ it follows from the second statement of Lemma 2.5, in which the coefficients $h^{-(2i-l)/2}$ are at most one. Now fix a real $\bar h$ and let $G(P) = \partial_{\bar h}^l[\frac12(F(P, \sqrt{\bar h}) + F(P, -\sqrt{\bar h}))]$, the derivative being taken in the variable $\bar h$ of the even evaluation in Q. Then G is entire, $\partial_h^i \partial_{\bar h}^l F^\natural(h, \bar h)$ is the i-th derivative at h of $\frac12(G(\sqrt h) + G(-\sqrt h))$, and $G^{(l')}(P)$ is the same expression as $G(P)$ with F replaced by $\partial_P^{l'} F$. Applying the claim first in the variable P and then, for each fixed $P \in \mathsf{J}(h)$, in the variable Q gives the lemma. □

Lemma 6.9 (Jets and weight tests)**.** *Let $X = (P, Q)$ be a real point, $k \ge 0$, and let $\phi(P, Q) = \Theta(P^2, Q^2)$ be a weight test. With the distributions $\Upsilon_X^{(k)}$ of (4.35),*

$$\langle \Upsilon_X^{(k)}, \phi\rangle = \frac{d^k}{d\zeta^k}\Big|_{\zeta=0} \Theta(P^2 + \tfrac{\zeta}{2}, Q^2 + \tfrac{\zeta}{2}).$$

Proof. By (4.35) and the chain rule it suffices to show that $\langle A_h^{(i)} \otimes A_{\bar h}^{(l)}, \phi\rangle = \partial_h^i \partial_{\bar h}^l \Theta(h, \bar h)$ for $h, \bar h \ge 0$, and for this it suffices to show that $\langle A_h^{(i)}, \phi_1\rangle = \theta^{(i)}(h)$ for $h \ge 0$, when θ is smooth on $\mathbb{R}$ and $\phi_1(P) = \theta(P^2)$. For $h > 0$ this is the definition of $A_h^{(i)}$, because $\langle A_h, \phi_1\rangle = \theta(h)$. For $h = 0$, Taylor's formula for θ at 0 gives $\phi_1(P) = \sum_{l \le i} \theta^{(l)}(0) P^{2l}/l! + O(P^{2i+2})$, hence $\phi_1^{(2i)}(0) = \frac{(2i)!}{i!}\theta^{(i)}(0)$, and $A_0^{(i)} = \frac{i!}{(2i)!}\delta_0^{(2i)}$. □

The Taylor array. Let Z be an admissible partition function of gap g at $c = 1 + 12a$, let $n \ge 1$ be an integer and $M > 0$, and suppose that $a < a_M$, the threshold of Corollary 5.24. For every state with $\Delta_j \le MH_a$ let $x_j = (P_j, Q_j) \in \Gamma$ be the grid point $[m/d, n'/d]$ which that corollary associates with the physical point $(\sqrt{h_j}, \sqrt{\bar h_j})$, and put

$$\zeta_j = \Delta_j - |x_j|^2; \qquad \text{thus} \qquad P_j^2 - Q_j^2 = s_j, \qquad |\zeta_j| \le C_M \, b_a H_a^{3/2}. \tag{6.24}$$

The *Taylor array of order n* is the distribution

$$\mathsf{T}_a = \nu_{\mathrm{vac}} + \sum_{j:\, \Delta_j \le MH_a} \sum_{l=0}^{n} \frac{\zeta_j^l}{l!} \Upsilon_{x_j}^{(l)}. \tag{6.25}$$

It differs from the distribution of Lemma 4.21 in that the expansion is about the physical weights: the central shift is not included in ζ_j, and the vacuum is the physical measure ν_{vac}. The terms with $l = 0$ are the state measures of the states $(|x_j|^2, s_j)$, so the coefficients of order zero of T_a count the states associated with each grid point; no comparison measure is subtracted.

Lemma 6.10 (The Taylor array)**.** *Let $n \ge 1$, $q = 2n$ and $D_0 > 0$, and let*

$$M \ge \max\Big\{8D_0 + 8(2q+4)^2 d^2 + 1, \ \frac{8(n+3)}{\pi}\Big\}. \tag{6.26}$$

There are $a_0 > 0$ and $C < \infty$, depending only on g, n, D_0 and M, such that for every admissible partition function of gap g at $c = 1 + 12a$ with $0 < a < a_0$ the Taylor array (6.25) has the following properties, where $b = b_a$ and $H = H_a$.

(i) T_a *is an even distribution with finite support contained in* Γ*, of local order at most* q*, and* $\mathcal{C}\mathsf{T}_a = \mathsf{T}_a$*.*

(ii) Its coefficients satisfy $|c_\alpha(z)| \le C_0(1+|z|)^2$ *for* $z \in \Gamma$ *and* $|\alpha| \le q$*, with a constant* C_0 *depending only on* g *and* n*.*

(iii) For every $f \in \mathcal{F}_q$ *and every* $f \in \mathcal{P}_q(\sqrt{D_0 H} + (q+3)d)$*,*

$$\big|\langle (\mathbb{S} - I)\mathsf{T}_a, f\rangle\big| \le C\,(a + b^{n+1})\,(1+H)^{3n+4}. \tag{6.27}$$

(iv) For every weight test ϕ *there are* C_ϕ *and* $a_\phi > 0$*, depending in addition on* ϕ*, with* $|\langle \mu_a, \phi\rangle - \langle \mathsf{T}_a, \phi\rangle| \le C_\phi\, b^{n+1}(1+H)^{3(n+1)/2}$ *for* $a < a_\phi$*.*

Proof. Throughout, $a < a_M$ is so small that $a \le \min\{\frac{1}{16}, \frac{1}{4}g^2\}$, $b < e^{-1}$ (so that $H \ge 1$ and $b^2 H \le 1$), $MH \ge 1$, $C_M b H^{3/2} \le 1$ and $2\pi a(MH+2) \le 1$; this is possible because $a \le b^2$ and $b \to 0$ uniformly. By (2.23) the number of states with $\Delta_j \le E$ is then at most eA_gE for $1 \le E \le MH+2$.

(i) The sum is finite. Each $\Upsilon^{(l)}_{x_j}$ with $l \le n$ is even, supported in the set of the reflections of x_j, which is contained in Γ, and of local order at most $2n$; the measure ν_{vac} is even and supported in $V \subset \Lambda_1 \subset \Gamma$. For the phase, let $X = (P,Q)$ be real with $P^2 - Q^2 \in \mathbb{Z}$ and let f be a polynomial. The even evaluation of $e^{2\pi i(P^2-Q^2)}f$ is $e^{2\pi i(h-\bar h)}f^\natural(h,\bar h)$, and along $(h,\bar h) = (P^2 + \frac{\zeta}{2}, Q^2 + \frac{\zeta}{2})$ the first factor is identically one; by (4.36), $\langle \mathcal{C}\Upsilon^{(l)}_X, f\rangle = \langle \Upsilon^{(l)}_X, f\rangle$. Both sides are distributions of finite order with the same finite support, and polynomials realize arbitrary jets at finitely many points; hence $\mathcal{C}\Upsilon^{(l)}_X = \Upsilon^{(l)}_X$. This applies to the points x_j and to the terms of ν_{vac}, all of which have integral spin.

(ii) For $h > 0$ and $k \ge 1$ the distribution $A^{(k)}_h$ is a combination, with coefficients depending only on k and l, of $h^{-(2k-l)/2}(\delta^{(l)}_{\sqrt{h}} + (-1)^l \delta^{(l)}_{-\sqrt{h}})$, $1 \le l \le k$, up to signs, by the second statement of Lemma 2.5; and $A^{(k)}_0 = \frac{k!}{(2k)!}\delta^{(2k)}_0$. A nonzero momentum coordinate of a point of Γ has square at least $(2d)^{-2}$. Hence the coefficients of $\Upsilon^{(l)}_{x_j}$, $l \le n$, are bounded by a constant depending only on d and n, and so are those of the contribution of the state j to (6.25), because $|\zeta_j| \le 1$. A state contributes at a point $z \in \Gamma$ only if z is a reflection of x_j; then $|z|^2 \le \Delta_j + 1 \le MH + 1$ and $\Delta_j \le |z|^2 + 1 \le MH + 2$, and there are at most $eA_g(|z|^2+1)$ such states. Adding the coefficients of ν_{vac} gives (ii).

(iii) *The tests.* Let g range over the finitely many functions $\psi_{0,\beta}$, $\mathbb{S}\psi_{0,\beta}$ ($\beta \in \mathcal{A}_q$) and f, $\mathbb{S}f$ ($f \in \mathcal{F}_q$). Each is a finite sum of functions $\mathfrak{m}[p,z]$, by Lemmas 4.18(ii) and 4.17(ii), and Lemma 4.17(iii) gives a constant C_1 with

$$|\partial^\alpha \mathsf{g}(X)| \le C_1\, e^{-\pi|\operatorname{Re} X|^2} \qquad \text{for } |\alpha| \le 2n+2,\ |\operatorname{Im} P|, |\operatorname{Im} Q| \le 1. \tag{6.28}$$

Let $f = \omega_\iota \mathbb{S}\psi_{y,\beta}$ be an element of $\mathcal{P}_q(\sqrt{D_0H} + (q+3)d)$. By (6.8), $\psi_{y,\beta} = \tau_{-y}\psi_{0,\beta}$, and by the rules $\mathbb{S}(\tau_{-w}f) = \chi_w \mathbb{S}f$, $\mathbb{S}(\chi_w f) = \tau_w \mathbb{S}f$ and $\mathbb{S}^2 f(X) = f(-X)$ of Section 4.1,

$$f = \sum_{k=0}^{q+1} (-1)^{q+1-k} \binom{q+1}{k} \chi_{y+k\mathsf{b}_\iota}\, \mathbb{S}\psi_{0,\beta},$$

$$(\mathbb{S}f)(X) = \sum_{k=0}^{q+1} (-1)^{q+1-k} \binom{q+1}{k} \psi_{0,\beta}(-X - y - k\mathsf{b}_\iota).$$

Thus, for every f in $\mathcal{F}_q$ or in $\mathcal{P}_q(\sqrt{D_0H} + (q+3)d)$, both f and $\mathbb{S}f$ are combinations, with coefficients of total modulus at most 2^{q+1}, of functions F of the following two types, where g is as above and $y', \xi \in \mathbb{R}^2$ satisfy

$$|y'|,\ |\xi| \le \varrho_H := \sqrt{D_0 H} + (2q+4)d : \tag{6.29}$$

(I) real translates and reflections, $F(X) = \mathsf{g}(X - y')$ or $F(X) = \mathsf{g}(-X - y')$;

(II) modulations, $F(X) = \chi_\xi(X)\,\mathsf{g}(X) = e^{2\pi i B(X,\xi)}\mathsf{g}(X)$.

Indeed the light-cone coordinates of $y + k\mathsf{b}_\iota$ are bounded by $\sqrt{D_0 H} + (2q+4)d$, and $|x|^2 = \frac{1}{2}(u_x^2 + v_x^2) \le \max\{u_x^2, v_x^2\}$. For type (I), (6.28) gives $|\partial^\alpha F(X)| \le C_1 e^{-\pi|\operatorname{Re} X \mp y'|^2}$ on the strip $|\operatorname{Im} P|, |\operatorname{Im} Q| \le 1$, uniformly in y'. For type (II), the derivatives of order k of $e^{2\pi i B(X,\xi)}$ are bounded by $(4\pi|\xi|)^k e^{4\pi|\operatorname{Im} X||\xi|}$, since $B(X,\xi) = 2(PP_\xi - QQ_\xi)$. On the thin strip $|\operatorname{Im} P|, |\operatorname{Im} Q| \le \sqrt{a/2}$ we have $|\operatorname{Im} X||\xi| \le \sqrt{a}\,\varrho_H \le C$, because $aH \le b^2 H \le 1$. Altogether, for both types and $|\alpha| \le 2n+2$,

$$|\partial^\alpha F(X)| \le C_2\,(1+H)^{n+1} \times \begin{cases} e^{-\pi|\operatorname{Re} X \mp y'|^2} & \text{(type I)},\\ e^{-\pi|\operatorname{Re} X|^2} & \text{(type II)}, \end{cases} \qquad |\operatorname{Im} P|, |\operatorname{Im} Q| \le \sqrt{a/2}. \tag{6.30}$$

The loss is polynomial in H and uniform in y' and ξ. All these functions belong to $\mathcal{G}_{2\pi}$, which is stable under real translations, reflections and multiplication by exponentials. By Lemma 2.4, $\mathcal{L}_a(f) = \mathcal{L}_a(\mathbb{S}f)$, and therefore

$$\langle(\mathbb{S} - I)\mathsf{T}_a, f\rangle = \big[\mathcal{L}_a(f) - \langle \mathsf{T}_a, f\rangle\big] - \big[\mathcal{L}_a(\mathbb{S}f) - \langle \mathsf{T}_a, \mathbb{S}f\rangle\big]. \tag{6.31}$$

It suffices to bound $\mathcal{L}_a(F) - \langle \mathsf{T}_a, F\rangle$ for F of type (I) or (II).

The decomposition. For a state with $\Delta_j \le MH$ let $\Phi_j(\zeta) = F^\natural(P_j^2 + \frac{\zeta}{2}, Q_j^2 + \frac{\zeta}{2})$. Since $P_j^2 = \frac{1}{2}(|x_j|^2 + s_j)$ and $h_j = \frac{1}{2}(\Delta_j + s_j)$, and likewise for Q_j^2 and $\bar h_j$, we have $\Phi_j(\zeta_j) = F^\natural(h_j, \bar h_j)$, and by (4.36) the pairing of the terms of the state j in (6.25) with F is the Taylor polynomial of degree n of Φ_j at 0, evaluated at ζ_j. Hence

$$\begin{aligned} \mathcal{L}_a(F) - \langle \mathsf{T}_a, F\rangle = {} & \sum_{k,l\in\{0,1\}} (-1)^{k+l}\Big[F^\natural(k - \tfrac{a}{2}, l - \tfrac{a}{2}) - F^\natural(k,l)\Big] \\ & + \sum_{j:\,\Delta_j \le MH} \Big[F^\natural(h_j - \tfrac{a}{2}, \bar h_j - \tfrac{a}{2}) - F^\natural(h_j, \bar h_j)\Big] \\ & + \sum_{j:\,\Delta_j \le MH} \operatorname{Rem}_n[\Phi_j](\zeta_j) + \sum_{j:\,\Delta_j > MH} F^\natural(h_j - \tfrac{a}{2}, \bar h_j - \tfrac{a}{2}), \end{aligned} \tag{6.32}$$

where Rem_n is the Taylor remainder of order n at 0.

Taylor remainders. For ζ' between 0 and ζ_j the point $(P_j^2 + \frac{\zeta'}{2}, Q_j^2 + \frac{\zeta'}{2})$ lies on the segment from (P_j^2, Q_j^2) to $(h_j, \bar h_j)$, so both its coordinates are nonnegative, and the sets J of Lemma 6.8 at these coordinates are real. The derivative $\Phi_j^{(n+1)}(\zeta')$ is an average of the derivatives $\partial_h^i \partial_{\bar h}^{n+1-i} F^\natural$ at this point. By Lemma 6.8 with $k = n+1$ and (6.30), $|\Phi_j^{(n+1)}(\zeta')| \le C(1+H)^{n+1}$. With (6.24) and the bound $eA_g MH$ for the number of states,

$$\sum_{j:\,\Delta_j \le MH} |\operatorname{Rem}_n[\Phi_j](\zeta_j)| \le C\,H\,(1+H)^{n+1}(C_M b H^{3/2})^{n+1} \le C\,b^{n+1}(1+H)^{3n+4}.$$

Central shifts. For real $h, \bar h \ge 0$,

$$F^\natural(h - \tfrac{a}{2}, \bar h - \tfrac{a}{2}) - F^\natural(h, \bar h) = -\frac{a}{2}\int_0^1 (\partial_h + \partial_{\bar h}) F^\natural(h - \tfrac{\tau a}{2}, \bar h - \tfrac{\tau a}{2})\,\mathrm{d}\tau.$$

The arguments are real and at least $-\frac{a}{2}$, so the points of the sets J at these arguments have imaginary parts of modulus at most $\sqrt{a/2}$. By Lemma 6.8 with $k = 1$ and (6.30) the difference is at most $Ca(1+H)^{n+1}$ in modulus. Summing over the states with $\Delta_j \le MH$ and the four vacuum weights, the first two sums of (6.32) are bounded by $Ca(1+H)^{n+2}$.

Tails. Let $\Delta_j > MH$ and let $X = X^{\sigma}_{j,a}$ be a shifted point of the state. By (2.16), $|\operatorname{Im} P|, |\operatorname{Im} Q| \le \sqrt{a/2}$ and $|\operatorname{Re} X|^2 \ge \Delta_j - a \ge MH - 1$. By (6.29) and (6.26), $4\varrho_H^2 \le 8D_0H + 8(2q+4)^2d^2 \le (M-1)H \le MH - 1$, so $|y'| \le \frac{1}{2}|\operatorname{Re} X|$. Hence $|\operatorname{Re} X \mp y'| \ge \frac{1}{2}|\operatorname{Re} X|$, and (6.30) gives, for both types, $|F(X)| \le C(1+H)^{n+1} e^{-\frac{\pi}{4}(\Delta_j - a)}$. By (5.2) the same bound holds for $F^{\natural}(h_j - \frac{a}{2}, \bar{h}_j - \frac{a}{2})$. By (2.24) with $\beta = \frac{\pi}{4}$ and $E_0 = MH$, where $\beta - 2\pi a \ge \frac{\pi}{8}$,

$$\sum_{j:\,\Delta_j > MH} e^{-\frac{\pi}{4}\Delta_j} \le C\,H\,e^{-\frac{\pi}{8}MH} = C\,H\,b^{\pi M/8} \le C\,H\,b^{n+3},$$

by (6.26). The last sum of (6.32) is therefore at most Cb^{n+2}.

Adding the three bounds and using (6.31) gives (6.27).

(iv) Let $\phi(P,Q) = \Theta(P^2, Q^2)$ and let E_ϕ be an upper bound for $h + \bar{h}$ on the support of Θ. For a state with $\Delta_j \le MH$ put $\Phi_j(\zeta) = \Theta(P_j^2 + \frac{\zeta}{2}, Q_j^2 + \frac{\zeta}{2})$. By Lemma 6.9 and (6.25),

$$\langle \mu_a, \phi\rangle - \langle \mathsf{T}_a, \phi\rangle = \sum_{j:\,\Delta_j \le MH} \operatorname{Rem}_n[\Phi_j](\zeta_j) + \sum_{j:\,\Delta_j > MH} \Theta(h_j, \bar{h}_j).$$

Let a be so small that $MH > E_\phi + 1$. Then the second sum vanishes. Since $|\zeta_j| \le 1$, a term of the first sum can be different from zero only if $\Delta_j \le E_\phi + 1$, which holds for at most $eA_g(E_\phi + 1)$ states, and each term is at most $|\zeta_j|^{n+1}$ times a bound for the derivatives of order $n+1$ of Θ. The claim follows from (6.24). □

Lemma 6.11 (Normal form)**.** *Let $0 < g \le \frac{1}{2}$ and let P_* be the operator of Lemma 6.7. For every integer $n \ge 1$ and every finite set $\mathcal{K}$ of weight tests there are constants C_n, K_n and $a_n > 0$, depending only on g, n and $\mathcal{K}$, such that every admissible partition function of gap g at $c = 1 + 12a$ with $0 < a < a_n$ satisfies*

$$\big|\langle \mu_a - P_*\mu_a, \phi\rangle\big| \le C_n\,(a + b_a^{n+1})\,(1 + H_a)^{K_n} \qquad (\phi \in \mathcal{K}). \tag{6.33}$$

One may take $K_n = 5n + 5$. Along an admissible sequence converging to $\sigma(R_1, R_2)$ with half-integers R_1, R_2, the measure $P_*\mu_a$ is, for small a, the positive combination (6.22) of at most four circle measures with total weight one.

Proof. The choices are made in the following order. First, g determines d, the grid Γ, the prime $\mathfrak{p}$, the cutoffs and P'_*. Secondly, n determines $q = 2n$ and, by Lemma 6.5, the family $\mathcal{F}_q$ and the projection Π_q. Thirdly, we apply Proposition 6.6 with this q, with $A_0 = 2$ and $A_1 = n + 2$; this fixes $D_0 = 4(n+2)/\pi$. Fourthly, we fix M satisfying (6.26) for this D_0. Only then do we invoke Corollary 5.24 with this M, which provides C_M and a_M, and Lemma 6.10. No earlier choice depends on a later one: $\mathcal{F}_q$, Π_q and D_0 do not depend on M, and by Lemma 6.10(ii) neither do the exponent $A_0 = 2$ and the constant C_0.

Let a be small and let T_a be the Taylor array (6.25) for these n and M. By Lemma 6.10(i)–(iii) the hypotheses of Proposition 6.6 hold for $\mathsf{T} = \mathsf{T}_a$ with $b = b_a$ and $\varepsilon = C(a + b_a^{n+1})(1 + H_a)^{3n+4}$. Hence there is $\mathsf{V}_a \in \mathsf{V}_q$ such that, for every compact K,

$$\max_{z \in \Gamma \cap K} \max_{|\alpha| \le q} |c_\alpha^{\mathsf{T}_a}(z) - c_\alpha^{\mathsf{V}_a}(z)| \le C_K\Big((a + b_a^{n+1})(1 + H_a)^{5n+5} + b_a^{n+2}\Big). \tag{6.34}$$

Let $\phi \in \mathcal{K}$ and $\chi = \phi - P'_*\phi$, a weight test by Lemma 6.7(i). Then

$$\langle \mu_a - P_*\mu_a, \phi\rangle = \langle \mu_a, \chi\rangle = \big[\langle \mu_a, \chi\rangle - \langle \mathsf{T}_a, \chi\rangle\big] + \langle \mathsf{T}_a - \mathsf{V}_a, \chi\rangle + \langle \mathsf{V}_a, \chi\rangle.$$

The last term vanishes by Lemma 6.7(iv). The first is at most $C_\chi b_a^{n+1}(1 + H_a)^{3(n+1)/2}$ by Lemma 6.10(iv). The second is the finite sum of the terms $(-1)^{|\alpha|}(c_\alpha^{\mathsf{T}_a} - c_\alpha^{\mathsf{V}_a})(z)\,\partial^\alpha\chi(z)$ over $z \in \Gamma \cap \operatorname{supp}\chi$ and $|\alpha| \le q$, and is bounded by (6.34) with $K = \operatorname{supp}\chi$. This proves (6.33). The last assertion is Lemma 6.7(v). □

The bound (6.33) holds for each fixed n and each fixed finite set of tests; it is not uniform in a window growing with H_a. The operator P_* does not depend on n or on the tests.

6.5 Pairing, pinning and the comparison measure

The combination (6.22) is not yet of the form $\frac{1}{2}\rho_R^{\mathrm{F}} + \frac{1}{2}\rho_t^{\mathrm{F}}$: at the self-dual radius it consists of circles of weight $\frac{1}{4}$, and none of its radii is an exact half-integer. Both defects are removed by comparing finitely many moments of clusters. The moments are converted into positions by the following elementary lemma, which requires no separation between the points.

Lemma 6.12 (Moment matching)**.** *Let $k \geq 1$ be an integer.*

(i) There is a constant C_k with the following property. Let $y_1 \leq \cdots \leq y_k$ and $y'_1 \leq \cdots \leq y'_k$ be real numbers in $[-1,1]$, let $0 \leq e \leq 1$, and suppose that

$$\Big|\sum_{i=1}^{k} y_i^l - \sum_{i=1}^{k} (y'_i)^l\Big| \leq e \qquad \text{for } 1 \leq l \leq 2k-1.$$

Then $|y_i - y'_i| \leq C_k\, e^{1/(2k-1)}$ for all i.

