hexagon:2611.00293v3

AN UPPER BOUND EXPONENT OF 0.9999873 FOR SQUARE-DIFFERENCE-FREE SETS

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ABSTRACT. Let $A \subseteq \{1, \dots, N\}$ contain no two distinct elements whose difference is a nonzero square, and let $s(N)$ be the largest possible size of such a set. We prove

$$s(N) \leq CN^{0.9999873}$$

with an absolute constant C.

A recent preprint of OpenAI proves $s(N) \leq CN^{1-\varepsilon}$ for some absolute $\varepsilon > 0$. Their proof utilizes a certain average of the Fourier transform over $(24!)^{161}$ points, and for the parameter choices displayed there we calculate that they obtained (Appendix F) $\varepsilon \approx \exp_{10}(-\exp_{10}(\exp_{10}(5782.77)))$, in the sense that $\log_{10}\log_{10}\log_{10}(1/\varepsilon) \approx 5782.77$.

We follow the same overall strategy, with new work at almost every quantitative step. The largest gain is the use of 72 slots (positions at which the Fourier transform is evaluated) in place of $(24!)^{161}$: the exponent is roughly a fixed constant divided by the number of slots, so this alone gains many orders of magnitude, but leaves far less room for loss elsewhere. Other savings come from the local construction at each prime. Small primes are handled jointly rather than one by one; and errors are measured in the actual half-distributions of the laws being compared, not bounded term by term against a product model. Where sums over the primes of a block of Weil-type error terms of size $O(p^{-1/2})$ would be too weak, we prove bounds of order $O(p^{-1})$ for the same error terms using étale cohomology over finite fields (Deligne's theorems and Katz–Sarnak monodromy for families of curves), the representation theory of $\mathrm{GL}_2(\mathbf{F}_p)$, exact counts at $p \equiv 3 \pmod 4$, and a modular evaluation of Ahlgren and Ono. Dependence in our distributions is controlled by the correlation inequalities of Peyre and Kassabov. The remaining finitely many numerical inequalities are verified by exact computer arithmetic, listed in the appendices; all analytic steps are proved in the text.

The companion repository [48] contains a partial Lean 4 formalization, totaling over 600,000 lines of lean. It establishes $s(N) \leq CN^{0.99999}$, conditional on five named analytic inputs and explicit numerical hypotheses. This formalizes the most critical parts of the proof.

*__AI use:__ This paper was written by Claude Opus 5.5, GPT Astra 6, and Muse Wonder-Full 1.0, under my prompting and supervision. The research was carried out by Claude Opus 5.5, GPT Astra 6, and ChatGPT Pro queries. I tried to have the AIs write a readable and comprehensible introduction that details the differences between this work and OpenAI's work, and the new ideas involved. This includes a dependency diagram showing the structure of the overall proof. In earlier drafts I found the introduction to be incomprehensible, and it was re-written under my supervision multiple times, so I hope these updates (primarily from Muse and Codex) help the reader get a sense of the overall work. I also recommend that the reader utilize their AI assistant of choice to help understand the paper.
The proof has undergone several intensive adversarial reviews by GPT Astra 6 and other agents; no errors or gaps were found in those reviews.
A Lean 4 formalization totaling over 600,000 lines of lean appears in the companion repository [48]. This formalization establishes $s(N) \leq CN^{0.99999}$, conditional on five named analytic inputs and explicit numerical hypotheses. This lean code was generated by Claude Opus 5.5 and GPT Astra 6, and covers many of the most critical parts of the proof.
The scope of the finite computer checks and internal reviews is stated in Section 1.7, Appendix D and Section 35.12.

Date: November 10, 2026.

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1. Introduction

1.1. **Statement and Context.** We say that a set $A \subseteq [N]$, where $[N] = \{1, \dots, N\}$, is *square-difference-free* if no difference of two distinct elements of A is the square of a positive integer. Let $s(N)$ be the largest possible size of such a set. Furstenberg [23] and Sárközy [65] proved that $s(N) = o(N)$. The quantitative bounds were subsequently improved by Pintz–Steiger–Szemerédi [56] and Bloom–Maynard [6]. Green and Sawhney [26] then showed that $s(N) \ll N \exp(-c_0\sqrt{\log N})$ for an absolute $c_0 > 0$, which saves more than every fixed power of $\log N$ but less than a fixed power of N. OpenAI [52] then gave the first power saving, proving that for some absolute $\varepsilon > 0$ we have $s(N) \le CN^{1-\varepsilon}$.

For the parameter choices displayed in OpenAI's paper, we calculate (Appendix F)

$$\varepsilon \approx 10^{-10^{10^{5782.77}}}, \qquad \text{in the sense that} \qquad \log_{10}\log_{10}\log_{10}(1/\varepsilon) \approx 5782.77.$$

This quantifies that one choice of parameters; it is not a claim about the best exponent their method could give.

This paper works within the OpenAI framework, and makes ε explicit, with new quantitative work at almost every step.

Theorem 1.1. *For every integer $N \ge 1$ and every square-difference-free set $A \subseteq [N]$,*

$$|A| \le CN^{1-\varepsilon}, \qquad \varepsilon = \frac{1814179}{141970826000},$$

so that the exponent is $1 - \varepsilon = 0.999987\ldots$, where C is an absolute constant.

Here $\varepsilon = a/(72 - \vartheta)$ with $a = \frac{1814179}{2000000000}$ and $\vartheta = \frac{1014587}{1000000}$.

The exponent $1 - \varepsilon$ is still far from the exponents in the known constructions of large square-difference-free sets. Ruzsa's construction [63], improved in [3, 44], gives exponents near 0.7334, and Krachun [43] gave an exponent exceeding 3/4. The strongest lower-bound construction is Naslund's, with exponent 0.7580758 [50]; its finite certificates are formalized in Lean [49].

The shape of the argument is as follows: let $\alpha = |A|/N$ be the density and let Y be the 72-point Fourier average defined in Section 1.2 below. The proof shows $\alpha^{72} \le Y$ (by reflection positivity) and $Y \le C\mathsf{L}^{-a}\alpha^{\vartheta}$, where $\mathsf{L} = N/K$ is the length of each of the K subintervals of $[N]$ (Step 1 below). Hence $\alpha^{72-\vartheta} \ll \mathsf{L}^{-a}$, which is the power saving. Here a measures decay with scale and ϑ is the power of α in the upper bound; for $0 < \alpha < 1$ a larger ϑ gives a smaller upper bound at fixed a. The word *slot* means one of the 72 positions at which the Fourier series is evaluated; a *law* is a probability or signed distribution on assignments of group elements to the slots. Full definitions are in Section 2 (informally) and Parts II–III (formally), and Section 1.8 collects the terminology in one place.

The paper proves the upper bound in two stages that share the same notation and must not be confused. The *baseline* stage, through Section 30, works with $\vartheta = 1$ and proves the proposition below. The *improved* stage, Sections 31 to 33, then strengthens the density estimate to $\vartheta = \frac{1014587}{1000000} > 1$, recomputes the prices (defined in Section 1.2) at the new parameters, and proves Theorem 1.1. Numerical tables before the improved stage belong to the baseline unless the caption says otherwise.

Proposition 1.2 (The baseline bound, with density exponent $\vartheta = 1$)**.** *For an absolute constant C_0, $s(N) \le C_0 N^{1-\frac{5422587}{433349609375}}$. This uses $a = \frac{5422587}{6103515625}$, $\vartheta = 1$ and $K = 890666$ intervals (the number K of intervals in Step 1 below).*

Section 30.3 proves this baseline. The improved stage then reproves the closing inequality at $a = \frac{1814179}{2000000000}$, $K = 863660$ and $\vartheta = \frac{1014587}{1000000}$; it keeps the same 72 slots, the same prime ranges and the same middle reference laws. It uses the new upper comparisons of Section 28.4, the joint mean

allocation of Section 26.6, the stronger density estimate, and the prefix law of Step 4 below. No result before the improved stage depends on $\vartheta > 1$. Section 34 (Part VI, after the proof) records how the explicit exponent developed; no proof in Parts I–V uses it.

The rest of this introduction explains the proof (Sections 1.2 and 1.3) and what is new in it (Sections 1.4 and 1.5), states the closing inequality with its baseline values (Section 1.6) and the kinds of input (Section 1.7), and ends with a reading guide (Section 1.9). It proves nothing; every statement in it is proved later.

1.2. **Overview.** The whole proof is organised around one average attached to A. We define it in this subsection, show that it is large when A is dense, and then explain how the induction proves that it must in fact be small. The rest of the paper either proves these two bounds or constructs the distributions that make the second bound explicit.

The average Y. Write $\alpha = |A|/N$ for the density of A, and write $N = K\mathsf{L}$, where K is a fixed number of intervals and L is the scale. There are 72 *slots*, forming a set V (Step 1 below gives the exact array). A *label* is an element of a finite group G_N built from residue classes at the primes; a *configuration* is a choice of one label z_v for each slot $v \in V$. Let F_A be the truncated Fourier series of $\mathbf{1}_A$ on G_N (the *Fourier lift* of A; see Section 6). It has mean α, but it may take negative values. Finally, let μ be a probability distribution on configurations, built as a product over blocks of primes. We call μ the *reference law*, and put

$$Y = \mathbb{E}_\mu \prod_{v \in V} F_A(z_v).$$

Thus Y is a 72-fold Fourier average of A: we choose a random configuration from μ and multiply the values of F_A at its 72 labels. If F_A were the constant α, this would be exactly α^{72}.

Why Y is large. The reference law μ is *reflection positive*: for each of several mirror symmetries of the slots, the matrix that couples the two halves of a configuration is positive semidefinite (Section 2 informally, Section 5 formally). Repeated Cauchy–Schwarz across these symmetries gives a signed Hölder inequality, and hence

$$\alpha^{72} \le Y.$$

The sign of F_A is the point. For a nonnegative function this would be an ordinary moment bound; here reflection positivity is what makes the same conclusion available for the signed lift. In particular, a dense set A forces Y to be large.

What we prove about Y. Against this, we prove an upper bound of the form

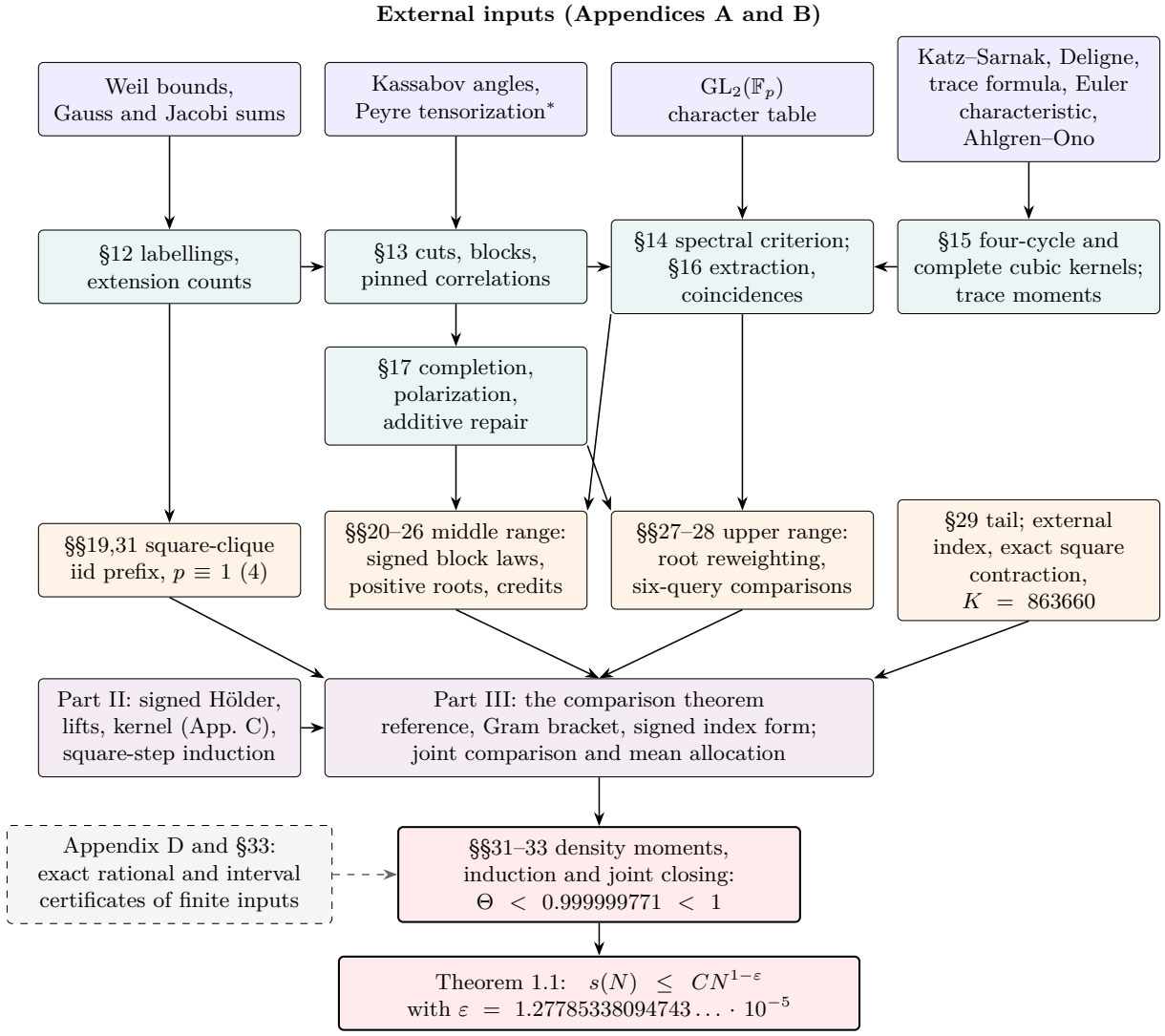
$$Y \le C\mathsf{L}^{-a}\alpha^{\vartheta},$$

where $a > 0$ is fixed and $\vartheta = 1$ in the baseline argument; the final argument obtains the stronger exponent $\vartheta = \frac{1014587}{1000000} > 1$. The two bounds together give $\alpha^{72-\vartheta} \ll \mathsf{L}^{-a}$, which is the power saving in Theorem 1.1.

The upper bound is proved by induction on the scale. Suppose it failed, and look at the smallest scale L where it could fail; call this a *least counterexample*. At every smaller scale the bound is already known, and the whole comparison below takes place at this one scale, with the smaller scales available as an induction hypothesis.

Forcing six square differences. At a least counterexample we change the reference law μ into a law that forces six specified pairs of slots, called *queries*, to receive labels differing by a square class at every prime. For such a law the average is negligible compared with Y: roughly, it counts pairs of elements of A at square distance, and A has none. Making this precise is the kernel estimate of Step 3 below; it is the only place where the hypothesis that A is square-difference-free is used directly.

The change of law is made in two stages, $\mu \to \lambda \to \nu$. The *first-stage law* λ is obtained from μ by a small number of explicit changes. (A *record* is one weighted summand in the explicit



*The finite Peyre theorem needed is proved in Appendix B. Blue: external inputs; green: local tools (Part IV); orange: the construction (Part V); violet: Parts II and III. Solid arrows are principal mathematical dependencies; the dashed arrow is arithmetic verification of finite inputs, and a passing certificate does not verify the reductions.

FIGURE 1. Principal dependencies of the proof. External inputs feed the local tools of Part IV; these build the laws of the construction in Part V, which verify the hypotheses (CP0)–(CP8) of the comparison theorem of Part III. The selected density moments and induction extend the analytic framework of Part II; the normalized closing in Section 33 then gives Theorem 1.1. The dashed arrow records finite arithmetic checks, whose scope is stated in Section 1.7 and Appendix D. The parts are described in Section 1.9.

expansion of a local law; a *component* of a signed combination is one nonnegative summand of that combination. The two words are distinct throughout.) The *comparison law* ν is a finite signed combination of nonnegative laws, each of which *accepts* the six queries: almost surely, each query pair differs by a nonzero square at every field prime (a prime whose coordinate in G_N is $\mathbb{F}_p$) and by a nonvanishing square class at the prime-power coordinates, and the summand has the uniform one-slot marginals required in (CP4). In informal prose we say such a summand *forces* a square difference when only the almost-sure square condition is meant; *accepts* always includes the marginal requirement. Their formal definitions are (CP1), (CP2) and (CP4) in Section 9.

The law ν is signed, so it need not be reflection positive, and the lower-bound argument does not apply to it. It is instead controlled at one distinguished mirror symmetry ι, the *reserved cut* (the other symmetries are used for the lower bound). A positive law N dominates λ at this cut, in the sense that the block matrix of cut forms $\left(\begin{smallmatrix} \mathsf{N} & \lambda \\ \lambda & \mathsf{N} \end{smallmatrix}\right)$ is positive semidefinite (written $\succeq_\iota 0$ below), and a second pair of positive laws dominates ν itself. The symmetric and *skew* (antisymmetric under ι) parts of each change $\nu - \lambda$ are dominated and priced separately.

What a price is. It remains to say what these changes cost. The primes are grouped into *blocks* $\mathcal{B}$; write $q_{\mathcal{B}}$ for the product of the primes in $\mathcal{B}$. Fixing the labels at one block fixes a residue class modulo $q_{\mathcal{B}}$, and splitting that class into progressions with square step restricts A to shorter progressions with square step (Step 2 below). Those shorter sets are still square-difference-free, so the induction hypothesis applies to them. This is the only use of the induction hypothesis, and its quantitative form is:

> Changing the law at one block $\mathcal{B}$ changes Y by at most (size of the change)$\times q_{\mathcal{B}}^{2a} \times Y$, up to a factor $1 + o(1)$ as $\mathsf{L} \to \infty$.

The product (size of the change) $\times\, q_{\mathcal{B}}^{2a}$ is called the *price* of the change. The "size" is a total variation or an operator norm, according to the kind of change, and norms are always taken relative to the true one-half distributions of the record being replaced (its *actual marginals*), not relative to a product approximation; the three precise forms are in Definition 9.2 and are summarised in Section 1.8. Grouping primes into blocks means the factor $q_{\mathcal{B}}^{2a}$ is paid once per block rather than once per prime. Character sums bound the sizes: they measure how far the local laws are from independent. The finite certificates of Part V then verify that, for one fully specified construction, all these sizes are small enough; they check finite inequalities only, not the analytic arguments (Section 1.7).

So the task of the paper can be stated plainly: construct laws that are reflection positive where the lower bound needs it, that force the six square differences where the kernel estimate needs it, and whose total price is strictly less than 1. These three requirements pull against each other, and most of the paper is about meeting them simultaneously.

The six indices and the number Θ. One point has been suppressed. The prime 2 and the small primes $p \equiv 3 \pmod 4$ are not coordinates of G_N. Each query therefore also carries an *external index*: an interval number and a residue modulo a fixed modulus. A signed average over the six indices expands into $2^6 = 64$ terms. Every term that couples a query across two adjacent intervals is negligible by the kernel estimate, except when the next interval wraps cyclically from the last interval back to the first; these exceptional terms are the *wrapping* terms, and they are priced separately.

Collecting the prices of all the changes above gives a single number, which in the baseline argument is

$$\Theta = \tfrac{63}{64}(L_0 + \mathcal{E}) + J + W + \tfrac{1-m}{64}. \tag{1.1}$$

Here L_0 is the total price of the first-stage change $\mu \to \lambda$; $\mathcal{E}$ is the total dominated price of the comparison change $\nu - \lambda$ at the reserved cut; J bounds the coupled terms coming from the external indices; W bounds the wrapping terms; and $m > 1$ measures how much of the mean

(constant-function) part of Y the dominating laws retain (Section 26), so the last term $(1-m)/64$ is a credit. The factor $63/64$ records that the empty term of the 64-term expansion is the comparison average itself. All symbols are defined fully in Section 1.6 and Section 9.

If $\Theta < 1$, then at a least counterexample the priced upper bound for Y is incompatible with the lower bound $\alpha^{72} \le Y$. Hence there is no least counterexample, and the upper bound holds at every scale. This implication is the comparison theorem, Theorem C (Theorem 9.21); its proof (Sections 10 and 11) uses nothing about the particular local laws, and Proposition 2.1 shows the whole mechanism with a single query. The rest of the paper verifies the hypotheses of Theorem C for one explicit construction and evaluates Θ. For Theorem 1.1 the improved stage then uses the joint form of this criterion (Corollary 11.2) together with the stronger density induction; (1.1) is its baseline form.

Why this shape gives a power saving. The conversion is worth isolating. Suppose that at some fixed $a > 0$, independent of the scale and of α, the construction above can be priced with $\Theta < 1$. Then $\alpha^{72-\vartheta} \ll \mathsf{L}^{-a}$ at once, so $\varepsilon = a/(72-\vartheta)$: no iteration is needed, and the saving is a fixed power for the elementary reason that the required win does not shrink as $\alpha \to 0$. This is also where the quantitative difficulty lies. With only the first-order (Weil) bounds $O(p^{-1/2})$ for the local character sums, the best certified value of Θ is still larger than 1 (Section 1.5); the second-order bounds $O(p^{-1})$ of Part IV, the relative (actual-marginal) form of the local estimates, and the variance and mean subtractions are what bring the total price below 1 at an explicit value of a.

Is this a density-increment argument? In ancestry, yes; in form, no. The classical arguments [56, 6] (and [26]) assume that A is dense and square-difference-free, count square differences by Fourier analysis, and deduce that A has larger relative density on a shorter structured set—a progression with square step in [56], a Bohr set in [6]. Iterating this increment until the density would exceed 1 bounds α. Each step gains only a small factor in density and loses length, so the final saving is smaller than any fixed power of N.

The present proof uses the same square-step rescaling, but it never produces a denser set. Instead, the induction hypothesis at all smaller scales is used at once, at a least counterexample, in the form of prices: passing to the shorter progressions is what makes a change of law at one block cost the factor $q_{\mathcal{B}}^{2a}$. And the quantity being bounded is not the density α itself but the 72-fold average Y. This is where the extra strength comes from. Since $\alpha^{72} \le Y \le C\mathsf{L}^{-a}\alpha^{\vartheta}$ with $\vartheta \approx 1$, a single constant-factor win—the inequality $\Theta < 1$ at one scale—already gives $\alpha^{72-\vartheta} \ll \mathsf{L}^{-a}$: the 72 slots amplify one rescaling gain into a power saving, with no iteration and no repeated loss of length. The cost of this amplification is that F_A changes sign, so the positivity that a density has for free must be manufactured (this is the role of reflection positivity), and the change of law must be priced in the actual marginals with second-order character-sum bounds, so that the total price converges at an explicit value of a.

The eleven ingredients, in order of first use. The genuine ideas, in the order in which the paper uses them, are: (i) reflection positivity across mirror symmetries of the slots; (ii) rescaling by progressions with square step, with an induction linear in the density; (iii) the kernel estimate of Step 3; (iv) handling small primes jointly through one signed external index per query; (v) graph laws made reflection positive by adding correction summands, the principal law itself sometimes reweighted; (vi) replacements that keep the true one-half distributions, so their differences pair to zero with constants and can be bounded by operator norms; (vii) controlling a signed, non-positive comparison law by domination at one cut, with symmetric and antisymmetric changes bounded separately; (viii) signed combinations over a whole block of primes whose first-order change is purely antisymmetric (adding an edge together with its mirror is the simplest case); (ix) subtracting variance and related savings and retaining the common mean, instead of bounding every term trivially; (x) square cliques for the primes $p \equiv 1 \pmod 4$; (xi) exact star factorisation: a *star* is a set of cross pairs sharing one endpoint; for a half law invariant under the square-affine maps

$z \mapsto rz + t$ with r a square unit, its Gram matrix is computed exactly, so star sums reduce to finite matrix tests.

What is routine verification rather than a new idea. The spectral certificates for specific graphs, the matching partitions and their extraction bookkeeping, the local constants of each prime range, the block partitions and their degree sums, the completion weights, the dominator witnesses, the tabulated middle prices, and the arithmetic that evaluates Θ. They sit behind lemmas stated once, with the finite inputs in the appendices: monotone majorants are checked at one endpoint, and polynomials with nonnegative coefficients are evaluated at upper prices.

1.3. **The proof in five steps.** The five steps are: (1) a 72-fold average at least α^{72}; (2) rescaling to shorter progressions with square step; (3) the kernel estimate, the only use of no square differences; (4) local distributions, by prime range; (5) summing the error bounds and completing the induction.

Step 1. A 72-fold average that is at least α^{72}. Let $N = K\mathsf{L}$, where $K = 890666$ is a fixed number of intervals and L is the scale. The labels live in the finite group $G_N = \prod_p \mathbb{Z}/p^{e_p}\mathbb{Z}$, the product over three kinds of primes: the primes $p \equiv 1 \pmod 4$ below $P_4 = 2^{512}$, with $e_p = 2m_{\mathrm{cl}}$ and $m_{\mathrm{cl}} = 8192$; the primes $p \equiv 3 \pmod 4$ in $[Q_3, P_4)$, where $Q_3 = 19\,951\,586$; and all primes in $[P_4, N^{1/20}]$; the last two kinds have $e_p = 1$. The prime 2 and the primes $p \equiv 3 \pmod 4$ below Q_3 are not coordinates; they enter through the external index of Step 4. The lift $F_A(z) = \sum_\xi \widehat{\mathbf{1}_A}(\xi) e(z \cdot \xi)$ sums, over the characters ξ of G_N of conductor at most $N^{1/20}$, the normalized exponential sums $\widehat{\mathbf{1}_A}(\xi) = N^{-1} \sum_{n \in A} e(-n\xi)$; it has mean $\alpha = |A|/N$, but it may be negative. The 72 slots are $V = (\mathbb{Z}/6\mathbb{Z})^2 \times \{0,1\}$: two six-chamber coordinates and a layer. Given a probability law μ on labellings $z \colon V \to G_N$, put $Y = \mathbb{E}_\mu \prod_{v \in V} F_A(z_v)$ (Definition 6.1).

Reflection positivity says that, at specified cuts of the slots, the matrix of the two halves is positive semidefinite. Here the cuts are the three chamber reflections of each six-chamber coordinate and the reserved cut ι, which flips the layer; their folds sort V (Lemma 5.13). Repeated Cauchy–Schwarz then gives a signed Hölder inequality. Unlike an ordinary moment, this still works for functions that change sign. Reflection positivity entered through statistical mechanics and constructive field theory [21, 15, 20, 19, 24]; in combinatorics the same positive-semidefinite formalism underlies graph-homomorphism inequalities and flag algebras [9, 29, 59, 45, 7]. The finite version needed here is Theorem 5.9. For the reference law $\mu = \bigotimes_{\mathcal{B}} \mu_{\mathcal{B}}$ of the construction, a product of laws on blocks of primes that are reflection positive at all these cuts and square-affine invariant, it gives $\alpha^{72} \le Y$ (Lemma 5.17).

Step 2. Rescaling to shorter progressions with square step. Fixing the labels at a block of primes specifies an integer residue class. Splitting it into progressions of square step rescales the problem to a shorter interval while preserving square-difference avoidance; the step is the least square divisible by the modulus (Lemma 8.2). The errors in this comparison are controlled in Sections 6 and 8. At a least counterexample to $Y \le C\mathsf{L}^{-a}\alpha$ (a smallest scale where the bound fails, so it holds at every smaller scale) this gives the prices of Section 1.2: each slot contributes a factor $(q_{\mathcal{B}}^2)^{a/72}$, hence $q_{\mathcal{B}}^{2a}$ over 72 slots, times the size of the change at $\mathcal{B}$. The moment bounds are linear in α, so the induction keeps the full density factor: from $\alpha^{72} \le Y \le C\mathsf{L}^{-a}\alpha$ one obtains $\alpha \le C' N^{-a/71}$, which is the conversion $\varepsilon = a/71$ (Theorem 8.12).

Step 3. The kernel estimate: the only use of no square differences. Suppose two slots are placed in adjacent intervals and, at every prime, their labels are forced to differ by a square class. Then the corresponding Fourier average is negligible: roughly, it counts pairs from A at square distance, and A has none. Making this precise is a circle-method calculation with a kernel supported on squares, for Fourier frequencies up to $N^{1/20}$ (Section 7 and Appendix C). This is the only step that directly uses the defining property of A. Everything else in the proof is about arranging, at small price, that six such forced pairs occur.

Step 4. Local distributions, by prime range. No single elementary estimate is efficient for every prime, and the class of p modulo 4 matters. If $p \equiv 1 \pmod 4$, then -1 is a square and the strict-square relation $y - x \in Q_p^*$ is symmetric. If $p \equiv 3 \pmod 4$, exactly one of $y - x$ and $x - y$ is a nonzero square, and the relation is a tournament, the Paley tournament. The construction has five ranges.

- *The small modulus.* The external modulus is $M_0 = 2^{129} \prod_{3 \le p < Q_3,\, p \equiv 3 \,(4)} p^{128}$. Each query carries an index (b, s): an interval number and a residue modulo M_0. Its coupled index is $(b + 1, s + r)$, where r is a random square residue. Averaging over r by its exact conductor gives one number R_I per query, which is how the small primes enter the closing sum (Lemma 29.6, Proposition 29.7, and Section 29).
- *The class-one prefix.* At the primes $p \equiv 1 \pmod 4$ below 2^{512} the labels are drawn conditionally independently from a random square clique of $\mathbb{Z}/p^{2m_{\mathrm{cl}}}\mathbb{Z}$, a set whose nonzero differences are even powers of p times unit squares. This law is reflection positive at every cut and accepts every query except on the event, of mass $2^{-m_{\mathrm{cl}}}$ at each such prime, that the two endpoint labels of a query coincide at that prime, which the comparison law deletes, at a total price below $8 \cdot 10^{-69}$ over all these primes (Section 19).
- *The middle range.* The primes $p \equiv 3 \pmod 4$ in $[Q_3, P_3)$, $P_3 = 167 \cdot 2^{40}$, are grouped into blocks $\mathcal{B}$ with $\sum_{p \in \mathcal{B}} \log p \le L = 570$, in 33 pieces. Each block carries a twelve-label law, completed to a reflection-positive law, and the comparison records built on it; on the seven largest pieces the principal twelve-label law is reweighted by a nonnegative function (Sections 20 to 26).
- *The upper range.* The primes $p \equiv 3 \pmod 4$ in $[P_3, 2^{512})$, in 310 bands, carry the completed two-layer law on all 72 slots, on blocks with $\sum \log p$ at most a band-dependent bound (≤ 886) (Sections 27 and 28).
- *Large primes.* Every prime $p \ge 2^{512}$, of either class, forms a block by itself, with price of order $p^{-3/2+2a}$, hence summable (Section 29).

The middle and upper laws start from a graph law: the uniform law on labellings of a small graph by distinct labels whose edges are strict-square constraints. A graph law is not reflection positive at chamber cuts: its negative cut spectrum, of order $p^{-1/2}$, is cancelled by adding folds, matchings, merges and constants with weights in powers of $p^{-1/2}$; this is a *completion* (Lemma 17.5). The weights are dictated by spectral bounds for cut operators (Theorem 14.21 and Section 14.2). Weil's bound controls them to first order; the second-order control needed in the middle and upper ranges comes from the bounds of Section 1.5. The comparison law keeps every summand that already forces the six square differences, replaces most others by forcing laws with the same one-half distributions, and deletes the rest; the price of each replacement is what Step 5 sums.

Step 5. Summing the error bounds and completing the induction. Each changed record of a block contributes its norm, relative to the actual half marginals, times $q_{\mathcal{B}}^{2a}$. On a block of logarithmic size at most L the factor $q_{\mathcal{B}}^{2a} \le e^{2aL}$ is bounded, while the local errors are those of its least prime. The prices are summed over each range with exact prime products and sieved counts below explicit thresholds and the explicit Chebyshev bound of Bennett, Martin, O'Bryant and Rechnitzer for primes $p \equiv 3 \pmod 4$ above them (Theorem 4.12; compare the explicit prime-number literature [61, 57]); the sum over the large primes is bounded by comparison with an integral. A signed product over the six disjoint queries amplifies the cancellation of Step 3. The comparison theorem turns the totals into the single number Θ, and the proof closes because $\Theta < 1$ (Section 1.6).

1.4. **What is inherited from the OpenAI preprint, and what is new here.** The overall architecture—signed functional, square-step comparison, actual-square kernel, separate treatment of the exceptional wrapping terms W, and induction—is inherited from [52]. Its Fourier-lift and density-comparison steps are descendants of the classical Roth–Gowers density-increment method

[62, 8, 25, 64, 71], but the reflection-positive functional replaces a single density increment by an induction on a much larger multilinear average. The explicit exponent comes from changing the local laws and the accounting at almost every quantitative interface.

Where the larger exponent comes from. In both proofs the final exponent has the form $\varepsilon =$ (scale exponent)/(number of slots), and two separate effects make ours larger; they multiply. The first is the denominator: the displayed OpenAI construction uses $d = (24!)^{161}$ slots, with $\log_{10} d \approx 3830.63$ (Appendix F), where we use 72; at an equal scale exponent this alone gains about 3829 orders of magnitude. The second is the numerator. The OpenAI argument does not quantify its local estimates, and for its displayed parameters the scale exponent is forced down by its mixing and modulus thresholds, making ε triple-exponentially small (Appendix F). Here every local law is rebuilt so that its price can be summed explicitly: relative rather than absolute graph estimates, primes grouped into blocks of bounded budget ($\sum \log p \le 570$ in the middle range), norms in the actual marginals, second-order character-sum bounds, and the variance and mean subtractions. The outcome is that $\Theta < 1$ already at $a = \frac{1814179}{2000000000} \approx 9.07 \cdot 10^{-4}$, and hence $\varepsilon = a/(72 - \vartheta) \approx 1.28 \cdot 10^{-5}$. The items below are the local changes behind this second effect; the most recent refinements (the joint mean allocation and the cross-weighted upper reweighting) improve ε only by a few percent on top.

Finite rectangle instead of an enormous one:
The displayed OpenAI construction uses $d = (24!)^{161}$ slots. Here $V = (\mathbb{Z}/6\mathbb{Z})^2 \times \{0, 1\}$ has 72 slots. This alone changes the conversion $\varepsilon \approx a/d$ by many orders of magnitude. It also leaves much less room for waste in the local estimates.

Relative, not merely absolute, graph estimates:
A square-constraint graph over $\mathbb{F}_p$ has very small density. Weil's bound estimates how many labels remain available after some neighbours are fixed. We divide by the graph's own expected density before normalising, so the relevant error is relative. This is what lets the same construction be used at explicit thresholds.

Small primes enter through a coupled external index:
The prime 2 and the primes $p \equiv 3 \pmod 4$ below the middle threshold enter a fixed modulus. Instead of estimating the rare event that all of their residues are simultaneously good, the proof couples the index of each query with a shifted copy, so that a selected difference is forced into a nonvanishing square class. The resulting price factor is the explicit conductor moment R_I. Exact norm envelopes at successive prime-power depths and the least square step determine its finite product (Lemma 29.6 and Proposition 29.7); the original prime moments remain an efficiently evaluated upper majorant for its large-prime part (Corollary 29.8 and Lemma 29.9).

Dependence after fixing labels is bounded as an operator norm:
After some labels are fixed, the remaining labels are not independent. Kassabov's subspace-angle inequalities [34] control dependence between two labels, and Peyre's maximal-correlation theorem [54] combines these into bounds between sets of labels; maximal correlation itself goes back to [30, 60, 2]. The finite form of Peyre's result needed here (Theorem 4.13) is proved in Appendix B. These estimates turn a heuristic "almost independent" into operator bounds that can be inserted into the budget.

Completions instead of independence:
Graph laws are not assumed independent across a cut, nor reflection positive. Their negative cut spectrum is small and explicit, and the positive corrections of a completion cancel it; the weights satisfy a few explicit inequalities dictated by the spectral bounds (Lemma 17.5).

Non-forcing records are replaced or deleted, not bounded by a union bound:
Records of the reference that do not accept the queries are repaired or deleted, not covered by a union bound over obstructions. Polarization and additive repairs produce positive records that impose the query constraints and keep the actual conditional marginals of the records

they replace. A record difference then annihilates constants, and a record of weight $O(p^{-1/2})$ costs $O(p^{-1})$.

A comparison law that is not reflection positive:

The accepting law is not assumed reflection positive, and in the middle range it is a signed combination of nonnegative accepting laws. It is compared with the first stage at one distinguished cut only, where a positive law dominates it in the 2×2 block sense; its changes are split into symmetric and antisymmetric parts with separate bounds, the antisymmetric price entering the closing sum only through $\cosh A$. The cancellation of Step 3 is applied to each nonnegative product component, so the signs cost only a constant in a vanishing remainder (Section 9).

Signed whole-block laws and positive envelopes:

In the middle range a fold record of the comparison law is a signed mixture, taken after the products over the primes of a block, of three positive accepting laws, each repaired to the actual half marginal of the record it replaces (Definition 23.9). Its first-order change against that record is purely skew: at a prime p, a three-arm star of norm about $1.2732\,p^{-1/2}$ at the parameter $\mathsf{t} = 127/500$. The case $\mathsf{t} = 0$ is mirror pairing, which adds to the record the query edge together with its mirror image under the reserved reflection; its star has two arms and norm about $\sqrt{2}\,p^{-1/2}$ (Lemma 23.5). The variation $1 + \mathsf{t}$ of the mixture does not grow with the number of primes. Products of two, four or six antisymmetric changes on distinct blocks are bounded by quantities smaller than the trivial norm estimate, using positive semidefinite dominating matrices, averaged over reflections (Sections 23 and 25).

Norms are taken in the marginals of the record being replaced:

Local laws are measured in the true one-half distribution of the record they replace, not in a product approximation. Leading terms and remainders (including coincident labels) are bounded separately, and simultaneous changes on several blocks by a power series with nonnegative coefficients in those bounds (Lemmas 10.14, 23.4 and 23.24). In the final upper range, extraction is cross-weighted: the conditional density of the witness labels is retained together with its cover marginal, rather than bounded separately and multiplied. The resulting smaller root caps lead to recomputed completion weights on 301 bands (Theorem 16.9 and Section 27.3.1).

Star-shaped features with exactly computable Gram matrices:

The star features of a square-affine invariant law have exact Gram matrices, so the leading part of a cut operator is a finite signed matrix of star incidences, certified by exact positivity tests (Section 14.2). This gives the bound at one chamber symmetry for the principal middle law, whose leading star matrix has norm exactly $\sqrt{30}$ (Section 20.4), and several upper-range bounds where old and new star matrices differ in only a few entries (Section 28 and Lemma 16.23).

Reweighting the principal twelve-label law by a nonnegative function:

On the seven largest middle subranges the principal twelve-label law (the weight-1 summand, called a *root* in Part V; the word has nothing to do with square roots) is reweighted, at every prime, by averages of squares of affine functions of three quartic trace statistics, or by a 39-trace square with a positive constant added (Definition 21.1). The reweighting keeps the symmetries, keeps the support inside the accepting set, and improves the one-sided spectral bound across a chamber symmetry; the changed one-half marginal is accounted for separately (Section 21).

Errors are tensored at the cut where they are largest:

The two repaired folded matching records of the first stage attain their largest cut norms (their sharp prices) at different cuts. All record differences whose sharp price is attained at one cut are tensored there, which makes the first-stage error and the error part of the bracket

about sixteen times smaller than a bound with one sharp factor per term (Lemma 10.8 and Corollary 10.9).

Subtractions in the closing inequality (variance, dominators, mean):
Three quantities are subtracted (or, for the mean, credited) in Θ rather than bounded trivially: the variance saved when a replacement preserves one-half distributions (Section 24); the saving from dominating matrices for products of changes (Section 25 and Proposition 26.4); and the common mean energy $m > 1$ retained from the six indices (Section 26). In the final closing, higher mean contributions are allocated jointly against the same labelled positive comparison coefficients before upper prices are substituted; only the residual series with nonnegative coefficients is then completed (Corollary 11.2 and Section 26.6). If only the variance and dominator savings were removed, with all other credits kept, Θ would exceed 1 by about 0.46 in the baseline; see Section 1.6.

Improving the power of α in the induction:
The baseline moment bounds are linear in the density, with conversion $\varepsilon = a/71$. The final argument strengthens this to $\alpha^{\frac{1014587}{1000000}}$. Most laws are controlled by grouping labels in pairs; rare cases where many labels coincide are instead handled by fixing (*pinning*) the coinciding labels and rescaling the resulting shorter progressions, and a new prefix law with independent labels removes repeated-label concentration. These steps are proved after the baseline closing (Section 31 and Section 32). The comparison error is bounded before the operator reduction, and the square-step main term is then estimated with the actual folded laws and their mixed moments, as in Proposition 32.9.

A kernel for larger cutoffs:
The actual-square kernel is proved for Fourier cutoffs up to $N^{1/20}$ and for prime-power moduli (Appendix C), so that the class-one primes can carry labels modulo $p^{2m_{\mathrm{cl}}}$.

Estimates are verified at explicit prime thresholds:
The middle range starts at $Q_3 = 19\,951\,586$ and the upper range at $P_3 = 167 \cdot 2^{40}$. These are not asymptotic placeholders: at the chosen thresholds, exact rational matrices and interval inequalities verify the hypotheses of the analytic lemmas, and monotone majorants carry each check to all larger primes. Appendix D states precisely which assertions are computational.

The general machinery and the earlier explicit design. The first explicit versions of this argument used the slot array $(\mathbb{Z}/6\mathbb{Z})^3$, with 216 slots, and a reserved cut that was a chamber reflection of the third coordinate. Most of the general machinery of Parts II–IV was developed for that 216-slot design: the chessboard calculus on products of reflection coordinates; lifts, moments, conductor decay and the actual-square kernel; the external index and its coupling; graph laws with Weil-type insertion counts and pinned influence; the spectral criterion with its coupling block, its witness extraction and the four-cycle bound; completions by folds, matchings, merges and constants; and the comparison of three laws, an all-cut reference, an intermediate law and an accepting law, by polarization and additive repairs that keep the actual conditional marginals. Parts II–IV state it once, in the generality the final design uses. The final proof is that design re-instantiated on 72 slots, with new local laws and four structural substitutions (Table 2): a two-point layer whose flip is the reserved cut replaces the third six-chamber coordinate, which changes the conversion at the same a from $a/(216 - \vartheta)$ to $a/(72 - \vartheta)$, hence to $a/71$ for the density-linear induction with $\vartheta = 1$; six disjoint one-way queries with a common signed index replace coupled families of amplified edges and their obstruction covers; the comparison law is not reflection positive, and is controlled at the reserved cut by a positive 2×2 block; and the primes are divided by their class modulo 4 into the ranges of Step 4. What the final theorem does not use are the instances of the 216-slot design: its particular laws, constants and certificates, and the proofs of its checkpoints. Section 3 explains each substitution as the answer to a quantitative failure, and Section 34.2 lists what survives of each part of that design.

1.5. **Where the Deligne bound enters.** It is useful to separate first-order and second-order character sums.

For one square constraint, or for counting extensions of one label at a time, the relevant sums contain one quadratic character of a polynomial with distinct roots. Weil's bound [73, 74, 69] gives square-root cancellation: after the normalisations of Section 12, the relative error is $O(p^{-1/2})$. Gauss and Jacobi sums evaluate the constant Fourier blocks exactly [5, 31]. These tools suffice for a large part of the construction and already appear, in a simpler form, in the OpenAI argument.

The difficulty is that the spectral estimates are conditional. One has fixed part of a graph labelling and must compare the joint law of two remaining labels with a product law. Errors from two one-variable counts can correlate. Bounding each separately by $O(p^{-1/2})$ is not enough for the negative spectral side: several terms must be $O(p^{-1})$ after normalisation, uniformly in the fixed labels and in the polynomials arising from the graph. A model sum is

$$\sum_{y,z\in\mathbb{F}_p} \chi(f(y)g(z)(y-z)),$$

where χ is the quadratic character and f, g have controlled degrees and distinct roots. This is the content of Theorem 15.13.

There is no single black box in the proof. Four algebraic-geometric facts are combined:

(i) The Grothendieck–Lefschetz trace formula rewrites the character sum as a trace of Frobenius on compactly supported cohomology.

(ii) For the family of curves $y^2 = f(x)(x-z)$, Katz–Sarnak [36] identify the relevant sheaf: it is lisse, tame, pure and geometrically irreducible, with large symplectic monodromy. In elementary language, as the parameter z moves there is no hidden invariant direction in which all the errors point the same way.

(iii) The Grothendieck–Ogg–Šafarevič formula bounds the dimension of the cohomology in terms of the degrees and ramification. This is where the explicit, degree-dependent constant comes from.

(iv) Deligne's theorem on weights [14, 13] bounds the Frobenius eigenvalues. After the normalisations in the spectral problem, this yields the required $O(p^{-1})$ rather than $O(p^{-1/2})$.

The precise external statements are isolated in Appendix A (Theorem A.6); their conversion into the estimates used by the local laws is carried out in Section 15. A reader willing to accept Theorem 15.13 and the second moments of Section 15 as named character-sum estimates can read the rest of the proof without following the sheaf argument.

A second, different second-order input concerns the complete bipartite graph $K_{2,2}$. Its character kernel is invariant under $\mathrm{PGL}_2(\mathbb{F}_p)$. The character table of $\mathrm{GL}_2(\mathbb{F}_p)$ therefore decomposes it into representation blocks, and Gauss–Jacobi identities bound those blocks. The resulting normalised estimate is $2/p$ (Theorem 15.8). This is representation theory, not the Katz–Sarnak sheaf input, although both feed the same spectral envelopes.

In the construction these bounds are used wherever two remaining labels must be compared with independent labels: in the spectral bounds for middle and upper laws (Section 14), in the non-star terms of the character expansion (Section 16), and in conditional distributions given a fixed middle label. The same sheaf family also gives mean-square bounds for trace functions with one free label, used for the two principal upper cuts below 2^{52} (Theorem 14.27). Precise entry points are cited in each Part V section.

Two later local estimates should be distinguished. The complete $K_{3,3}$ character kernel is bounded by an exact polynomial-resultant Gram identity, not by the cohomological input above (Theorem 15.12). The centred rectangle bound uses the symmetric square of the same Legendre sheaf, through the Mellin eigenvalue estimate of Lemma 20.18; its weaker $128q$ form avoids that symmetric-square step (Theorem 20.19).

This is the same circle of ideas as the modern theory of algebraic trace functions. Fouvry–Kowalski–Michel recast many character and exponential sums as Frobenius traces, then use Deligne's bounds and monodromy to show that no main term survives [16, 17]. Will Sawin's work is a prominent part of that programme, both over the integers—bilinear forms with Kloosterman and more general trace functions [41, 42, 18]—and over $\mathbf{F}_q[T]$, where the geometry is strong enough to prove function-field Chowla/twin-prime statements and near-square-root cancellation for factorization functions [68, 66, 40]. His quantitative sheaf theory [67] also addresses the same practical question faced here: after Deligne gives square-root size for Frobenius eigenvalues, how does one bound how many of them there are, uniformly and explicitly?

The overlap is methodological rather than a direct dependence. We do not quote a theorem of Sawin. Our sheaves come from one fixed family $y^2 = f(x)(x - z)$ over $\mathbb{F}_p$, arising from two remaining labels in the graphs of the local laws; the required output is not an asymptotic cancellation statement but an explicit $O(p^{-1})$ spectral envelope with a degree-dependent constant that can be inserted into the spectral caps at explicit thresholds. The Sawin–Shusterman results cited above concern increasing polynomial degree over a fixed finite field satisfying their size and characteristic hypotheses. What is shared is the mechanism: interpret a character sum cohomologically, use monodromy to exclude a hidden main term, bound conductors and cohomology dimensions, and let Deligne's weights supply the cancellation.

One further input is modular. The second moment of the neighbourhood count behind the variance credits of the middle range (Lemma 24.3) contains the twisted second moment $\sum_\lambda \chi(\lambda)T_\lambda^2$ of the point counts $T_\lambda = \sum_y \chi(y(y-1)(y-\lambda))$ of the Legendre family of elliptic curves, for which Weil's bound alone gives only $O(p^2)$. Ahlgren and Ono [1] evaluate it as $-p - a(p)$, where $a(p)$ is the p-th coefficient of the weight-four cusp form $\eta(2z)^4\eta(4z)^4$ of level 8, and Deligne's proof of the Ramanujan–Petersson conjecture [14] gives $|a(p)| \le 2p^{3/2}$ (Theorem A.7). With this evaluation the normalised second moment $\mathbb{E}[P_{\mathrm{adm}}\mathcal{W}^2]/p^2$ of Lemma 24.3 is at most $5/p + O(p^{-3/2})$, while the general bound for the graph moments it contains gives only $10/p + O(p^{-3/2})$. Through the root-half variances of the middle roots it enters the recomputation (restoration) of the variance credits after a tilt changes the root half marginal (Section 24) and the cut caps of the middle roots (Sections 20.4 and 21).

Why the qualitative result did not need second-order bounds. They were unnecessary, not unavailable. The OpenAI theorem asserts the existence of some $\varepsilon > 0$. For that purpose one may choose an enormous number of slots and an enormous prime threshold, and lose very large absolute constants, provided the errors eventually become small. The elementary one-edge Gauss-sum calculation and telescoping graph estimates in [52, Section 2] are enough for that qualitative conclusion.

An explicit exponent is stricter in three ways. First, $d = 72$, so there is no huge denominator in which to hide losses when passing from the functional exponent a to $\varepsilon = a/71$. Secondly, the laws must be certified at named finite thresholds, principally $Q_3 = 19\,951\,586$ in the middle range and $P_3 = 167 \cdot 2^{40}$ in the upper range. Thirdly, the closing sum has no room for a correlated first-order term: the certified spectral caps, completion variances and upper-root bounds are evaluated with the second-order estimates, and the baseline closing margin $1 - \Theta$ is only about $2.7 \cdot 10^{-11}$ (the final joint closing margin is $> 2.29 \cdot 10^{-7}$), so the certified exponents depend on them. The closing certificates use the four-cycle bound, Deligne's theorems, Katz–Sarnak and the evaluation of Ahlgren and Ono to obtain the second-order bounds entering that budget. This explains their role in this proof; it does not establish that these inputs are necessary for a comparable exponent by another argument, and whether a smaller exponent can be certified with first-order bounds alone has not been tested.

1.6. **The closing inequality $\Theta < 1$ in the baseline ($\vartheta = 1$).** The comparison theorem (Theorem 9.21, Theorem C) reduces the baseline ($\theta = 1$, where θ denotes ϑ in Part III) to nine hypotheses (CP0)–(CP8)—standing data, reference law, first-stage law, domination at the reserved cut, comparison law, optional sharper bounds for named families of simultaneous record changes, dominators for the comparison law, mean energies, and admissibility—and to the single inequality $\Theta < 1$ for the number (1.1) with $k = 6$ queries on $d = 72$ slots, where

$$\mathcal{E} = B(e^S \cosh A - 1) - D, \qquad J = Be^T \Big(\frac{R_I}{2}\Big)^6,$$
$$W = Be^{S+A}\Big[\Big(1 + \frac{R_I}{2}\, K^{-35/36}\Big)^6 - 1\Big].$$

Here S and A are the total symmetric and skew prices of $\nu - \lambda$, B is the energy factor of the law dominating λ (so $\mathcal{E}$ is the dominated price bound with savings D subtracted), T is the total dominator price entering through e^T (not an exponent of p), $R_I = \mathbb{E}(K \operatorname{sq}(D))^{2a/72}$ is the conductor factor of one external index (Definition 9.6), and the displayed formula for $\mathcal{E}$ is the baseline specialisation of the general complete-polynomial bound Ψ of Theorem C (that is, an admissible upper bound for Ψ at the baseline prices, in the notation of Theorem 9.21). The $63/64$ factor records that the empty term of the signed-index expansion is the comparison average itself; in J each query contributes $R_I/2$; W bounds the terms in which every active query wraps from the last interval to the first; and $m > 1$ lowers Θ through $(1 - m)/64$ (Remark 9.23).

Each quantity is constructed in Part V. The first stage λ differs from μ only on middle blocks; its price L_0 and the dominating law N, with energy factor $B = e^{U_0} + R_{\mathrm{err}}$ (where U_0 is the total price of the positive records added to μ in the bracket and R_{err} is the remaining two-cut first-stage error, Definition 9.13), are constructed in Section 22. The prices S and A are summed over all prime ranges: the deleted coincident-label event of the $p \equiv 1 \pmod 4$ prefix (Section 19, in the baseline counted in S only), the middle-range replacements (Section 23), the upper-range replacements (Section 28), and the large primes (Section 29). The savings D are the variance savings of Section 24 together with the dominator savings of Section 25 for even products of skew differences. The price total T and the mean constant m come from the dominators of ν in Section 26; R_I, J and W are computed from the index law at $K = 890666$ (Section 29). Section 30 assembles the baseline Θ; middle prices first computed at the evaluation exponent $a_0 = \frac{4442183}{5000000000} < a$ are converted to a by $t = e^{2(a-a_0)L}$ with $L = 570$ (this conversion is called *repricing* and needs $a \geq a_0$), and upper prices are converted band by band from $a_{\mathrm{u}} = 883/10^6 < a$. Table 1 lists the baseline values only. The improved closing in Section 33 uses $a = \frac{1814179}{2000000000}$, $K = 863660$ and $\vartheta = \frac{1014587}{1000000}$, replaces the wrap exponent by $-1 + \vartheta/36$, recomputes every price, and counts the new prefix deletion in both S and A; its value $\Theta < 0.999999771$ is the one used for Theorem 1.1. With those recomputed prices, the final closing number is

$$\Theta = \frac{63}{64}(L_0 + \mathcal{E}) + J + W + \frac{1 - m_1}{64} - \Delta_{\mathrm{mid}} - \Delta_u + \mathcal{R}_u < 0.999999771 < 1,$$

where m_1 retains only the one-block mean terms, after the first-stage error reserve, Δ_{mid} and Δ_u are the joint middle and finite-upper mean allocations, and $\mathcal{R}_u$ is the reserve for the latter allocation. The allocations are made inside the common positive completion before the shared energies are replaced by upper prices; they replace the higher-degree mean credits of Proposition 26.7 and are not additional credits to be added to the baseline m. This is (33.5); its complete ledger is in Section 33.

The certificate (Certificate 30.3) evaluates Θ in exact rational and outward interval arithmetic and gives $\Theta < 0.99999999997327$ (baseline), so that $1 - \Theta > 2.673 \cdot 10^{-11}$. In round numbers, $(63/64)(L_0 + \mathcal{E})$ contributes about 0.758, J about 0.251 and W about $4 \cdot 10^{-5}$, and $(1 - m)/64$ about -0.0092. The exponent a is chosen so that $\Theta < 1$ holds. Every price grows with a through

TABLE 1. Baseline closing only ($\vartheta = 1$, $a = \frac{5422587}{6103515625}$; this table does not prove Theorem 1.1, whose improved values are in Section 33): certified upper ($<$) and lower ($>$) bounds, rounded outward; ... marks an exactly defined number (Table 12 gives more digits).

term	meaning	value
t	repricing factor $e^{2(a-a_0)L}$ of the middle prices	$1.0000000616512\ldots$
U_0	exponent in e^{U_0} below (dominating law)	< 0.84691
B	energy factor $e^{U_0} + R_{\mathrm{err}}$ of that law	< 2.33263
L_0	total price of $\mu \to \lambda$	$< 4.7634 \cdot 10^{-4}$
S	total symmetric price of $\nu - \lambda$	< 0.014555
A	total skew price of $\nu - \lambda$	< 0.96928
D	savings; of these, variance part $D_{\mathrm{var}} > 0.0099585$	> 0.46685
$\mathcal{E}$	dominated bound $B(e^S \cosh A - 1) - D$	< 0.76897
T	dominator price total (in e^T)	< 0.60760
R_I	conductor factor of one index, $K = 890666$	< 1.24714
J	coupled term $Be^T(R_I/2)^6$	< 0.25177
W	wrapping bound	$< 3.8345 \cdot 10^{-5}$
m	mean constant (trivial value 1)	> 1.58994
Θ	closing number (1.1)	< 0.99999999997327

$q_{\mathcal{B}}^{2a}$, so a larger a needs smaller local bounds. The displayed point is feasible; no optimality is claimed. The construction itself is specified in Section 18 (Definition 18.1), and Table 5 lists, for each hypothesis of Theorem C, the statement of Part V that verifies it.

1.7. **Proof inputs and verification.** The proof has three kinds of input, and the paper keeps them separate.

First, there are the arguments proved in the text: Parts II–III (positivity, the kernel estimate, rescaling and induction, and the comparison theorem) and the local estimates of Part IV. They occupy most of the paper and do not depend on the numerical choices of Part V.

Secondly, there are named external theorems: Weil's bound for character sums of polynomials (Theorem 4.7); the explicit Chebyshev bound of Bennett, Martin, O'Bryant and Rechnitzer [4] for primes $p \equiv 3 \pmod 4$ (Theorem 4.12); Kassabov's angle inequality (Theorem 4.16); the character table of $\mathrm{GL}_2(\mathbb{F}_p)$; and, for the second-order sums, the ℓ-adic formalism: the Grothendieck–Lefschetz trace formula, the cohomology of curves, Deligne's theorem on weights, the Grothendieck–Ogg–Šafarevič formula and the Katz–Sarnak description of the family $y^2 = f(x)(x - z)$, including its local monodromy at the finite roots; and the evaluation by Ahlgren and Ono of a twisted second moment of the Legendre point counts, with Deligne's bound for the coefficients of a modular form (Theorem A.7). Their exact entry points are stated near the sections that use them and collected in Appendix A; Figure 1 is a compressed map. Peyre's theorem is proved in the finite form needed (Appendix B), the Gauss and Jacobi sums are evaluated in Section 4, and the other prime sums are exact finite computations or sums over all integers.

Thirdly, there are finite certificates. A *checked inequality* is a single finite matrix or numerical inequality verified by exact rational or outward-interval arithmetic; a *certificate* packages such verifications and proves nothing analytic. Examples are the thirteen completion inequalities for each of the 33 middle intervals, the positivity tests for the middle spectral bounds, and the 32 550 completion and domination inequalities for the 310 upper intervals, together with threshold checks and the evaluation of Θ. Appendix D lists every finite inequality used, with its verification method and data file (Table 19). A successful run verifies those finite assertions and the identity of their inputs; it is not a formal verification of the analytic reductions or of the external theorems.

This work was developed in parallel lines of work assisted by AI models (OpenAI Codex and Anthropic Claude models) and reviewed internally by AI models, including independent re-implementations of the finite computations; the closing number has been computed by two separately written programs; on each of the 45 compared scalar components of the closing evaluation (Certificate 30.3) their outward-interval enclosures overlap. The final joint closing also has author and independent complete assemblies (Certificate 33.3). These reviews are internal, not external referee reports; Section 35.12 states their scope.

1.8. **Terminology used throughout.** The proof uses several dozen nouns with precise, non-standard meanings. Each is defined formally where it first becomes a mathematical object (noted below); this subsection gives a one-sentence meaning usable from the start, so that roadmaps and theorem statements can be read in order. Where one English word has several meanings, the entry separates them; the bare ambiguous word is not used in titles.

Slots, configurations, and laws. A *slot* is one of the 72 positions $v \in V = (\mathbb{Z}/6\mathbb{Z})^2 \times \{0,1\}$ at which the Fourier lift F_A is evaluated in $Y = \mathbb{E}_\mu \prod_v F_A(z_v)$ (formally Section 5). A *configuration* is a map $z\colon V \to G_N$; z_v is its *label* at v. A *law* is a probability or finite signed distribution on configurations. The *reference law* μ is the reflection-positive product law defining Y; the *first-stage law* λ is μ after the small modifications of (CP2); the *comparison law* ν is the signed law of (CP4) whose nonnegative summands accept the queries (below). A *cut* is a mirror symmetry of V with a chosen plus half; the *reserved cut* ι flips the last coordinate and is reserved for the domination in Part III, the other cuts serving the lower bound. A law is *reflection positive* at a cut if the matrix coupling the two halves is positive semidefinite. A *query* is one of six fixed disjoint ordered slot pairs whose difference ν forces to be a square; *forces* means the square condition holds almost surely, while *accepts* (in (CP4)) means in addition the required uniform one-slot marginals and the conditions (R4) and (R5) of Section 6 hold, with nonvanishing square classes at the prime-power coordinates. Graph *edges* (Part IV) are different from queries and carry only the field-prime square condition.

Records and components (distinct). A *record* is one indexed weighted summand in the explicit expansion of a local law (formally Definition 9.9, where the phrase "nonnegative measure" must be read as this summand sense; differences of records are signed). A *component* of $\nu_{\mathcal{B}} = \sum_i c_i \nu_{\mathcal{B},i}$ is one nonnegative summand $\nu_{\mathcal{B},i}$ of the signed combination. Records are what is changed in $\nu - \lambda$; components are what must accept. Where the text below calls a computer file a "record" (in certificate descriptions), read *data file*; the mathematical meaning above is the only one used in statements. An *actual marginal* is a one-half distribution computed from the law itself, not from a product or uniform model; *shared labels* s are auxiliary conditioning variables in Definition 9.2(b) such that, conditional on s, a law splits across a cut. A record difference that *annihilates constants* has zero pairing with constant half-functions, which is why it can be priced by an operator norm.

Prices. Fix a block $\mathcal{B}$ of primes with product $q_{\mathcal{B}}$. The *price* of a signed measure ζ is $\|\zeta\|_{\mathrm{var}} q_{\mathcal{B}}^{2a}$; a *sharp price* is $w d_\zeta q_{\mathcal{B}}^{2a}$, where w is the weight of the record carrying ζ and d_ζ is the uniform bound in the shared-label decomposition of Definition 9.2(b); an *operator price* is a weight times an operator norm in the actual half marginals times $q_{\mathcal{B}}^{2a}$. An *upper price* is any number at least the price. The noun *cost* in the text below always means one of these prices or a class bound just defined; it is never a fourth, undefined quantity—when in doubt, replace "cost" by the price displayed in the same paragraph. In (CP5) a *class cost* $c_{\mathcal{C}}$ is the scalar in the special case $\varphi = c_{\mathcal{C}}\pi$, in which the class-bound polynomial φ is a multiple of the monomial π, the product of the prices of the selected records; the two are not the same. The superscripts in y^{arm}, y^{lead}, y^{rem} (and their totals A^{arm}, A^{lead}, A^{rem}) are labels for three auxiliary upper prices attached to one skew record in the degree-class replacement of Lemma 10.14(v) (the formal degree of a record type is the exponent of $r = p^{-1/2}$ in its graded completion weight, Definition 17.1): the first, the raw-arm price, is the auxiliary price whose product over the records of a selection is the first term of the class bound

there, and the second and third split the full skew price into its leading star part and a remainder. *Repricing* converts prices computed at an evaluation exponent a_0 to the theorem exponent $a \geq a_0$ by powers of $q_{\mathcal{B}}$; the inequality needs $a \geq a_0$. A *saving* is a proved amount subtracted from a price bound; the mean constant m is exceptional, being a lower bound added to an energy (hence $(1-m)/64$ in Θ). A *robust* bound (Definition 9.19) is one whose $o(1)$ error comes only from the two analytic remainders allowed in Part II; it is used in (CP5) and (CP7). A *complete polynomial* Ψ is a power series with nonnegative coefficients dominating every simultaneous-change bound coefficientwise; it is defined in the remark after Theorem C in Section 9 and developed in Part III—read that definition before applying Theorem C with families.

Dominations (four uses of "envelope"). A 2×2 block $\left(\begin{smallmatrix} M & K \\ K^* & M' \end{smallmatrix}\right) \succeq_\iota 0$ of cut matrices implies $|Q_K(x,y)| \leq Q_M(x,x)^{1/2} Q_{M'}(y,y)^{1/2}$ and is called a *bracket* at ι; the bracket of (CP3), dominating λ by N at the reserved cut, is also called the *reserved Gram bracket*. The *direct Gram* is the Gram matrix computed jointly for a reference fold record and its signed replacement (Section 26.1), rather than from separate norm bounds for the two. (CP3) brackets λ by N; (CP6) brackets ν itself by M_L, M_R (called *matching envelopes* in (CP6) and *dominators* in prose, to distinguish them from the *matching records* E_0, E_1); (CP4) record differences are bracketed by diagonal envelopes; Part V Section 25 uses *dominating matrices* D with $-D \preceq T \preceq D$ ("positive" means positive semidefinite there). Where *envelope* occurs without a qualifier below, the display in the same paragraph identifies which of the four is meant. Depth tails $\delta_{p,j}$ bound the norm of the index shift operator T on the exact-conductor sector of depth j at p; R_I is their averaged factor (Definition 9.6).

Prime packaging. An *atom* is one tensor factor of μ; a multi-prime middle or upper atom is a *prime block* $\mathcal{B}$ with $\sum_{p \in \mathcal{B}} \log p \leq L$ ($L = 570$ in the middle range; this roman L bounds $\log q_{\mathcal{B}}$ and is not the scale L). Consecutive blocks sharing constants form 33 middle prime intervals (called *pieces* in Part V displays) and 310 upper prime intervals (called *bands*); no law lives on a prime interval. Unqualified, *interval* in the index sections means one of the K subintervals of $[N]$, never a prime interval. "Piece" also occurs in Part II for a Fourier support component f_U; the two senses are distinguished as *middle interval* versus *support component*. The *prefix* is the range $p \equiv 1 \pmod 4$, $p < 2^{512}$; *large primes* $p \geq 2^{512}$ are one atom each (the word *tail* is used only with the qualifier *large primes* or *depth tail*). A *graph labelling* is a map $W \to \mathbf{F}_p$ sending each directed edge to a nonzero-square difference; *injective* is a separate adjective, and $\lambda_p(G)$ is uniform on the injective ones (Section 12). The weight-1 twelve-label summand is called the *principal law* (in Part V, the *root*); its nonnegative reweightings are *tilted laws*. *Completion* adds the four smaller summand types making a graph law reflection positive; *polarization* dominates the two nonaligned fold records at a chamber cut by the two matching records plus a merge correction, in the cut order at that cut; a *replacement at fixed marginals* (called a *repair* in Part V) substitutes a forcing law with exactly the replaced law's half marginals. Part V distinguishes *kept* (already forces), *deleted* (dropped, mass priced), *repaired one-way* (rebuilt from two half copies and the projected queries), and *replaced densely* (old cross graph kept, missing required edges added and the unrequired old edges meeting them softened); "repair" never means a small perturbation.

Patterns, folds, and channels. A *pattern* is a map $\phi\colon \mathbb{Z}/6\mathbb{Z} \to O$ into a small set of *objects* saying which object supplies the label at each of the six positions of one chamber coordinate (Definition 17.1): the identity (the root, type L), the three *folds* F_0, F_1, F_2 (type D), the two *matchings* E_0, E_1 (type S), the six *merges* (type R), and the constant maps (type C). A *fold record* is a record using a fold pattern, and its *fold chamber* is the chamber cut at which it is diagonal. A fold is *aligned* at a chamber cut when that cut is its fold chamber, so that it is diagonal, hence positive, there; at the other chamber cuts it is *nonaligned*. A *folded copy* (also *reserved-folded copy*) of a one-layer record is that record placed on the reserved diagonal, $\mathrm{Diag}_\iota(\,\cdot\,)$, with the same one-layer labels on both layers; this is the sense in names such as the folded matching records and

the measures F_{fold}, E_{fold} of Definition 22.4. This is a different use of the word from Part II, where the *folds* $\phi_\pm$ of a cut (Definition 5.7) are the two slot maps copying one half of V onto all of V, and a *folded energy* is the energy of such a copied input. At one prime $p \equiv 3 \pmod 4$ the four *Paley channels* $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$ are the explicit kernels χ(difference) between labels of the two halves of a fold record (23.1); a monomial in the channels is a *star monomial* if all its edges share one endpoint and a *nonstar* otherwise.

Further nouns in Parts IV–V. A *chamber* is one $\mathbb{Z}/6\mathbb{Z}$ coordinate value; a *chamber cut* is the reflection $u \mapsto 2a - 1 - u$ in one such coordinate. A *star* (above) is a set of cross pairs sharing an endpoint; a *spectator* is a tensor factor left unchanged. A *chart* is one fixed half law whose L^2 space is used to state and prove operator bounds for several laws at a cut (Remark 22.3); bounds proved in a chart are transported to the actual marginals by the lemmas named where they are used, so "in the fixed half chart" always means "in L^2 of the fixed half law displayed there". A *cell* is one of the geometric prime intervals $[x_j, x_{j+1})$ above the explicit Chebyshev threshold X_{Ch} in the block-charge estimate (Lemma 20.4), inside which primes are grouped into blocks of a fixed number n_j (not a group cell). A *gate* or *checked inequality* is one finite inequality verified in Appendix D; a *coupon* is one proved saving entering a single coefficient (in particular m), as defined in Section 24; a *cap* is a certified spectral bound $-gp^{-1/2}I \preceq A_p \preceq (1 + c^+/p)I$ for a named normalised cut operator A_p, with named constants g and c^+. The adjective *selected* means the Part V construction, except in Sections 8 and 11 where a *selected atom* is an atom of a structured energy and a *selected law* is a law occurring in an expansion, as defined there; a *selected diagonal* is $\mathrm{Diag}_\iota(m_\phi)$ for an eligible record ϕ (Section 24); *native* means computed directly from the law in its own actual marginals, not obtained by a conversion such as repricing from a_0 or transport from a fixed half chart; *final* means the construction of the closing under discussion—check whether baseline or improved parameters are in force; *anchor* and *evaluation exponent* both mean the smaller exponent a_0 at which a price was first computed. A *ledger* is an explicit list of the records of one prime range together with their weights and prices, whether displayed in the text or stored in a named data file; the *twelve-query ledger* is that list for the twelve queries $\mathcal{E}_{12}$, retained in the upper and tail ranges even though only six queries are used in the closing. The *frame* has two defined uses: the *frame data* of (CP0) in Part III (the standing data of Definition 9.1), and the *middle frame* (Definition 22.4), the package consisting of μ, its named records, the first stage λ and its dominator N on a middle block. Elsewhere "frame" is an informal word for the standing choices (slots, cuts, external index, exponents) within which an estimate is stated, and the display in the same paragraph always says which choices are meant. A *tag* is one of the two record differences of the first stage in (CP2)(b), one for each folded matching type. A *packet* occurs only in Section 32, where it means a set of atoms changed simultaneously.

Index and kernel. Each query carries an *external index* (b_e, s_e) (interval and residue); the *coupled index* is $(b_e + 1, s_e + r_e)$ with r_e a random square class. *Wrapping* is the case $b_e = K$, where $b_e + 1$ wraps to the first interval and kernel cancellation fails; W bounds the terms in which every active query wraps. The *kernel estimate* (Section 7 and Appendix C) is the statement that ordered pairs forced into a square class with no integer-square difference have negligible Fourier pairing.

Closings. Θ denotes the closing number for the stage being discussed: (1.1) in the baseline and (33.5) in the improved stage. The *baseline closing* (Section 30) has $\theta = \vartheta = 1$ and $\Theta < 0.99999999997327$; the *improved closing* (Section 33) has $\vartheta = \frac{1014587}{1000000} > 1$ and $\Theta < 0.999999771$ and uses an induction beyond Theorem C, whose hypothesis (CP0) requires $0 < \theta \le 1$. Appendices D and E tabulate the baseline unless a caption states the improved closing.

1.9. **Reading the proof.** The paper has six parts and six appendices. Part I is this introduction, the one-query miniature in Section 2 (Proposition 2.1), and the design explanation in Section 3. Part II (Sections 4 to 8) proves the lower bound $Y \ge \alpha^{72}$, defines Y, proves the kernel estimate, and proves rescaling and the baseline induction. Part III (Sections 9 to 11) states and proves Theorem C.

Part IV (Sections 12 to 17) proves the local estimates: counting labellings, cut operators, spectral bounds, second-order character sums, and how to make a graph law reflection positive. Part V (Sections 18 to 30) is the construction: it specifies the design, verifies the hypotheses of Theorem C range by range, each section ending with the rows of Table 5 that it verifies, and evaluates the baseline Θ in Section 30; the middle range takes seven of its sections (Sections 20 to 26: blocks and spectral bounds for the twelve-label law, its nonnegative reweightings, the reference and first-stage laws with their domination, the comparison records, the variance savings, the dominating matrices, and the dominators for ν with the mean constant). The improved closing follows in Sections 31 to 33. Part VI (Sections 34 and 35), placed after the proof, explains how the construction developed and records obstructions and limits. The appendices contain the external inputs (Appendix A), Peyre's theorem in finite form (Appendix B), the kernel proof (Appendix C), the finite certificates (Appendix D), the baseline data tables (Appendix E), and the OpenAI exponent calculation (Appendix F).

The logical dependence is simple: Part III uses Part II, Part V uses only the statements of Parts II–IV, and Parts I and VI cite nothing as a premise. A reader who wants the logical skeleton can read Sections 2 and 3, the statements of Part II, Section 9, Table 5, Section 18 and Section 30, accepting the local estimates. A reader who wants to see where the quantitative work occurs should then read Part IV and follow the prime ranges through Sections 19 to 29.

2. The whole argument with one query

What this section proves and how to read it. This section proves nothing new; it assembles, with one query instead of six, the four facts on which the whole proof rests, in the form Proposition 2.1: at a least counterexample, $1 \le \mathcal{E} + BR_I/2 + W_1 + o(1)$, where $W_1 = BR_I/(2K^{35/36})$ is the one-query wrapping term. The four facts are: (1) a reflection-positive law gives $\alpha^{72} \le Y$ for the 72-fold Fourier average Y (Section 2.1); (2) at a least counterexample every change of the law is bounded by its price times Y, because fixing labels restricts to shorter square-step progressions (Section 2.2); (3) small primes enter through an external index coupling the two ends of the query (Section 2.3); (4) coupled terms in which the query is accepted and intervals do not wrap are negligible by the kernel estimate (Section 2.4). Inputs: the definitions of Part II, quoted by label. Output: the one-query inequality, whose six-query form is Theorem C (Theorem 9.21); Part V applies it with six queries. A first reading should follow the statements and the sketch after Proposition 2.1 and skip the quoted hypothesis checks; nothing here is a premise of a later section. Section 2.6 explains why one query cannot give $\Theta < 1$, and Section 2.7 why conditioning independent labels does not work, which motivates Section 3.

2.1. **The 72-fold average and reflection positivity.** Let N be large (in particular so large that the prime ranges below are nonempty as stated), let $A \subseteq [N]$ be square-difference-free, and let $\alpha = |A|/N$. The labels of the proof live in the finite group $G_N = \prod_p \mathbb{Z}/p^{e_p}\mathbb{Z}$, a product over three kinds of primes (Definition 6.1): the primes $p \equiv 1 \pmod 4$ below $P_4 = 2^{512}$, with $e_p = 2m_{\mathrm{cl}}$ and $m_{\mathrm{cl}} = 8192$; the primes $p \equiv 3 \pmod 4$ in $[Q_3, P_4)$, $Q_3 = 19951586$; and all primes in $[P_4, N^{1/20}]$, the last two kinds with $e_p = 1$. The prime 2 and the primes $p \equiv 3 \pmod 4$ below Q_3 are not coordinates of G_N; they enter in Section 2.3. The *lift* of A is

$$F_A(z) = \sum_{q(\xi) \le N^{1/20}} \widehat{\mathbf{1}_A}(\xi)\, e(z \cdot \xi), \qquad \widehat{\mathbf{1}_A}(\xi) = \frac{1}{N} \sum_{n \in A} e(-n\xi),$$

the sum over the characters ξ of G_N whose conductor $q(\xi)$ is at most $N^{1/20}$. It is real, and its mean over the uniform law on G_N is the coefficient of the trivial character: $\mathbb{E}F_A = \alpha$.

Slots, cuts and reflection positivity. Let V be a finite set of *slots*, $d = |V|$. For a finite signed measure κ on the labellings $z = (z_v)_{v\in V} \in G_N^V$ and real functions g_v on G_N put

$$\Lambda_\kappa((g_v)_{v\in V}) = \int \prod_{v\in V} g_v(z_v)\, d\kappa(z), \qquad \Lambda_\kappa(g) = \Lambda_\kappa((g)_{v\in V}).$$

The average of the proof is $Y = \Lambda_\mu(F_A)$ for one fixed probability measure μ, the *reference*. A *cut* is a fixed-point-free involution ϑ of V with a decomposition $V = V_+ \sqcup V_-$, $V_- = \vartheta V_+$. The *cut matrix* of κ at ϑ is $C_\vartheta(\kappa)_{x,y} = \kappa(z_{V_+} = x,\ z_{\vartheta V_+} = y)$, indexed by $x, y \in G_N^{V_+}$, and its *cut form* is $Q_\kappa(X, X') = \langle X, C_\vartheta(\kappa) X'\rangle$ for *half-functions* $X, X' \colon G_N^{V_+} \to \mathbb{R}$. If X and X' are products of functions of the single labels, $Q_\kappa(X, X')$ is the average Λ_κ in which the factors of X sit on V_+ and those of X' are carried to V_- by ϑ (5.1). The measure κ is *reflection positive* at ϑ if $C_\vartheta(\kappa)$ is symmetric and positive semidefinite (Definitions 5.1 and 5.2). Reflection positivity at a cut is preserved by sums, positive multiples and tensor products (5.2).

One cut, and folding repeatedly until all slots agree. Let ϕ_+ be the map of V that fixes V_+ and sends $v \in V_-$ to ϑv, and ϕ_- the map that fixes V_- and sends $v \in V_+$ to ϑv. If κ is reflection positive at ϑ, the Cauchy–Schwarz inequality for its cut form gives, for every assignment $h \colon V \to \mathbb{R}^{G_N}$ of functions to slots (Lemma 5.8),

$$\big|\Lambda_\kappa(h)\big|^2 \le \Lambda_\kappa(h\circ\phi_+)\,\Lambda_\kappa(h\circ\phi_-), \qquad \Lambda_\kappa(h\circ\phi_\pm) \ge 0, \tag{2.1}$$

where $\Lambda_\kappa(h) = \Lambda_\kappa((h(v))_v)$: a mixed average is at most the geometric mean of the two averages in which one half has been copied onto the other. A sequence of cuts $\vartheta^1, \dots, \vartheta^T$ with signs $s_t \in \{\pm\}$ is *sorting* if the composition $\phi_{s_T}^{\vartheta^T} \circ \cdots \circ \phi_{s_1}^{\vartheta^1}$ of its folds is constant (Definition 5.7).

Theorem (Signed Hölder inequality)**.** *If κ is reflection positive at every cut of a sorting sequence, then for all real functions g, g_v on G_N* (Theorem 5.9)

$$\Lambda_\kappa(g) \ge 0 \qquad \textit{and} \qquad \big|\Lambda_\kappa((g_v)_{v\in V})\big| \le \prod_{v\in V} \Lambda_\kappa(g_v)^{1/d}.$$

Sketch. Fix finitely many functions $f_1, \dots, f_k$, and let M be the largest value of $|\Lambda_\kappa(h)|$ over the assignments h with values in $\{f_i\}$. If h attains M, both folded assignments $h\circ\phi_\pm$ again take values in $\{f_i\}$. By (2.1) they are nonnegative, at most M, and their product is at least M^2, so both equal M: folds of maximizers are maximizers. Folding along the whole sorting sequence produces a constant maximizer, so $M = \max_i \Lambda_\kappa(f_i)$. Applied to $f_v = g_v/(\Lambda_\kappa(g_v) + \epsilon)^{1/d}$, $\epsilon > 0$, multilinearity gives the inequality as $\epsilon \to 0$. This is the chessboard estimate of statistical mechanics in finite form. Consequently $\|g\|_\kappa = \Lambda_\kappa(g)^{1/d}$ is a seminorm, and if every one-slot marginal of κ is uniform, then $|\mathbb{E}g| \le \|g\|_\kappa$, by the inequality with g at one slot and 1 at the others (Corollary 5.10). With $g = F_A$ this is $\alpha^d \le \Lambda_\kappa(F_A)$. The inequality says more: every average of the same shape, with arbitrary real functions at the slots, is controlled by the diagonal averages of its inputs. This is what makes the prices of Section 2.2 possible.

The slot array. The slots form $V = \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \{0,1\}$, $d = 72$, with slots (u_0, u_1, ℓ) (Definition 5.12). On each chamber coordinate $\mathbb{Z}/6\mathbb{Z}$ there are the chamber reflections $\vartheta_a(j) = 2a - 1 - j$ with plus halves $H_a = \{a, a+1, a+2\}$, and $\vartheta_{c,a}$ is ϑ_a acting on u_c. The layer carries the reserved cut $\iota(u_0, u_1, \ell) = (u_0, u_1, 1-\ell)$, with plus half $V_+ = \{\ell = 0\}$. The plus folds of ϑ_0 and ϑ_4 are $f_0 = (0,1,2,2,1,0)$ and $f_4 = (0,0,5,4,4,5)$ (values at $0, \dots, 5$), and $f_0 \circ f_4 \circ f_0$ is constant. Folds of different coordinates commute, so the seven-step sequence $\vartheta_{0,0}, \vartheta_{0,4}, \vartheta_{0,0}, \vartheta_{1,0}, \vartheta_{1,4}, \vartheta_{1,0}, \iota$, all with sign $+$, maps every slot to $(0,0,0)$ (Lemma 5.13(ii),(iii)). The family $\mathfrak{F}$ of the chamber cuts and ι consists of seven involutions. The reference is a product $\mu = \bigotimes_{\mathcal{B}} \mu_{\mathcal{B}}$ over the *atoms* $\mathcal{B}$ of G_N (the single primes of the prefix, blocks of the primes $p \equiv 3 \pmod 4$ in $[Q_3, P_4)$, and the single tail primes) of probability measures that are reflection positive at every cut of $\mathfrak{F}$ and invariant under

the maps $z_{v,p} \mapsto r_p z_{v,p} + t_p$, r_p a square unit, chosen independently at each prime; in particular their one-slot marginals are uniform (Hypothesis 6.2). Hence (Lemma 5.17)

$$\alpha^{72} \le \mathcal{Y}(\mathsf{L}, A) = \Lambda_\mu(F_A) = Y. \tag{2.2}$$

Positivity turns density into size; the proof shows, conversely, that $Y \le C\mathsf{L}^{-a}\alpha$ at the scale L in the decomposition $N = K\mathsf{L}$ of Section 2.2 (the next subsection).

The six distinguished slot pairs (queries). The proof uses the six queries $e = (u_e, v_e) = ((0, j, 0), (2, \mathsf{c}j, 1))$, $j \in \mathbb{Z}/6\mathbb{Z}$, of the family $\mathcal{E}_6$, where $\mathsf{c}j = j + 2$ for even j and $\mathsf{c}j = j - 2$ for odd j. Their twelve endpoints are distinct, the sources lie in V_+, the targets in V_-, and the reflected targets $\iota v_e = (2, \mathsf{c}j, 0)$ in V_+ (Lemma 5.13(iv)). Each query crosses the reserved cut. In this section one query e is enough.

2.2. **Rescaling by square steps: the price of changing a local law.**

The induction. Write $N = K\mathsf{L}$ with a fixed integer $K \ge 2$ (in the baseline $K = 890666$; the improved stage uses $K = 863660$), and fix a price exponent a with $0 < a < \sigma = 1/825$, where σ is the error exponent of the analytic stage (Definition 8.1). The proof shows that for a suitable $C_0 \ge 1$

$$\mathcal{Y}(\mathsf{L}, A) \le C_0\mathsf{L}^{-a}\alpha$$

for every L and every square-difference-free $A \subseteq [K\mathsf{L}]$. With (2.2) this gives $\alpha^{71} \le C_0\mathsf{L}^{-a}$, that is $|A| \le CN^{1-a/71}$: the baseline saving is $\varepsilon = a/71$ (Theorem 8.12). If the bound fails, let L be the least scale at which it fails, for some A. Then (L, A) is a *least normalized counterexample* (Definition 8.4): $Y = \mathcal{Y}(\mathsf{L}, A) > C_0\mathsf{L}^{-a}\alpha$, while $\mathcal{Y}(\mathsf{L}', A') \le C_0(\mathsf{L}')^{-a}\alpha_{A'}$ for every scale $\mathsf{L}' < \mathsf{L}$ and every square-difference-free $A' \subseteq [K\mathsf{L}']$, $\alpha_{A'}$ its density. Every argument of the proof takes place at such a pair, and its aim is a contradiction once L is large. Throughout, $o(1)$ is a quantity that tends to zero as $\mathsf{L} \to \infty$, uniformly in A.

Fixing labels is rescaling. Let $\mathcal{B}$ be an atom of primes with $e_p = 1$, and $q_\mathcal{B} = \prod_{p\in\mathcal{B}} p$. Fixing, for one slot, its block label $z_\mathcal{B} \in X_\mathcal{B} = \prod_{p\in\mathcal{B}} \mathbb{F}_p$ fixes for that slot a residue class $\mathfrak{a}$ modulo $q_\mathcal{B}$ (the argument below is applied slot by slot through the signed Hölder inequality): by the Chinese remainder theorem the specialized lift $F_A(z_\mathcal{B}, \cdot)$ is, at every character ξ of the other coordinates with $q(\xi)q_\mathcal{B} \le N^{1/20}$, the lift of $q_\mathcal{B}\mathbf{1}_{A\cap(\mathfrak{a}+q_\mathcal{B}\mathbb{Z})}$ (Lemma 8.2(a)). Split the class into the $q_\mathcal{B}$ progressions with the square step $q_\mathcal{B}^2$. Since $q_\mathcal{B}^2k^2 = (q_\mathcal{B}k)^2$, the set A meets each progression $\{q_\mathcal{B}^2n' + c\}$ in a rescaled copy $A'_c = \{n' : q_\mathcal{B}^2n' + c \in A\}$, which is again square-difference-free and lives at the scale $N/q_\mathcal{B}^2$. So, up to the characters of large conductor, whose contribution is an error linear in α (Lemma 8.6), $F_A(z_\mathcal{B}, \cdot)$ is an average of lifts of square-difference-free sets at a shorter scale, composed with square-affine maps, which preserve μ (Lemma 8.2(b),(c)). By the triangle inequality for the seminorm $\Lambda_\mu(\cdot)^{1/72}$ and the induction hypothesis at the shorter scale, the specialized input has Λ_μ at most $(1 + o(1))(q_\mathcal{B}^2)^a(\alpha_\mathrm{f}/\alpha)Y$, α_f the density of A in the class; the average over the progressions introduces no further factor: by the triangle inequality for the seminorm $\Lambda_\mu(\cdot)^{1/72}$ its seminorm is at most the weighted average of the progression seminorms (the weights sum to one), and concavity of $t \mapsto t^{1/72}$ moves that average inside the power. A change of the law at $\mathcal{B}$ changes all 72 slots, and the signed Hölder inequality multiplies the 72 slot factors $(q_\mathcal{B}^2)^{a/72}$, while the fibre densities average to α under labels with uniform one-slot marginals.

The price of a change: total variation times $q_\mathcal{B}^{2a}$. The *price* of a finite signed measure ζ on $X_\mathcal{B}^V$ is $\|\zeta\|_{\mathrm{var}}\, q_\mathcal{B}^{2a}$ (Definition 9.2; at a prefix prime the factor is $p^{e_p a}$). The structured-energy estimate (Proposition 8.7) states that, at a least normalized counterexample, the *energy* $\Lambda_{\zeta\otimes\mu_{-\mathcal{B}}}(F_A)$ of ζ, in which the factor $\mu_\mathcal{B}$ of the reference is replaced by ζ, has absolute value at most $(1+o(1))\|\zeta\|_{\mathrm{var}}\, q_\mathcal{B}^{2a}\, Y$, provided the variation of ζ has uniform one-slot marginals; changes at several atoms multiply their prices. Inputs restricted to intervals and residue classes are priced in the same way. An input that sees only an interval of length $\mathsf{L} = N/K$ and a residue class modulo an integer D_0 (the

inputs $g^{I,D_0,s}$ of Section 2.3) is a lift at a scale shorter by the factor $K\,\mathrm{sq}(D_0)$, where $\mathrm{sq}(D_0)$ is the least square divisible by D_0 (for instance $\mathrm{sq}(24)=144$). Its slot costs $(K\,\mathrm{sq}(D_0))^{a/72}$, and a fixed interval adds the factor $K^{1/72}$, because the density of A on an interval of length N/K is at most $K\alpha$ (Definition 8.5). The product of these factors is the *coarse price* of the energy.
The only use of the induction hypothesis. Every comparison between laws in the proof is reduced to energies of this kind, and the induction hypothesis enters only through the rescaling above. Everything else is an exact identity at one scale, an estimate of a finite local law, or one of two analytic remainders: the cancellation of Section 2.4, and the selections of atoms with a large product (Lemmas 8.10 and 8.11). Both remainders are $O(\mathsf{L}^{-\sigma}\alpha)$, linear in the density. At a counterexample $Y > C_0\mathsf{L}^{-a}\alpha$, so such a remainder is at most $(C/C_0)\mathsf{L}^{a-\sigma}Y = o(1)Y$. This is why $a<\sigma$ is required, and why the induction can keep the full density factor α on the right: the density exponent $\theta=1$ of Theorem 8.12.

2.3. **One query and its external index** (b,s)**.** The prime 2 and the primes $p\equiv 3 \pmod 4$ below Q_3 form the fixed external modulus $M_0 = 2^{129}\prod_{3\le p<Q_3,\,p\equiv 3\,(4)} p^{128}$. They are not coordinates of G_N; their conductor depths are revealed by an external random index attached to each query.
Intervals, inputs and the index. Split $[N]$ into the K consecutive intervals $I_b = ((b-1)\mathsf{L}, b\mathsf{L}]\cap\mathbb{Z}$, $b\in\mathbb{Z}/K\mathbb{Z}$ (identified with $\{1,\dots,K\}$). For an interval I, a divisor D_0 of M_0 and $s\in\mathbb{Z}/D_0\mathbb{Z}$, the *input* $g^{I,D_0,s}$ is the lift of $D_0\mathbf{1}_{n\equiv s\,(D_0)}\mathbf{1}_A$ restricted to I, with coefficients normalized by $|I|$ (Definition 6.1(f)); its mean is the relative density $D_0\#\{n\in A\cap I : n\equiv s\ (D_0)\}/|I|$ of A in the class. Fix one query $e=(u_e,v_e)$. Its *index* is $\omega=(b,s)$, uniform on $\mathbb{Z}/K\mathbb{Z}\times\mathbb{Z}/M_0\mathbb{Z}$, and its *coupled index* is $\omega'=(b+1,s+r)$, with r independent of ω and distributed as the square of a uniform residue conditioned on root valuation less than 64 in each prime-power coordinate. Thus r is a nonvanishing square class (Definition 4.6 and Lemma 29.6). The *half-functions* X_ω and Y_ω carry the input $g^{I_b,M_0,s}$ at the source u_e, respectively at the reflected target ιv_e, and F_A at the other slots of V_+ (Definition 9.6; the half-function Y_ω is unrelated to the average Y).
The mean identity. The lift is linear, the intervals I_b partition $[N]$, and the residue classes partition $\mathbb{Z}$; so averaging an input over a uniform index returns the lift of A (Lemma 6.3):

$$\mathbb{E}_{b,s}\, g^{I_b,M_0,s} = F_A. \tag{2.3}$$

Hence $\mathbb{E}_\omega X_\omega = \mathbb{E}_\omega Y_\omega = \bar X$, the half-function carrying F_A at every slot of V_+, and $Q_\kappa(\bar X,\bar X) = \Lambda_\kappa(F_A) =: Y_\kappa$ for every measure κ; in particular $Y_\mu = Y$.
Coupling the two ends, and the signed combination $2P-T$. From now on cut forms are taken at the reserved cut ι. Let T be the Markov operator $(Tf)(b,s)=\mathbb{E}_r f(b+1,s+r)$ on functions of the index (this T is an operator, not the dominator price T of Theorem C), and P the average over the whole index. For a finite signed measure κ put

$$I_\kappa = \mathbb{E}_\omega Q_\kappa(X_\omega,(TY)_\omega), \qquad J_\kappa = \mathbb{E}_\omega Q_\kappa(X_\omega,((2P-T)Y)_\omega),$$

the operators acting on the index of Y_ω. Since $\iota(\iota v_e)=v_e$, the *coupled form* I_κ is the average of Λ_κ with $g^{I_b,M_0,s}$ at u_e, $g^{I_{b+1},M_0,s+r}$ at v_e and F_A at the other slots: it places the two ends of the query on the residue classes s and $s+r$ of the adjacent intervals I_b and I_{b+1}. By (2.3), $(PY)_\omega=\bar X$, so the *signed index form* is

$$J_\kappa = 2Y_\kappa - I_\kappa, \tag{2.4}$$

the case of one query of (9.1).
Decomposition by exact conductor. The Chinese remainder theorem splits the index into its interval coordinate and prime-power coordinates. In the p-coordinate, conditional expectation onto residues modulo p^j gives a nested filtration; its orthogonal differences are the exact-conductor sectors. Write $\delta_{p,j}$ for the decreasing envelope on the norm of the shift at conductor p^j proved in Lemma 29.6. Independently at the different primes choose J_p with $\mathbb{P}(J_p\ge j)=\delta_{p,j}$, and always

reveal the interval. On a nonconstant product sector of depths $\mathbf{j}$, both T and $2P-T$ have norm at most $\prod_p \delta_{p,j_p}$, with the interval factor 1. Both operators fix constants. The coefficient 2 used in the comparison comes instead from the expansion $2P-T$: its fully averaged term is $2P$, while the remaining term is the coupled form that cancels. There is no contradiction in the two numbers: 1 is the eigenvalue of both operators on constant index functions, while 2 is the coefficient of the fully averaged term Y_κ in the expansion (2.4).

Let $\mathcal{J}=(J_p)_p$ and $D_{\mathcal{J}}=\prod_{p\mid M_0} p^{J_p}$. Conditional expectation gives $X^{\mathcal{J}}=\mathbb{E}[X \mid b, s \bmod D_{\mathcal{J}}]$ and, for any bracket matrix M at the reserved cut (introduced in the next paragraph), $V_M(X)=\mathbb{E}_{\mathcal{J}}\mathbb{E}_\omega Q_M(X^{\mathcal{J}}_\omega, X^{\mathcal{J}}_\omega)$, and likewise for Y. The conditioned input is $g^{I_b, D_{\mathcal{J}}, s \bmod D_{\mathcal{J}}}$ (Lemma 6.3). The one affected slot in $X^{\mathcal{J}}$ and the one in $Y^{\mathcal{J}}$ each contribute the slot factor $(K\,\mathrm{sq}(D_{\mathcal{J}}))^{a/72}$, hence $(K\,\mathrm{sq}(D_{\mathcal{J}}))^{a/36}$ jointly after averaging their densities. Consequently

$$R_I=\mathbb{E}_{\mathcal{J}}(K\,\mathrm{sq}(D_{\mathcal{J}}))^{a/36}=M_{\mathrm{pad}}(K)<1.247139907407. \tag{2.5}$$

The numerical bound in (2.5) is at the baseline values $a=\frac{5422587}{6103515625}$ and $K=890666$. The finite product formula and its exact relation to the single-level square-shift moment are proved in Lemma 9.7 and Proposition 29.7. Only the conditioning used to bound an energy has the random modulus $D_{\mathcal{J}}$; the kernel cancellation itself always uses inputs at the full modulus M_0.

The index-support lemma. Let $\left(\begin{smallmatrix} M_{\mathrm{L}} & \Gamma \\ \Gamma^{\mathsf{T}} & M_{\mathrm{R}} \end{smallmatrix}\right) \succeq 0$ be a *bracket* of real matrices indexed by $G_N^{V_+}$, and let $\mathcal{T}$ be an operator on functions of the index that maps every exact-conductor sector into itself, is the identity on constants and has norm at most $\prod_p \delta_{p,j_p}$ on the sector of depths $\mathbf{j}$. Then Lemma 10.1 gives

$$\left|\mathbb{E}_\omega \langle X_\omega, \Gamma(\mathcal{T}Y)_\omega\rangle\right| \le V_{\mathrm{L}}^{1/2} V_{\mathrm{R}}^{1/2}, \tag{2.6}$$

where V_{L} and V_{R} are the depth-conditioned energies under M_{L} and M_{R}. The probability that every $J_p \ge j_p$ equals $\prod_p \delta_{p,j_p}$, exactly the weight of that sector in these energies. The same lemma keeps the constant component exactly. With $M_{\mathrm{L}}=M_{\mathrm{R}}=\Gamma \succeq 0$ and $\mathcal{T}=T$ this is the plain coupling inequality in the finite-depth form.

2.4. **Where square-difference-freeness enters.** In this one-query section a nonnegative law ν *accepts* the query e if, ν-almost surely, $z_{v_e}-z_{u_e}$ is a nonzero square at every prime of G_N with $e_p=1$, and (this is the almost-sure condition alone; in Part III *accepts* also includes the marginal requirements of (CP4), as in the glossary) at every prime of the prefix it is nonzero, of even valuation $2j<e_p$, with a square unit part (Definition 9.14(a)). Consider the part of the coupled form I_ν with $b\neq K$. The two ends of the query carry $g^{I_b,M_0,s}$ and $g^{I_{b+1},M_0,s+r}$. Every integer in the support of the first, $A\cap I_b\cap(s+M_0\mathbb{Z})$, is smaller than every integer in the support of the second; their residues differ by r, a nonvanishing square class modulo M_0; and at every prime of G_N the law forces the labels of the two ends to differ by a nonzero square. Fixing the labels at the primes below $P_5=2^{65536}$ places the two ends in residue classes whose difference is a square class of the kind the kernel admits (Lemma 8.2(a)); at the primes $p\ge P_5$ the endpoint pair is uniform on the strict-square pairs Ω_p, by the (R4) clause included in Definition 9.14(a) and not by the square condition alone, and integrating it turns the average into a Fourier pairing with the multiplier $g(\xi)=\varphi(q(\xi))^{-1}\sum_{m\in(\mathbb{Z}/q(\xi)\mathbb{Z})^\times} e(\xi m^2)$, the average of $e(\xi m^2)$ over the units modulo $q(\xi)$ (Lemma 4.11(d)).

Three of the hypotheses of the actual-square kernel (Proposition 7.3(iii)) are the form in which square-difference-freeness is used: the two functions F_1 and F_2 are supported in single residue classes a_1 and a_2 modulo D, with a_2-a_1 a nonvanishing square class; every $x\in\operatorname{supp}F_1$ is smaller than every $y\in\operatorname{supp}F_2$; and no difference $y-x$ is the square of a positive integer. Here F_1 and F_2 are built from the square-difference-free set A, which is what makes the third hypothesis true. Under these hypotheses the kernel bounds the Fourier pairing $D\sum_{\xi\in\Xi_{H,D}} g(\xi)\widehat{F}_1(-\xi)\widehat{F}_2(\xi)$,

truncated to the frequencies $\Xi_{H,D}$ of squarefree conductor at most $H = N^{1/40}$ prime to D, by $CH^{-1/3}\|F_1\|_{\ell^2}\|F_2\|_{\ell^2}/N$, where $H^{-1/3} = N^{-1/120}$. In the proof (Appendix C) this pairing is rewritten using a signed kernel whose support consists of positive integer squares; the third hypothesis makes that kernel's contribution exactly zero, and the factor $H^{-1/3}$ bounds the remainder of the rewriting. With conductor decay at the large primes and moment bounds for the other 70 slots, the part of I_ν with $b \neq K$ is $O(\mathsf{L}^{-\sigma}\alpha) = o(1)Y$ (Lemma 8.10). This is the only place where the absence of square differences yields a gain; the rescaling of Section 2.2 uses only that the property is preserved.

If $b = K$, the interval I_K is followed, cyclically, by I_1, and the two ends are not ordered. This *wrap* has probability $1/K$; its term is bounded by energies, and it becomes the term W of the closing.

2.5. **One query: $1 \le \mathcal{E} + BR_I/2 + W_1 + o(1)$ at a least counterexample.** This proposition is the whole mechanism with one query in place of six: if the reference law is reflection positive, a comparison law accepts the query and is dominated by a 2×2 block with energy factor B, then a least counterexample would force $1 \le \mathcal{E} + BR_I/2 + W_1 + o(1)$. Its proof is Remark 11.6, the case $k = 1$ of Theorem C.

Proposition 2.1 (One query)**.** *Consider the frame data of (CP0) (Definition 9.1) with the index law of Section 2.3 and a single query $e \in \mathcal{E}_6$, so $k = 1$. Let $0 < a < \sigma$, let (L, A) be a least normalized counterexample for $(a, 1, C_0)$ (Definition 8.4), and $Y = \mathcal{Y}(\mathsf{L}, A)$. Let the reference μ satisfy (CP1) (Definition 9.10): a product over the atoms of probability measures, each reflection positive at the seven cuts of $\mathfrak{F}$, square-affine invariant and mixing at the primes $p \ge P_5$. Let $\nu = \bigotimes_{\mathcal{B}} \nu_{\mathcal{B}}$ be a product of nonnegative measures, of total mass bounded independently of the scale, that accepts e in the sense of (CP4)(a) (Definition 9.14; in particular every one-slot marginal of $\nu_{\mathcal{B}}$ is a multiple of the uniform law). Suppose that*

- (H1) *$Y_\nu \ge (1 - \mathcal{E} - o(1))Y$ for a number $\mathcal{E}$; and*
- (H2) *there are nonnegative measures $M_{\mathcal{B}}$ on $X_{\mathcal{B}}^V$ with $\left(\begin{smallmatrix} M_{\mathcal{B}} & \nu_{\mathcal{B}} \\ \nu_{\mathcal{B}}^* & M_{\mathcal{B}} \end{smallmatrix}\right) \succeq_\iota 0$ for every atom, and a number B, such that the energy $\mathbb{E}_\omega \Lambda_M((g^{I_v, D_{0,v}, s_v})_v)$ of every structured slot datum (Definition 8.5(c); the expectation is over the index vector $\omega = (I_v, s_v)_v$ under the law $\mathbf{P}$ of that datum) under $M = \bigotimes_{\mathcal{B}} M_{\mathcal{B}}$ is at most $(B + o(1))$ times its coarse price times Y.*

Then, with the conductor moment R_I of (2.5)*,*

$$1 \le \mathcal{E} + \frac{BR_I}{2} + W_1 + o(1), \qquad W_1 = \frac{BR_I}{2K^{35/36}}.$$

Sketch of the proof. By (2.4), $2Y_\nu = J_\nu + I_\nu$. The brackets of (H2) tensorize over the atoms (Lemma 5.6), and (2.6), with Γ the cut matrix of ν at ι and $\mathcal{T} = 2P - T$, gives $|J_\nu| \le V_{\mathrm{L}}^{1/2} V_{\mathrm{R}}^{1/2}$. By (H2) and the slot prices of Section 2.3, each of the depth-conditioned energies V_{L} and V_{R} (the energies after conditioning on the random conductor depths $\mathcal{J}$) is at most $(BR_I + o(1))Y$, the conductors being held fixed and averaged over $\mathcal{J}$ only at the end. In I_ν the part with $b \neq K$ is $o(1)Y$ by the kernel (Section 2.4). The wrap $b = K$ has probability $1/K$, and (2.6) on the residue coordinates bounds it by energies at the fixed intervals I_K and I_1, which cost the density factor $K^{1/72}$ at each of the two changed slots: its term is at most $(BR_I K^{1/36}/K + o(1))Y = (2W_1 + o(1))Y$. With (H1), $2(1 - \mathcal{E} - o(1))Y \le 2Y_\nu \le (BR_I + 2W_1 + o(1))Y$; divide by $2Y > 0$. The complete proof is Remark 11.6, which checks the hypotheses of the lemmas used in the frame of Part III.

Keeping the common mean. Part (a) of Lemma 10.1 keeps exactly the component supported on no query and no conductor coordinate (the *empty-support*, that is constant, component): that of J_ν is Y_ν, and that of each energy is the mean energy $e_M = Q_M(\bar{X}, \bar{X})$ of the bracket measure. It

gives $J_\nu \le Y_\nu + (BR_I + o(1))Y - e_M$, and if $e_M \ge (m - o(1))Y$, the conclusion becomes

$$1 \le \tfrac{1}{2}\mathcal{E} + \frac{BR_I}{2} + W_1 + \frac{1-m}{2} + o(1) :$$

keeping the common mean halves the coefficient of $\mathcal{E}$, and a mean constant $m > 1$ lowers the right-hand side further through $(1-m)/2$. This is the case $k = 1$ of Theorem 9.21 without a first stage ($L_0 = 0$), with the one energy factor B of (H2) in place of both Be^T in J and Be^{S+A} in W.

Remark 2.2 (A positive comparison law). If every factor $\nu_{\mathcal{B}}$ were reflection positive at ι, the bracket of (H2) would hold with $M_{\mathcal{B}} = \nu_{\mathcal{B}}$, since $\left(\begin{smallmatrix} C & C \\ C & C \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \otimes C \succeq 0$ for $C = C_\iota(\nu_{\mathcal{B}}) \succeq 0$, and part (a) of the index-support lemma with $\mathcal{T} = T$ would be the plain coupling inequality $2Y_\nu - \bar{V}_\nu \le I_\nu$ of Corollary 10.3, where $\bar{V}_\nu = \frac{1}{2}(V_\nu(X) + V_\nu(Y))$ is a scalar (not the slot set V). The comparison laws of the construction accept the queries but are not reflection positive at ι (Section 3.6 explains why); the signed index form and the bracket are what replace positivity there. In the middle range the comparison laws are signed measures: a fold record of the comparison law is a signed combination of three nonnegative accepting laws with exact marginals (Definition 23.9). The cancellation of Section 2.4 is applied to each nonnegative product component, and the variation of the signed laws multiplies only the constant of a vanishing remainder (Definition 9.14 and Lemma 10.6). The bracket asks less than positivity: M has to dominate ν only at the one cut ι, and it need not accept anything.

2.6. **Why the sum of error bounds must be < 1, and why one query fails.** Proposition 2.1 says that a least counterexample forces $\mathcal{E} + BR_I/2 + W_1 \ge 1 - o(1)$. Hence if $\mathcal{E}$ and B do not depend on the scale and $\mathcal{E} + BR_I/2 + W_1 < 1$, no least normalized counterexample exists at large scales; once every $o(1)$ is shown to come only from structured energies and the two analytic remainders (Lemma 11.4), the induction of Theorem 8.12 gives $s(N) \le CN^{1-a/71}$. Each term has a direct meaning. The left side is the constant coefficient 2 of $2P - T$, divided by 2: in (2.4) the term without an active index is $2Y_\nu$, while the coupled form, in which the index is active, cancels by the kernel except for the wrap. The conductor moment $R_I \ge 1$ is the price of revealing the index: the squared slot price of an input restricted to an interval and to the residues kept by the inclusion law, averaged over the inclusions. And B is the energy factor of the bracket, the price of controlling by a positive measure a law that is not reflection positive at ι; in the comparison theorem $B = e^{U_0} + R_{\mathrm{err}} \ge 1$.

With one query the coupled term $BR_I/2$ is at least $R_I/2$, about 0.623 at the baseline R_I, so even with $B = 1$ the comparison price $\mathcal{E}$ would have to be below about 0.377; at the baseline energy factor $B < 2.332628665480$ it is about $1.454 > 1$, so one query cannot give $\Theta < 1$. Using k queries with disjoint endpoints and independent indices, the operator $\bigotimes_e (2P_e - T_e)$ has constant coefficient 2^k, while the energies factor over the queries, each contributing R_I (Lemma 10.17). The coupled term becomes

$$J = Be^T \Big(\frac{R_I}{2}\Big)^k,$$

where T is the total price of the dominators for ν in (CP6) (Definition 9.17). For $k = 6$, $(R_I/2)^6 < 0.0588$ at the baseline R_I, and $J < 0.25177$ at the baseline B and $T < 0.60760$ (baseline values; the improved values are in Section 33). Why six queries are used, and why these six, is explained in Section 3.4.

Theorem C is Proposition 2.1 for k queries with two refinements: a first-stage law λ with total price L_0 between μ and ν, dominated at ι with energy factor B, against which the symmetric and skew parts of $\nu - \lambda$ are priced separately; and retention of the common mean, which lowers the coefficient of $L_0 + \mathcal{E}$ to $1 - 2^{-k}$. Its comparison law may be, at a set of atoms whose number is bounded independently of the scale (the middle blocks), a real combination of nonnegative accepting laws (Definition 9.14, Remark 2.2). It gives $1 \le \Theta + o(1)$ with $\Theta =$

$(1-2^{-k})(L_0+\mathcal{E})+J+W+(1-m)/2^k$ at a least counterexample. Part V constructs ν with baseline $\Theta < 0.99999999997327$ (Section 30 and Table 1); the improved value $\Theta < 0.999999771$ is proved later by the separate $\vartheta > 1$ induction.

2.7. **Why conditioning independent labels on a square difference does not work.** Which law should accept the query? The obvious candidate starts from the law μ_0 of independent uniform labels, whose cut matrices have rank one and which is reflection positive at every cut, and conditions it on $z_{v_e} - z_{u_e} \in Q_p^*$ at every prime p of G_N with $e_p = 1$ (and on the corresponding event at the prefix primes). At one prime this replaces the uniform law of the pair (z_{u_e}, z_{v_e}) by the uniform law on the strict-square pairs Ω_p. The uniform law puts the mass $(p+1)/(2p)$ outside Ω_p, and conditioning moves all of it onto Ω_p; so the change $\nu_p - \mu_{0,p}$ has total variation norm $\|\nu_p - \mu_{0,p}\|_{\mathrm{var}} = (p+1)/p > 1$ (twice the moved mass), and its price exceeds $p^{2a} > 1$. In the expansion of $\nu = \bigotimes_p(\mu_{0,p} + (\nu_p - \mu_{0,p}))$ the prices of the primes multiply, and their total grows with the number of primes of G_N: it is not bounded independently of the scale, so $\mathcal{E}$ cannot be bounded. Even a change of size about $1/p$ at every prime is too large, since $\sum_p p^{2a-1}$ over the primes of G_N is unbounded.

Two ingredients are missing. First, reference laws whose records already give a square difference on most of their mass, so that an accepting law changes only a small part at each prime. Second, a way to bound many primes jointly: a change on a whole block of primes should be bounded by about its worst prime, with one factor $q_{\mathcal{B}}^{2a}$ for the block. Section 3 explains both (Sections 3.1 and 3.2 and Section 3.3 respectively).

Consequences and what to read next. This section quotes, by their labels, statements of Part II (the square shift Lemma 4.11, Section 5, the functional and the kernel of Definition 6.1 and Proposition 7.3, and Section 8) and of Part III (Sections 9 to 11), the comparison law of the middle range (Definition 23.9), and the final values of Section 30; its only new statement, Proposition 2.1, is proved in Remark 11.6.

3. From the one-query argument to the full construction

What this section proves and how to read it. This section proves nothing and is not a premise of any later section. Its purpose is to explain, before the formal construction in Part V, why each feature of that construction is needed. Section 2 reduced the theorem to: exhibit a signed law ν whose nonnegative summands accept six disjoint queries across ι, dominate it at ι as in Part III, and achieve $\Theta < 1$. Each subsection below follows the same pattern: *Problem* means "with the naive choice just described, $\Theta \geq 1$ (or no scale-independent bound exists), because [an explicit number]", and *Construction* describes the choice used in Part V and names the section proving it. The numerical illustrations use the density-linear baseline of Section 30. The final theorem uses the additional moments, normalized assignments and joint mean allocation in Section 33; Section 3.9 connects these ingredients. Historical comparisons are collected in Part VI.

Every version of the argument has the same three tasks: verify finitely many local operator inequalities, sum the resulting prices over blocks of primes, and check $\Theta < 1$. The subsections explain the local laws that make the first task possible with small constants, the block summation that makes the second finite, and the sharper inequalities that make the third true.

The earlier 216-slot construction used the same general tools. Table 2 records the four structural changes; Section 34.2 gives the historical instances and their certificates. Those instances are not premises of the present proof.

3.1. **Put square-difference constraints into the reference law (graph laws).**

TABLE 2. The four structural substitutions.

216-slot design	final design	explained in
$V = (\mathbb{Z}/6\mathbb{Z})^3$, sorted by nine folds; the reserved cut is a chamber reflection of the third coordinate; exponent $a/(216-\theta)$	$V = (\mathbb{Z}/6\mathbb{Z})^2 \times \{0,1\}$, sorted by seven folds; the reserved cut is the layer flip ι; exponent $a/(72-\theta)$ ($a/71$ at $\theta = 1$)	Section 3.4
nine coupled families and a singleton recurrence, with the records that do not accept covered by obstruction cylinders; later one fixed family of sixteen (at one checkpoint four) amplified edges	six disjoint one-way queries with a common signed index; records that do not accept are repaired with their actual marginals, or deleted	Sections 3.4 and 3.5
a comparison law that is reflection positive at the reserved cut	a comparison law that is a signed combination of nonnegative accepting laws, not reflection positive, controlled at ι by a reserved Gram bracket; symmetric and skew changes priced separately; signed whole-block fold records, variance and envelope credits, common mean	Sections 3.6 and 3.7
the same laws for both classes of primes (a map law, two middle arrays, a three-depth high range); small primes in the index with generic bounds	class one: a square-clique prefix; class three: a twelve-label middle law and a two-layer 72-slot upper law; small class-three primes in the index with the exact square-shift contraction	Sections 3.3, 3.6 and 3.8

Problem. The crude accepting law of Section 2.7 starts from independent labels and conditions them on a square difference. At every prime its change has variation more than one, and the prices of the primes multiply. The reference knows nothing about squares, so the comparison law has to create every square relation by itself.

Construction: graph laws. The square relations are put into the reference. At a prime p the labels are drawn from a *graph law* $\lambda_p(G)$ (Definition 12.1): the uniform law on the labellings of the vertices of a small oriented graph G by distinct elements of $\mathbb{F}_p$ in which every edge $u \to w$ is a strict-square constraint, $z_w - z_u \in Q_p^*$. The slots are labelled through maps from slots to vertices; the laws obtained in this way are the *records* of Section 3.2. A record that maps a query to an edge of G already forces that query at the prime and is kept unchanged, so it needs no replacement and contributes no replacement price. The middle law lives on twelve vertices, $\mathbb{Z}/6\mathbb{Z} \times \{0,1\}$ (one chamber coordinate and the layer), with all four layer lifts of the six edges $u \to \mathsf{c}u$: 24 directed edges, every vertex of outdegree two and indegree two (four incident constraints), and the edge $(0,0) \to (2,1)$ onto which every query projects. The upper law lives on all 72 slots, with 720 directed edges and every vertex of outdegree ten and indegree ten (twenty incident constraints), and its root contains every query (Section 20.3, Lemma 27.4).

Why a graph law is almost independent. The number of labels available to a vertex whose t neighbours carry distinct labels is $p2^{-t}$, up to a relative error of order $2^t p^{-1/2}$, by Weil's bound (Lemma 12.2). Iterating this count gives every marginal, conditional law and density of $\lambda_p(G)$, relative to the independent law, with explicit relative errors of order $2^t r$, $r = p^{-1/2}$ (Section 12); the relative form is what makes them usable at explicit thresholds (Section 1.4). The law is invariant under the maps $z \mapsto sz + c$ with s a nonzero square, so its one-slot marginals are uniform (Lemma 5.16). The errors grow like 2^t with the number of constraints at a vertex, which is one reason why graphs of small degree are used where the primes are small: the twelve-label law from $Q_3 = 19\,951\,586$ on, the 72-slot law only from $P_3 = 167 \cdot 2^{40}$ on (Section 3.3). What remains is

that a graph law is not reflection positive (Section 3.2), and that the records which do not map the queries to edges must be changed (Section 3.5).

3.2. **Graph laws are not reflection positive: adding correction summands.**
Problem. At a chamber cut the two halves of a graph law are coupled through its cross edges. Relative to the product of its half laws, its normalized cut operator is a combination of character kernels such as $\chi(y-x)$, each of normalized size about $p^{-1/2}$, and its spectrum reaches down to about $-gr$, for a constant g of the graph (Section 13). So $\lambda_p(G)$ is not reflection positive, and the signed Hölder inequality (Theorem 5.9) does not apply to it.
Construction: the five-type completion. The root law (type L, weight 1) is completed by records built from the laws of its half graphs and composed with maps of the chamber coordinate onto parts of itself: three folds (type D), two matchings (S), six merges (R) and two constants (C), with the graded weights 1, $w_{\mathrm{D}}r$, $w_{\mathrm{S}}r$, $w_{\mathrm{R}}r^2$, $w_{\mathrm{C}}r^3$ (Lemma 17.5). Positivity at a chamber cut is built in steps of one order in r each. The aligned fold is a copy of the half law placed symmetrically; it is fixed samplewise by the cut, and it supplies the positive diagonal that cancels the negative part $-dr$ of the root if $w_{\mathrm{D}} \geq d$ (here d, f, h are the spectral constants of Lemma 17.5; this d is not the slot count $d = 72$). A fold that is aligned at one cut is not aligned at the others, where it couples two corners of the half; the matchings pay for this if $w_{\mathrm{S}} \geq w_{\mathrm{D}}$. The row errors fr, hr of the folds and matchings are paid by the merges if $w_{\mathrm{R}} \geq f w_{\mathrm{D}} + h w_{\mathrm{S}}$, and the linear terms of the merges by the constants if $w_{\mathrm{C}} w_{\mathrm{S}} \geq w_{\mathrm{R}}^2$; the constants, of weight r^3, together with the matchings, of weight r, dominate the r^2 cross terms of the merges in the Schur complement at the cut, since the product of the r^3 and r weights has the same r-order as the square of the r^2 merge weight. The chamber group transports positivity from one chamber cut to all of them (Lemma 5.13(i)), and one-layer copies of the completion records, placed on the reserved diagonal, restore positivity at ι. (The history of this completion, through an earlier version on 216 slots, is in Section 34.) The twelve-label reference of the middle range (thirteen inequalities per piece, together with the inequality that protects the 2×2 block at ι, Lemma 22.1) and the 25-state two-layer reference of the upper range (105 inequalities per band, Lemma 27.17) are its two instances in the construction.
Reweighting the principal law by a nonnegative function. The root itself need not be the graph law. Reweighting it at every prime by a nonnegative weight built from trace scores—the average $\frac{1}{3}\sum_j (1+\mathsf{u}\mathsf{H}_j)^2$ of the individual squares for the three opposite base pairs, or $1/100+(1+\mathsf{u}H_{\mathcal{D}_{39}})^2$ for the coherent dictionary of 39 traces—keeps its accepting support, its symmetries and its uniform one-slot marginals, and lowers the constant d of its chamber cut below $\sqrt{30}$ (Definition 21.1, Lemmas 21.2 and 21.17, and Certificate 21.25). The price is a larger constant k_{L} of the layer cut, paid through the weight v_{L} of the one-layer copy (Proposition 21.9), and a changed half marginal, paid in the restorations of the variance credits and of the mean savings (Lemma 24.7(iv)). Both cut bounds are proved in the fixed half charts in which the completion inequalities are stated (Remark 22.3). The gain is only in the negative part of the chamber-cut operator. Indeed, every law satisfying the sixteen cross constraints of that cut, with the half marginal and chamber symmetries of the graph law, has centred norm — the operator norm on half-functions of mean zero in the stated half marginal — at least about $\sqrt{30}\,r$: with those marginals no improvement is possible. A nonnegative reweighting, however, changes the half marginal, and can thereby raise the least eigenvalue (Section 35.6). The construction tilts the root on the seven pieces from 2^{30}: averages of individual squares on six of them, and on the final piece the coherent 39-trace square with the constant $1/100$ added. Sharper collision bounds on two explicit subspaces, an interpolation between the two pair bounds, and a layer remainder summed without expanding all its monomials make these parameter values work (Section 21).
Inputs. The constants d, f, h are spectral caps and row bounds of graph laws (for d on the seven last middle pieces, of the positive roots). The spectral criterion certifies at one prime that the normalized cut operator of a graph law has spectrum in $[-gp^{-1/2}, 1 + O(p^{-1})]$, by a character

expansion of its kernel, an extraction of witnesses (Theorem 16.9) and a Schur test; monotone majorants carry the bound to all larger primes (Lemma 4.18). It has two routes: Route A, through conditional laws of the opposite half (Theorem 14.21), and, at the root of the upper law below 2^{52}, Route B, a direct decomposition into explicit uniform averages over allowed sets (Theorem 14.27). At the chamber cut of the uniform middle root, complete actual stars give the cap directly: the eighteen star features of the root have an exact signed star matrix of norm $\sqrt{30}$, and a translation-sector completion, a rectangular factorization of the root and balanced collisions bring the cap at Q_3 below 5.500058 (Section 20.4). The row bounds come from pinned influence, through Kassabov's and Peyre's inequalities (Proposition 13.16), and from density ratios (Lemmas 13.25 and 20.9).

Why second-order character sums are needed. On a block of primes (Section 3.3) the negative side of a cut operator is governed by the worst prime, but the positive side multiplies over the primes: if the factors have spectra in $[-g_p, 1+b_p]$, with $0 \le g_p \le 1+b_p$ and $b_p \ge 0$, the tensor product has spectrum in

$$\Big[-\max_p g_p \prod_{q\neq p}(1+b_q),\ \prod_p (1+b_p)\Big]$$

(Lemma 13.7). The density ratios of the conditional rows — the conditional distributions of one half given the labels of the other half, in the row notation of Section 14 — multiply over the block in the same way. These quantities must therefore be of order $1/p$, and a first-order bound, such as Weil's bound for one character, of order $p^{-1/2}$, is not enough for them. In the 216-slot design the first-order bound for a row of its middle completion would have given a bound of about 35 for the logarithm of the product of the row factors over its first middle piece, where the second-order bound gave $4.3 \cdot 10^{-3}$ for the same logarithm. In the final design some of the second-order quantities follow from exact counts at the primes $p \equiv 3 \pmod 4$: the density ratios of the middle rows lie between R_2^{-1} and $R_2 = (p-3)(p-7)/((p-11)(p-15)) = 1 + O(1/p)$, and between R_1^{-1} and $R_1 = (p-1)/(p-3)$ for one layer (Lemma 20.9). The others need genuinely second-order estimates (Section 15; Section 1.5 says where each enters): the four-cycle kernel, bounded through the character table of $\mathrm{GL}_2(\mathbb{F}_p)$ and Jacobi sums (Theorem 15.8); character sums in two variables and second moments of trace functions with a moving root λ (not the principal law), bounded through Deligne's theorem and the Katz–Sarnak description of a family of curves (Theorem 15.13 and Lemma 15.19); and, for the second moment of the neighbourhood count behind the half variances of the root, a twisted second moment of Legendre traces, evaluated through a cusp form of weight four (Theorem A.7 and Lemma 24.3).

3.3. **Summing a price at every prime is too large: group primes into blocks.**

Problem. With graph laws and repairs (Section 3.5) the comparison law changes a record of weight of order r by a measure of norm of order r, so the change at one prime costs about c/p, for a local constant c, and is priced $(c/p)\,p^{2a}$. The sum $c\sum_p p^{2a-1}$ over all primes is unbounded (Section 2.7), and on a finite range it is large: over the primes $p \equiv 3 \pmod 4$ of $[Q_3, P_3)$ the sum $\sum_p 1/p$ alone is about 0.33, while the local constants of the middle records are of order ten (twice a completion weight w_{D} between 5.43 and 5.51, times a norm coefficient of order one).

Construction: blocks. Consecutive primes are grouped into blocks $\mathcal{B}$ with $\log q_{\mathcal{B}} \le L$, $q_{\mathcal{B}} = \prod_{p\in\mathcal{B}} p$ (Section 20.1), and each block carries one law: the completion, with $r = b_{\mathcal{B}}^{-1/2}$ for the least prime $b_{\mathcal{B}}$ of $\mathcal{B}$, of the tensor product over its primes of the graph laws (for the root on the seven last middle pieces, of their positive tilts). A tensor product of operators that preserve the constants decomposes into exact-support sectors (Lemma 6.10). On a sector with one active prime, the difference of two tensor products is the difference at that prime; on a sector with several active primes it is a product of small factors. So a change made jointly at all primes of a block costs about as much as at its worst prime, $c/b_{\mathcal{B}}$, and its price is $c\,b_{\mathcal{B}}^{-1} q_{\mathcal{B}}^{2a} \le c\,e^{2aL}/b_{\mathcal{B}}$: one selection

factor e^{2aL} for the whole block. The negative spectrum of a block is likewise that of its worst prime, times a product $\prod_{p\in\mathcal{B}}(1+O(p^{-1}))$ that stays bounded (Section 3.2).
Choosing the block-size bound L. Near x about $L/\log x$ primes $p\equiv 3 \pmod 4$ fit into one block, so about $dx/(2L)$ blocks start in $[x, x+dx]$, and heuristically

$$\sum_{\mathcal{B}} \frac{c}{b_{\mathcal{B}}}\, q_{\mathcal{B}}^{2a} \approx c\,\frac{e^{2aL}}{2L}\,\log\frac{P_3}{Q_3}.$$

The factor e^{2aL}/L is least at $L=1/(2a)$, where the total is about $c\,a\,e\log(P_3/Q_3)$ and grows linearly with a. Hence a is limited by the local constants c: every price grows with a through $q_{\mathcal{B}}^{2a}$, and a can be raised only by lowering a local constant. At the baseline parameters $e^{2aL}\log(P_3/Q_3)/(2L)$ is about 0.039, against about 0.33 for single primes. The construction uses $L=570$, close to $1/(2a)\approx 562.8$ (Definition 20.1); although every block satisfies $\log q_{\mathcal{B}}\le L$, the local laws are verified at the slightly larger bound $L_{\mathrm{loc}}=585$, hence with the prime-count cap $\lfloor L_{\mathrm{loc}}/\log X\rfloor$, so they hold on every block (Lemma 20.3). Middle prices computed at the evaluation exponent $a_0=\frac{4442183}{5000000000}$ are converted to a by $t=e^{2(a-a_0)L}\approx 1.000000062$ (Lemma 9.3). Blocks are formed from exact prime counts below an explicit threshold and from fixed-size prime-counting intervals above it, controlled by Theorem 4.12.
The ranges. A block is priced with the local constants of its least prime, so different laws are used where different constants are smallest (Table 9). Below Q_3 the primes $p\equiv 3 \pmod 4$ are not coordinates of the labels but enter the index of the queries, each at a factor close to one in the conductor moment R_I (Section 3.8). In $[Q_3, P_3)$ the twelve-label law needs one added edge in each nonaccepting record of degree one, so its changes are of order $r^2=1/p$ per prime. From $P_3=167\cdot 2^{40}$ on, the two-layer law on all 72 slots contains every query in its root, every one of its records of degree at most one accepts. Repairs start at degree two and deletions at degree three, so its operator prices at one prime are of order $r^3=p^{-3/2}$ (Section 28.1); the Route B criterion supplies its root caps from P_3 to 2^{52}, with graded root counts and coefficients interpolated between certified prime thresholds, and the Route A criterion those from 2^{52} on (Section 27.1). From $P_4=2^{512}$ on, the primes of both classes are single atoms with a fixed two-layer tail law, at prices $O(p^{-3/2+2a})$ with a convergent sum (Section 29). The primes $p\equiv 1 \pmod 4$ below 2^{512} need no comparison at first order (Section 3.6).

3.4. **One query leaves an error term above 1: use six disjoint queries.**

Problem. With one query the coupled term $BR_I/2$ of Proposition 2.1 is about 1.45 at the baseline values, already above 1 before any comparison price is added (Section 2.6).
Construction: six disjoint queries. With k disjoint queries and independent indices the matching term is $Be^T(R_I/2)^k$ (Section 2.6, Theorem 9.21). But every retained record of the comparison law must accept every query, and acceptance costs added constraints. The geometry of V decides how many queries are cheap. The six queries $e_j=((0,j,0),(2,\mathsf{c}j,1))$, $j\in\mathbb{Z}/6\mathbb{Z}$, of Definition 5.12 all have the first-coordinate pair $(0,\mathsf{c}0)=(0,2)$; they differ only in the second chamber coordinate, on which they act by c. The middle laws ignore that coordinate (Lemma 5.13(v)), so they see one query, the edge from the label $(0,0)$ to the label $(2,1)$, and each nonaccepting fold or matching record of the middle law needs one added edge; the prefix law ignores the layer as well and sees one coincidence event (Section 18.2). Six is the largest number of disjoint queries with this projection, since there are only six slots $(0,j,0)$. So six queries cost the middle range no more than one, while the matching term carries $(R_I/2)^6<0.0588$; finally $J<0.25177$.

The larger family of twelve queries $((u,j,0),(\mathsf{c}u,\mathsf{c}j,1))$, $u\in\{0,5\}$, has two first-coordinate pairs, $(0,2)$ and $(5,3)$, and each nonaccepting fold or matching record then needs two added edges instead of one. The construction was first proved for these twelve queries and then specialized to the six, and only the six-query form was certified (row 8 of Table 14). The twelve queries

survive as the acceptance set of the middle merge repairs, of the one-way repairs and the dense EL comparison of the upper range, and of the tail, while the signed fold records of the middle, the LE mixture and the retained records $G_{01}L$, $G_{10}L$ of the upper range accept the six queries only (Remark 18.2 and Lemma 28.1). Twelve queries would need more than new edges: joint laws (couplings) of the two halves forcing both queries of a fold record do exist, but with the unchanged references the symmetric error of the fold record is then of order r, at least about $r/\sqrt{3}$, and that of a matching fork at least about $\sqrt{3}\,r$ in symmetric plus skew norm (Section 35.7).

Why 216 slots became 72. In the 216-slot design the laws of the lower ranges ignored the third chamber coordinate: its only role there was to carry the reserved cut. A two-point layer with one reflection serves as well. The folds $f_0 \circ f_4 \circ f_0$ in each of two chamber coordinates, followed by the one fold of the layer, map $V = (\mathbb{Z}/6\mathbb{Z})^2 \times \{0,1\}$ to a single slot (Lemma 5.13(iii)), so the chessboard estimate gives $\alpha^{72} \le Y$. Since the exponent of the theorem is $a/(|V| - \theta)$ (Theorem 9.21), the same price exponent a gives about three times the saving: $a/(216 - \theta)$ becomes $a/(72 - \theta)$, and the density-linear induction (Section 3.8) then gives $a/71$. The laws of the high range, which used the third coordinate in the 216-slot design (a hierarchy of three depths), became two-layer laws on the 72 slots. The first 72-slot design, which also treated the two classes of primes separately, raised the exponent from $2.288 \cdot 10^{-7}$ to $1.992 \cdot 10^{-6}$.

3.5. **Records that do not force a square difference: replace, delete, or dominate them.** Problem. The reference law is a positive combination of records. Most of its mass already forces the queries (in the middle range, the principal law and the aligned fold). Some records do not: in the middle range the other two fold records F_1, F_2 and the two matching records E_0, E_1 (degree one), and the merges and constants; in the upper range some records of degree at least two. Bounding a non-forcing record by its whole mass (a union bound over obstructions) charges order r for a degree-one record, which is first order and too large: summed over primes it cannot give $\Theta < 1$ (see Section 35.8 for the abandoned union-bound polynomials, which play no logical role). Each non-forcing record must instead be removed by domination, replaced at fixed marginals, or deleted, according to its type.

Construction: polarization and replacement at fixed marginals. Both are proved in Section 17. *Polarization* (Lemma 17.11): at a chamber cut the two nonaligned fold records are dominated, in the cut order, by the two matching records plus a merge correction of one more order in r, so they can be removed without losing positivity at that cut. This is an order of cut matrices, not of measures; only actual records are subtracted, so the law stays nonnegative. *Additive repair* (Lemma 17.14): a record whose graph lacks the query edge is replaced by the graph law with the edge added, reweighted by a factor $1 + f(x) + g(y)$ so that its half marginals at the cut are exactly those of the old record. The reweighting keeps the support, hence every constraint, and f, g solve two linear equations that are well conditioned when the conditional operator of the new record is a contraction. The new and the old record then have the same actual half marginals, so their difference annihilates the constants on both sides of the cut; its energy is controlled by its operator norm between the actual half spaces, which is of order r, not by its mass. A record of weight of order r costs $O(r) \cdot O(r) = O(1/p)$. Records of high degree are simply deleted and charged by their mass. The copies of records that the completion places on the reserved diagonal are handled by a *first stage*: polarization at the chamber cut ϑ_0 removes their nonaccepting fold records, and a one-stage repair keeps the actual conditional marginals of their matchings (Lemma 10.10, Section 22); its loss, with the errors of the two folded matching repairs tensored at their own first cuts (Corollary 10.9), is $L_0 < 4.7634 \cdot 10^{-4}$.

Replacements that keep the old cross edges. A one-way repair builds its new record from two half copies and the projected queries and discards the old cross graph, so that a degree-two record whose old cross graph is large changes by a kernel of the size of that whole graph. For such records the upper comparison keeps the old cross graph, adds the missing required edges, softens the

unrequired old edges that meet them, and repairs the result to the exact old actual marginals. Written in one common list of actual star feature functions, the old and the new star matrices then nearly cancel, and the difference is priced separately in its symmetric and skew parts. For the six LE records four such softened laws, each repaired to the same marginals, are mixed; the star matrix of the mixture is the mixture of their matrices, and at the selected weights its norm is below that of every component (Definition 28.4 and Propositions 28.5 and 28.6). At 2^{48} the symmetric and skew operator norms in the actual half marginals are about $1.84r$ and $1.83r$ per LE record and $3.20r$ and $2.31r$ per EL record, against about $9.2r$ per LE record and $9.7r$ per EL record, charged in full to both prices, for one-way repairs of the same records at 2^{49} in the construction of row 11 of Table 14.

3.6. **The forcing law is not reflection positive: domination by 2×2 blocks.**
Problem. In the 216-slot design the middle laws and all their changes depended only on the first two chamber coordinates, so every nonnegative law built from them was fixed samplewise by the reserved cut, and a nonnegative measure fixed samplewise by a cut is positive there (Lemma 5.4(i)): the comparison law stayed reflection positive at the reserved cut. The final middle law carries the layer: its twelve labels are a chamber coordinate times the layer, and its query edge runs from layer 0 to layer 1. So ι acts on its labels, and a change of a record is, at ι, a signed measure that is not positive. In the first 72-slot design the upper laws already carried the layer, and positivity at ι was kept by adding to each repair a positive diagonal of the size of its change, the one-sided bracket of Section 34.4; this diagonal was paid at first order.
Construction: domination at the reserved cut, with symmetric and skew parts bounded separately. The comparison law ν need not be positive at any cut, and on the middle blocks it is not even nonnegative: there it is a finite signed combination of nonnegative laws that accept the queries (Definition 9.14). It is compared with the first stage λ at ι alone, through a nonnegative measure N with $\left(\begin{smallmatrix} \mathsf{N} & \lambda \\ \lambda & \mathsf{N} \end{smallmatrix}\right) \succeq_\iota 0$, that is $-\mathsf{N} \preceq_\iota \lambda \preceq_\iota \mathsf{N}$ (Lemma 22.7); its energy factor is $B = e^{U_0} + R_{\mathrm{err}} < 2.33263$ (Lemma 10.12). Every change $\nu_{\mathcal{B}} - \lambda_{\mathcal{B}}$ is split into its symmetric and skew parts at ι, and each part is priced with its own positive envelope (Definition 9.14). A nonzero skew part is never positive semidefinite, since a cut matrix that is $\succeq 0$ is symmetric; keeping ν positive at ι means adding positive mass of the size of the change, which is what the first 72-slot design paid. But a skew part costs nothing at first order. In the expansion of $Y_\nu - Y_\lambda$ over the blocks, every term with an odd number of skew factors vanishes, because both halves carry the same half-function and λ is symmetric at ι (Lemma 10.14). The reserved cost is therefore $B(e^S \cosh A - 1)$: the symmetric price S enters linearly, the skew price A only through $\cosh A - 1 \approx A^2/2$. A comparison law that is positive at ι is its own bracket (Remark 2.2); the positive correction of the first 72-slot design is that case.
Signed combinations over a whole block of primes. So the first-order change of an accepting record should be skew. Write x_i and y_i for the labels of the object i of a two-layer fold record on the layers 0 and 1. At $p \equiv 3 \pmod 4$ the number -1 is a nonsquare, so the reserved reflection, which exchanges x and y, maps the channel $\mathsf{C} = \chi(y_0 - x_1)$ of the added accepting edge $x_1 \to y_0$ to $-\mathsf{D}$, where $\mathsf{D} = \chi(y_1 - x_0)$ is the channel of its mirror $x_0 \to y_1$. Writing K_{old} for the cut kernel of the reference fold record at ι, relative to the product of its two actual half laws, adding both edges multiplies this kernel by $(1+\mathsf{C})(1+\mathsf{D})$, so the change of kernel is

$$K_{\mathrm{old}}(1+\mathsf{C})(1+\mathsf{D}) - K_{\mathrm{old}} = K_{\mathrm{old}}(\mathsf{C}+\mathsf{D}) + K_{\mathrm{old}}\mathsf{CD}\,:$$

the first-order part $\mathsf{C}+\mathsf{D}$ is skew, and the symmetric part CD is the kernel of the two disjoint cross pairs (x_1, y_0) and (x_0, y_1), spanning only two of the four pairs of $\{x_0, x_1\} \times \{y_0, y_1\}$ (a nonstar two-edge kernel), of normalized norm $O(1/p)$; the other orientation, $(1+\mathsf{C})(1-\mathsf{D})$, would make the first-order change symmetric; this is not the mirror pairing of Lemma 23.5, which is the $(1+\mathsf{C})(1+\mathsf{D})$ orientation above. Within nonnegative combinations of the accepting sign patterns

a zero symmetric leading error forces this pairing, with skew leading norm $\sqrt{2}\,r$. The fold record of the construction allows signed weights: it is the signed mixture

$$\bigotimes_{p\in\mathcal{B}} P_{++,p} + \frac{\mathsf{t}}{2}\bigotimes_{p\in\mathcal{B}} P_{-+,p} - \frac{\mathsf{t}}{2}\bigotimes_{p\in\mathcal{B}} P_{--,p},$$

taken after the products over the primes of the block, of three nonnegative accepting laws, each repaired to the actual half marginal of the reference record by an accepting Markov coupling (Definition 23.9 and Lemma 23.7). Its symmetric leading change is still zero, its skew leading change is a three-arm star of norm about $\sqrt{1+\mathsf{t}^2+(1-\mathsf{t})^2}\,r = 1.2732\,r$ at $\mathsf{t} = 127/500$, and its variation $1+\mathsf{t}$ does not grow with the number of primes. In the complete sign model a zero symmetric leading error forces a one-parameter family with this skew norm, least at $\mathsf{t} = 1/2$, whose laws have variation at least $1+\mathsf{t}$ for $\mathsf{t} \geq 0$; the three-component law attains this variation, and the complete closing chooses $\mathsf{t} = 127/500$ (Remark 35.5). The comparison theorem allows such a law, because the cancellation of the kernel is applied to each nonnegative product component and the variation multiplies only the constant of a vanishing remainder (Definition 9.14). Mirror pairing is the case $\mathsf{t} = 0$. The even products of skew changes, which survive in $\cosh A$, are not positive semidefinite either; positive envelopes in the algebra generated by the features depending on a single label (the singleton features), averaged over the reflections of the slot array in the first chamber coordinate, price them well below the coefficient B that norm envelopes would give (Theorem 25.8 and Lemmas 25.3 and 25.5).

Including one fold diagonal in the dominating law. The bracket carries one copy of the fold diagonal, $\mathsf{N} = \lambda + F_{\mathrm{fold}} + H$ (Lemma 22.7), where F_{fold} is the sum of the two one-layer fold copies, H is the analogous diagonal copy of the two-layer matchings, E_{add} below is a further pair of one-layer matching copies, G is the polarization correction (two merge records), and $\mathsf{N}_0 = \mu + E_{\mathrm{add}} + G + H$ is the auxiliary dominator of Definition 22.4. Then the part $\mathsf{N}_0 - \mu$ of the bracket beyond the reference is invariant under the chamber group of the first coordinate except for the polarization correction G, a pair of merge records, and the envelopes use the invariant part as the positive background measure; its assigned price falls short of U_0 by only about $2.4\cdot 10^{-5}$ (Lemma 25.6). The price is a doubled protection of the fold records in the completion, $v_{\mathrm{D}} \geq 2w_{\mathrm{D}}k_{\mathrm{D}}$ (Lemma 22.1(iv)), which raises the prices of the first stage. It is affordable because the first-stage error is a series in the two first-stage record differences (the tags of (CP2)(b)), each tag being the difference whose largest cut norm is attained at its own cut, whose sharp bounds multiply there: $R_{\mathrm{err}} < 2.0445\cdot 10^{-4}$ and $L_0 < 4.7634\cdot 10^{-4}$, against about $3.3\cdot 10^{-3}$ and $7.6\cdot 10^{-3}$ with one sharp factor per term (Corollary 10.9).

The two residue classes modulo 4. At $p \equiv 1 \pmod 4$ the strict-square relation is symmetric. Labels drawn conditionally independently from a random square clique of $\mathbb{Z}/p^{2m_{\mathrm{cl}}}\mathbb{Z}$ (any two of its elements differ by an even power of p times a unit square) give a reference that is reflection positive at every cut, since a chamber cut form is the square of a conditional mean, and that accepts every query unless its two labels coincide (Lemma 19.2). This event has probability $2^{-m_{\mathrm{cl}}}$ and is deleted, at a total price below $8\cdot 10^{-69}$ (Lemma 19.3); so these primes cost nothing at first order, and they leave the small modulus. At $p \equiv 3 \pmod 4$ the relation is the Paley tournament, so a law that draws the two ends of a query exchangeably cannot accept it: these primes need the oriented graph laws and the bracket above, and the completion uses that the two-label middle law is uniform on distinct pairs, which holds at $p \equiv 3 \pmod 4$ and fails at $p = 13$ (Remark 20.10). So the laws that the 216-slot design used for both classes became a square-clique prefix below 2^{512} for class one and the middle and upper laws for class three (Sections 19 and 35.5). Class three also gives exact identities used throughout: the density ratios of the rows, the Paley identities $\mathsf{C}^* = -\mathsf{C}$, $\mathsf{C}\mathsf{C}^* = pI - J_{\mathrm{all}}$ of the channels of the fold records (Lemma 4.11(c)), and the exact contraction of the index (Section 3.8).

3.7. **The crude bound exceeds 1: five sharper inequalities.**
Problem. Bounding every simultaneous change by its norm, without sharper family bounds, without retaining the common mean, and with $T = S + A$, gives only the crude sufficient condition

$$L_0 + B\big(e^S \cosh A - 1\big) + Be^{S+A}\Big(\frac{R_I}{2}\Big)^6 + W < 1,$$

which is a different (stronger) requirement than $\Theta < 1$: it has no factor $63/64$, no savings D, and no mean term. At the baseline values its left-hand side is about $1.603 > 1$ (Remark 35.1), so this crude route cannot prove the theorem. Five separate sharper inequalities below replace it by $\Theta < 1$.
Construction: five sharper inequalities. Each credit below bounds a separately named term or coefficient in the closing quantities, and the five terms are distinct, so the five savings may be added without subtracting any quantity twice.

(a) *Variance credits* (Section 24). A repaired record has the actual half marginals of the record it replaces, so their difference annihilates the constants, and its form against the bracket is a variance: the energy of the diagonal copy at ι of its actual half marginal minus that of the product of its actual half marginals. Lower bounds for the latter, with positive backgrounds of the outside bracket, for one to four blocks and for the late pieces jointly, are subtracted from the reserved cost; finally $D_{\mathrm{var}} > 0.0099585$.

(b) *Positive joint envelopes* (Theorem 25.8 and Corollary 25.10). The even skew products are priced below their trivial coefficient B: by positive matrices built from the single-label and star feature functions and averaged over the reflections of the slot array. The part invariant under all reflections is priced separately at a smaller price, since an error depending on one changed coordinate can first be averaged over its orbit under the reflections; leading and remainder prices are kept apart. With (a) they form the discount $D > 0.46685$, and $\mathcal{E} = B(e^S \cosh A - 1) - D < 0.76897$, against about 1.2358 before the discount, of which about 1.2016 is the even skew part.

(c) *Bounding old and new records jointly* (the direct Gram matrix, Section 26.1). The matching envelopes of the fold records are built from a joint 2×2 Gram estimate of the old and the new record at ι, instead of their norm envelopes. For the signed law its ideal coefficient is about 2.3842 ($\sqrt{6}$ for pure pairing); each diagonal side coefficient of the joint Gram bound is the larger of the signed coefficient and a certified auxiliary coefficient bound for that side, and only its excess over the fold diagonal v_{D} that the bracket already carries is paid (Proposition 26.4). The matching exponent is $T < 0.60760$ instead of $S + A \approx 0.984$, and $J < 0.25177$ instead of about 0.3668. At the October 18 design this step alone took the closing from 1.1219 to 1.0122.

(d) *The common mean* (Lemma 10.1(a)). The component of empty support of the six signed indices is kept exactly, so the reserved cost is counted 63 times out of the 64 copies of the constant term: the coefficient of $L_0 + \mathcal{E}$ is $63/64$. It saves about 0.0120.

(e) *The mean credit* (Proposition 26.7). Positive means of records of the reference, and of the chosen side diagonals that the (CP6) matching dominators already include, at degrees up to four and on the late pieces jointly, give $m > 1.58994$ in the term $(1 - m)/64$, which saves about 0.0092.

The result is

$$\Theta = \tfrac{63}{64}(L_0 + \mathcal{E}) + J + W + \tfrac{1-m}{64} \approx 0.757418 + 0.251761 + 0.000038 - 0.009218,$$

certified as $\Theta < 0.99999999997327$ (Section 30). Without (d) and (e) it would be about 1.0212: in the baseline every listed credit is needed, and the margin $1 - \Theta$ is about $2.7 \cdot 10^{-11}$. Every credit is an exact relation minus upper bounds for the same quantities (Definition 9.20); an upper bound

is never used as a lower bound, and a certified joint bound is never replaced by a sum of looser ones (Section 35.11).

3.8. **Six parameters of the analytic stage and the estimates they control.** Six further parameters, from the analytic stage rather than the local laws, were also tightened; each removes a loss that would otherwise cost the final exponent.

The density exponent $\theta = 1$. The induction bounds Y by $C\mathsf{L}^{-a}\alpha^{\theta}$, and the exponent of the theorem is $a/(72-\theta)$. The analytic remainders of the cancellation and of the large products were first bounded absolutely, by $C\mathsf{L}^{-\sigma}$; against $Y > C_0\mathsf{L}^{-a}\alpha^{\theta}$ this needs α not too small, which forced $\theta < 1$ and a sparse side condition $as/(s-\theta) < \sigma$, with $s < 1$ the exponent of a bound for sparse sets. In the six-query design of exponent $7.057 \cdot 10^{-6}$, with $\theta = 29/50$, this condition was within one per cent of binding. Moment bounds that are linear in the density make every remainder $O(\mathsf{L}^{-\sigma}\alpha)$ (Lemmas 8.10 and 8.11), so the induction keeps $\theta = 1$ under the single condition $a < \sigma$ (Theorem 8.12); the baseline $a \approx 8.884 \cdot 10^{-4}$ is below $\sigma = 1/825 \approx 1.212 \cdot 10^{-3}$, and all analytic side conditions hold on the whole range $0 < a \le 9/10000$, independently of the thresholds Q_3 and P_3 (Certificate D.2).

