THE DAUGAVET PROPERTY OF BIDUAL SPACES

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Abstract. We construct a separable Banach space whose bidual has the Daugavet property. The construction proceeds by successively extending finite-dimensional spaces so that prescribed vectors are approximated by averages of newly added unit vectors. A simultaneous extension argument produces finitely many fixed functionals whose norm-attaining faces approximate the unit ball of the bidual.

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1. Introduction

Throughout the paper all spaces, scalars and operators are real. We say that a normed space X has the *Daugavet property*, shortly $X \in \mathrm{DPr}$, if

$$\|\mathrm{Id} + T\| = 1 + \|T\|,$$

for every rank-one operator $T\colon X \to X$. The following characterization was obtained in [3, Lemma 2.2].

Theorem 1.1. *Let X be a normed space. The following assertions are equivalent:*

(1) *X has the Daugavet property.*
(2) *For every $y \in S_X$, $x^* \in S_{X^*}$, and every $\varepsilon > 0$, there exists $x \in S_X$ such that*
$$x^*(x) \geq 1 - \varepsilon \quad \text{and} \quad \|x + y\| \geq 2 - \varepsilon.$$
(3) *For every $\varepsilon > 0$ and every $y \in S_X$, the closed convex hull of*
$$\{\, u \in (1+\varepsilon)B_X \mid \|y + u\| \geq 2 - \varepsilon \,\}$$
contains S_X.

We give a positive answer to the question of V. M. Kadets and D. Werner recorded in [1, Problem 2.1, p. 204], which asks whether there exists a Banach space whose bidual has the Daugavet property. The bidual problem is motivated in part by the fact that classical Daugavet spaces such as $C[0,1]$ and $L^1[0,1]$ lose the Daugavet property upon passing to their biduals.

Theorem 1.2. *There exists a separable Banach space X such that X^{**} has the Daugavet property.*

We use successive finite-dimensional extensions, as in [2, Sections 2–3]. The additional ingredient is a contractive extension into an auxiliary space $V_{C,M}$, with one prescribed coordinate. It yields a family of functionals chosen independently of the point of B_X being decomposed.

2. The finite-dimensional extension

Let E be a finite-dimensional normed space, $p \in B_E$, $\delta > 0$ and $n \in \mathbb{N}$. Put $q_n = n^{-1}(1, \ldots, 1)$ and define $F = F(E, p, \delta, n) = E \oplus \mathbb{R}^n$ by

$$\|(x, b)\|_F = \inf_{t \in \mathbb{R}} \big(\|x + tp\|_E + \|b - tq_n\|_1 + \delta|t| \big). \tag{2.1}$$

This is the quotient norm induced by the surjection

$$Q : E \oplus_1 \ell_1^n \oplus_1 (\mathbb{R}, \delta|\cdot|) \longrightarrow F, \qquad Q(y, b, t) = (y - tp, b + tq_n).$$

We identify E with $E \oplus \{0\}$ and put $h_j = (0, e_j) \in F$.

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Lemma 2.1. *The inclusion $E \hookrightarrow F$ is isometric, and*

$$h_j \in B_F \quad (1 \le j \le n), \qquad \left\| \frac{1}{n} \sum_{j=1}^{n} h_j - p \right\|_F \le \delta. \tag{2.2}$$

Under the pairing $(f,\alpha)(x,b) = f(x) + \sum_{j=1}^n \alpha_j b_j$, one has

$$\|(f,\alpha)\|_{F^*} = \max\left\{ \|f\|_{E^*},\ \|\alpha\|_\infty,\ \frac{1}{\delta}\left| \frac{1}{n}\sum_{j=1}^{n} \alpha_j - f(p) \right| \right\}. \tag{2.3}$$

If $n\delta \ge 2$, then

$$\|h_j\|_F = 1, \qquad \|x + h_j\|_F = 2 \quad (x \in S_E,\ 1 \le j \le n). \tag{2.4}$$

Proof. For $x \in E$,

$$\|x + tp\|_E + (1+\delta)|t| \ge \|x\|_E + (1 + \delta - \|p\|_E)|t| \ge \|x\|_E .$$