(ii) Let $y_1 \leq \cdots \leq y_k$ and $y'_1 \leq \cdots \leq y'_k$ be real, let $e \geq 0$, and suppose that $|y_i - y'_{\pi(i)}| \leq e$ for all i and some permutation π. Then $|y_i - y'_i| \leq e$ for all i.

For $k \leq 4$ the exponent in (i) *is at least $\frac{1}{7}$.*

Proof. (i) Suppose that $\varrho = |y_i - y'_i| > 0$ for some i; exchanging the two families we may assume $y_i < y'_i$. For $x \in [y_i, y'_i)$ at least i of the numbers y_l and at most $i-1$ of the numbers y'_l are at most x. At most $2k-2$ of the $2k$ numbers y_l, y'_l lie in the open interval (y_i, y'_i), and they divide it into at most $2k-1$ intervals. Hence there is an interval $(\alpha,\beta) \subset (y_i, y'_i)$ of length at least $\varrho/(2k-1)$ that contains none of the $2k$ numbers. Fix a smooth function s_0 on $\mathbb{R}$ with $\mathsf{s}_0 = 1$ on $(-\infty, 0]$ and $\mathsf{s}_0 = 0$ on $[1,\infty)$, and let $\mathsf{s}(x) = \mathsf{s}_0((x-\alpha)/(\beta-\alpha))$, so that $\|\mathsf{s}^{(l)}\|_\infty \leq C(\beta-\alpha)^{-l}$ for $l \leq 2k-1$. Let $\xi_1 < \cdots < \xi_m$, $m \leq 2k$, be the distinct values among the $2k$ numbers, and let q be the polynomial of degree at most $m-1$ that agrees with s at these points. In Newton form $\mathsf{q}(x) = \sum_{l=0}^{m-1} \mathsf{s}[\xi_1,\dots,\xi_{l+1}] \prod_{l' \leq l}(x - \xi_{l'})$, where the divided difference $\mathsf{s}[\xi_1,\dots,\xi_{l+1}]$ equals $\mathsf{s}^{(l)}/l!$ at some point and is therefore at most $C(\beta-\alpha)^{-l}$ in modulus. All $\xi_{l'}$ lie in $[-1,1]$ and $\beta - \alpha \leq 2$, so the coefficients of $\mathsf{q}(x) = \sum_{l<2k} \mathsf{q}_l x^l$ satisfy $|\mathsf{q}_l| \leq C'_k(\beta-\alpha)^{-(2k-1)} \leq C'_k(2k-1)^{2k-1}\varrho^{-(2k-1)}$. Now $\sum_l \mathsf{q}(y_l) - \sum_l \mathsf{q}(y'_l)$ is the number of y_l that are at most α minus the number of y'_l that are at most α, which is at least one. On the other hand it equals $\sum_{l=1}^{2k-1} \mathsf{q}_l(\sum_i y_i^l - \sum_i (y'_i)^l)$, the terms with $l=0$ cancelling, and is at most $(2k-1)C'_k(2k-1)^{2k-1}\varrho^{-(2k-1)}e$. Hence $\varrho^{2k-1} \leq C_k^{2k-1} e$.

(ii) Suppose that $y_i > y'_i + e$ for some i. For $l \geq i$ we have $y'_{\pi(l)} \geq y_l - e \geq y_i - e > y'_i$, hence $\pi(l) > i$. Thus π maps the $k-i+1$ indices $l \geq i$ injectively into the $k-i$ indices greater than i, which is impossible. The case $y'_i > y_i + e$ is symmetric, with π^{-1} in place of π. □

In the following theorem $m_r = 1 + \mathbf{1}_{\{r \in \{1,2\}\}}$ for a half-integer $r \geq 1$, as in (3.3), and the windows are those of (6.2). Recall from Lemma 6.1 that the states of the limit $\sigma(R_1,R_2)$ are those of the family $\mathfrak{S}(R_1,R_2)$. A family of states is determined by its measure, so $\mathfrak{S}(R_1,R_2)$ and $\mathfrak{S}(R_2,R_1)$ consist of the same states with the same multiplicities.

Theorem 6.13 (One comparison measure)**.** *Let $0 < g \leq \frac{1}{2}$ and let an admissible sequence of gap g converge to $\sigma(R_1,R_2)$, where $R_1 \leq R_2$ are half-integers. For all sufficiently small a in the sequence there are an index $\iota_a \in \{1,2\}$ and a radius $t_a \geq 1$, defined in* (6.38) *and* (6.43) *below from the dimensions of the states of ρ_a in the private clusters, with the following properties. Write $R = R_{\iota_a}$ and $T = R_{3-\iota_a}$; these depend on a only through ι_a.*

(i) $t_a - T \to 0$ as $a \to 0$ in the sequence.

(ii) For every weight test ϕ and every integer $n \geq 1$ there are constants C, K such that, for all sufficiently small a,

$$\big|\langle \rho_a - \sigma(R, t_a), \phi\rangle\big| \leq C\,\big(a + b_a^{n+1}\big)^{1/12}(1 + H_a)^K. \tag{6.35}$$

(iii) Let $(E_0, s_0) \in \mathbb{R} \times \mathbb{Z}$, let $\mathcal{W}_\eta(E_0, s_0)$ be isolated for $\mathfrak{S}(R_1, R_2)$, and let N_0 be the number of states of $\sigma(R_1, R_2)$ with dimension E_0 and spin s_0. For all sufficiently small a, each of the families $(\Delta_j, s_j)_j$ and $\mathfrak{S}(R, t_a)$ has exactly N_0 states with spin s_0 and dimension in $(E_0 - 2\eta, E_0 + 2\eta)$. If $y_1 \leq \cdots \leq y_{N_0}$ and $y_1' \leq \cdots \leq y_{N_0}'$ are their dimensions, then for every $n \geq 1$ there are C, K with

$$\max_{1 \leq i \leq N_0} |y_i - y_i'| \leq C\,\big(a + b_a^{n+1}\big)^{1/84}(1 + H_a)^K. \tag{6.36}$$

(iv) Let $\tilde{t}_a$ be radii with $|\tilde{t}_a - t_a| \leq e_a$, where $e_a \to 0$. Then (iii) *holds for $\mathfrak{S}(R, \tilde{t}_a)$ in place of $\mathfrak{S}(R, t_a)$, with the right side of* (6.36) *increased by Ce_a.*

The index ι_a and the radius t_a do not depend on n, on ϕ, or on the window; only the constants do.

Thus a single measure $\sigma(R, t_a)$, with an exact half-integer R, matches the states of every fixed cluster up to $(a + b_a^{n+1})^{1/84}$ and logarithms, for every n; once $a \leq \delta_a^m$ is known for every m (Theorem 5.25), this is every order in b_a. No statement is made that is uniform in a window growing with H_a.

Proof. Fix $n \geq 1$. We use Lemma 6.11 for a finite set $\mathcal{K}$ of weight tests consisting of the tests $\phi^{(1)}$, $\phi^{(3)}$ and $\phi_1, \dots, \phi_4$ introduced below, the test ϕ of (ii), and the tests (6.44) of the window considered in (iii). We write

$$\varepsilon_a = C_n(a + b_a^{n+1})(1 + H_a)^{K_n}$$

for the right side of (6.33); it tends to zero. All statements hold for sufficiently small a in the sequence. For a weight test ϕ the function $\mathsf{f}_\phi(t) = \langle \rho_t^{\mathrm{F}}, \phi\rangle$ is smooth, so

$$|\mathsf{f}_\phi(t) - \mathsf{f}_\phi(t')| \leq C_\phi\,|\log t - \log t'| \qquad (t, t' \in [\tfrac{1}{2d}, 2d]). \tag{6.37}$$

Step 1: the reconstructed radii. By Lemma 6.7(v), $P_*\mu_a = \sum_r \mathsf{c}_r \sum_{i \leq n_r} \rho^{\mathrm{F}}_{t_{r,i}}$ with $t_{r,i} = \mathsf{E}^{-1}(\Delta_{r,i}) \to r$, where only $r \in \{R_1, R_2\}$ occur. If $R_1 = 1$, let $\ell_1 \leq \cdots \leq \ell_{n_1}$ be the numbers $\log t_{1,i}$ in increasing order; here $n_1 = 2$ if $R_2 > 1$ and $n_1 = 4$ if $R_2 = 1$. We define two radii τ_1, τ_2 by

$$(\tau_1, \tau_2) = \begin{cases} (t_{R_1,1},\, t_{R_2,1}), & 1 < R_1 < R_2, \\ (t_{R_1,1},\, t_{R_1,2}), & 1 < R_1 = R_2, \\ (e^{(\ell_2 - \ell_1)/2},\, t_{R_2,1}), & 1 = R_1 < R_2, \\ (e^{(\ell_3 - \ell_2)/2},\, e^{(\ell_4 - \ell_1)/2}), & 1 = R_1 = R_2. \end{cases} \tag{6.38}$$

Then $\tau_i \to R_i$ for $i = 1, 2$, and $\tau_i \geq 1$ if $R_i = 1$. Put $\mu_a^{\mathrm{p}} = \frac{1}{2}\rho^{\mathrm{F}}_{\tau_1} + \frac{1}{2}\rho^{\mathrm{F}}_{\tau_2}$.

Step 2: pairing at the self-dual radius. We claim that for every weight test ϕ

$$\big|\langle P_*\mu_a - \mu_a^{\mathrm{p}}, \phi\rangle\big| \leq C_\phi\,\varepsilon_a^{1/3}. \tag{6.39}$$

If $R_1 > 1$ the two measures coincide, since $\mathsf{c}_r = \frac{1}{2}$ for $r > 1$. Let $R_1 = 1$. For $k \in \{1, 3\}$ let $\phi^{(k)}$ be the weight test (6.4) for $(E_0, s_0) = (\mathsf{E}(1), \mathfrak{p})$, $\eta = \eta_*$ and $\varphi(E) = (\log \mathsf{E}^{-1}(E))^k$; in the notation of Section 6.3,

$$\phi^{(k)}(X) = \vartheta^{\mathrm{sp}}\big(P^2 - Q^2 - \mathfrak{p}\big)\,\vartheta_1\big(|X|^2\big)\,\big(\log \mathsf{E}^{-1}(|X|^2)\big)^k.$$

It is smooth and compactly supported. By Lemma 6.7(v) and (6.5), $\langle \mu_a, \phi^{(k)} \rangle = \sum_{i=1}^{n_1} \ell_i^k$. By (6.19) and (6.5),

$$\langle \rho_t^{\mathrm{F}}, \phi^{(k)} \rangle = 2\,\vartheta_1(\mathsf{E}(t))\big(\log \mathsf{E}^{-1}(\mathsf{E}(t))\big)^k + 2\,\vartheta_1(\mathsf{E}(1/t))\big(\log \mathsf{E}^{-1}(\mathsf{E}(1/t))\big)^k.$$

For $t = t_{1,i}$, which lies in the interval I_1 of Lemma 6.7(iii) when a is small, this is $2(\log t)^k + 2(-\log t)^k = 0$, because k is odd. For $t = t_{R_2,i}$ with $R_2 > 1$, which lies in I_{R_2}, both cutoffs vanish, as shown in the proof of Lemma 6.7(iii). Hence $\langle P_* \mu_a, \phi^{(k)} \rangle = 0$, and (6.33) gives

$$\Big|\sum_{i=1}^{n_1} \ell_i\Big| \le \varepsilon_a, \qquad \Big|\sum_{i=1}^{n_1} \ell_i^3\Big| \le \varepsilon_a. \tag{6.40}$$

Let $n_1 = 2$ and $x = \frac{1}{2}(\ell_2 - \ell_1) \ge 0$. Then $|\ell_1 + x| = |\ell_2 - x| = \frac{1}{2}|\ell_1 + \ell_2| \le \varepsilon_a$. Since $\rho^{\mathrm{F}}_{e^{-x}} = \rho^{\mathrm{F}}_{e^{x}}$, the part of $P_*\mu_a$ belonging to $r = 1$ is $\frac{1}{4}(\rho^{\mathrm{F}}_{e^{\ell_1}} + \rho^{\mathrm{F}}_{e^{\ell_2}})$ and the corresponding part of μ_a^{p} is $\frac{1}{2}\rho^{\mathrm{F}}_{e^x} = \frac{1}{4}(\rho^{\mathrm{F}}_{e^{-x}} + \rho^{\mathrm{F}}_{e^x})$; the other parts coincide, and (6.37) gives (6.39). Let $n_1 = 4$ and put

$$\alpha = \tfrac{1}{2}(\ell_1 + \ell_4), \quad \lambda = \tfrac{1}{2}(\ell_4 - \ell_1), \qquad \alpha' = \tfrac{1}{2}(\ell_2 + \ell_3), \quad \lambda' = \tfrac{1}{2}(\ell_3 - \ell_2).$$

Then $\lambda \ge \lambda' \ge 0$, and $\ell_1 \le \ell_2$, $\ell_3 \le \ell_4$ give $\lambda - \lambda' \ge |\alpha - \alpha'|$. Let $e = \alpha + \alpha'$; by (6.40), $|e| = \frac{1}{2}|\sum_i \ell_i| \le \varepsilon_a$. Since $(\alpha - \lambda)^3 + (\alpha + \lambda)^3 = 2\alpha^3 + 6\alpha\lambda^2$,

$$\sum_i \ell_i^3 = 2\alpha^3 + 6\alpha\lambda^2 + 2\alpha'^3 + 6\alpha'\lambda'^2 = 6\alpha(\lambda^2 - \lambda'^2) + O(|e|),$$

where we inserted $\alpha' = e - \alpha$ and used that all numbers are bounded by one. Hence $|\alpha|(\lambda^2 - \lambda'^2) \le C\varepsilon_a$. If $|\alpha| \ge \varepsilon_a$, then $\lambda - \lambda' \ge |2\alpha - e| \ge |\alpha|$ and $\lambda + \lambda' \ge \lambda - \lambda'$, so $\lambda^2 - \lambda'^2 \ge \alpha^2$ and $|\alpha|^3 \le C\varepsilon_a$. In either case $|\alpha| \le C\varepsilon_a^{1/3}$, and $|\alpha'| \le |\alpha| + |e| \le C\varepsilon_a^{1/3}$. Thus the numbers $\ell_1, \ell_4, \ell_2, \ell_3$ are within $C\varepsilon_a^{1/3}$ of $-\lambda, \lambda, -\lambda', \lambda'$, and

$$\tfrac{1}{4}(\rho^{\mathrm{F}}_{e^{-\lambda}} + \rho^{\mathrm{F}}_{e^{\lambda}} + \rho^{\mathrm{F}}_{e^{-\lambda'}} + \rho^{\mathrm{F}}_{e^{\lambda'}}) = \tfrac{1}{2}\rho^{\mathrm{F}}_{e^{\lambda'}} + \tfrac{1}{2}\rho^{\mathrm{F}}_{e^{\lambda}} = \mu_a^{\mathrm{p}}.$$

With (6.37) this gives (6.39).

Step 3: the scalar at dimension two pins one radius. Put

$$x_i = \log(\tau_i/R_i) \to 0 \quad (i = 1, 2), \qquad k = m_{R_1} + m_{R_2} \in \{2, 3, 4\}.$$

Choose $\eta_{\mathrm{s}} \in (0, \frac{1}{4}]$ such that $\mathcal{W}_{\eta_{\mathrm{s}}}(2, 0)$ is isolated for $\mathfrak{S}(R_1, R_2)$. By Lemma 3.7(iv) the limit has $k - 1$ states of dimension two and spin zero, so by Lemma 6.2(ii) exactly $k - 1$ states of ρ_a have spin zero and dimension in $(2 - 2\eta_{\mathrm{s}}, 2 + 2\eta_{\mathrm{s}})$; we write their dimensions as $2 + z_i$, $1 \le i \le k - 1$, with $|z_i| < \eta_{\mathrm{s}}$. For $1 \le l \le 4$ let ϕ_l be the weight test (6.4) for $(E_0, s_0) = (2, 0)$, $\eta = \eta_{\mathrm{s}}$ and $\varphi(E) = (E - 2)^l$. Among the terms of ν_{vac}, the state $(0, 0)$ lies outside the support of ϕ_l, the states $(1, \pm 1)$ have nonzero spin, and the state $(2, 0)$ has $\varphi(2) = 0$. Hence

$$\langle \mu_a, \phi_l \rangle = \sum_{i=1}^{k-1} z_i^l = \mathsf{p}_l(0, z_1, \dots, z_{k-1}), \tag{6.41}$$

where p_l denotes the power sum of degree l of k numbers: the fixed vacuum state at dimension two contributes the node 0. By Lemma 6.1(i), μ_a^{p} is δ_0 plus the measure of the family of the states $(E_{m,n}(\tau_i), mn)$, $i = 1, 2$. A class with $mn = 0$ is the class of $(m, 0)$ or of $(0, n)$, with $E_{m,0}(\tau) = \frac{1}{2}m^2\tau^2$ and $E_{0,n}(\tau) = \frac{1}{2}n^2/\tau^2$. At $\tau = R_i$ these equal two exactly for $n = \pm 2R_i$ and, if $R_i \in \{1, 2\}$, for $m = \pm 2/R_i$; this gives m_{R_i} classes. Their states have the dimensions

$$E_{0,2R_i}(\tau_i) = 2e^{-2x_i}, \qquad E_{2/R_i,0}(\tau_i) = 2e^{2x_i} \quad (\text{if } R_i \in \{1, 2\}).$$

By Lemma 6.1(iii) and isolation, all other states of spin zero of the family have dimension outside $(2-2\eta_{\mathrm{s}}, 2+2\eta_{\mathrm{s}})$ when a is small. Therefore $\langle \mu_a^{\mathrm{p}}, \phi_l\rangle = \mathsf{p}_l(w_1,\dots,w_k)$, where $w_1,\dots,w_k$ are the numbers $2(e^{-2x_i}-1)$, $i=1,2$, together with $2(e^{2x_i}-1)$ for those i with $R_i \in \{1,2\}$. By (6.33), (6.39) and (6.41),

$$\bigl|\mathsf{p}_l(w_1,\dots,w_k) - \mathsf{p}_l(0,z_1,\dots,z_{k-1})\bigr| \le C\varepsilon_a^{1/3} \qquad (1\le l\le k).$$

By Newton's identities the product of k numbers is a polynomial with rational coefficients in their power sums of degrees $1,\dots,k$. All numbers involved are bounded by one, and the product of $0, z_1,\dots,z_{k-1}$ vanishes. Hence $|w_1\cdots w_k| \le C\varepsilon_a^{1/3}$. Since $|2(e^{\pm 2x}-1)| \ge |x|$ for $|x|\le \frac12$,

$$|x_1|^{m_{R_1}}|x_2|^{m_{R_2}} \le C\varepsilon_a^{1/3}, \qquad \text{hence} \qquad \min\{|x_1|,|x_2|\} \le \bigl(C\varepsilon_a^{1/3}\bigr)^{1/k} \le C\varepsilon_a^{1/12}. \tag{6.42}$$

We define

$$\iota_a = \begin{cases} 1, & |x_1| \le |x_2|, \\ 2, & |x_1| > |x_2|, \end{cases} \qquad t_a = \tau_{3-\iota_a}, \qquad R = R_{\iota_a}, \quad T = R_{3-\iota_a}, \tag{6.43}$$

and $\mu_a^{\mathrm{c}} = \frac12\rho_R^{\mathrm{F}} + \frac12\rho_{t_a}^{\mathrm{F}}$. Thus the radius with the smaller logarithmic displacement is replaced by its exact limit. By (6.37) and (6.42), $|\langle \mu_a^{\mathrm{p}} - \mu_a^{\mathrm{c}}, \phi\rangle| = \frac12|\mathsf{f}_\phi(\tau_{\iota_a}) - \mathsf{f}_\phi(R)| \le C_\phi\varepsilon_a^{1/12}$ for every weight test ϕ. Assertion (i) holds because $\tau_i \to R_i$, and $t_a \ge 1$ for small a: if $T>1$ this follows from (i), and if $T=1$ from Step 1.

Step 4: proof of (ii). By (2.12), $\rho_a - \sigma(R,t_a) = \mu_a - \mu_a^{\mathrm{c}}$. By (6.33), (6.39) and Step 3,

$$\bigl|\langle \mu_a - \mu_a^{\mathrm{c}}, \phi\rangle\bigr| \le \varepsilon_a + C_\phi\varepsilon_a^{1/3} + C_\phi\varepsilon_a^{1/12} \le C\,\varepsilon_a^{1/12},$$

which is (6.35) with $K = K_n/12$.

Step 5: proof of (iii). The statement about $(\Delta_j, s_j)_j$ is Lemma 6.2(ii). The statement about $\mathfrak{S}(R,t_a)$ follows from Lemma 6.2(i), applied to the pair (R,T) for each of the two possible values of ι_a, and from (i); note that $\sigma(R,T) = \sigma(R_1,R_2)$. If $N_0 = 0$ there is nothing more to prove. Otherwise $N_0 \le 4$ by Lemma 3.7(iii). For $1 \le l \le 2N_0 - 1$ let $\phi_l^{\mathcal{W}}$ be the weight test

$$\phi_l^{\mathcal{W}}(X) = \vartheta^{\mathrm{sp}}(P^2 - Q^2 - s_0)\,\vartheta\Bigl(\frac{|X|^2 - E_0}{\eta}\Bigr)\Bigl(\frac{|X|^2 - E_0}{\eta}\Bigr)^l, \tag{6.44}$$

which is of the form (6.4). By (6.5) and the first part of (iii), $\langle \rho_a, \phi_l^{\mathcal{W}}\rangle = \sum_i u_i^l$ and $\langle \sigma(R,t_a), \phi_l^{\mathcal{W}}\rangle = \sum_i (u_i')^l$ with $u_i = (y_i - E_0)/\eta$ and $u_i' = (y_i' - E_0)/\eta$, which lie in $(-1,1)$. By (ii) these power sums differ by at most $C\varepsilon_a^{1/12}$, and Lemma 6.12(i) with $k = N_0 \le 4$ gives $|y_i - y_i'| \le C(\varepsilon_a^{1/12})^{1/7} = C\varepsilon_a^{1/84}$. This is (6.36) with $K = K_n/84$.

Step 6: proof of (iv). By Lemma 6.2(i), for t close to T the states of $\mathfrak{S}(R,t)$ in the window are fixed states, which do not depend on t, and the moving states of finitely many classes that do not depend on t. By Lemma 6.1(iii) the dimension of a moving state changes by at most $C|\tilde t_a - t_a| \le Ce_a$ when t_a is replaced by $\tilde t_a$. Lemma 6.12(ii) shows that the ordered dimensions change by at most Ce_a. □

The exponent $\frac{1}{84} = \frac13\cdot\frac14\cdot\frac17$ is the product of the losses in the pairing of four self-dual nodes, in the pinning, where at most four scalar branches meet, and in the matching of at most four states. The bound four in the last step is Lemma 3.7(iii); it includes the clusters at the points of the vacuum support, where the limit has at most three long states.