The Fourier cutoff $N^{1/20}$. The error exponent is limited by the cutoff N^{β} of the lift. When the cutoff was raised, the comparison was pointwise in the selected labels and saved $\gamma\beta/2 - \tau - 72\delta$, and the large-product estimate saved $\tau/4 - \ell_0$, with $\gamma = 255/1024$ the mixing exponent at the tail primes and δ, ℓ_0 the moment losses (Definition 8.1); so $\sigma < \min(\gamma\beta/2 - \tau, \tau/4) \le \gamma\beta/10$, and at the cutoff $\beta = 1/36$ of the earlier versions this bound is about $6.9 \cdot 10^{-4}$, below the final a. The stage inequalities that the certificate checks keep this form ((S1) and (S4) of Certificate D.2). A single averaged comparison needs only $\gamma\beta/2 - 72\delta > \sigma$; summing the raw errors over the small selections in the improved induction uses the additional τ allowance in (S1), as proved in Proposition 32.9. The actual-square kernel for Fourier cutoffs up to $N^{1/20}$ and prime-power moduli (Appendix C) allows $\sigma = 1/825$; the prime powers are needed by the square-clique prefix.

The least square step. Fixing the labels at a set of atoms fixes a residue class modulo an integer D, which is split into progressions whose step must be a square divisible by D. The step is the least such square, $\mathrm{sq}(D)$ (8.1), where the earlier versions used the square of the index part of D. The single-level law therefore improved the step for its always-revealed modulus 24 from 576 to 144. The selected finite-depth law instead averages $\mathrm{sq}(\prod_p p^{J_p})$ over its conductor depths, with the same least-square principle at every level (Proposition 29.7).

A fixed number of intervals. Earlier versions took $K = 2^{64}$ intervals, making the wrap negligible. Every proof holds for arbitrary fixed $K \ge 2$ (Remark 8.13). The matching term grows like $K^{a/6}$, whereas the wrap decreases with K. The selected fixed value $K = 890666$ gives $W < 3.8345 \cdot 10^{-5}$. Feasibility at this integer is all the proof requires; the convexity calculation in Remark 29.10 guides its choice.

Square shifts at every conductor depth. At the first odd-prime level the exact square-shift contraction is $\sqrt{p+1}/(p-1)$, optimal among doubly stochastic matrices on the strict-square transitions (Lemma 4.11(b)). This single-level optimality does not rule out a better distribution across conductor depths. The selected law conditions a uniform root to have valuation less than m in each prime-power coordinate, and then squares it. Its depth envelopes and finite moment are proved in Lemma 29.6 and Proposition 29.7; the certified value is $R_I < 1.247139907407$ at the baseline exponent. The selected final closing uses $R_I < 1.252937308029$ (Section 33). The old prime-moment certificate remains an auxiliary majorant for this evaluation.

The digit pairs of the prefix. The square clique uses $m_{\mathrm{cl}} = 8192$ digit pairs. Its exception is priced below $2^{513-m_{\mathrm{cl}}+1024m_{\mathrm{cl}}a}$ (Lemma 19.3); with the earlier $m_{\mathrm{cl}} = 2048$ this bound would exceed one as soon as $a > 1535/2^{21}$, that is $\varepsilon > 1.0309 \cdot 10^{-5}$, below the final exponent. With 8192 pairs the bound stays below the allowance 10^{-12} on the whole analytic range $a \le 9/10000$ (Remark 35.6).

3.9. **From the local laws to the final closing.** The examples above explain the baseline prices. The same local interfaces support the main theorem, but its final assembly must recompute their assignments together. Table 3 gives the reading path; the baseline numerical table is Table 12, and the selected numerical ledger is in Section 33.

TABLE 3. The construction's interfaces and their final assembly.

Output	Proof and use
$\alpha^{72} \le Y$	Reflection positivity, Section 5
First-stage price L_0, dominator B	Middle reference and frame, Sections 20 to 22
Comparison prices S, A	Middle and upper accepting laws, Sections 23 and 28
Discount D	Fixed-marginal variance and positive envelopes, Sections 24 and 25
Matching price T and joint mean	Chosen sides and disjoint mean allocations, Sections 26 and 26.6
Conductor moment R_I	Finite index law and summable tail, Section 29
$Y \le C\mathsf{L}^{-a}\alpha^{\vartheta}$	Prefix moments and density induction, Sections 31 and 32
$\Theta < 1$	Normalized, compatible prices and the joint scalar, Section 33

Each arrow from a local improvement to this closing requires its actual marginals, completion weights, restoration bounds and normalizing factors. A smaller isolated operator constant does not by itself lower Θ. The final assembly keeps those dependencies explicit and evaluates every affected term at the selected parameters.

4. Preliminary lemmas: character sums, correlation, and monotone bounds

What this section proves and how to read it. This section proves no part of the main argument. It fixes conventions for finite probability spaces and operators, lists arithmetic notation, and states five inputs used as black boxes later: Weil's bound with Gauss and Jacobi evaluations (first-order $O(p^{-1/2})$ estimates), the spectrum of the square shift on $\mathbf{F}_p$, an explicit prime-counting bound, two correlation inequalities, and a method for extending a bound proved at the smallest prime of a range to every larger prime (monotone majorants). Inputs: only standard character-sum and probability facts, with Peyre's inequality proved in Appendix B. Outputs: the lemmas cited by Parts IV and V for counting, correlation after fixing labels, and prime summation. A first reading should note the conventions in Section 4.1 and the statements only; proofs can be skipped. The list in Section 4.2 is an *index of symbols* with pointers to defining sections, not a set of definitions; project terms are defined in Section 1.8 and at their formal sites. In particular "price" is defined in Definition 9.2, not here.

4.1. **Finite probability spaces, operators, and correlation.** All probability spaces in the paper are finite. This subsection fixes expectation, norms, operator order and correlation once, so that later sections can use them without repeating conventions. Asymptotic notation is the standard one, made precise as follows: $X \ll Y$, $X = O(Y)$ and $X \lesssim Y$ all mean $|X| \le CY$ for an absolute constant C, independent of N, of the scale L, of A and of p, unless a dependence is stated in the same statement or in the standing conventions of its section; $X \asymp Y$ means both $X \ll Y$ and $Y \ll X$; and $o(1)$ denotes a quantity tending to 0 as $\mathsf{L} \to \infty$ with all other parameters fixed, unless another limit is named. In statements at a least counterexample, $o(1)$ is this limit along counterexamples with $\mathsf{L} \to \infty$.

Definition 4.1 (Finite probability spaces and operator order)**.** For a finite probability space (X,μ), $\mathbb{E}_\mu f=\sum_{x\in X}\mu(x)f(x)$ and $\|f\|_{L^r(\mu)}=(\mathbb{E}_\mu|f|^r)^{1/r}$ for $1\le r<\infty$. The inner product is $\langle f,g\rangle_\mu=\mathbb{E}_\mu\overline{f}g$; $L^2_0(\mu)$ is the subspace of mean-zero functions. All unqualified averages on finite sets use uniform probability. A measure ν has density h with respect to μ if $\nu(x)=h(x)\mu(x)$ at each point; its mass is $\nu(X)$. The notation δ_x denotes the probability measure concentrated at x. For a self-adjoint operator T, the notation $T\succeq 0$ means $\langle f,Tf\rangle\ge 0$ for every f. We write $S\preceq T$ if $T-S\succeq 0$; this is the *Loewner order*. The operator norm is $\|T\|=\sup_{\|f\|_2=1}\|Tf\|_2$. For a numerical matrix, the operator norm uses the ordinary Euclidean norms on its coordinate spaces: matrix multiplication is a sum, with no probability normalization. In particular, this convention applies to the matrix of correlation bounds in Theorem 4.13. A *Markov operator* preserves nonnegative functions and the constant function 1.

Kernels, projections and centred norms. Probability spaces are finite and, after the points of mass zero are discarded, have full support. For such a space (Ω,λ), $\mathbb{J}f=\langle\mathbf{1},f\rangle_\lambda\,\mathbf{1}=(\mathbb{E}_\lambda f)\mathbf{1}$ and $P_0=I-\mathbb{J}$ (in Part IV also written Q; this Q is not the conductor cutoff Q of Section 6) are the orthogonal projections onto the constants and onto $L^2_0(\lambda)$. For finite probability spaces (X,μ), (Y,ν) and a kernel $K\colon X\times Y\to\mathbb{C}$ let $T_Kf(x)=\int K(x,y)f(y)\,d\nu(y)$, and let $\|K\|_{\mathrm{op}}$ be the norm of T_K from $L^2(\nu)$ to $L^2(\mu)$. Both spaces carry the uniform probability unless another measure is indicated; for the uniform probabilities, $\|K\|_{\mathrm{op}}$ is the operator norm of the entry matrix of K (Euclidean norms) divided by $\sqrt{|X||Y|}$. If $X=Y$, $\mu=\nu$ and $K(y,x)=\overline{K(x,y)}$, then T_K is self-adjoint. The all-ones kernel is J_{all}, so that $T_{J_{\mathrm{all}}}=\mathbb{J}$ on a uniform space; as a matrix acting on vectors indexed by n points, $J_{\mathrm{all}}v=(\sum_y v(y))\mathbf{1}=n\mathbb{J}v$.

Let T be a Markov operator from functions on Y to functions on X with $T\mathbf{1}=\mathbf{1}$ and $T^*\mathbf{1}=\mathbf{1}$, the adjoint taken with respect to μ and ν. Equivalently, $Tf(x)=\mathbb{E}_\pi[f(y)\mid x]$ for the coupling $\pi(x,y)=\mu(x)T(x,y)$ of μ and ν, $T(x,y)$ being the transition probability, so that $Tf(x)=\sum_{y\in Y}T(x,y)f(y)$; we call T the operator of the coupling. As $\langle\mathbf{1},Tf\rangle_\mu=\langle T^*\mathbf{1},f\rangle_\nu$, T maps $L^2_0(\nu)$ into $L^2_0(\mu)$, and its *centred norm* is $\|T|_{L^2_0(\nu)}\|=\|T-\mathbb{J}\|$, with $\mathbb{J}f=(\mathbb{E}_\nu f)\mathbf{1}$. By Definition 4.3 the centred norm of the operator of a coupling is the maximal correlation of its two coordinates. For a cut c of a slot array, $\kappa\succeq_c\kappa'$ means that the cut matrix of $\kappa-\kappa'$ is symmetric and positive semidefinite (Definition 5.2).

Definition 4.2 (Mean, fluctuation and compression)**.** On a finite probability space (Ω,λ) with full support let $\mathbb{J}f=(\mathbb{E}_\lambda f)\mathbf{1}$ and $P_0=I-\mathbb{J}$. For a real function U put $u=\mathbb{E}_\lambda U$, $U^\circ=P_0U$ and $\sigma_U=\|U^\circ\|_2$. Thus $\|U\|_2^2=u^2+\sigma_U^2$. For an operator T on $L^2(\lambda)$ with $\langle\mathbf{1},T\mathbf{1}\rangle=1$, its two *marginal errors* are $c=T\mathbf{1}-\mathbf{1}$ and $b=T^*\mathbf{1}-\mathbf{1}$; its *mean-zero compression* is $C=P_0TP_0$. Relative to the constants and their orthogonal complement,

$$T=\begin{pmatrix}1 & b^*\\ c & C\end{pmatrix},$$

where b^* is the functional $g\mapsto\langle b,g\rangle$ on $L^2_0(\lambda)$.

Both marginal errors are mean-zero, since $\langle\mathbf{1},c\rangle=\langle\mathbf{1},T\mathbf{1}\rangle-1=0=\langle T^*\mathbf{1},\mathbf{1}\rangle-1=\langle b,\mathbf{1}\rangle$, and for $f=\alpha\mathbf{1}+f_0$ with $f_0\in L^2_0(\lambda)$, $Tf=(\alpha+\langle b,f_0\rangle)\mathbf{1}+(\alpha c+Cf_0)$, which is the block form. The operator of a coupling of λ with itself has $b=c=0$ and centred norm $\|C\|$. The completion inequalities of Lemma 17.5 are written in this notation.

Definition 4.3 (Maximal and conditional maximal correlation)**.** For random variables X,Y on a finite probability space, the *maximal correlation* is

$$\rho(X;Y)=\sup\{|\mathbb{E}f(X)g(Y)|:\ \mathbb{E}f(X)=\mathbb{E}g(Y)=0,\ \mathbb{E}f(X)^2=\mathbb{E}g(Y)^2=1\},$$

with $\rho=0$ if either side has no non-constant functions. The intersection of the spaces $L^2(X)$ and $L^2(Y)$ of functions of X and of Y, inside L^2 of the underlying space, contains the constants. If it

consists of the constants only, then $\rho(X;Y)$ is the Friedrichs cosine of these two subspaces (the supremum of $|\langle f,g\rangle|$ over unit vectors f in one subspace and g in the other, after the constants are removed); if it is larger, then $\rho(X;Y)=1$, while the Friedrichs cosine may be smaller. In all cases $\rho(X;Y)$ is the norm of the conditional expectation operator $L^2_0(X)\to L^2_0(Y)$. For a family $(X_i)_{i\in I}$ and $S\subseteq I$, the *conditional* maximal correlation of X_i, X_j given $X_S=s$ is the maximal correlation computed under $\mathbb{P}(\cdot\mid X_S=s)$, for every value s of positive probability.

Definition 4.4 (Friedrichs angle)**.** For closed subspaces V, W of a finite-dimensional Hilbert space, the *cosine of the Friedrichs angle* is $\cos\angle(V,W)=\sup\{|\langle v,w\rangle| :\ v\in V,\ w\in W,\ \|v\|=\|w\|=1,\ v,w\perp V\cap W\}$. It is 0 if one space contains the other.

4.2. **Arithmetic notation, square classes, and index of symbols.**
Arithmetic notation. Throughout, p denotes a prime, $q=1/p$ and $r=p^{-1/2}$; for odd p, $\chi=\chi_p$ is the quadratic character. We write $[N]=\{1,\dots,N\}$ and $e(t)=e^{2\pi it}$. All logarithms are natural, an empty product is 1, and a positive integer is *squarefree* if no square of a prime divides it. For a positive integer d we write $v_p(d)$ for the exponent of p in d, $\varphi(d)$ for the number of invertible residues modulo d, $\omega(d)$ for the number of its distinct prime divisors (always with its argument) and $\mathrm{rad}(d)$ for their product; $\pi(x)$ is the number of primes up to x. For $\xi\in\mathbb{Q}/\mathbb{Z}$, $q(\xi)$ is its reduced denominator, the *conductor*, with $q(0)=1$; for a set U of primes, $q_U=\prod_{p\in U}p$. For an odd prime p we put

$$Q_p^*=\{u^2: u\in\mathbb{F}_p^\times\},\qquad A_p(t)=\mathbf{1}_{Q_p^*}(t),\qquad \Omega_p=\{(x,y)\in\mathbb{F}_p^2:\ y-x\in Q_p^*\};$$

$\chi_p(t)$ is 1 on nonzero squares, -1 on non-squares and 0 at $t=0$, and $G_p(k)=\sum_{u\in\mathbb{F}_p}e(ku^2/p)$. The space $L^2(\mathbb{F}_p)$ carries the uniform probability, $\langle f,g\rangle=\mathbb{E}_{a\in\mathbb{F}_p}\overline{f(a)}g(a)$, and $e_k(a)=e(ka/p)$, $k\in\mathbb{F}_p$, is an orthonormal basis; the e_k with $k\neq0$ span $L^2_0(\mathbb{F}_p)$. For a law ν on a finite set and functions F and G on that set, $\|F\|_\nu^2=\mathbb{E}_\nu|F|^2$ and $\langle F,G\rangle_\nu=\mathbb{E}_\nu\overline{F}G$; the index is dropped when the law is clear.

Definition 4.5 (Strict square classes)**.** Let $D=8d$ with d odd, not necessarily squarefree. A residue r mod D is a *strict square class* if $r\equiv1\pmod 8$ and r is a nonzero square modulo every prime dividing d.

By Hensel's lemma the second condition is equivalent to r being the square of a unit modulo every odd prime power dividing D: if p is odd, $p\nmid r$ and $c_j^2\equiv r\pmod{p^j}$, then $c_{j+1}=c_j+tp^j$ with $2c_jt\equiv(r-c_j^2)/p^j\pmod p$ satisfies $c_{j+1}^2\equiv r\pmod{p^{j+1}}$, and t exists because $p\nmid 2c_j$. Every unit modulo 8 has square 1, so by the Chinese remainder theorem the strict square classes modulo D are exactly the squares of units modulo D; in particular a strict square class is a unit modulo D.

The external index of Definition 9.6 also uses nonunit squares at a fixed prime-power modulus. We use the following more general class only when explicitly stated.

Definition 4.6 (Nonvanishing square classes)**.** Let $D=2^{h_2}\prod_{p\text{ odd}}p^{h_p}$ with $h_2\geq3$. A residue r mod D is a nonvanishing square class if, at each odd $p\mid D$, it has valuation $2j_p<h_p$ and square unit part, and at 2 it has valuation $2j_2$ with $h_2-2j_2\geq3$ and unit part congruent to 1 modulo 8. The valuations and unit parts are properties of the reductions of r modulo p^{h_p} for odd p and modulo 2^{h_2} at 2. In particular zero modulo a prime-power coordinate is excluded.

These conditions imply the existence of a square root modulo each full prime-power coordinate. At odd primes this is Hensel lifting after division by p^{2j_p}. At 2, begin with an odd root modulo 8: if $b^2\equiv u\pmod{2^k}$ with $k\geq3$, one of b and $b+2^{k-1}$ squares to u modulo 2^{k+1}, since their squares differ by 2^kb modulo 2^{k+1}. This lifts every unit $u\equiv1\pmod8$. Strict square classes with $v_2(D)=3$ are a special case.

The least square step. For a positive integer d put

$$\mathrm{sq}(d) = \prod_{p \mid d} p^{2\lceil v_p(d)/2 \rceil}, \tag{4.1}$$

the least positive square divisible by d: a square $\prod_p p^{2k_p}$ is divisible by d exactly when $2k_p \geq v_p(d)$ for every p. Thus $d \mid \mathrm{sq}(d) \mid d^2$, $\mathrm{sq}(de) = \mathrm{sq}(d)\,\mathrm{sq}(e)$ for coprime d, e, and $\mathrm{sq}(p) = p^2$, $\mathrm{sq}(8) = 16$, $\mathrm{sq}(24) = 144$.

Conventions on numbers. $o(1)$ is as defined in Section 4.1; in the analytic statements of Part II the convergence as $\mathsf{L} \to \infty$ is uniform in the set A. Numbers in the text are the numbers certified in Appendix D, written as in the certificate (outward-rounded decimals or exact rationals); a decimal that is not certified is marked "about". A *price* is $\|\zeta\|_{\mathrm{var}} q_{\mathcal{B}}^{2a}$ for a signed block measure ζ (Definition 9.2); an *operator price* is a weight times an operator norm relative to the actual half marginals times $q_{\mathcal{B}}^{2a}$; an *upper price* is any number at least the price, in practice a certified upper bound. Every coefficient that is *evaluated at an endpoint* is an admissible majorant in the sense of Lemma 4.18, written with its powers of r cancelled first (Remark 4.19).

Symbols. Table 4 lists the principal symbols and where they are defined. Its rules concern named quantities, written with the bare letter: t denotes only the repricing factor, τ and σ only the analytic-stage parameters, ι only the reserved cut and J only the matching term; and $\delta_{p,j}$ at a prime-power index coordinate denotes its depth tail, while δ_p, δ_j denote inclusion probabilities in the whole-coordinate special case. A letter used as a bound variable inside one statement or proof is local to it, for instance a number t of roots or the index of e_t, a type τ, a permutation σ, an index set J, the matrix $\mathcal{E}$ of correlation bounds in Theorem 4.13, or the point mass δ_x. In the same local sense m or m_0 may denote a probability measure on a set of labels (Sections 12, 23 and 25), m a multiset of middle pieces (Lemma 20.7 and Sections 24 to 26), and t may index the records of a block (Sections 9 and 10 and Part V); the mean credit m occurs only in the closing formulas, and the repricing factor t only in formulas that carry a price certified at the anchor a_0 to the exponent a: the repricing statements (Lemma 9.3 and Corollary 9.4), the totals and credits of Part V formed from such prices (for instance the middle prices, the variance credit and the class costs of the envelopes, Definition 30.1 and Corollary 25.10) and the closing formulas; these local uses do not occur there. Two rules are followed in every section: the letter B alone is the energy factor of the reserved Gram bracket, and a block of primes is written $\mathcal{B}$ wherever both can occur (Part IV, which never meets the energy factor, may write a block as B); and the layer coordinate of a slot is ℓ, so that t is free for the repricing factor. Several objects of the local laws carry sans-serif letters: the Paley channels A–D of the fold record, the scores H_j of the positive roots, the signed-law parameter t and the root-tilt parameter u; neither t nor u is ever the repricing factor t. The reserved cost is $\mathcal{E}$ (calligraphic, without subscript), and the query family is $\mathcal{E}_6$ (always with its subscript).

Table 4: Principal symbols. Numerical illustrations are at the baseline parameters unless the selected parameters are specified.

symbol	meaning (and where it is defined)
N, $[N]$, $s(N)$	the scale of the main theorem; $\{1, \dots, N\}$; the largest size of a subset of $[N]$ with no nonzero square difference
A, α	a square-difference-free set $A \subseteq [N]$ and its density $\alpha = \lvert A\rvert/N$; the letter A alone is also the skew total (row S, A, T)
ϵ	the exponent of the main theorem, $s(N) \leq CN^{1-\epsilon}$, $\varepsilon = a/(72-\theta) = \frac{1814179}{141970826000}$; the baseline has $\varepsilon = a/71 = \frac{5422587}{433349609375}$

symbol	meaning (and where it is defined)
a, a_0, a_{u}	the price exponent of the functional, baseline $a = \frac{5422587}{6103515625}$, selected $a = \frac{1814179}{2000000000}$; the anchor $a_0 = \frac{4442183}{5000000000}$, at which the middle prices are certified (Section 30.1); the upper ledger has its own anchor $a_{\mathrm{u}} = 883/10^6$ (Proposition 28.14)
θ	the density exponent of the induction: baseline 1, selected $\frac{1014587}{1000000}$ (also denoted ϑ in the closing); chamber reflections carry their cut indices, and $\theta(x;4,r)$ always carries its arguments
L, K	the interval scale, $N = K\mathsf{L}$; the number of intervals of the index: baseline $K = 890666$, selected $K = 863660$
L, L_{loc}, L_{band}, L_{u}	the actual block budget $L = 570$ of the middle range (Definition 20.1), used for the repricing and for the caps in the restorations of Lemma 24.8; the local cap $L_{\mathrm{loc}} = 585$, at which the local laws, the block densities, the row constants, the spectral caps and k_{D}, k_{new} are evaluated at every piece start (Lemma 20.3); the budget L_{band} of an upper band, an integer between 550 and 886 at which the local data of the band are evaluated (Definition 27.1); and the block length $L_{\mathrm{u}} = 50000/27$ of the local data of the tail datum, larger than every band budget (Definition 18.1)
G_N, e_p	the label group $\prod_p \mathbb{Z}/p^{e_p}\mathbb{Z}$ over the set $\mathcal{P}_N$ of primes of the scale N, with $e_p = 2m_{\mathrm{cl}}$ at the primes $p \equiv 1\ (4)$ below P_4 and $e_p = 1$ otherwise (Definition 6.1)
V, d	the slot array $(\mathbb{Z}/6\mathbb{Z})^2 \times \{0,1\}$ with slots (u_0, u_1, ℓ), $d = \lvert V\rvert = 72$; the layer coordinate is ℓ (Definition 5.12)
c	the rotation of $\mathbb{Z}/6\mathbb{Z}$: $\mathsf{c}j = j+2$ for even j and $\mathsf{c}j = j-2$ for odd j
$\vartheta_{c,a}$, H_a	the chamber reflection $\vartheta_a(j) = 2a-1-j$ in the coordinate $c \in \{0,1\}$, with plus half $H_a = \{a, a+1, a+2\}$
ι	the reserved cut $(u_0, u_1, \ell) \mapsto (u_0, u_1, 1-\ell)$, with plus half $\{\ell = 0\}$; ι is used for nothing else, and the class indicator is written $\mathbf{1}_{p\equiv 1\,(4)}$
$\mathcal{E}_6$, $e = (u_e, v_e)$	the six queries $u_e = (0, j, 0)$, $v_e = (\mathsf{c}0, \mathsf{c}j, 1)$, $j \in \mathbb{Z}/6\mathbb{Z}$ (Definition 5.12)
p, q, r; χ, Q_p^*, Ω_p	a prime, $q = 1/p$, $r = p^{-1/2}$; the quadratic character, the nonzero squares, the strict-square pairs
$G_p(k)$, G_χ, ε_G, Jac	Gauss sums, $G_\chi^2 = \chi(-1)p$, $\varepsilon_G = G_\chi/\sqrt{p}$ (Theorem 4.8); the Jacobi sum (Theorem 4.10)
$\delta_{p,j}$; δ_p, δ_j	the conductor-depth tail $\mathbb{P}(J_{e,p} \ge j)$ of Definition 9.6, with the selected $m = 64$ law of Lemma 29.6; the latter two symbols are inclusion probabilities in the whole-coordinate special case
$\omega_e = (b_e, s_e)$	the index of the query e: an interval and a residue modulo M_0 (Definition 9.6); count errors are written e_t, w_Δ, never ω
M_0, $D_{\mathcal{J}}$, $\mathrm{sq}(d)$	the external modulus $2^{129}\prod_{p<Q_3,\,p\equiv 3\,(4)} p^{128}$ at $m = 64$; the revealed modulus $\prod_p p^{J_{e,p}}$ for one query; the least square divisible by d, (4.1)
$\mathcal{J}$, R_I	the independent depth vector, with every interval revealed (Definition 9.6); the conductor moment $\mathbb{E}_{\mathcal{J}}(K\,\mathrm{sq}(D_{\mathcal{J}}))^{a/36} = M_{\mathrm{pad}}(K)$, bounded above by 1.247140 (Certificate 29.11)
Q_3, P_3, P_4, P_5	19951586, $167\cdot 2^{40}$, 2^{512}, 2^{65536}: the primes $p \equiv 3\ (4)$ of $[Q_3, P_3)$ form the middle range and those of $[P_3, P_4)$ the upper range; the large primes are those from P_4 on, one atom each; mixing in the sense of (R4) is asserted only from P_5 on (Definition 18.1)
m_{cl}, m	the number of digit pairs of the square-clique prefix, $m_{\mathrm{cl}} = 8192$ (Definition 18.1); the mean credit in Θ (Definition 9.18)
B	only the energy factor of the reserved Gram bracket, $B = e^{U_0} + R_{\mathrm{err}}$ (Definition 9.13); its certified upper end at the baseline parameters is 2.332628665480

symbol	meaning (and where it is defined)
$\mathcal{B}$, $q_{\mathcal{B}}$, $b_{\mathcal{B}}$	a block (atom) of primes; the product and the least of its primes
N, N_0	the reserved Gram bracket measures (Definition 9.13), never the scale N
S, A, T	the symmetric, skew and matching price totals (Theorem 9.21)
P, V, D, U_0, R_{err}, L_0	the first-stage and bracket totals, with the tag totals d_j, v_j; $L_0 = e^P \mathfrak{h}(d, v)$ and $R_{\mathrm{err}} = e^{U_0}\mathfrak{h}(d, v)$ (Definitions 9.11 and 9.13); for two tags with equal totals the series $\mathfrak{h}$ has the finite majorant $\mathfrak{h}_n^+$ (Corollary 10.9)
$\mathcal{E}$, J, W, Θ	the reserved cost, the matching term, the wrap and the closing number, $\Theta = \frac{63}{64}(L_0 + \mathcal{E}) + J + W + \frac{1-m}{64}$ for six queries in the baseline (Theorem 9.21); the selected joint formula is (33.5); $\Theta(z)$ is its inflated form
D, D_{var} (closing)	the total discount in $\mathcal{E} = B(e^S \cosh A - 1) - D$, the sum of the variance credit D_{var} and the savings of the envelope classes (Definition 30.1), never in one formula with the first-stage total D; the first-stage sharp total enters the closing only through its two equal tag totals $d_0 = d_1$, each one half of it (Definition 30.1)
A, B, C, D	the four Paley channels $\chi(y_2 - x_0)$, $\chi(x_2 - y_0)$, $\chi(y_0 - x_1)$, $\chi(y_1 - x_0)$ of the F record, sans-serif so as not to collide with the prices
t	only the repricing factor $e^{2(a-a_0)L}$, about 1.0000000616512 ($a_0 = \frac{4442183}{5000000000}$, $L = 570$; Definition 30.1)
β, η, γ, τ, t_{an}, σ	the analytic-stage parameters 1/20, 17/100, 255/1024, 1/201, 1/60, 1/825 (Definition 8.1)
τ_{side}, τ^{sp}, t_Y, Σ_i	the direct-Gram side weight 31/32; the free parameter of the spectral test; the mean (Young) allowance of a row of the upper completion, free in the joint rows of the bands (Lemma 17.5(f)) and 1/50 in the Schur rows of the tail datum (Lemma 17.5(c)); the block-start sums, certified lower bounds for $\sum_{\mathcal{B}} b_{\mathcal{B}}^{-1}$ over the blocks of piece i (Lemma 20.6)
J_{all}, $\mathbb{J}$	the all-ones kernel; the projection onto constants; J alone is the matching term
E_0, E_1, F_0, F_1, F_2, G, H, C	E_0 and E_1 matching records, F_0, F_1 and F_2 fold records, G merge records, H reserved half-diagonal records and C constant records of the local laws (Definitions 17.1 and 22.4)
$e_t(p, n)$, w_Δ, $e_t^{\pm}$, a_t, b_t, h_t	the coarse and graded insertion-count errors and their coefficients (Equations (12.1) and (12.2), Lemma 12.2)
A_F, A_F^{arm}, A_F^{lead}, A_F^{rem}, S_F (and E); S_{oth}, A_{oth}	the full skew price, the raw-arm price (coefficient $w_{\mathrm{D}} r^2$ per fold record, before the leading and remainder split), and the leading and remainder prices and the symmetric price of the fold (matching) records, with $A_F = A_F^{\mathrm{lead}} + A_F^{\mathrm{rem}}$, while the full skew price A_E of the matching records is bounded directly; the other middle symmetric and skew prices (Lemma 23.24, (23.7))
κ_{FF}	the cap of the envelope of the pairs of fold records, the fraction of their variance gain that is credited (Lemma 25.9)
t	the signed-law parameter $\mathsf{t} = \frac{127}{500}$ of the fold records (Definition 23.9)
$\mathfrak{v}_\nu$, n_ν	the variation factor of the comparison law and the number of its product components (Definition 9.14)
u; H_0, H_1, H_2, $\bar{\mathsf{H}}$	the root-tilt parameter, with its seven values for the seven tilted pieces in Definition 18.1; the three opposite-base-pair scores H_0, H_1, H_2 and their mean $\bar{\mathsf{H}}$. The last of the seven tilted pieces uses the signed coherent trace sum $H_{\mathcal{D}_{39}}$ and the positive floor 1/100 (Definition 21.1)
$\mathfrak{h}(d, v)$, d_j, v_j	the tagged first-stage series and the sharp and variation totals of the tag j (Definition 9.11)
U^{inv}	the assigned price of the invariant tags $E_{\mathrm{add}} + H$ of the outside law N_0 (Lemma 25.6)
$C_{n,m}$, $\mathcal{R}_{n,m}$, $D_{n,m}$	the leading class cost per raw-arm monomial, the remainder polynomial and the saving of the class with n fold and m matching skew records (Corollary 25.10)

symbol	meaning (and where it is defined)
$M_{n,m}$, $M^{\mathrm{inv}}_{n,m}$, δ_{s}	the collected coupling mass and the invariant orbit mass of a class; the price saving of one folded tag, sharp minus symmetric (Lemma 25.7)
m_F, m_Q	the fold and nonempty-Q coupons of the mean credit m (Proposition 26.7)
$\hat{P}$	the normalized Paley operator P/r at a class-three prime; $\hat{P}^* = -\hat{P}$, and $\hat{P}^2 = -I$ on L^2_0 (Lemma 4.11(c))
$s_{\mathtt{t}}$, $d_{\mathtt{t}}$, $\ell_{\mathtt{t}}$, e_{sg}	the ideal and actual coefficients of the signed direct Gram, the leading cap over r and the signed remainder constant (Lemma 26.1 and Proposition 23.13)
k_{D}, e_o; k_{new}, s_{fold}; $\bar{H}_X$	the negative cap and the centred remainder coefficient of the reference fold record (Lemma 23.2); the sharp and symmetric coefficients of the one-stage folded repair (Lemma 22.6); the block density factor $(1 - n_X e_X)^{-1}$ of a piece starting at X
$\delta^{+}_{s,i}$	the increment of the side $s \in \{L, R\}$ of the piece i over its auxiliary side, $\max(0, \bar{H}_{X_i} d_{\mathtt{t}}(X_i) - d^{\mathrm{aux}}_s(X_i))$ rounded up (Proposition 26.4); not to be confused with δ_{s}
g, c^+, B_{top}	the prime caps of a cut, $-gp^{-1/2} I \preceq \mathsf{A}_p \preceq (1 + c^+/p) I$ for its normalized cut operator A_p at p; the positive constant is written c^+ for the cuts of the middle range and B_{top} for the upper range, where the caps come from Route B (Lemmas 13.7 and 20.12 and Theorem 14.27)
piece, band	a range of the middle partition (Definition 20.1); a range of the upper partition (Definition 27.1)

4.3. **Character sums.** Let p be an odd prime. A *multiplicative character* is a homomorphism $\psi\colon \mathbb{F}_p^{\times} \to \mathbb{C}^{\times}$; we extend every multiplicative character, the trivial one ψ_0 included, by $\psi(0) = 0$, and only nontrivial characters will enter the sums below. The characters form a cyclic group of order $p-1$ (the dual of the cyclic group $\mathbb{F}_p^{\times}$); χ_p is its unique element of order 2. Hence for every $m \mid p-1$ there are characters of order exactly m, and if $4 \mid p-1$ every character ψ of order 4 satisfies $\psi^2 = \chi_p$. Since $|\psi(t)| = 1$ for $t \neq 0$, $\psi(t^{-1}) = \overline{\psi(t)} = \bar{\psi}(t)$. The element -1 is the element of order two of the cyclic group $\mathbb{F}_p^{\times}$, so it is a square exactly when $4 \mid p-1$: $\chi(-1) = 1$ if $p \equiv 1 \pmod 4$ and $\chi(-1) = -1$ if $p \equiv 3 \pmod 4$. For $s \in \mathbb{F}_p$ the *Gauss sum* of ψ is

$$g_s(\psi) = \sum_{t \in \mathbb{F}_p} \psi(t)\, e(st/p)\,, \qquad g(\psi) = g_1(\psi), \qquad G_\chi = g(\chi_p) = \sum_{t \in \mathbb{F}_p} \chi_p(t)\, e(t/p)\,,$$

and for multiplicative characters ψ_1, ψ_2 the *Jacobi sum* is $\mathrm{Jac}(\psi_1, \psi_2) = \sum_{t \in \mathbb{F}_p} \psi_1(t)\psi_2(1-t)$.

Theorem 4.7 (Weil's bound)**.** *Let p be an odd prime and $a_1, \dots, a_t \in \mathbb{F}_p$ distinct, $t \geq 1$.*

(a) $\bigl|\sum_{x \in \mathbb{F}_p} \chi_p\bigl((x-a_1)\cdots(x-a_t)\bigr)\bigr| \leq (t-1)\sqrt{p}$.

(b) *If $t \geq 2$ is even, then*

$$\Bigl|\sum_{x \in \mathbb{F}_p} \chi_p\bigl((x-a_1)\cdots(x-a_t)\bigr) + 1\Bigr| \leq (t-2)\sqrt{p},$$

so the sum has modulus at most $(t-2)\sqrt{p} + 1$.

Part (a) is Weil's theorem; see [33, Theorem 11.23], [69, Ch. II]. Part (b) follows from (a).

Proof of (b). Put $f(x) = \prod_{i \leq t}(x - a_i)$, $c = \prod_{i<t}(a_t - a_i) \neq 0$ and $b_i = 1/(a_i - a_t)$ for $i < t$; the b_i are distinct and nonzero. The map $u \mapsto a_t + 1/u$ is a bijection from $\mathbb{F}_p^{\times}$ onto $\mathbb{F}_p \setminus \{a_t\}$, and $x - a_i = (a_t - a_i)(u - b_i)/u$ for $x = a_t + 1/u$ and $i < t$. Hence

$$f(a_t + 1/u) = \frac{c\, g(u)}{u^t}, \qquad g(u) = \prod_{i<t}(u - b_i).$$

As t is even, $\chi(u^{-t}) = 1$ for $u \neq 0$. Since $f(a_t) = 0$ and $g(0) = \prod_{i<t}(-b_i) = 1/c$,

$$\sum_{x\in\mathbb{F}_p} \chi(f(x)) = \chi(c)\sum_{u\neq 0}\chi(g(u)) = \chi(c)\sum_{u\in\mathbb{F}_p}\chi(g(u)) - 1.$$

The polynomial g is a product of $t-1 \geq 1$ distinct linear factors, so by (a) the last sum has modulus at most $(t-2)\sqrt{p}$. □

Part (b) is the character sum over an even number of distinct roots, with the point at infinity retained: the substitution moves the root a_t to infinity, and the proof needs only (a). Its case $t = 2$ is the *two-root identity*, which we record with a direct proof: for distinct $a, b \in \mathbb{F}_p$,

$$\sum_{y\in\mathbb{F}_p} \chi((y-a)(y-b)) = -1, \qquad \text{hence} \qquad \mathbb{E}_{y\in\mathbb{F}_p}\,\chi(y-a)\chi(y-b) = \mathbf{1}_{a=b} - q \quad (a,b\in\mathbb{F}_p). \tag{4.2}$$

Indeed, for $y \neq a$, $\chi((y-a)(y-b)) = \chi((y-b)/(y-a)) = \chi(1-(b-a)/(y-a))$, and as y runs over $\mathbb{F}_p \setminus \{a\}$ the argument $1-(b-a)/(y-a)$ runs over $\mathbb{F}_p \setminus \{1\}$; the term $y = a$ vanishes. So the sum is $\sum_{s\neq 1}\chi(s) = -\chi(1) = -1$, because there are as many squares as non-squares in $\mathbb{F}_p^\times$. For $a = b$ the average is $(p-1)/p = 1-q$.

Theorem 4.8 (Gauss sums; see [31, Ch. 6, §§3–4 and Ch. 8, §2], [5, Ch. 1])**.** *Let p be an odd prime and ψ a nontrivial multiplicative character of $\mathbb{F}_p$.*

(i) *$g_0(\psi) = 0$, and $g_s(\psi) = \bar{\psi}(s)\,g(\psi)$ for every $s \in \mathbb{F}_p^\times$.*
(ii) *$|g(\psi)| = \sqrt{p}$.*
(iii) *For the quadratic character, $\overline{G_\chi} = \chi_p(-1)G_\chi$ and $G_\chi^2 = \chi_p(-1)\,p$; moreover $G_p(k) = \chi_p(k)G_\chi$ for every $k \in \mathbb{F}_p^\times$, and $\sum_{t\in\mathbb{F}_p}\chi_p(t)e(kt/p) = \chi_p(k)G_\chi$ for every $k \in \mathbb{F}_p$. We write $\varepsilon_G = G_\chi/\sqrt{p}$; thus $|\varepsilon_G| = 1$ and $\varepsilon_G^2 = \chi_p(-1)$, so G_χ is real if $p \equiv 1 \pmod 4$ and purely imaginary if $p \equiv 3 \pmod 4$.*

Proof. (i) $g_0(\psi) = \sum_{t\neq 0}\psi(t) = 0$, because ψ is a nontrivial character of the finite group $\mathbb{F}_p^\times$: if $\psi(t_0) \neq 1$ then multiplying the sum by $\psi(t_0)$ permutes its terms. For $s \neq 0$ substitute $t = s^{-1}u$: $g_s(\psi) = \sum_u \psi(s^{-1}u)e(u/p) = \psi(s^{-1})g(\psi) = \bar{\psi}(s)g(\psi)$.

(ii) On the one hand, by (i) and $|\bar{\psi}(s)| = 1$ for $s \neq 0$, $\sum_{s\in\mathbb{F}_p}|g_s(\psi)|^2 = (p-1)|g(\psi)|^2$. On the other hand, expanding,

$$\sum_{s\in\mathbb{F}_p}|g_s(\psi)|^2 = \sum_{x,y\in\mathbb{F}_p}\psi(x)\overline{\psi(y)}\sum_{s\in\mathbb{F}_p}e(s(x-y)/p) = p\sum_{x\in\mathbb{F}_p}|\psi(x)|^2 = p(p-1).$$

Hence $|g(\psi)|^2 = p$.

(iii) χ_p is real, so $\overline{G_\chi} = \sum_t \chi_p(t)e(-t/p) = g_{-1}(\chi_p) = \chi_p(-1)G_\chi$ by (i); with (ii), $p = G_\chi\overline{G_\chi} = \chi_p(-1)G_\chi^2$. The number of $u \in \mathbb{F}_p$ with $u^2 = t$ is $1+\chi_p(t)$ (this includes $t = 0$, since $\chi_p(0) = 0$), so for $k \neq 0$ $G_p(k) = \sum_t(1+\chi_p(t))e(kt/p) = \sum_t e(kt/p) + g_k(\chi_p) = 0 + \chi_p(k)G_\chi$. The last formula is (i) for $k \neq 0$ and $g_0(\chi_p) = 0$ for $k = 0$. Finally $|\varepsilon_G| = 1$ by (ii) and $\varepsilon_G^2 = G_\chi^2/p = \chi_p(-1)$, and $\chi_p(-1) = 1$ exactly when $p \equiv 1 \pmod 4$. □

Remark 4.9 (Gauss's sign). Gauss evaluated the sign: $G_\chi = \sqrt{p}$ if $p \equiv 1 \pmod 4$ and $G_\chi = i\sqrt{p}$ if $p \equiv 3 \pmod 4$ [31, Ch. 6, §4]. It is not used in this paper: wherever ε_G occurs, only $|\varepsilon_G| = 1$ and $\varepsilon_G^2 = \chi_p(-1)$ are needed, which is Theorem 4.8(iii).

Theorem 4.10 (Jacobi sums; see [31, Ch. 8, §3], [5, Ch. 2])**.** *Let p be an odd prime and ψ_1, ψ_2 multiplicative characters such that ψ_1, ψ_2 and $\psi_1\psi_2$ are nontrivial. Then*

$$\mathrm{Jac}(\psi_1,\psi_2) = \frac{g(\psi_1)\,g(\psi_2)}{g(\psi_1\psi_2)}, \qquad \textit{hence} \qquad |\,\mathrm{Jac}(\psi_1,\psi_2)| = \sqrt{p}.$$

Proof. Expanding the product and grouping by $s = x + y$,

$$g(\psi_1)g(\psi_2) = \sum_{s\in\mathbb{F}_p} e(s/p) \sum_{x+y=s} \psi_1(x)\psi_2(y).$$

For $s = 0$ the inner sum is $\sum_x \psi_1(x)\psi_2(-x) = \psi_2(-1)\sum_{x\neq 0}(\psi_1\psi_2)(x) = 0$, since $\psi_1\psi_2$ is nontrivial (Theorem 4.8(i), first part of its proof). For $s \neq 0$ substitute $x = st$, $y = s(1-t)$; this is a bijection of the solutions of $x+y = s$ with $t \in \mathbb{F}_p$, and the inner sum becomes $(\psi_1\psi_2)(s)\sum_t \psi_1(t)\psi_2(1-t) = (\psi_1\psi_2)(s)\,\mathrm{Jac}(\psi_1,\psi_2)$. Hence

$$g(\psi_1)g(\psi_2) = \mathrm{Jac}(\psi_1,\psi_2)\sum_{s\neq 0}(\psi_1\psi_2)(s)e(s/p) = \mathrm{Jac}(\psi_1,\psi_2)\,g(\psi_1\psi_2).$$

By Theorem 4.8(ii) all three Gauss sums have modulus $\sqrt{p} \neq 0$, which gives both claims. □

4.4. **The square shift and the Paley matrix.** The square shift adds a uniformly chosen nonzero square. At the primes $p \equiv 3 \pmod 4$ dividing the small modulus M_0 it is the one-coordinate kernel of the external index (Definition 9.6), so its centred norm is the inclusion probability of that coordinate; its tensor products over a set of primes give the strict-square pair law, which the cancellation along an accepted query expands in characters; and the Paley matrix is the signed part of both.

Lemma 4.11 (The square shift and the Paley matrix)**.** *Let p be an odd prime, $q = 1/p$ and $r = p^{-1/2}$. Let T_p be the* square shift *on $L^2(\mathbb{F}_p)$,*

$$(T_pf)(x) = \mathbb{E}_{u\in Q_p^*} f(x+u),$$

let C be the $p \times p$ Paley matrix *$C(x,y) = \chi(y-x)$, and let $P = T_C$ be the* Paley operator*, $(Pf)(x) = \mathbb{E}_{y\in\mathbb{F}_p}\chi(y-x)f(y)$.*

(a) *T_p is doubly stochastic: its matrix $\frac{2}{p-1}A_p(y-x)$ has $(p-1)/2$ nonzero entries in each row and in each column. It is diagonal in the additive characters: $T_pe_0 = e_0$, and $T_pe_k = \lambda_k e_k$ for $k \neq 0$ with*

$$\lambda_k = \frac{G_p(k)-1}{p-1} = \frac{\chi(k)G_\chi - 1}{p-1}.$$

Hence $\|T_p|_{L_0^2}\| \le (\sqrt{p}+1)/(p-1) = 1/(\sqrt{p}-1)$, with equality if $p \equiv 1 \pmod 4$. Likewise the matrices $(A_p(\pm(y-x)) - \frac{1}{2})_{x,y}$ have the eigenvalue $-\frac{1}{2}$ at e_0 and $(G_p(\pm k)-1)/2$ at e_k, $k \neq 0$, and norm at most $(\sqrt{p}+1)/2$.

(b) *If $p \equiv 3 \pmod 4$, then $|\lambda_k| = \sqrt{p+1}/(p-1)$ for every $k \neq 0$, so T_p restricted to $L_0^2(\mathbb{F}_p)$ is $\sqrt{p+1}/(p-1)$ times a unitary operator, and*

$$\|T_p|_{L_0^2}\| = \frac{\sqrt{p+1}}{p-1}.$$

The norm in this display is best possible among such transitions: for every odd p, every doubly stochastic matrix T on $\mathbb{F}_p$ with $T(x,y) = 0$ unless $y - x \in Q_p^*$ satisfies $\|T|_{L_0^2}\| \ge \sqrt{p+1}/(p-1)$. The number $\sqrt{p+1}/(p-1)$ is 1 at $p = 3$, less than 1 for $p \ge 5$, and less than $1/(\sqrt{p}-1)$ for every p.

(c) *$C\mathbf{1} = C^*\mathbf{1} = 0$, $C^* = \chi(-1)C$ and $CC^* = C^*C = pI - J_{\mathrm{all}}$, so $\|C\| = \sqrt{p}$. Hence $P\mathbf{1} = P^*\mathbf{1} = 0$, $P^* = \chi(-1)P$, $P^*P = PP^* = q(I - \mathbb{J})$ and $\|P\| = r$. Moreover, as matrices and as operators on $L^2(\mathbb{F}_p)$,*

$$T_p = \frac{J_{\mathrm{all}} - I + C}{p-1}, \qquad T_p = \frac{p\mathbb{J} - I + pP}{p-1}.$$

In particular $C^* = -C$ and $P^* = -P$ if $p \equiv 3 \pmod 4$.

(d) *Let $\mathcal{S}$ be a finite set of odd primes, $X_{\mathcal{S}} = \prod_{p\in\mathcal{S}} \mathbb{F}_p$ with the uniform probability, and $\pi_{\mathcal{S}}$ the uniform probability on $\prod_{p\in\mathcal{S}} \Omega_p \subseteq X_{\mathcal{S}} \times X_{\mathcal{S}}$, the strict-square pair law. The characters of $X_{\mathcal{S}}$ are $z \mapsto e(z\xi) = \prod_p e(a_p z_p/p)$ with $\xi = \sum_p a_p/p \in \mathbb{Q}/\mathbb{Z}$, and $\widehat{g}(\xi) = \mathbb{E}_z g(z)e(-z\xi)$. For functions g_1, g_2 on $X_{\mathcal{S}}$,*

$$\int g_1(x)g_2(y)\,d\pi_{\mathcal{S}}(x,y) = \sum_{\xi} \widehat{g_1}(-\xi)\,\widehat{g_2}(\xi)\,g(\xi),$$

the sum running over the characters of $X_{\mathcal{S}}$, where

$$g(\xi) = \prod_{p\in\mathcal{S},\ a_p\neq 0} \frac{G_p(a_p)-1}{p-1} = \frac{1}{\varphi(q(\xi))} \sum_{m\in(\mathbb{Z}/q(\xi)\mathbb{Z})^\times} e\Big(\xi m^2\Big)$$

is the unit quadratic average of the actual-square kernel (Definition 7.1). Moreover

$$|g(\xi)| \le \prod_{p\mid q(\xi)} \frac{1}{\sqrt{p}-1}, \qquad \text{and} \qquad |g(\xi)| \le q(\xi)^{-1/3} \text{ if every prime factor of } q(\xi) \text{ is at least } 64.$$

Proof. (a) Since $|Q_p^*| = (p-1)/2$, the matrix of T_p is $T_p(x,y) = \frac{2}{p-1}A_p(y-x)$. For fixed x there are $(p-1)/2$ values of y with $y-x \in Q_p^*$, and for fixed y there are as many values of x; so T_p is doubly stochastic. It is a convolution, and $T_p e_k(x) = e_k(x)\,\mathbb{E}_{u\in Q_p^*} e(ku/p)$. Every nonzero square has exactly two square roots, so

$$\mathbb{E}_{u\in Q_p^*} e(ku/p) = \frac{1}{p-1}\sum_{m\in\mathbb{F}_p^\times} e\Big(km^2/p\Big) = \frac{G_p(k)-1}{p-1}.$$

This is 1 at $k=0$, and $(\chi(k)G_\chi - 1)/(p-1)$ at $k\neq 0$ by Theorem 4.8(iii). The e_k with $k\neq 0$ form an orthonormal basis of $L_0^2(\mathbb{F}_p)$, so $\|T_p|_{L_0^2}\| = \max_{k\neq 0}|\lambda_k| \le (|G_\chi|+1)/(p-1) = (\sqrt{p}+1)/(p-1)$. If $p\equiv 1 \pmod 4$, then $G_\chi = \pm\sqrt{p}$ is real, χ takes both signs on $\mathbb{F}_p^\times$, and some $k\neq 0$ has $\chi(k)G_\chi = -\sqrt{p}$, so that $|\lambda_k| = (\sqrt{p}+1)/(p-1)$. For the second statement, $\sum_y (A_p(y-x)-\frac12)e_k(y) = e_k(x)(\sum_{u\in Q_p^*} e(ku/p) - \frac{p}{2}\mathbf{1}_{k=0})$, which is $-\frac12 e_0$ at $k=0$ and $\frac12(G_p(k)-1)e_k$ at $k\neq 0$; for $A_p(x-y)$ replace k by $-k$. Both matrices are diagonal in the orthonormal basis $(p^{-1/2}e_k)_k$ of $\mathbb{C}^{\mathbb{F}_p}$, so their norms are the largest moduli of their eigenvalues, at most $(\sqrt{p}+1)/2$, since $|G_p(\pm k)| = \sqrt{p}$ for $k\neq 0$.

(b) If $p\equiv 3 \pmod 4$, then $G_\chi^2 = -p$ (Theorem 4.8(iii)), so $\chi(k)G_\chi = \pm i\sqrt{p}$ for $k\neq 0$ and $|\chi(k)G_\chi - 1|^2 = p+1$. Hence $|\lambda_k| = \sqrt{p+1}/(p-1)$ for every $k\neq 0$, and T_p is diagonal on $L_0^2(\mathbb{F}_p)$ in an orthonormal basis with eigenvalues of this common modulus. For the optimality, let T be doubly stochastic with $T(x,y) = 0$ unless $y-x\in Q_p^*$. Since $T\mathbf{1} = \mathbf{1}$ and $T^*\mathbf{1} = \mathbf{1}$, T maps L_0^2 into itself, and the constants carry the singular value 1. Hence the squared Hilbert–Schmidt norm satisfies $\|T\|_{\mathrm{HS}}^2 = 1 + \|T|_{L_0^2}\|_{\mathrm{HS}}^2 \le 1 + (p-1)\|T|_{L_0^2}\|^2$. Each row of T has at most $(p-1)/2$ nonzero entries, summing to 1, so by Cauchy–Schwarz its squared entries sum to at least $2/(p-1)$, and $\|T\|_{\mathrm{HS}}^2 \ge 2p/(p-1)$. Combining, $\|T|_{L_0^2}\|^2 \ge (2p/(p-1)-1)/(p-1) = (p+1)/(p-1)^2$. Finally $\sqrt{p+1} < \sqrt{p}+1$ for every p, and $\sqrt{p+1} < p-1$ exactly when $p>3$.

(c) The row sums $\sum_y \chi(y-x)$ and the column sums vanish, so $C\mathbf{1} = C^*\mathbf{1} = 0$. The adjoint has entries $C^*(x,y) = \overline{C(y,x)} = \chi(x-y) = \chi(-1)C(x,y)$. The (x,z) entry of CC^* is $\sum_y \chi(y-x)\chi(y-z)$, which is $p-1$ for $x=z$ and -1 for $x\neq z$ by (4.2); so $CC^* = pI - J_{\mathrm{all}}$, and $C^*C = \chi(-1)C^2 = CC^*$. On the constants CC^* vanishes and on their orthogonal complement it is pI, so $\|C\|^2 = p$. As a matrix acting on vectors, P is C/p, and the uniform probability multiplies every inner product by the same factor $1/p$, so the adjoint of P is C^*/p. Hence $P^* = \chi(-1)P$, $P^*P = C^*C/p^2 = (pI - J_{\mathrm{all}})/p^2 = q(I-\mathbb{J})$ (as $J_{\mathrm{all}}/p = \mathbb{J}$ on vectors), $PP^* = P^*P$ and $\|P\| = \|C\|/p = r$. Finally $2A_p(y-x) = 1 + \chi(y-x) - \mathbf{1}_{y=x}$ for all x,y: both sides are 2 if

$y-x \in Q_p^*$ and 0 otherwise. So $T_p = \frac{2}{p-1}(A_p(y-x))_{x,y} = (J_{\mathrm{all}} - I + C)/(p-1)$; applied to f this is $(p\mathbb{E}f - f + pPf)/(p-1)$.

(d) The map $(x,u) \mapsto (x, x+u)$ is a bijection of $\mathbb{F}_p \times Q_p^*$ onto Ω_p. So under $\pi_{\mathcal{S}}$ the pair (x,y) is $(x, x+u)$, with x uniform on $X_{\mathcal{S}}$ and $u = (u_p)_p$ independent of x, its coordinates independent and uniform on the sets Q_p^*. Hence $\int g_1(x)g_2(y)\,d\pi_{\mathcal{S}} = \mathbb{E}_x g_1(x)(T_{\mathcal{S}}g_2)(x)$ with $T_{\mathcal{S}} = \bigotimes_{p\in\mathcal{S}} T_p$. Expand $g_2(y) = \sum_\xi \widehat{g_2}(\xi)e(y\xi)$. The character $e(\cdot\,\xi)$ is the tensor product of the characters e_{a_p} of the factors, so by (a) $T_{\mathcal{S}}e(\cdot\,\xi) = g(\xi)e(\cdot\,\xi)$ with $g(\xi) = \prod_{a_p\neq 0}(G_p(a_p)-1)/(p-1)$; and $\mathbb{E}_x g_1(x)e(x\xi) = \widehat{g_1}(-\xi)$. This is the identity. By the computation in (a), $(G_p(a)-1)/(p-1) = \frac{1}{p-1}\sum_{m\in\mathbb{F}_p^\times} e\big(am^2/p\big)$. By the Chinese remainder theorem, $m \bmod q(\xi)$ runs over the units exactly when $(m \bmod p)_{p\mid q(\xi)}$ runs over $\prod_p \mathbb{F}_p^\times$, and $e(\xi m^2) = \prod_p e(a_p m^2/p)$; this gives the second expression for $g(\xi)$. By (a) each factor has modulus at most $1/(\sqrt{p}-1)$, and for $p \geq 64$, $\sqrt{p}-1 = p^{1/3}(p^{1/6}-p^{-1/3}) \geq p^{1/3}(2-\frac{1}{4}) \geq p^{1/3}$. Multiply over $p \mid q(\xi)$. □

The optimality in (b) concerns this specified class of transitions, the doubly stochastic kernels supported on square shifts; it places no restriction on a different construction of the indices. The index law uses (b), the Paley channels of the local laws use (c) (Lemma 12.10), the pinned pair bounds (bounds with prescribed labels fixed) use the second statement of (a) (Lemma 13.12), and the cancellation uses (d).

4.5. **Primes in the classes modulo four.** For primes in a residue class modulo four we use the following explicit Chebyshev estimate. Write $\theta(x;4,r) = \sum_{p\leq x,\ p\equiv r\ (4)} \log p$.

Theorem 4.12 (Bennett–Martin–O'Bryant–Rechnitzer [4, Theorem 1.2 and Table 1])**.** *For $r \in \{1,3\}$ and every real $x \geq 4\,800\,162\,889$,*

$$\left|\theta(x;4,r) - \frac{x}{2}\right| < c_\theta(4)\,\frac{x}{\log x}, \qquad c_\theta(4) = 0.0004822 = \frac{2411}{5000000}.$$

Consequently, if $X \geq 4\,800\,162\,889$, then $|\theta(t;4,r) - t/2| \leq \mathtt{e}_X t$ for every $t \geq X$, where $\mathtt{e}_X = 2411/(5000000\log X)$.

The consequence holds because $t/\log t \leq t/\log X$ for $t \geq X$. The constants are those of the published version (Section A.4). In the construction this theorem supplies the prime counts for the cells of the middle range above $4\,800\,162\,890$, from which their prices and the analytic lower start sums of the mean credit are computed (Equation (20.1) and Lemma 20.6); the same cell charge prices the cells of the upper bands (Proposition 28.14). Below $4\,800\,162\,890$, one more than the threshold of the theorem, no Chebyshev estimate is used (Remark 20.8).

4.6. **Two correlation inequalities (Peyre and Kassabov).** Maximal correlation and the Friedrichs angle are defined in Definitions 4.3 and 4.4. When the intersection of the two function spaces consists of the constants, the maximal correlation is the cosine of the Friedrichs angle between them; this is why an inequality for angles (Kassabov's) and one for correlations (Peyre's) combine in the pinned influence bounds of the local laws.

Theorem 4.13 (Peyre's tensorization [54, Theorem 3.3.1])**.** *Let A, B be disjoint nonempty finite sets, and let $(X_v)_{v\in A\cup B}$ be random variables on a common probability space, each taking finitely many values. Let $\varepsilon_{ij} \geq 0$ for $i \in A$, $j \in B$, and $\mathcal{E} = (\varepsilon_{ij})$. Suppose that for all $i \in A$, $j \in B$, every $S \subseteq (A\cup B)\setminus\{i,j\}$ and every value s of $X_S = (X_v)_{v\in S}$ with $\mathbb{P}(X_S = s) > 0$,*

$$\rho(X_i; X_j \mid X_S = s) \leq \varepsilon_{ij}. \tag{4.3}$$

Then

$$\rho\big((X_a)_{a\in A}; (X_b)_{b\in B}\big) \leq \min\{1, \|\mathcal{E}\|_{\mathrm{op}}\}. \tag{4.4}$$

More precisely, (4.4) *holds as soon as there are enumerations* $A = \{a_1, \dots, a_N\}$ *and* $B = \{b_1, \dots, b_M\}$ *such that* (4.3) *holds for* $i = a_k$, $j = b_l$ *and the single set* $S = \{a_1, \dots, a_{k-1}\} \cup \{b_1, \dots, b_{l-1}\}$, *for all* $1 \le k \le N$ *and* $1 \le l \le M$.

Here $\|\mathcal{E}\|_{\mathrm{op}}$ is the norm of $\mathcal{E}$ as an operator $\ell^2(B) \to \ell^2(A)$ (Euclidean norms, Definition 4.1), its largest singular value. The second statement has a weaker hypothesis, because the sets S it uses are among those allowed in the first: they avoid a_k and b_l. Appendix B gives the complete proof, which is Peyre's argument in finite form; the proof establishes the second statement directly. The paper applies only the first.

Corollary 4.14 (Independent pairs)**.** *Let* (X_k, Y_k), $1 \le k \le n$, *be pairs of finite-valued random variables, the* n *pairs independent of each other. Then*

$$\rho((X_k)_k; (Y_k)_k) = \max_k \rho(X_k; Y_k).$$

The same holds conditionally: if the triples (X_k, Y_k, Z_k) *are independent across* k *and* $z = (z_k)_k$ *is a value of* $(Z_k)_k$ *of positive probability, then* $\rho((X_k)_k; (Y_k)_k \mid (Z_k)_k = z) = \max_k \rho(X_k; Y_k \mid Z_k = z_k)$.

Proof. Apply Theorem 4.13 with $X_{a_k} = X_k$ and $X_{b_k} = Y_k$. Let S avoid a_k and b_l, and let s be a value of X_S of positive probability. The event $\{X_S = s\}$ is an intersection of events, one in each pair, so under $\mathbb{P}(\cdot \mid X_S = s)$ the pairs are still independent. If $k \ne l$, then X_k and Y_l lie in different pairs, are conditionally independent, and $\rho(X_k; Y_l \mid X_S = s) = 0$. If $k = l$, the event depends only on the other pairs, the conditional law of (X_k, Y_k) is its law, and $\rho(X_k; Y_k \mid X_S = s) = \rho(X_k; Y_k)$. So the hypothesis holds with the diagonal matrix $\mathcal{E} = \mathrm{diag}(\rho(X_k; Y_k))_k$, whose norm is $\max_k \rho(X_k; Y_k)$. The reverse inequality holds because functions of X_k alone and of Y_k alone are admissible in the definition of $\rho((X_k)_k; (Y_k)_k)$. For the conditional statement apply this under $\mathbb{P}(\cdot \mid (Z_k)_k = z)$: by the same product computation the pairs remain independent, with the conditional laws of (X_k, Y_k) given $Z_k = z_k$. □

In operator form: the operator of a product of couplings is the tensor product of the operators of the factors, and its centred norm is the largest centred norm of a factor (Lemma 6.10(c)).

Corollary 4.15 (Conditional decorrelation)**.** *Let* X, Y, Z *be finite-valued random variables on a common probability space and* $\varepsilon \ge 0$, *and assume* $\rho(X; Y \mid Z = z) \le \varepsilon$ *for every value* z *with* $\mathbb{P}(Z = z) > 0$. *If a real function* $f \in L^2(X, Z)$ *satisfies* $\mathbb{E}[f \mid Z] = 0$ *and* $g \in L^2(Y, Z)$ *is real, then*

$$|\mathbb{E}[fg]| \le \varepsilon \, \|f\| \, \|g - \mathbb{E}[g \mid Z]\| \le \varepsilon \, \|f\| \, \|g\| .$$

Proof. This is Lemma B.1(b), proved in Appendix B on a finite space of full support (Section B.1); every quantity here depends only on the joint law of (X, Y, Z), so this is no restriction. The proof is fibrewise: on each set $\{Z = z\}$ of positive probability the hypothesis bounds $|\mathbb{E}[fg \mid Z = z]|$ by ε times the conditional norms of f and of $g - \mathbb{E}[g \mid Z]$, and Cauchy–Schwarz for the sum over z gives the claim. □

The two corollaries are used together when a law on a whole block is built from feature functions (the block step of Lemma 25.1): given a function of all the labels of a block, the conditional correlation tensorizes over the primes of the block (Corollary 4.14), and a function with conditional mean zero is then decorrelated from every function of the other side (Corollary 4.15).

Theorem 4.16 (Kassabov's angle inequality [34, Lemma 4.2(b)])**.** *Let* V_1, V_2, V_3 *be subspaces of a finite-dimensional Hilbert space and put* $\varepsilon_{ij} = \cos \angle(V_i, V_j)$. *If* $\varepsilon_{13} < 1$ *and* $\varepsilon_{23} < 1$, *then*

$$\cos \angle(V_1 \cap V_3, V_2 \cap V_3) \le \frac{\varepsilon_{12} + \varepsilon_{13}\varepsilon_{23}}{\sqrt{(1 - \varepsilon_{13}^2)(1 - \varepsilon_{23}^2)}}.$$

It enters through the elimination of pinned vertices in Lemma 13.15, and through it in the pinned influence theorem Proposition 13.16.

4.7. **Monotone majorants: a bound at the smallest prime implies the bound at every larger prime.** Every local constant of Part V is an explicit expression in a prime p. It is computed once, at the start X of a range of primes (a piece of the middle range, an upper band, the tail threshold), and then used at every prime $p \geq X$ and on every block of primes of the range. This is valid because each such constant, written as a function of the real variables $r = p^{-1/2}$, $q = 1/p$, $\lambda = 1/\log p$ and a block count (the number of primes in the block), is nondecreasing in each of them. To write a constant in these variables replace p by r^{-2}: for instance $p/(p-1) = 1/(1-r^2)$, $1/(p-1) = r^2/(1-r^2)$, $k/(p-k) = kr^2/(1-kr^2)$ and $k/p = kr^2$ for a fixed number k. "Nondecreasing" always refers to these variables. The definition and the lemma below record the calculus once; every section that evaluates a constant at an endpoint cites the lemma and displays the constant in a form built from its operations.

Definition 4.17 (Admissible and small functions)**.** A *box* is a product $\Delta = I_1 \times \cdots \times I_k$ of intervals $I_j \subseteq [0,\infty)$, each containing its supremum $x_j^+ < \infty$; the point $x^+ = (x_1^+, \ldots, x_k^+)$ is its *endpoint*, and Δ is ordered coordinatewise. A function $\varphi\colon \Delta \to [0,\infty)$ is *admissible* if $\varphi(x) \leq \varphi(x')$ whenever $x \leq x'$. If r is a coordinate whose interval lies in $(0,\infty)$, then φ is *small* if φ/r is admissible. A map $F\colon \mathcal{D} \to [0,\infty)$ on a set $\mathcal{D} \subseteq [0,\infty)^m$ is *monotone* if $F(y) \leq F(y')$ whenever $y \leq y'$ in $\mathcal{D}$.

Lemma 4.18 (Admissible majorants)**.** *Let Δ be a box with endpoint x^+.*

(i) *Nonnegative constants, the coordinates, and every power series in the coordinates with nonnegative coefficients that converges on Δ are admissible; such a series in which every monomial contains r is small. In particular every polynomial in r with nonnegative coefficients, such as a count error of Lemma 12.2, and $p/(p-1) = 1/(1-r^2)$ are admissible.*

(ii) *If $F\colon \mathcal{D} \to [0,\infty)$ is monotone and $\varphi_1, \ldots, \varphi_m$ are admissible with $(\varphi_1(x), \ldots, \varphi_m(x)) \in \mathcal{D}$ for every $x \in \Delta$, then $F(\varphi_1, \ldots, \varphi_m)$ is admissible. Hence sums, products, maxima, minima and square roots of admissible functions, and powers φ^ψ with $\varphi \geq 1$, are admissible; so are e^φ, $e^\varphi - 1$, $(e^\varphi - 1)/\varphi$ (with value 1 where $\varphi = 0$) and $\log(1+\varphi)$, and, where $\varphi < 1$ on Δ, also $1/(1-\varphi)$, $(1+\varphi)/(1-\varphi)$ and $-\log(1-\varphi)$; and so are the operator norm of a matrix whose entries are admissible and the largest eigenvalue of a real symmetric matrix whose entries are admissible. Finally, $(G, D, r) \mapsto G + D^2 r/(1+Gr)$ is monotone on $\{G, D, r \geq 0,\ Dr \leq 1\}$, so $\varphi_G + \varphi_D^2 \varphi_r/(1+\varphi_G\varphi_r)$ is admissible for admissible $\varphi_G, \varphi_D, \varphi_r$ with $\varphi_D\varphi_r \leq 1$ on Δ.*

(iii) *Every small function is admissible. If φ, ψ are small and h is admissible, then $\varphi + \psi$, $\max(\varphi,\psi)$, $\min(\varphi,\psi)$, $h\varphi$ and $\sqrt{\varphi\psi}$ are small. If $F(y) = yG(y)$ with $G \geq 0$ monotone, then $F \circ \varphi$ is small for small φ (with values in the domain of G); in particular $e^\varphi - 1$, and, where $\varphi < 1$, $-\log(1-\varphi)$, $\varphi/(1-\varphi)$ and $(1+\varphi)/(1-\varphi) - 1$ are small. If $\varphi_1, \ldots, \varphi_n$ are small, then $\prod_i (1+\varphi_i) - 1$ is small. The function $\log(1+\varphi)$ need not be small: $\log(1+y)/y$ decreases.*

(iv) *One fixed candidate, chosen independently of x (for instance by its value at the endpoint), among finitely many admissible candidates is admissible. If φ is admissible and $\psi \leq \varphi$ on Δ, then $\max(\varphi,\psi) = \varphi$ is admissible, whatever ψ is.*

Consequently the following hold.

(a) Endpoint. *If φ is admissible, then $\varphi(x) \leq \varphi(x^+)$ for every $x \in \Delta$. In particular, let $P > 1$ and $L \geq 0$ be real, and let $\Delta = (0, P^{-1/2}] \times (0, 1/P] \times (0, 1/\log P] \times [0, L/\log P]$ in the variables r, q, λ and the block count (or a product of some of these factors). Then every inequality of the form $\varphi(x^+) < c$ (or $\leq c$) checked at the endpoint holds at every prime $p \geq P$ and every block count at most $L/\log P$, and every admissible coefficient at such a prime is at most its value at the endpoint.*

(b) Piece start. *Let $X > 1$, $q_X = 1/X$, $r_X \geq X^{-1/2}$ and $n_X \geq 0$ (the certificates use $r_X = 1/\lfloor\sqrt{X}\rfloor$), and let $\varphi(q, r, l)$ be admissible on the box $(0, q_X] \times (0, r_X] \times [0, n_X]$. If a*

quantity at the prime p with block count $l \le n_X$ is $r^d\varphi(q,r,l)$, where $d \ge 0$, $r = p^{-1/2}$ and $q = 1/p$, then for every prime $p \ge X$ it is at most $p^{-d/2}\varphi(q_X, r_X, n_X)$.

(c) Block inflation. *Let $c \ge 0$ and $L > 0$. The function $X \mapsto \exp(cL/(X\log X))$ is nonincreasing on $(1,\infty)$ (decreasing if $c > 0$). If $\mathcal{B}$ is a set of primes, all at least $X > 1$, with $q_{\mathcal{B}} \le e^L$, then $|\mathcal{B}| \le L/\log X$, $\sum_{p\in\mathcal{B}} 1/p \le L/(X\log X)$ and $\prod_{p\in\mathcal{B}}(1 + c/p) \le \exp(cL/(X\log X))$.*

(d) Matrix tests. *Let M_0 be a fixed Hermitian matrix and $E_1, \dots, E_m \succeq 0$ fixed positive semidefinite matrices of the same size, and let $c_1, \dots, c_m$ be real functions on Δ with $c_i(x) \ge c_i(x^+)$ for every x (for instance $c_i = -\varphi_i$, or $c_i = g/\varphi_i$ with a constant $g \ge 0$ independent of x, where $\varphi_i > 0$ is admissible). If $M_0 + \sum_i c_i(x^+)E_i \succeq 0$, then $M_0 + \sum_i c_i(x)E_i \succeq 0$ for every $x \in \Delta$.*

(e) Several thresholds. *Let $\mathcal{A}(p,g)$ be a statement about a prime p and a number g that remains true when g is increased (for instance $C_p \succeq -gp^{-1/2}I$, $C_p \preceq (1 + g/p)I$ or $c(p) \le g$). If, for each s in a finite set, $\mathcal{A}(p, g_s)$ holds for every prime $p \ge y_s$, then $\mathcal{A}(p, \min\{g_s : y_s \le y\})$ holds for every prime $p \ge y$ and every real $y \ge \min_s y_s$.*

Proof. (i) For $0 \le x \le x'$ every monomial satisfies $x^\alpha \le x'^\alpha$, so a series with nonnegative coefficients is admissible. If every monomial contains r, the series is r times a series with nonnegative coefficients, which converges on Δ because $r > 0$ there; so it is small. The count errors are polynomials in r with nonnegative coefficients, and $1/(1 - r^2) = \sum_{k\ge0} r^{2k}$ for $r < 1$.

(ii) If $x \le x'$, then $\varphi_i(x) \le \varphi_i(x')$ for every i, and the monotonicity of F gives the claim. The listed maps are monotone on their domains: $(y,z) \mapsto y + z$, yz, $\max(y,z)$, $\min(y,z)$ on $[0,\infty)^2$; $y \mapsto \sqrt{y}$; $(y,z) \mapsto y^z$ on $[1,\infty) \times [0,\infty)$; $y \mapsto e^y$, $e^y - 1$, $(e^y - 1)/y = \sum_{k\ge0} y^k/(k+1)!$ and $\log(1 + y)$ on $[0,\infty)$; and $y \mapsto 1/(1 - y) = \sum_k y^k$, $(1 + y)/(1 - y) = 1 + 2\sum_{k\ge1} y^k$ and $-\log(1-y) = \sum_{k\ge1} y^k/k$ on $[0,1)$. For real matrices with $0 \le M \le M'$ entrywise and unit vectors u, v, $|v^*Mu| \le |v|^{\mathsf{T}}M|u| \le |v|^{\mathsf{T}}M'|u| \le \|M'\|$, where $|u|$ is the vector of moduli; so $\|M\| \le \|M'\|$. For real symmetric $M \ge 0$, $u^{\mathsf{T}}Mu \le |u|^{\mathsf{T}}M|u|$, so the largest eigenvalue is the maximum of $u^{\mathsf{T}}Mu$ over nonnegative unit vectors u, which is monotone in the entries in the same way. Finally, on $\mathcal{D} = \{G, D, r \ge 0,\ Dr \le 1\}$ the map $F(G,D,r) = G + D^2r/(1 + Gr)$ has the partial derivatives $1 - D^2r^2/(1+Gr)^2 \ge 0$ (as $Dr \le 1 \le 1 + Gr$), $2Dr/(1+Gr) \ge 0$ and $D^2/(1+Gr)^2 \ge 0$. If $y \le y'$ in $\mathcal{D}$, the path from y to y' that raises first G, then D, then r stays in $\mathcal{D}$, because the product Dr only increases along it, up to its value at y'; so $F(y) \le F(y')$.

(iii) A small $\varphi = r\varphi'$ is a product of admissible functions. Write also $\psi = r\psi'$. Then $\varphi + \psi = r(\varphi' + \psi')$, $\max(\varphi,\psi) = r\max(\varphi',\psi')$, $\min(\varphi,\psi) = r\min(\varphi',\psi')$, $h\varphi = rh\varphi'$ and $\sqrt{\varphi\psi} = r\sqrt{\varphi'\psi'}$, and (ii) applies to the cofactors. If $F(y) = yG(y)$, then $F(\varphi)/r = G(\varphi)\varphi'$, admissible by (ii); this covers $e^y - 1$ with $G(y) = (e^y - 1)/y$, $-\log(1-y)$ with $G(y) = \sum_{k\ge1} y^{k-1}/k$, $y/(1-y)$ with $G(y) = 1/(1-y)$, and $(1+y)/(1-y) - 1 = 2y/(1-y)$. Finally $\prod_i(1 + \varphi_i) - 1 = \sum_{\emptyset\ne I}\prod_{i\in I}\varphi_i$, and each term is a small function times admissible ones.

(iv) A choice made once is a fixed admissible function, and $\max(\varphi,\psi) = \varphi$ when $\psi \le \varphi$.

(a) Every $x \in \Delta$ satisfies $x \le x^+$. For a prime $p \ge P > 1$, $p^{-1/2} \le P^{-1/2}$, $1/p \le 1/P$ and $1/\log p \le 1/\log P$.

(b) For a prime $p \ge X$, $1/p \le q_X$, $p^{-1/2} \le X^{-1/2} \le r_X$ and $l \le n_X$, so $\varphi(1/p, p^{-1/2}, l) \le \varphi(q_X, r_X, n_X)$ by (a); multiply by $p^{-d/2}$.

(c) The function $X\log X$ is positive on $(1,\infty)$, with derivative $\log X + 1 > 0$. Every prime of $\mathcal{B}$ is at least X, so $|\mathcal{B}|\log X \le \log q_{\mathcal{B}} \le L$; then $\sum_{p\in\mathcal{B}} 1/p \le |\mathcal{B}|/X \le L/(X\log X)$, and $\prod_{p\in\mathcal{B}}(1 + c/p) \le \exp(c\sum_{p\in\mathcal{B}} 1/p)$.

(d) The difference of the two matrices is $\sum_i (c_i(x) - c_i(x^+))E_i \succeq 0$.

(e) Every s with $y_s \le y$ gives $\mathcal{A}(p, g_s)$ at every prime $p \ge y \ge y_s$, hence $\mathcal{A}(p,g)$ for the smallest such g_s. □

Remark 4.19 (How the lemma is applied). (1) *Cancel the powers of r first.* A remainder of the form $q \cdot c(q,r)$ is used as $r \cdot (rc(q,r))$, with the admissible coefficient $rc(q,r)$, before the piece start is used. At the piece start the coefficient is evaluated at $q_X = 1/X$ and $r_X = 1/\lfloor\sqrt{X}\rfloor \geq X^{-1/2}$, and the bound used for a block with least prime $b \geq X$ is $b^{-1/2}$ times this coefficient (part (b) with $d = 1$). A remainder computed at q_X and then divided by the larger r_X would not in general bound the coefficient, since $q_X/r_X < X^{-1/2}$ when $r_X > X^{-1/2}$; this is never done.

(2) *Logarithmic counts.* Inserting the vertices of a labelling in a fixed order, with the count errors of Lemma 12.2 at each step, gives logarithmic count bounds. Where such a bound must be small in the sense of Definition 4.17 it is used in the form $\sum e$, an upper bound for $\sum \log(1+e)$, or $\sum -\log(1-e)$; both are small when the count errors e are, whereas $\sum \log(1+e)$ is admissible but in general not small.

(3) *Choices and gates.* Orders, partitions, Schur-test parameters and the branch of a maximum are fixed at the endpoint before (a) is applied. A minimum of bounds is taken only among bounds whose own hypotheses, which are gates, hold at the endpoint, and hence everywhere by (a). No monotonicity of an eigenvector or of an optimized parameter is used. When one branch of a maximum is not monotone, (iv) applies once it is shown to be dominated by an admissible branch; this is how the two branches of the maximum in the spectral criterion are treated (Lemmas 14.29 and 27.7).

(4) *Classes.* A matrix test whose matrices depend on p only through $p \bmod 4$ is checked for both classes, and (d) is applied within each class. The threshold hypotheses of spectral data are gates built from the operations of (i) and (ii), and persist by (a). A spectral cap valid for every prime $p \geq P$, times the block inflation of (c) evaluated at the start $X \geq P$ of a piece or band, bounds the corresponding factor on every block of that piece or band (Corollary 14.22).

Consequences and what to read next. External inputs: Weil's theorem (Theorem 4.7(a)), the explicit Chebyshev bound of Bennett, Martin, O'Bryant and Rechnitzer (Theorem 4.12) and Kassabov's inequality (Theorem 4.16); Section A.4 gives their sources. Peyre's theorem (Theorem 4.13) and the conditional decorrelation inequality (Corollary 4.15) are proved in Appendix B. Everything else is proved here from standard facts: the cyclicity of $\mathbb{F}_p^\times$, the Chinese remainder theorem, Hensel's lemma, Fourier analysis on finite abelian groups and the Cauchy–Schwarz inequality.

5. Reflection positivity and a Hölder inequality for signed laws

What this section proves and how to read it. This section proves that a signed 72-fold average can still be bounded below by α^{72} and above by diagonal averages, provided the averaging law is positive across mirror symmetries. Precisely, it defines cuts and reflection positivity, proves that domination by a positive 2×2 block at one cut permits Cauchy–Schwarz for signed laws, iterates this over seven cuts to obtain the signed Hölder inequality, and constructs the 72 slots with their seven cuts and six queries. Signed laws appear because the comparison law ν of Part III cannot be reflection positive; the block domination is what replaces positivity for ν. Inputs: only finite linear algebra. Outputs: Theorem 5.9 (hence $\alpha^{72} \leq Y$) and the block-domination calculus used in Part III. A first reading should follow the definitions, Theorem 5.9, and the slot construction in Section 5.4; the detailed one-cut lemmas can be treated as reference. Until Section 5.4, V is an arbitrary finite slot set; from there on $V = (\mathbb{Z}/6\mathbb{Z})^2 \times \{0,1\}$.

Notation. Throughout, V is a finite set of *slots*, $d = |V|$, and $\mathcal{X}$ is a finite set of labels. For a finite signed measure ν on $\mathcal{X}^V$ and real functions g_v on $\mathcal{X}$ we write

$$\Lambda_\nu((g_v)_{v\in V}) = \int \prod_{v\in V} g_v(z_v)\, d\nu(z), \qquad \Lambda_\nu(g) = \Lambda_\nu((g)_{v\in V}).$$

A map $h\colon V \to \mathbb{R}^{\mathcal{X}}$ is an *assignment*, and $\Lambda_\nu(h) = \Lambda_\nu((h(v))_{v\in V})$. For a map Ψ between finite sets, $\Psi_*\kappa$ is the image of a measure κ; for a permutation σ of V we write $\sigma_*\kappa$ for the image of κ under the map $z \mapsto z\circ\sigma = (z_{\sigma v})_{v\in V}$ of $\mathcal{X}^V$.

5.1. **Cuts and cut matrices.** Laws obtained by restricting a law to an event, by repair, or as comparison laws are in general not reflection positive. Two facts replace positivity at a single cut: measures that the cut fixes samplewise are ordered in the Loewner order at that cut (Lemma 5.4), and a measure bracketed at one cut by a reflection positive law inherits that law's mixed Hölder bound (Lemma 5.6, Remark 5.11). This subsection fixes the language.

Definition 5.1 (Cuts and reflection positivity)**.** A *cut* of V is a fixed-point-free involution ϑ of V together with a decomposition $V = V_+ \sqcup V_-$ with $V_- = \vartheta V_+$; V_+ is its *plus half*. We name a cut by its involution when the plus half is understood. A finite signed measure ν on $\mathcal{X}^V$ is *reflection positive* (RP) at ϑ if it is invariant under $(z_v)_v \mapsto (z_{\vartheta v})_v$ and

$$\int F(z_{V_+})\, F\big((z_{\vartheta u})_{u\in V_+}\big)\, d\nu(z) \;\geq\; 0 \qquad \text{for every } F\colon \mathcal{X}^{V_+} \to \mathbb{R}.$$

Equivalently (Definition 5.2), its cut matrix $\big(\nu(z_{V_+} = x,\ z_{\vartheta V_+} = y)\big)_{x,y\in\mathcal{X}^{V_+}}$ is symmetric and positive semidefinite.

Positive multiples and sums of measures that are RP at ϑ are RP at ϑ, and so are tensor products of such measures (5.2). Since $|V| = 2|V_+|$, a slot set with a cut has an even number of slots.

Definition 5.2 (Cut matrix, cut order and cut form)**.** Fix a cut ϑ with halves $V_\pm$. For a finite signed measure ν on $\mathcal{X}^V$ its *cut matrix* at ϑ is

$$C_\vartheta(\nu)_{x,y} = \nu(z_{V_+} = x,\ z_{\vartheta V_+} = y) \qquad (x, y \in \mathcal{X}^{V_+}),$$

where $z_{V_+} = x$ means $z_u = x_u$ and $z_{\vartheta V_+} = y$ means $z_{\vartheta u} = y_u$ for all $u \in V_+$. For real symmetric matrices, $A \preceq B$ means that $B - A$ is positive semidefinite (the Loewner order of Definition 4.1). For finite signed measures κ, κ' on $\mathcal{X}^V$ we write $\kappa \succeq_\vartheta \kappa'$, or $\kappa' \preceq_\vartheta \kappa$, if $C_\vartheta(\kappa - \kappa')$ is symmetric and positive semidefinite; thus ν is RP at ϑ if and only if $\nu \succeq_\vartheta 0$. The *cut form* of κ at ϑ is the bilinear form

$$Q_\kappa(X, Y) = \langle X, C_\vartheta(\kappa)Y\rangle = \int X(z_{V_+})\, Y\big((z_{\vartheta u})_{u\in V_+}\big)\, d\kappa(z)$$

on real functions X, Y on $\mathcal{X}^{V_+}$ (*half-functions*), with the Euclidean inner product of $\mathbb{R}^{\mathcal{X}^{V_+}}$.

The map $z \mapsto (z_{V_+}, (z_{\vartheta u})_{u\in V_+})$ is a bijection of $\mathcal{X}^V$ onto $\mathcal{X}^{V_+} \times \mathcal{X}^{V_+}$, so a measure is determined by its cut matrix at any one cut. The cut form $Q_\kappa(X, Y)$ is bilinear in (X, Y) and linear in κ; if $\kappa \succeq_\vartheta 0$ it is a symmetric positive semidefinite form, and $\kappa \preceq_\vartheta \kappa'$ implies $Q_\kappa(X, X) \leq Q_{\kappa'}(X, X)$ for every X.

The bilinear identity. For real g_v on $\mathcal{X}$ put $F(x) = \prod_{u\in V_+} g_u(x_u)$ and $G(y) = \prod_{u\in V_+} g_{\vartheta u}(y_u)$. Since $V = V_+ \sqcup \vartheta V_+$,

$$\Lambda_\nu\big((g_v)_{v\in V}\big) = \sum_{x,y} F(x)\, C_\vartheta(\nu)_{x,y}\, G(y) = \langle F, C_\vartheta(\nu)G\rangle = Q_\nu(F, G). \tag{5.1}$$

When every g_v equals g, both F and G are the half-function $F_g(x) = \prod_{u\in V_+} g(x_u)$ carrying g at every slot of the plus half; so $\Lambda_\nu(g) = Q_\nu(F_g, F_g)$, at every cut of V.

Products. If ν_1, ν_2 are finite signed measures on $\mathcal{X}_1^V$, $\mathcal{X}_2^V$, we write $\nu_1 \otimes \nu_2$ for the image of the product measure under $(z^1, z^2) \mapsto ((z_v^1, z_v^2))_{v\in V}$, a measure on $(\mathcal{X}_1 \times \mathcal{X}_2)^V$. Under the identification $(\mathcal{X}_1 \times \mathcal{X}_2)^{V_+} = \mathcal{X}_1^{V_+} \times \mathcal{X}_2^{V_+}$,

$$C_\vartheta(\nu_1 \otimes \nu_2) = C_\vartheta(\nu_1) \otimes C_\vartheta(\nu_2) \qquad \text{(Kronecker product)}, \tag{5.2}$$

because the entry at $((x^1,x^2),(y^1,y^2))$ factors. A Kronecker product of symmetric positive semidefinite matrices is symmetric positive semidefinite (its eigenvalues are the products of those of the factors). Hence $\kappa_i \succeq_\vartheta 0$ for all i implies $\bigotimes_i \kappa_i \succeq_\vartheta 0$.

Remark 5.3 (Reflections, halves and permutations). Let κ be a finite signed measure on $\mathcal{X}^V$ and ϑ a cut.

(a) *Reflection.* $C_\vartheta(\vartheta_*\kappa) = C_\vartheta(\kappa)^{\mathsf{T}}$, since $\vartheta_*\kappa(z_{V_+} = x,\ z_{\vartheta V_+} = y) = \kappa(z_{\vartheta V_+} = x,\ z_{V_+} = y)$. As κ is determined by its cut matrix, κ is invariant under the reflection if and only if $C_\vartheta(\kappa)$ is symmetric. The *symmetric* and *skew parts* $\frac{1}{2}(\kappa \pm \vartheta_*\kappa)$ have the symmetric and the antisymmetric part of $C_\vartheta(\kappa)$ as cut matrices. The diagonal cut form of a measure with antisymmetric cut matrix vanishes, and so does that of its product with any measure with symmetric cut matrix (5.2).

(b) *Halves.* (ϑ, V_-) is again a cut. Identify $\mathcal{X}^{V_-}$ with $\mathcal{X}^{V_+}$ by $x \mapsto (x_{\vartheta u})_{u\in V_+}$; then the cut matrix at (ϑ, V_-) is the transpose of the cut matrix at (ϑ, V_+). Hence $\kappa \succeq \kappa'$ at one choice of the plus half if and only if at the other; in particular reflection positivity does not depend on the choice of the plus half.

(c) *Permutations.* For a permutation σ of V, the involution $\sigma^{-1}\vartheta\sigma$ with plus half $\sigma^{-1}V_+$ is a cut, and $s(x) = (x_{\sigma u})_{u\in\sigma^{-1}V_+}$ is a bijection $\mathcal{X}^{V_+} \to \mathcal{X}^{\sigma^{-1}V_+}$ with

$$C_{\sigma^{-1}\vartheta\sigma}(\sigma_*\kappa)_{s(x),s(y)} = C_\vartheta(\kappa)_{x,y}.$$

Indeed, for $w = z\circ\sigma$ we have $w_{\sigma^{-1}V_+} = s(z_{V_+})$ and $(w_{\sigma^{-1}\vartheta\sigma u})_{u\in\sigma^{-1}V_+} = s((z_{\vartheta v})_{v\in V_+})$. Hence $\kappa \succeq_\vartheta \kappa'$ if and only if $\sigma_*\kappa \succeq_{\sigma^{-1}\vartheta\sigma} \sigma_*\kappa'$.

Lemma 5.4 (Common-cut order)**.** *Let ϑ be a cut of V with halves $V_\pm$, and let* $\mathrm{Fix}(\vartheta) = \{z \in \mathcal{X}^V : z_{\vartheta v} = z_v \text{ for all } v \in V\}$.

(i) Diagonal measures. *Let κ be a nonnegative measure on $\mathcal{X}^V$ supported on* $\mathrm{Fix}(\vartheta)$*; we say that κ is* diagonal at ϑ*, or* fixed samplewise by ϑ*. Then $C_\vartheta(\kappa)$ is diagonal with entries $\kappa(z_{V_+} = x) \ge 0$; in particular $\kappa \succeq_\vartheta 0$, and κ is RP at ϑ. Every nonnegative measure whose support lies in that of κ is again diagonal at ϑ: a restriction of κ to an event, a nonnegative reweighting of κ, and, if $\kappa = Z_*\pi$ for a finite set Ω with weights $\pi \ge 0$ and a map $Z\colon \Omega \to \mathcal{X}^V$ (a* latent representation*), every $Z_*(w\pi)$ with $w \ge 0$ on Ω (a restriction to latent events if w is an indicator). For a finite nonnegative measure m on $\mathcal{X}^{V_+}$, the* diagonal copy $\mathrm{Diag}_\vartheta(m)$*, the image of m under the map $x \mapsto z$ with $z_{V_+} = x$ and $z_{\vartheta u} = x_u$ for $u \in V_+$ (for a probability measure m: the law of z with $z_{V_+} \sim m$ and $z_{\vartheta u} = z_u$), is diagonal at ϑ with cut matrix* $\mathrm{diag}(m)$.

(ii) Order. *If $\kappa, \kappa_1, \dots, \kappa_k$ are diagonal at ϑ and $\kappa \le \sum_i \kappa_i$, then $0 \preceq_\vartheta \kappa \preceq_\vartheta \sum_i \kappa_i$.*

(iii) Adjoining a factor. *If $\kappa \preceq_\vartheta \kappa'$ on $\mathcal{X}^V$ and ρ is a finite signed measure on $\mathcal{Y}^V$ with $\rho \succeq_\vartheta 0$, then $\kappa\otimes\rho \preceq_\vartheta \kappa'\otimes\rho$ on $(\mathcal{X}\times\mathcal{Y})^V$. In particular (ii) persists after adjoining ρ.*

(iv) Diagonal functionals. *Consequently, in the situation of (ii) and (iii), $0 \le \Lambda_{\kappa\otimes\rho}(g) \le \sum_i \Lambda_{\kappa_i\otimes\rho}(g)$ for every real function g on $\mathcal{X}\times\mathcal{Y}$.*

(v) One kernel on both halves. *Let ϑ' be a cut of a finite set W with halves $W_\pm$, $\mathcal{Y}$ a finite set, and P a Markov kernel from $\mathcal{X}^{V_+}$ to $\mathcal{Y}^{W_+}$ (a nonnegative matrix with unit row sums). For a finite signed measure κ on $\mathcal{X}^V$ let κ^P be the measure on $\mathcal{Y}^W$ obtained by applying P independently to z_{V_+} and to the reflected minus half $(z_{\vartheta u})_{u\in V_+}$, and placing the results on W_+ and, through ϑ', on W_-:*

$$\kappa^P(w) = \sum_z \kappa(z)\, P\big(z_{V_+}, w_{W_+}\big)\, P\big((z_{\vartheta u})_{u\in V_+}, (w_{\vartheta' u'})_{u'\in W_+}\big).$$

Then $C_{\vartheta'}(\kappa^P) = P^{\mathsf{T}} C_\vartheta(\kappa) P$; so $\kappa \succeq_\vartheta \kappa'$ implies $\kappa^P \succeq_{\vartheta'} \kappa'^P$. The measure κ^P is nonnegative if κ is, has the mass of κ, and has plus-half marginal mP if κ has plus-half marginal

m. For a finite nonnegative measure m on $\mathcal{X}^{V_+}$, $P^{\mathsf{T}}\operatorname{diag}(m)P \preceq \operatorname{diag}(mP)$; hence $\pi + k\operatorname{Diag}_\vartheta(m) \succeq_\vartheta 0$ with $k \geq 0$ implies $\pi^P + k\operatorname{Diag}_{\vartheta'}(mP) \succeq_{\vartheta'} 0$. If P is deterministic, $P(x,\cdot) = \delta_{\phi(x)}$, then $\kappa^P = \Phi_*\kappa$ for the half-local *map $\Phi\colon \mathcal{X}^V \to \mathcal{Y}^W$ with*

$$\Phi(z)_{W_+} = \phi(z_{V_+}), \qquad \big(\Phi(z)_{\vartheta' u'}\big)_{u' \in W_+} = \phi\big((z_{\vartheta u})_{u \in V_+}\big);$$

the map Φ intertwines the two reflections, it maps $\operatorname{Fix}(\vartheta)$ into $\operatorname{Fix}(\vartheta')$, and $\Phi_*\operatorname{Diag}_\vartheta(m) = \operatorname{Diag}_{\vartheta'}(\phi_* m)$.

Proof. (i) $C_\vartheta(\kappa)_{x,y} = \kappa(z_{V_+} = x,\ z_{\vartheta V_+} = y)$. On $\operatorname{Fix}(\vartheta)$ we have $(z_{\vartheta u})_{u \in V_+} = z_{V_+}$, so the entry vanishes unless $y = x$, and the diagonal entries are $\kappa(z_{V_+} = x) \geq 0$. A nonnegative diagonal matrix is symmetric and positive semidefinite. The support of $Z_*(w\pi)$ is contained in $Z(\{\pi > 0\})$, which is the support of κ. The diagonal copy is supported on $\operatorname{Fix}(\vartheta)$, with diagonal entries $m(x)$.

(ii) $\sum_i \kappa_i - \kappa$ is a nonnegative measure supported in $\bigcup_i \operatorname{supp}\kappa_i \subseteq \operatorname{Fix}(\vartheta)$. By linearity of C_ϑ and (i), $C_\vartheta(\sum_i \kappa_i) - C_\vartheta(\kappa)$ is nonnegative diagonal, and $C_\vartheta(\kappa) \succeq 0$ by (i).

(iii) By (5.2), $C_\vartheta(\kappa' \otimes \rho) - C_\vartheta(\kappa \otimes \rho) = C_\vartheta(\kappa' - \kappa) \otimes C_\vartheta(\rho)$, a Kronecker product of two symmetric positive semidefinite matrices.

(iv) Apply (5.1) with the same real vector F_g on both sides of the inequalities of (ii) and (iii).

(v) For $x', y' \in \mathcal{Y}^{W_+}$,

$$C_{\vartheta'}(\kappa^P)_{x',y'} = \sum_z \kappa(z) P(z_{V_+}, x') P\big((z_{\vartheta u})_u, y'\big) = \sum_{x,y} C_\vartheta(\kappa)_{x,y} P(x,x') P(y,y'),$$

which is the matrix identity. Thus $\Big\langle G, C_{\vartheta'}(\kappa^P) G \Big\rangle = \langle PG, C_\vartheta(\kappa) PG \rangle$ for every real G, and $P^{\mathsf{T}} C P$ is symmetric when C is; this gives the order statement. Nonnegativity, the mass and the marginal follow from the unit row sums. For the diagonal, by Jensen's inequality for each row of P,

$$\Big\langle G, P^{\mathsf{T}} \operatorname{diag}(m) P G \Big\rangle = \sum_x m(x)\,(PG)(x)^2 \leq \sum_x m(x)\,(PG^2)(x) = \langle G, \operatorname{diag}(mP) G \rangle\,,$$

so $C_{\vartheta'}(\pi^P) + k\operatorname{diag}(mP) \succeq P^{\mathsf{T}}\big(C_\vartheta(\pi) + k\operatorname{diag}(m)\big)P \succeq 0$, and the left side is symmetric. If P is deterministic, the defining sum collapses to $\Phi_*\kappa$. Since $(z \circ \vartheta)_{V_+} = (z_{\vartheta u})_u$ and $((z \circ \vartheta)_{\vartheta u})_u = z_{V_+}$, we get $\Phi(z \circ \vartheta) = \Phi(z) \circ \vartheta'$; for $z \in \operatorname{Fix}(\vartheta)$ both halves of $\Phi(z)$ are $\phi(z_{V_+})$, so $\Phi(z) \in \operatorname{Fix}(\vartheta')$. Finally $P^{\mathsf{T}} \operatorname{diag}(m) P = \operatorname{diag}(\phi_* m)$, and two measures with the same cut matrix coincide. □

Remark 5.5 (Limits of the common-cut order). (a) Part (iv) is a union bound in the cone of positive semidefinite cut matrices, not a pointwise inequality for the integrand: a product of Fourier lifts can be negative. A Boolean cover $\mathbf{1}_B \leq \sum_F \mathbf{1}_{I_F}$ of events of a latent representation therefore becomes the inequality $0 \leq \Lambda_{Z_*(\mathbf{1}_B \pi) \otimes \rho}(g) \leq \sum_F \Lambda_{Z_*(\mathbf{1}_{I_F} \pi) \otimes \rho}(g)$ of diagonal functionals.

(b) The samplewise hypothesis is essential. Being supported on $\operatorname{Fix}(\vartheta)$ is a property of the samples; invariance of a measure under the reflection does not imply it, and reflection positivity of a measure at every cut does not make its restrictions to events positive semidefinite at a chosen cut without an additional argument. This is one reason why the residues modulo the small modulus and the position of an integer among the intervals are external indices of the comparison (Definition 9.6) and not coordinates of the labels: a reference law carrying correlated residues would have to be restricted to events of the residues, and such restrictions need not remain positive semidefinite at the cuts where positivity is required.

5.2. **Controlling a law that is not reflection positive by two reflection-positive laws (2×2 blocks).** If a block matrix of two positive semidefinite diagonal blocks is positive semidefinite, its off-diagonal block is bounded by the diagonal ones. Such a *bracket* controls a measure that

is not reflection positive at a cut by two positive ones, and brackets tensorize. This is how the comparison law, which is not reflection positive, is controlled at the reserved cut.

Lemma 5.6 (Block-Gram calculus)**.** *Let $\mathcal{H}, \mathcal{H}', \mathcal{H}_1, \dots, \mathcal{H}_n$ be finite-dimensional Hilbert spaces.*

(a) Brackets. *For operators M on $\mathcal{H}$, M' on $\mathcal{H}'$ and $K\colon \mathcal{H}' \to \mathcal{H}$, the block operator $\left(\begin{smallmatrix} M & K \\ K^* & M' \end{smallmatrix}\right)$ on $\mathcal{H} \oplus \mathcal{H}'$ is positive semidefinite if and only if $M \succeq 0$, $M' \succeq 0$ and $K = M^{1/2} C M'^{1/2}$ for an operator C with $\|C\| \le 1$. Then $|\langle x, Ky\rangle| \le \langle x, Mx\rangle^{1/2} \langle y, M'y\rangle^{1/2}$.*

(b) Tensorization. *If $\left(\begin{smallmatrix} M_i & K_i \\ K_i^* & M_i' \end{smallmatrix}\right) \succeq 0$ on $\mathcal{H}_i \oplus \mathcal{H}_i$ for each i, then the same holds for $M = \bigotimes_i M_i$, $M' = \bigotimes_i M_i'$ and $K = \bigotimes_i K_i$, and $|\langle x, Ky\rangle| \le \langle x, Mx\rangle^{1/2} \langle y, M'y\rangle^{1/2}$.*

(c) Modulus. *Every operator T on a finite-dimensional Hilbert space satisfies*

$$\begin{pmatrix} |T^*| & T \\ T^* & |T| \end{pmatrix} \succeq 0, \qquad |T| = (T^*T)^{1/2}, \quad |T^*| = (TT^*)^{1/2}.$$

(d) Symmetric brackets. *For self-adjoint M, K, $\left(\begin{smallmatrix} M & K \\ K & M \end{smallmatrix}\right) \succeq 0$ if and only if $M \pm K \succeq 0$. If A_i, B_i are self-adjoint with $A_i \pm B_i \succeq 0$ for every i, then $\bigotimes_i A_i \pm \bigotimes_i B_i \succeq 0$; for two factors, explicitly,*

$$2\,(A \otimes N \pm D \otimes G) = (A \pm D) \otimes (N + G) + (A \mp D) \otimes (N - G).$$

In particular $0 \preceq A_i \preceq A_i'$ for every i implies $\bigotimes_i A_i \preceq \bigotimes_i A_i'$.

Proof. (a) The diagonal blocks of a positive semidefinite block operator are positive semidefinite. If $K = M^{1/2} C M'^{1/2}$ with $\|C\| \le 1$, the form of the block operator at (x, y) is at least $\|M^{1/2}x\|^2 - 2\|M^{1/2}x\|\,\|M'^{1/2}y\| + \|M'^{1/2}y\|^2 \ge 0$. Conversely, for $\epsilon > 0$ conjugate the block operator plus ϵ by $\operatorname{diag}((M+\epsilon)^{-1/2}, (M'+\epsilon)^{-1/2})$: the result $\left(\begin{smallmatrix} I & C_\epsilon \\ C_\epsilon^* & I \end{smallmatrix}\right)$, $C_\epsilon = (M+\epsilon)^{-1/2} K (M'+\epsilon)^{-1/2}$, is positive definite, and its form at $(x, -C_\epsilon^* x)$ is $\|x\|^2 - \|C_\epsilon^* x\|^2$, so $\|C_\epsilon\| \le 1$. Since $K = (M+\epsilon)^{1/2} C_\epsilon (M'+\epsilon)^{1/2}$ for every ϵ, any limit point C of C_ϵ as $\epsilon \to 0$ is a contraction with $K = M^{1/2} C M'^{1/2}$, by continuity of the square root. The inequality is Cauchy–Schwarz: $|\langle x, Ky\rangle| = |\left\langle M^{1/2}x, CM'^{1/2}y \right\rangle|$.

(b) Write $K_i = M_i^{1/2} C_i M_i'^{1/2}$ by (a). The square root of a tensor product of positive semidefinite operators is the tensor product of the square roots, so $K = M^{1/2}(\bigotimes_i C_i) M'^{1/2}$, and $\|\bigotimes_i C_i\| = \prod_i \|C_i\| \le 1$; apply (a).

(c) Let $T = U|T|$ with U unitary. Then $|T^*| = U|T|U^*$, and the block operator equals $X|T|X^*$, where X is the column operator $h \mapsto (Uh, h)$ into $\mathcal{H} \oplus \mathcal{H}$; indeed $X^*(a, b) = U^*a + b$, $U|T| = T$ and $|T|U^* = T^*$.

(d) Conjugation by the unitary $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} I & I \\ I & -I \end{smallmatrix}\right)$ turns $\left(\begin{smallmatrix} M & K \\ K & M \end{smallmatrix}\right)$ into $\left(\begin{smallmatrix} M+K & 0 \\ 0 & M-K \end{smallmatrix}\right)$. If $A_i \pm B_i \succeq 0$, the first part gives the brackets $\left(\begin{smallmatrix} A_i & B_i \\ B_i & A_i \end{smallmatrix}\right) \succeq 0$, (b) tensorizes them, $\bigotimes_i B_i$ is self-adjoint, and the first part applies again. The two-factor identity is expanded directly, and each summand on its right is a tensor product of positive semidefinite operators. Finally $0 \preceq A_i \preceq A_i'$ gives $A_i' \pm A_i \succeq 0$. □

Brackets of measures. For finite signed measures κ, κ', ζ on $\mathcal{X}^V$ and a cut ϑ we write $\left(\begin{smallmatrix} \kappa & \zeta \\ \zeta^* & \kappa' \end{smallmatrix}\right) \succeq_\vartheta 0$, a *bracket at ϑ*, if the block matrix with blocks $C_\vartheta(\kappa)$, $C_\vartheta(\zeta)$, $C_\vartheta(\zeta)^{\mathsf{T}}$ (the block denoted ζ^*; ζ^* is not a separate measure), $C_\vartheta(\kappa')$ is positive semidefinite. By (a), $|Q_\zeta(X, Y)| \le Q_\kappa(X, X)^{1/2} Q_{\kappa'}(Y, Y)^{1/2}$ for all half-functions; by (b) and (5.2), brackets of the factors of product measures tensorize to a bracket of the products; and by (d), if $C_\vartheta(\zeta)$ is symmetric, then $\left(\begin{smallmatrix} \kappa & \zeta \\ \zeta & \kappa \end{smallmatrix}\right) \succeq_\vartheta 0$ if and only if $\kappa \pm \zeta \succeq_\vartheta 0$.

5.3. **From one cut to all slots: the Hölder inequality.** Reflection positivity turns a quadratic-form inequality at one cut into a mixed inequality across all slots. We first prove the single-cut Cauchy–Schwarz step, then explain why a sorting sequence gives the signed Hölder inequality. The method comes from infrared and chessboard estimates in statistical mechanics [21, 15, 20, 19, 24];

the same positive semidefinite formalism appears in graph-density inequalities and flag algebras [9, 29, 59, 45, 7]. We use only the finite cut-matrix form proved below.

Definition 5.7 (Folds; sorting sequences)**.** For a cut ϑ define $\phi^{\vartheta}_{\pm}\colon V \to V$ by $\phi^{\vartheta}_{+}(v) = v$ for $v \in V_+$ and $\phi^{\vartheta}_{+}(v) = \vartheta v$ for $v \in V_-$; and $\phi^{\vartheta}_{-}(v) = \vartheta v$ for $v \in V_+$ and $\phi^{\vartheta}_{-}(v) = v$ for $v \in V_-$. A finite sequence of cuts $\vartheta^1, \dots, \vartheta^T$ with chosen signs $s_t \in \{\pm\}$ is *sorting* if $\phi^{\vartheta^T}_{s_T} \circ \cdots \circ \phi^{\vartheta^1}_{s_1}$ is constant on V.

Lemma 5.8 (Cauchy–Schwarz at one cut)**.** *Suppose ν is reflection positive at ϑ. Then for every assignment $h\colon V \to \mathbb{R}^{\mathcal{X}}$,*

$$|\Lambda_\nu(h)|^2 \le \Lambda_\nu(h \circ \phi^{\vartheta}_{+})\, \Lambda_\nu(h \circ \phi^{\vartheta}_{-}), \qquad \Lambda_\nu(h \circ \phi^{\vartheta}_{\pm}) \ge 0. \tag{5.3}$$

In particular $\Lambda_\nu(g) \ge 0$ for a constant assignment.

Proof. Put $F(x) = \prod_{u \in V_+} h(u)(x_u)$ and $G(x) = \prod_{u \in V_+} h(\vartheta u)(x_u)$ for $x \in \mathcal{X}^{V_+}$. By (5.1), $\Lambda_\nu(h)$ is the cut form of ν at (F, G), and its values at (F, F) and (G, G) are $\Lambda_\nu(h \circ \phi^{\vartheta}_{+})$ and $\Lambda_\nu(h \circ \phi^{\vartheta}_{-})$: the first folded assignment puts $h(u)$ at u and at ϑu, the second puts $h(\vartheta u)$ at both ($u \in V_+$). Cauchy–Schwarz for the positive semidefinite form of $C_\vartheta(\nu)$ gives (5.3). For a constant assignment $F = G$. □

Theorem 5.9 (Signed Hölder inequality)**.** *Let ν be a finite signed measure on $\mathcal{X}^V$ that is RP at every cut of a sorting sequence. Then for all real functions g, g_v on $\mathcal{X}$,*

$$\Lambda_\nu(g) \ge 0 \qquad \textit{and} \qquad |\Lambda_\nu((g_v)_{v \in V})| \le \prod_{v \in V} \Lambda_\nu(g_v)^{1/d}.$$

Proof. By Lemma 5.8, diagonal values are nonnegative. Fix real functions $f_1, \dots, f_k$ and let $M = \max_h |\Lambda_\nu(h)|$ over the assignments $h\colon V \to \{f_1, \dots, f_k\}$. If $M > 0$, let h attain it. Both folded assignments again take values in $\{f_i\}$, so by (5.3) they are nonnegative, at most M, and have product at least M^2. Hence both equal M: folds of maximisers are maximisers. Applying the folds in the order $\vartheta^T, \vartheta^{T-1}, \dots, \vartheta^1$ turns h into $h \circ \phi^{\vartheta^T}_{s_T} \circ \cdots \circ \phi^{\vartheta^1}_{s_1}$, which is a constant maximiser. Therefore $M = \max_i \Lambda_\nu(f_i)$, which also holds when $M = 0$. For $\epsilon > 0$ apply this to $f_v = g_v / (\Lambda_\nu(g_v) + \epsilon)^{1/d}$, which satisfy $\Lambda_\nu(f_v) \le 1$. Multilinearity gives $|\Lambda_\nu((g_v))| \le \prod_v (\Lambda_\nu(g_v) + \epsilon)^{1/d}$; let $\epsilon \downarrow 0$. □

The proof uses only reflection positivity at the cuts of the sequence; it does not use that ν is nonnegative or normalized. Recall that a *seminorm* is a nonnegative, absolutely homogeneous function satisfying the triangle inequality; it may vanish on nonzero vectors.

Corollary 5.10 (Seminorm)**.** *Under the hypotheses of Theorem 5.9, $\|g\|_\nu = \Lambda_\nu(g)^{1/d}$ is a seminorm on $\mathbb{R}^{\mathcal{X}}$. If every one-slot marginal of ν is the uniform probability measure on $\mathcal{X}$, then $|\mathbb{E}_{x \in \mathcal{X}} g(x)| \le \|g\|_\nu$.*

Proof. Homogeneity holds because d is even. For the triangle inequality, expand $\Lambda_\nu(g_1 + g_2)$ into 2^d mixed terms and bound each by Theorem 5.9:

$$0 \le \Lambda_\nu(g_1 + g_2) \le \sum_{S \subseteq V} \|g_1\|_\nu^{|S|} \|g_2\|_\nu^{d - |S|} = (\|g_1\|_\nu + \|g_2\|_\nu)^d.$$

For the mean, put g in one slot and 1 in the others; then $\Lambda_\nu(1) = 1$ and the uniform marginal give $|\mathbb{E} g| = |\Lambda_\nu(g, 1, \dots, 1)| \le \|g\|_\nu \|1\|_\nu^{d-1} = \|g\|_\nu$. □

Remark 5.11 (Domination at one cut). Let μ satisfy the hypotheses of Theorem 5.9, let ϑ be a cut at which μ is RP (it need not belong to the sorting sequence), and let D be a finite signed measure with a bracket $\left(\begin{smallmatrix} \mu & D \\ D^* & \mu \end{smallmatrix}\right) \succeq_\vartheta 0$, for instance $0 \preceq_\vartheta D \preceq_\vartheta \mu$ (Lemma 5.6(d)). With F, G as

in (5.1), Lemma 5.6(a) and Theorem 5.9 for the two folded assignments, in which each g_u with $u \in V_+$, respectively $u \in V_-$, occurs twice, give

$$|\Lambda_D((g_v)_v)| \le Q_\mu(F,F)^{1/2} Q_\mu(G,G)^{1/2} \le \prod_{v \in V} \Lambda_\mu(g_v)^{1/d}.$$

The measure D need not be positive, RP at any other cut, a product, or normalized, and no seminorm is attached to it; only the dominating law carries the Hölder inequality. Normalizing D by its mass would in general destroy the domination, and we never do it.

5.4. **The 72 slots $(\mathbb{Z}/6\mathbb{Z})^2 \times \{0,1\}$, their symmetries, and the six queries.** The slots of the functional are indexed by a product of reflection coordinates: two six-chamber coordinates and a two-point layer. A product of coordinates, each with its own cuts and its own sorting sequence, is sorted by the concatenation of the sequences, because folds of different coordinates commute; this is the reason for working with products. A chamber coordinate carries the rotation c, the relation R and the group $\mathcal{G}_6$ of chamber automorphisms defined below, which transport positivity from one chamber cut to the others.

Definition 5.12 (Reflection coordinates; the slot array)**.**

(a) A *reflection coordinate* is a finite set with a family of cuts that contains a sorting sequence. For reflection coordinates $V^{(i)}$, $i \in \mathcal{C}$ (a finite index set), a cut ϑ of $V^{(i)}$ with plus half $V^{(i)}_+$ induces the cut $\vartheta^{[i]}$ of $V = \prod_{i \in \mathcal{C}} V^{(i)}$ that applies ϑ to the coordinate v_i and fixes the other coordinates, with plus half $\{v : v_i \in V^{(i)}_+\}$; these are the *cuts of the coordinate* i.

(b) The *chamber coordinate* is $\mathbb{Z}/6\mathbb{Z}$, whose elements are *chambers*, with the *rotation* $\mathsf{c}(j) = j+2$ for even j and $\mathsf{c}(j) = j-2$ for odd j, the relation $\mathsf{R}(x,y) \Leftrightarrow y \in \{x, x+1, x-1, \mathsf{c}(x)\}$, and, for $a \in \mathbb{Z}/6\mathbb{Z}$, the *chamber reflection* $\vartheta_a(j) = 2a-1-j$ with plus half $H_a = \{a, a+1, a+2\}$; its family of cuts consists of the six *chamber cuts* (ϑ_a, H_a), $a \in \mathbb{Z}/6\mathbb{Z}$ (they are cuts by Lemma 5.13(i) and contain a sorting sequence by Lemma 5.13(ii)). A pair $(j, \mathsf{c}j)$ is a c-*step*. The plus fold of ϑ_a is f_a: $f_a(j) = j$ for $j \in H_a$ and $f_a(j) = \vartheta_a(j)$ otherwise.

(c) The *layer* is $\{0,1\}$ with the single cut $\ell \mapsto 1-\ell$ and plus half $\{0\}$; its plus fold $\ell \mapsto 0$ is constant, so this cut is a sorting sequence.

(d) The *slot array* is $V = \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \{0,1\}$, $d = |V| = 72$, with slots $v = (u_0, u_1, \ell)$; the coordinates 0 and 1 are the *chamber coordinates* and the coordinate 2, with value ℓ, is the *layer*. The *chamber cut* $\vartheta_{c,a}$ ($c \in \{0,1\}$, $a \in \mathbb{Z}/6\mathbb{Z}$) is the cut of the coordinate c induced by ϑ_a. The *reserved cut* is the cut of the layer,

$$\iota(u_0, u_1, \ell) = (u_0, u_1, 1-\ell), \qquad V_+ = \{\ell = 0\}, \qquad V_- = \iota V_+.$$

The *family of cuts* of V is $\mathfrak{F} = \{\vartheta_{c,a} : c \in \{0,1\},\ a \in \mathbb{Z}/6\mathbb{Z}\} \cup \{\iota\}$: thirteen cuts, with seven distinct involutions, since $\vartheta_{a+3} = \vartheta_a$ with complementary halves (three involutions in each chamber coordinate, and ι). At ι we write C_ι, $\succeq_\iota$ and Q_κ for the cut matrix, the cut order and the cut form (Definition 5.2).

(e) The *queries* are the six ordered pairs of slots

$$e^{(j)} = \big((0,j,0),\, (\mathsf{c}0, \mathsf{c}j, 1)\big) = \big((0,j,0),\, (2, \mathsf{c}j, 1)\big), \qquad j \in \mathbb{Z}/6\mathbb{Z},$$

forming the family $\mathcal{E}_6$. For $e = (u_e, v_e) \in \mathcal{E}_6$, u_e is the *source* and v_e the *target*.

Lemma 5.13 (The slot array)**.**

(i) Chamber geometry and transport. *Each ϑ_a is a fixed-point-free involution of $\mathbb{Z}/6\mathbb{Z}$ commuting with c, with $\vartheta_a(H_a) = \mathbb{Z}/6\mathbb{Z} \setminus H_a$; moreover $\vartheta_{a+3} = \vartheta_a$ and $H_{a+3} = \mathbb{Z}/6\mathbb{Z} \setminus H_a$, so $\vartheta_{c,a+3}$ is $\vartheta_{c,a}$ with its halves exchanged. The* chamber automorphisms*, the permutations α of $\mathbb{Z}/6\mathbb{Z}$ with $\alpha \circ \mathsf{c} = \mathsf{c} \circ \alpha$ and $\mathsf{R}(\alpha x, \alpha y) \Leftrightarrow \mathsf{R}(x,y)$, form a group $\mathcal{G}_6$ of order 6: the translations $j \mapsto j + 2k$ ($k = 0,1,2$) and the reflections $\vartheta_0, \vartheta_1, \vartheta_2$. For $a \in \mathbb{Z}/6\mathbb{Z}$ the*

element $\alpha_a \in \mathcal{G}_6$ given by $\alpha_a(j) = j - a$ for even a and $\alpha_a(j) = a + 2 - j$ for odd a satisfies $\alpha_a(H_a) = H_0$ and $\alpha_a \vartheta_a \alpha_a^{-1} = \vartheta_0$. There is exactly one bijection $H_a \to H_0$ that preserves R *(the* half isomorphism *of H_a); it is the restriction of every $\alpha \in \mathcal{G}_6$ with $\alpha(H_a) = H_0$, and it maps the unique* c*-step inside H_a to $(0, 2)$.* Transport: *let $V = \prod_{i \in \mathcal{C}} V^{(i)}$ be a product of reflection coordinates whose coordinate c is a chamber coordinate (for instance $\mathbb{Z}/6\mathbb{Z}$ itself, $\mathbb{Z}/6\mathbb{Z} \times \{0, 1\}$, or the slot array with $c \in \{0, 1\}$), write $\vartheta_{c,a}$ for the cut of the coordinate c induced by ϑ_a, and for $\alpha \in \mathcal{G}_6$ let $\alpha^{(c)}$ be the permutation of V that applies α to the coordinate c. If a finite signed measure κ on $\mathcal{X}^V$ is invariant under $\alpha_a^{(c)}$, then $C_{\vartheta_{c,a}}(\kappa) = S_a C_{\vartheta_{c,0}}(\kappa) S_a^{\mathsf{T}}$ for the permutation matrix S_a of the bijection $x \mapsto (x_{\alpha_a^{(c)} u})_u$ of half-labellings; so a lower bound $C_{\vartheta_{c,0}}(\kappa) \succeq Q$ gives $C_{\vartheta_{c,a}}(\kappa) \succeq S_a Q S_a^{\mathsf{T}}$, and $\kappa \succeq_{\vartheta_{c,0}} \kappa'$ implies $\kappa \succeq_{\vartheta_{c,a}} (\alpha_a^{(c)})_* \kappa'$. In particular a measure invariant under $\mathcal{G}_6$ acting on the coordinate c that is RP at $\vartheta_{c,0}$ is RP at every chamber cut of the coordinate c; and on the slot array a measure invariant under $\mathcal{G}_6$ acting on each chamber coordinate that is RP at $\vartheta_{0,0}$, $\vartheta_{1,0}$ and ι is RP at every cut of $\mathfrak{F}$.*

(ii) Folds. *$f_0 = (0, 1, 2, 2, 1, 0)$ and $f_4 = (0, 0, 5, 4, 4, 5)$ (values at $0, \dots, 5$), and $f_0 \circ f_4 \circ f_0$ is the constant map 0; so $(\vartheta_0, \vartheta_4, \vartheta_0)$, all with sign $+$, is a sorting sequence of $\mathbb{Z}/6\mathbb{Z}$. The minus fold of ϑ_a is f_{a+3}; in particular f_4 is the plus fold of ϑ_4, the involution $\vartheta_1 = \vartheta_4$ with plus half $H_4 = \{4, 5, 0\}$.*

(iii) Products and sorting. *In a product $V = \prod_{i \in \mathcal{C}} V^{(i)}$ of reflection coordinates every induced cut is a cut; its folds act on its own coordinate by the folds of the factor and fix the other coordinates; reflections and folds of cuts of different coordinates commute; and every interleaving of sorting sequences of the factors that keeps the order within each factor is a sorting sequence of V. For the slot array, ι is a cut, every chamber reflection and chamber fold commutes with ι and its folds, the seven-step sequence*

$$\vartheta_{0,0},\ \vartheta_{0,4},\ \vartheta_{0,0},\ \vartheta_{1,0},\ \vartheta_{1,4},\ \vartheta_{1,0},\ \iota,$$

all with sign $+$, maps every slot to $(0, 0, 0)$, and $\mathfrak{F}$ consists of seven distinct involutions. Consequently a finite signed measure ν on $\mathcal{X}^V$ that is RP at the cuts of this sequence satisfies Theorem 5.9 and Corollary 5.10 with $d = 72$; if its one-slot marginals are uniform probabilities, $(\mathbb{E}g)^{72} \le \Lambda_\nu(g)$ whenever $\mathbb{E}g \ge 0$, which is $\alpha^{72} \le Y$ for the functional (Lemma 5.17(e)).

(iv) Queries. *The twelve endpoints of the six queries are distinct; every source lies in V_+ and every target in V_-; the reflected targets $\iota v_e = (2, \mathsf{c}j, 0)$ lie in V_+; and the twelve slots u_e, ιv_e ($e \in \mathcal{E}_6$) of V_+ are distinct. Each query applies* c *to both chamber coordinates and flips the layer.*

(v) Cylinder pullback. *Let $V = \prod_{i \in \mathcal{C}} V^{(i)}$ be a product of reflection coordinates, $\emptyset \ne \mathcal{C}' \subseteq \mathcal{C}$ a set of* active *coordinates, $W = \prod_{i \in \mathcal{C}'} V^{(i)}$ with the induced cuts, and $\pi \colon V \to W$ the projection. The* pullback *ρ^* of a finite signed measure ρ on $\mathcal{X}^W$ is its image under $\zeta \mapsto \zeta \circ \pi$: the slot v receives the label of $\pi(v)$.*

(a) Inactive cuts. *If ϑ is a cut of a coordinate $i \notin \mathcal{C}'$, every $\zeta \circ \pi$ lies in $\mathrm{Fix}(\vartheta^{[i]})$; so every nonnegative measure supported on such labellings, for instance the pullback of any nonnegative measure on $\mathcal{X}^W$, is diagonal at $\vartheta^{[i]}$ and has a nonnegative diagonal cut matrix there (Lemma 5.4(i)).*

(b) Active cuts. *If ϑ is a cut of a coordinate $i \in \mathcal{C}'$, then $\zeta \mapsto \zeta \circ \pi$ is half-local for the cuts induced by ϑ on W and on V (Lemma 5.4(v)); so an inequality $\rho_1 \succeq \rho_2$ at the cut of W gives $\rho_1^* \succeq \rho_2^*$ at the cut of V, and in particular reflection positivity passes from ρ to ρ^*.*

(c) Marginals. *For $u, v \in V$ the law of (z_u, z_v) under ρ^* is the law of $(\zeta_{\pi u}, \zeta_{\pi v})$ under ρ.*

(d) Equivariance. *If ρ is invariant under $\zeta \mapsto (g(\zeta_w))_w$ for a bijection g of $\mathcal{X}$, then ρ^* is invariant under $z \mapsto (g(z_v))_v$.*

For the slot array and $\mathcal{C}' = \{0,2\}$ (a law on labels indexed by (u_0,ℓ)), the cuts of the second chamber coordinate are inactive, and by (c) the ordered pair of labels at every query of $\mathcal{E}_6$ under the pullback ρ^* has the law of the single ordered pair at the slots $(0,0)$ and $(2,1)$ of $\mathbb{Z}/6\mathbb{Z} \times \{0,1\}$ under the original law ρ.

Proof. (i) $\vartheta_a(\vartheta_a(j)) = 2a-1-(2a-1-j) = j$. A fixed point would satisfy $2j \equiv 2a-1 \pmod 6$, which is impossible because $2j-2a+1$ is odd. For $i \in \{0,1,2\}$, $\vartheta_a(a+i) = a-1-i$, so $\vartheta_a(H_a) = \{a-1, a-2, a-3\} = \mathbb{Z}/6\mathbb{Z} \setminus H_a$; and $\vartheta_{a+3}(j) = 2a+5-j \equiv \vartheta_a(j)$, $H_{a+3} = \{a+3, a+4, a+5\} = \mathbb{Z}/6\mathbb{Z} \setminus H_a$. The map ϑ_a reverses parity. For even j, $\vartheta_a(j)$ is odd and $\mathsf{c}(\vartheta_a(j)) = 2a-3-j = \vartheta_a(j+2) = \vartheta_a(\mathsf{c}j)$; for odd j, $\mathsf{c}(\vartheta_a(j)) = 2a+1-j = \vartheta_a(j-2) = \vartheta_a(\mathsf{c}j)$.

Chamber automorphisms. Let α be one. For every x, the elements $y \neq x$ with $\mathsf{R}(x,y)$ and $y \neq \mathsf{c}(x)$ are $x \pm 1$, because $\mathsf{c}(x) = x \pm 2$. Since α is a bijection commuting with c and preserving R, it maps this set for x onto the set for αx; so it preserves the hexagon adjacency $y = x \pm 1$, and α is a rotation $j \mapsto j+k$ or a reflection $j \mapsto m-j$. A rotation commutes with c if and only if k is even: then it preserves parity and $\mathsf{c}(j+k) = \mathsf{c}(j)+k$, while for odd k, $\mathsf{c}(k) = k-2 \neq k+2 = \mathsf{c}(0)+k$. A reflection commutes with c if and only if m is odd: then it reverses parity and $\mathsf{c}(m-j) = m-\mathsf{c}(j)$, while for even m, $\mathsf{c}(m) = m+2 \neq m-2 = m-\mathsf{c}(0)$. The odd values $m = 2a-1$ give $\vartheta_0, \vartheta_1, \vartheta_2$. Conversely these six maps commute with c and change the differences $y-x \in \{0, \pm 1\}$ at most by a sign, so they preserve R, and so do their inverses. The chamber automorphisms form a group by definition; it has these six elements.

The elements α_a. For even a, α_a is a translation by an even number, $\alpha_a(H_a) = H_0$, and $\alpha_a \vartheta_a \alpha_a^{-1}(j) = 2a-1-(j+a)-a = -1-j = \vartheta_0(j)$. For odd a, α_a is the reflection $j \mapsto m-j$ with $m = a+2$ odd, hence lies in $\mathcal{G}_6$; it is an involution with $\alpha_a(H_a) = \{2,1,0\}$, and $\alpha_a \vartheta_a \alpha_a(j) = a+2-(2a-1-(a+2-j)) = 5-j = \vartheta_0(j)$.

Half isomorphisms. Restricted to $H_0 = \{0,1,2\}$, the relation R holds on all ordered pairs except $(2,0)$: indeed $\mathsf{c}(0) = 2$, while $0 \notin \{2,1,3,4\}$. So $(0,2)$ is the unique pair (x,y) of H_0 with $\mathsf{R}(x,y)$ and not $\mathsf{R}(y,x)$, and it is the only c-step inside H_0 ($\mathsf{c}(1) = 5$ and $\mathsf{c}(2) = 4$ lie outside H_0). The automorphism α_a maps H_a onto H_0 preserving R and c-steps, so H_a also has a unique asymmetric pair, which is its unique c-step. A R-preserving bijection $H_a \to H_0$ must map the asymmetric pair of H_a to $(0,2)$ and therefore the third element to 1. This proves uniqueness, and the restriction of every $\alpha \in \mathcal{G}_6$ with $\alpha(H_a) = H_0$ is such a bijection.

Transport. Let $\sigma = \alpha_a^{(c)}$. Since $\alpha_a^{-1} \vartheta_0 \alpha_a = \vartheta_a$, we have $\sigma^{-1} \vartheta_{c,0} \sigma = \vartheta_{c,a}$, and $\sigma^{-1}\{v : v_c \in H_0\} = \{v : v_c \in \alpha_a^{-1} H_0\} = \{v : v_c \in H_a\}$ is the plus half of $\vartheta_{c,a}$. If $\sigma_* \kappa = \kappa$, then Remark 5.3(c) gives the relabelling of the cut matrix, and $\kappa \succeq_{\vartheta_{c,0}} \kappa'$ implies $\kappa = \sigma_* \kappa \succeq_{\vartheta_{c,a}} \sigma_* \kappa'$. For the last two assertions apply this with $\kappa' = 0$ to each $a \in \mathbb{Z}/6\mathbb{Z}$ and each chamber coordinate.

(ii) For f_0: $H_0 = \{0,1,2\}$ and $\vartheta_0(3,4,5) = (2,1,0)$. For f_4: $H_4 = \{4,5,0\}$ and $\vartheta_4(j) = 7-j \equiv 1-j$, so $\vartheta_4(1,2,3) = (0,5,4)$. Hence

$$f_0(\mathbb{Z}/6\mathbb{Z}) = \{0,1,2\}, \qquad f_4(\{0,1,2\}) = \{0,5\}, \qquad f_0(\{0,5\}) = \{0\},$$

and $f_0 \circ f_4 \circ f_0$ is the constant map 0. The minus fold of ϑ_a maps H_a by ϑ_a onto H_{a+3} and fixes H_{a+3}; as $\vartheta_{a+3} = \vartheta_a$, it is the plus fold f_{a+3} of ϑ_{a+3}.

(iii) An induced cut $\vartheta^{[i]}$ is a fixed-point-free involution because ϑ is one, and it maps $\{v : v_i \in V_+^{(i)}\}$ onto $\{v : v_i \in V_-^{(i)}\}$. If $v_i \in V_+^{(i)}$, then v lies in the plus half and is fixed by the plus fold; otherwise the plus fold sends v to $\vartheta^{[i]} v$, whose coordinate i is $\vartheta(v_i)$. So the folds of $\vartheta^{[i]}$ replace v_i by its fold and fix the other coordinates, and maps acting on different coordinates commute. In the composition of the folds of an interleaved sequence, the coordinate i is changed only by the folds of the coordinate i, in their order, and their composition is constant on $V^{(i)}$; so the composition is

constant on V. For the slot array the factor sequences are $(\vartheta_0, \vartheta_4, \vartheta_0)$ for both chamber coordinates, by (ii), and the single cut of the layer, whose plus fold $\ell \mapsto 0$ is constant; ι is the cut of the layer. The induced involutions $\vartheta_{c,a}$ and $\vartheta_{c,a'}$ coincide if and only if $\vartheta_a = \vartheta_{a'}$, that is $2a \equiv 2a' \pmod 6$, that is $a' \in \{a, a+3\}$; involutions of different coordinates differ; so $\mathfrak{F}$ consists of $3+3+1=7$ involutions. The last assertion is Theorem 5.9 and Corollary 5.10.

(iv) The sources $(0, j, 0)$ are six distinct slots with $u_0 = 0$ and $\ell = 0$; the targets $(2, \mathsf{c}j, 1)$ are six distinct slots, since c is a bijection, with $u_0 = 2$ and $\ell = 1$. As $0 \neq 2$, the twelve endpoints are distinct. Sources lie in V_+ and targets in V_-, and $\iota v_e = (2, \mathsf{c}j, 0) \in V_+$; the slots u_e and ιv_e are distinct, since their first coordinates are 0 and 2. The last sentence is the definition, with $\mathsf{c}0 = 2$.

(v) (a) If $i \notin \mathcal{C}'$, then $\vartheta^{[i]}v$ differs from v only in the coordinate i, so $\pi(\vartheta^{[i]}v) = \pi(v)$ and $(\zeta \circ \pi)_{\vartheta^{[i]}v} = (\zeta \circ \pi)_v$. (b) Write ϑ_V, ϑ_W for the cuts induced by ϑ on V and on W, with plus halves V_+, W_+. Since $i \in \mathcal{C}'$, $\pi \circ \vartheta_V = \vartheta_W \circ \pi$, $\pi(V_+) \subseteq W_+$ and $\pi^{-1}(W_+) = V_+$. Put $\phi(\xi) = (\xi_{\pi v})_{v \in V_+}$ for $\xi \in \mathcal{X}^{W_+}$. For $z = \zeta \circ \pi$ we have $z_{V_+} = \phi(\zeta_{W_+})$ and $(z_{\vartheta_V v})_{v \in V_+} = (\zeta_{\vartheta_W \pi v})_{v \in V_+} = \phi\big((\zeta_{\vartheta_W w})_{w \in W_+}\big)$; so the pullback is the half-local map of ϕ, and Lemma 5.4(v) applies. (c) and (d) are immediate from $z_v = \zeta_{\pi v}$. For the slot array with $\mathcal{C}' = \{0, 2\}$, $\pi(0, j, 0) = (0, 0)$ and $\pi(2, \mathsf{c}j, 1) = (2, 1)$. □

Remark 5.14. Nothing in the transport of (i), in (iii) and in (v) depends on the particular factors. For instance the array $(\mathbb{Z}/6\mathbb{Z})^3$, with the nine-step sequence formed from $(\vartheta_0, \vartheta_4, \vartheta_0)$ in each of its three chamber coordinates, is sorted in the same way; Part VI uses this array.

5.5. **Square-affine invariance.** Many laws below are invariant under relabelling all field elements at once by maps $x \mapsto rx + c$ with r a square unit, chosen independently at each prime. Such a map acts only on the sampled field labels, not on the discrete choices in a record (which component or type was selected). Conditioning on a component or a type therefore preserves this invariance, while conditioning on a specified field label need not preserve it, and no invariance is claimed after such conditioning below. This subsection defines the invariance and lists the operations that preserve it.

Definition 5.15 (Square-affine invariance and pair law)**.** Let $\mathcal{B}$ be a finite set of odd primes with exponents $e_p \geq 1$ ($p \in \mathcal{B}$), and $X_{\mathcal{B}} = \prod_{p \in \mathcal{B}} \mathbb{Z}/p^{e_p}\mathbb{Z}$. Let $\mathfrak{G}_{\mathcal{B}}$ be the group of maps $g = (g_p)_{p \in \mathcal{B}}$ of $X_{\mathcal{B}}$, $g(x)_p = r_p x_p + c_p$, where r_p is a square unit modulo p^{e_p} and $c_p \in \mathbb{Z}/p^{e_p}\mathbb{Z}$, chosen *independently* for each prime (for $e_p = 1$: $r_p \in Q_p^*$ and $c_p \in \mathbb{F}_p$). It acts on tuples $z \in X_{\mathcal{B}}^S$ (S any finite index set) by $(gz)_s = g(z_s)$, the same g for every index. A finite signed measure μ on $\Omega \times X_{\mathcal{B}}^S$, where the finite set Ω records discrete data fixed by the group (maps, component and type indices, tags), is *square-affine invariant* if it is invariant under $(\omega, z) \mapsto (\omega, gz)$ for all $g \in \mathfrak{G}_{\mathcal{B}}$. If $e_p = 1$ for every $p \in \mathcal{B}$, write $\Omega_{\mathcal{B}} = \prod_{p \in \mathcal{B}} \Omega_p \subseteq X_{\mathcal{B}}^2$ (the pairs (x, y) with $y_p - x_p \in Q_p^*$ for every $p \in \mathcal{B}$); the *product strict-square pair law* is the uniform probability on $\Omega_{\mathcal{B}}$, which is $\bigotimes_{p \in \mathcal{B}} \mathrm{Unif}(\Omega_p)$.

Lemma 5.16 (Marginals and pair laws)**.** *Let $\mathcal{B}$ be as in Definition 5.15.*

(1) Invariance is preserved *by the following operations on square-affine invariant measures: sums and real multiples; normalization of a nonzero nonnegative measure; passing to the variation $|\mu|$; multiplication by a nonnegative function $\Phi(\omega, z)$ with $\Phi(\omega, gz) = \Phi(\omega, z)$ (for instance the restriction to an event defined by the latent record ω alone, or to injectivity at every prime, or, when $e_p = 1$, a density that is a product of functions $A_p(\pm(z_s - z_{s'})_p)$); images under $(\omega, z) \mapsto (\omega', z \circ \varsigma_\omega)$, where ω' is a function of ω and $\varsigma_\omega \colon S' \to S$ is a map of index sets depending on ω (including the cylinder pullback of Lemma 5.13(v), forgetting coordinates); products $\mu_1 \otimes \mu_2$ of invariant measures over disjoint prime sets $\mathcal{B}_1, \mathcal{B}_2$ (as a measure over $\mathcal{B}_1 \cup \mathcal{B}_2$, with latent record (ω_1, ω_2)); and mixtures $\sum_{\omega_0 \in \Omega_0} \kappa(\omega_0)\, \delta_{\omega_0} \otimes \mu^{\omega_0}$ on $(\Omega_0 \times \Omega) \times X_{\mathcal{B}}^S$ of invariant measures μ^{ω_0} indexed by an additional latent variable ω_0 with law κ (for example a map shared by all primes of a block, drawn before the field labels, which are then independent across the primes). For a prime p with $e_p = 1$ and an*

oriented graph G, the uniform law on the injective strict-square labellings of G over $\mathbb{F}_p$ (the law $\lambda_p(G)$ of Definition 12.1) and the uniform law on all its strict-square labellings are invariant ($\mathcal{B} = \{p\}$).

(2) One-slot marginals. *If μ is an invariant finite signed measure on $\Omega \times X_{\mathcal{B}}^S$, then for every $s \in S$ and $\omega \in \Omega$ the measure $\mu(\{\omega\} \times \{z : z_s \in \cdot\})$ on $X_{\mathcal{B}}$ is $\mu(\{\omega\} \times X_{\mathcal{B}}^S)\ \mathrm{Unif}(X_{\mathcal{B}})$. In particular the one-slot marginals of μ and of $|\mu|$ are $\mu(1)\,\mathrm{Unif}(X_{\mathcal{B}})$ and $|\mu|(1)\,\mathrm{Unif}(X_{\mathcal{B}})$, where $\mu(1)$ is the total mass; and if μ is nonnegative, the law of z_s under $\mu(\cdot \mid \omega)$ is uniform for every ω of positive mass. These conclusions need only invariance under the translations $(\omega, z) \mapsto (\omega, (z_s + c)_s)$, $c \in X_{\mathcal{B}}$.*

(3) Pair laws. *Suppose $e_p = 1$ for every $p \in \mathcal{B}$. If μ is a nonzero nonnegative invariant measure, $s \neq s'$, and μ-almost surely $(z_s, z_{s'}) \in \Omega_{\mathcal{B}}$, then the law of $(z_s, z_{s'})$ under $\mu(\cdot \mid \omega)$, and under $\mu/\mu(1)$, is the product strict-square pair law* $\mathrm{Unif}(\Omega_{\mathcal{B}})$.

Proof. (1) Each operation commutes with the action $(\omega, z) \mapsto (\omega, gz)$. For images, $(gz) \circ \varsigma_\omega = g(z \circ \varsigma_\omega)$ because g acts slotwise; for products, $\mathfrak{G}_{\mathcal{B}_1 \cup \mathcal{B}_2} = \mathfrak{G}_{\mathcal{B}_1} \times \mathfrak{G}_{\mathcal{B}_2}$ acts factorwise; for the variation, the image under the bijection g of the Jordan decomposition $\mu = \mu_+ - \mu_-$ is again a decomposition into mutually singular nonnegative measures, so by uniqueness g preserves $\mu_\pm$ and $|\mu|$; the other cases are immediate, injectivity being preserved because each g_p is a bijection. The uniform laws are invariant because $z \mapsto rz + c$ ($r \in Q_p^*$) is a bijection of the respective sets of labellings: it multiplies every difference by a nonzero square and preserves injectivity. Functions $A_p(\pm(z_s - z_{s'})_p)$ are invariant for the same reason.

(2) The measure $\mu(\{\omega\} \times \{z : z_s \in \cdot\})$ is invariant under the translations $x \mapsto x + c$, $c \in X_{\mathcal{B}}$, which lie in $\mathfrak{G}_{\mathcal{B}}$ and act transitively on $X_{\mathcal{B}}$; so it gives the same mass to every point. Summing over ω gives the unconditional statement, and $|\mu|$ is invariant under the same translations by the Jordan argument of (1). Only the translations were used.

(3) $\mathfrak{G}_{\mathcal{B}}$ acts transitively on $\Omega_{\mathcal{B}}$: given $(x, y), (x', y') \in \Omega_{\mathcal{B}}$, put $g(w)_p = \frac{y'_p - x'_p}{y_p - x_p}(w_p - x_p) + x'_p$; the ratio is a quotient of nonzero squares, so $g \in \mathfrak{G}_{\mathcal{B}}$, and $g(x) = x'$, $g(y) = y'$. Conditioning on ω preserves invariance, so the law of $(z_s, z_{s'})$ under $\mu(\cdot \mid \omega)$ is an invariant probability on the finite set $\Omega_{\mathcal{B}}$, on which $\mathfrak{G}_{\mathcal{B}}$ acts transitively; hence it is uniform. Mixing over ω gives the statement for $\mu/\mu(1)$. Finally $\Omega_{\mathcal{B}} = \prod_p \Omega_p$, so its uniform law is the product of the uniform laws of the factors. □

5.6. **Independent atoms: product laws and $\alpha^{72} \leq Y$.** The reference law of the functional is a product of laws of independent atoms, finite sets of primes. The next lemma collects what the product inherits from its factors.

An *atom* is a finite set $\mathcal{B}$ of odd primes with exponents $e_p \geq 1$; $X_{\mathcal{B}} = \prod_{p \in \mathcal{B}} \mathbb{Z}/p^{e_p}\mathbb{Z}$ carries its uniform probability measure. Let V be a slot set with a family $\mathfrak{F}$ of cuts that contains a sorting sequence, $d = |V|$, and let $\mathcal{A}$ be a set of pairwise disjoint atoms, each with a measure $\mu_{\mathcal{B}}$ on $X_{\mathcal{B}}^V$. For a finite set $\mathcal{S} \subseteq \mathcal{A}$ put $X_{\mathcal{S}} = \prod_{\mathcal{B} \in \mathcal{S}} X_{\mathcal{B}}$, identify a family $(z_{v,\mathcal{B}})_{v \in V, \mathcal{B} \in \mathcal{S}}$ with $(z_v)_{v \in V}$, $z_v = (z_{v,\mathcal{B}})_{\mathcal{B}} \in X_{\mathcal{S}}$, and let

$$\mu_{\mathcal{S}} = \bigotimes_{\mathcal{B} \in \mathcal{S}} \mu_{\mathcal{B}}, \qquad Z_{\mathcal{S}}(g) = \Lambda_{\mu_{\mathcal{S}}}(g) \quad (g \colon X_{\mathcal{S}} \to \mathbb{R}).$$

For $\mathcal{S}_1 \subseteq \mathcal{S}$, $\mathbb{E}_{\mathcal{S}_1} g$ is the uniform average of g over the coordinates of the atoms of $\mathcal{S}_1$: $\mathbb{E}_{\mathcal{S}_1} g(x) = \mathbb{E}_{y \in X_{\mathcal{S}_1}} g(x + y)$, with y extended by 0 at the other atoms.

Lemma 5.17 (Signed Hölder inequality, mean and averaging)**.** *Suppose that every $\mu_{\mathcal{B}}$ is a probability measure on $X_{\mathcal{B}}^V$, reflection positive at every cut of $\mathfrak{F}$, and square-affine invariant (Definition 5.15, for the primes of $\mathcal{B}$). Let $\mathcal{S} \subseteq \mathcal{A}$ be finite and let g, g_v ($v \in V$) be real functions on $X_{\mathcal{S}}$.*

(a) *$\mu_{\mathcal{S}}$ is a probability measure, RP at every cut of $\mathfrak{F}$. Hence $Z_{\mathcal{S}}(g) \geq 0$, $|\Lambda_{\mu_{\mathcal{S}}}((g_v)_v)| \leq \prod_v Z_{\mathcal{S}}(g_v)^{1/d}$, and $\|g\|_{\mathcal{S}} = Z_{\mathcal{S}}(g)^{1/d}$ is a seminorm.*

(b) *$\mu_{\mathcal{S}}$ is invariant under $\mathfrak{G}_{\mathcal{S}} = \prod_{\mathcal{B}\in\mathcal{S}} \mathfrak{G}_{\mathcal{B}}$; its one-slot marginals are* $\mathrm{Unif}(X_{\mathcal{S}})$*; and for $u, w \in V$ the law of (z_u, z_w) is the product over $\mathcal{B} \in \mathcal{S}$ of the laws of $(z_{u,\mathcal{B}}, z_{w,\mathcal{B}})$ under $\mu_{\mathcal{B}}$. Consequently $|\mathbb{E}_x g(x)| \leq \|g\|_{\mathcal{S}}$.*

(c) *$\|g \circ \psi\|_{\mathcal{S}} = \|g\|_{\mathcal{S}}$ for every primewise map $\psi \in \mathfrak{G}_{\mathcal{S}}$, $\psi(z)_p = r_p z_p + c_p$.*

(d) *If $\mathcal{S}_1 \subseteq \mathcal{S}$, then $Z_{\mathcal{S}}(\mathbb{E}_{\mathcal{S}_1} g) = Z_{\mathcal{S}\setminus\mathcal{S}_1}(\mathbb{E}_{\mathcal{S}_1} g) \leq Z_{\mathcal{S}}(g)$. If g depends only on the coordinates of the atoms of $\mathcal{S}_2 \subseteq \mathcal{S}$, then $Z_{\mathcal{S}}(g) = Z_{\mathcal{S}_2}(g)$.*

(e) *If $\mathbb{E}_x g(x) = \alpha \geq 0$, then $\alpha^d \leq Z_{\mathcal{S}}(g)$. In particular, for the lift, the inputs and the functional of Definition 6.1 (with $\mathcal{S} = \mathcal{A}_N$, $d = 72$, and I, D_0, s as there),*

$$\mathbb{E}_{z\in G_N} g^{I,D_0,s}(z) = \frac{D_0\,\#\{n \in A \cap I :\ n \equiv s\ (D_0)\}}{|I|},$$

so $\mathbb{E}F_A = \alpha := |A|/N$ and $\alpha^{72} \leq \mathcal{Y}(\mathsf{L}, A)$.