Together with $t = 0$, this proves isometry. Taking $t = 0$ at h_j and $t = 1$ at $(-p, q_n)$ proves (2.2). The adjoint of the quotient map is

$$Q^*(f,\alpha) = \left(f, \alpha, \frac{1}{n}\sum_{j=1}^{n} \alpha_j - f(p) \right),$$

which gives (2.3). If $n\delta \ge 2$, taking $f = 0$ and $\alpha = e_j$ in (2.3) gives $\|h_j\|_F = 1$. Finally, fix $x \in S_E$ and j, and choose $f \in S_{E^*}$ with $f(x) = 1$. Set $\alpha_j = 1$ and $\alpha_k = f(p)$ for $k \ne j$. Then

$$\left| \frac{1}{n}\sum_{k=1}^{n} \alpha_k - f(p) \right| = \frac{1 - f(p)}{n} \le \frac{2}{n} \le \delta.$$

Thus $(f,\alpha) \in B_{F^*}$ and $(f,\alpha)(x + h_j) = 2$. The triangle inequality gives (2.4). □

Lemma 2.2. *Let Y be a normed space, $T : E \to Y$ linear and $b_1, \dots, b_n \in Y$. The unique linear extension $\widetilde{T} : F \to Y$ satisfying $\widetilde{T}|_E = T$ and $\widetilde{T}h_j = b_j$ has norm*

$$\left\|\widetilde{T}\right\|_{\mathcal{L}(F,Y)} = \max\left\{ \|T\|_{\mathcal{L}(E,Y)},\ \max_{1\le j\le n} \|b_j\|_Y,\ \frac{1}{\delta}\left\| \frac{1}{n}\sum_{j=1}^{n} b_j - Tp \right\|_Y \right\}. \tag{2.5}$$

Proof. Apply (2.3) to $y^* \circ \widetilde{T} = (y^* \circ T, (y^*(b_j))_{j=1}^n)$ and take the supremum over $y^* \in B_{Y^*}$. □

3. The inductive construction

Put $E_1 = \mathbb{R}$ and $P_0 = \varnothing$. Given E_s, choose a finite 2^{-s}-net $N_s \subset B_{E_s}$ and set

$$P_s = P_{s-1} \cup N_s, \qquad n_s = (s+2)!, \qquad m_s = s|P_s|.$$

Enumerate

$$P_s \times \{1, \dots, s\} = \{(p_{s,r}, k_{s,r}) : 1 \le r \le m_s\}.$$

Starting with $F_{s,0} = E_s$, define

$$F_{s,r} = F\left(F_{s,r-1}, p_{s,r}, \frac{1}{k_{s,r}}, n_s \right) \quad (1 \le r \le m_s), \qquad E_{s+1} = F_{s,m_s}. \tag{3.1}$$

Here every $p_{s,r}$ is chosen in E_s before the extensions in (3.1). Lemma 2.1 gives isometric inclusions

$$E_s = F_{s,0} \subset F_{s,1} \subset \cdots \subset F_{s,m_s} = E_{s+1}.$$

Let X be the completion of $X_0 = \bigcup_{s\ge1} E_s$, and put $P = \bigcup_{s\ge1} P_s$. Then X is separable, and

$$\overline{P}^{\|\cdot\|_X} = B_X. \tag{3.2}$$

Indeed, $\bigcup_s B_{E_s}$ is dense in B_X, and $\operatorname{dist}(x, P_s) \le 2^{-s}$ for $x \in B_{E_r}$ and $s \ge r$.

For each fixed $p \in P$ and $k, M \in \mathbb{N}$, the construction contains arbitrarily late elementary extensions

$$F(E, p, 1/k, n) \qquad \text{with } M \mid n. \tag{3.3}$$

In fact, $p \in P_s$, $k \le s$ and $M \mid (s+2)!$ for all sufficiently large s.