Corollary 6.14 (Matching of the low states)**.** *In the situation of Theorem 6.13, let $E_* \geq 2$ and $n \geq 1$. There are constants C, K and, for all sufficiently small a in the sequence, an injective map $j \mapsto j'$ from the set of states of ρ_a with $\Delta_j \leq E_*$ into the family $\mathfrak{S}(R, t_a)$, whose image contains every state of $\mathfrak{S}(R, t_a)$ of dimension at most $E_* - 1$, such that j' has the spin of j and the dimensions satisfy*

$$|\Delta_j - \Delta'_{j'}| \leq C\,(a + b_a^{n+1})^{1/84}(1 + H_a)^K. \tag{6.45}$$

The same holds for $\mathfrak{S}(R, \tilde{t}_a)$ when $|\tilde{t}_a - t_a| \leq e_a \to 0$, with the right side increased by Ce_a.

Proof. Let $\mathfrak{C}$ be the finite set of pairs (E_0, s_0) that occur as states of $\sigma(R_1, R_2)$ with $E_0 \leq E_* + 1$, and choose $\eta \in (0, \frac{1}{4}]$ such that all windows $\mathcal{W}_\eta(E_0, s_0)$, $(E_0, s_0) \in \mathfrak{C}$, are isolated for $\mathfrak{S}(R_1, R_2)$. These windows are pairwise disjoint. Choose $E' \in (E_*, E_* + 1)$ that is not the dimension of a state of $\sigma(R_1, R_2)$, and let U be the set of X with $|X|^2 < E'$ that lie in the closure of none of the windows $\mathcal{W}_{\eta/2}(E_0, s_0)$, $(E_0, s_0) \in \mathfrak{C}$. This is a bounded open set, invariant under the reflections, its boundary does not meet $\operatorname{supp}\sigma(R_1, R_2)$, and no state of $\sigma(R_1, R_2)$ is supported in it. By Corollary 2.11, for small a no state of ρ_a is supported in U. The four physical points of a state have the same squared norm and the same spin, so a state with $\Delta_j < E'$ is either supported in U or supported in the closure of one of the windows $\mathcal{W}_{\eta/2}(E_0, s_0)$. Hence every state of ρ_a with $\Delta_j \leq E_*$ is supported in one of the windows $\mathcal{W}_\eta(E_0, s_0)$, $(E_0, s_0) \in \mathfrak{C}$. The same holds for every state of $\mathfrak{S}(R, t_a)$ of dimension at most $E_* - 1$: a fixed state is a state of $\sigma(R_1, R_2)$, and a moving state of a class $\{\pm(m, n)\}$ satisfies $|E_{m,n}(t_a) - E_{m,n}(T)| < \eta$ for small a, by Lemma 6.1(iii), while $(E_{m,n}(T), mn)$ is a state of $\sigma(R_1, R_2)$ of dimension at most E_*. In each window, Theorem 6.13(iii) provides the same number of states of the two families and the bound (6.36) for their ordered dimensions. We let j' be the state of $\mathfrak{S}(R, t_a)$ in the window of j with the same position in the order by dimension, ties being resolved by a fixed enumeration. The last assertion follows in the same way from Theorem 6.13(iv). □

6.6 Kernel data at a pair of half-integers and their collisions

We now fix the kernels that will be used at the moving radius. Since the radius is not a limit of the sequence, the hypotheses (H1)–(H4) and (S1)–(S5) of Section 5.6 are replaced by quantitative statements on a punctured neighbourhood of T.

Collision data. Let R, T be half-integers with $1 \leq R, T \leq R_{\max}$. We put

$$I = [\tfrac{1}{2},\, R_{\max} + 1], \qquad \ell = 1, \qquad p = \text{the smallest odd integer greater than } (R_{\max} + 1)^2 + 2, \tag{6.46}$$

so that (5.30) holds, T is an interior point of I also when $T = 1$, and $p \geq 7$. As in Section 5.6, $p_k = p + k$, $Y_\nu(t) = [t, \nu/t]$, $B_k(t) = Y_{p_k}(t)$ and $E_\nu(t) = \frac{1}{2}(t^2 + \nu^2/t^2) = E_{1,\nu}(t)$; the function E_p is strictly decreasing on I, with $E_p'(t) = t - p^2/t^3 \leq -4/t$ there.

- If $R \notin \{1, 2\}$, the *product data* are

$$U = 4R, \qquad V_j = 2(4R + 3 + j), \qquad A_j = [U, V_j] \quad (j = 0, 1, 2).$$

  Then $A_j = [4\cdot R,\, 2R(4R+3+j)/R] \in \Lambda_R$, $U \geq 6$, $V_j \geq 18$ and $V_j - U = 4R + 6 + 2j \geq 3$, so (5.33) holds, with $\Delta V = 2$. The polynomials are the ten functions Φ_t^*, $\Phi_t^{j,k}$ of (5.34).

- If $R \in \{1, 2\}$, the *sum data* are $w = 3$, so that $W(t) = Y_3(t) = [t, 3/t]$ and $\mathcal{N} = \{p_0, p_1, p_2, 3\}$. The polynomials are the four functions $\Phi_t^{(\nu)} = \Omega\, S_{Y_\nu(t)} S_{Y_\nu(t)^\sharp}$, $\nu \in \mathcal{N}$. Since $T^2 \neq 3$ and $T^2 < p$, the points $Y_\nu(T)$, $\nu \in \mathcal{N}$, are off both axes.

In both cases the polynomials vanish on the set $\Sigma(t)$ of (5.31) for every $t > 0$, as shown in Section 5.6. Their centres are written as in (5.35) and Lemma 5.1(ii),

$$\xi = \eta_1 Y + \eta_2 Y^\sharp + \eta_3 B_k(t) + \eta_4 B_k(t)^\sharp \qquad (Y \in \{C, A_0, A_1, A_2\}),$$
$$\xi = \zeta + \eta_1 Y_\nu(t) + \eta_2 Y_\nu(t)^\sharp \qquad (\zeta \in \{0, \pm C, \pm C^\sharp\}),$$

with signs $\eta_i \in \{-1, 0, 1\}$; the number $|\eta|$ of nonzero signs is the *length* of the centre. By Lemma 5.1, the coefficient of a centre of length $|\eta|$ is $\frac{1}{16}(-\frac{1}{2})^{|\eta|}$ for the product polynomials; for the sum polynomials it is positive for $(\zeta = 0, |\eta| \in \{0, 2\})$ and $(\zeta \neq 0, |\eta| = 1)$ and negative for $(\zeta = 0, |\eta| = 1)$ and $(\zeta \neq 0, |\eta| \in \{0, 2\})$.

All these centres, and the row preimages $x_{l,k,j}(t)$ of (5.36) in the case of sum data, have the form (5.37),

$$\xi(t) = \big[a_1 + b_1 t + b_2 \nu/t,\ a_2 + b_2 t + b_1 \nu/t\big], \qquad a_1, a_2, b_1, b_2 \in \mathbb{Z}, \quad \nu \geq 1, \tag{6.47}$$

with

$$\begin{aligned} &[a_1, a_2] = \eta_1 Y + \eta_2 Y^\sharp, \quad (b_1, b_2) = (\eta_3, \eta_4), \quad \nu = p_k \quad \text{(product centres)}, \\ &[a_1, a_2] = \zeta, \quad (b_1, b_2) = (\eta_1, \eta_2) \qquad\qquad\quad \text{(sum centres)}, \\ &[a_1, a_2] = [2,\ 2 - 2l], \quad (b_1, b_2) = (-k, -j), \quad \nu = 3 \quad \text{(row preimages)}. \end{aligned} \tag{6.48}$$

We call these finitely many maps $t \mapsto \xi(t)$ the *objects* of the data and denote their set by $\mathfrak{O}$. Let $\varrho_1 \geq 2$ be a bound for $|\xi(t)|$, $\xi \in \mathfrak{O}$, $t \in I$, and let

$$E_* = (\varrho_1 + 3)^2. \tag{6.49}$$

The *reference branches* are the constant maps with value in $(V \cup \Lambda_R \cup \Lambda_{1/R}) \cap \overline{D(0, \varrho_1 + 2)}$ and the maps $t \mapsto [mt, n/t]$ and $t \mapsto [m/t, nt]$ with integers $|m|, |n| \leq N_1$, where N_1 is so large that every point of $\Lambda_t \cup \Lambda_{1/t}$ of norm at most $\varrho_1 + 2$, for some $t \in I$, is the value at t of one of them. Their set $\mathfrak{B}$ is finite, and

$$\Sigma(t) \cap \overline{D(0, \varrho_1 + 2)} \subset \{\lambda(t) : \lambda \in \mathfrak{B}\} \subset \Sigma(t) \qquad (t \in I). \tag{6.50}$$

Lemma 6.15 (Zeros of order at most two)**.** *Let $T > 0$ and let $\mathfrak{F}$ be a finite set of real functions on $(0, \infty)$, each of one of the two forms*

$$f(t) = \alpha t + \beta + \gamma/t \qquad \text{or} \qquad f(t) = \alpha t^2 + \beta + \gamma/t^2, \qquad \alpha, \beta, \gamma \in \mathbb{R}.$$

There are $\varepsilon_0 \in (0, \frac{1}{2}T]$ and $c > 0$ such that for every $f \in \mathfrak{F}$ and every t with $0 < |t - T| \leq \varepsilon_0$:

(i) either f vanishes identically, or $|f(t)| \geq c\,|t - T|^2$;

(ii) either f is constant with an integer value, or $|\sin(\pi f(t))| \geq c\,|t - T|^2$.

Proof. (i) Let $f \not\equiv 0$. In the first case $tf(t) = \alpha t^2 + \beta t + \gamma$ is a nonzero polynomial of degree at most two in t; in the second, $t^2 f(t) = \alpha s^2 + \beta s + \gamma$ is a nonzero polynomial of degree at most two in $s = t^2$. Such a polynomial vanishes at T, respectively at $s = T^2$, to an order $k \in \{0, 1, 2\}$, so its quotient by $(t - T)^k$, respectively $(s - T^2)^k$, is continuous and nonzero at that point, and bounded below in modulus on a neighbourhood. Since $|s - T^2| = |t - T|(t + T) \geq T|t - T|$ and $|t - T| \leq 1$ may be assumed, this gives $|f(t)| \geq c|t - T|^k \geq c|t - T|^2$ near T.

(ii) If $f(T) \notin \mathbb{Z}$, then $\operatorname{dist}(f(t), \mathbb{Z})$ is bounded below near T. If $f(T) = N \in \mathbb{Z}$, then $f - N$ is of the same form and does not vanish identically unless f is the constant N; by (i), $|f(t) - N| \geq c|t - T|^2$, and $|f(t) - N| = \operatorname{dist}(f(t), \mathbb{Z})$ near T. In both cases (5.26) gives the claim. □

The exponent two cannot be improved for functions of the first form:

$$2 - t - \frac{1}{t} = -\frac{(t-1)^2}{t}. \tag{6.51}$$

Differences of this kind occur for the data above when $T = 1$. For instance, the first light-cone coordinate of the centre $C - Y_\nu(t) - Y_\nu(t)^\sharp = [2 - t - \nu/t,\ -t - \nu/t]$ differs from that of the point $[(1-\nu)/t,\ -(1+\nu)t] \in \Lambda_{1/t}$ by (6.51). We do not decide in which cases the distance between two points is in fact of first order, and use the quadratic bound throughout.

Lemma 6.16 (Quadratic bounds for collisions)**.** *Let collision data be given. There are $\varepsilon_0 \in (0, \frac{1}{4}]$, $c_0 \in (0,1]$, $\varpi_0 \in (0,1]$ and $\lambda_0 > 0$ such that $[T - \varepsilon_0, T + \varepsilon_0]$ is contained in the interior of I and the following hold for every t with $0 < \theta := |t - T| \le \varepsilon_0$.*

(i) Separations. *For every object $\xi \in \mathfrak{O}$ and every $\lambda \in \mathfrak{O} \cup \mathfrak{B}$, either ξ and λ coincide identically on I, or $|\xi(t) - \lambda(t)| \ge c_0\,\theta^2$. Consequently*

$$\operatorname{dist}(\xi(t),\ \Sigma(t) \setminus \{\xi(t)\}) \ge c_0\,\theta^2 \qquad (\xi \in \mathfrak{O}), \tag{6.52}$$

and $\xi(t) \in \Sigma(t)$ if and only if ξ coincides identically with a reference branch.

(ii) Sines. *For each of the phases f listed in Table 1, $|\sin(\pi f(t))| \ge c_0\,\theta^2$; that is, the factor S_Y in the second column satisfies*

$$S_Y(X_0) \ge c_0^2\,\theta^4 \tag{6.53}$$

at the target X_0 in the first column.

(iii) Anchors. *The points $(\pm 1, \pm 1)$, A_j, $B_k(t)$ and $Y_\nu(t)$, $\nu \in \mathcal{N}$, that occur in the data have momentum coordinates of modulus at least ϖ_0, and the rates (5.25) of the intended zeros satisfy*

$$\lambda(A_j, C),\ \lambda(A_j, C^\sharp),\ \lambda(B_k(t), B_0(t))\ (k = 1,2),\ \lambda(Y_\nu(t), B_0(t))\ (\nu \in \{p_1, p_2, 3\})\ \ge \lambda_0.$$

data	target X_0	factor S_Y	phase $f(t) = B(X_0, Y)$
product	A_j $(j = 0,1,2)$	$S_{B_0(t)}$	$pU/t + V_j t$
product	A_j $(j = 0,1,2)$	$S_{B_0(t)^\sharp}$	$Ut + pV_j/t$
product	$B_k(t)$ $(k = 1,2)$	S_C	$2p_k/t$
product	$B_k(t)$ $(k = 1,2)$	$S_{C^\sharp}$	$2t$
product	$B_k(t)$ $(k = 1,2)$	$S_{B_0(t)^\sharp}$	$t^2 + pp_k/t^2$
sum	$Y_\nu(t)$ $(\nu \in \{p_1, p_2, 3\})$	$S_{C^\sharp}$	$2t$
sum	$Y_\nu(t)$ $(\nu \in \{p_1, p_2, 3\})$	$S_{B_0(t)^\sharp}$	$t^2 + p\nu/t^2$
sum	$[2,2]$	$S_{B_0(t)}$ and $S_{B_0(t)^\sharp}$	$2(t + p/t)$

Table 1: The factors that are required to be different from zero at the targets of the local-zero arguments, and their phases. The intended zeros are not listed: they are $B(A_j, C) = 2V_j$, $B(A_j, C^\sharp) = 2U$, $B(B_k(t), B_0(t)) = p + p_k$ and $B(Y_\nu(t), B_0(t)) = p + \nu$, which are integers for all t.

Proof. (i) The light-cone coordinates of the elements of $\mathfrak{O} \cup \mathfrak{B}$ are functions of the first form in Lemma 6.15. We apply that lemma to the finitely many coordinate differences of pairs (ξ, λ). If $\xi \not\equiv \lambda$, one coordinate difference f does not vanish identically, and $|\xi(t) - \lambda(t)| \ge$

$|f(t)|/\sqrt{2} \geq c\theta^2/\sqrt{2}$, because $|x|^2 = \frac{1}{2}(u_x^2 + v_x^2)$. For (6.52), let $y \in \Sigma(t)$, $y \neq \xi(t)$. If $|y| > \varrho_1 + 2$, then $|y - \xi(t)| > 2 \geq c_0\theta^2$. Otherwise $y = \lambda(t)$ for some $\lambda \in \mathfrak{B}$ by (6.50), and $\lambda \not\equiv \xi$. The last assertion follows from (6.50) in the same way.

(ii) The phases are computed from $B(X,Y) = u_X v_Y + v_X u_Y$ with $B_0(t) = [t, p/t]$, $B_0(t)^\sharp = [p/t, t]$, $C = [2,0]$, $C^\sharp = [0,2]$. Each is of one of the two forms of Lemma 6.15 and is not constant, since its coefficients of t, $1/t$, respectively t^2, $1/t^2$, are positive integers or vanish, and not all of them vanish. Lemma 6.15(ii) applies.

(iii) The points $(\pm 1, \pm 1)$ and A_j are fixed and off both axes, because $U \neq V_j$. For $t \in I$ the momentum coordinates of $B_k(t) = [t, p_k/t]$ are $\frac{1}{2}(p_k/t \pm t)$, of modulus at least $1/t$, because $p_k > t^2 + 2$. For $\nu = 3$ the coordinates of $Y_3(t)$ are $\frac{1}{2}(t \pm 3/t)$, which do not vanish at $t = T$ because $T^2 \neq 3$, hence are bounded away from zero near T. The rates are

$$\frac{2V_j}{|U^2 - V_j^2|}, \qquad \frac{2U}{|U^2 - V_j^2|}, \qquad \frac{k}{p_k^2/t^2 - t^2}, \qquad \frac{|p - \nu|}{|t^2 - \nu^2/t^2|},$$

by (5.25); they are positive and continuous near T, since $p \neq \nu$ for $\nu \in \{p_1, p_2, 3\}$. We choose ε_0 so small that all the statements hold. □

Two of these factors are bounded below by a positive constant near a radius that is not a half-integer, as in Section 5, but vanish at $t = T$ in the present situation. First, the phase $2t$ of $S_{C^\sharp}$ is an integer at every half-integer T, so $\sin^2(2\pi t)$ vanishes at $t = T$. Secondly, if $A_j \in \Lambda_T$ or $A_j \in \Lambda_{1/T}$, then the phase of $S_{B_0(t)}$, respectively of $S_{B_0(t)^\sharp}$, at A_j is an integer at $t = T$, because $B_0(T) \in \Lambda_T$, $B_0(T)^\sharp \in \Lambda_{1/T}$ and these lattices are integral for B. Neither factor is bounded below by a constant near T; both are covered by the bound $c_0^2\theta^4$.

The next lemma determines which incidences between the centres and the reference lattices hold at a radius close to T and different from T. By Lemma 6.16(i) these are exactly the incidences that hold identically in t, and those are read off from the integers (6.48). We write

$$\mu_t^{\mathrm{c}} = \tfrac{1}{2}\rho_R^{\mathrm{F}} + \tfrac{1}{2}\rho_t^{\mathrm{F}} = \tfrac{1}{4}(e_R + e_{1/R} + e_t + e_{1/t}), \qquad \sigma(R,t) = \mu_t^{\mathrm{c}} - \nu_{\mathrm{vac}}.$$

Lemma 6.17 (Incidences at the moving radius)**.** *Let collision data be given and let* $0 < |t - T| \leq \varepsilon_0$. *Let* $\xi \in \mathfrak{O}$ *be given by* (6.47).

(i) $\xi(t) \in \Lambda_t$ *if and only if* $a_1 = a_2 = 0$ *and* $b_2 = 0$; $\xi(t) \in \Lambda_{1/t}$ *if and only if* $a_1 = a_2 = 0$ *and* $b_1 = 0$; $\xi(t) \in V \cup \Lambda_R \cup \Lambda_{1/R}$ *if and only if* $b_1 = b_2 = 0$ *and* $[a_1, a_2] \in V \cup \Lambda_R \cup \Lambda_{1/R}$. *Consequently*

$$\mu_t^{\mathrm{c}}(\{\xi(t)\}) = \tfrac{1}{4}\Big(\mathbf{1}_{\{a=0,\, b_2=0\}} + \mathbf{1}_{\{a=0,\, b_1=0\}} + \mathbf{1}_{\{b=0\}}\,(e_R + e_{1/R})(\{[a_1, a_2]\})\Big). \tag{6.54}$$

(ii) The centres of the polynomials have the positions and masses listed in Table 2. In particular: the only centre at the origin is the one of length zero; every other centre with a positive coefficient lies outside V; *every centre with a negative coefficient either lies outside* $\Sigma(t)$ *and has mass zero, or is one of the points* $\pm C$, $\pm C^\sharp$, $\pm A_j$, $\pm A_j^\sharp$, $\pm Y_\nu(t)$, $\pm Y_\nu(t)^\sharp$, *with the mass given in the table.*

(iii) The row preimages $x_{l,k,j}(t)$ *with* $(k,j) \neq (0,0)$ *lie outside* $\Sigma(t)$. *The three others are* $[2,2] = (2,0)$, $[2,0] = (1,1)$ *and* $[2,-2] = (0,2)$; *for* $R \in \{1,2\}$ *they have* μ_t^{c}*-mass* $\frac{1}{2}$.

Proof. (i) Suppose that $\xi(t) = [mt, n/t] \in \Lambda_t$. Since $|\xi(t)| \leq \varrho_1$, the map $\lambda(t') = [mt', n/t']$ is a reference branch with $\lambda(t) = \xi(t)$, and Lemma 6.16(i) shows that $\xi \equiv \lambda$ on I. By the first line of (5.38), whose proof uses only that the two coordinate equations hold identically and that $\nu \neq 0$, this implies $a_1 = a_2 = 0$ and $b_2 = 0$. Conversely, if $a = 0$ and $b_2 = 0$, then

centres	sign	$[a_1, a_2]$, (b_1, b_2)	position at t	μ_t^c-mass
Product data, $Y \in \{C, A_0, A_1, A_2\}$, $B = B_k(t)$				
0	+	$a = 0$, $b = 0$	origin	1
$\pm C$, $\pm C^\sharp$	−	$a \neq 0$, $b = 0$	in V, off both axes	$\frac{1}{4}$
$\pm A_j$, $\pm A_j^\sharp$	−	$a \neq 0$, $b = 0$	in $\Lambda_R \cup \Lambda_{1/R}$, not in V	$\frac{1}{4} n_j^0$
$\pm B$	−	$a = 0$, $b = (\pm 1, 0)$	in Λ_t only	$\frac{1}{4}$
$\pm B^\sharp$	−	$a = 0$, $b = (0, \pm 1)$	in $\Lambda_{1/t}$ only	$\frac{1}{4}$
$\pm Y \pm Y^\sharp$	+	$a \neq 0$, $b = 0$	fixed, not in V	$\frac{1}{4}(e_R + e_{1/R})$
$\pm B \pm B^\sharp$	+	$a = 0$, $b_1 b_2 \neq 0$	not in $\Sigma(t)$	0
other centres of length 2	+	$a \neq 0$, $b \neq 0$	not in $\Sigma(t)$	0
length 3	−	$a \neq 0$, $b \neq 0$	not in $\Sigma(t)$	0
length 4	+	$a \neq 0$, $b \neq 0$	not in $\Sigma(t)$	0
Sum data, $\nu \in \mathcal{N}$				
0	+	$a = 0$, $b = 0$	origin	1
$\pm Y_\nu$	−	$a = 0$, $b = (\pm 1, 0)$	in Λ_t only	$\frac{1}{4}$
$\pm Y_\nu^\sharp$	−	$a = 0$, $b = (0, \pm 1)$	in $\Lambda_{1/t}$ only	$\frac{1}{4}$
$\pm Y_\nu \pm Y_\nu^\sharp$	+	$a = 0$, $b_1 b_2 \neq 0$	not in $\Sigma(t)$	0
$\pm C$, $\pm C^\sharp$	−	$a \neq 0$, $b = 0$	in $V \cap \Lambda_R \cap \Lambda_{1/R}$	$\frac{1}{2}$
$\zeta \neq 0$, $\lvert\eta\rvert = 1$	+	$a \neq 0$, $b \neq 0$	not in $\Sigma(t)$	0
$\zeta \neq 0$, $\lvert\eta\rvert = 2$	−	$a \neq 0$, $b \neq 0$	not in $\Sigma(t)$	0

Table 2: The centres of the collision kernels at a radius t with $0 < |t - T| \leq \varepsilon_0$. Here $n_j^0 = 1$ if $A_j \notin \Lambda_{1/R}$ and $n_j^0 = 2$ if $A_j \in \Lambda_{1/R}$, and $a = [a_1, a_2]$, $b = (b_1, b_2)$ are the integers of (6.48).

$\xi(t) = [b_1 t,\, b_1 \nu/t] \in \Lambda_t$. The statement for $\Lambda_{1/t}$ follows in the same way from the second line of (5.38), with $\xi(t) = [b_2 \nu/t,\, b_2 t]$ in the converse, and the statement for fixed points from the third line. Formula (6.54) follows, because each of the four combs gives mass one to each of its points.