Proof. (a) A product of probability measures is a probability measure. At a cut $\vartheta \in \mathfrak{F}$, $C_\vartheta(\mu_{\mathcal{S}}) = \bigotimes_{\mathcal{B}} C_\vartheta(\mu_{\mathcal{B}})$ by (5.2), a Kronecker product of symmetric positive semidefinite matrices. Since $\mathfrak{F}$ contains a sorting sequence, Theorem 5.9 and Corollary 5.10 apply; d is even because V has a cut. (b) The invariance is the product clause of Lemma 5.16(1), and the one-slot marginals are uniform by Lemma 5.16(2). Under a product measure the coordinates of different factors are independent, and (z_u, z_w) is the family of the pairs $(z_{u,\mathcal{B}}, z_{w,\mathcal{B}})$. The mean bound is the second part of Corollary 5.10. (c) By (b), the map $(z_v)_v \mapsto (\psi(z_v))_v$ preserves $\mu_{\mathcal{S}}$. (d) $\mathbb{E}_{\mathcal{S}_1} g$ is the average of the translates $g(\cdot + y)$, $y \in X_{\mathcal{S}_1}$; by (c) with $r_p = 1$ they all have seminorm $\|g\|_{\mathcal{S}}$, and the triangle inequality of (a) gives the inequality. The two equalities follow from Fubini, since the factors $\mu_{\mathcal{B}}$ are probability measures and the integrand does not depend on the labels of the atoms integrated out. (e) By (b), $\alpha \leq \|g\|_{\mathcal{S}}$. For the lift, averaging over z annihilates every nontrivial character, and the trivial character has conductor 1, at most the cutoff Q of the lift (Section 6), so it is retained and the mean of $\mathcal{L}_{I,N} f$ is $\widehat{f}_I(0) = |I|^{-1} \sum_{n\in I} f(n)$. With $f = (D_0 \mathbf{1}_{n\equiv s\,(D_0)} \mathbf{1}_A)|_I$ this is the stated number; with $I = [N]$ and $D_0 = 1$ it is α. Apply the first part to $g = F_A$, for which $Z_{\mathcal{A}_N}(F_A) = \mathcal{Y}(\mathsf{L}, A)$. □

For the slot array and its family $\mathfrak{F}$ the hypotheses of the lemma are contained in (R1)–(R3) of Hypothesis 6.2, which Part V verifies atom by atom at every cut of $\mathfrak{F}$, not only at the sorting cuts (see the discussion after Hypothesis 6.2).

Consequences and what to read next. Only linear algebra (the Loewner order of Definition 4.1, Kronecker products, polar decomposition, Cauchy–Schwarz and Jensen's inequality) and the arithmetic notation of Section 4: the nonzero squares Q_p^*, the strict-square pairs Ω_p and the indicator A_p. Later definitions are named only as examples or applications: the laws $\lambda_p(G)$ (Definition 12.1) in Lemma 5.16, the coupled index (Definition 9.6) in Remark 5.5, and the lift, the functional and the reference hypotheses (Definition 6.1 and Hypothesis 6.2) in and after Lemma 5.17(e). No property of square-difference-free sets is used in this section.

6. The functional Y: Fourier truncation and estimates preserving the density

What this section proves and how to read it. This section defines $Y = \mathcal{Y}(\mathsf{L}, A)$, the 72-fold average of a truncated Fourier series F_A of $\mathbf{1}_A$ (the *lift*: F_A retains characters of conductor at most $N^{1/20}$ so that fixing labels can be analysed prime by prime), and proves three properties used throughout: averaging interval-and-residue inputs returns F_A exactly; L^{2r} norms of lifts are linear in the density α (so remainders in Section 8 are $O(\mathsf{L}^{-\sigma}\alpha)$ with $\sigma = 1/825$ (Definition 8.1)); and Fourier terms of large conductor are damped at large primes. It also proves that tensor products preserve exact Fourier support (the resulting orthogonal components of a function, indexed by exact Fourier

support, are called *sectors*; they are not prime pieces; a *spectator* is a factor left unchanged in such a tensor product). Inputs: Section 5 and the reference hypotheses (R1)–(R5) (Section 6.2), verified in Part V. Outputs: the definition Definition 6.1 and the moment and decay estimates cited in Parts III and V. A first reading should follow the definition and the statements of the moment and decay results; the tensor-product linear algebra can be used as reference. (R4) uses *mixing* (damping of nonconstant functions at large primes, Definition 6.13 below) and (R5) uses the forcing terminology of Section 1.8.

6.1. **Definition of the lift F_A and the functional $Y(\mathsf{L}, A)$.** The slot array is $V = (\mathbb{Z}/6\mathbb{Z})^2 \times \{0,1\}$ of Definition 5.12, $d = |V| = 72$, with the family $\mathfrak{F}$ of its cuts: the chamber cuts $\vartheta_{c,a}$ of the two chamber coordinates and the reserved cut ι. The family contains a sorting sequence (Definition 5.7, Lemma 5.13(iii)). The multilinear averages $\Lambda_\kappa((g_v)_v)$ and $\Lambda_\kappa(g)$ of a finite signed measure κ on X^V are those of Section 5.

Definition 6.1 (The functional)**.** Let $\beta = 1/20$, let $K \geq 2$ be a fixed integer, and let $Q_3 = 19951586$, $P_4 = 2^{512}$ and $P_5 = 2^{65536}$. Put

$$\mathcal{P}_1 = \{p < P_4 : \ p \equiv 1 \pmod 4\}, \qquad \mathcal{P}_3 = \{Q_3 \leq p < P_4 : \ p \equiv 3 \pmod 4\},$$

fix integers $m_p \geq 1$ for $p \in \mathcal{P}_1$, and put $e_p = 2m_p$ for $p \in \mathcal{P}_1$ and $e_p = 1$ for every other prime. For a finite set $\mathcal{S}$ of primes let $X_\mathcal{S} = \prod_{p\in\mathcal{S}} \mathbb{Z}/p^{e_p}\mathbb{Z}$, with its uniform probability measure, and call $D_\mathcal{S} = |X_\mathcal{S}| = \prod_{p\in\mathcal{S}} p^{e_p}$ its *full order*.

(a) *Scales.* For an integer $\mathsf{L} \geq 1$ put $N = K\mathsf{L}$, $Q = N^\beta$, $\mathcal{T}_N = \{p \text{ prime} : \ P_4 \leq p \leq N^\beta\}$, $\mathcal{P}_N = \mathcal{P}_1 \cup \mathcal{P}_3 \cup \mathcal{T}_N$ and $G_N = X_{\mathcal{P}_N}$.

(b) *Characters.* A character of $X_\mathcal{S}$ is $z \mapsto e(z \cdot \xi)$ with $\xi = \sum_{p\in\mathcal{S}} a_p/p^{e_p} \in \mathbb{Q}/\mathbb{Z}$, $a_p \in \mathbb{Z}/p^{e_p}\mathbb{Z}$, and $z \cdot \xi = \sum_p a_p z_p/p^{e_p}$; we identify it with ξ. Its *exact support* is $\{p : a_p \neq 0\}$, and its *conductor* $q(\xi)$ is the reduced denominator of ξ, that is, $q(\xi) = \prod_p p^{e_p - \min(v_p(a_p), e_p)}$, the product over $p \in \mathcal{S}$, with a_p represented by an integer between 0 and $p^{e_p} - 1$ and $v_p(0)$ taken as $+\infty$. For $n \in \mathbb{Z}$ the number $e(n\xi)$ is well defined. The Fourier coefficient of $G\colon X_\mathcal{S} \to \mathbb{C}$ is $\widehat{G}(\xi) = \mathbb{E}_z G(z) e(-z \cdot \xi)$.

(c) *Lift.* For a nonempty interval I of integers and $f\colon I \to \mathbb{C}$ put

$$\widehat{f_I}(\xi) = \frac{1}{|I|} \sum_{n\in I} f(n) e(-n\xi) \quad (\xi \in \mathbb{Q}/\mathbb{Z}), \qquad \mathcal{L}_{I,N} f(z) = \sum_{\xi\in\Xi_N} \widehat{f_I}(\xi)\, e(z \cdot \xi) \quad (z \in G_N),$$

where Ξ_N, the set of *retained characters*, consists of the characters ξ of G_N with $q(\xi) \leq Q$. It is defined by the *total* conductor and depends neither on I nor on f. It contains every character supported on the prefix coordinates ($p \in \mathcal{P}_1$) of conductor $p^j \leq Q$, $1 \leq j \leq e_p$, and it is closed under negation, so the lift of a real function is real.

(d) *Atoms and reference laws.* The set $\mathcal{A}_N$ of *atoms* at scale N consists of the clique atoms $\{p\}$, $p \in \mathcal{P}_1$; the blocks of a fixed partition of $\mathcal{P}_3$ into finite sets (the middle blocks of Definition 20.1 and the upper blocks of Definition 27.1); and the tail atoms $\{p\}$, $p \in \mathcal{T}_N$. For an atom $\mathcal{B}$ we write $X_\mathcal{B}$ for $X_\mathcal{S}$ with $\mathcal{S}$ the set of primes of $\mathcal{B}$, and $\mu_\mathcal{B}$ for its reference law, a probability measure on $X_\mathcal{B}^V$ (constructed in Part V; Parts II and III use only Hypothesis 6.2). For a set $\mathcal{S}$ of atoms put $\mu_\mathcal{S} = \bigotimes_{\mathcal{B}\in\mathcal{S}} \mu_\mathcal{B}$ and $Z_\mathcal{S}(g) = \Lambda_{\mu_\mathcal{S}}(g)$ for $g\colon X_\mathcal{S} \to \mathbb{R}$; and $\mu_N = \mu_{\mathcal{A}_N}$, $Z_N = Z_{\mathcal{A}_N}$.

(e) *Functional.* For $A \subseteq [N]$ let $F_A = \mathcal{L}_{[N],N}\mathbf{1}_A$ and $\mathcal{Y}(\mathsf{L}, A) = Z_N(F_A)$, and let $\mathfrak{Y}(\mathsf{L})$ be the maximum of $\mathcal{Y}(\mathsf{L}, A)$ over the square-difference-free sets $A \subseteq [N]$.

(f) *Inputs.* For $b \in \mathbb{Z}/K\mathbb{Z}$, identified with $\{1, \dots, K\}$ through the least positive residues, let $I_b = ((b-1)\mathsf{L}, b\mathsf{L}] \cap \mathbb{Z}$, and put $\mathcal{I}_\mathsf{L} = \{[N], I_1, \dots, I_K\}$. The intervals I_b have length L and partition $[N]$; every element of I_b is smaller than every element of I_{b+1} for $1 \leq b < K$, and the

shift $b \mapsto b+1$ wraps I_K to I_1. The *small modulus* is $M_0 = 2^{2m_{\mathrm{ext}}+1} \prod_{3\le p<Q_3,\ p\equiv 3 \pmod 4} p^{2m_{\mathrm{ext}}}$, with fixed $m_{\mathrm{ext}} = 64$. For $I \in \mathcal{I}_{\mathsf{L}}$, a positive divisor D_0 of M_0 and $s \in \mathbb{Z}/D_0\mathbb{Z}$,

$$g^{I,D_0,s} = \mathcal{L}_{I,N}\big((D_0 \mathbf{1}_{n\equiv s\,(D_0)} \mathbf{1}_A)|_I\big).$$

Throughout, Y and $\mathcal{Y}$ denote this functional. The small modulus is coprime to $D_{\mathcal{P}_N}$: its primes are 2 and the primes $p \equiv 3 \pmod 4$ below Q_3, none of which lies in $\mathcal{P}_N$. So M_0 is not a coordinate of G_N; it enters only through the inputs and the external index of the queries (Definition 9.6). The atoms of $\mathcal{P}_1 \cup \mathcal{P}_3$ are present at every scale, also when their primes exceed N^β; a clique prime $p > Q$ then carries no nonconstant retained character. Two fixed integers control them,

$$Q_{\mathrm{cl}} = \prod_{p\in\mathcal{P}_1} p^{e_p}, \qquad Q_{\mathrm{pre}} = Q_{\mathrm{cl}} \prod_{\substack{Q_3\le p<P_5\\ p\notin\mathcal{P}_1}} p,$$

the full order of the clique prefix and a common multiple of the full orders of all atoms below P_5. The data K and $(m_p)_{p\in\mathcal{P}_1}$ are fixed, and constants may depend on them and on M_0, Q_{cl}, Q_{pre} (Remark 8.13). The baseline uses $K = 890666$, and the main theorem uses $K = 863660$; both use $m_p = m_{\mathrm{cl}} = 8192$ at every $p \in \mathcal{P}_1$ (Definition 18.1).

Support components ("pieces") of a lift. Let $\mathcal{P}$ be a finite set of primes with $e_p = 1$ for $p \in \mathcal{P}$, so that $X_{\mathcal{P}} = \prod_{p\in\mathcal{P}} \mathbb{F}_p$, and put $q_U = \prod_{p\in U} p$ for $U \subseteq \mathcal{P}$; a character of $X_{\mathcal{P}}$ with exact support U has conductor q_U. For $f\colon I \to \mathbb{C}$ and $U \subseteq \mathcal{P}$, the *piece of exact support* U of the lift of f on $X_{\mathcal{P}}$ is $f_U(z) = \sum_\xi \widehat{f_I}(\xi) e(z\cdot\xi)$, the sum over the characters ξ of $X_{\mathcal{P}}$ of exact support U. Each f_U depends only on z_U and has mean zero in every coordinate of U; if f is real, so is f_U. The mean of a lift is its coefficient at $\xi = 0$, $\mathbb{E}_z \mathcal{L}_{I,N} f(z) = \widehat{f_I}(0) = \mathbb{E}_I f$, since $q(0) = 1 \le Q$. A lift of a nonnegative function can take negative values: all positivity in this paper comes from the reflection positivity of the laws, not from the inputs.

6.2. **Assumptions (R1)–(R5) on the per-atom laws (proved in Part V).** The analytic stage uses the following properties of the reference laws and of the factors of the comparison law, and nothing else about the local laws.

Hypothesis 6.2 (Reference atom laws)**.** For every interval scale $\mathsf{L} \ge 1$, with $N = K\mathsf{L}$:

- (R1) every $\mu_{\mathcal{B}}$ is a probability measure on $X_{\mathcal{B}}^V$, and the law of an atom does not depend on N;
- (R2) every $\mu_{\mathcal{B}}$ is reflection positive (Definition 5.1) at every cut of the family $\mathfrak{F}$, that is, at the chamber cuts of both chamber coordinates and at the reserved cut ι; in particular at the seven cuts of the sorting sequence of Lemma 5.13;
- (R3) every $\mu_{\mathcal{B}}$ is square-affine invariant (Definition 5.15): it is invariant under the simultaneous maps $z_{v,p} \mapsto r_p z_{v,p} + c_p$, where r_p is a square unit and c_p an element of $\mathbb{Z}/p^{e_p}\mathbb{Z}$, chosen independently for $p \in \mathcal{B}$; and every signed measure $\rho_{\mathcal{B}}$ on $X_{\mathcal{B}}^V$ that occurs in Parts III and V at $\mathcal{B}$ as a component, a replacement, a one-layer folded copy, or a difference of such measures, has a variation $|\rho_{\mathcal{B}}|$ whose one-slot marginals are all $|\rho_{\mathcal{B}}|(1)\,\mathrm{Unif}(X_{\mathcal{B}})$. Both hold for every measure that is invariant under these maps, since the variation of an invariant measure is invariant; in particular every one-slot marginal of $\mu_{\mathcal{B}}$ is $\mathrm{Unif}(X_{\mathcal{B}})$;
- (R4) for every tail prime $p \ge P_5$, with $\gamma = 255/1024$: the law μ_p and its accepted part $\bar\nu_p = \nu_p/\nu_p(1)$, where ν_p is the factor at the one-prime atom $\{p\}$ of the comparison law ν of Definition 9.14 (at a tail prime the comparison law is nonnegative, Lemma 29.5), are (γ, v)-mixing (Definition 6.13) for every slot v; and for each query (u, v) of the query family $\mathcal{E}_k$ of Definition 9.1(c) (in Part V, $\mathcal{E}_6$) the (u, v)-marginal of $\bar\nu_p$ is $\mathrm{Unif}(\Omega_p)$ and $\bar\nu_p$ is $(\gamma, (u, v))$-mixing: its conditional operator from the pair (u, v) to the other 70 slots fixes constants, maps mean-zero functions to mean-zero functions and has norm at most $p^{-\gamma}$ on them;

(R5) under every accepted part, that is every normalized component $\bar{\nu}_{\mathcal{B},i} = \nu_{\mathcal{B},i}/\nu_{\mathcal{B},i}(1)$ of positive mass of the comparison law (Definition 9.14) (components of mass zero are omitted), almost surely each query pair is a strict-square pair at every prime of $\mathcal{B}$ with $e_p = 1$, and at a clique prime $p \in \mathcal{P}_1$ its difference is nonzero, of even valuation $2j < e_p$, with a square unit part.

Part V verifies these properties: (R1)–(R3) for the clique atoms by Lemma 19.2, for the middle blocks by Lemma 22.1, for the upper blocks by Lemma 27.17 and for the tail atoms by Lemma 29.2; (R4) is Lemma 29.3; (R5) is the acceptance property (a) of Definition 9.14, checked range by range and component by component; and the variation clause of (R3) is part of the admissibility of every selected law (Definition 8.9). Each of the four reference lemmas proves reflection positivity at every cut of $\mathfrak{F}$, not only at the sorting cuts. This is needed: the one-cut arguments of Parts III and V also use first-coordinate chamber cuts outside the sorting sequence, for instance $\vartheta_{0,2}$, with plus half $\{2,3,4\}$, which is neither a sorting cut nor a sorting cut with its halves exchanged.

6.3. **Averaging a lift over intervals and residues is exact.** The external index replaces the lift by averages over subintervals and residue classes. The next lemma says these averages return the original lift exactly, with no error term; this is why conditioning on the index preserves the mean α.

Lemma 6.3 (Index averages)**.** *Let I be a nonempty interval of integers and $I = J_1 \sqcup \cdots \sqcup J_k$ a partition of I into nonempty intervals. For every $f\colon I \to \mathbb{C}$,*

$$\mathcal{L}_{I,N} f = \sum_{j=1}^{k} \frac{|J_j|}{|I|}\, \mathcal{L}_{J_j,N}(f|_{J_j}).$$

Consequently, for $\mathsf{L} \geq 1$, $N = K\mathsf{L}$, $A \subseteq [N]$ and a positive divisor D_0 of M_0:

(i) *$K^{-1}\sum_{b\in\mathbb{Z}/K\mathbb{Z}} g^{I_b,D_0,s} = g^{[N],D_0,s}$ for every $s \in \mathbb{Z}/D_0\mathbb{Z}$;*

(ii) *if s is uniform on $\mathbb{Z}/D_0\mathbb{Z}$ and D' is a positive divisor of D_0, then $\mathbb{E}[g^{I,D_0,s} \mid s \bmod D'] = g^{I,D',s \bmod D'}$ for every $I \in \mathcal{I}_{\mathsf{L}}$; in particular ($D' = 1$) $\mathbb{E}_s g^{I,D_0,s} = \mathcal{L}_{I,N}(\mathbf{1}_A|_I)$;*

(iii) *if b and s are independent and uniform on $\mathbb{Z}/K\mathbb{Z}$ and on $\mathbb{Z}/D_0\mathbb{Z}$, then $\mathbb{E}_{b,s} g^{I_b,D_0,s} = F_A$ and $\mathbb{E}_s g^{[N],D_0,s} = F_A$, identically on G_N.*

These identities are exact and hold for every fixed $K \geq 2$.

Proof. For every $\xi \in \mathbb{Q}/\mathbb{Z}$, $\widehat{f}_I(\xi) = \sum_j (|J_j|/|I|)\widehat{(f|_{J_j})}_{J_j}(\xi)$, because the J_j partition I. The lift sums these coefficients over the set Ξ_N, which depends neither on the interval nor on the function, so it is linear in the coefficient vector and the identity follows. (i) Apply it to the partition of $[N]$ into the I_b, for which $|I_b|/N = 1/K$, and $f = D_0 \mathbf{1}_{n\equiv s}\mathbf{1}_A$. (ii) Let $t \in \mathbb{Z}/D'\mathbb{Z}$ and $n \in \mathbb{Z}$. Given $s \equiv t \pmod{D'}$, s is uniform among the D_0/D' residues modulo D_0 above t, of which exactly one is $\equiv n \pmod{D_0}$ if $n \equiv t \pmod{D'}$, and none otherwise; hence the conditional expectation of $D_0\mathbf{1}_{n\equiv s\,(D_0)}$ is $D'\mathbf{1}_{n\equiv t\,(D')}$. Multiply by $\mathbf{1}_A(n)$, restrict to I and apply the linear map $\mathcal{L}_{I,N}$. (iii) By (ii), $\mathbb{E}_s g^{I,D_0,s} = \mathcal{L}_{I,N}(\mathbf{1}_A|_I)$, which is F_A for $I = [N]$; averaging over b and using the partition identity gives $\mathbb{E}_{b,s} g^{I_b,D_0,s} = K^{-1}\sum_b \mathcal{L}_{I_b,N}(\mathbf{1}_A|_{I_b}) = F_A$. □

Only the linearity of the lift and the independence of Ξ_N from the interval and the input are used; the prime-power coordinates of G_N play no role. In the index law (Definition 9.6) the residue $s \in \mathbb{Z}/M_0\mathbb{Z}$ has prime-power coordinates by the Chinese remainder theorem. Revealing a residue to depth J_p at each prime is conditioning on $s \bmod D_{\boldsymbol{J}}$, where $D_{\boldsymbol{J}} = \prod_p p^{J_p}$. When every revealed depth J_p is either full (the whole p-coordinate of M_0) or zero, write W for the set of fully revealed coordinates and D_W for the corresponding modulus; in either case (ii) turns $g^{I,M_0,s}$ into $g^{I,D_{\boldsymbol{J}},s \bmod D_{\boldsymbol{J}}}$. For inputs with independent indices at distinct slots, such as the pairwise disjoint

endpoints of the queries, the expectation and the conditional expectations of the product are the products of those of the factors.

6.4. L^{2r} **bounds for lifts, linear in the density.** The moment bounds have three ingredients: a conditional Parseval bound for the pieces of a lift, a recursion for their square function, and decoupling over the prime coordinates (compare the general decoupling formalism for multilinear and U-statistic averages in [11]; the coordinate-colour argument needed here is proved directly below). We state them separately, so that the dependence on the conductor cutoff and on the number of prime factors remains visible. The first ingredient is arithmetic. A generic hypercontractive bound would not do: on $\mathbb{F}_p$ the mean-zero function $\mathbf{1}_{x=0}-1/p$ has $\|f\|_{2r}/\|f\|_2$ of order $p^{1/2-1/(2r)}$. The pieces of a lift come from an interval, and the Gram (large-sieve) bound uses this.

Lemma 6.4 (Gram bound and conditional square sums)**.** *Let I be an interval of integers and $Q\ge 1$.*

(a) If Ξ is a set of distinct elements of $\mathbb{Q}/\mathbb{Z}$ with reduced denominators at most Q, then for every $f\colon I\to\mathbb{C}$

$$\sum_{\xi\in\Xi}|\widehat{f}_I(\xi)|^2\le\Big(1+\frac{Q^4}{2|I|}\Big)\mathbb{E}_I|f|^2.$$

(b) Let $|I|\ge 10Q^6$ and $\|f\|_\infty\le M$, let $\mathcal{P}$ be a finite set of primes with $e_p=1$, and let $\mathcal{F}$ be a family of subsets $U\subseteq\mathcal{P}$ with $q_U\le Q$. For every $W\subseteq\mathcal{P}$ with $q_W\le Q$ and every $z_W\in X_W$, the pieces f_U of the lift of f on $X_{\mathcal{P}}$ satisfy

$$\sum_{U\in\mathcal{F},\,U\supseteq W}\mathbb{E}\big[|f_U|^2\;\big|\;z_W\big]\le 4M^2\,4^{|W|}. \tag{6.1}$$

Proof. (a) Distinct elements of $\mathbb{Q}/\mathbb{Z}$ with reduced denominators at most Q are at distance at least Q^{-2}, and there are at most $\sum_{q\le Q}\varphi(q)\le Q^2$ of them. Write $\|t\|$ for the distance from $t\in\mathbb{Q}/\mathbb{Z}$ to 0. For $\xi\ne\xi'$ in Ξ, $|I|^{-1}|\sum_{n\in I}e(n(\xi-\xi'))|\le(2|I|\,\|\xi-\xi'\|)^{-1}\le Q^2/(2|I|)$. So the Gram matrix G of the unit vectors $\phi_\xi=(|I|^{-1/2}e(n\xi))_{n\in I}$, $\xi\in\Xi$, has norm at most $1+Q^4/(2|I|)$ (Schur's test), and Bessel's inequality in Gram form, $\sum_\xi|\langle f,\phi_\xi\rangle|^2\le\|G\|\,\|f\|^2$, is the claim, since $|\langle f,\phi_\xi\rangle|^2=|I|\,|\widehat{f}_I(\xi)|^2$ and $\|f\|^2=|I|\mathbb{E}_I|f|^2$.

(b) Write $U=W\sqcup U'$ and $\xi=\xi_W+\eta$ with exact supports W and U'. After fixing z_W, the coefficient of $e(z_{U'}\cdot\eta)$ in f_U is

$$c_W(\eta;z_W)=\frac{1}{|I|}\sum_{n\in I}f(n)e(-n\eta)\prod_{p\in W}\big(p\mathbf{1}_{n\equiv z_p\,(p)}-1\big),$$

since $\sum_{a=1}^{p-1}e(a(z_p-n)/p)=p\mathbf{1}_{n\equiv z_p}-1$. By conditional Parseval the left side of (6.1) is $\sum_{\eta\in\Omega_W}|c_W(\eta;z_W)|^2$, summed over the union Ω_W of the sets of characters of exact support $U\setminus W$, $U\in\mathcal{F}$, $U\supseteq W$ (different U give disjoint sets); the elements of Ω_W have reduced denominators at most Q/q_W, all coprime to q_W. Expanding the product over the subsets $J\subseteq W$ gives $c_W=\sum_J(-1)^{|W\setminus J|}c_J$ with $c_J(\eta)=(q_J/|I|)\sum_{n\in I,\,n\equiv z_J\,(q_J)}f(n)e(-n\eta)$. Write $n=n_J+q_Jn'$ with n' in an interval T_J of length L_J, where $|L_J-|I|/q_J|\le 1$; since $|I|/q_J\ge 10Q^5$, $L_J\ge 0.9|I|/q_J$ and $q_JL_J\le 1.1|I|$. The frequencies $q_J\eta$, $\eta\in\Omega_W$, are distinct, because q_J is a unit modulo every denominator in Ω_W, and have reduced denominators at most Q. With $F_J(n')=f(n_J+q_Jn')$, $|c_J(\eta)|=(q_JL_J/|I|)\,|\widehat{(F_J)}_{T_J}(q_J\eta)|$, and (a) on T_J, with $1+Q^4/(2L_J)\le 1+Q^5/(1.8|I|)\le 19/18$, gives $(\sum_\eta|c_J(\eta)|^2)^{1/2}\le 1.1\sqrt{19/18}\,M\le 2M$. The triangle inequality over the $2^{|W|}$ sets J gives $(\sum_\eta|c_W(\eta;z_W)|^2)^{1/2}\le 2^{|W|}\cdot 2M$, which is (6.1). □

Lemma 6.5 (Square-function recursion)**.** *Let $X=\prod_{p\in\mathcal{P}}X_p$ be a finite product of finite probability spaces, $\mathcal{F}$ a finite family of subsets of $\mathcal{P}$ of size at most k, and $h_U\colon X\to\mathbb{C}$ $(U\in\mathcal{F})$ functions such that h_U depends only on z_U. Suppose that for some $c\ge 0$, every $W\subseteq\mathcal{P}$ contained in a member of $\mathcal{F}$, and every z_W,*

$$\sum_{U\in\mathcal{F},\,U\supseteq W}\mathbb{E}\big[|h_U|^2\mid z_W\big]\le c\,4^{|W|}.$$

Put $S=(\sum_{U\in\mathcal{F}}|h_U|^2)^{1/2}$ and $T_j=\mathbb{E}S^{2j}$, so $T_0=1$. Then $T_j\le c\,4^k2^{(j-1)k}\,T_{j-1}$ for every $j\ge1$; hence, for all integers $0\le j_0\le r$,

$$T_r\le(c\,4^k)^{r-j_0}\,2^{k(r(r-1)-j_0(j_0-1))/2}\,T_{j_0}.$$

Proof. Let $j\ge1$. Expand $S^{2j}=\sum_{U_1,\dots,U_j}\prod_i|h_{U_i}|^2$ and fix $U_2,\dots,U_j$, with union $\mathcal{U}$, so $|\mathcal{U}|\le(j-1)k$. The product $\prod_{i\ge2}|h_{U_i}|^2$ depends only on $z_{\mathcal{U}}$, and since the coordinates are independent, $\mathbb{E}[|h_{U_1}|^2\mid z_{\mathcal{U}}]=\mathbb{E}[|h_{U_1}|^2\mid z_W]$ with $W=U_1\cap\mathcal{U}$. Grouping the supports U_1 by W, which takes at most $2^{|\mathcal{U}|}\le2^{(j-1)k}$ values, each contained in a member of $\mathcal{F}$ and of size at most k, gives pointwise $\sum_{U_1}\mathbb{E}[|h_{U_1}|^2\mid z_{\mathcal{U}}]\le2^{(j-1)k}c\,4^k$. Multiplying by $\prod_{i\ge2}|h_{U_i}|^2$, taking expectations and summing over $U_2,\dots,U_j$ gives $T_j\le c4^k2^{(j-1)k}T_{j-1}$. Iterate from j_0+1 to r, using $\sum_{j=j_0+1}^{r}(j-1)=(r(r-1)-j_0(j_0-1))/2$. □

For the pieces of a lift with $\|f\|_\infty\le M$, (6.1) gives the hypothesis with $c=4M^2$, and the case $j_0=0$ reads $\|S\|_{2r}\le2M\,2^{k(r+3)/4}$.

Lemma 6.6 (Decoupling by coordinate colours)**.** *Let $X=\prod_{p\in\mathcal{P}}X_p$ be a finite product of finite probability spaces, $\mathcal{F}$ a finite family of subsets of $\mathcal{P}$ of size at most k, and $f_U\colon X\to\mathbb{C}$ $(U\in\mathcal{F})$ functions such that f_U depends only on the coordinates in U and has mean zero in each of them. Put $S=(\sum_{U\in\mathcal{F}}|f_U|^2)^{1/2}$ and, for an integer $r\ge1$, $K_{2r}=((2r)!/(2^rr!))^{1/(2r)}$. Then*

$$\Big\|\sum_{U\in\mathcal{F}}f_U\Big\|_{2r}\le(k+1)(2eK_{2r})^k\|S\|_{2r}.$$

Proof. We use Khintchine's inequality $\|\sum a_i\epsilon_i\|_{2r}\le K_{2r}(\sum|a_i|^2)^{1/2}$ for independent uniform signs ϵ_i. For real a_i the even moment is at most that of $\sum a_i\mathrm{g}_i$ with independent standard Gaussians (expand: only even exponents survive, with nonnegative coefficients, and $\mathbb{E}\epsilon^{2m}=1\le\mathbb{E}\mathrm{g}^{2m}$), which is $(2r-1)!!(\sum a_i^2)^r=K_{2r}^{2r}(\sum a_i^2)^r$. For complex a_i apply this to the real and imaginary parts and Minkowski in L^r to $|X|^2=(\mathrm{Re}X)^2+(\mathrm{Im}X)^2$.

The same constant applies to Hilbert-space vectors v_i. Work in their finite-dimensional real span (realifying a complex Hilbert space), and let G be a standard Gaussian vector there. With $c_r=(2r-1)!!=K_{2r}^{2r}$, the scalar inequality and then Minkowski in L^r give

$$\mathbb{E}_\epsilon\Big\|\sum_i\epsilon_iv_i\Big\|^{2r}=c_r^{-1}\mathbb{E}_G\mathbb{E}_\epsilon\Big\langle G,\sum_i\epsilon_iv_i\Big\rangle^{2r}\le\mathbb{E}_G\Big(\sum_i\langle G,v_i\rangle^2\Big)^r\le c_r\Big(\sum_i\|v_i\|^2\Big)^r.$$

This also holds conditionally when the vectors depend on other variables.

Fix $j\ge1$ and colour the coordinates independently and uniformly with j colours; a set of size j is *transversal* if its elements have distinct colours, which happens with probability $\pi_j=j!/j^j$. Fix a colouring and let $\mathcal{T}$ be the transversal members of $\mathcal{F}$ of size j. Take independent copies X^0,X^1 of the coordinates and put $D_U=\prod_{p\in U}(I-\mathrm{swap}_p)f_U(X^0)$, where swap_p exchanges the p-coordinates of X^0 and X^1. Then $D_U=\sum_{U'\subseteq U}(-1)^{|U'|}f_U(X^{U'})$, where $X^{U'}$ takes the coordinates in U' from X^1 and the others from X^0; integrating out X^1 makes every term with $U'\neq\emptyset$ vanish, since f_U has mean zero in each coordinate of U. So $\mathbb{E}_{X^1}D_U=f_U(X^0)$, and by Jensen $\|\sum_{\mathcal{T}}f_U\|_{2r}\le\|\sum_{\mathcal{T}}D_U\|_{2r}$. Each swap_p preserves the joint law of (X^0,X^1) and changes the sign of D_U exactly when $p\in U$; hence $\|\sum_{\mathcal{T}}D_U\|_{2r}=\|\sum_{\mathcal{T}}\epsilon_UD_U\|_{2r}$, where $\epsilon_U=\prod_{p\in U}\epsilon_p$ for

independent uniform signs ϵ_p. A transversal U has exactly one coordinate $p_c(U)$ of each colour c, so $\epsilon_U = \prod_{c=1}^{j} \epsilon_{p_c(U)}$, and we integrate the signs one colour at a time. With the signs of colour 1 free and the rest fixed, $\sum_{\mathcal{T}} \epsilon_U D_U = \sum_p \epsilon_p A_p$ over the coordinates of colour 1, and Khintchine gives the bound $K_{2r} \| \sum_p |A_p|^2 \|_r^{1/2}$. Each A_p is in turn $\sum_{p'} \epsilon_{p'} A_{p,p'}$ over the coordinates of colour 2, and the Hilbert-space inequality applied to the vectors $(A_{p,p'})_p$, conditionally on all remaining variables, gives $\| \sum_p |A_p|^2 \|_r \le K_{2r}^2 \| \sum_{p,p'} |A_{p,p'}|^2 \|_r$. After the j colours,

$$\Big\| \sum_{\mathcal{T}} \epsilon_U D_U \Big\|_{2r} \le K_{2r}^{j} \Big\| \Big(\sum_{U \in \mathcal{T}} |D_U|^2 \Big)^{1/2} \Big\|_{2r}.$$

For transversal U the sets $U' \subseteq U$ correspond to sets C of colours, so $D_U = \sum_{C \subseteq [j]} (-1)^{|C|} f_U(X^{(C)})$, where $X^{(C)}$ takes the coordinates of the colours in C from X^1 and the others from X^0 and does not depend on U. As every $X^{(C)}$ has the law of X^0, Minkowski in $\ell^2(\mathcal{T})$ and in L^{2r} gives $\|(\sum_{\mathcal{T}} |D_U|^2)^{1/2}\|_{2r} \le 2^j \|S\|_{2r}$. So $\| \sum_{\mathcal{T}} f_U \|_{2r} \le (2K_{2r})^j \|S\|_{2r}$ for every colouring. Every U of size j is transversal with probability π_j, so $\sum_{|U|=j} f_U = \pi_j^{-1} \mathbb{E}_{\mathrm{col}} \sum_{\mathcal{T}} f_U$, and Minkowski over the colourings gives $\| \sum_{|U|=j} f_U \|_{2r} \le \pi_j^{-1} (2K_{2r})^j \|S\|_{2r} \le (2eK_{2r})^j \|S\|_{2r}$, as $j^j/j! \le e^j$. The term with $|U| = 0$ is constant and bounded in modulus by S. Sum over $0 \le j \le k$, using $2eK_{2r} \ge 1$. $\square$

The moment theorem combines the three lemmas. It keeps the ℓ^2 mass $\mathbb{E}_I |f|^2$ of the integer input, and with it the density.

Lemma 6.7 (Moments that retain the ℓ^2 mass)**.** *Let $\mathcal{S}' \subseteq \mathcal{P}_1$ and let $\mathcal{P}$ be a finite set of primes with $e_p = 1$; put $X = X_{\mathcal{S}'} \times X_{\mathcal{P}}$. Let $Q \ge 1$, let I be an interval with $|I| \ge 10Q^6$, and let $f \colon I \to \mathbb{C}$ with $\|f\|_\infty \le M$. For each character χ of $X_{\mathcal{S}'}$ with $q(\chi) \le Q$ let $\mathcal{F}_\chi$ be a family of sets $U \subseteq \mathcal{P}$ with $q(\chi) q_U \le Q$, and put*

$$G(z', z) = \sum_{\chi} e(z' \cdot \chi) \sum_{U \in \mathcal{F}_\chi} (f_\chi)_U(z), \qquad f_\chi(n) = f(n) e(-n\chi),$$

where $(f_\chi)_U$ is the piece of exact support U of the lift of f_χ on $X_{\mathcal{P}}$. We call such a G a whole-support piece *of the lift of f on X with cutoff Q. Let $k = \max\{|U| : U \subseteq \mathcal{P},\ q_U \le Q\}$, and let K_{2r} be as in Lemma 6.6.*

(a) $\|G\|_2^2 \le (1 + Q^4/(2|I|))\, \mathbb{E}_I |f|^2 \le \frac{21}{20} \mathbb{E}_I |f|^2$.

(b) For every integer $r \ge 1$,

$$\|G\|_{2r} \le D_{\mathcal{S}'} (k+1) (2eK_{2r})^k (2M)^{1-1/r} 2^{k(r^2+3r-4)/(4r)} \big(\tfrac{21}{20} \mathbb{E}_I |f|^2 \big)^{1/(2r)}.$$

(c) For fixed r and $\epsilon > 0$ (an auxiliary exponent, not the exponent of the main theorem) there is $C_{r,\epsilon}$, depending only on r, ϵ and Q_{cl}, with $\|G\|_{2r}^{2r} \le C_{r,\epsilon} Q^\epsilon M^{2r-2} \mathbb{E}_I |f|^2$. In particular, for $f = D_0 \mathbf{1}_{n \equiv s\,(D_0)} \mathbf{1}_A$ on I with $D_0 \mid M_0$, the right-hand side is at most $C_{r,\epsilon} Q^\epsilon M_0^{2r} \alpha_I$, where $\alpha_I = |A \cap I|/|I|$.

No division by $\mathbb{E}_I |f|^2$, M or a density occurs; if $\mathbb{E}_I |f|^2 = 0$ then $G = 0$.

The lift itself is such a piece: a character of G_N is $\chi + \xi$ with χ a character of $X_{\mathcal{P}_1}$ and ξ one of $X_{\mathcal{P}_3 \cup \mathcal{T}_N}$, $q(\chi+\xi) = q(\chi) q(\xi)$ and $\widehat{f_I}(\chi + \xi) = \widehat{(f_\chi)_I}(\xi)$; so $\mathcal{L}_{I,N} f$ is the whole-support piece with $\mathcal{S}' = \mathcal{P}_1$, $\mathcal{P} = \mathcal{P}_3 \cup \mathcal{T}_N$ and $\mathcal{F}_\chi = \{U : q(\chi) q_U \le Q\}$.

Proof. Step 1: the Gram bound. The Fourier coefficient of G at $\chi + \xi''$ is 0 or $\widehat{f_I}(\chi + \xi'')$, and the characters with a nonzero coefficient are distinct elements of $\mathbb{Q}/\mathbb{Z}$ with reduced denominators at most Q; so Parseval and Lemma 6.4(a) prove (a), as $1 + Q^4/(2|I|) \le 21/20$. Applied to f_χ, whose modulus is that of f, with the cutoff $Q/q(\chi)$, the same argument gives for each χ

$$E_\chi := \sum_{U \in \mathcal{F}_\chi} \|(f_\chi)_U\|_2^2 \le \tfrac{21}{20} \mathbb{E}_I |f|^2. \tag{6.2}$$

Step 2: one prefix character. Fix χ and write $h_U = (f_\chi)_U$, $S^2 = \sum_{U\in\mathcal{F}_\chi} |h_U|^2$ and $T_j = \mathbb{E}S^{2j}$, so $T_1 = E_\chi$. By (6.1) for f_χ, with $\|f_\chi\|_\infty \le M$ and $|I| \ge 10Q^6$, the hypothesis of Lemma 6.5 holds with $c = 4M^2$; the case $j_0 = 1$ gives $T_r \le (2M)^{2r-2}2^{k(r^2+3r-4)/2}E_\chi$, because $(4M^2 4^k)^{r-1}2^{kr(r-1)/2} = (2M)^{2r-2}2^{k(r-1)(r+4)/2}$. The h_U depend only on z_U and have mean zero in each coordinate of U, so Lemma 6.6 gives

$$\Big\|\sum_U h_U\Big\|_{2r} \le (k+1)(2eK_{2r})^k\, T_r^{1/(2r)} \le (k+1)(2eK_{2r})^k (2M)^{1-1/r}2^{k(r^2+3r-4)/(4r)}E_\chi^{1/(2r)}.$$

Step 3: the prefix. $|e(z'\cdot\chi)| = 1$, and there are at most $D_{\mathcal{S}'}$ characters χ. Minkowski over χ, Step 2 and (6.2) prove (b).

Step 4: asymptotics. A product of k distinct primes is at least $(k+1)!$, so $(k+1)! \le Q$ and $k = O(\log Q/\log\log Q)$. Hence every factor of (b) that is exponential in k is $Q^{o(1)}$ for fixed r, and $D_{\mathcal{S}'} \le Q_{\mathrm{cl}}$. Raising (b) to the power $2r$ gives the first part of (c). For $f = D_0\mathbf{1}_{n\equiv s}\mathbf{1}_A$ we have $M = D_0 \le M_0$ and $\mathbb{E}_I|f|^2 = D_0^2\#\{n\in A\cap I : n\equiv s\}/|I| \le M_0^2\alpha_I$. □

In Section 8 the lemma is applied to the inputs on G_N, whose specializations at fixed labels on an arbitrary set of atoms are integrated over the labels by Fubini, $\mathbb{E}_{z_\mathcal{W}}\|G(z_\mathcal{W},\cdot)\|_s^s = \|G\|_s^s$ for every set $\mathcal{W}$ of coordinates and every $s \ge 1$ (the fibres of a finite uniform product have equal mass); and to the lifts of rescaled sets on the complement of the selected atoms (Lemma 8.2(e)). A fixed label, without averaging, satisfies the explicit bound of the next corollary.

Corollary 6.8 (Conditional moments)**.** *In the setting of Lemma 6.7, let $\mathcal{W}\subseteq\mathcal{P}$, let $z_\mathcal{W}\in X_\mathcal{W}$, and fix arbitrary values on any subset of the coordinates of $X_{\mathcal{S}'}$; the bound below holds for every such choice. Then for every integer $r\ge 1$, with $A_r = 2eK_{2r}2^{(r+3)/4}$,*

$$\|G(z_\mathcal{W},\cdot)\|_{2r} \le D_{\mathcal{S}'}\, 2M\, 3^{|\mathcal{W}|}(k+1)A_r^k,$$

the norm being taken over the remaining coordinates with their uniform law. If $Q\le N^\beta$ and $q_\mathcal{W}\le N^{C_0}$ for a fixed C_0, the right-hand side is at most $C_{r,\epsilon}MN^\epsilon$ for every $\epsilon>0$, uniformly in the labels.

Proof. Fixing prefix coordinates multiplies the term of χ by a unimodular constant or leaves it a function of the free prefix coordinates, so by Minkowski over χ it suffices to treat one χ, with $h_U = (f_\chi)_U$. Group the supports by $U_0 = U\cap\mathcal{W}$. For fixed U_0 the functions $h'_{U'}(y) = h_{U_0\cup U'}(z_{U_0}, y)$, $U'\subseteq\mathcal{P}\setminus\mathcal{W}$, depend only on $y_{U'}$ and have mean zero in each coordinate of U'; for $W'\subseteq\mathcal{P}\setminus\mathcal{W}$ their conditional square sum over $U'\supseteq W'$ given $y_{W'}$ is part of the sum in (6.1) with pinned set $U_0\cup W'$, hence at most $4M^2 4^{|U_0|}4^{|W'|}$. Lemma 6.5 with $c = 4M^2 4^{|U_0|}$ and $j_0 = 0$, and Lemma 6.6, give $\|\sum_{U'} h'_{U'}\|_{2r} \le 2M2^{|U_0|}(k+1)A_r^k$. Summing over $U_0\subseteq\mathcal{W}$, with $\sum_{U_0}2^{|U_0|} = 3^{|\mathcal{W}|}$, proves the bound. A product of $|\mathcal{W}|$ distinct primes is at least $(|\mathcal{W}|+1)!$, so $3^{|\mathcal{W}|}$ and the factors exponential in k are $N^{o(1)}$. □

Remark 6.9 (Averages and suprema over fibres). The estimates of Section 8 use only averaged moments. Averaging the fibre norms over $z_\mathcal{W}$ returns the full norm exactly. For a general function a single fixed fibre can be larger by the factor $q_\mathcal{W}^{1/s}$, the reciprocal of the s-th root of the fibre probability $q_\mathcal{W}^{-1}$: $\|G(z_\mathcal{W},\cdot)\|_s \le q_\mathcal{W}^{1/s}\|G\|_s$; for a lift a fixed label costs only $3^{|\mathcal{W}|} = N^{o(1)}$. This does not bound a pinned lift by a pinned density, and a pinned lift need not vanish on an empty fibre (Remark 8.14).

6.5. **Tensor products preserve exact Fourier support.** A tensor product of operators that fix the constants on both sides preserves the orthogonal decomposition of a product space by exact nonconstant support, so its norm, the norm of a difference of two such products, and their symmetric and skew parts can be computed support by support.

Setting. Let $\mathcal{K}$ be a finite set. For $j \in \{0\} \cup \mathcal{K}$ let (Ω_j, π_j) and (Ω'_j, π'_j) be finite probability spaces, $L^2(\pi_j)$ the (real or complex) functions on the support of π_j, and $L^2_0(\pi_j) = \{f : \int f\, d\pi_j = 0\}$; put $\Omega = \prod_j \Omega_j$, $\pi = \bigotimes_j \pi_j$ and $\pi' = \bigotimes_j \pi'_j$. The index 0 is a *spectator*, possibly a one-point space. For $S \subseteq \mathcal{K}$ the *sector* of exact support S is

$$H_S(\pi) = L^2(\pi_0) \otimes \bigotimes_{j \in S} L^2_0(\pi_j) \otimes \bigotimes_{j \in \mathcal{K} \setminus S} \mathbb{C}\mathbf{1},$$

and X_S is the component of $X \in L^2(\pi)$ in $H_S(\pi)$, taken coordinatewise for maps into a finite-dimensional space. The sectors are mutually orthogonal and sum to $L^2(\pi)$, and for $W \subseteq \mathcal{K}$ the conditional expectation $\mathbb{E}_\pi[X \mid \omega_0, \omega_W]$ is the orthogonal projection onto $\bigoplus_{S \subseteq W} H_S(\pi)$, that is, it equals $\sum_{S \subseteq W} X_S$. A linear operator $T_j \colon L^2(\pi_j) \to L^2(\pi'_j)$ is *sector-preserving* if $T_j\mathbf{1} = \mathbf{1}$ and $T_j^*\mathbf{1} = \mathbf{1}$. Then T_j maps $L^2_0(\pi_j)$ into $L^2_0(\pi'_j)$, as $\langle \mathbf{1}, T_j f\rangle = \left\langle T_j^*\mathbf{1}, f\right\rangle = \langle \mathbf{1}, f\rangle$; we write T_j° for this restriction and $\delta_j = \|T_j^\circ\|$ (this local δ_j is a norm, not one of the depth tails $\delta_{p,j}$). The Markov operator $T_j f(x') = \mathbb{E}[f(x) \mid x']$ of a coupling of π'_j and π_j (a probability measure on $\Omega'_j \times \Omega_j$ with these marginals) is sector-preserving, with $\delta_j \le 1$, its maximal correlation; so is a doubly stochastic matrix on a finite set with the uniform law. The spectator operator $T_0 \colon L^2(\pi_0) \to L^2(\pi'_0)$ is any linear operator.

Lemma 6.10 (Sector lemma)**.** *In this setting let T_j ($j \in \mathcal{K}$) be sector-preserving and $T = T_0 \otimes \bigotimes_{j \in \mathcal{K}} T_j$; put $\delta_S = \prod_{j \in S} \delta_j$.*

(a) Sectors. *T maps $H_S(\pi)$ into $H_S(\pi')$ and equals $T_0 \otimes \bigotimes_{j \in S} T_j^\circ \otimes \bigotimes_{j \notin S} 1$ there, of norm $\|T_0\|\, \delta_S$; the same holds for T^*. Consequently, if R is a linear combination of such tensor products and their adjoints, all acting between the same two spaces, then R maps every sector into the corresponding sector and $\|R\| = \max_S \|R|_{H_S(\pi)}\|$; when $\pi' = \pi$ this applies to the symmetric and skew parts $\frac{1}{2}(R \pm R^*)$, and an $n \times n$ operator matrix whose entries are such combinations maps every $H_S(\pi) \otimes \mathbb{C}^n$ into itself; it is positive semidefinite if and only if each of these restrictions is.*

(b) Differences. *Let T'_j ($j \in \mathcal{K}$) be sector-preserving as well, with $\delta'_j = \|T_j'^\circ\|$, $\delta'_S = \prod_{j \in S} \delta'_j$, $\|T_j'^\circ - T_j^\circ\| \le d_j$ and $\max(\delta_j, \delta'_j) \le \kappa_j$, and $T' = T_0 \otimes \bigotimes_j T'_j$. Then $T' - T$ vanishes on $H_\emptyset(\pi)$ and, on $H_S(\pi)$,*

$$\|(T' - T)|_{H_S}\| \le \|T_0\| \min\Big(\sum_{j \in S} d_j \prod_{k \in S \setminus \{j\}} \kappa_k,\ \delta'_S + \delta_S \Big).$$

Let moreover $\pi' = \pi$ and $T_0 = I$. On $H_{\{j\}}$ the difference and its symmetric and skew parts are those of $T_j'^\circ - T_j^\circ$. If every T_j is self-adjoint, the skew part of $T' - T$ on H_S is that of $T'|_{H_S}$, of norm at most δ'_S. If $d_j \le d$ and $\kappa_j \le \kappa \le 1/2$ for all j, the first bound is at most $|S| d \kappa^{|S|-1} \le 2d\kappa \le d$ for $|S| \ge 2$; so if also $\|\mathrm{Sym}(T'_j - T_j)\| \le d_{\mathrm{s}}$ and $\|\mathrm{Skew}(T'_j - T_j)\| \le d_{\mathrm{a}}$ for all j, then

$$\|\mathrm{Sym}(T' - T)\| \le \max(d_{\mathrm{s}}, 2d\kappa), \qquad \|\mathrm{Skew}(T' - T)\| \le \max(d_{\mathrm{a}}, 2d\kappa), \qquad \|T' - T\| \le d.$$

More generally, with $\pi' = \pi$ and $T_0 = I$, the norms of $T' - T$ and of its symmetric and skew parts are the maxima over $S \ne \emptyset$ of the norms of their restrictions to H_S; on $H_{\{j\}}$ these restrictions are the corresponding parts of $T_j'^\circ - T_j^\circ$, and on H_S with $|S| \ge 2$ their norms are at most $\delta_S + \delta'_S$, which is at most $\delta_j\delta_k + \delta'_j\delta'_k$ for any $j \ne k$ in S when all $\delta_i, \delta'_i \le 1$. Hence, in that case,

$$\|\mathrm{Sym}(T' - T)\| \le \max\Big(\max_j \|\mathrm{Sym}(T_j'^\circ - T_j^\circ)\|,\ \max_{j \ne k} (\delta_j\delta_k + \delta'_j\delta'_k) \Big),$$

the second maximum being 0 *if* $|\mathcal{K}| = 1$*, and the same holds for* $\mathrm{Skew}(T' - T)$ *and for* $T' - T$*.*

(c) Markov tensor maximum. *Let the spectator be trivial,* $\mathcal{K} \neq \emptyset$*, and* $\delta_j \leq 1$ *for all* j *(for instance, Markov operators of couplings with exact marginals). Then* $\|T - \mathbb{J}\| = \max_j \delta_j$*, where* $\mathbb{J}f = (\int f\, d\pi)\mathbf{1}$*. If moreover* $\pi' = \pi$ *and each* T_j *is self-adjoint with spectrum in* $[-c_j, 1]$*,* $0 \leq c_j \leq 1$*, then* $T \succeq -(\max_j c_j)I$*.*

(d) Random inclusion. *Let* $\mathcal{J} \subseteq \mathcal{K}$ *be a random set containing each* j *independently with probability* $\delta_j \in [0,1]$*, where here the* δ_j *are any numbers (in the applications, bounds for the norms of (a)), and* $\delta_S = \prod_{j \in S} \delta_j$*. Let* E *be a finite-dimensional space with a positive semidefinite Hermitian form* M*, and* $\|Y\|_M^2 = \int M(Y(\omega), Y(\omega))\, d\pi(\omega)$ *for* $Y \colon \Omega \to E$*. Then components of different supports are orthogonal for this form, and for every* $X \colon \Omega \to E$

$$\mathbb{E}_{\mathcal{J}} \big\| \mathbb{E}_\pi[X \mid \omega_0, \omega_{\mathcal{J}}] \big\|_M^2 = \sum_{S \subseteq \mathcal{K}} \delta_S \|X_S\|_M^2.$$

(e) Exact-support decay. *For every* $X \in L^2(\pi)$*,* $\|TX\|^2 = \sum_S \|TX_S\|^2 \leq \|T_0\|^2 \sum_S \delta_S^2 \|X_S\|^2$*.*

(f) Product densities. *If the spectator is trivial and* $\varphi_j \in L^2(\pi_j)$ *with* $\int \varphi_j\, d\pi_j = 1$ *(*$j \in \mathcal{K}$*), then* $\| \bigotimes_j \varphi_j - \mathbf{1}\|^2 = \prod_j (1 + \|\varphi_j - \mathbf{1}\|^2) - 1$*.*

(g) Tensor spectrum *(no sector hypothesis). If* $A_1, \dots, A_r$ *are self-adjoint operators on finite-dimensional Hilbert spaces with spectra in* $[-c_i, 1 + b_i]$*, where* $b_i \geq 0$ *and* $0 \leq c_i \leq 1 + b_i$*, then the spectrum of* $\bigotimes_i A_i$ *lies in*

$$\Big[-\max_i \Big(c_i \prod_{k \neq i} (1 + b_k) \Big),\ \prod_i (1 + b_i) \Big];$$

in particular $\bigotimes_i A_i \succeq -(\max_i c_i) \prod_i (1 + b_i) I \succeq -(\max_i c_i) e^{\sum_i b_i} I$*.*

(h) Two supports. *Let* $\pi' = \pi$*,* $T_0 = I$*, let* T_j *and* T_j' *be as in* (b) *with every* T_j *self-adjoint, and write* $T_j' = T_j + S_j + A_j$ *with* $S_j = \mathrm{Sym}(T_j' - T_j)$ *and* $A_j = \mathrm{Skew}(T_j' - T_j)$*, so that* $T_j + S_j = \mathrm{Sym}\, T_j'$ *and* $A_j = \mathrm{Skew}\, T_j'$*. Then* S_j *and* A_j *vanish on the constants and map* $L_0^2(\pi_j)$ *into itself, and on* $H_{\{j,k\}}$*,* $j \neq k$*, with the identity on the spectator and on the constants of the other coordinates,*

$$\begin{aligned} \mathrm{Sym}(T' - T) &= T_j^\circ \otimes S_k^\circ + S_j^\circ \otimes T_k^\circ + S_j^\circ \otimes S_k^\circ + A_j^\circ \otimes A_k^\circ, \\ \mathrm{Skew}(T' - T) &= (T_j + S_j)^\circ \otimes A_k^\circ + A_j^\circ \otimes (T_k + S_k)^\circ; \end{aligned}$$

in particular the skew part of $T' - T$ *on* $H_{\{j,k\}}$ *has norm at most* $\|(T_j + S_j)^\circ\|\, \|A_k^\circ\| + \|A_j^\circ\|\, \|(T_k + S_k)^\circ\|$*.*

Proof. The setting. For $S \neq S'$ pick j in the symmetric difference; on pure tensors the factors at j are $\mathbf{1}$ and a function of mean zero, so $H_S \perp H_{S'}$. Expanding $L^2(\pi_j) = \mathbb{C}\mathbf{1} \oplus L_0^2(\pi_j)$ for $j \in \mathcal{K}$ shows that the sectors sum to $L^2(\pi)$ and that the functions of (ω_0, ω_W) form $\bigoplus_{S \subseteq W} H_S$.

(a) On a pure tensor $f_0 \otimes \bigotimes_{j \in S} f_j \otimes \bigotimes_{j \notin S} \mathbf{1}$ with $f_j \in L_0^2(\pi_j)$, T acts as $T_0 f_0 \otimes \bigotimes_{j \in S} T_j^\circ f_j \otimes \bigotimes_{j \notin S} \mathbf{1} \in H_S(\pi')$, and such tensors span $H_S(\pi)$; tensoring with $\mathbb{C}\mathbf{1}$ is isometric, and the norm of a tensor product of operators is the product of the norms. The adjoint $T^* = T_0^* \otimes \bigotimes_j T_j^*$ has the same form, each T_j^* being sector-preserving with $(T_j^*)^\circ = (T_j^\circ)^*$. A combination R is the orthogonal sum of its restrictions to the sectors, which are orthogonal on both sides; so is an operator matrix with such entries, on the spaces $H_S(\pi) \otimes \mathbb{C}^n$, and its quadratic form is the sum of theirs.

(b) By (a), $T' - T$ is $T_0 \otimes (\bigotimes_{j \in S} T_j'^\circ - \bigotimes_{j \in S} T_j^\circ) \otimes 1$ on H_S, and 0 on $H_\emptyset$, where both act as $T_0 \otimes 1$. Telescoping over an ordering $j_1, \dots, j_s$ of S,

$$\bigotimes_i T_{j_i}'^\circ - \bigotimes_i T_{j_i}^\circ = \sum_{i=1}^{s} T_{j_1}'^\circ \otimes \cdots \otimes T_{j_{i-1}}'^\circ \otimes (T_{j_i}'^\circ - T_{j_i}^\circ) \otimes T_{j_{i+1}}^\circ \otimes \cdots \otimes T_{j_s}^\circ,$$

of norm at most $\sum_i d_{j_i}\prod_{k\neq j_i}\kappa_k$; the triangle inequality gives the second bound. With $T_0 = I$ and $\pi' = \pi$, $T' - T$ and its adjoint preserve every H_S, so its symmetric and skew parts restrict to those of the restrictions; on $H_{\{j\}}$ the restriction is $T_j'^{\circ} - T_j^{\circ}$. If every T_j is self-adjoint, so is $T|_{H_S}$, and the skew part of $(T'-T)|_{H_S}$ is that of $\bigotimes_{j\in S} T_j'^{\circ}$, of norm at most δ_S'. For $s = |S| \geq 2$, $sd\kappa^{s-1} = \kappa d\, s\kappa^{s-2} \leq \kappa d\, s2^{2-s} \leq 2d\kappa \leq d$, and the three displayed bounds are the maxima over the sectors. For the final assertion of (b), (a) applies to the combinations $T'-T$, $\mathrm{Sym}(T'-T)$ and $\mathrm{Skew}(T'-T)$, whose restrictions to H_S are $(T'-T)|_{H_S}$ and its two parts. On H_S with $|S| \geq 2$ the second bound of the display gives $\delta_S + \delta_S'$, and the symmetric and skew parts of an operator have norm at most its own; for $j \neq k$ in S every factor of δ_S and of δ_S' other than those at j and k is at most one.

(c) $\mathbb{J} = \bigotimes_j \mathbb{J}_j$, with $\mathbb{J}_j f = (\int f\, d\pi_j)\mathbf{1}$, is a tensor product of sector-preserving operators with $\mathbb{J}_j^{\circ} = 0$; so it agrees with T on $H_\emptyset$ and vanishes on the other sectors, and $T - \mathbb{J}$ vanishes on $H_\emptyset$ and has norm $\delta_S \leq \min_{j\in S}\delta_j$ on H_S, $S \neq \emptyset$, with equality at singletons; by (a) its norm is $\max_j \delta_j$. If each T_j is self-adjoint with spectrum in $[-c_j, 1]$, then on H_S the operator $\bigotimes_{j\in S} T_j^{\circ}$ has eigenvalues $\prod_{j\in S}\lambda_j$, $\lambda_j \in [-c_j, 1]$; a negative product has a negative factor $\lambda_i \geq -c_i$ and other factors of modulus at most one, so it is at least $-\max_j c_j$. On $H_\emptyset$ the operator is the identity.

(d) For $X_S \in H_S \otimes E$, $X_{S'} \in H_{S'} \otimes E$ and $S \neq S'$, $\int M(X_S, X_{S'})\, d\pi = \sum_{a,b} M_{ab} \int \overline{X_{S,a}} X_{S',b}\, d\pi = 0$ in a basis of E. Hence $\|\mathbb{E}[X \mid \omega_0, \omega_W]\|_M^2 = \sum_{S\subseteq W} \|X_S\|_M^2$ for fixed W; averaging over $\mathcal{J}$ gives $\sum_S \mathbb{P}(S \subseteq \mathcal{J})\|X_S\|_M^2$, and $\mathbb{P}(S \subseteq \mathcal{J}) = \delta_S$.

(e) The images TX_S lie in the orthogonal sectors $H_S(\pi')$, and $\|TX_S\| \leq \|T_0\|\delta_S\|X_S\|$ by (a).

(f) With $\varphi_j = \mathbf{1} + c_j$, $c_j \in L_0^2(\pi_j)$, $\bigotimes_j \varphi_j - \mathbf{1} = \sum_{S\neq\emptyset} \bigotimes_{j\in S} c_j \otimes \bigotimes_{j\notin S} \mathbf{1}$, the term of S lying in H_S; by orthogonality the squared norm is $\sum_{S\neq\emptyset}\prod_{j\in S}\|c_j\|^2 = \prod_j(1 + \|c_j\|^2) - 1$.

(g) $\bigotimes_i A_i$ is self-adjoint with an orthonormal eigenbasis of product vectors and eigenvalues $\prod_i \lambda_i$, $\lambda_i \in [-c_i, 1 + b_i]$, so $|\lambda_i| \leq 1 + b_i$. A product is at most $\prod_i(1 + b_i)$, and a negative one has a negative factor $\lambda_i \geq -c_i$, so it is at least $-c_i \prod_{k\neq i}(1 + b_k)$. Finally $1 + b \leq e^b$.

(h) As T_j and T_j' are sector-preserving, $T_j' - T_j$ and its adjoint vanish on the constants, hence so do S_j, A_j and their adjoints; and $\langle \mathbf{1}, S_j f\rangle = \left\langle S_j^*\mathbf{1}, f\right\rangle = 0$, likewise for A_j, so both map $L_0^2(\pi_j)$ into itself. Thus $T_j'^{\circ} = T_j^{\circ} + S_j^{\circ} + A_j^{\circ}$ on $L_0^2(\pi_j)$, with T_j°, S_j° self-adjoint and A_j° skew. By (a), $T' - T$ acts on $H_{\{j,k\}}$ as $T_j'^{\circ} \otimes T_k'^{\circ} - T_j^{\circ} \otimes T_k^{\circ}$, with the identity on the other factors. Put $U_j = (T_j + S_j)^{\circ}$ and $U_k = (T_k + S_k)^{\circ}$, both self-adjoint. The adjoint of $(U_j + A_j^{\circ}) \otimes (U_k + A_k^{\circ})$ is $(U_j - A_j^{\circ}) \otimes (U_k - A_k^{\circ})$, so its symmetric part is $U_j \otimes U_k + A_j^{\circ} \otimes A_k^{\circ}$ and its skew part is $U_j \otimes A_k^{\circ} + A_j^{\circ} \otimes U_k$. The operator $T_j^{\circ} \otimes T_k^{\circ}$ is self-adjoint; subtracting it and expanding $U_j \otimes U_k - T_j^{\circ} \otimes T_k^{\circ}$ gives the two identities. The norm bound follows from the triangle inequality and $\|X \otimes Y\| = \|X\|\,\|Y\|$. □

Remark 6.11 (How the sector lemma is used). Part (c) is why the comparison records of Part V keep their actual marginals: Markov operators of couplings with exact marginals, taken in those marginals, have centred norm equal to the *maximum* of the centred norms of their factors under tensor products, not the sum or product of those norms. The maximal correlation of a product of couplings is the largest factor correlation. For self-adjoint Markov operators the product is bounded below by the minimum of zero and the smallest eigenvalues of its factors; if at least one factor has a negative eigenvalue, the smallest product eigenvalue equals the smallest factor eigenvalue. Operators normalized in a reference space are not Markov; when their factor spectra lie in $[-c_i, 1 + b_i]$, part (g) bounds every product eigenvalue above by $\prod_i(1 + b_i)$ and below by $-\max_i c_i \prod_{k\neq i}(1 + b_k)$ (Lemma 13.7). Part (b) prices a block difference: on a support of at least two primes the difference of two products is of second order, and when the old operator is self-adjoint only the new one contributes to the skew part. Parts (a), (d) and (e) are the whole content of the index estimates (Corollary 10.4(i), Lemma 10.1) and of conductor decay; part (f) says that for a product density $\bigotimes_j(\mathbf{1} + c_j)$ the squared L^2 distance from $\mathbf{1}$ is $\prod_j(1 + \|c_j\|^2) - 1$.

The closing clause of (b) prices a block whose prime differences have separate symmetric and skew coefficients (the upper comparisons of Section 28; for the signed fold records of Section 23, which are combinations of products, Remark 6.12 gives the same decomposition), and (h) is the two-prime step of the matching records, whose reference factor is self-adjoint (Lemma 23.18).

Remark 6.12 (Signed mixtures of products). Let the spectator be trivial, let $T_{i,j}$ (i in a finite index set, $j \in \mathcal{K}$) be sector-preserving, and put $K = \sum_i c_i \bigotimes_{j\in\mathcal{K}} T_{i,j}$ with real c_i, $\sum_i c_i = 1$. By (a), K maps every sector into the corresponding sector: it is the identity on $H_\emptyset$, it acts on $H_{\{j\}}$ as the combination $\sum_i c_i T^\circ_{i,j}$ at the single coordinate j, and on H_S its norm is at most $\sum_i |c_i| \prod_{j\in S} \|T^\circ_{i,j}\|$. The difference of two such combinations decomposes in the same way: it vanishes on $H_\emptyset$, it is the difference of the two coordinate combinations on $H_{\{j\}}$, and on H_S its norm is at most the sum of the two bounds; when $\pi' = \pi$, its symmetric and skew parts restrict to the sectors in the same way, with the same bound on H_S, and all these norms are maxima over the sectors. If the $T_{i,j}$ are the operators of couplings $\pi_{i,j}$ of π'_j and π_j, then K is the operator of the signed measure $\sum_i c_i \bigotimes_j \pi_{i,j}$, which has the marginals π' and π and total variation at most $\sum_i |c_i|$, whatever $|\mathcal{K}|$. A combination taken at every coordinate, $\bigotimes_j(\sum_i c_i T_{i,j})$, has the same marginals, but its total variation is the product of those of the coordinate combinations, for which only the bound $(\sum_i |c_i|)^{|\mathcal{K}|}$ is available. The fold records of the middle range are combinations of the first kind, of three products with $\sum_i |c_i| = 1 + \mathsf{t}$ (Definition 23.9).

6.6. **Fourier terms of large conductor are damped at the mixing primes $p \ge P_5$.** A Fourier term supported on many prime coordinates is small after conditioning on the other slots, provided each such prime supplies a mixing factor. The orthogonality of distinct Fourier supports lets these factors multiply without counting the supports. Reflection positivity is not an assumption in this step.

Definition 6.13 (Conditional mixing)**.** Let X be a finite set, ρ a probability measure on X^V, and $Z \subseteq V$ a nonempty block of slots, with ρ_Z the ρ-law of z_Z and λ the ρ-law of $z_{V\setminus Z}$. The *conditional operator* from Z to the other slots is $T^{\rho,Z} f(y) = \mathbb{E}_\rho[f(z_Z) \mid z_{V\setminus Z} = y]$, from $L^2(\rho_Z)$ to $L^2(\lambda)$; it fixes the constants, maps ρ_Z-mean-zero functions to λ-mean-zero functions and is a contraction. For a prime p and $\gamma > 0$, a probability measure ρ on $\mathbb{F}_p^V$ is (γ, Z)-*mixing* if $\|T^{\rho,Z} f\|_{L^2(\lambda)} \le p^{-\gamma} \|f\|_{L^2(\rho_Z)}$ for every f with $\int f \, d\rho_Z = 0$. For $Z = \{v\}$ we say (γ, v)-*mixing* (when ρ has uniform one-slot marginals, $\rho_{\{v\}} = \mathrm{Unif}(\mathbb{F}_p)$), and for a query pair $Z = \{u, v\}$, ordered as (u, v), $(\gamma, (u, v))$-*mixing*.

Lemma 6.14 (Mixtures)**.** *Let Z and Z' be finite sets and $\kappa = \sum_{i=1}^n \varpi_i \kappa_i$ with $\varpi_i > 0$, $\sum_i \varpi_i = 1$, where the κ_i are probability measures on $Z \times Z'$ with the same Z-marginal m. Let κ' and κ'_i be the Z'-marginals, and $T^\kappa f(z') = \mathbb{E}_\kappa[f(z) \mid z']$, T^{κ_i} likewise, for $f \in L^2(m)$. Then $T^\kappa f = \sum_i h_i T^{\kappa_i} f$ κ'-almost everywhere, with $h_i = \varpi_i \, d\kappa'_i / d\kappa'$ and $\sum_i h_i = 1$, and*

$$\|T^\kappa f\|^2_{L^2(\kappa')} \le \sum_i \varpi_i \|T^{\kappa_i} f\|^2_{L^2(\kappa'_i)}.$$

Each T^{κ_i}, and T^κ, fixes the constants, is a contraction and maps m-mean-zero functions to mean-zero functions. Consequently, if $\|T^{\kappa_i} f\| \le c_i \|f\|$ for every m-mean-zero f, then $\|T^\kappa f\|^2 \le \sum_i \varpi_i c_i^2 \|f\|^2$: a mixture of (γ, Z)-mixing laws with a common Z-marginal is (γ, Z)-mixing, and a component of weight ϖ costs at most $\varpi^{1/2}$ in the norm.

Proof. For z' with $\kappa'(z') > 0$, $T^\kappa f(z') = \sum_z f(z) \kappa(z, z') / \kappa'(z') = \sum_i (\varpi_i \kappa'_i(z') / \kappa'(z')) \, T^{\kappa_i} f(z')$, the terms with $\kappa'_i(z') = 0$ being absent. By convexity $|T^\kappa f|^2 \le \sum_i h_i |T^{\kappa_i} f|^2$ pointwise, and $\int h_i |T^{\kappa_i} f|^2 \, d\kappa' = \varpi_i \int |T^{\kappa_i} f|^2 \, d\kappa'_i$. Conditional Jensen gives $\|T^{\kappa_i} f\|^2_{L^2(\kappa'_i)} \le \int |f|^2 \, dm$, and $\int T^{\kappa_i} f \, d\kappa'_i = \int f \, dm$. □

The atoms below P_5, forming the finite low part of the functional, have no quantitative mixing bound in (R4); in the lemma below they contribute only the factor C'^{γ}. The conditional operator of the low part has norm at most one, and, in the notation of the lemma, a frequency of total conductor above H has conductor above H/C' at the tail primes $p \ge P_5$.

Lemma 6.15 (Conductor decay with a finite prefix)**.** *Let X' be a finite product of cyclic groups of prime-power order, and let C' be a positive integer divisible by the reduced denominator of every character of X'. Let $\mathcal{T}$ be a finite set of primes not dividing C'. Let ρ' be a probability measure on X'^{V} with uniform one-slot marginals. Fix a slot $v \in V$ and $\gamma > 0$, and for $p \in \mathcal{T}$ let ρ_p be a probability measure on $\mathbb{F}_p^V$ with uniform one-slot marginals that is (γ, v)-mixing at this same slot v (Definition 6.13). Put $\rho = \rho' \otimes \bigotimes_{p\in\mathcal{T}} \rho_p$. Let $g_w \colon X' \times X_{\mathcal{T}} \to \mathbb{C}$ $(w \in V)$ and $H \ge 1$, and suppose that $\widehat{g_v}(\xi) = 0$ whenever $q(\xi) \le H$. Then*

$$\Big| \int \prod_{w\in V} g_w(z_w)\, d\rho(z) \Big| \le \Big(\frac{C'}{H}\Big)^{\gamma} \|g_v\|_2 \prod_{w\ne v} \|g_w\|_{2(d-1)},$$

with norms for the uniform probability measure. If instead $\rho' = m\bar{\rho}'$ with $m \ge 0$ and $\bar{\rho}'$ as above, the right-hand side is multiplied by m.

Proof. The conditional operator. Let λ', λ_p be the laws of $z'_{V\setminus\{v\}}$ under ρ' and of $z_{p,V\setminus\{v\}}$ under ρ_p, and T', T_p the conditional operators of ρ', ρ_p from slot v (Definition 6.13). By conditional Jensen and the uniform marginal of z'_v,

$$\|T'f\|^2_{L^2(\lambda')} = \mathbb{E}_{\rho'} \big|\mathbb{E}_{\rho'}[f(z'_v) \mid z'_{V\setminus\{v\}}]\big|^2 \le \mathbb{E}_{\rho'} |f(z'_v)|^2 = \|f\|^2_{L^2(X')}, \tag{6.3}$$

so $\|T'\| \le 1$; no mixing of ρ' is used. Each T_p fixes the constants and $\int T_p f\, d\lambda_p = \mathbb{E}_{x\in\mathbb{F}_p} f(x)$, as $z_{p,v}$ is uniform; so T_p is sector-preserving, with $\|T_p^{\circ}\| \le p^{-\gamma}$ by mixing. The factors of ρ are independent, so the conditional law of z_v given $z_{V\setminus\{v\}}$ is the product of theirs, and the conditional operator of ρ from slot v is $T = T' \otimes \bigotimes_{p\in\mathcal{T}} T_p$, from the uniform space $X' \times X_{\mathcal{T}}$ to $L^2(\lambda)$, $\lambda = \lambda' \otimes \bigotimes_p \lambda_p$.

The support condition. Apply Lemma 6.10 with the spectator X', $T_0 = T'$, and the coordinates $p \in \mathcal{T}$. A character of $X' \times X_{\mathcal{T}}$ is $\xi' + \xi_{\mathcal{T}}$, with ξ' a character of X' and $\xi_{\mathcal{T}}$ of exact tail support $U \subseteq \mathcal{T}$; it lies in H_U, and since $q(\xi')$ divides C' and is coprime to $q(\xi_{\mathcal{T}}) = q_U$, its conductor is $q(\xi')q_U \le C'q_U$. The component $g_{v,U}$ of g_v in H_U collects the Fourier terms of g_v of tail support U; if it is nonzero, then $H < q(\xi) \le C'q_U$ for some such term, so $q_U > H/C'$. By Lemma 6.10(e),

$$\|Tg_v\|^2_{L^2(\lambda)} \le \sum_{U\colon q_U > H/C'} q_U^{-2\gamma} \|g_{v,U}\|_2^2 \le \Big(\frac{C'}{H}\Big)^{2\gamma} \|g_v\|_2^2.$$

Conclusion. Conditioning on $z_{V\setminus\{v\}}$, then applying Cauchy–Schwarz in $L^2(\lambda)$ and Hölder with $d-1$ factors,

$$\Big| \int \prod_{w} g_w(z_w)\, d\rho \Big| = \Big| \int (Tg_v)(y) \prod_{w\ne v} g_w(y_w)\, d\lambda(y) \Big| \le \|Tg_v\|_{L^2(\lambda)} \prod_{w\ne v} \Big(\mathbb{E}_{\lambda} |g_w(y_w)|^{2(d-1)}\Big)^{\frac{1}{2(d-1)}},$$

and the λ-law of y_w is the ρ-law of z_w, uniform on $X' \times X_{\mathcal{T}}$. The factor m follows by multilinearity. □

With X' trivial and $C' = 1$ this is the same estimate with no low part. The factor C'^{γ} from the low part does not decay with H, and all decay comes from the tail primes; the lemma is useful only because C' is bounded independently of the scale. Section 8 applies it with X' the product of the groups of the atoms below P_5 that are present, $C' = Q_{\mathrm{pre}}$, and $\mathcal{T}$ the tail primes $p \ge P_5$; (R3) and (R4) supply its hypotheses, and no mixing is used below P_5. The cancellation needs the same decay with a pair of slots as the source, and a truncation of the pair supports.

Lemma 6.16 (Decay by exact pair support)**.** *Let $\mathcal{T}$ be a finite set of primes, (u,v) an ordered pair of distinct slots and $\gamma>0$. For $p\in\mathcal{T}$ let ρ_p be a probability measure on $\mathbb{F}_p^V$ whose (u,v)-marginal is $\mathrm{Unif}(\Omega_p)$ and which is $(\gamma,(u,v))$-mixing, and put $\rho=\bigotimes_p\rho_p$. For $U\subseteq\mathcal{T}$ let E_U be the conditional expectation, under $\bigotimes_p\mathrm{Unif}(\Omega_p)$, onto the pair coordinates at the primes of U, and $q_U=\prod_{p\in U}p$. Then for $T_0\geq 1$, every pair function φ on $\prod_p\Omega_p$ and every function W of the labels of the other slots,*

$$\Big|\int\varphi(z_u,z_v)W(z_{V\setminus\{u,v\}})\,d\rho(z)\Big|^2\leq\|W\|_2^2\Big[T_0^{-2\gamma}\|\varphi\|_2^2+\sum_{q_U\leq T_0}q_U^{-2\gamma}\|E_U\varphi\|_2^2\Big],$$

with $\|\varphi\|_2$ and $\|E_U\varphi\|_2$ in $L^2(\bigotimes_p\mathrm{Unif}(\Omega_p))$ and $\|W\|_2$ in L^2 of the ρ-law λ of the other slots.

Proof. The factors of ρ are independent, so the conditional operator of ρ from the pair (u,v) to the other slots is $T=\bigotimes_p T_p$, with T_p that of ρ_p; each T_p fixes the constants and integrates against λ_p as against $\mathrm{Unif}(\Omega_p)$, so it is sector-preserving, with $\|T_p^\circ\|\leq p^{-\gamma}$. Conditioning on the other slots and Cauchy–Schwarz give $|\int\varphi W\,d\rho|\leq\|T\varphi\|_{L^2(\lambda)}\|W\|_2$, and $\|T\varphi\|^2\leq\sum_U q_U^{-2\gamma}\|\varphi_U\|_2^2$ by Lemma 6.10(e), with a trivial spectator. The supports with $q_U>T_0$ contribute at most $T_0^{-2\gamma}\|\varphi\|^2$. For the others, E_U is the orthogonal projection onto $\bigoplus_{S\subseteq U}H_S$, so φ_U is the U-component of $E_U\varphi$ and $\|\varphi_U\|_2\leq\|E_U\varphi\|_2$. □

Consequences and what to read next. The slot array, its cuts and its six queries (Definition 5.12 and Lemma 5.13), reflection positivity (Definition 5.1), sorting sequences (Definition 5.7) and the multilinear averages of Section 5, in the definitions only; the large sieve in Gram form and Khintchine's inequality, both proved here. The reference laws, the comparison law and admissibility (Hypothesis 6.2 and Definitions 8.9 and 9.14) are interfaces whose instances are proved in Part V.

7. Cancellation from square-difference-freeness: statement of the kernel estimate

What this section proves and how to read it. Square-difference-freeness of A produces cancellation in exactly one place of the analytic stage: a bound for a truncated Fourier pairing of two functions that live on single residue classes of a modulus D, on ordered intervals, with no difference a positive square. This section states that bound; elsewhere, in the rescaling of Section 8, only the fact that the property passes to progressions with square step is used. The bound is applied in the accepted-pair cancellation (Lemma 8.10), and through it in the wrap of the signed index form (Lemma 10.6). The proof, a quadratic circle-method argument, is in Appendix C. The statement carries symbolic exponents because the cutoff of the functional and the range of the kernel have to match: the cancellation applies the kernel at the cutoff $H=N^{1/40}$ with moduli up to $N^{1/20}$, and a kernel bound proved for a smaller cutoff does not apply to it. The proof is the same for every admissible choice of exponents, so it is given once.

Definition 7.1 (Kernel notation)**.** For $\xi\in\mathbb{Q}/\mathbb{Z}$ let $q(\xi)$ be its reduced denominator, and let

$$g(\xi)=\frac{1}{\varphi(q(\xi))}\sum_{m\in(\mathbb{Z}/q(\xi)\mathbb{Z})^\times}e(\xi m^2),\qquad g(0)=1,$$

be the *unit quadratic average* at ξ; thus $|g(\xi)|\leq 1$, and for squarefree odd $q(\xi)$ this is the product formula of Lemma 4.11(d). For $H\geq 1$ and any positive integer D put

$$\Xi_{H,D}=\{\xi\in\mathbb{Q}/\mathbb{Z}:\ q(\xi)\leq H,\ q(\xi)\text{ squarefree},\ (q(\xi),D)=1\}.$$

When $D=8d$ with d odd, strict square classes modulo D are those of Definition 4.5; by the paragraph after that definition they are exactly the squares of units modulo D, so a strict square class is the square of a unit modulo every odd prime power dividing D. For an integer $N\geq 1$ and a finitely supported $F\colon\mathbb{Z}\to\mathbb{C}$ write $\widehat{F}(\xi)=N^{-1}\sum_n F(n)e(-n\xi)$ and $\|F\|_{\ell^2}=(\sum_n|F(n)|^2)^{1/2}$.

Three exponents enter the proof: the modulus satisfies $D \lesssim N^{\kappa}$, the cutoff is $H \asymp N^{\beta/2}$, and the major arcs of the circle method are the neighbourhoods (of the radius fixed in Appendix C) of the fractions with denominator at most N^{η}; the saving is $H^{-1/3} \asymp N^{-\beta/6}$.

Definition 7.2 (Kernel exponents)**.** A triple (β, κ, η) of positive numbers is *admissible for the kernel* if the following seven strict inequalities hold:

(E1) $\kappa + \frac{\beta}{2} - \frac{\eta}{2} < -\frac{\beta}{6}$, (E2) $\kappa + \beta - \eta < -\frac{\beta}{6}$, (E3) $\eta + 2\kappa + \frac{3\beta}{2} - 1 < -\frac{\beta}{6}$,
(E4) $2\eta + 2\kappa + \beta - \frac{1}{2} < -\frac{\beta}{6}$, (K0) $3\eta < 1$, (K1) $\kappa + \frac{\beta}{2} < \eta$,
(K2) $2\eta + \kappa + \frac{\beta}{2} < 1$.

Here (K0) makes the major arcs disjoint, (K1) puts every centre of the model kernel (Lemma C.3) on a major arc, and (K2) separates distinct centres; (E1)–(E4) say that the four error terms of the circle method (the actual kernel on the minor arcs, the model kernel on the minor arcs, and the model and the actual kernel on the major arcs) are smaller than the saving $N^{-\beta/6}$.

Proposition 7.3 (The actual-square kernel)**.** *Let (β, κ, η) be admissible for the kernel, and fix $C_D, C_H \geq 1$. There are C and N_0, depending only on β, κ, η, C_D and C_H, with the following property. Let $N \geq N_0$, let $D = 8d \leq C_D N^{\kappa}$ with d odd (not necessarily squarefree), and let $C_H^{-1} N^{\beta/2} \leq H \leq C_H N^{\beta/2}$. Let $F, J \colon \mathbb{Z} \to \mathbb{C}$ be supported in $[1, N]$, in residue classes a and b modulo D. Suppose that $x < y$ for all $x \in \operatorname{supp} F$ and $y \in \operatorname{supp} J$, and that no such difference $y - x$ is the square of a positive integer.*

(i) If $b - a$ is a strict square class modulo D, then

$$\Big| D \sum_{\xi \in \Xi_{H,D}} g(\xi) \widehat{F}(-\xi) \widehat{J}(\xi) \Big| \leq C\, N^{-\beta/6}\, \frac{\|F\|_{\ell^2} \|J\|_{\ell^2}}{N}. \tag{7.1}$$

(ii) Let $\mathcal{P}_0$ be a finite set of odd primes and $m_p \geq 1$ ($p \in \mathcal{P}_0$) integers such that the exponent of p in D is exactly $2m_p$. Suppose that $b - a \equiv 1 \pmod 8$, that $b - a$ is a nonzero square modulo every prime of d outside $\mathcal{P}_0$, and that at each $p \in \mathcal{P}_0$ either $b - a$ is a unit square modulo p, or $v_p(b - a) = 2j_p$ with $1 \leq j_p < m_p$ and the unit part of the representative a nonzero square modulo p (where $v_p(b - a)$ is the valuation of any integer representative of the class $b - a$ modulo p^{2m_p}, independent of the choice when it is $< 2m_p$). Then (7.1) holds with the same C and N_0 as in (i), as a special case of (iii). The corresponding rescaling identity is as follows. With $R = \prod_p p^{2j_p}$ over the primes of $\mathcal{P}_0$ at which $v_p(b - a) \geq 2$, $D_R = D/R$ and $N' = \lceil N/R \rceil$, the expression inside the absolute value in (7.1) (the pairing*) equals $R(N'/N)^2$ times the corresponding expression with D replaced by D_R, at length N', for the functions F', J' on $[1, N']$ constructed in Lemma C.9, to which (i) applies with the constants $C_D R^{\kappa}$ and $C_H R^{\beta/2}$.*

(iii) With the condition $D = 8d$, d odd, replaced by $D = 2^{h_2} \prod_{p \text{ odd}} p^{h_p} \leq C_D N^{\kappa}$, $h_2 \geq 3$, (7.1) holds whenever $b - a$ is a nonvanishing square class (Definition 4.6). The constants C and N_0 may be taken depending only on $\beta, \kappa, \eta, C_D, C_H$, as in (i). The root-class construction of Lemma C.1 supplies for these classes the same arithmetic coefficient and the same period and Fourier-coefficient bounds as for the strict classes of (i), for the proof in Appendix C.

In particular, if $|F|, |J| \leq 1$ and F, J are supported in intervals I_1, I_2, the right-hand side of (7.1) is at most $C N^{-\beta/6} N^{-1} ((|I_1|/D + 1)(|I_2|/D + 1))^{1/2}$.

The paper applies the proposition at one triple.

Corollary 7.4 (The kernel at $\beta = 1/20$)**.** *The triple $(\beta, \kappa, \eta) = (1/20, 1/20, 17/100)$ is admissible for the kernel. The slacks of (E1), (E2), (E3), (E4), (K0), (K1), (K2), that is, the differences*

between the right-hand and the left-hand sides, are

$$\frac{1}{600},\quad \frac{37}{600},\quad \frac{97}{150},\quad \frac{1}{600},\quad \frac{49}{100},\quad \frac{19}{200},\quad \frac{117}{200}.$$

Hence Proposition 7.3 holds with $D \le C_D N^{1/20}$, $C_H^{-1} N^{1/40} \le H \le C_H N^{1/40}$ and the saving $N^{-\beta/6} = N^{-1/120}$. These seven numbers are (S6) of Certificate D.2, checked there in exact rational arithmetic; this is the form of the kernel that Lemma 8.10 uses.

Proof. With $\beta = \kappa = 1/20$ and $\eta = 17/100$, so that $\beta/2 = 1/40$ and $\beta/6 = 1/120$, the seven slacks are, over the common denominator 600,

$$\tfrac{51-30-15-5}{600} = \tfrac{1}{600},\quad \tfrac{102-30-30-5}{600} = \tfrac{37}{600},\quad \tfrac{600-102-60-45-5}{600} = \tfrac{388}{600} = \tfrac{97}{150},$$
$$\tfrac{300-204-60-30-5}{600} = \tfrac{1}{600},\quad 1-\tfrac{51}{100} = \tfrac{49}{100},\quad \tfrac{17}{100}-\tfrac{3}{40} = \tfrac{19}{200},\quad 1-\tfrac{34}{100}-\tfrac{3}{40} = \tfrac{117}{200}.$$

All are positive, and $\beta/6 = 1/120$. □

How square-difference-freeness enters. The multiplier $g(\xi)$ is the average of $e(\xi m^2)$ over the units modulo $q(\xi)$. Three hypotheses of the proposition are the form in which square-difference-freeness is used: F and J are supported in single residue classes a and b modulo D whose difference is a nonvanishing square class (strict in (i), of controlled valuation in (ii)); every $x \in \operatorname{supp} F$ is smaller than every $y \in \operatorname{supp} J$; and no difference $y-x$ is the square of a positive integer. In the applications F and J are built from a square-difference-free set A, restricted to two adjacent intervals and to two residue classes that differ by a square, and this makes the third hypothesis true. The proof compares two kernels on the positive integers: a model kernel k_0, the multiplier $\sum_{\xi\in\Xi_{H,D}} g(\xi)e(-\xi n)$ restricted to $1 \le n \le N$ and to the class $n \equiv b-a \pmod D$, and an actual-square kernel k, supported on the squares m^2, $1 \le m \le \sqrt{N}$, weighted by the root-class weights of period dividing D and by the truncated indicator, expanded in additive characters in Definition C.2, that the root is a unit at every prime outside D up to the cutoff H. The root classes of Lemma C.1 give one construction for all three parts of the proposition. Put $B_h(F,J) = N^{-1}\sum_{x,n} h(n)F(x)J(x+n)$. The ordering and the class condition give $B_{k_0}(F,J) = D\sum_{\xi\in\Xi_{H,D}} g(\xi)\widehat{F}(-\xi)\widehat{J}(\xi)$, the pairing of the proposition, while $B_k(F,J) = 0$ because no difference of the supports is a positive square. Plancherel bounds $|B_{k_0-k}(F,J)|$ by $N^{-1}\sup_\alpha |S_0(\alpha)-S(\alpha)|\,\|F\|_{\ell^2}\|J\|_{\ell^2}$, where $S_0(\alpha) = \sum_n k_0(n)e(\alpha n)$ and $S(\alpha) = \sum_n k(n)e(\alpha n)$ are the Fourier series of the two kernels, and the circle method shows that this supremum is $O(H^{-1/3})$ with a constant depending only on $\beta, \kappa, \eta, C_D, C_H$. This is the only place in the analytic stage where the defining property of A yields a gain. The argument is a quadratic circle-method input in the tradition of Davenport and Vaughan [10, 72]; for the large-sieve and Fourier background see also [46, 33]. It follows [52, Prop. 6.1], with prime-power moduli, a modulus and a cutoff that are only comparable to powers of N, the division of (ii) and symbolic exponents.

The ℓ^2 form and the division. No bound on $|F|$ or $|J|$ is needed for (7.1). The ℓ^2 norms make the estimate homogeneous of degree one in each input, which is what lets the pair cancellation of Lemma 8.10 retain the counts of A in the two residue classes in the two intervals, and so gives a remainder bounded by a constant multiple of the density. A zero difference is not covered by (ii), and it does not occur under (R5): the exceptional clique records, on which a query difference vanishes, are removed from the comparison law, and Lemma 19.3 bounds the price of this removal, which is counted in the symmetric part only. Every divided prime keeps an exponent $2m_p - 2j_p \ge 2$ in D_R, so D_R has the same prime divisors as D and the frequency set is unchanged, $\Xi_{H,D_R} = \Xi_{H,D}$. The factor $R(N'/N)^2$ in (ii) is essential: without it the two pairings differ by the factor $R(N'/N)^2$ of (ii) itself, and equality of the ratios $D/|I|$ and $(D/R)/(|I|/R)$ for an interval I and its rescaling alone is not the normalized Fourier identity of (ii).

The range of the exponents. With $-\beta/6$ moved to the left, each of the seven conditions says that an affine function of (β, κ) with nonnegative coefficients is smaller than a number depending only on η. So with $\kappa = \beta$ and $\eta = 17/100$ they hold for every $\beta \in (0, 1/20]$, since they hold at $\beta = 1/20$: the proposition contains the kernel at every smaller cutoff exponent, in particular at $\beta \le 1/36$ with squarefree d. For $\kappa = \beta$ the admissible η form the window $(10\beta/3,\ 1/4 - 19\beta/12)$, nonempty only for $\beta < 3/59$ (Remark C.10); so $\beta = 1/20$ is close to the largest cutoff exponent that this method allows when the modulus is as large as the cutoff.

Consequences and what to read next. This section has used the strict square classes of Definition 4.5 and the product formula of Lemma 4.11(d) only in its definitions. The proof of the proposition is Appendix C.

8. Induction on the scale: reducing pinned averages to smaller instances

What this section proves and how to read it. This section proves that, at a least counterexample, every pinned average (an average with labels fixed at a few atoms and inputs restricted to residue classes, called a *structured energy*) is at most its price times Y, and that the induction then converts $\Theta < 1$ into $s(N) \le CN^{1-a/71}$ in the baseline ($\theta = 1$). The mechanism is the single rescaling already used in Section 2: fixing labels restricts A to shorter square-step progressions, where the induction hypothesis holds. Inputs: $Y = \mathcal{Y}(\mathsf{L}, A)$ of Definition 6.1 with atom laws satisfying (R1)–(R5). Outputs: the pinned-average bound, the rules (*admissible for the energy estimates*, Definition 8.9—one of three qualified uses of "admissible" in the paper) for applying sharp bounds without counting a price twice, negligibility of forced pairs and of large pinnings, and Theorem 8.12. Part III uses only these outputs. A first reading should follow the rescaling in Section 8.2 and the induction in Section 8.6; the folded-energy order of operations can be read as a specified order (sharp bound, then average over shared labels, then rescale).

Remainders linear in the density. Two features of the estimates matter. First, every analytic remainder is $O(\mathsf{L}^{-\sigma}\alpha)$, with the density $\alpha = |A|/N$ to the first power. The density enters through the moment bounds of Lemma 6.7, which are linear in α: an L^{2r} norm of an input costs $\alpha^{1/(2r)}$, and the density exponents of every Hölder split used below add up to exactly one ($1/2 + 71/142$ in the comparison, $2 \cdot 2/4$ and $70 \cdot 2/140$ in the cancellation, $72/72$ for large products). At a counterexample, where $Y > C_0\mathsf{L}^{-a}\alpha^{\theta}$, a remainder $C\mathsf{L}^{-\sigma}\alpha$ is at most $(C/C_0)\mathsf{L}^{a-\sigma}\alpha^{1-\theta}Y$, uniformly in α, and the induction closes at the density exponent $\theta = 1$ under the single condition $a < \sigma$. An absolute remainder $C\mathsf{L}^{-\sigma}$ would be small compared with Y only if α were not too small, which forces a sparseness condition and $\theta < 1$ (Remark 8.15).

Second, the averaging is done in a fixed order. A sharp operator bound is applied at fixed shared labels, with a constant that does not depend on them; the Cauchy–Schwarz inequality is then integrated over the law of the shared labels; and only then is the comparison applied, to the two resulting unconditional energies (Lemma 8.8). No comparison is pointwise in the selected labels, and no conditional distribution of the non-shared labels of one half, given the shared labels, is assumed to be uniform.

The induction hypothesis is used only through the rescaling of Lemma 8.2. Fixing the labels at a set of atoms fixes a residue class; splitting the class into progressions with a square step returns the same problem at a shorter scale, because a square-difference-free set meets a progression of square step in a rescaled square-difference-free set.

8.1. Numerical exponents and the inequalities they must satisfy.

The exponents of the analytic estimates are fixed once.

Definition 8.1 (Analytic stage)**.** $\beta = 1/20$ is the exponent of the functional; $\eta = 17/100$ is the major-arc exponent of Appendix C; $\gamma = 255/1024$ is the one-slot and pair mixing exponent at the tail primes $p \ge P_5$; selected sets of field primes have product at most N^{τ}, $\tau = 1/201$; $T_0 = N^{t_{\mathrm{an}}}$,

$t_{\mathrm{an}} = 1/60$, is the truncation of pair supports in the cancellation; $H = N^{\beta/2} = N^{1/40}$ is the outside cutoff of the comparison and of the kernel; $\sigma = 1/825$ is the error exponent; $\delta = 10^{-10}$ is the moment loss per slot and $\ell_0 = 10^{-8}$ the moment loss of the large-product estimate. A *constant* depends only on these parameters, on K, M_0, Q_{pre}, the reference laws and the moment constants of Lemma 6.7; it never depends on L, A, the selected atoms, their labels, the intervals or the residues.

The stage inequalities. Certificate D.2 (Appendix D) checks in exact rational arithmetic that these parameters satisfy the following strict inequalities:

(S1) $\gamma\beta/2 - \tau - 72\delta - \sigma > 0$;
(S2) $\gamma t_{\mathrm{an}} - 72\delta - \sigma > 0$;
(S3) $\beta/6 - t_{\mathrm{an}}(1/2 - \gamma) - 72\delta - \sigma > 0$;
(S4) $\tau/4 - \ell_0 - \sigma > 0$;
(S5) $1 - 2\tau - 6\beta > 0$, $\beta/2 - \tau > 0$, $\beta/2 - t_{\mathrm{an}} > 0$ and $\beta(1 - 2\tau) - \beta/2 > 0$;
(S6) the seven kernel slacks of Lemma C.5, computed in Corollary 7.4, are positive;
(S7) the tail coefficients and the tail gates of Lemma 29.3 hold; only the gate at P_5 is used.

Here (S1) is used in the comparison, where only the weaker $\gamma\beta/2 - 72\delta > \sigma$ is needed for an individual averaged estimate (Lemma 8.6); its full slack also absorbs the sum of raw comparison errors in Proposition 32.9. Conditions (S2) and (S3) are used in the accepted-pair cancellation (Lemma 8.10), where they are the two saving exponents $\gamma t_{\mathrm{an}} = 17/4096$ and $1/120 - t_{\mathrm{an}}(1/2 - \gamma) = 17/4096$ less the moment loss; (S4) in the large-product estimate (Lemma 8.11); the four inequalities of (S5), in this order, give the moment hypothesis at the target scale, complete selected fibres ($HD_E \le Q$ when $q_E \le N^{\tau}$), complete pair fibres and the kernel modulus ($HN^{t_{\mathrm{an}}}Q_{\mathrm{pre}} \le Q$), and the target cutoff ($n^{\beta} \ge H$), through Lemma 8.3; (S6) is the admissibility of the kernel triple of Corollary 7.4, used in Appendix C, and (S7) is used in Lemma 29.3. None of (S1)–(S7) involves K or the clique prefix. Apart from the slot prices ξ_v and the density factors κ_v of Definition 8.5, in which K, divisors of M_0 and the clique powers p^{e_p} appear explicitly, the fixed integers K, M_0, Q_{cl} and Q_{pre} enter the estimates of this section only through constants, through the term $u_{\mathsf{L}} \to 0$ and through the starting scale of Lemma 8.3; this is possible because every inequality is strict.

8.2. **Residue-class restriction and square-step rescaling.** For a set E of atoms of $\mathcal{A}_N$ we write X_E for the product of their groups, $D_E = |X_E|$ for its full order, q_E for the product of the primes of the atoms of E with $e_p = 1$, and $E_{\mathrm{cl}} = E \cap \mathcal{P}_1$ for its clique primes. Thus $D_E = q_E \prod_{p \in E_{\mathrm{cl}}} p^{e_p} \le q_E Q_{\mathrm{cl}}$. For a divisor D_0 of M_0 put $D = D_0 D_E$. The two factors are coprime: the primes of M_0 are 2 and the primes $p \equiv 3 \pmod 4$ below Q_3, none of which is a prime of an atom. The *square step* of E and D_0 is the least perfect square divisible by D (4.1),

$$R = \mathrm{sq}(D) = \mathrm{sq}(D_0)\, q_E^2 \prod_{p \in E_{\mathrm{cl}}} p^{e_p}; \tag{8.1}$$

the second form holds because sq is multiplicative on coprime factors, q_E is squarefree and every e_p is even. The primes of R are those of D, so R is coprime to every prime of an atom outside E.

Lemma 8.2 (Exact fibres and the square-step comparison input)**.** *Let $\mathsf{L} \ge 2$, $N = K\mathsf{L}$, let $A \subseteq [N]$ be square-difference-free and let $E \subseteq \mathcal{A}_N$, with $\mathcal{S} = \mathcal{A}_N \setminus E$. Let $I \in \mathcal{I}_{\mathsf{L}}$, $D_0 \mid M_0$, $s \in \mathbb{Z}/D_0\mathbb{Z}$, $z \in X_E$ and $g = g^{I,D_0,s}$. Put $D = D_0 D_E$, let $\mathfrak{a} \bmod D$ be the class with $\mathfrak{a} \equiv s \pmod{D_0}$ and $\mathfrak{a} \equiv z_p \pmod{p^{e_p}}$ for the primes p of E, and let*

$$\alpha_{\mathrm{f}} = \frac{D}{|I|}\, \#\{x \in A \cap I :\ x \equiv \mathfrak{a} \pmod{D}\}$$

be the density of A in the fibre of z.

(a) (Exact fibre.) *For every $f : I \to \mathbb{C}$ and every character ξ of $X_{\mathcal{S}}$ with $q(\xi)D_E \le Q$, the coefficient at ξ of the specialization $(\mathcal{L}_{I,N}f)(z,\cdot)$ is $(D_E/|I|)\sum_{x\in I,\, x\equiv z\,(D_E)} f(x)e(-x\xi)$, where $x \equiv z\ (D_E)$ means $x \equiv z_p \pmod{p^{e_p}}$ for each prime p of E. In particular*

$$\widehat{g(z,\cdot)}(\xi) = \frac{D}{|I|}\sum_{\substack{x\in A\cap I\\ x\equiv \mathfrak{a}\,(D)}} e(-x\xi).$$

(b) (Square step.) *Let $R = \mathrm{sq}(D)$ be the square step of E and D_0, $\ell = \lceil |I|/(KR)\rceil$, $n = K\ell$ and $\lambda = Rn/|I|$. Write $I = c_I + [|I|]$, $A_I = (A\cap I) - c_I \subseteq [|I|]$ and $J_{\mathfrak{a}} = \{j \in [R] : j \equiv \mathfrak{a} - c_I \pmod{D}\}$ (so $|J_{\mathfrak{a}}| = R/D$), and for $j \in J_{\mathfrak{a}}$ put $c_j = j - R$,*

$$A'_j = \{n' \in [n] :\ Rn' + c_j \in A_I\}, \qquad T_j(y) = R^{-1}(y - c_I - c_j) \quad (y \in X_{\mathcal{S}}),$$

$h_j = \mathcal{L}^{\mathcal{S}}_{[n],n^\beta}\mathbf{1}_{A'_j}$, the lift of $\mathbf{1}_{A'_j}$ on $X_{\mathcal{S}}$ with cutoff n^β, and

$$\widetilde{g} = \frac{\lambda D}{R}\sum_{j\in J_{\mathfrak{a}}} h_j \circ T_j,$$

the square-step model *of $g(z,\cdot)$. Each A'_j is square-difference-free, each T_j is a map of the form in (R3), and*

$$\frac{D}{R}\sum_{j\in J_{\mathfrak{a}}}\alpha_j = \frac{\alpha_{\mathrm{f}}}{\lambda}, \qquad \alpha_j = \frac{|A'_j|}{n}.$$

If $H \le n^\beta$ and $HD_E \le Q$, then $\widetilde{g}$ and $g(z,\cdot)$ have the same Fourier coefficient at every character ξ of $X_{\mathcal{S}}$ with $q(\xi) \le H$.

(c) (Restoration.) *$Z_{\mathcal{S}}(h_j) \le \mathcal{Y}(\ell, A'_j)$ for every $j \in J_{\mathfrak{a}}$.*

(d) (Scales.) *$1 \le \lambda < 1 + KR/|I|$ and $\ell \ge \mathsf{L}/\xi_I$ with $\xi_I = (N/|I|)R$. Moreover $\ell < \mathsf{L}$, unless $E = \emptyset$, $D_0 = 1$ and $I = [N]$.*

(e) (Moments.) *If $n \ge 10n^{6\beta}$, then for every integer $r \ge 1$ and every $\epsilon > 0$, $\|\widetilde{g}\|_{2r}^{2r} \le C_{r,\epsilon}N^\epsilon\lambda^{2r-1}\alpha_{\mathrm{f}}$, with the constant $C_{r,\epsilon}$ of Lemma 6.7(c).*

Proof. (a) Every prime of D_0 divides M_0 and none divides D_E, so the class $\mathfrak{a}$ exists and is unique. Write a character of G_N as $\xi + \zeta$ with ξ a character of $X_{\mathcal{S}}$ and ζ one of X_E; their conductors are coprime, so $q(\xi+\zeta) = q(\xi)q(\zeta)$, and $q(\zeta)$ divides D_E. If $q(\xi)D_E \le Q$, every character ζ of X_E occurs together with ξ in the lift, so the coefficient of $e(y\cdot\xi)$ in $(\mathcal{L}_{I,N}f)(z,\cdot)$ is

$$\frac{1}{|I|}\sum_{x\in I} f(x)e(-x\xi)\sum_{\zeta} e((z-x)\cdot\zeta) = \frac{D_E}{|I|}\sum_{\substack{x\in I\\ x\equiv z\,(D_E)}} f(x)e(-x\xi),$$

by orthogonality of the characters of each cyclic group $\mathbb{Z}/p^{e_p}\mathbb{Z}$. For $f = D_0\mathbf{1}_{x\equiv s\,(D_0)}\mathbf{1}_A$ the Chinese remainder theorem gives the second formula, and it holds also when the fibre is empty. Completeness of the selected group is necessary here: for the cyclic group $\mathbb{Z}/9\mathbb{Z}$ with cutoff 3, $I = [1,9]$, $A = \{9\}$ and label 3, the retained characters have numerators $0, 3, 6$ and the specialized lift equals $1/3$, although the fibre $\{x \equiv 3\ (9)\} \cap A$ is empty.

(b) Since D divides R, exactly R/D residues $j \in [R]$ satisfy $j \equiv \mathfrak{a} - c_I \pmod{D}$. The map $(n', j) \mapsto Rn' + c_j = R(n'-1) + j$ is a bijection $[n]\times[R] \to [Rn]$, and $[|I|] \subseteq [Rn]$ because $Rn \ge |I|$; restricted to $j \in J_{\mathfrak{a}}$ it is a bijection onto the integers of $[Rn]$ in the class $\mathfrak{a} - c_I \bmod D$. The integers of $(|I|, Rn]$ do not lie in A_I, so the padding contributes nothing, and $\sum_{j\in J_{\mathfrak{a}}}|A'_j| = \#\{x \in A\cap I : x \equiv \mathfrak{a}\ (D)\}$. Therefore $(D/R)\sum_j \alpha_j = D\,\#\{x \in A\cap I : x \equiv \mathfrak{a}\ (D)\}/(Rn) = \alpha_{\mathrm{f}}|I|/(Rn) = \alpha_{\mathrm{f}}/\lambda$. If $n'_1 < n'_2$ lie in A'_j and $n'_2 - n'_1 = k^2$ with $k \ge 1$, then two elements of A differ by $Rk^2 = (\sqrt{R}\,k)^2$, which is impossible because R is a perfect square. The integer R is coprime to every prime of $\mathcal{S}$, and as R is a perfect square, R^{-1} is a square unit modulo every p^{e_p}, p a prime of $\mathcal{S}$; so T_j is of the

form in (R3). Let ξ be a character of $X_{\mathcal{S}}$ with $q(\xi) \le H$. Multiplication by the unit R permutes the characters of $X_{\mathcal{S}}$ and preserves conductors, and $e(T_j(y)\cdot R\xi) = e((y - c_I - c_j)\cdot\xi)$. Since $q(R\xi) = q(\xi) \le H \le n^\beta$, the character $R\xi$ is retained in h_j, and the coefficient of $e(y\cdot\xi)$ in $\widetilde{g}$ is

$$\frac{\lambda D}{R}\sum_{j\in J_{\mathfrak{a}}} e(-(c_I+c_j)\xi)\,\frac{1}{n}\sum_{n'\in A'_j} e(-n'R\xi) = \frac{\lambda D}{Rn}\sum_{\substack{x\in A\cap I\\ x\equiv \mathfrak{a}\,(D)}} e(-x\xi) = \frac{D}{|I|}\sum_{\substack{x\in A\cap I\\ x\equiv \mathfrak{a}\,(D)}} e(-x\xi).$$

By (a), which applies since $q(\xi)D_E \le HD_E \le Q$, this is the coefficient of $g(z,\cdot)$.

(c) Let $\mathcal{A}_n$ be the atoms of the functional at the scale ℓ, whose group has cutoff n^β ($K\ell = n$); so $\mathcal{A}_n \subseteq \mathcal{A}_N$, and put $E' = E \cap \mathcal{A}_n$. A character with $q \le n^\beta$ involves only primes at most n^β, and every such prime of $\mathcal{S}$ lies in an atom of $\mathcal{A}_n$ (the atoms of $\mathcal{P}_1 \cup \mathcal{P}_3$ belong to every scale), so h_j depends only on the coordinates of $\mathcal{S}' = \mathcal{S}\cap\mathcal{A}_n = \mathcal{A}_n \setminus E'$, and $Z_{\mathcal{S}}(h_j) = Z_{\mathcal{S}'}(h_j)$ by Fubini (Lemma 5.17(d)). Let $u_j = \mathcal{L}_{[n],n}\mathbf{1}_{A'_j}$ be the full lift at scale n. Averaging u_j over the coordinates of E' deletes exactly the characters that are nonconstant on an atom of E' and leaves the others; the remaining characters are those of h_j, with the same coefficients. Hence $\mathbb{E}_{E'}u_j = h_j$. Averaging over atoms does not increase Z (Lemma 5.17(d)), and the atom laws at scale ℓ are those at scale L by (R1), so

$$Z_{\mathcal{S}'}(h_j) = Z_{\mathcal{A}_n}(\mathbb{E}_{E'}u_j) \le Z_{\mathcal{A}_n}(u_j) = \mathcal{Y}(\ell, A'_j).$$

(d) $n = K\lceil |I|/(KR)\rceil$ lies in $[|I|/R, |I|/R+K)$, which gives the bounds for λ, and $\ell \ge |I|/(KR) = \mathsf{L}/\xi_I$. If $D > 1$, then $R \ge 4$, because R is a perfect square, and $\ell \le \lceil N/(4K)\rceil = \lceil \mathsf{L}/4\rceil < \mathsf{L}$. If $D = 1$, then $E = \emptyset$, $D_0 = 1$ and $R = 1$; for $I = I_b$, $\ell = \lceil \mathsf{L}/K\rceil < \mathsf{L}$, since $K \ge 2$ and $\mathsf{L} \ge 2$.

(e) T_j preserves the uniform measure, so by Minkowski $\|\widetilde{g}\|_{2r} \le \lambda\frac{D}{R}\sum_j \|h_j\|_{2r}$. Each h_j is a whole-support piece of the lift of $\mathbf{1}_{A'_j}$ on $X_{\mathcal{S}}$ with cutoff n^β, and $|[n]| \ge 10(n^\beta)^6$, so $\|h_j\|_{2r}^{2r} \le C_{r,\epsilon}N^\epsilon\alpha_j$ by Lemma 6.7(c) and $n \le N$. Concavity of $x \mapsto x^{1/(2r)}$ and (b) give $\|\widetilde{g}\|_{2r} \le \lambda(C_{r,\epsilon}N^\epsilon\alpha_{\mathrm{f}}/\lambda)^{1/(2r)}$. □

The hypotheses of (b) and (e) hold for every selection used below, from a fixed starting scale on. We collect once all the conditions on the scale that the comparison and the cancellation need.

Lemma 8.3 (Scale conditions)**.** *There is $\mathsf{L}_* \ge 2$, depending only on the analytic stage, K, M_0, Q_{cl} and Q_{pre}, such that for every $\mathsf{L} \ge \mathsf{L}_*$, with $N = K\mathsf{L}$, $Q = N^\beta$, $H = N^{\beta/2}$ and $T_0 = N^{t_{\mathrm{an}}}$:*

(i) $\mathsf{L} \ge 10N^{6\beta}$;
(ii) $HQ_{\mathrm{cl}}N^\tau \le Q$;
(iii) $n_0 := N^{1-2\tau}/(KM_0^2Q_{\mathrm{cl}})$ *satisfies* $n_0^\beta \ge H$ *and* $n_0 \ge 10N^{6\beta}$;
(iv) $K^2M_0^2Q_{\mathrm{cl}}N^{2\tau-1} \le 1$;
(v) $N^\beta \ge P_5$, *so that every prime of* $[P_4, P_5)$ *is a coordinate of* G_N;
(vi) $HQ_{\mathrm{pre}}T_0 \le Q$ *and* $M_0Q_{\mathrm{pre}}T_0 \le \min(N^{1/20}, \mathsf{L})$;
(vii) N *is at least the thresholds* N_0 *and* N_0' *of Proposition 7.3(i)–(iii) at the triple of Corollary 7.4,* $(\beta,\kappa,\eta) = (1/20, 1/20, 17/100)$, *with* $C_D = C_H = 1$, $\mathcal{P}_0 = \mathcal{P}_1$ *and* $m_p = e_p/2$.

Consequently, let $E \subseteq \mathcal{A}_N$ with $q_E \le N^\tau$, $D_0 \mid M_0$ and $I \in \mathcal{I}_{\mathsf{L}}$, and let R, n and λ be as in Lemma 8.2(b). Then $R \le M_0^2Q_{\mathrm{cl}}N^{2\tau}$, $HD_E \le Q$, $n_0 \le n \le N$, $n^\beta \ge H$ and $n \ge 10n^{6\beta}$; and $1 \le \lambda \le 2$ and $\lambda^{72} \le 1 + u_{\mathsf{L}}$, where

$$u_{\mathsf{L}} = \bigl(1 + K^2M_0^2Q_{\mathrm{cl}}N^{2\tau-1}\bigr)^{72} - 1$$

tends to zero as $\mathsf{L} \to \infty$. In particular the hypotheses of Lemma 8.2(b) and (e) hold.

Proof. Each of (i)–(iv) and (vi) compares two powers of N, up to fixed constants, with a positive gap in the exponents: $1-6\beta > 0$ for (i), $\beta/2-\tau > 0$ for (ii), $\beta(1-2\tau)-\beta/2 > 0$ and $1-2\tau-6\beta > 0$ for (iii), $1-2\tau > 0$ for (iv), and $\beta/2 - t_{\mathrm{an}} > 0$, hence also $\beta - t_{\mathrm{an}} > 0$ and $1 - t_{\mathrm{an}} > 0$, for (vi); all of them follow from (S5). Conditions (v) and (vii) are fixed thresholds. So all seven hold for large L.

For the consequences, $\mathrm{sq}(D_0) \le D_0^2 \le M_0^2$ and $\prod_{p\in E_{\mathrm{cl}}} p^{e_p} \le Q_{\mathrm{cl}}$ give $R \le M_0^2 Q_{\mathrm{cl}} N^{2\tau}$, and $D_E \le q_E Q_{\mathrm{cl}}$ with (ii) gives $HD_E \le Q$. Next $n = K\lceil |I|/(KR)\rceil \le K\lceil N/K\rceil = N$, and $n \ge |I|/R \ge (N/K)/(M_0^2 Q_{\mathrm{cl}} N^{2\tau}) = n_0$ because $|I| \ge \mathsf{L} = N/K$; by (iii), $n^\beta \ge n_0^\beta \ge H$ and $n \ge n_0 \ge 10N^{6\beta} \ge 10n^{6\beta}$. Finally Lemma 8.2(d) gives $1 \le \lambda < 1 + KR/|I| \le 1 + K^2 M_0^2 Q_{\mathrm{cl}} N^{2\tau-1}$, which is at most 2 by (iv), and raising to the power 72 gives $\lambda^{72} \le 1 + u_{\mathsf{L}}$. □

Conditions (i)–(iv) are used in the comparison (Lemma 8.6 and Proposition 8.7), (i) and (v)–(vii) in the cancellation (Lemma 8.10), and (i) in the large-product estimate (Lemma 8.11).

8.3. **Pinned averages at a least counterexample.** All energy estimates are made at a least counterexample to the bound that the induction proves.

Definition 8.4 (Least normalized counterexample)**.** Let $a > 0$, $0 < \theta \le 1$ and $C_0 \ge 1$. A pair (L, A), with $A \subseteq [K\mathsf{L}]$ square-difference-free, is a *least normalized counterexample* for (a, θ, C_0) if $\mathcal{Y}(\mathsf{L}, A) > C_0 \mathsf{L}^{-a}\alpha^\theta$, and $\mathcal{Y}(\ell, A') \le C_0 \ell^{-a}\alpha_{A'}^\theta$ for every $\ell < \mathsf{L}$ and every square-difference-free $A' \subseteq [K\ell]$, where $\alpha_{A'} = |A'|/(K\ell)$ and $0^\theta = 0$. Then $\alpha > 0$, since $F_\emptyset = 0$.

Definition 8.5 (Structured energies and prices)**.** Let $a > 0$ and $0 < \theta \le 1$ be the exponents of the induction. Let $\mathsf{L} \ge 1$, $N = K\mathsf{L}$, let $A \subseteq [N]$ be square-difference-free and $\alpha = |A|/N > 0$. A *structured energy* at (L, A) is given by the following data.

(a) A set $E \subseteq \mathcal{A}_N$ of *selected* atoms with $q_E \le N^\tau$; put $\mathcal{S} = \mathcal{A}_N \setminus E$.
(b) A finite signed measure ρ on X_E^V whose variation $|\rho|$ has every one-slot marginal equal to $\|\rho\|\,\mathrm{Unif}(X_E)$, where $\|\rho\| = |\rho|(X_E^V)$.
(c) For each $v \in V$ a divisor $D_{0,v}$ of M_0 and a length $M_v \in \{\mathsf{L}, N\}$, and a probability law $\mathbf{P}$ on index vectors $\omega = (I_v, s_v)_{v\in V}$ with $I_v \in \mathcal{I}_{\mathsf{L}}$, $|I_v| = M_v$ and $s_v \in \mathbb{Z}/D_{0,v}\mathbb{Z}$.

The energy is

$$\mathfrak{E} = \mathbb{E}_{\omega\sim\mathbf{P}} \int \Lambda_{\mu_{\mathcal{S}}}\big((g_v^\omega(z_v,\cdot))_{v\in V}\big)\, d\rho(z), \qquad g_v^\omega = g^{I_v, D_{0,v}, s_v}.$$

A slot is *unchanged* if $E = \emptyset$, $D_{0,v} = 1$ and $M_v = N$ (then $g_v^\omega = F_A$), and *changed* otherwise. With $R_E = q_E^2 \prod_{p\in E_{\mathrm{cl}}} p^{e_p}$, the slot prices are

$$\xi_v = \frac{N}{M_v}\,\mathrm{sq}(D_{0,v})R_E, \qquad \bar\alpha_v = \mathbb{E}_{\mathbf{P}}\,\frac{D_{0,v}}{M_v}\#\{x \in A \cap I_v : x \equiv s_v \ (D_{0,v})\}.$$

The *main price* is $\Pi_{\mathrm{main}} = \prod_v \xi_v^{a/72} \prod_{v\ \mathrm{changed}} (\bar\alpha_v/\alpha)^{\theta/72}$. If $\kappa_v \ge 1$ are numbers with $\bar\alpha_v \le \kappa_v\alpha$ for every changed slot, the *coarse price* is $\Pi = \prod_v \xi_v^{a/72} \prod_{v\ \mathrm{changed}} \kappa_v^{\theta/72}$; thus $\Pi \ge \max(1, \Pi_{\mathrm{main}})$.

One may take $\kappa_v = 1$ if (I_v, s_v) is uniform on $\{I_1, \dots, I_K\} \times \mathbb{Z}/D_{0,v}\mathbb{Z}$, or if $M_v = N$ and s_v is uniform; $\kappa_v = K$ if s_v is uniform given a fixed interval I_v, since then $\bar\alpha_v = \alpha_{I_v} \le K\alpha$; and always $\kappa_v = M_0 K$. Since $\mathrm{sq}(D_{0,v})R_E = \mathrm{sq}(D_{0,v}D_E)$, the price ξ_v is ξ_{I_v} of Lemma 8.2(d): for $M_v = N$ it is the square step of E and $D_{0,v}$, and for $M_v = \mathsf{L}$ it is K times it. If a clique prime p belongs to E, its contribution to $\prod_v \xi_v^{a/72}$ over the 72 slots is $p^{e_p a}$; a prefix atom not in E contributes no factor.

Lemma 8.6 (Averaged comparison with a density factor)**.** *There is a constant C_{dc} with the following property. Let $\mathsf{L} \ge \mathsf{L}_*$ (Lemma 8.3) and consider a structured energy at (L, A). Put $H = N^{1/40}$. For every ω, $z \in X_E^V$ and changed slot v, let $\widetilde{g}_v^\omega(z_v,\cdot)$ be the square-step model of Lemma 8.2(b) for $(E, I_v, D_{0,v}, s_v, z_v)$; for an unchanged slot put $\widetilde{g}_v^\omega = F_A$. Let $\widetilde{\mathfrak{E}}$ be the energy with $\widetilde{g}_v^\omega$ in place of g_v^ω. Then*

$$\mathbb{E}_{\mathbf{P}} \int \big|\Lambda_{\mu_{\mathcal{S}}}((g_v^\omega(z_v,\cdot))_v) - \Lambda_{\mu_{\mathcal{S}}}((\widetilde{g}_v^\omega(z_v,\cdot))_v)\big|\, d|\rho|(z)$$
$$\le C_{\mathrm{dc}}\|\rho\| N^{-\gamma\beta/2+72\delta}\alpha \le C_{\mathrm{dc}}\|\rho\|\mathsf{L}^{-\sigma}\alpha;$$

in particular $|\mathfrak{E}-\widetilde{\mathfrak{E}}|\le C_{\mathrm{dc}}\|\rho\|\mathsf{L}^{-\sigma}\alpha$.

Proof. Step 0: the scale. By Lemma 8.3, applied to E and each $D_{0,v}$, I_v, the hypotheses $H\le n^{\beta}$, $HD_E\le Q$ and $n\ge 10n^{6\beta}$ of Lemma 8.2(b),(e) hold at every changed slot, and $1\le\lambda_v\le 2$.

Step 1: telescoping. Fix ω and z, and let $v_1,\dots,v_k$ be the changed slots. Then $\Lambda((g_v)_v)-\Lambda((\widetilde g_v)_v)=\sum_{i=1}^{k}\Lambda(Y^{(i)})$, where $Y^{(i)}$ carries $\widetilde g_{v_j}$ at v_j for $j<i$, $g_{v_i}-\widetilde g_{v_i}$ at v_i, g_{v_j} at v_j for $j>i$, and F_A at the unchanged slots (all specialized at z). By Lemma 8.2(b), the input at v_i has no character of conductor at most H. The law $\mu_{\mathcal S}$ is the product of the laws of the atoms of $\mathcal S$ below P_5, a probability measure with uniform one-slot marginals by (R3), and of the tail laws μ_p, $p\ge P_5$, which satisfy the mixing hypothesis (R4) at every slot (in the notation of Definition 6.13). Every character of the first factor has conductor dividing Q_{pre}. By Lemma 6.15 with $C'=Q_{\mathrm{pre}}$,

$$|\Lambda(Y^{(i)})|\le\Big(\frac{Q_{\mathrm{pre}}}{H}\Big)^{\gamma}\|(g_{v_i}-\widetilde g_{v_i})(z_{v_i},\cdot)\|_2\prod_{w\ne v_i}\|Y_w^{(i)}(z_w,\cdot)\|_{142}.$$

Step 2: averaging. Integrate over $\omega\sim\mathbf P$ and $z\sim|\rho|/\|\rho\|$ and apply Hölder's inequality with the exponents 2 and 142 (71 times), so that $1/2+71/142=1$:

$$\mathbb E\Big[\|X\|_2\prod_w\|Y_w\|_{142}\Big]\le(\mathbb E\|X\|_2^2)^{1/2}\prod_w(\mathbb E\|Y_w\|_{142}^{142})^{1/142}.$$

Under $|\rho|/\|\rho\|$ each z_w is uniform on X_E and independent of ω. Hence, by Fubini and Lemma 6.7(c) (the lift on G_N is a whole-support piece of the kind treated there, with $\mathcal S'=\mathcal P_1$, and $|I_w|\ge\mathsf L\ge 10N^{6\beta}$), for $2r\in\{2,142\}$ and every $\epsilon>0$,

$$\mathbb E\|g_w^{\omega}(z_w,\cdot)\|_{2r}^{2r}=\mathbb E_\omega\|g_w^{\omega}\|_{L^{2r}(G_N)}^{2r}\le C_{r,\epsilon}N^{\epsilon}M_0^{2r}\,\mathbb E_\omega\alpha_{I_w}\le C_{r,\epsilon}N^{\epsilon}M_0^{2r}K\alpha,$$

because $\alpha_I\le K\alpha$ for $|I|\ge N/K$; the same holds for F_A. By Lemma 8.2(e), with $\lambda_w\le 2$, $\mathbb E\|\widetilde g_w(z_w,\cdot)\|_{2r}^{2r}\le C_{r,\epsilon}N^{\epsilon}2^{2r-1}\mathbb E\alpha_{\mathrm{f},w}$, where $\alpha_{\mathrm{f},w}$ is the fibre density of z_w, and $\mathbb E_z\alpha_{\mathrm{f},w}=D_{0,w}\#\{x\in A\cap I_w:x\equiv s_w\}/|I_w|\le M_0K\alpha$, because the class $\mathfrak a(z_w)$ is uniform among the D_E classes modulo D compatible with s_w. Finally $\|X\|_2^2\le 2\|g\|_2^2+2\|\widetilde g\|_2^2$. Thus every factor is at most $(C_\epsilon N^{\epsilon}\alpha)^{1/(2r)}$, and each of the at most 72 terms is bounded by $Q_{\mathrm{pre}}^{\gamma}H^{-\gamma}C_\epsilon N^{\epsilon}\alpha$, the density exponents adding to $1/2+71/142=1$. Take $\epsilon=72\delta$. Finally $N^{-\gamma\beta/2+72\delta}\le N^{-\sigma}\le\mathsf L^{-\sigma}$ by (S1). □

Proposition 8.7 (Density-structured energy estimate)**.** *Let $0<\theta\le 1$, $a>0$ and $C_0\ge 1$, and let $(\mathsf L,A)$ be a least normalized counterexample for (a,θ,C_0) (Definition 8.4), with $\mathsf L\ge\mathsf L_*$ and $Y=\mathcal Y(\mathsf L,A)$. Then every structured energy at $(\mathsf L,A)$ satisfies*

$$|\mathfrak E|\le\|\rho\|\big[(1+u_{\mathsf L})\,\Pi_{\mathrm{main}}\,Y+C_{\mathrm{dc}}\mathsf L^{-\sigma}\alpha\big]\le\|\rho\|\Big(1+u_{\mathsf L}+\frac{C_{\mathrm{dc}}}{C_0}\mathsf L^{a-\sigma}\alpha^{1-\theta}\Big)\Pi\,Y,$$

with $u_{\mathsf L}\to 0$ as in Lemma 8.3.

Proof. By Lemma 8.6 it suffices to bound $\widetilde{\mathfrak E}$ by $\|\rho\|(1+u_{\mathsf L})\Pi_{\mathrm{main}}Y$. Fix ω and z. By the signed Hölder inequality of Lemma 5.17(a) for $\mu_{\mathcal S}$, $|\Lambda_{\mu_{\mathcal S}}((\widetilde g_v)_v)|\le\prod_v Z_{\mathcal S}(\widetilde g_v(z_v,\cdot))^{1/72}$. At an unchanged slot $E=\emptyset$, $\mathcal S=\mathcal A_N$ and $Z_{\mathcal S}(F_A)=Y$. At a changed slot v, with the data $D,R,J_{\mathfrak a},h_j,A'_j,\lambda_v,\ell_v$ of Lemma 8.2(b) for that slot, the triangle inequality for the seminorm $Z_{\mathcal S}^{1/72}$ and its invariance under the maps T_j (Lemma 5.17(a),(c)), then Lemma 8.2(c), give

$$Z_{\mathcal S}(\widetilde g_v)^{1/72}\le\lambda_v\frac{D}{R}\sum_{j\in J_{\mathfrak a}}Z_{\mathcal S}(h_j)^{1/72}\le\lambda_v\frac{D}{R}\sum_{j\in J_{\mathfrak a}}\mathcal Y(\ell_v,A'_j)^{1/72}.$$

Here $\ell_v<\mathsf L$ by Lemma 8.2(d), so the hypothesis on smaller scales gives $\mathcal Y(\ell_v,A'_j)\le C_0\ell_v^{-a}\alpha_j^{\theta}$, also when $A'_j=\emptyset$. Concavity of $x\mapsto x^{\theta/72}$ and Lemma 8.2(b) give $\frac{D}{R}\sum_j\alpha_j^{\theta/72}\le(\alpha_{\mathrm{f},v}/\lambda_v)^{\theta/72}$, where

$\alpha_{\mathrm{f},v}$ is the fibre density of z_v, hence

$$Z_{\mathcal{S}}(\widetilde{g}_v) \le \lambda_v^{72-\theta} C_0 \ell_v^{-a} \alpha_{\mathrm{f},v}^{\theta} \le (1+u_{\mathsf{L}}) C_0 \mathsf{L}^{-a} \xi_v^a \alpha_{\mathrm{f},v}^{\theta},$$

using $\ell_v \ge \mathsf{L}/\xi_v$ and $\lambda_v^{72} \le 1+u_{\mathsf{L}}$ (Lemma 8.2(d), Lemma 8.3). The price ξ_v does not depend on ω or z. Every factor is nonnegative, so Hölder's inequality with 72 equal exponents on the probability space of (ω, z), with law $\mathbf{P} \otimes |\rho|/\|\rho\|$, gives

$$|\widetilde{\mathfrak{E}}| \le \|\rho\| \prod_v \left(\mathbb{E}\, Z_{\mathcal{S}}(\widetilde{g}_v(z_v,\cdot))\right)^{1/72}.$$

At a changed slot, Jensen's inequality ($\theta \le 1$) and $\mathbb{E}\alpha_{\mathrm{f},v} = \bar{\alpha}_v$ (as in Step 2 of Lemma 8.6) give $\mathbb{E} Z_{\mathcal{S}}(\widetilde{g}_v) \le (1+u_{\mathsf{L}}) C_0 \mathsf{L}^{-a} \alpha^{\theta} \xi_v^a (\bar{\alpha}_v/\alpha)^{\theta} \le (1+u_{\mathsf{L}}) Y \xi_v^a (\bar{\alpha}_v/\alpha)^{\theta}$, since $C_0 \mathsf{L}^{-a} \alpha^{\theta} < Y$. Multiplying over the slots proves the first inequality. For the second, $\Pi_{\mathrm{main}} \le \Pi$, $\Pi \ge 1$ and $C_{\mathrm{dc}} \mathsf{L}^{-\sigma} \alpha = (C_{\mathrm{dc}}/C_0) \mathsf{L}^{a-\sigma} \alpha^{1-\theta} \cdot C_0 \mathsf{L}^{-a} \alpha^{\theta} < (C_{\mathrm{dc}}/C_0) \mathsf{L}^{a-\sigma} \alpha^{1-\theta}\, \Pi\, Y$. □

The additive error is bounded before the main term is estimated, and the main term carries no additive error, so no density ratio is pulled through an additive error; in particular a density ratio below one is never factored out of the displayed bound, whose additive error was estimated before any ratio is formed (an example of the error this avoids is in Section 34.8). The second form uses only the coarse factors $\kappa_v \ge 1$: a fixed interval index costs at most $K^{\theta/72}$ per changed slot, and the finer factor $(\alpha_{I_v}/\alpha)^{\theta/72}$ is never multiplied into the additive term (Remark 10.7).

The average over the R/D progressions of the square step is essential, and their number costs nothing: the weights D/R of the triangle inequality add up to one, and concavity moves the average inside the power $\theta/72$. A changed slot therefore costs only $\xi_v^{a/72}$, where ξ_v bounds the factor by which the child scale $\ell_v \ge \mathsf{L}/\xi_v$ is shorter, together with the density of its fibre.

8.4. **Differences of laws at one cut, and the order for applying bounds.** At one cut, the comparison theorem bounds a signed difference of records K_s through the inequality (8.2) below: its pairing with half-functions on the two sides is at most a constant d times the product of their norms in the half marginals. (This displayed form is what "sharp" means throughout the paper.) Cauchy–Schwarz across the cut then replaces the signed difference by the geometric mean of two nonnegative energies, obtained by copying one half onto all slots—the folds $\phi_\pm$ of Definition 5.7. The next lemma states this reduction, the order in which such steps may be composed, and the marginals and prices of the two energies.

Lemma 8.8 (Folded energies)**.** *Let ϑ be a cut of V with halves $V_\pm$ and folds $\phi_\pm$ (Definition 5.7), at which every $\mu_{\mathcal{B}}$ is reflection positive. Let $E \subseteq \mathcal{A}_N$, $\mathcal{S} = \mathcal{A}_N \setminus E$, and write $z = (z_+, z_-)$ with $z_\pm \in X_E^{V_\pm}$. Let π be a probability measure on a finite set Ω_0 of shared labels; for $s \in \Omega_0$ let κ_s be a probability measure on X_E^V with half marginals $m_\pm^s$ on $X_E^{V_\pm}$, and let K_s be a signed measure on X_E^V such that, for all real functions φ on $X_E^{V_+}$ and ψ on $X_E^{V_-}$,*

$$\Big| \int \varphi(z_+)\psi(z_-)\, dK_s(z) \Big| \le d\, \|\varphi\|_{L^2(m_+^s)} \|\psi\|_{L^2(m_-^s)}, \tag{8.2}$$

with d independent of s. Let g_v $(v \in V)$ be real functions on $X_E \times X_{\mathcal{S}}$. Put

$$T = \int \pi(ds) \int \Lambda_{\mu_{\mathcal{S}}}\big((g_v(z_v,\cdot))_{v\in V}\big)\, dK_s(z),$$
$$\mathfrak{E}_\pm = \int \pi(ds) \int \Lambda_{\mu_{\mathcal{S}}}\big((g_{\phi_\pm(u)}(z_{\phi_\pm(u)},\cdot))_{u\in V}\big)\, dm_\pm^s(z_\pm).$$

(a) $\mathfrak{E}_\pm \ge 0$ and $|T| \le d\, \mathfrak{E}_+^{1/2} \mathfrak{E}_-^{1/2}$.

(b) $\mathfrak{E}_\pm$ *is the energy of the inputs* $g_{\phi_\pm(u)}$ *with the nonnegative selected measure* $\kappa^\pm$*, the image of* $\pi(ds)m^s_\pm(dz_\pm)$ *under* $z_\pm \mapsto (z_{\phi_\pm(u)})_{u\in V}$*. The one-slot marginal of* $\kappa^\pm$ *at* u *is the marginal at* $\phi_\pm(u)$ *of* $\kappa = \int \pi(ds)\kappa_s$*. In particular, if* κ *has uniform one-slot marginals on* V_+ *(resp.* V_-*), so has* κ^+ *(resp.* κ^-*) on all of* V*.*

(c) If the inputs also depend on an external index ω *with a probability law* $\mathbf{P}$ *independent of the labels, then* $\mathbb{E}_{\mathbf{P}}|T^\omega| \le d\,(\mathbb{E}_{\mathbf{P}}\mathfrak{E}^\omega_+)^{1/2}(\mathbb{E}_{\mathbf{P}}\mathfrak{E}^\omega_-)^{1/2}$*.*

(d) If $K_s = K'_s \otimes \pi'_s$*, where* π'_s *is a probability measure on the labels of further atoms (a coupling of its two half marginals) and* K'_s *satisfies* (8.2) *relative to* κ'_s*, then* K_s *satisfies* (8.2) *with the same* d *relative to* $\kappa'_s \otimes \pi'_s$*.*

(d′) (Common cut.) *If* $K_s = K^1_s \otimes K^2_s$*, where* K^1_s *and* K^2_s *are signed measures on the labels of two disjoint sets of atoms, each* K^i_s *satisfying* (8.2) *at the same cut* ϑ*, relative to a probability measure* κ^i_s*, with the constant* d_i*, then* K_s *satisfies* (8.2) *relative to* $\kappa^1_s \otimes \kappa^2_s$ *with the constant* d_1d_2*; if the two factors have their own laws of shared labels, take the product law. By induction, the same holds for finitely many factors, with the product of their constants.*

(e) (Copying.) *For every function* $b : V \to \mathbb{R}$*,*

$$\sum_{u\in V} b(\phi_+(u)) + \sum_{u\in V} b(\phi_-(u)) = 2\sum_{v\in V} b(v).$$

Hence, for structured inputs, the slot data of $\mathfrak{E}_\pm$ *at* u *are those of the slot* $\phi_\pm(u)$ *of* T*, and* $\Pi(\mathfrak{E}_+)\Pi(\mathfrak{E}_-) = \Pi^2$*, where* Π *is the coarse price of the slot data of* T*; likewise for the main prices and for every product* $\prod_v c_v$ *of nonnegative numbers attached to the slots. If the slot data depend on further data, such as the conductors of an index support, that are held fixed through a finite sequence of such steps, each applied to an energy produced by the previous ones, then for each fixed value of these data the final energies* $\mathfrak{E}_i$*, which enter the resulting bound with the exponents* 2^{-j_i} *(*j_i *the number of steps leading to* $\mathfrak{E}_i$*), satisfy* $\prod_i \Pi(\mathfrak{E}_i)^{2^{-j_i}} = \Pi$*.*

Proof. (a) A sharp bound is a block-Gram bracket. To see this, fix s, identify minus-half labels with plus-half labels through the reflection, and let $D^s_\pm$ be the diagonal matrices of the half marginals $m^s_\pm$ and $C_s = C_\vartheta(K_s)$. Inequality (8.2) is equivalent to

$$\begin{pmatrix} dD^s_+ & C_s \\ C_s^{\mathsf{T}} & dD^s_- \end{pmatrix} \succeq 0 :$$

the sharp bound makes its quadratic form nonnegative by $2xy \le x^2 + y^2$; conversely, applying a nonnegative quadratic form to $(t\varphi, \psi)$ and varying t gives the sharp bound. This also covers zero marginal masses and $d = 0$.

Let $Q = C_\vartheta(\mu_{\mathcal{S}}) \succeq 0$, by reflection positivity and (5.2). The bracket with every block equal to Q is positive semidefinite. Tensorizing it with the preceding bracket (Lemma 5.6(b)) gives

$$\begin{pmatrix} dD^s_+ \otimes Q & C_s \otimes Q \\ C_s^{\mathsf{T}} \otimes Q & dD^s_- \otimes Q \end{pmatrix} \succeq 0.$$

Evaluate at the two half-functions carrying their respective inputs g_v. By (5.1), the off-diagonal form is the integrand defining T at fixed s; the diagonal forms, without the factor d, are exactly the two folded energies at fixed s, since a diagonal copy repeats the same half assignment across the cut. They are nonnegative, and the block-Gram inequality bounds the absolute value of the first form by d times their geometric mean. Integrating over s and applying Cauchy–Schwarz in $L^2(\pi)$ proves (a).

(b) The first statement is the definition; for the marginal, the law of $z_{\phi_+(u)}$ under $\pi(ds)m^s_+(dz_+)$ is the marginal at $\phi_+(u) \in V_+$ of $\int \pi(ds)\kappa_s$, because m^s_+ is the V_+-marginal of κ_s.

(c) Apply (a) for each ω and Cauchy–Schwarz in $L^2(\mathbf{P})$.

(d′) Tensorize the two sharp brackets from (a). Their diagonal blocks are $d_1 d_2 D^1_{\pm} \otimes D^2_{\pm}$ and their off-diagonal block is the cut matrix of $K^1_s \otimes K^2_s$, by (5.2). The same equivalence therefore gives the sharp bound with constant $d_1 d_2$. The argument is pointwise in all shared labels, so independent shared-label laws can be combined by taking their product, and induction gives finitely many factors. For (d), a probability coupling satisfies the sharp bound with constant one by Cauchy–Schwarz under that coupling; apply (d′).

(e) Each $v \in V_+$ occurs exactly twice among $\phi_+(u)$, $u \in V$ (for $u = v$ and $u = \vartheta v$), and no $v \in V_-$ occurs; symmetrically for ϕ_-. This is the identity, and the same count gives $\prod_u c_{\phi_+(u)} \cdot \prod_u c_{\phi_-(u)} = \prod_{v \in V_+} c_v^2 \prod_{v \in V_-} c_v^2 = (\prod_v c_v)^2$ for all numbers $c_v \geq 0$. With c_v the slot factors $\xi_v^{a/72}$, $\kappa_v^{\theta/72}$ and $(\bar{\alpha}_v/\alpha)^{\theta/72}$ (the last two at changed slots), this is the statement on prices; the changed or unchanged status of a slot is copied with its input. For a sequence of steps, induct on the number of steps: replacing an energy $\mathfrak{E}$ of exponent 2^{-j} by the two energies of a further step, each of exponent 2^{-j-1}, does not change the product, because $\Pi(\mathfrak{E}_+)^{1/2}\Pi(\mathfrak{E}_-)^{1/2} = \Pi(\mathfrak{E})$ by the first part. □

By (a), (e) and Proposition 8.7, a sharp difference with constant d costs at most $d(1 + u_{\mathsf{L}} + (C_{\mathrm{dc}}/C_0)\mathsf{L}^{a-\sigma}\alpha^{1-\theta})\Pi\, Y$ at a least normalized counterexample: the geometric mean of the two energies retains the price Π, not Π^2. The same holds for the weights carried by the slots of an index support. In the index energies of Part III each query conductor occurs at two slots, so it enters with the exponent $2a/72 = a/36$. A later chamber fold can give four occurrences on one energy and none on the other, but the geometric mean keeps the original two; the conductors are therefore held fixed through every Cauchy–Schwarz step and averaged only after the complementary energies are combined, and no higher conductor moment is used (Lemma 10.17).

The two energy estimates apply to the energies of the comparison once the selected laws and the bounds applied to them have the following properties. They are properties of the construction, verified where the laws are built.

Definition 8.9 (Admissible laws)**.** Consider a reduction of a multilinear form to structured energies by expansions over selected atoms, sharp operator bounds at cuts (Lemma 8.8) and Proposition 8.7. Its closure obligations are:

- (O1) every selected law that remains after the operator-to-energy reduction is nonnegative, or an invariant signed measure, with uniform whole-slot marginals on the selected atoms ((R3), Definition 9.14), and it is independent of the interval, residue and inclusion indices;
- (O2) every sharp operator bound is uniform in the shared labels at which it is applied (the constant d of (8.2));
- (O3) index-support projections are taken before the field folds, and the inclusion conductors stay fixed through all folds and geometric means; they are averaged only afterwards (Lemma 8.8(e));
- (O4) the chamber folds used in a geometric mean commute with the reserved cut ι, so that complementary folds have the copying property of Lemma 8.8(e).

A signed measure on the labels of a finite set of atoms is an *admissible law* if it satisfies (O1) and every sharp operator bound applied to it satisfies (O2). A reduction is *admissible* if its selected laws are admissible and it satisfies (O3) and (O4).

For the slot array V every chamber fold commutes with the reserved cut ι (Lemma 5.13(iii)); so a law fixed samplewise by ι, and an assignment of inputs invariant under ι, stay so after a chamber fold. Whole-slot uniformity does not imply uniformity of the conditional distributions of the non-shared labels of one half given the shared labels, and none of the statements of this section needs it. A pointwise statement with arbitrary pinned values of the non-shared labels would need

a further theorem and is not used. Each construction lemma of Part V states that its laws are admissible; Table 10 collects them.

8.5. **Forced pairs and very large pinnings are negligible.** The accepted queries enter through one estimate: an average in which an ordered pair of slots carries inputs from adjacent intervals, with residues differing by a nonvanishing square class, and whose law forces a strict-square difference on that pair at every prime with $e_p = 1$ (at a clique prime, a nonzero difference of even valuation with a square unit part), is negligible. Square-difference-freeness is used here, through the actual-square kernel.

Lemma 8.10 (Accepted-pair cancellation with a density factor)**.** *There is a constant C_{can} with the following property. Let $\mathsf{L} \ge \mathsf{L}_*$ (Lemma 8.3), $N = K\mathsf{L}$, let $A \subseteq [N]$ be square-difference-free and $\alpha = |A|/N$. Let X_{pre} be the product of the groups of the atoms below P_5, and let $\varrho = \varrho_{\mathrm{pre}} \otimes \bigotimes_{P_5 \le p \le N^\beta} \varrho_p$ be a product of probability measures on X_{pre}^V and on $\mathbb{F}_p^V$, each with uniform one-slot marginals. Let (u, v) be an ordered pair of slots such that:*

(a) for every $p \ge P_5$, ϱ_p satisfies the hypothesis of Lemma 6.16 at (u, v) with the exponent γ: the (u, v)-marginal of ϱ_p is $\mathrm{Unif}(\Omega_p)$, and its conditional operator from the pair (u, v) to the other slots fixes constants, maps mean-zero functions to mean-zero functions and has norm at most $p^{-\gamma}$ on them;

(b) ϱ_{pre}-almost surely, $z_{v,p} - z_{u,p}$ is a nonzero square at every prime with $e_p = 1$ below P_5, and at every clique prime it is nonzero, of even valuation $2j < e_p$, with a square unit part.

Let $1 \le b \le K-1$, $s \in \mathbb{Z}/M_0\mathbb{Z}$, and let $r \in \mathbb{Z}/M_0\mathbb{Z}$ be a nonvanishing square class (Definition 4.6). Let $G_u = g^{I_b, M_0, s}$, $G_v = g^{I_{b+1}, M_0, s+r}$, and let $G_w = g^{I_w, D_{0,w}, s_w}$ $(w \ne u, v)$ be inputs of Definition 6.1. Then

$$\Big| \int \prod_{w \in V} G_w(z_w)\, d\varrho(z) \Big| \le C_{\mathrm{can}} N^{72\delta} \alpha \big(N^{-\gamma t_{\mathrm{an}}} + N^{-1/120 + t_{\mathrm{an}}(1/2-\gamma)} \big) \le 2 C_{\mathrm{can}} \mathsf{L}^{-\sigma} \alpha.$$

The bound is uniform in b, s, r and the other indices, so it persists after integration against any probability law of the indices; for $c\varrho$ with $0 \le c \le 1$ it is multiplied by c.