4. Extensions with a prescribed coordinate

Fix $C \geq 1$ and $M \in \mathbb{N}$. On $V = V_{C,M} = \mathbb{R} \oplus \mathbb{R}^M$ put

$$\|(a,c)\|_V = \max\left\{|a|,\ \max_{1\leq i\leq M}|a+c_i|,\ \max_{A\subseteq\{1,\dots,M\}}\left|a+\frac{1}{C}\sum_{i\in A}c_i\right|\right\}. \tag{4.1}$$

Write $K = B_V$, $L(a,c) = a$, $\pi_i(a,c) = c_i$ and $c_i^{\pm} = \max\{\pm c_i, 0\}$. Then

$$(a,c)\in K \iff \begin{cases}|a|\leq 1, \quad -1-a\leq c_i\leq 1-a & (1\leq i\leq M),\\ \|c^+\|_1\leq C(1-a), \quad \|c^-\|_1\leq C(1+a).\end{cases} \tag{4.2}$$

In particular,

$$\|c\|_\infty \leq 2, \qquad \|c\|_1 \leq 2C \quad ((a,c)\in K). \tag{4.3}$$

If $(a,c)\in K$ and $|a|<1$, then

$$v_+ = \left(1, -\frac{2c^-}{1+a}\right)\in K, \qquad v_- = \left(-1, \frac{2c^+}{1-a}\right)\in K, \qquad (a,c) = \frac{1+a}{2}v_+ + \frac{1-a}{2}v_-. \tag{4.4}$$

All assertions follow directly from (4.2).

Lemma 4.1. *Let $F = F(E,p,\delta,n)$, $f_F \in B_{F^*}$ and $0<\theta<1$. Every contraction $T_E : E \to V_{C,M}$ with $L\circ T_E = \theta f_F|_E$ admits a contraction $T_F : F \to V_{C,M}$ such that*

$$T_F|_E = T_E, \qquad L\circ T_F = \theta f_F. \tag{4.5}$$

If, in addition,

$$M \mid n, \qquad M\geq C, \qquad \frac{4}{C}\leq(1-\theta)\delta, \tag{4.6}$$

then, for any partition $\{1,\dots,n\} = J_1\sqcup\cdots\sqcup J_M$ with $|J_i| = n/M$, the extension may be chosen to satisfy

$$(L+\pi_i)(T_F h_j) = 1 \qquad (j\in J_i,\ 1\leq i\leq M). \tag{4.7}$$

Proof. Put $a_0 = \theta f_F(p)$ and $a_j = \theta f_F(h_j)$. Since $|a_0|\leq\theta<1$, apply (4.4) to $T_E p$ and write

$$T_E p = \frac{1+a_0}{2}v_+ + \frac{1-a_0}{2}v_-, \qquad b_j = \frac{1+a_j}{2}v_+ + \frac{1-a_j}{2}v_-.$$

Then $b_j\in K$, $Lb_j = a_j$, and (2.2) gives

$$\left\|\frac{1}{n}\sum_{j=1}^n b_j - T_E p\right\|_V = \left|\frac{1}{n}\sum_{j=1}^n a_j - a_0\right|\left\|\frac{v_+-v_-}{2}\right\|_V \leq \theta\delta. \tag{4.8}$$

Thus $T_F|_E = T_E$, $T_F h_j = b_j$ defines a contraction by (2.5), and satisfies (4.5).

Assume (4.6). For $(a,c)\in K$, define

$$R_i(a,c) = \left(a, \left(1-\frac{1}{C}\right)c + \beta_i e_i\right), \qquad \beta_i = 1-a-\left(1-\frac{1}{C}\right)c_i \in [0,2].$$

If $R_i(a,c) = (a,c')$, then $c_i' = 1-a$, the coordinate bounds in (4.2) hold, and

$$\left\|(c')^+\right\|_1 \leq \left(1-\frac{1}{C}\right)C(1-a) + (1-a) = C(1-a), \qquad \left\|(c')^-\right\|_1 \leq \left(1-\frac{1}{C}\right)C(1+a).$$

Hence $R_i(K)\subset K$. Write $b_j = (a_j, c^{(j)})$, let $\ell(j) = i$ for $j\in J_i$, and put

$$\beta_i^{(j)} = 1-a_j-\left(1-\frac{1}{C}\right)c_i^{(j)}, \qquad b_j' = R_{\ell(j)}b_j.$$

The mean change has the form

$$\frac{1}{n}\sum_{j=1}^n (b_j'-b_j) = (0,d), \qquad d = -\frac{1}{nC}\sum_{j=1}^n c^{(j)} + \frac{1}{n}\sum_{j=1}^n \beta_{\ell(j)}^{(j)} e_{\ell(j)}.$$

Since $|J_i| = n/M$, (4.3) yields

$$\|d\|_\infty \le \frac{2}{C} + \frac{2}{M}, \qquad \|d\|_1 \le 4, \qquad \|(0,d)\|_V \le \max\left\{\|d\|_\infty, \frac{\|d\|_1}{C}\right\} \le \frac{4}{C}.$$