(ii) For a product centre, $[a_1, a_2] = \eta_1 Y + \eta_2 Y^\sharp = [\eta_1 u_Y + \eta_2 v_Y,\ \eta_1 v_Y + \eta_2 u_Y]$ with $Y = [u_Y, v_Y]$ equal to $[2, 0]$ or to $[U, V_j]$. Since $u_Y^2 \neq v_Y^2$, this vector vanishes only for $\eta_1 = \eta_2 = 0$; for $\eta_1 \eta_2 \neq 0$ its coordinates are $\pm(u_Y + v_Y)$ twice or $\pm(u_Y - v_Y)$ with opposite signs. The columns of the table for $[a_1, a_2]$ and (b_1, b_2) follow from (6.48), and the positions and masses from (i), as follows.

Length one, fixed. The point $C = [2, 0]$ lies in Λ_r if and only if $2/r \in \mathbb{Z}$ and in $\Lambda_{1/r}$ if and only if $2r \in \mathbb{Z}$. For product data $2R \in \mathbb{Z}$ and $2/R \notin \mathbb{Z}$, because $R \notin \{1, 2\}$; so $(e_R + e_{1/R})(\{C\}) = 1$ and the mass is $\frac{1}{4} = \nu_{\text{vac}}(\{C\})$. For sum data both numbers are integers and the mass is $\frac{1}{2}$. The same holds at $-C$ and $\pm C^\sharp$ by the invariance of μ_t^c under the reflections. The points A_j belong to Λ_R by construction, so $(e_R + e_{1/R})(\{A_j\}) = n_j^0 \in \{1, 2\}$, and they are not in V because $U, V_j \geq 3$. By (i) none of these fixed points lies in $\Lambda_t \cup \Lambda_{1/t}$, since $a \neq 0$.

Length one, moving. For $\pm B_k(t)$ and $\pm Y_\nu(t)$ we have $a = 0$ and $b = (\pm 1, 0)$, and for their images under $\sharp$, $b = (0, \pm 1)$. By (i) the first lie in Λ_t and not in $\Lambda_{1/t} \cup V \cup \Lambda_R \cup \Lambda_{1/R}$, the second in $\Lambda_{1/t}$ only, and the mass is $\frac{1}{4}$.

Fixed centres of length two. For $\pm Y \pm Y^\sharp$ the coordinates are ± 2 for $Y = C$, so the point is $\pm(2, 0)$ or $\pm(0, 2)$ in momentum coordinates, and they have modulus at least 3 for $Y = A_j$, by (5.33); in both cases the point is not in V. Its mass may be positive.

All other centres. If $a = 0$ and $b_1 b_2 \neq 0$, or if $a \neq 0$ and $b \neq 0$, none of the three conditions of (i) holds, so $\xi(t) \notin \Sigma(t)$ and the mass vanishes. This applies to the centres $\pm B \pm B^\sharp$ and $\pm Y_\nu \pm Y_\nu^\sharp$, to the product centres of length two with one fixed and one moving vector, to those of length three and four, in which at least one of η_1, η_2 and at least one of η_3, η_4 is different from zero, and to the sum centres with $\zeta \neq 0$ and $\eta \neq 0$.

The summary in (ii) follows by inspection of the table and of the signs of the coefficients recalled before (6.47).

(iii) For $(k,j) \neq (0,0)$ we have $a = [2, 2-2l] \neq 0$ and $b \neq 0$, so (i) applies. For $k = j = 0$ the points are fixed, and for $R \in \{1,2\}$ each of $[2,2]$, $[2,0]$, $[2,-2]$ lies in Λ_R and in $\Lambda_{1/R}$, because $2R$ and $2/R$ are integers. □

The hypotheses on the integer data that enter this analysis are that $R \neq 2$ and $R \neq 1$ for product data (so that $C \notin \Lambda_R$), that $U \neq V_j$, and that the integers ν are different from zero; the parity of p is used only in Lemma 5.9. In the limit $t \to T$ further incidences appear: $B_k(T)$ lies in Λ_R when $R = T$, the centres $\pm C$ lie in $\Lambda_{1/T}$, and $\pm C^\sharp$ in Λ_T, for every half-integer T (and in both lattices when $T \in \{1,2\}$), and A_j may lie in $\Lambda_T \cup \Lambda_{1/T}$. These are the collisions. The masses in Table 2 are those at the moving radius, not the larger limiting ones.

Lemma 6.18 (Mass balance at an arbitrary radius)**.** *Let $r_1, \dots, r_k > 0$ and let $\Phi = \sum_{i \in \Xi} c_i e^{2\pi i B(\cdot, \xi_i)}$ be a trigonometric polynomial as in (5.5) that vanishes on $\Lambda_{r_1} \cup \dots \cup \Lambda_{r_k}$. Then $\sum_{i \in \Xi} c_i \, e_{r_l}(\{\xi_i\}) = 0$ for every l. In particular, for collision data, every polynomial Φ_t of the data and every $t > 0$,*

$$\sum_{i \in \Xi} c_i \, \mu_t^c(\{\xi_i(t)\}) = 0. \tag{6.55}$$

Proof. Let $r = r_l$ and $0 < \epsilon \leq 1$. The function Φ_ϵ of (5.6) is a Schwartz function vanishing on Λ_r, and $\mathbb{S} e_r = e_r$ by (2.11). By (5.10),

$$0 = \epsilon^2 \langle e_r, \Phi_\epsilon \rangle = \epsilon^2 \langle e_r, \mathbb{S}\Phi_\epsilon \rangle = \sum_{i \in \Xi} c_i \sum_{\lambda \in \Lambda_r} e^{-2\pi |\lambda - \xi_i|^2/\epsilon^2}.$$

The inner sums are dominated by the convergent sums $\sum_\lambda e^{-2\pi|\lambda-\xi_i|^2}$ and converge to $e_r(\{\xi_i\})$ as $\epsilon \to 0$. For collision data the polynomials vanish on $\Sigma(t) \supset \Lambda_R \cup \Lambda_{1/R} \cup \Lambda_t \cup \Lambda_{1/t}$. □

6.7 The resolved peak budget

The kernel estimates at the moving radius use the following three properties of a sequence. The exponents 1000 and 60 are convenient fixed numbers; what is needed of them is recorded in Remark 6.23.

Definition 6.19 (Resolved sequences)**.** Let collision data be given, with ε_0 as in Lemma 6.16 and E_* as in (6.49). A *resolved sequence* for these data is an admissible sequence of gap g converging to $\sigma(R,T)$, together with radii $t_a \in I$, such that for all sufficiently small a in the sequence, with $\theta_a = |t_a - T|$:

(K1) $0 < \theta_a \leq \varepsilon_0$, $\theta_a \to 0$, and $a \leq \theta_a^{1000}$;

(K2) there is an injective map $j \mapsto j'$ from the set of states of ρ_a with $\Delta_j \leq E_*$ into the family $\mathfrak{S}(R, t_a)$, whose image contains every state of $\mathfrak{S}(R,t_a)$ of dimension at most $E_* - 1$, such that j' has the spin of j and $|\Delta_j - \Delta'_{j'}| \leq \theta_a^{60}$;

(K3) some state of ρ_a has spin p and dimension exactly $E_p(t_a)$.

We call θ_a the *collision scale*, and put

$$\varrho_a = c_\varrho \, \theta_a^2, \qquad c_\varrho = \tfrac{1}{4} c_0, \qquad \kappa_a = c_0^2 \, \theta_a^4, \tag{6.56}$$

with c_0 as in Lemma 6.16.

In (K2), if a state j has weights $(h, \bar{h})$ and j' has weights $(h', \bar{h}')$, then $|h - h'| = |\bar{h} - \bar{h}'| = \frac{1}{2}|\Delta_j - \Delta'_{j'}|$, and $|\sqrt{x} - \sqrt{x'}| \le \sqrt{|x - x'|}$ for $x, x' \ge 0$. Hence, for each of the four sign patterns σ,

$$|X_j^\sigma - X_{j'}'^\sigma| \le \theta_a^{30}, \tag{6.57}$$

where $X_{j'}'^\sigma$ denotes the physical point of j' with the sign pattern σ. This holds also when a weight of j' vanishes.

Lemma 6.20 (Exact counts on the resolving scale)**.** *Let a resolved sequence be given. For all sufficiently small a, every object $\xi \in \mathfrak{O}$, and $x = \xi(t_a)$:*

(i) a pair (j, σ) satisfies $X_j^\sigma \in D(x, \varrho_a)$ if and only if $\Delta_j \le E_*$ and $X_{j'}'^\sigma = x$; in this case $|X_j^\sigma - x| \le \theta_a^{30}$. Consequently

$$\rho_a(D(x, \varrho_a)) = \sigma(R, t_a)(\{x\}) = \mu_{t_a}^{\mathrm{c}}(\{x\}) - \nu_{\mathrm{vac}}(\{x\}), \tag{6.58}$$

and $D(x, 2\varrho_a)$ contains no point of V other than possibly x.

(ii) If x is off both axes, then exactly $N_x = 4\sigma(R, t_a)(\{x\})$ states of ρ_a have a physical point in $D(x, \varrho_a)$. They have the spin $P_x^2 - Q_x^2$ of x, and their dimensions satisfy $|\Delta_j - |x|^2| \le \theta_a^{60}$. We call them the states at x.

Proof. Let a be so small that (K1) and (K2) hold and $\theta_a^{30} < \varrho_a \le \frac{1}{2}$. By (6.52) and the choice of c_ϱ,

$$D(x, 4\varrho_a) \cap \Sigma(t_a) \subset \{x\}, \tag{6.59}$$

and $V \cup \operatorname{supp} \sigma(R, t_a) \subset \Sigma(t_a)$ by (5.32).

(i) Let $X_j^\sigma \in D(x, \varrho_a)$. Then $\Delta_j = |X_j^\sigma|^2 \le (\varrho_1 + 1)^2 \le E_*$, so j' is defined, and by (6.57) the point $X_{j'}'^\sigma$ of $\operatorname{supp} \sigma(R, t_a)$ lies in $D(x, 2\varrho_a)$; by (6.59) it equals x. Conversely, let s be a state of $\mathfrak{S}(R, t_a)$ with a physical point at x. Its dimension is $|x|^2 \le \varrho_1^2 \le E_* - 1$, so $\mathsf{s} = j'$ for a unique state j with $\Delta_j \le E_*$, and $|X_j^\sigma - x| \le \theta_a^{30} < \varrho_a$ for the corresponding sign pattern. Thus $(j, \sigma) \mapsto (j', \sigma)$ is a bijection between the pairs with $X_j^\sigma \in D(x, \varrho_a)$ and the pairs consisting of a state of $\mathfrak{S}(R, t_a)$ and a sign pattern with physical point x. Each pair carries mass $\frac{1}{4}$ in ρ_a, respectively in $\sigma(R, t_a)$ by Lemma 6.1(ii), which gives (6.58). The statement about V is contained in (6.59).

(ii) If x is off both axes, the four physical points of a state with a physical point at x are distinct, so exactly one sign pattern of such a state gives x, its weights are (P_x^2, Q_x^2), and there are N_x such states in $\mathfrak{S}(R, t_a)$. By the bijection in (i), the states of ρ_a with a physical point in $D(x, \varrho_a)$ are the preimages of these N_x states under $j \mapsto j'$, and each of them has exactly one physical point in the disc. They have the spin of x, and the dimension bound is (K2). □

Lemma 6.21 (Resolved peak budget)**.** *Let a resolved sequence be given, and let Φ_t be one of the polynomials of the data, with centres $\xi_i(t)$ and coefficients c_i, $i \in \Xi$. For a in the sequence put*

$$\Phi = \Phi_{t_a}, \qquad \varrho = \varrho_a, \qquad m_i = \mu_{t_a}^{\mathrm{c}}(\{\xi_i(t_a)\}).$$

For all sufficiently small a the following hold.

(i) The numbers m_i lie in $[0, 1]$ and satisfy the balance (5.19). Conditions (B0), (B1) *and* (B2) *of Lemma 5.6 hold, with equality $\rho_a(D(\xi_i(t_a), \varrho_a)) = m_i$ in* (B2)*, and condition* (B3)(b) *holds at every index with $c_i < 0$ and $m_i = 0$.*

(ii) *At every index with $c_i < 0$ and $m_i > 0$ the point $x = \xi_i(t_a)$ is one of $\pm C$, $\pm C^\sharp$, $\pm A_j$, $\pm A_j^\sharp$, $\pm Y_\nu(t_a)$, $\pm Y_\nu(t_a)^\sharp$. The geometric requirements of* (B3)(a) *hold at x, and with $w_i = \nu_{\mathrm{vac}}(\{x\})$ and $n_i = N_x$ the number of states at x in the sense of Lemma 6.20(ii),*

$$w_i + \tfrac14 n_i = m_i, \qquad n_i = \begin{cases} 0 & \text{at } \pm C, \pm C^\sharp \text{ for product data,} \\ 1 & \text{at } \pm C, \pm C^\sharp \text{ for sum data,} \\ n_j^0 & \text{at } \pm A_j, \pm A_j^\sharp, \\ 1 & \text{at } \pm Y_\nu(t_a),\ \pm Y_\nu(t_a)^\sharp. \end{cases}$$

(iii) *Let $\varepsilon \geq 0$ be such that every state j at a centre x of* (ii) *satisfies $|\Delta_j - |x|^2| \leq \varepsilon$, and suppose that $\varepsilon \leq \varrho_a^2$. Then for every ϵ with $0 < \epsilon \leq \frac12$ and $a \leq \frac14\epsilon^2$,*

$$\sum_j \Phi^\natural_{t_a,\epsilon}(h_j, \bar h_j) \leq \frac{C_{\mathrm{B}}}{\theta_a^4}\Big((a+\varepsilon^2)\epsilon^{-4} + e^{-\pi\varrho_a^2/(4\epsilon^2)}\Big), \tag{6.60}$$

where C_{B} depends only on g and the data. All terms on the left are nonnegative.

The masses and their cancellation are taken at the moving radius t_a. At the limiting radius T the masses at some centres are larger. The additional states of ρ_a are, by (6.57) and (6.59), within θ_a^{30} of points of $\Sigma(t_a)$ at distance at least $4\varrho_a$ from the centre, and they are counted at no centre. The atoms of ν_{vac} enter only at the origin and at the centres $\pm C$, $\pm C^\sharp$, where they are positive; no centre comes within $4\varrho_a$ of a negative atom. Lemma 5.6 has no separate hypothesis on the points outside the discs $D(\xi_i, \varrho)$: they are estimated by Lemma 5.5 in its proof, where the displacement of the shifted vacuum point near $(1,1)$, which is of order a, is accounted for as well.

Proof. Let a be so small that Lemma 6.20 applies, $\varrho_a \leq \min\{\frac14, \frac12\sqrt{g}, \frac12\varpi_0\}$ and $a \leq \min\{\frac14 g^2, \frac14\varrho_a^2\}$; the last condition holds by (K1).

(i) Each of the four combs gives mass at most one to a point, so $0 \leq m_i \leq 1$, and (5.19) is (6.55) at $t = t_a$. (B0) holds because the polynomials vanish on $\Sigma(t) \supset V$. By Lemma 6.17(ii) the only centre at the origin has length zero, a positive coefficient and mass one, which is (B1). Let $c_i > 0$ and $x = \xi_i(t_a) \neq 0$. Then $x \notin V$ by Lemma 6.17(ii), hence $\operatorname{dist}(x, V) \geq 4\varrho_a$ by (6.59), and $\rho_a(D(x,\varrho_a)) = \mu^{\mathrm{c}}_{t_a}(\{x\}) = m_i$ by (6.58); this is (B2). Let $c_i < 0$ and $m_i = 0$. By Lemma 6.17(ii), $x = \xi_i(t_a) \notin \Sigma(t_a)$, so $\operatorname{dist}(x, V) \geq 4\varrho_a$ by (6.59) and $\rho_a(D(x,\varrho_a)) = 0$ by (6.58); this is (B3)(b).

(ii) The list of points is that of Lemma 6.17(ii). By Lemma 6.16(iii) their momentum coordinates have modulus at least $\varpi_0 \geq 2\varrho_a$. By Lemma 6.20(i) the disc $D(x,\varrho_a)$ contains no point of V other than possibly x, and $x \in V$ only for $x \in \{\pm C, \pm C^\sharp\} = \{(\pm1, \pm1)\}$, where $w_i = \frac14$ by (5.1); otherwise $w_i = 0$. By (6.58), $\frac14 N_x = \sigma(R, t_a)(\{x\}) = m_i - w_i$, and the values of $n_i = N_x$ follow from Table 2. By Lemma 6.20(ii) the n_i states at x are distinct and have the spin of x.

(iii) By (ii) and the hypothesis on ε, condition (B3)(a) holds at every index with $c_i < 0$ and $m_i > 0$, with the states at $\xi_i(t_a)$. Together with (i), all hypotheses of Lemma 5.6 hold for $\Phi = \Phi_{t_a}$, with $\varrho = \varrho_a$, the bound ϱ_1 for the centres, $m_* = 1$ and the fixed coefficients. The range (5.21) holds by the choice of a and the hypotheses on ϵ and ε. Hence (5.20) holds with a constant independent of ϱ, and $\varrho_a^{-2} = c_\varrho^{-2}\theta_a^{-4}$. The terms are nonnegative by Lemma 5.2(ii) and (iii). □

The local-zero arguments below use Lemma 5.8(iii) in the following form.

Lemma 6.22 (Anchoring at the moving radius)**.** *Let a resolved sequence be given. Let $X_0(t)$ be one of the maps $t \mapsto A_j$, $t \mapsto B_k(t)$ $(k = 1, 2)$, $t \mapsto Y_\nu(t)$ $(\nu \in \{p_1, p_2, 3\})$, and let Ψ_t be a function on $\mathbb{R}^2$ with*

$$\Psi_t \geq \prod_{i=1}^{k} S_{Y_i} \prod_{i=1}^{k'} S_{Y_i'},$$

where $k \geq 1$, the factors S_{Y_i} are intended zeros at $X_0(t)$ listed in the caption of Table 1, and the factors $S_{Y_i'}$ are factors listed in that table for the target $X_0(t)$. There is a constant C such that, for all sufficiently small a, every state j at $X_0(t_a)$, with dimension $|X_0(t_a)|^2 + \delta$ and physical point X in $D(X_0(t_a), \varrho_a)$, satisfies

$$|\delta| \leq C\, \theta_a^{-2k'/k}\, \Psi_{t_a}(X)^{1/(2k)}. \tag{6.61}$$

Proof. We apply Lemma 5.8(iii) with $X_0 = X_0(t_a)$, $\varrho_* = \varrho_1$, which bounds X_0 and the vectors C, $C^\sharp$, $B_0(t_a)$, $B_0(t_a)^\sharp$ because these are centres of the data, the constant $\varpi = \varpi_0$ and the rate $\lambda = \lambda_0$ of Lemma 6.16(iii), and the lower bound $\varkappa = \kappa_a = c_0^2\theta_a^4$ of (6.53) for the factors $S_{Y_i'}$ at X_0. The number δ_1 of that lemma is then at least $c\,\theta_a^4$ with a constant $c > 0$. By Lemma 6.20(ii) the state has the spin of X_0 and $|\delta| \leq \theta_a^{60} \leq \delta_1$ for small a, and its physical point in $D(X_0, \varrho_a)$ has the signs of X_0, so that it is the point $X(\delta)$ of (5.24). The lemma gives $|\delta| \leq \lambda_0^{-1}\big((2/\kappa_a)^{k'}\Psi_{t_a}(X)\big)^{1/(2k)}$, which is (6.61). □

Remark 6.23 (The exponents). The geometric input is the exponent two of Lemma 6.16: discs of radius $\varrho_a \asymp \theta_a^2$ and factors bounded below by $\kappa_a \asymp \theta_a^4$. All further losses are derived from it: the prefactor θ_a^{-4} in (6.60), the factor $\epsilon^{-4} \asymp \theta_a^{-8}H^2$ for a resolving width $\epsilon^2 \asymp \varrho_a^2/H$, and the powers of κ_a^{-1} in (6.61). In the proofs of Propositions 6.24 and 6.25 the largest loss is θ_a^{-36}. Of the exponents in Definition 6.19, the arguments use that the matching accuracy θ_a^{60} is smaller than ϱ_a^2, than the radius $\delta_1 \asymp \theta_a^4$ of validity of the local-zero estimate, and than θ_a^{52}, which is needed in the scalar row; and that $a \leq \theta_a^{1000}$ makes $a^{1/16}\theta_a^{-9/2}$ and $a^{1/9}\theta_a^{-16/3}$ positive powers of a.

6.8 Exclusion of resolved sequences

In the two propositions of this subsection a resolved sequence is given, $L = \log(1/a)$, constants called C depend only on g and the data, and all statements hold for sufficiently small a in the sequence. By (K1),

$$\theta_a^{-1} \leq a^{-1/1000}, \qquad aL \leq a^{1/2} \leq \theta_a^{500}. \tag{6.62}$$

For a logarithmic scale $H \geq 1$ we shall use resolving widths ϵ with ϵ^2 a fixed multiple of ϱ_a^2/H. For such a width and a state with $\Delta_j \leq E_*$ the broad Gaussian $e^{-2\pi\epsilon^2\Delta_j}$ is at least $\frac{1}{2}$ when a is small, because $\epsilon \to 0$. All kernels of the data are even, so their value at a physical point X_j^σ of a state does not depend on σ and equals the even evaluation at $(h_j, \bar{h}_j)$; the left side of (6.60) is written $\sum_j \Phi_{t_a,\epsilon}(X_j^\sigma)$ below.

Proposition 6.24 (Pinned radius different from one and two)**.** *Let collision data with $R \notin \{1, 2\}$ be given. There is no resolved sequence for these data.*

Proof. Suppose that a resolved sequence exists. We use the width

$$\epsilon^2 = \frac{\pi\varrho_a^2}{4L}, \qquad \text{so that} \qquad e^{-\pi\varrho_a^2/(4\epsilon^2)} = a, \qquad \epsilon^{-4} = \frac{16L^2}{\pi^2 c_\varrho^4\, \theta_a^8}. \tag{6.63}$$

By (6.62), $\epsilon \leq \frac{1}{2}$ and $a \leq \frac{1}{4}\epsilon^2$ for small a.