Proof. Step 0. Use the constant of Proposition 7.3(iii) at the triple of Corollary 7.4, with $C_D = C_H = 1$. The external modulus and the clique moduli are fixed independently of the scale. Put $H = N^{1/40}$, $T_0 = N^{t_{\mathrm{an}}}$ and $D_{\mathrm{pre}} = |X_{\mathrm{pre}}|$, which divides Q_{pre}. By Lemma 8.3(v), every prime of $[P_4, P_5)$ is a coordinate of G_N; the other primes below P_5 are coordinates of G_N or divide M_0.

Step 1: the pair decomposition. Condition on the prefix labels $x = (x_w)_{w \in V}$ of all slots, distributed by ϱ_{pre}. Put $\varphi_x(y_u, y_v) = G_u(x_u, y_u) G_v(x_v, y_v)$ and $W_x = \prod_{w \ne u,v} G_w(x_w, \cdot)$, functions of the tail labels. By (a), Lemma 6.16 applies to the tail factors, so

$$\Big| \int \varphi_x W_x \Big|^2 \le \|W_x\|_2^2 \Big[T_0^{-2\gamma} \|\varphi_x\|_2^2 + \sum_{q_U \le T_0} q_U^{-2\gamma} \|E_U \varphi_x\|_2^2 \Big],$$

with $\|\varphi_x\|_2$ for $\bigotimes_p \mathrm{Unif}(\Omega_p)$, $\|W_x\|_2$ for the law of the other slots, and U running over sets of tail primes $p \ge P_5$. Integrate over x and apply Cauchy–Schwarz:

$$\Big| \int \prod_w G_w\, d\varrho \Big| \le \Big(\int \|W_x\|_2^2 \Big)^{1/2} \Big(T_0^{-2\gamma} \int \|\varphi_x\|_2^2 + \sum_{q_U \le T_0} q_U^{-2\gamma} \int \|E_U \varphi_x\|_2^2 \Big)^{1/2}.$$

Step 2: moments. Under ϱ every z_w is uniform on G_N. By Hölder with 70 factors and Lemma 6.7(c) with moments of order 140 (all inputs have intervals of length at least $\mathsf{L} \ge 10N^{6\beta}$, by Lemma 8.3(i)),

$$\int \|W_x\|_2^2\, d\varrho_{\mathrm{pre}} = \int \prod_{w \ne u,v} |G_w(z_w)|^2\, d\varrho \le \prod_{w \ne u,v} \|G_w\|_{140}^2 \le C_\epsilon N^\epsilon \alpha,$$

and by Cauchy–Schwarz and the fourth moments, $\int \|\varphi_x\|_2^2 \le \|G_u\|_4^2\|G_v\|_4^2 \le C_\epsilon N^\epsilon\alpha$. The density exponents are $70\cdot 2/140 = 1$ and $2\cdot 2/4 = 1$.

Step 3: a low pair support. Fix U with $q_U \le T_0$. The function $E_U\varphi_x$ averages both pair variables over the tail primes outside U. Fix also the pair labels $y = (y_u, y_v)$ at U, with $(y_{u,p}, y_{v,p}) \in \Omega_p$; let $\mathcal{S}'$ be the set of tail primes $P_5 \le p \le N^\beta$ outside U, and $g_1 = G_u(x_u, y_u, \cdot)$, $g_2 = G_v(x_v, y_v, \cdot)$ on $X_{\mathcal{S}'}$. By the pair multiplier identity of Lemma 4.11(d),

$$E_U\varphi_x(y) = \Psi_{\le H} + \Psi_{>H}, \qquad \Psi_{\le H} = \sum_{q(\xi)\le H} \widehat{g_1}(-\xi)\widehat{g_2}(\xi)g(\xi),$$

the sums running over the characters of $X_{\mathcal{S}'}$, with $g(\xi)$ the unit quadratic average (Definition 7.1). Since every prime of $\mathcal{S}'$ exceeds 64, $|g(\xi)| \le q(\xi)^{-1/3} < H^{-1/3}$ for $q(\xi) > H$, and Cauchy–Schwarz and Parseval give $|\Psi_{>H}| \le H^{-1/3}\|g_1\|_2\|g_2\|_2$.

For the low part put $D = M_0 D_{\mathrm{pre}} q_U$. By Lemma 8.3(vi), $HD_{\mathrm{pre}}q_U \le Q$, so Lemma 8.2(a), with the conditioned atoms in the role of E, gives for $q(\xi) \le H$

$$\widehat{g_1}(\xi) = \frac{D}{\mathsf{L}} \sum_{\substack{n\in A\cap I_b\\ n\equiv \mathfrak{a}_1\ (D)}} e(-n\xi), \qquad \widehat{g_2}(\xi) = \frac{D}{\mathsf{L}} \sum_{\substack{n\in A\cap I_{b+1}\\ n\equiv \mathfrak{a}_2\ (D)}} e(-n\xi),$$

where $\mathfrak{a}_1 \equiv s$, $\mathfrak{a}_2 \equiv s + r \pmod{M_0}$, $\mathfrak{a}_1 \equiv x_u$, $\mathfrak{a}_2 \equiv x_v$ on X_{pre} and $\mathfrak{a}_1 \equiv y_u$, $\mathfrak{a}_2 \equiv y_v$ on X_U. Every prime below P_5 and every prime of U divides D; hence the characters of $X_{\mathcal{S}'}$ with $q(\xi) \le H$ are exactly the elements of $\Xi_{H,D}$ (Definition 7.1), since a squarefree $q \le H \le N^\beta$ coprime to D has all its primes in $\mathcal{S}'$. The difference $\mathfrak{a}_2 - \mathfrak{a}_1$ is r modulo M_0, a nonzero square at the primes of U and at the primes below P_5 with $e_p = 1$, and at the clique primes nonzero of even valuation below e_p with square unit part, by (b). With $F_i = \mathbf{1}_A \mathbf{1}_{I_{b+i-1}} \mathbf{1}_{n\equiv \mathfrak{a}_i\ (D)}$ on $[N]$ and $\widehat{F_i}$ normalized by N, we have $\widehat{g_1}(-\xi) = (DN/\mathsf{L})\widehat{F_1}(-\xi)$ and $\widehat{g_2}(\xi) = (DN/\mathsf{L})\widehat{F_2}(\xi)$. The supports are ordered, since I_b lies below I_{b+1}, and no difference is a positive square. So Proposition 7.3(iii) applies in the form of Corollary 7.4, with $D \le N^{1/20}$ (Lemma 8.3(vi)), $H = N^{1/40}$ and N above the thresholds of the proposition (Lemma 8.3(vii)):

$$\begin{aligned} |\Psi_{\le H}| &= \frac{DN^2}{\mathsf{L}^2}\Big|D \sum_{\xi\in\Xi_{H,D}} g(\xi)\widehat{F_1}(-\xi)\widehat{F_2}(\xi)\Big| \\ &\le \frac{DN^2}{\mathsf{L}^2}\,\frac{CN^{-1/120}\|F_1\|_{\ell^2}\|F_2\|_{\ell^2}}{N} = CKN^{-1/120}\sqrt{\alpha_{\mathrm{f},1}\alpha_{\mathrm{f},2}}, \end{aligned}$$

where $\alpha_{\mathrm{f},i} = (D/\mathsf{L})\|F_i\|_{\ell^2}^2 = (D/\mathsf{L})\#\{n \in A\cap I_{b+i-1} : n \equiv \mathfrak{a}_i\ (D)\}$ are the two fibre densities. The fibre modulus D cancels. Since $D \le \mathsf{L}$ (Lemma 8.3(vi)), $\alpha_{\mathrm{f},i} \le (D/\mathsf{L})(\mathsf{L}/D + 1) \le 2$.

Now average over $x \sim \varrho_{\mathrm{pre}}$ and $y \sim \bigotimes_{p\in U}\mathrm{Unif}(\Omega_p)$. The labels (x_u, y_u) are uniform on $X_{\mathrm{pre}}\times X_U$, so the class $\mathfrak{a}_1$ is uniform among the $D_{\mathrm{pre}}q_U$ classes modulo D above s, and $\mathbb{E}\alpha_{\mathrm{f},1} = M_0\#\{n \in A\cap I_b : n \equiv s\ (M_0)\}/\mathsf{L} \le M_0K\alpha$. Hence $\mathbb{E}(\alpha_{\mathrm{f},1}\alpha_{\mathrm{f},2}) \le 2\mathbb{E}\alpha_{\mathrm{f},1} \le 2M_0K\alpha$; no independence of the two classes is used. For the high part, Cauchy–Schwarz and Jensen's inequality ($\|g_1\|_2^4 \le \mathbb{E}|g_1|^4$ over $X_{\mathcal{S}'}$) give $\mathbb{E}(\|g_1\|_2^2\|g_2\|_2^2) \le \|G_u\|_4^2\|G_v\|_4^2 \le C_\epsilon N^\epsilon\alpha$, because (x_u, y_u) together with the uniform outside label is uniform on G_N. Since $\|E_U\varphi_x\|_2^2 = \mathbb{E}_y|E_U\varphi_x(y)|^2$ and $|\Psi|^2 \le 2|\Psi_{\le H}|^2 + 2|\Psi_{>H}|^2$,

$$\int \|E_U\varphi_x\|_2^2\, d\varrho_{\mathrm{pre}}(x) \le 4C^2K^3M_0N^{-1/60}\alpha + 2H^{-2/3}C_\epsilon N^\epsilon\alpha \le C'_\epsilon N^{-1/60+\epsilon}\alpha,$$

uniformly in U.

Step 4: summation. Distinct sets U have distinct products, and $0 < \gamma < 1/2$, so $\sum_{q_U\le T_0} q_U^{-2\gamma} \le 1 + T_0^{1-2\gamma}/(1-2\gamma)$. Inserting Steps 2 and 3 into Step 1,

$$\Big|\int \prod_w G_w\, d\varrho\Big| \le C_\epsilon N^\epsilon\alpha\big(N^{-\gamma t_{\mathrm{an}}} + N^{-1/120 + t_{\mathrm{an}}(1/2-\gamma)}\big).$$

Take $\epsilon = 72\delta$; both exponents are below $-\sigma$ by (S2) and (S3), which proves the lemma. □

At a nonunit clique or external-index difference the kernel is applied in the root-class form of Proposition 7.3(iii), which keeps every pinned prime in the modulus. Its constants satisfy the same uniform bounds. For the odd prefix with a unit external index, Proposition 7.3(ii), where an even-valuation square is divided out of the modulus, may be used in place of (iii). The fourth moment enters with the exponent exactly $1/2$, and no strict inequality between moment exponents is needed.

Lemma 8.11 (Large products with a density factor)**.** *Let $(\eta_{\mathcal{B}})$ be nonnegative numbers indexed by the atoms of all scales, with $\Pi_{\mathrm{H}} = \prod_{\mathcal{B}}(1+\eta_{\mathcal{B}} q_{\mathcal{B}}^{1/4}) < \infty$, where $q_{\mathcal{B}}$ is the product of the primes of $\mathcal{B}$ with $e_p = 1$. For $\epsilon > 0$ there is C_ϵ with the following property. Let $\mathsf{L} \ge 1$ with $\mathsf{L} \ge 10N^{6\beta}$, $c \ge 1$, and for every set $E \subseteq \mathcal{A}_N$ with $q_E > N^\tau$ let κ'_E be a finite nonnegative measure on G_N^V with $\kappa'_E(1) \le c\prod_{\mathcal{B}\in E}\eta_{\mathcal{B}}$ whose one-slot marginals are multiples of the uniform law, and let $(G_v^E)_v$ be inputs $g^{I,D_0,s}$ of Definition 6.1, possibly averaged over indices by a probability law. Then*

$$\sum_{E:\, q_E > N^\tau} \Big| \int \prod_v G_v^E(z_v)\, d\kappa'_E(z) \Big| \le c\, C_\epsilon \Pi_{\mathrm{H}} N^{-\tau/4+\epsilon}\alpha.$$

The same bound holds if the term of E is replaced by $a_E (I_E^+ I_E^-)^{1/2}$ with $0 \le a_E \le c\prod_{\mathcal{B}\in E}\eta_{\mathcal{B}}$ and $I_E^\pm$ energies of such inputs under probability measures with uniform one-slot marginals. With $\epsilon = \ell_0$ the right-hand side is at most $c\,C\,\Pi_{\mathrm{H}}\mathsf{L}^{-\sigma}\alpha$, by (S4).

Proof. For fixed E and indices, Hölder's inequality with 72 factors in $L^{72}(\kappa'_E)$ and the uniform marginals give $|\int \prod_v G_v\, d\kappa'_E| \le \kappa'_E(1)\prod_v \|G_v\|_{L^{72}(G_N)}$; after averaging over the indices, Hölder over their law gives the same product of averaged norms. By Lemma 6.7(c) with moments of order 72, each norm is at most $(C_\epsilon N^\epsilon M_0^{72} K\alpha)^{1/72}$, so the product is $C_\epsilon N^\epsilon \alpha$: the density exponents are $72/72 = 1$. The same bound holds for $I_E^\pm$, hence for $(I_E^+ I_E^-)^{1/2}$. Finally, by Rankin's trick,

$$\sum_{E:\, q_E > N^\tau} \prod_{\mathcal{B}\in E} \eta_{\mathcal{B}} \le N^{-\tau/4} \sum_E \prod_{\mathcal{B}\in E} \eta_{\mathcal{B}} q_{\mathcal{B}}^{1/4} \le N^{-\tau/4}\Pi_{\mathrm{H}}.$$

□

In the comparison proof the coefficient $\eta_{\mathcal{B}}$ of an atom is taken to be the sum of the prices at $\mathcal{B}$ listed in (CP8)(c), after the operator-to-energy reduction (Lemma 11.4(ii), Definition 9.20); a spectral price entering the sum is the price produced by this reduction, not the total variation of the original signed law. The product Π_{H} is then finite by the summability (c) of Definition 9.20: the atoms below P_4 are finitely many, each with a finite factor in Π_{H}, and at a tail prime p the prices of the selected laws are s_p, a_p and the matching prices $T_{\mathrm{L},p} = T_{\mathrm{R},p}$, which add up to at most three times $s_p + a_p \le C_{\mathrm{tail}} p^{-3/2+2a}$, so that their sum weighted by $p^{1/4}$ converges (Lemma 29.5(iii), (iv)).

8.6. **From $\Theta < 1$ to $s(N) \le CN^{1-a/71}$ (baseline induction).** It remains to convert a closing inequality at a least counterexample into a bound for every A. The theorem does this by induction on the scale: if a least counterexample would satisfy $Y \le \Theta Y$ with $\Theta(0) < 1$ and only the two allowed kinds of remainder, then no counterexample exists and $Y \le C\mathsf{L}^{-a}\alpha^\theta$ holds in general.

Theorem 8.12 (Linear-density induction)**.** *Let $0 < a < \sigma$ and $0 < \theta \le 1$, and assume (R1)–(R5). Suppose there are $\mathsf{L}_1 \ge 1$, $z_0 > 0$ and a function $\Theta : [0, z_0] \to [0,\infty)$, continuous at 0 with $\Theta(0) < 1$, such that the following recurrence hypothesis holds.*

(Rec) Let $C_0 \ge 1$, let (L, A) be a least normalized counterexample for (a, θ, C_0) with $\mathsf{L} \ge \mathsf{L}_1$, and let $z \in [0, z_0]$. If every structured energy that the recurrence bounds at (L, A) satisfies $|\mathfrak{E}| \le (1+z)\|\rho\|\,\Pi\,\mathcal{Y}(\mathsf{L}, A)$ with its coarse price Π, and the sum of the absolute values

of its cancellation and large-product remainders is at most $z\,\mathcal{Y}(\mathsf{L},A)$, then $\mathcal{Y}(\mathsf{L},A)\le\Theta(z)\,\mathcal{Y}(\mathsf{L},A)$.

Suppose further that these remainders consist of a number, bounded independently of L and A, of integrals of the form bounded in Lemma 8.10, integrated against probability laws of the indices and multiplied by coefficients bounded independently of L and A, together with finitely many sums of the form bounded in Lemma 8.11, whose number and whose constants c and Π_{H} are independent of L and A. Then there is C with

$$\mathcal{Y}(\mathsf{L},A)\le C\,\mathsf{L}^{-a}\alpha^{\theta}$$

for every $\mathsf{L}\ge 1$ and every square-difference-free $A\subseteq[K\mathsf{L}]$. Consequently $\alpha^{72-\theta}\le C\mathsf{L}^{-a}$, and $s(N)\le C'N^{1-a/(72-\theta)}$ for every $N\ge 1$. At $\theta=1$ the conversion is $\varepsilon=a/71$.

Proof. Let C_{R} be a constant with: the remainders at any least normalized counterexample (L,A) with $\mathsf{L}\ge\mathsf{L}_*$ have total absolute value at most $C_{\mathrm{R}}\mathsf{L}^{-\sigma}\alpha$. It exists by Lemmas 8.10 and 8.11 and the hypothesis on their number and coefficients, and it does not depend on C_0, because the remainders do not involve the induction.

Choice of constants. By continuity choose $z_1\in(0,z_0]$ with $\Theta(z)\le(1+\Theta(0))/2<1$ for $0\le z\le z_1$. Choose $\mathsf{L}_0\ge\max(\mathsf{L}_1,\mathsf{L}_*)$ such that $u_{\mathsf{L}}\le z_1/2$ and $(C_{\mathrm{dc}}+C_{\mathrm{R}})\mathsf{L}^{a-\sigma}\le z_1/2$ for $\mathsf{L}\ge\mathsf{L}_0$; this is possible because $a<\sigma$. For the finitely many $\mathsf{L}<\mathsf{L}_0$, every retained coefficient of F_A has modulus at most α, so $|F_A|\le n_{\mathsf{L}}\alpha$, with n_{L} the number of retained characters of G_N, and $\mathcal{Y}(\mathsf{L},A)\le(n_{\mathsf{L}}\alpha)^{72}\le n_{\mathsf{L}}^{72}\alpha$. Put $C_{\mathrm{base}}=\max_{\mathsf{L}<\mathsf{L}_0}n_{\mathsf{L}}^{72}$ and $C_0=\max(1,C_{\mathrm{base}}\mathsf{L}_0^{a})$. Then $\mathcal{Y}(\mathsf{L},A)\le C_{\mathrm{base}}\alpha\le C_0\mathsf{L}^{-a}\alpha^{\theta}$ for $\mathsf{L}<\mathsf{L}_0$, since $\alpha\le\alpha^{\theta}$.

The least counterexample. Suppose the bound $\mathcal{Y}(\mathsf{L},A)\le C_0\mathsf{L}^{-a}\alpha^{\theta}$ fails for some pair, and let L be the least scale at which it fails, for some A. Then (L,A) is a least normalized counterexample, $\mathsf{L}\ge\mathsf{L}_0$, $\alpha>0$ and $Y=\mathcal{Y}(\mathsf{L},A)>0$. By Proposition 8.7, every structured energy satisfies $|\mathfrak{E}|\le(1+u_{\mathsf{L}}+C_{\mathrm{dc}}\mathsf{L}^{a-\sigma}\alpha^{1-\theta}/C_0)\|\rho\|\Pi Y\le(1+z_1)\|\rho\|\Pi Y$, because $\alpha\le 1$ and $C_0\ge 1$. The remainders are at most $C_{\mathrm{R}}\mathsf{L}^{-\sigma}\alpha=(C_{\mathrm{R}}/C_0)\mathsf{L}^{a-\sigma}\alpha^{1-\theta}\cdot C_0\mathsf{L}^{-a}\alpha^{\theta}<z_1Y$. By (Rec), $Y\le\Theta(z_1)Y<Y$, a contradiction. Hence the bound holds with $C=C_0$. No lower bound on α is used: the relative error $C\mathsf{L}^{a-\sigma}\alpha^{1-\theta}/C_0$ is uniform in α because $\theta\le 1$.

Conversion. By Lemma 5.17(e), $\alpha^{72}\le\mathcal{Y}(\mathsf{L},A)\le C_0\mathsf{L}^{-a}\alpha^{\theta}$, so $\alpha^{72-\theta}\le C_0\mathsf{L}^{-a}$ when $\alpha>0$, and $|A|=\alpha N\le C_0^{1/(72-\theta)}K^{a/(72-\theta)}N^{1-a/(72-\theta)}$. For an arbitrary $N\ge K$ put $N'=K\lceil N/K\rceil<2N$; a square-difference-free subset of $[N]$ is one of $[N']$, which gives $s(N)\le C'N^{1-a/(72-\theta)}$. For $N<K$, $s(N)\le N\le K$. □

The hypothesis (Rec) is supplied by the comparison theorem: in the proof of Theorem 9.21, Lemma 11.4 shows that (Rec) holds for the density exponent θ of the frame data, with $\Theta(z)$ the closing number of Theorem 9.21 evaluated with every energy bound inflated by the factor $1+z$ and every remainder bounded by zY; it also proves the continuity at $z=0$ and that the remainders have the forms of Lemmas 8.10 and 8.11. In the baseline theorem $\theta=1$, and $\Theta(0)$ is the certified closing value. The induction uses only the moments of orders 2, 4, 72, 140 and 142, and no limit in an interpolation parameter.

Remark 8.13 (Fixed parameters). Every statement and proof of Section 8 holds for an arbitrary fixed integer $K\ge 2$, with the least square step (8.1), and for arbitrary fixed even exponents $e_p=2m_p$ at the clique primes, and so do the statements of Part II that it uses: the functional is defined for these data (Definition 6.1); the index averages hold for every $K\ge 2$ (Lemma 6.3); the moment constants depend on the prefix only through Q_{cl} (Lemma 6.7(c)); conductor decay allows any finite prefix (Lemma 6.15), and the pair-support decay involves only the tail primes (Lemma 6.16); and the kernel allows any m_p, with constants depending on $\prod_{p\in\mathcal{P}_1}p^{e_p}$ (Proposition 7.3(iii), Lemma C.1). Constants and the starting scale depend on these fixed data, and nothing is asserted uniformly as K or the prefix grow with N. Each interval pair I_b,I_{b+1} with $1\le b<K$ is ordered, which, with their

common length L, is all that the cancellation uses of the intervals, and the child scales are strictly smaller by Lemma 8.2(d). The baseline theorem uses $K = 890666$, $e_p = 2m_{\mathrm{cl}}$ with $m_{\mathrm{cl}} = 8192$ at the class-one primes below P_4, $\theta = 1$ and the price exponent $a = \frac{5422587}{6103515625}$, which satisfies the gate $a < \sigma$ of Theorem 8.12, with $\sigma - a = 65195254/201416015625$, and gives $\varepsilon = a/71 = \frac{5422587}{433349609375}$. The gates of the analytic transfer that involve a (the analytic, density, fixed-index and fixed-prefix gates, Certificate D.2(S0)) hold on the whole range $0 < a \le 9/10000$ and for every fixed $K \ge 2$; in particular, there the induction slack is $\sigma - a \ge 103/330000$, and the exponent of the bound $2^{513-m_{\mathrm{cl}}+1024m_{\mathrm{cl}}a}$ for the price of the deleted clique exception (Lemma 19.3) is at most $-80783/625$, so that this price together with the tail prices beyond P_4 stays below 10^{-12} (Certificate D.3(a), (f); Remark 19.4).

Remark 8.14 (Empty and concentrated fibres).

(i) If $A = \emptyset$, every lift and every energy vanishes; the induction treats this case before any division by α.

(ii) If A is a single point, $\mathbb{E}_I|f|^2 = 1/|I|$, and Lemma 6.7 bounds every fixed moment by $C_\epsilon N^\epsilon/|I|$; no hypothesis $\alpha|I| \to \infty$ is used.

(iii) A pinned lift need not vanish on an empty fibre. Let $|I|$ be divisible by 15, $A = \{n \in I : 15 \mid n\}$, $\mathcal{P} = \{3, 5\}$ and $Q = 5$. The lift is $G(z_3, z_5) = (3 \cdot \mathbf{1}_{z_3=0} + 5 \cdot \mathbf{1}_{z_5=0} - 1)/15$. At $z_3 = 1$ the fibre is empty, but $G(1, z_5) = (5 \cdot \mathbf{1}_{z_5=0} - 1)/15 \ne 0$. Its part of conductor at most $Q/3$ is zero, as the exact fibre identity (Lemma 8.2(a)) requires. The comparison never asserts more than this: the square-step model $\widetilde{g}$ vanishes on an empty fibre, and the difference $g - \widetilde{g}$ is controlled by averaged moments.

(iv) If A is one residue class modulo a retained conductor, the lift concentrates on one class of the product group, and its moments of every fixed order are of order α; so the linear dependence on α in Lemma 6.7 cannot be improved in general.

Remark 8.15 (The density exponent). At $\theta = 1$ the density saving $a/(72 - \theta)$ is $a/71$. Without the density-linear moments of Lemma 6.7, a remainder $C\mathsf{L}^{-\sigma}$ is compared with $Y > C_0\mathsf{L}^{-a}\alpha^\theta$ through the sparse-set bound $Y \le C_\epsilon \mathsf{L}^\epsilon \alpha^s$ (every $\epsilon > 0$), $s = (10^6 - 36)/(10^6 - 1)$, which follows from Hölder, Parseval and a moment of order $2 \cdot 10^6$ by interpolation. This needs the sparse gate $as/(s - \theta) < \sigma$, which allows exactly $\theta < s(\sigma - a)/\sigma$: about 0.2680 at the price exponent of the baseline theorem, so a saving of less than $a/71.73$. That route also needs the uniform whole-slot marginals: its main-term Jensen step uses them (Definition 8.9). Only $\theta = 0$, with saving $a/72$, needs no density averaging at all. The density-linear moments thus gain the factor $72/71$ over $\theta = 0$, and about 1.010 over the sparse gate.

Consequences and what to read next. Inputs used in this section: the functional, its atoms, inputs and reference hypotheses (Definition 6.1 and Hypothesis 6.2); the least square step and the strict square classes ((4.1), Definitions 4.5 and 4.6); the products of atom laws with the signed Hölder inequality, the seminorm, invariance and averaging (Lemma 5.17); the density-linear moment bound (Lemma 6.7); conductor decay with a finite prefix and the pair-support decay (Lemmas 6.15 and 6.16, with Definition 6.13); the pair multiplier of the strict-square pair law (Lemma 4.11(d)); the actual-square kernel and its notation (Definition 7.1, Proposition 7.3, and Corollary 7.4); the tail prices, for the finiteness of Π_{H} (Lemma 29.5); reflection positivity, folds and the one-cut Cauchy–Schwarz inequality (Definitions 5.1 and 5.7 and Lemma 5.8); the slot array (Lemma 5.13); and the stage inequalities of Certificate D.2.

9. The comparison theorem: standing data, prices, and hypotheses

What this section proves and how to read it. This section states Theorem C: if distributions μ, λ, ν satisfy explicit hypotheses (CP0)–(CP8) and the number Θ formed from their prices is < 1, then

$s(N) \le C' N^{1-a/(72-\theta)}$ with $0 < \theta \le 1$. No particular local law is used; Part V later exhibits laws satisfying the hypotheses. In words: μ is the reflection-positive reference law; λ is obtained from μ by record differences with the same conditional one-half distributions, grouped by the cut at which each difference has a uniform operator bound (a *tag*); a positive law N dominates λ at the reserved cut in the 2×2 block sense; ν is a finite signed combination of nonnegative laws each accepting the queries (in the bundled sense of Section 1.8), with symmetric and skew differences priced separately; second positive laws dominate ν itself. Inputs: Part II only. Output: Theorem 9.21, proved in Sections 10 and 11 and applied with $k = 6$ in Part V (Table 5). A first reading should follow (CP0), the price definitions, the statement of Theorem C with the reading guide preceding it, and treat (CP1)–(CP8) as a checklist to be consulted in Part V. Two reading-order warnings: the repricing corollary in Section 9.1 mentions totals L_0, B defined in (CP2)–(CP3) below and can be skipped until then; and *robust* bounds, used in (CP5) and (CP7), are defined in Definition 9.19 immediately after (CP7)—in brief, bounds whose $o(1)$ errors come only from the two allowed analytic remainders. Theorem C is the k-query form of Proposition 2.1.

Letters. In this part B is the energy factor of the law dominating λ, A the total skew price, $\mathcal{E}$ the dominated price bound for $\nu - \lambda$ and J the coupled index term; an atom (a block of primes) is written $\mathcal{B}$. The square-difference-free set of integers of the main theorem is also written A, but only in its lift F_A, as the second entry of a pair (L, A) (in $\mathcal{Y}(\mathsf{L}, A)$, in a counterexample, and in "uniformly in A"), and where such a set is quantified (as in Definition 9.6 and Theorem C); its density is $\alpha = |A|/N$.

9.1. **Standing data (CP0), prices, and conversion between exponents.** Theorem C is stated for abstract comparison data, which Part V later verifies for one concrete construction. This subsection lists those data as (CP0), then defines the price of changing the law at one atom and how a price computed at a smaller exponent a_0 is converted to the exponent a of the theorem.

Definition 9.1 ((CP0) Standing data: slots, scales, queries, index)**.** *Comparison data* consist of the following objects.

(a) *Slots and cuts.* The slot array V of Definition 5.12, $d = |V| = 72$, with the family $\mathfrak{F}$ of its cuts: the chamber cuts $\vartheta_{c,a}$ of the two coordinates and the reserved cut $\iota(u_0, u_1, \ell) = (u_0, u_1, 1 - \ell)$, with plus half $V_+ = \{\ell = 0\}$ and minus half $V_- = \iota V_+$. The family contains the sorting sequence of Lemma 5.13(iii) (Definition 5.7).

(b) *Scales, atoms and the analytic stage.* A fixed integer $K \ge 2$ and, for every scale $\mathsf{L} \ge 1$ with $N = K\mathsf{L}$, the group G_N, its set $\mathcal{A}_N$ of atoms, the lift F_A and the inputs $g^{I,D_0,s}$ of Definition 6.1, with fixed prefix exponents e_p; the analytic stage of Definition 8.1; a density exponent $0 < \theta \le 1$; and a price exponent a with $0 < a < \sigma$. We write $\mathcal{A} = \bigcup_N \mathcal{A}_N$ for the set of atoms of all scales. Every law attached below to an atom $\mathcal{B}$ is a measure on $X_{\mathcal{B}}^V$ that does not depend on the scale.

(c) *Queries.* A family $\mathcal{E}_k$ of $k \ge 1$ directed queries $e = (u_e, v_e)$ with pairwise disjoint endpoints (the $2k$ slots u_e, v_e are distinct), sources $u_e \in V_+$ and reflected targets $\iota v_e \in V_+$.

(d) *Index law.* A fixed modulus $M_0 = \prod_p p^{h_p}$, $h_2 \ge 3$, coprime to every prime of G_N; a product convolution law on nonvanishing square increments at these coordinates; and independent conductor-depth envelopes and moment R_I, as in Definition 9.6. Each exact-conductor space is preserved by the coordinate transition, and the norm of the transition restricted to it is bounded by the corresponding decreasing depth tail. The interval transition is the cyclic shift.

Only (a) refers to the particular array: the arguments of this part use of V only that its family of cuts contains a sorting sequence and the reserved cut, and that the chamber folds commute with ι ((O4) of Definition 8.9; Lemma 5.13(iii)). So the scheme of (CP1)–(CP8) is not tied to this array, and Section 34.2 reads the 216-slot design, with $V = (\mathbb{Z}/6\mathbb{Z})^3$ and the reserved cut $\vartheta_{2,0}$, as

its first instance. The analytic stage of Part II, however, is formulated for 72 slots, and Theorem C is stated for the array of (a), so $d = 72$ throughout; the letter d marks where the number of slots enters an exponent (the slot price exponent a/d and the density factor θ/d). In Part V, $k = 6$ and $\mathcal{E}_6$ is the family of the six queries of Definition 5.12(e), which satisfies (c) by Lemma 5.13(iv).
Energies, the counterexample and $o(1)$. For a finite signed measure $\kappa = \bigotimes_{\mathcal{B}} \kappa_{\mathcal{B}}$ on G_N^V let $C_\iota(\kappa)$ be its cut matrix at ι and $Q_\kappa(X, Y) = \langle X, C_\iota(\kappa) Y\rangle$ its cut form for half-functions X, Y on $G_N^{V_+}$ (Definition 5.2), and put

$$Y_\kappa = Q_\kappa(\bar{X}, \bar{X}) = \Lambda_\kappa(F_A, \dots, F_A),$$

where $\bar{X}$ is the half-function carrying F_A at every slot of V_+. Since every slot carries the same input, Y_κ is also the value at $(\bar{X}, \bar{X})$ of the cut form of κ at any other cut. For the reference μ of (CP1), $Y_\mu = \mathcal{Y}(\mathsf{L}, A)$ and $\alpha^d \le Y_\mu$ (Lemma 5.17). Every argument of this part takes place at a least normalized counterexample (L, A) for (a, θ, C_0) (Definition 8.4), and $Y = Y_\mu > 0$ there. Energies whose selected atoms have a union E with $q_E \le N^\tau$ are structured energies, bounded by their coarse prices (Proposition 8.7), with sharp bounds applied in the order of Lemma 8.8. Every expansion over selected atoms also has terms with $q_E > N^\tau$; their sum is bounded by Lemma 8.11, with the post-reduction prices of the selected laws as the coefficients $\eta_{\mathcal{B}}$, and is $O(\mathsf{L}^{-\sigma}\alpha) = o(1)Y$. Throughout, $o(1)$ denotes a quantity that tends to zero as $\mathsf{L} \to \infty$, uniformly in A. Every $o(1)$ of this part is produced by structured energies and by the two kinds of analytic remainders, the large selections just described and the cancellations of Lemma 8.10; this is the robustness of Definition 9.19, which is what the hypothesis (Rec) of Theorem 8.12 asks for (Lemma 11.4).

At a least counterexample, fixing the labels at an atom $\mathcal{B}$ fixes residue classes, and splitting a class into progressions whose step is a square divisible by $q_{\mathcal{B}}$ returns the problem at a shorter scale (Lemma 8.2); by the induction hypothesis this costs the factor $(q_{\mathcal{B}}^2)^{a/d}$ at each of the d slots. This is the origin of the prices.

Definition 9.2 (Prices)**.** Let $\mathcal{B}$ be an atom and $q_{\mathcal{B}} = \prod_{p \in \mathcal{B}} p$ its block modulus. Its *selection factor* is $q_{\mathcal{B}}^{2a}$; for a prefix atom $\{p\}$ with e_p even we read $q_{\mathcal{B}}^{2a}$ as $p^{e_p a}$, the factor of selecting the full group $\mathbb{Z}/p^{e_p}\mathbb{Z}$ (Definition 8.5).

(a) The *price* of a finite signed measure ζ on $X_{\mathcal{B}}^V$ is $\|\zeta\|_{\mathrm{var}}\, q_{\mathcal{B}}^{2a}$: selecting $\mathcal{B}$ changes every slot.
(b) Let $c \in \mathfrak{F}$ be a cut, κ a nonnegative measure on $X_{\mathcal{B}}^V$ of mass w and ζ a signed measure on $X_{\mathcal{B}}^V$. Then ζ has *sharp bound* d_ζ *relative to* κ *at* c if, for a probability law π on a finite set of shared labels, $\kappa = w \int \kappa_s\, \pi(ds)$ and $\zeta = w \int K_s\, \pi(ds)$ with probability measures κ_s and signed measures K_s that satisfy (8.2) at c relative to the half marginals of κ_s, with the constant d_ζ, for every s. The *sharp price* of ζ is $w\, d_\zeta\, q_{\mathcal{B}}^{2a}$. More generally an *operator price* is a weight, times an operator norm relative to actual half marginals, times $q_{\mathcal{B}}^{2a}$.
(c) An *upper price* is any number at least the price. A *total* is a sum of prices over the atoms of $\mathcal{A}$. In an expansion over selected atoms each selected atom is charged its price once per expanded term.

The unpriced part of a price (a mass, a weight, a weighted norm) does not depend on the exponent, and an atom's laws do not depend on the scale; so a total over $\mathcal{A}$ bounds the corresponding sum over $\mathcal{A}_N$ at every scale.

Lemma 9.3 (Repricing)**.** *Let $0 < a' \le a''$ and $L \ge 0$, and let $\mathcal{P}$ be a family of atoms of primes with $e_p = 1$ such that $\log q_{\mathcal{B}} \le L$ for every $\mathcal{B} \in \mathcal{P}$. Then $q_{\mathcal{B}}^{2a''} \le e^{2(a''-a')L} q_{\mathcal{B}}^{2a'}$ for $\mathcal{B} \in \mathcal{P}$, and for all numbers $c_{\mathcal{B}} \ge 0$ that do not depend on the exponent,*

$$\sum_{\mathcal{B} \in \mathcal{P}} c_{\mathcal{B}}\, q_{\mathcal{B}}^{2a''} \le e^{2(a''-a')L} \sum_{\mathcal{B} \in \mathcal{P}} c_{\mathcal{B}}\, q_{\mathcal{B}}^{2a'}.$$

In particular every price, sharp price or operator price of a law of an atom of $\mathcal{P}$ at the exponent a'' is at most $e^{2(a''-a')L}$ times its value at a'; an upper bound at a' for a sum of such prices (for

instance the block charge (20.1) *of Part V, in the notation there), multiplied by* $e^{2(a''-a')L}$*, bounds the sum at* a''*; and two such steps compose to one step from the first to the last exponent.*

Proof. For each atom, $q_{\mathcal{B}}^{2a''} = q_{\mathcal{B}}^{2a'} e^{2(a''-a')\log q_{\mathcal{B}}} \le q_{\mathcal{B}}^{2a'} e^{2(a''-a')L}$, because $a''-a' \ge 0$ and $\log q_{\mathcal{B}} \le L$. Multiply by $c_{\mathcal{B}} \ge 0$ and sum. A price is its unpriced part times $q_{\mathcal{B}}^{2a}$, and the partition and the laws are the same at both exponents; only the selection factor changes. Two steps with budgets L_1, L_2 multiply to the factor $e^{2(a_1-a_0)L_1+2(a_2-a_1)L_2}$, which is $e^{2(a_2-a_0)L}$ for a common budget L. □

The argument needs $a'' \ge a'$: for a negative difference the upper bound L for $\log q_{\mathcal{B}}$ would point the wrong way.

Corollary 9.4 (The first stage at a larger exponent)**.** *Let* $a_0 \le a$*, and let every first-stage atom of (CP2) be a block of primes with* $e_p = 1$ *and* $\log q_{\mathcal{B}} \le L$*. Let* U_0'*,* P' *and* d_j'*,* v_j' $(j \in \mathfrak{T})$ *be upper bounds for the totals* U_0*,* P *and the tag totals* d_j*,* v_j *of (CP2) and (CP3), computed for the same laws at the exponent* a_0*, and put* $t = e^{2(a-a_0)L}$*. Then at the exponent* a

$$U_0 \le tU_0', \qquad P \le tP', \qquad d_j \le td_j', \qquad v_j \le tv_j' \quad (j \in \mathfrak{T}),$$

and

$$L_0 \le e^{tP'}\mathfrak{h}(td', tv'), \qquad R_{\mathrm{err}} \le e^{tU_0'}\mathfrak{h}(td', tv'), \qquad B \le e^{tU_0'}(1 + \mathfrak{h}(td', tv')).$$

Proof. Every first-stage price is a price of a law of a first-stage atom: $p_{\mathcal{B}}$ and $u_{\mathcal{B}}$ are prices of positive records, $d_{\mathcal{B},i}$ is a sharp price and $v_{\mathcal{B},i}$ the price of two positive records. So Lemma 9.3, applied to the sums over the first-stage atoms and, for the tag totals, to the sums over the differences of each tag, gives the first four bounds. The series $\mathfrak{h}$ and the exponential are nondecreasing in every argument (Remark 9.12), which gives the bounds for $L_0 = e^P\mathfrak{h}(d, v)$ and $R_{\mathrm{err}} = e^{U_0}\mathfrak{h}(d, v)$, and then $B = e^{U_0} + R_{\mathrm{err}} \le e^{tU_0'} + e^{tU_0'}\mathfrak{h}(td', tv')$. □

In the closing of Part V the corollary is applied with the totals of the middle range certified at the anchor $a_0 = \frac{4442183}{5000000000}$ (Lemma 22.9 and Certificate D.4) and the actual middle block budget $L = 570$ (Definition 30.1); the upper prices have their own anchor $a_{\mathrm{u}} = 883/10^6$ and are repriced band by band, each with its own logarithmic product bound (Proposition 28.14).

9.2. **Queries and the random interval-and-residue index.** The primes dividing M_0 are not coordinates of G_N. They enter through a fixed external residue index on each query, together with an interval index. The coupled index is $(b_e + 1, s_e + r_e)$, where r_e has the prescribed product law on nonvanishing square classes. On a nonwrapping interval, accepted-square cancellation follows from Lemma 8.10 and the prime-power kernel extension Proposition 7.3(iii). Uniform averaging still gives exactly the original lift. The residue depths enter the estimates through the conductor moment R_I (Definition 9.6(g)). The following elementary unit-square case also explains the earlier squarefree index and remains useful when bounding its moment.

Lemma 9.5 (Unit squares)**.** *Let* $M_0 = 8d_0$ *with* d_0 *odd and squarefree, and let* $\mathcal{Q}_{M_0}$ *be the set of squares of the unit group* $(\mathbb{Z}/M_0\mathbb{Z})^\times$*.*

- *(i)* $\mathcal{Q}_{M_0} = \{r \in \mathbb{Z}/M_0\mathbb{Z} : \ r \equiv 1 \pmod 8,\ r \bmod p \in Q_p^* \text{ for every prime } p \mid d_0\}$*. Every element of* $\mathcal{Q}_{M_0}$ *is a strict square class modulo* M_0 *(Definition 4.5).*
- *(ii) Under the Chinese remainder isomorphism* $\mathbb{Z}/M_0\mathbb{Z} \cong \mathbb{Z}/8\mathbb{Z} \times \prod_{p|d_0}\mathbb{F}_p$*, the uniform probability on* $\mathcal{Q}_{M_0}$ *is the product of the point mass at* 1 *in* $\mathbb{Z}/8\mathbb{Z}$ *and the uniform probabilities on the sets* Q_p^**,* $p \mid d_0$*. In particular* $|\mathcal{Q}_{M_0}| = \prod_{p|d_0}(p-1)/2$*, and if* $3 \mid d_0$ *the factor at* 3 *is the point mass at* 1*.*

Proof. Since d_0 is odd and squarefree, the Chinese remainder theorem gives $\mathbb{Z}/M_0\mathbb{Z} \cong \mathbb{Z}/8\mathbb{Z} \times \prod_{p|d_0}\mathbb{F}_p$ and $(\mathbb{Z}/M_0\mathbb{Z})^\times \cong (\mathbb{Z}/8\mathbb{Z})^\times \times \prod_{p|d_0}\mathbb{F}_p^\times$. The squares of a product of groups are the products of the squares. Every odd square is $\equiv 1 \pmod 8$, so the squares of $(\mathbb{Z}/8\mathbb{Z})^\times = \{1, 3, 5, 7\}$

form $\{1\}$, and the squares of $\mathbb{F}_p^\times$ form Q_p^*; this is the description in (i), and its elements are $\equiv 1 \pmod 8$ and nonzero squares modulo every prime of d_0, which is the definition of a strict square class. The uniform probability on a product set is the product of the uniform probabilities, $|Q_p^*| = (p-1)/2$, and $Q_3^* = \{1\}$; this is (ii). □

Definition 9.6 (The coupled index)**.** Let $\mathcal{E}_k$ be the query family of (CP0), $\mathsf{L} \geq 1$, $N = K\mathsf{L}$, $A \subseteq [N]$ square-difference-free, and let I_b, $b \in \mathbb{Z}/K\mathbb{Z}$, be the intervals of Definition 6.1: consecutive intervals of length L partitioning $[N]$, each element of I_b smaller than each element of I_{b+1} for $1 \leq b < K$, while $b+1$ wraps I_K to I_1.

(a) *Index space.* $\Omega = \prod_{e \in \mathcal{E}_k} \Omega_e$ with $\Omega_e = \mathbb{Z}/K\mathbb{Z} \times \mathbb{Z}/M_0\mathbb{Z}$ and the uniform probability measure; $\omega = (\omega_e)_e$, $\omega_e = (b_e, s_e)$. By the Chinese remainder theorem $\Omega = \prod_{j \in \mathcal{K}} \Omega_j$ is a product of cyclic groups indexed by the *coordinates* $\mathcal{K} = \bigsqcup_e \mathcal{K}_e$: the interval coordinate b_e and the prime-power residues $s_e \bmod p^{h_p}$ for $p \mid M_0$.

(b) *Half-functions.* For $\omega \in \Omega$ and $x \in G_N^{V_+}$,

$$X_\omega(x) = \prod_e g^{I_{b_e}, M_0, s_e}(x_{u_e}) \prod_{v \in V_+ \setminus \{u_e\}_e} F_A(x_v),$$
$$Y_\omega(x) = \prod_e g^{I_{b_e}, M_0, s_e}(x_{\iota v_e}) \prod_{v \in V_+ \setminus \{\iota v_e\}_e} F_A(x_v),$$

and $\bar{X}(x) = \prod_{v \in V_+} F_A(x_v)$. (The half-function Y_ω always carries an index; it is not one of the energies Y_κ.)

(c) *Coupling.* Independently for every query and prime, choose the specified square increment $r_{e,p} \bmod p^{h_p}$; combine them by the Chinese remainder theorem to obtain r_e. The coupled index is $\omega' = (b+1, s+r)$. Its Markov operator $T_e f(b_e, s_e) = \mathbb{E}_{r_e} f(b_e + 1, s_e + r_e)$ is the tensor product of the interval shift and the coordinate convolutions. Let P_e average over the entire index ω_e, interval coordinate included. All the coordinate transitions are doubly stochastic. The finite laws of the square increments $r_{e,p}$ used in the application are given in Lemma 29.6; the uniform unit-square law of Lemma 9.5 is a special alternative.

(d) *Coupled and signed forms.* For a finite signed measure κ on G_N^V and $\mathcal{S} \subseteq \mathcal{E}_k$ put

$$I_{\kappa,\mathcal{S}} = \mathbb{E}_\omega\, Q_\kappa\Big(X_\omega, \Big(\bigotimes_{e \in \mathcal{S}} T_e \bigotimes_{e \notin \mathcal{S}} P_e\, Y\Big)_\omega\Big), \qquad J_\kappa = \mathbb{E}_\omega\, Q_\kappa(X_\omega, (\mathcal{T}Y)_\omega), \qquad \mathcal{T} = \bigotimes_e (2P_e - T_e),$$

the operators acting on the index variable of Y_ω; they commute with the field labels. J_κ is the *signed index form.*

(e) *Depth law.* On a residue coordinate modulo p^{h_p} let $E_{p,j}$ be conditional expectation given the residue modulo p^j, $0 \leq j \leq h_p$, with $E_{p,0}$ the average. Choose decreasing tails $1 = \delta_{p,0} \geq \delta_{p,1} \geq \cdots \geq \delta_{p,h_p} \geq 0$ bounding the transition norm on the exact-conductor spaces $\operatorname{ran}(E_{p,j} - E_{p,j-1})$. Set $\delta_{p,h_p+1} = 0$ and choose a random depth with $\mathbb{P}(J_{e,p} \geq j) = \delta_{p,j}$, independently across primes and queries. Always reveal the interval. Write $\mathcal{J}$ for this whole depth vector and $Z^{\mathcal{J}}$ for the corresponding conditional expectation. Bernoulli inclusion is the special case of a chain containing only the constant and full-coordinate projections.

(f) *Energies.* For a nonnegative measure M on G_N^V with $M \succeq_\iota 0$ the *single-half random-inclusion energies* of X and Y under M are

$$V_M(X) = \mathbb{E}_{\mathcal{J}} \mathbb{E}_\omega\, Q_M(X_\omega^{\mathcal{J}}, X_\omega^{\mathcal{J}}), \qquad V_M(Y) = \mathbb{E}_{\mathcal{J}} \mathbb{E}_\omega\, Q_M(Y_\omega^{\mathcal{J}}, Y_\omega^{\mathcal{J}}),$$

the two summands of the energy of the one-cut coupling inequality (Corollary 10.3) without its factor $\frac{1}{2}$.

(g) *Conductor moment.* Put $D_{\mathcal{J}_e} = \prod_{p|M_0} p^{J_{e,p}}$. With $\mathrm{sq}(D)$ the least square divisible by D, the one-query moment is

$$R_I = \mathbb{E}_{\mathcal{J}}(K\ \mathrm{sq}(D_{\mathcal{J}_e}))^{2a/d}.$$

It does not depend on e. Depths are held fixed through every fold and structured-energy estimate, and averaged only afterwards.

The coupled forms place the two ends of every query of $\mathcal{S}$ on the residue classes s_e and $s_e + r_e$ of the intervals I_{b_e} and I_{b_e+1}. Indeed, by the bilinear identity (5.1), $Q_\kappa(X_\omega, Y_{\omega''}) = \Lambda_\kappa((k_v)_{v\in V})$ with $k_{u_e} = g^{I_{b_e},M_0,s_e}$, $k_{v_e} = g^{I_{b_e''},M_0,s_e''}$ and $k_v = F_A$ otherwise, because the slot $\iota(\iota v_e) = v_e$ carries the factor of $Y_{\omega''}$ at ιv_e; and an averaged index $e \notin \mathcal{S}$ leaves F_A at both of its ends. In $V_M(Y)$ the map Y is evaluated at a uniform index; since ω' is uniform as well (the T_j are doubly stochastic), this is the natural normalization. The depth law gives the exact norm envelope on each conductor vector $\mathbf{j}$: the envelope $\prod_p \delta_{p,j_p}$ is exactly the probability that all the depths j_p are revealed. The index-support lemma, with its filtration form below, applies to both the transition and its adjoint. By Lemma 6.3, $X^{\mathcal{J}}_\omega$ carries $g^{I_{b_e},D_{\mathcal{J}_e},s_e \bmod D_{\mathcal{J}_e}}$ at u_e and F_A at the other slots of V_+, and likewise $Y^{\mathcal{J}}_\omega$ at the slots ιv_e (Corollary 10.4(ii)).

Lemma 9.7 (Conductor moment)**.** *For $w \ge 0$ and every query e,*

$$\mathbb{E}_{\mathcal{J}}(K\ \mathrm{sq}(D_{\mathcal{J}_e}))^w = K^w \prod_{p|M_0} \left[1 + (p^{2w}-1) \sum_{l=1}^{\lceil h_p/2\rceil} p^{2w(l-1)} \delta_{p,2l-1}\right].$$

The expectation of the product over k queries is the k-th power of this expression. Every factor is nondecreasing in w and in the chosen tails. For the earlier squarefree law $M_0 = 8d_0$, with 8 always revealed and Bernoulli odd-prime probabilities δ_p, the expression is $(16K)^w \prod_{p|d_0}(1 + \delta_p(p^{2w}-1))$.

Proof. The local least square is $p^{2\lceil J_{e,p}/2\rceil}$. For an integer-valued nonnegative variable J,

$$\mathbb{E}z^{\lceil J/2\rceil} = 1 + (z-1)\sum_{l\ge 1} z^{l-1}\mathbb{P}(J \ge 2l-1).$$

Apply this finite summation-by-parts identity with $z = p^{2w}$; independence across primes and queries proves the assertions. The Bernoulli specialization follows from $\mathrm{sq}(8) = 16$ and $\mathrm{sq}(p) = p^2$. In the selected construction $w = 2a/72 = a/36$ and the finite-depth moment is evaluated in Proposition 29.7. □

The signed index form. Expanding the product of the commuting operators $2P_e - T_e$, $\bigotimes_e(2P_e - T_e) = \sum_{\mathcal{S}}(-1)^{|\mathcal{S}|}2^{k-|\mathcal{S}|}\bigotimes_{e\in\mathcal{S}} T_e \bigotimes_{e\notin\mathcal{S}} P_e$, so for every finite signed measure κ

$$J_\kappa = \sum_{\mathcal{S}\subseteq\mathcal{E}_k} (-1)^{|\mathcal{S}|}2^{k-|\mathcal{S}|} I_{\kappa,\mathcal{S}}, \qquad I_{\kappa,\varnothing} = Y_\kappa. \tag{9.1}$$

The second identity holds because every unused index averages its slot function to the full lift: $\mathbb{E}_{b,s} g^{I_b,M_0,s} = F_A$ for uniform (b,s) (Lemma 6.3), and X_ω, Y_ω are products of functions of the independent ω_e at the distinct slots u_e, respectively ιv_e; so $(\bigotimes_e P_e Y)_\omega = \mathbb{E}Y = \bar{X}$, $\mathbb{E}X = \bar{X}$, and $I_{\kappa,\varnothing} = Q_\kappa(\mathbb{E}X, \bar{X}) = Y_\kappa$ by bilinearity. The constant coefficient 2^k of $\mathcal{T}$ is the factor 2^k of the closing: the empty-support component of J_ν is Y_ν itself.

Remark 9.8 (The choice of the coupling). Nothing in the proofs depends on the following two facts. No doubly stochastic kernel on $\mathbb{F}_p$ supported on nonzero square shifts has a smaller centred norm than the uniform square shift (Lemma 4.11(b)). Reusing one index for two queries generally correlates their inputs, and the product mean identity $\mathbb{E}X = \bar{X}$ then has no justification; this is why the queries have pairwise disjoint endpoints and independent indices.

9.3. **Hypotheses (CP1)–(CP8) on the three laws.** The hypotheses are conditions, atom by atom, on the reference law μ, the first-stage law λ and the signed comparison law ν. We first fix the words *record*, *diagonal* and *bracket*, then state (CP1)–(CP8) in order.

Definition 9.9 (Records, diagonals and brackets)**.** A *record* of an atom $\mathcal{B}$ is a nonnegative measure on $X_{\mathcal{B}}^{V}$, and a *positive record* is a record whose normalization is admissible (Definition 8.9); in particular its one-slot marginals are uniform. A record is *diagonal at a cut c* if it is fixed samplewise by the reflection of c; at ι we say *reserved diagonal*. A diagonal record has a nonnegative diagonal cut matrix at its cut, so it is $\succeq_c 0$ (Lemma 5.4(i)). For a probability law m on $X_{\mathcal{B}}^{V_+}$, $\mathrm{Diag}_\iota(m)$ is its *diagonal copy* (Lemma 5.4(i)): the plus half has law m and the minus half is its reflection; its cut matrix is $\mathrm{diag}(m)$. For a measure ζ with image $\iota_*\zeta$ under the reflection, $\frac{1}{2}(\zeta+\iota_*\zeta)$ and $\frac{1}{2}(\zeta-\iota_*\zeta)$ are the *symmetric* and *skew parts* of ζ at ι; their cut matrices are the symmetric and the antisymmetric part of $C_\iota(\zeta)$ (Remark 5.3(a)). A *bracket* at ι is a bracket of measures $\left(\begin{smallmatrix} M & K \\ K^* & M' \end{smallmatrix}\right) \succeq_\iota 0$ in the sense of Section 5.2, a positive semidefinite block matrix of cut matrices at ι. It gives $|Q_K(x,y)| \le Q_M(x,x)^{1/2} Q_{M'}(y,y)^{1/2}$, brackets tensorize over atoms, and for K symmetric at ι, $\left(\begin{smallmatrix} M & K \\ K & M \end{smallmatrix}\right) \succeq_\iota 0$ if and only if $M \pm K \succeq_\iota 0$ (Lemma 5.6(a),(b),(d)).

Definition 9.10 ((CP1) Reference)**.** The *reference* at scale N is a product $\mu = \bigotimes_{\mathcal{B}\in\mathcal{A}_N} \mu_{\mathcal{B}}$ of probability measures that satisfies (R1)–(R3) of Hypothesis 6.2, with (R2) at every cut of $\mathfrak{F}$, and the part of (R4) that concerns μ: at every tail prime $p \ge P_5$, μ_p is (γ, v)-mixing for every slot v.

Reflection positivity is required at every cut of the family, not only along the sorting sequence (see the remark after Hypothesis 6.2): the first comparison and the bracket energies of the next section, and the restorations (Lemma 24.8) and companion energies (Lemma 26.6) of Part V, use chamber cuts outside the sorting sequence.

Definition 9.11 ((CP2) First stage)**.** A set $\mathcal{A}^{\mathrm{fs}} \subseteq \mathcal{A}$ of *first-stage atoms*, a finite set $\mathfrak{T}$ of *tags* with a cut $c_j \in \mathfrak{F}$ for every $j \in \mathfrak{T}$, and, for $\mathcal{B} \in \mathcal{A}^{\mathrm{fs}}$, signed measures $\rho_{\mathcal{B}}$ and $\delta_{E,\mathcal{B}}$ on $X_{\mathcal{B}}^{V}$ (E anticipates the application, where the tagged differences below are those of the matching records; see Remark 9.12) such that:

(a) $\rho_{\mathcal{B}} - \mu_{\mathcal{B}} = \sum_i c_i \kappa_i$ is a finite real combination of positive records κ_i, with *variation price* $p_{\mathcal{B}} = \sum_i |c_i| \kappa_i(1)\, q_{\mathcal{B}}^{2a}$, and $\rho_{\mathcal{B}} \succeq_{\vartheta_0} \mu_{\mathcal{B}}$ at one cut $\vartheta_0 \in \mathfrak{F}$, the same for all first-stage atoms;

(b) $\delta_{E,\mathcal{B}} = \sum_i (\kappa'_{\mathcal{B},i} - \kappa_{\mathcal{B},i})$ is a finite sum of differences of positive records of equal mass; each difference carries a tag $j(\mathcal{B},i) \in \mathfrak{T}$ and has a sharp bound relative to $\kappa_{\mathcal{B},i}$ at the cut $c_{j(\mathcal{B},i)}$ of its tag (Definition 9.2(b)); $d_{\mathcal{B},i}$ is its sharp price and $v_{\mathcal{B},i} = (\kappa_{\mathcal{B},i}(1) + \kappa'_{\mathcal{B},i}(1)) q_{\mathcal{B}}^{2a}$ its *variation price*, the price of both positive parts of the difference.

The *first stage* is $\lambda = \bigotimes_{\mathcal{B}} \lambda_{\mathcal{B}}$ with $\lambda_{\mathcal{B}} = \rho_{\mathcal{B}} + \delta_{E,\mathcal{B}}$ on $\mathcal{A}^{\mathrm{fs}}$ and $\lambda_{\mathcal{B}} = \mu_{\mathcal{B}}$ otherwise. Its totals are $P = \sum_{\mathcal{B}} p_{\mathcal{B}}$, the *tag totals*

$$d_j = \sum_{(\mathcal{B},i):\, j(\mathcal{B},i)=j} d_{\mathcal{B},i}, \qquad v_j = \sum_{(\mathcal{B},i):\, j(\mathcal{B},i)=j} v_{\mathcal{B},i} \qquad (j \in \mathfrak{T}),$$

$D = \sum_j d_j$ and $V = \sum_j v_j$, and its *cost* is

$$L_0 = e^P \mathfrak{h}(d,v), \qquad \mathfrak{h}(d,v) = \sum_{\boldsymbol{k} \in \mathbb{Z}_{\ge 0}^{\mathfrak{T}} \setminus \{0\}} \min_{j:\, k_j \ge 1} \frac{d_j^{k_j} \prod_{j' \ne j} v_{j'}^{k_{j'}}}{\prod_{j' \in \mathfrak{T}} k_{j'}!},$$

where $d = (d_j)_j$, $v = (v_j)_j$ and $\boldsymbol{k} = (k_j)_{j\in\mathfrak{T}}$ runs over the nonzero *count vectors*.

Remark 9.12 (Tags). The series $\mathfrak{h}$ has nonnegative terms, and it is nondecreasing in every argument, a minimum of nondecreasing functions being nondecreasing. It converges: with $m_j = \max(d_j, v_j)$ every term at $\boldsymbol{k}$ is at most $\prod_j m_j^{k_j}/k_j!$, so $\mathfrak{h}(d,v) \le e^{\sum_j m_j} - 1$. If $d_j \le v_j$ for every tag, then

$\mathfrak{h}(d,v) \le De^V$, so $L_0 \le De^{P+V}$, the cost of expanding with one sharp difference per term: bound the minimum at $\boldsymbol{k}$ by the term of one tag j with $k_j \ge 1$, use $d_j^{k_j} \le d_j v_j^{k_j-1}$ and $k_j! \ge (k_j-1)!$, and sum over the $\boldsymbol{k}$ assigned to each j. The condition $d_j \le v_j$ can always be arranged when the two positive parts of every difference have the same half marginals in every shared label: a difference of two probability couplings with common half marginals satisfies (8.2) with the constant 2, each coupling being a contraction between its marginal spaces, so its sharp constant may be taken at most 2 and then $d_{\mathcal{B},i} \le v_{\mathcal{B},i}$. In that sense tags only sharpen. A tag collects the differences that are sharp at a common cut, and all of them are applied sharply together (Lemma 10.8); with a single tag the series is $\mathfrak{h} = e^{d_0} - 1$, and its gain over De^V comes from the differences being paid sharply at once instead of one at a time. In Part V there are two tags, the two folded matching records of the one-stage repair, each sharp at its own first cut, with $d_0 = d_1 = D/2$ and $v_0 = v_1 = V/2$ (Lemma 22.6 and Corollary 10.9).

Definition 9.13 ((CP3) Gram bracket)**.** The *bracket* is $\mathsf{N} = \bigotimes_{\mathcal{B}} \mathsf{N}_{\mathcal{B}}$ with $\mathsf{N}_{\mathcal{B}} = \mu_{\mathcal{B}}$ off the first-stage atoms and $\mathsf{N}_{\mathcal{B}} = \mathsf{N}_{0,\mathcal{B}} + \delta_{E,\mathcal{B}}$ on them, where $\mathsf{N}_{0,\mathcal{B}} - \mu_{\mathcal{B}}$ is a finite sum of positive reserved-diagonal records, of total price $u_{\mathcal{B}}$, such that for every atom $\lambda_{\mathcal{B}}$ is symmetric at ι, $\mathsf{N}_{\mathcal{B}}$ is nonnegative, and

$$\begin{pmatrix} \mathsf{N}_{\mathcal{B}} & \lambda_{\mathcal{B}} \\ \lambda_{\mathcal{B}} & \mathsf{N}_{\mathcal{B}} \end{pmatrix} \succeq_\iota 0, \qquad \text{equivalently} \qquad \mathsf{N}_{\mathcal{B}} \pm \lambda_{\mathcal{B}} \succeq_\iota 0.$$

We also write $\mathsf{N}_0 = \bigotimes_{\mathcal{B}} \mathsf{N}_{0,\mathcal{B}}$ for the record-difference-free part of the bracket, with $\mathsf{N}_{0,\mathcal{B}} = \mu_{\mathcal{B}}$ off the first-stage atoms. Its totals are $U_0 = \sum_{\mathcal{B}} u_{\mathcal{B}}$ and $R_{\mathrm{err}} = e^{U_0}\mathfrak{h}(d,v)$, with the tags and prices of (CP2)(b), and the *energy factor* of the bracket is

$$B = e^{U_0} + R_{\mathrm{err}}.$$

The bracket is the substitute for reflection positivity of λ at the reserved cut: λ need not be positive there, but it is dominated in absolute value by a positive measure whose energies are priced. An energy with outside law N on the atoms outside its selected set X costs at most B times its price (Lemma 10.12): e^{U_0} for the part without record differences and R_{err} for the rest.

Definition 9.14 ((CP4) Comparison law)**.** A *comparison law* is a product $\nu = \bigotimes_{\mathcal{B}} \nu_{\mathcal{B}}$ of finite signed measures on $X_{\mathcal{B}}^V$, each a finite real combination $\nu_{\mathcal{B}} = \sum_{i \in I_{\mathcal{B}}} c_{\mathcal{B},i}\nu_{\mathcal{B},i}$ of nonnegative measures $\nu_{\mathcal{B},i}$, its *components*, with $\nu_{\mathcal{B}}(1) > 0$, and with $|I_{\mathcal{B}}| = 1$, $c_{\mathcal{B},1} = 1$ at all but finitely many atoms of $\mathcal{A}$, such that:

(a) *Acceptance.* Every query of $\mathcal{E}_k$ is accepted by every component: $\nu_{\mathcal{B},i}(1) > 0$ and, $\nu_{\mathcal{B},i}$-almost surely, for every e, the difference $z_{v_e} - z_{u_e}$ is a nonzero square at every prime of $\mathcal{B}$ with $e_p = 1$, and at a prefix prime it is nonzero, of even valuation $2j < e_p$, with a square unit part. All one-slot marginals of every component are multiples of the uniform law. With the normalized components $\bar\nu_{\mathcal{B},i} = \nu_{\mathcal{B},i}/\nu_{\mathcal{B},i}(1)$ as the accepted parts, (R5) and the part of (R4) that concerns $\bar\nu$ hold, the latter for the queries of $\mathcal{E}_k$ (in Part V, $\mathcal{E}_k = \mathcal{E}_6$).

(b) *Records and brackets.* We have

$$\nu_{\mathcal{B}} - \lambda_{\mathcal{B}} = \sum_{t \in \mathcal{T}_{\mathcal{B}}} (\zeta^{\mathrm{s}}_{\mathcal{B},t} + \zeta^{\mathrm{a}}_{\mathcal{B},t}),$$

a finite sum over the records t of $\mathcal{B}$ that the comparison law changes (repairs, replaces or deletes), where $\zeta^{\mathrm{s}}_{\mathcal{B},t}$ and $\zeta^{\mathrm{a}}_{\mathcal{B},t}$ are the symmetric and skew parts at ι of the change of t; and there are reserved-diagonal probability laws $D^{\mathrm{s}}_{\mathcal{B},t}$, $D^{\mathrm{a}}_{\mathcal{B},t}$ with uniform one-slot marginals (diagonal copies of actual half marginals of t, in the applications) and numbers $\tilde\pi^{\mathrm{s}}_{\mathcal{B},t}, \tilde\pi^{\mathrm{a}}_{\mathcal{B},t} \ge 0$ with

$$\begin{pmatrix} \tilde\pi^{\mathrm{s}}_{\mathcal{B},t} D^{\mathrm{s}}_{\mathcal{B},t} & \zeta^{\mathrm{s}}_{\mathcal{B},t} \\ \zeta^{\mathrm{s}}_{\mathcal{B},t} & \tilde\pi^{\mathrm{s}}_{\mathcal{B},t} D^{\mathrm{s}}_{\mathcal{B},t} \end{pmatrix} \succeq_\iota 0, \qquad \begin{pmatrix} \tilde\pi^{\mathrm{a}}_{\mathcal{B},t} D^{\mathrm{a}}_{\mathcal{B},t} & \zeta^{\mathrm{a}}_{\mathcal{B},t} \\ \zeta^{\mathrm{a}*}_{\mathcal{B},t} & \tilde\pi^{\mathrm{a}}_{\mathcal{B},t} D^{\mathrm{a}}_{\mathcal{B},t} \end{pmatrix} \succeq_\iota 0.$$

The diagonal entries are the *norm envelopes* of the change; their coefficients carry no price. The prices are $\pi^{\epsilon}_{\mathcal{B},t} = \tilde{\pi}^{\epsilon}_{\mathcal{B},t} q_{\mathcal{B}}^{2a}$ ($\epsilon \in \{\mathrm{s},\mathrm{a}\}$), $s_{\mathcal{B}} = \sum_t \pi^{\mathrm{s}}_{\mathcal{B},t}$, $a_{\mathcal{B}} = \sum_t \pi^{\mathrm{a}}_{\mathcal{B},t}$, and the totals are

$$S = \sum_{\mathcal{B}} s_{\mathcal{B}}, \qquad A = \sum_{\mathcal{B}} a_{\mathcal{B}}.$$

(c) *Selections.* A *selection* ω is a nonempty finite set $X \subseteq \mathcal{A}_N$ of atoms with a record $t_{\mathcal{B}} \in \mathcal{T}_{\mathcal{B}}$ and a type $\epsilon_{\mathcal{B}} \in \{\mathrm{s},\mathrm{a}\}$ for every $\mathcal{B} \in X$, the number of skew types being even. Its signed measure, its monomial and its term are

$$\zeta_\omega = \bigotimes_{\mathcal{B}\in X} \zeta^{\epsilon_{\mathcal{B}}}_{\mathcal{B},t_{\mathcal{B}}}, \qquad \pi_\omega = \prod_{\mathcal{B}\in X} \pi^{\epsilon_{\mathcal{B}}}_{\mathcal{B},t_{\mathcal{B}}}, \qquad I_\omega = Y_{\zeta_\omega \otimes \lambda_{-X}}, \qquad \lambda_{-X} = \bigotimes_{\mathcal{B}\notin X} \lambda_{\mathcal{B}}.$$

The *variation factor* of ν is $\mathfrak{v}_\nu = \prod_{\mathcal{B}\in\mathcal{A}} |\nu|_{\mathcal{B}}/\nu_{\mathcal{B}}(1)$, with $|\nu|_{\mathcal{B}} = \sum_{i\in I_{\mathcal{B}}} |c_{\mathcal{B},i}|\,\nu_{\mathcal{B},i}(1)$, and $n_\nu = \prod_{\mathcal{B}\in\mathcal{A}} |I_{\mathcal{B}}|$ is the number of its *product components* $\bigotimes_{\mathcal{B}} c_{\mathcal{B},i_{\mathcal{B}}}\nu_{\mathcal{B},i_{\mathcal{B}}}$. Both are finite, do not depend on the scale, and equal one when every $\nu_{\mathcal{B}}$ is represented by a single nonnegative component with coefficient one; every factor $|\nu|_{\mathcal{B}}/\nu_{\mathcal{B}}(1)$ is at least one.

Summing the brackets of (b) over t gives blockwise brackets with the diagonal laws $D^{\epsilon}_{\mathcal{B}} = \sum_t \tilde{\pi}^{\epsilon}_{\mathcal{B},t} D^{\epsilon}_{\mathcal{B},t} / \sum_t \tilde{\pi}^{\epsilon}_{\mathcal{B},t}$ and the coefficients $\sum_t \tilde{\pi}^{\epsilon}_{\mathcal{B},t}$. A term of the expansion of $\bigotimes_{\mathcal{B}}(\lambda_{\mathcal{B}} + \sum_t(\zeta^{\mathrm{s}}_{\mathcal{B},t} + \zeta^{\mathrm{a}}_{\mathcal{B},t}))$ chooses at most one record and one type at each atom; since $\lambda_{\mathcal{B}}$ is symmetric at ι, a term with an odd number of skew factors has a skew cut matrix and vanishes at $(\bar{X}, \bar{X})$. This is why selections carry an even number of skew types. In the construction of Part V, $\nu_{\mathcal{B}}$ is normalized by the normalizing mass of $\mu_{\mathcal{B}}$, so that retained records keep their weights. In Part V the middle blocks carry signed combinations, through their fold records (three components each, Definition 23.9); every other atom law is nonnegative. The signs enter the theorem only through the cancellation of the accepted queries, which is applied to each product component separately (Lemma 10.6); the brackets of (b) and every energy bound concern the record changes and the envelopes, not the components.

Remark 9.15 (Charging rules). (i) Let a record $w\kappa$, with κ a probability measure symmetric at ι and m its half marginal, be deleted, so that $-w\kappa$ is its change. Then $\zeta^{\mathrm{s}} = -w\kappa$, $\zeta^{\mathrm{a}} = 0$, $D^{\mathrm{s}} = \mathrm{Diag}_\iota(m)$ and $\tilde{\pi}^{\mathrm{s}} = w$ satisfy (b): a deleted positive record is charged its mass in $\tilde{\pi}^{\mathrm{s}}$ only. Indeed $|Q_\kappa(f,f)| \le \int |f(z_+)||f(\iota z_-)|\,d\kappa \le \|f\|^2_{L^2(m)} = Q_{\mathrm{Diag}_\iota(m)}(f,f)$, where ιz_- denotes the minus-half labels read on V_+, so $\mathrm{Diag}_\iota(m) \pm \kappa \succeq_\iota 0$. (ii) Let a record $w\Pi$ be replaced by $w\Pi'$, where Π, Π' are probability measures with the same half marginal m on both halves of ι, and let Δ be the operator on $L^2(m)$ with $\langle \varphi, \Delta\psi\rangle_m = \int \varphi(z_+)\psi(\iota z_-)\,d(\Pi' - \Pi)(z)$, with symmetric and skew parts $\Delta^{\mathrm{s}}, \Delta^{\mathrm{a}}$. Then (b) holds with $D^{\mathrm{s}} = D^{\mathrm{a}} = \mathrm{Diag}_\iota(m)$, $\tilde{\pi}^{\mathrm{s}} = w\|\Delta^{\mathrm{s}}\|$ and $\tilde{\pi}^{\mathrm{a}} = w\|\Delta^{\mathrm{a}}\|$: such a repair is charged its weight times its operator norms. Indeed the cut matrix of $\Pi' - \Pi$ is $\mathrm{diag}(m)\Delta$ and that of $\mathrm{Diag}_\iota(m)$ is $\mathrm{diag}(m)$, so the form of the symmetric bracket at (f,g) is $c\|f\|^2_m + 2\langle f, \Delta^{\mathrm{s}} g\rangle_m + c\|g\|^2_m$, $c = \|\Delta^{\mathrm{s}}\|$, which is nonnegative; likewise for the skew part. (iii) *Skew-only charging.* In (ii) let Π be symmetric at ι. Then Δ^{a} is the skew part of the operator Δ' of Π' alone, $\langle \varphi, \Delta'\psi\rangle_m = \int \varphi(z_+)\psi(\iota z_-)\,d\Pi'(z)$: the operator of $\Pi' - \Pi$ is $\Delta' - \Delta_\Pi$, with Δ_Π the operator of Π, which is self-adjoint. So a replacement of a symmetric record is charged in the skew price only by the skew part of the new law, $\tilde{\pi}^{\mathrm{a}} = w\|\Delta'^{\mathrm{a}}\|$, whatever the symmetric record it replaces; and a deleted symmetric record has no skew charge, as in (i).

Definition 9.16 ((CP5) Class bounds)**.** Optionally: for every atom $\mathcal{B}$ finitely many numbers $y_{\mathcal{B},j} \ge 0$ ($j \in J_{\mathcal{B}}$), the *auxiliary prices*, each attached to one record $t \in \mathcal{T}_{\mathcal{B}}$ of (CP4)(b) and one type $\epsilon \in \{\mathrm{s},\mathrm{a}\}$ (in Part V, operator prices of parts of the record changes: raw-arm, leading and remainder prices, Lemma 23.24); a family of pairwise disjoint *classes* $\mathcal{C}$ of selections, each selection of a class having at most $n_{\max}$ atoms, with $n_{\max}$ a fixed integer independent of the scale, of A and of the class; and for every selection ω of a class a polynomial ϕ_ω with nonnegative coefficients in

the auxiliary prices, its *class bound*, each of whose monomials is a product, over the atoms $\mathcal{B}$ of ω, of one auxiliary price attached to the record and type $(t_{\mathcal{B}}, \epsilon_{\mathcal{B}})$ of ω at $\mathcal{B}$; such that robustly and uniformly in $\mathcal{C}$

$$\sum_{\omega\in\mathcal{C}} |I_\omega| \le \sum_{\omega\in\mathcal{C}} (\phi_\omega + o(1)\pi_\omega) Y.$$

A *class cost* $0 \le c_{\mathcal{C}} \le B$ is the case $\phi_\omega = c_{\mathcal{C}}\pi_\omega$, the prices $\pi^{\epsilon_{\mathcal{B}}}_{\mathcal{B},t_{\mathcal{B}}}$ of the selected records serving as auxiliary prices; its *gain* is $B - c_{\mathcal{C}}$.

Every selection costs at most B times its monomial (Lemma 10.14); a class cost below B, or class bounds below B times the monomials of their class, record a saving proved for that class: a variance credit (Section 24), a bound by positive dominating matrices (Section 25), or both. The requirement $c_{\mathcal{C}} \le B$ is no restriction: since every class satisfies the bound with the cost B, a proved cost $c \ge 0$ may always be replaced by $\min(c, B)$. In particular a cost certified below an upper bound B' of B gives the class cost $\min(c, B) \le c$, and the closing may be evaluated with c and B' (Remark 9.22). A class bound in auxiliary prices is used when it improves, in the evaluated totals, on B times the full monomials of its class; the degree-class replacement of Lemma 10.14(v) is the case used in Part V. Since a monomial of a class bound contains one auxiliary price of each record of its selection, it determines the selection; this is what lets the class bounds of all selections be added up as one power series.

Definition 9.17 ((CP6) Matching envelopes)**.** For $s \in \{\mathrm{L}, \mathrm{R}\}$ and every atom, nonnegative measures $M_{s,\mathcal{B}}$ such that $M_{s,\mathcal{B}} - \mathsf{N}_{\mathcal{B}}$ is a finite nonnegative combination of positive reserved-diagonal records, of price $T_{s,\mathcal{B}}$, and

$$\begin{pmatrix} M_{\mathrm{L},\mathcal{B}} & \nu_{\mathcal{B}} \\ \nu^*_{\mathcal{B}} & M_{\mathrm{R},\mathcal{B}} \end{pmatrix} \succeq_\iota 0$$

for every atom. The *matching exponent* is a number T with $\sum_{\mathcal{B}} T_{s,\mathcal{B}} \le T$ for both sides s.

Definition 9.18 ((CP7) Mean energies)**.** A number m, the *mean credit*, such that robustly

$$\tfrac{1}{2}(e_{\mathrm{L}} + e_{\mathrm{R}}) \ge (m - o(1))\, Y, \qquad e_s = Q_{\bigotimes_{\mathcal{B}} M_{s,\mathcal{B}}}(\bar{X}, \bar{X}),$$

where e_s is the *mean energy* of the envelope of the side s.

The envelopes are allowed to differ on the two sides of the reserved cut, and each half-function is measured only under its own envelope. The mean energies enter the closing through the common mean of the signed index (Lemma 10.1): the empty-support component of J_ν is Y_ν, and that of the envelope energies is e_s.

Definition 9.19 (Robust bounds)**.** Let $z \ge 0$. A least normalized counterexample (L, A), for (a, θ, C_0) with any $C_0 \ge 1$, is *z-inflated* for an argument if every structured energy $\mathfrak{E}$ that the argument bounds at (L, A) satisfies $|\mathfrak{E}| \le (1+z)\|\rho\|\,\Pi\, Y$ with its coarse price Π, and the absolute values of its *remainders* (the integrals of the form bounded in Lemma 8.10 and the sums of the form bounded in Lemma 8.11 that it uses) add up to at most zY. An inequality $\Xi \le (c + o(1))Y$, with Ξ a quantity attached to (L, A) and c a number, is *robust* if it has an argument such that:

(a) its remainders are a number, bounded independently of L and A, of integrals of the form of Lemma 8.10, each integrated against a probability law of the indices and multiplied by a coefficient bounded independently of L and A, and finitely many sums of the form of Lemma 8.11 whose constants c and Π_{H} (in the notation of that lemma) do not depend on L and A;

(b) there are $z_0 > 0$, L_1 and a function $c(\cdot)$ on $[0, z_0]$, continuous at 0 with $c(0) = c$, such that $\Xi \le c(z)Y$ at every z-inflated least normalized counterexample with $\mathsf{L} \ge \mathsf{L}_1$ and $z \le z_0$.

Robust lower bounds $\Xi \geq (c - o(1))Y$ are defined in the same way. A family of bounds is robust *uniformly* if one argument, one z_0 and L_1, and one modulus of continuity at 0 serve for all of them; in particular the remainders of the whole family together satisfy (a).

A robust bound holds with $o(1)$ in the ordinary sense. By Proposition 8.7, every least normalized counterexample with L large is z-inflated with $z = u_{\mathsf{L}} + (C_{\mathrm{dc}} + C_{\mathrm{R}})\mathsf{L}^{a-\sigma}$, which tends to zero uniformly in A because $a < \sigma$ and $C_0 \geq 1$; here C_{R} bounds the remainders allowed by (a), through Lemmas 8.10 and 8.11, as in the proof of Theorem 8.12.

Definition 9.20 ((CP8) Admissibility, robustness, summability)**.**

(a) *Admissibility.* Every law named in (CP2)–(CP6) (the positive records of (CP2)(a), the records $\kappa_{\mathcal{B},i}, \kappa'_{\mathcal{B},i}$, the records of $\mathsf{N}_{0,\mathcal{B}} - \mu_{\mathcal{B}}$, the laws $D^{\epsilon}_{\mathcal{B},t}$ and the records of $M_{s,\mathcal{B}} - \mathsf{N}_{\mathcal{B}}$) is admissible, and so is every selected law that the reductions of this part produce from them; every reduction of this part satisfies (O3) and (O4) of Definition 8.9.

(b) *Robustness.* The bounds of (CP5) and (CP7) are robust (Definition 9.19), those of (CP5) uniformly in the class.

(c) *Summability.* With $q_{\mathcal{B}}$ here denoting the product of the primes of $\mathcal{B}$ with $e_p = 1$ only (so the weight $q_{\mathcal{B}}^{1/4}$ is 1 at a prefix atom, whose prices in Definition 9.2 nevertheless carry the factor $p^{e_p a}$), $v_{\mathcal{B}} = \sum_i v_{\mathcal{B},i}$ and $d_{\mathcal{B}} = \sum_i d_{\mathcal{B},i}$ as in (CP2), $s_{\mathcal{B}}, a_{\mathcal{B}}$ as in (CP4) and the auxiliary prices of (CP5),

$$\sum_{\mathcal{B}\in\mathcal{A}} q_{\mathcal{B}}^{1/4}\Big(p_{\mathcal{B}} + v_{\mathcal{B}} + d_{\mathcal{B}} + u_{\mathcal{B}} + s_{\mathcal{B}} + a_{\mathcal{B}} + T_{\mathrm{L},\mathcal{B}} + T_{\mathrm{R},\mathcal{B}} + \sum_{j\in J_{\mathcal{B}}} y_{\mathcal{B},j}\Big) < \infty.$$

Part (c) makes the totals P, V, D (and the tag totals), U_0, S, A, T and the totals of the auxiliary prices finite and independent of the scale, and it makes the product Π_{H} of Lemma 8.11 finite for the coefficients that the expansions produce: these are post-reduction prices of the selected laws, and only the tail atoms form an infinite family. A bound is shown to be robust by the criterion of Lemma 11.4(i): it is built from structured-energy bounds by sums, products and monotone operations, or it is an exact relation minus such bounds, and every strict inequality it needs holds strictly at $z = 0$.

9.4. **Theorem C: if $\Theta < 1$ then $s(N) \leq C' N^{1-a/(72-\theta)}$.**

How to read the statement. Theorem C is conditional: it assumes laws satisfying (CP0)–(CP8) and concludes the power saving from the single numerical inequality $\Theta < 1$. The symbols in Θ are: $d = 72$ slots and k queries ($k = 6$ in Part V); L_0 the total price of $\mu \to \lambda$ (CP2); B the energy factor of the law dominating λ (CP3); S, A the total symmetric and skew prices of $\nu - \lambda$ (CP4), with $\hat{S}, \hat{A}$ any finite upper bounds for them; $\mathcal{E}$ any number at least $\Psi(\hat{\pi}, \hat{y})$, where Ψ is a *complete polynomial*—a power series with nonnegative coefficients dominating every simultaneous-change bound, defined in Remark 9.22 below and constructed in Lemma 10.14 (without (CP5) families, $\Psi(\hat{\pi}) = B(e^{\hat{S}}\cosh\hat{A} - 1)$; the simpler display $\mathcal{E} = B(e^{S}\cosh A - 1) - D$ in Section 1.6 is the Part V evaluation with savings D subtracted); T the total price of the dominators for ν (CP6), entering through e^T; R_I the conductor factor of one index (Definition 9.6); m the mean-energy lower bound (CP7), whose trivial admissible value is $1 - L_0$ (Remark 10.18); and J, W as displayed. Hatted quantities are certified upper bounds: by monotonicity (Remark 9.22) the theorem may be applied with upper bounds for prices and a lower bound for m. Note $0 < \theta \leq 1$ in (CP0): the improved exponent $\vartheta = \frac{1014587}{1000000} > 1$ of Theorem 1.1 is not obtained by substituting ϑ for θ here, but by the separate induction of Sections 31 to 33. In (CP4) the letter ω denotes a *selection* (a choice of changed records), not the random index ω of Definition 9.6; the context (prices versus intervals) distinguishes them.

Theorem 9.21 (Theorem C. If $\Theta < 1$ then $s(N) \le C' N^{1-a/(72-\theta)}$)**.** *Let comparison data satisfy (CP0)–(CP8) at every scale. Let $\hat\pi \ge \pi$ be upper prices for the records of (CP4), with totals $\hat S \ge S$ and $\hat A \ge A$ finite; let $\hat y \ge y$ be upper auxiliary prices of (CP5), of finite total $\sum_{\mathcal{B},j} \hat y_{\mathcal{B},j}$; let Ψ be a complete polynomial of Lemma 10.14(ii) for the classes and class bounds of (CP5) (for instance in the completed form (iv) or the replaced form (v) of that lemma) that converges at $(\hat\pi, \hat y)$, and let $\mathcal{E} \ge \Psi(\hat\pi, \hat y)$. Put*

$$J = Be^T\Big(\frac{R_I}{2}\Big)^k, \qquad W = Be^{S+A}\Big[\Big(1 + \frac{R_I}{2K^{1-2\theta/d}}\Big)^k - 1\Big],$$
$$\Theta = (1-2^{-k})(L_0 + \mathcal{E}) + J + W + \frac{1-m}{2^k}.$$

If $\Theta < 1$, there is a constant C, depending only on the comparison data, such that $\mathcal{Y}(\mathsf{L}, A) \le C\mathsf{L}^{-a}\alpha^\theta$ for every $\mathsf{L} \ge 1$ and every square-difference-free set $A \subseteq [K\mathsf{L}]$, $\alpha = |A|/(K\mathsf{L})$. Consequently

$$s(N) \le C' N^{1-a/(d-\theta)} \qquad \text{for every } N \ge 1.$$

The proof is in Section 11. Without classes, the complete polynomial of Lemma 10.14(ii) is $\Psi = B(e^{S(x)}\cosh A(x) - 1)$ and $\Psi(\hat\pi) = B(e^{\hat S}\cosh \hat A - 1)$; a class replaces B times its monomials by its class bounds.

Remark 9.22 (Monotonicity). Θ is nondecreasing in L_0, $\mathcal{E}$, B, S, A, T and R_I, and nonincreasing in m: $1 - 2^{-k} > 0$, J is nondecreasing in B, T and R_I, and W in B, $S + A$ and R_I, all of these being nonnegative. A complete polynomial is any power series with nonnegative coefficients that dominates, coefficient by coefficient, the series $\sum_\omega b_\omega$ of Lemma 10.14(ii), in which a selection in no class contributes B on its monomial, a selection of a class with a class cost that cost on its monomial, and a selection of a class with class bounds its class bound. So every power series with nonnegative coefficients dominating these, for instance one built from certified upper bounds for B, for the class costs and for the coefficients of the class bounds, gives an admissible $\mathcal{E}$ when it is evaluated at upper prices and upper auxiliary prices. Hence the theorem holds with certified upper bounds for L_0, B, S, A, T, R_I, the prices, the auxiliary prices, the class costs and the coefficients of the class bounds, and a certified lower bound for m, in place of the quantities themselves; this is how the closing of Part V evaluates Θ.

Remark 9.23 (Reading the closing number). The sum $L_0 + \mathcal{E}$ bounds $|Y_\nu - Y_\mu|/Y$ up to $o(1)$, and since the empty term of the signed-index expansion is Y_ν itself, the bound $L_0 + \mathcal{E}$ enters Θ with the coefficient $(2^k - 1)/2^k$ (see the opening of Section 10). In J each query contributes $R_I/2$; W bounds the terms in which every query to which the coupled transition, rather than the average, is applied in that term wraps; and $m > 1$ lowers Θ through $(1 - m)/2^k$. In the baseline application $k = 6$, $d = 72$, $\theta = 1$, so $1 - 2^{-k} = 63/64$ and $K^{1-2\theta/d} = K^{35/36}$ with $K = 890666$, giving $\varepsilon = a/71$ and $\Theta < 0.99999999997327$ (Section 30). These are baseline values, quoted here only as an illustration; Theorem C itself uses no Part V number.

Consequences and what to read next. From Part II: the slot array and its cuts (Lemma 5.13, Definition 5.7), the cut calculus (Definition 5.2, Equation (5.1), Remark 5.3, Lemma 5.4, Lemma 5.6), the functional and its inputs (Definition 6.1, Lemma 6.3), the reference hypotheses (Hypothesis 6.2), the square shift (Lemma 4.11), the sector lemma (Lemma 6.10), and the analytic notions of Definitions 8.1, 8.4, 8.5 and 8.9, Equation (8.2), Lemmas 8.10 and 8.11.

10. Five estimates at a least counterexample

What this section proves and how to read it. The comparison theorem (Theorem 9.21) reduces the baseline theorem to the inequality $\Theta < 1$ between finitely many prices. Its proof, in the next section, composes five inequalities proved here at a least normalized counterexample: the first

comparison $Y_\lambda \ge (1-L_0-o(1))Y$ (Lemma 10.10); the reserved cost $|Y_\nu - Y_\lambda| \le (\mathcal{E}+o(1))Y$, with $\mathcal{E}$ a complete polynomial in the prices and the auxiliary prices (Lemma 10.14); the wrap $|J_\nu - 2^k Y_\nu| \le (2^k W + o(1))Y$ (Lemma 10.6); the index-support inequality $J_\nu \le Y_\nu + \frac{1}{2}[(V_{\mathrm{L}} - e_{\mathrm{L}}) + (V_{\mathrm{R}} - e_{\mathrm{R}})]$ (Lemma 10.1 and Corollary 10.4); and the envelope bound $V_{\mathrm{L}}, V_{\mathrm{R}} \le (2^k J + o(1))Y$ (Lemma 10.17). They rest on one calculus of energies under the reserved Gram bracket (Lemmas 10.8, 10.12 and 10.13), which Part V uses as well: for the variance credits and envelope class bounds of (CP5), together with the variance form of Lemma 10.16 (Lemmas 24.12 and 25.9, Proposition 24.17, Theorem 25.8, and Corollary 25.10), for the mean credit (CP7) (Proposition 26.7), and in the closing, which evaluates the complete polynomial (Certificate 30.3). Every statement uses only the frame data and the hypotheses of Theorem 9.21 that it lists, and Part II; none refers to a particular local law.

How the inequalities fit together. The reference μ is reflection positive at every cut and defines $Y = Y_\mu$. The first-stage law λ is a law that dominates μ in the cut order at one cut of the family, plus record differences grouped in tags, each sharp at the cut of its tag, so it loses only L_0. The comparison law ν accepts every query and is a finite signed combination of nonnegative accepting laws, but it is not reflection positive; the reserved Gram bracket controls it record by record. The signed index form J_ν of (9.1) carries the constant coefficient 2^k on Y_ν, and every other term of its expansion couples a nonempty set of queries and cancels by the actual-square kernel unless all its intervals wrap. On the other side, J_ν is bounded by the random-inclusion energies of the matching envelopes. Decompose by the exact support of the index functions: the component of J_ν with empty support is Y_ν itself, and that of each energy is the mean energy of the envelope, which dominates Y_λ. Keeping these two components exactly, rather than bounding J_ν by the full envelope energies without subtracting their mean energies, saves one of the 2^k copies of the reserved comparison cost: in Θ the coefficient of $\mathcal{E}$ is $1 - 2^{-k}$, while the matching term and the wrap keep coefficient one (Remark 10.18). The index step bounds each half-function only under its own envelope and uses no symmetry of ν and no odd-skew cancellation; skewness enters only through the exact identity of the empty component.

Conventions. Notation, energies Y_κ, the least normalized counterexample and $o(1)$ are those of the previous section: atoms are written $\mathcal{B}$, $B = e^{U_0} + R_{\mathrm{err}}$, $\mathsf{N}_{\mathcal{B}} = \mathsf{N}_{0,\mathcal{B}} = \lambda_{\mathcal{B}} = \mu_{\mathcal{B}}$ off the first-stage atoms, $d = |V| = 72$, (L, A) is a least normalized counterexample with L larger than the thresholds of the lemmas used in this section (every $o(1)$ is as $\mathsf{L} \to \infty$) and $Y = Y_\mu > 0$. A positive measure on the labels X_X^V of a finite set X of atoms is *admissible* in the sense of Definition 8.9: its one-slot marginals are multiples of the uniform law, it does not depend on the indices, and every sharp bound applied to it is uniform in the shared labels (on one atom, its normalization is a positive record). Prices are those of Definition 9.2, and $C_X = \prod_{\mathcal{B}\in X} q_{\mathcal{B}}^{2a}$ is the price of selecting every atom of X.

Remainders. Every bound below is robust in the sense of Definition 9.19: its $o(1)$ is a finite sum, with coefficients bounded independently of L and A, of *energy errors* (the relative error of Proposition 8.7, multiplying a structured-energy bound once), *cancellation remainders* (integrals of the form bounded in Lemma 8.10) and *large-product remainders* (the terms of an expansion whose union E of selected atoms has $q_E > N^\tau$, bounded together by Lemma 8.11 with the post-reduction prices of the selected laws as coefficients). Each statement lists which kinds occur; this is what Lemma 11.4(ii) uses.

Fixed selections and summed remainders. An individual asymptotic bound below with a selected set X is a bound for fixed X; its large-product remainder need not be uniform as X varies. In a sum over selections we first expand the outside laws and split by the *full* union E of selected atoms. On $q_E \le N^\tau$ the structured-energy error is uniform in E and its labels. All terms with $q_E > N^\tau$ are instead summed by Lemma 8.11, with their post-reduction coefficients. Thus no summed argument below uses a pointwise $o(1)$ for an arbitrary large selected set. For fixed X, the

same procedure also proves the individual asymptotic statements: if Z is the further selected set, then

$$\sum_{Z:\, q_X q_Z > N^\tau} \prod_{\mathcal{B}\in Z} \eta_{\mathcal{B}} \le N^{-\tau/4} q_X^{1/4} \prod_{\mathcal{B}} (1 + \eta_{\mathcal{B}} q_{\mathcal{B}}^{1/4}).$$

The factor $q_X^{1/4}$ is fixed in that assertion. When X varies in a priced expansion it is included in the full union before Rankin's inequality is applied, and (CP8)(c) bounds the resulting product.

10.1. **Bounding a bilinear form by energies of random restrictions.** The small primes enter the comparison only through the external index. The next lemma is the inequality behind every index estimate: the one-cut coupling inequality, the bound of the signed index form by random-inclusion energies, and the common mean are its cases.

Lemma 10.1 (Index-support lemma)**.** *Let $\mathcal{X}_+$ be a finite set, and M_{L}, M_{R}, Γ real matrices indexed by $\mathcal{X}_+ \times \mathcal{X}_+$, with $M_{\mathrm{L}}, M_{\mathrm{R}}$ symmetric and*

$$\begin{pmatrix} M_{\mathrm{L}} & \Gamma \\ \Gamma^{\mathsf{T}} & M_{\mathrm{R}} \end{pmatrix} \succeq 0. \tag{10.1}$$

Let $\Omega = \prod_{j\in\mathcal{K}} \Omega_j$ be a finite product of finite sets with the uniform probability measure. On coordinate j, let $E_{j,0} \le \cdots \le E_{j,h_j} = I$ be a chain of conditional-expectation projections, with $h_j \ge 1$ and $E_{j,0}$ the projection onto constants. Put $\Pi_{j,0} = E_{j,0}$, $\Pi_{j,l} = E_{j,l} - E_{j,l-1}$ for $l \ge 1$, and

$$H_{\boldsymbol{l}} = \bigotimes_{j\in\mathcal{K}} \operatorname{ran} \Pi_{j,l_j}, \qquad 0 \le l_j \le h_j.$$

Choose tails $1 = \delta_{j,0} \ge \delta_{j,1} \ge \cdots \ge \delta_{j,h_j} \ge 0$ and put $\delta_{\boldsymbol{l}} = \prod_j \delta_{j,l_j}$. Let $\mathcal{T}$ on $L^2(\Omega)$ preserve every $H_{\boldsymbol{l}}$, act as the identity on $H_{\mathbf{0}}$, and have norm at most $\delta_{\boldsymbol{l}}$ on $H_{\boldsymbol{l}}$. It acts componentwise on vector-valued maps. Independently of ω, reveal independent depths $J_j \in \{0,\dots,h_j\}$ with $\mathbb{P}(J_j \ge l) = \delta_{j,l}$, write $\mathcal{J} = (J_j)_j$, and set $Z^{\mathcal{J}} = (\bigotimes_j E_{j,J_j})Z$. For $X, Y : \Omega \to \mathbb{R}^{\mathcal{X}_+}$ put

$$\mathsf{J} = \mathbb{E}_\omega \langle X(\omega), \Gamma(\mathcal{T}Y)(\omega)\rangle, \qquad e_{\mathrm{L}} = \langle \mathbb{E}X, M_{\mathrm{L}}\, \mathbb{E}X\rangle, \qquad e_{\mathrm{R}} = \langle \mathbb{E}Y, M_{\mathrm{R}}\, \mathbb{E}Y\rangle,$$
$$V_{\mathrm{L}} = \mathbb{E}_{\mathcal{J}} \mathbb{E}_\omega \left\langle X^{\mathcal{J}}(\omega), M_{\mathrm{L}} X^{\mathcal{J}}(\omega)\right\rangle, \qquad V_{\mathrm{R}} = \mathbb{E}_{\mathcal{J}} \mathbb{E}_\omega \left\langle Y^{\mathcal{J}}(\omega), M_{\mathrm{R}} Y^{\mathcal{J}}(\omega)\right\rangle.$$

Then $V_{\mathrm{L}} \ge e_{\mathrm{L}} \ge 0$, $V_{\mathrm{R}} \ge e_{\mathrm{R}} \ge 0$, and

(a)

$$\left|\mathsf{J} - \langle \mathbb{E}X, \Gamma\, \mathbb{E}Y\rangle\right| \le \tfrac{1}{2}\left[(V_{\mathrm{L}} - e_{\mathrm{L}}) + (V_{\mathrm{R}} - e_{\mathrm{R}})\right]; \tag{10.2}$$

(b) $|\mathsf{J}| \le V_{\mathrm{L}}^{1/2} V_{\mathrm{R}}^{1/2}$.

The whole-coordinate case is $h_j = 1$: identify $\mathcal{J}$ with the set of coordinates whose depth is one. Then $Z^{\mathcal{J}} = \mathbb{E}[Z \mid \omega_{\mathcal{J}}]$, each coordinate is included independently with probability $\delta_j = \delta_{j,1}$, and the sector indexed by $S = \{j : l_j = 1\}$ is $H_S = \bigotimes_{j\in S} L_0^2(\Omega_j) \otimes \bigotimes_{j\notin S} \mathbb{R}\mathbf{1}$, with weight $\delta_S = \prod_{j\in S} \delta_j$.

Proof. Factorization. The diagonal blocks of (10.1) are positive semidefinite, and $\Gamma = M_{\mathrm{L}}^{1/2} C M_{\mathrm{R}}^{1/2}$ with $\|C\| \le 1$ (Lemma 5.6). Put $\|Z\|_M^2 = \mathbb{E}_\omega \langle Z(\omega), MZ(\omega)\rangle$ for symmetric $M \succeq 0$. Matrices acting on $\mathbb{R}^{\mathcal{X}_+}$ commute with $\mathcal{T}$ and with (conditional) expectations over ω.

Decomposition. Nested orthogonal projections satisfy $E_{j,l}E_{j,l'} = E_{j,\min(l,l')}$. Their differences $\Pi_{j,l}$ are therefore mutually orthogonal projections summing to I. Tensorizing gives $L^2(\Omega) = \bigoplus_{\boldsymbol{l}} H_{\boldsymbol{l}}$, so componentwise $Z = \sum_{\boldsymbol{l}} Z_{\boldsymbol{l}}$ and $Z_{\mathbf{0}} = \mathbb{E}Z$. These components are orthogonal for every form $\mathbb{E}_\omega \langle \cdot, M\,\cdot\rangle$ with $M \succeq 0$, and

$$Z^{\mathcal{J}} = \sum_{\boldsymbol{l}:\, l_j \le J_j \text{ for every } j} Z_{\boldsymbol{l}}.$$

This is the depth version of the exact-support decomposition of Lemma 6.10.

The form and its energy envelope. Since $\mathcal{T}$ preserves the sectors, only equal sectors pair in J. On the sector $H_{\boldsymbol{l}}$ the factorization gives

$$\left|\mathbb{E}_\omega \langle X_{\boldsymbol{l}}, \Gamma\mathcal{T} Y_{\boldsymbol{l}}\rangle\right| = \left|\mathbb{E}_\omega \left\langle M_{\mathrm{L}}^{1/2} X_{\boldsymbol{l}}, (C\otimes\mathcal{T}) M_{\mathrm{R}}^{1/2} Y_{\boldsymbol{l}}\right\rangle\right| \le \delta_{\boldsymbol{l}} \|X_{\boldsymbol{l}}\|_{M_{\mathrm{L}}} \|Y_{\boldsymbol{l}}\|_{M_{\mathrm{R}}}.$$

The zero-sector term is $\langle \mathbb{E}X, \Gamma\,\mathbb{E}Y\rangle$. The same coefficient $\delta_{\boldsymbol{l}} = \mathbb{P}(J_j \ge l_j \text{ for every } j)$ is the probability that the random projection retains this sector. Orthogonality therefore gives the energy identity

$$V_{\mathrm{L}} = \sum_{\boldsymbol{l}} \delta_{\boldsymbol{l}} \|X_{\boldsymbol{l}}\|_{M_{\mathrm{L}}}^2 = e_{\mathrm{L}} + \sum_{\boldsymbol{l}\neq\boldsymbol{0}} \delta_{\boldsymbol{l}} \|X_{\boldsymbol{l}}\|_{M_{\mathrm{L}}}^2,$$

and likewise for V_{R}. In the whole-coordinate case this is Lemma 6.10(d).

Conclusion. For (a), sum the bound for the form over $\boldsymbol{l}\neq\boldsymbol{0}$ and use $2xy \le x^2+y^2$. For (b), sum over all $\boldsymbol{l}$ and apply Cauchy–Schwarz with weights $\delta_{\boldsymbol{l}}$. The energy identity also gives $V_s \ge e_s \ge 0$ for $s\in\{\mathrm{L},\mathrm{R}\}$. □

Remark 10.2 (The inclusion and depth laws). In the whole-coordinate special case, the inclusion law makes the energy exactly the norm envelope of the index operator: $\mathbb{P}(S\subseteq\mathcal{J}) = \delta_S$ turns the bound δ_S on H_S into the weight of H_S in V_{L} and V_{R}. The proof uses only $\mathbb{P}(S\subseteq\mathcal{J}) \ge \delta_S$, and among such laws the independent one minimizes every moment later paid for the inclusions: for $w_j \ge 1$, $\mathbb{E}\prod_{j\in\mathcal{J}} w_j = \sum_S \mathbb{P}(S\subseteq\mathcal{J}) \prod_{j\in S}(w_j-1) \ge \prod_j (1+\delta_j(w_j-1))$. For the finite-depth law of (CP0)(d), independence instead gives $\mathbb{P}(J_j \ge k_j \text{ for every } j) = \prod_j \delta_{j,k_j}$. These joint tails give the sector weights in the filtration form of the lemma and the conductor moment R_I of Lemma 9.7.

The instructive base case is the plain coupling inequality, in which one positive form plays all three roles.

Corollary 10.3 (One-cut coupling inequality)**.** *In the whole-coordinate case of Lemma 10.1, let $C\succeq 0$ be symmetric, $\|F\|_C^2 = \langle F, CF\rangle$, and let T_j be doubly stochastic matrices on Ω_j with $\|T_j f\| \le \delta_j \|f\|$ for f of mean zero. Let ω be uniform, $\mathbb{P}(\omega' = \eta \mid \omega) = \prod_j T_j(\omega_j,\eta_j)$, and $\mathcal{J}$ independent of (ω,ω'). With $\mathsf{I} = \mathbb{E}\langle X(\omega), CY(\omega')\rangle$ and $\mathsf{V} = \frac12 \mathbb{E}_{\mathcal{J}}\mathbb{E}_\omega[\|X^{\mathcal{J}}\|_C^2 + \|Y^{\mathcal{J}}\|_C^2]$,*

$$\tfrac12\|\mathbb{E}X+\mathbb{E}Y\|_C^2 - \mathsf{V} \;\le\; \mathsf{I} \;\le\; \mathsf{V} - \tfrac12\|\mathbb{E}X-\mathbb{E}Y\|_C^2; \tag{10.3}$$

in particular $|\mathsf{I}|\le\mathsf{V}$. No relation between $\mathbb{E}X$ and $\mathbb{E}Y$ is assumed.

Proof. $\mathbb{E}[Y(\omega') \mid \omega] = (TY)(\omega)$ with $T = \bigotimes_j T_j$. Each T_j fixes the constants (its rows sum to one) and maps $L_0^2(\Omega_j)$ into itself (its columns sum to one), so T satisfies the hypotheses on $\mathcal{T}$ (Lemma 6.10(a)). Apply (a) with $M_{\mathrm{L}} = M_{\mathrm{R}} = \Gamma = C$, for which (10.1) holds as $\left(\begin{smallmatrix} C & C \\ C & C\end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1\end{smallmatrix}\right)\otimes C$: since $\mathsf{V} = \frac12(V_{\mathrm{L}}+V_{\mathrm{R}})$, it gives $|\mathsf{I} - \langle \mathbb{E}X, C\,\mathbb{E}Y\rangle| \le \mathsf{V} - \frac12(\|\mathbb{E}X\|_C^2 + \|\mathbb{E}Y\|_C^2)$, which is (10.3). □

Corollary 10.4 (The signed index form and the common mean)**.** *Assume (CP0)(c),(d) of Theorem 9.21. Let $\nu = \bigotimes_{\mathcal{B}} \nu_{\mathcal{B}}$ be a product of finite signed measures on the $X_{\mathcal{B}}^V$ (for instance the comparison law of (CP4)), and let $M_{\mathrm{L},\mathcal{B}}$, $M_{\mathrm{R},\mathcal{B}}$ be nonnegative measures with $\left(\begin{smallmatrix} M_{\mathrm{L},\mathcal{B}} & \nu_{\mathcal{B}} \\ \nu_{\mathcal{B}}^* & M_{\mathrm{R},\mathcal{B}}\end{smallmatrix}\right) \succeq_\iota 0$ for every atom, as in (CP6). Let J_ν be the signed index form of* (9.1)*, X_ω, Y_ω, $\bar X$ the half-functions of Definition 9.6, $e_s = Q_{\otimes M_s}(\bar X,\bar X)$ for $s\in\{\mathrm{L},\mathrm{R}\}$, and $V_{\mathrm{L}} = V_{\otimes M_{\mathrm{L}}}(X)$, $V_{\mathrm{R}} = V_{\otimes M_{\mathrm{R}}}(Y)$ the single-half random-inclusion energies of Definition 9.6. Then:*

- *(i) $\mathcal{T} = \bigotimes_e (2P_e - T_e)$, and likewise $\mathcal{T}^*$, maps every exact-conductor sector H_S of the index space into itself, with S a vector of conductor depths, is the identity on $H_{\boldsymbol{0}}$ and has norm at most δ_S on H_S;*
- *(ii) $\mathbb{E}_\omega X_\omega = \mathbb{E}_\omega Y_\omega = \bar X$, and for every value of $\mathcal{J}$, $X_\omega^{\mathcal{J}}$ carries $g^{I_{b_e}, D_{\mathcal{J}_e}, s_e \bmod D_{\mathcal{J}_e}}$ at u_e and F_A at the other slots of V_+, likewise $Y_\omega^{\mathcal{J}}$ at the slots ιv_e;*

(iii) $V_s \ge e_s \ge 0$,

$$|J_\nu - Y_\nu| \le \tfrac{1}{2}\big[(V_{\mathrm{L}} - e_{\mathrm{L}}) + (V_{\mathrm{R}} - e_{\mathrm{R}})\big], \tag{10.4}$$

and $|J_\nu| \le V_{\mathrm{L}}^{1/2} V_{\mathrm{R}}^{1/2}$.

Proof. (i) The coordinates are the interval and the full prime-power residues of Definition 9.6. Convolution preserves each exact conductor space, as does its adjoint. The interval has the two-sector chain of constants and their orthogonal complement, with tail one. On a conductor vector S, put S_e for its coordinates belonging to query e. If $S_e = \mathbf{0}$, then $P_e = T_e = I$ on this sector. Otherwise $P_e = 0$ and $2P_e - T_e = -T_e$, whose norm is bounded by the product of the tails belonging to S_e. Tensoring over queries proves (i), for both $\mathcal{T}$ and its adjoint. For whole-coordinate Bernoulli inclusion the same argument is the ordinary support proof.

(ii) For uniform (b, s), $\mathbb{E}_{b,s} g^{I_b, M_0, s} = F_A$, and conditioning a uniform residue on its revealed depths $\mathcal{J}_e$ turns $g^{I, M_0, s}$ into $g^{I, D_{\mathcal{J}_e}, s \bmod D_{\mathcal{J}_e}}$, with $D_{\mathcal{J}_e}$ the product of the revealed prime powers (Lemma 6.3). The queries have pairwise disjoint endpoints and independent indices, so X_ω and Y_ω are products of functions of the independent ω_e at distinct slots, with F_A at the remaining slots; expectations and conditional expectations factorize over e, and the interval coordinates lie in $\mathcal{J}$ with probability one.

(iii) Let M_{L}, M_{R}, Γ be the cut matrices at ι of $\bigotimes_{\mathcal{B}} M_{\mathrm{L},\mathcal{B}}$, $\bigotimes_{\mathcal{B}} M_{\mathrm{R},\mathcal{B}}$ and ν; by (5.2) and the tensorization of brackets (Lemma 5.6) they satisfy (10.1). Then $\mathsf{J} = J_\nu$, $\left\langle \bar{X}, \Gamma \bar{X} \right\rangle = Y_\nu$, and the depth envelope of (CP0)(d) is that of the filtration form of Lemma 10.1; apply that lemma with (i) and (ii). □

Remark 10.5 (Where the gain comes from). With $M_{\mathrm{L}} = M_{\mathrm{R}} = \Gamma \succeq 0$, Lemma 10.1(a) is the plain inequality $|\mathsf{J} - e| \le V - e$ for the common mean energy e of one positive form, and gives nothing new: the empty components on the two sides coincide. The gain comes from envelopes whose mean energies $e_{\mathrm{L}}, e_{\mathrm{R}}$ exceed the mean energy Y_ν of the law itself. Only single-half energies occur, X under M_{L} and Y under M_{R}, which is the order in which (10.1) holds; the bracket with the envelopes exchanged is neither proved nor needed.

10.2. **The signed average over intervals: cancellation except when intervals wrap.** Recall from Definition 9.6 that P_e averages over the whole index of the query e and that T_e is the coupled transition to the next interval with a square residue increment. The signed index form uses the signed combination $2P_e - T_e$ in place of T_e. Expanding the product over the k queries (9.1) isolates the term $2^k Y_\nu$, in which every query is fully averaged. Every other term applies the actual coupled transition T_e to a nonempty set of queries; such a term pairs two points in ordered intervals forced into a square class, so it is negligible by the kernel estimate—unless an interval wraps ($b_e = K$, Section 9.2), where the two intervals are no longer ordered and cancellation fails. The lemma bounds the wrapping terms by the quantity W. The signed combination is controlled by the same 2×2 domination as the ordinary coupling (Corollary 10.4), through Lemma 10.17 below, whose proof does not depend on this lemma.

Lemma 10.6 (Expansion and wrap)**.** *Assume (CP0)–(CP4) and (CP8) of Theorem 9.21, and put*

$$W = B e^{S+A}\Big[\Big(1 + \frac{R_I}{2K^{1-2\theta/d}}\Big)^k - 1\Big].$$

Then $|J_\nu - 2^k Y_\nu| \le (2^k W + o(1))Y$. *The* $o(1)$ *consists of at most* $3^k n_\nu$ *cancellation remainders, with coefficients at most* $2^k \mathfrak{v}_\nu e^{P+S}$, *and of the energy errors and large-product remainders of the wrapping terms.*

Proof. By (9.1), $J_\nu - 2^k Y_\nu = \sum_{\mathcal{S} \ne \varnothing} (-1)^{|\mathcal{S}|} 2^{k-|\mathcal{S}|} I_{\nu,\mathcal{S}}$, where $I_{\nu,\mathcal{S}}$ carries the ordinary coupling on the queries of $\mathcal{S} \subseteq \mathcal{E}_k$ and the full average on the others. Split $I_{\nu,\mathcal{S}} = \sum_{\mathcal{W} \subseteq \mathcal{S}} I_{\nu,\mathcal{S},\mathcal{W}}$ by the set $\mathcal{W} = \{e \in \mathcal{S} : b_e = K\}$ of wrapping active queries; there are 3^k pairs $\mathcal{W} \subseteq \mathcal{S}$.

The mass of ν. On a first-stage atom $\lambda_{\mathcal{B}}(1) = 1+(\rho_{\mathcal{B}}-\mu_{\mathcal{B}})(1) \le 1+p_{\mathcal{B}}$, as the record differences of (CP2) have mass zero. The cut matrix of $\zeta^{\mathrm{a}}_{\mathcal{B},t}$ is skew, so $\zeta^{\mathrm{a}}_{\mathcal{B},t}(1) = 0$, and the bracket of $\zeta^{\mathrm{s}}_{\mathcal{B},t}$ in (CP4) at the all-ones vector gives $|\zeta^{\mathrm{s}}_{\mathcal{B},t}(1)| \le \tilde{\pi}^{\mathrm{s}}_{\mathcal{B},t}$. As $q_{\mathcal{B}}^{2a} \ge 1$, $0 < \nu(1) \le \prod_{\mathcal{B}}(1+p_{\mathcal{B}}+s_{\mathcal{B}}) \le e^{P+S}$. Hence, every factor $|\nu|_{\mathcal{B}}/\nu_{\mathcal{B}}(1)$ of (CP4) being at least one, $\prod_{\mathcal{B}\in\mathcal{A}_N} |\nu|_{\mathcal{B}} \le \mathfrak{v}_\nu\, \nu(1) \le \mathfrak{v}_\nu e^{P+S}$.

Nonwrapping terms. If $\mathcal{W} \ne \mathcal{S}$, pick $e \in \mathcal{S}\setminus\mathcal{W}$. Then $b_e \le K-1$, the inputs $g^{I_{b_e},M_0,s_e}$ at u_e and $g^{I_{b_e+1},M_0,s_e+r_e}$ at v_e lie on adjacent ordered intervals, and r_e is a nonvanishing square class modulo M_0, by (CP0)(d) and Lemma 29.6. Expand every $\nu_{\mathcal{B}}$ into its components: ν becomes a sum of at most n_ν product components $\bigotimes_{\mathcal{B}} c_{\mathcal{B},i_{\mathcal{B}}}\nu_{\mathcal{B},i_{\mathcal{B}}}$. By (CP4)(a) the normalized factors $\bar{\nu}_{\mathcal{B},i_{\mathcal{B}}}$ of each product component accept every query, have uniform one-slot marginals and satisfy (R4) and (R5) of Hypothesis 6.2; their product is a probability measure of the form required in Lemma 8.10, and these are the hypotheses of that lemma for (u_e, v_e); the other slots carry inputs of Definition 6.1(f). At fixed indices each product component contributes $\prod_{\mathcal{B}} c_{\mathcal{B},i_{\mathcal{B}}}\nu_{\mathcal{B},i_{\mathcal{B}}}(1)$ times an integral bounded in that lemma, uniformly in the indices, so the bound persists after integration against their conditional law. Summing the absolute values of the coefficients over the product components gives $\prod_{\mathcal{B}\in\mathcal{A}_N}|\nu|_{\mathcal{B}}$. So $I_{\nu,\mathcal{S},\mathcal{W}}$, with its expansion coefficient, is a sum of at most n_ν cancellation remainders whose coefficients add up to at most $2^k\prod_{\mathcal{B}}|\nu|_{\mathcal{B}} \le 2^k\mathfrak{v}_\nu e^{P+S}$. No other step of this proof uses the components: the wrapping terms below are bounded through the brackets of (CP3) and (CP4), which hold for the signed $\nu_{\mathcal{B}}$, and not through the variation of ν.

Wrapping terms. Let $\mathcal{W} = \mathcal{S}$ and $j = |\mathcal{S}|$. All j intervals wrap with probability K^{-j}. Given this, the averages P_e ($e \notin \mathcal{S}$) replace those inputs by F_A on both sides, and the term is $K^{-j}\mathbb{E}_s\,\langle X'(s), \Gamma(T'Y')(s)\rangle$: $X'(s)$ carries g^{I_K,M_0,s_e} at u_e and $Y'(s)$ carries g^{I_1,M_0,s_e} at $\iota(v_e)$ ($e \in \mathcal{S}$), both carry F_A elsewhere, s is uniform, $\Gamma = C_\iota(\nu)$, and T' is the residue part of the coupling on $\mathcal{S}$, a tensor product of doubly stochastic kernels with the norms given by the conductor tails of (CP0)(d) on exact-conductor spaces. On every atom the brackets of (CP3) and (CP4) add up to

$$\begin{pmatrix} M_{\mathcal{B}} & \nu_{\mathcal{B}} \\ \nu^*_{\mathcal{B}} & M_{\mathcal{B}} \end{pmatrix} \succeq_\iota 0, \qquad M_{\mathcal{B}} = \mathsf{N}_{\mathcal{B}} + \sum_t \big(\tilde{\pi}^{\mathrm{s}}_{\mathcal{B},t} D^{\mathrm{s}}_{\mathcal{B},t} + \tilde{\pi}^{\mathrm{a}}_{\mathcal{B},t} D^{\mathrm{a}}_{\mathcal{B},t}\big),$$

and these tensorize (Lemma 5.6). As in the proof of Corollary 10.3, T' satisfies the hypotheses of Lemma 10.1, and part (b) bounds the term by $K^{-j}(V'_{\mathrm{L}}V'_{\mathrm{R}})^{1/2}$, with $V'_{\mathrm{L}}, V'_{\mathrm{R}}$ the single-half random-inclusion energies of X' and Y' under $\bigotimes_{\mathcal{B}} M_{\mathcal{B}}$. Here $M_{\mathcal{B}} - \mathsf{N}_{\mathcal{B}}$ is a nonnegative combination of the admissible positive reserved-diagonal laws $D^{\epsilon}_{\mathcal{B},t}$, of total price $S+A$, so Lemma 10.17, with $T_Q = S+A$ and the fixed intervals I_K, I_1 ($\kappa_e = K$, Remark 10.7), gives $V'_{\mathrm{L}}, V'_{\mathrm{R}} \le (Be^{S+A}(K^{2\theta/d}R_I)^j + o(1))Y$. Hence $|I_{\nu,\mathcal{S},\mathcal{S}}| \le (Be^{S+A}(R_IK^{2\theta/d-1})^j + o(1))Y$.

Summation. Summing 2^{k-j} times this over the $\binom{k}{j}$ sets of size j,

$$\sum_{j=1}^{k}\binom{k}{j}2^{k-j}Be^{S+A}(R_IK^{2\theta/d-1})^j = Be^{S+A}\big[(2+R_IK^{2\theta/d-1})^k - 2^k\big] = 2^kW. \qquad \square$$

Remark 10.7 (The wrap uses the coarse interval factor). In the wrapping terms the intervals are fixed, so each changed index slot has the coarse factor $\kappa_v = K$ of Definition 8.5 ($\alpha_I \le K\alpha$), which Proposition 8.7 charges as $K^{\theta/d}$. An active query changes two slots of each single-half energy, so the wrap pays $K^{2\theta/d}$ per active query; at $d = 72$, $\theta = 1$ the ratio per query is $R_I/K^{35/36}$, as in Theorem 9.21. The finer factor $(\bar{\alpha}_v/\alpha)^{\theta/d}$ of the main price is not used: it may not be multiplied into the additive error of Proposition 8.7.

10.3. **Energies of the dominating law, and $Y_\lambda \ge (1-L_0-o(1))Y$.** Neither the comparison law nor the bracket measure need be reflection positive along a sorting sequence, as the structured-energy estimate requires of the outside law (for the bracket, (CP3) gives positivity at ι only). Their energies are bounded by expanding them into admissible positive laws, whose energies

Proposition 8.7 bounds by their prices, and into record differences grouped in tags; at the cut of a tag all its differences are applied sharply together (Lemma 8.8(d′)), and the folded-energy lemma bounds them by the geometric mean of two folded energies. This is done once (Lemma 10.8); its consequences are the first comparison, the factor B under the bracket, and the order relations that Part V uses throughout.

Energies of selected laws. Fix structured slot data: divisors $D_{0,v}$ of M_0, lengths $M_v \in \{\mathsf{L}, N\}$ and a law $\mathbf{P}$ of index vectors $\omega = (I_v, s_v)_{v\in V}$ independent of the labels, as in Definition 8.5(c), with inputs $g_v^\omega = g^{I_v, D_{0,v}, s_v}$. For a finite set X of atoms let Π_X be the coarse price of these data with selected set X; for unchanged inputs (F_A at every slot) $\Pi_X = C_X$. An *outside law* is one of the products μ, N_0, N (with N_0 as in Definition 9.13); write $O_{-X} = \bigotimes_{\mathcal{B}\notin X} O_{\mathcal{B}}$. For a finite signed measure κ on X_X^V put

$$\mathfrak{E}_O[\kappa] = \mathbb{E}_{\omega\sim\mathbf{P}} \int \Lambda_{O_{-X}}\big((g_v^\omega(z_v,\cdot))_{v\in V}\big)\, d\kappa(z), \qquad \mathfrak{E} = \mathfrak{E}_\mu,$$

and, for unchanged inputs, $Y^O[\kappa] = Y_{\kappa\otimes O_{-X}}$, $Y[\kappa] = Y^\mu[\kappa]$. At any cut an energy is the value of the cut form of the measure at the half-function carrying F_A at every slot of the plus half, whose reflection is the same half-function (5.1). Thus $Y[\bigotimes_{\mathcal{B}\in X}\mu_{\mathcal{B}}] = Y$, and for admissible positive κ with $q_X \le N^\tau$, $\mathfrak{E}[\kappa]$ is a structured energy.

Lemma 10.8 (Tagged sharp-first expansion)**.** *Assume (CP1) of Theorem 9.21. Let F be a finite set of atoms and $\mathfrak{T}$ a finite set of tags with cuts $c_j \in \mathfrak{F}$, and for $\mathcal{B}\in F$ let $\rho_{\mathcal{B}}$ and $\delta_{\mathcal{B},i}$ (i in a finite set) be finite signed measures on $X_{\mathcal{B}}^V$ such that:*

- *(a) $\rho_{\mathcal{B}} - \mu_{\mathcal{B}} = \sum_h c_h\rho_h$ is a finite real combination of admissible positive measures, of* variation price *$p_{\mathcal{B}} = \sum_h |c_h|\rho_h(1) q_{\mathcal{B}}^{2a}$;*
- *(b) $\delta_{\mathcal{B},i} = \kappa'_{\mathcal{B},i} - \kappa_{\mathcal{B},i}$ with admissible positive measures of equal mass $w_{\mathcal{B},i}$; it carries a tag $j(\mathcal{B},i)\in\mathfrak{T}$ and has a sharp bound $\tilde d_{\mathcal{B},i}$ relative to $\kappa_{\mathcal{B},i}$ at the cut $c_{j(\mathcal{B},i)}$ of its tag, in the sense of Definition 9.2(b): for a probability law π of shared labels, $\kappa_{\mathcal{B},i} = w_{\mathcal{B},i}\int \kappa_s\,\pi(ds)$ and $\delta_{\mathcal{B},i} = w_{\mathcal{B},i}\int K_s\,\pi(ds)$, each K_s satisfying (8.2) at $c_{j(\mathcal{B},i)}$ relative to the half marginals of κ_s with the constant $\tilde d_{\mathcal{B},i}$.*

Let $d_{\mathcal{B},i} = \tilde d_{\mathcal{B},i} w_{\mathcal{B},i} q_{\mathcal{B}}^{2a}$ be the sharp price of $\delta_{\mathcal{B},i}$ and $v_{\mathcal{B},i} = 2w_{\mathcal{B},i}q_{\mathcal{B}}^{2a}$ the price of its two positive parts (as in (CP2)(b)), $P = \sum_{\mathcal{B}} p_{\mathcal{B}}$, and let d_j and v_j be the sums of the $d_{\mathcal{B},i}$ and of the $v_{\mathcal{B},i}$ over the differences of tag j. Assume that these prices satisfy the summability condition of (CP8)(c), as they do in the applications below; in particular P and the d_j, v_j are bounded independently of the scale. Let X be a fixed finite set of atoms disjoint from F and κ an admissible positive measure on X_X^V. Then, for structured slot data and with the selected set $X\cup F$,

$$\Big|\mathfrak{E}\Big[\kappa\otimes\bigotimes_{\mathcal{B}\in F}\Big(\rho_{\mathcal{B}} + \sum_i \delta_{\mathcal{B},i}\Big)\Big] - \mathfrak{E}\Big[\kappa\otimes\bigotimes_{\mathcal{B}\in F}\rho_{\mathcal{B}}\Big]\Big| \le (e^P\mathfrak{h}(d,v) + o(1))\|\kappa\|\,\Pi_X Y,$$

with $\mathfrak{h}$ the series of (CP2). The $o(1)$ consists of energy errors and one large-product remainder.

Proof. We expand the product over F and bound each nonempty choice of differences at the cut of its tag; summing over count vectors gives the series $\mathfrak{h}(d,v)$. A term of the difference chooses a nonempty set $S\subseteq F$ of atoms and one difference $\delta_{\mathcal{B},i_{\mathcal{B}}}$ at each $\mathcal{B}\in S$, and carries $\rho_{\mathcal{B}}$ at the atoms of $F\setminus S$; at most one difference of an atom enters a term. Its *count vector* is $\boldsymbol{k}\in\mathbb{Z}_{\ge0}^{\mathfrak{T}}\setminus\{0\}$, $k_j = |\{\mathcal{B}\in S : j(\mathcal{B},i_{\mathcal{B}}) = j\}|$. The terms with count vector $\boldsymbol{k}$ form the *category* of $\boldsymbol{k}$. Fix $\boldsymbol{k}$ and a tag j with $k_j\ge1$; we bound the sum of the absolute values of the terms of the category, using the cut c_j.

In a term of the category let $S_j\subseteq S$ be the k_j atoms whose difference has tag j. Expand the difference at every atom of $S\setminus S_j$ into its two given positive parts (not into its Jordan parts, whose marginals need not be uniform), and every $\rho_{\mathcal{B}}$, $\mathcal{B}\in F\setminus S$, as $\mu_{\mathcal{B}} + \sum_h c_h\rho_h$. The term becomes a

signed sum of energies T' in which the atoms of S_j carry their tag-j differences, a set $X' \subseteq F \setminus S_j$ carries admissible positive measures, X carries κ, and every other atom carries $\mu_{\mathcal{B}}$.

In T' apply Lemma 8.8 at c_j with the selected set $E = X \cup X' \cup S_j$. The outside $\mu_{\mathcal{B}}$ are reflection positive at c_j by (R2) at every cut of the family (CP1). Each $\delta_{\mathcal{B},i_{\mathcal{B}}}$, $\mathcal{B} \in S_j$, is $w_{\mathcal{B},i_{\mathcal{B}}} \int K_s \, \pi_{\mathcal{B}}(ds)$ with kernels that satisfy (8.2) at the common cut c_j; by part (d′) of that lemma, under the product law of the shared labels, their tensor product over S_j satisfies (8.2) at c_j relative to the product of the κ_s, with the constant $\prod_{\mathcal{B}\in S_j} \tilde{d}_{\mathcal{B},i_{\mathcal{B}}}$. The normalized positive measures on $X \cup X'$ form a coupling π'' of its half marginals, so by part (d) the tensor product with π'' keeps this constant: positive couplings are contractions between their marginal spaces, and no positivity of these factors at c_j is used. By parts (a), (c), $|T'|$ is at most $\prod_{\mathcal{B}\in S_j} \tilde{d}_{\mathcal{B},i_{\mathcal{B}}} w_{\mathcal{B},i_{\mathcal{B}}}$ times the masses of the positive factors times the geometric mean of two folded energies $\mathfrak{E}_{\pm}$; by (b) these are energies of admissible positive measures, since a fold only copies an existing slot, and by (e) their coarse prices multiply to $\Pi(T')^2$, $\Pi(T') = \Pi_X \prod_{\mathcal{B}\in X'\cup S_j} q_{\mathcal{B}}^{2a}$. If $q_E \le N^{\tau}$, Proposition 8.7, applied once to each folded energy with the union E of the selected moduli, gives

$$|T'| \le (1+o(1)) \prod_{\mathcal{B}\in S_j} d_{\mathcal{B},i_{\mathcal{B}}} \prod_{\mathcal{B}\in X'} \big(\text{price of the positive factor at } \mathcal{B}\big)\, \|\kappa\|\, \Pi_X Y :$$

the geometric mean retains $\Pi(T')$, not $\Pi(T')^2$, no atom is charged twice, and $1+o(1)$ is not raised to the number of selected atoms. The terms with $q_E > N^{\tau}$ have the form $a_E (I_E^+ I_E^-)^{1/2}$ of Lemma 8.11.

Summing over the energies T' of one term: at an atom of $S \setminus S_j$ the two positive parts have total price $v_{\mathcal{B},i_{\mathcal{B}}}$, and at an atom of $F \setminus S$ the factor is $\mu_{\mathcal{B}}$ (weight one) or a positive record of $\rho_{\mathcal{B}} - \mu_{\mathcal{B}}$ (total weight $p_{\mathcal{B}}$). Summing then over the terms of the category, and dropping the requirement that the atoms be distinct, which only adds nonnegative terms, the category is at most

$$(1+o(1)) \, \frac{d_j^{k_j} \prod_{j'\ne j} v_{j'}^{k_{j'}}}{\prod_{j'\in\mathfrak{T}} k_{j'}!} \prod_{\mathcal{B}\in F} (1+p_{\mathcal{B}})\, \|\kappa\|\, \Pi_X Y,$$

because the sum, over the sets of $k_{j'}$ differences of tag j' at distinct atoms, of the product of their prices is at most the $k_{j'}$-th power of the total price of the tag divided by $k_{j'}!$. This holds for every tag j with $k_j \ge 1$, and we use the tag that gives the smallest bound. The minimum is taken after the whole category has been bounded with each tag, not inside the sum over its terms, so it needs no relation between the ratios $d_{\mathcal{B},i}/v_{\mathcal{B},i}$ of the individual differences; the two bounds refer to the same terms, and their intermediate folded couplings need not coincide. The categories partition the terms. Summing over $\boldsymbol{k} \ne 0$ and using $\prod_{\mathcal{B}}(1+p_{\mathcal{B}}) \le e^P$ gives the main term $e^P \mathfrak{h}(d,v) \|\kappa\| \Pi_X Y$, the tag totals over F being at most the totals of the statement and $\mathfrak{h}$ being nondecreasing. The terms with $q_E > N^{\tau}$ of all categories form one large-product remainder: such a term is determined by the role of each atom of $E \setminus X$ (a difference applied sharply, a positive part of a difference, or a positive record of $\rho_{\mathcal{B}} - \mu_{\mathcal{B}}$), so for each E their coefficients add up to at most $\|\kappa\|\, \Pi_X$ times the product over $E \setminus X$ of $p_{\mathcal{B}} + v_{\mathcal{B}} + d_{\mathcal{B}}$, with $v_{\mathcal{B}} = \sum_i v_{\mathcal{B},i}$, $d_{\mathcal{B}} = \sum_i d_{\mathcal{B},i}$. □

Corollary 10.9 (Two tags). *Let $\mathfrak{T} = \{0,1\}$ and $d_0 = d_1 \le v_0 = v_1$, and write d and v for the common values (here d is a tag total, not the number of slots) and $\mathfrak{h}(d,v)$ for the series of (CP2) at these totals. Then*

$$\mathfrak{h}(d,v) = \sum_{k_0+k_1>0} \frac{d^{\max(k_0,k_1)} v^{\min(k_0,k_1)}}{k_0!\, k_1!}.$$

For every integer $n \ge 1$ with $d < n+2$ and $dv < (n+2)^2$,

$$\mathfrak{h}(d,v) \le \mathfrak{h}_n^+(d,v) := \sum_{i=1}^{n}\Big[\frac{2d^i}{i!}\sum_{l=0}^{i-1}\frac{v^l}{l!} + \frac{(dv)^i}{(i!)^2}\Big] + \frac{2e^v d^{n+1}}{(n+1)!\,(1-d/(n+2))} + \frac{(dv)^{n+1}}{\big((n+1)!\big)^2(1-dv/(n+2)^2)},$$

every term of which is nonnegative and nondecreasing in d and in v where the two gates hold.

Proof. With $d \le v$, $\min(d^{k_0}v^{k_1}, d^{k_1}v^{k_0}) = d^{\max(k_0,k_1)}v^{\min(k_0,k_1)}$, and a count vector with $k_1 = 0$ (or $k_0 = 0$) allows one tag only, with the same value; this is the series. Group its terms by $i = \max(k_0,k_1) \ge 1$: the two terms with $\{k_0,k_1\} = \{i,l\}$, $l < i$, give $(2d^i/i!)\sum_{l<i} v^l/l!$, and the term $k_0 = k_1 = i$ gives $(dv)^i/(i!)^2$. The groups with $i \le n$ form the finite sum. For $i \ge n+1$ bound $\sum_{l<i} v^l/l!$ by e^v; the successive ratios of $2d^i/i!$ are $d/(i+1) \le d/(n+2)$, and those of $(dv)^i/(i!)^2$ are $dv/(i+1)^2 \le dv/(n+2)^2$, so the two geometric series give the two tails. Every term is a power series in d and v with nonnegative coefficients, or such a series divided by a positive nonincreasing function of d or of dv, and is therefore nonnegative and nondecreasing in d and in v. □

In Part V, $n = 4$ (Definition 30.1). The checked inequalities $d < 6$, $dv < 36$ and $d \le v$ are listed in Appendix D; these inequalities justify the displayed geometric bound for the tail of the two-tag series, which itself converges for every finite nonnegative d, v. At the baseline parameters its value is the one used in Remark 9.12 (no comparison with a one-sharp bound is needed).

Lemma 10.10 (First comparison)**.** *Assume (CP1), (CP2) and (CP8) of Theorem 9.21. Then $Y_\lambda \ge (1 - L_0 - o(1))Y$, with $L_0 = e^P\mathfrak{h}(d,v)$ of (CP2), where the $o(1)$ consists of energy errors and one large-product remainder.*

Proof. Step 1: $Y_\rho \ge Y$, where $\rho = \bigotimes_{\mathcal{B}} \rho_{\mathcal{B}}$, $\rho_{\mathcal{B}} = \mu_{\mathcal{B}}$ off the first-stage atoms. At the cut ϑ_0 of (CP2)(a), $\mu_{\mathcal{B}} \succeq_{\vartheta_0} 0$ for every atom by (R2) at every cut of the family, and $\rho_{\mathcal{B}} \succeq_{\vartheta_0} \mu_{\mathcal{B}}$ on the first-stage atoms. Tensor products preserve the Loewner order of positive semidefinite matrices (Lemma 5.6(d)), and the cut matrix of a product measure is the tensor product of the cut matrices (5.2); so $\rho \succeq_{\vartheta_0} \mu$, and evaluation at the half-function carrying F_A at every slot gives $Y_\rho \ge Y_\mu = Y$ (5.1).

Step 2: $|Y_\lambda - Y_\rho| \le (L_0 + o(1))Y$. This is Lemma 10.8 with $X = \varnothing$, F the first-stage atoms of $\mathcal{A}_N$, the $\rho_{\mathcal{B}}$ and the tagged record differences of (CP2), and unchanged inputs ($\Pi_\varnothing = 1$); the tag totals over $\mathcal{A}_N$ are at most those of (CP2). □

Remark 10.11 (Why the first stage is cheap). Step 1 is an exact relation at one scale; in the construction (CP2)(a) is the polarization inequality Lemma 17.11 (Section 22.2). Step 2 pays only for record differences between laws with the same actual conditional marginals at the cut of their tag, whose sharp price (an operator distance times a mass) is smaller than their variation price by a factor $O(p^{-1/2})$ at the least prime p of the atom; and the differences of one tag are paid together at their common cut, so that a term with k_0 differences of one tag and k_1 of the other costs the smaller of $d^{k_0}v^{k_1}/(k_0!k_1!)$ and $v^{k_0}d^{k_1}/(k_0!k_1!)$ instead of one sharp price times the variation of all the others. At the baseline parameters the two tags have $d = D/2$, about $2.37 \cdot 10^{-5}$, and $v = V/2$, about 1.69, and $L_0 < 4.76332373 \cdot 10^{-4}$; the one-sharp cost De^{P+V} would be about $7.6 \cdot 10^{-3}$, larger than the margin of the closing by eight orders of magnitude (Table 12).

Lemma 10.12 (Energies under the bracket)**.** *Assume (CP1), (CP2)(b), (CP3) and (CP8) of Theorem 9.21. Let X be a fixed finite set of atoms, κ an admissible positive measure on X_X^V, and*

fix structured slot data. Then

$$|\mathfrak{E}[\kappa]| \le (1+o(1))\|\kappa\|\,\Pi_X Y, \qquad |\mathfrak{E}_{\mathsf{N}_0}[\kappa]| \le (e^{U_0}+o(1))\|\kappa\|\,\Pi_X Y,$$
$$|\mathfrak{E}_{\mathsf{N}}[\kappa]-\mathfrak{E}_{\mathsf{N}_0}[\kappa]| \le (R_{\mathrm{err}}+o(1))\|\kappa\|\,\Pi_X Y,$$

and hence $|\mathfrak{E}_{\mathsf{N}}[\kappa]| \le (B+o(1))\|\kappa\|\,\Pi_X Y$. *When such an energy is one of the two factors of a Cauchy–Schwarz step, the same bound holds for both, and their geometric mean retains* B, *not* B^2. *The* $o(1)$ *consists of energy errors and large-product remainders.*

Proof. For fixed X, eventually $q_X \le N^{\tau}$, and the first bound is Proposition 8.7. In a sum over X, terms with $q_X > N^{\tau}$ are retained in the aggregate large-product remainder described above. On a first-stage atom $\mathcal{B} \notin X$ write $\mathsf{N}_{\mathcal{B}} = \mathsf{N}_{0,\mathcal{B}} + \delta_{E,\mathcal{B}}$ and $\mathsf{N}_{0,\mathcal{B}} = \mu_{\mathcal{B}} + (\mathsf{N}_{0,\mathcal{B}} - \mu_{\mathcal{B}})$, the second summand being an admissible positive measure of price $u_{\mathcal{B}}$ by (CP3) and (CP8). Expanding the latter, $\mathfrak{E}_{\mathsf{N}_0}[\kappa]$ is the sum over the sets X' of first-stage atoms outside X of the energies with selected measures $\kappa \otimes \bigotimes_{\mathcal{B}\in X'}(\mathsf{N}_{0,\mathcal{B}} - \mu_{\mathcal{B}})$, each at most $(1+o(1))\|\kappa\|\,\Pi_X \prod_{X'} u_{\mathcal{B}}\, Y$; the sum is at most $(1+o(1))\prod_{\mathcal{B}}(1+u_{\mathcal{B}})\|\kappa\|\,\Pi_X Y \le (1+o(1))e^{U_0}\|\kappa\|\,\Pi_X Y$. The difference is Lemma 10.8 with F the first-stage atoms outside X, $\rho_{\mathcal{B}} = \mathsf{N}_{0,\mathcal{B}}$ (variation price $u_{\mathcal{B}}$) and the tagged record differences of (CP2)(b): it is at most $(e^{U_0}\mathfrak{h}(d,v)+o(1))\|\kappa\|\,\Pi_X Y = (R_{\mathrm{err}}+o(1))\|\kappa\|\,\Pi_X Y$, the tag totals over the first-stage atoms outside X being at most those of (CP2) and $\mathfrak{h}$ being nondecreasing. For the last assertion, the two factors of a Cauchy–Schwarz step have coarse prices whose product is the square of the original price (Lemma 8.8(e)). □

The next lemma collects the order, contraction and interpolation estimates for energies that Part V applies repeatedly: its parts (E1)–(E3) are the basic comparisons, and (P1)–(P5), (P3′) are the product estimates used when a bound proved in a fixed chart is transported to actual marginals (restoration), when centred norms are interpolated, and when class bounds are formed, in the places named in Part V.

Lemma 10.13 (Energy toolkit: order, contraction and interpolation estimates)**.** *Assume (CP1)–(CP3) and (CP8) of Theorem 9.21. Let* X *be a finite set of atoms and* κ, κ' *finite signed measures on* X_X^V. *In (E2), (P1)–(P3) and (P3′),* ϑ *is a cut of the family and* $O = \mu$, *or* $\vartheta = \iota$ *and* O *is any outside law.*

- *(E1) If* κ *is positive and admissible, then* $|Y[\kappa]| \le (1+o(1))\|\kappa\|C_X Y$, $|Y^{\mathsf{N}_0}[\kappa]| \le (e^{U_0}+o(1))\|\kappa\|C_X Y$, $|Y^{\mathsf{N}}[\kappa]| \le (B+o(1))\|\kappa\|C_X Y$ *and* $|Y^{\mathsf{N}}[\kappa]-Y^{\mathsf{N}_0}[\kappa]| \le (R_{\mathrm{err}}+o(1))\|\kappa\|C_X Y$.
- *(E2) If* $\kappa \succeq_\vartheta \kappa'$, *then* $Y^O[\kappa] \ge Y^O[\kappa']$; *more generally, at any cut* ϑ, $\kappa \succeq_\vartheta \kappa'$ *implies* $Y_{\kappa\otimes O'} \ge Y_{\kappa'\otimes O'}$ *for every finite signed measure* O' *on the labels of the atoms outside* X *with* $O' \succeq_\vartheta 0$. *If* $\kappa \succeq_\iota 0$, *then* $Y[\kappa] \le Y^{\mathsf{N}_0}[\kappa]$.
- *(E3) A positive measure fixed samplewise by the reflection of* ϑ *(*diagonal at ϑ*) has a nonnegative diagonal cut matrix at* ϑ, *so it is* $\succeq_\vartheta 0$.

For a positive measure m *on the plus-half labels at* ϑ *let* $\mathrm{Diag}_\vartheta(m)$ *be its diagonal copy of Lemma 5.4(i) (the plus half has law* m, *the minus half is its reflection) and* $J_\vartheta(m) = (m\otimes m)/m(1)$ *the measure with independent halves of laws proportional to* m.

- *(P1)* Contraction. *If* π *is a positive measure on* X_X^V *with half marginals* $m_\pm$ *at* ϑ, *then* $|Y^O[\pi]| \le Y^O[\mathrm{Diag}_\vartheta(m_+)]^{1/2} Y^O[\mathrm{Diag}_\vartheta(m_-)]^{1/2}$, *where* $\mathrm{Diag}_\vartheta(m_-)$ *is the diagonal measure whose minus half has law* m_-.
- *(P2)* Density domination. *If* $m \le Mm'$ *for positive measures on the plus-half labels, then* $Y^O[\mathrm{Diag}_\vartheta(m)] \le M Y^O[\mathrm{Diag}_\vartheta(m')]$.
- *(P3)* Centred interpolation. *Let* π *be a positive measure symmetric at* ϑ *with half marginal* m *and Markov operator* T *on* $L^2(m)$. *If* $\|T-\mathbb{J}\| \le \kappa_0 \le 1$, *then* $\pi \preceq_\vartheta (1-\kappa_0)J_\vartheta(m) + \kappa_0 \,\mathrm{Diag}_\vartheta(m)$, *hence* $Y^O[\pi] \le (1-\kappa_0)Y^O[J_\vartheta(m)] + \kappa_0 Y^O[\mathrm{Diag}_\vartheta(m)]$. *The case* $\kappa_0 = 1$ *holds for every such* π.

(P3′) Product centred interpolation. *Let π be as in (P3), on the labels of X, with half marginal m and constant $\kappa_0 \le 1$, and let π' be a positive measure on the labels of a finite set Z of atoms disjoint from X, symmetric at ϑ, with half marginal m'. Then*

$$\pi \otimes \pi' \preceq_\vartheta (1-\kappa_0)\, J_\vartheta(m) \otimes \mathrm{Diag}_\vartheta(m') + \kappa_0\, \mathrm{Diag}_\vartheta(m \otimes m'),$$

hence $Y^O[\pi \otimes \pi'] \le (1-\kappa_0) Y^O[J_\vartheta(m) \otimes \mathrm{Diag}_\vartheta(m')] + \kappa_0 Y^O[\mathrm{Diag}_\vartheta(m \otimes m')]$. No order is tensored with an indefinite factor: π' enters only through its Markov operator, a self-adjoint contraction.

(P4) Products. *If $\pi_\mathcal{B}$ ($\mathcal{B} \in X$) are as in (P3) with half marginals $m_\mathcal{B}$ and constants $\kappa_\mathcal{B}$, then $\bigotimes_\mathcal{B} \pi_\mathcal{B}$ is as in (P3) with half marginal $\bigotimes_\mathcal{B} m_\mathcal{B}$ and constant $\max_\mathcal{B} \kappa_\mathcal{B}$.*

(P5) $\pm$ tensorization. *If ζ is a signed measure on X_X^V with $\kappa \pm \zeta \succeq_\iota 0$, then $|Y_{\zeta \otimes \lambda_{-X}}| \le Y^{\mathsf{N}}[\kappa]$. If $\left(\begin{smallmatrix} \kappa & \zeta \\ \zeta^* & \kappa' \end{smallmatrix}\right) \succeq_\iota 0$, then $|Y_{\zeta \otimes \lambda_{-X}}| \le Y^{\mathsf{N}}[\kappa]^{1/2} Y^{\mathsf{N}}[\kappa']^{1/2}$. If the cut matrix of ζ at ι is skew, then $Y_{\zeta \otimes \lambda_{-X}} = 0$.*

Proof. (E1) is Lemma 10.12 for unchanged inputs. (E2): for O' as stated, the cut matrix of $(\kappa - \kappa') \otimes O'$ at ϑ is $C_\vartheta(\kappa - \kappa') \otimes C_\vartheta(O')$ (5.2), a Kronecker product of symmetric positive semidefinite matrices; evaluate at the half-function carrying F_A (5.1). The outside laws are such measures: $\mu_{-X} \succeq_\vartheta 0$ at every cut of the family by (R2), as tensor products of reflection positive measures are reflection positive (Definition 5.1); $\mathsf{N}_{-X} \succeq_\iota 0$ because each $\mathsf{N}_\mathcal{B}$ is a diagonal block of the bracket of (CP3); and $\mathsf{N}_{0,-X} \succeq_\iota 0$ because $\mathsf{N}_{0,\mathcal{B}} = \mu_\mathcal{B} + (\mathsf{N}_{0,\mathcal{B}} - \mu_\mathcal{B})$, with $\mu_\mathcal{B} \succeq_\iota 0$ and the second summand positive and reserved diagonal, hence $\succeq_\iota 0$ by (E3). For the last claim, $0 \preceq_\iota \mu_\mathcal{B} \preceq_\iota \mathsf{N}_{0,\mathcal{B}}$, so $\mu_{-X} \preceq_\iota \mathsf{N}_{0,-X}$ (Lemma 5.6(d)), and $C_\iota(\kappa) \otimes (C_\iota(\mathsf{N}_{0,-X}) - C_\iota(\mu_{-X})) \succeq 0$. (E3) is Lemma 5.4(i).