Together with (4.8), this gives

$$\left\|\frac{1}{n}\sum_{j=1}^{n} b'_j - T_E p\right\|_V \le \theta\delta + \frac{4}{C} \le \delta.$$

Now prescribe $T_F|_E = T_E$ and $T_F h_j = b'_j$. Formula (2.5) proves contractivity; $Lb'_j = a_j$ and $(L+\pi_i)b'_j = 1$ for $j \in J_i$ prove the rest. □

5. Fixed functionals and bidual decompositions

The space X and its chain of finite-dimensional subspaces are now fixed.

Proposition 5.1. *Let $f \in B_{X^*}$, $0 < \theta < 1$, $k_0 \in \mathbb{N}$, $\delta_0 = 1/k_0$, $C \ge 1$ and $M \in \mathbb{N}$ satisfy*

$$M \ge C, \qquad \frac{4}{C} \le (1-\theta)\delta_0. \tag{5.1}$$

There exist $g_1, \dots, g_M \in X^*$ such that, with $\phi_i = \theta f + g_i$, one has

$$\phi_i \in S_{X^*}, \qquad \left\|\sum_{i=1}^{M} \sigma_i g_i\right\|_{X^*} \le 2C \quad (\sigma_i \in \{-1,1\}). \tag{5.2}$$

For these same functionals, put $\mathcal{F}_i = \{z \in B_{X^{**}} : z(\phi_i) = 1\}$. Then

$$B_{X^{**}} \subset \frac{1}{M}(\mathcal{F}_1 + \cdots + \mathcal{F}_M) + \delta_0 B_{X^{**}}. \tag{5.3}$$

The functionals depend on $f, \theta, \delta_0, C, M$, but are chosen independently of the vector in $B_{X^{**}}$ to be decomposed.

Proof. Set $V = V_{C,M}$ and $T_{E_1}x = (\theta f(x), 0)$. Along the fixed chain (3.1), apply Lemma 4.1 to each inclusion $E \subset F = F(E, p, \delta, n)$, always with $f_F = f|_F$. Use the additional conclusion whenever $\delta = \delta_0$ and $M \mid n$. The resulting contractions satisfy

$$T_F|_E = T_E, \qquad L \circ T_F = \theta f|_F.$$

They define a contraction on X_0 and hence a contraction $T : X \to V$ with $L \circ T = \theta f$. Put

$$g_i = \pi_i \circ T, \qquad \phi_i = (L + \pi_i) \circ T.$$

By (4.1) and (4.3),

$$\|\phi_i\|_{X^*} \le 1, \qquad \sup_{x \in B_X} \sum_{i=1}^{M} |g_i(x)| \le 2C,$$

which implies the sign estimate in (5.2).

Fix $p \in P$. By (3.3), choose an elementary extension at p with $\delta = \delta_0$ and $M \mid n$. For its partition $J_1, \dots, J_M$, set

$$z_i(p) = \frac{M}{n}\sum_{j \in J_i} h_j \in B_X.$$

Equations (2.2) and (4.7) give

$$\phi_i(z_i(p)) = 1, \qquad \left\|p - \frac{1}{M}\sum_{i=1}^{M} z_i(p)\right\|_X \le \delta_0. \tag{5.4}$$

In particular, $\|\phi_i\|_{X^*} = 1$ and each $\mathcal{F}_i$ is nonempty.

Identify X with its canonical image in X^{**}. The sets $\mathcal{F}_i$ are weak-star compact. Therefore

$$Q = \frac{1}{M}(\mathcal{F}_1 + \cdots + \mathcal{F}_M) + \delta_0 B_{X^{**}}$$

is weak-star compact, being the image of a finite product of weak-star compact sets under addition and scalar multiplication. By (5.4) and the fact that ϕ_i are chosen before the selection of p, $P \subset Q$. Goldstine's theorem and (3.2) now give

$$B_{X^{**}} = \overline{P}^{w^*} \subset Q,$$

as required. □

6. Proof of the main theorem

Proof of Theorem 1.2. Fix $u \in S_{X^{**}}$ and $0 < \varepsilon, \tau < 1$. Choose, in this order,

$$\eta = \frac{\varepsilon}{8}, \qquad \theta = 1 - \frac{\varepsilon}{4}, \qquad f \in S_{X^*} \text{ with } u(f) > 1 - \eta,$$
$$k_0 \in \mathbb{N} \text{ with } \delta_0 = \frac{1}{k_0} < \frac{\tau}{2},$$
$$C \geq \max\left\{1, \frac{4}{(1-\theta)\delta_0}\right\}, \qquad M \in \mathbb{N}, \quad M \geq \max\left\{C, \frac{8C}{\eta\tau}\right\}.$$