Step 1: the budget of Φ^*. By Lemma 6.21(ii) the centres of $\Phi^*_{t_a}$ with negative coefficient and positive mass are $\pm C$, $\pm C^\sharp$, at which there are no states, and $\pm B_0(t_a)$, $\pm B_0(t_a)^\sharp$, at which there is exactly one state. The state of (K3) has spin p and dimension $E_p(t_a) = |B_0(t_a)|^2$, so one of its physical points is $B_0(t_a)$; it is therefore the state at $B_0(t_a)$, and its displacement is zero. Lemma 6.21(iii) with $\varepsilon = 0$ and (6.63) give

$$\sum_j \Phi^*_{t_a,\epsilon}(X_j^\sigma) \le C\theta_a^{-4}(a\,\theta_a^{-8}L^2 + a),$$

and hence, all terms being nonnegative,

$$\Phi^*_{t_a}(X_j^\sigma) \le \Gamma_0 := C\,a\,\theta_a^{-12}L^2 \qquad (\Delta_j \le E_*). \tag{6.64}$$

Step 2: anchoring. Let j be one of the n_i^0 states at A_i, $i \in \{0,1,2\}$, with dimension $|A_i|^2 + \delta$. We apply Lemma 6.22 to $\Psi_t = \Phi^*_t = S_C S_{C^\sharp} S_{B_0(t)} S_{B_0(t)^\sharp}$ with the two intended zeros S_C, $S_{C^\sharp}$ and the two factors $S_{B_0(t)}$, $S_{B_0(t)^\sharp}$ of Table 1; thus $k = k' = 2$. By (6.61) and (6.64),

$$|\delta| \le C\,\theta_a^{-2}\,\Gamma_0^{1/4} \le C\,a^{1/4}\theta_a^{-5}L^{1/2} \qquad \text{for every state at } A_i. \tag{6.65}$$

Let $k \in \{1,2\}$ and let $E_{p_k}(t_a) + \delta_{k,a}$ be the dimension of the state at $B_k(t_a)$. We apply Lemma 6.22 to the same function with the intended zero $S_{B_0(t)}$ and the three factors S_C, $S_{C^\sharp}$, $S_{B_0(t)^\sharp}$ of Table 1: one intended zero and three factors, so the exponents in (6.61) are 6 and $\frac12$, and

$$|\delta_{k,a}| \le C\,\theta_a^{-6}\,\Gamma_0^{1/2} \le C\,a^{1/2}\theta_a^{-12}L \qquad (k = 1,2). \tag{6.66}$$

Put $\varepsilon_a = C\,a^{1/4}\theta_a^{-12}L$, which bounds the right sides of (6.65) and (6.66). By (6.62), $\varepsilon_a \le Ca^{1/4-12/1000}L \le \varrho_a^2$ for small a.

Step 3: the nine budgets. Fix $i,k \in \{0,1,2\}$. By Lemma 6.21(ii) the centres of $\Phi^{i,k}_{t_a}$ with negative coefficient and positive mass are $\pm A_i$, $\pm A_i^\sharp$, $\pm B_k(t_a)$, $\pm B_k(t_a)^\sharp$, and by Steps 1 and 2 the states at these centres have displacements at most ε_a. Lemma 6.21(iii) with $\varepsilon = \varepsilon_a$ gives

$$\sum_j \Phi^{i,k}_{t_a,\epsilon}(X_j^\sigma) \le C\theta_a^{-4}\Big((a + a^{1/2}\theta_a^{-24}L^2)\theta_a^{-8}L^2 + a\Big) \le C\,a^{1/2}\theta_a^{-36}L^4. \tag{6.67}$$

Step 4: localization. Let j be a state with $\Delta_j \le L$ and $X = [u,v]$ one of its physical points. By (6.63), $2\pi\epsilon^2\Delta_j \le \frac12\pi^2\varrho_a^2 \le 1$, so the broad Gaussian at X is at least e^{-1}, and by (6.67)

$$S_{A_i}(X)S_{A_i^\sharp}(X)\,S_{B_k(t_a)}(X)S_{B_k(t_a)^\sharp}(X) \le \Gamma_a := C\,a^{1/2}\theta_a^{-36}L^4 \qquad (0 \le i,k \le 2).$$

By (6.62), $\Gamma_a \le Ca^{1/2-36/1000}L^4 \le 1$. Moreover $uv = s_j \in \mathbb{Z}$ and $|u|, |v| \le \Upsilon := \sqrt{2L}$. We apply Lemma 5.9(c) with $t = t_a$, $t_- = \frac12$, $t_+ = R_{\max} + 1$, $\ell = 1$, the integer p of (6.46), $\Delta V = 2$, $D = 2U\Delta V = 16R$, and

$$\zeta = \tfrac12\Gamma_a^{1/8} = C\,a^{1/16}\theta_a^{-9/2}L^{1/2}, \qquad \zeta\,\Upsilon \le C\,a^{1/16}\theta_a^{-9/2}L \le C\,a^{29/500}L,$$

where the last inequality is (6.62). Since $\zeta\Upsilon \to 0$, the smallness conditions of parts (a) and (b) of that lemma hold. Hence there is a real point X_j', equal either to a point of $\Lambda_{t_a} \cup \Lambda_{1/t_a}$ or to a point $[m/D, n/D]$, whose spin is exactly s_j, with

$$|\Delta_j - |X_j'|^2| \le C\,\zeta\,\Upsilon \le C\,a^{1/16}\theta_a^{-9/2}L \le C\,a^{29/500}L. \tag{6.68}$$

In the second case X_j' lies in the set Σ_D of (5.29).

Step 5: contradiction. The set Σ_D is the union of the circle lattices with radii in $\{k/D : k \mid D^2\}$, which contains 1 and is closed under inversion by Lemma 4.12; the radii t_a lie in the compact interval I; and by Step 4 every state with $\Delta_j \le L$ has a real point $X_j' \in \Sigma_D \cup \Lambda_{t_a} \cup \Lambda_{1/t_a}$ of spin exactly s_j satisfying (6.68), for all sufficiently small a in the sequence. This contradicts Theorem 4.23, applied with $u_* = 1$ to the elements of the sequence below a suitable threshold. □

Theorem 4.23 makes no assumption on the limit of the radii t_a. Here $t_a \to T$ is a half-integer, so that Λ_{t_a} approaches a lattice contained in the rational grid, and for $T = 1$ the squared transverse momenta of points of Λ_{t_a} near the momentum axes tend to zero. Both situations are covered by that theorem, whose statement and proof place no restriction on the limit t_0 of the radii: if $|t_a - T| \ge a^\eta$ along a subsequence it uses the moving certificate of Proposition 4.20 at $t_0 = T$, and otherwise it transports the points to the radius T. For the sequences produced in the proof of Theorem A the first alternative occurs, because $|t_a - T|$ is comparable to δ_a.

Proposition 6.25 (Pinned radius one or two)**.** *Let collision data with $R \in \{1, 2\}$ be given. There is no resolved sequence for these data.*

Proof. Suppose that a resolved sequence exists.

Step 0: the states used. We apply Lemmas 6.17 and 6.20 to several objects. At $C = (1, 1)$ the measure $\mu^{\mathrm{c}}_{t_a}$ has mass $\frac{1}{2}$ and ν_{vac} has mass $\frac{1}{4}$, so there is exactly one state at $(1, 1)$; it has spin zero and dimension $2 + z_a$ with

$$\varsigma_a := |z_a| \le \theta_a^{60}.$$

We call it the *long scalar*; its physical point in $D((1, 1), \varrho_a)$ is $(\sqrt{h}, \sqrt{h})$ with $h = 1 + \frac{1}{2}z_a$, and this is the only physical point of a state in that disc. For $\nu \in \mathcal{N}$ there is exactly one state at $Y_\nu(t_a)$; it has spin ν and dimension $E_\nu(t_a) + \delta_{\nu,a}$, and $\delta_{p,a} = 0$ as in Step 1 of the proof of Proposition 6.24. The row preimage $x_{0,0,0} = [2, 2] = (2, 0)$ lies on the P-axis and outside V, and $\sigma(R, t_a)(\{(2, 0)\}) = \frac{1}{2}$ by Lemma 6.17(iii). A state with a physical point at $(2, 0)$ has weights $(4, 0)$ and gives mass $\frac{1}{2}$ to this point, so $\mathfrak{S}(R, t_a)$ contains exactly one such state, whose sign patterns $(+, +)$ and $(+, -)$ give the point $(2, 0)$. By Lemma 6.20(i) there is one state j_+ of ρ_a such that the physical points of states in $D((2, 0), \varrho_a)$ are exactly the two points $(\sqrt{h}, \pm\sqrt{h})$ of j_+; it has spin 4 and dimension $4 + \delta_a^+$, where $0 < \delta_a^+ \le \theta_a^{60}$, the lower bound because $h > 0$. In the same way there is one state j_-, of spin -4 and dimension $4 + \delta_a^-$, such that the physical points of states in $D((0, 2), \varrho_a)$ are the two points $(\pm\sqrt{h}, \sqrt{h})$ of j_-. Finally, for $(k, j) \ne (0, 0)$ the preimage $x_{l,k,j}(t_a)$ is not in $\Sigma(t_a)$, and by Lemma 6.20(i) the disc $D(x_{l,k,j}(t_a), \varrho_a)$ contains no physical point of a state and no point of V.

Step 1: the scale. Put

$$b = \varsigma_a + a, \qquad H = \log(1/b) \le L, \qquad \beta = a + \varsigma_a^2, \qquad \epsilon^2 = \frac{\pi \varrho_a^2}{16H}. \tag{6.69}$$

Then $b \le 2\theta_a^{60} \to 0$,

$$e^{-\pi\varrho_a^2/(4\epsilon^2)} = b^4 \le 8\beta, \qquad e^{-\pi\varrho_a^2/(2\epsilon^2)} = b^8, \qquad \epsilon^{-4} = \frac{256H^2}{\pi^2 c_\varrho^4 \theta_a^8},$$

and, since $x \log^{5/2}(1/x) \le C\sqrt{x}$ for $0 < x \le 1$,

$$\varsigma_a H^{5/2} \le b \log^{5/2}(1/b) \le C\, b^{1/2} \le C\, \theta_a^{30}. \tag{6.70}$$

By (6.62), $aH \le aL \le \theta_a^{500}$, so $\epsilon \le \frac{1}{2}$ and $a \le \frac{1}{8}\epsilon^2$ for small a; thus Lemma 6.21(iii) and Lemma 5.13(iii) apply with the widths ϵ and $\epsilon/\sqrt{2}$.

Step 2: the budget of $\Phi^{(p)}$. By Lemma 6.21(ii) the centres of $\Phi^{(p)}_{t_a}$ with negative coefficient and positive mass are $\pm C$, $\pm C^\sharp$, at which the only state is the long scalar, with displacement $\varsigma_a \le \varrho_a^2$, and $\pm B_0(t_a)$, $\pm B_0(t_a)^\sharp$, at which the state has displacement zero. Lemma 6.21(iii) with $\varepsilon = \varsigma_a$ gives

$$\sum_j \Phi^{(p)}_{t_a,\epsilon}(X^\sigma_j) \le C\theta_a^{-4}\big((a+\varsigma_a^2)\,\theta_a^{-8}H^2 + b^4\big) \le C\,\beta\,\theta_a^{-12}H^2,$$

and therefore

$$\Phi^{(p)}_{t_a}(X^\sigma_j) \le \Gamma_0 := C\,\beta\,\theta_a^{-12}H^2 \qquad (\Delta_j \le E_*). \tag{6.71}$$

Step 3: the state at $W(t_a)$. Since $\Omega \ge S_{C^\sharp}$, we have $\Phi^{(p)}_t \ge S_{C^\sharp}S_{B_0(t)}S_{B_0(t)^\sharp}$. We apply Lemma 6.22 at $X_0(t) = Y_3(t) = W(t)$ with the intended zero $S_{B_0(t)}$ and the two factors $S_{C^\sharp}$, $S_{B_0(t)^\sharp}$ of Table 1; thus $k=1$, $k'=2$, and by (6.71)

$$|\delta_{3,a}| \le C\,\theta_a^{-4}\,\Gamma_0^{1/2} \le C\,\beta^{1/2}\theta_a^{-10}H. \tag{6.72}$$

Step 4: the state near $(2,0)$. Let $X=[u,v]$ be the physical point $(\sqrt{h},\sqrt{\bar h})$ of j_+, so that $u,v>0$, $uv=4$ and $|X-(2,0)| \le \theta_a^{30}$ by Lemma 6.20(i). Then, as $|u-2|, |v-2| \le \frac14$,

$$\delta_a^+ = \tfrac12(u^2+v^2) - uv = \tfrac12(u-v)^2 \le (u-2)^2 + (v-2)^2 \le \tfrac{1}{16}\,\Omega(X),$$

by (5.26). By (6.53) for the target $[2,2]$, both $S_{B_0(t_a)}$ and $S_{B_0(t_a)^\sharp}$ are at least κ_a at $(2,0)$. For real Y the function S_Y has gradient bounded by $2\pi|Y|$, since $|B(Z,Y)| \le 2|Z||Y|$; hence these two factors are at least $\kappa_a - 2\pi\varrho_1\theta_a^{30} \ge \frac12\kappa_a$ at X. With (6.71),

$$0 < \delta_a^+ \le \frac{1}{16}\,\frac{4\,\Gamma_0}{\kappa_a^2} \le C\,\beta\,\theta_a^{-20}H^2. \tag{6.73}$$

Step 5: the budget of $\Phi^{(3)}$ at the width $\epsilon/\sqrt{2}$. By Lemma 6.21(ii) the centres of $\Phi^{(3)}_{t_a} = \Omega\,S_{W(t_a)}S_{W(t_a)^\sharp}$ with negative coefficient and positive mass are $\pm C$, $\pm C^\sharp$ and $\pm W(t_a)$, $\pm W(t_a)^\sharp$. Put $\varepsilon = C\beta^{1/2}\theta_a^{-10}H$ with the constant of (6.72), enlarged so that $C \ge 1$. It bounds $|\delta_{3,a}|$ and $\varsigma_a \le \beta^{1/2}$, and $\beta^{1/2}H \le a^{1/2}L + bH \le a^{1/4} + Cb^{1/2} \le C\theta_a^{25}$ by (K1) and the inequality used in (6.70), so $\varepsilon \le \varrho_a^2$ for small a. Lemma 6.21(iii) with the width $\epsilon/\sqrt{2}$ gives

$$\sum_j \Phi^{(3)}_{t_a,\epsilon/\sqrt{2}}(X^\sigma_j) \le C\theta_a^{-4}\Big((a+\beta\,\theta_a^{-20}H^2)\theta_a^{-8}H^2 + b^8\Big) \le C\,\beta\,\theta_a^{-32}H^4. \tag{6.74}$$

Step 6: the row from above. By (5.51) with $t=t_a$ and $w=3$, (6.74), and $\epsilon^2 = \pi c_\varrho^2\theta_a^4/(16H)$,

$$\big|\langle \mu_a, \mathsf{D}_{t_a}\psi_\epsilon\rangle\big| \le C\,\epsilon^3\beta\,\theta_a^{-32}H^4 + C\,a\,\epsilon^{-1} \le C\,\beta\,\theta_a^{-26}H^{5/2}. \tag{6.75}$$

Step 7: the row from below. By Lemma 5.12(i), $\langle \mu_a, \mathsf{D}_{t_a}\psi_\epsilon\rangle$ is the combination, with coefficients $b_lb_kb_j$ of total modulus 64, of the numbers $\langle \mu_a, \psi_{\epsilon,x}\rangle$, $x = x_{l,k,j}(t_a)$. Each of these is the sum of $\psi_{\epsilon,x}$ over the physical points of all states, with weights $\frac14$, and over the points of V, with the weights (5.1). By Lemma 5.12(ii) and (5.17), the points at distance at least ϱ_a from x contribute at most $(C_1+4)e^{-\pi\varrho_a^2/(2\epsilon^2)} = (C_1+4)b^8$. The points within ϱ_a of x were determined in Step 0. For $(k,j) \ne (0,0)$ there are none. For $x_{1,0,0} = (1,1)$ they are the atom of $\nu_{\rm vac}$ at $(1,1)$, at which $\psi_{\epsilon,(1,1)}$ vanishes, and the point $(\sqrt{h},\sqrt{h})$ of the long scalar, with weight $\frac14$; by Lemma 5.12(iii),

$$\Big|\langle \mu_a, \psi_{\epsilon,(1,1)}\rangle - \frac{z_a}{16}\Big| \le \frac{\varsigma_a^2}{32} + \frac{\varsigma_a^3}{2\epsilon^2} + (C_1+4)b^8 \le C\,\beta,$$

where $\varsigma_a^3\epsilon^{-2} \le C\varsigma_a^2\,\varsigma_a H\theta_a^{-4} \le C\varsigma_a^2$ by (6.70); the expansion of that lemma is valid for all $|z_a| \le 1$ and requires no comparison between ς_a and ϵ. For $x_{0,0,0} = (2,0)$ they are the two points of j_+, each with weight $\frac14$, and by Lemma 5.12(iv) and (6.73) they contribute at most $\frac{1}{16}\delta_a^+ \le C\beta\theta_a^{-20}H^2$ in modulus. For $x_{2,0,0} = (0,2)$ they are the two points of j_-, whose contributions cancel by Lemma 5.12(iv). The coefficients of the three preimages with $k = j = 0$ are $1, -2, 1$. Hence

$$\Big|\langle \mu_a, \mathsf{D}_{t_a}\psi_\epsilon\rangle + \frac{z_a}{8}\Big| \le C\,\beta\,\theta_a^{-20}H^2. \tag{6.76}$$

The pairing here is the physical one (5.4), in which the vacuum atom at $(1,1)$ is not shifted; the central shift is contained in (5.51).

Step 8: the long scalar. By (6.75) and (6.76),

$$\tfrac18\varsigma_a \le C\,(a+\varsigma_a^2)\,\theta_a^{-26}H^{5/2}.$$

By (6.70), $C\varsigma_a\theta_a^{-26}H^{5/2} \le C\theta_a^4 \le \frac{1}{16}$ for small a, so the quadratic term is at most $\frac{1}{16}\varsigma_a$, and with (6.62)

$$\varsigma_a \le C\,a\,\theta_a^{-26}L^{5/2} \le C\,a^{974/1000}L^{5/2} < a^{1/3}. \tag{6.77}$$

Step 9: the final budgets. We now use the width (6.63). As in Step 2, with $\varepsilon = \varsigma_a < a^{1/3} \le \varrho_a^2$, Lemma 6.21(iii) gives

$$\sum_j \Phi^{(p)}_{t_a,\epsilon}(X_j^\sigma) \le C\theta_a^{-4}((a+a^{2/3})\theta_a^{-8}L^2 + a) \le C\,a^{2/3}\theta_a^{-12}L^2,$$

so that $\Phi^{(p)}_{t_a}(X_j^\sigma) \le Ca^{2/3}\theta_a^{-12}L^2$ for $\Delta_j \le E_*$. For $k \in \{1,2\}$ we apply Lemma 6.22 at $X_0(t) = B_k(t) = Y_{p_k}(t)$ to $\Phi^{(p)}_t \ge S_{C^\sharp}S_{B_0(t)}S_{B_0(t)^\sharp}$, with the intended zero $S_{B_0(t)}$ and the two factors of Table 1:

$$|\delta_{p_k,a}| \le C\,\theta_a^{-4}(a^{2/3}\theta_a^{-12}L^2)^{1/2} = C\,a^{1/3}\theta_a^{-10}L \qquad (k=1,2).$$

Put $\varepsilon_a = C\,a^{1/3}\theta_a^{-10}L$ with $C \ge 1$; it bounds ς_a and $|\delta_{p_k,a}|$, and $\varepsilon_a \le \varrho_a^2$ for small a by (6.62). For $k = 0,1,2$ the centres of $\Phi^{(p_k)}_{t_a}$ with negative coefficient and positive mass are $\pm C$, $\pm C^\sharp$, $\pm B_k(t_a)$, $\pm B_k(t_a)^\sharp$, and Lemma 6.21(iii) with $\varepsilon = \varepsilon_a$ gives

$$\sum_j \Phi^{(p_k)}_{t_a,\epsilon}(X_j^\sigma) \le C\theta_a^{-4}\Big((a+a^{2/3}\theta_a^{-20}L^2)\theta_a^{-8}L^2 + a\Big) \le C\,a^{2/3}\theta_a^{-32}L^4. \tag{6.78}$$

Step 10: localization. Let j be a state with $\Delta_j \le L$ and $X = [u,v]$ one of its physical points. As in Step 4 of the proof of Proposition 6.24, the broad Gaussian at X is at least e^{-1}, so by (6.78)

$$\Omega(X)\,S_{B_k(t_a)}(X)S_{B_k(t_a)^\sharp}(X) \le \Gamma_a := C\,a^{2/3}\theta_a^{-32}L^4 \le 1 \qquad (k=0,1,2),$$

and $uv = s_j \in \mathbb{Z}$, $|u|, |v| \le \Upsilon = \sqrt{2L}$. We apply Lemma 5.9(d) and (a) with $t = t_a$, $t_- = \frac12$, $t_+ = R_{\max}+1$, $\ell = 1$ and

$$\zeta = \tfrac12\Gamma_a^{1/6} = C\,a^{1/9}\theta_a^{-16/3}L^{2/3}, \qquad \zeta\,\Upsilon \le C\,a^{1/9}\theta_a^{-16/3}L^{7/6} \le C\,a^{119/1125}L^{7/6},$$

by (6.62). Since $\zeta\Upsilon \to 0$, the smallness conditions hold, and there is a real point X_j', either in $\Lambda_{t_a} \cup \Lambda_{1/t_a}$ or of the form $[m/2, n/2]$, whose spin is exactly s_j, with

$$\big|\Delta_j - |X_j'|^2\big| \le C\,\zeta\,\Upsilon \le C\,a^{1/9}\theta_a^{-16/3}L^{7/6} \le C\,a^{119/1125}L^{7/6}. \tag{6.79}$$

In the second case $X_j' \in \Sigma_2 = \Lambda_{1/2} \cup \Lambda_1 \cup \Lambda_2$ by (5.29).

Step 11: contradiction. As in Step 5 of the proof of Proposition 6.24, the localization (6.79) of all states with $\Delta_j \le L$ contradicts Theorem 4.23, applied with the radii $\{\frac12, 1, 2\}$, the interval I, the radii t_a and $u_* = 1$. □

In Step 8 the quadratic term is absorbed because the comparison theorem already gives $\varsigma_a \le \theta_a^{60}$, which is much smaller than the loss θ_a^{-26}. The case $\varsigma_a = 0$ needs no separate treatment, since the scale $b = \varsigma_a + a$ is positive.

6.9 Proof of Theorem A

Proof of Theorem A. The two formulations in the theorem are equivalent, and by Remark 2.2 it suffices to consider $0 < g \le \frac{1}{2}$. Suppose that the theorem fails for such a g. Then there are admissible partition functions of gap g at central charges $c = 1 + 12a$ with a arbitrarily small, hence an admissible sequence of gap g with $a \to 0$. By Theorem 2.9 a subsequence converges, and by Theorem 5.21 its limit is $\sigma(R_1, R_2)$ with half-integers $1 \le R_1 \le R_2 \le R_{\max}$. We show that a further subsequence, with suitable radii, is a resolved sequence for collision data (R, T), $\{R, T\} = \{R_1, R_2\}$. This contradicts Proposition 6.24 if $R \notin \{1, 2\}$ and Proposition 6.25 if $R \in \{1, 2\}$. Subsequences of admissible sequences are admissible sequences, and all statements below hold for sufficiently small a in the sequence under consideration.

Step 1: the comparison measure. Theorem 6.13 provides the indices ι_a and the radii $t_a \ge 1$. Passing to a subsequence we may assume that ι_a is constant. Then $R = R_{\iota_a}$ and $T = R_{3-\iota_a}$ are fixed half-integers, $\sigma(R, T) = \sigma(R_1, R_2)$, and $t_a \to T$. We fix the collision data of Section 6.6 for (R, T), and with them the interval I, the integer p, the constants ε_0, c_0 of Lemma 6.16, and E_*.