(P1) Let $D_\pm$ be the diagonal cut matrices of $\mathrm{Diag}_\vartheta(m_\pm)$, the minus half read on the plus half through the reflection. For real f, g on the plus-half labels, $\langle f, D_+ f\rangle + 2\langle f, C_\vartheta(\pi) g\rangle + \langle g, D_- g\rangle = \int (f(z_+) + g(\tilde z_-))^2\, d\pi \ge 0$, with $\tilde z_-$ the reflected minus half, so $\left(\begin{smallmatrix} D_+ & C_\vartheta(\pi) \\ C_\vartheta(\pi)^\mathsf{T} & D_- \end{smallmatrix}\right) \succeq 0$. Tensorize with $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \otimes C_\vartheta(O_{-X})$ and apply the block-Gram inequality (Lemma 5.6) at the half-function carrying F_A; for $O = \mu$ this is Lemma 8.8(a) with $d = 1$. (P2): $\mathrm{Diag}_\vartheta(Mm' - m)$ is positive and diagonal at ϑ; apply (E3), (E2).

(P3) Relative to m the cut matrices of π, $J_\vartheta(m)$ and $\mathrm{Diag}_\vartheta(m)$ are $D_m T$, $D_m \mathbb{J}$ and D_m (D_m the diagonal matrix of m), so the cut order between measures of half marginal m is the Loewner order of their operators on $L^2(m)$. T is self-adjoint and fixes the constants, so $T - \mathbb{J} = (I - \mathbb{J})(T - \mathbb{J})(I - \mathbb{J}) \preceq \kappa_0 (I - \mathbb{J})$, that is, $T \preceq (1 - \kappa_0)\mathbb{J} + \kappa_0 I$; apply (E2). The Markov operator of a positive coupling is a contraction on $L^2(m)$ (Jensen), so $\|T - \mathbb{J}\| \le 1$ always.

(P3′) Relative to $m \otimes m'$ the cut matrices of $\pi \otimes \pi'$, $J_\vartheta(m) \otimes \mathrm{Diag}_\vartheta(m')$ and $\mathrm{Diag}_\vartheta(m \otimes m')$ are $D(T \otimes T')$, $D(\mathbb{J} \otimes I)$ and D, with D the diagonal matrix of $m \otimes m'$ and T' the Markov operator of π' on $L^2(m')$, which is self-adjoint, because π' is symmetric, and a contraction. The operator $T \otimes T'$ maps $\operatorname{ran} \mathbb{J} \otimes L^2(m')$ and its orthogonal complement $L_0^2(m) \otimes L^2(m')$ into themselves. On the first it is $I \otimes T' \preceq I$; on the second it is $(T - \mathbb{J}) \otimes T'$, self-adjoint of norm at most κ_0, hence $\preceq \kappa_0 I$. So $T \otimes T' \preceq \mathbb{J} \otimes I + \kappa_0 (I - \mathbb{J}) \otimes I = (1 - \kappa_0)\mathbb{J} \otimes I + \kappa_0 I$, which is the order claimed; apply (E2).

(P4): the Markov operator of the product is $\bigotimes_\mathcal{B} T_\mathcal{B}$, self-adjoint, and $\| \bigotimes_\mathcal{B} T_\mathcal{B} - \mathbb{J}\| = \max_\mathcal{B} \|T_\mathcal{B} - \mathbb{J}\|$ (Lemma 6.10(c)).

(P5) By (CP3), $\mathsf{N}_\mathcal{B} \pm \lambda_\mathcal{B} \succeq_\iota 0$ for every atom (off the first-stage atoms these are $2\mu_\mathcal{B}$ and 0). With $\kappa \pm \zeta \succeq_\iota 0$, Lemma 5.6(d) gives $\kappa \otimes \mathsf{N}_{-X} \pm \zeta \otimes \lambda_{-X} \succeq_\iota 0$, and evaluation at the half-function carrying F_A gives the first claim. For the bracket form tensorize the bracket of ζ with those of (CP3) (Lemma 5.6(b)) and apply part (a) of that lemma. If $C_\iota(\zeta)$ is skew, so is $C_\iota(\zeta) \otimes C_\iota(\lambda_{-X})$, whose form vanishes at two equal arguments (Remark 5.3(a)). □

10.4. **Expanding $\nu - \lambda$ over changed blocks: the bound as a power series in prices.** The difference $Y_\nu - Y_\lambda$ at the reserved cut is expanded by choosing, at any set of atoms, which records

were changed; each such choice is a *selection*. A selection in no special class is bounded by B times the product of the prices of its changed records. Selections in a class of (CP5) may have a smaller proved bound. Summing over all selections gives a power series in the prices; any series with nonnegative coefficients that dominates it coefficient by coefficient (a *complete polynomial*, Section 1.8) may be used in the closing. This subsection makes the expansion precise.

Selections. Recall from (CP4)(b), (c) (Definition 9.14) the decomposition $\nu_{\mathcal{B}} - \lambda_{\mathcal{B}} = \sum_{t\in\mathcal{T}_{\mathcal{B}}}(\zeta^{\mathrm{s}}_{\mathcal{B},t} + \zeta^{\mathrm{a}}_{\mathcal{B},t})$ over the changed records t of $\mathcal{B}$, the brackets of $\zeta^{\epsilon}_{\mathcal{B},t}$ with the norm envelopes $\tilde{\pi}^{\epsilon}_{\mathcal{B},t}D^{\epsilon}_{\mathcal{B},t}$, the prices $\pi^{\epsilon}_{\mathcal{B},t}$ with totals S and A, and the selections ω (a set X of atoms with a record $t_{\mathcal{B}}$ and a type $\epsilon_{\mathcal{B}}$ at each, an even number of them skew) with their measures ζ_ω, monomials π_ω and terms $I_\omega = Y_{\zeta_\omega\otimes\lambda_{-X}}$. In formal variables $x = (x^{\epsilon}_{\mathcal{B},t})$ for the prices and $y = (y_{\mathcal{B},j})$ for the auxiliary prices of (CP5) let $S(x) = \sum x^{\mathrm{s}}_{\mathcal{B},t}$, $A(x) = \sum x^{\mathrm{a}}_{\mathcal{B},t}$ and $e^{S(x)}\cosh A(x) - 1 = \sum_{\boldsymbol{n}} c_{\boldsymbol{n}}x^{\boldsymbol{n}}$, the sum over the nonconstant monomials of even skew degree. Every $c_{\boldsymbol{n}}$ is positive (it is $1/\boldsymbol{n}!$), and a squarefree monomial has coefficient one; so has, in particular, the monomial $x^{\boldsymbol{n}(\omega)}$ of a selection, for which $\pi^{\boldsymbol{n}(\omega)} = \pi_\omega$. A selection determines its atoms, records and types, hence its monomial; selections of different patterns of types, also those without skew types, are different monomials, and (CP5) may be proved for every pattern with an even number of skew types. For a selection of a class $\mathcal{C}$ of (CP5) with a class cost let $c_\omega = c_{\mathcal{C}}$, and $c_\omega = B$ for a selection in no class; for these put $b_\omega = c_\omega x^{\boldsymbol{n}(\omega)}$. For a selection of a class with class bounds put $b_\omega = \phi_\omega$, a polynomial in y. A monomial in x is the monomial of at most one selection, and a monomial of a class bound determines its selection (CP5); so $\sum_\omega b_\omega$, the sum over all selections of atoms of $\mathcal{A}$, is a power series in (x, y) with nonnegative coefficients, each of whose coefficients is a finite sum of coefficients of the b_ω.

Lemma 10.14 (Record expansion and complete polynomial)**.** *Assume (CP0)–(CP4) and (CP8) of Theorem 9.21.*

(i) $Y_\nu - Y_\lambda = \sum_\omega I_\omega$, *and* $|I_\omega| \le (B + o(1))\pi_\omega Y$ *for every fixed selection* ω.

(ii) Complete polynomials. *Assume (CP5). Let* $\Psi(x, y)$ *be a power series with nonnegative coefficients such that* $\Psi - \sum_\omega b_\omega$ *has nonnegative coefficients. Then* $|Y_\nu - Y_\lambda| \le (\Psi(\hat{\pi}, \hat{y}) + o(1))Y$ *for all upper prices* $\hat{\pi} \ge \pi$ *and upper auxiliary prices* $\hat{y} \ge y$ *at which* Ψ *converges. Such a* Ψ *is a* complete polynomial. *Without classes,* $\Psi = B(e^{S(x)}\cosh A(x) - 1)$, *and this reads* $|Y_\nu - Y_\lambda| \le (B(e^S\cosh A - 1) + o(1))Y$; *with* $B' \ge B$ *(for instance a certified upper bound for* B*) every* B *may be replaced by* B'. *For classes with class costs,* $\Psi = \sum_{\boldsymbol{n}}(B' - \hat{g}_{\boldsymbol{n}})c_{\boldsymbol{n}}x^{\boldsymbol{n}}$ *is complete whenever* $0 \le \hat{g}_{\boldsymbol{n}} \le B'$ *for every* $\boldsymbol{n}$ *and* $\hat{g}_{\boldsymbol{n}(\omega)} \le B' - c_\omega$ *for every selection* ω *(the* $\hat{g}_{\boldsymbol{n}}$ *are arbitrary in* $[0, B']$ *on the other monomials, which have a repeated atom).*

(iii) Ψ *is nondecreasing in each price variable and in each auxiliary variable. In particular an upper bound for the total of a nonempty set of variables may be distributed over them in proportion to their actual values when their total is positive, and arbitrarily as nonnegative upper prices when their total is zero, giving a sum exactly equal to the given bound.*

(iv) Completion. *Assume (CP5) with class costs, and let* $B' \ge B$. *Let* $\mathcal{D}(x) = \sum_{\boldsymbol{n}} d_{\boldsymbol{n}}x^{\boldsymbol{n}}$ *be a power series with nonnegative coefficients, convergent at* $\hat{\pi}$, *with* $d_{\boldsymbol{n}} \le B' - c_\omega$ *on the monomial of every selection and* $d_{\boldsymbol{n}} \le B'c_{\boldsymbol{n}}$ *on every other monomial (*$c_{\boldsymbol{n}} = 0$ *if* $x^{\boldsymbol{n}}$ *does not occur). Then, with* $\hat{S} = S(\hat{\pi})$, $\hat{A} = A(\hat{\pi})$,

$$|Y_\nu - Y_\lambda| \le (B'(e^{\hat{S}}\cosh\hat{A} - 1) - \mathcal{D}(\hat{\pi}) + o(1))Y.$$

That is, $\Psi = B'(e^{S(x)}\cosh A(x) - 1) - \mathcal{D}(x)$ *is a complete polynomial in the sense of (ii).*

(v) Degree-class replacement. *Divide the skew records into kinds* $1, \dots, r$: *for every kind* l *a set* $\mathcal{R}_l$ *of pairs* $(\mathcal{B}, t)$, $t \in \mathcal{T}_{\mathcal{B}}$, *the sets being disjoint. Let* $A_l(x) = \sum_{(\mathcal{B},t)\in\mathcal{R}_l} x^{\mathrm{a}}_{\mathcal{B},t}$, *and let every* $(\mathcal{B}, t) \in \mathcal{R}_l$ *carry auxiliary prices* $y^{\mathrm{arm}}_{\mathcal{B},t}, y^{\mathrm{lead}}_{\mathcal{B},t}, y^{\mathrm{rem}}_{\mathcal{B},t} \ge 0$ *(raw-arm, leading and remainder),*

attached to (t,a)*, with totals* A_l^{arm}*,* A_l^{lead}*,* A_l^{rem} *over* $\mathcal{R}_l$*. For* $\boldsymbol{n}\in\mathbb{Z}^r_{\geq 0}$ *with* $|\boldsymbol{n}|\geq 2$ *even let* $\mathcal{C}_{\boldsymbol{n}}$ *be the class of the selections consisting of* n_l *skew records of kind* l *for every* l *and of no other record, and suppose that* $\mathcal{C}_{\boldsymbol{n}}$ *satisfies (CP5) with*

$$\phi_\omega = C_{\boldsymbol{n}}\prod_{\mathcal{B}\in X} y^{\mathrm{arm}}_{\mathcal{B},t_{\mathcal{B}}} + B'\Big(\prod_{\mathcal{B}\in X}(y^{\mathrm{lead}}_{\mathcal{B},t_{\mathcal{B}}}+y^{\mathrm{rem}}_{\mathcal{B},t_{\mathcal{B}}}) - \prod_{\mathcal{B}\in X} y^{\mathrm{lead}}_{\mathcal{B},t_{\mathcal{B}}}\Big)$$

for a number $C_{\boldsymbol{n}}\geq 0$*, where* $X=X(\omega)$*. Let* Ψ_0 *be a complete polynomial for the other classes, the selections of* $\mathcal{C}_{\boldsymbol{n}}$ *being in no class, whose coefficients on the monomials of degree* n_l *in the skew variables of kind* l *for every* l *and of degree zero in all other variables are those of* $B'(e^{S(x)}\cosh A(x)-1)$*. Then, with* $\boldsymbol{n}!=\prod_l n_l!$*,*

$$\Psi = \Psi_0 - \frac{1}{\boldsymbol{n}!}\Big[B'\prod_l A_l^{n_l} - C_{\boldsymbol{n}}\prod_l (A_l^{\mathrm{arm}})^{n_l} - B'\Big(\prod_l (A_l^{\mathrm{lead}}+A_l^{\mathrm{rem}})^{n_l} - \prod_l (A_l^{\mathrm{lead}})^{n_l}\Big)\Big],$$

read as a power series in the variables of the individual records, is a complete polynomial for all the classes. For a set of such degree vectors the replacements add. Credits: *if moreover the selections* ω *of* $\mathcal{C}_{\boldsymbol{n}}$ *satisfy (CP5) with the smaller class bounds* $\phi_\omega-\gamma_\omega$*, where the credits* γ_ω *are polynomials with nonnegative coefficients (for instance variance credits on the raw-arm monomials), then* $\Psi-\Gamma$ *is a complete polynomial for every power series* Γ *with nonnegative coefficients whose coefficient is at most that of* γ_ω *on every monomial of a selection* ω *of* $\mathcal{C}_{\boldsymbol{n}}$ *and at most that of* $\Psi-\sum_\omega b_\omega$ *on every other monomial, as in (iv).*

The $o(1)$ *consists of energy errors and large-product remainders.*

In (ii) the bounds of (CP5) hold robustly and uniformly in the class ((CP8)(b)); this is where the bounded arity $n_{\max}$ enters, since the bounds of the classes then come from finitely many types of operator reductions. No argument uses a particular arity. In Part V every selection of the variance, background and late-pool credits and of the envelope classes has at most six atoms, and so has every selection of the mean savings of (CP7) (Proposition 24.17 and Lemma 24.16).

The replacement (v) applies to coarser kinds as well: if $C_{\boldsymbol{n}}$ depends only on the sums of the n_l over groups of kinds, then by the multinomial expansion the replacements of the degree vectors with given group sums add up to the replacement for the merged kinds, whose totals are the sums of the totals. In Part V the kinds are the fold records and the two matching records E_0, E_1: the general classes, with n fold and m matching records, are replaced with the merged kind of the matching records, and the matching pairs with E_0 and E_1 separate, the aligned pairs (two records of one kind) and the crossed pairs (one of each) each having the total $A_E^2/4$, the two matching records having equal totals (Corollary 25.10 and Definition 30.1).

Proof. (i) Expand $\nu=\bigotimes_{\mathcal{B}}(\lambda_{\mathcal{B}}+\sum_t(\zeta^{\mathrm{s}}_{\mathcal{B},t}+\zeta^{\mathrm{a}}_{\mathcal{B},t}))$. A term with an odd number of skew factors has a skew cut matrix at ι, because $\lambda_{\mathcal{B}}$ is symmetric there by (CP3), and vanishes by (P5). The remaining terms are Y_λ and the I_ω. For a selection, the tensor product of the brackets of its factors and (P5) give $|I_\omega|\leq\prod_{\mathcal{B}\in X}\tilde\pi^{\epsilon_{\mathcal{B}}}_{\mathcal{B},t_{\mathcal{B}}}\,Y^{\mathsf{N}}[\bigotimes_{\mathcal{B}\in X}D^{\epsilon_{\mathcal{B}}}_{\mathcal{B},t_{\mathcal{B}}}]\leq(B+o(1))\pi_\omega Y$ by (E1), the laws $D^{\epsilon}_{\mathcal{B},t}$ being admissible and normalized.

(ii) Fully expand the bracket outside each selection as in the proof of Lemma 10.12. For selections in no class, apply the uniform structured-energy bound only to the resulting terms whose full union E satisfies $q_E\leq N^\tau$; the other terms, summed over the selections and the outside expansion together, are one large-product remainder. Their coefficients are bounded by products of the sums of the local prices in (CP8)(c), so Lemma 8.11 makes their total $o(1)Y$. The small terms have one uniform relative energy error. Combine these bounds with the class sums of (CP5), which are uniform by (CP8)(b). Thus, after adding this aggregate remainder, $|Y_\nu-Y_\lambda|\leq\sum_\omega|I_\omega|\leq\sum_\omega(b_\omega(\pi,y)+o(1)\pi_\omega)Y+o(1)Y$, with $b_\omega(\pi,y)=c_\omega\pi_\omega$ off the classes with class bounds and $b_\omega(\pi,y)=\phi_\omega(y)$ on them, the bounds of (CP5) being uniform in the class ((CP8)(b)). The arguments (π,y) and $(\hat\pi,\hat y)$ are nonnegative and $(\pi,y)\leq(\hat\pi,\hat y)$, so $\sum_\omega b_\omega(\pi,y)\leq$

$\Psi(\pi, y) \le \Psi(\hat\pi, \hat y)$ by the dominance and by (iii). Also $\sum_\omega \pi_\omega \le e^S \cosh A - 1$ is bounded, so the errors add up to $o(1)Y$. Without classes $\sum_\omega b_\omega = B \sum_\omega x^{\boldsymbol{n}(\omega)}$, which $B(e^{S(x)} \cosh A(x) - 1)$ dominates, its coefficient being B on every squarefree monomial and nonnegative on the others. For the last sentence of (ii): the coefficient of Ψ on the monomial of a selection is $B' - \hat g_{\boldsymbol{n}(\omega)} \ge c_\omega$, since $c_{\boldsymbol{n}(\omega)} = 1$, and on the other monomials it is nonnegative; distinct selections have distinct monomials, so this is the dominance.

(iii) Every coefficient of Ψ is nonnegative. If $0 < \sum_G \pi \le \hat\Pi_G$, then $\hat\pi_g = \pi_g \hat\Pi_G / \sum_G \pi \ge \pi_g$ and $\sum_G \hat\pi_g = \hat\Pi_G$. If G is nonempty and the actual total is zero, all its prices are zero; assign the upper total to any one variable and zero to the others. Empty groups need no allocation.

(iv) On the monomial of a selection the coefficient of $B'(e^{S(x)} \cosh A(x) - 1) - \mathcal{D}(x)$ is $B' - d_{\boldsymbol{n}} \ge c_\omega$, and on every other monomial it is $B' c_{\boldsymbol{n}} - d_{\boldsymbol{n}} \ge 0$; this is the dominance of (ii), both series converging absolutely at $\hat\pi$.

(v) By the multinomial expansion of $\cosh A(x)$, the monomials of $B'(e^{S(x)} \cosh A(x) - 1)$ of degree n_l in the skew variables of kind l for every l and of degree zero in all other variables sum to $B' \prod_l A_l(x)^{n_l} / \boldsymbol{n}!$: the coefficient of such a monomial $x^{\boldsymbol{m}}$ in $\prod_l A_l^{n_l}$ is $\boldsymbol{n}!/\boldsymbol{m}!$, and in $e^S \cosh A$ it is $1/\boldsymbol{m}!$. These monomials are the monomials of the selections of $\mathcal{C}_{\boldsymbol{n}}$ and of the tuples of that degree with a repeated atom. By the hypothesis on Ψ_0, subtracting $B' \prod_l A_l^{n_l} / \boldsymbol{n}!$ sets their coefficients to zero and leaves all other coefficients of Ψ_0, which dominate the b_ω of the selections outside $\mathcal{C}_{\boldsymbol{n}}$ (those of $\mathcal{C}_{\boldsymbol{n}}$, in no class for Ψ_0, contributed only to these monomials). The added polynomial $\big[C_{\boldsymbol{n}} \prod_l (A_l^{\mathrm{arm}})^{n_l} + B'\big(\prod_l (A_l^{\mathrm{lead}} + A_l^{\mathrm{rem}})^{n_l} - \prod_l (A_l^{\mathrm{lead}})^{n_l}\big)\big] / \boldsymbol{n}!$ is a polynomial in the auxiliary variables with nonnegative coefficients, so Ψ has nonnegative coefficients. It dominates $\sum_{\omega \in \mathcal{C}_{\boldsymbol{n}}} \phi_\omega$. A selection of $\mathcal{C}_{\boldsymbol{n}}$ is a set of distinct atoms with one record of kind l at n_l of them; it corresponds to exactly $\boldsymbol{n}!$ tuples $((\mathcal{B}_1, t_1), \dots, (\mathcal{B}_{|\boldsymbol{n}|}, t_{|\boldsymbol{n}|}))$ whose first n_1 entries lie in $\mathcal{R}_1$, the next n_2 in $\mathcal{R}_2$, and so on, with distinct atoms. So $\sum_{\omega \in \mathcal{C}_{\boldsymbol{n}}} \phi_\omega$ is $1/\boldsymbol{n}!$ times the sum of $C_{\boldsymbol{n}} \prod_i y^{\mathrm{arm}}_{\mathcal{B}_i, t_i} + B'\big(\prod_i (y^{\mathrm{lead}}_{\mathcal{B}_i, t_i} + y^{\mathrm{rem}}_{\mathcal{B}_i, t_i}) - \prod_i y^{\mathrm{lead}}_{\mathcal{B}_i, t_i}\big)$ over these tuples. The sum of the same expression over all tuples, the atoms not required to be distinct, is the added polynomial, by expanding the products of the totals; and every expression in the sum has nonnegative coefficients, the bracket being the sum of the products in which at least one remainder price occurs. So dropping the distinctness adds only nonnegative coefficients, and Ψ dominates $\sum_\omega b_\omega$. Several degree vectors concern disjoint sets of monomials in x and add disjoint nonnegative polynomials in y, so their replacements add. For the credits: on a monomial of a selection ω of $\mathcal{C}_{\boldsymbol{n}}$ the coefficient of $\Psi - \Gamma$ is at least that of $\phi_\omega - \gamma_\omega$, and on every other monomial at least that of $\sum_\omega b_\omega$; this is the dominance of (ii) for the credited class bounds, and it implies that $\Psi - \Gamma$ has nonnegative coefficients. □

Remark 10.15 (Completion and subtraction). Parts (iv) and (v) are the only ways in which a discount is subtracted from the reserved cost, and they fix the order of operations of the closing (Certificate 30.3, which takes for B' the certified upper end of B): every genuine selection first receives its nonnegative coefficient, the cost of its class after its credits ($B' - d_{\boldsymbol{n}} \ge c_\omega$ in (iv)) or its credited class bound (in (v)); the monomials with a repeated atom receive nonnegative coefficients ($[0, B' c_{\boldsymbol{n}}]$ in (iv)); only then is the series rewritten as an exponential minus homogeneous polynomials in the totals of the price variables and of the auxiliary variables and evaluated at the upper prices. The homogeneous terms are not lower bounds for a distinct-block discount, and no claim is made that a subtracted polynomial, considered alone, is a lower bound for the saving from distinct blocks. The finite gates of the closing which say that every coefficient stays nonnegative after the credits (every leading class cost $C_{n,m} \ge 0$, every credited coefficient nonnegative, every restoration gain — the factor in $[0, 1)$ by which a restored variance credit (Section 24) multiplies a class coefficient — in $[0, 1)$) are exactly the conditions of (iv) and (v); that a class is replaced only when its saving is positive is a choice, not a condition of (v), which holds whatever the sign of the saving. Two rules govern the subtractions in this paper: an upper bound for a quantity is

never used as a lower bound for it; and a discount is subtracted only inside an expression with nonnegative coefficients in the same chosen variables, as in (iv) and (v). In the replaced form (v) the remainder polynomial contains only the separately proved leading prices y^{lead} and the remainder prices y^{rem}; neither a raw arm price nor the difference of a full price and a remainder price may replace a leading price there, since neither bounds the leading part of a record change.

The same completion bounds other polynomials with nonnegative coefficients. If $0 \le \kappa_b \le \kappa_H$ and $v_{\mathcal{B}}, w_{\mathcal{B}} \ge 0$, then

$$\kappa_H \prod_{\mathcal{B}} (1 + v_{\mathcal{B}} + w_{\mathcal{B}}) - (\kappa_H - \kappa_b) \prod_{\mathcal{B}} (1 + v_{\mathcal{B}})$$

has coefficient κ_b on the monomials in the $v_{\mathcal{B}}$ alone and κ_H on all others, all squarefree, and it is nondecreasing in every $v_{\mathcal{B}}$ and every $w_{\mathcal{B}}$ separately: its derivative in $w_{\mathcal{B}}$ is $\kappa_H \prod_{\mathcal{B}' \ne \mathcal{B}} (1 + v_{\mathcal{B}'} + w_{\mathcal{B}'}) \ge 0$, and its derivative in $v_{\mathcal{B}}$ is at least $\kappa_b \prod_{\mathcal{B}' \ne \mathcal{B}} (1 + v_{\mathcal{B}'}) \ge 0$. Hence, for chosen upper prices $m_{\mathcal{B}} \ge v_{\mathcal{B}}$ and $n_{\mathcal{B}} \ge w_{\mathcal{B}}$ with $\sum_{\mathcal{B}} (m_{\mathcal{B}} + n_{\mathcal{B}}) \le U$, it is at most its value at (m, n), which is at most $\kappa_H e^U - (\kappa_H - \kappa_b) e^{\sum_{\mathcal{B}} m_{\mathcal{B}}}$: the expansion of the latter has the same coefficients κ_b, κ_H on the squarefree monomials and nonnegative ones on all others. An upper price of the invariant tags may thus stand in the subtracted exponential, because the form is monotone in the tag variables; it is never used as a lower bound of a norm (Lemma 25.6 and Theorem 25.8).

10.5. **When a change preserves half-marginals its bound is a variance.** When a comparison law replaces a record by a law with the same actual half marginals at ι, the difference annihilates the constants on both sides, and its form against the bracket is a *variance*: the energy of the record's selected diagonal minus that of the product of its actual half marginals. This is the interface between the variance credits of Part V and the class bounds (CP5): a lower bound for the product energy lowers the coefficient of the class below B.

Lemma 10.16 (Variance form at the reserved cut)**.** *Assume (CP1)–(CP3) and (CP8) of Theorem 9.21. Let X be a nonempty finite set of atoms and, for $\mathcal{B} \in X$, let $m_{\mathcal{B}}$ be a probability measure on the plus-half labels $X_{\mathcal{B}}^{V_+}$ at ι with $\mathrm{Diag}_\iota(m_{\mathcal{B}})$ and $J_\iota(m_{\mathcal{B}})$ admissible, and $\Delta_{\mathcal{B}}$ a self-adjoint or skew-adjoint operator on $L^2(m_{\mathcal{B}})$ with $\Delta_{\mathcal{B}}\mathbf{1} = \Delta_{\mathcal{B}}^*\mathbf{1} = 0$ and $\|\Delta_{\mathcal{B}}\| \le \gamma_{\mathcal{B}}$, the number of skew-adjoint $\Delta_{\mathcal{B}}$ being even. Put $m_X = \bigotimes_{\mathcal{B}} m_{\mathcal{B}}$, $\mathrm{Diag}_X = \bigotimes_{\mathcal{B}} \mathrm{Diag}_\iota(m_{\mathcal{B}})$, $J_X = \bigotimes_{\mathcal{B}} J_\iota(m_{\mathcal{B}})$, and let ζ be the signed measure on X_X^V whose cut operator at ι relative to m_X is $\Delta_X = \bigotimes_{\mathcal{B}} \Delta_{\mathcal{B}}$. Then:*

(i) $Y^{\mathsf{N}}[\mathrm{Diag}_X] \ge Y^{\mathsf{N}}[J_X]$ and $|Y_{\zeta \otimes \lambda_{-X}}| \le \prod_{\mathcal{B}} \gamma_{\mathcal{B}} \, (Y^{\mathsf{N}}[\mathrm{Diag}_X] - Y^{\mathsf{N}}[J_X])$;

(ii) $Y^{\mathsf{N}}[\mathrm{Diag}_X] \le (B + o(1)) C_X Y$ and $Y^{\mathsf{N}}[J_X] \ge Y[J_X] - (R_{\mathrm{err}} + o(1)) C_X Y$;

(iii) consequently, if $Y^{\mathsf{N}_0}[J_X] \ge (\ell - o(1))Y$ with $\ell \ge 0$ and $C_X \le \bar{C}$, then

$$|Y_{\zeta \otimes \lambda_{-X}}| \le \prod_{\mathcal{B}} \gamma_{\mathcal{B}} \, C_X (B - g + o(1)) Y, \qquad g = \max(0, \ell/\bar{C} - R_{\mathrm{err}}).$$

By (E2), as in the proof of (ii), $Y^{\mathsf{N}_0}[J_X] \ge Y[J_X]$, so the hypothesis of (iii) holds for every lower bound ℓ of $Y[J_X]$; positive background terms of the expansion of N_0 may be added to such a bound (Proposition 24.17). Only $\mathsf{N}_0 \succeq_\iota \mu$ and the priced difference $\mathsf{N} - \mathsf{N}_0$ are used, not an order between N and μ. The $o(1)$ consists of energy errors and large-product remainders.

Proof. (i) The number of skew-adjoint factors is even, so Δ_X is self-adjoint. As $\Delta_{\mathcal{B}}\mathbf{1} = \Delta_{\mathcal{B}}^*\mathbf{1} = 0$, Δ_X and Δ_X^* annihilate every vector with a constant factor, in particular the constants of $L^2(m_X)$; so $\Delta_X = (I - \mathbb{J})\Delta_X(I - \mathbb{J})$ and, with $\gamma = \prod_{\mathcal{B}} \gamma_{\mathcal{B}} \ge \|\Delta_X\|$, $\gamma(I - \mathbb{J}) \pm \Delta_X \succeq 0$. Relative to m_X the cut operators of Diag_X and J_X are I and $\mathbb{J}$, so $\gamma(\mathrm{Diag}_X - J_X) \pm \zeta \succeq_\iota 0$, and (P5) gives the second assertion; the first is the case $\Delta_X = 0$.

(ii) Diag_X and J_X are admissible probability measures, so (E1) gives the first bound and $|Y^{\mathsf{N}}[J_X] - Y^{\mathsf{N}_0}[J_X]| \le (R_{\mathrm{err}} + o(1)) C_X Y$; moreover $J_X \succeq_\iota 0$, its cut operator being a projection, so $Y^{\mathsf{N}_0}[J_X] \ge Y[J_X]$ by (E2).

(iii) By (i), the first bound of (ii), (E1) for $\mathsf{N}-\mathsf{N}_0$ at J_X and the hypothesis, $Y^{\mathsf{N}}[J_X] \ge Y^{\mathsf{N}_0}[J_X] - (R_{\mathrm{err}}+o(1))C_X Y \ge (\ell-o(1))Y-(R_{\mathrm{err}}+o(1))C_X Y$, and so $|Y_{\zeta\otimes\lambda_{-X}}| \le \gamma(BC_X+R_{\mathrm{err}}C_X-\ell+o(1))Y$. If $\ell/\bar{C} \le R_{\mathrm{err}}$, use instead $|Y_{\zeta\otimes\lambda_{-X}}| \le \gamma Y^{\mathsf{N}}[\mathrm{Diag}_X] \le \gamma(B+o(1))C_X Y$, valid as $Y^{\mathsf{N}}[J_X] \ge 0$ by (E2). Otherwise $\ell \ge C_X\ell/\bar{C}$, whence the claim, as $C_X \ge 1$. □

10.6. **Energies of the dominators for ν are at most $Be^T R_I^k Y$.** Two places need energies not of ν itself but of the laws $M_{\mathcal{B}}$ that dominate it in (CP6): the index-support inequality, which bounds the signed form J_ν by such energies with random inclusion of the index coordinates, and the wrapping terms, which need the same energies at a fixed interval. The lemma bounds both: writing $M_{\mathcal{B}} = \mathsf{N}_{\mathcal{B}} + Q_{\mathcal{B}}$, the excess $Q_{\mathcal{B}}$ with total price T_Q contributes a factor e^{T_Q}, and each coupled query contributes its conductor moment R_I.

Lemma 10.17 (Envelope energies)**.** *Assume (CP0), (CP1), (CP2)(b), (CP3) and (CP8) of Theorem 9.21. For every atom let $M_{\mathcal{B}} = \mathsf{N}_{\mathcal{B}} + Q_{\mathcal{B}}$, with $Q_{\mathcal{B}}$ a nonnegative combination of admissible positive reserved-diagonal laws and $\sum_{\mathcal{B}} \|Q_{\mathcal{B}}\| q_{\mathcal{B}}^{2a} \le T_Q$, the prices $\|Q_{\mathcal{B}}\| q_{\mathcal{B}}^{2a}$ satisfying the summability condition of (CP8)(c) (as those of (CP4)(b) and (CP6) do). Let $\mathcal{S} \subseteq \mathcal{E}_k$, and for $e \in \mathcal{S}$ let b_e be uniform on $\mathbb{Z}/K\mathbb{Z}$ ($\kappa_e = 1$) or fixed ($\kappa_e = K$) and s_e uniform on $\mathbb{Z}/M_0\mathbb{Z}$, all independent. Let X carry $g^{I_{b_e},M_0,s_e}$ at the source u_e of each $e \in \mathcal{S}$ and F_A at every other slot of V_+, and let $V = \mathbb{E}_{\mathcal{J}}\mathbb{E}_\omega \left\langle X^{\mathcal{J}}, C_\iota(\bigotimes_{\mathcal{B}} M_{\mathcal{B}})X^{\mathcal{J}}\right\rangle$ be its single-half random-inclusion energy, where $\mathcal{J}$ reveals every uniform interval coordinate and independent prime-power depths with the tails $\delta_{p,j}$ of (CP0)(d) (for $\mathcal{S} = \mathcal{E}_k$ and uniform intervals, $V = V_{\otimes M}(X)$ of Definition 9.6). Then*

$$V \le \Big(Be^{T_Q} \prod_{e\in\mathcal{S}} (\kappa_e^{2\theta/d} R_I) + o(1)\Big) Y,$$

and the same holds with the inputs at the reflected targets $\iota(v_e)$. The $o(1)$ consists of energy errors and large-product remainders.

With $\mathcal{S} = \mathcal{E}_k$, uniform intervals and $Q_{\mathcal{B}} = M_{s,\mathcal{B}} - \mathsf{N}_{\mathcal{B}}$ of (CP6), this is $V_s \le (Be^T R_I^k + o(1))Y = (2^k J + o(1))Y$ with $J = Be^T(R_I/2)^k$, as in Theorem 9.21.

Proof. Fix $\mathcal{J}$ and the intervals. The revealed depths determine s_e modulo $D_{\mathcal{J}_e} = \prod_p p^{J_{e,p}}$, a divisor of M_0. By Lemma 6.3 and the disjointness of the endpoints, as in Corollary 10.4(ii), $X^{\mathcal{J}}$ carries $g^{I_{b_e},D_{\mathcal{J}_e},s_e \bmod D_{\mathcal{J}_e}}$ at u_e ($e \in \mathcal{S}$); these support projections come before any fold, as (O3) requires. By (5.1), $\left\langle X^{\mathcal{J}}, C_\iota(\bigotimes_{\mathcal{B}} M_{\mathcal{B}})X^{\mathcal{J}}\right\rangle$ is the energy of $\bigotimes_{\mathcal{B}} M_{\mathcal{B}}$ with the input of u at u and at ιu ($u \in V_+$): each $e \in \mathcal{S}$ has the two changed slots u_e, ιu_e, each of coarse price $(K\,\mathrm{sq}(D_{\mathcal{J}_e}))^{a/d}\kappa_e^{\theta/d}$ (Definition 8.5, with the square step of Lemma 8.2). Expand the outside laws fully and split by the total selected union as above; the large unions contribute one aggregate $o(1)Y$. For the small unions, start with $\bigotimes_{\mathcal{B}} M_{\mathcal{B}} = \sum_Z \bigotimes_{\mathcal{B}\in Z} Q_{\mathcal{B}} \otimes \mathsf{N}_{-Z}$ over the finite sets Z of atoms. Each $\bigotimes_{\mathcal{B}\in Z} Q_{\mathcal{B}}$ is admissible and positive, so the small-union part of the proof of Lemma 10.12 bounds its contribution by $(B+o(1))\prod_{\mathcal{B}\in Z}\|Q_{\mathcal{B}}\|\,\Pi_Z Y$, where $\Pi_Z = C_Z \prod_{e\in\mathcal{S}} (K\,\mathrm{sq}(D_{\mathcal{J}_e}))^{2a/d}\kappa_e^{2\theta/d}$. Since $\sum_Z \prod_{\mathcal{B}\in Z}\|Q_{\mathcal{B}}\| q_{\mathcal{B}}^{2a} = \prod_{\mathcal{B}}(1+\|Q_{\mathcal{B}}\| q_{\mathcal{B}}^{2a}) \le e^{T_Q}$, the energy at fixed $\mathcal{J}$ is at most $(B+o(1))e^{T_Q}\prod_{e\in\mathcal{S}}(K\,\mathrm{sq}(D_{\mathcal{J}_e}))^{2a/d}\kappa_e^{2\theta/d}\,Y$. In Lemma 10.12 the conductors stay fixed through every fold and geometric mean, and complementary folds keep two occurrences of each conductor (Lemma 8.8(e), (O4)). Only now average over $\mathcal{J}$: the $\mathcal{J}_e$ are independent with the law of (CP0)(d), so $\mathbb{E}_{\mathcal{J}}\prod_{e\in\mathcal{S}}(K\,\mathrm{sq}(D_{\mathcal{J}_e}))^{2a/d} = R_I^{|\mathcal{S}|}$, and the errors are uniform in $\mathcal{J}$. For the reflected targets use the slots $\iota(v_e)$ and v_e. □

Remark 10.18 (The trivial mean credit and the bookkeeping of the closing). Assume (CP1)–(CP3), (CP6) and (CP8). For every atom and $s \in \{\mathrm{L},\mathrm{R}\}$, $M_{s,\mathcal{B}} \pm \lambda_{\mathcal{B}} = (\mathsf{N}_{\mathcal{B}} \pm \lambda_{\mathcal{B}}) + (M_{s,\mathcal{B}} - \mathsf{N}_{\mathcal{B}}) \succeq_\iota 0$: the first summand by (CP3) (off the first-stage atoms $\mathsf{N}_{\mathcal{B}} \pm \lambda_{\mathcal{B}}$ is $2\mu_{\mathcal{B}}$ or 0), the second by (E3). By

Lemma 5.6(d), $\bigotimes_{\mathcal{B}} M_{s,\mathcal{B}} \pm \bigotimes_{\mathcal{B}} \lambda_{\mathcal{B}} \succeq_\iota 0$, and at $\bar{X}$ this is $e_s \geq Y_\lambda \geq (1 - L_0 - o(1))Y$ (Lemma 10.10). Hence (CP7) holds with the trivial credit $m = 1 - L_0$, and then $\Theta = (1 - 2^{-k})(L_0 + \mathcal{E}) + J + W + 2^{-k}L_0 = L_0 + (1 - 2^{-k})\mathcal{E} + J + W$: the first-stage loss enters $2^k - 1$ times through Y_ν and once through the mean credit, so it keeps coefficient one, as do the matching term and the wrap, and only the reserved cost drops to $1 - 2^{-k}$. No resource is counted twice: the class bounds and credits enter only through the reserved cost, a separate inequality between Y_ν and Y_λ (Lemma 10.14); the matching prices enter only through the bounds for V_{L}, V_{R} (Lemma 10.17); and this credit uses no price at all, only positivity, and no order between N and μ. The construction uses the larger credit of Proposition 26.7 for (CP7); this remark is not one of its premises.

Consequences and what to read next. From Part II: cut matrices, their order and the Kronecker rule (Definition 5.2, (5.1), (5.2), Definition 5.1, Remark 5.3), the common-cut order (Lemma 5.4), the block-Gram calculus (Lemma 5.6) and the sector lemma (Lemma 6.10); the functional, its inputs, the index averages and the reference hypotheses (Definition 6.1, Lemma 6.3, Hypothesis 6.2); structured energies and their estimate, folded energies, admissibility, the square step, the cancellation and the large products (Definitions 8.5 and 8.9, Proposition 8.7, and Lemmas 8.2, 8.8, 8.10 and 8.11), at a least normalized counterexample (Definition 8.4). From Part III: the frame data and hypotheses of Theorem 9.21, the index law, the nonvanishing square classes and the signed index form (Definition 9.6, Lemma 9.5, (9.1)), the comparison law (Definition 9.14) and the prices (Definition 9.2).

11. From the five estimates to Theorem C

What this section proves and how to read it. This section proves Theorem C. The closing inequality (Lemma 11.1) combines the wrapping bound W, the mean energies $e_{\mathrm{L}}, e_{\mathrm{R}}$ and the envelope energies $V_{\mathrm{L}}, V_{\mathrm{R}}$ of the previous section with the first comparison: at a least counterexample, $1 \leq \Theta + o(1)$. The robustness lemma (Lemma 11.4) shows that every $o(1)$ in this chain is produced by structured energies and by the two analytic remainders, so that the density-linear induction of Theorem 8.12 applies. The section ends with the proof of the one-query proposition (Remark 11.6) and with Table 5, which records where Part V verifies each hypothesis.

11.1. At a least counterexample, $1 \leq \Theta + o(1)$.

This lemma is where the comparison closes: the five estimates of Section 10 combine to show that a least counterexample forces $1 \leq \Theta + o(1)$. Hence $\Theta < 1$ rules out a counterexample at large scale, which is the contradiction behind Theorem C.

Lemma 11.1 (Closing inequality). *Assume (CP0)–(CP4), (CP6) and (CP8). Let $\mathcal{E}$ and m be numbers such that, at every least normalized counterexample,*

(i) $Y_\nu \geq Y_\lambda - (\mathcal{E} + o(1))Y$, *and*
(ii) $\frac{1}{2}(e_{\mathrm{L}} + e_{\mathrm{R}}) \geq (m - o(1))Y$.

Then, at every least normalized counterexample,

$$1 \leq \Theta(\mathcal{E}, m) + o(1), \qquad \Theta(\mathcal{E}, m) = (1 - 2^{-k})(L_0 + \mathcal{E}) + J + W + \frac{1 - m}{2^k}, \tag{11.1}$$

with J and W as in Theorem 9.21. Consequently, if $\Theta(\mathcal{E}, m) < 1$, no least normalized counterexample exists with L sufficiently large.

Proof. By Lemma 10.6,

$$2^k Y_\nu \leq J_\nu + (2^k W + o(1))Y. \tag{11.2}$$

The index-support lemma, in the form of Corollary 10.4, applies to the signed index operator $\mathcal{T} = \bigotimes_e (2P_e - T_e)$, the joint bracket of the envelopes of (CP6) tensorized over the finitely many atoms of $\mathcal{A}_N$, and the half-functions X_ω, Y_ω, whose means are $\bar{X}$. Its mean term is Y_ν, its

mean energies are the e_s of (CP7), and its random-inclusion energies are $V_{\mathrm{L}} = V_{\bigotimes M_{\mathrm{L}}}(X)$ and $V_{\mathrm{R}} = V_{\bigotimes M_{\mathrm{R}}}(Y)$ (Definition 9.6(f)). By (10.4),

$$J_\nu \le Y_\nu + \tfrac{1}{2}\big[(V_{\mathrm{L}} - e_{\mathrm{L}}) + (V_{\mathrm{R}} - e_{\mathrm{R}})\big]. \tag{11.3}$$

By Lemma 10.17, applied with $Q_{\mathcal{B}} = M_{s,\mathcal{B}} - \mathsf{N}_{\mathcal{B}}$ and $T_Q = T$ (CP6), $V_{\mathrm{L}}, V_{\mathrm{R}} \le (Be^T R_I^k + o(1))Y = (2^k J + o(1))Y$. With (ii), (11.2) and (11.3) give

$$(2^k - 1)\, Y_\nu \le \big(2^k J - m + 2^k W + o(1)\big)Y.$$

By (i) and Lemma 10.10, $Y_\nu \ge (1 - L_0 - \mathcal{E} - o(1))Y$. Since $Y > 0$ and $2^k - 1 > 0$, $(2^k - 1)(1 - L_0 - \mathcal{E}) \le 2^k J - m + 2^k W + o(1)$, that is, $2^k \le (2^k - 1)(L_0 + \mathcal{E}) + 2^k J + 2^k W + (1 - m) + o(1)$; dividing by 2^k gives (11.1). The last assertion follows because $o(1) \to 0$ as $\mathsf{L} \to \infty$, uniformly over square-difference-free $A \subseteq [K\mathsf{L}]$. □

Corollary 11.2 (Joint comparison and mean allocation)**.** *The comparison loss $Y_\lambda - Y_\nu$ and the mean energies $e_{\mathrm{L}}, e_{\mathrm{R}}$ enter the preceding proof with opposite signs. It suffices to bound their combined contribution. The separate bounds (i) and (ii) in Lemma 11.1 may be replaced by a single robust bound. Suppose that, for some real constant $\mathcal{H}$,*

$$(2^k - 1)(Y_\lambda - Y_\nu) - \tfrac{1}{2}(e_{\mathrm{L}} + e_{\mathrm{R}}) \le (2^k \mathcal{H} - 1 + o(1))Y.$$

Then the resulting closing number is

$$\Theta = (1 - 2^{-k})L_0 + J + W + \mathcal{H}.$$

In particular, positive comparison energies and negative mean energies may first be combined on the same labelled diagonal. If their residual coefficients are nonnegative, their common energies may then be bounded by upper prices and completed to exponential series.

Proof. Keep the mean energies in (11.3) until after combining that inequality with (11.2). This gives

$$(2^k - 1)Y_\lambda \le (2^k(J + W) + o(1))Y + (2^k - 1)(Y_\lambda - Y_\nu) - \tfrac{1}{2}(e_{\mathrm{L}} + e_{\mathrm{R}}).$$

Use the displayed joint bound and $Y_\lambda \ge (1 - L_0 - o(1))Y$, and divide by 2^k. Nonnegative residual coefficients permit exactly the positive completion argument of Lemma 10.14; applying an upper price to an isolated negative mean term would not permit it. The robustness proof of Lemma 11.4 applies to the combined positive series, with the same two kinds of analytic remainder. □

Remark 11.3 (The trivial value of the mean constant). By Remark 10.18, hypothesis (ii) always holds with $m = 1 - L_0$, and then $\Theta(\mathcal{E}, 1 - L_0) = L_0 + (1 - 2^{-k})\mathcal{E} + J + W$; that remark explains this bookkeeping. A larger m lowers Θ by $(m - 1 + L_0)/2^k$; Part V proves $m > 1$ (Proposition 26.7).

11.2. **Every error term comes from the two allowed remainders.** The hypothesis (Rec) of Theorem 8.12 asks for the closing inequality with every analytic error made explicit: every structured-energy bound inflated by the factor $1 + z$, every remainder bounded by zY, and a closing number $\Theta(z)$ continuous at $z = 0$. The next lemma supplies it and records how each $o(1)$ above arises.

Lemma 11.4 (The recurrence hypothesis)**.** *Assume (CP0)–(CP8).*

(i) Robust bounds. *Exact same-scale relations (identities and inequalities between quantities at the same scale, with no error term) hold unchanged at every z-inflated counterexample. A bound $\Xi \le (c + o(1))Y$ obtained by applying Proposition 8.7, directly or through Lemma 8.8, (E1) or Lemma 10.12, to finitely many families of structured energies, summing with nonnegative coefficients bounded by convergent products of prices, and adding remainders of the forms of Definition 9.19(a), is robust: at a z-inflated counterexample its constant c becomes $(1 + z)c + c'z$, where c' is a bound, independent of L and A, for the coefficients*

multiplying the remainder integrals and sums in its derivation. An exact relation minus robust upper bounds is a robust lower bound. Let a bound be derived from finitely many robust bounds, with functions $c_1(z), \dots, c_n(z)$, by an argument that is valid whenever finitely many strict inequalities between the $c_i(z)$ and fixed numbers hold, and let its constant be $F(c_1, \dots, c_n)$, with F continuous at $(c_1(0), \dots, c_n(0))$. If the strict inequalities hold at $z = 0$, the bound is robust, with $c(z) = F(c_1(z), \dots, c_n(z))$.

(ii) The lemmas. *The bounds of Lemma 10.13 and of Lemmas 10.1, 10.6, 10.8, 10.10, 10.12, 10.14, 10.16 and 10.17 and Corollary 10.4 are exact or robust for fixed selected sets. For the summed families used here, the full-union split of Section 10 gives uniform z_0, L_1 and aggregate remainder coefficients. Their remainders are of two kinds:*

(a) the nonwrapping terms of Lemma 10.6: for each of the at most $3^k n_\nu$ triples of a product component of ν, an index subset and a subset of wrapping queries that does not contain every query of the subset (a wrap pattern with a nonwrapping query*), an integral of the form bounded in Lemma 8.10, against a probability law of the indices, the coefficients of the product components for one pair (X_ω, Y_ω) of half-functions adding up to at most $2^k \mathfrak{v}_\nu e^{P+S}$;*

(b) the terms of the selections whose union E of atoms has $q_E > N^\tau$, in finitely many sums of the form bounded in Lemma 8.11 whose coefficients $\eta_{\mathcal{B}}$ are the prices of the selected laws after the operator-to-energy reduction of Lemma 11.4(ii) (the post-reduction prices*); together they have the bound of one such sum, with c the sum of their constants, and with $\Pi_{\mathrm{H}} = \prod_{\mathcal{B}}(1 + \eta_{\mathcal{B}} q_{\mathcal{B}}^{1/4})$ as in Lemma 8.11; neither c nor Π_{H} depends on L or A.*

(iii) The recurrence. *Let $\mathcal{E}$ and m be numbers for which (i) and (ii) of Lemma 11.1 hold robustly. Then there are $z_0 > 0$, L_1 and a function $\Theta\colon [0, z_0] \to [0, \infty)$, continuous at 0 with $\Theta(0) = \max(\Theta(\mathcal{E}, m), 0)$, such that the hypothesis (Rec) of Theorem 8.12 holds; its remainders are those of (ii) together with those of the robust arguments for (i) and (ii) of Lemma 11.1, and they have the forms that theorem requires.*

Proof. (i) An exact relation holds at every scale with no error term, so it needs no inflation, and a bound of the form $\Xi \le cY$ that it gives has $c(z) = c$. In a bound of the second kind each structured energy is at most $1 + z$ times its main bound at a z-inflated counterexample; the coefficients are nonnegative, so the sum of the main bounds is multiplied by at most $1 + z$; and the remainders, at most zY in total, enter with coefficients at most c', which by Definition 9.19(a) is independent of L and A. If $\Xi \ge \Xi_0 - U$ with Ξ_0 exact and $U \le u(z)Y$, then $\Xi \ge \Xi_0 - u(z)Y$. In the last case, by continuity there is $z_0 > 0$ such that the strict inequalities hold for $0 \le z \le z_0$; the argument then applies at every z-inflated counterexample with $z \le z_0$, and $F(c_1(z), \dots, c_n(z)) \to F(c_1(0), \dots, c_n(0))$ as $z \to 0$. The number of remainders and their coefficients remain bounded, being those of finitely many arguments.

(ii) We go through the proofs of these statements. The following are *exact relations*: $Y[\bigotimes_{\mathcal{B}} \mu_{\mathcal{B}}] = Y$; the orders (E2) and (E3) and the relations (P1)–(P5) of Lemma 10.13, among them the product centred interpolation (P3′); the cut-order and bracket manipulations (Equation (5.2), Lemma 5.6, part (i) of Lemma 10.16); the vanishing of the terms with an odd number of skew factors in Lemma 10.14; Step 1 of the first comparison, which is (CP2)(a); the identity (9.1); and the inequalities of Lemma 10.1 and Corollary 10.4, which hold for arbitrary vectors and envelopes. The following are upper bounds of the second kind of (i): (E1); the tagged expansion Lemma 10.8, which bounds each term of the tagged series, indexed by the numbers of summands of the two tags, by the product of the sharp norms of one tag times the geometric mean of two folded energies (Lemma 8.8(a), (d′), (e)), the tag used for a category being chosen from the prices alone, so that the bound is again a sum of structured energies with nonnegative coefficients, and is used for Step 2 of Lemma 10.10; Lemma 10.12, the same expansion with the bracket outside, in which the

geometric mean of two inflated factors of a Cauchy–Schwarz step is again inflated by $1+z$, so that B becomes $(1+z)B$ and not $(1+z)^2B$; the small-union bound underlying $|I_\omega| \le (B+o(1))\pi_\omega Y$ of Lemma 10.14(i); the energy bounds of Lemma 10.17 and of the wrapping terms of Lemma 10.6, in which the conductors are held fixed through every fold and averaged only at the end ((O3) of Definition 8.9); and the upper bound in Lemma 10.16(ii). Each of these is a sum of structured energies with nonnegative coefficients bounded by convergent price products such as $e^P\mathfrak{h}(d,v)$, $e^{U_0}\mathfrak{h}(d,v)$, e^{S+A} and e^T. The following are lower bounds of the third kind: the first comparison $Y_\lambda \ge (1-L_0-o(1))Y$, which is Step 1 minus Step 2; the bound $Y^{\mathsf{N}}[J_X] \ge Y[J_X]-(R_{\mathrm{err}}+o(1))C_XY$ of Lemma 10.16(ii), which is (E2) minus (E1); and the gain $\max(0,\ell/\bar{C}-R_{\mathrm{err}})$ of Lemma 10.16(iii), a continuous function of robust bounds. Finally, Lemma 10.14(ii), (iii) combine the bounds of (i) of that lemma with the class bounds of (CP5), which are robust uniformly in the class by (CP8)(b), through the complete polynomial Ψ. The series $\sum_\omega b_\omega$ that Ψ dominates has as coefficients B, the scalars $c_{\mathcal{C}}$ and the coefficients of the class bounds; at a z-inflated counterexample these become functions of z that converge to them uniformly over the classes, and we may replace each by its positive part, while the errors $o(1)\pi_\omega$ of (CP5) become $\epsilon(z)\pi_\omega$ with one function $\epsilon(z)\to 0$. Adding to Ψ the positive parts of the changes of the b_ω, and $\epsilon(z)(e^{S(x)}\cosh A(x)-1)$, gives a series Ψ_z with nonnegative coefficients that dominates the inflated bounds. The difference $\Psi_z(\hat\pi,\hat y)-\Psi(\hat\pi,\hat y)$ is at most the largest change of a coefficient times the sum of the values at $(\hat\pi,\hat y)$ of the monomials of the b_ω, plus $\epsilon(z)(e^{\hat S}\cosh\hat A-1)$. That sum is finite: the monomials in the prices sum to at most $e^{\hat S}\cosh\hat A-1$, since distinct selections have distinct squarefree monomials, and those of the class bounds to at most $\prod_{\mathcal{B}}(1+\sum_j \hat y_{\mathcal{B},j})$, since each of them contains one auxiliary price of each atom of its selection and determines the selection (CP5); this product is finite because $\hat y$ has a finite total. So $\Psi_z(\hat\pi,\hat y)$ tends to $\Psi(\hat\pi,\hat y)$ as $z\to 0$.

The remainders. By Lemma 10.6, whose proof splits the term of an index subset $\mathcal{S}$ by the set of its wrapping queries (each query is of exactly one of three kinds: in $\mathcal{S}$ and wrapping, in $\mathcal{S}$ and nonwrapping, or outside $\mathcal{S}$), the nonwrapping terms are at most $3^k n_\nu$ integrals of the form bounded in Lemma 8.10, one for each product component of ν, against the conditional law of the indices, with coefficients adding up to at most $2^k\mathfrak{v}_\nu e^{P+S}$ for each pair; that proof also bounds the mass of ν by (CP2) and the brackets of (CP4). This is (a). The selections with $q_E > N^\tau$ occur in finitely many expansions, each of the form bounded in Lemma 8.11 with post-reduction prices of the selected laws as the $\eta_{\mathcal{B}}$. Each law chosen at an atom in these expansions is one of the laws of (CP2)–(CP6) or a fold or normalization of one, so its price is at most the sum of the prices of the atom listed in (CP8)(c); the components of a signed comparison law are never selected laws, since they occur only in the cancellation of (a), so the variation factor $\mathfrak{v}_\nu$ does not enter here. Taking that sum as $\eta_{\mathcal{B}}$ for all the expansions, $\Pi_{\mathrm{H}}=\prod_{\mathcal{B}}(1+\eta_{\mathcal{B}}q_{\mathcal{B}}^{1/4})$ is finite by (CP8)(c); summing the finitely many sums gives the bound of one such sum. The number of terms and their coefficients do not depend on L or A. This is (b).

(iii) Let (L,A) be a z-inflated least normalized counterexample, with L and z in the ranges of the finitely many robust bounds used. By (ii), the steps of the proof of Lemma 11.1 hold with each $o(1)$ replaced by the deviation of a function continuous at 0: $2^kY_\nu \le J_\nu + 2^kW(z)Y$; (11.3) exactly; $V_{\mathrm{L}},V_{\mathrm{R}} \le 2^kJ(z)Y$; and $Y_\lambda \ge (1-L_0(z))Y$, with $W(0)=W$, $J(0)=J$ and $L_0(0)=L_0$. By hypothesis, $Y_\nu \ge Y_\lambda - \mathcal{E}(z)Y$ and $\frac12(e_{\mathrm{L}}+e_{\mathrm{R}}) \ge m(z)Y$, with $\mathcal{E}(0)=\mathcal{E}$ and $m(0)=m$. The algebra of that proof gives $Y \le \Theta(z)Y$ with

$$\Theta(z) = \bigl(1-2^{-k}\bigr)\bigl(L_0(z)+\mathcal{E}(z)\bigr) + J(z) + W(z) + \frac{1-m(z)}{2^k},$$

which is continuous at 0 with $\Theta(0)=\Theta(\mathcal{E},m)$; since $Y>0$, the inequality persists when $\Theta(z)$ is replaced by $\max(\Theta(z),0)$. This is (Rec), with z_0 the minimum and L_1 the maximum of the numbers z_0,L_1 of the robust bounds involved. Its remainders are those of (ii) and those of the

robust arguments for (i) and (ii) of Lemma 11.1; by Definition 9.19(a) they are finitely many integrals of the form of Lemma 8.10, with coefficients bounded independently of L and A, and finitely many sums of the form of Lemma 8.11, which together have the bound of one such sum with the sum of their constants c and the largest of their products Π_{H}. This is the second hypothesis of Theorem 8.12. □

Remark 11.5 (Robustness of the final chain). In Part V every bound that enters (CP5) and (CP7) is of one of the kinds of Lemma 11.4(i).

(a) *Upper bounds of structured energies*, with nonnegative coefficients over certified prices: the direct prices of the middle records and their repricing (Lemmas 9.3 and 23.24); the repriced first-stage totals and the tagged series, which give U_0, B, R_{err} and L_0 (Corollaries 9.4 and 10.9); every leading class cost (with n, m and $\kappa_H, \kappa_b, U^{\mathrm{inv}}, \delta_{\mathrm{s}}$ as in Corollary 25.10)

$$C_{n,m} = \kappa_H e^{U_0} - (\kappa_H - \kappa_b) e^{tU^{\mathrm{inv}}} + M_{n,m} R_{\mathrm{err}} - M^{\mathrm{inv}}_{n,m} e^{tU^{\mathrm{inv}}} t\delta_{\mathrm{s}}$$

whose terms are an outside energy, the invariant-background polynomial with nonnegative coefficients (Lemma 25.6 and Remark 10.15), normalized couplings bounded by (E1), and an orbit coupon that lowers the coupling term by an exact orbit decomposition (Lemma 25.7); the remainder polynomials $B\mathcal{R}_{n,m}$; J and W; the upper prices $S_{\mathrm{u}}(a)$, $A_{\mathrm{u}}(a)$ (Proposition 28.14); and the complete polynomial after the repeated-block monomials are added with nonnegative coefficients (Lemma 10.14(iv), (v)).

(b) *Exact same-scale relations*: the orders (E2), (E3) and (P3′); the operator comparison of the pair projections with their positive substitutes (Lemma 25.2); the matrix-envelope orders and their mask form (Lemma 25.3); the changes of variables of Lemma 25.5; the orbit decomposition of Lemma 25.7; the positive compressions of the aligned matching bridge (Lemma 24.11(a), (c)); the all-degree polarization identity (Lemma 17.11); and the contraction and density steps of Lemma 26.6.

(c) *Exact relations minus upper bounds*: the restorations l with the individual piece caps (Lemmas 20.7 and 24.8; under inflation $l = \max(0, 1 - (C-1)h)$ becomes $\max(0, 1 - ((1+z)C - 1)h)$); the gains with positive backgrounds and the late-pool gains (Proposition 24.17 and Lemmas 24.12 and 24.16), positive parts of continuous expressions; the selected-only aligned matching gain (Lemmas 24.11 and 25.9); the credited class bounds (Lemma 25.9); the variance credit D_{var} (Corollary 24.18); and the mean credit m (Proposition 26.7), which subtracts $R_{\mathrm{err}} e^{T_K + T_{QE}}$.

The strict inequalities used are: every saving used in (a)–(c) has a stated positive lower bound; every restoration gain lies in $[0, 1)$ with a positive denominator; the two-cut conditions $d < 6$, $dv < 36$ of Corollary 10.9; and the positive-mean branch; the closing and restoration scripts assert them with strict margins (Table 19). Three further gates are nonnegativity conditions that need no margin. The orbit-coupon gate $M^{\mathrm{inv}}_{n,m} e^{tU^{\mathrm{inv}}} t\delta_{\mathrm{s}} \le M_{n,m} R_{\mathrm{err}}$ concerns the coefficient $M_{n,m} R_{\mathrm{err}} - M^{\mathrm{inv}}_{n,m} e^{tU^{\mathrm{inv}}} t\delta_{\mathrm{s}}$ of the coupling term, which is itself a bound of the kind (a) and so only grows under inflation. Every subtracted term cg on a monomial in the auxiliary prices y^{arm} of a replaced class, a fixed envelope cap c times a gain $g < 1$, is below the coefficient $C_{n,m}$ of that monomial (Lemma 25.9); this gate is proved in the text, from $c \le \kappa_b e^{tU^{\mathrm{inv}}} \le C_{n,m}$ (Proposition 30.2), not asserted by a script. Under inflation the coefficient, a bound of the kind (a), only grows, and the credit, its gain being of the kind (c), only shrinks. The restoration regime $0 \le l \le C/2$ is kept by inflation, which lowers $l = \max(0, 1 - (C-1)h)$ and raises C. The side coefficients $k_s \ge 0$, the side increments $\delta^+_{s,X}$ of (26.1) and the bracket condition (Propositions 26.4 and 26.5 and Equation (22.1)) are inequalities between fixed numbers that enter only exact bracket inequalities, so inflation does not affect them; the piece caps $1 \le C_i \le e^{2aL}$ (Lemma 20.7) and the domain $0 < a \le 9/10000$ of the analytic gates are fixed numbers as well. Under inflation the class

costs and B increase together and every normalized restoration stays at most one. The selections of the classes and of the credits have at most six atoms, the envelope expansions have finitely many couplings, the comparison law has finitely many product components, and K is fixed, so the remainders are finitely many.

11.3. **End of the proof: applying the baseline induction.** Everything is now in place: the record expansion bounds $|Y_\nu - Y_\lambda|$, hypothesis (CP7) supplies the mean credit m, and the closing lemma together with the induction of Section 8 finishes Theorem C.

Proof of Theorem 9.21. Let (L, A) be a least normalized counterexample. By Lemma 10.14(ii), applied with the classes and class bounds of (CP5), the upper prices $\hat\pi$ and the upper auxiliary prices $\hat y$,

$$|Y_\nu - Y_\lambda| \le (\Psi(\hat\pi, \hat y) + o(1))Y \le (\mathcal{E} + o(1))Y,$$

which is hypothesis (i) of Lemma 11.1; (CP7) is hypothesis (ii). Both hold robustly: the first by Lemma 11.4(ii), with $\mathcal{E}(z) = \mathcal{E} + |\Psi_z(\hat\pi, \hat y) - \Psi(\hat\pi, \hat y)| + c'z$, where Ψ_z is the inflated series of that proof and c' bounds the coefficients of the remainders (Lemma 11.4(i)), and the second by (CP8)(b). By Lemma 11.4(iii) there are $z_0 > 0$, L_1 and a function $\Theta(\cdot)$, continuous at 0 with $\Theta(0) = \max(\Theta, 0) < 1$, for which (Rec) of Theorem 8.12 holds, with remainders of the forms that theorem requires. Its other hypotheses hold: $0 < a < \sigma$ and $0 < \theta \le 1$ by (CP0)(b), and (R1)–(R5) by (CP1) and (CP4)(a), with the clauses of (R4) and (R5) on the accepted parts read for the normalized components of ν and the queries of $\mathcal{E}_k$, as (CP4)(a) states them; these clauses enter only through the hypotheses of Lemma 8.10, which Lemma 10.6 checks for every product component. It gives a constant C, depending only on the comparison data, with $\mathcal{Y}(\mathsf{L}, A) \le C\mathsf{L}^{-a}\alpha^\theta$ for every $\mathsf{L} \ge 1$ and every square-difference-free $A \subseteq [K\mathsf{L}]$, and $s(N) \le C'N^{1-a/(d-\theta)}$ for every $N \ge 1$, with $d = 72$. □

The proof is the composition of Lemma 11.1, Lemma 10.10, Lemma 10.12 (inside Lemmas 10.14 and 10.17), Lemma 10.6, Lemma 10.14, Lemma 10.1 (through Corollary 10.4), Lemma 10.17 and Lemma 11.4, followed by Theorem 8.12 of Part II, and of nothing else. No symmetry of ν and no positivity of ν is used, except that each of its finitely many product components is a nonnegative accepting law in the cancellation of the nonwrapping terms (Lemma 10.6); the skew parts enter only through the vanishing of the terms with an odd number of skew factors in the record expansion and through the separate pricing of skew changes.

11.4. **The case of one query (Proposition 2.1).** For completeness we record how the one-query proposition of Part I follows from the same argument with $k = 1$. Nothing later depends on this remark.

Remark 11.6 (Proof of the one-query proposition). We read the hypotheses of Proposition 2.1 with the notation of this part: $k = 1$ and $\mathcal{E}_1 = \{e\}$; μ satisfies (CP1); ν is a product of nonnegative atom measures of bounded total mass satisfying (CP4)(a) for the query e; (H1) is $Y_\nu \ge (1 - \mathcal{E} - o(1))Y$; and (H2) gives nonnegative measures $M_\mathcal{B}$ with $\left(\begin{smallmatrix} M_\mathcal{B} & \nu_\mathcal{B} \\ \nu_\mathcal{B}^* & M_\mathcal{B} \end{smallmatrix}\right) \succeq_\iota 0$ such that the energy under $\bigotimes_\mathcal{B} M_\mathcal{B}$ of every structured slot datum (Definition 8.5(c)) is at most $(B + o(1))$ times its coarse price times Y. With $k = 1$, (9.1) reads $2Y_\nu = J_\nu + I_{\nu,\{e\}}$.

(1) Part (b) of Lemma 10.1, applied as in Corollary 10.4 with $M_\mathrm{L} = M_\mathrm{R}$ the cut matrix of $\bigotimes M_\mathcal{B}$, Γ that of ν, and $\mathcal{T} = 2P_e - T_e$, gives $|J_\nu| \le V_\mathrm{L}^{1/2} V_\mathrm{R}^{1/2}$ with $V_\mathrm{L} = V_{\bigotimes M}(X)$ and $V_\mathrm{R} = V_{\bigotimes M}(Y)$. By Lemma 6.3 the energy of $X^\mathcal{J}$ under $\bigotimes M_\mathcal{B}$ is the energy under this measure of a structured slot datum whose changed slots are u_e and ιu_e, each with the slot price $(K\,\mathrm{sq}(D_{\mathcal{J}_e}))^{a/d}$ and the coarse factor 1 of a uniform index; by (H2) it is at most $(B + o(1))(K\,\mathrm{sq}(D_{\mathcal{J}_e}))^{2a/d}Y$, the conductor being held fixed and averaged at the end. Averaging over $\mathcal{J}$ gives $V_\mathrm{L} \le (BR_I + o(1))Y$, and likewise for V_R; so $|J_\nu| \le (BR_I + o(1))Y$.

(2) Split $I_{\nu,\{e\}}$ according to b_e. If $b_e \neq K$, the query e is accepted, its two ends carry $g^{I_{b_e},M_0,s_e}$ and $g^{I_{b_e+1},M_0,s_e+r_e}$ with ordered supports and a nonvanishing square class r_e (Lemma 29.6 and Definition 4.6), and the term is a cancellation remainder of the form of Lemma 8.10, hence $o(1)Y$. The event $b_e = K$ has probability $1/K$. At this fixed interval, the form of Lemma 10.1 for the filtration by residue coordinates, with the operator T_e, bounds the term by the geometric mean of two energies as in (1), in which the fixed interval adds the coarse factor $K^{\theta/d}$ at each of the two changed slots. Hence $|I_{\nu,\{e\}}| \le (BR_IK^{2\theta/d}/K + o(1))Y = (2W_1 + o(1))Y$ with $W_1 = BR_I/(2K^{1-2\theta/d})$.

(3) Therefore $2(1 - \mathcal{E} - o(1))Y \le 2Y_\nu = J_\nu + I_{\nu,\{e\}} \le (BR_I + 2W_1 + o(1))Y$, that is, $1 \le \mathcal{E}+BR_I/2+W_1+o(1)$, which is the assertion of Proposition 2.1, where $\theta = 1$ and $K^{1-2\theta/d} = K^{35/36}$.

If part (a) of Lemma 10.1 is used in (1) instead, then $J_\nu \le Y_\nu + (BR_I + o(1))Y - \frac{1}{2}(e_{\mathrm{L}} + e_{\mathrm{R}})$, and with $\frac{1}{2}(e_{\mathrm{L}} + e_{\mathrm{R}}) \ge (m - o(1))Y$ one obtains $1 \le \frac{1}{2}\mathcal{E} + BR_I/2 + W_1 + (1-m)/2 + o(1)$: keeping the common mean halves the coefficient of $\mathcal{E}$. This is the case $k = 1$ of Theorem 9.21, without a first stage ($L_0 = 0$) and with the one energy factor B of (H2) in place of both Be^T in J and Be^{S+A} in W.

11.5. **Where the hypotheses are verified (table for Theorem C).** Table 5 lists each hypothesis of Theorem C, what Part V shows for the construction of Definition 18.1, and the statements that show it. Each section of Part V ends with a paragraph naming the rows it verifies.

Table 5: Where the hypotheses of Theorem C are verified.

Hypothesis	What is shown	Proved in
(CP0)(a) slots, cuts	V sorted by seven folds; ι fixed-point free; the family $\mathfrak{F}$	Definition 5.12 and Lemma 5.13
(CP0)(b) atoms, exponents, $a < \sigma$	$a = \frac{5422587}{6103515625} < 9/10000 < \sigma = 1/825$; inequalities (S1)–(S7); $K = 890666$; checked inequalities involving a hold for $0 < a \le 9/10000$	Definitions 6.1 and 18.1 and Certificates D.2 and D.3; data files in Appendix D
(CP0)(c) queries	six queries with disjoint endpoints, sources and reflected targets in V_+	Lemma 5.13, Section 18.2
(CP0)(d) index law	finite conductor depths $m_{\mathrm{ext}} = 64$; nonvanishing square increments modulo M_0; $R_I = M_{\mathrm{pad}}(K) < 1.247139907407$	Lemmas C.1, 9.7 and 29.6, Proposition 29.7, Corollary 29.8, and Certificate 29.11
(CP1) prefix	probability law, reflection positive at every cut, square-affine invariant	Lemma 19.2
(CP1) middle	the same for the twelve-label reference; the uniform root below 2^{30}, and seven positive roots from 2^{30}	Lemma 22.1 via Lemma 17.5; root caps by Lemma 20.12 (route A, Theorem 14.21), Propositions 20.21, 21.7 and 21.9; the fold cap Lemma 23.2; rows by Proposition 13.16 and Lemmas 12.7 and 13.25
(CP1) upper	the same for the two-layer reference in 310 bands	Lemma 27.17, Lemma 17.5 (with (f)); Theorems 14.21 and 14.27, Lemmas 16.12, 16.23 and 27.15, and Proposition 27.14
(CP1) tail	the same for the large-prime law, at primes of both classes mod 4	Lemma 29.2 (the tail law of Definition 29.1); class-one caps, Section 14.11
(R4)	(γ, v)-mixing, $\gamma = 255/1024$, at $p \ge P_5$; query pairs	Lemmas 6.14, 12.8 and 29.3; (S7) of Certificate D.2

Hypothesis	What is shown	Proved in
(R5), (CP4)(a)	accepted query pairs are strict-square pairs for every component; clique differences of even valuation with square unit part; uniform one-slot marginals	Lemmas 19.2, 23.7, 23.15 and 23.20, Certificate 23.10, Definition 23.9, and Corollary 23.21, Section 28.1, Lemma 28.1, Definition 28.4, and Proposition 28.6, Lemma 29.5(i)
(CP2)(a)	$E_{\mathrm{add}} + G - F_{\mathrm{fold}} \succeq_{\vartheta_0} 0$	Lemma 22.5: Lemma 17.11 with the rows of Lemma 22.1(i) (Proposition 13.16 and Lemma 13.25)
(CP2)(b)	E_{new} keeps the actual conditional marginals; two tags, E_0 sharp at its first cut and E_1 at its own, with equal sharp and equal variation prices; sharp coefficient k_{new}	Lemma 22.6 and Certificate D.5
(CP3)	$\mathsf{N} \pm \lambda \succeq_\iota 0$ for $\mathsf{N} = \lambda + F_{\mathrm{fold}} + H$; $\mathsf{N}_0 - \mu = E_{\mathrm{add}} + G + H$ positive reserved diagonal; the protection gate $v_D \geq 2w_D k_D$	Lemma 22.7(i), (ii); Lemma 22.1(iii), and (iv) for the protection gate (Equation (22.1)); Lemma 12.11(c)
P, V, D, U_0, tags	the middle-frame prices P, V, D, the block prices, repricing, the tagged series	Lemmas 9.3, 20.4 and 22.9, Certificate D.4, and Corollaries 9.4 and 10.9; Proposition 30.2(a)
(CP4) middle F	the signed whole-block law: three positive accepting components repaired to the actual marginal, exact marginals, variation $1 + \mathsf{t}$; direct prices	Lemma 23.24 with Definition 23.9, Lemmas 23.4 and 23.7, Proposition 23.13, and Corollary 17.17
(CP4) middle E	the law K_1 with exact marginals; prices; degree-four remainder	Lemma 23.24 with Lemmas 23.15, 23.17 and 23.18
(CP4) merges, deletions	asymmetric repairs; deleted positive records	Lemma 23.24 with Lemma 23.20, Corollary 23.21, and Remark 9.15
(CP4) upper	84 one-way repairs (skew-only charging; forest records in the bands from 2^{48}), comparisons by the dense EL law, the LE mixture, deletions; $S_{\mathrm{u}}(a)$, $A_{\mathrm{u}}(a)$	Propositions 28.3, 28.5, 28.6 and 28.14 and Lemmas 17.14 and 28.1, Remark 9.15(iii); Lemma 28.2 (tail)
(CP4) prefix	the deleted exception, reserved diagonal, charged to S	Lemma 19.3
(CP4) tail	repairs with the all-prime caps and deletions; $s_p + a_p \leq C_{\mathrm{tail}} p^{-3/2+2a}$; S_{tail}, A_{tail} at a	Lemma 29.5
(CP5) variance classes	the variance savings with positive backgrounds, of degrees one to four, and the savings pooled over the middle intervals with least prime at least 2^{31}, with the piece caps	Corollary 24.18 via Lemmas 10.16, 20.7, 24.8, 24.11, 24.12 and 24.16 and Proposition 24.17
(CP5) envelope classes	the class bounds $C_{n,m}(A_F^{\mathrm{arm}})^n(A_E^{\mathrm{arm}})^m + B\mathcal{R}_{n,m}$ ($n \geq 1$, $n + m \in \{2, 4, 6\}$), of the aligned and of the crossed matching pairs, and of $(0, 4)$, $(0, 6)$	Corollary 25.10 via Theorem 25.8 and Lemmas 25.3 and 25.5 to 25.7; Lemma 10.14(v)
(CP5) credited classes	$\kappa_{FF} g$ on fold pairs; the selected-only $\kappa_b g$ on aligned matching pairs; the quartic alternatives	Lemma 25.9

Hypothesis	What is shown	Proved in
(CP6)	the matching bracket with the signed direct Gram, the sides $\max(\bar{H}_X d_{\mathfrak{t}}, d_s^{\mathrm{aux}})$ and the chosen diagonals; T	Propositions 26.4 and 26.5 and Lemmas 26.1 and 26.3
(CP7)	the mean credit $m = 1 + m_F + m_Q - R_{\mathrm{err}} e^{T_K + T_{QE}}$	Proposition 26.7 and Lemma 26.6
(CP8)(a)	(O1)–(O4) for every selected law	Definition 8.9 and Table 10
(CP8)(b)	every bound is of a kind of Lemma 11.4(i); strict gates	Lemma 11.4 and Remark 11.5; Theorem 25.8(iv)
(CP8)(c)	at a tail atom only s_p, a_p and $T_{\mathrm{L},p} = T_{\mathrm{R},p}$ are nonzero, with $T_{\mathrm{L},p} \le s_p + a_p$ ($M_{s,p} - \mu_p$ is the sum of the norm envelopes), and $\sum_p (s_p + a_p) p^{1/4} < \infty$; auxiliary prices only at middle atoms; the other atoms are finitely many	Lemma 29.5(iii), (iv); Proposition 26.5(i)
$\Theta < 1$ (baseline)	$\Theta < 0.99999999997327$	Proposition 30.2 and Certificate 30.3; data files in Appendix D

Consequences and what to read next. The lemmas of the previous section and (E1)–(E3); (CP0)–(CP8) with Definitions 9.6 and 9.19 and Lemma 9.5 and (9.1); and from Part II Proposition 8.7, Lemmas 8.8, 8.10 and 8.11, and Theorem 8.12, Definition 8.9 and Lemma 5.6 and Equation (5.2).

12. Square-constraint labellings over $\mathbb{F}_p$

Part IV.. Section 12 and the five sections after it prove the local operator estimates with which Part V verifies the hypotheses (CP1)–(CP7) of the comparison theorem (Theorem 9.21). The statements of Part IV are made for arbitrary oriented graphs, spectral data or couplings, and their hypotheses are inequalities that a finite check verifies at one prime and that the admissible majorants of Lemma 4.18 extend to all larger primes; an evaluation at a particular graph and threshold, such as Lemma 12.8(d), is stated as an instance.

What this section proves and how to read it. This section proves the counting estimates behind every local law. A *graph labelling* is a map from the vertices of a small oriented graph G to $\mathbf{F}_p$ sending each directed edge to a nonzero-square difference (*injective* is a separate adjective; $\lambda_p(G)$ is uniform on the injective labellings). The main estimate is: once neighbours of a new vertex carry distinct labels, the number of choices for the new label is $p2^{-t}$ up to relative error $O(t2^t p^{-1/2})$ for t constraints (Weil's bound). Iterating gives sizes, marginals and conditional laws; comparing two graphs gives a ratio of four counts; at $p \equiv 3 \pmod 4$ exact identities for directed triples are proved. Inputs: Weil's bound (Section 4). Outputs: the counts used for cut operators, spectral bounds and all Part V local laws. A first reading should follow the definitions and Lemma 12.2; sequential counts can be used as reference. Terminology for Part IV: a *cut operator* A_p represents $E[F(x)H(x')]$ as $\langle F, A_p H\rangle$ in the half law (defined in Section 13); a *spectral cap* is a certified bound $-gp^{-1/2} I \preceq A_p \preceq (1 + c^+/p) I$ for all $p \ge p_0$ in a stated residue class, for a stated threshold p_0; *pinning* means conditioning on prescribed labels; the *soft kernel* is the kernel obtained by expanding every square indicator as $(1+\chi)/2$ and dropping the condition that labels be distinct; the resulting terms in which two labels that must be distinct coincide are the *coincidence terms*, and the difference between the true kernel and the soft kernel is a sum of such terms. Full definitions are at the sites noted; this paragraph only allows the statements to be read in order.

Conventions. Throughout, p is an odd prime, $q = 1/p$ and $r = p^{-1/2}$; $\chi = \chi_p$ is the quadratic character of $\mathbb{F}_p$ with $\chi(0) = 0$, and Q_p^* is the set of nonzero squares. $L^2(\mathbb{F}_p)$ carries the uniform probability, $\mathbb{E}_x$ is the average over $x \in \mathbb{F}_p$, $H_0 \subseteq L^2(\mathbb{F}_p)$ is the space of mean-zero functions and $\mathbb{J}$ is the projection onto the constants. A strict-square pair is a pair (x, x') with $x' - x \in Q_p^*$; Ω_p is the set of them.

12.1. **Oriented graphs, labellings, and count errors.** The local laws of Part V are uniform distributions on assignments of elements of $\mathbb{F}_p$ to the vertices of a small directed graph, with prescribed differences required to be nonzero squares. This subsection defines those assignments (called labellings) and the error terms that measure how close their counts are to what independent uniform labels would give.

Definition 12.1 (Oriented graphs and strict-square labellings)**.** An *oriented graph* $G = (W, E)$ is a finite directed graph without loops and with at most one edge on each unordered pair of vertices. The degree $\deg_G(w)$ counts incoming and outgoing edges, and for $W' \subseteq W$, $G[W']$ is the induced oriented graph and $G - w = G[W \setminus \{w\}]$. A *strict-square labelling* of G over $\mathbb{F}_p$ is a map $z\colon W \to \mathbb{F}_p$ with $z_v - z_u \in Q_p^*$ for every edge $(u, v) \in E$; it is *injective* if $z_u \neq z_w$ for $u \neq w$. Let $\mathrm{Inj}_p(G)$ be the set of injective strict-square labellings and, when it is nonempty, let $\lambda_p(G)$ (also written λ_G) be the uniform probability on it: the *graph law* of G at p. For a block $\mathcal{B}$ (a finite set of odd primes), $\lambda_{\mathcal{B}}(G) = \bigotimes_{p\in\mathcal{B}} \lambda_p(G)$ is a law on $(\prod_{p\in\mathcal{B}} \mathbb{F}_p)^W$; the labels at different primes are independent. A labelling of a set $A \subseteq W$ is *feasible* if it is the restriction of an element of $\mathrm{Inj}_p(G)$.

A vertex is labelled by a field element. The extension count below counts the possible labels of one new vertex after its neighbours have distinct fixed labels. Each neighbour imposes a condition on one quadratic character and so halves the count, up to a character sum of size $O(\sqrt{p})$: the main term is $p2^{-t}$ for t constraints. For integers $t, \Delta \geq 0$ and $n, z \geq 0$ put

$$h_t = \frac{t}{2} - 1 + 2^{-t}, \qquad e_t(p, n) = 2^t\big(h_t r + n r^2\big), \qquad w_\Delta(p, n) = 2^\Delta\Big(\frac{\Delta}{2} r + n r^2\Big), \tag{12.1}$$

the *coarse* count error and its uniform majorant, and the *graded* coefficients and count errors

$$\begin{gathered} a_t = (t-3)2^{t-1} + 2, \qquad b_t = 2^{t-1} - 1 \quad (t \geq 1), \qquad a_0 = b_0 = 0, \\ e_t^+ = a_t r + b_t r^2, \qquad e_t^-(z) = e_t^+ + z 2^t r^2. \end{gathered} \tag{12.2}$$

Thus $a_0 = a_1 = a_2 = 0$, $a_3 = 2$, $a_4 = 10$, $a_5 = 34$, and $b_2 = 1$, $b_4 = 7$. We use the following elementary properties.

- (E1) $a_{t+1} - a_t = (t-1)2^{t-1}$ and $b_{t+1} - b_t = 2^{t-1}$ for $t \geq 1$, $a_1 - a_0 = b_1 - b_0 = 0$, and $h_{t+1} - h_t = \frac{1}{2}(1 - 2^{-t})$; so a_t, b_t, h_t are nondecreasing in t, and $h_t \leq t/2$.
- (E2) $a_t + b_t = 2^t h_t$. Hence $e_t^+ \leq e_t^-(z) \leq e_t(p, z)$, with $e_t(p, z) - e_t^-(z) = b_t(r - r^2) \geq 0$, and $e_t(p, n) \leq w_\Delta(p, n)$ for $t \leq \Delta$.
- (E3) For fixed t, z, n, Δ, each of e_t^+, $e_t^-(z)$, $e_t(p, n)$, $w_\Delta(p, n)$ and its quotient by r is a polynomial in r with nonnegative coefficients; so all of them are nondecreasing in r, that is, nonincreasing in p. All are nondecreasing in t (w_Δ in Δ), z and n.
- (E4) $2^{-t}(1 - e_t^-(z))$ and $2^{-t}(1 - e_t(p, z))$ are nonincreasing in t and in z.

Indeed $a_t + b_t = (t-2)2^{t-1} + 1 = 2^t h_t$, which gives (E2) with (E1); (E3) is read off (12.1) and (12.2); and for (E4), $2^{-t}(1 - e_t^-(z)) = 2^{-t} - 2^{-t} a_t r - 2^{-t} b_t r^2 - z r^2$ and $2^{-t}(1 - e_t(p, z)) = 2^{-t} - h_t r - z r^2$, where $2^{-t} a_t$ and $2^{-t} b_t$ are nondecreasing (for $t \geq 1$ their increments are $\frac{1}{2} - 2^{-t}$ and 2^{-t-1}, and from $t = 0$ to $t = 1$ they are zero).

12.2. **Counting choices for one new vertex.** An edge between a new vertex, labelled x, and a vertex labelled β imposes $x-\beta \in Q_p^*$ (edge into the new vertex) or $\beta - x \in Q_p^*$ (edge out of it), that is, $\chi(x-\beta)=1$ or $\chi(x-\beta)=\chi(-1)$. The admissible labels of a vertex whose t neighbours carry distinct labels, and which must avoid a set of used labels, therefore form a set Φ as in the following lemma, with the neighbour labels as roots. More generally, for $\epsilon_i \in \{\pm 1\}$ the condition $\epsilon_i(x-\beta_i) \in Q_p^*$ is $\chi(x-\beta_i)=\chi(\epsilon_i)$, so counts of this form are the case $s_i=\chi(\epsilon_i)$. At $p \equiv 3 \pmod 4$ the sign records the orientation of the edge; at $p \equiv 1 \pmod 4$ every sign is $+1$.

Lemma 12.2 (Graded extension count)**.** *Let $t \ge 0$, let $\beta_1,\dots,\beta_t \in \mathbb{F}_p$ be distinct, let $s_1,\dots,s_t \in \{1,-1\}$, and let $\mathcal{X} \subseteq \mathbb{F}_p$ contain $\beta_1,\dots,\beta_t$. Put*

$$\Phi = \{x \in \mathbb{F}_p \setminus \mathcal{X} :\ \chi(x-\beta_i)=s_i\ (1 \le i \le t)\}, \qquad \mathfrak{S} = \sum_{x\in\mathbb{F}_p} \prod_{i=1}^{t} (1+s_i\chi(x-\beta_i)).$$

(a) $|\mathfrak{S}-p| \le a_t\sqrt{p}+b_t$.
(b) $|\Phi| \le p2^{-t}(1+e_t^+)$, *whatever the size of $\mathcal{X}$.*
(c) *If $|\mathcal{X}| \le z$, then $|\Phi| \ge p2^{-t}(1-e_t^-(z))$.*
(d) *If $|\mathcal{X}| \le n$, then $p2^{-t}-h_t\sqrt{p}-n \le |\Phi| \le p2^{-t}+h_t\sqrt{p}$; hence $|\Phi| \in p2^{-t}[1-e_t(p,n), 1+e_t(p,n)]$.*

Proof. For $y \neq 0$ the indicator of $\chi(y)=s$ is $\frac12(1+s\chi(y))$. Let $f(x)=2^{-t}\prod_i(1+s_i\chi(x-\beta_i))$, so that $\mathfrak{S}=2^t\sum_x f(x)$. Each factor $\frac12(1+s_i\chi(x-\beta_i))$ lies in $\{0,\frac12,1\}$, so $0 \le f \le 1$, and for $x \notin \{\beta_1,\dots,\beta_t\}$, $f(x)$ is the indicator of the t conditions. Since the roots lie in $\mathcal{X}$,

$$|\Phi| = \sum_{x\notin\mathcal{X}} f(x) = 2^{-t}\mathfrak{S} - \sum_{x\in\mathcal{X}} f(x), \qquad 0 \le \sum_{x\in\mathcal{X}} f(x) \le |\mathcal{X}|. \tag{12.3}$$

Expanding the product and using the multiplicativity of χ,

$$\mathfrak{S} = \sum_{S\subseteq\{1,\dots,t\}} s_S W_S, \qquad s_S=\prod_{i\in S} s_i, \qquad W_S = \sum_{x\in\mathbb{F}_p} \chi\Big(\prod_{i\in S}(x-\beta_i)\Big).$$

Here $W_\emptyset = p$; $W_S = \sum_x \chi(x-\beta) = 0$ for $|S|=1$; $W_S=-1$ for $|S|=2$ by (4.2); for odd $|S|=k\ge 3$, $|W_S| \le (k-1)\sqrt{p}$ by Theorem 4.7(a), the roots being distinct; and for even $|S|=k\ge 4$, $|W_S+1| \le (k-2)\sqrt{p}$ by Theorem 4.7(b), so $|W_S| \le (k-2)\sqrt{p}+1$. Summing,

$$|\mathfrak{S}-p| \le \sqrt{p}\Big[\sum_{k \text{ odd},\,k\ge3}\binom{t}{k}(k-1) + \sum_{k \text{ even},\,k\ge4}\binom{t}{k}(k-2)\Big] + \sum_{k \text{ even},\,k\ge2}\binom{t}{k}.$$

For $t \ge 1$ the last sum is $2^{t-1}-1=b_t$. The bracket equals $\sum_{k\ge3}\binom{t}{k}(k-1) - \sum_{k \text{ even},\,k\ge4}\binom{t}{k}$; since $\sum_{k\ge0}\binom{t}{k}(k-1) = (t-2)2^{t-1}$ and $\sum_{k \text{ even}}\binom{t}{k} = 2^{t-1}$, removing the terms $k\le 2$ gives $[(t-2)2^{t-1}+1-\binom{t}{2}] - [2^{t-1}-1-\binom{t}{2}] = a_t$. This proves (a); for $t=0$, $\mathfrak{S}=p$.

By (a), $2^{-t}\mathfrak{S} \in p2^{-t}[1-e_t^+, 1+e_t^+]$. The upper bound of (12.3) uses only $f\ge 0$, which gives (b); its lower bound gives $|\Phi| \ge p2^{-t}(1-e_t^+) - z = p2^{-t}(1-e_t^-(z))$, which is (c). Finally $2^{-t}(a_t\sqrt{p}+b_t) \le 2^{-t}(a_t+b_t)\sqrt{p} = h_t\sqrt{p}$ by (E2), and (d) follows from (b) and (c). □

Two features of the count recur below. First, the excluded labels are handled collectively, through (12.3): the exact value $W_S=-1$ of a pair sum is available only for the sum over all of $\mathbb{F}_p$, and the partial sums of $\chi(x-\beta_i)\chi(x-\beta_j)$ over $x \notin \mathcal{X}$ are not -1 when $\mathcal{X}$ contains more than the roots. Second, Weil's bound needs distinct roots. This is why every count in this paper is a count of *injective* extensions: the pinned labels are then distinct, and injectivity contributes only the excluded-label term $z2^tr^2$ of $e_t^-(z)$, which is of lower order. The coarse error $e_t(p,n)$ results from bounding $|W_S|$ for $|S|=2$ by $\sqrt{p}$ instead of using $W_S=-1$, and for even $|S|=k\ge4$ by

$(k-1)\sqrt{p}$; it is larger than the graded error by (E2), so every statement proved or certified with $e_t(p,n)$ remains valid, and the certificates use both.

Corollary 12.3 (Signed neighbourhoods)**.** *Let $x_1,\dots,x_4 \in \mathbb{F}_p$ be distinct, $s \in \{\pm1\}^4$, and put $\upsilon_+ = 11r+6q$ and $\upsilon_- = 11r+70q$. The soft count $\mathfrak{S} = \sum_{z\in\mathbb{F}_p}\prod_i(1+s_i\chi(z-x_i))$ satisfies $|\mathfrak{S}-p| \le 10\sqrt{p}+7 \le 11\sqrt{p}+6$, and the signed neighbourhood $\{z \notin \{x_1,\dots,x_4\} : \chi(z-x_i)=s_i\ \forall i\}$ has between $\frac{p}{16}(1-\upsilon_-)$ and $\frac{p}{16}(1+\upsilon_+)$ elements. If k further labels are excluded, at least $\frac{p}{16}(1-\upsilon_- - 16kq)$ of its elements remain.*

Proof. Take $t=4$ in Lemma 12.2, so $a_4=10$, $b_4=7$, and $\mathcal{X}=\{x_1,\dots,x_4\}$. Part (a) gives $|\mathfrak{S}-p| \le 10\sqrt{p}+7 \le 11\sqrt{p}+6$, part (b) gives at most $\frac{p}{16}(1+10r+7q) \le \frac{p}{16}(1+\upsilon_+)$ elements and part (c) with $z=4$ at least $\frac{p}{16}(1-10r-71q) \ge \frac{p}{16}(1-\upsilon_-)$, since $q \le r$. Removing k labels costs at most $k = \frac{p}{16}\cdot 16kq$. □

12.3. **Sequential counts and densities.** Labelling the vertices of a graph one at a time and applying Lemma 12.2 at each step bounds the number of completions of a partial labelling. The lemmas of this subsection hold with any count error that the extension count supplies; we isolate in Definition 12.4 the property of the error that the lemmas use.

Definition 12.4 (One-step error pairs)**.** Let $n \ge 1$. A *one-step error pair* at (p,n) consists of two sequences $(\epsilon_t^+)_{t\ge0}$, $(\epsilon_t^-)_{t\ge0}$ of nonnegative numbers, nondecreasing in t, such that $p2^{-t}(1-\epsilon_t^-) \le |\Phi| \le p2^{-t}(1+\epsilon_t^+)$ for every set Φ of Lemma 12.2 with t roots and an excluded set of at most n labels.

By Lemma 12.2 and (E1)–(E3), two one-step error pairs at (p,n) are the *graded pair*, $\epsilon_t^+ = e_t^+$ and $\epsilon_t^- = e_t^-(n)$, and the *coarse pair*, $\epsilon_t^\pm = e_t(p,n)$; so is every pair of larger nondecreasing sequences.

Lemma 12.5 (Sequential counts)**.** *Let $G=(W,E)$ be an oriented graph, $n \ge |W|$, $A \subseteq W$, and let $v_1,\dots,v_s$ be an ordering of $W\setminus A$. Let t_i be the number of G-neighbours of v_i in $A\cup\{v_1,\dots,v_{i-1}\}$, so that $\sum_i t_i = |E| - |E(G[A])|$, and put* $\mathrm{main} = p^s 2^{-\sum_i t_i}$. *Let (ϵ^+,ϵ^-) be a one-step error pair at (p,n) with $\epsilon^-_{t_i} < 1$ for all i, and define the* logarithmic count errors *of the ordering,*

$$\mathrm{up} = \sum_i \log(1+\epsilon^+_{t_i}), \qquad \mathrm{lo} = \sum_i -\log(1-\epsilon^-_{t_i}).$$

For $x \in \mathrm{Inj}_p(G[A])$ let $C(x)$ be the number of $y \in \mathrm{Inj}_p(G)$ with $y|_A = x$.

(a) $\mathrm{main}\cdot e^{-\mathrm{lo}} \le C(x) \le \mathrm{main}\cdot e^{\mathrm{up}}$ *for every $x \in \mathrm{Inj}_p(G[A])$; the upper bound holds without the hypothesis on ϵ^-. In particular every element of $\mathrm{Inj}_p(G[A])$ is feasible, and $\mathrm{Inj}_p(G)\neq\emptyset$ if $\mathrm{Inj}_p(G[A])\neq\emptyset$ (for instance if $A=\emptyset$).*

(b) *The marginal of $\lambda_p(G)$ on A is supported on $\mathrm{Inj}_p(G[A])$, and its density with respect to $\lambda_p(G[A])$ takes values in $[e^{-(\mathrm{up}+\mathrm{lo})}, e^{\mathrm{up}+\mathrm{lo}}]$.*

(c) Anchored extension probability. *Let $Z = W\setminus A$ and let H be the set of edges of G between A and Z. For every $x \in \mathrm{Inj}_p(G[A])$, the probability under $y \sim \lambda_p(G[Z])$ that $(x,y) \in \mathrm{Inj}_p(G)$ lies in $2^{-|H|}[e^{-(\mathrm{up}+\mathrm{lo})}, e^{\mathrm{up}+\mathrm{lo}}]$.*

(d) *For the coarse pair, $e^{\mathrm{up}+\mathrm{lo}} = R := \prod_i \frac{1+e_{t_i}(p,n)}{1-e_{t_i}(p,n)}$. If G has maximum degree Δ and $e_\Delta(p,n)<1$, the hypotheses hold for the coarse pair, for every A and every ordering.*

(e) Thresholds. *For the graded and the coarse pairs, the quantities $\epsilon^\pm_{t_i}$, $\mathfrak{u} = \sum_i \epsilon^+_{t_i}/r$ and $\mathfrak{l} = \mathrm{lo}/r$, and for the coarse pair also $\log R/r$, are nondecreasing functions of r on the range where every $\epsilon^-_{t_i}<1$. Hence, if $\epsilon^-_{t_i}<1$ at a real threshold $P>1$, then the same holds at every prime $p \ge P$, and there $\log(\max_x C(x)/\mathrm{main}) \le r_p\,\mathfrak{u}(P)$ and $\log(\mathrm{main}/\min_x C(x)) \le r_p\,\mathfrak{l}(P)$, where $r_p = p^{-1/2}$ and $\mathfrak{u}(P)$, $\mathfrak{l}(P)$ are evaluated at $r = P^{-1/2}$.*

Proof. Every edge not inside A is counted exactly once in $\sum_i t_i$, at its endpoint of largest index; this gives $\sum_i t_i = |E| - |E(G[A])|$.

(a) The elements of $\mathrm{Inj}_p(G)$ extending x are obtained by choosing $y(v_1), y(v_2), \dots$ in turn. When v_i is labelled, its t_i labelled neighbours carry pairwise distinct labels (injectivity), and, since G has at most one edge on each unordered pair, each of them imposes exactly one condition $\chi(y(v_i) - \beta) = s$, with β its label. The new label must avoid the $|A| + i - 1 \le n - 1$ labels used so far, which contain the roots. Conversely every sequence of choices satisfying these conditions gives an element of $\mathrm{Inj}_p(G)$ extending x: the edges inside A are satisfied by x, every other edge at the step of its later endpoint. So the number of choices at step i is the size of a set Φ of Lemma 12.2 with t_i roots and at most $n-1$ excluded labels, and it lies in $p2^{-t_i}[1 - \epsilon^-_{t_i}, 1 + \epsilon^+_{t_i}]$ whatever the earlier choices were. Multiplying over i gives (a).

(b) The marginal of $\lambda_p(G)$ at x is $C(x)/\sum_{x'} C(x')$ and $\lambda_p(G[A])(x) = 1/|\mathrm{Inj}_p(G[A])|$, so the density is $C(x)/\operatorname{mean}_{x'} C(x')$, which lies between $\min C/\max C$ and $\max C/\min C$, hence in the stated interval by (a); it is positive everywhere on $\mathrm{Inj}_p(G[A])$.

(c) The probability is $C(x)/|\mathrm{Inj}_p(G[Z])|$. By (a), $C(x) \in p^s 2^{-|H|-|E(G[Z])|}[e^{-\mathrm{lo}}, e^{\mathrm{up}}]$. By (a) for $G[Z]$ with no anchors and the same ordering, $|\mathrm{Inj}_p(G[Z])| \in p^s 2^{-|E(G[Z])|} \prod_i [1 - \epsilon^-_{t_i^0}, 1 + \epsilon^+_{t_i^0}]$, where $t_i^0 \le t_i$ is the number of earlier neighbours of v_i in Z; since $\epsilon^\pm$ are nondecreasing in t, this interval lies in $p^s 2^{-|E(G[Z])|}[e^{-\mathrm{lo}}, e^{\mathrm{up}}]$. Divide.

(d) The identity is the definition. If $e_\Delta(p,n) < 1$ then $e_{t_i}(p,n) \le e_\Delta(p,n) < 1$ by (E3), since $t_i \le \Delta$.

(e) By (E3), $\epsilon^\pm_t$ and $\epsilon^\pm_t/r$ are polynomials in r with nonnegative coefficients. The functions $x \mapsto -\log(1-x)/x = \sum_{k\ge1} x^{k-1}/k$ and $x \mapsto \operatorname{artanh}(x)/x = \sum_{k\ge0} x^{2k}/(2k+1)$ are increasing on $(0,1)$, so $-\log(1-\epsilon^-)/r = \big(-\log(1-\epsilon^-)/\epsilon^-\big)(\epsilon^-/r)$ and $\log\frac{1+e}{1-e}/r = 2\big(\operatorname{artanh}(e)/e\big)(e/r)$ are products of nonnegative nondecreasing functions of r (this is an instance of Lemma 4.18). Since $r_p \le P^{-1/2}$ for $p \ge P$, the hypothesis transfers from P to p, and by (a) and $\log(1+x) \le x$, $\log(\max C/\mathrm{main}) \le \sum_i \epsilon^+_{t_i}(p) = r_p \mathfrak{u}(p) \le r_p \mathfrak{u}(P)$; the other bound is the same argument. □

The next lemma collects the facts about one vertex, about subsets and about conditional laws of a graph law that the cut and spectral estimates use; its part (e) is the density bound behind the extraction of Theorem 16.9.

Lemma 12.6 (Vertices, subsets and conditional laws)**.** *Let $G = (W, E)$ be an oriented graph with $|W| \le n$, maximum degree Δ and $e_\Delta(p,n) < 1$. Let $\lambda = \lambda_p(G)$, $d_w = \deg_G(w)$, and for $Y \subseteq W$ put*

$$R_Y = \prod_{i \in Y} \frac{1 + e_{d_i}(p,n)}{1 - e_{d_i}(p,n)}, \qquad d_i = \deg_G(i).$$

(a) *λ is invariant under the maps $z \mapsto (\alpha z_w + \beta)_{w \in W}$ with $\alpha \in Q_p^*$, $\beta \in \mathbb{F}_p$, and its one-vertex marginals are uniform (Lemma 5.16).*

(b) *Let $w \in W$. Conditionally on $z_{W\setminus\{w\}}$, the label z_w is uniform on its admissible set $\Phi_w(z_{W\setminus\{w\}})$, which has at least $p2^{-d_w}(1 - e_{d_w}(p,n))$ elements; so its conditional density with respect to the uniform probability on $\mathbb{F}_p$ is at most $2^{d_w}/(1 - e_{d_w}(p,n))$. If m is a probability measure on $\mathrm{Inj}_p(G)$ whose density with respect to λ takes values in an interval $[c, Rc]$, $c > 0$, then under m the conditional probability of each value of z_w given $z_{W\setminus\{w\}}$ is at most $R/|\Phi_w|$.*

(c) *The marginal of λ on $W \setminus \{w\}$ is supported on $\mathrm{Inj}_p(G - w)$, and its density with respect to $\lambda_p(G - w)$ takes values in $[R^{-1}_{\{w\}}, R_{\{w\}}]$.*

(d) *Let $J \subseteq W$. The marginal λ_J of λ on J is supported on $\mathrm{Inj}_p(G[J])$, and its density with respect to $\lambda_p(G[J])$ takes values in an interval $[c', R_{W\setminus J}c']$, $c' > 0$ (in particular in*

$[R_{W\setminus J}^{-1}, R_{W\setminus J}]$*). For* m *as in* (b)*, the density of the marginal* m_J *with respect to* $\lambda_p(G[J])$ *takes values in an interval* $[c'', R\,R_{W\setminus J}c'']$*.*

(e) *Let* $W = V \sqcup C \sqcup U$ *with* $V \neq \emptyset$*, let* x_C *be a feasible labelling of* C*, order* V*, and let* t_i *be the number of neighbours of the* i*-th vertex of* V *in* C *and among the earlier vertices of* V*, so that* $t_V = \sum_i t_i$ *is the number of edges inside* V *or between* V *and* C*. The conditional law of* z_V *given* $z_C = x_C$ *is supported on injective tuples, and its density with respect to the uniform probability on* $\mathbb{F}_p^V$ *is at most*

$$\frac{2^{t_V} R_U}{\prod_i (1 - e_{t_i}(p,n))} \;\le\; \frac{2^{t_V} R_U}{(1 - e_\Delta(p,n))^{|V|}}.$$

Proof. (a) is Lemma 5.16, parts (1) and (2): the map $z \mapsto \alpha z + \beta$ is a bijection of $\mathrm{Inj}_p(G)$.

(b) For fixed $z_{W\setminus\{w\}} = y'$ the elements of $\mathrm{Inj}_p(G)$ extending y' correspond to the admissible values of z_w, all equally likely under λ. They form a set Φ of Lemma 12.2 whose roots are the d_w (distinct) labels of the neighbours of w and whose excluded set consists of the other $|W| - 1 \le n$ labels, so Lemma 12.2(d) gives the lower bound. Under m the conditional probability of $a \in \Phi_w$ is $(dm/d\lambda)(y', a) / \sum_{a' \in \Phi_w} (dm/d\lambda)(y', a') \le Rc/(|\Phi_w| c)$.

(d) The marginal of λ at $x \in \mathbb{F}_p^J$ is $C(x)/|\mathrm{Inj}_p(G)|$, $C(x)$ the number of completions. By Lemma 12.5(a),(d) for the anchors J and any ordering of $W \setminus J$, whose numbers t_i of earlier neighbours satisfy $t_i \le d_i$, C is positive exactly on $\mathrm{Inj}_p(G[J])$ and takes values in an interval whose endpoints have ratio at most $\prod(1 + e_{t_i})/(1 - e_{t_i}) \le R_{W\setminus J}$ (by (E3)). The density with respect to the uniform law on $\mathrm{Inj}_p(G[J])$ is proportional to C. For m, $m_J(x)$ is the sum of m over the completions of x, which lies between $cC(x)/|\mathrm{Inj}_p(G)|$ and $RcC(x)/|\mathrm{Inj}_p(G)|$.

(c) is (d) with $J = W \setminus \{w\}$.

(e) The conditional probability of x_V is $C(x_C, x_V) / \sum_{x'_V} C(x_C, x'_V)$, where $C(x_C, x_V)$ is the number of completions of (x_C, x_V) on U. By Lemma 12.5(a) with the anchors $C \cup V$, this count is positive exactly when $(x_C, x_V) \in \mathrm{Inj}_p(G[C \cup V])$, and it takes values in an interval with endpoint ratio at most R_U; so the admissible tuples are injective, and the conditional probability of each is at most R_U divided by their number. Since x_C is feasible, $x_C \in \mathrm{Inj}_p(G[C])$, and Lemma 12.5(a) for $G[C \cup V]$ with the anchors x_C and the given ordering of V shows that there are at least $p^{|V|} 2^{-t_V} \prod_i (1 - e_{t_i}(p,n))$ admissible tuples. Multiplying by $p^{|V|}$ gives the bound, and the last inequality is (E3) with $t_i \le \Delta$. □

For a block $\mathcal{B}$ all these statements hold at each prime of $\mathcal{B}$, and the corresponding statements for $\lambda_{\mathcal{B}}(G)$ follow because its marginals, conditional laws and densities are products over the primes of $\mathcal{B}$.

12.4. **Comparing two graph laws: a ratio of four counts.** In Part V (Lemmas 22.1 and 27.17) the conditional laws given a middle label (the rows of the middle and upper references) of two graph laws whose graphs differ only away from the conditioned and queried vertices are compared. Their density is then a ratio of four completion counts, whose main terms cancel.

Lemma 12.7 (Four-count comparison)**.** *Let* G *and* G' *be oriented graphs with vertex sets* W *and* W'*, and let* M, S *be disjoint subsets of* $W \cap W'$ *such that* $G[M \cup S] = G'[M \cup S]$ *(the same edges with the same orientations). Fix orderings and one-step error pairs for which Lemma 12.5 applies to* G *with the anchors* M *and with the anchors* $M \cup S$*, and to* G' *with the same anchors, and write* $\mathrm{up}_G(A), \mathrm{lo}_G(A)$*,* $\mathrm{up}_{G'}(A), \mathrm{lo}_{G'}(A)$ *for the logarithmic count errors with the anchors* A*; write* $C_G^A(x_A)$ *for the number of elements of* $\mathrm{Inj}_p(G)$ *extending* x_A*, so that* $C_G^{\emptyset} = |\mathrm{Inj}_p(G)|$*. Let* $\lambda = \lambda_p(G)$ *and* $\lambda' = \lambda_p(G')$*.*

(a) *For every* $x_M \in \mathrm{Inj}_p(G[M])$*, the conditional laws of* z_S *given* $z_M = x_M$ *under* λ *and under* λ' *are both supported on* $\Sigma(x_M) = \{x_S : (x_M, x_S) \in \mathrm{Inj}_p(G[M \cup S])\}$*, and the density of*

the first with respect to the second is

$$\varrho(x_M, x_S) = \frac{C_G^{M\cup S}(x_M, x_S)\, C_{G'}^{M}(x_M)}{C_G^{M}(x_M)\, C_{G'}^{M\cup S}(x_M, x_S)}.$$

(b) *The main terms cancel, and*

$$\log \varrho \le \mathrm{up}_G(M\cup S) + \mathrm{lo}_G(M) + \mathrm{lo}_{G'}(M\cup S) + \mathrm{up}_{G'}(M),$$
$$-\log \varrho \le \mathrm{lo}_G(M\cup S) + \mathrm{up}_G(M) + \mathrm{up}_{G'}(M\cup S) + \mathrm{lo}_{G'}(M).$$

With $M = \emptyset$ this compares the marginals of λ and λ' on S.

(c) *For a block $\mathcal{B}$, the density of the conditional law of z_S given z_M under $\lambda_{\mathcal{B}}(G)$ with respect to that under $\lambda_{\mathcal{B}}(G')$ is the product over $p \in \mathcal{B}$ of the densities of* (a)*, and its logarithm lies in the sum over $p \in \mathcal{B}$ of the intervals of* (b).

Proof. (a) Under λ the law of $z_{M\cup S}$ is $C_G^{M\cup S}/|\operatorname{Inj}_p(G)|$ and that of z_M is $C_G^M/|\operatorname{Inj}_p(G)|$, so the conditional law of z_S given $z_M = x_M$ is $C_G^{M\cup S}(x_M,\cdot)/C_G^M(x_M)$. By Lemma 12.5(a) it is positive exactly on $\Sigma(x_M)$. The same holds for G', with the same set $\Sigma(x_M)$ because $G'[M\cup S] = G[M\cup S]$, and the ratio is ϱ.

(b) For a graph H with vertex set W_H write $\mathrm{main}_H(A) = p^{|W_H|-|A|}2^{-(|E(H)|-|E(H[A])|)}$ for the main term of Lemma 12.5. In the quotient of the four main terms of ϱ the powers of p cancel, and the powers of 2 combine to $2^{|E(G[M\cup S])|-|E(G[M])|-|E(G'[M\cup S])|+|E(G'[M])|} = 1$, since G and G' have the same induced graphs on $M\cup S$ and on M. By Lemma 12.5(a) each count C_H^A lies in $\mathrm{main}_H(A)\,[e^{-\mathrm{lo}_H(A)}, e^{\mathrm{up}_H(A)}]$; inserting these four intervals gives the bounds.

(c) The labels at different primes are independent under both product laws, so the conditional laws and their density are products. □

In Part V the lemma is applied to the field graphs G_F and G_E of a fold record and of a matching record on three objects (the fold and matching patterns of Definition 17.1), whose induced graphs on any two adjacent objects coincide (Lemma 20.11(d) in the middle range, Lemma 27.4(d) in the upper range): on two adjacent objects the fold relation and the matching relation both relate every pair of distinct object values and neither has a forward pair. With M the middle object and S one of the outer objects, part (a) writes the density of the conditional outer marginals of the rows as a ratio of completion counts, and part (b) bounds its logarithm: in the middle range the counts are exact (Lemma 20.9(iii)), and in the upper range part (b) with the graded insertion counts gives the pointwise marginal ratio of the rows (F5). With $M = \emptyset$ and S the middle object, part (b) bounds the density of the middle marginals at first order. On a block whose primes are at least b and whose logarithmic length $\sum \log p$ is at most L (any such bound; $L = 570$ in the construction), the errors, which are nondecreasing in r by Lemma 12.5(e), add over at most $L/\log b$ primes by part (c).

12.5. **How uniform is one label, or a queried pair, given the others?** The mixing hypothesis (R4) of Hypothesis 6.2 needs, at large primes, that conditioning on all other slots leaves the maximal correlation of one slot, or of a queried pair of slots, with the rest small (Definition 6.13). For a graph law this follows from the symmetry of $\lambda_p(G)$ alone: the operator T^*T commutes with translations and with square dilations, so it is diagonal, or block diagonal, in the additive characters with equal traces along the dilation orbits, and its trace is an average of reciprocal counts.

Lemma 12.8 (Trace lemma for conditional mixing)**.** *Let $G = (W, E)$ be an oriented graph with $n = |W| \ge 3$ vertices and maximum degree Δ, let p be an odd prime with $\operatorname{Inj}_p(G) \ne \emptyset$, and let $\lambda = \lambda_p(G)$.*

(a) One slot. *Let $v \in W$, $d = \deg_G(v)$, and let $w < 1$ with $w \geq e_d(p, n-1)$ (for instance $w = w_\Delta(p,n)$ if this is below one). The conditional operator $T = T^{\lambda,v}$ of Definition 6.13 satisfies, for every $f\colon \mathbb{F}_p \to \mathbb{C}$ with $\mathbb{E}f = 0$,*

$$\|Tf\|^2 \leq \frac{2}{p-1}\Big(\frac{2^d}{1-w} - 1\Big)\|f\|^2.$$

(b) A pair, trace form. *Let $(u,v) \in E$ and $W' = W \setminus \{u,v\}$. The pair (z_u, z_v) has law $\mathrm{Unif}(\Omega_p)$. For a labelling z' of W' in the support of the marginal of λ, let $\Phi(z') \subseteq \Omega_p$ be the set of pairs (x,x') such that z' together with $z_u = x$, $z_v = x'$ lies in $\mathrm{Inj}_p(G)$; for $y \in Q_p^*$ let $c_y(z')$ be the number of x with $(x, x+y) \in \Phi(z')$, and put $\varphi_y(z') = \frac{p-1}{2}\, c_y(z')/|\Phi(z')| - 1$. The conditional operator $(T_{uv}f)(z') = \mathbb{E}_\lambda[f(z_u,z_v) \mid z_{W'} = z']$ satisfies, for every $f \in L^2(\mathrm{Unif}(\Omega_p))$ with mean zero,*

$$\|T_{uv}f\|^2 \leq \max\Big\{\mathbb{E}_\lambda\Big[\frac{2}{p-1}\sum_{y\in Q_p^*}\varphi_y^2\Big],\ p\,\mathbb{E}_\lambda\Big[\frac{1}{|\Phi|}\Big]\Big\}\|f\|^2.$$

(c) A pair, explicit form. *In* (b) *let $d'_u = \deg_G(u) - 1$, $d'_v = \deg_G(v) - 1$, $t = d'_u + d'_v$, $s = \max(d'_u, d'_v)$, $n_\times = d'_u d'_v$, and let $e < 1$ with $e \geq e_t(p, 2n-4)$ and $e \geq e_{s+1}(p,n)$. Put*

$$\eta = \frac{n_\times 2^{t-s+1}}{p-1}, \qquad \varphi_{\mathrm{gen}} = \frac{3e}{(1-e)^2} + (1+e)\eta, \qquad \varphi_{\mathrm{exc}} = \frac{2^{t-s}(1+e)}{(1-e)^2}.$$

Then for every mean-zero $f \in L^2(\mathrm{Unif}(\Omega_p))$,

$$\|T_{uv}f\|^2 \leq \max\Big\{\varphi_{\mathrm{gen}}^2 + \frac{2n_\times}{p-1}\varphi_{\mathrm{exc}}^2,\ \frac{2^{t+1}}{(p-1)(1-e)^2}\Big\}\|f\|^2.$$

(d) Seventy-two vertices of degree at most twenty. *If $n = 72$ and $\Delta \leq 20$, then for every prime $p \geq 2^{512}$, every vertex v and every edge (u,v), the squared norms in* (a) *and* (b) *on the mean-zero functions are at most $2^{22}/p$ and $2^{92}/p$ respectively.*

Proof. Let $\mathfrak{G}$ be the group of the maps $x \mapsto \alpha x + \beta$ ($\alpha \in Q_p^*$, $\beta \in \mathbb{F}_p$), acting on labellings coordinatewise. It preserves $\mathrm{Inj}_p(G)$ (it multiplies every difference by a nonzero square), hence λ and all its marginals (Lemma 5.16); its translations act transitively on $\mathbb{F}_p$, and $\mathfrak{G}$ acts transitively on Ω_p (Lemma 5.16, part (3)), so the one-slot marginals are uniform and, the edge (u,v) forcing $(z_u, z_v) \in \Omega_p$, the pair has law $\mathrm{Unif}(\Omega_p)$.

(a) For a labelling z' of $W \setminus \{v\}$ in the support of the marginal λ', let $\Phi_v(z')$ be the set of admissible labels of v; since λ is uniform, $Tf(z') = |\Phi_v(z')|^{-1}\sum_{x\in\Phi_v(z')} f(x)$. The labels of the d neighbours of v are distinct and lie among the $n-1$ excluded labels, so Lemma 12.2(d) gives $|\Phi_v(z')| \geq p2^{-d}(1 - e_d(p,n-1)) \geq p2^{-d}(1-w)$. For $g \in \mathfrak{G}$, $\Phi_v(gz') = g\Phi_v(z')$, hence $T(f\circ g) = (Tf)\circ g$; the maps $f \mapsto f\circ g$ are unitary on $L^2(\mathbb{F}_p)$ and on $L^2(\lambda')$, so $S = T^*T$ commutes with them. Commuting with the translations, S is diagonal in the characters $e_\xi(x) = e(\xi x/p)$, each of which spans the joint eigenspace of the translations for its own character: $Se_\xi = \kappa_\xi e_\xi$ with $\kappa_\xi = \|Te_\xi\|^2 \geq 0$. Since $e_\xi(\alpha x) = e_{\alpha\xi}(x)$, commuting with the dilations gives $\kappa_{\alpha\xi} = \kappa_\xi$ for $\alpha \in Q_p^*$: κ takes one value κ_+ on Q_p^* and one value κ_- on the nonsquares, and $\kappa_0 = \|T\mathbf{1}\|^2 = 1$. In the orthonormal basis $(\sqrt{p}\,\mathbf{1}_{\{x\}})_x$,

$$\operatorname{tr} S = \sum_x p\,\|T\mathbf{1}_{\{x\}}\|^2 = \sum_x p\,\mathbb{E}_{\lambda'}\Big[\frac{\mathbf{1}\{x\in\Phi_v\}}{|\Phi_v|^2}\Big] = p\,\mathbb{E}_{\lambda'}\Big[\frac{1}{|\Phi_v|}\Big] \leq \frac{2^d}{1-w},$$

and $\operatorname{tr} S = 1 + \frac{p-1}{2}(\kappa_+ + \kappa_-)$. Hence $\max(\kappa_+,\kappa_-) \leq \frac{2}{p-1}(\frac{2^d}{1-w} - 1)$, and for $f = \sum_{\xi\neq 0}\widehat{f}(\xi)e_\xi$, $\|Tf\|^2 = \sum_{\xi\neq0}\kappa_\xi|\widehat{f}(\xi)|^2 \leq \max(\kappa_+,\kappa_-)\|f\|^2$. Finally $e_d(p,n-1) \leq e_\Delta(p,n) \leq w_\Delta(p,n)$ by (E2), (E3).

(b) Use the coordinates (x,y), $x'=x+y$, which identify Ω_p with $\mathbb{F}_p\times Q_p^*$ and $\mathrm{Unif}(\Omega_p)$ with the uniform law there; write $\mathbb{E}_y$ for the average over $y\in Q_p^*$ and λ'' for the marginal of $z_{W'}$. Since λ is uniform, $T_{uv}f(z')=|\Phi(z')|^{-1}\sum_{\omega\in\Phi(z')}f(\omega)$, and as in (a), $\Phi(gz')=g\Phi(z')$ for $g\in\mathfrak{G}$ (acting diagonally on Ω_p), so $S=T_{uv}^*T_{uv}$ commutes with the unitaries $f\mapsto f\circ g$. The translation $x\mapsto x+\beta$ acts by $(x,y)\mapsto(x+\beta,y)$ and the dilation by $\alpha\in Q_p^*$ by $(x,y)\mapsto(\alpha x,\alpha y)$. For $k\in\mathbb{F}_p$ let $\mathcal{H}_k=\{e_k(x)\psi(y)\}$; these spaces are orthogonal, they are the joint eigenspaces of the translations (with distinct characters), and $L_0^2(\mathrm{Unif}(\Omega_p))=(\mathcal{H}_0\ominus\mathbb{C}\mathbf{1})\oplus\bigoplus_{k\neq0}\mathcal{H}_k$. So S preserves each $\mathcal{H}_k$, and also $\mathcal{H}_0\ominus\mathbb{C}\mathbf{1}$ (it fixes $\mathbf{1}$ and is self-adjoint), and $\|T_{uv}|_{L_0^2}\|^2$ is the largest of the norms of the restrictions S_k of S to $\mathcal{H}_k$ ($k\neq0$) and to $\mathcal{H}_0\ominus\mathbb{C}\mathbf{1}$.

Nonzero frequencies. The dilation by α maps $\mathcal{H}_k$ onto $\mathcal{H}_{\alpha k}$ and commutes with S, so $\operatorname{tr}S_k$ takes one value σ_+ for $k\in Q_p^*$ and one value σ_- for the nonsquares. Since $S_k\succeq0$, $\|S_k\|\le\operatorname{tr}S_k$, and $\frac{p-1}{2}(\sigma_++\sigma_-)\le\operatorname{tr}S$. In the orthonormal basis $(|\Omega_p|^{1/2}\mathbf{1}_{\{\omega\}})_\omega$, as in (a), $\operatorname{tr}S=|\Omega_p|\,\mathbb{E}_{\lambda''}[1/|\Phi|]$. With $|\Omega_p|=p(p-1)/2$ this gives $\|S_k\|\le\sigma_++\sigma_-\le p\,\mathbb{E}[1/|\Phi|]$.

Frequency zero. For $\psi\in\mathcal{H}_0$ with $\mathbb{E}_y\psi=0$,

$$T_{uv}\psi(z')=\sum_{y\in Q_p^*}\frac{c_y(z')}{|\Phi(z')|}\psi(y)=\mathbb{E}_y\big[(1+\varphi_y(z'))\psi(y)\big]=\mathbb{E}_y\big[\varphi_y(z')\psi(y)\big],$$

so by Cauchy–Schwarz $|T_{uv}\psi(z')|^2\le\mathbb{E}_y[\varphi_y(z')^2]\,\mathbb{E}_y|\psi|^2$, and $\|T_{uv}\psi\|^2\le\mathbb{E}_{\lambda''}\big[\frac{2}{p-1}\sum_y\varphi_y^2\big]\|\psi\|^2$.

(c) Fix z' in the support; let $\mathcal{L}=z'(W')$, a set of $n-2$ labels, and let B_u, B_v be the sets of labels of the neighbours of u, respectively v, in W'; $|B_u|=d'_u$, $|B_v|=d'_v$. A pair $(x,x+y)$ lies in $\Phi(z')$ iff $x,x+y\notin\mathcal{L}$ and x satisfies the d'_u conditions $\chi(x-\beta)=s_\beta$ ($\beta\in B_u$) from the edges at u and the d'_v conditions $\chi(x-(\beta'-y))=s'_{\beta'}$ ($\beta'\in B_v$) from the edges at v; the edge (u,v) is the condition $y\in Q_p^*$. So c_y is the size of a set Φ of Lemma 12.2 with excluded set $\mathcal{L}\cup(\mathcal{L}-y)$, of size at most $2n-4$, which contains all the roots. Call y *generic* if $B_u\cap(B_v-y)=\emptyset$; at most $n_\times$ values of y are not.

Counts. For generic y there are t distinct roots, and Lemma 12.2(d) gives $c_y\in p2^{-t}[1-e,1+e]$. For every y, keeping only the conditions of an endpoint with s conditions, Lemma 12.2(b) gives $c_y\le p2^{-s}(1+e_s^+)\le p2^{-s}(1+e)$, since $e_s^+\le e_s(p,n)\le e_{s+1}(p,n)\le e$. Labelling first the endpoint with s conditions and then the other endpoint, which has $t-s+1\le s+1$ conditions (its own and the edge (u,v)), Lemma 12.5(a) with the coarse pair gives $|\Phi(z')|\ge p^2 2^{-(t+1)}(1-e)^2$.

Bounds. Hence $p\,\mathbb{E}[1/|\Phi|]\le 2^{t+1}/(p(1-e)^2)$, which is at most the second term. Recall $|Q_p^*|=\frac{p-1}{2}$. For generic y, $|Q_p^*|c_y/|\Phi|\le|Q_p^*|p2^{-t}(1+e)/(p^22^{-(t+1)}(1-e)^2)\le(1+e)/(1-e)^2$, so $\varphi_y\le3e/(1-e)^2$; and $|\Phi|=\sum_y c_y\le|Q_p^*|p2^{-t}(1+e)+n_\times p2^{-s}(1+e)=|Q_p^*|p2^{-t}(1+e)(1+\eta)$, so $|Q_p^*|c_y/|\Phi|\ge(1-e)/((1+e)(1+\eta))$ and

$$-\varphi_y\le\frac{2e+(1+e)\eta}{(1+e)(1+\eta)}\le2e+(1+e)\eta.$$

Thus $|\varphi_y|\le\varphi_{\mathrm{gen}}$ for generic y. For the other y, $-1\le\varphi_y\le|Q_p^*|p2^{-s}(1+e)/(p^22^{-(t+1)}(1-e)^2)\le\varphi_{\mathrm{exc}}$, and $\varphi_{\mathrm{exc}}\ge1$, so $|\varphi_y|\le\varphi_{\mathrm{exc}}$. Therefore $\frac{2}{p-1}\sum_y\varphi_y^2\le\varphi_{\mathrm{gen}}^2+\frac{2n_\times}{p-1}\varphi_{\mathrm{exc}}^2$, and (c) follows from (b).

(d) In (a) take $w=w_{20}(p,72)=2^{20}(10r+72r^2)$, which is at least $e_d(p,71)$ for $d\le20$ by (E2), (E3); the bound is nondecreasing in d, so $d=20$ is the worst case. In (c) take $e=e_{38}(p,140)=2^{38}((18+2^{-38})r+140r^2)$, which is at least $e_t(p,140)$ and $e_{s+1}(p,72)$ since $t\le38$ and $s+1\le20$; the right side of (c) is nondecreasing in $n_\times\le361$, $t-s\le19$ and $t+1\le39$, so the worst case is $d'_u=d'_v=19$, with $\eta=361\cdot2^{20}/(p-1)$ and $\varphi_{\mathrm{exc}}=2^{19}(1+e)/(1-e)^2$. Multiplied

by p, the three resulting expressions are

$$\frac{2}{1-r^2}\Big(\frac{2^{20}}{1-w}-1\Big), \qquad \Big(\frac{\varphi_{\mathrm{gen}}}{r}\Big)^2+\frac{722\,\varphi_{\mathrm{exc}}^2}{1-r^2}, \qquad \frac{2^{39}}{(1-r^2)(1-e)^2},$$

and e/r, w/r and $\eta/r=361\cdot 2^{20}r/(1-r^2)$ are nondecreasing in r; so all three are nondecreasing in r (Lemma 4.18), and it suffices to bound them at $p=2^{512}$, which Certificate 12.9 does. □

Certificate 12.9 (Mixing constants of the two-layer root at $p=2^{512}$)**.** At $p=2^{512}$, the quantities $e=e_{38}(p,140)$, $\eta=361\cdot 2^{20}/(p-1)$, $\varphi_{\mathrm{gen}}=3e/(1-e)^2+(1+e)\eta$, $\varphi_{\mathrm{exc}}=2^{19}(1+e)/(1-e)^2$ and $w=2^{20}(10p^{-1/2}+72p^{-1})$ of the proof of Lemma 12.8(d) satisfy $e<\frac12$, $w<\frac12$ and the three inequalities below. They are evaluated in exact rational arithmetic by the function `leading_trace` of `research/push_2026_10_17/structural/layer72_analytic.py`:

$$p\Big(\varphi_{\mathrm{gen}}^2+\frac{722}{p-1}\varphi_{\mathrm{exc}}^2\Big)<2^{92}, \qquad \frac{p\,2^{39}}{(p-1)(1-e)^2}<2^{92}, \qquad \frac{2p}{p-1}\Big(\frac{2^{20}}{1-w}-1\Big)<2^{22}.$$

For orientation only (not part of the check), the three left sides are approximately $2^{87.51}$, 2^{39} and 2^{21}. How the check is run and authenticated is recorded in its row of Table 19.

12.6. **Primes $p\equiv 3 \pmod 4$: the Paley tournament, triples, and operators.** At a prime $p\equiv 3 \pmod 4$, -1 is a nonsquare, so for $x\neq y$ exactly one of $y-x$ and $x-y$ is a square: the strict-square relation is a tournament, the Paley tournament, and the character kernel $\chi(t-s)$ is antisymmetric. This is why the middle and upper primes are taken with $p\equiv 3 \pmod 4$: the laws of directed configurations there have exact counts and exact correlation identities, which the following lemma collects. (At $p\equiv 1 \pmod 4$ the relation is symmetric and the tournament statements of the lemma fail; its part (b) holds for every odd p, and the disjoint-triple operator at the end of the subsection uses no character sum and is treated at every prime $p\geq 7$.)

Lemma 12.10 (Paley-triple calculus)**.** *Let $p\equiv 3 \pmod 4$ be prime and write $a\to b$ if $\chi(b-a)=1$. Put $k=(p-3)/4$ and let P be the Paley operator of Lemma 4.11, $(Ph)(s)=\mathbb{E}_t\chi(t-s)h(t)$ on $L^2(\mathbb{F}_p)$.*

(a) Tournament counts. *$\to$ is a tournament on $\mathbb{F}_p$ in which every label has $(p-1)/2$ out-neighbours and $(p-1)/2$ in-neighbours. Two distinct labels $a\neq a'$ have exactly k common out-neighbours and exactly k common in-neighbours, and $\#\{c: a\to c\to a'\}=k+1-\mathbf{1}[a\to a']$.*

(b) Character correlations. *$\mathbb{E}_t\chi(t-a)\chi(t-a')=\mathbf{1}[a=a']-q$ for all $a,a'\in\mathbb{F}_p$ (for every odd p; this is (4.2)).*

(c) The Paley operator. *$P^*=-P$, $P\mathbf{1}=0$, $P^*P=PP^*=q(I-\mathbb{J})$ and $\|P\|=r$ (Lemma 4.11(c)). Hence P maps H_0 into itself, and on H_0 we have $P^2=-qI$, i.e. $pP^2=-I$, and P/r is unitary with eigenvalues $\pm i$.*

(d) Directed triples. *Let ν_F be the uniform law on $H_F=\{(x_0,x_1,x_2): x_0,x_1,x_2$ distinct, $x_0\to x_2\}$, and $V_i\colon L^2(\mathbb{F}_p)\to L^2(\nu_F)$, $(V_ih)(x)=h(x_i)$. The coordinates of ν_F are uniform, each V_i is an isometry, and V_i^* is the conditional expectation onto x_i. On H_0,*

$$V_0^*V_2=\frac{P-qI}{1-q}, \qquad V_0^*V_1=V_1^*V_2=-\frac{I+V_0^*V_2}{p-2}=-\frac{I}{p-1}-\frac{pP}{(p-1)(p-2)},$$

and so $V_2^*V_0=-(P+qI)/(1-q)$ and $V_1^*V_0=V_2^*V_1=-\frac{I}{p-1}+\frac{pP}{(p-1)(p-2)}$. The norms on H_0 are exactly

$$\|V_0^*V_2\|=\varrho:=\frac{r\sqrt{1+q}}{1-q}=\frac{\sqrt{p+1}}{p-1}, \qquad \|V_0^*V_1\|=\|V_1^*V_2\|=\varrho_1:=\frac{q}{1-q}\sqrt{1+\frac{q}{(1-2q)^2}},$$

and $\varrho_1 \le \varrho' := q(1+\varrho)/(1-2q)$. *For* $p \ge 7$, $\varrho' \le \varrho$, *and the Gram operator* $(V_i^* V_j)_{i,j}$ *of* (V_0, V_1, V_2) *on* H_0^3 *has norm at most* $\Gamma := 1 + \varrho + \varrho'$.

(e) Features. *Let* $A_1 = V_2 P$, $A_2 = V_1 P$ *and* $(A_3 h)(x) = -\mathbb{E}_s[\chi(s-x_2)\chi(s-x_1)h(s)] - q\,\mathbb{E}h$, *operators* $L^2(\mathbb{F}_p) \to L^2(\nu_F)$. *Each* A_j *annihilates the constants and maps into the* ν_F*-mean-zero functions, and on* H_0

$$A_1^*A_1 = A_2^*A_2 = qI, \qquad A_3^*A_3 = \frac{1-2q-q^2}{p-1} I \preceq qI, \qquad A_1^*A_2 = -\frac{q}{p-1} I + \frac{P}{(p-1)(p-2)},$$
$$A_1^*A_3 = -\frac{q}{(p-1)(p-2)} I + \frac{q}{p-1} P, \qquad A_2^*A_3 = \frac{q}{(p-1)(p-2)} I + \frac{q}{p-1} P,$$
$$V_1^*A_3 = -\frac{q}{p-1} I + \frac{P}{(p-1)(p-2)}, \qquad V_2^*A_3 = -\frac{q}{p-1} I - \frac{P}{(p-1)(p-2)},$$
$$V_0^*A_1 = -\frac{I+P}{p-1}, \qquad V_0^*A_2 = \frac{I}{(p-1)(p-2)} - \frac{P}{p-1}, \qquad V_0^*A_3 = \frac{2I}{p(p-1)(p-2)}.$$

Hence, for $\xi \in \mathbb{R}^3$ *and* $A_\xi = \xi_1 A_1 + \xi_2 A_2 + \xi_3 A_3$, $V_0^* A_\xi = a_\xi I + b_\xi P$ *on* H_0 *with*

$$a_\xi = -\frac{\xi_1 q}{1-q} + \frac{\xi_2 q^2}{(1-q)(1-2q)} + \frac{2\xi_3 q^3}{(1-q)(1-2q)}, \qquad b_\xi = -\frac{(\xi_1+\xi_2)q}{1-q},$$

and $\|(A_1, A_2, A_3)\| \le r\gamma$ *on* H_0^3, *with* $\gamma^2 = 1 + q(1+2r)/(1-2q)$.

(f) Density perturbation. *Let* $m = g\nu_F$ *be a probability measure on* H_F *with uniform coordinate marginals and* $|g-1| \le e$. *Then* $|\langle F, G\rangle_m - \langle F, G\rangle_{\nu_F}| \le e\|F\|_{\nu_F}\|G\|_{\nu_F}$ *and* $(1-e)\|F\|_{\nu_F}^2 \le \|F\|_m^2 \le (1+e)\|F\|_{\nu_F}^2$ *for all* F, G. *In particular, for* $f, h \in H_0$ *the correlations* $\langle V_i f, V_j h\rangle$ *under* m *and under* ν_F *differ by at most* $e\|f\|\|h\|$, *the norms* $\|V_i f\|$ *being equal under both laws; and for every operator* $A\colon L^2(\mathbb{F}_p) \to L^2(\nu_F)$, *the adjoints of* V_i *in* $L^2(m)$ *and in* $L^2(\nu_F)$ *satisfy* $\|V_{i,m}^* A - V_{i,\nu_F}^* A\| \le e\|A\|$.

Proof. (b) is (4.2).

(a) Since $\chi(-1) = -1$, $\chi(a-b) = -\chi(b-a)$, so exactly one of $a \to b$, $b \to a$ holds for $a \ne b$; out- and in-neighbours of a are the $(p-1)/2$ labels b with $\chi(b-a) = 1$, respectively -1. For $a \ne a'$,

$$4\#\{c : a \to c,\ a' \to c\} = \sum_{c \ne a, a'} (1+\chi(c-a))(1+\chi(c-a')) = (p-2) - \chi(a'-a) - \chi(a-a') - 1 = p-3,$$

using $\sum_{c \ne a} \chi(c-a) = 0$ and (4.2); the in-neighbours are the same computation with $1-\chi$, and give $p-3$ again. For the paths, $4\#\{c\} = \sum_{c \ne a, a'} (1+\chi(c-a))(1+\chi(a'-c)) = (p-2) - 2\chi(a'-a) + 1$, because both linear sums equal $-\chi(a'-a)$ and $\sum_c \chi(c-a)\chi(a'-c) = -\sum_c \chi((c-a)(c-a')) = 1$; this is $4k$ if $a \to a'$ and $4k+4$ otherwise.

(c) The first assertions are Lemma 4.11(c), since $\chi(-1) = -1$. As $P^*\mathbf{1} = -P\mathbf{1} = 0$, P maps H_0 into itself; there $P^*P = PP^* = qI$, so P/r is unitary, and $P^2 = -P^*P = -qI$, so its eigenvalues are $\pm i$.

(d) The translations preserve H_F and act transitively on each coordinate, so the coordinates are uniform, V_i is an isometry, and $\langle V_i h, F\rangle_{\nu_F} = \mathbb{E}_s \overline{h(s)}\, \mathbb{E}_{\nu_F}[F \mid x_i = s]$. Given $x_0 = s$, the label x_2 is uniform on the set $s + Q_p^*$ of the out-neighbours of s, and given (x_0, x_2) the label x_1 is uniform on the other $p-2$ labels. So $V_0^* V_2$ is the square shift T_p of Lemma 4.11, which is $(p\mathbb{J} - I + pP)/(p-1)$ by part (c) there, that is $(pP - I)/(p-1)$ on H_0; and for $g \in H_0$, $\mathbb{E}[g(x_1) \mid x_0, x_2] = -(g(x_0) + g(x_2))/(p-2)$, which gives $V_0^* V_1$; conditioning on x_2 instead gives $V_2^* V_1 = -(I + V_2^* V_0)/(p-2)$, whose adjoint is $V_1^* V_2$. The closed forms follow from $q/(1-q) = 1/(p-1)$, and the adjoints from $P^* = -P$. By (c), P is normal on H_0 with eigenvalues $\pm ir$, so $V_0^* V_2$ has eigenvalues $(-q \pm ir)/(1-q)$, of modulus ϱ, and $V_0^* V_1$ has eigenvalues $-\frac{1}{p-1} \mp \frac{ipr}{(p-1)(p-2)}$, of modulus $\frac{1}{p-1}(1 + \frac{p}{(p-2)^2})^{1/2} = \varrho_1$; and $\varrho_1 = \|I + V_0^* V_2\|/(p-2) \le (1+\varrho)/(p-2) = \varrho'$. The inequality $\varrho' \le \varrho$ is $q \le \varrho(1-3q)$,

i.e. $1 \le (p-3)\sqrt{p+1}/(p-1)$, which holds for $p \ge 7$. The Schur test applied to the symmetric matrix of the norms of the blocks of the Gram operator bounds its norm by the largest row sum, $1 + \max(\varrho + \varrho_1, 2\varrho_1) \le \Gamma$.

(e) By (c), $A_1\mathbf{1} = A_2\mathbf{1} = 0$, and $A_3\mathbf{1} = -(\mathbf{1}[x_1 = x_2] - q) - q = 0$ by (b). The functions A_1h, A_2h have mean $\mathbb{E}Ph = 0$, and $\mathbb{E}_{\nu_F}[\chi(s-x_2)\chi(s-x_1)]$ does not depend on s (translation invariance), hence equals its average over s, which is $-q$; so $\mathbb{E}_{\nu_F} A_3h = q\mathbb{E}h - q\mathbb{E}h = 0$.

Let $M_{1|2} = V_2^*V_1$, the conditional expectation of a function of x_1 given x_2. For distinct a, b the number of x_0 with $(x_2, x_1) = (a, b)$ is $(p - 2 - \chi(a-b))/2$; summing $g(b)$ against these weights gives, on all of $L^2(\mathbb{F}_p)$,

$$M_{1|2} = \frac{p\mathbb{J} - I}{p-1} + \frac{pP}{(p-1)(p-2)}, \qquad M_{1|2}^* = V_1^*V_2 = \frac{p\mathbb{J} - I}{p-1} - \frac{pP}{(p-1)(p-2)}.$$

On H_0, where $\mathbb{J} = 0$ and P maps H_0 into itself, $A_1^*A_1 = A_2^*A_2 = P^*P = qI$ and $A_1^*A_2 = P^*M_{1|2}P = -\frac{P^*P}{p-1} + \frac{pP^*PP}{(p-1)(p-2)} = -\frac{q}{p-1}I + \frac{P}{(p-1)(p-2)}$, since $pq = 1$. For A_3 write $\Psi_h(a,b) = \mathbb{E}_s\chi(s-a)\chi(s-b)h(s)$, so $A_3h(x) = -\Psi_h(x_2, x_1)$ on H_0, and $\varphi_s(b) = \chi(s-b)$. By (b), $(P\varphi_s)(a) = q - \mathbf{1}[a = s]$, and $\mathbb{J}\varphi_s = 0$, so

$$\mathbb{E}[\chi(s-x_1) \mid x_2 = a] = (M_{1|2}\varphi_s)(a) = -\frac{\chi(s-a)}{p-1} - \frac{p\mathbf{1}[a=s] - 1}{(p-1)(p-2)}.$$

Multiplying by $\chi(s-a)h(s)$ and averaging over s, with $\chi(s-a)^2 = \mathbf{1}[s \ne a]$ and $\chi(s-a)\mathbf{1}[a=s] = 0$, gives $\mathbb{E}[\Psi_h(a, x_1) \mid x_2 = a] = \frac{q}{p-1}h(a) + \frac{Ph(a)}{(p-1)(p-2)}$ for $h \in H_0$, which is $-V_2^*A_3h(a)$. Conditioning on x_1 with $M_{1|2}^*$ gives $V_1^*A_3$ in the same way. Then $A_1^*A_3 = P^*V_2^*A_3$ and $A_2^*A_3 = P^*V_1^*A_3$, and $P^* = -P$, $P^2 = -qI$ give the stated values. For $A_3^*A_3$: the law of (x_2, x_1) is $(p - 2 - \chi(a-b))/(p(p-1)(p-2))$ on distinct pairs, Ψ_h is symmetric and the antisymmetric part $\chi(a-b)$ integrates to zero against $\Psi_{h'}\Psi_h$; with $\mathbb{E}_{a,b}\Psi_{h'}\Psi_h = \mathbb{E}_{s,s'}h'(s)h(s')(\mathbf{1}[s=s'] - q)^2 = q(1-2q)\langle h', h\rangle$ and $\Psi_h(a,a) = -qh(a)$, one gets $\frac{p}{p-1}[q(1-2q) - q^3]\langle h', h\rangle = \frac{1-2q-q^2}{p-1}\langle h', h\rangle$, which is at most $q\langle h, h\rangle$ for $h' = h$. Next $V_0^*A_1 = V_0^*V_2P = (P^2 - qP)/(1-q) = -(I+P)/(p-1)$ and $V_0^*A_2 = V_0^*V_1P = -\frac{P}{p-1} - \frac{pP^2}{(p-1)(p-2)}$. Finally, given $x_0 = s$ and $x_2 = a$, $\mathbb{E}[\chi(t-x_1) \mid x_0, x_2] = -(\chi(t-s) + \chi(t-a))/(p-2)$; so

$$\begin{aligned}\mathbb{E}[\chi(t-x_2)\chi(t-x_1) \mid x_0 = s] &= -\frac{\chi(t-s)\mathbb{E}[\chi(t-x_2) \mid x_0 = s] + \mathbb{P}(x_2 \ne t \mid x_0 = s)}{p-2} \\ &= -\frac{1 - 2\cdot\mathbf{1}[t \ne s]/(p-1)}{p-2},\end{aligned}$$

by $\mathbb{E}[\chi(t-x_2) \mid x_0 = s] = (1 - \chi(t-s) - p\mathbf{1}[t=s])/(p-1)$ (the computation of $V_0^*V_2$ applied to φ_t) and $\mathbb{P}(x_2 = t \mid x_0 = s) = (\mathbf{1}[t \ne s] + \chi(t-s))/(p-1)$. Averaging against $h \in H_0$ gives $V_0^*A_3h = -\mathbb{E}[\Psi_h(x_2, x_1) \mid x_0] = \frac{2q}{(p-1)(p-2)}h$.

The formula for $V_0^*A_\xi$ is linearity with $1/(p-1) = q/(1-q)$ and $1/((p-1)(p-2)) = q^2/((1-q)(1-2q))$. For the norm, the Schur test applied to the matrix of the norms $\|A_i^*A_j\|$ (with $\|P\| = r$) bounds $\|(A_1, A_2, A_3)\|^2$ by its largest row sum. The first two rows sum to $q + \frac{q^2}{1-q} + \frac{rq^2}{(1-q)(1-2q)} + \frac{q^3}{(1-q)(1-2q)} + \frac{q^2r}{1-q} = q(1 + \frac{q(1+2r)}{1-2q}) = q\gamma^2$, and the third row is smaller by $\frac{2q^2(1-2q+qr-q^2)}{(1-q)(1-2q)} > 0$.