Thus η, θ depend only on ε; δ_0 depends only on τ; and C, M may be chosen as functions of (ε, τ) alone. Apply Proposition 5.1 to obtain g_i and ϕ_i. Choosing σ_i according to the sign of $u(g_i)$ gives

$$\sum_{i=1}^{M} |u(g_i)| = u\left(\sum_{i=1}^{M} \sigma_i g_i\right) \leq 2C. \tag{6.1}$$

Put

$$I = \{i : |u(g_i)| \leq \eta\}, \qquad b = M - |I|.$$

Then $b \leq 2C/\eta$ and $b/M \leq \tau/4 < 1$. For every $i \in I$,

$$u(\phi_i) = \theta u(f) + u(g_i) > \theta(1-\eta) - \eta. \tag{6.2}$$

Let $v \in B_{X^{**}}$. By (5.3), there are $z_i \in \mathcal{F}_i$ with

$$\left\| v - \frac{1}{M}\sum_{i=1}^{M} z_i \right\|_{X^{**}} \leq \delta_0.$$

For $i \in I$, (6.2) yields

$$\|u + z_i\|_{X^{**}} \geq (u + z_i)(\phi_i) > 1 + \theta(1-\eta) - \eta = 2 - \big((1-\theta) + (1+\theta)\eta\big) > 2 - \varepsilon,$$

since $(1-\theta) + (1+\theta)\eta = \varepsilon/2 - \varepsilon^2/32 < \varepsilon$. Consequently

$$w = \frac{1}{|I|}\sum_{i\in I} z_i \in \operatorname{conv}\{z \in B_{X^{**}} : \|u + z\| \geq 2 - \varepsilon\}.$$

Splitting the original average into the retained and discarded summands, we obtain

$$\frac{1}{M}\sum_{i=1}^{M} z_i = \frac{1}{M}\sum_{i\in I} z_i + \frac{1}{M}\sum_{i\notin I} z_i = \frac{M-b}{M}\, w + \frac{1}{M}\sum_{i\notin I} z_i.$$

Subtracting w therefore yields

$$\frac{1}{M}\sum_{i=1}^{M} z_i - w = \frac{1}{M}\sum_{i\notin I} z_i - \frac{b}{M}\, w.$$

There are exactly b discarded indices. Hence

$$\begin{aligned}\left\| \frac{1}{M}\sum_{i=1}^{M} z_i - w \right\|_{X^{**}} &\leq \frac{1}{M}\sum_{i\notin I} \|z_i\|_{X^{**}} + \frac{b}{M}\|w\|_{X^{**}} \\ &\leq \frac{b}{M} + \frac{b}{M} = \frac{2b}{M}.\end{aligned}$$

Combining this estimate with

$$\left\| v - \frac{1}{M}\sum_{i=1}^{M} z_i \right\|_{X^{**}} \leq \delta_0,$$

we conclude that

$$\begin{aligned} \|v - w\|_{X^{**}} &\leq \left\| v - \frac{1}{M}\sum_{i=1}^{M} z_i \right\|_{X^{**}} + \left\| \frac{1}{M}\sum_{i=1}^{M} z_i - w \right\|_{X^{**}} \\ &\leq \delta_0 + \frac{2b}{M} \\ &\leq \delta_0 + \frac{4C}{\eta M} \\ &< \tau. \end{aligned}$$

Here the penultimate inequality uses $b \leq 2C/\eta$, and the last inequality follows from the choice of the parameters. Since v and τ were arbitrary, $B_{X^{**}} = \overline{\operatorname{conv}}\{z \in B_{X^{**}} : \|u + z\| \geq 2 - \varepsilon\}$ with respect to the norm topology in X^{**}. The characterization Theorem 1.1 completes the proof. □

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