Step 2: the finite orders. Put

$$m = 20000, \qquad n = 20000.$$

These integers are chosen after the data of Step 1, on which they do not depend, and before any estimate involving a. By Theorem 5.25 with the exponent $1/m$ and $k = 1$,

$$a \le \delta_a^m \tag{6.80}$$

for all sufficiently small a. (The alternative $\delta_a < a^{1/m}$ along a subsequence is the case excluded in the proof of that theorem by Proposition 5.23 and Theorem 4.22.) Since $\delta_a \to 0$, it follows that $\sqrt{a} \le \delta_a$, hence

$$\delta_a \le b_a \le 2\delta_a, \qquad H_a \le \log(1/\delta_a), \qquad \bigl(a + b_a^{n+1}\bigr)^{1/84} \le \bigl(\delta_a^{20000} + (2\delta_a)^{20001}\bigr)^{1/84} \le 2\,\delta_a^{20000/84}.$$

As $20000/84 > 238$, for every pair of constants C, K we have

$$C\,\bigl(a + b_a^{n+1}\bigr)^{1/84}(1 + H_a)^K \le 2C\,\delta_a^{238}\bigl(1 + \log(1/\delta_a)\bigr)^K \le \delta_a^{200} \tag{6.81}$$

for small a. Thus the bounds (6.36) and (6.45), for this n and for the finitely many windows and the number E_* used below, are at most δ_a^{200}.

Step 3: the collision scale. Consider the window $\mathcal{W}_{\eta_d}(E_d, 0)$. A point of spin zero of $\operatorname{supp}\sigma(R_1, R_2) \subset \Gamma$ has dimension in $\frac{1}{2d^2}\mathbb{Z}$, and $3\eta_d < \frac{1}{2d^2}$, so the window is isolated for $\mathfrak{S}(R_1, R_2)$. By Lemma 5.22 the states of ρ_a in it have the dimensions $E_d + z_{i,a}$, $1 \le i \le 4$, and $\delta_a = \max_i |z_{i,a}|$. By (5.67) the numbers d/R, dR, d/T, dT are integers. The classes with $mn = 0$ and $E_{m,n}(R) = E_d$ are those of $(d/R, 0)$ and $(0, dR)$, neither of which is the class of $(0, 2R)$, since $d \ge 4$; the classes with $mn = 0$ and $E_{m,n}(T) = E_d$ are those of $(d/T, 0)$ and $(0, dT)$. By Lemma 6.2(i) the states of $\mathfrak{S}(R, t_a)$ in the window have the dimensions

$$E_d, \qquad E_d, \qquad E_{d/T,0}(t_a) = E_d\,(t_a/T)^2, \qquad E_{0,dT}(t_a) = E_d\,(T/t_a)^2.$$

For two ordered quadruples the largest distances to E_d differ by at most the largest distance between corresponding entries. Hence, by Theorem 6.13(iii) and (6.81),

$$\bigl|\delta_a - E_d\,\mathfrak{d}(t_a)\bigr| \le \delta_a^{200}, \qquad \mathfrak{d}(t) = \max\bigl\{|(t/T)^2 - 1|,\ |(T/t)^2 - 1|\bigr\}.$$

For $|t-T| \le \frac12 T$ we have $|t-T|/T \le \mathfrak{d}(t) \le 10\,|t-T|/T$. Since $\delta_a^{200} \le \frac12\delta_a$, we get $\frac12\delta_a \le E_d\,\mathfrak{d}(t_a) \le 2\delta_a$ and

$$\frac{T}{20E_d}\,\delta_a \le |t_a - T| \le \frac{2T}{E_d}\,\delta_a. \tag{6.82}$$

In particular $t_a \neq T$, because $\delta_a > 0$ by (6.80).

Step 4: anchoring the radius on an actual state. By Corollary 6.14 and (6.81) there is, for small a, a map $j \mapsto j'$ from the states of ρ_a with $\Delta_j \le E_*$ into $\mathfrak{S}(R,t_a)$ as described there, with $|\Delta_j - \Delta'_{j'}| \le \delta_a^{200}$. The moving state of the class of $(1,p)$ has spin p and dimension $E_{1,p}(t_a) = E_p(t_a) \le \varrho_1^2 \le E_* - 1$, so it is the image of a state j_0 of ρ_a, which has spin p and

$$|\Delta_{j_0} - E_p(t_a)| \le \delta_a^{200}. \tag{6.83}$$

This state does not depend on the choices made in Corollary 6.14: it is the only state of ρ_a of spin p whose dimension is within $c\,\delta_a$ of $E_p(t_a)$, for a suitable constant $c > 0$. Indeed, the states of spin p of $\mathfrak{S}(R,t)$ are, by Lemma 6.1(ii), the moving states of the classes of $(1,p)$ and $(p,1)$, of dimensions $E_p(t)$ and $E_p(1/t)$, and the fixed states of the same two classes, of dimensions $E_p(R)$ and $E_p(1/R)$. The equation $E_p(x) = E_p(T)$, $x > 0$, holds only for $x = T$ and $x = p/T$, and $p/T > R_{\max}$. Hence the dimensions other than $E_p(t)$ that tend to $E_p(T)$ as $t \to T$ are $E_p(1/t)$ when $T = 1$, and the constants $E_p(R) = E_p(T)$ when $R = T$ and $E_p(1/R) = E_p(T)$ when $R = T = 1$. Since $|E_p'| \ge 4/(R_{\max}+1)$ on I and $E_p(1/t) - E_p(t) = \frac12(p^2-1)(t^2 - t^{-2})$, each of them differs from $E_p(t)$ by at least a constant multiple of $|t-T|$ near T; so $E_p(t_a)$ is a simple node of the comparison and is separated from the other nodes of spin p by at least $c'|t_a - T| \ge 2c\,\delta_a$, by (6.82). As every state of ρ_a of spin p with dimension near $E_p(T)$ is within δ_a^{200} of the node to which it is mapped, the claim follows.

Since E_p is strictly decreasing on I, T is interior to I and $\Delta_{j_0} \to E_p(T)$, there is a unique $\tilde t_a \in I$ with

$$E_p(\tilde t_a) = \Delta_{j_0}, \qquad |\tilde t_a - t_a| \le \tfrac14(R_{\max}+1)\,\delta_a^{200}, \tag{6.84}$$

by (6.83) and the lower bound for $|E_p'|$. Let $\tilde\theta_a = |\tilde t_a - T|$. By (6.82) and (6.84),

$$\frac{T}{40E_d}\,\delta_a \le \tilde\theta_a \le \frac{4T}{E_d}\,\delta_a. \tag{6.85}$$

Step 5: the sequence with the radii $\tilde t_a$ is resolved. (K1): by (6.85), $0 < \tilde\theta_a \to 0$, and by (6.80) and (6.85), $a \le \delta_a^{20000} \le (40E_d\tilde\theta_a/T)^{20000} \le \tilde\theta_a^{1000}$ for small a. (K2): by the last assertion of Corollary 6.14, with $e_a = \frac14(R_{\max}+1)\delta_a^{200}$, and by (6.81), there is a map $j \mapsto j'$ into $\mathfrak{S}(R,\tilde t_a)$ with the required properties and

$$|\Delta_j - \Delta'_{j'}| \le C\,\delta_a^{200} \le C\,(40E_d\tilde\theta_a/T)^{200} \le \tilde\theta_a^{60}.$$

(K3): the state j_0 has spin p and dimension $E_p(\tilde t_a)$ by (6.84). Hence the sequence, with the radii $\tilde t_a$, is a resolved sequence for the collision data (R,T), which is excluded by Propositions 6.24 and 6.25. □

Remark 6.26 (The order of the choices). The proof uses the following finite quantities, fixed in this order.

(1) The gap g determines d, the grid Γ, the prime $\mathfrak{p}$ and the reconstruction operator P_*.

(2) The limit of the sequence and the rule (6.43), which involve no Taylor order, determine the pinned radius R and the limit T of the free radius. The pair (R,T) determines the collision data and the constants of Lemma 6.16. The collision exponent is two, the losses in Propositions 6.24 and 6.25 are at most θ_a^{-36}, and Definition 6.19 requires the accuracy θ_a^{60} and the bound $a \le \theta_a^{1000}$. These exponents do not depend on g or on the data.

(3) The exponent $m = 20000$ of the comparison between a and δ_a and the Taylor order $n = 20000$ are chosen so that $\min\{m, n+1\}/84 > 200 \geq 60$ and $m > 1000$.

(4) The order n determines the local order $q = 2n$, the tests $\mathcal{F}_q$ and the projection Π_q, then the number $D_0 = 4(n+2)/\pi$, then the window constant M of (6.26), as explained in the proof of Lemma 6.11.

(5) Finally a is taken small. The certificates of Theorem 4.23, which are used for the exponents $\frac{29}{500}$ and $\frac{119}{1125}$ of (6.68) and (6.79), are fixed inside the proof of that theorem.

No quantity depends on one that is chosen later, and no Taylor order depends on a. With $a \leq \tilde{\theta}_a^{m'}$ for a larger exponent m' in place of 1000, the same proofs give the localization exponents $\frac{1}{16} - \frac{9}{2m'}$ and $\frac{1}{9} - \frac{16}{3m'}$; the proof of Theorem A provides every $m' < 20000$. The only input from Theorem 5.25 is the inequality (6.80); no upper bound for δ_a by a power of a is used, and none is available.

7 Remarks on effectivity

Theorem A asserts that a threshold $\epsilon(g)$ exists, and its proof gives no value for it. In this section we locate the non-effective steps of that proof, show that those of a structural nature can be replaced by terminating searches, and state precisely which quantitative information is missing. The outcome is a conditional statement, Theorem 7.7; it is not an effective form of Theorem A.

The proof of Theorem A is non-effective in four places.

(a) *Approach to the boundary family.* Theorems 2.9 and 3.5 show that along a sequence $a \downarrow 0$ the long measures accumulate only on $\mathcal{B}_g$. This is a compactness statement and comes without a modulus.

(b) *Stabilization.* The arguments of Sections 5 and 6 begin by fixing the numbers of states in finitely many windows by Corollary 2.11, which holds for all sufficiently small a in a given sequence, below an unspecified threshold.

(c) *Finite certificates.* The tests of Theorem 4.19 and Proposition 4.20 are built from a linear identity supplied by Lemma 4.16, which asserts that a row lies in a span without exhibiting the combination. The finite-dimensional estimate of Proposition 6.6 likewise comes with a constant that is not exhibited.

(d) *Constants.* The analytic estimates carry unnamed constants C, c, κ.

We remove (a) in the sense of recursion theory (Proposition 7.5), turn (b) into an explicit hypothesis (Definition 7.3), and remove (c) for the origin certificates (Propositions 7.12 and 7.13). Source (d) is not addressed; Subsection 7.6 lists the estimates concerned.

Throughout this section an *algorithm* is a Turing machine acting on finite data: integers, rational numbers, and real algebraic numbers, the latter given by their minimal polynomial over $\mathbb{Q}$ and an isolating interval with rational endpoints. "Computable" is meant in this sense only, and no bound on the running time of any search below is implied.

7.1 Observables and certified local exclusions

Let $\mathcal{Y} = \{(\Delta, s) \in \mathbb{R} \times \mathbb{Z} : \Delta \geq |s|\}$. By a *state measure* we mean a positive measure $\rho = \sum_j A_{h_j} \otimes A_{\bar{h}_j}$ on $\mathbb{R}^2$ with $h_j, \bar{h}_j \geq 0$, integer spins, and finitely many states below every dimension; its states are points $(\Delta_j, s_j) \in \mathcal{Y}$, repeated according to multiplicity. The long measure ρ_a of an admissible partition function is a state measure, and so is $\sigma(R, t)$ for every

half-integer $R \geq 1$ and every real $t \geq 1$, by Theorem 3.5(ii) applied with a sufficiently small gap.

Definition 7.1 (Rational tests and observables)**.** A *rational test* is a function $\varphi : \mathbb{R} \times \mathbb{Z} \to \mathbb{R}$ such that $\varphi(\cdot, s) = 0$ for all but finitely many s and each $\varphi(\cdot, s)$ is continuous, compactly supported and piecewise polynomial, with rational breakpoints and rational coefficients. For a state measure ρ we put $\langle \rho, \varphi \rangle = \sum_j \varphi(\Delta_j, s_j)$, and for a finite family Φ of rational tests we write $\Phi(\rho) = (\langle \rho, \varphi \rangle)_{\varphi \in \Phi} \in \mathbb{R}^\Phi$, with the norm $\| \cdot \|_\infty$ on $\mathbb{R}^\Phi$.

The number $\langle \rho, \varphi \rangle$ is the integral against ρ of the continuous, compactly supported, even function $X \mapsto \sum_s \varphi(|X|^2, s) \max\{0, 1 - 2|P^2 - Q^2 - s|\}$, so the observables are continuous for the vague convergence of Theorem 2.9(a). A rational test is a finite rational datum, from which one computes a rational $E_\varphi \geq 1$ with $\varphi(\Delta, s) = 0$ for $\Delta > E_\varphi$ and a rational Lipschitz constant for φ in Δ; we put $E_\Phi = \max_{\varphi \in \Phi} E_\varphi$. Any class of continuous compactly supported tests with computable values and computable moduli of continuity would serve equally well.

For the boundary measures, (2.10), (2.3) and (2.12) give

$$\langle \sigma(R,t), \varphi \rangle = \tfrac{1}{2} \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} \big[\varphi(E_{m,n}(R), mn) + \varphi(E_{m,n}(t), mn) \big] + \varphi(1,1) + \varphi(1,-1) - \varphi(2,0), \tag{7.1}$$

where $E_{m,n}(r) = \frac{1}{2}(m^2 r^2 + n^2 r^{-2})$; the lattices Λ_r and $\Lambda_{1/r}$ contribute equally. Only charges with $m^2 \leq 2E_\varphi$ and $n^2 \leq 2E_\varphi r^2$ contribute. Hence, for a half-integer R, the map $t \mapsto \Phi(\sigma(R,t))$ is continuous on $[1, \infty)$, has a computable Lipschitz constant on every interval $[1, \bar{r}]$ with $\bar{r}$ rational, and takes rational, exactly computable values at rational t. We put $r_g = (2g)^{-1/2}$ and

$$\Pi_g = \{(R,t) : 2R \in \mathbb{Z},\ 1 \leq R \leq r_g,\ 1 \leq t \leq r_g\},$$

so that $\mathcal{B}_g = \{\sigma(R,t) : (R,t) \in \Pi_g\}$, because $\sigma(R,t) = \sigma(t,R)$.

Sharp cluster counts are not continuous observables, but they are determined by open conditions on continuous ones.

Lemma 7.2 (Cluster counts from continuous tests)**.** *Let $s \in \mathbb{Z}$, let $I \subset \mathbb{R}$ be a bounded open interval, and let χ^-, χ^+ be rational tests vanishing off the spin s with*

$$0 \leq \chi^-(\cdot, s) \leq \mathbf{1}_I \leq \mathbf{1}_{\overline{I}} \leq \chi^+(\cdot, s).$$

Let ρ be a state measure and $n_0 \geq 0$ an integer. If $|\langle \rho, \chi^- \rangle - n_0| < \frac{1}{4}$ and $|\langle \rho, \chi^+ \rangle - n_0| < \frac{1}{4}$, then ρ has exactly n_0 states of spin s with dimension in I, and no further state of spin s with dimension in $\overline{I}$. For $n_0 = 0$ the condition on χ^+ alone suffices.

Proof. Let n^- and n^+ be the numbers of states of spin s with dimension in I and in $\overline{I}$. Then $n^- \geq \langle \rho, \chi^- \rangle > n_0 - \frac{1}{4}$ and $n^+ \leq \langle \rho, \chi^+ \rangle < n_0 + \frac{1}{4}$, while $n^- \leq n^+$ are integers. Hence $n^- = n^+ = n_0$. □

Definition 7.3 (Certified local exclusions)**.** Let $g \in (0, \frac{1}{2}]$ be rational. A *local exclusion datum* is a triple (Φ, O, α) consisting of a finite family Φ of rational tests, an open box $O = \prod_{\varphi \in \Phi} (o_\varphi^-, o_\varphi^+) \subset \mathbb{R}^\Phi$ with $o_\varphi^- < o_\varphi^+$ in $\mathbb{Q} \cup \{\pm\infty\}$, and a rational $\alpha > 0$. It is a *certified local exclusion for the gap g* if no admissible partition function of gap g at $c = 1 + 12a$ with $0 < a < \alpha$ satisfies $\Phi(\rho_a) \in O$. It *covers* $\sigma \in \mathcal{B}_g$ if $\Phi(\sigma) \in O$.

"Certified" refers to the truth of the exclusion, which is a statement about all admissible partition functions; it cannot be read off from the finite datum. By Lemma 7.2, a box can prescribe the number of states in each of finitely many windows and the emptiness of others. Such a box contains $\Phi(\sigma)$, with n_0 the number of states of σ in the window, whenever no state of σ of the spin in question lies in the support of $\chi^+ - \chi^-$; for then both pairings equal n_0. In this way the conclusions of Corollary 2.11, available along a sequence below an unknown threshold, become hypotheses on a single partition function.

7.2 A computable modulus for the approach to the boundary

The following elementary fact is the only verification device used by the searches of this section.

Lemma 7.4 (Certifying strict positivity)**.** *Let Q be a finite union of closed boxes $\prod_i [p_i, p_i']$ with rational $p_i \le p_i'$, possibly of different dimensions, and let $G : \mathsf{Q} \to \mathbb{R}$. Suppose we are given a rational λ with $|G(w) - G(w')| \le \lambda \|w - w'\|_\infty$ whenever w, w' lie in the same box, and an algorithm which, for a rational point $w \in \mathsf{Q}$ and an integer $n \ge 1$, returns a rational $G_n(w)$ with $|G_n(w) - G(w)| \le 2^{-n}$. Then there is a procedure, uniform in these data, which halts if and only if $G > 0$ everywhere on Q.*

Proof. Let ℓ be the largest side length of the boxes. At stage n, consider in each box the finitely many points whose coordinates are $p_i + k2^{-n}(p_i' - p_i)$, $0 \le k \le 2^n$; every point of the box is within $2^{-n}\ell$ of one of them. The procedure halts at stage n if $G_n(w) > (1 + \lambda\ell)2^{-n}$ at all these points w. If it halts, then for w' in a box and w a nearest such point, $G(w') \ge G_n(w) - 2^{-n} - \lambda\ell 2^{-n} > 0$. Conversely G is continuous, so if $G > 0$ on Q then $G \ge \mu$ for some $\mu > 0$, and the procedure halts as soon as $(2 + \lambda\ell)2^{-n} < \mu$. □

Proposition 7.5 (A computable modulus)**.** *There is an algorithm which, given a rational $g \in (0, \frac{1}{2}]$, a finite family Φ of rational tests and a rational $\gamma > 0$, halts and returns a rational $a_\gamma \in (0, g^2/4]$ with the following property: for every admissible partition function of gap g at $c = 1 + 12a$ with $0 < a \le a_\gamma$ there is $\sigma \in \mathcal{B}_g$ with $\|\Phi(\rho_a) - \Phi(\sigma)\|_\infty < \gamma$.*

Proof. Put $a_0 = g^2/4$ and $\mathsf{c}_g = 3^8(1 + \lceil 18 g^{-2} \rceil)$. For $j \ge 1$ let k_j be the largest integer $k \ge 0$ with $\mathsf{c}_g 2^k \le j$, or $k_j = 0$ if $j < \mathsf{c}_g$, and let $\lambda_j = \max\{g, k_j^2/256\}$. At most $\mathsf{c}_g 2^{k+1}$ indices j have $k_j = k$.

Step 1: a compact container. Let Z be admissible of gap g with $0 < a \le a_0$, and number its states so that $\Delta_1 \le \Delta_2 \le \cdots$. We claim that $\Delta_j \ge \lambda_j$. For $E \ge 1$, (2.22) with $y = E^{-1/2}$ gives

$$N_a(E) \le e^{2\pi y E} P_a(y) \le (1 + C_g)\sqrt{E}\, e^{2\pi(1+a)\sqrt{E}} \le (1 + C_g)\, e^{8\sqrt{E}},$$

because $a \le \frac{1}{16}$, $2\pi \cdot \frac{17}{16} < 7$ and $\sqrt{E} \le e^{\sqrt{E}}$; for $E < 1$ we use $N_a(E) \le N_a(1)$. Since $e^8 < 3^8$ and $C_g = 2e^{\pi/2} g^{-2} < 18 g^{-2}$, we get $N_a(E) \le \mathsf{c}_g e^{8\sqrt{E}}$ for all $E \ge 0$. Now $j \le N_a(\Delta_j)$, so $\mathsf{c}_g 2^k \le j$ implies $8\sqrt{\Delta_j} \ge k \log 2 \ge k/2$, that is, $\Delta_j \ge k^2/256$. Together with $\Delta_j \ge g$ this proves the claim.

Let $Y_j = \{\infty\} \cup \{(\Delta, s) \in \mathcal{Y} : \Delta \ge \lambda_j\}$, the one-point compactification of a closed subset of $\mathbb{R} \times \mathbb{Z}$, and let

$$\mathfrak{K}_g = [0, a_0] \times \prod_{j \ge 1} Y_j,$$

a compact metrizable space. We write its points as $\xi = (a, (w_j)_{j \ge 1})$ and call the entries $w_j \ne \infty$ the states of ξ. By the claim, every admissible Z of gap g with $0 < a \le a_0$ defines a point $\xi_Z \in \mathfrak{K}_g$, by listing its states in nondecreasing order of dimension and filling up with ∞ if there are finitely many. For a rational test φ put $\langle \xi, \varphi \rangle = \sum_{w_j \ne \infty} \varphi(w_j)$; only the finitely many j with $\lambda_j \le E_\varphi$ contribute, so this is a continuous function on $\mathfrak{K}_g$, and $\Phi(\xi_Z) = \Phi(\rho_a)$. For $\tau = x + iy \in \mathbb{H}$ put

$$\widehat{Z}(\tau; \xi) = \sqrt{y}\, e^{2\pi a y} \Big(\big|1 - e^{2\pi i \tau}\big|^2 + \sum_{w_j = (\Delta_j, s_j) \ne \infty} e^{2\pi i x s_j - 2\pi y \Delta_j} \Big), \qquad \mathsf{F}_\tau(\xi) = \widehat{Z}(\tau; \xi) - \widehat{Z}(-1/\tau; \xi).$$

The jth term is continuous on Y_j, with value 0 at ∞, and is bounded by $e^{-2\pi y \lambda_j}$; since $\sum_j e^{-\beta\lambda_j} \le \mathsf{c}_g \sum_{k \ge 0} 2^{k+1} e^{-\beta k^2/256} < \infty$ for every $\beta > 0$, the series converges uniformly on $\mathfrak{K}_g$,

locally uniformly in τ. Thus F_τ is continuous on $\mathfrak{K}_g$, and $\tau \mapsto \mathsf{F}_\tau(\xi)$ is continuous on $\mathbb{H}$. By (2.2), $\widehat{Z}(\tau;\xi_Z) = \widehat{Z}_a(\tau)$, so that

$$\mathsf{F}_\tau(\xi_Z) = 0 \qquad \text{for all } \tau \in \mathbb{H}. \tag{7.2}$$

Step 2: truncation. Let $k \geq 12$ be an integer, $E_k = k^2/256 \geq g$ and $J_k = \mathsf{c}_g 2^{k+1} - 1$, so that $\lambda_j \leq E_k$ exactly for $j \leq J_k$. Put

$$\mathfrak{K}_{g,k} = \prod_{j=1}^{J_k} Y_{j,k}, \qquad Y_{j,k} = \{*\} \cup \{(\Delta, s) \in \mathcal{Y} : \lambda_j \leq \Delta \leq E_k\}.$$

Each $Y_{j,k}$ is the union of an isolated point and finitely many closed intervals with rational endpoints, one for each admissible spin, so $[0,a_0] \times \mathfrak{K}_{g,k}$ is a finite union of boxes with rational vertices. Let $\iota_k : [0,a_0] \times \mathfrak{K}_{g,k} \to \mathfrak{K}_g$ replace $*$ by ∞ and put $w_j = \infty$ for $j > J_k$, and let $\pi_k : \mathfrak{K}_g \to [0,a_0] \times \mathfrak{K}_{g,k}$ keep a and the states w_j, $j \leq J_k$, with $\Delta_j \leq E_k$, replacing the other entries by $*$. A state of ξ with $\Delta_j \leq E_k$ has $\lambda_j \leq E_k$, hence $j \leq J_k$; so $\iota_k\pi_k\xi$ is obtained from ξ by deleting exactly the states with $\Delta_j > E_k$. Consequently, if $E_k \geq E_\Phi$ then

$$\Phi(\iota_k\pi_k\xi) = \Phi(\xi). \tag{7.3}$$

Among the deleted states, fewer than $\mathsf{c}_g 2^{k+1}$ have $j \leq J_k$, each with $\Delta_j > k^2/256$, and those with $j > J_k$ have $\Delta_j \geq k_j^2/256$ with $k_j > k$. Therefore, with $\vartheta(\beta,k) = \sum_{k' \geq k} 2^{k'+1} e^{-\beta k'^2/256}$ and $y' = \operatorname{Im}(-1/\tau)$,

$$|\mathsf{F}_\tau(\xi) - \mathsf{F}_\tau(\iota_k\pi_k\xi)| \leq \mathsf{c}_g\Big(\sqrt{y}\, e^{2\pi a_0 y}\vartheta(2\pi y,k) + \sqrt{y'}\, e^{2\pi a_0 y'}\vartheta(2\pi y',k)\Big). \tag{7.4}$$

The right side is a computable real number which tends to zero as $k \to \infty$.