(f) $|\langle F, G\rangle_m - \langle F, G\rangle_{\nu_F}| = |\mathbb{E}_{\nu_F}[(g-1)\overline{F}G]| \le e\,\mathbb{E}_{\nu_F}|F||G|$, and Cauchy–Schwarz gives the first claim; the second is the case $F = G$. The coordinate marginals of m are uniform, so $\|V_if\|_m = \|f\| = \|V_if\|_{\nu_F}$ and the means of V_if, $f \in H_0$, vanish under both laws. For the adjoints, $\left\langle f, (V_{i,m}^* - V_{i,\nu_F}^*)Ah\right\rangle = \langle V_if, Ah\rangle_m - \langle V_if, Ah\rangle_{\nu_F}$, which is at most $e\|f\|\|A\|\|h\|$ in modulus. □

In the middle range these identities enter three times: the counts of (a) make the middle pair law exact (Lemma 20.9) and give the actual pair laws of the fold records (Lemma 23.4); the Grams of (d) give the channel norms of the reference fold record (Lemma 23.2) and, with (f), the block-extension estimates for the actual features (Lemma 25.1); and (e) is the Gram-matrix calculus behind the joint bound for the reference and the signed comparison fold records (Lemma 26.1), with A_3 the second-order feature of the comparison record.

The matching records of the middle range are uniform on pairs of disjoint injective triples, and at the reserved cut their Markov operator relative to the actual half law is the following operator (the identification is Lemma 22.7). Its negativity is exact and uses no character sum.

Lemma 12.11 (The disjoint-triple operator)**.** *Let $p \ge 7$, let $\mathcal{T}$ be the set of ordered triples of distinct elements of $\mathbb{F}_p$ with its uniform law, and let T be the Markov operator on $L^2(\mathcal{T})$ which, given a triple x, chooses a triple $y \in \mathcal{T}$ disjoint from x uniformly: $Tf(x) = \frac{1}{(p-3)(p-4)(p-5)} \sum_{y \cap x = \emptyset} f(y)$.*

(a) *T is self-adjoint, annihilates the functions whose average over the orderings of each unordered triple vanishes, and on functions of the unordered triple it has the eigenvalues*

$$\lambda_j = (-1)^j \binom{p-3-j}{3-j} \Big/ \binom{p-3}{3} \qquad (j = 0, 1, 2, 3), \tag{12.4}$$

that is 1, $-\frac{3}{p-3}$, $\frac{6}{(p-3)(p-4)}$ and $-\frac{6}{(p-3)(p-4)(p-5)}$, the eigenvalue 1 having the constants as its eigenspace.

(b) *Hence $-\frac{3}{p-3} I \preceq T \preceq I$, the lower constant is attained, and $\|T - \mathbb{J}\| = \frac{3}{p-3}$.*

(c) *For a block $\mathcal{B}$ of primes with least prime $b \ge 7$, the tensor product $T_{\mathcal{B}} = \bigotimes_{p \in \mathcal{B}} T$ on L^2 of the product of the uniform laws satisfies $-\frac{3}{b-3} I \preceq T_{\mathcal{B}} \preceq I$ and $\|T_{\mathcal{B}} - \mathbb{J}\| \le \frac{3}{b-3}$.*

Proof. (a) The kernel of T with respect to the uniform law is symmetric, so T is self-adjoint, and $T\mathbf{1} = \mathbf{1}$. For f on $\mathcal{T}$ let $\bar{f}$ be its average over the six orderings of each triple. Then $Tf(x) = \binom{p-3}{3}^{-1} \sum_Y \bar{f}(Y)$, the sum over the unordered triples Y disjoint from $\{x_0, x_1, x_2\}$; so $Tf = 0$ if $\bar{f} = 0$, and on the functions of the unordered triple T acts as the operator $\bar{T}$ on functions of 3-subsets of $\mathbb{F}_p$ given by the same formula.

For $0 \le j \le 3$ let $\mathcal{F}_j$ be the space of functions on j-subsets, with the counting inner product. Put $(U_j f)(B) = \sum_{x \in B} f(B \setminus \{x\})$ for $f \in \mathcal{F}_j$ and $|B| = j+1$, and $D_{j+1} = U_j^*$, so $(D_{j+1} g)(A) = \sum_{y \notin A} g(A \cup \{y\})$. Counting the terms that add and remove the same element gives $D_{j+1} U_j - U_{j-1} D_j = (p-2j) I$ on $\mathcal{F}_j$ ($U_{-1} D_0 = 0$). For $j \le 2 < p/2$ this gives $D_{j+1} U_j \succeq (p-2j) I$, so U_j is injective and $\mathcal{F}_{j+1} = U_j(\mathcal{F}_j) \oplus \ker D_{j+1}$ (the range of U_j is orthogonal to $\ker U_j^*$). Iterating, $\mathcal{F}_3$ is the direct sum of the four spaces of *lifts* $f_h(Y) = \sum_{A \subseteq Y, |A| = j} h(A)$ of *harmonic* functions $h \in \ker D_j$ ($j = 0, \dots, 3$, with $\ker D_0 = \mathcal{F}_0$ the constants); indeed the lift $U_2 \cdots U_j h$ is $(3-j)!\, f_h$.

If h is harmonic on j-subsets, the sum of $h(A)$ over the j-sets A containing a fixed set B with $|B| < j$ is zero: for $|B| = j-1$ it is $(D_j h)(B)$, and for smaller B it is $(j - |B|)^{-1}$ times the sum of these over the $(j-1)$-sets $B' \supseteq B$, since each $A \supseteq B$ contains $j - |B|$ of them. Hence inclusion–exclusion over the elements of a 3-set S gives

$$\sum_{|A| = j,\ A \cap S = \emptyset} h(A) = \sum_{B \subseteq S} (-1)^{|B|} \sum_{A \supseteq B} h(A) = (-1)^j \sum_{A \subseteq S,\ |A| = j} h(A).$$

Each j-set A disjoint from S lies in $\binom{p-3-j}{3-j}$ of the 3-sets disjoint from S, so $\sum_{Y \cap S = \emptyset} f_h(Y) = \binom{p-3-j}{3-j} (-1)^j f_h(S)$, that is $\bar{T} f_h = \lambda_j f_h$. The values (12.4) follow by evaluating the binomial coefficients; the four eigenvalues are distinct, so the four spaces are the eigenspaces of the self-adjoint operator $\bar{T}$, and the $j = 0$ space is the constants.

(b) The spectrum of T lies in $\{1, \lambda_1, \lambda_2, \lambda_3, 0\}$. For $p \ge 7$, $|\lambda_3| \le \lambda_2 \le \frac{3}{p-3} = |\lambda_1|$ (these are $1 \le p-5$ and $2 \le p-4$), and the $j = 1$ space is not zero.

(c) The eigenvalues of $T_{\mathcal{B}}$ are the products of eigenvalues of the factors, with tensor products of eigenvectors. Every factor has modulus at most one. A negative product has a negative factor, of modulus at most $\frac{3}{p-3} \le \frac{3}{b-3}$; and a product other than the one with all factors equal to 1 (whose eigenvector is the constant) has a factor of modulus at most $\frac{3}{p-3}$. □

The operator $\bar{T}$ is the normalized adjacency operator of the Kneser graph of 3-subsets of a p-set, two subsets being adjacent when they are disjoint, and (12.4) is its classical spectrum; the proof above is self-contained. In the middle range (Lemma 22.7) the bound is used as follows: for a block with least prime $b \ge 7$ (writing $r = b^{-1/2}$ in this paragraph), an unfolded empty-triple record of mass $w_E r$ has reserved negative part at least $-\frac{3}{b-3}$ times its own half diagonal, so it is compensated by a folded actual-half diagonal of mass $\frac{3}{b-3} w_E r = \frac{3w_E}{1-3/b} r^3$; in particular the doubled correction used to replace a removed folded empty-triple diagonal has mass at most $\frac{6w_E}{1-3/b} r^3$. The actual half law of the record is exactly the uniform law on injective triples, so no change of reference density is required.

Consequences and what to read next. Weil's bound in odd and even degree (Theorem 4.7) and the two-root identity (4.2); the square shift and the Paley operator (Lemma 4.11); square-affine invariance and its consequences for one-slot and pair marginals (Lemma 5.16); the conditional operator of Definition 6.13; and the admissible majorants (Lemma 4.18) for the transfer of thresholds.

13. Cuts of graph laws: the cut operator and correlations after pinning

What this section proves and how to read it. This section defines the cut operator A_p of a graph law (the operator representing dependence across a mirror symmetry), shows how bounds at one prime extend to blocks, and bounds correlations remaining after some labels are fixed (*pinned*). It then proves conversion lemmas between the true one-half distributions (*actual marginals*) and product reference measures. Here *block* in "block operator" means a matrix block, not a prime block; a *row* is a conditional law of outer labels given a middle label (defined at first use below). Inputs: Sections 5 and 12. Outputs: the setting for the spectral criteria of Section 14. A first reading should follow the definition of A_p and the statements of the pinning bounds; the reference-measure conversions can be consulted as needed; each Part V section cites the lemma it uses. Statements concern one prime or one block, with hypotheses in $r = p^{-1/2}$ verified at a threshold and extended by Lemma 4.18.

13.1. **A graph with an involution, and its cut operator.** Throughout this section p is an odd prime, and q, r, χ, Q_p^* and $A_p = \mathbf{1}_{Q_p^*}$ are as in Section 4.2. Oriented graphs, the sets $\mathrm{Inj}_p(H)$ and the graph laws $\lambda_p(H)$ are those of Definition 12.1, and $e_t(p, n')$ is the coarse count error of (12.1), nondecreasing in t, n' and r ((E3)). Inner products and norms $\langle F, H\rangle_\nu$, $\|F\|_\nu$ are those of $L^2(\nu)$ (the index is dropped when the law is clear), and maximal correlations are those of Definition 4.3.

Definition 13.1 (Spectral data)**.** *Spectral data* $(G, \vartheta, U^+, \mathfrak{o})$ consist of

(S0) an oriented graph $G = (U, E)$, an involutive automorphism ϑ of G that preserves orientations ($(u, w) \in E$ iff $(\vartheta u, \vartheta w) \in E$), and a set $U^+ \subseteq U$ with $\vartheta(U^+) = U \setminus U^+ =: U^-$;

and an ordering $\mathfrak{o} = (v_1, \dots, v_n)$ of U^+, called the *deletion order* because labels are chosen by successively deleting $v_1, v_2, \dots$. Thus (G, U^+, ϑ) is a *reflection cut*; its *cross edges* are the edges with one end in each half. We use the following notation.

- $n = |U^+|$, so $|U| = 2n$; $G^+ = G[U^+]$; for $j, k \in U^+$ we write $j \sim k$ if jk is an edge of G^+ (either orientation); d_k^0 is the degree of k in G^+ and $\Delta^+ = \max_k d_k^0$; for $L \subseteq U^+$, $e(L)$ is the number of edges of G^+ inside L.

- The *cross signs* ς_{jk} $(j,k \in U^+)$: $\varsigma_{jk} = +1$ if $(j, \vartheta k) \in E$, $\varsigma_{jk} = -1$ if $(\vartheta k, j) \in E$, and $\varsigma_{jk} = 0$ otherwise. $N(k) = \{j \in U^+ : \varsigma_{jk} \neq 0\}$, $r_k = |N(k)|$ (a degree, not $r = p^{-1/2}$), $d_k = d_k^0 + r_k$, $\Delta = \max_k d_k$; $E_\times = \{(j,k) \in U^+ \times U^+ : \varsigma_{jk} \neq 0\}$ is the set of *cross pairs* and $m_\times = |E_\times|$. In a pair (j,k), j is the *x-endpoint* and k the *x'-endpoint*.
- The *deletion degrees* t_i $(1 \le i \le n)$: t_i is the number of G-neighbours of v_i in $U \setminus \{v_1, \dots, v_{i-1}\}$ (the vertices of U^- are never deleted). The *row ratio*, an upper bound for the ratio between the largest and smallest numbers of extensions of a half labelling, is

$$R_{\mathrm{cut}}(p) = \prod_{i=1}^{n} \frac{1 + e_{t_i}(p, 2n)}{1 - e_{t_i}(p, 2n)}. \tag{13.1}$$

Let $p_0 > 0$ be real and $r_0 = p_0^{-1/2}$, with $e_\Delta(p_0, 2n)$ denoting the polynomial in r_0 of (12.1). The data satisfy the *threshold hypothesis at p_0* if

(S1) $e_\Delta(p_0, 2n) < 1$ and $p_0 > 4n^2$.

Since $e_\Delta(\cdot, 2n)$ is a polynomial in r with nonnegative coefficients, (S1) at p_0 implies (S1) at every prime $p \ge p_0$; this is the simplest instance of the monotone-majorant principle (Lemma 4.18) by which one exact check at a threshold covers all larger primes. The pinned correlations of the half graph G^+ are governed by the influence conditions (13.5) of Section 13.3, which are not part of the definition.

Lemma 13.2 (Elementary facts)**.** *Under* (S0)*: ς is well defined, $\varsigma_{kj} = -\varsigma_{jk}$ and $\varsigma_{kk} = 0$; hence $(j,k) \in E_\times$ iff $(k,j) \in E_\times$, and $\#\{k : \varsigma_{jk} \neq 0\} = r_j$. The map $(j,k) \mapsto$ (the edge of G between j and ϑk) is a bijection from $E_\times$ onto the set of cross edges. Every vertex $k \in U^+$ and its image ϑk have degree d_k in G, so Δ is the maximum degree of G. Finally ϑ maps G^+ isomorphically, with orientations, onto $G[U^-]$.*

Proof. ς_{jk} is well defined because G contains at most one of the two opposite directed edges on each unordered pair. Applying ϑ (an orientation-preserving involution) gives $(k, \vartheta j) \in E$ iff $(\vartheta k, j) \in E$, so $\varsigma_{kj} = +1$ iff $\varsigma_{jk} = -1$, and likewise with the signs exchanged. An edge $(k, \vartheta k)$ would give the opposite edge $(\vartheta k, k)$; hence $\varsigma_{kk} = 0$. Every cross edge joins some $j \in U^+$ to some ϑk, $k \in U^+$, in one orientation, which is the bijection. The degree of k is d_k^0 plus the number of k' with $\varsigma_{kk'} \neq 0$, which is r_k by antisymmetry; ϑ is an automorphism, so ϑk has the same degree. The last claim is the automorphism property restricted to U^+. □

Fix a prime p. Put $\mathcal{I}^+ = \mathrm{Inj}_p(G^+)$, $\mu = \mu_p = \lambda_p(G^+)$ (uniform on $\mathcal{I}^+$; the *half law*) and $\lambda = \lambda_p(G)$ (throughout this section μ, λ are these local graph laws, not the reference and first-stage laws of Part III). For $y \in \mathbb{F}_p^U$ write $x = y|_{U^+}$ $(x_k = y_k)$ and $x' = (y \circ \vartheta)|_{U^+}$ $(x'_k = y_{\vartheta k})$. For $(j,k) \in E_\times$ the cross constraint of the edge between j and ϑk reads $\varsigma_{jk}(x'_k - x_j) \in Q_p^*$. The identification of a labelling y of the whole graph with the pair (x, x') of labellings of G^+, and this form of the cross constraints, are what the whole spectral method rests on.

Lemma 13.3 (Cut kernel)**.** *Let spectral data satisfy $e_\Delta(p, 2n) < 1$.*

(a) *$\mathrm{Inj}_p(G)$ and $\mathcal{I}^+$ are nonempty, and $y \mapsto (x, x')$ is a bijection of $\mathrm{Inj}_p(G)$ onto $\{(x, x') \in \mathcal{I}^+ \times \mathcal{I}^+ : K(x, x') = 1\}$, where*

$$K(x, x') = \mathbf{1}\{x(U^+) \cap x'(U^+) = \emptyset\} \prod_{(j,k) \in E_\times} A_p(\varsigma_{jk}(x'_k - x_j)). \tag{13.2}$$

Moreover $K(x, x') = K(x', x)$.

(b) *Under λ the pair (x, x') has law $\mu \otimes \mu \cdot K/Z$ with $Z = Z_p = \mathbb{E}_{\mu\otimes\mu} K > 0$, and x, x' have the same law m_p, the* actual half marginal*. The* normalized cut operator A_p *(the*

operator A_p of the roadmaps) on $L^2(\mu)$, $(\mathsf{A}_pH)(x)=\mathbb{E}_{x'\sim\mu}[K(x,x')H(x')]/Z$, is self-adjoint, $\mathbb{E}_\lambda[F(x)H(x')]=\langle F,\mathsf{A}_pH\rangle_\mu$ for real F,H, and $\mathsf{A}_p\mathbf{1}=\varrho:=dm_p/d\mu$.

(c) *For $x\in\mathcal{I}^+$ let $N(x)=\#\{x'\in\mathcal{I}^+:K(x,x')=1\}$. Then $N_-\le N(x)\le N_+$ for every x, with $N_\pm=\prod_{i=1}^n p2^{-t_i}(1\pm e_{t_i}(p,2n))$ and $N_+/N_-=R_{\mathrm{cut}}(p)$. Hence $R_{\mathrm{cut}}(p)^{-1}\le\varrho\le R_{\mathrm{cut}}(p)$ on $\mathcal{I}^+$, and $2^{-m_\times}R_{\mathrm{cut}}(p)^{-1}\le Z\le 2^{-m_\times}R_{\mathrm{cut}}(p)$.*

(d) *λ, μ and m_p are invariant under the diagonal maps $z\mapsto(az_u+c)_u$ ($a\in Q_p^*$, $c\in\mathbb{F}_p$), every one-vertex marginal of λ, μ and m_p is uniform on $\mathbb{F}_p$, and λ is invariant under $(x,x')\mapsto(x',x)$.*

(e) *Let B be a finite set of odd primes with $e_\Delta(p,2n)<1$ for $p\in B$, $X=\prod_{p\in B}\mathbb{F}_p$, $\lambda_B(G)=\bigotimes_p\lambda_p(G)$, $\mu_B=\bigotimes_p\mu_p$ and $\mathsf{A}_B=\bigotimes_{p\in B}\mathsf{A}_p$ on $L^2(\mu_B)=\bigotimes_p L^2(\mu_p)$. Then for all real $F,H:X^{U^+}\to\mathbb{R}$, $\mathbb{E}_{y\sim\lambda_B(G)}[F(x)H(x')]=\langle F,\mathsf{A}_BH\rangle_{\mu_B}$.*

(f) (Positive reweightings.) *Let $w\ge0$ be a function on $\mathrm{Inj}_p(G)$ with $\mathbb{E}_\lambda w>0$, write $w(x,x')$ for its value at the labelling with halves (x,x') (and put $w(x,x')K(x,x')=0$ where $K(x,x')=0$), and let $\lambda^w=w\lambda/\mathbb{E}_\lambda w$. Under λ^w the pair (x,x') has law $\mu\otimes\mu\cdot wK/(Z\mathbb{E}_\lambda w)$. The operator A_p^w on $L^2(\mu)$ with kernel $wK/(Z\mathbb{E}_\lambda w)$ relative to $\mu\otimes\mu$ satisfies $\mathbb{E}_{\lambda^w}[F(x)H(x')]=\left\langle F,\mathsf{A}_p^wH\right\rangle_\mu$ for real F,H, and $\mathsf{A}_p^w\mathbf{1}=dm_p^w/d\mu$, where m_p^w is the law of x under λ^w; A_p^w is self-adjoint if $w(x,x')=w(x',x)$. For a block B as in (e) and $w_B=\prod_{p\in B}w_p$, the identity of (e) holds for $\lambda_B(G)^{w_B}=\bigotimes_p\lambda_p(G)^{w_p}$ with $\bigotimes_p\mathsf{A}_p^{w_p}$ in place of A_B.*

Proof. (a) G has $2n$ vertices and maximum degree Δ (Lemma 13.2), so $\mathrm{Inj}_p(G)\ne\emptyset$ by Lemma 12.5, and restriction gives $\mathcal{I}^+\ne\emptyset$. Since ϑ is a bijection, y is determined by (x,x'). The constraints defining $\mathrm{Inj}_p(G)$ are: the edges inside U^+ and injectivity on U^+, which say $x\in\mathcal{I}^+$; the edges inside U^- and injectivity on U^-, which say $x'\in\mathcal{I}^+$, because ϑ maps G^+ onto $G[U^-]$ with orientations; the cross edges, which are the factors of (13.2) by Lemma 13.2; and injectivity across the cut, the first factor. The map $y\mapsto y\circ\vartheta$ is a bijection of $\mathrm{Inj}_p(G)$ (an automorphism preserves injectivity and the constraints), and it maps (x,x') to (x',x); hence the image set is symmetric, and K, its indicator, is symmetric.

(b) λ is uniform, so the law of (x,x') is the uniform law on $\{K=1\}$, which is $\mu\otimes\mu\cdot K/Z$ with $Z=|\mathrm{Inj}_p(G)|/|\mathcal{I}^+|^2>0$. The swap symmetry of (a) shows that x and x' have the same law. A_p has a real symmetric kernel, so it is self-adjoint, and $\langle F,\mathsf{A}_pH\rangle_\mu$ equals $\mathbb{E}_{\mu\otimes\mu}[F(x)K(x,x')H(x')]/Z=\mathbb{E}_\lambda[F(x)H(x')]$. Finally $m_p(x)=N(x)/|\mathrm{Inj}_p(G)|$ and $\mu(x)=1/|\mathcal{I}^+|$, so $(\mathsf{A}_p\mathbf{1})(x)=N(x)/(Z|\mathcal{I}^+|)=m_p(x)/\mu(x)$.

(c) By the automorphism ϑ, $N(x)$ is the number of labellings of U^+ that extend the labelling $x\circ\vartheta$ of U^- to an element of $\mathrm{Inj}_p(G)$. Label U^+ in the order $v_n,v_{n-1},\dots,v_1$. When v_i is labelled, its labelled neighbours are its t_i neighbours in $U\setminus\{v_1,\dots,v_i\}$, their labels are distinct, and fewer than $2n$ labels are in use. By Lemma 12.2 (with the used labels as excluded set) the number of choices lies in $p2^{-t_i}[1-e_{t_i}(p,2n),1+e_{t_i}(p,2n)]$, whatever the earlier choices; here $e_{t_i}(p,2n)\le e_\Delta(p,2n)<1$. Multiplying gives $N_-\le N(x)\le N_+$, and $N_+/N_-=R_{\mathrm{cut}}(p)$ by (13.1). By (b), $\varrho=N|\mathcal{I}^+|/|\mathrm{Inj}_p(G)|=N/\mathbb{E}_\mu N$, so it lies in $[N_-/N_+,N_+/N_-]$. For Z, label U^+ alone in the same order: v_i then has $t_i^0\le t_i$ labelled neighbours (those in G^+) and fewer than n labels are in use, so $|\mathcal{I}^+|\in\prod_i p2^{-t_i^0}[1-e_{t_i^0}(p,n),1+e_{t_i^0}(p,n)]$ with $e_{t_i^0}(p,n)\le e_{t_i}(p,2n)$. Since $\sum_i t_i=e(U^+)+m_\times$ (each edge inside U^+ is counted at its earlier-deleted endpoint, each cross edge at its end in U^+, and there are $m_\times$ cross edges by Lemma 13.2) and $\sum_i t_i^0=e(U^+)$, comparing the two products factor by factor shows that the quotient $Z=|\mathrm{Inj}_p(G)|/|\mathcal{I}^+|^2=\mathbb{E}_\mu N/|\mathcal{I}^+|$ lies in $2^{-m_\times}[R_{\mathrm{cut}}^{-1},R_{\mathrm{cut}}]$.

(d) λ and μ are uniform laws on injective strict-square labellings, hence square-affine invariant, and m_p is the image of λ under the coordinate map $y \mapsto x$; so all three are invariant and have uniform one-vertex marginals by Lemma 5.16(1),(2). The swap invariance is (a).

(e) All laws and kernels are products over $p \in B$, and the identity follows from (b) for product functions and then for all F, H by bilinearity, since product functions span $\bigotimes_p L^2(\mu_p)$.

(f) Through the bijection of (a), λ^w has the density $w/\mathbb{E}_\lambda w$ relative to λ, so by (b) the pair (x, x') has the law $\mu \otimes \mu \cdot wK/(Z\mathbb{E}_\lambda w)$ under λ^w. Hence

$$\left\langle F, \mathsf{A}_p^w H \right\rangle_\mu = \frac{\mathbb{E}_{\mu\otimes\mu}[F(x)\,(wK)(x,x')\,H(x')]}{Z\mathbb{E}_\lambda w} = \mathbb{E}_{\lambda^w}[F(x)H(x')],$$

and $(\mathsf{A}_p^w \mathbf{1})(x)$ is the density of the law of x under λ^w relative to μ. The kernel is real, and it is symmetric when w is, because K is symmetric. For a block, $\mathbb{E}_{\lambda_B(G)} w_B = \prod_p \mathbb{E}_{\lambda_p(G)} w_p$, so $\lambda_B(G)^{w_B}$ is the product of the laws $\lambda_p(G)^{w_p}$, and the identity follows from the prime identity for product functions and bilinearity, as in (e). □

Thus the full labelling law is the product of two copies of the half law reweighted by the cross kernel; every cut-operator estimate starts from this identity, and parts (a)–(c) use the reflection only to identify the two halves. Part (f) is the form in which the positive roots of the late middle pieces are measured (Section 21): the reference stays the half law μ, and the reweighting enters only the kernel and the constant image $\mathsf{A}_p^w \mathbf{1}$. The factor $2^{-m_\times}$ in (c) is the expected mass of the cross constraints; normalizing by the exact mass Z instead is what makes the later estimates relative. The norm matters too: A_p acts on L^2 of the half law μ, not of the actual half marginal m_p, and by (c) the two norms differ by a factor in $[R_{\mathrm{cut}}^{-1/2}, R_{\mathrm{cut}}^{1/2}]$. The comparison records of Part V are measured in the actual marginals (Sections 13.4 and 13.5).

We write $Q(F) = \langle F, \mathsf{A}_p F\rangle_\mu = \mathbb{E}_\lambda[F(x)F(x')]$ for real F, and $\mathcal{B}(F,H) = \mathbb{E}_\lambda[F(x)H(x')]$ for the symmetric bilinear form (symmetric by the swap invariance). For a set $S \subseteq U^+$ and a value s with $\mu(x_S = s) > 0$ (a *feasible pinning*) the conditional law is $\mu(\cdot \mid x_S = s)$.

Lemma 13.4 (Transport along isomorphisms)**.** *Let (G, ϑ, U^+) and (G', ϑ', U'^+) satisfy* (S0)*, and let $\phi\colon U \to U'$ be an isomorphism of oriented graphs with $\phi(U^+) = U'^+$ and $\phi \circ \vartheta = \vartheta' \circ \phi$. Then $y \mapsto y \circ \phi^{-1}$ is a bijection $\mathrm{Inj}_p(G) \to \mathrm{Inj}_p(G')$ which maps the halves (x, x') of y to the halves $(x \circ \phi^{-1}, x' \circ \phi^{-1})$ of $y \circ \phi^{-1}$. Consequently $\Phi F(\xi) = F(\xi \circ \phi)$ defines a unitary operator $L^2(\mu_p) \to L^2(\mu'_p)$ with $\Phi \mathsf{A}_p \Phi^* = \mathsf{A}'_p$ and $\Phi\varrho = \varrho'$; Φ maps the functions of the labels of $A \subseteq U^+$ onto the functions of the labels of $\phi(A)$, so the compressions of A_p and A'_p to corresponding subspaces are unitarily equivalent; the same holds over a block of primes with $\bigotimes_p \Phi$; and if ϕ maps $\mathfrak{o}$ to a deletion order $\mathfrak{o}'$, the row ratios agree. In particular, if ϕ is an automorphism of G, the cut operators of $\lambda_p(G)$ at the cuts (U^+, ϑ) and $(\phi(U^+), \phi\vartheta\phi^{-1})$ are unitarily equivalent.*

Proof. ϕ maps edges to edges with orientations, and $(y\circ\phi^{-1})(\phi w) - (y\circ\phi^{-1})(\phi u) = y(w) - y(u)$, so $y \circ \phi^{-1} \in \mathrm{Inj}_p(G')$ iff $y \in \mathrm{Inj}_p(G)$. The new halves are $(y\circ\phi^{-1})|_{U'^+} = x \circ \phi^{-1}$ and $(y \circ \phi^{-1} \circ \vartheta')|_{U'^+} = (y \circ \vartheta \circ \phi^{-1})|_{U'^+} = x' \circ \phi^{-1}$, using $\phi^{-1} \circ \vartheta' = \vartheta \circ \phi^{-1}$. Since ϕ maps G^+ onto G'^+, the half law μ'_p is the image of μ_p under $x \mapsto x \circ \phi^{-1}$, so Φ is unitary, and

$$\left\langle \Phi F, \mathsf{A}'_p \Phi H \right\rangle_{\mu'_p} = \mathbb{E}_{\lambda_p(G')}\big[(\Phi F)(x \circ \phi^{-1})(\Phi H)(x' \circ \phi^{-1})\big] = \mathbb{E}_{\lambda_p(G)}[F(x)H(x')] = \langle F, \mathsf{A}_p H\rangle_{\mu_p}$$

by Lemma 13.3(b). The actual half marginals correspond in the same way, which gives $\Phi\varrho = \varrho'$. The statements on functions of a vertex set and on blocks are immediate, and the deletion degrees are preserved because ϕ preserves adjacency. □

13.2. **From mean-zero bounds to operator bounds.** Several facts turn a spectral bound for one normalized operator at one prime into the bounds the construction uses. An upper

bound on the mean-zero compression, together with a small constant image, gives a positive envelope $1 + O(1/p)$ (Proposition 13.6); a lower bound on the compression gives the matching lower envelope; and for an operator commuting with translations the centred bound may be split between the translation-invariant functions and their complement. The envelopes at the primes of a block tensorize, the upper envelopes contributing the factor $\prod_p (1 + c^+/p) \le \exp(c^+ \sum_p 1/p)$ (Lemma 13.7). The operators A_p act on the reference half law and are not Markov: their upper spectrum exceeds 1 by about c^+/p, and that excess is the source of the inflation factor. Markov operators of couplings, taken in their actual marginals, pay only the maximum of their factors under tensor products (Lemma 6.10(c)); this is one reason why the comparison records of Part V keep their actual marginals. Two more tools bound syntheses of many feature maps: weighted frames (finite families of feature maps with a positive weight vector, bounded through the largest eigenvalue of a symmetric majorant matrix; Lemma 13.8), and swapped rectangular kernels, whose nonzero eigenvalues are, up to sign, products of two singular values of the rectangle (Lemma 13.9). The projections $\mathbb{J}$ and $P_0 = I - \mathbb{J}$ are those of Section 4.1.

The following identity is the common reason that eliminating one diagonal block loses at most the square of the coupling norm divided by the lower bound of that block. It will be used both for the constant direction of a normalized kernel and for smaller compressions in Part V.

Lemma 13.5 (Block completion)**.** *Let $T = \left(\begin{smallmatrix} A & B^* \\ B & D \end{smallmatrix}\right)$ be self-adjoint on a finite-dimensional orthogonal sum $\mathcal{H}_1 \oplus \mathcal{H}_2$, and put $b = \|B\|$.*

(a) *If $A \succeq sI$ with $s > 0$, then*

$$\langle (x,y), T(x,y)\rangle = \left\| A^{1/2}x + A^{-1/2}B^*y \right\|^2 + \left\langle y, (D - BA^{-1}B^*)y \right\rangle .$$

Consequently $T \succeq 0$ if and only if $D - BA^{-1}B^* \succeq 0$, and $D - BA^{-1}B^* \succeq D - (b^2/s)I$.

(b) *If $A \succeq aI$, $D \succeq dI$ and $a > d$, then*

$$T \succeq \left(d - \frac{b^2}{a-d} \right) I.$$

Dually, if $A \preceq aI$, $D \preceq dI$ and $a > d$, then $T \preceq \left(a + \frac{b^2}{a-d}\right) I$.

Proof. Expanding the square gives (a); choosing $x = -A^{-1}B^*y$ proves the necessity of its positivity criterion, and $A^{-1} \preceq s^{-1}I$ gives the residual bound. For (b), put $\lambda = d - b^2/(a-d)$. The first diagonal block of $T - \lambda I$ is at least $(a-d)I$, and the second is at least $b^2/(a-d)I$, so (a) gives $T - \lambda I \succeq 0$. Apply this lower bound to $-T$ with the two summands interchanged to obtain the upper bound. □

Proposition 13.6 (Positive envelope from the mean-zero compression)**.** *Let (Ω, ν) be a finite probability space with full support and A a self-adjoint operator on $L^2(\nu)$ with $\langle \mathbf{1}, A\mathbf{1}\rangle = 1$.*

(a) *If $\|P_0 A\mathbf{1}\| \le e$ and $P_0 A P_0 \preceq \beta P_0$ with $\beta < 1$, then $A \preceq (1 + \frac{e^2}{1-\beta})I$.*

(a′) (Lower envelope.) *If $\|P_0 A\mathbf{1}\| \le e$ and $P_0 A P_0 \succeq -gP_0$ with $g \ge 0$, then $A \succeq -(g + \frac{e^2}{1+g})I$.*

(a″) (Translation sectors.) *Let a finite group act on Ω preserving ν, let A commute with the induced unitary operators $f \mapsto f \circ \gamma^{-1}$, and let $\mathcal{I} \ni \mathbf{1}$ be the space of invariant functions. Then A preserves $\mathcal{I}$ and $\mathcal{I}^\perp$, and $u = P_0 A\mathbf{1} \in \mathcal{I}$. If $\|A|_{\mathcal{I}^\perp}\| \le C$, $\|P_0 A P_0|_{\mathcal{I}}\| \le H < 1$ and $\|u\|^2 \le \vartheta_u$, then*

$$-\max(C, H + \vartheta_u)\, I \preceq A \preceq \max\Big(C,\, 1 + \frac{\vartheta_u}{1-H}\Big) I.$$

(b) (Normalized kernels.) *Let $K \ge 0$ be a symmetric kernel on $\Omega \times \Omega$ with operator $T_K f(x) = \mathbb{E}_{y\sim\nu} K(x,y) f(y)$, and suppose that $\|T_K - k\mathbb{J}\| \le k\varepsilon$ for some $k > 0$ and $0 \le \varepsilon < 1$.*

Put $z = \langle \mathbf{1}, T_K \mathbf{1}\rangle / k$, $A = T_K/(kz)$ *and* $\hat{\varepsilon} = \varepsilon/(1-\varepsilon)$. *Then* $|z-1| \le \varepsilon$, $\langle \mathbf{1}, A\mathbf{1}\rangle = 1$, $\|P_0 A \mathbf{1}\| \le \hat{\varepsilon}$, $\|P_0 A P_0\| \le \hat{\varepsilon}$ *and* $A \succeq -\hat{\varepsilon} I$; *if* $\hat{\varepsilon} < 1$, *then* $A \preceq (1 + \hat{\varepsilon}^2/(1-\hat{\varepsilon}))I$.

(c) (All primes above a threshold.) *Suppose that for every prime* $p \ge p_0$ *an operator* $A = A_p$ *as above satisfies (a) with* $e = e(r)$ *and* $\beta = gr$, *where* $e(r)/r$ *is nondecreasing in* $r = p^{-1/2}$ *and* $gr_0 < 1$, $r_0 = p_0^{-1/2}$. *Then* $A_p \preceq (1 + B^+/p)I$ *for every prime* $p \ge p_0$, *with* $B^+ = p_0\, e(r_0)^2/(1 - gr_0)$.

Proof. (a), (a′) Relative to $\langle \mathbf{1}\rangle \oplus \mathbf{1}^\perp$, the operator has diagonal blocks 1 and $P_0 A P_0$, and coupling vector $P_0 A \mathbf{1}$ of norm at most e. Apply the upper and lower bounds of Lemma 13.5(b), respectively, with diagonal bounds $(1, \beta)$ and $(1, -g)$.

(a″) For a group element γ the operator $U_\gamma f = f \circ \gamma^{-1}$ is unitary on $L^2(\nu)$ and fixes $\mathbf{1}$, so it commutes with $\mathbb{J}$ and P_0. If $f \in \mathcal{I}$, then $U_\gamma A f = A U_\gamma f = A f$ for every γ; so A preserves $\mathcal{I}$, and, being self-adjoint, also $\mathcal{I}^\perp$. Hence $A\mathbf{1} \in \mathcal{I}$, $u = P_0 A \mathbf{1}$ lies in $\mathcal{I}_0 = \mathcal{I} \cap \mathbf{1}^\perp$, and $D_{\mathcal{I}} = P_0 A P_0|_{\mathcal{I}_0}$ is a self-adjoint operator on $\mathcal{I}_0$ with $\|D_{\mathcal{I}}\| \le H$. As $L^2(\nu) = \mathcal{I} \oplus \mathcal{I}^\perp$ and A preserves both summands, it suffices to bound A on each; on $\mathcal{I}^\perp$, $-CI \preceq A \preceq CI$. On $\mathcal{I}$, parts (a) and (a′), with $e^2 = \vartheta_u$ and $\beta = g = H$, give the upper bound $1 + \vartheta_u/(1-H)$ and the lower bound $-H - \vartheta_u/(1+H) \ge -H - \vartheta_u$. Combining these with the bounds on $\mathcal{I}^\perp$ proves the claim.

(b) Write $T_K = k\mathbb{J} + D$ with $\|D\| \le k\varepsilon$. Then $kz = \langle \mathbf{1}, T_K \mathbf{1}\rangle = k + \langle \mathbf{1}, D\mathbf{1}\rangle$, so $|z-1| \le \varepsilon$, and clearly $\langle \mathbf{1}, A\mathbf{1}\rangle = 1$. Since $P_0 \mathbb{J} = \mathbb{J} P_0 = 0$, $P_0 A \mathbf{1} = P_0 D \mathbf{1}/(kz)$ and $P_0 A P_0 = P_0 D P_0/(kz)$ have norms at most $\varepsilon/z \le \hat{\varepsilon}$. Next $A = \mathbb{J}/z + D/(kz) \succeq D/(kz) \succeq -(\varepsilon/z) I \succeq -\hat{\varepsilon} I$. Finally $P_0 A P_0 \preceq \hat{\varepsilon} P_0$, and (a) with $e = \beta = \hat{\varepsilon}$ gives the upper bound.

(c) For $p \ge p_0$, $r \le r_0$, and $e(r)^2/(1-gr) = r^2 (e(r)/r)^2/(1-gr) \le r^2 (e(r_0)/r_0)^2/(1-gr_0) = B^+/p$. Insert this in (a). □

Part (a) is the positive envelope of both spectral criteria: with $e = \varepsilon_\varrho$ (the bound on $\|P_0 \mathsf{A}_p \mathbf{1}\|$ of (c)) and $\beta = gr$ it gives $A \preceq (1 + \varepsilon_\varrho^2/(1-gr))I$ at one prime, which (c) turns into the threshold bound $1 + B^+/p$ of Theorem 14.21, and with $\beta = g_{\mathrm{root}} r$ it gives that of Theorem 14.27. It also covers an operator written as $A = \langle w, \cdot\rangle\, w + R$ with w real, $\langle \mathbf{1}, w\rangle = 1$, $\|w - \mathbf{1}\| \le v$, and $R = P_0 R P_0$ self-adjoint with $\|R\| \le d$: then $\langle \mathbf{1}, A\mathbf{1}\rangle = 1$, $P_0 A \mathbf{1} = w - \mathbf{1}$ and $P_0 A P_0 \preceq (d + v^2) P_0$, so $A \preceq (1 + v^2/(1 - d - v^2))I$ when $d + v^2 < 1$.

With (a), parts (a′) and (a″) give the two-sided caps $-gp^{-1/2} I \preceq T_p \preceq (1 + c^+/p)I$ of the cuts whose centred blocks are bounded directly: (a″) gives the upper cap of the cut L20 of the uniform root (Section 20.4); (a) and (a′) give both caps of the chamber cut of a positive root on one of the seven middle intervals with least prime at least 2^{30} (Proposition 21.7), and (a) the upper cap of its layer cut (Proposition 21.9); and (a′) gives the lower caps of the merge and constant compressions of the upper range (Proposition 27.14). At the cut L20 the group of (a″) is the group of simultaneous translations of all labels: it preserves the half law (Lemma 13.3(d)) and commutes with the cut operator, because the kernel (13.2) depends only on differences of labels.

Lemma 13.7 (Tensor products of compressions)**.**

(a) (Tensor spectrum, Lemma 6.10(g).) *If* $C_1, \dots, C_k$ *are self-adjoint operators on finite-dimensional Hilbert spaces with spectra in* $[-g_i, 1 + b_i]$, *where* $b_i \ge 0$ *and* $0 \le g_i \le 1 + b_i$, *then* $\bigotimes_i C_i \succeq -\max_i \big(g_i \prod_{j \ne i}(1 + b_j)\big) I \succeq -(\max_i g_i) \exp(\sum_i b_i) I$.

(b) *Let* B *be a finite set of odd primes,* $b = \min B$, *and for* $p \in B$ *let* T_p *be a self-adjoint operator on a finite-dimensional Hilbert space with*

$$-gp^{-1/2} I \preceq T_p \preceq (1 + c^+/p) I, \qquad gp^{-1/2} < 1,$$

where $g, c^+ \ge 0$. *Then*

$$\bigotimes_{p \in B} T_p \succeq -\frac{g}{\sqrt{b}} \prod_{p \in B} \Big(1 + \frac{c^+}{p}\Big) I \succeq -\frac{g}{\sqrt{b}} \exp\Big(c^+ \sum_{p \in B} \frac{1}{p}\Big) I,$$

and if $B = \{p\}$ the bound is $-gp^{-1/2}I$. In particular, let spectral data be given with $e_\Delta(p, 2n) < 1$ for $p \in B$, and for $p \in B$ let $\mathcal{V}_p \subseteq L^2(\mu_p)$ be a subspace with

$$-gp^{-1/2}\,\|F\|^2 \le \langle F, \mathsf{A}_p F\rangle_{\mu_p} \le (1 + c^+/p)\,\|F\|^2 \qquad (F \in \mathcal{V}_p),$$

with $g, c^+ \ge 0$ and $gp^{-1/2} < 1$ as above. Then for every $F \in \bigotimes_{p\in B} \mathcal{V}_p$

$$\langle F, \mathsf{A}_B F\rangle_{\mu_B} \ge -\frac{g}{\sqrt{b}} \prod_{p\in B} \Big(1 + \frac{c^+}{p}\Big)\,\|F\|^2_{\mu_B} \ge -\frac{g}{\sqrt{b}} \exp\Big(c^+ \sum_{p\in B} \frac{1}{p}\Big)\,\|F\|^2_{\mu_B},$$

and if $B = \{p\}$ the bound is $-gp^{-1/2}\,\|F\|^2$. For $A \subseteq U^+$ the functions of x_A in $L^2(\mu_B)$ form the subspace $\bigotimes_p \mathcal{V}_{A,p}$, where $\mathcal{V}_{A,p}$ is the space of functions of $x_{A,p}$ in $L^2(\mu_p)$; so (b) applies to the compression to the functions of the labels of a vertex set.

(c) *If $\log \prod_{p\in B} p \le L$, then $|B| \le L/\log b$ and $\sum_{p\in B} 1/p \le L/(b \log b)$.*

The lower and the upper bound in (b) may come from different arguments. The upper bound for $\mathcal{V}_p = L^2(\mu_p)$, which bounds every compression, is supplied by Proposition 13.6(c), with $c^+ = B^+$.

Proof. (a) is Lemma 6.10(g) with $c_i = g_i$. (b) Apply (a) to the operators T_p, with $g_p = gp^{-1/2} \le gb^{-1/2}$ and $b_p = c^+/p$; the condition $g_p \le 1 + b_p$ holds since $gp^{-1/2} < 1$, and $g_p \prod_{p'\ne p}(1 + b_{p'}) \le gb^{-1/2} \prod_{p'\in B}(1 + b_{p'})$. For a single prime the product over $p' \ne p$ is empty. For spectral data let P_p be the orthogonal projection of $L^2(\mu_p)$ onto $\mathcal{V}_p$ and $C_p = P_p \mathsf{A}_p P_p|_{\mathcal{V}_p}$, a self-adjoint operator on $\mathcal{V}_p$ whose spectrum lies in its numerical range, hence in $[-gp^{-1/2}, 1 + c^+/p]$. On $\bigotimes_p \mathcal{V}_p \subseteq L^2(\mu_B)$, $(\bigotimes_p P_p)(\bigotimes_p \mathsf{A}_p)(\bigotimes_p P_p) = \bigotimes_p C_p$, so $\langle F, \mathsf{A}_B F\rangle = \Big\langle F, (\bigotimes_p C_p)F\Big\rangle$ for $F \in \bigotimes_p \mathcal{V}_p$, and the first statement applies with $T_p = C_p$. The last statement holds because μ_B is a product and $x_A = (x_{A,p})_{p\in B}$, so the functions of x_A are spanned by the products $\prod_p f_p(x_{A,p})$. (c) is Lemma 4.18(c) with $X = b$, since $q_B = \prod_{p\in B} p \le e^L$. □

By (c), a block of logarithmic length at most L whose least prime b is at least X has $\sum_{p\in B} 1/p \le L/(b \log b) \le L/(X \log X)$. So if the caps of (b) hold with the same g, c^+ at every prime $p \ge X$, then

$$\bigotimes_{p\in B} T_p \succeq -g \exp\Big(\frac{c^+ L}{X \log X}\Big)\, b^{-1/2} I :$$

so the displayed lower bound, with the number $g \exp(c^+ L/(X \log X))$ in place of g, holds for every such block. Part (b) uses only the two caps of each T_p, so it applies as well to the cut operators of positive reweightings (Lemma 13.3(f)) and to their compressions to subspaces.

Weighted frames. Many estimates of Part V bound a sum of feature maps whose ranges are not orthogonal, through a matrix of bounds for their pairwise overlaps. The next lemma is the general form of that step; its part (c) reduces the largest eigenvalue of a symmetric majorant to a small matrix when the majorant has symmetries.

Lemma 13.8 (Weighted frames)**.** *Let $V_i \colon \mathcal{H}_i \to \mathcal{H}$ ($i \in I$, I finite) be operators between finite-dimensional Hilbert spaces, and let $G = (G_{ij})_{i,j\in I}$ be a symmetric matrix with nonnegative entries such that $\|V_i^* V_j\| \le G_{ij}$ for all i, j (for isometries one may take $G_{ii} = 1$).*

(a) *For $d_i \ge 0$ and $D = \operatorname{diag}(d_i)$,*

$$\Big\|\sum_i d_i V_i V_i^*\Big\| \le \lambda_{\max}(D^{1/2} G D^{1/2}) \le \max_i \sum_j G_{ij} d_j;$$

equivalently, the synthesis map $(z_i)_i \mapsto \sum_i \sqrt{d_i} V_i z_i$ from $\bigoplus_i \mathcal{H}_i$ to $\mathcal{H}$ has squared norm at most these numbers. In particular $\|\sum_i V_i V_i^*\| \le \lambda_{\max}(G)$.

(b) *For finitely many such families* $(V_i^{(k)})_{i\in I_k}$ *with matrices* $G^{(k)}$*, the tensor products* $\bigotimes_k V_{i_k}^{(k)}$*, indexed by* $\prod_k I_k$*, satisfy the hypothesis with the Kronecker product* $\bigotimes_k G^{(k)}$*.*

(c) (Perron gate.) *Let* G *be any symmetric matrix with nonnegative entries indexed by* I*, preserved by a group of permutations of* I *(*$G_{\sigma i,\sigma j} = G_{ij}$*) with orbits* $O_1, \dots, O_m$*, and put* $\bar G_{ab} = (|O_a||O_b|)^{-1/2} \sum_{i\in O_a,\, j\in O_b} G_{ij}$*. Then* $\lambda_{\max}(G) = \lambda_{\max}(\bar G)$*. Hence* $\lambda_{\max}(G) \le \lambda$ *as soon as* $\lambda I - \bar G \succeq 0$*, which may be verified by eliminating positive diagonal entries (Schur complements).*

In applications of (a), any smaller bound for $\lambda_{\max}(G)$ *may replace the row-sum bound.*

Proof. (a) The synthesis map $S(z_i)_i = \sum_i \sqrt{d_i} V_i z_i$ has the adjoint $S^*h = (\sqrt{d_i} V_i^* h)_i$, so $SS^* = \sum_i d_i V_i V_i^*$ and $\|\sum_i d_i V_i V_i^*\| = \|S^*S\|$. For $z = (z_i)_i$ put $w_i = \|z_i\|$; then

$$\langle z, S^*Sz\rangle = \sum_{i,j} \sqrt{d_i d_j}\, \langle z_i, V_i^* V_j z_j\rangle \le \sum_{i,j} \sqrt{d_i d_j}\, G_{ij} w_i w_j = \left\langle w, D^{1/2} G D^{1/2} w\right\rangle,$$

which is at most $\lambda_{\max}(D^{1/2}GD^{1/2})\,\|z\|^2$, since $\|w\| = \|z\|$, and $S^*S \succeq 0$ has norm equal to the supremum of this form over unit vectors. Deleting the indices with $d_i = 0$ removes a zero block of $D^{1/2}GD^{1/2}$ and does not increase the row sums; for $d_i > 0$ the matrix $D^{1/2}GD^{1/2} = D^{1/2}(GD)D^{-1/2}$ is similar to GD, whose eigenvalues are at most its largest row sum $\max_i \sum_j G_{ij} d_j$ in modulus, because its entries are nonnegative.

(b) For multi-indices (i_k), (j_k), $(\bigotimes_k V_{i_k}^{(k)})^*(\bigotimes_k V_{j_k}^{(k)}) = \bigotimes_k V_{i_k}^{(k)*} V_{j_k}^{(k)}$, whose norm is the product of the norms of the factors, at most $\prod_k G^{(k)}_{i_k j_k}$, the entry of the Kronecker product; this matrix is symmetric with nonnegative entries.

(c) Let v be a real unit eigenvector of G for $\lambda_1 = \lambda_{\max}(G)$, and let $|v|$ be its entrywise modulus. Since the entries of G are nonnegative, $\langle |v|, G|v|\rangle \ge \langle v, Gv\rangle = \lambda_1$, so $|v|$ maximizes the Rayleigh quotient and is an eigenvector for λ_1. The permutation matrices of the group commute with G, so the average $\bar v$ of the images of $|v|$ under the group is again an eigenvector for λ_1; it is nonzero, since it is nonnegative with the same positive entry sum as $|v|$, and it is invariant. The invariant vectors are those constant on every orbit, with the orthonormal basis $e_a = \mathbf{1}_{O_a}/|O_a|^{1/2}$; G maps them to invariant vectors, and $\langle e_a, Ge_b\rangle = \bar G_{ab}$. So $\bar G$ is the matrix of the restriction of G to the invariant vectors, a compression of G, which gives $\lambda_{\max}(\bar G) \le \lambda_1$; and $\bar v$ is an eigenvector of that restriction for λ_1, which gives the reverse inequality. □

When G has unit diagonal, $\lambda_{\max}(G) = 1 + \lambda_{\max}(G - I)$, and (c) may be applied to the off-diagonal part $G - I$. For example, for arbitrary nonnegative numbers $\varrho_{01}, \varrho_{02}, \varrho_{03}, \varrho_{34}$, five features with the off-diagonal bounds $G_{01} = G_{12} = \varrho_{01}$, $G_{02} = \varrho_{02}$, $G_{03} = G_{24} = \varrho_{03}$, $G_{34} = \varrho_{34}$ (and zero elsewhere), as the fold features of Lemma 25.1, admit the symmetry that exchanges $0 \leftrightarrow 2$ and $3 \leftrightarrow 4$; its orbits are $\{0,2\}$, $\{3,4\}$ and $\{1\}$, and the Perron block of the off-diagonal part is

$$\begin{pmatrix} \varrho_{02} & \varrho_{03} & \sqrt{2}\,\varrho_{01} \\ \varrho_{03} & \varrho_{34} & 0 \\ \sqrt{2}\,\varrho_{01} & 0 & 0 \end{pmatrix}.$$

Eliminating its last two diagonal entries shows that the synthesis map of the five features has squared norm at most $1 + \lambda$ for every λ with

$$\lambda > \varrho_{34}, \qquad \lambda - \varrho_{02} - \frac{\varrho_{03}^2}{\lambda - \varrho_{34}} - \frac{2\varrho_{01}^2}{\lambda} \ge 0.$$

The lemma also gives the diagonal caps of the positive envelopes (Theorem 25.8) and the frames of the complete actual stars (Section 14.2).

Swapped rectangular kernels. At a root cut whose half factors into two independent parts, the soft kernel is a product of one rectangular kernel with its own transpose, the second coordinates of the two halves being exchanged. Its spectrum is then read off from the singular values of the rectangle, and products with kernel factors depending on a common auxiliary label factor through the same exchange.

Lemma 13.9 (Swapped rectangular kernels)**.** *Let (X,α) and (Y,β) be finite probability spaces with full support, let $A\colon L^2(\beta)\to L^2(\alpha)$ have the real kernel $A(x;y)$, $(Ah)(x)=\mathbb{E}_{y\sim\beta}A(x;y)h(y)$, and let $T[A,A]$ be the operator on $L^2(\alpha\otimes\beta)$ whose kernel relative to $\alpha\otimes\beta$ is $A(x;y')A(x';y)$ at $((x,y),(x',y'))$.*

(a) *$T[A,A](f\otimes g)=Ag\otimes A^*f$, that is, $T[A,A]=(A\otimes A^*)\,\mathrm{Swap}$ with $\mathrm{Swap}(f\otimes g)=g\otimes f$. Let (u_i) and (v_j) be orthonormal bases of $L^2(\alpha)$ and $L^2(\beta)$, and $\sigma_i\ge 0$, with $Av_i=\sigma_iu_i$ and $A^*u_i=\sigma_iv_i$ for $i\le k=\min(|X|,|Y|)$, and $Av_j=0$, $A^*u_i=0$ for $j>k$, $i>k$ (a singular value decomposition). Then $T[A,A](u_i\otimes v_j)=\sigma_i\sigma_j\,u_j\otimes v_i$ for $i,j\le k$, and $T[A,A](u_i\otimes v_j)=0$ otherwise. So $T[A,A]$ is self-adjoint, its eigenvalues are the σ_i^2, the numbers $\pm\sigma_i\sigma_j$ $(i<j\le k)$ and zero, and $\|T[A,A]\|=\|A\|^2$. If $A=\sigma_0u_0\,\langle v_0,\cdot\rangle+A'$ with unit vectors u_0,v_0, $\sigma_0\ge0$, $A'v_0=0$, $A'^*u_0=0$ and $\|A'\|\le c$, and if $s\ge\max(\sigma_0,c)$, then $T[A,A]\succeq -sc\,I$, and $\|T[A,A]h\|\le sc\,\|h\|$ for every h orthogonal to $u_0\otimes v_0$.*

(b) (Products with a bounded row.) *Let (Z,γ), (S,π) and (S',π') be finite probability spaces with full support, let $P\colon L^2(\pi)\to L^2(\gamma)$ have the real kernel $P(z;s)$, and let $R(z;t)$ be a real kernel on $Z\times S'$ with $\sup_z\|R(z;\cdot)\|_{L^2(\pi')}\le\xi$. The product synthesis $(P\cdot R)F(z)=\mathbb{E}_{s\sim\pi,\,t\sim\pi'}P(z;s)R(z;t)F(s,t)$ has norm at most $\xi\,\|P\|$ from $L^2(\pi\otimes\pi')$ to $L^2(\gamma)$. More generally, let Γ be a real positive semidefinite kernel on Z (the matrix $(\Gamma(z,z'))$ is positive semidefinite) with $\Gamma(z,z)\le c$, let W be an operator on $L^2(\gamma)$ with $0\preceq W\preceq aI$, and let $\Gamma\circ W$ be the operator whose kernel relative to $\gamma\otimes\gamma$ is the pointwise product of Γ with the kernel of W. Then $0\preceq\Gamma\circ W\preceq caI$.*

(c) (Entangled swap.) *Let $\mathcal{H}=L^2_0(\mathbb{F}_p)$ for the uniform law on $\mathbb{F}_p$ and $e_0(a,s)=(p\mathbf{1}_{a=s}-1)/\sqrt{p-1}$. Then e_0 is a unit vector of $\mathcal{H}\otimes\mathcal{H}$, and if $e_0\otimes I\colon\mathcal{H}\to\mathcal{H}^{\otimes3}$ maps h to $e_0\otimes h$ and Swap_{13} exchanges the first and the third factor, then $(e_0\otimes I)^*\,\mathrm{Swap}_{13}(e_0\otimes I)=I/(p-1)$ on $\mathcal{H}$.*

(d) (Common auxiliary label.) *Let (S,π) be a finite probability space with full support, let $x\mapsto\phi_x$ and $x\mapsto\phi'_x$ be maps from X to real functions in $L^2(\pi)$, put $M(x,x')=\langle\phi_x,\phi'_{x'}\rangle_\pi$, and let $\mathcal{Y},\mathcal{Y}'\colon L^2(\beta\otimes\pi)\to L^2(\alpha)$ have the kernels $A(x;y)\phi_x(s)$ and $A(x;y)\phi'_x(s)$. Let $T[A,A]\circ M$ be the operator with kernel $A(x;y')A(x';y)M(x,x')$. Then*

$$T[A,A]\circ M=(\mathcal{Y}\otimes I_\beta)\,\mathrm{Swap}_{\beta\beta}(\mathcal{Y}'^*\otimes I_\beta),$$

where $\mathrm{Swap}_{\beta\beta}$ exchanges the two Y-labels of $L^2(\beta\otimes\pi\otimes\beta)$ and the auxiliary label s is shared. So $\|T[A,A]\circ M\|\le\|\mathcal{Y}\|\,\|\mathcal{Y}'\|$, and $\|\mathcal{Y}\|\le\|\Phi\|\sup_x\|A(x;\cdot)\|_{L^2(\beta)}$ by (b), where $\Phi\colon L^2(\pi)\to L^2(\alpha)$ has the kernel $\phi_x(s)$; likewise for $\mathcal{Y}'$. If $\phi_x=\phi^1_x+\phi^2_x$ and $\phi'_x=\phi'^1_x+\phi'^2_x$ for all x, with $\phi^1_x,\phi'^1_x\in E$ and $\phi^2_x,\phi'^2_x\in E^\perp$ for one subspace $E\subseteq L^2(\pi)$, then $T[A,A]\circ M=T[A,A]\circ M^{11}+T[A,A]\circ M^{22}$ with $M^{kk}(x,x')=\left\langle\phi^k_x,\phi'^k_{x'}\right\rangle$: there are no cross terms.

All of this holds with the two factors exchanged, since $T[A,A]$ is $T[A^*,A^*]$ written on $L^2(\beta\otimes\alpha)$; multipliers $M(y,y')$ of the Y-labels factor through A^*.

Proof. (a) For $f\in L^2(\alpha)$ and $g\in L^2(\beta)$,

$$(T[A,A](f\otimes g))(x,y)=\mathbb{E}_{x',y'}A(x;y')A(x';y)f(x')g(y')=(Ag)(x)\,(A^*f)(y),$$

which is the first identity. Hence $T[A,A](u_i\otimes v_j)=Av_j\otimes A^*u_i$, which is $\sigma_j\sigma_i\,u_j\otimes v_i$ for $i,j\le k$ and zero otherwise. In the orthonormal basis $(u_i\otimes v_j)$ the operator therefore has the eigenvectors

$u_i \otimes v_i$ ($i \le k$, eigenvalue σ_i^2), $u_i \otimes v_j \pm u_j \otimes v_i$ ($i < j \le k$, eigenvalues $\pm\sigma_i\sigma_j$) and the remaining basis vectors (eigenvalue zero); it is self-adjoint, and its norm is $\max_i \sigma_i^2 = \|A\|^2$. In the last statement $Av_0 = \sigma_0 u_0$ and $A^* u_0 = \sigma_0 v_0$, and $A = A'$ maps $v_0^\perp$ into $u_0^\perp$; so a singular value decomposition may be chosen in which (σ_0, u_0, v_0) is one of the triples, with index 0 say, and all other singular values are at most $\|A'\| \le c$. Every negative eigenvalue is some $-\sigma_i\sigma_j$ with $i \ne j$: it is at most $\sigma_0 c$ in modulus if $0 \in \{i, j\}$ and at most c^2 otherwise, and both numbers are at most sc. The orthogonal complement of the eigenvector $u_0 \otimes v_0$ is spanned by the other eigenvectors, whose eigenvalues are σ_i^2 ($i \ne 0$), $\pm\sigma_0\sigma_j$ ($j \ne 0$) and $\pm\sigma_i\sigma_j$ ($i, j \ne 0$), all at most $\max(c^2, \sigma_0 c) \le sc$ in modulus.

(b) Write $F_t = F(\cdot, t)$. By the Cauchy–Schwarz inequality in t,

$$|(P \cdot R)F(z)|^2 = \Big|\mathbb{E}_t R(z;t)\,(PF_t)(z)\Big|^2 \le \xi^2\,\mathbb{E}_t|(PF_t)(z)|^2,$$

and averaging over z gives $\|(P \cdot R)F\|^2 \le \xi^2 \mathbb{E}_t \|PF_t\|^2 \le \xi^2 \|P\|^2 \|F\|^2$. For the general statement write $W(z, z')$ for the kernel of W. Since $\langle f, Wf\rangle_\gamma = \sum_{z,z'} \gamma(z) f(z)\, W(z, z')\, \gamma(z') f(z')$, the operator W is positive semidefinite exactly when the matrix $(W(z, z'))$ is, and the identity has the kernel $\mathbf{1}_{z=z'}/\gamma(z)$. By the Schur product theorem the matrices $(\Gamma(z, z')W(z, z'))$ and $(\Gamma(z, z')(a\mathbf{1}_{z=z'}/\gamma(z) - W(z, z')))$ are positive semidefinite, and the operator with kernel $a\Gamma(z, z)\mathbf{1}_{z=z'}/\gamma(z)$ is multiplication by $a\Gamma(z, z) \le ca$. The first statement is also the case $\Gamma(z, z') = \langle R(z;\cdot), R(z';\cdot)\rangle_{\pi'}$, $W = PP^*$ of the second, because the Gram operator $(P \cdot R)(P \cdot R)^*$ has the kernel $\mathbb{E}_s P(z;s)P(z';s) \cdot \mathbb{E}_t R(z;t)R(z';t)$.

(c) The kernel of the identity of $L^2(\mathbb{F}_p)$ relative to the uniform law is $p\mathbf{1}_{a=s} = \sum_i f_i(a) f_i(s)$ for every real orthonormal basis $(f_i)_{0 \le i < p}$ with $f_0 = \mathbf{1}$; so $\sqrt{p-1}\, e_0 = \sum_{i \ge 1} f_i \otimes f_i$, an element of $\mathcal{H} \otimes \mathcal{H}$ of squared norm $p - 1$. For $h = f_k$ ($k \ge 1$), $\mathrm{Swap}_{13}(e_0 \otimes f_k) = (p-1)^{-1/2} \sum_{i \ge 1} f_k \otimes f_i \otimes f_i$, and since $\langle e_0, f_k \otimes f_i\rangle = (p-1)^{-1/2}\mathbf{1}_{i=k}$, contracting the first two factors with e_0 leaves $f_k/(p-1)$.

(d) The kernel of $T[A, A] \circ M$ is $\mathbb{E}_{s\sim\pi}[A(x; y')\phi_x(s)][A(x'; y)\phi'_{x'}(s)]$. For $F \in L^2(\alpha \otimes \beta)$, $(\mathcal{Y}'^* \otimes I_\beta)F$ is the function $(y_1, s, y_2) \mapsto \mathbb{E}_{x'} A(x'; y_1)\phi'_{x'}(s)F(x', y_2)$; the swap exchanges y_1 and y_2, and $\mathcal{Y} \otimes I_\beta$ then integrates $A(x; y_1)\phi_x(s)$ against the variables (y_1, s). Renaming $y_1 = y'$ and $y_2 = y$ gives $\mathbb{E}_{x',y',s} A(x; y')\phi_x(s)A(x'; y)\phi'_{x'}(s)F(x', y')$, the action of $T[A, A] \circ M$. The swap is unitary, which gives the norm bound, and $\mathcal{Y}$ is the product synthesis of (b) with $P = \Phi$ and $R = A$, up to the order of the variables. Finally $M = \sum_{k,l} M^{kl}$ with $M^{kl}(x, x') = \left\langle \phi_x^k, \phi_{x'}'^l \right\rangle$, and $M^{12} = M^{21} = 0$ because $E \perp E^\perp$; the map $M \mapsto T[A, A] \circ M$ is linear. For the last statement, the kernel of $T[A^*, A^*]$ at $((y, x), (y', x'))$ is $A^*(y; x')A^*(y'; x) = A(x'; y)A(x; y')$, the kernel of $T[A, A]$ at $((x, y), (x', y'))$. □

If a score has the form $\eta(x, x') = r^{-1}\langle \psi_x, \psi'_{x'}\rangle$, its square is the multiplier $q^{-1}M$ of (d) with the rows $\phi_x = \psi_x \otimes \psi_x$ and $\phi'_x = \psi'_x \otimes \psi'_x$. Splitting these rows along the diagonal unit vector d of Corollary 15.10(b) ($E = \mathbb{C}d$), parts (b) and (d) give the square identities of the positive roots (Lemma 21.4), and part (c) evaluates the leading terms of their linear traces (Lemma 21.5). At the cut L20 of the uniform root the distinguished vector of (a) is $u_0 \otimes v_0$ with $v_0 = \mathbf{1}$ and u_0 the normalised row-sum function of the rectangle (as in Proposition 20.17); it is invariant under simultaneous translations, so on the independent half chart (a) bounds the negative part of the soft kernel and its norm on the nontrivial translation sectors. After the compression to the actual half and the collision and normalizer corrections, these bounds give the constant C of Proposition 13.6(a″) at that cut (Section 20.4).

13.3. **Correlations after fixing coordinates.** The pinned influence theorem turns a local count into a bound for all conditional correlations of a graph law at once. Pin every vertex but two: the two free labels then range over explicit sets of sizes about $p2^{-d_i}$ and $p2^{-d_j}$ (d_i, d_j the degrees of the two vertices), coupled by one square constraint or by injectivity alone, and their maximal

correlation is of order r or r^2 (Lemma 13.12). Correlations under coarser pinnings are reached by releasing one vertex at a time. In the language of subspaces, passing to a coarser pinning intersects the coordinate subspaces with V_k (Lemma 13.14), and Kassabov's inequality bounds the new angles by the old ones through exactly the formula by which one step of Gaussian elimination in the normalized matrix $I-\mathcal{E}$ produces the next Schur complement. So the matrix $\mathcal{E}$ of fully pinned correlations bounds every partially pinned correlation through $(I-\mathcal{E})^{-1}$ (Lemma 13.15), and Peyre's tensorization (Theorem 4.13) passes from pairs of vertices to pairs of disjoint sets of vertices (Proposition 13.16).

Definition 13.10 (Influence data)**.** Let $G=(U,E)$ be an oriented graph with $n=|U|$ vertices, d_i the degree of $i\in U$ in G, and $\mu=\lambda_p(G)$ (so $\mathrm{Inj}_p(G)\neq\emptyset$), with coordinates $y=(y_u)_{u\in U}$. A *pinning* (S,s), $S\subseteq U$, $s\in\mathbb{F}_p^S$, is *feasible* if $\mu(y_S=s)>0$; then $\mu_{S,s}=\mu(\cdot\mid y_S=s)$ is uniform on the set $\Omega_{S,s}$ of completions, $F=U\setminus S$ is the set of *free* vertices, and $\Delta_F=\max_{i\in F}d_i$. A pinning (S',s') *extends* (S,s) if $S'\supseteq S$ and $s'|_S=s$.

(i) *Count errors.* Numbers $w,w^{\mathrm{conn}}\in[0,1)$ are *count errors for* (G,F) *at* p if for all distinct $a_1,\dots,a_t\in\mathbb{F}_p$, all signs $\epsilon_1,\dots,\epsilon_t$ and every $E'\supseteq\{a_1,\dots,a_t\}$, the number $N=\#\{x\in\mathbb{F}_p\setminus E':\epsilon_i(x-a_i)\in Q_p^*\ (1\le i\le t)\}$ satisfies

$$\frac{2^tN}{p}\ge 1-w\quad\text{if } t\le\Delta_F,\ |E'|\le n,\qquad \frac{2^tN}{p}\ge 1-w^{\mathrm{conn}}\quad\text{if } t\le\Delta_F+1,\ |E'|\le 2n. \tag{13.3}$$

By Lemma 12.2(d) and (E2), the coarse errors $w=w_{\Delta_F}(p,n)$ and $w^{\mathrm{conn}}=w_{\Delta_F+1}(p,2n)$ of (12.1) are count errors whenever they are below one; by Lemma 12.2(c) and (E3), so are the graded errors $w=e^-_{\Delta_F}(n)$ and $w^{\mathrm{conn}}=e^-_{\Delta_F+1}(2n)$ of (12.2). The two numbers are separate choices: one may be coarse and the other graded.

(ii) *Influence entries.* With $H_w=(1+w)/(1-w)^2$, for distinct $i,j\in F$

$$\varepsilon_{ij}=\begin{cases}H_w\,2^{(d_i+d_j-2)/2}(r+r^2), & ij \text{ an edge (either orientation)},\\ H_w\,2^{(d_i+d_j)/2}r^2, & \text{otherwise},\end{cases} \tag{13.4}$$

and $\varepsilon_{ii}=0$. The *influence matrix* is $\mathcal{E}=(\varepsilon_{ij})_{i,j\in F}$.

Only the free vertices are indexed, but their degrees d_i include the edges to pinned vertices; the degrees of pinned vertices do not enter. The matrix $\mathcal{E}$ is symmetric with nonnegative entries, so $\|\mathcal{E}\|$ is at most its maximal row sum. For either choice of count errors in (i), the numbers w, w^{conn}, H_w and ε_{ij}/r are admissible functions of r in the sense of Lemma 4.18 (polynomials with nonnegative coefficients, and $x\mapsto 1/(1-x)$ of them). The proof below gives (13.4) with $1/(1-w)$ in place of H_w (Lemma 13.12); we state (13.4) with H_w, the constant the certificates evaluate (Table 19).

Definition 13.11 (Influence conditions)**.** An oriented graph with n vertices and maximum degree Δ satisfies the *influence conditions at* p_0 if

$$w_\Delta(p_0,n)<\frac12,\qquad w_{\Delta+1}(p_0,2n)<1,\qquad e<\frac14, \tag{13.5}$$

with w_Δ as in (12.1), where e is the maximal row sum, at $r=p_0^{-1/2}$, of the influence matrix of the empty pinning ($F=U$) with $w=w_\Delta(p_0,n)$.

The three quantities are admissible functions of r, so the conditions at p_0 hold at every prime $p\ge p_0$, and there they give the hypotheses of Proposition 13.16 for the empty pinning ($\mathrm{Inj}_p(G)\neq\emptyset$ by Lemma 12.5, since $e_\Delta(p,n)\le w_\Delta(p,n)<1$). For spectral data, the certificates assert (13.5) for the half graph G^+ at every threshold at which Theorem 14.21 is applied (Table 19); no theorem of this paper assumes (13.5).

Lemma 13.12 (Fully pinned pairs)**.** *In the setting of Definition 13.10, let w satisfy the first inequality of* (13.3)*. Let $i, j \in F$ be distinct and let z be a feasible value of $y_{U\setminus\{i,j\}}$ that extends s. Then $\rho(y_i; y_j \mid y_{U\setminus\{i,j\}} = z) \le \varepsilon_{ij}$.*

Proof. Let S_i be the set of $x \in \mathbb{F}_p \setminus z(U \setminus \{i,j\})$ satisfying the constraints between i and the pinned vertices, and similarly S_j. Let t_i be the number of pinned neighbours of i: $t_i = d_i - 1$ if ij is an edge and $t_i = d_i$ otherwise; similarly t_j. Given the pinning, (y_i, y_j) is uniform on $\{(x,x') \in S_i \times S_j : K(x,x') = 1\}$, where $K(x,x') = A_p(\pm(x'-x))$ (sign by the orientation) if ij is an edge, and $K(x,x') = \mathbf{1}\{x \ne x'\}$ otherwise. Let $a(x) = \sum_{x' \in S_j} K(x,x')$ and $b(x') = \sum_{x \in S_i} K(x,x')$. If ij is an edge, $a(x)$ counts the labels of j outside the $n-1$ values $z(U \setminus \{i,j\}) \cup \{x\}$ that satisfy $t_j + 1 = d_j \le \Delta_F$ constraints with distinct roots (the pinned neighbours of j and $x \notin z(U\setminus\{i,j\})$), so $a(x) \ge p2^{-d_j}(1-w) = \frac{1}{2}p2^{-t_j}(1-w)$ by (13.3); if not, $a(x)$ counts the labels outside the same $n-1$ values that satisfy the $t_j = d_j$ constraints of j, so $a(x) \ge p2^{-t_j}(1-w)$. The same holds for b with i in place of j. Put $k = \frac{1}{2}$ for an edge and $k = 1$ otherwise; then $\min a \cdot \min b \ge k^2 p^2 2^{-t_i - t_j}(1-w)^2$.

The conditional expectation operator from $L^2(y_j)$ to $L^2(y_i)$ is unitarily equivalent to the matrix $\widetilde{K} = D_a^{-1/2} K D_b^{-1/2} \colon \ell^2(S_j) \to \ell^2(S_i)$; its top singular value is 1, with singular vectors $\sqrt{b}, \sqrt{a}$, and the maximal correlation is its second singular value $s_2(\widetilde{K})$. Since $D_a^{-1/2}(kJ_{\mathrm{all}})D_b^{-1/2}$ has rank one (J_{all} the all-ones matrix),

$$s_2(\widetilde{K}) \le \left\| D_a^{-1/2}(K - kJ_{\mathrm{all}})D_b^{-1/2} \right\| \le \frac{\|K - kJ_{\mathrm{all}}\|_{\ell^2(S_j)\to\ell^2(S_i)}}{\sqrt{\min a \cdot \min b}}.$$

For an edge, $K - kJ_{\mathrm{all}}$ is a submatrix of the $\mathbb{F}_p \times \mathbb{F}_p$ matrix $(A_p(\pm(x'-x)) - \frac{1}{2})$, whose norm is at most $\frac{\sqrt{p}+1}{2}$ by Lemma 4.11(a), for either orientation. For a non-edge, $K - J_{\mathrm{all}}$ is minus a partial identity, of norm at most 1. Hence, for an edge, $\rho \le 2^{(t_i+t_j)/2}\frac{\sqrt{p}+1}{p(1-w)} = 2^{(d_i+d_j-2)/2}\frac{r+r^2}{1-w}$, and for a non-edge $\rho \le 2^{(d_i+d_j)/2}\frac{r^2}{1-w}$. These are $\varepsilon_{ij}/(H_w(1-w)) \le \varepsilon_{ij}$, since $H_w(1-w) = (1+w)/(1-w) \ge 1$. □

Lemma 13.13 (Connectivity of pinned state spaces)**.** *In the setting of Definition 13.10, let w^{conn} satisfy the second inequality of* (13.3)*. For every feasible pinning (S', s') that extends (S, s), any two elements of $\Omega_{S',s'}$ are joined by a sequence of elements of $\Omega_{S',s'}$ in which consecutive elements differ in exactly one free coordinate.*

Proof. Every free vertex of (S', s') is free for (S, s), so it has degree at most Δ_F. By (13.3), for every $t \le \Delta_F + 1$ and every set of at most $2n$ excluded values containing the roots, a vertex with t root constraints at distinct roots has at least $p2^{-t}(1 - w^{\mathrm{conn}}) > 0$ admissible labels. Let $x, y \in \Omega_{S',s'}$ and let $T = y(U \setminus S')$ be the set of target labels; $|T| \le n$.

Phase 1. For each free vertex in turn, change its label to a value outside T and outside all labels currently in use that satisfies its constraints with all its (currently labelled) neighbours: at most Δ_F distinct roots and at most $2n$ exclusions, so this is possible. Each step stays in $\Omega_{S',s'}$. Afterwards no free vertex carries a target label.

Phase 2. Process the free vertices i one by one. Call a free vertex *placed* once it carries its target label. Invariant: placed vertices carry their targets; unplaced free vertices carry labels outside T. Before placing i, change each unplaced neighbour $j \ne i$ of i to a value that satisfies all constraints of j with its current neighbours (including the current label of i), satisfies the constraint of the edge between j and i for the prospective label y_i, and avoids T and all labels in use. The prospective root y_i differs from every current label: pinned and placed vertices carry target labels $\ne y_i$ (injectivity of y), and unplaced vertices carry labels outside T. So there are at most $\Delta_F + 1$ distinct roots and at most $2n$ exclusions, and the move is possible; the invariant is kept. Now change i to y_i: its constraints with pinned and placed neighbours hold because y is feasible, those with unplaced neighbours hold by construction, and y_i is not in use. Later moves

only touch unplaced vertices and respect the constraints with all current labels. When all free vertices are placed we have reached y. □

Lemma 13.14 (Coordinate subspaces)**.** *In the setting of Lemma 13.13, fix a feasible pinning (S', s') that extends (S, s), with free set F'. In $L^2(\mu_{S',s'})$ let V_i ($i \in F'$) be the subspace of functions of $y_{F'\setminus\{i\}}$, and put $V_A = \bigcap_{i\in A} V_i$ for $A \subseteq F'$ ($V_\emptyset = L^2(\mu_{S',s'})$).*

(i) *V_A is the space of functions of $y_{F'\setminus A}$; in particular $V_{F'}$ is the space of constants. Moreover $V_A \cap V_{A'} = V_{A\cup A'}$.*

(ii) *Let $\beta, \beta' \subseteq F'$ be disjoint and nonempty, and let $\bar\varepsilon \geq 0$ be such that $\rho(y_\beta; y_{\beta'} \mid y_{U\setminus(\beta\cup\beta')} = z) \leq \bar\varepsilon$ for every feasible value z of $y_{U\setminus(\beta\cup\beta')}$ that extends s'. Then $\cos\angle(V_\beta, V_{\beta'}) \leq \bar\varepsilon$. In particular, if w satisfies the first inequality of* (13.3)*, then $\cos\angle(V_i, V_j) \leq \varepsilon_{ij}$ for distinct $i, j \in F'$.*

(iii) *For disjoint nonempty $\beta, \beta' \subseteq F'$, $\cos\angle(V_{F'\setminus\beta'}, V_{F'\setminus\beta}) = \rho(y_\beta; y_{\beta'} \mid y_{S'} = s')$.*

Proof. (i) Functions of $y_{F'\setminus A}$ lie in V_A. Conversely let $f \in V_A$, so f is unchanged by any change of one coordinate in A that stays in $\Omega_{S',s'}$. For a value u of $y_{F'\setminus A}$, the completions with $y_{F'\setminus A} = u$ form $\Omega_{S'\cup(F'\setminus A),(s',u)}$, which is connected by single-coordinate moves in A (Lemma 13.13, applied to this pinning, which extends (S, s)). So f is constant there. The last statement is the definition of V_A.

(ii) By (i), $V_\beta \cap V_{\beta'} = V_{\beta\cup\beta'}$ consists of the functions of $u = y_{F'\setminus(\beta\cup\beta')}$. Let $f \in V_\beta \ominus V_{\beta\cup\beta'}$ and $g \in V_{\beta'} \ominus V_{\beta\cup\beta'}$: f is a function of $(y_{\beta'}, u)$ with $\mathbb{E}[f \mid u] = 0$, and g a function of (y_β, u) with $\mathbb{E}[g \mid u] = 0$. Conditionally on u the law is $\mu(\cdot \mid y_{U\setminus(\beta\cup\beta')} = (s', u))$, a full pinning of $U \setminus (\beta \cup \beta')$ that extends s', under which f is a centred function of $y_{\beta'}$ and g a centred function of y_β. So $|\mathbb{E}[fg \mid u]| \leq \bar\varepsilon\, \mathbb{E}[f^2 \mid u]^{1/2}\mathbb{E}[g^2 \mid u]^{1/2}$, and averaging over u with Cauchy–Schwarz gives $|\langle f, g\rangle| \leq \bar\varepsilon \|f\| \|g\|$ (Corollary 4.15). For single vertices, Lemma 13.12 supplies $\bar\varepsilon = \varepsilon_{ij}$.

(iii) By (i), $V_{F'\setminus\beta'}$ and $V_{F'\setminus\beta}$ are the functions of $y_{\beta'}$ and of y_β, and their intersection is $V_{F'}$, the constants, since $\beta \cap \beta' = \emptyset$. So the Friedrichs cosine (Definition 4.4) is the supremum over centred functions, which is the maximal correlation (Definition 4.3). □

Lemma 13.15 (All partial pinnings: Kassabov–Schur elimination)**.** *In the setting of Lemma 13.14, let $\mathcal{P}$ be a partition of F' into nonempty parts and $\bar E = (\bar E_{\beta\beta'})_{\beta,\beta'\in\mathcal{P}}$ a symmetric matrix with nonnegative entries, zero diagonal and $\left\|\bar E\right\| < 1$, such that $\cos\angle(V_\beta, V_{\beta'}) \leq \bar E_{\beta\beta'}$ for all distinct $\beta, \beta' \in \mathcal{P}$ (for instance, by Lemma 13.14(ii), if every fully pinned correlation $\rho(y_\beta; y_{\beta'} \mid y_{U\setminus(\beta\cup\beta')} = z)$ is at most $\bar E_{\beta\beta'}$). Then for all distinct $\beta, \beta' \in \mathcal{P}$*

$$\rho(y_\beta; y_{\beta'} \mid y_{S'} = s') \leq \left[(I - \bar E)^{-1}\right]_{\beta\beta'}.$$

In particular, if w satisfies the first inequality of (13.3) *and $\|\mathcal{E}\| < 1$, then for distinct $i, j \in F'$*

$$\rho(y_i; y_j \mid y_{S'} = s') \leq \left[(I - \mathcal{E}_{F'F'})^{-1}\right]_{ij} \leq \left[(I - \mathcal{E})^{-1}\right]_{ij}.$$

Proof. For $\mathcal{Y} \subseteq \mathcal{P}$ put $V[\mathcal{Y}] = V_{\cup\mathcal{Y}}$; by Lemma 13.14(i) it is the space of functions of $y_{F'\setminus\cup\mathcal{Y}}$, and $V[\mathcal{Y}] \cap V[\mathcal{Y}'] = V[\mathcal{Y} \cup \mathcal{Y}']$. Let $M = I - \bar E$, positive definite since $\left\|\bar E\right\| < 1$. For $\mathcal{Y} \subseteq \mathcal{P}$ and distinct $\beta, \beta' \in \mathcal{P} \setminus \mathcal{Y}$ put $c_\mathcal{Y}(\beta, \beta') = \cos\angle(V[\mathcal{Y} \cup \{\beta\}], V[\mathcal{Y} \cup \{\beta'\}])$. Let $M^{(\mathcal{Y})}$ be the Schur complement of $M_{\mathcal{Y}\mathcal{Y}}$ in M, a positive definite matrix on $\mathcal{P} \setminus \mathcal{Y}$, let $N^{(\mathcal{Y})}$ be its normalization $N^{(\mathcal{Y})}_{\beta\beta'} = M^{(\mathcal{Y})}_{\beta\beta'}/(M^{(\mathcal{Y})}_{\beta\beta} M^{(\mathcal{Y})}_{\beta'\beta'})^{1/2}$, and put $e_\mathcal{Y}(\beta, \beta') = -N^{(\mathcal{Y})}_{\beta\beta'}$.

Claim: $0 \leq e_\mathcal{Y}(\beta, \beta') < 1$ *and* $c_\mathcal{Y}(\beta, \beta') \leq e_\mathcal{Y}(\beta, \beta')$. For $\mathcal{Y} = \emptyset$, $e_\emptyset(\beta, \beta') = \bar E_{\beta\beta'}$, and the claim is the hypothesis. Since $N^{(\mathcal{Y})}$ is positive definite with unit diagonal, its off-diagonal entries have modulus < 1. Let $\mathcal{Y}' = \mathcal{Y} \cup \{\gamma\}$. Eliminating γ from $N^{(\mathcal{Y})}$ (Schur complements are transitive, and conjugating by a positive diagonal matrix commutes with taking Schur complements and with

normalizing) gives

$$e_{\mathcal{Y}'}(\beta,\beta') = \frac{e_{\mathcal{Y}}(\beta,\beta') + e_{\mathcal{Y}}(\beta,\gamma)e_{\mathcal{Y}}(\beta',\gamma)}{\sqrt{(1-e_{\mathcal{Y}}(\beta,\gamma)^2)(1-e_{\mathcal{Y}}(\beta',\gamma)^2)}} \ \geq 0.$$

On the other hand, Kassabov's inequality (Theorem 4.16) with $V_1 = V[\mathcal{Y}\cup\{\beta\}]$, $V_2 = V[\mathcal{Y}\cup\{\beta'\}]$, $V_3 = V[\mathcal{Y}\cup\{\gamma\}]$, for which $V_1\cap V_3 = V[\mathcal{Y}'\cup\{\beta\}]$ and $V_2\cap V_3 = V[\mathcal{Y}'\cup\{\beta'\}]$, gives

$$c_{\mathcal{Y}'}(\beta,\beta') \leq \frac{c_{\mathcal{Y}}(\beta,\beta') + c_{\mathcal{Y}}(\beta,\gamma)c_{\mathcal{Y}}(\beta',\gamma)}{\sqrt{(1-c_{\mathcal{Y}}(\beta,\gamma)^2)(1-c_{\mathcal{Y}}(\beta',\gamma)^2)}},$$

the denominators being positive by induction. The right side is nondecreasing in each of its three arguments on $[0,1)$; by induction on $|\mathcal{Y}|$, $c_{\mathcal{Y}} \leq e_{\mathcal{Y}}$ entrywise, which proves the claim. Thus the cosines of the coordinate subspaces are bounded by the recursion of Gaussian elimination in $I-\bar{E}$.

Take $\mathcal{Y} = \mathcal{P}\setminus\{\beta,\beta'\}$. By Lemma 13.14(iii), $c_{\mathcal{Y}}(\beta,\beta') = \rho(y_\beta; y_{\beta'} \mid y_{S'} = s')$. The Schur complement of $M_{\mathcal{Y}\mathcal{Y}}$ is the inverse of the $\{\beta,\beta'\}$ block of $\mathcal{Q} = M^{-1}$, so $e_{\mathcal{Y}}(\beta,\beta') = \mathcal{Q}_{\beta\beta'}/\sqrt{\mathcal{Q}_{\beta\beta}\mathcal{Q}_{\beta'\beta'}}$. The Neumann series $\mathcal{Q} = \sum_{k\geq 0}\bar{E}^k$ has nonnegative entries and $\mathcal{Q}_{\beta\beta} \geq 1$; hence $e_{\mathcal{Y}}(\beta,\beta') \leq \mathcal{Q}_{\beta\beta'}$. For the second statement take the partition of F' into singletons and $\bar{E} = \mathcal{E}_{F'F'}$, which is admissible by Lemma 13.14(ii) and has $\|\mathcal{E}_{F'F'}\| \leq \|\mathcal{E}\| < 1$; finally $(\mathcal{E}^k_{F'F'})_{ij} \leq (\mathcal{E}^k)_{ij}$ entrywise. □

Proposition 13.16 (Pinned influence theorem)**.** *Let G be an oriented graph with n vertices, p an odd prime, $\mu = \lambda_p(G)$, (S,s) a feasible pinning with free set F, w and w^{conn} count errors for (G,F) at p* (13.3)*, and $\mathcal{E}$ the influence matrix* (13.4)*, with $\|\mathcal{E}\| \leq e < 1$. Let (S',s') be a feasible pinning that extends (S,s), with free set F'.*

(a) *For distinct $i,j \in F'$,*

$$\rho(y_i; y_j \mid y_{S'} = s') \leq \big[(I-\mathcal{E}_{F'F'})^{-1}\big]_{ij} \leq \big[(I-\mathcal{E})^{-1}\big]_{ij} \leq \varepsilon_{ij} + \frac{e^2}{1-e}.$$

(b) *For disjoint nonempty $X,Y \subseteq F'$,*

$$\rho(y_X; y_Y \mid y_{S'} = s') \leq \min\Big\{1,\ \|\mathcal{E}_{XY}\| + \frac{e^2}{1-e}\Big\}, \qquad \mathcal{E}_{XY} = (\varepsilon_{uv})_{u\in X,\, v\in Y}.$$

Proof. (a) The first two inequalities are Lemma 13.15. Next $(I-\mathcal{E})^{-1} = I + \mathcal{E} + \mathcal{E}^2(I-\mathcal{E})^{-1}$, and the last matrix is symmetric with spectrum in $\{\lambda^2/(1-\lambda) : |\lambda| \leq e\} \subseteq [0, e^2/(1-e)]$; so its norm, and with it every entry and the norm of every submatrix, is at most $e^2/(1-e)$. For $i \neq j$ the identity contributes nothing.

(b) Apply Theorem 4.13 (first statement) in the probability space $(\Omega_{S',s'}, \mu_{S',s'})$ to the families $(y_u)_{u\in X}$ and $(y_v)_{v\in Y}$, with $\varepsilon'_{uv} = [(I-\mathcal{E})^{-1}]_{uv} \geq 0$. Its hypothesis asks for $\rho(y_u; y_v \mid y_{S''} = s'') \leq \varepsilon'_{uv}$ in this space, for $S'' \subseteq (X\cup Y)\setminus\{u,v\}$ and every value s'' of positive probability; this is $\rho(y_u; y_v \mid y_{S'\cup S''} = (s',s''))$ under μ, a feasible pinning that extends (S,s), so (a) supplies it. The theorem gives $\rho \leq \min\{1, \big\|([(I-\mathcal{E})^{-1}]_{uv})_{u\in X, v\in Y}\big\|_{\mathrm{op}}\}$. Since $X\cap Y = \emptyset$, this matrix is $\mathcal{E}_{XY}$ plus a submatrix of $\mathcal{E}^2(I-\mathcal{E})^{-1}$, so its norm is at most $\|\mathcal{E}_{XY}\| + e^2/(1-e)$. □

The certificates evaluate the following form, which is weaker only through the harmless factor $\sqrt{|X||Y|} \geq 1$ inserted in its proof, with the Schur test for the cross block.

Corollary 13.17 (Schur-test form)**.** *In the setting of Proposition 13.16, without its assumption on $\|\mathcal{E}\|$, put $c_{ij} = \varepsilon_{ij}/r$, let $q_{\mathrm{inf}} \geq \max_{i\in F}\sum_{j\in F} c_{ij}$ with $rq_{\mathrm{inf}} < 1$, and for disjoint nonempty $X,Y \subseteq F'$ let*

$$c_\times^2 = \Big(\max_{u\in X}\sum_{v\in Y} c_{uv}\Big)\Big(\max_{v\in Y}\sum_{u\in X} c_{uv}\Big).$$

Then

$$\rho(y_X; y_Y \mid y_{S'} = s') \leq r\Big(c_\times + \sqrt{|X||Y|}\,\frac{rq_{\mathrm{inf}}^2}{1-rq_{\mathrm{inf}}}\Big).$$

Proof. $\mathcal{E} = r(c_{ij})$ is symmetric and nonnegative, so $\|\mathcal{E}\| \le rq_{\inf} < 1$, and Proposition 13.16 applies with $e = rq_{\inf}$, for which $e^2/(1-e) = r \cdot rq_{\inf}^2/(1-rq_{\inf})$. The Schur test gives $\|\mathcal{E}_{XY}\| \le rc_\times$. So Proposition 13.16(b) gives the bound without the factor $\sqrt{|X||Y|}$, which is at least one. □

Remark 13.18 (Uses and the certified forms). (1) The conditional influence bound (F2) and the row correlations ρ_F, ρ_E of the twelve-label reference (Lemma 22.1) are the bound of Corollary 13.17 with the graded count errors $w = e^-_{\Delta_F}(n)$, $w^{\mathrm{conn}} = e^-_{\Delta_F+1}(2n)$, and with Δ_F, $q_{\inf}$ taken over the free vertices: for the rows, X and Y are the labels of the two outer objects and the middle object is pinned; for the layer repairs, X and Y are the layer halves and nothing is pinned. The certificates assert $w < \frac{1}{2}$, $w^{\mathrm{conn}} < 1$, $rq_{\inf} < 1$ and that the bound is below one; the bound without $\sqrt{|X||Y|}$ is not used. (2) The frame of the upper joint stars (Lemma 28.2) uses Proposition 13.16(a), and (b) for disjoint vertex sets, for the empty pinning of an induced half graph, with $w = w_\Delta(p,n)$ and $w^{\mathrm{conn}} = e^-_{\Delta+1}(2n)$. (3) *Monotonicity.* For fixed G, pinning pattern and X, Y, the quantities $q_{\inf}$, $c_\times$ and $rq_{\inf}^2/(1-rq_{\inf})$ of Corollary 13.17 are admissible functions of r (Lemma 4.18); so hypotheses verified at p_0 hold at every prime $p \ge p_0$, and the bound at p is at most r_p times its value divided by r at p_0. (4) For a reflection cut with halves $U^\pm$ and the empty pinning, Proposition 13.16(b) bounds $\rho(y_{U^+}; y_{U^-})$, the centred norm of the cut coupling in its actual half marginal.

13.4. **Recentring: from reference means to actual marginal means.** A coupling given by a kernel on a product of reference laws is centred in the reference, but its correlation must be measured in its own marginals. The exact identity (a) below separates the two: the cross term of the actual centring is a product of two marginal defects. It is used with a normalized kernel (part (b): the exact-support row correlations and the spectral bound for the replaced records of the one-way repairs), with an additive repair (the exact marginal cancellation Lemma 17.13), and for differences of normalized marginals (part (c)).

Lemma 13.19 (Recentering)**.** *Let (X, μ_X) and (Y, μ_Y) be finite probability spaces with full support, and let P_0 denote the projections onto the mean-zero functions.*

(a) *Let $\mathfrak{b}$ be a bilinear form on real functions on X times real functions on Y with $\mathfrak{b}(\mathbf{1},\mathbf{1}) = 1$, and define ϱ_L, ϱ_R by $\mathfrak{b}(f,\mathbf{1}) = \langle f, \varrho_L\rangle_{\mu_X}$ and $\mathfrak{b}(\mathbf{1},h) = \langle \varrho_R, h\rangle_{\mu_Y}$. If $\langle f, \varrho_L\rangle_{\mu_X} = 0$ and $\langle \varrho_R, h\rangle_{\mu_Y} = 0$, then*

$$\mathfrak{b}(f,h) = \mathfrak{b}(P_0 f, P_0 h) - \langle \varrho_L - \mathbf{1}, P_0 f\rangle_{\mu_X} \langle \varrho_R - \mathbf{1}, P_0 h\rangle_{\mu_Y}.$$

(b) *Let $K \ge 0$ be a kernel on $X \times Y$ relative to $\mu_X \otimes \mu_Y$ (operator $(Kh)(x) = \mathbb{E}_{y\sim\mu_Y} K(x,y)h(y)$), $z = \langle \mathbf{1}, K\mathbf{1}\rangle > 0$, and $\pi = (K/z)\,\mu_X \otimes \mu_Y$, with marginals $\varrho_L\mu_X$ and $\varrho_R\mu_Y$, so that $\varrho_L - \mathbf{1} = P_0 K\mathbf{1}/z$ and $\varrho_R - \mathbf{1} = P_0 K^*\mathbf{1}/z$. Suppose $|z-1| \le \varepsilon < 1$, $\|P_0 K\mathbf{1}\| \le M_L$, $\|P_0 K^*\mathbf{1}\| \le M_R$, $\|P_0 K P_0\| \le c$, $\varrho_L \ge e^{-l_L}$ and $\varrho_R \ge e^{-l_R}$. Then for f, h centred under the marginals of π,*

$$\frac{\langle f, Kh\rangle}{z} = \frac{\langle f, P_0 K P_0 h\rangle}{z} - \langle \varrho_L - \mathbf{1}, f\rangle \langle \varrho_R - \mathbf{1}, h\rangle,$$

and the maximal correlation of π is at most

$$e^{(l_L+l_R)/2}\Big(\frac{c}{1-\varepsilon} + \frac{M_L M_R}{(1-\varepsilon)^2}\Big); \qquad \text{for } l_L = l_R = l,\ M_L = M_R = M: \quad e^{l}\Big(\frac{c}{1-\varepsilon} + \frac{M^2}{(1-\varepsilon)^2}\Big).$$

(c) *For two kernels K_1, K_2 as in (b), with $z_k = \langle \mathbf{1}, K_k\mathbf{1}\rangle$, $m_k = P_0 K_k \mathbf{1}$ and left marginal densities $\varrho_k = K_k\mathbf{1}/z_k$,*

$$\varrho_1 - \varrho_2 = \frac{m_1 - m_2}{z_1} - \frac{(z_1 - z_2)\, m_2}{z_1 z_2};$$

hence if $|z_k - 1| \le \varepsilon_k < 1$, $\|m_1 - m_2\| \le M_D$, $|z_1 - z_2| \le \Delta_z$ *and* $\|m_2\| \le M_2$, *then* $\|\varrho_1 - \varrho_2\| \le (M_D + \Delta_z M_2/(1-\varepsilon_2))/(1-\varepsilon_1)$.

Proof. (a) Write $f = f_0 + a$ and $h = h_0 + b$ with $f_0 = P_0 f$, $h_0 = P_0 h$ and constants a, b. Since $\langle \mathbf{1}, \varrho_L \rangle = \mathfrak{b}(\mathbf{1}, \mathbf{1}) = 1$ and $f_0 \perp \mathbf{1}$, $0 = \langle f, \varrho_L \rangle = \langle f_0, \varrho_L - \mathbf{1} \rangle + a$, so $a = -\langle f_0, \varrho_L - \mathbf{1} \rangle$ and $\mathfrak{b}(f_0, \mathbf{1}) = \langle f_0, \varrho_L \rangle = -a$; likewise $b = -\langle \varrho_R - \mathbf{1}, h_0 \rangle$ and $\mathfrak{b}(\mathbf{1}, h_0) = -b$. Hence $\mathfrak{b}(f, h) = \mathfrak{b}(f_0, h_0) + b\,\mathfrak{b}(f_0, \mathbf{1}) + a\,\mathfrak{b}(\mathbf{1}, h_0) + ab = \mathfrak{b}(f_0, h_0) - ab$.

(b) Apply (a) to $\mathfrak{b}(f, h) = \langle f, Kh \rangle / z$: then $\mathfrak{b}(\mathbf{1}, \mathbf{1}) = 1$, $\mathfrak{b}(f, \mathbf{1}) = \langle f, K\mathbf{1} \rangle / z$, so $\varrho_L = K\mathbf{1}/z$ and $\varrho_L - \mathbf{1} = P_0 K \mathbf{1}/z$ (because $\mathbb{E}_{\mu_X} K\mathbf{1} = z$), and $\mathfrak{b}(P_0 f, P_0 h) = \langle f, P_0 K P_0 h \rangle / z$; moreover $\langle \varrho_L - \mathbf{1}, P_0 f \rangle = \langle \varrho_L - \mathbf{1}, f \rangle$. The centring conditions of (a) are the centring of f, h under the marginals of π. With $z \ge 1 - \varepsilon$, $\|\varrho_L - \mathbf{1}\| \le M_L/(1-\varepsilon)$ and $\|\varrho_R - \mathbf{1}\| \le M_R/(1-\varepsilon)$, so $|\int f(x)h(y)\,d\pi| = |\mathfrak{b}(f,h)| \le \left(\frac{c}{1-\varepsilon} + \frac{M_L M_R}{(1-\varepsilon)^2}\right) \|f\|_{\mu_X} \|h\|_{\mu_Y}$. Finally $\|f\|_{\mu_X}^2 \le e^{l_L} \mathbb{E}_{\mu_X}[\varrho_L f^2]$, the squared norm of f under the left marginal of π, and likewise for h.

(c) $\varrho_k = \mathbf{1} + m_k/z_k$ by (b), and $m_1/z_1 - m_2/z_2 = (m_1 - m_2)/z_1 + m_2(z_2 - z_1)/(z_1 z_2)$. □

Remark 13.20 (Instances). (1) The exact-support correlation bound of Section 27.6.3 is (b) verbatim, with c the bound of the centred strict compression and M the bound of the constant images. (2) A normalized cut operator A is self-adjoint with $\langle \mathbf{1}, A\mathbf{1} \rangle = 1$, so it is (b) with $z = 1$, $\varepsilon = 0$ and $M_L = M_R$; if $\|P_0 A P_0\| \le gr$ and $\|A\mathbf{1} - \mathbf{1}\|^2 \le B^+ r^2$ (for instance with $B^+ \ge p\varepsilon_\varrho^2/(1-gr)$ when $\|P_0 A \mathbf{1}\| \le \varepsilon_\varrho$), and the actual half marginal has density at least e^{-l}, its correlation relative to the actual marginal is at most $e^l(gr + B^+ r^2)$. This is the spectral bound for the replaced records of the one-way repairs ((F7)). (3) For an additive repair $d\tilde{\nu} = (1 + f(x) + g(y))\,d\nu$ of a coupling ν of ν_X, ν_Y, apply (a) with the references ν_X, ν_Y and $\mathfrak{b}(h, k) = \int hk\,d\tilde{\nu}$: for h, k centred under the marginals $(1+u)\nu_X$, $(1+v)\nu_Y$ of $\tilde{\nu}$, $\int hk\,d\tilde{\nu} = \int h_0 k_0 (1 + f + g)\,d\nu - \langle u, h_0 \rangle \langle v, k_0 \rangle$, which is the identity behind Lemma 17.13. (4) Part (c) is the normalizer comparison for the upper product rows (display (PR13) in Section 27.6.2).

13.5. **Marginals of the law itself (actual marginals).** Bounds are often proved in a convenient reference probability measure, while prices require norms in the marginals of the law itself—its *actual marginals*. The next lemma moves between the two when the actual marginals differ from the reference by a density factor between $1 - \eta$ and $1 + \eta$ (the reference is called ν inside the lemma; this is a local notation, not the comparison law of Part III). Part (a) bounds a correlation computed in the actual marginals by one computed in the reference; part (b) says that the operator of a difference of two records with the same actual marginal m is, after the unitary change of measure, the same operator computed relative to the reference, with norm increased by at most $1/(1-\eta)$.

Lemma 13.21 (Actual marginals)**.** *Let ν be a probability measure on a finite set X with full support, let $0 \le \eta < 1$, and let $\mathcal{H}$ be a space of real functions on X that contains the constants. Write $\Pi f = f - \mathbb{E}_\nu f$.*

(a) *Let $K \ge 0$ be the density, relative to $\nu \otimes \nu$, of a probability measure on $X \times X$ with marginals $m_L = (1 + \delta_L)\nu$ and $m_R = (1 + \delta_R)\nu$, where $|\delta_L|, |\delta_R| \le \eta$. Suppose that $|\langle \Pi f, K \Pi g \rangle_\nu| \le \kappa \|\Pi f\|_\nu \|\Pi g\|_\nu$ for all $f, g \in \mathcal{H}$. Then for all $f, g \in \mathcal{H}$ with $\mathbb{E}_{m_L} f = \mathbb{E}_{m_R} g = 0$,*

$$\Big| \int f(x) g(y) K(x,y)\,d\nu(x)\,d\nu(y) \Big| \le \frac{\kappa + \eta^2}{1 - \eta}\, \|f\|_{L^2(m_L)}\, \|g\|_{L^2(m_R)}\,.$$

(b) (Density transport.) *Let $m = \psi\nu$ be a probability measure with $\psi \ge 1 - \eta$, let t be a real function on $X \times X$, let T be the operator on $L^2(\nu)$ with kernel t relative to $\nu \otimes \nu$, and let T_m be the operator on $L^2(m)$ with $\langle f, T_m h \rangle_m = \int f(x) t(x,y) h(y)\,d\nu(x)\,d\nu(y)$. Let $W \colon L^2(m) \to L^2(\nu)$, $Wf = \psi^{1/2} f$, and let M_ψ be multiplication by ψ. Then W is unitary and $T_m = W^* M_\psi^{-1/2} T M_\psi^{-1/2} W$; hence $\|T_m\| \le \|T\| / \min \psi \le \|T\|/(1-\eta)$, the transposed*

kernel gives T_m^*, and the symmetric and skew parts of T_m are those of T transported in the same way, with the same bound.

Proof. (a) Apply Lemma 13.19(a) to $\mathfrak{b}(f,g) = \int f(x)g(y)K(x,y)\,d\nu(x)\,d\nu(y)$:

$$\mathfrak{b}(f,g) = \langle \Pi f, K\Pi g\rangle_\nu - \langle \delta_L, \Pi f\rangle_\nu \langle \delta_R, \Pi g\rangle_\nu .$$

Since $\mathcal{H}$ contains the constants, $\Pi f, \Pi g \in \mathcal{H}$. The first term is bounded by the hypothesis, and the second by $\eta^2 \|\Pi f\|_\nu \|\Pi g\|_\nu$. Finally $\|\Pi f\|_\nu \le \|f\|_\nu \le (1-\eta)^{-1/2}\|f\|_{m_L}$, and likewise for g.

(b) $\langle Wf, Wh\rangle_\nu = \mathbb{E}_\nu[\psi f h] = \langle f, h\rangle_m$, and W is onto since $\psi > 0$. For $f, h \in L^2(m)$, $\langle f, T_m h\rangle_m = \langle f, Th\rangle_\nu = \left\langle M_\psi^{-1/2} Wf, T M_\psi^{-1/2} Wh\right\rangle_\nu$, which is the identity, and $\left\|M_\psi^{-1/2}\right\|^2 = 1/\min\psi$. The adjoint statement is the symmetry of the defining formula. □

Part (a) is the exactly normalized case of the recentering Lemma 13.19(b) ($z = 1$, $\varepsilon = 0$, $M_L, M_R \le \eta$, $e^l = 1/(1-\eta)$), restricted to a space $\mathcal{H}$. It bounds the centred norm of a coupling in its actual marginals when only its centred norm in a nearby reference is known.

Positive quadratic tilts. A positive reweighting of a symmetric kernel by a square changes its marginal. The next lemma bounds the new mass, the centred part of the new marginal and its pointwise range in terms of a few operator norms of the pointwise products of the kernel with the scores, where η_j are arbitrary real functions on $X \times X$.

Lemma 13.22 (Marginals of a positive quadratic tilt)**.** *Let (X,ν) be a finite probability space with full support and $K \ge 0$ a symmetric kernel on $X \times X$ with $\mathbb{E}_{\nu\otimes\nu} K = 1$, marginal density $m = K\mathbf{1} > 0$, $\sup m/\inf m \le R$ and $\|m - \mathbf{1}\|_\nu \le \delta_0$. Let η_0, η_1, η_2 be real functions on $X \times X$ that are permuted among themselves by $(x,x') \mapsto (x',x)$, let $\bar\eta = \frac{1}{3}\sum_j \eta_j$, and let $V = \frac{1}{3}\sum_j \eta_j^2$ or $V = \bar\eta^2$. Write $K_{\bar\eta} = K \circ \bar\eta$ and $K_V = K \circ V$ for the pointwise products, and use the same letters for their operators on $L^2(\nu)$. For constants $c_V, e, \ell, h, c, b \ge 0$ suppose*

$$\|K_V - c_V K\| \le e, \quad \|P_0 K_{\bar\eta}\mathbf{1}\|_\nu \le \ell, \quad |\langle \mathbf{1}, K_{\bar\eta}\mathbf{1}\rangle_\nu| \le h, \quad |K_{\bar\eta}\mathbf{1}| \le c\,K\mathbf{1}, \quad |\eta_j| \le b \text{ on } \{K > 0\}.$$

For $\mathsf{u} \ge 0$ put $s = 1 + c_V\mathsf{u}^2$, and assume $\mathsf{u}c < 1$ and $z = s - 2\mathsf{u}h - \mathsf{u}^2 e > 0$. Then $T = (K + 2\mathsf{u}K_{\bar\eta} + \mathsf{u}^2 K_V)/Z$, with Z the $\nu\otimes\nu$-mass of the numerator, is a symmetric nonnegative kernel of mass one, and

$$Z \ge z, \qquad \|T\mathbf{1} - \mathbf{1}\|_\nu \le \frac{s\delta_0 + 2\mathsf{u}\ell + \mathsf{u}^2 e}{z}, \qquad \frac{\sup T\mathbf{1}}{\inf T\mathbf{1}} \le R\,\frac{(1+\mathsf{u}b)^2}{(1-\mathsf{u}c)^2}.$$

If $T_1, \dots, T_n$ are symmetric nonnegative kernels of mass one on finite probability spaces (X_i, ν_i) with $\|T_i\mathbf{1} - \mathbf{1}\| \le \delta_i$ and $\sup T_i\mathbf{1}/\inf T_i\mathbf{1} \le R_i$, then $T = \bigotimes_i T_i$ has range ratio at most $\prod_i R_i$ and marginal chi-square $\|T\mathbf{1} - \mathbf{1}\|^2 \le \prod_i(1+\delta_i^2) - 1$.

Proof. Put $w = 1 + 2\mathsf{u}\bar\eta + \mathsf{u}^2 V$, so that the numerator is $K \circ w$. In both cases $V \ge \bar\eta^2$ (for the mean of the squares by the Cauchy–Schwarz inequality), so $w \ge (1 + \mathsf{u}\bar\eta)^2 \ge 0$; the kernel T is nonnegative, and it is symmetric because K, $\bar\eta$ and V are.

Mass. $Z = \langle \mathbf{1}, K\mathbf{1}\rangle + 2\mathsf{u}\langle \mathbf{1}, K_{\bar\eta}\mathbf{1}\rangle + \mathsf{u}^2\langle \mathbf{1}, K_V\mathbf{1}\rangle$, with $\langle \mathbf{1}, K\mathbf{1}\rangle = 1$, $\langle \mathbf{1}, K_{\bar\eta}\mathbf{1}\rangle \ge -h$, and $\langle \mathbf{1}, K_V\mathbf{1}\rangle \ge c_V - e$ because $\langle \mathbf{1}, (K_V - c_V K)\mathbf{1}\rangle \ge -e$. So $Z \ge s - 2\mathsf{u}h - \mathsf{u}^2 e = z > 0$.

Centred marginal. $T\mathbf{1}$ has mean one, so

$$T\mathbf{1} - \mathbf{1} = \frac{1}{Z}\,P_0\big(sK\mathbf{1} + 2\mathsf{u}K_{\bar\eta}\mathbf{1} + \mathsf{u}^2(K_V - c_V K)\mathbf{1}\big),$$

with $\|P_0 K\mathbf{1}\| = \|m - \mathbf{1}\| \le \delta_0$, $\|P_0 K_{\bar\eta}\mathbf{1}\| \le \ell$ and $\|P_0(K_V - c_V K)\mathbf{1}\| \le \|K_V - c_V K\| \le e$, since P_0 is a contraction and $\|\mathbf{1}\| = 1$.

Range. $(T\mathbf{1})(x) = m(x)\,\mathbb{E}_K[w \mid x]/Z$, where $\mathbb{E}_K[\cdot \mid x]$ is the average in x' against the probability $K(x,x')\,d\nu(x')/m(x)$; thus $K_{\bar\eta}\mathbf{1} = m\,\mathbb{E}_K[\bar\eta \mid x]$ and $|\mathbb{E}_K[\bar\eta \mid x]| \le c$. By conditional Jensen,

$$\mathbb{E}_K[w \mid x] \ge \mathbb{E}_K[(1+\mathsf{u}\bar\eta)^2 \mid x] \ge \big(1 + \mathsf{u}\mathbb{E}_K[\bar\eta \mid x]\big)^2 \ge (1-\mathsf{u}c)^2,$$

and on $\{K>0\}$, $w \le (1+\mathsf{u}b)^2$ in both cases (for the mean of the squares, $w = \frac{1}{3}\sum_j (1+\mathsf{u}\eta_j)^2$). The common factor $1/Z$ cancels from the ratio.

Products. The marginal density of $\bigotimes_i T_i$ is $\bigotimes_i T_i\mathbf{1}$, so its range ratio is the product of the ratios, and, every $T_i\mathbf{1}$ having mean one, its chi-square is $\prod_i(1+\|T_i\mathbf{1}-\mathbf{1}\|^2)-1$ (Lemma 6.10(f)). □

Proposition 21.7 applies the lemma to the positive roots of the late middle pieces (Definition 21.1), with $\eta_j = \mathsf{H}_j$ and $c_V = 1$ for the mean of the individual squares, and the parameters of the selected laws. The square-of-the-mean option is a second instance of the same identity. In addition, Lemma 24.7(iv) uses its product clause for the range ratio and the marginal chi-square of a block of such primes.

13.6. **Restriction to an injectivity event.** Conditional row laws are first computed against a product of laws on the allowed label sets, in which the labels of one row are independent, and then restricted to the event that they are distinct. The next lemma transfers the three numbers that the spectral and correlation estimates need (the centred compression, the constant image and the total mass) through such a restriction.

Lemma 13.23 (Restriction to an injectivity event)**.** *Let (X,λ_X) and (Y,λ_Y) be finite probability spaces with full support, $I_X \subseteq X$ and $I_Y \subseteq Y$ nonempty events, $P_X = \lambda_X(I_X)$, $P_Y = \lambda_Y(I_Y)$, and $\mu_X = \lambda_X(\cdot \mid I_X)$, $\mu_Y = \lambda_Y(\cdot \mid I_Y)$. For a kernel S on $X\times Y$ let $T_S\colon L^2(\lambda_Y)\to L^2(\lambda_X)$, $T_Sh(x) = \mathbb{E}_{y\sim\lambda_Y} S(x,y)h(y)$, and let $S_\mu\colon L^2(\mu_Y)\to L^2(\mu_X)$ be the operator with kernel $S|_{I_X\times I_Y}$ relative to $\mu_X\otimes\mu_Y$. Let $\iota_X f = \mathbf{1}_{I_X} f/\sqrt{P_X}$ (f extended by zero) and similarly ι_Y (these ι_X,ι_Y are embeddings, not the reserved cut ι).*

(a) *$\iota_X\colon L^2(\mu_X)\to L^2(\lambda_X)$ is an isometry that maps μ_X-centred functions to λ_X-centred functions, $\iota_X^* g = \sqrt{P_X}\, g|_{I_X}$, and $\iota_X\mathbf{1} = \sqrt{P_X}\mathbf{1} + v_X$ with v_X centred and $\|v_X\| = \sqrt{1-P_X}$; likewise for Y. Moreover $S_\mu = \iota_X^* T_S \iota_Y/\sqrt{P_XP_Y}$.*

(b) *With P_0 the projections onto mean-zero functions, $\|S_\mu\| \le \|T_S\|/\sqrt{P_XP_Y}$ and $\|P_0S_\mu P_0\| \le \|P_0T_SP_0\|/\sqrt{P_XP_Y}$. Let M'_X be the supremum of $|\langle \iota_X f, T_S\mathbf{1}\rangle|$ over μ_X-centred f of norm one; thus $M'_X \le \|P_0T_S\mathbf{1}\|$. Then*

$$\|P_0S_\mu\mathbf{1}\| \le \frac{M'_X}{\sqrt{P_X}} + \|P_0T_SP_0\|\sqrt{\frac{1-P_Y}{P_XP_Y}},$$

the same holds with X,Y exchanged for S_μ^*, and

$$|\langle\mathbf{1},S_\mu\mathbf{1}\rangle - \langle\mathbf{1},T_S\mathbf{1}\rangle| \le \|P_0T_S^*\mathbf{1}\|\sqrt{\frac{1-P_Y}{P_Y}} + \|P_0T_S\mathbf{1}\|\sqrt{\frac{1-P_X}{P_X}} + \|P_0T_SP_0\|\sqrt{\frac{(1-P_X)(1-P_Y)}{P_XP_Y}}.$$

(c) *In particular, if $P_X,P_Y \ge I_0 > 0$ and C bounds $\|P_0T_S\mathbf{1}\|$, $\|P_0T_S^*\mathbf{1}\|$ and $\|P_0T_SP_0\|$ (for instance $C \ge \|T_{S-a}\|$ for some constant kernel a), then $\|P_0S_\mu P_0\| \le \|P_0T_SP_0\|/I_0$, $\|P_0S_\mu\mathbf{1}\| \le M'_X/\sqrt{I_0} + C\sqrt{1-I_0}/I_0 \le (M + C\sqrt{1-I_0})/I_0$ for every $M \ge M'_X$, and*

$$|\langle\mathbf{1},S_\mu\mathbf{1}\rangle - a_0| \le \frac{Z + C(2\sqrt{1-I_0}+1-I_0)}{I_0} \qquad \text{whenever } |\langle\mathbf{1},T_S\mathbf{1}\rangle - a_0| \le Z.$$

(d) *If λ_X is a product of laws on $\mathbb{F}_p$ indexed by an m-element set, each with density at most κ relative to the uniform law, and I_X is the event that the m coordinates are distinct, then $P_X \ge 1 - \binom{m}{2}\kappa/p$.*

Proof. (a) $\|\iota_X f\|_{\lambda_X}^2 = \mathbb{E}_{\lambda_X}[\mathbf{1}_{I_X}f^2]/P_X = \mathbb{E}_{\mu_X}f^2$ and $\mathbb{E}_{\lambda_X}\iota_X f = \sqrt{P_X}\mathbb{E}_{\mu_X}f$. For the adjoint, $\langle\iota_X f, g\rangle_{\lambda_X} = \mathbb{E}_{\lambda_X}[\mathbf{1}_{I_X}fg]/\sqrt{P_X} = \sqrt{P_X}\,\mathbb{E}_{\mu_X}[fg]$. Put $v_X = \mathbf{1}_{I_X}/\sqrt{P_X} - \sqrt{P_X}\mathbf{1}$; then $\mathbb{E}v_X = 0$ and $\|v_X\|^2 = 1 - 2P_X + P_X = 1 - P_X$. Finally $\langle f, S_\mu h\rangle_{\mu_X} = \mathbb{E}_{\lambda_X\otimes\lambda_Y}[\mathbf{1}_{I_X} f\, S\, \mathbf{1}_{I_Y}h]/(P_XP_Y) = \langle\iota_X f, T_S\iota_Y h\rangle/\sqrt{P_XP_Y}$.

(b) The first two bounds follow from (a), since ι_X, ι_Y are isometries preserving centring and $\langle \iota_X f, T_S \iota_Y h\rangle = \langle \iota_X f, P_0 T_S P_0 \iota_Y h\rangle$ for centred f, h. For a centred unit f,

$$\langle f, S_\mu \mathbf{1}\rangle = \frac{\langle \iota_X f, T_S(\sqrt{P_Y}\mathbf{1} + v_Y)\rangle}{\sqrt{P_X P_Y}} = \frac{\langle \iota_X f, T_S \mathbf{1}\rangle}{\sqrt{P_X}} + \frac{\langle \iota_X f, P_0 T_S P_0 v_Y\rangle}{\sqrt{P_X P_Y}},$$

and $|\langle \iota_X f, T_S\mathbf{1}\rangle| = |\langle \iota_X f, P_0 T_S \mathbf{1}\rangle| \le \|P_0 T_S \mathbf{1}\|$. Expanding $\langle \iota_X \mathbf{1}, T_S \iota_Y \mathbf{1}\rangle / \sqrt{P_X P_Y}$ in the same way gives $\langle \mathbf{1}, S_\mu \mathbf{1}\rangle = \langle \mathbf{1}, T_S \mathbf{1}\rangle + \langle P_0 T_S^* \mathbf{1}, v_Y\rangle / \sqrt{P_Y} + \langle v_X, P_0 T_S \mathbf{1}\rangle / \sqrt{P_X} + \langle v_X, P_0 T_S P_0 v_Y\rangle / \sqrt{P_X P_Y}$.

(c) Insert $P_X, P_Y \ge I_0$, $\sqrt{1-P} \le \sqrt{1-I_0}$ and $1/\sqrt{I_0} \le 1/I_0$ in (b): the last display of (b) is at most $C(2\sqrt{1-I_0}/\sqrt{I_0} + (1-I_0)/I_0) \le C(2\sqrt{1-I_0} + 1 - I_0)/I_0$, and $|\langle \mathbf{1}, T_S \mathbf{1}\rangle - a_0| \le Z \le Z/I_0$. The three compressions do not change when a constant kernel is added to S, since $T_a \mathbf{1} = a\mathbf{1}$ and $T_a P_0 = 0$.

(d) Two independent coordinates with laws λ_i, λ_j of density at most κ coincide with probability $\sum_x \lambda_i(x)\lambda_j(x) \le \max_x \lambda_j(x) \le \kappa/p$. Take a union bound over the $\binom{m}{2}$ pairs. □

In the upper product rows the lemma is applied to the star part of a conditional kernel, with $a_0 = 0$ for its mean (the normalizer errors and the centred constant images of Section 27.6.1), and to the difference of the F and E star parts, with C the sum of their full-operator bounds (display (PR12) in Section 27.6.2); in the exact-support row correlations of Section 27.6.3, part (b) gives the bound of the centred strict compression.

13.7. **Kernels depending on few coordinates.** Many kernels below depend on only a few of the labels. The next lemma says their operator norm can then be computed using those labels alone, up to explicit factors comparing the true marginals with the uniform ones.

Lemma 13.24 (Projected kernels)**.** *Let ν be a probability measure on a finite product $\Omega = \prod_{i\in\Lambda} \Omega_i$; for $I \subseteq \Lambda$ let ν_I be the law of $x_I = (x_i)_{i\in I}$ under ν, υ_I the uniform law on $\Omega_I = \prod_{i\in I}\Omega_i$, and $\varrho_I \ge \max d\nu_I/d\upsilon_I$.*

(a) *Let $I, J \subseteq \Lambda$ be nonempty and let κ be a function of (x_I, y_J) only. The norm of the operator with kernel κ on $L^2(\nu)$ (relative to $\nu\otimes\nu$) equals the norm of the operator with kernel κ from $L^2(\nu_J)$ to $L^2(\nu_I)$ (relative to $\nu_I\otimes\nu_J$), and is at most $(\varrho_I\varrho_J)^{1/2}$ times the norm of κ from $L^2(\upsilon_J)$ to $L^2(\upsilon_I)$. In particular, if a probability λ on Ω has density at most H relative to the uniform law, every kernel has norm on $L^2(\lambda)$ at most H times its norm for independent uniform labels.*

(b) *Let $\Omega = \mathbb{F}_p^{\{0,1,2\}}$. For the uniform law ν_F on ordered triples of distinct labels with $x_2 - x_0 \in Q_p^*$ (the law λ_p of the graph on $\{0,1,2\}$ with the single edge $0\to 2$) one may take $\varrho_I = 1$ for one label, $\varrho_{\{0,2\}} = 2/(1-q)$, $\varrho_{\{0,1\}} = \varrho_{\{1,2\}} = 1/(1-2q)$ and $\varrho_{\{0,1,2\}} = 2/((1-q)(1-2q))$. For the uniform law ν_E on ordered triples of distinct labels one may take $\varrho_I = 1$ for one label, $\varrho_I = 1/(1-q)$ for two labels and $\varrho_{\{0,1,2\}} = 1/((1-q)(1-2q))$.*

Proof. (a) The operator factors as $\iota_I \widetilde{\kappa}\, \iota_J^*$, where $\iota_I\colon L^2(\nu_I) \to L^2(\nu)$ is the isometric embedding of functions of x_I, ι_J^* is the conditional expectation onto functions of x_J, and $\widetilde{\kappa}$ is the operator with kernel κ from $L^2(\nu_J)$ to $L^2(\nu_I)$; since ι_I is isometric and $\iota_J^*\iota_J$ is the identity, the norms agree. If ψ_I is the density of ν_I, then $\langle \varphi, \widetilde{\kappa}\phi\rangle = \mathbb{E}_{\upsilon_I\otimes\upsilon_J}[\varphi\psi_I\,\kappa\,\phi\psi_J]$, and $\mathbb{E}_{\upsilon_I}[\varphi^2\psi_I^2] \le \varrho_I \|\varphi\|_{\nu_I}^2$. The last statement is the case $I = J = \Lambda$.

(b) Every coordinate of ν_F and of ν_E is uniform, by translation invariance. ν_F has $p(p-1)(p-2)/2$ equally likely triples, so its density is $2p^2/((p-1)(p-2))$. The pair (x_0, x_2) is uniform on the $p(p-1)/2$ directed pairs, of density $2p/(p-1)$. Given distinct x_0, x_1 there are at most $(p-1)/2$ completions x_2, and given distinct x_1, x_2 at most $(p-1)/2$ completions x_0, which gives the density bound $p/(p-2)$. Under ν_E every pair is uniform on distinct pairs, of density $p/(p-1)$, and the triple has density $p^2/((p-1)(p-2))$. □

With (a), a kernel norm computed for independent uniform labels, for instance by the kernel bound of Lemma 16.4, transfers to every such law at the cost of the displayed density factors.

13.8. **Conditional laws given the middle label, under a change of reference.** The five-type completions (Lemma 17.5) need, at each label of the middle object, a bound for the fold and matching row operators relative to one common reference law of the outer rows. The fold row law has its own marginals; the next lemma bounds the effect of the change of reference.

Lemma 13.25 (Row comparison)**.** *Let ν be a probability measure on a finite set X with full support, π a probability measure on $X\times X$ with marginals $\pi_L = r_L\nu$ and $\pi_R = r_R\nu$ and maximal correlation ρ_π, and T the operator on $L^2(\nu)$ with kernel $d\pi/d(\nu\otimes\nu)$. Let $H_L \ge \max r_L$, $H_R \ge \max r_R$, $v_L \ge \|r_L - \mathbf{1}\|_\nu$ and $v_R \ge \|r_R - \mathbf{1}\|_\nu$.*

(a) *For real $f, h \in L^2(\nu)$ with ν-means a, b and centred parts f_0, h_0,*

$$\begin{aligned}\langle f, (T-\mathbb{J})h\rangle_\nu &= \mathrm{Cov}_\pi(f(x),h(y)) + (\mathbb{E}_{\pi_L} f - a)\mathbb{E}_{\pi_R} h + a(\mathbb{E}_{\pi_R} h - b)\\ &= a\,\langle r_R - \mathbf{1}, h_0\rangle + b\,\langle r_L - \mathbf{1}, f_0\rangle + \langle f_0, Th_0\rangle\,,\end{aligned}$$

so the two mean terms are at most $v_R|a|\,\|h_0\|$ and $v_L|b|\,\|f_0\|$, and the fluctuation term satisfies $|\langle f_0, Th_0\rangle| \le (\sqrt{H_L H_R}\,\rho_\pi + v_L v_R)\,\|f_0\|\,\|h_0\|$. Hence

$$\|T-\mathbb{J}\| \le \sqrt{H_L H_R}\,\rho_\pi + v_L \min\{\sqrt{H_R}, 1+v_R\} + v_R \le \sqrt{H_L H_R}\,\rho_\pi + v_L + v_R + v_L v_R.$$

(b) *If $H_0 \ge 1$ and $m \ge 0$ satisfies $m \le H_0$ and $|m-1| \le H_0 - 1$ (for instance $m \in [1/H_0, H_0]$), then $\|mT - \mathbb{J}\| \le H_0(\sqrt{H_L H_R}\,\rho_\pi + v_L + v_R + v_L v_R) + H_0 - 1$.*

(c) *Let $X = \prod_{p\in B} X_p$, $\nu = \bigotimes_p \nu_p$ and $\pi = \bigotimes_p \pi_p$. Then $r_L = \prod_p r_{L,p}$, $1 + \|r_L - \mathbf{1}\|^2 = \prod_p (1 + \|r_{L,p} - \mathbf{1}\|^2)$, likewise for r_R, and $\rho_\pi = \max_p \rho_{\pi_p}$. A density ρ with $\mathbb{E}_\nu \rho = 1$ and values in $[l, u]$ has $\|\rho - \mathbf{1}\|^2 \le (u-1)(1-l)$.*

Proof. (a) $\langle f, Th\rangle_\nu = \mathbb{E}_\pi[f(x)h(y)]$ and $\langle f, \mathbb{J}h\rangle_\nu = ab$, which gives the first expression. For the second, $\mathbb{E}_\pi[f(x)h(y)] = ab + a\mathbb{E}_{\pi_R} h_0 + b\mathbb{E}_{\pi_L} f_0 + \mathbb{E}_\pi[f_0(x)h_0(y)]$, with $\mathbb{E}_{\pi_R} h_0 = \langle r_R, h_0\rangle = \langle r_R - \mathbf{1}, h_0\rangle$ and likewise for f_0. Next $\mathbb{E}_\pi[f_0(x)h_0(y)] = \mathrm{Cov}_\pi(f_0, h_0) + \mathbb{E}_{\pi_L} f_0\,\mathbb{E}_{\pi_R} h_0$; the covariance is at most $\rho_\pi\,\|f_0\|_{\pi_L}\,\|h_0\|_{\pi_R} \le \rho_\pi\sqrt{H_L H_R}\,\|f_0\|_\nu\,\|h_0\|_\nu$, and the product of the means at most $v_L v_R\,\|f_0\|\,\|h_0\|$. For the norm bound (the kernel is real, so real f, h suffice) use the first expression: the covariance term is bounded in the same way, with f, h in place of f_0, h_0; $|\mathbb{E}_{\pi_L} f - a| = |\langle r_L - \mathbf{1}, f\rangle| \le v_L\,\|f\|$, $|\mathbb{E}_{\pi_R} h| \le \min\{\|h\|_{\pi_R}, \|r_R\|\,\|h\|\} \le \min\{\sqrt{H_R}, 1+v_R\}\,\|h\|$ (as $\|r_R\|^2 = 1 + \|r_R - \mathbf{1}\|^2$), $|a| \le \|f\|$ and $|\mathbb{E}_{\pi_R} h - b| \le v_R\,\|h\|$.

(b) $\|mT - \mathbb{J}\| \le m\,\|T - \mathbb{J}\| + |m-1|\,\|\mathbb{J}\|$.

(c) The marginal densities of a product are products, and the identity for $\|r_L - \mathbf{1}\|^2$ is Lemma 6.10(f). Maximal correlations of product couplings are maxima over the factors by Lemma 6.10(c). Finally $(u-r)(r-l) \ge 0$ gives $\mathbb{E}r^2 \le (u+l)\mathbb{E}r - ul = u + l - ul$, so $\mathbb{E}(r-1)^2 \le (u-1)(1-l)$. □

Remark 13.26 (Instances). In the applications ν is the common conditional reference of the outer rows at a fixed middle label, π the conditional fold row law, $\rho_\pi \le \rho_F$ a bound for its maximal correlation (Corollary 13.17, or in the upper rows its minimum with (27.6)), and m the density of the fold middle law relative to the common middle law at that label. (1) On a block of n_B primes with per-prime marginal densities in $[R^{-1}, R]$ and $m = 1$, (c) gives $H_L, H_R \le R^{n_B}$ and $v^2 \le \exp(n_B(R-1)^2) - 1$, whence the row coefficient $fr = R^{n_B}\rho_F + 2v + v^2$ of the twelve-label reference, with R the density-ratio bound R_j of Lemma 20.9(iii) (Lemma 22.1); a matching row has exact marginals, so $hr = \rho_E$. (2) With pointwise log-ratio bounds $|\log r_L|, |\log r_R| \le \ell_{\max}$ and $|\log m| \le \ell_{\mathrm{mid}}$, (a) and (b) give $e^{\ell_{\mathrm{mid}}}(e^{\ell_{\max}}\rho_F + e^{2\ell_{\max}}) - 1$, the common-reference fold row bound of the two-layer rows. (3) With middle ratio H, pointwise marginal inflation R_{m} and

chi-square defect υ, (b) gives the full-norm coefficient $f = [H(R_{\mathrm{m}}\rho_F + 2\upsilon + \upsilon^2) + H - 1]/r$ of the rows at large primes (Lemma 29.5), the alternative to their Schur coefficient ((F5)(b)). (4) With $|r_L - 1|, |r_R - 1| \le c_0$ pointwise, the first bound of (a) gives $\rho_\pi(1 + c_0) + c_0\sqrt{1 + c_0} + c_0$. (5) In the rows of the upper bands the fold and matching rows are bounded jointly on the actual matching chart (Lemma 16.23 and Corollary 16.24) and enter the completions through Lemma 17.5(f); part (b) is not used there.

Consequences and what to read next. The labellings, the count errors and the sequential counts (Definition 12.1, (12.1), (12.2), Lemma 12.2, Lemma 12.5); maximal correlation, Friedrichs angles, Kassabov's angle inequality, Peyre's tensorization (first statement) and conditional decorrelation (Definitions 4.3 and 4.4, Theorems 4.13 and 4.16, and Corollary 4.15); the norm of the square-shift matrices (Lemma 4.11(a)); square-affine invariance and uniform marginals (Lemma 5.16); the Markov tensor maximum, product densities and the tensor spectrum (Lemma 6.10(c),(f),(g)); and admissible majorants and block budgets (Lemma 4.18). Three standard facts are used without statement: the Schur product theorem, conditional Jensen, and the Rayleigh quotient characterization of the largest eigenvalue.

14. Spectral criteria for cut operators

What this section proves and how to read it. This section proves that a lower bound $A_p \succeq -gp^{-1/2}I$ for a cut operator (the normalized cut operator A_p of Lemma 13.3), for all $p \ge p_0$, can be verified by one finite positive-semidefiniteness test at p_0. Two routes are given: Route A via conditional laws of the opposite half, centring at the half-law mean; Route B via explicit uniform averages over allowed label sets, needing no completed law. A third method computes star-shaped parts exactly in a fixed invariant half law. *Centring at the half-law mean* means subtracting the mean under the half law itself, not under a uniform model; a *matrix part* is a block of an operator matrix, never a prime block. Inputs: Sections 13, 15 and 16. Outputs: the spectral caps used for completions in Part V. A first reading should follow the statements of the two criteria and Section 14.10; Fourier and coupling details are reference material. The list of Part V cuts using each route is in Section 3; it is not needed to read the criteria. Both routes also bound the mean-zero compression and give $\mathsf{A}_p \preceq (1 + O(1/p))I$ (A_p and $r = p^{-1/2}$ are as in the Conventions paragraph below), which is what allows tensorisation over a prime block. Cut codes such as L20 denote specific mirror symmetries of the twelve- or 72-label graphs; each is expanded where it is first used in Part V and need not be memorised here.

The method. The criteria split $L^2(\mu)$ into the constants, the *degree-one space* $\mathcal{V}$ of sums of mean-zero functions of one label, and the rest. Fourier analysis diagonalizes A_p on $\mathcal{V}$: on the mode ξ it acts as $r\mathbf{A}_0(\xi)$ up to an error of operator norm $O(r^2)$, quantified in Section 14.3, where $\mathbf{A}_0(\xi) = \chi(\xi)\varepsilon_G\Delta^E$ is the signed matrix of the cross edges (Sections 14.1 and 14.3). The coupling of $\mathcal{V}$ to the rest is controlled by a Gram matrix of the conditional densities of one opposite label; these densities lie within the factor R'_k of uniform on the explicit allowed sets $\Phi_k(x)$ (Definition 14.9) and are compared entrywise with an explicit model matrix (Sections 14.4 and 14.5). On the rest the cross kernel is expanded in characters of sets of cross pairs (Section 14.6). Centring the test function at its mean under the half law makes the empty set and every star vanish exactly there (Section 14.7). What remains are the nonstar sets, which are of second order and are bounded by the extraction lemma and by bounding the terms with a repeated label separately (Theorem 16.9 and Lemma 16.15). A Schur complement over the two remaining blocks gives the finite test (Section 14.8). Route B (Section 14.9) works with explicit Markov stars instead of the conditional law of the opposite half, and therefore needs no information on the completed cut law. Every error term is an admissible function of $r = p^{-1/2}$, so one exact test at a threshold covers all larger primes (Section 14.10). Class one ($p \equiv 1 \pmod 4$) needs the extra terms $\mathbf{H}^{\mathrm{cov}}$ and $\varepsilon_{\mathrm{par}}$, collected in Section 14.11. When the half law is one explicit invariant law, the star part can instead be

computed exactly in that law (Section 14.2); this is how the cut L20 of the uniform middle root, the cuts of the positive roots and the observed caps of the upper range are bounded.

Conventions. Throughout, $(G, \vartheta, U^+, \mathfrak{o})$ are spectral data (Definition 13.1; here ϑ is the cut involution, not the density exponent ϑ of the closings), with the notation of Section 13.1: $n = |U^+|$, the cross signs ς_{jk}, $N(k)$, $r_k = |N(k)|$, the internal degrees d_k^0, $d_k = d_k^0 + r_k$, $\Delta = \max_k d_k$, $\Delta^+ = \max_k d_k^0$, the cross pairs $E_\times$, $m_\times = |E_\times|$, the deletion degrees t_i and R_{cut}. Let p be an odd prime with $e_\Delta(p, 2n) < 1$, $r = p^{-1/2}$ and $q = 1/p$. Then $\mathcal{I}^+ = \mathrm{Inj}_p(G^+)$, $\mu = \lambda_p(G^+)$ is the half law, $\lambda = \lambda_p(G)$, m_p is the actual half marginal, K is the cut kernel (13.2), $Z = \mathbb{E}_{\mu\otimes\mu} K$, $\varrho = dm_p/d\mu = \mathsf{A}_p\mathbf{1}$ and $Q(F) = \langle F, \mathsf{A}_p F\rangle_\mu$ (Lemma 13.3). Inner products are conjugate-linear in the first variable. A_p has a real symmetric kernel, so its quadratic form is determined by real functions. $P_0 = I - \mathbb{J}$ is the projection onto the μ-mean-zero functions, and $u \otimes v$ is the operator $f \mapsto \langle v, f\rangle_\mu u$. The class of p is $\mathfrak{c} \in \{1, 3\}$, $p \equiv \mathfrak{c} \pmod 4$, and $\mathbf{1}_{p\equiv 1(4)}$ is the indicator of class one. Every error term below is an explicit expression in p; replacing p by r^{-2} makes it a function of r, and *admissible* means admissible in the sense of Definition 4.17, that is, nondecreasing in r.

14.1. **Single-variable Fourier analysis.** On $L^2(\mathbb{F}_p)$ we use the uniform probability, with inner product $\langle f, g\rangle = \mathbb{E}_a \overline{f(a)} g(a)$. The characters $e_\xi(a) = e(\xi a/p)$ ($\xi \in \mathbb{F}_p$) form an orthonormal basis, $\hat g(\xi) = \langle e_\xi, g\rangle$, and $L_0^2(\mathbb{F}_p)$ is the space of mean-zero functions. Let $\varepsilon_G = G_\chi/\sqrt{p}$ (Theorem 4.8(iii)), let C be the *Gauss operator* $(Ch)(a) = \mathbb{E}_{b\in\mathbb{F}_p}\chi(b-a)h(b)$ (the Paley operator P of Lemma 4.11, written C here and in Section 16), and for $\varsigma \in \{\pm 1\}$ let $(C_\varsigma h)(a) = \mathbb{E}_b \chi(\varsigma(b-a))h(b)$.

Lemma 14.1 (The Gauss operator)**.** *$Ce_\xi = c_\xi e_\xi$ with $c_0 = 0$ and $c_\xi = \chi(\xi)\varepsilon_G r$ for $\xi \neq 0$. Moreover $\varepsilon_G^2 = \chi(-1)$, so $\varepsilon_G \in \{\pm 1\}$ if $p \equiv 1 \pmod 4$ and $\varepsilon_G \in \{\pm i\}$ if $p \equiv 3 \pmod 4$; $|c_\xi| = r$ for $\xi \neq 0$; $\|C\| = r$; $C\mathbf{1} = C^*\mathbf{1} = 0$; $\overline{c_\xi} = c_{-\xi} = \chi(-1)c_\xi$; and $C_\varsigma = \chi(\varsigma)C$.*

Proof. $(Ce_\xi)(a) = \mathbb{E}_b\chi(b-a)e(\xi b/p) = e_\xi(a)\, p^{-1}\sum_t \chi(t)e(\xi t/p)$, and by Theorem 4.8(iii) the last sum is $\chi(\xi)G_\chi$ for every $\xi \in \mathbb{F}_p$. So $c_\xi = \chi(\xi)G_\chi/p = \chi(\xi)\varepsilon_G r$. By the same part of Theorem 4.8, $\varepsilon_G^2 = \chi(-1)$ and $|\varepsilon_G| = 1$, so $|c_\xi| = r$ for $\xi \neq 0$; $\overline{G_\chi} = \chi(-1)G_\chi$ gives $\overline{c_\xi} = \chi(-1)c_\xi$, and $\chi(-\xi) = \chi(-1)\chi(\xi)$ gives $c_{-\xi} = \chi(-1)c_\xi$. Since C is the Paley operator P, Lemma 4.11(c) gives $\|C\| = r$ and $C\mathbf{1} = C^*\mathbf{1} = 0$. Finally $\chi(\varsigma(b-a)) = \chi(\varsigma)\chi(b-a)$. □

The group $\mathrm{Aff}_p = \{z \mapsto az + c :\ a \in Q_p^*,\ c \in \mathbb{F}_p\}$ of square-affine maps acts diagonally on $\mathbb{F}_p^2$ and on $\mathbb{F}_p^W$ for every finite set W. Its orbits on $\mathbb{F}_p^2$ are the diagonal, $\Omega_p = \{(a,b) : b - a \in Q_p^*\}$ and $\Omega_p' = \{(a,b) : b - a \notin Q_p^* \cup \{0\}\}$. Indeed the diagonal is one orbit by translation; given $(a,b), (a',b') \in \Omega_p$, the map $z \mapsto sz + c$ with $s = (b'-a')/(b-a) \in Q_p^*$ and $c = a' - sa$ sends (a,b) to (a',b'); and likewise on Ω_p'. An invariant kernel is therefore a combination of three indicator functions, and its eigenvalues are the formulas of Lemma 14.2(a) below, through the c_ξ of Lemma 14.1.

Lemma 14.2 (Aff_p-invariant kernels)**.**

(a) *Let $\kappa\colon \mathbb{F}_p^2 \to \mathbb{C}$ satisfy $\kappa(\gamma a, \gamma b) = \kappa(a,b)$ for $\gamma \in \mathrm{Aff}_p$. Then $\kappa = \alpha\mathbf{1}_{\mathrm{diag}} + \beta_+\mathbf{1}_{\Omega_p} + \beta_-\mathbf{1}_{\Omega_p'}$ for constants $\alpha, \beta_\pm$, and the operator $(\mathcal{K}_\kappa h)(a) = \mathbb{E}_b\kappa(a,b)h(b)$ satisfies $\mathcal{K}_\kappa e_\xi = \lambda_\xi e_\xi$ with*

$$\lambda_0 = \frac{\alpha}{p} + \frac{p-1}{p}\cdot\frac{\beta_+ + \beta_-}{2}, \qquad \lambda_\xi = \frac{1}{p}\Big(\alpha - \frac{\beta_+ + \beta_-}{2}\Big) + c_\xi\,\frac{\beta_+ - \beta_-}{2} \quad (\xi \neq 0).$$

(b) (Pair laws.) *Let (u,v) be an $\mathbb{F}_p^2$-valued random pair whose law is Aff_p-invariant, $\pi^= = \mathbb{P}(u = v)$ and $D = \mathbb{E}\chi(v-u)$. The law of (u,v) is $\pi^=\,\mathrm{Unif}(\mathrm{diag}) + \pi^+\,\mathrm{Unif}(\Omega_p) + \pi^-\,\mathrm{Unif}(\Omega_p')$ with $\pi^+ + \pi^- = 1 - \pi^=$ and $\pi^+ - \pi^- = D$, and for all $g, h \in L_0^2(\mathbb{F}_p)$*

$$\mathbb{E}\big[\overline{g(u)}h(v)\big] = \pi^=\langle g, h\rangle - \frac{1-\pi^=}{p-1}\langle g, h\rangle + \frac{p}{p-1}D\langle g, Ch\rangle\,. \tag{14.1}$$

(c) (Covariant fields.) *Let ν be a probability measure on a finite set $\mathcal{Y}$ on which Aff_p acts preserving ν, and let $\mathcal{V}_1,\dots,\mathcal{V}_s\colon \mathcal{Y}\times\mathbb{F}_p\to\mathbb{C}$ satisfy $\mathcal{V}_k(\gamma y,\gamma a)=\mathcal{V}_k(y,a)$ for $\gamma\in\mathrm{Aff}_p$. Put $\Gamma_{kl}=\mathbb{E}_\nu\mathbb{E}_a\overline{\mathcal{V}_k}\mathcal{V}_l$, $M_{kl}=\mathbb{E}_\nu\overline{\mathbb{E}_a\mathcal{V}_k}\,\mathbb{E}_a\mathcal{V}_l$ and $\mathcal{P}_{kl}=\mathbb{E}_\nu\langle\mathcal{V}_k(y,\cdot),C\mathcal{V}_l(y,\cdot)\rangle$. Then for every $\xi\neq0$ and $w\in\mathbb{C}^s$, with $\mathcal{V}_w=\sum_k w_k\mathcal{V}_k$,*

$$\mathbb{E}_\nu\bigl|\mathbb{E}_a[\mathcal{V}_w(y,a)e_\xi(a)]\bigr|^2=\frac{w^*(\Gamma-M+\chi(\xi)G_\chi\mathcal{P})w}{p-1}.$$

(d) (Commutant and invariant functions, $p\equiv3\pmod4$.) *(i) Every real operator T on $L^2(\mathbb{F}_p)$ that commutes with the operators $h\mapsto h\circ\gamma$, $\gamma\in\mathrm{Aff}_p$, is $\alpha_TI+\beta_T\mathbb{J}+\gamma_TC$ with real $\alpha_T,\beta_T,\gamma_T$. On $L^2_0(\mathbb{F}_p)$ it is $\alpha_TI+\gamma_TC$, a normal operator of norm $(\alpha_T^2+q\gamma_T^2)^{1/2}$, with $\alpha_T=\mathrm{tr}(T)/(p-1)$ and $\gamma_T=\mathrm{tr}(C^*T)/(1-q)$ (traces on $L^2_0(\mathbb{F}_p)$), and it is self-adjoint if and only if $\gamma_T=0$. (ii) Hence, if Aff_p acts on $(\mathcal{Y},\nu)$ as in (c) and $Y,Y'\colon L^2_0(\mathbb{F}_p)\to L^2(\nu)$ map real functions to real functions and intertwine the two actions, then $T=Y^*Y'$ is $\alpha_TI+\gamma_TC$ on $L^2_0(\mathbb{F}_p)$ as in (i), and $Y^*Y=\mathrm{tr}(Y^*Y)/(p-1)\cdot I$. (iii) If the law of a random $x=(x_w)_{w\in W}\in\mathbb{F}_p^W$ is invariant under the simultaneous translations $x\mapsto x+c\mathbf{1}$ and f is a function on $\mathbb{F}_p^W$ with $f(x+c\mathbf{1})=f(x)$ for all c and $\mathbb{E}f(x)=0$, then $\mathbb{E}[f(x)\mid x_w]=0$ for every $w\in W$.*

Proof. (a) An invariant κ is constant on the three orbits. By translation invariance $\kappa(a,a+d)=\tilde\kappa(d)$ depends only on d, so $(\mathcal{K}_\kappa e_\xi)(a)=e_\xi(a)\mathbb{E}_d\tilde\kappa(d)e(\xi d/p)$. For $\xi\neq0$, $\sum_{d\in Q_p^*}e(\xi d/p)=\frac12\sum_{d\neq0}(1+\chi(d))e(\xi d/p)=\frac12(-1+\chi(\xi)G_\chi)$ by Theorem 4.8(iii), and likewise with $-\chi$ for the non-squares; with $\chi(\xi)G_\chi/p=c_\xi$ this gives λ_ξ. For $\xi=0$ the sum is $\alpha+\frac{p-1}{2}(\beta_++\beta_-)$.