Step 3: certificates. Fix enumerations $\tau_1, \tau_2, \ldots$ of $\mathbb{Q}(i) \cap \mathbb{H}$ and $t_1, t_2, \ldots$ of $\mathbb{Q} \cap [1, r_g]$; membership in the latter set is the decidable condition $t \geq 1$, $2gt^2 \leq 1$. For $i \leq k$ compute a rational $\varepsilon_{i,k}$ which is at least the right side of (7.4) at $\tau = \tau_i$ and exceeds it by at most 2^{-k}; thus $\varepsilon_{i,k} \to 0$ as $k \to \infty$. Let $k_\Phi = \max\{12, \lceil 16\sqrt{E_\Phi}\,\rceil\}$, so that $E_k \geq E_\Phi$ for $k \geq k_\Phi$. For integers $k \geq k_\Phi$ and l with $2^{-l} \leq a_0$ define, on $\mathsf{Q}_{l,k} = [0, 2^{-l}] \times \mathfrak{K}_{g,k}$,

$$G_{l,k}(a,w) = \max\Big\{\gamma - \min_{R,\, i \leq k} \|\Phi(\iota_k(a,w)) - \Phi(\sigma(R,t_i))\|_\infty,\ \max_{i \leq k}\big(|\mathsf{F}_{\tau_i}(\iota_k(a,w))| - \varepsilon_{i,k}\big)\Big\},$$

where R ranges over the half-integers in $[1, r_g]$. We claim: *if $G_{l,k} > 0$ on $\mathsf{Q}_{l,k}$, then $a_\gamma = 2^{-l}$ has the property stated in the proposition.* Indeed, let Z be admissible of gap g with $0 < a \leq 2^{-l}$ and put $(a,w) = \pi_k\xi_Z \in \mathsf{Q}_{l,k}$. By (7.2) and (7.4), $|\mathsf{F}_{\tau_i}(\iota_k(a,w))| \leq \varepsilon_{i,k}$ for all $i \leq k$, so the first entry of the maximum is positive: there are R and i with $\|\Phi(\iota_k(a,w)) - \Phi(\sigma(R,t_i))\|_\infty < \gamma$. By (7.3) the first vector is $\Phi(\rho_a)$, and $\sigma(R,t_i) \in \mathcal{B}_g$.

Step 4: a certificate exists. Fix l_0 with $2^{-l_0} \leq a_0$. We show that $G_{l_0+k,k} > 0$ on $\mathsf{Q}_{l_0+k,k}$ for some $k \geq k_\Phi$. Suppose not, and choose for every $k \geq k_\Phi$ a point (a_k, w_k) with $G_{l_0+k,k}(a_k,w_k) \leq 0$. Let $\xi_k = \iota_k(a_k,w_k)$. By compactness of $\mathfrak{K}_g$ a subsequence converges to some $\xi_\infty = (a_\infty, (w_j))$. Since $a_k \leq 2^{-l_0-k}$, we have $a_\infty = 0$. Since $|\mathsf{F}_{\tau_i}(\xi_k)| \leq \varepsilon_{i,k}$ for $k \geq i$ and $\varepsilon_{i,k} \to 0$, continuity gives $\mathsf{F}_{\tau_i}(\xi_\infty) = 0$ for every i, hence $\mathsf{F}_\tau(\xi_\infty) = 0$ for all $\tau \in \mathbb{H}$. Since $\|\Phi(\xi_k) - \Phi(\sigma(R,t_i))\|_\infty \geq \gamma$ for $k \geq i$, continuity in ξ and in t and density of the t_i give

$$\|\Phi(\xi_\infty) - \Phi(\sigma(R,t))\|_\infty \geq \gamma \qquad \text{for all } (R,t) \in \Pi_g. \tag{7.5}$$

On the other hand, let $(h_j, \bar{h}_j)$ be the weights of the states of ξ_∞ and $H_0 = (1-q)(1-\bar{q}) + \sum_j q^{h_j}\bar{q}^{\bar{h}_j}$. Then $h_j, \bar{h}_j \geq 0$, the spins are integers, $\Delta_j \geq \lambda_j \geq g$, and the sum converges on

$\mathbb{H}$, so H_0 has the form (3.1). The function $Z_0 = H_0/|\eta|^2$ is invariant under $\tau \mapsto \tau+1$ because the spins are integers, and under $\tau \mapsto -1/\tau$ because $\sqrt{y}\,H_0(\tau) = \widehat{Z}(\tau;\xi_\infty)$ and $\mathsf{F}_\tau(\xi_\infty) = 0$. The proofs of Lemma 2.4 at $a = 0$, of Lemmas 3.3 and 3.4, and of Theorem 3.5(i) use no property of a limit other than these. Hence the state measure of ξ_∞ equals $\sigma(R,t)$ for some $(R,t) \in \Pi_g$, which contradicts (7.5).

Step 5: the search. On each box of $\mathsf{Q}_{l,k}$ the function $G_{l,k}$ is obtained by finitely many operations max, min, $|\cdot|$ from rational constants, from the piecewise polynomial functions $\langle \iota_k(a,w), \varphi\rangle$, from the rational numbers $\langle \sigma(R,t_i), \varphi\rangle$ given by (7.1), and from the finite sums $\mathsf{F}_{\tau_i}(\iota_k(a,w))$ of exponentials. A Lipschitz constant and approximate values at rational points are therefore computable, and Lemma 7.4 applies. The algorithm runs the procedures of that lemma for all pairs (l,k) with $2^{-l} \le a_0$ and $k \ge k_\Phi$ in an interleaved fashion and returns $a_\gamma = 2^{-l}$ for the first pair at which one of them halts. It halts by Step 4 and its output is correct by Step 3. □

Remark 7.6. The algorithm of Proposition 7.5 is an unbounded search, and "computable" has only its recursion-theoretic meaning. The correctness of the output (Step 3) is elementary. That the search halts (Step 4) is proved by compactness together with Theorem 3.5, and therefore rests on the reduction quoted from [2] in Proposition 3.1. The proof provides no bound for the halting time and no lower bound for a_γ in terms of g, Φ and γ; the dimension of $\mathsf{Q}_{l,k}$ is of order $\mathsf{c}_g 2^k$. The proposition concerns the approach to the whole of $\mathcal{B}_g$ and cannot be improved in this respect by the same method: every element of $\mathcal{B}_g$ is a common zero of the functions F_τ at $a = 0$.

7.3 Gluing certified local exclusions

Theorem 7.7 (Conditional threshold)**.** *There is an algorithm $\mathcal{A}$ with the following property. Its input is a rational $g \in (0, \frac{1}{2}]$ together with a sequence $(\Phi_i, O_i, \alpha_i)_{i\ge1}$ of local exclusion data, presented by a program, or an oracle, that returns the ith datum on input i. Suppose that*

(i) every (Φ_i, O_i, α_i) is a certified local exclusion for the gap g, and

(ii) every $\sigma \in \mathcal{B}_g$ is covered by at least one of them: there is i with $\Phi_i(\sigma) \in O_i$.

Then $\mathcal{A}$ halts, and its output is a rational $\epsilon(g) > 0$ such that no admissible partition function of gap g has central charge in $(1, 1+\epsilon(g))$.

Proof. For a box O as in Definition 7.3 and $v \in \mathbb{R}^\Phi$ let

$$\operatorname{depth}_O(v) = \min\{1,\ \min_\varphi (v_\varphi - o^-_\varphi),\ \min_\varphi (o^+_\varphi - v_\varphi)\}.$$

This function is 1-Lipschitz for $\|\cdot\|_\infty$, and $v \in O$ if and only if $\operatorname{depth}_O(v) > 0$.

First search. For an integer $i_0 \ge 1$, a rational $\gamma \in (0,1)$ and a rational $\bar r$ with $2g\bar r^2 > 1$, that is, $\bar r > r_g$, consider the function

$$G(R,t) = \max_{i \le i_0} \operatorname{depth}_{O_i}\big(\Phi_i(\sigma(R,t))\big) - \gamma$$

on the finite union of the intervals $\{R\} \times [1, \bar r]$, R a half-integer in $[1, r_g]$. By the discussion following (7.1) it has a computable Lipschitz constant and computable values at rational t. The algorithm interleaves the procedures of Lemma 7.4 for all triples $(i_0, \gamma, \bar r)$ and stops at the first triple for which $G > 0$ is certified. Such a triple exists. Indeed, by (ii) the open sets $\{(R,t) \in \Pi_g : \Phi_i(\sigma(R,t)) \in O_i\}$ cover the compact set Π_g; hence for some i_0 the continuous function $\max_{i\le i_0} \operatorname{depth}_{O_i}(\Phi_i(\sigma(R,t)))$ is positive on Π_g, so it exceeds some rational 2γ there,

and by uniform continuity it exceeds γ on $\{R\} \times [1, \bar{r}]$ for every R once $\bar{r}$ is close enough to r_g.

Second search. Let Φ be the disjoint union of $\Phi_1, \dots, \Phi_{i_0}$, apply the algorithm of Proposition 7.5 to (g, Φ, γ), and let a_γ be its output. The algorithm $\mathcal{A}$ returns

$$\epsilon(g) = 12 \min\{a_\gamma, \alpha_1, \dots, \alpha_{i_0}\}.$$

Correctness. Suppose Z is admissible of gap g at $c = 1 + 12a$ with $0 < 12a < \epsilon(g)$. Since $a \le a_\gamma$, there is $\sigma = \sigma(R,t)$ with $(R,t) \in \Pi_g$ and $\|\Phi(\rho_a) - \Phi(\sigma)\|_\infty < \gamma$. By the first search there is $i \le i_0$ with $\operatorname{depth}_{O_i}(\Phi_i(\sigma)) > \gamma$. As $\|\Phi_i(\rho_a) - \Phi_i(\sigma)\|_\infty < \gamma$ and the depth is 1-Lipschitz, $\operatorname{depth}_{O_i}(\Phi_i(\rho_a)) > 0$, that is, $\Phi_i(\rho_a) \in O_i$. Since $a < \alpha_i$, this contradicts (i). □

Remark 7.8. For an arbitrary real $g > 0$ one applies the theorem to a rational $g' < \min\{g, \frac{1}{2}\}$: an admissible partition function of gap g is admissible of gap g' (compare Remark 2.2). Rationality of the gap is used only so that the input and the conditions $2gR^2 \le 1$, $2gt^2 \le 1$ are handled exactly.

Remark 7.9. For a single fixed g, the existence of an algorithm that prints a valid threshold is a trivial consequence of Theorem A; likewise (i) and (ii) can be met trivially once a valid threshold is known, by a single datum whose box is all of $\mathbb{R}^\Phi$. The content of Theorem 7.7 is that one explicitly described algorithm, with g as input, converts local information near each point of $\mathcal{B}_g$ into a threshold, and that the validity of its output is established without Theorem A: it uses hypothesis (i) and Step 3 of the proof of Proposition 7.5, while the classification enters only in the proof that $\mathcal{A}$ halts. The algorithm verifies (ii) for the data it uses, with a margin; it does not and cannot verify (i). If (ii) fails it does not halt. If (ii) holds and (i) fails it halts, and its output need not be valid.

7.4 A smallness cutoff

Once an estimate has an explicit constant, the absorption of logarithms into a power can be made explicit as well.

Lemma 7.10 (A rational smallness cutoff)**.** *Let $p > 0$, $C \ge 1$ and $0 < \gamma \le 1$ be rational, let $b \ge 0$ be an integer, and let k be an integer with*

$$k \ge \max\Big\{2,\ \frac{8b^2}{p^2},\ \frac{4C2^b}{p\gamma}\Big\}.$$

Then $C\,x^p\,(1 + \log(1/x))^b < \gamma$ for every $0 < x \le 2^{-k}$.

Proof. Let $\ell = \log(1/x) \ge k \log 2 \ge k/2 \ge 1$. Since $\log \ell \le \sqrt{\ell}$ and $\ell \ge 4b^2/p^2$,

$$(1 + \ell)^b \le (2\ell)^b \le 2^b e^{b\sqrt{\ell}} \le 2^b e^{p\ell/2}.$$

With $\zeta = C2^b/\gamma$ we have $p\ell/2 \ge pk/4 \ge \zeta$, so

$$C\,x^p(1+\ell)^b = C\,e^{-p\ell}(1+\ell)^b \le C2^b e^{-p\ell/2} \le C2^b e^{-\zeta} = \gamma\,\zeta e^{-\zeta} < \gamma. \qquad \square$$

A finite list of such inequalities is satisfied below 2^{-k} for the largest of the corresponding integers k. The third entry of the maximum is wasteful: the proof shows that any integer exceeding $(4/p)\log(C2^b/\gamma)$ may take its place. The form stated avoids logarithms of the data.

7.5 Computable origin certificates

We turn to source (c). Fix integers $d, q \geq 1$. Let $\mathcal{R}_d$ be the set of radii of Lemma 4.12, $\Sigma_0 = \bigcup_{r\in\mathcal{R}_d} \Lambda_r \subset d^{-1}\Lambda_1$, and

$$\Sigma_d(t) = d^{-1}\Lambda_1 \cup \Lambda_t \cup \Lambda_{1/t} \supset \Sigma_0 \cup \Lambda_t \cup \Lambda_{1/t}, \qquad t > 0;$$

in light-cone coordinates $d^{-1}\Lambda_1 = d^{-1}\mathbb{Z}^2$. The single grid $d^{-1}\Lambda_1$ is a transverse family adapted to $\mathcal{R}_d$, with phase exponent $\nu = d^2$ (Definition 4.8). We use the conventions of Section 4 for translations τ_w, multi-indices and the sets $\mathcal{A}_q$, write $s(X) = P^2 - Q^2$, $e_P = (1,0)$, $e_Q = (0,1)$, and denote multi-indices by $\beta = (\beta_P, \beta_Q)$ and distributions by T.

The phase rows (4.22) used in Section 4 have coefficients involving powers of πi. They can be replaced by equations with rational coefficients.

Lemma 7.11 (The local phase equation)**.** *Let $x = (x_P, x_Q) \in \mathbb{R}^2$, let $\mathsf{T}_x = \sum_{|\beta|\leq q} c_\beta\, \partial^\beta \delta_x$, and put $\varpi_x = s - s(x)$. If ν is a nonzero integer with $\nu s(x) \in \mathbb{Z}$, then $\mathcal{C}^\nu \mathsf{T}_x = \mathsf{T}_x$ if and only if $\varpi_x \mathsf{T}_x = 0$. Moreover, with $c_{\beta'} = 0$ for $|\beta'| > q$,*

$$\begin{aligned}\varpi_x \mathsf{T}_x = \sum_{|\beta|\leq q-1} \big[&-2x_P(\beta_P+1)c_{\beta+e_P} + (\beta_P+2)(\beta_P+1)c_{\beta+2e_P} \\ &+ 2x_Q(\beta_Q+1)c_{\beta+e_Q} - (\beta_Q+2)(\beta_Q+1)c_{\beta+2e_Q}\big]\partial^\beta\delta_x.\end{aligned} \tag{7.6}$$

The coefficients of this equation are polynomials over $\mathbb{Q}$ in the coordinates of x; no power of π occurs.

Proof. Let V_q be the space of distributions supported at x of order at most q. For a smooth function f, $f\,\partial^\beta\delta_x = \sum_{\beta'\leq\beta}(-1)^{|\beta-\beta'|}\binom{\beta}{\beta'}(\partial^{\beta-\beta'}f)(x)\,\partial^{\beta'}\delta_x$. Hence multiplication by f on V_q depends only on the Taylor polynomial of f of degree q at x, and multiplication by a function vanishing at x lowers the order. In particular the operator m of multiplication by ϖ_x satisfies $\mathsf{m}^{q+1} = 0$ on V_q. Write $e^{2\pi i\nu z} - 1 = 2\pi i\nu z\,\mathsf{E}(z)$ with E entire and $\mathsf{E}(0) = 1$. Since $e^{2\pi i\nu s(x)} = 1$, on V_q

$$\mathcal{C}^\nu - 1 = e^{2\pi i\nu\varpi_x} - 1 = 2\pi i\nu\,\mathsf{m}\,\mathsf{E}(\mathsf{m}), \qquad \mathsf{E}(\mathsf{m}) = \sum_{k\leq q} \frac{\mathsf{E}^{(k)}(0)}{k!}\,\mathsf{m}^k,$$

and $\mathsf{E}(\mathsf{m})$ is the identity plus a nilpotent operator, hence invertible, and commutes with m. This proves the equivalence. For the formula, write $\varpi_x = 2x_P(P-x_P) + (P-x_P)^2 - 2x_Q(Q-x_Q) - (Q-x_Q)^2$ and use $(P-x_P)^k\partial^\beta\delta_x = (-1)^k\frac{\beta_P!}{(\beta_P-k)!}\,\partial^{\beta-ke_P}\delta_x$ for $k \leq \beta_P$, and zero otherwise, together with the same identity in Q. □

Labels and rows. We recall the set-up of Steps 1–3 of the proof of Proposition 4.20 in the present case, in which all data are rational. The classes are $\{0,+,-\}$, with $\Gamma_0(t) = d^{-1}\Lambda_1$, $\Gamma_+(t) = \Lambda_t$, $\Gamma_-(t) = \Lambda_{1/t}$. A *Laurent vector* is an element of $\mathbb{Q}[t,t^{-1}]^2$, regarded as a map $(0,\infty) \to \mathbb{R}^2$. The labels are $\mathfrak{L} = \{0\} \sqcup (\{0,+,-\} \times (\mathbb{Z}^2 \setminus \{0\}))$, with the Laurent vectors, in light-cone coordinates,

$$x_0 = 0, \qquad x_{0,m,n} = (m/d,\, n/d), \qquad x_{+,m,n} = (mt,\, n/t), \qquad x_{-,m,n} = (m/t,\, nt).$$

For every $t > 0$ the map $\lambda \mapsto x_\lambda(t)$ sends $\mathfrak{L}$ onto $\Sigma_d(t)$. A *dual vector* of the class i is a Laurent vector $x_{i,m,n}$ with $(m,n) \neq 0$ and, for $i = 0$, $m, n \in d^2\mathbb{Z}$; its values lie in $\Gamma_i(t)^\vee \setminus \{0\}$. A *label array* is an element $\hat{c} \in \mathbb{C}^{\mathfrak{L}\times\mathcal{A}_q}$, and a *row* is a finitely supported function on $\mathfrak{L} \times \mathcal{A}_q$ with values in $\mathbb{Q}[t,t^{-1}]$; for each $t > 0$ it acts on label arrays. We consider two kinds of rows.

(1) *Translation rows.* Let $\mathbf{w} = (w_0, w_+, w_-)$ consist of dual vectors of the three classes, let $k \geq q+1$ be an integer, Y a Laurent vector and $\beta \in \mathcal{A}_q$. Put

$$\mathsf{r}_{\mathbf{w},k,Y,\beta}(\hat{c}) = \sum_{\kappa \in \{0,\dots,k\}^3} \prod_i (-1)^{k-\kappa_i} \binom{k}{\kappa_i} \sum_{\substack{\lambda \in \mathfrak{L} \\ x_\lambda \equiv Y + \kappa_0 w_0 + \kappa_+ w_+ + \kappa_- w_-}} \hat{c}_\beta(\lambda), \tag{7.7}$$

where $\equiv$ is equality of Laurent vectors; the inner sum has at most one term. This is the generic row of Step 3 of the proof of Proposition 4.20 for the operator $\mathsf{D}_{\mathbf{w},k}(t) = \prod_i (\tau_{w_i(t)} - I)^k$.

(2) *Rational phase rows.* For $\lambda \in \mathfrak{L}$ and $|\beta| \leq q-1$, the bracket in (7.6), formed with $c_{\beta'} = \hat{c}_{\beta'}(\lambda)$ and $x = x_\lambda$.

Let $\mathcal{R}_{d,q}$ be the set of all these rows. Equality in $\mathbb{Q}[t, t^{-1}]$ is decidable, so each row is computable from its parameters, and $\mathcal{R}_{d,q}$ can be enumerated as $\mathsf{r}_1, \mathsf{r}_2, \dots$.

Proposition 7.12 (A terminating search for the row identities)**.** *For every β with $1 \leq |\beta| \leq q$ the coordinate row $\hat{c} \mapsto \hat{c}_\beta(0)$ is a $\mathbb{Q}(t)$-linear combination of finitely many rows of $\mathcal{R}_{d,q}$. Consequently the search which, for $i = 1, 2, \dots$, decides by exact elimination over $\mathbb{Q}(t)$ whether all these coordinate rows lie in the span of $\mathsf{r}_1, \dots, \mathsf{r}_i$ halts, and returns i_0 and $a_{\beta,i} \in \mathbb{Q}(t)$ such that*

$$\hat{c}_\beta(0) = \sum_{i \leq i_0} a_{\beta,i}(t)\, \mathsf{r}_i(\hat{c}) \qquad (1 \leq |\beta| \leq q) \tag{7.8}$$

for all label arrays $\hat{c}$ and all $t > 0$ that are not poles of the $a_{\beta,i}$.

Proof. Fix a real $t_* > 0$ transcendental over $\mathbb{Q}$. Evaluation at t_* is injective on $\mathbb{Q}[t, t^{-1}]$, so two Laurent vectors that agree at t_* agree identically. Hence $\lambda \mapsto x_\lambda(t_*)$ is a bijection of $\mathfrak{L}$ onto $\Sigma_d(t_*)$, and label arrays are the same as arrays on $\Sigma_d(t_*) \times \mathcal{A}_q$. The radii 1, t_*, $1/t_*$ are pairwise incommensurable, so the three lattices $\Gamma_i(t_*)$ form a transverse family of hyperbolic grids with support $\Sigma_d(t_*)$ and phase exponent d^2. Lemma 4.15 provides finitely many operators $\mathsf{D}_l = \prod_i (\tau_{w_{i,l}(t_*)} - I)^{m_l}$, with dual vectors $w_{i,l}$ and $m_l \geq q+1$, such that every array c on $\Sigma_d(t_*)$, of arbitrary growth, whose distribution T_c satisfies $\mathsf{D}_l \mathsf{T}_c = 0$ for all l and $\mathcal{C}^{d^2}\mathsf{T}_c = \mathsf{T}_c$ has $c_\beta(0) = 0$ for $1 \leq |\beta| \leq q$. As in Step 4 of the proof of Proposition 4.20, the coefficient (4.21) of $\partial^\beta \delta_y$ in $\mathsf{D}_l \mathsf{T}_c$ is either zero for all c or the value at t_* of a row (7.7) with $Y(t_*) = y$. Since $d^2 s$ is integer valued on $\Sigma_d(t_*)$, Lemma 7.11 shows that $\mathcal{C}^{d^2}\mathsf{T}_c = \mathsf{T}_c$ is equivalent to the vanishing on c of all rational phase rows at t_*. So $c \mapsto c_\beta(0)$ vanishes on every array annihilated by all rows of $\mathcal{R}_{d,q}$ evaluated at t_*, and by Lemma 4.16 it is a complex linear combination of finitely many of them, say of the values at t_* of $\mathsf{r}_1, \dots, \mathsf{r}_i$.

Let M be the matrix of $\mathsf{r}_1, \dots, \mathsf{r}_i$ in the finitely many columns in which these rows or the coordinate row have a nonzero entry, and M' the matrix with the coordinate row adjoined. Their entries lie in $\mathbb{Q}[t, t^{-1}]$, so a minor vanishes if and only if its value at t_* does. Hence the ranks of M and M' over $\mathbb{Q}(t)$ equal the ranks over $\mathbb{C}$ of their values at t_*, which coincide. Therefore the coordinate row lies in the $\mathbb{Q}(t)$-span of $\mathsf{r}_1, \dots, \mathsf{r}_i$, and the search halts at the first index for which this holds for all β. An identity between rows with coefficients in $\mathbb{Q}(t)$ may be evaluated at every $t > 0$ that is not a pole of a coefficient. □

The number t_* occurs only in the proof that the search halts; the search itself is exact linear algebra over $\mathbb{Q}(t)$. The identity (7.8) can take the place of (4.32) in the proof of Proposition 4.20: by Lemma 7.11 the rational phase rows vanish, like the rows $\mathsf{p}^t_{\lambda,\beta}$ used there, on the label array (4.29) of every distribution T supported in $\Sigma_d(t)$, of local order at most q, with $\mathcal{C}^{d^2}\mathsf{T} = \mathsf{T}$. The next proposition makes the remaining data of that proof computable at an algebraic parameter.

Proposition 7.13 (Algebraic parameters)**.** *Let $t_0 > 0$ be a real algebraic number.*

(i) One can compute a transverse family $\Gamma_i = d_i^{-1}\Lambda_{r_i}$, with $r_i \in \mathbb{Q}(t_0)$, whose support S_0 contains $\Sigma_d(t_0)$. The search analogous to that of Proposition 7.12, over the translation rows (4.21) of all operators $\prod_i(\tau_{w_i} - I)^k$ with $w_i \in \Gamma_i^\vee \setminus \{0\}$ and $k \geq q+1$, and the rational phase rows at the points of S_0, with exact arithmetic in $\mathbb{Q}(t_0)$, halts and returns a row identity (4.24) for S_0 with coefficients in $\mathbb{Q}(t_0)$, in which the phase rows are the rational ones.

(ii) Given (7.8), one can compute a rational $\varrho \in (0, t_0/2)$, an integer $K \geq 0$ and a rational C_1 with the following properties. Let r_μ range over the translation rows occurring in (7.8), with coefficients a_μ, and let $\mathfrak{d}_\mu$ be the function defined for r_μ in Step 6 of the proof of Proposition 4.20. For $0 < |t - t_0| < \varrho$ the number t is neither a pole of a coefficient of (7.8) nor a zero of a function $\mathfrak{d}_\mu$, and

$$|a_\mu(t)|\,|\mathfrak{d}_\mu(t)|^{-(q+1)^2} \leq C_1\,|t - t_0|^{-K}.$$

Consequently assertions (ii) and (iii) of Proposition 4.20 hold, for distributions supported in $\Sigma_d(t)$, with $\varepsilon_0 = \varrho$, with this K, and with the tests $f_{\mu,t} = a_\mu(t)\,\omega_\mu(t)\,\mathbb{S}\psi_{\mu,t}$ formed from (7.8) as in Step 7 of that proof, $\omega_\mu(t)$ being the multiplier of the translation operator of r_μ.

Proof. (i) Equality and integrality of elements of $\mathbb{Q}(t_0)$ are decidable, hence so are the rationality of t_0 and of t_0^2 and the membership of a point with coordinates in $\mathbb{Q}(t_0)$ in a lattice $d_i^{-1}\Lambda_{r_i}$. If $t_0 = p_1/p_2$ is rational, take the single grid $\operatorname{lcm}(d, p_1p_2)^{-1}\Lambda_1$. If t_0 is irrational and $t_0^2 = p_1/p_2$, take $d^{-1}\Lambda_1$ and $(p_1p_2)^{-1}\Lambda_{t_0}$; the latter contains Λ_{1/t_0} because $(m/t_0, nt_0) = (mp_2^2\,t_0/(p_1p_2),\, np_1^2/(p_1p_2t_0))$. If t_0^2 is irrational, take $d^{-1}\Lambda_1$, Λ_{t_0} and Λ_{1/t_0}. In each case the radii are pairwise incommensurable. The rows have entries in $\mathbb{Q}(t_0)$ and are computable. As in the proof of Proposition 7.12, Lemmas 4.15, 7.11 and 4.16 show that each coordinate row is a complex combination of finitely many of them, and the rank of a matrix with entries in $\mathbb{Q}(t_0)$ is the same over $\mathbb{Q}(t_0)$ and over $\mathbb{C}$.

(ii) Let $\mathsf{r}_\mu = \mathsf{r}_{\mathbf{w},k,Y,\beta}$ be one of the finitely many translation rows in (7.8). In Step 6 of the proof of Proposition 4.20 the function $\mathfrak{d}_\mu(t)$ is the product of the factors $e^{2\pi i g(t)} - 1$ over the functions $g = \mathsf{v}_{i,\mathbf{k}}^\star$ attached to the pairs $(i, \mathbf{k})$ of type two; each g is the value of one of the coordinate forms (du, dv), $(u/t, tv)$, $(tu, v/t)$ on the Laurent vector $Y + \sum_i k_i w_i$. Thus $g \in \mathbb{Q}[t, t^{-1}]$, and g is not an integer constant. Types and the choices $\iota(i, \mathbf{k})$ are decided by exact computation in $\mathbb{Q}[t, t^{-1}]$. On $[t_0/2, 2t_0]$ each g is bounded by a computable constant, so $g(t) \in \mathbb{Z}$ holds there only at the roots of the finitely many nonzero polynomial equations $g(t) = m$, m an integer below that bound. These roots, together with the poles of the $a_{\beta,i}$, form a finite set of real algebraic numbers whose isolating intervals can be computed and whose equality with t_0 can be decided. Choose ϱ smaller than the distance from t_0 to the other elements of this set. For each factor of $\mathfrak{d}_\mu$, either $g(t_0) \notin \mathbb{Z}$, and then $|e^{2\pi i g(t)} - 1| \geq 4\,\operatorname{dist}(g(t), \mathbb{Z})$ has a computable positive lower bound near t_0; or $g(t_0) = m \in \mathbb{Z}$, and then $g(t) - m = (t - t_0)^{k'} e(t)$ with a computable $k' \geq 1$ and a rational function e over $\mathbb{Q}(t_0)$ with $e(t_0) \neq 0$, so that $|e^{2\pi i g(t)} - 1| \geq 4|g(t) - m| \geq 2|e(t_0)|\,|t - t_0|^{k'}$ after decreasing ϱ by means of a bound for e'. The poles of the a_μ at t_0 are treated in the same way. This gives the displayed bound, with K computed from the orders of vanishing as in Step 9 of that proof. For $0 < |t - t_0| < \varrho$ the number t avoids the sets E' and E'' of Steps 5 and 6, and Steps 6–9 apply without change, with (7.8) in place of (4.32); they use only that the translation vectors are dual vectors, that the exponents are at least $q+1$, and that the phase rows vanish on the label arrays of the distributions considered. □

With these data the obstruction of Theorem 4.23 acquires a computable threshold, relative to bounds for three families of constants that are fixed by their proofs but are not displayed in this paper: the constants C_k of Lemma 2.6, the constants of Lemma 4.17(ii) and (iii), and the constant of Lemma 4.18(ii). Each is a function of finitely many integer and rational arguments. By a *system of bounds* $\mathfrak{b}$ we mean computable functions of these arguments that are upper bounds for the three families.

Proposition 7.14 (A computable obstruction to localization)**.** *Let $\mathfrak{b}$ be a system of bounds. There is an algorithm, using $\mathfrak{b}$, which, given a rational $g \in (0, \frac{1}{2}]$, an integer $d \geq 1$, a real algebraic $t_0 > 0$, a rational $p > 0$, a rational $C \geq 1$ and an integer $A \geq 0$, returns rationals $\varrho > 0$ and $\alpha > 0$ with the following property. There do not exist an admissible partition function of gap g at $c = 1 + 12a$ with $0 < a < \alpha$ and a real t with $|t - t_0| < \varrho$ such that every state with $\Delta_j \leq 2L$, $L = \log(1/a)$, admits a real point $X_j \in \Sigma_d(t)$ with*

$$s(X_j) = s_j, \qquad \big|\Delta_j - |X_j|^2\big| \leq C\, a^p (1 + L)^A. \tag{7.9}$$

Proof. Certificates. Let $p' = \min\{p, 1\}$, let N be the least integer with $(N + 1)p' > 2$, and $q = 2N$. Run the search of Proposition 7.12 and compute ϱ and K by Proposition 7.13(ii). By Step 9 of the proof of Proposition 4.20, the bounds of Proposition 7.13(ii) and $\mathfrak{b}$, one computes a rational M_1 such that the tests $f_{\mu,t}$, say n in number, satisfy (4.38) with $2N + 2$ derivatives, $b = 1$ and $M = M_1|t-t_0|^{-K}$, for $0 < |t-t_0| < \varrho$. Put $\vartheta = \min\{p, 1, 1/(8(K+1))\}$, let N_0 be the least integer with $(N_0 + 1)\vartheta > 2$, and apply Proposition 7.13(i) with $q_0 = 2N_0$ in place of q. As in the proof of Theorem 4.19, its row identity yields finitely many tests f_l^0 and a constant C_0 such that (4.23) holds for the even tempered distributions supported in $S_0 \supset \Sigma_d(t_0)$, of local order at most q_0, that are fixed by $\mathcal{C}$; here C_0 is a rational bound for the sums of the absolute values of the coefficients, and by $\mathfrak{b}$ one computes a rational M_0 such that the f_l^0 satisfy (4.38) with $2N_0 + 2$ derivatives, $b = 1$ and $M = M_0$. The denominators that occur are numbers $e^{2\pi i g} - 1$ with $g \in \mathbb{Q}(t_0) \setminus \mathbb{Z}$, and are bounded below by $4\,\mathrm{dist}(g, \mathbb{Z})$.

Preliminary bounds. Let Z, a, t and the points X_j be as in the statement, with $a \leq \min\{g^2/4, 1/64\}$, put $\vartheta_a = Ca^p(1 + L)^A$, and assume $\vartheta_a < g$. Since $aL \leq \frac{1}{2}$ and $2L \geq 1$, (2.23) gives $\#\{j : \Delta_j \leq 2L\} \leq 2e^{2\pi} A_g(1 + L)$. Since $2\pi a \leq \frac{1}{8}$, (2.24) with $\beta = 1$ and $E_0 = 2L$ gives

$$\sum_{\Delta_j > 2L} e^{-\Delta_j} \leq A_g\Big(\frac{16L}{7} + \frac{64}{49}\Big) a^{2-4\pi a} \leq 3A_g(1 + L)\, a^{7/4}.$$

The set $\Sigma_d(t)$ is invariant under both reflections and contains Λ_1.

First regime: $|t-t_0| \geq a^\vartheta$. Let T_a be the distribution (4.37) for $\Sigma = \Sigma_d(t)$, $E_* = 2L$, $\varepsilon = \vartheta_a$ and the order N. By Lemma 4.21(i),(ii), by (4.26) and by (4.39) with $M = M_1|t - t_0|^{-K} \leq M_1 a^{-K\vartheta}$,

$$\frac{a}{4} \leq n\, C_N e^5 M_1\, a^{-K\vartheta}\Big[2e^{2\pi} A_g(1 + L)\big((C + 1)a^{p'}(1 + L)^A\big)^{N+1} + a^{N+1} + 3A_g(1 + L)a^{7/4}\Big],$$

where C_N is the constant of Lemma 4.21, bounded through $\mathfrak{b}$. As $(N+1)p' > 2$ and $K\vartheta < \frac{1}{8}$, every power of a on the right is at most $a^{13/8}$. Hence $1 \leq C_1'\, a^{5/8}(1 + L)^{1+A(N+1)}$ with a computable rational C_1'.

Second regime: $|t - t_0| < a^\vartheta$. As in the second case of the proof of Theorem 4.23, replace each X_j on $\Lambda_t \cup \Lambda_{1/t}$ by the point X_j' of $\Lambda_{t_0} \cup \Lambda_{1/t_0}$ with the same charges, and put $X_j' = X_j$ otherwise. The spin is unchanged, and since $t, t_0 \geq t_0/2$ and $|X_j|^2 \leq \Delta_j + \vartheta_a \leq 2L + 1$, the estimate given there yields $\big||X_j'|^2 - |X_j|^2\big| \leq (2L + 1)(e^{4a^\vartheta/t_0} - 1) \leq C_2(1 + L)a^\vartheta$ with a computable C_2, provided $4a^\vartheta \leq t_0$. Thus (7.9) holds for the points $X_j' \in \Sigma_d(t_0)$ with the right side $\vartheta_a' = (C + C_2)a^\vartheta(1 + L)^{\max\{A,1\}}$, because $\vartheta \leq p$. Assume $\vartheta_a' < g$ and let T_a' be the

distribution (4.37) for $\Sigma = \Sigma_d(t_0)$, $E_* = 2L$, $\varepsilon = \vartheta'_a$ and the order N_0. By Lemma 4.21 and (4.23),

$$\frac{a}{4} \le C_0 C_{N_0} e^5 M_0 \Big[2e^{2\pi} A_g (1+L) (\vartheta'_a + a)^{N_0+1} + a^{N_0+1} + 3A_g(1+L) a^{7/4} \Big].$$

As $(N_0+1)\vartheta > 2$, every power of a on the right is at most $a^{7/4}$, and $1 \le C'_3\, a^{3/4}(1+L)^{1+\max\{A,1\}(N_0+1)}$ with a computable rational C'_3.

The threshold. The requirements $\vartheta_a < g$, $4a^{\vartheta} \le t_0$, $\vartheta'_a < g$ and the negations of the two final inequalities are finitely many conditions of the form treated in Lemma 7.10, after the constants have been enlarged to rational numbers. Let 2^{-k} be the resulting cutoff and put $\alpha = \min\{g^2/4, 1/64, 2^{-k}\}$. For $0 < a < \alpha$ both regimes lead to a contradiction; the case $t = t_0$ belongs to the second. □

A system of bounds exists, since the proofs of the three lemmas consist of finitely many explicit estimates; but none is written down here, so the algorithm of Proposition 7.14 is specified only up to this choice. By Lemma 4.12 a point of $d^{-1}\Lambda_1$ of integer spin lies in Σ_0, so the proposition is a counterpart, effective relative to $\mathfrak{b}$, of Corollary 4.24, and thus of Theorems 4.22 and 4.23, for $\mathcal{R}_0 = \mathcal{R}_d$, $u_* = 2$ and t near t_0.

7.6 What remains

To derive an effective form of Theorem A from Theorem 7.7 one must exhibit, for rational g, an effective enumeration of local exclusion data covering $\mathcal{B}_g$, together with a proof that each datum is certified. The proof of Theorem A indicates how this might be done. It excludes a limit $\sigma(R, t_0)$ by three different arguments.

(1) If t_0 is not a half-integer and lies outside the finite exceptional set of Lemma 5.10, the fixed kernel families of Propositions 5.11 (for $R > 1$, $R \ne 2$) and 5.15 (for $R \in \{1, 2\}$) apply. Here one box would have to serve a whole interval of values of t_0, with constants uniform on the interval.

(2) At the remaining non-half-integer values of t_0, which are algebraic, the kernels are adapted to the arithmetic of t_0 as in the proof of Theorem 5.21.

(3) At the finitely many pairs of half-integers, the argument of Section 6 applies, with its case distinctions.

In each case the proof considers a sequence converging to $\sigma(R, t_0)$, fixes finitely many cluster counts by Corollary 2.11, derives a localization of all states with $\Delta \le \log(1/a)$ within a power of a, and concludes by Theorem 4.23. Three changes would turn such a proof into a certified local exclusion covering $\sigma(R, t_0)$. First, the stabilized counts are imposed as a box by Lemma 7.2; this is unproblematic, since only finitely many windows occur and the reference clusters are located by rational functions of the radii. Secondly, Theorem 4.23 is replaced by Proposition 7.14; since that proposition is formulated for the window $\Delta_j \le 2L$, whereas Sections 5 and 6 localize the states with $\Delta_j \le L$, the localization steps must first be repeated in the window $2L$, which changes only constants. Thirdly, every intermediate estimate is proved with an explicit constant, valid for all admissible partition functions whose observables lie in the box, after which the finitely many smallness conditions are met by Lemma 7.10.

The third change has not been carried out. The estimates concerned are the following.

(i) *Peak budgets.* In Lemmas 5.6 and 6.21: the constants C of (5.20) and C_{B} of (6.60) (the exponent $\pi\varrho^2/4$ and the restriction $a \le \frac{1}{4}\epsilon^2$ on the Gaussian width ϵ are explicit), and the radii ϱ, ϱ_a of the neighbourhoods of the Fourier centres in which states are counted or excluded.

(ii) *Local zeros.* The lower bounds for the kernels near the states that anchor their negative peaks, in terms of the displacement of the state, and the radii on which these bounds hold; in the collision case, the constant in the lower bound $c\,|t-T|^2$ of Lemma 6.16 for separations and sine factors.

(iii) *Taylor remainders and central shifts.* The constants of Lemma 2.6 and Corollary 2.7, and the envelope constants of each test function to which they are applied. The lemma makes these explicit in principle, since all tests are explicit Gaussian mixtures; this includes the system of bounds assumed in Proposition 7.14.

(iv) *Row estimates.* The constants of the scalar-row estimate of Proposition 5.14; the constants and smallness conditions of Lemma 5.9 are already explicit.

(v) *The normal form.* The constants of Proposition 6.6, Lemmas 6.11 and 6.12 and Theorem 6.13 at the fixed Taylor order n used, including the power of the logarithm. Source (c) was removed above only for the origin certificates; the finite-dimensional estimate of Proposition 6.6 would require a certified search of its own.

(vi) *The collision scale.* The two constants in the comparison $c\,\delta_a \le \theta_a \le C\delta_a$ between the collision scale and the four-scalar displacement, and the threshold in δ_a below which Theorem 6.13 places the selected states in the neighbourhoods of (i) and (ii).

Apart from the explicit ones just noted, none of these constants is derived in this paper. Consequently no certified local exclusion has been exhibited at any point of $\mathcal{B}_g$, hypothesis (i) of Theorem 7.7 has not been verified for any g, and no value of $\epsilon(g)$ has been computed. Deriving the constants is a finite task, but until it is done the effectivity of $\epsilon(g)$ is open.

Even with these constants the output of the algorithm of Theorem 7.7 would not be self-certifying. The searches of Propositions 7.5, 7.12 and 7.13 and the first search of Theorem 7.7 verify what they find: a cover with a margin, or an exact linear identity. The thresholds α_i, by contrast, enter as data, and their validity rests on the proofs of the local exclusions; an error in one of those proofs would not be detected by the algorithm (Remark 7.9).

Two features of the bookkeeping are sound independently of the missing constants. First, no step of the procedures compares transcendental quantities for exact equality. Exact decisions are made only in $\mathbb{Q}$, $\mathbb{Q}(t)$ and $\mathbb{Q}(t_0)$; real numbers such as values of the functions F_τ enter only through strict inequalities verified with a margin by Lemma 7.4, and a verification that does not succeed is never read as an equality. Secondly, no step requires deciding the order of the dimensions of states of an unknown partition function. The boxes are open conditions on continuous observables. The ordering of the states in Step 1 of the proof of Proposition 7.5, and the case distinctions of Sections 5 and 6 (for instance which of the two radii is pinned, or whether δ_a is below a fixed power of a), occur only in correctness proofs, where the exclusion must be established in every case; they are not branches of an algorithm.

Finally, the finite orders involved indicate that no practical bound should be expected from this route. The argument of Section 6 is run with the following fixed choices (Remark 6.26): separations bounded below by $c\,\theta_a^2$ (Lemma 6.16); losses of at most θ_a^{-36} in the resolved estimates; matching of the selected clusters to accuracy θ_a^{60} and the bound $a \le \theta_a^{1000}$ (Definition 6.19); the inequality $a \le \delta_a^{20000}$ from Theorem 5.25; and Taylor order $n = 20000$ in the normal form, the matching bound (6.36) of Theorem 6.13 carrying the exponent $\frac{1}{84} = \frac{1}{3}\cdot\frac{1}{4}\cdot\frac{1}{7}$ (the last factor being the loss in Lemma 6.12), and $20000/84$ exceeding 200. The final localization exponents are then $\frac{29}{500}$ and $\frac{119}{1125}$, so that Proposition 7.14 would be used with $p = \frac{29}{500}$, that is, with Taylor order $N = 34$ and moving certificates of local order 68, and with certificates at t_0 of local order $2N_0$, where $(N_0+1)\min\{\frac{29}{500}, \frac{1}{8(K+1)}\} > 2$. The normal form at $n = 20000$ concerns distributions of local order 40000 on the rational grid. These orders are fixed before a and serve an existence proof; they were not chosen with a view to computation.

Statement on the use of AI

The proof presented here was developed by GPT-6 Astra under the direction of the author. The writeup was examined and edited by Claude Fable 5.2: five independent referee passes, one for each part of the proof, checked the arguments line by line, and a further pass adjudicated their findings.

References

[1] N. Benjamin, K. Langhoff, W. Li and J. Thaler, *Constraining the spectral gap of 2d CFTs just above $c = 1$*, arXiv:2611.07011.

[2] L. Eberhardt and N. Levine, *Classification of $c = 1$ CFTs*, arXiv:2609.16111.

[3] S. Hellerman, *A universal inequality for CFT and quantum gravity*, J. High Energy Phys. **08** (2011) 130, arXiv:0902.2790.

[4] D. Friedan and C. A. Keller, *Constraints on 2d CFT partition functions*, J. High Energy Phys. **10** (2013) 180, arXiv:1307.6562.

[5] S. Collier, Y.-H. Lin and X. Yin, *Modular bootstrap revisited*, J. High Energy Phys. **09** (2018) 061, arXiv:1608.06241.

[6] T. Hartman, D. Mazáč and L. Rastelli, *Sphere packing and quantum gravity*, J. High Energy Phys. **12** (2019) 048, arXiv:1905.01319.

[7] J. Kaidi and E. Perlmutter, *Discreteness and integrality in conformal field theory*, J. High Energy Phys. **02** (2021) 064, arXiv:2008.02190.

[8] N. Afkhami-Jeddi, H. Cohn, T. Hartman and A. Tajdini, *Free partition functions and an averaged holographic duality*, J. High Energy Phys. **01** (2021) 130, arXiv:2006.04839.

[9] P. J. Cohen, *On a conjecture of Littlewood and idempotent measures*, Amer. J. Math. **82** (1960), 191–212.