(b) The law is invariant, so it is constant on the orbits; $\chi(v-u)$ is 0, +1, −1 on the diagonal, on Ω_p and on Ω_p', which gives the decomposition. The function $\kappa(a,b)=p^2\mathbb{P}(u=a,v=b)$ is invariant, with $\alpha=p\pi^=$ and $\beta_\pm=2p\pi^\pm/(p-1)$, since $|\Omega_p|=|\Omega_p'|=p(p-1)/2$. For $g,h\in L^2_0$, $\mathbb{E}[\overline{g(u)}h(v)]=\langle g,\mathcal{K}_\kappa h\rangle=\sum_{\xi\neq0}\lambda_\xi\hat g(\xi)\hat h(\xi)$, and by (a) and Lemma 14.1, $\lambda_\xi=\pi^=-\frac{1-\pi^=}{p-1}+\frac{p}{p-1}Dc_\xi$. Since $\langle g,h\rangle=\sum_\xi\hat g\hat h$ and $\langle g,Ch\rangle=\sum_\xi c_\xi\hat g\hat h$, this is (14.1).

(c) Put $\kappa(a,b)=\mathbb{E}_\nu[\overline{\mathcal{V}_w(y,a)}\mathcal{V}_w(y,b)]$. By the invariance of ν, κ is Aff_p-invariant, and the left side is $\mathbb{E}_{a,b}\kappa(a,b)\overline{e_\xi(a)}e_\xi(b)=\langle e_\xi,\mathcal{K}_\kappa e_\xi\rangle=\lambda_\xi$. Now $w^*\Gamma w=\mathbb{E}_a\kappa(a,a)=\alpha$; $w^*Mw=\mathbb{E}_{a,b}\kappa(a,b)=\lambda_0$; and $w^*\mathcal{P}w=\mathbb{E}_{a,b}\chi(b-a)\kappa(a,b)=\frac{p-1}{2p}(\beta_+-\beta_-)$. Eliminating $\beta_\pm$ in (a), $\lambda_\xi=(\alpha-\lambda_0+\chi(\xi)G_\chi w^*\mathcal{P}w)/(p-1)$.

(d) (i) $T=\mathcal{K}_\kappa$ with $\kappa(a,b)=pT(a,b)$, the matrix of T scaled by p; T commutes with the square-affine maps exactly when κ is invariant, and κ is real when T is. By (a), $\kappa=\alpha\mathbf{1}_{\mathrm{diag}}+\beta_+\mathbf{1}_{\Omega_p}+\beta_-\mathbf{1}_{\Omega_p'}$ with real coefficients, and since $\mathbf{1}_{\Omega_p}(a,b)=\frac12(1-\mathbf{1}_{a=b}+\chi(b-a))$ and $\mathbf{1}_{\Omega_p'}(a,b)=\frac12(1-\mathbf{1}_{a=b}-\chi(b-a))$,

$$T=\Bigl(\alpha-\frac{\beta_++\beta_-}{2}\Bigr)\frac{I}{p}+\frac{\beta_++\beta_-}{2}\mathbb{J}+\frac{\beta_+-\beta_-}{2}\,C.$$

On $L^2_0(\mathbb{F}_p)$, $\mathbb{J}=0$, and C has the eigenvalues $c_\xi=\chi(\xi)\varepsilon_G r\in\{\pm ir\}$, $\xi\neq0$, each value occurring $(p-1)/2$ times (Lemma 14.1). So $\alpha_TI+\gamma_TC$ is normal with the eigenvalues $\alpha_T\pm i\gamma_Tr$, of modulus $(\alpha_T^2+q\gamma_T^2)^{1/2}$; $\mathrm{tr}\,C=\sum_{\xi\neq0}c_\xi=0$ and $\mathrm{tr}(C^*C)=(p-1)q=1-q$ give the two traces; and $C^*=-C$ shows that $(\alpha_TI+\gamma_TC)^*=\alpha_TI-\gamma_TC$, which equals $\alpha_TI+\gamma_TC$ only if $\gamma_T=0$. (ii) For $\gamma\in\mathrm{Aff}_p$ let U_γ and $\tilde U_\gamma$ be the two unitary actions, with $YU_\gamma=\tilde U_\gamma Y$ and $Y'U_\gamma=\tilde U_\gamma Y'$. Then $Y^*\tilde U_\gamma=U_\gamma Y^*$, so Y^*Y' commutes with every U_γ and is real; extended by zero on the constants (which the U_γ fix) it is an operator as in (i). As Y^*Y is self-adjoint, its coefficient γ_T vanishes. (iii) Let $g(t)=\mathbb{E}[f(x)\mid x_w=t]$. The law of x_w is translation invariant, hence uniform, and replacing x by $x+c\mathbf{1}$ gives $g(t)=g(t-c)$ for all c; so g is constant, equal to $\mathbb{E}f(x)=0$. □

By Lemma 13.3(d) the laws λ, μ and m_p are invariant under Aff_p and have uniform one-vertex marginals, and so has every pair law obtained from them by taking two coordinates; part (b) applies to all of them. Put

$$\delta_l^{\times} = \frac{e_{d_l}(p,2n) + e_{d_l+1}(p,2n)}{1 - e_{d_l}(p,2n)}, \qquad \delta_l^{\mu} = \frac{e_{d_l^0}(p,n) + e_{d_l^0+1}(p,n)}{1 - e_{d_l^0}(p,n)} \qquad (l \in U^+). \tag{14.2}$$

Lemma 14.3 (Small pair constants)**.** *Assume $e_\Delta(p,2n) < 1$.*

(a) *For $k,l \in U^+$ with $\varsigma_{kl} = 0$ (including $k = l$), $|\mathbb{E}_\lambda \chi(x'_l - x_k)| \le \min(\delta_k^{\times}, \delta_l^{\times})$. If $\varsigma_{kl} \neq 0$, then $\mathbb{E}_\lambda \chi(x'_l - x_k) = \chi(\varsigma_{kl})$. In all cases $\mathbb{E}_\lambda \chi(x'_k - x_l) = \chi(-1)\mathbb{E}_\lambda \chi(x'_l - x_k)$.*

(b) *For distinct non-adjacent $k,l \in U^+$, $|\mathbb{E}_\mu \chi(x_l - x_k)| \le \min(\delta_k^{\mu}, \delta_l^{\mu})$. If (k,l) is an edge of G^+ the value is 1; if (l,k) is an edge it is $\chi(-1)$.*

Proof. (a) Let $\varsigma_{kl} = 0$ and condition on all labels y_w, $w \neq \vartheta l$. By Lemma 12.6(b) for G ($2n$ vertices), the label $x'_l = y_{\vartheta l}$ is uniform on its admissible set $\mathcal{A}$: the values outside the other $2n-1$ labels that satisfy the d_l constraints with the distinct labels of the neighbours of ϑl. By Lemma 12.2(d), $|\mathcal{A}| \in p2^{-d_l}[1 - e_{d_l}, 1 + e_{d_l}]$, with $e_t = e_t(p,2n)$. The vertex k is not a neighbour of ϑl (this is $\varsigma_{kl} = 0$; for $k = l$ it is $\varsigma_{ll} = 0$), so x_k is distinct from the roots of these constraints and lies in the excluded set. The subset $\mathcal{A}^+ = \{a \in \mathcal{A} : a - x_k \in Q_p^*\}$ is cut out by $d_l + 1$ constraints with distinct roots, so $2|\mathcal{A}^+| \in p2^{-d_l}[1 - e_{d_l+1}, 1 + e_{d_l+1}]$. Since $x'_l \neq x_k$, the conditional expectation of $\chi(x'_l - x_k)$ is $(2|\mathcal{A}^+| - |\mathcal{A}|)/|\mathcal{A}|$, of modulus at most $(e_{d_l} + e_{d_l+1})/(1 - e_{d_l}) = \delta_l^{\times}$; average over the conditioning. The swap invariance of λ gives $\mathbb{E}\chi(x'_l - x_k) = \mathbb{E}\chi(x_l - x'_k) = \chi(-1)\mathbb{E}\chi(x'_k - x_l)$, the last identity of (a); applying the first bound to the pair (l,k) ($\varsigma_{lk} = -\varsigma_{kl} = 0$) gives the bound $\delta_k^{\times}$. If $\varsigma_{kl} \neq 0$, the cross constraint $\varsigma_{kl}(x'_l - x_k) \in Q_p^*$ holds λ-almost surely, so $\chi(x'_l - x_k) = \chi(\varsigma_{kl})$. (b) The same argument for G^+ (n vertices, maximum degree Δ^+, $e_{\Delta^+}(p,n) \le e_\Delta(p,2n) < 1$), with d_l^0 constraints and at most $n-1$ excluded labels. On an edge (k,l), $x_l - x_k \in Q_p^*$ almost surely. □

14.2. **Exact star factorisation in a fixed invariant half law.** The criteria of the following subsections compare a cut kernel with product laws: route A through the conditional laws of single labels of the opposite half and an explicit model matrix, route B through explicit normalized stars on allowed sets. When the half law is one fixed law invariant under Aff_p, the stars need no comparison. Every Gram $\mathcal{F}_\alpha^* \mathcal{F}_\beta$ of the features defined below is then a combination of I and C (Lemma 14.2(d)), with coefficients given by two traces, and the complete star part of a soft cross kernel (defined in Corollary 14.5 below) factors exactly through these features. This subsection records the Grams (Lemma 14.4), the star matrix (Corollary 14.5), the projections of unobserved features (Lemma 14.6) and the real form of the phase gates (the positive-semidefiniteness tests at the phases $\pm i$; Remark 14.7). Its uses are listed in the opening of this section; it uses neither criterion.

Setting. In this subsection no spectral data are fixed. Let $p \equiv 3 \pmod 4$, $\mathcal{H} = L_0^2(\mathbb{F}_p)$ and $\hat{P} = C/r$, the normalized Paley operator of Section 4; by Lemma 14.1, $\hat{P} e_\xi = \pm i e_\xi$ for $\xi \neq 0$, so $\hat{P}^* = -\hat{P}$ and $\hat{P}^2 = -I$ on $\mathcal{H}$. Let U be a finite set and μ a law on the injective tuples $x = (x_i)_{i \in U} \in \mathbb{F}_p^U$ that is invariant under the diagonal action of Aff_p, for instance the uniform law on the injective strict-square labellings of an oriented graph (Definition 12.1) or a marginal of such a law. As in the rest of the section, $P_0 = I - \mathbb{J}$ is the projection onto the μ-mean-zero functions. For $R \subseteq U$, $z \in \mathbb{F}_p$ and $h \in \mathcal{H}$ put

$$\psi_R(z,x) = \prod_{j \in R} \chi(z - x_j), \qquad w_R(x) = \mathbb{E}_z \psi_R(z,x),$$
$$(W_R h)(x) = \mathbb{E}_z \psi_R(z,x) h(z), \qquad (V_i h)(x) = h(x_i),$$

with z uniform on $\mathbb{F}_p$. The sets $S \subseteq U$ with $|S| \geq 2$ are the *supports*, w_S is the *star mean* of S, and the maps V_i ($i \in U$) and W_S/r are the *actual features* of μ, the singletons and the stars. We write $\mathcal{F}_\alpha$, $\alpha \in U \sqcup \mathcal{S}$, for the features of a set $\mathcal{S}$ of supports ($\mathcal{F}_i = V_i$, $\mathcal{F}_S = W_S/r$), and call $\mathcal{F}_\alpha^* \mathcal{F}_\beta$ their *Gram*.

Lemma 14.4 (Complete actual star synthesis)**.** *In this setting the V_i are isometries of $\mathcal{H}$ into $L_0^2(\mu)$, the W_S map $\mathcal{H}$ into $L_0^2(\mu)$, and the ranges of all these maps are orthogonal to the functions of x that are invariant under the simultaneous translations $x \mapsto x + c\mathbf{1}$.*

(a) (Diagonal Grams.) *On $\mathcal{H}$,*

$$W_S^* W_S = \frac{q(1 - |S|q - \mathbb{E}_\mu w_S^2)}{1-q}\, I \preceq qI. \tag{14.3}$$

(b) (Cross Grams.) *For supports S, T put $\mathcal{A}_{ST} = \mathbb{E}_\mu \mathbb{E}_z \psi_S \psi_T$, $\mathcal{C}_{ST} = \mathbb{E}_\mu w_S w_T$ and $\mathcal{B}_{ST} = \mathbb{E}_\mu \langle \psi_S(\cdot, x), C\psi_T(\cdot, x) \rangle$. Then, on $\mathcal{H}$,*

$$W_S^* W_T = \frac{q(\mathcal{A}_{ST} - \mathcal{C}_{ST})}{1-q}\, I + \frac{\mathcal{B}_{ST}}{1-q}\, C, \qquad \|W_S^* W_T\| = \frac{q}{1-q}\Big((\mathcal{A}_{ST} - \mathcal{C}_{ST})^2 + \frac{\mathcal{B}_{ST}^2}{q} \Big)^{1/2}. \tag{14.4}$$

(c) (Pivot bound.) *Let $S \neq T$, $k = |S|$, $l = |T|$, $c = |S \cap T|$, $d = |S \triangle T|$, and $a_1 = 0$, $a_2 = q$, $a_d = (d-1)r$ for $d \geq 3$. Let $j \in S \triangle T$ be a* pivot*: given the other coordinates, every value of x_j has conditional probability at most D_j/p. Then*

$$\frac{\|W_S^* W_T\|}{q} \leq \frac{\big([a_d + (c + (k-1)(l-1))q]^2 + q(D_j - 1) \big)^{1/2}}{1-q}. \tag{14.5}$$

(d) (Singleton entries.) *Let $k = |S|$. Then*

$$\frac{(1-q)\,\|V_i^* W_S\|}{r} \leq \begin{cases} \big(r^2(1 + (k-1)r)^2 + k^2 q \big)^{1/2}, & i \notin S, \\ q\sqrt{1+q}, & i \in S,\ k = 2, \\ 2\sqrt{2}\, q, & i \in S,\ k = 3, \\ \big((k-1)^2 q^2 + ((k-2)r + q)^2 \big)^{1/2}, & i \in S,\ k \geq 4, \end{cases} \tag{14.6}$$

and $\|V_i^* V_j\| \leq r\sqrt{1+q}/(1-q)$ for $i \neq j$.

(e) (Disjoint supports and overlapping pair supports.) *If ρ_S bounds the maximal correlation (Definition 4.3) of x_S and $x_{U \setminus S}$, then $\|(W_S/r)^*(W_T/r)\| \leq \rho_S$ for every support T disjoint from S. Let $S = \{a, b\}$ and $T = \{a, b'\}$ with a, b, b' distinct, and let ρ bound, for every value of x_a, the conditional maximal correlation of x_b and $x_{U \setminus \{a,b\}}$ given x_a. Then*

$$\|(W_S/r)^*(W_T/r)\| \leq \frac{2\rho}{1-q} + \frac{q(q+r)^2}{(1-q)^2}. \tag{14.7}$$

(f) (Synthesis.) *Let $\mathcal{S}$ be a set of supports and $\mathcal{F} \colon \mathcal{H}^U \oplus \mathcal{H}^{\mathcal{S}} \to L_0^2(\mu)$, $\mathcal{F}((h_\alpha)_\alpha) = \sum_\alpha \mathcal{F}_\alpha h_\alpha$, the synthesis of its features. If a symmetric nonnegative matrix N indexed by $U \sqcup \mathcal{S}$ satisfies $\|\mathcal{F}_\alpha^* \mathcal{F}_\beta\| \leq N_{\alpha\beta}$ for $\alpha \neq \beta$, then $\|\mathcal{F}\|^2 \leq 1 + \lambda_{\max}(N) \leq 1 + \max_\alpha \sum_\beta N_{\alpha\beta}$.*

Proof. Equivariance and traces. The group Aff_p acts unitarily on $\mathcal{H}$ by $h \mapsto h \circ \gamma^{-1}$ and, since μ is invariant, on $L^2(\mu)$ by $F \mapsto F \circ \gamma^{-1}$ (diagonally). For $\gamma(z) = sz + c$ with $s \in Q_p^*$, $\chi(\gamma z - \gamma y) = \chi(s)\chi(z-y) = \chi(z-y)$, so $\psi_S(\gamma z, \gamma x) = \psi_S(z, x)$, and V_i, W_S intertwine the two actions and map real functions to real functions. Both are of the form $(Yh)(x) = \mathbb{E}_z \phi(z, x) h(z)$ with a real kernel ϕ: $\phi = \psi_S$ for W_S, and $\phi(z, x) = p\mathbf{1}_{z = x_i}$ for V_i. By Lemma 14.2(d), the Gram $T = Y^* Y'$ of two such maps is, on $\mathcal{H}$, $\alpha_T I + \gamma_T C$ with $\alpha_T = \mathrm{tr}(T)/(p-1)$, $\gamma_T = \mathrm{tr}(C^* T)/(1-q)$ and $\|T\| = (\alpha_T^2 + q\gamma_T^2)^{1/2}$, and $\gamma_T = 0$ if $Y = Y'$. If (e_m) is an orthonormal basis of $\mathcal{H}$, then

$(Ye_m)(x) = \langle \phi(\cdot, x), e_m\rangle$, and $\sum_m \overline{\langle f, e_m\rangle}\,\langle g, e_m\rangle = \langle g, f - \mathbb{E}_z f(z)\rangle$; with $\langle C^*\phi, e_m\rangle = \langle \phi, Ce_m\rangle$ this gives, for kernels ϕ, ϕ' of Y, Y' and traces on $\mathcal{H}$,

$$\operatorname{tr}(Y^*Y') = \mathbb{E}_\mu(\mathbb{E}_z\phi\phi' - \mathbb{E}_z\phi\,\mathbb{E}_z\phi'), \qquad \operatorname{tr}(C^*Y^*Y') = \mathbb{E}_\mu\,\langle \phi(\cdot, x), C\phi'(\cdot, x)\rangle\,. \tag{14.8}$$

Ranges. By translation invariance every one-vertex marginal of μ is uniform, so V_i is an isometry and $\mathbb{E}_\mu V_i h = \mathbb{E}h = 0$. Let $F \in L^2(\mu)$ satisfy $F(x + c\mathbf{1}) = F(x)$. By Lemma 14.2(d) applied to $F - \mathbb{E}_\mu F$, $\mathbb{E}_\mu[F \mid x_i] = \mathbb{E}_\mu F$, so $\langle F, V_i h\rangle_\mu = \overline{\mathbb{E}_\mu F}\,\mathbb{E}h = 0$. Next $\langle F, W_S h\rangle_\mu = \mathbb{E}_z h(z)\,\mathbb{E}_\mu[\overline{F(x)}\psi_S(z, x)]$, and the inner expectation does not depend on z, since $\psi_S(z + c, x + c\mathbf{1}) = \psi_S(z, x)$ and μ and F are translation invariant; as $\mathbb{E}h = 0$, $\langle F, W_S h\rangle_\mu = 0$. For $F = \mathbf{1}$ this says that $W_S h$ has μ-mean zero.

(a) In (14.8) take $\phi = \phi' = \psi_S$. As x is injective, $\psi_S(z, x)^2 = 1$ except at the $|S|$ labels x_j, $j \in S$, so $\operatorname{tr}(W_S^* W_S) = 1 - |S|q - \mathbb{E}_\mu w_S^2$; divide by $p - 1 = (1 - q)/q$. The bound by qI holds already for $|S| \ge 1$, since $\mathbb{E}_\mu w_S^2 \ge 0$.

(b) Take $\phi = \psi_S$, $\phi' = \psi_T$ in (14.8): the traces are $\mathcal{A}_{ST} - \mathcal{C}_{ST}$ and $\mathcal{B}_{ST}$.

(c) Since $\chi(z - x_j)^2 = 1 - \mathbf{1}_{z=x_j}$ and the labels are distinct,

$$\psi_S\psi_T = \psi_{S\triangle T}(z, x) - \sum_{j \in S\cap T} \mathbf{1}_{z=x_j}\,\psi_{S\triangle T}(x_j, x),$$

so $|\mathbb{E}_z\psi_S\psi_T| \le |\mathbb{E}_z\psi_{S\triangle T}| + cq$. The average of a product of d characters with distinct roots is 0 for $d = 1$, $-q$ for $d = 2$ (4.2), and at most $(d - 1)r$ in modulus for $d \ge 3$ (Theorem 4.7); so $|\mathcal{A}_{ST}| \le a_d + cq$. The same facts give $|w_S| \le (k - 1)r$ for every support ($w_S = -q$ if $k = 2$), so $|\mathcal{C}_{ST}| \le (k - 1)(l - 1)q$. For $\mathcal{B}_{ST}$ let first $j \in S \setminus T$. Conditionally on $x_{U\setminus\{j\}}$ let ν_j be the law of x_j and $\phi_j(z) = \mathbb{E}[\chi(z - x_j) \mid x_{U\setminus\{j\}}] = \sum_t \nu_j(t)\chi(z - t)$. As ψ_T and $\psi_{S\setminus\{j\}}$ do not depend on x_j, $\mathcal{B}_{ST} = \mathbb{E}_\mu\left\langle \phi_j\psi_{S\setminus\{j\}}, C\psi_T\right\rangle$. By (4.2), $\|\phi_j\|^2 = \sum_{t,t'} \nu_j(t)\nu_j(t')(\mathbf{1}_{t=t'} - q) = \sum_t \nu_j(t)^2 - q \le q(D_j - 1)$; with $|\psi_{S\setminus\{j\}}| \le 1$, $\|C\psi_T\| \le r$ (Lemma 14.1) and Cauchy–Schwarz, $|\mathcal{B}_{ST}| \le q(D_j - 1)^{1/2}$. If $j \in T \setminus S$, write $\mathcal{B}_{ST} = \mathbb{E}_\mu\,\langle C^*\psi_S, \psi_T\rangle$ and argue in the same way, as $\|C^*\| = r$. Insert these bounds in the norm of (b).

(d) In (14.8) take $\phi = p\mathbf{1}_{z=x_i}$ and $\phi' = \psi_S$: $\operatorname{tr}(V_i^* W_S) = \mathbb{E}_\mu(\psi_S(x_i, x) - w_S)$ and $\operatorname{tr}(C^* V_i^* W_S) = \mathbb{E}_\mu(C\psi_S(\cdot, x))(x_i) = \mathbb{E}_\mu\mathbb{E}_z\chi(z - x_i)\psi_S(z, x) =: \beta_i$. So $(1 - q)\,\|V_i^* W_S\|/r = \big(q(\mathbb{E}_\mu\psi_S(x_i, x) - \mathbb{E}_\mu w_S)^2 + \beta_i^2\big)^{1/2}$. If $i \notin S$, $|\psi_S(x_i, x)| \le 1$, $|w_S| \le (k - 1)r$, and $\beta_i = \mathbb{E}_\mu w_{S\cup\{i\}}$ has modulus at most kr by Theorem 4.7. If $i \in S$, $\psi_S(x_i, x) = 0$, and $\chi(z - x_i)\psi_S(z, x) = (1 - \mathbf{1}_{z=x_i})\psi_{S\setminus\{i\}}(z, x)$ gives $\beta_i = \mathbb{E}_\mu\big(w_{S\setminus\{i\}} - q\psi_{S\setminus\{i\}}(x_i, x)\big)$: for $k = 2$, $w_S = -q$ and $w_{S\setminus\{i\}} = 0$, so the trace is q and $|\beta_i| \le q$; for $k = 3$, $|w_S| \le 2r$ and $w_{S\setminus\{i\}} = -q$, so $|\beta_i| \le 2q$; for $k \ge 4$, $|w_S| \le (k - 1)r$ and $|\beta_i| \le (k - 2)r + q$. These give the four cases. For $i \ne j$ take $\phi = p\mathbf{1}_{z=x_i}$, $\phi' = p\mathbf{1}_{z=x_j}$: the traces are -1 (injectivity) and $\mathbb{E}_\mu\chi(x_j - x_i) \in [-1, 1]$, so $\|V_i^* V_j\| \le (q^2 + q)^{1/2}/(1 - q)$.

(e) For disjoint S, T and $h, h' \in \mathcal{H}$, $W_S h$ and $W_T h'$ are functions of x_S and of $x_{U\setminus S}$ with μ-mean zero, of norms at most $r\,\|h\|$ and $r\,\|h'\|$ by (a); so $|\,\langle W_S h, W_T h'\rangle_\mu\,| \le \rho_S q\,\|h\|\,\|h'\|$, which is the first claim. Now let $S = \{a, b\}$, $T = \{a, b'\}$, and write $(W_S h)(x) = F_{x_a}(x_b)$ and $(W_T h')(x) = G_{x_a}(x_{b'})$. Fix $x_a = \alpha$. The pair (x_a, x_b) has an invariant law without diagonal mass, so by Lemma 14.2(b) the conditional law of x_b has the density $(1 - \mathbf{1}_{\beta=\alpha} + s_{ab}\chi(\beta - \alpha))/(1 - q) \le 2/(1 - q)$ in β relative to the uniform law, with $s_{ab} = \mathbb{E}\chi(x_b - x_a) \in [-1, 1]$. As $F_\alpha(\beta) = \mathbb{E}_t\chi(t - \beta)\chi(t - \alpha)h(t) = \big(C(\chi(\cdot - \alpha)h)\big)(\beta)$, $\mathbb{E}[|F_\alpha(x_b)|^2 \mid x_a = \alpha] \le 2r^2\,\|h\|^2/(1 - q)$. Summing the density against F_α with (4.2) and $\chi(-1) = -1$ gives the conditional mean

$$m_F(\alpha) = \mathbb{E}[F_\alpha(x_b) \mid x_a = \alpha] = \frac{q^2 h(\alpha) + q s_{ab}(Ch)(\alpha)}{1 - q},$$

of norm at most $q(q + r)\,\|h\|/(1 - q)$ in α; likewise for G and h'. Split $F_\alpha = m_F(\alpha) + F_\alpha^\circ$ and $G_\alpha = m_G(\alpha) + G_\alpha^\circ$ into conditional means and fluctuations. The fluctuations are functions of

x_b and of $x_{b'}$ with conditional mean zero given x_a, so their conditional inner product is at most $\rho \cdot 2r^2 \|h\| \|h'\| / (1-q)$ in modulus, and the means contribute at most $q^2(q+r)^2 \|h\| \|h'\| / (1-q)^2$. Divide the sum by $r^2 = q$.

(f) The diagonal Grams satisfy $V_i^* V_i = I$ and $(W_S/r)^*(W_S/r) \preceq I$ by (a), so the matrix $I + N$ bounds the norms of all Grams. As $\|\mathcal{F}\|^2 = \|\sum_\alpha \mathcal{F}_\alpha \mathcal{F}_\alpha^*\|$, Lemma 13.8(a) with unit weights gives the bound by $\lambda_{\max}(I+N) = 1 + \lambda_{\max}(N)$, and by the largest row sum of $I + N$. □

Lemma 14.4 is an operator inequality for the whole synthesis: nested and overlapping supports are allowed, and no orthogonality of distinct supports is assumed; the only loss beyond the diagonal Grams is the term $\lambda_{\max}(N)$ in (f). For the uniform law on the injective strict-square labellings of a graph on n vertices, the label x_j is conditionally uniform on its admissible set (Lemma 12.6(b)), so Lemma 12.2(d) gives the pivot density $D_j = 2^t/(1 - e_t(p,n))$, with t the degree of j, whenever $e_t(p,n) < 1$; the pivot minimizing the right side of (14.5) is chosen for each pair of supports. The next statement is the reason for the frame: the star part of a soft cross kernel is a fixed matrix in these features.

Corollary 14.5 (Complete star matrix)**.** *In this setting let $D = (D_{ij})_{i,j \in U}$ be a real matrix with $|D_{ij}| \le 1$, and let*

$$K_D(x,y) = \prod_{i,j \in U} (1 + D_{ij} \chi(y_j - x_i))$$

be the soft cross kernel *of D between two copies x, y of the half, and write K_D also for the operator on $L^2(\mu)$ with kernel K_D relative to $\mu \otimes \mu$. In the expansion of the product, the* complete star part *is the sum of the terms of the single pairs (i,j), of the* left stars $\prod_{j \in S} D_{ij} \chi(y_j - x_i)$ *with centre i and leaves S, and of the* right stars $\prod_{i \in S} D_{ij} \chi(y_j - x_i)$ *with centre j and leaves S ($|S| \ge 2$). Let $\mathcal{F}$ be the synthesis of Lemma 14.4(f) for a set $\mathcal{S}$ that contains every set of leaves. Then the complete star part, compressed by P_0 on both sides, is*

$$r\, \mathcal{F} M(\hat{P}) \mathcal{F}^*,$$

where $M(\hat{P})$ is the block operator matrix indexed by $U \sqcup \mathcal{S}$ with the singleton block $D \otimes \hat{P}$, the entries $(-1)^{|S|} \prod_{j \in S} D_{ij}$ (row i, column S) and $\prod_{i \in S} D_{ij}$ (row S, column j), each times $I_{\mathcal{H}}$, and zero block between supports. Let $M(i)$ be the complex matrix obtained by replacing $\hat{P}$ with the imaginary unit i. Then $\left\| r \mathcal{F} M(\hat{P}) \mathcal{F}^* \right\| \le r \|\mathcal{F}\|^2 \|M(i)\|$; the same holds for the symmetric and the skew part, with $\operatorname{Sym} M(i)$ *and* $\operatorname{Skew} M(i)$*, and for the difference of two soft kernels in one synthesis, with the difference of their matrices.*

Proof. For $f \in L^2(\mu)$ let $g_j = V_j^* f$, so that $\mathbb{E}_\mu[f(y) \mid y_j = t] = \mathbb{E}_\mu f + g_j(t)$, the marginal of y_j being uniform. *Single pairs.* As $\mathbb{E}_t \chi(t - x_i) = 0$, $\mathbb{E}_y \chi(y_j - x_i) f(y) = \mathbb{E}_t \chi(t - x_i) g_j(t) = (C g_j)(x_i)$: the pair (i,j) gives $D_{ij} V_i C V_j^* = r D_{ij} V_i \hat{P} V_j^*$. *Right stars.* $\mathbb{E}_y \psi_S(y_j, x) f(y) = \mathbb{E}_t \psi_S(t,x)(\mathbb{E}_\mu f + g_j(t)) = w_S(x) \mathbb{E}_\mu f + (W_S g_j)(x)$: the right star gives $\prod_{i \in S} D_{ij}\, (W_S V_j^* + w_S \otimes \mathbf{1})$. *Left stars.* As $\chi(-1) = -1$, $\prod_{j \in S} \chi(y_j - x_i) = (-1)^{|S|} \psi_S(x_i, y)$. The function $z \mapsto \mathbb{E}_y \psi_S(z,y) f(y)$ has mean $\langle w_S, f \rangle_\mu$ over z, and its projection onto $\mathcal{H}$ is $W_S^* f$; so the left star gives $(-1)^{|S|} \prod_{j \in S} D_{ij}\, (V_i W_S^* + \mathbf{1} \otimes w_S)$.

Centring. $P_0(\mathbf{1} \otimes w_S) = 0$ and $(w_S \otimes \mathbf{1}) P_0 = 0$, while V_i and W_S map into $L^2_0(\mu)$ and V_j^*, W_S^* vanish on the constants (Lemma 14.4). So the compression is the sum of the terms $r D_{ij} V_i \hat{P} V_j^*$, $(-1)^{|S|} \prod_{j \in S} D_{ij}\, r V_i (W_S/r)^*$ and $\prod_{i \in S} D_{ij}\, r (W_S/r) V_j^*$, which is $r \mathcal{F} M(\hat{P}) \mathcal{F}^*$.

Norms. By Lemma 14.1, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with $\hat{P} = \pm i I$ on $\mathcal{H}_\pm$. Write $M(\hat{P}) = M_0 \otimes I + M_1 \otimes \hat{P}$ with real matrices M_0, M_1. On $\mathbb{C}^{U \sqcup \mathcal{S}} \otimes \mathcal{H}_\pm$ it acts as $M(\pm i) \otimes I$, and $M(-i) = \overline{M(i)}$ has the norm of $M(i)$; so $\left\| M(\hat{P}) \right\| = \|M(i)\|$, and $\left\| r \mathcal{F} M(\hat{P}) \mathcal{F}^* \right\| \le r \|\mathcal{F}\|^2 \|M(i)\|$. Since $\hat{P}^* = -\hat{P}$, $M(\hat{P})^*$ acts as $M(\pm i)^* \otimes I$ on the same spaces, and the symmetric and skew parts of $\mathcal{F} M(\hat{P}) \mathcal{F}^*$ are

$\mathcal{F}\operatorname{Sym} M(\hat{P})\mathcal{F}^*$ and $\mathcal{F}\operatorname{Skew} M(\hat{P})\mathcal{F}^*$; the same argument applies to them. For two soft kernels on the same half law both star parts have this form with one synthesis, so their difference is $r\mathcal{F}(M_D(\hat{P}) - M_{D'}(\hat{P}))\mathcal{F}^*$. □

If D is antisymmetric, as is the matrix Δ^E of the factors of a cut kernel at $p \equiv 3 \pmod 4$ (Lemma 14.18, Section 14.3), then $\prod_{k\in S} D_{kj} = (-1)^{|S|}\prod_{k\in S} D_{jk}$ for every singleton j and support S: the block of the right stars is the transpose of the block of the left stars, and $M(i) = \left(\begin{smallmatrix} iD & A \\ A^{\mathsf{T}} & 0 \end{smallmatrix}\right)$, with $A_{jS} = (-1)^{|S|}\prod_{k\in S} D_{jk}$, is Hermitian, of the form tested in Remark 14.7.

The star means w_S are not lost: they sit in the rank-one terms $\mathbf{1}\otimes w_S$ and $w_S\otimes\mathbf{1}$, which only the double centring removes. For $|S|\ge 3$, w_S is in general a nonconstant translation-invariant function with pointwise bound $|w_S| \le (|S|-1)r$, so it remains in the kernel through these rank-one terms; only the doubly centred matrix omits it. The raw scalar means, the nonstar terms and the coincidences of the kernel are not covered by the corollary and are bounded separately in each application. For two soft kernels on the same half law one synthesis over the union of their supports cancels every common star coefficient exactly.

When a cut operator is compressed to the functions of a set O of observed coordinates, as for the merge and constant marginals of the upper range (Proposition 27.14), the star matrix splits into its principal part on the features measurable in x_O and the rest. A feature that involves an unobserved coordinate is small after the compression:

Lemma 14.6 (Unobserved features)**.** *In this setting let $O \subseteq U$, let Π_O be the conditional expectation onto the functions of x_O, and let $j \in U\setminus O$ be such that, given the other coordinates, every value of x_j has conditional probability at most D_j/p. Then*

$$\|\Pi_O V_j\| \le r\Big(\frac{D_j-1}{1-q}\Big)^{1/2}, \qquad \|\Pi_O(W_S/r)\| \le r\Big(\frac{D_j-1}{1-q}\Big)^{1/2} \quad (j\in S).$$

No independence of the unobserved coordinates is used.

Proof. Let Π_j be the conditional expectation onto the functions of $x_{U\setminus\{j\}}$, and $\nu_j(\cdot \mid x)$ the conditional law of x_j given them. As $O \subseteq U\setminus\{j\}$, $\Pi_O = \Pi_O\Pi_j$ and $\|\Pi_O\| \le 1$, so it suffices to bound $\Pi_j V_j$ and $\Pi_j W_S$. Conditional expectation commutes with the diagonal action of Aff_p, since μ is invariant; so these maps intertwine the two actions, and their Grams are real self-adjoint operators on $\mathcal{H}$ commuting with Aff_p, that is, scalars equal to their traces over $p-1 = (1-q)/q$ (Lemma 14.2(d)). The map $\Pi_j V_j$ has the kernel $\phi = p\nu_j(z \mid x)$, and by (14.8) its Gram has the trace $\mathbb{E}_\mu(p\sum_z \nu_j(z\mid x)^2 - 1) \le D_j - 1$. The map $\Pi_j W_S$ has the kernel $\phi = \phi_j\psi_{S\setminus\{j\}}$ with $\phi_j(z,x) = \mathbb{E}[\chi(z-x_j)\mid x_{U\setminus\{j\}}] = \sum_t \nu_j(t\mid x)\chi(z-t)$, as $\psi_{S\setminus\{j\}}$ does not depend on x_j. By (14.8) and $|\psi_{S\setminus\{j\}}| \le 1$, its Gram has trace at most $\mathbb{E}_\mu\mathbb{E}_z\phi_j^2$, and as in the proof of Lemma 14.4(c), $\mathbb{E}_z\phi_j^2 = \sum_t \nu_j(t\mid x)^2 - q \le q(D_j-1)$. Dividing by $p-1$ gives $\|\Pi_j V_j\|^2 \le q(D_j-1)/(1-q)$ and $\|\Pi_j W_S\|^2 \le q^2(D_j-1)/(1-q)$. □

The division by $\dim\mathcal{H} = p-1$ is where the invariance enters: a Hilbert–Schmidt bound alone would give only $\|\Pi_j W_S\| \le r(D_j-1)^{1/2}$. A star with all its leaves observed is measurable in x_O and is not changed by Π_O; for a star with an unobserved leaf the best unobserved pivot may be chosen.

Remark 14.7 (Real forms of the phase gates). On $\mathcal{H}$ the operator $\hat{P}$ has the eigenvalues $\pm i$, so a coefficient operator built from real matrices and $\hat{P}$ is the direct sum of its values at i and at $-i$, which are complex conjugate matrices; one test at $\hat{P} = i$ covers both phases. A Hermitian matrix $X_0 + iX_1$ (X_0 real symmetric, X_1 real antisymmetric) is positive semidefinite if and only if the real symmetric matrix $\left(\begin{smallmatrix} X_0 & -X_1 \\ X_1 & X_0 \end{smallmatrix}\right)$ of twice the size is, since $(u+iv)^*(X_0+iX_1)(u+iv)$ is its

quadratic form at (u, v) for real u, v; the certificates test this real form by exact rational LDL^{T} elimination. A bound $\|M(i)\| \le k$ for a Hermitian $M(i) = \left(\begin{smallmatrix} iD & A \\ A^* & 0 \end{smallmatrix}\right)$, with D real antisymmetric and A real, is $kI \pm M(i) \succeq 0$, that is, after eliminating the lower block kI, $k^2 I - AA^* \pm ikD \succeq 0$; a non-Hermitian M uses its singular-value dilation $\left(\begin{smallmatrix} kI & M \\ M^* & kI \end{smallmatrix}\right) \succeq 0$ in the same way. The same real forms serve the matrix test (14.21) at class three (Remark 14.30), the conditional star and joint row gates (Proposition 16.22 and Lemma 16.23) and the observed caps (Proposition 27.14).

14.3. **The degree-one part.** Let $\mathcal{V}_k = \{g(x_k) : g \in L^2_0(\mathbb{F}_p)\} \subseteq L^2(\mu)$ and $\mathcal{V} = \sum_{k \in U^+} \mathcal{V}_k$, the *degree-one space*, with the μ-orthogonal projection $\Pi_{\mathcal{V}}$. Since the one-vertex marginals of μ and of m_p are uniform, every $v \in \mathcal{V}$ has $\mathbb{E}_\mu v = \mathbb{E}_{m_p} v = 0$. For $v = \sum_k g_k(x_k)$ with $g_k \in L^2_0(\mathbb{F}_p)$ put $\hat{g}(\xi) = (\hat{g}_k(\xi))_{k \in U^+} \in \mathbb{C}^{U^+}$ and $\|g\|^2 = \sum_k \|g_k\|^2 = \sum_{\xi \ne 0} |\hat{g}(\xi)|^2$. Then $v = \sum_{\xi \ne 0} v_\xi$ with the *mode components* $v_\xi = \sum_k \hat{g}_k(\xi) e_\xi(x_k)$; if the g_k are real, $\hat{g}(-\xi) = \overline{\hat{g}(\xi)}$.

The translations $x \mapsto x + c\mathbf{1}$ ($c \in \mathbb{F}_p$) act unitarily on $L^2(\mu)$ and commute with A_p, with $\Pi_{\mathcal{V}}$ and with $\mathbb{J}$ (μ and K are translation invariant, and $\mathcal{V}$ is a translation-invariant subspace). Hence $L^2(\mu)$ is the orthogonal sum of the *isotypic subspaces* $\mathcal{W}_\xi = \{F : F(x + c\mathbf{1}) = e_\xi(c) F(x) \text{ for all } c\}$, $\xi \in \mathbb{F}_p$; each is preserved by A_p, $\Pi_{\mathcal{V}}$ and P_0, and $v_\xi \in \mathcal{W}_\xi$.

Let Δ^E be the $U^+ \times U^+$ matrix with $\Delta^E_{kl} = \chi(\varsigma_{kl})$ if $\varsigma_{kl} \ne 0$ and 0 otherwise. It depends on p only through $\chi(-1)$: it is the 0/1 matrix of the cross pairs (symmetric) if $p \equiv 1 \pmod 4$, and the antisymmetric matrix (ς_{kl}) if $p \equiv 3 \pmod 4$. Let $\Lambda_D \ge \left\|\Delta^E\right\|$ be a certified bound (the certificates use an exact rational one). Put

$$\mathbf{A}_0(\xi) = \chi(\xi) \varepsilon_G \, \Delta^E, \qquad \eta_Q = \frac{\Lambda_D}{p-1} + \frac{p}{p-1} \max_k \sum_{l \colon \varsigma_{kl} = 0} \min(\delta^\times_k, \delta^\times_l) + \frac{n}{(p-1)r}, \tag{14.9}$$

$$\varepsilon_V = \frac{n-1}{p-1} + \frac{p}{p-1} \, r \, \max_k \Big(d^0_k + \sum_{l \ne k,\, l \not\sim k} \min(\delta^\mu_k, \delta^\mu_l) \Big). \tag{14.10}$$

$\mathbf{A}_0(\xi)$ is Hermitian: for $p \equiv 1 \pmod 4$, ε_G is real and Δ^E real symmetric; for $p \equiv 3 \pmod 4$, ε_G is imaginary and Δ^E real antisymmetric. Also $\mathbf{A}_0(\xi)^* \mathbf{A}_0(\xi) = (\Delta^E)^{\mathsf{T}} \Delta^E$.

Lemma 14.8 (The degree-one block)**.** *Assume $e_\Delta(p, 2n) < 1$.*

(a) *For $v = \sum_k g_k(x_k)$ and $v' = \sum_k h_k(x_k)$ in $\mathcal{V}$ (complex coefficients allowed), $\mathbb{E}_\lambda[\overline{v(x)} v'(x')] = \langle v, \mathsf{A}_p v' \rangle_\mu = \sum_{\xi \ne 0} \hat{g}(\xi)^* \mathbf{Q}(\xi) \hat{h}(\xi)$, where $\mathbf{Q}(\xi)_{kl} = -\frac{1}{p-1} + \frac{p}{p-1} c_\xi \delta^\times_{kl}$ and $\delta^\times_{kl} = \mathbb{E}_\lambda \chi(x'_l - x_k)$. The matrix $\mathbf{Q}(\xi)$ is Hermitian and $\mathbf{Q}(-\xi) = \overline{\mathbf{Q}(\xi)}$.*

(b) *$\mathbf{Q}(\xi)/r = \mathbf{A}_0(\xi) + E_Q(\xi)$ with $E_Q(\xi)$ Hermitian, $\|E_Q(\xi)\| \le \eta_Q$, and $\|\mathbf{A}_0(\xi)\| = \left\|\Delta^E\right\|$.*

(c) *For $\xi \ne 0$ and $w \in \mathbb{C}^{U^+}$, $\|\sum_k w_k e_\xi(x_k)\|^2_\mu = w^* \mathbf{N}(\xi) w$ with $\mathbf{N}(\xi) = I - \frac{\mathbf{1}\mathbf{1}^{\mathsf{T}} - I}{p-1} + \frac{p}{p-1} c_\xi \Delta^\mu$, where $\Delta^\mu_{kl} = \mathbb{E}_\mu \chi(x_l - x_k)$ ($k \ne l$) and $\Delta^\mu_{kk} = 0$; and $\|\mathbf{N}(\xi) - I\| \le \varepsilon_V$. Different mode components are orthogonal in $L^2(\mu)$. Hence $(1 - \varepsilon_V)\|g\|^2 \le \|v\|^2_\mu \le (1 + \varepsilon_V)\|g\|^2$ for every $v \in \mathcal{V}$.*

Proof. (a) The identity $\mathbb{E}_\lambda[\overline{v(x)} v'(x')] = \langle v, \mathsf{A}_p v' \rangle_\mu$ is Lemma 13.3(b), extended to complex functions by linearity. The pair (x_k, x'_l) has an Aff_p-invariant law with $\pi^= = 0$ (injectivity across the cut) and $D = \delta^\times_{kl}$, so by Lemma 14.2(b) $\mathbb{E}_\lambda[\overline{g_k(x_k)} h_l(x'_l)] = -\frac{1}{p-1} \langle g_k, h_l \rangle + \frac{p}{p-1} \delta^\times_{kl} \langle g_k, C h_l \rangle$. By Parseval and Lemma 14.1, $\langle g_k, h_l \rangle = \sum_\xi \overline{\hat{g}_k(\xi)} \hat{h}_l(\xi)$ and $\langle g_k, C h_l \rangle = \sum_\xi c_\xi \overline{\hat{g}_k(\xi)} \hat{h}_l(\xi)$, and the term $\xi = 0$ vanishes. Here $\delta^\times_{kl}$ is real and $\delta^\times_{lk} = \chi(-1) \delta^\times_{kl}$ (Lemma 14.3(a)); with $\overline{c_\xi} = \chi(-1) c_\xi$ this gives $\overline{\mathbf{Q}(\xi)_{lk}} = \mathbf{Q}(\xi)_{kl}$, and $\overline{c_\xi} = c_{-\xi}$ gives $\mathbf{Q}(-\xi) = \overline{\mathbf{Q}(\xi)}$.

(b) Write $\delta^{\times} = \Delta^{E} + \Delta^{\mathrm{sm}}$, where $\Delta^{\mathrm{sm}}_{kl} = \delta^{\times}_{kl}$ if $\varsigma_{kl} = 0$ and 0 otherwise (Lemma 14.3(a)). With $c_\xi/r = \chi(\xi)\varepsilon_G$ and $\frac{p}{p-1} = 1 + \frac{1}{p-1}$,

$$\mathbf{Q}(\xi)/r = \mathbf{A}_0(\xi) + \frac{\mathbf{A}_0(\xi)}{p-1} + \frac{p}{p-1}\chi(\xi)\varepsilon_G\Delta^{\mathrm{sm}} - \frac{\mathbf{1}\mathbf{1}^{\mathsf{T}}}{(p-1)r}.$$

$|\chi(\xi)\varepsilon_G| = 1$, so $\|\mathbf{A}_0(\xi)\| = \left\|\Delta^{E}\right\| \le \Lambda_D$. The matrix of moduli $|\Delta^{\mathrm{sm}}|$ is symmetric with entries at most $\min(\delta^{\times}_k, \delta^{\times}_l)$; by the Schur test its norm, and hence $\|\Delta^{\mathrm{sm}}\|$, is at most the maximal row sum in (14.9). $\left\|\mathbf{1}\mathbf{1}^{\mathsf{T}}\right\| = n$. This gives $\|E_Q\| \le \eta_Q$, and E_Q is Hermitian as a difference of Hermitian matrices.

(c) For $k \ne l$, Lemma 14.2(b) for (x_k, x_l) under μ ($\pi^{=} = 0$, $D = \Delta^{\mu}_{kl}$) gives $\mathbb{E}_\mu[\overline{e_\xi(x_k)}e_\xi(x_l)] = -\frac{1}{p-1} + \frac{p}{p-1}c_\xi\Delta^{\mu}_{kl}$, and the diagonal terms are 1; this is the formula for $\mathbf{N}(\xi)$. As in (a), $c_\xi\Delta^{\mu}$ is Hermitian. Its norm is at most r times the maximal row sum of the symmetric matrix $|\Delta^{\mu}|$, whose entries are 1 on the d^0_k internal edges at k and at most $\min(\delta^{\mu}_k, \delta^{\mu}_l)$ otherwise (Lemma 14.3(b)); and $\left\|\mathbf{1}\mathbf{1}^{\mathsf{T}} - I\right\| = n - 1$. So $\|\mathbf{N}(\xi) - I\| \le \varepsilon_V$. The mode components v_ξ lie in the mutually orthogonal spaces $\mathcal{W}_\xi$, and $\|g\|^2 = \sum_\xi |\hat{g}(\xi)|^2$. □

14.4. **Coupling degree-one functions to the remainder.** The spectral criteria split test functions into the degree-one space $\mathcal{V}$, handled by Fourier analysis in the preceding subsection, and its orthogonal complement. What remains is the leakage from $\mathcal{V}$ into the complement under the cut operator, namely $\|(I - \Pi_{\mathcal{V}})\mathsf{A}_pF_V\|$ for $F_V \in \mathcal{V}$; this subsection bounds it. Write $\mathsf{A}_p = \varrho \cdot \mathcal{P}$, i.e. $(\mathsf{A}_pH)(x) = \varrho(x)(\mathcal{P}H)(x)$, with $(\mathcal{P}H)(x) = \mathbb{E}_\lambda[H(x') \mid x]$. Then A_pF_V is controlled by the conditional laws of the single opposite labels x'_k given x. These are probability vectors on explicit allowed sets with almost constant densities, and the model matrix $\mathbf{H}$ below is the Gram matrix that their densities would have if they were exactly the normalized indicators of the allowed sets. Since route B (Section 14.9) needs the same estimates for the uniform laws on the allowed sets, we state them for a family of such densities.

For $x \in \mathcal{I}^+$ and $k \in U^+$ the *allowed set* is

$$\Phi_k(x) = \{a \in \mathbb{F}_p \setminus x(U^+) :\ \varsigma_{jk}(a - x_j) \in Q_p^* \text{ for all } j \in N(k)\}.$$

It is a set as in Lemma 12.2 with the r_k distinct roots x_j, $j \in N(k)$, and the excluded set $x(U^+)$ of n labels, so $|\Phi_k(x)| \in p2^{-r_k}[1 - \hat{e}_k, 1 + \hat{e}_k]$ with $\hat{e}_k = e_{r_k}(p, n)$ (Lemma 12.2(d)). For $\gamma \in \mathrm{Aff}_p$, $\Phi_k(\gamma x) = \gamma\Phi_k(x)$.

Definition 14.9 (Coupling data)**.** *Coupling data* at p consist of a law ν on $\mathcal{I}^+$ and, for every $k \in U^+$ and $x \in \mathcal{I}^+$, a probability vector $P_k(\cdot \mid x)$ on $\mathbb{F}_p$ supported on $\Phi_k(x)$, such that:

(i) ν is Aff_p-invariant, and its density with respect to μ takes values in an interval $[c, R_\nu c]$ with $c > 0$ and $R_\nu \ge 1$;
(ii) $P_k(\gamma a \mid \gamma x) = P_k(a \mid x)$ for $\gamma \in \mathrm{Aff}_p$;
(iii) $P_k(a \mid x) = (1 + \delta_k(a; x))/|\Phi_k(x)|$ on $\Phi_k(x)$, with $1 + \delta_k \in [1/R'_k, R'_k]$ for numbers $R'_k \ge 1$; we put $\delta_k = 0$ off $\Phi_k(x)$ and $\delta'_k = R'_k - 1$, so $|\delta_k| \le \delta'_k$.

Put $\Psi_k(x, a) = pP_k(a \mid x) - 1$, a function with $\mathbb{E}_a\Psi_k(x, a) = 0$ for a uniform on $\mathbb{F}_p$, and

$$\widetilde{\mathbf{H}}_{kl} = \mathbb{E}_\nu\left\langle \Psi_k(x, \cdot), \Psi_l(x, \cdot)\right\rangle, \qquad \mathbf{D}_{kl} = \mathbb{E}_\nu\left\langle \Psi_k(x, \cdot), C\Psi_l(x, \cdot)\right\rangle, \qquad \mathbf{M}(\xi) = \frac{\widetilde{\mathbf{H}}}{p-1} + \frac{p}{p-1}c_\xi\mathbf{D}.$$

For $F_V = \sum_k g_k(x_k) \in \mathcal{V}$ let $(\mathcal{P}F_V)(x) = \sum_k \sum_a P_k(a \mid x)g_k(a)$.

The two instances used below are these.

(A) *The cut.* $\nu = m_p$, and $P_k(\cdot \mid x)$ is the conditional law of x'_k given x under λ; then $\mathcal{P}F_V(x) = \mathbb{E}_\lambda[F_V(x') \mid x]$. Here $R_\nu = R_{\mathrm{cut}}$: by Lemma 13.3(b),(c) and its proof, $\varrho = N/\mathbb{E}_\mu N$ with $N_- \le N \le N_+$ and $N_+/N_- = R_{\mathrm{cut}}$. Condition (ii) holds because λ is Aff_p-invariant. For (iii), let $\mathfrak{o}_k$ be the order of U^+ that starts with k and then lists $U^+ \setminus \{k\}$ in the reverse of the deletion order $\mathfrak{o}$; for $i \ne k$ let $t'_{k,i}$ be r_i plus the number of G^+-neighbours of i preceding i in $\mathfrak{o}_k$, and

$$R'_k = \prod_{i \ne k} \frac{1 + e_{t'_{k,i}}(p, 2n)}{1 - e_{t'_{k,i}}(p, 2n)}. \tag{14.11}$$

Let $N_k(a)$ be the number of $x' \in \mathcal{I}^+$ with $K(x, x') = 1$ and $x'_k = a$, so that $P_k(a \mid x) = N_k(a)/\sum_b N_k(b)$. If $a \notin \Phi_k(x)$ then $N_k(a) = 0$ (injectivity across the cut, or a cross constraint at k fails). If $a \in \Phi_k(x)$, label the vertices $i \ne k$ of x' in the order $\mathfrak{o}_k$: when x'_i is chosen, its constraints are the r_i cross constraints with the roots x_j, $j \in N(i)$, and the internal constraints with its already labelled x'-neighbours (including k), $t'_{k,i} \le d_i \le \Delta$ constraints in all; their roots are distinct, and the excluded set $x(U^+) \cup \{\text{labels already chosen}\}$ has fewer than $2n$ elements and contains them. By Lemma 12.2(d) the number of choices lies in $p2^{-t'_{k,i}}[1 - e_{t'_{k,i}}(p, 2n), 1 + e_{t'_{k,i}}(p, 2n)]$, whatever the earlier choices. Hence $N_k(a) \in [L_k, U_k]$ on $\Phi_k(x)$, with $L_k > 0$ and $U_k/L_k = R'_k$, and $1 + \delta_k(a) = N_k(a)/\operatorname{avg}_{\Phi_k} N_k \in [1/R'_k, R'_k]$, where $\operatorname{avg}_{\Phi_k} N_k := |\Phi_k(x)|^{-1} \sum_{a \in \Phi_k(x)} N_k(a)$.

(B) *Explicit stars.* $\nu = \mu$, and $P_k(\cdot \mid x)$ is the uniform law on $\Phi_k(x)$: $R_\nu = 1$ and $R'_k = 1$, $\delta_k = 0$.

Lemma 14.10 (Modes of coupling data)**.** *For coupling data and $F_V = \sum_k g_k(x_k) \in \mathcal{V}$, $\|\mathcal{P}F_V\|_\nu^2 = \sum_{\xi \ne 0} \hat{g}(\xi)^* \mathbf{M}(\xi) \hat{g}(\xi)$, and $\mathbf{M}(\xi)$ is Hermitian. In instance* (A)*, $\|\mathsf{A}_p F_V\|_\mu^2 \le (\sup \varrho)\, \|\mathcal{P}F_V\|_{m_p}^2$.*

Proof. Since $\mathbb{E}_a[pP_k(a \mid x)] = 1$ and $\mathbb{E}_a g_k = 0$, $\sum_a P_k(a \mid x) g_k(a) = \mathbb{E}_a[\Psi_k(x, a) g_k(a)]$. Hence $\mathcal{P}F_V = \sum_{\xi \ne 0} Y_\xi$ with $Y_\xi(x) = \mathbb{E}_a[\sum_k \hat{g}_k(\xi) \Psi_k(x, a) e_\xi(a)]$. By (ii) for the translations, $Y_\xi(x + c\mathbf{1}) = e_\xi(c) Y_\xi(x)$, and ν is translation invariant; so the Y_ξ are orthogonal in $L^2(\nu)$. By Lemma 14.2(c) with $\mathcal{Y} = \mathcal{I}^+$ and the covariant fields Ψ_k, for which $M = 0$, $\|Y_\xi\|_\nu^2 = \hat{g}^*(\widetilde{\mathbf{H}} + \chi(\xi) G_\chi \mathbf{D}) \hat{g}/(p-1) = \hat{g}^* \mathbf{M}(\xi) \hat{g}$, because $\chi(\xi) G_\chi = p c_\xi$. $\mathbf{M}(\xi)$ is Hermitian, since the left side is real for every $\hat{g}$. In instance (A), $\mathsf{A}_p H = \varrho \cdot \mathcal{P}H$ (Lemma 13.3(b)), so $\|\mathsf{A}_p F_V\|_\mu^2 = \mathbb{E}_\mu[\varrho^2 |\mathcal{P}F_V|^2] = \mathbb{E}_{m_p}[\varrho |\mathcal{P}F_V|^2]$. □

The model. For $J \subseteq U^+$ let $\varepsilon_J \in \mathbb{R}^{U^+}$ have the entries $(\varepsilon_J)_k = \prod_{j \in J} \chi(\varsigma_{jk})$ if $J \subseteq N(k)$ and 0 otherwise, and write $\varepsilon_J \cdot w = \sum_k (\varepsilon_J)_k w_k$. The *star Gram model* is

$$\mathbf{H} = \sum_{\emptyset \ne J \subseteq U^+} \varepsilon_J \varepsilon_J^{\mathsf{T}}, \qquad \mathbf{H}_{kl} = \prod_{j \in N(k) \cap N(l)} (1 + \chi(\varsigma_{jk}\varsigma_{jl})) - 1. \tag{14.12}$$

Thus $\mathbf{H} \succeq 0$, $\mathbf{H}_{kk} = 2^{r_k} - 1$, and $\|\mathbf{H}\|_{\mathrm{row}} = \max_k \sum_l |\mathbf{H}_{kl}| \ge \|\mathbf{H}\|$. Distinct k, l are *inconsistent* if $\chi(\varsigma_{jk}\varsigma_{jl}) = -1$ for some $j \in N(k) \cap N(l)$ (then $\mathbf{H}_{kl} = -1$), and *consistent* otherwise.

Definition 14.11 (The coupling model)**.** A set $J \subseteq U^+$ with $|J| \ge 2$ is *coverable* if $G^+[J]$ has no isolated vertex. Put

$$\mathbf{H}^{\mathrm{cov}} = \sum_{\substack{|J| \ge 2 \\ J \text{ coverable}}} \varepsilon_J \varepsilon_J^{\mathsf{T}}, \qquad \mathbf{H}^{\mathrm{nc}} = \sum_{\substack{|J| \ge 2 \\ J \text{ not coverable}}} \varepsilon_J \varepsilon_J^{\mathsf{T}}, \qquad \mathbf{H}^+ = \mathbf{H} + \mathbf{1}_{p \equiv 1(4)} \mathbf{H}^{\mathrm{cov}}.$$

These matrices, Δ^E and $\mathbf{H}$ depend on p only through its class $\mathfrak{c}$; we write $\mathbf{H}_{(\mathfrak{c})}$, $\mathbf{H}^+_{(\mathfrak{c})}$, $\Delta^E_{(\mathfrak{c})}$ when the class matters, and put

$$\mathbf{A}_{1,\pm} = \pm \Delta^E_{(1)}, \qquad \mathbf{A}_{3,\pm} = \pm i\, \Delta^E_{(3)}.$$

By Lemma 14.1, for every $\xi \neq 0$ the matrix $\mathbf{A}_0(\xi)$ is one of $\mathbf{A}_{\mathfrak{c},+}$, $\mathbf{A}_{\mathfrak{c},-}$, both values occur, and $\mathbf{A}_{\mathfrak{c},\pm}^* \mathbf{A}_{\mathfrak{c},\pm} = (\Delta_{(\mathfrak{c})}^E)^{\mathsf{T}} \Delta_{(\mathfrak{c})}^E$. For a real symmetric matrix X, $\bar{\lambda}(X)$ denotes a fixed certified number with $\bar{\lambda}(X) \geq \lambda_{\max}(X)$; every formula below uses this fixed number (an exact LDL^{T} certificate of $\bar{\lambda} I - X \succeq 0$, or the maximal absolute row sum).

$\mathbf{H}^{\mathrm{cov}}$ and $\mathbf{H}^{\mathrm{nc}}$ are Gram sums, hence positive semidefinite. For $p \equiv 1 \pmod 4$, $\chi(-1) = 1$, so $(\varepsilon_J)_k = [J \subseteq N(k)]$, $\Delta_{kl}^E = [\varsigma_{kl} \neq 0]$, $\mathbf{H}_{kl} = 2^{|N(k) \cap N(l)|} - 1$, and every pair is consistent.
The Gram comparison. Put $\beta_{kk} = 2^{r_k} \hat{e}_k / (1 - \hat{e}_k)$, and for consistent $k \neq l$, with $u_{kl} = |N(k) \cup N(l)|$ and $e_u = e_{u_{kl}}(p, n)$,

$$\beta_{kl} = 2^{|N(k) \cap N(l)|} \max\Big(\frac{1 + e_u}{(1 - \hat{e}_k)(1 - \hat{e}_l)} - 1,\ 1 - \frac{1 - e_u}{(1 + \hat{e}_k)(1 + \hat{e}_l)} \Big); \tag{14.13}$$

$\beta_{kl} = 0$ for inconsistent $k \neq l$, and $\varepsilon'_H = \max_k \sum_l \beta_{kl}$. For consistent $k \neq l$ put $\Upsilon_{kl} = 2^{|N(k) \cap N(l)|}(1 + e_u)/((1 - \hat{e}_k)(1 - \hat{e}_l))$, and $\Upsilon_{kl} = 0$ otherwise (in particular for inconsistent $k \neq l$; the diagonal values Υ_{kk} are not used). For coupling data put

$$\varepsilon_X = \max_k \Big[\frac{2^{r_k} \delta_k'^2}{1 - \hat{e}_k} + \sum_{l \neq k} \Upsilon_{kl} (\delta'_k + \delta'_l + \delta'_k \delta'_l) \Big].$$

Lemma 14.12 (Gram comparison)**.** *Assume $e_\Delta(p, 2n) < 1$ and let coupling data be given. Let $\psi_k(x, \cdot) = (p/|\Phi_k(x)|)\mathbf{1}_{\Phi_k(x)} - 1$ and $\mathbf{H}^0_{kl} = \mathbb{E}_\nu \langle \psi_k, \psi_l \rangle$. Then $\mathbf{H}^0 \preceq \mathbf{H} + \varepsilon'_H I$ and $\widetilde{\mathbf{H}} \preceq \mathbf{H} + \varepsilon_H I$ with $\varepsilon_H := \varepsilon'_H + \varepsilon_X$. In instance* (B)*, $\widetilde{\mathbf{H}} = \mathbf{H}^0$ and $\varepsilon_X = 0$.*

Proof. The model. Fix x. Then $\langle \psi_k, \psi_l \rangle = p|\Phi_k \cap \Phi_l|/(|\Phi_k||\Phi_l|) - 1$. For $k = l$ this is $p/|\Phi_k| - 1$, while $\mathbf{H}_{kk} + 1 = 2^{r_k}$ and $p/|\Phi_k| \in 2^{r_k}[(1 + \hat{e}_k)^{-1}, (1 - \hat{e}_k)^{-1}]$; so the two differ by at most β_{kk}. For inconsistent $k \neq l$ some $j \in N(k) \cap N(l)$ requires $a - x_j \in \varsigma_{jk} Q_p^*$ and $a - x_j \in \varsigma_{jl} Q_p^*$, which are disjoint sets; so $\Phi_k \cap \Phi_l = \emptyset$ and both entries are -1. For consistent $k \neq l$ the two constraints at a common j coincide, so $\Phi_k \cap \Phi_l$ is cut out by u_{kl} constraints with distinct roots and has $p2^{-u_{kl}}[1 - e_u, 1 + e_u]$ elements; since $2^{r_k + r_l - u_{kl}} = 2^{|N(k) \cap N(l)|} = \mathbf{H}_{kl} + 1$, the entries differ by at most β_{kl}. These bounds hold for every x; average over ν and apply Gershgorin to the real symmetric matrix $\mathbf{H}^0 - \mathbf{H}$.

The deviations. By (iii), $pP_k = (1 + \psi_k)(1 + \delta_k)$, so $\Psi_k = \psi_k + (1 + \psi_k)\delta_k$ and $\widetilde{\mathbf{H}} = \mathbf{H}^0 + X + X^{\mathsf{T}} + Y$ with $X_{kl} = \mathbb{E}_\nu \langle \psi_k, (1 + \psi_l)\delta_l \rangle$ and $Y_{kl} = \mathbb{E}_\nu \langle (1 + \psi_k)\delta_k, (1 + \psi_l)\delta_l \rangle$. Since P_l is a probability on Φ_l, $\sum_{a \in \Phi_l} \delta_l(a) = 0$, and therefore

$$\langle \psi_k, (1 + \psi_l)\delta_l \rangle = \frac{p^2}{|\Phi_k||\Phi_l|} \mathbb{E}_a[\mathbf{1}_{\Phi_k \cap \Phi_l} \delta_l] - \frac{p}{|\Phi_l|} \mathbb{E}_a[\mathbf{1}_{\Phi_l} \delta_l] = \frac{p^2}{|\Phi_k||\Phi_l|} \mathbb{E}_a[\mathbf{1}_{\Phi_k \cap \Phi_l} \delta_l].$$

For $k = l$ this vanishes, so $X_{kk} = 0$. For inconsistent $k \neq l$, $X_{kl} = Y_{kl} = 0$. For consistent $k \neq l$, $p|\Phi_k \cap \Phi_l|/(|\Phi_k||\Phi_l|) \leq \Upsilon_{kl}$, so $|X_{kl}| \leq \Upsilon_{kl} \delta'_l$ and $|Y_{kl}| \leq \Upsilon_{kl} \delta'_k \delta'_l$. Finally $Y_{kk} \leq \mathbb{E}_\nu (p/|\Phi_k|) \delta_k'^2 \leq 2^{r_k} \delta_k'^2 / (1 - \hat{e}_k)$. The matrix of these bounds for $|X_{kl}| + |X_{lk}| + |Y_{kl}|$ is symmetric, so by Gershgorin $X + X^{\mathsf{T}} + Y \preceq \varepsilon_X I$. In instance (B), $\delta_k = 0$. □

The Paley part. For $k \in U^+$, $J \subseteq N(k)$, $a \in \mathbb{F}_p$ and $x \in \mathcal{I}^+$ put

$$\chi_{J,k}(a; x) = \prod_{j \in J} \chi(\varsigma_{jk}(a - x_j)), \qquad \varphi_k(a; x) = \prod_{j \in N(k)} (1 + \chi(\varsigma_{jk}(a - x_j))) = \sum_{J \subseteq N(k)} \chi_{J,k}(a; x),$$

$\varphi_k^\circ = \varphi_k - 1$ and, for coupling data, $E_k = pP_k(\cdot \mid x) - \varphi_k$, so that $\Psi_k = \varphi_k^\circ + E_k$. For $J \subseteq U^+$ let $g_J(a; x) = \prod_{j \in J} \chi(a - x_j)$, so that $\chi_{J,k} = (\varepsilon_J)_k g_J$, and

$$S_J(x_J) = \langle g_J, C g_J \rangle = \mathbb{E}_{a,b}\, g_J(a) \chi(b - a) g_J(b), \qquad t_J = \mathbb{E}_\nu S_J(x_J).$$

Put

$$\theta_j = \Big(\frac{R_\nu 2^{d_j^0}}{1 - e_{d_j^0}(p,n)} - 1\Big)^{1/2}, \qquad \theta_{\max} = \max_j \theta_j,$$

$$\rho_{\mathrm{row}} = \max_k (2^{r_k} - 1) \sum_l (2^{r_l} - 1), \qquad \varepsilon_{\neq} = r\, \theta_{\max} \rho_{\mathrm{row}},$$

$$\Delta_k = \max\Big(\frac{R'_k}{1-\hat e_k} - 1,\ 1 - \frac{1}{(1+\hat e_k) R'_k}\Big), \qquad \hat E^2 = \sum_k \Big(2^{r_k}(1+\hat e_k)\Delta_k^2 + \frac{n 4^{r_k}}{p}\Big),$$

and, for numbers $\Gamma_E \ge 0$ and ε_H with $\widetilde{\mathbf{H}} \preceq \mathbf{H} + \varepsilon_H I$,

$$\varepsilon_{\mathrm{rem}}(\Gamma_E) = 2\big(\Gamma_E(\bar\lambda(\mathbf{H}) + \varepsilon_H)\big)^{1/2} + \Gamma_E. \tag{14.14}$$

The notation $\varepsilon_{\mathrm{rem}}(\Gamma_E)$ suppresses the dependence on ε_H.

Lemma 14.13 (The Paley part)**.** *Assume $e_\Delta(p, 2n) < 1$ and let coupling data be given.*

(a) *For every $x \in \mathcal{I}^+$ and $w \in \mathbb{C}^{U^+}$, $\|\sum_k w_k E_k(\cdot; x)\| \le \hat E |w|$.*

(b) *Let Γ_E satisfy $\mathbb{E}_\nu \|\sum_k w_k E_k(\cdot;x)\|^2 \le \Gamma_E |w|^2$ for all w (by (a), $\Gamma_E = \hat E^2$ is admissible), and ε_H as above. Then for $w \in \mathbb{C}^{U^+}$*

$$\Big| w^* \mathbf{D} w - \sum_{|J| \ge 2} t_J\, |\varepsilon_J \cdot w|^2 \Big| \le r(\varepsilon_{\neq} + \varepsilon_{\mathrm{rem}}(\Gamma_E)) |w|^2.$$

(c) *$S_J = 0$ if $|J| = 1$, $|S_J| \le r$ for every J, and $S_J = 0$ (so $t_J = 0$) for every J if $p \equiv 3 \pmod 4$.*

Proof. (a) Let $\lambda_k = p 2^{-r_k}/|\Phi_k| \in [(1+\hat e_k)^{-1}, (1-\hat e_k)^{-1}]$. For $a \notin x(U^+)$ every factor of φ_k is 0 or 2, so $\varphi_k(a;x) = 2^{r_k} \mathbf{1}_{\Phi_k}(a)$; for $a \in x(U^+)$, $0 \le \varphi_k \le 2^{r_k}$ and $P_k(a \mid x) = 0$. On Φ_k, $E_k = 2^{r_k}(\lambda_k(1+\delta_k) - 1)$ with $\lambda_k(1+\delta_k) \in [1/((1+\hat e_k)R'_k), R'_k/(1-\hat e_k)]$, so $|E_k| \le 2^{r_k}\Delta_k$ there; $E_k = 0$ off $\Phi_k \cup x(U^+)$, and $|E_k| \le 2^{r_k}$ on the n points of $x(U^+)$. Since $|\Phi_k|/p \le 2^{-r_k}(1+\hat e_k)$, $\|E_k\|^2 \le 2^{r_k}(1+\hat e_k)\Delta_k^2 + n 4^{r_k}/p$, and Cauchy–Schwarz gives $\|\sum_k w_k E_k\| \le \hat E |w|$.

(b) *Splitting.* With $\Psi_w = \sum_k w_k \Psi_k$, $\varphi^\circ_w = \sum_k w_k \varphi^\circ_k$ and $E_w = \sum_k w_k E_k$,

$$w^* \mathbf{D} w = \mathbb{E}_\nu \langle \Psi_w, C\Psi_w\rangle = \mathbb{E}_\nu \langle \varphi^\circ_w, C\varphi^\circ_w\rangle + \mathbb{E}_\nu \langle \varphi^\circ_w, C E_w\rangle + \mathbb{E}_\nu \langle E_w, C\Psi_w\rangle .$$

As $\|\varphi^\circ_w\| \le \|\Psi_w\| + \|E_w\|$ and $\|C\| = r$, the last two terms are at most $r\mathbb{E}_\nu[\|E_w\|\,(2\|\Psi_w\| + \|E_w\|)]$, which by Cauchy–Schwarz in x is at most $r\big(2\Gamma_E^{1/2}|w|(\mathbb{E}_\nu\|\Psi_w\|^2)^{1/2} + \Gamma_E|w|^2\big)$; and $\mathbb{E}_\nu\|\Psi_w\|^2 = w^*\widetilde{\mathbf{H}} w \le (\bar\lambda(\mathbf{H}) + \varepsilon_H)|w|^2$. This is $r\varepsilon_{\mathrm{rem}}(\Gamma_E)|w|^2$. The main term is

$$\mathbb{E}_\nu \langle \varphi^\circ_w, C\varphi^\circ_w\rangle = \sum_{k,l} \overline{w_k} w_l \sum_{\emptyset \ne J \subseteq N(k)} \sum_{\emptyset \ne J' \subseteq N(l)} \mathbb{E}_\nu \langle \chi_{J,k}, C\chi_{J',l}\rangle .$$

Terms with $J \ne J'$. Suppose $j \in J \setminus J'$ (if $J \subseteq J'$ take $j \in J' \setminus J$ and use C^*, which also has norm r). Then $\chi_{J',l}$ and $\chi_{J\setminus\{j\},k}$ do not depend on x_j, so $\mathbb{E}_\nu\langle \chi_{J,k}, C\chi_{J',l}\rangle = \mathbb{E}_\nu\big\langle \phi_j \chi_{J\setminus\{j\},k}, C\chi_{J',l}\big\rangle$ with $\phi_j(a) = \sum_t \nu_j(t)\chi(\varsigma_{jk}(a-t))$, where ν_j is the conditional law of x_j given the other labels under ν. By (4.2), $\|\phi_j\|^2 = \|\nu_j\|_{\ell^2}^2 - \frac{1}{p} \le \max \nu_j - \frac{1}{p}$. Under μ the conditional law of x_j is uniform on a set of at least $p 2^{-d_j^0}(1 - e_{d_j^0}(p,n))$ values, and the density of ν with respect to μ has ratio at most R_ν, so $\max \nu_j \le R_\nu 2^{d_j^0}/(p(1 - e_{d_j^0}(p,n)))$ (Lemma 12.6(b) for G^+). Hence $\|\phi_j\| \le r\theta_j$, and as $|\chi| \le 1$ the term is at most $r^2\theta_{\max}$ in modulus. There are at most $(2^{r_k}-1)(2^{r_l}-1)$ pairs (J, J'), and the Schur test for the symmetric matrix $((2^{r_k}-1)(2^{r_l}-1))_{kl}$ bounds their total by $r^2\theta_{\max}\rho_{\mathrm{row}}|w|^2 = r\varepsilon_{\neq}|w|^2$.

Terms with $J = J'$. For $J \subseteq N(k) \cap N(l)$, $\mathbb{E}_\nu\langle \chi_{J,k}, C\chi_{J,l}\rangle = (\varepsilon_J)_k (\varepsilon_J)_l t_J$, so these terms sum to $\sum_J t_J |\varepsilon_J \cdot w|^2$; the singletons contribute nothing by (c).

(c) If $p \equiv 3 \pmod 4$, C is real antisymmetric ($\chi(-1) = -1$), so $\langle g, Cg\rangle = 0$ for real g. If $J = \{j\}$, substituting $a \mapsto a + x_j$, $b \mapsto b + x_j$ and then $b = au$ ($a \neq 0$), $p^2 S_J = \sum_{a\neq 0} \chi(a)^3 \sum_u \chi(u)\chi(u-1) = -\sum_{a\neq 0}\chi(a) = 0$. Always $|S_J| \le \|g_J\|^2 \, \|C\| \le r$. □

For $p \equiv 3 \pmod 4$ the lemma says that the Paley part of the coupling is of lower order: $|w^*\mathbf{D}w| \le r(\varepsilon_{\neq} + \varepsilon_{\mathrm{rem}})|w|^2$. For $p \equiv 1 \pmod 4$ the diagonal terms t_J with $|J| \ge 2$ survive, and Section 14.11 bounds them.

Lemma 14.14 (The coupling block)**.** *Assume $e_\Delta(p, 2n) < 1$ and $\varepsilon_V < 1$, and take the coupling data of instance* (A)*. Let the following numbers be given.*

- *$R_+ \ge \sup_{\mathcal{I}^+} \varrho$; for instance $R_+ = R_{\mathrm{cut}}$ (Lemma 13.3(c)), or R_+ of Proposition 14.20*(c2)*.*
- *ε_H with $\widetilde{\mathbf{H}} \preceq \mathbf{H} + \varepsilon_H I$; for instance $\varepsilon'_H + \varepsilon_X$ (Lemma 14.12), or its minimum with $\varepsilon_H^{\mathrm{loc}}$ of Lemma 14.17.*
- *Γ_E with $\mathbb{E}_{m_p} \|\sum_k w_k E_k(\cdot; x)\|^2 \le \Gamma_E |w|^2$ for all w; for instance $\hat{E}^2$ (Lemma 14.13(a)), or its minimum with the completion-variance bound Γ_E^{loc} of Lemma 14.16.*
- *If $p \equiv 1 \pmod 4$, a number $\bar\zeta \ge 0$ with $|t_J| \le \bar\zeta r$ for every J with $|J| \ge 2$ that is not coverable; Section 14.11 supplies admissible values.*
- *$\Lambda_+ \ge \lambda_{\max}(\mathbf{H}^+)$, $\Lambda_{\mathrm{nc}} \ge \lambda_{\max}(\mathbf{H}^{\mathrm{nc}})$, $\Lambda_D \ge \left\|\Delta^E\right\|$ and $\bar\lambda(\mathbf{H}) \ge \lambda_{\max}(\mathbf{H})$.*

Put $\varepsilon_{\mathrm{par}} = \mathbf{1}_{p\equiv 1(4)}\bar\zeta\Lambda_{\mathrm{nc}}$, $\varepsilon_{\mathrm{rem}} = \varepsilon_{\mathrm{rem}}(\Gamma_E)$ of (14.14) *with this ε_H, and*

$$\eta_B = \Big(\frac{R_+ p}{p-1} - 1\Big)\Lambda_+ + \frac{R_+ p}{p-1}(\varepsilon_H + \varepsilon_{\neq} + \varepsilon_{\mathrm{rem}} + \varepsilon_{\mathrm{par}}) + \varepsilon_V \Lambda_D^2 + 2\Lambda_D \eta_Q. \tag{14.15}$$

Then for every $\xi \neq 0$

$$\mathbf{M}(\xi) \preceq \frac{1}{p-1}\Big[\widetilde{\mathbf{H}} + \mathbf{1}_{p\equiv 1(4)}\mathbf{H}^{\mathrm{cov}} + (\varepsilon_{\neq} + \varepsilon_{\mathrm{rem}} + \varepsilon_{\mathrm{par}})I\Big] \preceq \frac{\Lambda_M}{p-1} I,$$

with $\Lambda_M = \Lambda_+ + \varepsilon_H + \varepsilon_{\neq} + \varepsilon_{\mathrm{rem}} + \varepsilon_{\mathrm{par}}$, and for every complex $F_V = \sum_k g_k(x_k) \in \mathcal{V}$

$$\|(I - \Pi_{\mathcal{V}})\mathsf{A}_p F_V\|_\mu^2 \le r^2 \sum_{\xi\neq 0} \hat g(\xi)^* \big(\mathbf{H}^+ - \mathbf{A}_0(\xi)^*\mathbf{A}_0(\xi) + \eta_B I\big)\hat g(\xi). \tag{14.16}$$

Proof. The matrix $\mathbf{M}$*.* By Lemma 14.13(b),(c), and since $|t_J| \le r$ for every J, $|t_J| \le \bar\zeta r$ for the non-coverable J, and both $\mathbf{H}^{\mathrm{cov}}$ and $\mathbf{H}^{\mathrm{nc}}$ are Gram sums,

$$|w^*\mathbf{D}w| \le r\big(\mathbf{1}_{p\equiv 1(4)}(w^*\mathbf{H}^{\mathrm{cov}}w + \bar\zeta\, w^*\mathbf{H}^{\mathrm{nc}}w) + (\varepsilon_{\neq} + \varepsilon_{\mathrm{rem}})|w|^2\big),$$

and $\bar\zeta w^*\mathbf{H}^{\mathrm{nc}}w \le \bar\zeta\Lambda_{\mathrm{nc}}|w|^2$. Multiply by $\frac{p}{p-1}|c_\xi| = \frac{p}{p-1}r$ and use $pr^2 = 1$; as $\mathbf{M}(\xi)$ is Hermitian (Lemma 14.10) this gives the first inequality, and Lemma 14.12 ($\widetilde{\mathbf{H}} \preceq \mathbf{H} + \varepsilon_H I$) with $\mathbf{H}^+ \preceq \Lambda_+ I$ gives the second.

Step 1. By Lemma 14.10 and the first part, $\|\mathsf{A}_p F_V\|_\mu^2 \le R_+ \sum_\xi \hat g^* \mathbf{M}\hat g \le r^2 \frac{R_+ p}{p-1}\sum_\xi \hat g^*\big[\mathbf{H}^+ + (\varepsilon_H + \varepsilon_{\neq} + \varepsilon_{\mathrm{rem}} + \varepsilon_{\mathrm{par}})I\big]\hat g$, and $\frac{R_+ p}{p-1}\mathbf{H}^+ \preceq \mathbf{H}^+ + (\frac{R_+ p}{p-1} - 1)\Lambda_+ I$, because $\mathbf{H}^+ \succeq 0$ and $R_+ p/(p-1) \ge 1$ ($\sup\varrho \ge \mathbb{E}_\mu \varrho = 1$).

Step 2. Let $v = \sum_k v_k(x_k)$ with $\hat v(\xi) = \mathbf{Q}(\xi)\hat g(\xi)$ for $\xi \neq 0$ and $\hat v(0) = 0$. Then $v \in \mathcal{V}$, with no reality condition on its coefficients. By Lemma 14.8(a), $\langle v, \mathsf{A}_p F_V\rangle_\mu = \sum_\xi |\mathbf{Q}\hat g|^2$, and by Lemma 14.8(c), $\|v\|^2 \le (1 + \varepsilon_V)\sum_\xi |\mathbf{Q}\hat g|^2$. Hence $\|\Pi_{\mathcal{V}}\mathsf{A}_p F_V\|^2 \ge |\langle v, \mathsf{A}_p F_V\rangle|^2 / \|v\|^2 \ge (1 + \varepsilon_V)^{-1}\sum_\xi |\mathbf{Q}(\xi)\hat g(\xi)|^2$ (trivially if the sum is 0).

Conclusion. By Lemma 14.8(b), $|\mathbf{Q}\hat g|^2/r^2 = |(\mathbf{A}_0 + E_Q)\hat g|^2 \ge |\mathbf{A}_0\hat g|^2 - 2\Lambda_D\eta_Q|\hat g|^2 =: X - Y$, and since $\varepsilon_V < 1$ and $X \le \Lambda_D^2|\hat g|^2$,

$$(1+\varepsilon_V)^{-1}|\mathbf{Q}\hat g|^2/r^2 \ge (1-\varepsilon_V)\max(0, X-Y) \ge X - Y - \varepsilon_V X \ge \hat g^*\mathbf{A}_0^*\mathbf{A}_0\hat g - (2\Lambda_D\eta_Q + \varepsilon_V\Lambda_D^2)|\hat g|^2.$$

By Pythagoras, $\|(I - \Pi_{\mathcal{V}})\mathsf{A}_p F_V\|^2 = \|\mathsf{A}_p F_V\|^2 - \|\Pi_{\mathcal{V}}\mathsf{A}_p F_V\|^2$; collecting the terms gives (14.16). □

The proof uses $\sup \varrho$ only in Step 1, where $\mathsf{A}_p F_V = \varrho \cdot \mathcal{P} F_V$. The other occurrences of R_{cut}, through R_ν in θ_j and in the constant ζ of Section 14.11, bound ratios of counts; the number R_+ used in Step 1 is not substituted for R_{cut} in them. For $p \equiv 3 \pmod 4$ the model is $\mathbf{H}^+ = \mathbf{H}$ and $\varepsilon_{\mathrm{par}} = 0$.

14.5. **Variance of completion counts.** The coupling inputs ε_H and Γ_E measure how far the conditional density $pP_k(\cdot \mid x)$ of instance (A) is from the normalized indicator of $\Phi_k(x)$, that is, how much the number of completions of the opposite half varies with the value $a = x'_k$. The bounds of Lemmas 14.12 and 14.13 use only the range of the completion counts, through R'_k, which is a product over all vertices. At the completed-root cuts LL20 (the chamber cut in coordinate 0 of the two-layer root) and LL22 (its layer cut), defined in Section 27.2.1 (see also Lemma 27.10), this range bound is too crude for the test (14.21). The estimate below integrates the internal neighbourhood S of k exactly: conditionally on the labels z of the vertices outside $S \cup \{k\}$, whose law does not depend on a, the count is, up to the edges inside S and to coincidences, a product of one-variable counts, and each depends on a only through one Paley factor of norm r; the edges inside S are expanded in characters and bounded by matchings or by the two-variable sum of Theorem 15.13. Minkowski's inequality then averages over z. The result is a bound for the relative variance of the counts, which enters the coupling block through the two lemmas that follow it.

Fix $x \in \mathcal{I}^+$ and $k \in U^+$, and let $C_k(a)$ be the number of $x' \in \mathcal{I}^+$ with $K(x, x') = 1$ and $x'_k = a$ (the count $N_k(a)$ of instance (A)); it vanishes off $\Phi_k = \Phi_k(x)$. Write $\mathbb{E}_{\Phi_k}$, Var_{Φ_k} for the uniform probability on Φ_k, and let $v_k(x) = \mathrm{Var}_{\Phi_k}(C_k)/(\mathbb{E}_{\Phi_k} C_k)^2$ be the *relative completion variance*. We use count errors $(\epsilon_t^+)_{t \ge 0}$ and $(\epsilon_t^-(z))_{t,z \ge 0}$, nondecreasing in t and z, such that every set Φ of Lemma 12.2 with t roots satisfies $|\Phi| \le p2^{-t}(1 + \epsilon_t^+)$, whatever its excluded set, and $|\Phi| \ge p2^{-t}(1 - \epsilon_t^-(z))$ if it excludes at most z labels. By Lemma 12.2(b),(c) the graded errors e_t^+, $e_t^-(z)$ of (12.2) are such errors, and so is every larger choice. The certificates use $\epsilon_t^+ = c_t r + \binom{t}{2} r^2$ and $\epsilon_t^-(z) = \epsilon_t^+ + z 2^t r^2$ with $c_t = \sum_{j=3}^t \binom{t}{j}(j-1)$. These are larger: $c_t - a_t = b_t - \binom{t}{2}$ is the number of even-size subsets of $\{1, \dots, t\}$ with at least four elements (with a_t, b_t as in (12.2); proof of Lemma 12.2), so $\epsilon_t^+ - e_t^+ = \epsilon_t^-(z) - e_t^-(z) = (c_t - a_t)(r - r^2) \ge 0$; and c_t, $\binom{t}{2}$ are nondecreasing in t.

Lemma 14.15 (Completion variance)**.** *Assume $e_\Delta(p, 2n) < 1$, and let $x \in \mathcal{I}^+$ and $k \in U^+$.*

(a) (Range.) *If $0 < l \le C \le u$ on a probability space and $\bar{C} = \mathbb{E}C$, then $\mathrm{Var}(C)/\bar{C}^2 \le (u - \bar{C})(\bar{C} - l)/\bar{C}^2 \le (u - l)^2/(4ul)$. In particular $v_k(x) \le \delta_k'^2/(4(1 + \delta'_k))$ with $\delta'_k = R'_k - 1$ of (14.11).*

(b) (The internal neighbourhood.) *Let $S = \{i \in U^+ : i \sim k\}$, let $R = U^+ \setminus (\{k\} \cup S)$, and suppose that $G^+[S]$ is bipartite, with parts S_0, S_1. For $i \in S$ put $m_i = d_i - 1 - \deg_{G^+[S]}(i)$ and*

$$\alpha_i^- = 1 - \epsilon_{m_i}^-(2n), \qquad \alpha_i^+ = 1 + \epsilon_{m_i}^+, \qquad \alpha_i^{++} = 1 + \epsilon_{m_i+1}^+,$$
$$\kappa_i = \frac{2^{m_i+1}}{1 - \epsilon_{m_i+1}^-(2n)}, \qquad \kappa_\Phi = \frac{2^{r_k}}{1 - \epsilon_{r_k}^-(n)}, \qquad L_i = (\kappa_\Phi 2^{m_i} \alpha_i^+)^{1/2} r + 2^{m_i} r^2,$$

and assume that all displayed denominators are positive. For a nonempty set H of edges of $G^+[S]$ let P_H be the minimum of the numbers in (i), over all matchings satisfying the stated component condition (this set may be empty), and in (ii), over all edges $uv \in H$ (nonempty, since $H \ne \emptyset$).

(i) Matching bounds. *Let $M = \{u_1 v_1, \dots, u_m v_m\} \subseteq H$ be a nonempty matching, $u_i \in S_0$, $v_i \in S_1$, and let its* contracted digraph *on $\{1, \dots, m\}$ have an arc $i \to l$ for every edge of H between u_i and v_l, $i \ne l$. If its strongly connected components have size at most*

two, b of them of size two, put

$$B_H(M) = 2^b r^{|M|} \prod_{i\in V(M)} (2^{m_i+1}\alpha_i^{++})^{1/2} \prod_{i\in S\setminus V(M)} \alpha_i^{++}.$$

(ii) Two-variable bounds. *For an edge $uv \in H$, with $d_u = m_u + 1$, $d_v = m_v + 1$, f_u and f_v the numbers of other edges of H at u and at v, $s_u = \alpha_u^{++} + d_u 2^{d_u} r^2$, $s_v = \alpha_v^{++} + d_v 2^{d_v} r^2$ and $(C_0, C_1, C_2) = \sum_{i\le d_u,\, j\le d_v} \binom{d_u}{i}\binom{d_v}{j}\mathbf{c}(i + f_u, j + f_v)$,*

$$\Big(r^2(C_0 + rC_1 + r^2C_2) + 2nr^2\big(2^{d_u}s_v + 2^{d_v}s_u\big)\Big) \prod_{i\in S\setminus\{u,v\}} \alpha_i^{++}.$$

Here $\mathbf{c}(a, b) = (0, 0, 0)$ if $ab = 0$, $\mathbf{c}(1, 1) = (1, 0, 0)$, and otherwise $\mathbf{c}(a, b)$ is the vector of the coefficients of $1, r, r^2$ in $\mathcal{B}(a, b, 0)$ of (15.7) if $a \ge 2$ (in particular if both are at least two), and in $\mathcal{B}(b, a, 0)$ if $a < 2 \le b$.

Put

$$\mathrm{Lin}_k = \sum_{i\in S} L_i \prod_{j\in S\setminus\{i\}} \alpha_j^{++}, \qquad \mathrm{Pair}_k = \sum_{\emptyset\ne H\subseteq E(G^+[S])} P_H,$$

$$\mathrm{Coll}_k = 2^{|E(G^+[S])|} \prod_{i\in S} \alpha_i^{++} \sum_{\{i,j\}\subseteq S} \min(\kappa_i, \kappa_j)\, r^2, \qquad \mathrm{Exc}_k = \prod_{i\in S} \alpha_i^{+}\, (|R|\kappa_\Phi)^{1/2} r,$$

$\varepsilon_k^{\mathrm{loc}} = \mathrm{Lin}_k + \mathrm{Pair}_k + \mathrm{Coll}_k + \mathrm{Exc}_k$ *and* $\ell_k^{\mathrm{loc}} = \prod_{i\in S} \alpha_i^{-} - \varepsilon_k^{\mathrm{loc}}$. *If $\ell_k^{\mathrm{loc}} > 0$, then*

$$v_k(x) \le \big(\varepsilon_k^{\mathrm{loc}}/\ell_k^{\mathrm{loc}}\big)^2. \tag{14.17}$$

The bound depends on x only through the graph data, so it is uniform in x. If $G^+[S]$ is a forest, every nonempty matching is admissible in (i) with $b = 0$: a directed cycle of the contracted graph, followed through the matched edges, would be a cycle of $G^+[S]$. If $G^+[S]$ is a four-cycle together with isolated vertices, $b = 1$ exactly when H is the four-cycle and $|M| = 2$, and $b = 0$ otherwise. These are the two geometries of the root cuts: each internal neighbourhood of LL20 is a $K_{2,2}$ together with isolated vertices, and each internal neighbourhood of LL22 is a forest (the evaluator of Remark 14.30 checks that every internal neighbourhood is of one of these two kinds).

Proof. (a) $(u - C)(C - l) \ge 0$ gives $\mathbb{E}C^2 \le (u + l)\bar{C} - ul$, so $\operatorname{Var} C \le (u - \bar{C})(\bar{C} - l)$, and $(u - l)^2\bar{C}^2 - 4ul(u - \bar{C})(\bar{C} - l) = ((u + l)\bar{C} - 2ul)^2 \ge 0$. For the second statement take the uniform probability on Φ_k and $C = C_k$, which lies in $[L_k, U_k]$ with $U_k/L_k = R'_k$ (instance (A) of Section 14.4).

(b) *Conditioning.* The vertices of R are not adjacent to k. Let $\mathcal{Z}$ be the set of labellings z of R that are injective, avoid $x(U^+)$, and satisfy the edges inside R and the cross constraints of R with x; it does not depend on a. A completion counted by $C_k(a)$ is a triple (a, y, z) with $z \in \mathcal{Z}$, $a \notin z(R)$ and a labelling y of S satisfying the remaining constraints. For $i \in S$ the constraints of y_i with the fixed labels are its r_i cross constraints with x and its edges to R, m_i in all, with distinct roots; let Φ_i^z be the set of values outside $x(U^+) \cup z(R)$ (at most $2n$ labels) satisfying them, and $A_i = 2^{m_i}|\Phi_i^z|/p$, so that $\alpha_i^- \le A_i \le \alpha_i^+$. For $a \in \Phi_k \setminus z(R)$ let $\mathcal{A}_i(a) = \{y \in \Phi_i^z : y \ne a,$ the constraint of the edge between i and k holds$\}$, a set with $m_i + 1$ distinct roots, and $\rho_i = 2^{m_i+1}\mathbf{1}_{\mathcal{A}_i(a)}$ on $\mathbb{F}_p$. Then $q_i := \mathbb{E}\rho_i = 2^{m_i+1}|\mathcal{A}_i(a)|/p \le \alpha_i^{++}$ and $\|\rho_i\|_2^2 = 2^{m_i+1}q_i \le 2^{m_i+1}\alpha_i^{++}$. Define the normalized count $\mathcal{H}(a, z) = 2^{|E(G^+[S])|}\prod_{i\in S}(2^{m_i+1}/p) \cdot \#\{y\}$ for $a \notin z(R)$, and $\mathcal{H}(a, z) = 0$ if $a \in z(R)$. Then $C_k(a) = c_0 \sum_{z\in\mathcal{Z}} \mathcal{H}(a, z)$ with a constant $c_0 > 0$ independent of a and z, and $\mathcal{Z} \ne \emptyset$ because $C_k \not\equiv 0$ (Lemma 13.3(c)). For $a \notin z(R)$, with y uniform on $\mathbb{F}_p^S$,

$$\mathcal{H}(a, z) = \mathbb{E}_y\Big[\prod_{i\in S} \rho_i(y_i)\, \mathbf{1}[y \text{ injective}] \prod_{e\in E(G^+[S])} (1 + \chi_e(y))\Big],$$

where $\chi_e(y) = \chi(y_w - y_{w'})$ for the edge $e = (w', w)$: on injective y, $1 + \chi_e(y)$ is twice the indicator of the edge constraint $y_w - y_{w'} \in Q_p^*$.

The linear part. For $a \in \Phi_k \setminus z(R)$, $q_i(a) = \frac{2^{m_i}}{p} \sum_{y \in \Phi_i^z} (1 + s_i \chi(y - a) - \mathbf{1}_{y=a}) = A_i + f_i(a) - \frac{2^{m_i}}{p} \mathbf{1}_{\Phi_i^z}(a)$, with $s_i \in \{\pm 1\}$ the sign of the constraint of the edge between i and k and $f_i = s_i 2^{m_i} C \mathbf{1}_{\Phi_i^z}$. By Lemma 14.1, $\|f_i\|_2^2 \le 4^{m_i} r^2 |\Phi_i^z|/p = 2^{m_i} A_i r^2$. The uniform probability on Φ_k has density at most κ_Φ with respect to the uniform probability on $\mathbb{F}_p$, so $\|(q_i - A_i)\mathbf{1}_E\|_{L^2(\Phi_k)} \le L_i$ on $E = \Phi_k \setminus z(R)$. There $q_j \le \alpha_j^{++}$ and $A_j \le \alpha_j^+ \le \alpha_j^{++}$; telescoping $\prod_i q_i - \prod_i A_i$ and Minkowski give $\|(\prod_i q_i - \prod_i A_i)\mathbf{1}_E\|_{L^2(\Phi_k)} \le \mathrm{Lin}_k$.

Coincidences. Let $\mathcal{H}'$ be the expression for $\mathcal{H}$ with the indicator of injectivity replaced by the indicator of injectivity inside S_0 and inside S_1. The product over $E(G^+[S])$ lies in $[0, 2^{|E(G^+[S])|}]$, and on injective y the two agree. Hence $0 \le \mathcal{H}' - \mathcal{H} \le 2^{|E(G^+[S])|} \sum \mathbb{E}_y[\prod_i \rho_i(y_i) \mathbf{1}_{y_i = y_j}]$ over the pairs $\{i, j\}$ with one vertex in each part. For any pair, $\mathbb{E}_y[\prod_l \rho_l \mathbf{1}_{y_i = y_j}] = \prod_l q_l \cdot |\mathcal{A}_i \cap \mathcal{A}_j|/(|\mathcal{A}_i||\mathcal{A}_j|) \le \prod_l q_l \cdot \min(\kappa_i, \kappa_j)/p$, since $\mathcal{A}_i$ excludes at most $2n$ labels and so $|\mathcal{A}_i| \ge p/\kappa_i$. Expanding the product over the edges, $\mathcal{H}' = \sum_{H \subseteq E(G^+[S])} T_H$ with

$$T_H = \mathbb{E}_y\Big[\prod_{i \in S} \rho_i(y_i)\, \mathbf{1}[y \text{ injective inside } S_0 \text{ and inside } S_1] \prod_{e \in H} \chi_e(y) \Big],$$

and $\prod_i q_i - T_\emptyset$ lies in $[0, \prod_l q_l \sum \min(\kappa_i, \kappa_j)/p]$, the sum over the pairs inside one part. These two lists of pairs are disjoint, so on E

$$\Big| \mathcal{H} - \prod_i q_i - \sum_{\emptyset \ne H} T_H \Big| \le \mathrm{Coll}_k.$$

Matching bounds. Let $M \subseteq H$ be a matching as in (i). Freeze the labels of $S \setminus V(M)$. The factors of T_H that involve a matched vertex and an unmatched one (characters of edges of H, and the indicators of distinctness inside a part) are functions of the matched label of modulus at most one; absorb them into ρ_i, $i \in V(M)$, which does not increase $\|\rho_i\|_2$. The remaining kernel between the labels of $V(M) \cap S_0$ and of $V(M) \cap S_1$ is a product of signed characters of the core $H[V(M)]$, with the matching edges as diagonal; all edges of H join the two parts, and the distinctness inside each part restricted to $V(M)$ is the projection onto injective tuples. By Lemma 16.4(a) (if $b = 0$) and (b), the compression of this kernel by these projections has norm at most $2^b r^{|M|}$, so the inner expectation over the matched labels is at most $2^b r^{|M|} \prod_{i \in V(M)} \|\rho_i\|_2$ in modulus, by Cauchy–Schwarz. The remaining factors among the unmatched labels have modulus at most one, and $\mathbb{E}_y \prod_{i \notin V(M)} \rho_i(y_i) = \prod_{i \notin V(M)} q_i \le \prod_{i \notin V(M)} \alpha_i^{++}$. Hence $|T_H| \le B_H(M)$.

Two-variable bounds. Let $uv \in H$, $u \in S_0$, $v \in S_1$, and freeze all labels except y_u, y_v. Off $x(U^+) \cup z(R) \cup \{a\}$, which contains its roots, $\rho_u(y_u)$ equals the soft product $\tilde\rho_u(y_u) = \prod_\beta (1 + s_\beta \chi(y_u - \beta))$ over its d_u constraints with the fixed labels and a (the *optional centres*; a centre here is a label β in a factor $\chi(y_u - \beta)$, not a principal law); the characters of the other edges of H at u have the frozen labels of S_1 as centres (the f_u *forced centres*). The optional centres lie in $x(U^+) \cup z(R) \cup \{a\}$ and the forced centres are labels of $\mathcal{A}_w$, $w \in S_1$, distinct by the injectivity inside S_1; so all $d_u + f_u$ centres at u are distinct, and likewise at v. Expanding the optional factors, the average over $(y_u, y_v) \in \mathbb{F}_p^2$ of $\tilde\rho_u \tilde\rho_v$ times the characters is a sum of $\binom{d_u}{i}\binom{d_v}{j}$ averages $\mathbb{E}_{y,y'} \chi(f(y)g(y')(y' - y))$ with $\deg f = i + f_u$, $\deg g = j + f_v$, each multiplied by a sign. If $\deg g = 0$ the sum over y' vanishes, and similarly for $\deg f = 0$; if $\deg f = \deg g = 1$ the two-root identity (4.2) gives modulus at most p; otherwise Theorem 15.13 gives modulus at most $p\,\mathcal{B}(\cdot, \cdot, s) \le p\,\mathcal{B}(\cdot, \cdot, 0)$. So this average is at most $r^2(C_0 + rC_1 + r^2 C_2)$. Restoring the excluded values: $\tilde\rho_u \ge 0$, it agrees with 2^{d_u} times the indicator of its constraints off its d_u roots and is at most 2^{d_u} at the roots, so $\mathbb{E}\tilde\rho_u \le s_u$; each of y_u, y_v must avoid at most $2n$ labels (the excluded labels and the frozen labels of its own part); so the restriction changes the average by at most $\frac{2n}{p}(2^{d_u} s_v + 2^{d_v} s_u)$. The edge uv

vanishes on the diagonal, and the coincidences between the two parts are charged in Coll_k. The other factors are bounded as in the matching case. Hence $|T_H|$ is at most the bound (ii), and so $|T_H| \le P_H$.

Averaging. On E, $|\mathcal{H} - \prod_i A_i| \le |\prod_i q_i - \prod_i A_i| + \mathrm{Pair}_k + \mathrm{Coll}_k$. On $\Phi_k \setminus E$ (where $a \in z(R)$), $\mathcal{H} = 0$ and $|\mathcal{H} - \prod_i A_i| \le \prod_i \alpha_i^+$, while the uniform probability of this set is at most $|R|/|\Phi_k| \le |R|\kappa_\Phi/p$. Hence $\|\mathcal{H}(\cdot,z) - \prod_i A_i\|_{L^2(\Phi_k)} \le \varepsilon_k^{\mathrm{loc}}$ for every $z \in \mathcal{Z}$. Let $\bar{\mathcal{H}} = \mathbb{E}_z \mathcal{H}(\cdot,z)$ and $\bar{A} = \mathbb{E}_z \prod_i A_i \ge \prod_i \alpha_i^-$, with z uniform on $\mathcal{Z}$. By Minkowski, $\left\|\bar{\mathcal{H}} - \bar{A}\right\|_{L^2(\Phi_k)} \le \varepsilon_k^{\mathrm{loc}}$, so $\mathbb{E}_{\Phi_k}\bar{\mathcal{H}} \ge \bar{A} - \varepsilon_k^{\mathrm{loc}} \ge \ell_k^{\mathrm{loc}} > 0$ and $\mathrm{Var}_{\Phi_k}\bar{\mathcal{H}} \le \left\|\bar{\mathcal{H}} - \bar{A}\right\|^2 \le (\varepsilon_k^{\mathrm{loc}})^2$. As $C_k = c_0|\mathcal{Z}|\bar{\mathcal{H}}$, the relative variance of C_k is that of $\bar{\mathcal{H}}$. □

The relative variance enters the coupling block in two ways: through the error functions E_k (the input Γ_E) and through the Gram matrix of the conditional densities (the input ε_H). In Lemmas 14.16 and 14.17 below, $\bar{v}_k$ is any number with $v_k(x) \le \bar{v}_k$ for every x, for instance the minimum of the bounds of Lemma 14.15(a) and (b).

Lemma 14.16 (Projection-error diagonal)**.** *In instance* (A)*, for every $x \in \mathcal{I}^+$ and $k \in U^+$,*

$$\|E_k(\cdot;x)\|^2 \le \frac{n4^{r_k}}{p} + \frac{2^{r_k}}{1-\hat{e}_k}(\bar{v}_k + \hat{e}_k^2), \tag{14.18}$$

with $\hat{e}_k = e_{r_k}(p,n)$. Consequently Γ_E^{loc}, the sum over k of the right side, satisfies $\|\sum_k w_k E_k(\cdot;x)\|^2 \le \Gamma_E^{\mathrm{loc}}|w|^2$ for every x, and $\min(\Gamma_E^{\mathrm{loc}}, \hat{E}^2)$ is an admissible input Γ_E of Lemma 14.14.

Proof. Let $\alpha = |\Phi_k|/p$ and $\bar{C}_k = \mathbb{E}_{\Phi_k} C_k$. On Φ_k, $pP_k = \alpha^{-1}C_k/\bar{C}_k$ and $\varphi_k = 2^{r_k}$, so $E_k = \alpha^{-1}(C_k/\bar{C}_k - 1) + (\alpha^{-1} - 2^{r_k})$; off $\Phi_k \cup x(U^+)$, $E_k = 0$; and $|E_k| \le 2^{r_k}$ on $x(U^+)$. Since $\sum_{\Phi_k}(C_k/\bar{C}_k - 1) = 0$, the cross term vanishes and $\|E_k\|^2 \le \alpha^{-1}v_k + \alpha^{-1}(1 - 2^{r_k}\alpha)^2 + n4^{r_k}/p$. Finally $\alpha^{-1} \le 2^{r_k}/(1-\hat{e}_k)$ and $|1 - 2^{r_k}\alpha| \le \hat{e}_k$. The second statement is Cauchy–Schwarz over k, and the minimum of two admissible inputs is admissible. □

A *weighted Schur bound* of a nonnegative symmetric matrix Y is a number $\max_k (Yw)_k/w_k$ for a vector w with positive entries. It bounds the norm of every Hermitian matrix X with $|X_{kl}| \le Y_{kl}$, since $|u^*Xu| \le \sum Y_{kl}|u_k||u_l| \le \sum_k |u_k|^2 (Yw)_k/w_k$.

Lemma 14.17 (Conditional-density Gram refinement)**.** *In instance* (A) *put $y_k = 2^{r_k}\bar{v}_k/(1-\hat{e}_k)$, let Y be the matrix with $Y_{kk} = y_k$, $Y_{kl} = \min\{(y_k y_l)^{1/2}, \Upsilon_{kl}\delta_k'\delta_l'\}$ for consistent $k \ne l$ and $Y_{kl} = 0$ otherwise, and let Γ_D be the smaller of $\sum_k y_k$ and a weighted Schur bound of Y. Let h_0 be a weighted Schur bound of the matrix (β_{kl}) of* (14.13)*. Then*

$$\widetilde{\mathbf{H}} \preceq \mathbf{H} + \varepsilon_H^{\mathrm{loc}} I, \qquad \varepsilon_H^{\mathrm{loc}} = h_0 + 2\big(\Gamma_D(\bar{\lambda}(\mathbf{H}) + h_0)\big)^{1/2} + \Gamma_D. \tag{14.19}$$

So $\min(\varepsilon_H' + \varepsilon_X, \varepsilon_H^{\mathrm{loc}})$ is an admissible input ε_H of Lemma 14.14.

Proof. Fix x. As in the proof of Lemma 14.12, $\Psi_k = \psi_k + D_k$ with $D_k = (1+\psi_k)\delta_k = \alpha_k^{-1}\mathbf{1}_{\Phi_k}(C_k/\bar{C}_k - 1)$, $\alpha_k = |\Phi_k|/p$ and $\bar{C}_k = \mathbb{E}_{\Phi_k} C_k$. The Gram matrix of the ψ_k differs from $\mathbf{H}$ entrywise by at most β_{kl}, so its norm distance from $\mathbf{H}$ is at most h_0, and the synthesis operator $A_x\colon w \mapsto \sum_k w_k\psi_k$ satisfies $\|A_x w\|^2 \le w^*\mathbf{H}w + h_0|w|^2$, and so $\|A_x w\|^2 \le (\bar{\lambda}(\mathbf{H}) + h_0)|w|^2$. Next $\|D_k\|^2 = \alpha_k^{-1}v_k \le y_k$; for consistent $k \ne l$, $|\langle D_k, D_l\rangle| \le \min\{\|D_k\|\,\|D_l\|, \Upsilon_{kl}\delta_k'\delta_l'\}$, by the intersection count of the proof of Lemma 14.12; and for inconsistent pairs the supports are disjoint. So the synthesis operator $D_x\colon w \mapsto \sum_k w_k D_k$ has $\|D_x\|^2 \le \Gamma_D$ (the trace bound $\|D_x\|^2 \le \sum_k \|D_k\|^2$ needs no positivity). Finally $w^*\widetilde{\mathbf{H}}w = \mathbb{E}_{m_p}\|A_x w + D_x w\|^2 \le \mathbb{E}(\,\|A_x w\|^2 + 2\,\|A_x w\|\,\|D_x w\| + \|D_x w\|^2\,)$, which is at most $w^*\mathbf{H}w + \varepsilon_H^{\mathrm{loc}}|w|^2$. □

At the root cuts the certificates take, at each threshold, the smaller of each refined input and the standard one: $\Gamma_E = \min(\Gamma_E^{\mathrm{loc}}, \hat{E}^2)$, with $\bar{v}_k$ the bound (b) of Lemma 14.15 (each P_H the smaller of its matching and two-variable bounds), and $\varepsilon_H = \min(\varepsilon'_H + \varepsilon_X, \varepsilon_H^{\mathrm{loc}})$, with $\bar{v}_k$ the minimum of the bounds (a) and (b); they then recompute $\varepsilon_{\mathrm{rem}}$ and η_B and repeat the exact positive-semidefiniteness tests of (14.21). All these inputs are admissible (Lemma 14.29(d)), with the Schur weights fixed at the threshold.

14.6. **The character expansion.** The part of the cut kernel orthogonal to the degree-one space is handled by expanding it in the characters of its cross pairs. Each set S of cross pairs contributes one operator T_S; sets whose pairs all share one endpoint (*stars*) are computed exactly below, and the remaining sets are grouped into a nonstar term and a collision term, bounded in Section 16. For a cross pair $e = (j,k) \in E_\times$ put $\chi_e(x,x') = \chi(\varsigma_{jk}(x'_k - x_j))$, and for $S \subseteq E_\times$ put $\chi_S = \prod_{e\in S}\chi_e$; let T_S be the operator on $L^2(\mu)$ with kernel χ_S relative to $\mu\otimes\mu$, $(T_S h)(x) = \mathbb{E}_{x'\sim\mu}[\chi_S(x,x')h(x')]$. Let $J(S)$ and $K(S)$ be the sets of x-endpoints and x'-endpoints of S. A nonempty S is a *column star* if $|K(S)| = 1$ (all its pairs have the same x'-endpoint), a *row star* if $|J(S)| = 1$, and a *nonstar* otherwise; single pairs are stars of both kinds. The *transpose* of S is $S^{\mathsf{t}} = \{(k,j) : (j,k) \in S\} \subseteq E_\times$ (Lemma 13.2). Put

$$K_0(x,x') = \mathbf{1}\{x(U^+) \cap x'(U^+) = \emptyset\} \prod_{e\in E_\times} (1 + \chi_e(x,x')), \qquad Z_0 = \mathbb{E}_{\mu\otimes\mu} K_0,$$

let $T_{\mathrm{ns}} = \sum_{S \text{ nonstar}} T_S$, and let T_{coll} be the operator with the kernel $\mathbf{1}\{x(U^+) \cap x'(U^+) \neq \emptyset\}\prod_{e\in E_\times}(1+\chi_e(x,x'))$. We use the same letter for a kernel and its operator relative to $\mu\otimes\mu$.

Lemma 14.18 (Expansion)**.** *On $\mathcal{I}^+ \times \mathcal{I}^+$, $K_0 = 2^{m_\times} K$, so $Z_0 = 2^{m_\times} Z$ and $\mathsf{A}_p = K_0/Z_0$; and, as operators on $L^2(\mu)$,*

$$K_0 = \mathbf{1}\otimes\mathbf{1} + \sum_{S \text{ star}} T_S + T_{\mathrm{ns}} - T_{\mathrm{coll}}. \tag{14.20}$$

All these kernels are real; the kernel of T_{coll} is nonnegative and symmetric, and $T_S^* = T_{S^{\mathsf{t}}}$ for every S.

Proof. $\prod_e(1+\chi_e) = \sum_S \chi_S$ identically, and $\chi_\emptyset = 1$ gives $\mathbf{1}\otimes\mathbf{1}$. If $x(U^+)\cap x'(U^+) = \emptyset$, then $x'_k \neq x_j$ for every pair, and $1 + \chi(y) = 2A_p(y)$ for $y \neq 0$, so the product is $2^{m_\times}K$ by (13.2); otherwise $K = 0$ and $K_0 = 0$. Each factor of the collision kernel lies in $\{0,1,2\}$. Exchanging x and x' maps χ_e, $e = (j,k)$, to $\chi(\varsigma_{jk}(x_k - x'_j)) = \chi(\varsigma_{kj}(x'_j - x_k)) = \chi_{(k,j)}$, because $\varsigma_{kj} = -\varsigma_{jk}$ (Lemma 13.2); this gives the symmetry of the collision kernel and $T_S^* = T_{S^{\mathsf{t}}}$. □

For $x \in \mathcal{I}^+$ and $t \in \mathbb{F}_p$ let $\varphi_k(t;x) = \prod_{j\in N(k)}(1 + \chi(\varsigma_{jk}(t - x_j)))$ as in Section 14.4, $h_{\star,k}(x) = \mathbb{E}_t\varphi_k(t;x) - 1$, and let $h_\star = \sum_k h_{\star,k}$; this is the star function of (15.9) in Section 15, with its centre t integrated out. The *standard star constants* are

$$P_{\mathrm{st}} = r^2 \sum_{k\in U^+} \binom{r_k}{2}, \qquad W_{\mathrm{st}} = r \sum_{k\in U^+} \sum_{s=3}^{r_k} \binom{r_k}{s}(s-1).$$

Lemma 14.19 (Stars)**.** *Assume $e_\Delta(p,2n) < 1$.*

(a) *Let $S = J\times\{k\}$ be a column star, $\emptyset \neq J \subseteq N(k)$, and $h \in L^2(\mu)$. Then $(T_S h)(x) = \mathbb{E}_t[\chi_{J,k}(t;x)\,\mathbb{E}_\mu[h \mid x_k = t]]$. In particular $T_S h = (\mathbb{E}_\mu h)\,T_S\mathbf{1}$ if $\mathbb{E}_\mu[h \mid x_k]$ is constant, and $T_S\mathbf{1} = (\varepsilon_J)_k w_J$ with $w_J(x) = \mathbb{E}_t\prod_{j\in J}\chi(t - x_j)$; on $\mathcal{I}^+$, $w_J = 0$ if $|J| = 1$, $w_J = -q$ if $|J| = 2$, and $|w_J| \le (|J|-1)r$ if $|J| \ge 3$.*

(b) *For a row star S, $T_S = (T_{S^{\mathsf{t}}})^*$ with S^{t} a column star, and $T_S\mathbf{1}$ is a constant.*

(c) *The column stars at k together map $\mathbf{1}$ to $h_{\star,k}$, and $\sum_{S \text{ star}} T_S\mathbf{1} = h_\star + \mathbb{E}_\mu h_\star$; hence $\langle \mathbf{1}, \sum_{S \text{ star}} T_S\mathbf{1}\rangle_\mu = 2\mathbb{E}_\mu h_\star$.*

(d) $|\mathbb{E}_\mu h_\star| \le P_{\rm st} + W_{\rm st}$ *and* $\|h_\star - \mathbb{E}_\mu h_\star\|_\mu \le W_{\rm st}$.

Proof. (a) $\chi_S(x,x') = \chi_{J,k}(x'_k;x)$ depends on x' only through x'_k, which is uniform under μ. The values of w_J are Lemma 15.18: for $|J|=1$ the sum $\sum_t \chi(t-x_j)$ vanishes, for $|J|=2$ use (4.2) (x is injective), and for $|J|\ge 3$ Theorem 4.7. The identity $\chi_{J,k} = (\varepsilon_J)_k \prod_j \chi(t-x_j)$ is the multiplicativity of χ. (b) The first statement is Lemma 14.18. $(T_S\mathbf{1})(x) = \mathbb{E}_{x'} \prod_{k\in K(S)} \chi(\varsigma_{jk}(x'_k - x_j))$ with $\{j\} = J(S)$; replacing x' by $x' + x_j\mathbf{1}$, which preserves μ, shows that it does not depend on x. (c) Summing (a) over $\emptyset \ne J \subseteq N(k)$ gives $\mathbb{E}_t[(\varphi_k(t;x)-1)\cdot 1] = h_{\star,k}$. The single pairs contribute $w_{\{j\}} = 0$, so the row stars (with at least two pairs) contribute a constant equal to $\sum \langle \mathbf{1}, T_S\mathbf{1}\rangle = \sum \langle T_{S^{\rm t}}\mathbf{1}, \mathbf{1}\rangle$, which is the total mean of the column stars with at least two pairs, $\mathbb{E}_\mu h_\star$. (d) By (a), $h_\star = -q\sum_k \sum_{|J|=2} (\varepsilon_J)_k + h^{\ge 3}$ with $h^{\ge 3} = \sum_k \sum_{|J|\ge 3} (\varepsilon_J)_k w_J$, where $|(\varepsilon_J)_k| \le 1$. The first term is a constant of modulus at most $P_{\rm st}$, and $\|h^{\ge 3}\|_\infty \le W_{\rm st}$. Hence $h_\star - \mathbb{E}_\mu h_\star = h^{\ge 3} - \mathbb{E}_\mu h^{\ge 3}$, of norm at most $\|h^{\ge 3}\|_\mu \le W_{\rm st}$. □

14.7. **Centring at the half-law mean.** The test functions are centred at their mean under the half law μ. Centring at the mean under m_p instead leaves $\mathbb{E}_\mu F_\perp \ne 0$, and then every star and every four-cycle can contribute through $\mathbb{E}_\mu F_\perp$; centring at $\mathbb{E}_\mu F$ removes these terms exactly, in exchange for one cross term $2c\,\langle \varrho - 1, F_\perp\rangle_\mu$ between the constant and the remainder, which is of second order. Let $\mathcal{H}_\perp = (\mathbb{C}\mathbf{1} \oplus \mathcal{V})^\perp \subseteq L^2(\mu)$.

Proposition 14.20 (Centring at the μ-mean)**.** *Assume $e_\Delta(p,2n) < 1$ and $\varepsilon_V < 1$.*

(a) *For real $F \in L^2(\mu)$ let $c = \mathbb{E}_\mu F$, $F_V = \Pi_{\mathcal{V}}(F-c) = \sum_k g_k(x_k)$ and $F_\perp = F - c - F_V \in \mathcal{H}_\perp$. Every $h \in \mathcal{H}_\perp$ has $\mathbb{E}_\mu[h \mid x_k] = 0$ for all k; hence $\langle h', T_S h\rangle_\mu = 0$ for $h, h' \in \mathcal{H}_\perp$ if $S = \emptyset$ or S is a star. Moreover $\langle \mathbf{1}, \mathsf{A}_p v\rangle_\mu = 0$ for $v \in \mathcal{V}$, and $\|F\|_\mu^2 = c^2 + \|F_V\|_\mu^2 + \|F_\perp\|_\mu^2 \ge (1-\varepsilon_V)(c^2 + \|g\|^2 + \|F_\perp\|_\mu^2)$.*

(b) $Q(F) = c^2 + 2c\,\langle \varrho - 1, F_\perp\rangle_\mu + Q(F_V) + 2\,\langle F_\perp, \mathsf{A}_p F_V\rangle_\mu + Q(F_\perp)$.

(c) (Inputs.) *Let $N_{\rm ns}$, $D_{\rm col}$, $M_\star$ satisfy $\|T_{\rm ns}\| \le N_{\rm ns}$, $\|T'_{\rm coll}\| \le D_{\rm col}$ and $\|h_\star - \mathbb{E}_\mu h_\star\|_\mu \le M_\star$, and let $\varepsilon_Z < 1$ satisfy $Z_0 \ge 1 - \varepsilon_Z$; for instance $\varepsilon_Z = Z_\star + N_{\rm ns} + D_{\rm col}$ with $2|\mathbb{E}_\mu h_\star| \le Z_\star$, or $\varepsilon_Z = \min(\varepsilon'_Z, \varepsilon''_Z)$ for two numbers $\varepsilon'_Z, \varepsilon''_Z$ formed in this way from different admissible pairs $(Z_\star, D_{\rm col})$. Put $R_Z = (1-\varepsilon_Z)^{-1}$ and $\varepsilon_\varrho = R_Z(M_\star + N_{\rm ns} + D_{\rm col})$.*

(c1) $Z_0 - 1 = 2\mathbb{E}_\mu h_\star + \langle \mathbf{1}, (T_{\rm ns} - T_{\rm coll})\mathbf{1}\rangle_\mu$, *and* $2^{-m_\times}/Z \le R_Z$.

(c2) $\sup_{\mathcal{I}^+} \varrho \le R_+ := \dfrac{\prod_{i=1}^n (1 + e_{t_i}(p,2n))}{(1-\varepsilon_Z)\prod_{i\in U^+}(1 - e_{d_i^0}(p,n))}$, *with the deletion degrees t_i.*

(c3) $\|\varrho - 1\|_\mu = \|P_0 \mathsf{A}_p \mathbf{1}\|_\mu \le \varepsilon_\varrho$.

(c4) $|\,\langle h, \mathsf{A}_p h\rangle_\mu\,| \le R_Z(N_{\rm ns} + D_{\rm col})\,\|h\|_\mu^2$ *for $h \in \mathcal{H}_\perp$.*

(c5) $c^2 + 2c\,\langle \varrho - 1, F_\perp\rangle_\mu + Q(F_\perp) \ge -c_\perp r\,\|F_\perp\|_\mu^2$ *with $c_\perp = (R_Z(N_{\rm ns} + D_{\rm col}) + \varepsilon_\varrho^2)/r$.*

(d) (Standard star inputs.) *The numbers $M_\star = W_{\rm st}$ and $Z_\star = 2(P_{\rm st} + W_{\rm st})$ are admissible in* (c)*, by Lemma 14.19(d).*

Proof. (a) $h \perp \mathbf{1}$ and $h \perp \mathcal{V}_k$, and every function of x_k is a constant plus an element of $\mathcal{V}_k$; so h is orthogonal to all functions of x_k. For $S = \emptyset$, $\langle h', T_\emptyset h\rangle = \overline{\mathbb{E}h'}\,\mathbb{E}h = 0$. For a column star at k, $T_S h = 0$ by Lemma 14.19(a); for a row star, $\langle h', T_S h\rangle = \langle T_{S^{\rm t}} h', h\rangle = 0$ by (b) of that lemma. Next $\langle \mathbf{1}, \mathsf{A}_p v\rangle = \mathbb{E}_\lambda v(x') = \mathbb{E}_{m_p} v = 0$ (Lemma 13.3(b),(d)). The three functions c, F_V, $F_\perp$ are pairwise orthogonal, and Lemma 14.8(c) gives $\|F_V\|^2 \ge (1-\varepsilon_V)\,\|g\|^2$. (b) Put $G = F_V + F_\perp$, so $\mathbb{E}_\mu G = 0$. As A_p is self-adjoint with $\mathsf{A}_p\mathbf{1} = \varrho$, $Q(c+G) = c^2\mathbb{E}_\mu\varrho + 2c\,\langle \varrho, G\rangle + Q(G)$, with $\mathbb{E}_\mu\varrho = 1$, $\langle \varrho, G\rangle = \langle \varrho - 1, G\rangle$ and $\langle \varrho - 1, F_V\rangle = \mathbb{E}_{m_p} F_V - \mathbb{E}_\mu F_V = 0$. (c1) Pair (14.20) with $\mathbf{1}$ and use Lemma 14.19(c). By Lemma 14.18, $2^{-m_\times}/Z = 1/Z_0$. (c2) By the proof of Lemma 13.3(b),(c), $\varrho(x) = N(x)/(Z|\mathcal{I}^+|)$ with $N(x) \le \prod_i p 2^{-t_i}(1 + e_{t_i}(p,2n))$, and $\sum_i t_i = |E(G^+)| + m_\times$. By

Lemma 12.5(a) for G^+ (any order, $t_i^0 \le d_i^0$), $|\mathcal{I}^+| \ge p^n 2^{-|E(G^+)|}\prod_i(1-e_{d_i^0}(p,n))$. With $Z \ge 2^{-m_\times}(1-\varepsilon_Z)$ the powers of p and 2 cancel. (c3) $\mathbb{E}_\mu \varrho = 1$, so $\varrho - 1 = P_0\mathsf{A}_p\mathbf{1}$. By (14.20) and Lemma 14.19(c), $P_0K_0\mathbf{1} = (h_\star - \mathbb{E}_\mu h_\star) + P_0(T_{\mathrm{ns}} - T_{\mathrm{coll}})\mathbf{1}$, of norm at most $M_\star + N_{\mathrm{ns}} + D_{\mathrm{col}}$; divide by $Z_0 \ge 1-\varepsilon_Z$. (c4) By (a) only $T_{\mathrm{ns}} - T_{\mathrm{coll}}$ contributes to $\langle h, K_0 h\rangle$; divide by Z_0. (c5) $c^2 + 2c\langle \varrho - 1, F_\perp\rangle \ge -\langle \varrho - 1, F_\perp\rangle^2 \ge -\varepsilon_\varrho^2 \|F_\perp\|^2$, and (c4) bounds $Q(F_\perp)$. □

The proposition reduces the perpendicular block to the four numbers N_{ns}, D_{col}, $M_\star$ and ε_Z: after centring at the μ-mean no star term survives on $\mathcal{H}_\perp$, and every nonstar set, a four-cycle included, lies in a class of a nonstar partition, whose classes together are bounded by N_{ns} (Lemma 16.2 and Theorem 16.9). The two remaining inputs are the coincidence bound D_{col} (Lemma 16.15) and the star marginal $M_\star$ and mean $Z_\star$: the standard constants of (d), or the sharper ordered-star bounds of Proposition 15.26 and Lemma 15.29.

14.8. **Route A (conditional laws, centring at the half-law mean).** Route A centres test functions at their mean under the half law and works through the conditional law of the opposite half. The theorem packages the estimates of the previous subsections into one finite test at the smallest prime p_0; hypotheses (R1)–(R3) name exactly which earlier estimates must be supplied.

Theorem 14.21 (Spectral criterion centred at the half-law mean (Route A))**.** *Let $(G, \vartheta, U^+, \mathfrak{o})$ be spectral data satisfying* (S1) *of Definition 13.1 at p_0, put $r_0 = p_0^{-1/2}$, and fix a class $\mathfrak{c} \in \{1,3\}$. Suppose that the following hold at every prime $p \ge p_0$ with $p \equiv \mathfrak{c} \pmod 4$, with $r = p^{-1/2}$, where every input is a function of $r \in (0, r_0]$, and N_{ns}/r, D_{col}/r, $M_\star/r$, ε_Z, Γ_E, ε_H and $\bar\zeta$ are admissible.*

- (R1) (Degree one.) *η_Q and ε_V are the quantities* (14.9) *and* (14.10)*, with a fixed $\Lambda_D \ge \big\|\Delta^E_{(\mathfrak{c})}\big\|$.*
- (R2) (Perpendicular inputs.) *N_{ns}, D_{col}, $M_\star$ and $\varepsilon_Z < 1$ are as in Proposition 14.20*(c)*, and $\varepsilon_Z(r_0) < 1$.*
- (R3) (Coupling.) *ε_H, Γ_E and, if $\mathfrak{c} = 1$, $\bar\zeta$ satisfy the hypotheses of Lemma 14.14 at p, and η_B is the quantity* (14.15) *formed with them, with R_+ of Proposition 14.20*(c2) *(or $R_+ = R_{\mathrm{cut}}$) and with fixed bounds Λ_+, Λ_{nc}, $\bar\lambda(\mathbf{H})$ for the matrices of class $\mathfrak{c}$.*

Put $R_Z = (1-\varepsilon_Z)^{-1}$, $\varepsilon_\varrho = R_Z(M_\star + N_{\mathrm{ns}} + D_{\mathrm{col}})$ and $c_\perp = \big(R_Z(N_{\mathrm{ns}} + D_{\mathrm{col}}) + \varepsilon_\varrho^2\big)/r$. Let $\bar c_\perp, \bar\eta_Q, \bar\eta_B, \bar\varepsilon_V$ be upper bounds of the values of $c_\perp, \eta_Q, \eta_B, \varepsilon_V$ at r_0, with $\bar\varepsilon_V < 1$. Suppose that for each sign $\pm$ there is $\tau^{\mathrm{sp}}_{\mathfrak{c},\pm} > 0$ with

$$\mathbf{M}_{\mathfrak{c},\pm} := \mathbf{A}_{\mathfrak{c},\pm} - \frac{\mathbf{H}^+_{(\mathfrak{c})} - (\Delta^E_{(\mathfrak{c})})^{\mathsf{T}}\Delta^E_{(\mathfrak{c})}}{\tau^{\mathrm{sp}}_{\mathfrak{c},\pm}} + \Big(\bar c_\perp + \tau^{\mathrm{sp}}_{\mathfrak{c},\pm} - \bar\eta_Q - \frac{\bar\eta_B}{\tau^{\mathrm{sp}}_{\mathfrak{c},\pm}}\Big) I \succeq 0. \tag{14.21}$$

For $\mathfrak{c} = 3$, if $\tau^{\mathrm{sp}}_{3,+} = \tau^{\mathrm{sp}}_{3,-}$ the two matrices are complex conjugate, so one of the two tests suffices. Let

$$g_{\mathfrak{c}} \;\ge\; \max_{\pm} \frac{\bar c_\perp + \tau^{\mathrm{sp}}_{\mathfrak{c},\pm}}{1 - \bar\varepsilon_V}.$$

Then for every prime $p \ge p_0$ with $p \equiv \mathfrak{c} \pmod 4$:

- (a) *$\mathsf{A}_p \succeq -g_{\mathfrak{c}}\, p^{-1/2} I$ on $L^2(\mu_p)$;*
- (b) *$\|P_0\mathsf{A}_pP_0\| \le g_{\mathfrak{c}}\, p^{-1/2}$;*
- (c) *$\|\varrho - 1\|_\mu = \|P_0\mathsf{A}_p\mathbf{1}\|_\mu \le \varepsilon_\varrho(r)$, and $p\,\varepsilon_\varrho(r)^2 \le p_0\,\varepsilon_\varrho(r_0)^2$;*
- (d) *for every $g \ge g_{\mathfrak{c}}$ with $gr_0 < 1$, $\mathsf{A}_p \preceq (1 + B^+/p)I$ with $B^+ = p_0\,\varepsilon_\varrho(r_0)^2/(1-gr_0)$.*

Corollary 14.22 (Blocks)**.** *In the setting of Theorem 14.21, let $g \ge g_{\mathfrak{c}}$ with $gr_0 < 1$ and B^+ as in* (d)*, and let $\mathcal{B}$ be a finite set of primes $p \equiv \mathfrak{c} \pmod 4$, all at least $P \ge p_0$, with $\sum_{p\in\mathcal{B}} \log p \le L$, and $b = \min \mathcal{B}$. Then $\bigotimes_{p\in\mathcal{B}} \mathsf{A}_p \succeq -d\, b^{-1/2} I$ with $d = g_{\mathfrak{c}} \exp\big(B^+ L/(P\log P)\big)$, and so also with g in place of $g_{\mathfrak{c}}$. If the hypotheses hold for both classes and $g \ge \max(g_1, g_3)$, the same holds for blocks of primes of both classes.*

Proof. By (a) and (d), at every $p \in \mathcal{B}$, $-g_{\mathfrak{c}} p^{-1/2} \le \mathsf{A}_p \le 1 + B^+/p$ and $g_{\mathfrak{c}} p^{-1/2} \le g r_0 < 1$. By Lemma 13.7(b) with $\mathcal{V}_p = L^2(\mu_p)$, $\bigotimes_p \mathsf{A}_p \succeq -g_{\mathfrak{c}} b^{-1/2} \exp(B^+ \sum_{p \in \mathcal{B}} 1/p)$, and $\sum_{p \in \mathcal{B}} 1/p \le L/(b \log b) \le L/(P \log P)$ by Lemma 13.7(c) and since $X \log X$ increases. For both classes use $\max(g_1, g_3)$ in place of $g_{\mathfrak{c}}$, and the larger of the two values of B^+. $\square$

Proof of Theorem 14.21. Fix a prime $p \ge p_0$ with $p \equiv \mathfrak{c} \pmod 4$, so that $r \le r_0$.

Uniformity. (S1) at p_0 gives (S1) at p. By Lemma 14.29 and the admissibility of the inputs, $\varepsilon_V(r) \le \bar{\varepsilon}_V < 1$, $\varepsilon_Z(r) \le \varepsilon_Z(r_0) < 1$, $\eta_Q(r) \le \bar{\eta}_Q$, $\eta_B(r) \le \bar{\eta}_B$, $c_\perp(r) \le \bar{c}_\perp$, and ε_ϱ / r is nondecreasing; the matrices depend only on $\mathfrak{c}$. So Lemma 14.14 and Proposition 14.20 apply at p. Let $\Xi_\pm$ be the set of $\xi \neq 0$ with $\chi(\xi)\varepsilon_G = \pm 1$ if $\mathfrak{c} = 1$, and with $\chi(\xi)\varepsilon_G = \pm i$ if $\mathfrak{c} = 3$; the two sets partition $\mathbb{F}_p^\times$, and $\mathbf{A}_0(\xi) = \mathbf{A}_{\mathfrak{c},\pm}$ on $\Xi_\pm$ (Lemma 14.1). For $s > 0$ write $\mathbf{M}_{\mathfrak{c},\pm}(s)$ for the matrix of (14.21) with s in place of $\tau^{\mathrm{sp}}_{\mathfrak{c},\pm}$. For every $s > 0$ and every $\xi \in \Xi_\pm$, the matrix

$$\mathcal{M}_s(\xi) = \mathbf{A}_0(\xi) - \frac{\mathbf{H}^+ - \mathbf{A}_0(\xi)^* \mathbf{A}_0(\xi)}{s} - \Big(\eta_Q(r) + \frac{\eta_B(r)}{s}\Big) I$$

dominates $\mathbf{M}_{\mathfrak{c},\pm}(s) - (\bar{c}_\perp + s) I$ (Lemma 4.18(d)), so $\mathcal{M}_s(\xi) \succeq -(\bar{c}_\perp + s) I$ whenever $\mathbf{M}_{\mathfrak{c},\pm}(s) \succeq 0$, in particular for $s = \tau^{\mathrm{sp}}_{\mathfrak{c},\pm}$.

Centring at the half-law mean. Let F be real, with the decomposition $F = c + F_V + F_\perp$ of Proposition 14.20. By parts (b) and (c5) of that proposition,

$$Q(F) \ge -\bar{c}_\perp r \, \|F_\perp\|^2 + Q(F_V) + 2 \, \langle F_\perp, \mathsf{A}_p F_V \rangle \, ,$$

and $Q(F_V) \ge r \sum_{\xi \neq 0} \hat{g}(\xi)^* (\mathbf{A}_0(\xi) - \eta_Q I) \hat{g}(\xi)$ (Lemma 14.8(a),(b)).

Splitting by translation characters. Work in the complexification of $L^2(\mu)$ and let $F_{V,\pm}$ and $F_{\perp,\pm}$ be the components of F_V and $F_\perp$ in $\bigoplus_{\xi \in \Xi_\pm} \mathcal{W}_\xi$. These summands are orthogonal for either prime class. The components need not be real: $F_{V,\pm} \in \mathcal{V}$ has coefficients $\hat{g} \mathbf{1}_{\Xi_\pm}$, $F_{\perp,\pm} \perp \mathcal{V}$, and $\|F_{\perp,+}\|^2 + \|F_{\perp,-}\|^2 \le \|F_\perp\|^2$. As A_p and $\Pi_{\mathcal{V}}$ preserve every translation sector,

$$\langle F_\perp, \mathsf{A}_p F_V \rangle = \sum_{\pm} \langle F_{\perp,\pm}, (I - \Pi_{\mathcal{V}}) \mathsf{A}_p F_{V,\pm} \rangle \, .$$

Put $s_\pm = \tau^{\mathrm{sp}}_{\mathfrak{c},\pm}$. Young's inequality on each orthogonal summand and the complex form of (14.16) give

$$2 |\, \langle F_\perp, \mathsf{A}_p F_V \rangle \,| \le \sum_{\pm} \Big(s_\pm r \, \|F_{\perp,\pm}\|^2 + \frac{r}{s_\pm} \sum_{\xi \in \Xi_\pm} \hat{g}^* (\mathbf{H}^+ - \mathbf{A}_0^* \mathbf{A}_0 + \eta_B I) \hat{g} \Big).$$

Hence $Q(F) \ge r \sum_\pm \sum_{\xi \in \Xi_\pm} \hat{g}^* \mathcal{M}_{s_\pm}(\xi) \hat{g} - \bar{c}_\perp r \, \|F_\perp\|^2 - \sum_\pm s_\pm r \, \|F_{\perp,\pm}\|^2$, and by the uniformity step

$$Q(F) \ge -(\bar{c}_\perp + \max_\pm s_\pm) r (\, \|g\|^2 + \|F_\perp\|^2 \,) \ge -g_{\mathfrak{c}} (1 - \bar{\varepsilon}_V) \, r \, (c^2 + \|g\|^2 + \|F_\perp\|^2) \ge -g_{\mathfrak{c}} r \, \|F\|^2 \, ,$$

by Proposition 14.20(a). This proves (a) for both classes. The difference between the classes remains entirely in their finite matrices: for class three the two tests with a common parameter are complex conjugate, as stated in the theorem.

The compression (b). For real $h \perp \mathbf{1}$, $c = 0$ in the decomposition, and Proposition 14.20(c4) is two-sided: $-Q(h_\perp) \ge -R_Z(N_{\mathrm{ns}} + D_{\mathrm{col}}) \, \|h_\perp\|^2 \ge -\bar{c}_\perp r \, \|h_\perp\|^2$. Apply the argument above to $-\mathsf{A}_p$: only the sign of $\mathbf{A}_0(\xi)$ changes, and on the modes $\Xi_\pm$ the bracket of $-\mathsf{A}_p$ is $\mathcal{M}_s$ with $\mathbf{A}_{\mathfrak{c},\mp}$; use $s = \tau^{\mathrm{sp}}_{\mathfrak{c},\mp}$ on those modes. This gives $-Q(h) \ge -g_{\mathfrak{c}} r \, \|h\|^2$, and with (a) $|Q(h)| \le g_{\mathfrak{c}} r \, \|h\|^2$; as $P_0 \mathsf{A}_p P_0$ is real symmetric, (b) follows. This is why both signs are needed: on a given mode the lower bound uses one sign and the upper bound the other.

(c) and (d). (c) is Proposition 14.20(c3) together with $p \varepsilon_\varrho(r)^2 = (\varepsilon_\varrho(r)/r)^2 \le (\varepsilon_\varrho(r_0)/r_0)^2$. (d) is Proposition 13.6(c) with $e = \varepsilon_\varrho$ and $\beta = gr \ge g_{\mathfrak{c}} r$, using (b) and (c). $\square$

Remark 14.23 (Pinned correlations of the half law are not used). The proof uses (S1) only, through the extension counts of the degree-one and coupling blocks (Lemmas 14.3, 14.12 and 14.13), the star means, the nonstar extraction (Theorem 16.9), the coincidence bound and the normalization. After μ-centring the empty set and the stars vanish on $F_\perp$, and every other character set, each $K_{2,2}$ included, is a leaf of the nonstar partition. The influence conditions (13.5) of G^+ were needed in the perpendicular estimates of those earlier 216-slot criteria that charged the $K_{2,2}$ terms separately through a pure-pair frame bound $\sum_J \|\Pi_{H_J} F\|^2 \le \mathrm{Fr}\, \|F\|^2$ (Section 35.10); the criterion here does not use that bound, although the certificates still assert (13.5) for G^+ (Remark 14.30).

Remark 14.24 (The row-norm criterion). If $s > 0$ satisfies $s^2 - (\Lambda_D + \bar\eta_Q)s - (\bar\lambda(\mathbf{H}^+) + \bar\eta_B) \ge 0$, then (14.21) holds with $\tau^{\mathrm{sp}}_{\mathfrak{c},\pm} = s$, because $\mathbf{A}_{\mathfrak{c},\pm} \succeq -\Lambda_D I$ and $\mathbf{H}^+ - (\Delta^E)^{\mathsf{T}}\Delta^E \preceq \bar\lambda(\mathbf{H}^+) I$. A scalar test of this kind, with row norms and the earlier model $2\mathbf{H}$, was the acceptance test of an earlier version of the criterion. No certificate of the final construction uses it: every cap is certified by the matrix test (14.21) itself.

Remark 14.25 (The earlier criteria). The criteria of the 216-slot design (its Theorem S and its sharpenings at its middle cuts; see Section 34) are built from the same blocks, those of Sections 14.1, 14.3, 14.4 and 14.6, and the same Schur complement. Theorem S centres at $\mathbb{E}_{m_p} F$, uses the model $2\mathbf{H}$ and bounds the perpendicular block by decision trees and bicliques. Its sharpenings centre at $\mathbb{E}_\mu F$ as in Proposition 14.20 and treat the class-one Paley terms with dilation parity and an exceptional-clique term (here $\mathbf{H}^{\mathrm{cov}}$ and $\bar\zeta$ suffice, Section 14.11); they charge the $K_{2,2}$ terms first through a pure-pair frame and finally, in the matching test, through a matching partition and coincidence charging as in Section 14.7. All of them use one test parameter for all signs and assume (13.5). So they are not literally cases of Theorem 14.21, and no cap of the final construction uses them; Section 34.2 records them.

Remark 14.26 (Where route A is applied). The three applications evaluate the same test and differ only in the inputs.

(i) *Twelve-label middle cuts* (Lemma 20.12): the cuts L21, E21 and L10 of the uniform graph laws. The cuts F21 and L20 are computed as well, but the middle uses for F21 the negative cap k_{D} of the reference fold record (Lemma 23.2, Certificate D.6) and for L20 the caps of Section 20.4, from the complete actual stars; on the three late middle pieces the two-layer root is a positive reweighting of the uniform root (the tilted law of Section 21), and its cuts L20 and L21 have the caps of Propositions 21.7 and 21.9. The two grids of Certificate 20.13 have 26 and 9 thresholds and 520 and 180 exact tests. N_{ns} from a compressed matching partition, $D_{\mathrm{col}} = D_{\mathrm{nat}}$ of (16.8), the standard star inputs ($M_\star = W_{\mathrm{st}}$, $Z_\star = 2(P_{\mathrm{st}} + W_{\mathrm{st}})$), and the standard coupling inputs ($\varepsilon_H = \varepsilon'_H + \varepsilon_X$, $\Gamma_E = \hat E^2$, $\bar\zeta = \zeta$). The block caps use g_3 only; B^+ is computed with the maximum over both classes, which is allowed by (d).

(ii) *Ten nonroot upper cuts* ((F6)): FL21, EL21, FL22, EL22, FF22, FE22, EE22, LL10, FL11, EL11, at the thresholds 2^{47} (for the upper bands below 2^{48}) and 2^{48} (for the bands from 2^{48} on) and at the thresholds of the tail datum; inputs as in (i). Below $P_4 = 2^{512}$ only class three occurs, and g is the larger of the caps of the two conjugate class-three tests; for the tail datum g is the maximum over both classes and both signs. The merge and constant marginals of EL22 in four reserved states have the observed caps of Proposition 27.14, which are not route-A applications.

(iii) *Completed roots* LL20, LL22 from 2^{52} on (Lemma 27.10): $N_{\mathrm{ns}} = \sum_{k=2}^{4} C_k r^k$ from the augmented partition (a refinement of the compressed partition of Lemma 16.2), with the nonstar coefficients C_k of Corollary 16.11(b); $D_{\mathrm{col}} = \min(D_{\mathrm{ord}}, D_{\mathrm{nat}})$ (Lemma 16.15(e)); $M_\star$ and $Z_\star$ of Proposition 15.26 and Lemma 15.29, maximized over the two classes; ε_Z the smaller of $2(P_{\mathrm{st}} + W_{\mathrm{st}}) + N_{\mathrm{ns}} + D_{\mathrm{nat}}$ and $Z_\star + N_{\mathrm{ns}} + D_{\mathrm{col}}$; R_+ recomputed with this ε_Z; Γ_E and ε_H the minima of Section 14.5; and, at class one, $\bar\zeta$ the minimum of Lemma 14.32.

14.9. **Route B (uniform averages over allowed sets).** Route A tests the cut operator through the conditional law of the opposite half, whose completion counts it must control. At the two root cuts this needs the completion-variance denominator ℓ_k^{loc} of Lemma 14.15 to be positive, which holds only for p above approximately $2^{49.96}$ for LL20 and $2^{49.30}$ for LL22 — well above P_3, which is why Route B is used at these cuts; no bound in the paper uses these approximate values as thresholds. Route B expands the kernel around the product of the half laws and compares its star part with explicit normalized stars: averages over the allowed sets $\Phi_k(x)$, not conditional laws of the completed cut law. Their star synthesis is controlled by a finite Schur test. It is stated for $p \equiv 3 \pmod 4$, the only class at which it is applied.

Setting. Let $p \equiv 3 \pmod 4$ be a prime with $e_\Delta(p, 2n) < 1$ and $p > 4n^2$. Then $\mathbf{H}^+ = \mathbf{H}$, $\mathbf{A}_0(\xi) = \chi(\xi)\varepsilon_G\Delta^E = \pm i\Delta^E$ is Hermitian, $\mathbf{A}_0(\xi)^2 = (\Delta^E)^{\mathsf{T}}\Delta^E$, and both signs occur. Let $\Lambda_D \ge \left\|\Delta^E\right\|$ and $\mathbf{R} = \mathbf{H} - (\Delta^E)^{\mathsf{T}}\Delta^E$. Let $U = L_0^2(\mathbb{F}_p)$ and $\mathbf{E}\colon U^n \to L^2(\mu)$, $(\mathbf{E}g)(x) = \sum_k g_k(x_k)$; the *explicit star* is $(B_k g)(x) = |\Phi_k(x)|^{-1}\sum_{t\in\Phi_k(x)} g(t)$, with $\mathbf{B}g = \sum_k B_k g_k$. Put $(S_k g)(x) = \mathbb{E}_t\varphi_k(t;x)g(t)$ and $\mathbf{S}g = \sum_k S_k g_k$. Translations act unitarily on U^n and diagonally on $L^2(\mu)$, and $\mathbf{E}$, $\mathbf{B}$, $\mathbf{S}$ intertwine these actions ($\Phi_k(x + c\mathbf{1}) = \Phi_k(x) + c$). Hence $\mathbf{E}^*\mathbf{E}$, $\mathbf{E}^*\mathbf{B}$ and $\mathbf{B}^*\mathbf{B}$ preserve each mode $\mathbb{C}^n e_\xi$, $\xi \ne 0$, and act there by $n \times n$ matrices; distinct modes have orthogonal images.

Theorem 14.27 (Spectral criterion from uniform averages over allowed sets (Route B))**.** *In this setting assume the following.*

(H1) Frame. *On every mode $\xi \ne 0$,*

$$\|\mathbf{E}^*\mathbf{E} - I\| \le \varepsilon_V, \qquad \|\mathbf{E}^*\mathbf{B}/r - \mathbf{A}_0(\xi)\| \le \varepsilon_\times, \qquad \mathbf{B}^*\mathbf{B}/r^2 \preceq \mathbf{H} + \varepsilon_{\mathbf{B}} I.$$

(H2) Schur test. *With $\varepsilon_{\mathbf{W}} = \varepsilon_{\mathbf{B}} + 2\Lambda_D\varepsilon_\times + \Lambda_D^2\varepsilon_V$ and $\varepsilon_{\mathbf{EW}} = \varepsilon_\times + \Lambda_D\varepsilon_V$, there are rationals $s_\star, g_\star > 0$ such that, with $\alpha_\star = 1 + \varepsilon_V + s_\star$ and $\beta_\star = \varepsilon_{\mathbf{W}} + \varepsilon_{\mathbf{EW}}^2/s_\star$, the matrix $\mathbf{R} + \beta_\star I$ is positive definite and*

$$\pm\varepsilon_G\Delta^E + \frac{g_\star}{\alpha_\star} I - \frac{\mathbf{R} + \beta_\star I}{g_\star} \succeq 0 \qquad \text{for both signs.} \tag{14.22}$$

(H3) Inputs. *Nonnegative numbers N_{ns}, D_{col}, $M_\star$, $z_\star$ and $Z_\star$ satisfy $\|T_{\mathrm{ns}}\| \le N_{\mathrm{ns}}$, $\|T_{\mathrm{coll}}\| \le D_{\mathrm{col}}$, $\|h_\star - \mathbb{E}_\mu h_\star\|_\mu \le M_\star$, $|\mathbb{E}_\mu h_\star| \le z_\star$ and $2|\mathbb{E}_\mu h_\star| \le Z_\star$, and $\varepsilon_Z := Z_\star + N_{\mathrm{ns}} + D_{\mathrm{col}} < 1$.*

Put

$$\varepsilon_{\mathrm{soft}} = 2(1+\varepsilon_V)^{1/2}\Big[r\Big(\sum_k \hat{e}_k^2(\mathbf{H}_{kk} + \varepsilon_{\mathbf{B}})\Big)^{1/2} + nr^2\Big(\sum_k 4^{r_k}\Big)^{1/2}\Big],$$

$$g_{\mathrm{root}} = \frac{g_\star + (N_{\mathrm{ns}} + D_{\mathrm{col}} + M_\star^2 + z_\star^2 + \varepsilon_{\mathrm{soft}})/r}{1 - \varepsilon_Z}, \qquad \varepsilon_\varrho = \frac{M_\star + N_{\mathrm{ns}} + D_{\mathrm{col}}}{1 - \varepsilon_Z}.$$

Then, with $\mathbf{W} = \mathbf{B}/r - \mathbf{E}\mathbf{A}_0$ and $\mathbf{L} = r(\mathbf{E}\mathbf{A}_0\mathbf{E}^* + \mathbf{E}\mathbf{W}^* + \mathbf{W}\mathbf{E}^*)$:

(a) $-g_\star r I \preceq \mathbf{L} \preceq g_\star r I$;
(b) $K_0 = (\mathbf{1} + h_\star) \otimes (\mathbf{1} + h_\star) + \mathbf{L} + \mathcal{R}$ *with* $\|\mathcal{R}\| \le N_{\mathrm{ns}} + D_{\mathrm{col}} + M_\star^2 + z_\star^2 + \varepsilon_{\mathrm{soft}}$;
(c) $|Z_0 - 1| \le \varepsilon_Z$ *and* $\mathsf{A}_p \succeq -g_{\mathrm{root}} r I$;
(d) $\|P_0\mathsf{A}_p\mathbf{1}\| \le \varepsilon_\varrho$ *and* $\|P_0\mathsf{A}_pP_0\| \le g_{\mathrm{root}} r$;
(e) *if* $g_{\mathrm{root}} r < 1$, *then* $\mathsf{A}_p \preceq (1 + B_{\mathrm{top}}/p)I$ *with* $B_{\mathrm{top}} = p\varepsilon_\varrho^2/(1 - g_{\mathrm{root}} r)$.

The two signs in (14.22) are the two values of $\mathbf{A}_0(\xi)$. Each direction of (a) needs both: on a given mode the lower bound uses the sign $\chi(\xi)$, and the upper bound the opposite one. The upper bound in (a) enters through (d), and (d) enters (e). The two matrices of (14.22) are complex conjugates ($\varepsilon_G\Delta^E$ is $\pm i$ times a real matrix and $\mathbf{R}$ is real), so either test implies the other; the certificate checks both.

Proof. (a) Let $\mathbf{F} = [\mathbf{E}, \mathbf{W}] \colon U^n \oplus U^n \to L^2(\mu)$ and, on the mode ξ, $\mathcal{C}(\xi) = \left(\begin{smallmatrix} \mathbf{A}_0(\xi) & I \\ I & 0 \end{smallmatrix}\right)$. Then $\mathbf{F}\mathcal{C}\mathbf{F}^* = \mathbf{E}\mathbf{A}_0\mathbf{E}^* + \mathbf{W}\mathbf{E}^* + \mathbf{E}\mathbf{W}^* = \mathbf{L}/r$. Write $\Gamma = \mathbf{E}^*\mathbf{B}/r - \mathbf{A}_0$ and $\Delta_V = \mathbf{E}^*\mathbf{E} - I$, so $\|\Gamma\| \le \varepsilon_\times$ and $\|\Delta_V\| \le \varepsilon_V$. Then $\mathbf{E}^*\mathbf{W} = \Gamma - \Delta_V\mathbf{A}_0$ has norm at most $\varepsilon_{\mathbf{EW}}$, and

$$\mathbf{W}^*\mathbf{W} = \frac{\mathbf{B}^*\mathbf{B}}{r^2} - \mathbf{A}_0^2 - \Gamma^*\mathbf{A}_0 - \mathbf{A}_0\Gamma + \mathbf{A}_0\Delta_V\mathbf{A}_0 \preceq \mathbf{R} + \varepsilon_{\mathbf{W}} I.$$

For (u, v) on one mode, $\|\mathbf{E}u + \mathbf{W}v\|^2 \le (1+\varepsilon_V)\|u\|^2 + 2\varepsilon_{\mathbf{EW}}\|u\|\,\|v\| + \langle v, (\mathbf{R} + \varepsilon_{\mathbf{W}})v\rangle$, and $2\varepsilon_{\mathbf{EW}}\|u\|\,\|v\| \le s_\star\|u\|^2 + \varepsilon_{\mathbf{EW}}^2\|v\|^2/s_\star$ gives $\mathbf{F}^*\mathbf{F} \preceq \mathcal{M} := \operatorname{diag}(\alpha_\star I, \mathbf{R} + \beta_\star I)$, which is positive definite. Thus $\mathcal{Q} = \mathbf{F}\mathcal{M}^{-1/2}$ has norm at most one, and $\mathbf{F}\mathcal{C}\mathbf{F}^* = \mathcal{Q}(\mathcal{M}^{1/2}\mathcal{C}\mathcal{M}^{1/2})\mathcal{Q}^*$. By congruence, $\mathcal{M}^{1/2}\mathcal{C}\mathcal{M}^{1/2} + g_\star I \succeq 0$ is equivalent to $\mathcal{C} + g_\star\mathcal{M}^{-1} \succeq 0$, that is, to

$$\begin{pmatrix} \mathbf{A}_0(\xi) + (g_\star/\alpha_\star)I & I \\ I & g_\star(\mathbf{R} + \beta_\star I)^{-1} \end{pmatrix} \succeq 0,$$

and, the lower right block being positive definite, to the Schur complement condition (14.22) with the sign of $\chi(\xi)$. Hence $\mathbf{L} \succeq -\mathcal{Q}(g_\star r I)\mathcal{Q}^* \succeq -g_\star r I$. For the upper bound conjugate by the unitary $\operatorname{diag}(I, -I)$, which commutes with $\mathcal{M}$: it carries $-\mathcal{C}(\xi)$ to the matrix $\mathcal{C}$ with $-\mathbf{A}_0(\xi)$ in place of $\mathbf{A}_0(\xi)$, and the same argument with the other sign of (14.22) gives $-\mathbf{L} \succeq -g_\star r I$.

(b) *Stars.* The uniform one-vertex marginals of μ give $(\mathbf{E}^*f)_k(t) = \mathbb{E}_\mu[f \mid x_k = t] - \mathbb{E}_\mu f$. By Lemma 14.19(a), summed over $\emptyset \ne J \subseteq N(k)$, the column stars at k act by

$$f \mapsto \mathbb{E}_t\big[(\varphi_k(t;x) - 1)(\mathbb{E}_\mu f + (\mathbf{E}^*f)_k(t))\big] = h_{\star,k}(x)\,\mathbb{E}_\mu f + (S_k(\mathbf{E}^*f)_k)(x),$$

since $\mathbb{E}_t(\mathbf{E}^*f)_k = 0$. So the column stars give $h_\star \otimes \mathbf{1} + \mathbf{S}\mathbf{E}^*$, and the row stars, their adjoints by Lemma 14.19(b), give $\mathbf{1} \otimes h_\star + \mathbf{E}\mathbf{S}^*$. Row and column stars overlap exactly in the single pairs. The single pair $\{(j,k)\}$ acts by $f \mapsto \chi(\varsigma_{jk})(C(\mathbf{E}^*f)_k)(x_j)$, since its constant part integrates to zero; as $Ce_\xi = r\chi(\xi)\varepsilon_G e_\xi$, the single pairs sum to $r\mathbf{E}\mathbf{A}_0\mathbf{E}^*$. By (14.20),

$$K_0 = (\mathbf{1} + h_\star) \otimes (\mathbf{1} + h_\star) - h_\star \otimes h_\star + \mathbf{S}\mathbf{E}^* + \mathbf{E}\mathbf{S}^* - r\mathbf{E}\mathbf{A}_0\mathbf{E}^* + T_{\mathrm{ns}} - T_{\mathrm{coll}}.$$

Since $\mathbf{B} = r(\mathbf{E}\mathbf{A}_0 + \mathbf{W})$ and $\mathbf{A}_0$ is Hermitian, $\mathbf{B}\mathbf{E}^* + \mathbf{E}\mathbf{B}^* - r\mathbf{E}\mathbf{A}_0\mathbf{E}^* = \mathbf{L}$. Hence (b) holds with $\mathcal{R} = (\mathbf{S} - \mathbf{B})\mathbf{E}^* + \mathbf{E}(\mathbf{S} - \mathbf{B})^* - h_\star \otimes h_\star + T_{\mathrm{ns}} - T_{\mathrm{coll}}$, and $\|h_\star \otimes h_\star\| = \|h_\star\|_\mu^2 = \|h_\star - \mathbb{E}_\mu h_\star\|^2 + (\mathbb{E}_\mu h_\star)^2 \le M_\star^2 + z_\star^2$.

Soft stars. For $t \notin x(U^+)$ each factor of $\varphi_k(t;x)$ is 0 or 2, so $\varphi_k(t;x) = 2^{r_k}\mathbf{1}_{\Phi_k(x)}(t)$. Hence

$$S_k g = a_k\, B_k g + \mathrm{coll}_k g, \qquad a_k(x) = \frac{2^{r_k}|\Phi_k(x)|}{p}, \qquad (\mathrm{coll}_k g)(x) = \frac{1}{p}\sum_{i \in U^+} \varphi_k(x_i; x) g(x_i),$$

with $|a_k - 1| \le \hat{e}_k$ and $0 \le \varphi_k \le 2^{r_k}$. By (H1) on the modes, $\|B_k g\| \le r(\mathbf{H}_{kk} + \varepsilon_{\mathbf{B}})^{1/2}\,\|g\|$, and each x_i is uniform, so $\|\mathrm{coll}_k g\| \le n2^{r_k}p^{-1}\,\|g\|$. Cauchy–Schwarz over k gives

$$\|\mathbf{S} - \mathbf{B}\| \le r\Big(\sum_k \hat{e}_k^2(\mathbf{H}_{kk} + \varepsilon_{\mathbf{B}})\Big)^{1/2} + \frac{n}{p}\Big(\sum_k 4^{r_k}\Big)^{1/2},$$

and $\|\mathbf{E}\| \le (1+\varepsilon_V)^{1/2}$, so the two soft terms of $\mathcal{R}$ have norm at most $\varepsilon_{\mathrm{soft}}$.

(c) The images of $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{S}$ have μ-mean zero: the first because the one-vertex marginals are uniform, the others because the law of t in B_k, and the weight $\mathbb{E}_\mu\varphi_k(t;x)$ in S_k, are translation invariant in t. Also $\mathbf{E}^*\mathbf{1} = 0$. Hence $\mathbf{L}\mathbf{1} = 0$, $\langle \mathbf{1}, \mathbf{L}\,\cdot\rangle = 0$ and $(\mathbf{S} - \mathbf{B})^*\mathbf{1} = 0$. Pairing (b) with $\mathbf{1}$, $Z_0 = \langle \mathbf{1}, K_0\mathbf{1}\rangle = 1 + 2\mathbb{E}_\mu h_\star + \langle \mathbf{1}, (T_{\mathrm{ns}} - T_{\mathrm{coll}})\mathbf{1}\rangle$, so $|Z_0 - 1| \le \varepsilon_Z$; moreover $Z_0 > 0$. As $(\mathbf{1} + h_\star) \otimes (\mathbf{1} + h_\star) \succeq 0$, (a) and (b) give $K_0 \succeq -(g_\star r + \|\mathcal{R}\|)I$, and dividing by $Z_0 \ge 1 - \varepsilon_Z$ gives $\mathsf{A}_p \succeq -g_{\mathrm{root}} r I$.

(d) Applying (b) to $\mathbf{1}$ and projecting, $P_0K_0\mathbf{1} = (h_\star - \mathbb{E}_\mu h_\star) + P_0(T_{\mathrm{ns}} - T_{\mathrm{coll}})\mathbf{1}$, because the soft terms vanish on $\mathbf{1}$ and the term $-h_\star\mathbb{E}_\mu h_\star$ of $\mathcal{R}$ cancels against $(\mathbf{1} + h_\star)\mathbb{E}_\mu h_\star$. Hence $\|P_0K_0\mathbf{1}\| \le M_\star + N_{\mathrm{ns}} + D_{\mathrm{col}}$. Compressing (b) by P_0 on both sides removes $\mathbf{1} \otimes \mathbf{1}$, $h_\star \otimes \mathbf{1}$ and

$\mathbf{1}\otimes h_\star$; the two terms $\pm P_0(h_\star\otimes h_\star)P_0$ cancel, and $P_0\mathbf{L}P_0=\mathbf{L}$. Thus $P_0K_0P_0=\mathbf{L}+P_0[(\mathbf{S}-\mathbf{B})\mathbf{E}^*+\mathbf{E}(\mathbf{S}-\mathbf{B})^*+T_{\rm ns}-T_{\rm coll}]P_0$, of norm at most $g_\star r+\varepsilon_{\rm soft}+N_{\rm ns}+D_{\rm col}$ by (a). Divide by Z_0.

(e) A_p is self-adjoint with $\langle\mathbf{1},\mathsf{A}_p\mathbf{1}\rangle=1$; use Proposition 13.6(a) with $e=\varepsilon_\varrho$ and $\beta=g_{\rm root}r$, by (d). □

The proof never uses the completed conditional law of the root, its row ratio $R_{\rm cut}$, or the completion-variance denominator. The rank-one term $h_\star\otimes h_\star$ is included in the remainder bound of (b) through $M_\star^2+z_\star^2$, not omitted, and it does not enter the centred block.

Lemma 14.28 (Inputs for the star synthesis at the root cuts)**.** *In the setting of Theorem 14.27,* (H1) *holds with the following numbers, in which $\hat E^2$, $\varepsilon_{\rm rem}$ and $\varepsilon_{\neq}$ are the quantities so named in Section 14.4, specialised to the explicit-stars mode ($R'_k=1$ and $R_\nu=1$, so that $\Delta_k=\hat e_k/(1-\hat e_k)$ and θ_j^μ has no R_ν factor).*

(i) *ε_V is the quantity* (14.10).

(ii) *With $\phi_k=(\hat e_k+e_{r_k+1}(p,n))/(1-\hat e_k)$,*

$$\varepsilon_\times=\frac{n}{(p-1)r}+\frac{\Lambda_D}{p-1}+\frac{p}{p-1}\Big(\max_i\sum_{k:\,\varsigma_{ik}=0}\phi_k\cdot\max_k\sum_{i:\,\varsigma_{ik}=0}\phi_k\Big)^{1/2}.$$

(iii) *With ε'_H of* (14.13), *$\theta_j^\mu=(2^{d_j^0}/(1-e_{d_j^0}(p,n))-1)^{1/2}$, $\rho_{\rm row}$ as in Section 14.4, and*

$$\hat E^2=\sum_k\Big(2^{r_k}(1+\hat e_k)\Big(\frac{\hat e_k}{1-\hat e_k}\Big)^2+\frac{n4^{r_k}}{p}\Big),\qquad \varepsilon_{\rm rem}=2\big(\hat E^2(\|\mathbf{H}\|_{\rm row}+\varepsilon'_H)\big)^{1/2}+\hat E^2,$$

$$\varepsilon_{\neq}=r\max_j\theta_j^\mu\,\rho_{\rm row},\qquad \varepsilon_{\mathbf{B}}=\Big(\frac{p}{p-1}-1\Big)\|\mathbf{H}\|_{\rm row}+\frac{p}{p-1}\big(\varepsilon'_H+\varepsilon_{\neq}+\varepsilon_{\rm rem}\big).$$

Proof. (i) On the mode ξ, $\mathbf{E}^*\mathbf{E}$ is the matrix $\mathbf{N}(\xi)$ of Lemma 14.8(c).

(ii) The pair (x_i,t), with $x\sim\mu$ and t uniform on $\Phi_k(x)$, has an Aff_p-invariant law, and $t\neq x_i$ always. By Lemma 14.2(b) the entry (i,k) of $\mathbf{E}^*\mathbf{B}$ on the mode ξ is $-1/(p-1)+\frac{p}{p-1}m_{ik}c_\xi$, where $m_{ik}=\mathbb{E}\chi(t-x_i)$. If $\varsigma_{ik}\neq0$, every $t\in\Phi_k(x)$ satisfies $\chi(t-x_i)=\chi(\varsigma_{ik})$, so $m_{ik}=\Delta_{ik}^E$. If $\varsigma_{ik}=0$ (including $i=k$), x_i is not a root of the constraints of $\Phi_k(x)$, and the $t\in\Phi_k(x)$ with $\chi(t-x_i)=1$ are cut out by r_k+1 constraints with distinct roots; their number lies in $p2^{-r_k-1}[1\pm e_{r_k+1}(p,n)]$. Since $\sum_{t\in\Phi_k}\chi(t-x_i)=2\#\{\chi=1\}-|\Phi_k|$, this sum is at most $p2^{-r_k}(e_{r_k+1}+\hat e_k)$ in modulus, and $|m_{ik}|\le\phi_k$ pointwise in x. Therefore

$$\frac{\mathbf{E}^*\mathbf{B}}{r}-\mathbf{A}_0(\xi)=-\frac{\mathbf{1}\mathbf{1}^{\mathsf T}}{(p-1)r}+\frac{\mathbf{A}_0(\xi)}{p-1}+\frac{p}{p-1}\chi(\xi)\varepsilon_GM^{\rm off},$$

where $M^{\rm off}$ has the entries m_{ik} with $\varsigma_{ik}=0$ and zeros elsewhere. Bound the three norms by n, Λ_D and the Schur test.

(iii) This is Section 14.4 for the explicit stars, instance (B), in which $R_\nu=1$ and $R'_k=1$: then $\theta_j=\theta_j^\mu$, $\Delta_k=\hat e_k/(1-\hat e_k)$ and $\varepsilon_X=0$. For $w\in\mathbb{C}^n$, $\mathbf{B}(we_\xi)=\mathcal{P}(\sum_k w_ke_\xi(x_k))$ in the notation of Definition 14.9, so by Lemma 14.10

$$\frac{w^*(\mathbf{B}^*\mathbf{B})(\xi)w}{r^2}=\frac{w^*\mathbf{M}(\xi)w}{r^2}=\frac{p}{p-1}(w^*\mathbf{H}^0w+pc_\xi\,w^*\mathbf{D}^0w),$$

with $\mathbf{H}^0=\widetilde{\mathbf{H}}$ and $\mathbf{D}^0=\mathbf{D}$ for these data. By Lemma 14.12, $\mathbf{H}^0\preceq\mathbf{H}+\varepsilon'_HI$; by Lemma 14.13, with $\Gamma_E=\hat E^2$, $\bar\lambda(\mathbf{H})=\|\mathbf{H}\|_{\rm row}$ and $\varepsilon_H=\varepsilon'_H$, and since $t_J=0$ for $p\equiv3\pmod4$, $|w^*\mathbf{D}^0w|\le r(\varepsilon_{\neq}+\varepsilon_{\rm rem})|w|^2$. As $|pc_\xi|r=1$ and $\mathbf{H}\succeq0$, $\|\mathbf{H}\|\le\|\mathbf{H}\|_{\rm row}$,

$$\frac{w^*\mathbf{B}^*\mathbf{B}w}{r^2}\le\frac{p}{p-1}(w^*\mathbf{H}w+(\varepsilon'_H+\varepsilon_{\neq}+\varepsilon_{\rm rem})|w|^2)\le w^*\mathbf{H}w+\varepsilon_{\mathbf{B}}|w|^2.$$ □

The same proofs give Theorem 14.27 and Lemma 14.28 at $p \equiv 1 \pmod 4$, with $\mathbf{H}^+ = \mathbf{H} + \mathbf{H}^{\mathrm{cov}}$ in place of $\mathbf{H}$ everywhere except inside $\varepsilon_{\mathrm{rem}}$, with $\pm\Delta^E$ in place of $\pm\varepsilon_G\Delta^E$ (now two distinct tests), and with $\varepsilon_{\mathrm{par}} = (R_{\mathrm{full}} - 1 + n/((p-n)r))\,\|\mathbf{H}^{\mathrm{nc}}\|_{\mathrm{row}}$ (R_{full} is defined in Lemma 14.31 below) added inside the last bracket of $\varepsilon_{\mathbf{B}}$ (Lemma 14.31 with $R_\nu = 1$); the caps of the final construction use only $p \equiv 3 \pmod 4$. The application to the root cuts, with the inputs of (H3), the monotone extension from an endpoint, the exact frame and Schur tests and the certified class-three caps, is Lemma 27.6, Lemma 27.7, Certificate 27.8 and Certificate 27.9.

14.10. **From one prime to all primes and blocks.** Both criteria are checked at one threshold and used at every larger prime and on blocks of primes. The first step is the monotonicity of the error terms.

Lemma 14.29 (Monotone spectral quantities)**.** *Let spectral data satisfy* (S1) *at* p_0, $r_0 = p_0^{-1/2}$, *and write every quantity as a function of* $r \in (0, r_0]$. *In part* (a), *quantities involving coupling data use instance* (A) *or* (B) *of Section 14.4.*

(a) *The following are admissible: every* $e_t(p, n')$ *with* $t \le \Delta$, $n' \le 2n$, *and* $\hat{e}_k$; $R_{\mathrm{cut}} - 1$, $R_{\mathrm{full}} - 1$, $R'_k - 1$, $R_\nu R_{\mathrm{full}} - 1$ *and their quotients by* r; $\delta_l^{\times}$, δ_l^{μ}, ε_V *and* η_Q; β_{kl}, ε'_H, Υ_{kl} *and* ε_X; θ_j, θ_j^{μ}, $\varepsilon_{\neq}$, Δ_k, $\hat{E}^2$, ϕ_k, $\varepsilon_{\times}$, *and* ζ *of Lemma 14.31.*

(b) *If the inputs* Γ_E, ε_H, $\bar{\zeta}$ *and* R_+ *are admissible, so are* $\varepsilon_{\mathrm{rem}}$, $\varepsilon_{\mathrm{par}}$, η_B *and* Λ_M.

(c) *If* N_{ns}/r, D_{col}/r, $M_\star/r$ *and* ε_Z *are admissible and* $\varepsilon_Z(r_0) < 1$, *then* R_Z, R_+ *of Proposition 14.20*(c2), ε_ϱ/r, ε_ϱ *and* $c_\perp$ *are admissible, and so are the bound* $R_{\mathrm{full}} - 1 + n/((p-n)r) + \varepsilon_\varrho$ *of Lemma 14.32 and its minimum with* ζ; P_{st}/r^2 *and* W_{st}/r *are constants.*

(d) *If the count errors* $\epsilon_t^{\pm}$ *of Section 14.5 are polynomials in* r *with nonnegative coefficients, and at* r_0 *the denominators of Lemma 14.15*(b) *are positive and* $\ell_k^{\mathrm{loc}} > 0$, *then* $\ell_k^{\mathrm{loc}} > 0$ *on* $(0, r_0]$, *and the bound* $(\varepsilon_k^{\mathrm{loc}}/\ell_k^{\mathrm{loc}})^2$ *of Lemma 14.15, the bound (a) there,* Γ_E^{loc}, *and* $\varepsilon_H^{\mathrm{loc}}$ *with fixed Schur weights are admissible.*

The matrices Δ^E, $\mathbf{H}$, $\mathbf{H}^{\mathrm{cov}}$, $\mathbf{H}^{\mathrm{nc}}$ *and* $\mathbf{H}^+$ *depend on* p *only through its class modulo* 4.

Proof. We use Lemma 4.18. Every $e_t(p, n') = 2^t(h_t r + n' r^2)$ is a polynomial in r with nonnegative coefficients and no constant term; so is e_u for every u; and by (S1) every $e_t(p, n')$ with $t \le \Delta$, $n' \le 2n$ is below one at r_0, hence on $(0, r_0]$. Further $p/(p-1) = 1/(1-r^2)$, $1/(p-1) = r^2/(1-r^2)$, $1/((p-1)r) = r/(1-r^2)$ and $n/((p-n)r) = nr/(1-nr^2)$, where $nr^2 < 1$ by (S1). A product of factors $(1+e)/(1-e)$ with small $e < 1$ is one plus a small function. This gives the ratios and the constants of (14.2), (14.9), (14.10), Υ_{kl}, ε_X, $\hat{E}^2$ (whose last term is $n4^{r_k}r^2$), ϕ_k, $\varepsilon_\times$ and $\zeta = (R_\nu R_{\mathrm{full}} - 1) + nr/(1-nr^2)$. In θ_j the argument of the square root is at least 0 and nondecreasing, since $R_\nu \ge 1$ and $2^{d_j^0}/(1-e) \ge 1$, and $\varepsilon_{\neq} = r\theta_{\max}\rho_{\mathrm{row}}$. Both branches of Δ_k are nondecreasing. In (14.13) the second branch of the maximum never exceeds the first: for $0 \le a, b < 1$ and $e_u \ge 0$,

$$\frac{1+e_u}{(1-a)(1-b)} + \frac{1-e_u}{(1+a)(1+b)} - 2 = \frac{2\big(a^2+b^2+ab-a^2b^2+e_u(a+b)\big)}{(1-a^2)(1-b^2)} \ge 0,$$

so β_{kl} equals its first branch, which is nondecreasing; this avoids any condition on e_u. For (b), $\varepsilon_{\mathrm{rem}}$, $\varepsilon_{\mathrm{par}} = \bar{\zeta}\Lambda_{\mathrm{nc}}$, Λ_M and η_B are built from these by sums, products and square roots, and $R_+ p/(p-1) - 1 = R_+/(1-r^2) - 1 \ge 0$ is nondecreasing. For (c), $R_Z = 1/(1-\varepsilon_Z)$ and R_+ are nondecreasing when $\varepsilon_Z < 1$ is; $\varepsilon_\varrho/r = R_Z(M_\star/r + N_{\mathrm{ns}}/r + D_{\mathrm{col}}/r)$; $c_\perp = R_Z(N_{\mathrm{ns}}/r + D_{\mathrm{col}}/r) + r(\varepsilon_\varrho/r)^2$; $\varepsilon_\varrho = r \cdot (\varepsilon_\varrho/r)$; and $R_{\mathrm{full}} - 1$, $nr/(1-nr^2)$ and ζ are admissible by (a). For (d), the denominators $1 - \epsilon_t^-(z)$ of κ_Φ and κ_i are nonincreasing in r and positive at r_0, so κ_Φ and κ_i are admissible; every term of $\varepsilon_k^{\mathrm{loc}}$ is built from r, κ_Φ, κ_i, α_i^+, α_i^{++} and fixed integers by sums, products and square roots, so it is admissible, while $\prod_i \alpha_i^-$ is nonincreasing; hence ℓ_k^{loc} is nonincreasing and $\varepsilon_k^{\mathrm{loc}}/\ell_k^{\mathrm{loc}}$ nondecreasing. The bound (a) is $\delta_k'^2/(4(1+\delta'_k))$ with δ'_k admissible, and $y \mapsto y^2/(4(1+y))$

increases. The remaining quantities are sums, products, minima and square roots of these, and a weighted Schur bound with fixed weights is a fixed nonnegative combination of matrix entries. □

Remark 14.30 (How a cap is certified). For a cut and a threshold p_0 the route-A evaluator computes, at $r_0 = p_0^{-1/2}$ and in outward interval arithmetic, the quantities of (R1)–(R3). It asserts $e_{d_l}(p_0, 2n) < 1$ for every l, hence the first half of (S1) (the second, $p_0 > 4n^2$, holds at every threshold used, since $p_0 \ge 2^{24} > 4 \cdot 36^2 \ge 4n^2$ for the values $n \le 36$ occurring in Remark 14.26), $\varepsilon_Z < \frac{1}{2}$ and $\bar\varepsilon_V < 1$ (and asserts the influence conditions (13.5) of G^+, which no statement uses). For each class $\mathfrak{c} \in \{1, 3\}$ and each sign, a floating-point bisection proposes a rational $\tau^{\mathrm{sp}}_{\mathfrak{c},\pm}$, which is accepted only by an exact rational LDL^{T} test of (14.21); for $\mathfrak{c} = 3$ the test runs on the real $2n \times 2n$ embedding of the Hermitian matrix (Remark 14.7). In this test $\bar c_\perp, \bar\eta_Q, \bar\eta_B, \bar\varepsilon_V$ are the upper ends of the enclosures, rounded up to denominator 10^{10}. The evaluator records $g_{\mathfrak{c},\pm} = (\lceil 10^8(\bar c_\perp + \tau^{\mathrm{sp}}_{\mathfrak{c},\pm})/(1-\bar\varepsilon_V)\rceil + 1)/10^8$ and $g = \max_{\mathfrak{c},\pm} g_{\mathfrak{c},\pm} \ge \max(g_1, g_3)$, asserts $gr_0 < 1$, and records $B^+ = p_0\varepsilon_\varrho(r_0)^2/(1-gr_0)$ as an interval. An argument applying Corollary 14.22 at a block start $Q \ge p_0$ with block length at most L uses $d = g' \exp(B^+L/(Q\log Q))$ and asserts $d/\sqrt{Q} < 1$, where $g' = g_3$ in the middle range and on the bands of the upper range, where B^+ is also recomputed with g_3 in place of g, as (d) allows, and $g' = g$ for the tail datum (Corollary 14.22). Every input is an admissible function fixed at p_0 together with the partitions and the test matrices, so its value at p_0 is a valid input at every prime $p \ge p_0$ (Lemmas 4.18 and 14.29). Floating point only proposes the rational parameters; the exact tests prove them. The route-B caps are extended from their endpoints in the same way (Lemma 27.7), with the two branches of (14.13) checked as above. The twelve-label grids are listed in Table 34, and the grid of the cut L20 of Section 20.4 in Table 35. The caps of the upper range are recorded in the certificates of Section 27 and in Certificate 28.12, and Table 32 lists the band inputs.

14.11. **Primes $p \equiv 1 \pmod 4$: extra terms (large primes only).** This subsection is used only at the tail primes $p \ge P_4 = 2^{512}$ with $p \equiv 1 \pmod 4$ (Section 29.1). At every other prime of the construction where a spectral cap is used, $p \equiv 3 \pmod 4$ and $t_J = 0$ (Lemma 14.13(c)). At $p \equiv 1 \pmod 4$ the degree-one matrices $\mathbf{A}_{1,\pm} = \pm\Delta^E$ are real symmetric, the two signs of (14.21) are distinct tests, the model is $\mathbf{H}^+ = \mathbf{H} + \mathbf{H}^{\mathrm{cov}}$ (Definition 14.11), and the non-coverable sets enter η_B through $\varepsilon_{\mathrm{par}} = \bar\zeta\Lambda_{\mathrm{nc}}$ (Lemma 14.14). It remains to supply admissible values of $\bar\zeta$.

Lemma 14.31 (Class-one Paley terms). *Let $p \equiv 1 \pmod 4$, assume $e_\Delta(p, 2n) < 1$, and let coupling data be given. For every $J \subseteq U^+$ with $|J| \ge 2$ and every $x \in \mathcal{I}^+$, $|S_J(x_J)| \le r$; in particular $|t_J| \le r$. If J is not coverable, then*

$$|t_J| \le \zeta r, \qquad \zeta = (R_\nu R_{\mathrm{full}} - 1) + \frac{n}{(p-n)r}, \qquad R_{\mathrm{full}} = \prod_{i \in U^+} \frac{1 + e_{d_i^0}(p,n)}{1 - e_{d_i^0}(p,n)}.$$

In instance (A), $R_\nu R_{\mathrm{full}} = R_{\mathrm{cut}} R_{\mathrm{full}}$.

Proof. $|S_J| \le \|g_J\|^2 \, \|C\| \le r$. Let $j \in J$ be isolated in $G^+[J]$, and let u_{cons} be the uniform law on $\mathrm{Inj}_p(G^+[J])$. By Lemma 12.6(d) for G^+ and the density bound (i) of the coupling data, the law ν_J of x_J under ν is $w\,u_{\mathrm{cons}}/\mathbb{E}_{u_{\mathrm{cons}}} w$ with $\max w / \min w \le R_\nu R_{U^+\setminus J} \le R := R_\nu R_{\mathrm{full}}$. Since $w/\mathbb{E}w \in [1/R, R]$ and $|S_J| \le r$, $|t_J - \mathbb{E}_{u_{\mathrm{cons}}} S_J| = |\mathbb{E}_{u_{\mathrm{cons}}}[(w/\mathbb{E}w - 1)S_J]| \le (R-1)r$. Under u_{cons}, given $x_{J\setminus\{j\}}$ the label x_j is uniform on the $p - |J| + 1$ values not used, because j has no edge in $G^+[J]$. Write $S_J = \mathbb{E}_{a,b} \chi(b-a)\chi(a-x_j)\chi(b-x_j) g_{J\setminus\{j\}}(a) g_{J\setminus\{j\}}(b)$. The terms $a = b$ vanish, and for $a \ne b$, $|\sum_{t \notin x(J\setminus\{j\})} \chi((a-t)(b-t))| \le 1 + (|J|-1)$ by (4.2); so $|\mathbb{E}_{u_{\mathrm{cons}}} S_J| \le |J|/(p - |J| + 1) \le n/(p-n)$. Hence $|t_J| \le \zeta r$. □

So $\bar{\zeta} = \zeta$ is admissible in Lemma 14.14; the ten nonroot cuts use it. At the root cuts the constant ζ (with $R_\nu = R_{\mathrm{cut}}$ in instance (A)) is larger at every stored threshold than the bound of the transfer below, which replaces it there.

Lemma 14.32 (Transfer of the non-coverable Paley averages)**.** *Let $p \equiv 1 \pmod 4$, take instance* (A)*, and let $\varepsilon_\varrho \geq \|\varrho - 1\|_\mu$. Then for every J with $|J| \geq 2$ that is not coverable,*

$$\frac{|t_J|}{r} \leq R_{\mathrm{full}} - 1 + \frac{n}{(p-n)r} + \varepsilon_\varrho.$$

Consequently $\bar{\zeta} = \min\{\zeta,\ R_{\mathrm{full}} - 1 + n/((p-n)r) + \varepsilon_\varrho\}$ is admissible in Lemma 14.14.

Proof. Here $t_J = \mathbb{E}_{m_p} S_J$. Since $|S_J| \leq r$ pointwise, Cauchy–Schwarz gives $|\mathbb{E}_{m_p} S_J - \mathbb{E}_\mu S_J| = |\mathbb{E}_\mu[(\varrho - 1)S_J]| \leq \varepsilon_\varrho r$. The isolated-vertex argument of Lemma 14.31 for $\nu = \mu$ ($R_\nu = 1$) gives $|\mathbb{E}_\mu S_J| \leq (R_{\mathrm{full}} - 1 + n/((p-n)r))r$. □

The certificate applies the transfer only at the root cuts LL20 and LL22, with $\varepsilon_\varrho = R_Z(M_\star + N_{\mathrm{ns}} + D_{\mathrm{col}})$, the constant coupling of Proposition 14.20(c3); there it gives the smaller value at every stored threshold, and the stored class-one root caps, which are inputs of the tail law, are computed with it. Every quantity is admissible (Lemma 14.29), so the value at the threshold covers all larger primes of the class. Sharper class-one bounds recorded in Section 35.10 are not used by any certificate of the construction.

Consequences and what to read next. This section used the following inputs; its outputs are the two criteria and the exact star synthesis above. From Part II: Weil's bound, the two-root identity, Gauss sums and the Paley operator (Theorems 4.7 and 4.8 and Lemma 4.11, (4.2)); maximal correlation (Definition 4.3); admissible majorants (Definition 4.17 and Lemma 4.18). From Part IV: the extension count, the sequential counts and the conditional laws of graph laws (Lemmas 12.2, 12.5 and 12.6, (12.2)); spectral data, the cut kernel, the positive envelope and the block lift (Definition 13.1, Lemmas 13.2, 13.3 and 13.7, and Proposition 13.6); weighted frames (Lemma 13.8), in Section 14.2; the star-trace dictionary and the star function (Lemma 15.18, (15.9)); the two-variable sum Theorem 15.13 with (15.7); and the witness kernel Lemma 16.4. As inputs of the criteria (not used in their proofs): the nonstar and coincidence bounds (Lemmas 16.2 and 16.15, Theorem 16.9, and Corollary 16.11) and the ordered-star bounds (Proposition 15.26 and Lemma 15.29). No input of Section 14.2 comes from the two criteria.

15. Second-order character sums: $O(p^{-1})$ where Weil gives $O(p^{-1/2})$

What this section proves and how to read it. A bound is *second order* if it is $O(r^2) = O(1/p)$ where one Weil bound gives $O(r)$. Three such bounds are proved, for a four-cycle kernel, a two-variable sum, and trace functions with one free label (called *moving pinned values* below; "root" in lemma names refers to pinned values, not to the Part V principal law). A reader willing to accept Theorems 15.8 and 15.13 as stated can skip the sheaf arguments; external inputs are collected in Appendix A. Weil's bound gives square-root cancellation in a sum of one quadratic character of one polynomial; after the normalizations of the local laws this square-root cancellation is a relative error of order $r = p^{-1/2}$. The spectral criteria of Section 14, as they are applied in Part V, are conditional: part of a labelling is fixed, and the joint law of the remaining labels is compared with a product law. Errors of two one-variable counts can then correlate, and several terms must be of order r^2, uniformly in the fixed labels. This section proves the three second-order estimates the construction needs, in the generality in which they are used. No graph occurs in Sections 15.1 and 15.3, and Section 15.4 uses a half law only through its one-vertex conditional laws, except for the density bound of Lemma 15.28(a), which uses the tuple densities of a graph law.

(1) *The four-cycle kernel* $\chi(\prod_{i,j\le 2}(y_j - x_i))$ on ordered pairs of distinct labels has norm at most $2/p = 2r^2$ for uniform probability (Theorem 15.8 and Corollary 15.9), where Cauchy–Schwarz gives only r. This is the norm of a decided complete rectangle (one all of whose pairs are present or absent; Section 16.1) in the witness kernels (Lemma 16.4) and of the $K_{2,2}$ core of the completion variance (Lemma 14.15). Split along the diagonal of two auxiliary labels, it bounds the remainder of the raw pair maps by order $q = 1/p$ (Corollary 15.10): this bound is used for the positive roots of the late middle pieces (Section 21).
(2) *The two-variable sum* $\sum_{y,z}\chi(f(y)g(z)(y-z))$ with split squarefree f, g is $O(p)$ rather than $O(p^{3/2})$ (Theorem 15.13). It bounds the complete conditional means of the product rows (Section 27.6.2) and the pair term of the completion variance. It also bounds the conditional pivot means of weighted star marginals (Lemma 15.28), which bound the higher-star marginals ($|T| \ge 3$ in Lemma 15.28) of the dense comparisons and of the six-query mixture of the upper range (Propositions 28.5 and 28.6).
(3) *The moving-root traces* τ_S describe the integrated-centre stars of a cut kernel once all labels but one are fixed. They are almost orthogonal for distinct root sets, also against character weights (Lemma 15.19). Ordered pivot groups turn this into the star marginal $M_\star$ and the star mean $z_\star$ of the upper root (Proposition 15.26 and Lemma 15.29), which enter Lemmas 27.6 and 27.10; the uniform forms of the moments enter the conditional row bounds of the product rows (Corollaries 15.20 and 15.21).

The first estimate is representation theory of $\mathrm{PGL}_2(\mathbb{F}_p)$ together with Jacobi sums. The second and third come from Deligne's theorem applied to the Katz–Sarnak sheaves of the family of curves $y^2 = f(x)(x-z)$; the external inputs are stated in Appendix A. One further second-order input is not proved here: the twisted second moment $\sum_\lambda \chi(\lambda)T_\lambda^2$ of the Legendre traces $T_\lambda = \sum_y \chi(y(y-1)(y-\lambda))$, to which the graph moment M_{G_5} behind the variance savings (the graph G_5 and its moment are defined in Lemma 24.3) reduces, is evaluated through a cusp form of weight four (Theorem A.7) and enters Lemma 24.3 directly. Section 15.5 lists where each estimate enters. Throughout, p is an odd prime, $r = p^{-1/2}$, $\chi = \chi_p$ is the quadratic character with $\chi(0) = 0$, and $\mathbb{E}_x$ is the average over $x \in \mathbb{F}_p$.

15.1. **The four-cycle kernel.** For four distinct points, the character kernel of a complete rectangle $K_{2,2}$ is the quadratic character of a cross-ratio (it is 0 when points coincide), so it is invariant under $\mathrm{PGL}_2(\mathbb{F}_p)$ acting on pairs of points of the projective line. The permutation representation on ordered pairs of distinct points is induced from the split torus and has multiplicities at most two, so the kernel acts by computable scalars on all but one isotypic component. The scalars are twisted sums over the split and the nonsplit torus: products of two Jacobi sums and squares of sums of modulus $\sqrt{p}$. This gives spectral norm at most $2p$ for the character matrix (Theorem 15.8), that is $2/p$ after normalisation by counting measure (Corollary 15.9).
Notation. Points of $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{F}_p)$ are represented by the column vectors $(u,1)^{\mathsf{T}}$, $u \in \mathbb{F}_p$, and $\infty = (1,0)^{\mathsf{T}}$; for column vectors $a = (a_0,a_1)^{\mathsf{T}}$, $b = (b_0,b_1)^{\mathsf{T}}$ put $a \wedge b = a_0b_1 - a_1b_0$. Let $\mathcal{X} = \{(x_1,x_2) \in \mathbb{P}^1 \times \mathbb{P}^1 : x_1 \ne x_2\}$, so $|\mathcal{X}| = p(p+1)$, and for $x, z \in \mathcal{X}$ let

$$M(x,z) = \chi\Big(\prod_{i,j\in\{1,2\}} (z_i \wedge x_j)\Big), \tag{15.1}$$

computed with the chosen representatives. For finite points $z_i \wedge x_j = z_i - x_j$, so on pairs of finite points $M(x,z) = \chi((z_1-x_1)(z_1-x_2)(z_2-x_1)(z_2-x_2))$. We also write M for the matrix $(M(x,z))_{x,z\in\mathcal{X}}$ and for the operator $(Mh)(x) = \sum_{z\in\mathcal{X}} M(x,z)h(z)$ on $\mathbb{C}^{\mathcal{X}}$ with the standard inner product $\langle h,k\rangle = \sum_x \overline{h(x)}k(x)$.

If $\{z_1,z_2\}\cap\{x_1,x_2\}=\emptyset$, the product in (15.1) is the cross-ratio $[z_1,z_2;x_1,x_2]=\frac{(z_1\wedge x_1)(z_2\wedge x_2)}{(z_1\wedge x_2)(z_2\wedge x_1)}$ times the nonzero square $\big((z_1\wedge x_2)(z_2\wedge x_1)\big)^2$, so $M(x,z)=\chi([z_1,z_2;x_1,x_2])$; otherwise $M(x,z)=0$. The cross-ratio does not depend on the representatives and is invariant under $\mathrm{GL}_2(\mathbb{F}_p)$. Thus $M(x,z)$ depends only on the orbit of the configuration (x,z), which is why its spectrum can be computed by representation theory.

Let $\Gamma=\mathrm{PGL}_2(\mathbb{F}_p)$, of order $p(p^2-1)$, acting on $\mathbb{P}^1$ by matrices on column vectors and diagonally on $\mathcal{X}$, and let $\Lambda(g)h(y)=h(g^{-1}y)$ be the permutation representation on $\mathbb{C}^{\mathcal{X}}$. We fix a nonsquare $D\in\mathbb{F}_p^\times$ and an element $\sqrt{D}\in\mathbb{F}_{p^2}$ with $(\sqrt{D})^2=D$, so $\mathbb{F}_{p^2}=\mathbb{F}_p(\sqrt{D})$; for $\zeta\in\mathbb{F}_{p^2}$, $\bar\zeta=\zeta^p$, $N\zeta=\zeta\bar\zeta\in\mathbb{F}_p$ and $\operatorname{tr}\zeta=\zeta+\bar\zeta\in\mathbb{F}_p$. Let $\mathsf{U}=\{s\in\mathbb{F}_{p^2}:Ns=1\}$, a cyclic group of order $p+1$ (the kernel of the norm, which maps $\mathbb{F}_{p^2}^\times$ onto $\mathbb{F}_p^\times$), and η its unique character of order 2; for $s\in\mathsf{U}$, $s^{-1}=\bar s$ and $\operatorname{tr}s=s+s^{-1}$. The map $\zeta\mapsto\zeta/\bar\zeta=\zeta^{1-p}$ is a homomorphism $\mathbb{F}_{p^2}^\times\to\mathsf{U}$ with kernel $\mathbb{F}_p^\times$, hence onto (orders p^2-1 and $p+1$); the elements $\zeta=m+\sqrt{D}$, $m\in\mathbb{F}_p$, together with $\zeta=1$ represent the cosets of $\mathbb{F}_p^\times$, and $\zeta=1$ is the one with $\zeta/\bar\zeta=1$. In Γ we use the elements $u=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $h_a=\left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)$ $(a\in\mathbb{F}_p^\times)$, and, for $\xi\in\mathbb{F}_{p^2}\setminus\mathbb{F}_p$, the image g_ξ in Γ of the $\mathbb{F}_p$-linear map $\zeta\mapsto\xi\zeta$ of $\mathbb{F}_{p^2}=\mathbb{F}_p\oplus\mathbb{F}_p\sqrt{D}$ (in the basis $1,\sqrt{D}$); g_ξ has the eigenvalues $\xi,\bar\xi$. Multiplicative characters vanish at 0.

Lemma 15.1 (Invariance). (a) *The value $M(x,z)$ does not depend on the chosen representatives. Moreover $M(x,z)=M(z,x)$, and $M(gx,gz)=M(x,z)$ for every $g\in\mathrm{GL}_2(\mathbb{F}_p)$. Hence M is real symmetric and commutes with $\Lambda(g)$ for every $g\in\Gamma$.*

(b) *Γ acts transitively on $\mathcal{X}$, and the stabilizer of $x_0=(0,\infty)$ is the image $\mathsf{T}=\{h_a:a\in\mathbb{F}_p^\times\}$ of the diagonal matrices, of order $p-1$. Hence $\mathbb{C}^{\mathcal{X}}\cong\mathrm{Ind}_{\mathsf{T}}^{\Gamma}\mathbf{1}$.*

Proof. (a) For column vectors and $g\in\mathrm{GL}_2(\mathbb{F}_p)$, $(ga)\wedge(gb)=\det g\,(a\wedge b)$, and $(ca)\wedge b=c(a\wedge b)=a\wedge(cb)$. Each of the four points x_1,x_2,z_1,z_2 occurs in exactly two of the four factors $z_i\wedge x_j$. Replacing a representative a by ca $(c\in\mathbb{F}_p^\times)$ therefore multiplies the product by c^2, and replacing every point by its image under g multiplies it by $(\det g)^4$ times a product of squares of the normalizing constants of the images; in all cases the product is multiplied by a nonzero square, which does not change χ. Moreover $\prod_{i,j}(x_i\wedge z_j)=\prod_{i,j}(-(z_j\wedge x_i))=\prod_{i,j}(z_i\wedge x_j)$, so M is symmetric; it is real. If $g\in\Gamma$, then $(M\Lambda(g)h)(x)=\sum_z M(x,z)h(g^{-1}z)=\sum_z M(x,gz)h(z)=\sum_z M(g^{-1}x,z)h(z)=(\Lambda(g)Mh)(x)$. (b) Given $(x_1,x_2)\in\mathcal{X}$ with representatives a_1,a_2, the matrix with columns a_1,a_2 is invertible and maps $\infty=(1,0)^{\mathsf{T}}$ to x_1 and $0=(0,1)^{\mathsf{T}}$ to x_2; so Γ is transitive on $\mathcal{X}$. A matrix fixing the points 0 and ∞ has both standard basis vectors as eigenvectors, so it is diagonal, and its image in Γ is some h_a. Since $|\Gamma|/|\mathsf{T}|=p(p+1)=|\mathcal{X}|$, the permutation representation on $\mathcal{X}$ is $\mathrm{Ind}_{\mathsf{T}}^{\Gamma}\mathbf{1}$. □

Lemma 15.2 (Classes and characters of Γ). (a) *For $\xi\in\mathbb{F}_{p^2}\setminus\mathbb{F}_p$, the conjugacy class of g_ξ in Γ depends only on $s=\xi/\bar\xi\in\mathsf{U}\setminus\{1\}$; we write g_s for it. Moreover h_a is conjugate to $h_{1/a}$ and g_s to $g_{1/s}$, and for every class function F on Γ*

$$\sum_{g\in\Gamma}F(g)=F(1)+(p^2-1)F(u)+\frac{p(p+1)}{2}\sum_{a\in\mathbb{F}_p^\times\setminus\{1\}}F(h_a)+\frac{p(p-1)}{2}\sum_{s\in\mathsf{U}\setminus\{1\}}F(g_s). \tag{15.2}$$

(b) *The irreducible complex representations of Γ are, up to isomorphism, the following, with the characters shown at u, h_a $(a\neq1)$ and g_s $(s\neq1)$:*

representation	*dimension*	u	h_a	g_s	*number*
$\mathbf{1}$	1	1	1	1	1
sgn	1	1	$\chi(a)$	$\eta(s)$	1
St	p	0	1	-1	1
$\mathrm{St}\otimes\mathrm{sgn}$	p	0	$\chi(a)$	$-\eta(s)$	1
$\pi(\mu)$, $\mu\in\widehat{\mathbb{F}_p^\times}$, $\mu^2\neq 1$	$p+1$	1	$\mu(a)+\bar\mu(a)$	0	$(p-3)/2$
D_ν, $\nu\in\widehat{\mathsf{U}}$, $\nu^2\neq 1$	$p-1$	-1	0	$-(\nu(s)+\bar\nu(s))$	$(p-1)/2$

Here $\pi(\mu)\cong\pi(\bar\mu)$ and $D_\nu\cong D_{\bar\nu}$, and the last column is the number of isomorphism classes of each kind; all characters are real. St *is the complement of* $\mathbf{1}$ *in the permutation representation of* Γ *on* $\mathbb{P}^1$.

Proof. We use Theorem A.1 with $q=p$. Scaling a matrix does not change its type, so every $g\neq 1$ in Γ has a well-defined type (b), (c) or (d).

(a) Let $\xi,\xi'\in\mathbb{F}_{p^2}\setminus\mathbb{F}_p$ with $\xi'/\bar\xi'\in\{\xi/\bar\xi,\bar\xi/\xi\}$. The multiplications by ξ' and by $\bar\xi'$ have the same eigenvalues $\{\xi',\bar\xi'\}$, so they are conjugate in $\mathrm{GL}_2(\mathbb{F}_p)$ (Theorem A.1; explicitly, the $\mathbb{F}_p$-linear map $\zeta\mapsto\bar\zeta$ conjugates one into the other). Replacing ξ' by $\bar\xi'$ if necessary, we may therefore assume $\xi'/\bar\xi'=\xi/\bar\xi$. Then $c=\xi'/\xi$ satisfies $\bar c=c$, so $c\in\mathbb{F}_p^\times$, and the multiplication by $\xi'=c\xi$ is c times the multiplication by ξ; hence $g_{\xi'}=g_\xi$ in Γ. This shows that the class depends only on s and that $g_s\sim g_{1/s}$. Similarly $h_{1/a}$ is the image of $\mathrm{diag}(1,a)$, which is conjugate to $\mathrm{diag}(a,1)$.

For (15.2) we count the elements of Γ of each type; each element of Γ has exactly $p-1$ preimages in $\mathrm{GL}_2(\mathbb{F}_p)$. By Theorem A.1(iv), a conjugacy class of $\mathrm{GL}_2(\mathbb{F}_p)$ of type (b), (c) or (d) has p^2-1, $p(p+1)$ or $p(p-1)$ elements. *Type (b).* For $x\in\mathbb{F}_p^\times$ the non-scalar matrices with characteristic polynomial $(T-x)^2$ form one class, so there are $(p-1)(p^2-1)$ such matrices and p^2-1 elements of type (b) in Γ. All of them are conjugate to u: the matrix $\left(\begin{smallmatrix}x&1\\0&x\end{smallmatrix}\right)$ represents the same element as $\left(\begin{smallmatrix}1&1/x\\0&1\end{smallmatrix}\right)=\mathrm{diag}(1/x,1)\,u\,\mathrm{diag}(1/x,1)^{-1}$. *Type (c).* For $a\in\mathbb{F}_p^\times\setminus\{1\}$ let $C_a\subseteq\Gamma$ be the set of elements with a preimage whose eigenvalues are x,ax for some x; then $C_a=C_{1/a}$, every element of C_a is conjugate to h_a, and the C_a partition the elements of type (c). The eigenvalue sets $\{x,ax\}$, $x\in\mathbb{F}_p^\times$, are $p-1$ distinct sets if $a\neq-1$ and $(p-1)/2$ if $a=-1$. So $|C_a|=p(p+1)$ for $a\neq\pm1$ and $|C_{-1}|=p(p+1)/2$. In the sum $\frac{p(p+1)}{2}\sum_{a\neq1}F(h_a)$ each set C_a with $a\neq-1$ appears twice (as a and $1/a$) with weight $\frac{p(p+1)}{2}$, and C_{-1} once; so this sum is $\sum_{g\text{ of type (c)}}F(g)$. *Type (d).* For $s\in\mathsf{U}\setminus\{1\}$ let $C'_s\subseteq\Gamma$ be the set of elements with a preimage whose eigenvalues $\xi,\bar\xi$ satisfy $\xi/\bar\xi\in\{s,1/s\}$; by Theorem A.1 and the first part of (a), every element of C'_s is conjugate to g_s. The equation $\xi/\bar\xi=s$ has $p-1$ solutions, and the eigenvalue sets $\{\xi,\bar\xi\}$ with $\xi/\bar\xi=s$ are $p-1$ distinct sets if $s\neq-1$ (since then $\bar\xi/\xi=1/s\neq s$) and $(p-1)/2$ if $s=-1$. So $|C'_s|=p(p-1)$ for $s\neq\pm1$ and $|C'_{-1}|=p(p-1)/2$, and as for type (c) the last sum in (15.2) is $\sum_{g\text{ of type (d)}}F(g)$. (The count $1+(p^2-1)+\frac{p(p+1)}{2}(p-2)+\frac{p(p-1)}{2}p=p^3-p$ confirms that every element is counted once.)

(b) The irreducible representations of Γ are those of $\mathrm{GL}_2(\mathbb{F}_p)$ on which the scalar matrices act trivially. By Theorem A.1(iii) and Schur's lemma (a scalar matrix acts on an irreducible representation by its character value divided by the dimension), the scalar xI acts on U_α and on V_α by $\alpha(x)^2$, on $W_{\alpha,\beta}$ by $\alpha\beta(x)$ and on X_φ by $\varphi(x)$. Hence the representations of Γ are: U_α, V_α with $\alpha^2=1$, that is $\alpha\in\{1,\chi\}$ (we put $\mathbf{1}=U_1$, $\mathrm{sgn}=U_\chi$, $\mathrm{St}=V_1$, $\mathrm{St}\otimes\mathrm{sgn}=V_\chi$); $W_{\mu,\bar\mu}=:\pi(\mu)$ with $\mu\neq\bar\mu$, that is $\mu^2\neq1$; and X_φ with $\varphi|_{\mathbb{F}_p^\times}=1$ and $\varphi^p\neq\varphi$. A character φ of $\mathbb{F}_{p^2}^\times$ trivial on $\mathbb{F}_p^\times$ factors through the surjection $\zeta\mapsto\zeta/\bar\zeta$ onto U: $\varphi(\zeta)=\nu(\zeta/\bar\zeta)$ for a unique $\nu\in\widehat{\mathsf{U}}$; since $(\zeta/\bar\zeta)^p=\bar\zeta/\zeta$, $\varphi^p=\varphi$ iff $\bar\nu=\nu$ iff $\nu^2=1$; we put $D_\nu=X_\varphi$. The isomorphisms are those of Theorem A.1(ii), and the numbers follow ($p-3$ characters μ of $\mathbb{F}_p^\times$ and $p-1$ characters ν of U

have square $\neq 1$). The values: u is of type (b) with $x=1$, h_a of type (c) with $(x,y)=(a,1)$, and g_ξ of type (d) with $\zeta=\xi$; so $\varphi(\xi)+\varphi(\bar\xi)=\nu(s)+\nu(\bar s)=\nu(s)+\bar\nu(s)$. Finally $\xi\mapsto\chi(N\xi)$ is a character of $\mathbb{F}_{p^2}^\times$ that is trivial on $\mathbb{F}_p^\times$ ($Nc=c^2$) and nontrivial (the norm is onto $\mathbb{F}_p^\times$), so it is $\nu_0(\xi/\bar\xi)$ with ν_0 of order 2, that is $\chi(N\xi)=\eta(s)$; this gives the values of sgn and $\mathrm{St}\otimes\mathrm{sgn}$ at g_s. All values are real. □

Lemma 15.3 (Decomposition)**.** *As a representation of* Γ*,*

$$\mathbb{C}^{\mathcal{X}} \cong \mathbf{1}\oplus \mathrm{St}^{\oplus 2}\oplus(\mathrm{St}\otimes\mathrm{sgn})\oplus\bigoplus_{\{\mu,\bar\mu\}}\pi(\mu)\oplus\bigoplus_{\{\nu,\bar\nu\}}D_\nu,$$

the sums over the unordered pairs with $\mu^2\neq 1$*, resp.* $\nu^2\neq 1$*;* sgn *does not occur.*

Proof. By Lemma 15.1(b) and Frobenius reciprocity, the multiplicity of an irreducible π is $m_\pi=\dim\pi^{\mathsf T}=\frac{1}{p-1}\sum_{a\in\mathbb{F}_p^\times}\chi_\pi(h_a)=\frac{1}{p-1}\big(\dim\pi+\sum_{a\neq1}\chi_\pi(h_a)\big)$. With the table of Lemma 15.2(b) and $\sum_{a\neq1}\chi(a)=-1$, $\sum_{a\neq1}(\mu+\bar\mu)(a)=-2$ ($\mu\neq1$): $m_{\mathbf 1}=1$, $m_{\mathrm{sgn}}=0$, $m_{\mathrm{St}}=\frac{p+p-2}{p-1}=2$, $m_{\mathrm{St}\otimes\mathrm{sgn}}=1$, $m_{\pi(\mu)}=\frac{p+1-2}{p-1}=1$, $m_{D_\nu}=1$. (Check: $1+2p+p+\frac{p-3}{2}(p+1)+\frac{p-1}{2}(p-1)=p(p+1)$.) □

Lemma 15.4 (Eigenvalue on a multiplicity-one constituent)**.** *Let* π *be an irreducible representation of* Γ *of multiplicity one in* $\mathbb{C}^{\mathcal{X}}$*, and* V_π *its isotypic subspace. Then* M *acts on* V_π *as the scalar*

$$\lambda_\pi=\frac{1}{|\Gamma|}\sum_{g\in\Gamma}\chi_\pi(g)\,\Phi(g),\qquad \Phi(g)=\sum_{x\in\mathcal{X}}M(x,gx).$$

Φ *is a class function.*

Proof. M commutes with $\Lambda(\Gamma)$ (Lemma 15.1(a)), so it preserves $V_\pi\cong\pi$ and acts there by a scalar λ_π (Schur's lemma). $P_\pi=\frac{\dim\pi}{|\Gamma|}\sum_g\overline{\chi_\pi(g)}\Lambda(g)$ is the orthogonal projection onto V_π, so $\lambda_\pi\dim\pi=\mathrm{Tr}(MP_\pi)=\frac{\dim\pi}{|\Gamma|}\sum_g\overline{\chi_\pi(g)}\,\mathrm{Tr}(M\Lambda(g))$. The matrix of $\Lambda(g)$ has the entries $[y=gx]$, so $\mathrm{Tr}(M\Lambda(g))=\sum_x M(x,gx)=\Phi(g)$; the characters are real (Lemma 15.2(b)). $\Phi(kgk^{-1})=\Phi(g)$ by the substitution $x\mapsto kx$ and Lemma 15.1(a). □

For $a\in\mathbb{F}_p^\times$ put $\Upsilon(a)=\sum_{t\in\mathbb{F}_p}\chi\big(t(a-t)(at-1)\big)$, and for $s\in\mathsf U$ put $R(s)=\sum_{t\in\mathsf U\setminus\{1\}}\chi(\mathrm{tr}\,s-\mathrm{tr}\,t)$.

Lemma 15.5 (The function Φ)**.** (i) $\Phi(1)=0$ *and* $\Phi(u)=-p(1+\chi(-1))$.
(ii) $\Phi(h_a)=(p-1)(\Upsilon(a)-1)$ *for* $a\neq1$*;* $\Upsilon(1/a)=\Upsilon(a)$ *and* $\Upsilon(1)=-\chi(-1)$.
(iii) $\Phi(g_s)=-(p+1)\,\eta(s)\,R(s)$ *for* $s\in\mathsf U\setminus\{1\}$.

Proof. (i) $x_1\wedge x_1=0$, so $M(x,x)=0$. The element u maps $(v,1)^{\mathsf T}$ to $(v+1,1)^{\mathsf T}$ and fixes ∞. If x contains ∞, the factor $(u\infty)\wedge\infty$ vanishes. For finite x with $d=x_1-x_2\neq0$ the four factors $(ux)_i-x_j$ are 1, $1+d$, $1-d$, 1, so $\Phi(u)=p\sum_{d\neq0}\chi(1-d^2)$. Now $\sum_{d\in\mathbb{F}_p}\chi(1-d^2)=\chi(-1)\sum_d\chi((d-1)(d+1))=-\chi(-1)$ by the two-root identity (4.2), and the term $d=0$ is 1; so $\Phi(u)=p(-\chi(-1)-1)$. (ii) h_a fixes 0 and ∞, so pairs containing one of them contribute 0. For $x_1,x_2\in\mathbb{F}_p^\times$ write $x_2=tx_1$, $t\notin\{0,1\}$; the four factors are $(a-1)x_1$, $(a-t)x_1$, $(at-1)x_1$, $(a-1)tx_1$, with product $(a-1)^2x_1^4\,t(a-t)(at-1)$. Hence $\Phi(h_a)=(p-1)\sum_{t\neq0,1}\chi(t(a-t)(at-1))=(p-1)(\Upsilon(a)-1)$: the term $t=0$ of $\Upsilon(a)$ is 0 and the term $t=1$ is $\chi((a-1)^2)=1$. Next, $\Upsilon(1/a)=\sum_t\chi\big(t(1-at)(t-a)a^{-2}\big)=\Upsilon(a)$, and $\Upsilon(1)=\sum_t\chi(-t(t-1)^2)=\sum_{t\neq1}\chi(-t)=-\chi(-1)$. (iii) Identify $\mathbb{F}_p^2$ with $\mathbb{F}_{p^2}$ by $(a_0,a_1)\mapsto\alpha=a_0+a_1\sqrt{D}$. For α, β corresponding to a, b one checks $\bar\alpha\beta-\alpha\bar\beta=2\sqrt{D}(a_0b_1-a_1b_0)$, so $a\wedge b=(\bar\alpha\beta-\alpha\bar\beta)/(2\sqrt{D})$. The points of $\mathbb{P}^1$ are the cosets $\mathbb{F}_{p^2}^\times/\mathbb{F}_p^\times$, and g_ξ acts by multiplication by ξ. The multiplications by elements of $\mathbb{F}_{p^2}^\times$ form a subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$ whose image $\mathsf T'\subseteq\Gamma$ acts simply transitively on $\mathbb{P}^1$ and commutes with g_ξ. For $k\in\mathsf T'$, $M(kx,g_\xi kx)=M(kx,kg_\xi x)=M(x,g_\xi x)$ by Lemma 15.1(a); every $x\in\mathcal{X}$ is mapped by exactly

one $k \in \mathsf{T}'$ to a pair with first point [1], so $\Phi(g_\xi) = (p+1)\sum_w M(([1],[w]),([\xi],[\xi w]))$, the sum over the p cosets $[w] \neq [1]$. The numerators $2\sqrt{D}(z_i \wedge x_j)$ of the four factors are $\bar\xi - \xi$, $\bar\xi w - \xi\bar w$, $\bar\xi\bar w - \xi w$ and $N(w)(\bar\xi - \xi)$, and the common denominator $(2\sqrt{D})^4 = 16D^2$ is a square. Write $\xi = \xi_0 + \xi_1\sqrt{D}$ with $\xi_1 \neq 0$; then $(\bar\xi - \xi)^2 = 4\xi_1^2 D$ is a nonsquare. Further

$$(\bar\xi w - \xi\bar w)(\bar\xi\bar w - \xi w) = N(w)\operatorname{tr}(\xi^2) - N(\xi)\operatorname{tr}(w^2) = N(w)N(\xi)(\operatorname{tr} s - \operatorname{tr} t), \qquad s = \xi/\bar\xi,\ t = w/\bar w,$$

since $\operatorname{tr}(\zeta^2)/N\zeta = \zeta/\bar\zeta + \bar\zeta/\zeta$. Hence the summand is $-\chi(N\xi)\chi(\operatorname{tr} s - \operatorname{tr} t) = -\eta(s)\chi(\operatorname{tr} s - \operatorname{tr} t)$ (Lemma 15.2(b) for $\chi(N\xi) = \eta(s)$), and as $[w]$ runs over the cosets $\neq [1]$, $t = w/\bar w$ runs over $\mathsf{U} \setminus \{1\}$ bijectively. □

We also need two identities. Let $s \in \mathsf{U} \setminus \{1\}$ and write $s = w/\bar w$ with $w = m + \sqrt{D}$. Then $\operatorname{tr} s - 2 = -(s-1)(\bar s - 1) = -N(s-1)$ and $s - 1 = (w - \bar w)/\bar w = 2\sqrt{D}/\bar w$, so $N(s-1) = -4D/N(w)$ and $\chi(\operatorname{tr} s - 2) = \chi(D)\chi(Nw) = -\eta(s)$. Since $\chi(2 - \operatorname{tr} t) = \chi(-1)\chi(\operatorname{tr} t - 2)$ and $\sum_{t\in\mathsf{U}} \eta(t) = 0$,

$$\sum_{t\in\mathsf{U}} \chi(2 - \operatorname{tr} t) = \chi(-1), \qquad \sum_{s\in\mathsf{U}} \psi(s)\chi(\operatorname{tr} s - 2) = -\sum_{s\neq 1} (\psi\eta)(s) = 1 \quad (\psi \in \widehat{\mathsf{U}},\ \psi \neq \eta). \tag{15.3}$$

For $\psi \in \widehat{\mathsf{U}}$ and $\mu \in \widehat{\mathbb{F}_p^\times}$ put

$$\mathcal{N}(\psi) = \sum_{s,t\in\mathsf{U}} \psi(s)\chi(\operatorname{tr} s - \operatorname{tr} t), \qquad \mathcal{S}(\mu) = \sum_{a\in\mathbb{F}_p^\times} \mu(a)\Upsilon(a),$$

and let $\operatorname{Jac}(\psi_1,\psi_2) = \sum_{v\in\mathbb{F}_p} \psi_1(v)\psi_2(1-v)$ be the Jacobi sum of Theorem 4.10. Since $\operatorname{tr}(s^{-1}) = \operatorname{tr} s$ and $\Upsilon(1/a) = \Upsilon(a)$, $\mathcal{N}(\bar\psi) = \mathcal{N}(\psi)$ and $\mathcal{S}(\bar\mu) = \mathcal{S}(\mu)$. Since $\mathbb{F}_p^\times \subseteq \mathbb{F}_{p^2}^\times$ is a subgroup of a finite abelian group, χ extends to a character $\tilde\chi$ of $\mathbb{F}_{p^2}^\times$; we fix one ($\tilde\chi(0) = 0$). For $\kappa \in \widehat{\mathsf{U}}$ put $\mathcal{C}(\kappa) = \sum_{v\in\mathsf{U}} \kappa(v)\tilde\chi(v-1)$.

Lemma 15.6 (Line sums)**.** *Let Ξ be a character of $\mathbb{F}_{p^2}^\times$ whose restriction to $\mathbb{F}_p^\times$ is nontrivial, and $\alpha \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$. Then $\bigl|\sum_{m\in\mathbb{F}_p} \Xi(m+\alpha)\bigr| = \sqrt{p}$.*

Proof. $m + \alpha \neq 0$ for all m, and $|\sum_m \Xi(m+\alpha)|^2 = p + \sum_{m\neq n} \Xi(\frac{m+\alpha}{n+\alpha})$. We claim that $(m,n) \mapsto w = \frac{m+\alpha}{n+\alpha}$ is a bijection from $\{(m,n) \in \mathbb{F}_p^2 : m \neq n\}$ onto $\mathbb{F}_{p^2} \setminus \mathbb{F}_p$. If $w = c \in \mathbb{F}_p$, then $(1-c)\alpha = cn - m \in \mathbb{F}_p$, which forces $c = 1$ and $m = n$; so the map lands in $\mathbb{F}_{p^2} \setminus \mathbb{F}_p$. Given $w \notin \mathbb{F}_p$, the elements $1, w$ are a basis of $\mathbb{F}_{p^2}$ over $\mathbb{F}_p$, so $m - wn = (w-1)\alpha$ has exactly one solution $(m,n) \in \mathbb{F}_p^2$, and $m \neq n$ (otherwise $(w-1)(m+\alpha) = 0$, impossible); this solution is the unique preimage. Hence the last sum is $\sum_{\mathbb{F}_{p^2}^\times} \Xi - \sum_{\mathbb{F}_p^\times} \Xi = 0 - 0$, because Ξ is nontrivial on $\mathbb{F}_p^\times$, hence on $\mathbb{F}_{p^2}^\times$. □

Lemma 15.7 (Torus sums)**.** (a) *Let C be a cyclic group of even order, -1 its element of order two, ψ a character of C and $F : C \to \mathbb{C}$. If $\psi(-1) = -1$ then $\sum_{y\in C} \psi(y)F(y^2) = 0$. If $\psi(-1) = 1$ then $\sum_{y\in C} \psi(y)F(y^2) = \sum_\kappa \sum_{w\in C} \kappa(w)F(w)$, the sum over the two characters κ of C with $\kappa^2 = \psi$.*

(b) (Split torus.) *If $\mu(-1) = -1$ then $\mathcal{S}(\mu) = 0$. If $\mu(-1) = 1$, then*

$$\mathcal{S}(\mu) = \chi(-1)\sum_\kappa \operatorname{Jac}(\kappa,\chi)\operatorname{Jac}(\bar\kappa\chi,\chi),$$

the sum over the two characters κ of $\mathbb{F}_p^\times$ with $\kappa^2 = \bar\mu$. If $\mu^2 \neq 1$, or if $\mu = \chi$, then $|\mathcal{S}(\mu)| \le 2p$.

(c) (Nonsplit torus.) *Let $\psi \in \widehat{\mathsf{U}}$ and $\rho = (\psi\bar{\tilde\chi})|_{\mathsf{U}}$. If $\rho(-1) = -1$ then $\mathcal{N}(\psi) = 0$; otherwise $\mathcal{N}(\psi) = \sum_\kappa \mathcal{C}(\kappa)^2$, the sum over the two $\kappa \in \widehat{\mathsf{U}}$ with $\kappa^2 = \rho$. Moreover $|\mathcal{C}(\kappa)| = \sqrt{p}$ for every $\kappa \in \widehat{\mathsf{U}}$; hence $|\mathcal{N}(\psi)| \le 2p$ for every ψ.*

Proof. (a) If $\psi(-1) = -1$, the substitution $y \mapsto -y$ multiplies the sum by $\psi(-1) = -1$, so it vanishes. If $\psi(-1) = 1$: the characters of C form a cyclic group of even order, in which the characters with value 1 at -1 form the subgroup of index two, that is the squares; so $\psi = \kappa_0^2$, and the characters with square ψ are κ_0 and $\kappa_0\chi_C$, where χ_C is the character of order two of C. As $y \mapsto y^2$ is two-to-one onto the subgroup C^2 of squares and $\kappa_0 + \kappa_0\chi_C = 2\kappa_0\mathbf{1}_{C^2}$, $\sum_y \psi(y)F(y^2) = \sum_y \kappa_0(y^2)F(y^2) = 2\sum_{w\in C^2}\kappa_0(w)F(w) = \sum_{w\in C}(\kappa_0+\kappa_0\chi_C)(w)F(w)$.

(b) The terms $t = 0$ of Υ vanish. For $t \neq 0$ write $a = tb$; then $t(a-t)(at-1) = t^2(b-1)(t^2b-1)$. With $c = t^2b$ we have $\mu(a) = \bar\mu(t)\mu(c)$ and $\chi(b-1) = \chi(c-t^2)$, so

$$\mathcal{S}(\mu) = \sum_{c\neq 0} \mu(c)\chi(c-1)\, I(c), \qquad I(c) = \sum_{t\neq 0} \bar\mu(t)\chi(c-t^2).$$

Apply (a) with $C = \mathbb{F}_p^\times$, $\psi = \bar\mu$ (so $\psi(-1) = \mu(-1)$) and $F(y) = \chi(c-y)$: if $\mu(-1) = -1$ then $I(c) = 0$, and otherwise $I(c) = \sum_\kappa \sum_{y\neq 0}\kappa(y)\chi(c-y)$. For $c \neq 0$ the substitution $y = cv$ gives $\sum_y \kappa(y)\chi(c-y) = (\kappa\chi)(c)\,\mathrm{Jac}(\kappa,\chi)$. Hence $\mathcal{S}(\mu) = \sum_\kappa \mathrm{Jac}(\kappa,\chi)\sum_{c\neq 0}(\mu\kappa\chi)(c)\chi(c-1)$, and $\mu\kappa = \bar\kappa$ (as $\kappa^2 = \bar\mu$), while $\sum_c(\bar\kappa\chi)(c)\chi(c-1) = \chi(-1)\sum_c(\bar\kappa\chi)(c)\chi(1-c) = \chi(-1)\,\mathrm{Jac}(\bar\kappa\chi,\chi)$. Moduli: if $\mu^2 \neq 1$ then $\kappa^4 \neq 1$, so $\kappa \neq 1$, $\kappa \neq \chi$; hence κ, χ, $\kappa\chi$ and $\bar\kappa\chi$, χ, $\bar\kappa\chi\cdot\chi = \bar\kappa$ are nontrivial, and Theorem 4.10 gives $|\,\mathrm{Jac}\,| = \sqrt{p}$ for both Jacobi sums. If $\mu = \chi$ is even, i.e. $\chi(-1) = 1$ ($p \equiv 1 \pmod 4$), κ has order 4, so $\kappa\chi = \bar\kappa$ and $\bar\kappa\chi = \kappa$ are nontrivial, and again $|\,\mathrm{Jac}\,| = \sqrt{p}$. In both cases $|\mathcal{S}(\mu)| \le 2p$; if $\mu = \chi$ is odd ($\chi(-1) = -1$), $\mathcal{S}(\chi) = 0$.

(c) For $s,t \in \mathsf{U}$, $\mathrm{tr}\,s - \mathrm{tr}\,t = (s-t)(st-1)/(st) \in \mathbb{F}_p$. Since $\tilde\chi$ restricts to χ on $\mathbb{F}_p^\times$ and is multiplicative, $\chi(\mathrm{tr}\,s - \mathrm{tr}\,t) = \tilde\chi(s-t)\tilde\chi(st-1)\bar{\tilde\chi}(st)$ (both sides vanish iff $t \in \{s, s^{-1}\}$). Put $t = sw$, $w \in \mathsf{U}$: the summand $\psi(s)\chi(\mathrm{tr}\,s - \mathrm{tr}\,t)$ becomes $\rho(s)\cdot\bar{\tilde\chi}(w)\tilde\chi(1-w)\cdot\tilde\chi(s^2w-1)$. So $\mathcal{N}(\psi) = \sum_w \bar{\tilde\chi}(w)\tilde\chi(1-w)\sum_{s\in\mathsf{U}}\rho(s)\tilde\chi(s^2w-1)$. Apply (a) with $C = \mathsf{U}$ (cyclic of even order $p+1$; note $-1 \in \mathsf{U}$) and $F(y) = \tilde\chi(yw-1)$: the inner sum vanishes if $\rho(-1) = -1$, and otherwise it is $\sum_\kappa\sum_{y\in\mathsf{U}}\kappa(y)\tilde\chi(yw-1) = \sum_\kappa \bar\kappa(w)\mathcal{C}(\kappa)$ ($v = yw$). Finally $\bar{\tilde\chi}(w)\tilde\chi(1-w) = \tilde\chi(w^{-1}-1)$, and $\sum_w \bar\kappa(w)\tilde\chi(w^{-1}-1) = \mathcal{C}(\kappa)$ ($w \mapsto w^{-1}$).

For $|\mathcal{C}(\kappa)|$: the map $z \mapsto z^{p-1} = \bar z/z$ sends $\mathbb{F}_{p^2}^\times$ onto U with kernel $\mathbb{F}_p^\times$, the coset of 1 goes to 1 (whose term $\tilde\chi(1-1) = 0$ vanishes), and the cosets of $z = m + \sqrt{D}$, $m \in \mathbb{F}_p$, go bijectively to $\mathsf{U}\setminus\{1\}$. For such z, $\bar z/z - 1 = -2\sqrt{D}/z$. Hence, with the character $\Xi(z) = \kappa(z^{p-1})\bar{\tilde\chi}(z)$ of $\mathbb{F}_{p^2}^\times$,

$$\mathcal{C}(\kappa) = \tilde\chi(-2\sqrt{D})\sum_{m\in\mathbb{F}_p}\Xi(m+\sqrt{D}),$$

and Ξ restricts to $\bar\chi = \chi \neq 1$ on $\mathbb{F}_p^\times$, because $z \mapsto \kappa(z^{p-1})$ is trivial there. Lemma 15.6 with $\alpha = \sqrt{D}$ gives $|\mathcal{C}(\kappa)| = \sqrt{p}$. □

Theorem 15.8 (Four-cycle operator norm)**.** *For every odd prime p, every eigenvalue of the real symmetric matrix M of* (15.1) *lies in $[-2p, 2p]$. In particular every principal submatrix of M (for instance the one indexed by the ordered pairs of distinct elements of $\mathbb{F}_p$) has spectral norm at most $2p$.*

Proof. By Lemma 15.3 $\mathbb{C}^{\mathcal{X}}$ is the orthogonal sum of the isotypic subspaces, which M preserves (it commutes with $\Lambda(\Gamma)$). We determine the eigenvalues on each.

$\pi(\mu)$, $\mu^2 \neq 1$. By Lemma 15.4, (15.2), Lemma 15.5 and Lemma 15.2(b),

$$\begin{aligned}\lambda &= \frac{1}{|\Gamma|}\Big[(p^2-1)\Phi(u) + \frac{p(p+1)}{2}\sum_{a\neq 1}(\mu+\bar\mu)(a)(p-1)(\Upsilon(a)-1)\Big]\\ &= -(1+\chi(-1)) + \frac{1}{2}\sum_{a\neq 1}(\mu+\bar\mu)(a)(\Upsilon(a)-1).\end{aligned}$$

Since $\sum_{a\neq1}\mu(a)=-1$ and $\Upsilon(1)=-\chi(-1)$, $\sum_{a\neq1}\mu(a)(\Upsilon(a)-1)=\mathcal{S}(\mu)+\chi(-1)+1$, and likewise for $\bar\mu$ with $\mathcal{S}(\bar\mu)=\mathcal{S}(\mu)$. So $\lambda=\mathcal{S}(\mu)$, and $|\lambda|\le 2p$ by Lemma 15.7(b).

D_ν, $\nu^2\neq1$. Here $\chi_\pi(u)=-1$, $\chi_\pi(h_a)=0$, so $\lambda=(1+\chi(-1))+\frac12\sum_{s\neq1}(\nu+\bar\nu)(s)\eta(s)R(s)$. For $\psi\in\widehat{\mathrm{U}}$ with $\psi\neq\eta$, removing from $\mathcal{N}(\psi)$ the row $s=1$ and the column $t=1$ (whose common term $\chi(0)$ is 0) gives, by (15.3), $\sum_{s\neq1}\psi(s)R(s)=\mathcal{N}(\psi)-\chi(-1)-1$. With $\psi=\nu\eta$ and $\psi=\bar\nu\eta$ (both $\neq\eta$ as $\nu\neq1$) and $\mathcal{N}(\bar\nu\eta)=\mathcal{N}(\overline{\nu\eta})=\mathcal{N}(\nu\eta)$, we get $\lambda=\mathcal{N}(\nu\eta)$, and $|\lambda|\le 2p$ by Lemma 15.7(c).

$\mathrm{St}\otimes\mathrm{sgn}$. Here $\chi_\pi(u)=0$, $\chi_\pi(h_a)=\chi(a)$, $\chi_\pi(g_s)=-\eta(s)$, so

$$\begin{aligned}\lambda&=\frac12\sum_{a\neq1}\chi(a)(\Upsilon(a)-1)+\frac12\sum_{s\neq1}R(s)\\&=\frac12(\mathcal{S}(\chi)+\chi(-1)+1)+\frac12(\mathcal{N}(\mathbf{1})-\chi(-1)-1)=\frac12(\mathcal{S}(\chi)+\mathcal{N}(\mathbf{1})),\end{aligned}$$

and $|\lambda|\le 2p$ by Lemma 15.7(b),(c).

The trivial representation. It is spanned by the constant function, with eigenvalue the row sum $\sum_z M(x_0,z)$, which is the same for all x_0 (transitivity). For $x_0=(0,\infty)$: with the representatives $(z_i,1)$, $(0,1)$, $(1,0)$ fixed above, $z_i\wedge0=z_i$ and $z_i\wedge\infty=-1$ for finite z_i, while $\infty\wedge\infty=0$; so $M(x_0,z)=\chi(z_1z_2)$ if $z_1,z_2\in\mathbb{F}_p$ and 0 otherwise, and $\sum_{z_1\neq z_2}\chi(z_1z_2)=(\sum_v\chi(v))^2-\sum_v\chi(v^2)=-(p-1)$.

St *(multiplicity 2).* Let $C(z,x)=\chi(z\wedge x)$ for $z,x\in\mathbb{P}^1$. Then $\sum_{z\in\mathbb{P}^1}C(z,x)C(z,x')=p\,[x=x']$: for $x=x'$ there are p points $z\neq x$; for $x\neq x'$ both finite, $\sum_{z\in\mathbb{F}_p}\chi((z-x)(z-x'))=-1$ by (4.2), and the term $z=\infty$ is $\chi(1)^2=1$; for $x'=\infty$, x finite, $\sum_{z\in\mathbb{F}_p}\chi(z-x)\chi(-1)=0$ and the term $z=\infty$ vanishes. Now $M(x,z)=\phi_x(z_1)\phi_x(z_2)$ with $\phi_x(z)=C(z,x_1)C(z,x_2)$, so $\sum_{z\in\mathbb{P}^1}\phi_x(z)=0$ and $\phi_x(z)^2=\mathbf{1}[z\notin\{x_1,x_2\}]$. For $f:\mathbb{P}^1\to\mathbb{C}$ with $\sum f=0$ put $F_1(x)=f(x_1)$, $F_2(x)=f(x_2)$. Then

$$(MF_1)(x)=\sum_{z_1}f(z_1)\phi_x(z_1)\sum_{z_2\neq z_1}\phi_x(z_2)=-\sum_{z_1}f(z_1)\phi_x(z_1)^2=-\sum_{z_1\notin\{x_1,x_2\}}f(z_1)=f(x_1)+f(x_2),$$

and, $M(x,z)$ being symmetric in (z_1,z_2), also $MF_2=F_1+F_2$. The space $V_1=\{F_1+F_2'\colon\ \sum f=\sum f'=0\}$ (with $F_2'(x)=f'(x_2)$) has dimension $2p$ (if $f(x_1)+f'(x_2)=0$ for all $x_1\neq x_2$, then f is constant, as $p+1\ge3$, hence 0, and then $f'=0$), is Γ-invariant and is isomorphic to $\mathrm{St}\oplus\mathrm{St}$ (Lemma 15.2(b): St is the space of functions on $\mathbb{P}^1$ with sum 0). Since $m_{\mathrm{St}}=2$, V_1 is the St-isotypic subspace, and M acts on it as the identity on St tensored with $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$, with eigenvalues 2 and 0.

So every eigenvalue of M is one of $\mathcal{S}(\mu)$, $\mathcal{N}(\nu\eta)$, $\frac12(\mathcal{S}(\chi)+\mathcal{N}(\mathbf{1}))$, $-(p-1)$, 2, 0, all of modulus at most $2p$. A principal submatrix of a real symmetric matrix is a compression, so its norm is at most $\|M\|\le2p$. □

Every application of Theorem 15.8 in this paper needs it for normalized counting measure, for functions that vanish on noninjective tuples, and sometimes against conditional laws with bounded densities. We record these forms once.

Corollary 15.9 (The four-cycle kernel for uniform probability)**.** *Let $a,b\ge2$, let $\varsigma_{ij}\in\{\pm1\}$ $(i\le a,\ j\le b)$, and put*

$$K(x,y)=\prod_{i\le a,\ j\le b}\chi(\varsigma_{ij}(y_j-x_i))\qquad(x\in\mathbb{F}_p^a,\ y\in\mathbb{F}_p^b).$$

(a) *If $f:\mathbb{F}_p^a\to\mathbb{C}$ vanishes on the tuples with $x_1=x_2$ and $h:\mathbb{F}_p^b\to\mathbb{C}$ on the tuples with $y_1=y_2$ (in particular, if both vanish on noninjective tuples), then, for uniform probability on $\mathbb{F}_p^a$ and $\mathbb{F}_p^b$,*

$$\left|\mathbb{E}_{x,y}\overline{f(x)}\,K(x,y)\,h(y)\right|\le\frac{2}{p}\,\|f\|_2\,\|h\|_2\,. \tag{15.4}$$

For $a = b = 2$: the operator with kernel K relative to uniform probability on $\mathbb{F}_p^2$, compressed to the functions that vanish on the diagonal, has norm at most $2/p = 2r^2$.

(b) *Let ν_1, ν_2 be probability measures on $\mathbb{F}_p^a$, $\mathbb{F}_p^b$ with $\nu_1\{x_1 = x_2\} = \nu_2\{y_1 = y_2\} = 0$ and densities relative to uniform probability at most D_1, D_2. Then for $f \in L^2(\nu_1)$, $h \in L^2(\nu_2)$,*

$$\left|\mathbb{E}_{x\sim\nu_1,\, y\sim\nu_2} \overline{f(x)}\, K(x,y)\, h(y)\right| \le \frac{2}{p}\sqrt{D_1 D_2}\, \|f\|_{\nu_1}\, \|h\|_{\nu_2}\,.$$

Proof. Since $\chi(\varsigma(y-x)) = \chi(\varsigma)\chi(y-x)$, the signs multiply K by a constant of modulus one; we take them equal to 1.

(a) First let $a = b = 2$. For finite pairs $K(x,y) = M(x,y)$, and f, h are supported on the set $\mathcal{X}_{\text{aff}}$ of ordered pairs of distinct elements of $\mathbb{F}_p$. So the left side of (15.4) is $p^{-4}|\mathbf{f}^* M_{\text{aff}} \mathbf{h}|$, where M_{aff} is the principal submatrix of M on $\mathcal{X}_{\text{aff}}$ and $\mathbf{f}$, $\mathbf{h}$ are the vectors of values. By Theorem 15.8 this is at most $p^{-4} \cdot 2p\, \|\mathbf{f}\|_{\ell^2}\, \|\mathbf{h}\|_{\ell^2} = p^{-4} \cdot 2p \cdot p\, \|f\|_2 \cdot p\, \|h\|_2 = \frac{2}{p}\, \|f\|_2\, \|h\|_2$. For general a, b fix $x' = (x_3, \dots, x_a)$ and $y' = (y_3, \dots, y_b)$. Then $K(x,y) = K_{2,2}\big((x_1,x_2),(y_1,y_2)\big)\, \alpha(x_1,x_2)\beta(y_1,y_2)c$, where $K_{2,2}$ is the kernel for $a = b = 2$, $\alpha = \prod_{i\le 2<j} \chi(y_j - x_i)$, $\beta = \prod_{j\le 2<i} \chi(y_j - x_i)$ and $c = \prod_{i,j>2} \chi(y_j - x_i)$ have modulus at most one and depend on (x', y'). The sliced functions $f(\cdot, x')\alpha$ and $h(\cdot, y')\beta$ vanish on the diagonal, so the case $a = b = 2$ bounds the inner average over (x_1, x_2, y_1, y_2) by $\frac{2}{p}\, \|f(\cdot, x')\|_2\, \|h(\cdot, y')\|_2$. Averaging over the independent x' and y' and applying Cauchy–Schwarz separately in x' and in y' gives (15.4).

(b) Let d_i be the density of ν_i. The left side is $|\mathbb{E}_{x,y} \overline{(f d_1)(x)} K(x,y) (h d_2)(y)|$ for uniform probability, and $f d_1$, $h d_2$ vanish where $x_1 = x_2$, resp. $y_1 = y_2$. By (a) it is at most $\frac{2}{p}\, \|f d_1\|_2\, \|h d_2\|_2$, and $\|f d_1\|_2^2 = \mathbb{E}[d_1^2 |f|^2] \le D_1 \mathbb{E}[d_1 |f|^2] = D_1\, \|f\|_{\nu_1}^2$, and likewise for h. □

The positive roots of the late middle pieces (Section 21) multiply a root kernel by a polynomial of degree two in quartic traces $\mathbb{E}_s \psi_y(s)\psi_{y'}(s)$ of two pairs of labels, built from the *pair traces* $\psi_y(s) = \chi(s - y_0)\chi(s - y_1)$. The maps defined by pair traces are close to rank one in the sense of (15.5) below ($\|R\|^2 \le c_0 q^2$ there): their diagonal part is explicit, and their off-diagonal part is the four-cycle kernel, of norm $O(1/p)$ by Corollary 15.9.

Corollary 15.10 (Raw pair maps)**.** *Let p be an odd prime and $q = 1/p$. Unless said otherwise every label is uniform on $\mathbb{F}_p$, and $L^2(\mathbb{F}_p^k)$ carries the uniform probability. For a pair $y = (y_0, y_1)$ of labels put $\psi_y(s) = \chi(s - y_0)\chi(s - y_1)$.*

(a) *The maps $h \mapsto \mathbb{E}_s \chi(s - a) h(s)$ ($h \in L^2(\mathbb{F}_p)$) and $F \mapsto \mathbb{E}_{s,t} \chi(s - a)\chi(t - a) F(s,t)$ ($F \in L^2(\mathbb{F}_p^2)$), into functions of $a \in \mathbb{F}_p$, have norms r and $\sqrt{q(1-q)} \le r$. The* raw pair map *$h \mapsto \mathbb{E}_s \psi_y(s) h(s)$ has norm $\sqrt{q(1-q)}$ into functions of $y \in \mathbb{F}_p^2$, and norm at most r into functions of y uniform on the ordered pairs of distinct labels. A tensor product of such maps on independent labels has the product of their norms as its norm.*

(b) *Let y be uniform on the ordered pairs of distinct labels, let $S\colon L^2(\mathbb{F}_p^2) \to L^2(y)$ be the map $(SF)(y) = \mathbb{E}_{s,u} \psi_y(s)\psi_y(u) F(s,u)$, and let $d = \sqrt{p}\, \mathbf{1}_{s=u}$, a unit vector of $L^2(\mathbb{F}_p^2)$. Then*

$$S = r(1-2q)\, \mathbf{1}\, \langle d, \cdot \rangle + R, \qquad Rd = 0, \qquad \|R\|^2 \le c_0 q^2, \qquad c_0 = \frac{4}{1-q} + 2q. \tag{15.5}$$

In detail, the space $L^2(\mathbb{F}_p^2)$ is the orthogonal sum of $\mathbb{C}d$, of the functions Df, where $(Df)(s,u) = \sqrt{p}\, \mathbf{1}_{s=u} f(s)$ and $f \in L_0^2(\mathbb{F}_p)$, and of the functions that vanish on the diagonal $\{s = u\}$. On the second space $(RDf)(y) = -qr\big(f(y_0) + f(y_1)\big)$ and $\|RDf\| \le \sqrt{2}\, qr\, \|f\|$; on the third, R agrees with S and has norm at most $2q/\sqrt{1-q}$. For every y the row $(s,u) \mapsto \psi_y(s)\psi_y(u) - r(1-2q)\, d(s,u)$ of R has norm at most one in $L^2(\mathbb{F}_p^2)$.

Proof. By (4.2), $\mathbb{E}_a \chi(s - a)\chi(s' - a) = \mathbf{1}_{s=s'} - q$ for all s, s'.

(a) The first map is the Paley operator P of Lemma 4.11, of norm r by part (c) there. Write R_2 for the second map and R_3 for the raw pair map into functions of $y \in \mathbb{F}_p^2$. Then

$$(R_2R_2^*g)(a) = \mathbb{E}_{a'}\big(\mathbb{E}_s\chi(s-a)\chi(s-a')\big)^2 g(a'), \qquad (R_3^*R_3h)(s) = \mathbb{E}_{s'}\big(\mathbb{E}_{y_0}\chi(s-y_0)\chi(s'-y_0)\big)^2 h(s'),$$

and both kernels are $(\mathbf{1}_{a=a'}-q)^2 = (1-2q)\mathbf{1}_{a=a'}+q^2$ (with s, s' in place of a, a' in the second). The operator with this kernel is $q(1-2q)$ times the identity on $L_0^2(\mathbb{F}_p)$ and $q(1-q)$ on the constants, so $\|R_2\| = \|R_3\| = \sqrt{q(1-q)}$. For y uniform on distinct pairs: the uniform probability on $\mathbb{F}_p^2$ is $1-q$ times the law of y plus q times the uniform law on the diagonal, and on the diagonal $\mathbb{E}_{y_0}\psi_{(y_0,y_0)}(s)\psi_{(y_0,y_0)}(s') = \mathbb{E}_{y_0}\mathbf{1}_{y_0\notin\{s,s'\}} = 1-2q+q\mathbf{1}_{s=s'}$; hence

$$\mathbb{E}_y\psi_y(s)\psi_y(s') = \frac{(1-2q)\mathbf{1}_{s=s'}+q^2-q(1-2q+q\mathbf{1}_{s=s'})}{1-q} = \frac{1-2q-q^2}{1-q}\mathbf{1}_{s=s'} - \frac{q(1-3q)}{1-q},$$

and the Gram operator of the raw pair map is $q(1-2q-q^2)/(1-q) \le q$ on $L_0^2(\mathbb{F}_p)$ and q^2 on the constants. Finally, the norm of a tensor product of operators is the product of their norms.

(b) The map $D\colon L^2(\mathbb{F}_p) \to L^2(\mathbb{F}_p^2)$ is an isometry onto the functions supported on the diagonal, and $D\mathbf{1} = d$; this gives the orthogonal decomposition. Since $y_0 \neq y_1$, $\psi_y(s)^2 = \mathbf{1}_{s\notin\{y_0,y_1\}}$, so

$$(SDf)(y) = \sqrt{p}\,\mathbb{E}_{s,u}\mathbf{1}_{s=u}\psi_y(s)\psi_y(u)f(s) = r\,\mathbb{E}_s\mathbf{1}_{s\notin\{y_0,y_1\}}f(s) = r\,\mathbb{E}f - qr\big(f(y_0)+f(y_1)\big).$$

For $f=\mathbf{1}$ this is $Sd = r(1-2q)\mathbf{1}$, and $\langle d,d\rangle = 1$ gives $Rd = 0$. For $f \in L_0^2(\mathbb{F}_p)$, $\langle d, Df\rangle = \mathbb{E}f = 0$, so $RDf = SDf = -qr(f(y_0)+f(y_1))$; as y_0 and y_1 are uniform and $\mathbb{E}f(y_0)\overline{f(y_1)} = -\|f\|^2/(p-1)$, $\|RDf\|^2 = 2q^3(1-\frac{1}{p-1})\|f\|^2 \le 2q^3\|f\|^2$.

A function F that vanishes on the diagonal has $\langle d, F\rangle = 0$, so $RF = SF$. The kernel $\psi_y(s)\psi_y(u) = \prod_{i\in\{0,1\}}\prod_{v\in\{s,u\}}\chi(v-y_i)$ is the kernel K of Corollary 15.9 with $a=b=2$, the pair y in the role of x and all signs $+1$. For $G \in L^2(y)$ let $\tilde G$ be its extension by zero to $\mathbb{F}_p^2$, so that $\big\|\tilde G\big\|_{L^2(\mathbb{F}_p^2)} = (1-q)^{1/2}\|G\|$. By Corollary 15.9(a), $\tilde G$ and F vanishing on the diagonals,

$$|\,\langle G, SF\rangle_{L^2(y)}\,| = \frac{1}{1-q}\Big|\mathbb{E}_{y\in\mathbb{F}_p^2}\overline{\tilde G(y)}\,(SF)(y)\Big| \le \frac{1}{1-q}\cdot\frac{2}{p}\,(1-q)^{1/2}\,\|G\|\,\|F\| = \frac{2q}{\sqrt{1-q}}\,\|G\|\,\|F\|\,.$$

For a general $F = \lambda d + Df + F'$, with $f \in L_0^2(\mathbb{F}_p)$ and F' vanishing on the diagonal, $RF = RDf + RF'$, and Cauchy–Schwarz gives $\|RF\|^2 \le (2q^3 + \frac{4q^2}{1-q})(\,\|f\|^2 + \|F'\|^2\,) \le c_0q^2\,\|F\|^2$.

Finally, the row of S at y is $k_y(s,u) = \psi_y(s)\psi_y(u)$, of modulus at most one, so $\|k_y\| \le 1$; and $\langle d, k_y\rangle = (Sd)(y) = r(1-2q)$, so the row of R at y, $k_y - r(1-2q)d = k_y - \langle d,k_y\rangle\,d$, is the orthogonal projection of k_y onto $d^\perp$. □

For two pairs y, y' of distinct labels the normalized quartic trace $H = r^{-1}\mathbb{E}_s\psi_y(s)\psi_{y'}(s)$ (the scores of the positive roots, Definition 21.1, are of this form) satisfies $H^2 = q^{-1}\langle k_y, k_{y'}\rangle$, where k_y is the row of S at y, and so, by (b),

$$H^2 = (1-2q)^2 + q^{-1}\langle \rho_y, \rho_{y'}\rangle, \qquad \rho_y = k_y - r(1-2q)\,d.$$

The square of a score thus differs from the constant $(1-2q)^2$ by a Gram kernel of the rows of R, divided by q. With the notation of Lemma 13.9 (in particular that this Gram kernel is positive semidefinite), the bound $\|R\|^2 \le c_0q^2$ turns this into an error of order q in the square estimates of the positive roots (Lemma 21.4).

Remark 15.11 (The constant 2). The formulas of the proof of Theorem 15.8 reproduce the full spectrum of M for every odd prime $p \le 73$ (floating-point computations of two internal referees, the second up to $p \le 89$; evaluating the formulas for every odd prime $p < 2000$ gives $\max|\lambda| < 2p$ throughout), and $\max|\lambda|/p$ reaches $2 - 4.4\cdot10^{-12}$ at $p = 71$. These floating-point computations suggest that the constant 2 is optimal; they do not prove optimality (and are not used). The exact

finite consistency tests of Theorem 15.8 are described in Remark D.13; they are not used in any proof.

15.2. **The complete three-by-three character kernel.** The four-cycle estimate has a counterpart for three labels on each side. Here a polynomial Gram identity replaces the character table. The bound applies to the complete bipartite graph $K_{3,3}$; deleting one of its edges does not preserve the argument.

Theorem 15.12 (Squarefree cubic resultant)**.** *Let p be an odd prime and let $\mathcal{S}_d$ be the monic squarefree polynomials of degree d over $\mathbb{F}_p$. On counting-measure spaces, set*

$$B_d(f,g) = \chi(\mathrm{Res}(g,f)), \qquad f \in \mathcal{S}_d, \quad g \in \mathcal{S}_3.$$

Then $B_3^* B_3 \preceq p^3 I$. Consequently the kernel

$$\prod_{i,j=1}^{3} \chi(y_j - x_i)$$

on uniform $\mathbb{F}_p^3$, compressed on both sides to internally distinct triples, has norm at most $6p^{-3/2}$.

Proof. We first record the polynomial character identity needed for the Gram. For a monic squarefree polynomial D of degree $2m$, put $\chi_D(f) = \chi(\mathrm{Res}(D,f))$ and $S_j(D) = \sum_{f \text{ monic}, \deg f = j} \chi_D(f)$. On the finite étale algebra $\mathbb{F}_p[X]/D$, use the additive character $\psi_D(z) = e(\mathrm{res}(z/D)/p)$, where the residue is the coefficient of X^{-1} at infinity. In the monomial basis $X^{2m-1}, \dots, X, 1$, ordered by decreasing degree, the matrix of $(u,v) \mapsto \mathrm{res}(uv/D)$ has antidiagonal entries 1 and zeros below that antidiagonal, so its determinant is $(-1)^{m(2m-1)}$. Factoring the algebra into fields and evaluating the quadratic Gauss sums gives

$$\sum_z \chi_D(z)\psi_D(bz) = p^m \chi_D(b).$$

This holds also for nonunits b: a field factor then has a zero multiplicative-character sum. The annihilator of the polynomials of degree less than k consists of those of degree less than $2m-k$. Summing the displayed Fourier identity over monic degree-k polynomials, and then over the nonzero leading coefficient, yields

$$S_k(D) = p^{k-m}\left((p-1)\sum_{j=0}^{2m-k-2} S_j(D) - S_{2m-k-1}(D)\right), \qquad k < 2m.$$

For $j \geq 2m$, $S_j(D) = 0$ by periodicity modulo D: the sum over a complete residue system modulo D vanishes. In particular,

$\deg D$	identities
6	$S_3 = (p-1)(1+S_1) - S_2$,
4	$S_2 = p - 1 - S_1, \quad S_3 = -p$,
2	$S_1 = -1, \quad S_2 = S_3 = 0$.

Write $G_d = B_d^* B_d$, let r_g be the number of rational roots of g, put $R = \mathrm{diag}(r_g)$ and $L = G_1 + J$, and introduce the features

$$C(a,g) = \mathbf{1}[g(a) = 0], \qquad V(a,t;g) = \mathbf{1}[g(a) = 0]\chi((g/(X-a))(t)).$$

Here $a, t \in \mathbb{F}_p$ and $g \in \mathcal{S}_3$; in V, the quotient is understood only when a is a root of g, and $V(a,t;g) = 0$ otherwise. Set $T = C^*C + V^*V$ and $E = \sum_a V(a,a)^* V(a,a)$. The exact identity is

$$G_3 = p^3 I + RL + LR - L - G_2 - (p+1)T + E. \tag{15.6}$$

Here is an entrywise verification, including nonsplit cubics. Fix g, h, put $U = \gcd(g, h)$, $D = gh/U^2$, and let u be the number of $a \in \mathbb{F}_p$ such that $g(a)h(a) = 0$. With $M_j = \sum_{\deg f=j,\ f \text{ monic}} \chi_g(f)\chi_h(f)$, the nonsquarefree quadratics and cubics are, respectively, $(X-a)^2$ and $(X-a)^2(X-b)$. Thus

$$(G_1)_{g,h} = M_1, \quad (G_2)_{g,h} = M_2 - (p-u), \quad (G_3)_{g,h} = M_3 - (p-u)M_1.$$

For $g \neq h$, the generating series of the M_j is that of the $S_j(D)$ multiplied by $\prod_{P|U}(1-\chi_D(P)z^{\deg P})$. If $U = 1$, the degree-six identity gives zero correction in (15.6). If $U = X - a$, put $\eta = \chi_D(X-a)$ and $\ell = 1 + S_1(D)$. Then

$$M_1 = \ell - 1 - \eta, \quad M_2 = p - (1+\eta)\ell + \eta, \quad M_3 = -p(1+\eta) + \eta\ell,$$

and $T_{g,h} = \ell$, $E_{g,h} = \eta$, giving the correction $-(p+1)\ell + \eta$. If $U = (X-a)(X-b)$, write $e_1 = \chi_D(X-a)+\chi_D(X-b)$, $e_2 = \chi_D(X-a)\chi_D(X-b)$. Then $(M_1, M_2, M_3) = (-1-e_1, e_1+e_2, -e_2)$ and $(T_{g,h}, E_{g,h}) = (-e_1, e_1)$, giving $(p+2)e_1$. If U is an irreducible quadratic, the same calculation with $\eta = \chi_D(U)$ gives $(M_1, M_2, M_3) = (-1, -\eta, \eta)$ and $T_{g,h} = E_{g,h} = 0$. On the diagonal, listing the degrees of the irreducible factors of g, the three factorization types give

type	r_g	M_1	M_2	M_3	$T_{g,g}$
111	3	$p-3$	p^2-3p+3	p^3-3p^2+3p-1	$3(p-1)$
12	1	$p-1$	p^2-p-1	p^3-p^2-p+1	$p+1$
3	0	p	p^2	p^3-1	0

and $E_{g,g} = r_g$. Substitution proves (15.6) in every case.

Finally $0 \preceq E \preceq T$. The synthesis identities

$$\sum_a C(a,g) = r_g, \qquad \sum_a \chi(t-a)V(a,t;g) = r_g\chi(g(t))$$

and Cauchy–Schwarz imply $C^*C \succeq p^{-1}RJR$ and $V^*V \succeq (p-1)^{-1}RG_1R$, hence $pT \succeq RLR$. Since $RLR - RL - LR + L = (R-I)L(R-I) \succeq 0$, (15.6) gives $G_3 \preceq p^3 I$. Restriction to split cubics is a compression. Each split cubic has six ordered root triples; lifting both variables therefore multiplies the counting norm by 6. Dividing by p^3 for uniform-cube measure proves the last assertion. □

As with the four-cycle bound, separate input and output restrictions are contractions. A jointly imposed missing-edge mask is a different operator and needs its own proof. The complete nine-edge case is used in the finite upper comparisons of Proposition 28.7. Other three-by-three subgraphs there retain matching, path or four-cycle witnesses.

15.3. **A two-variable character sum.** The second estimate concerns two labels whose characters are coupled through one factor $y-z$. For fixed z the inner sum over y is, up to an explicit correction, minus the trace of Frobenius at z on a sheaf of rank $2\lfloor \deg f/2 \rfloor$ attached to the family of curves $y^2 = f(x)(x-z)$ (Katz–Sarnak). Twisting by the Kummer sheaf of g keeps this sheaf geometrically irreducible of rank at least two. So the H^2_c coinvariants vanish (Lemma 15.15 below), i.e. no term of order $p^{3/2}$ occurs; the compactly supported cohomology lives in degree one, its dimension is the tame Euler characteristic, and Deligne's theorem bounds its Frobenius eigenvalues by p. This is square-root cancellation in both variables.

For an integer $m \geq 1$ put $\gamma(m) = m$ if m is even and $\gamma(m) = m-1$ if m is odd ($\gamma(m) = 2\lfloor m/2 \rfloor$; for z distinct from the roots of f, this is twice the genus of the smooth projective model of $y^2 = f(x)(x-z)$ with $\deg f = m$), and for integers $m \geq 2$, $n \geq 1$, $s \geq 0$

$$\mathcal{B}(m,n,s) = \gamma(m)(m+n-s-1) + [m \text{ odd}]\big((n-1)r + mr^2\big) + m\big((m-2)r + r^2\big). \tag{15.7}$$

Theorem 15.13 (Two-variable character sum)**.** *Let p be an odd prime, $r = p^{-1/2}$, $m \ge 2$ and $n \ge 1$ integers, $a_1, \dots, a_m \in \mathbb{F}_p$ distinct, $b_1, \dots, b_n \in \mathbb{F}_p$ distinct, $c, c' \in \mathbb{F}_p^\times$,*

$$f(X) = c\prod_{i=1}^{m}(X - a_i), \qquad g(X) = c'\prod_{k=1}^{n}(X - b_k), \qquad s = \bigl|\{a_1, \dots, a_m\} \cap \{b_1, \dots, b_n\}\bigr|.$$

Then $\Sigma = \sum_{y,z\in\mathbb{F}_p} \chi\bigl(f(y)\, g(z)\, (y - z)\bigr)$ satisfies $|\Sigma| \le p\,\mathcal{B}(m, n, s)$, and if $n \ge 2$ also $|\Sigma| \le p\,\mathcal{B}(n, m, s)$.

The main term $\gamma(m)(m + n - s - 1)$ is the dimension of a cohomology group of weight at most 2; all other terms are the displayed multiples of r and r^2 in (15.7), with coefficients depending only on m and n. (Only split polynomials occur in this paper. For even m the same proof works for squarefree f, g that do not split, with Weil's bound for general squarefree polynomials in place of Theorem 4.7; for odd m the construction in Lemma 15.14 uses a root of f in $\mathbb{F}_p$.) The proof rests on the following lemma, in which $U_f = \mathbb{A}^1_{\mathbb{F}_p} \setminus \{a_1, \dots, a_m\}$ and traces and Frobenius eigenvalues are regarded as complex numbers through the isomorphism of $\overline{\mathbb{Q}}_\ell$ with $\mathbb{C}$ fixed in Section A.2. Its parts (v) and (vi) are used only in Section 15.4.

Lemma 15.14 (The sheaf of the family $y^2 = f(x)(x - z)$)**.** *Let $f = \prod_{i=1}^{m}(X - a_i)$ with $m \ge 2$ and distinct $a_i \in \mathbb{F}_p$ (so f is monic), and put*

$$\tau_f(z) = \frac{1}{p}\Bigl(\sum_{y\in\mathbb{F}_p} \chi\bigl(f(y)(y - z)\bigr) + [m \text{ odd}]\Bigr) \qquad (z \in \mathbb{F}_p).$$

If m is odd, fix a root a of f (any root may be chosen). The construction below, applied to this fixed a, gives a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}'$ on U_f such that:

(i) *$\mathcal{F}'$ has rank $\gamma(m)$ and is pure of weight 1;*
(ii) *$\mathcal{F}'$ is geometrically irreducible;*
(iii) *$\mathcal{F}'$ is tame at every point of $\mathbb{P}^1 \setminus U_f = \{a_1, \dots, a_m, \infty\}$;*
(iv) *$p\,\tau_f(z) = -\mathrm{Tr}(\mathrm{Frob}_z \mid \mathcal{F}')$ for every $z \in U_f(\mathbb{F}_p)$;*
(v) *there is a perfect alternating pairing $\mathcal{F}' \otimes \mathcal{F}' \to \overline{\mathbb{Q}}_\ell(-1)$ of lisse sheaves on U_f;*
(vi) *for every root a_i of f if m is even, and for every root $a_i \ne a$ if m is odd, the inertia group at a_i acts on $\mathcal{F}'$ through unipotent automorphisms, not all trivial.*

Proof. m even. Let $\mathcal{F}' = \mathcal{F}_f$ be the sheaf of Theorem A.6 with $k = \mathbb{F}_p$ and $h = f$, a polynomial of degree $m = 2 \cdot \frac{m}{2}$, $\frac{m}{2} \ge 1$, with m distinct roots; then $U_h = U_f$. Parts (i), (ii), (iii) and (v) are Theorem A.6(i), (iii), (ii) and (v), and Theorem A.6(iv) with $E = \mathbb{F}_p$, $t = z$, gives $-\mathrm{Tr}(\mathrm{Frob}_z \mid \mathcal{F}_f) = \sum_x \chi((x - z)f(x)) = p\,\tau_f(z)$. For (vi), $\mathcal{F}_f$ is tame at a_i (Theorem A.6(ii)), so its local monodromy there factors through the pro-cyclic tame quotient ((E3)), and the inertia image is the closure of the group generated by the element u of Theorem A.6(vi). As $(u - 1)^2 = 0$, the powers $u^k = 1 + k(u - 1)$ are unipotent, hence so is every element $1 + c(u - 1)$ of the closure, and $u \ne 1$.

m odd. Then $m \ge 3$. Put $f_1 = f/(X - a)$, so $\deg f_1 = m - 1 \ge 2$ is even and $f_1(a) \ne 0$, and

$$f_1^\sharp(X) = X^{m-1} f_1(a + X^{-1}) = \prod_{a_i \ne a} (1 + (a - a_i)X).$$

$f_1^\sharp$ has degree $m - 1$ (leading coefficient $f_1(a) \ne 0$), the $m - 1$ distinct roots $1/(a_i - a) \in \mathbb{F}_p^\times$ ($a_i \ne a$), and $f_1^\sharp(0) = 1$. Let $\varphi_a : \mathbb{A}^1 \setminus \{a\} \to \mathbb{A}^1 \setminus \{0\}$, $\varphi_a(z) = 1/(z - a)$, an isomorphism of $\mathbb{F}_p$-schemes that extends to the automorphism $z \mapsto 1/(z - a)$ of $\mathbb{P}^1$ (with $a \mapsto \infty$, $\infty \mapsto 0$). It maps U_f onto $W = \mathbb{A}^1 \setminus (\{0\} \cup Z(f_1^\sharp))$, since $\varphi_a(a_i) = 1/(a_i - a)$ for $a_i \ne a$. Let $\mathcal{F}_{f_1^\sharp}$ be the sheaf of Theorem A.6 with $k = \mathbb{F}_p$, $h = f_1^\sharp$ (degree $m - 1 = 2 \cdot \frac{m-1}{2}$, $\frac{m-1}{2} \ge 1$, distinct roots), lisse on

$U_{f_1^\sharp} = \mathbb{A}^1 \setminus Z(f_1^\sharp) \supseteq W$, and let $\mathcal{L} = \mathcal{L}_{\chi(a-z)}$ be the Kummer sheaf (E4) of the polynomial $a - z$, lisse on $\mathbb{A}^1 \setminus \{a\} \supseteq U_f$. Put

$$\mathcal{F}' = \varphi_a^*(\mathcal{F}_{f_1^\sharp}|_W) \otimes \mathcal{L}|_{U_f}.$$

Transport of (i)–(iii) and (v). Restriction to a dense open, pull-back by an isomorphism, and tensoring with a rank-one sheaf preserve geometric irreducibility ((E1)). Thus Theorem A.6(i),(iii) and (E2), (E4) give rank $m - 1 = \gamma(m)$, purity of weight 1 and geometric irreducibility for $\mathcal{F}'$. For tameness, the automorphism φ_a sends the roots other than a to $Z(f_1^\sharp)$, sends a to infinity, and sends infinity to 0. The original sheaf is tame at $Z(f_1^\sharp)$ and infinity and lisse at 0; hence its restriction to W and pull-back are tame at every missing point. The Kummer sheaf is tame there too, so their tensor product is tame ((E3)). Finally, $\mathcal{L} \otimes \mathcal{L}$ is the constant sheaf, since its monodromy character has order dividing two ((E4)). Pulling back the pairing of Theorem A.6(v) and using this trivialization gives the perfect alternating pairing for $\mathcal{F}'$, with the same target $\overline{\mathbb{Q}}_\ell(-1)$. This proves (i)–(iii) and (v).

(iv) Let $z \in U_f(\mathbb{F}_p)$ and $w = \varphi_a(z) = 1/(z - a)$, so $-w = 1/(a - z)$. By (E1), (E4) and Theorem A.6(iv) with $E = \mathbb{F}_p$,

$$-\mathrm{Tr}(\mathrm{Frob}_z \mid \mathcal{F}') = -\chi(a - z)\,\mathrm{Tr}(\mathrm{Frob}_w \mid \mathcal{F}_{f_1^\sharp}) = \sum_{x' \in \mathbb{F}_p} \chi\big((a - z)(x' - w) f_1^\sharp(x')\big).$$

On the other hand, the term $y = a$ of $\sum_y \chi(f(y)(y-z))$ vanishes ($f(a) = 0$), and $y \mapsto x' = 1/(y-a)$ is a bijection of $\mathbb{F}_p \setminus \{a\}$ onto $\mathbb{F}_p^\times$ with $y = a + 1/x'$. For such y, $f(y) = x'^{-1} f_1(a + 1/x') = x'^{-m} f_1^\sharp(x')$ and $y - z = x'^{-1}((a-z)x' + 1) = x'^{-1}(a-z)(x'-w)$, so $f(y)(y-z) = x'^{-(m+1)}(a-z)(x'-w) f_1^\sharp(x')$ with $m + 1$ even. Hence

$$\begin{aligned}\sum_y \chi(f(y)(y-z)) &= \sum_{x' \neq 0} \chi\big((a-z)(x'-w) f_1^\sharp(x')\big) \\ &= -\mathrm{Tr}(\mathrm{Frob}_z \mid \mathcal{F}') - \chi\big((a-z)(-w) f_1^\sharp(0)\big) = -\mathrm{Tr}(\mathrm{Frob}_z \mid \mathcal{F}') - 1,\end{aligned}$$

since $(a - z)(-w) = 1$ and $f_1^\sharp(0) = 1$; that is, $p\,\tau_f(z) = -\mathrm{Tr}(\mathrm{Frob}_z \mid \mathcal{F}')$. (vi) Let $a_i \neq a$ be a root of f. The automorphism of $\mathbb{P}^1$ extending φ_a maps a_i to the root $1/(a_i - a)$ of $f_1^\sharp$ and, by (E3), the inertia groups at a_i onto those at $1/(a_i - a)$; so the inertia at a_i acts on $\varphi_a^*(\mathcal{F}_{f_1^\sharp}|_W)$ as the inertia at $1/(a_i - a)$ acts on $\mathcal{F}_{f_1^\sharp}|_W$, that is, as it acts on $\mathcal{F}_{f_1^\sharp}$ (restriction to the dense open W keeps the inertia groups and the local monodromy at this point, (E3)). This is as stated in (vi), by the argument of the even case applied to $\mathcal{F}_{f_1^\sharp}$ (Theorem A.6(ii), (vi)). The sheaf $\mathcal{L}$ is lisse at a_i, so the inertia at a_i acts trivially on it ((E3)), and the tensor product has the same local monodromy at a_i as $\varphi_a^* \mathcal{F}_{f_1^\sharp}$. □

For odd m we have $m \geq 3$, so the root a can be chosen different from any given root of f. Thus every root of f is, for some allowed choice of the root a in the construction of $\mathcal{F}'$, a point at which the inertia acts through nontrivial unipotent automorphisms; this is how part (vi) is used in Lemma 15.19.

The cohomological step used below is the same for one trace and for a product of two traces. Once the geometric coinvariants vanish, only the rank, the weight and the number of punctures enter the bound.

Lemma 15.15 (Tame trace bound)**.** *Let $B \subseteq \mathbb{F}_p$ have $m \geq 1$ elements, let $U = \mathbb{A}^1_{\mathbb{F}_p} \setminus B$, and let $\mathcal{G}$ be a lisse sheaf on U of rank R, mixed of weights at most w, and tame at every point of*

$B \cup \{\infty\}$. *Suppose its generic stalk V has zero coinvariants under $\pi_1^{\mathrm{geom}}(U)$. Then*

$$\dim H_c^1(U \otimes \bar{\mathbb{F}}_p, \mathcal{G}) = R(m-1), \qquad \left| \sum_{x \in U(\mathbb{F}_p)} \mathrm{Tr}(\mathrm{Frob}_x \mid \mathcal{G}) \right| \le R(m-1)p^{(w+1)/2}.$$

In particular the coinvariant hypothesis holds if $\mathcal{G}$ is geometrically irreducible of rank at least two.

Proof. By Theorem A.3, compactly supported cohomology vanishes outside degrees one and two, and $H_c^2 \cong V_{\pi_1^{\mathrm{geom}}(U)}(-1) = 0$. Tameness at all $m+1$ missing points makes every Swan conductor zero, so Theorem A.5 gives $-\dim H_c^1 = R(2-(m+1))$. By Theorem A.2, the trace sum is $-\mathrm{Tr}(\mathrm{Frob}_p \mid H_c^1)$, and every eigenvalue in this trace has modulus at most $p^{(w+1)/2}$ by Theorem A.4. Finally, the span of the vectors $gv - v$ is invariant. For an irreducible representation it is zero or all of V; the first possibility makes the action trivial, which is incompatible with irreducibility in rank at least two. Thus the coinvariants vanish in the stated special case. □

Proof of Theorem 15.13. Step 0 (normalization). Replacing f, g by f/c, g/c' multiplies Σ by $\chi(cc') = \pm 1$, and $\mathcal{B}$ does not depend on c, c'; so we may assume f and g monic. Write $Z(f) = \{a_i\}$, $Z(g) = \{b_k\}$, so $|Z(f) \cup Z(g)| = m + n - s$.

Step 1 (the three parts). $\Sigma = \sum_{z \in \mathbb{F}_p} \chi(g(z)) \sum_y \chi\big(f(y)(y-z)\big)$. For $z = a_j \in Z(f)$, $f(y)(y-a_j) = (y-a_j)^2 f_j(y)$ with $f_j = \prod_{i \ne j}(X - a_i)$, $\deg f_j = m-1 \ge 1$, so $\chi(f(y)(y-a_j)) = \chi(f_j(y))$ for $y \ne a_j$ and 0 for $y = a_j$; by Theorem 4.7, $|\sum_y \chi(f(y)(y-a_j))| \le |\sum_y \chi(f_j(y))| + 1 \le (m-2)\sqrt{p} + 1$. For $z \notin Z(f)$, Lemma 15.14(iv) gives $\sum_y \chi(f(y)(y-z)) = p\,\tau_f(z) - [m \text{ odd}] = -\mathrm{Tr}(\mathrm{Frob}_z \mid \mathcal{F}') - [m \text{ odd}]$. Hence

$$\Sigma = -\sum_{z \in U_f(\mathbb{F}_p)} \chi(g(z)) \,\mathrm{Tr}(\mathrm{Frob}_z \mid \mathcal{F}') \;-\; [m \text{ odd}] \sum_{z \in U_f(\mathbb{F}_p)} \chi(g(z)) + \sum_{z \in Z(f)} \chi(g(z)) \sum_y \chi\big(f(y)(y-z)\big).$$

The last part is at most $m\big((m-2)\sqrt{p}+1\big)$ in modulus. The middle sum differs from $\sum_{z \in \mathbb{F}_p} \chi(g(z))$ by at most m terms of modulus at most 1, and $|\sum_z \chi(g(z))| \le (n-1)\sqrt{p}$ by Theorem 4.7; so the middle part is at most $(n-1)\sqrt{p} + m$ (and occurs only for odd m).

Step 2 (the sheaf $\mathcal{G}$). Let $U' = U_f \setminus Z(g) = \mathbb{A}^1 \setminus (Z(f) \cup Z(g))$, $\mathcal{L}_{\chi(g)}$ the Kummer sheaf of g of (E4), lisse on $\mathbb{A}^1 \setminus Z(g) \supseteq U'$, and $\mathcal{G} = \mathcal{L}_{\chi(g)}|_{U'} \otimes \mathcal{F}'|_{U'}$. As $\chi(g(z)) = 0$ for $z \in Z(g)$, the first part is $-\sum_{z \in U'(\mathbb{F}_p)} \mathrm{Tr}(\mathrm{Frob}_z \mid \mathcal{G})$ by (E1) and (E4). $\mathcal{G}$ is lisse of rank $\gamma(m) \ge 2$ and pure of weight 1 (Lemma 15.14(i), (E2)); it is geometrically irreducible (Lemma 15.14(ii): the restriction to the dense open U' and the tensor product with a rank-one sheaf preserve this, (E1)); and it is tame at every point of $\mathbb{P}^1 \setminus U' = Z(f) \cup Z(g) \cup \{\infty\}$ by (E3) (restriction to a dense open keeps tameness, a sheaf is tame at a point where it is lisse, and tensor products of tame sheaves are tame): $\mathcal{F}'$ is tame at $Z(f) \cup \{\infty\}$ (Lemma 15.14(iii)) and lisse at the points of $Z(g) \setminus Z(f) \subseteq U_f$, and $\mathcal{L}_{\chi(g)}$ is tame at the points of $Z(g) \cup \{\infty\}$ ((E4)) and lisse at those of $Z(f) \setminus Z(g)$.

Step 3 (cohomology). The sheaf of Step 2 satisfies Lemma 15.15: its rank is $\gamma(m) \ge 2$, its weight is 1, it is geometrically irreducible, and it is tame at all $m + n - s + 1$ missing points, including infinity. Therefore

$$\left| \sum_{z \in U'(\mathbb{F}_p)} \mathrm{Tr}(\mathrm{Frob}_z \mid \mathcal{G}) \right| \le \gamma(m)(m+n-s-1)p.$$

The trace sum is $-p\sum_{z \in U'(\mathbb{F}_p)} \chi(g(z))\tau_f(z)$, a rational integer by Lemma 15.14(iv), so the bound is independent of the identification of $\overline{\mathbb{Q}}_\ell$ with $\mathbb{C}$. It bounds the first part of Step 1.

Step 4 (conclusion). Adding the three parts,

$$|\Sigma| \le \gamma(m)(m+n-s-1)p + [m \text{ odd}]\big((n-1)\sqrt{p}+m\big) + m\big((m-2)\sqrt{p}+1\big) = p\,\mathcal{B}(m,n,s).$$

Finally, $\Sigma = \chi(-1)\sum_{z,y}\chi(g(z)f(y)(z-y))$ is, up to sign, the sum of the theorem for the pair (g,f); if $n \ge 2$ the bound just proved applies to it and gives $|\Sigma| \le p\,\mathcal{B}(n,m,s)$. □

Remark 15.16 (Sharpness and the coefficients of $\mathcal{B}$). (i) The main term is nearly attained (numerically): for f proportional to g of degree two ($m=n=s=2$) the group H^1_c has dimension 2, and exact computations for primes up to $9\cdot 10^4$ give $|\Sigma|/(p\mathcal{B})$ up to 0.998 (numerical, not used). The consistency test of Theorem 15.13 against exact double sums is described in Remark D.13; it is not an input of any proof. (ii) As a polynomial in r, $\mathcal{B}(m,n,0) = C_0 + C_1 r + C_2 r^2$ with $(C_0,C_1,C_2) = (\gamma(m)(m+n-1),\ m(m-2)+[m \text{ odd}](n-1),\ m(1+[m \text{ odd}]))$; these are the coefficient triples with which the completion variance (Lemma 14.15) uses the theorem. (iii) $\mathcal{B}(m,n,s)$ is nonincreasing in s and nondecreasing in n. Since $\mathcal{B}(2,4,0) = 10+2r^2$, $\mathcal{B}(3,4,0) = 12+6r+6r^2$ and $\mathcal{B}(4,4,0) = 28+8r+4r^2$, and $r \le 1/\sqrt{3}$, we get $\mathcal{B}(m,n,s) \le \mathcal{B}(4,4,0) < 40$ for $2 \le m \le 4$, $1 \le n \le 4$, $s \ge 0$ and every odd prime; the product rows use this.

15.4. **Trace functions with one free label.** At a root cut of the upper range the star part of the cut kernel is an average over the label z of an integrated centre of a product of characters $\chi(z-x_j)$ over the leaves j of the star. Fixing all half labels except the label x_i of one leaf turns such a star into an affine function of a single *moving-root trace* $\tau_S(x_i)$, whose fixed roots S are the labels of the other leaves (Lemma 15.18). The second moments of combinations of these traces against the conditional law of x_i (a uniform law on an allowed set, whose density is a product of characters with the neighbouring labels as roots) are governed by the almost-orthogonality of traces with distinct root sets against character weights (Lemma 15.19). This is a statement about sheaves: distinct root sets give non-isomorphic sheaves, even after a quadratic twist, because their local monodromy at a root in the symmetric difference differs. First moments satisfy (15.11), with its factor r^2 in place of the pointwise factor r in $|\tau_S| \le sr$ (Lemma 15.17(b)), and they make pivot groups almost orthogonal without any independence assumption (Proposition 15.26). For a graph law the same conditioning, with the conditional means bounded through Theorem 15.13 and the diagonal terms through densities, bounds star marginals with arbitrary real coefficients (Lemma 15.28). The subsection ends with the exact mean of the complete star (Lemma 15.29).

15.4.1. *Trace functions and stars.* For a set $S \subseteq \mathbb{F}_p$ of $s = |S| \ge 2$ distinct elements put $f_S = \prod_{c\in S}(X-c)$ and

$$\tau_S(x) = \frac{1}{p}\Big(\sum_{z\in\mathbb{F}_p}\chi\Big((z-x)\prod_{c\in S}(z-c)\Big) + \mathbf{1}_{|S| \text{ odd}}\Big) \qquad (x \in \mathbb{F}_p). \tag{15.8}$$

Thus $\tau_S = \tau_{f_S}$ in the notation of Lemma 15.14. We write $\mathcal{F}_S$ for a sheaf on $\mathbb{A}^1 \setminus S$ given by Lemma 15.14 for $f = f_S$ (for odd s with some choice of the root a), and $h(s) = 2\lfloor s/2\rfloor = \gamma(s)$ for its rank. The correction $\mathbf{1}_{|S| \text{ odd}}$ accounts for the points at infinity: for $x \notin S$ and odd s the polynomial $(z-x)f_S(z)$ is squarefree of even degree with leading coefficient one, so the smooth projective curve $y^2 = (z-x)f_S(z)$ has two rational points at infinity, which the affine sum misses. With the correction, $p\,\tau_S(x)$ is, for every s and $x \notin S$, minus the trace of Frobenius on the first cohomology of that curve.

Lemma 15.17 (Moving-root traces)**.** *Let $S \subseteq \mathbb{F}_p$ with $|S| = s \ge 2$.*

(a) *τ_S is real, and $\tau_S(x) = -p^{-1}\operatorname{Tr}(\mathrm{Frob}_x \mid \mathcal{F}_S)$ for $x \notin S$.*
(b) *$|\tau_S(x)| \le h(s)r$ for $x \notin S$, and $|\tau_S(x)| \le \big((s-2)\sqrt{p}+2\big)/p \le sr$ for $x \in S$. In particular $|\tau_S| \le sr$ on $\mathbb{F}_p$.*

Proof. (a) is Lemma 15.14(iv). (b) For $x \notin S$, $\mathcal{F}_S$ has rank $h(s)$ and is pure of weight 1 (Lemma 15.14(i)), so each Frobenius eigenvalue at x has modulus $\sqrt{p}$, and $|p\,\tau_S(x)| \le h(s)\sqrt{p}$. For $x \in S$, $(z-x)f_S(z) = (z-x)^2 f_{S\setminus\{x\}}(z)$; the term $z = x$ vanishes, and for $z \ne x$ the summand is $\chi(f_{S\setminus\{x\}}(z))$. Hence $p\,\tau_S(x) = \sum_z \chi(f_{S\setminus\{x\}}(z)) - \chi(f_{S\setminus\{x\}}(x)) + \mathbf{1}_{s \text{ odd}}$. The first sum has modulus at most $(s-2)\sqrt{p}$ by Theorem 4.7 (a product of $s-1 \ge 1$ distinct linear factors), and the two other terms have modulus at most one; finally $2 \le 2\sqrt{p}$. □

Lemma 15.18 (Star-trace dictionary)**.** *Let $x : U \to \mathbb{F}_p$ be injective, $T \subseteq U$ nonempty, and $w_T(x) = \mathbb{E}_z \prod_{j\in T} \chi(z - x_j)$. Then $w_T(x) = 0$ if $|T| = 1$, $w_T(x) = -1/p$ if $|T| = 2$, and for $|T| \ge 3$ and every $i \in T$,*

$$w_T(x) = \tau_{x(T\setminus\{i\})}(x_i) - \frac{\mathbf{1}_{|T| \text{ even}}}{p}, \qquad x(T\setminus\{i\}) = \{x_j : j \in T\setminus\{i\}\}.$$

For signs $\varsigma_j \in \{\pm 1\}$, $\mathbb{E}_z \prod_{j\in T} \chi(\varsigma_j(z - x_j)) = \big(\prod_{j\in T} \chi(\varsigma_j)\big) w_T(x)$.

Proof. For $|T| = 1$, $\sum_z \chi(z - x_j) = 0$. For $|T| = 2$, $\sum_z \chi((z-x_j)(z-x_{j'})) = -1$ by (4.2), as $x_j \ne x_{j'}$. For $|T| \ge 3$ and $i \in T$, $\prod_{j\in T}(z - x_j) = (z - x_i)\prod_{c\in S}(z - c)$ with $S = x(T\setminus\{i\})$, a set of $|T| - 1 \ge 2$ distinct elements, and $|S|$ is odd iff $|T|$ is even; so $w_T(x) = \tau_S(x_i) - \mathbf{1}_{|T| \text{ even}}/p$ by (15.8). The signs factor out since χ is multiplicative. □

The star function. Consider spectral data (Definition 13.1) with half vertex set U^+, half graph G^+, cross signs ς_{jk} and $N(k) = \{j \in U^+ : \varsigma_{jk} \ne 0\}$. The column stars of the cut kernel centred at k have kernels $\prod_{j\in T} \chi(\varsigma_{jk}(x'_k - x_j))$, $\emptyset \ne T \subseteq N(k)$; averaging the centre label z uniformly on $\mathbb{F}_p$ gives the *integrated-centre star function*

(15.9)
$$h_\star(x) = \sum_{k\in U^+} \Big(\mathbb{E}_z \prod_{j\in N(k)} \big(1+\chi(\varsigma_{jk}(z-x_j))\big) - 1\Big) = \sum_{\emptyset\ne T\subseteq U^+} a_T w_T(x), \qquad a_T = \sum_{k:\, T\subseteq N(k)} \prod_{j\in T} \chi(\varsigma_{jk}).$$

We fix this sign convention: the signs $\chi(\varsigma_{jk})$ belong to the centre k, and no factor $\chi(-1)^{|T|}$ appears. (Writing the characters as $\chi(x_j - z)$ multiplies a_T by $\chi(-1)^{|T|}$; the bounds below use $|a_T|$ for odd $|T|$, and for even $|T|$ the factor is one, so they do not depend on the convention.) By Lemma 15.18, on injective labellings $h_\star$ is a constant plus a linear combination of the traces $\tau_{x(T\setminus\{i\})}(x_i)$, $|T| \ge 3$, for any choice of $i \in T$. The row stars map constants to constants (Lemma 14.19(b)), so they disappear under projection off constants.

15.4.2. *Means and mean squares of the traces τ_S.* For a finite set $A \subseteq \mathbb{F}_p$ put $\chi_A(x) = \prod_{a\in A} \chi(x-a)$, the squarefree character factor with root set A ($\chi_\emptyset = 1$).

Lemma 15.19 (Moving-root moments)**.** *Let p be an odd prime, $r = p^{-1/2}$, $h(s) = 2\lfloor s/2\rfloor$, and let $S, T, A \subseteq \mathbb{F}_p$ with $|S| = s \ge 2$, $|T| = t \ge 2$ and $|A| = u \ge 0$; the three sets may overlap arbitrarily. Then*

$$\Big|\mathbb{E}_x \chi_A \tau_S \tau_T - p^{-1}\mathbf{1}_{S=T,\ A=\emptyset}\Big| \le r^3\Big[(s+t+u-1)h(s)h(t) + 1 + r(s+t+u)st\Big], \tag{15.10}$$

$$\big|\mathbb{E}_x \chi_A \tau_S\big| \le r^2\Big[(s+u-1)h(s) + r(s+u)s\Big]. \tag{15.11}$$

Proof. Let $U = \mathbb{A}^1 \setminus (S \cup T \cup A)$ and $\pi = \pi_1^{\mathrm{geom}}(U)$. We use sheaves $\mathcal{F}_S$, $\mathcal{F}_T$ of Lemma 15.14 for f_S, f_T, restricted to U, with $\mathcal{F}_T = \mathcal{F}_S$ if $T = S$; when s or t is odd, the root a of that construction (the *inversion root*) is chosen in Step 3 (Lemma 15.14 and Lemma 15.17 hold for every choice). Let $\mathcal{L}$ be the Kummer sheaf of $g_A = \prod_{a\in A}(X - a)$ restricted to U (the constant sheaf if $A = \emptyset$), with monodromy character λ; so $\mathrm{Tr}(\mathrm{Frob}_x \mid \mathcal{L}) = \chi_A(x)$ for $x \in U(\mathbb{F}_p)$ ((E4)). Put $\mathcal{G} = \mathcal{F}_S \otimes \mathcal{F}_T \otimes \mathcal{L}$, and write V_S, V_T for the generic stalks of $\mathcal{F}_S$, $\mathcal{F}_T$ as representations ρ_S, ρ_T of π.

Step 1 (traces and omitted points). For every $x \in U(\mathbb{F}_p)$, Lemma 15.17(a) and (E1) give

$$\chi_A(x)\tau_S(x)\tau_T(x) = p^{-2}\operatorname{Tr}(\mathrm{Frob}_x \mid \mathcal{G}).$$

For $x \in A$, $\chi_A(x) = 0$. The remaining points $x \in (S \cup T) \setminus A$ number at most $s+t$, and there $|\chi_A\tau_S\tau_T| \le st\,r^2$ by Lemma 15.17(b). Hence

$$\Big|\mathbb{E}_x \chi_A\tau_S\tau_T - p^{-3}\sum_{x\in U(\mathbb{F}_p)} \operatorname{Tr}(\mathrm{Frob}_x \mid \mathcal{G})\Big| \le p^{-1}(s+t)st\,r^2 \le r^4(s+t+u)st. \tag{15.12}$$

Step 2 (the top cohomology). $\mathcal{G}$ is lisse on U of rank $h(s)h(t)$, pure of weight 2 ((E2); $\mathcal{L}$ has weight 0), and tame at every point of $\mathbb{P}^1 \setminus U$ (Lemma 15.14(iii), (E3), (E4)). By Theorems A.2 and A.3,

$$\sum_{x\in U(\mathbb{F}_p)} \operatorname{Tr}(\mathrm{Frob}_x \mid \mathcal{G}) = \operatorname{Tr}(\mathrm{Frob}_p \mid H_c^2) - \operatorname{Tr}(\mathrm{Frob}_p \mid H_c^1), \qquad H_c^i = H_c^i(U \otimes \bar{\mathbb{F}}_p, \mathcal{G}),$$

and $H_c^2 \cong (V_S \otimes V_T \otimes \lambda)_\pi(-1)$. The dual of the space of coinvariants is the space of π-invariant linear forms on $V_S \otimes V_T \otimes \lambda$, that is the π-invariants of $V_S^\vee \otimes V_T^\vee \otimes \lambda^{-1}$. The pairing of Lemma 15.14(v) identifies V_S with $V_S^\vee$ as π-representations (π acts trivially on $\overline{\mathbb{Q}}_\ell(-1)$), and $\lambda^{-1} = \lambda$; so this space is $(V_S \otimes V_T^\vee \otimes \lambda)^\pi = \operatorname{Hom}_\pi(V_T, V_S \otimes \lambda)$. The representations V_T and $V_S \otimes \lambda$ are irreducible (Lemma 15.14(ii); restriction to a dense open and twisting by a rank-one sheaf preserve geometric irreducibility, (E1)). By Schur's lemma, $\dim H_c^2 = 1$ if $V_S \cong V_T \otimes \lambda$ (equivalently $V_T \cong V_S \otimes \lambda$, as $\lambda^2 = 1$), and $H_c^2 = 0$ otherwise.

Step 3 (local monodromy: $H_c^2 = 0$ unless $S = T$ and $A = \emptyset$). For a finite point $c \in \mathbb{P}^1 \setminus U$ let $I_c \subseteq \pi$ be an inertia group at c ((E3)). A sheaf lisse at c has trivial inertia image there, and a rank-one sheaf has scalar inertia image. Restriction to the dense open U keeps the local monodromy at c ((E3)), so Lemma 15.14(vi) for $\mathcal{F}_S$ on $\mathbb{A}^1 \setminus S$ holds for its restriction to U. For every $c \in A$ the character λ is nontrivial on I_c: c is a simple root of g_A, so the double covering $w^2 = g_A(z)$ is ramified at c and the local monodromy of $\mathcal{L}$ at c is the character of order 2 of I_c (the case of odd multiplicity in (E4), and (E3) for the restriction to U).

Suppose $S \ne T$. Choose $c_0 \in S \triangle T$; exchanging S and T if necessary, $c_0 \in S \setminus T$. If s is odd, construct $\mathcal{F}_S$ with an inversion root different from c_0. By Lemma 15.14(vi), $\rho_S(I_{c_0})$ is not contained in the scalars (a unipotent scalar is the identity), while $\mathcal{F}_T$ is lisse at c_0 and $\mathcal{L}$ has rank one, so $\rho_T \otimes \lambda$ maps I_{c_0} into the scalars. Hence $V_S \ncong V_T \otimes \lambda$.

Suppose $S = T$ and $A \ne \emptyset$, so that $\mathcal{F}_T = \mathcal{F}_S$. Choose $c_0 \in A$ and, if $c_0 \in S$ and s is odd, construct $\mathcal{F}_S$ with an inversion root different from c_0. Take $g \in I_{c_0}$ with $\lambda(g) = -1$. If $c_0 \notin S$, then $\rho_S(g) = 1$ ($\mathcal{F}_S$ is lisse at c_0), while $(\rho_S \otimes \lambda)(g) = -1$. If $c_0 \in S$, then $\rho_S(g)$ is unipotent by Lemma 15.14(vi), so all its eigenvalues are 1, while all eigenvalues of $(\rho_S \otimes \lambda)(g) = -\rho_S(g)$ are -1. In both cases $\rho_S(g)$ and $(\rho_S \otimes \lambda)(g)$ are not conjugate, so $V_S \ncong V_S \otimes \lambda$.

Thus $H_c^2 = 0$ unless $S = T$ and $A = \emptyset$. (In that case $\lambda = 1$, $V_T = V_S$ and H_c^2 is one-dimensional; the case is computed directly in Step 4.)

Step 4 (the diagonal case). Let $S = T$ and $A = \emptyset$. Write $p\,\tau_S(x) = Y(x) + \mathbf{1}_{s \text{ odd}}$ with $Y(x) = \sum_z \chi(z-x)\chi(f_S(z))$. Then $\sum_x Y(x) = 0$, since $\sum_x \chi(z-x) = 0$. By the two-root identity (4.2), $\sum_x \chi\big((z-x)(z'-x)\big)$ is $p-1$ if $z = z'$ and -1 otherwise, and $\chi(f_S(z))^2 = \mathbf{1}_{z\notin S}$; so

$$\sum_x Y(x)^2 = p(p-s) - \Big(\sum_z \chi(f_S(z))\Big)^2,$$

$$\mathbb{E}_x \tau_S^2 = \frac{\sum_x Y(x)^2 + p\mathbf{1}_{s \text{ odd}}}{p^3} = \frac{1}{p} - \frac{h(s)}{p^2} - \frac{1}{p^3}\Big(\sum_z \chi(f_S(z))\Big)^2,$$

as $s - \mathbf{1}_{s \text{ odd}} = h(s)$. By Theorem 4.7(a) the last sum has modulus at most $(s-1)\sqrt{p}$, so $|\mathbb{E}_x\tau_S^2 - p^{-1}| \le (h(s) + (s-1)^2)r^4 \le 2s^3r^4$, which is the last term of (15.10) for $T = S$, $u = 0$.

Conceptually, the main term $1/p$ is the one invariant of $\mathcal{F}_S \otimes \mathcal{F}_S$, the pairing of Lemma 15.14(v), on which Frobenius acts by p; the computation does not use this. The sheaves are needed for distinct root sets and nontrivial weights.

Step 5 (dimension and weights). If $S \neq T$ or $A \neq \emptyset$, Steps 2 and 3 show that the geometric coinvariants of $\mathcal{G}$ vanish. Its rank is $h(s)h(t)$, its weight is 2, and it is tame at the $|S \cup T \cup A| + 1$ missing points, including infinity. Thus Lemma 15.15 gives

$$\Big| \sum_{x \in U(\mathbb{F}_p)} \mathrm{Tr}(\mathrm{Frob}_x \mid \mathcal{G}) \Big| \leq h(s)h(t)(|S \cup T \cup A| - 1)p^{3/2} \leq (s+t+u-1)h(s)h(t)p^{3/2}.$$

Dividing by p^3 and adding (15.12) gives (15.10), with the term 1 of the bracket to spare (it allows for the invariant of the diagonal case).

Step 6 (the first moment). Let $U' = \mathbb{A}^1 \setminus (S \cup A)$ and $\mathcal{G}' = \mathcal{F}_S \otimes \mathcal{L}'$ on U', with $\mathcal{L}'$ the Kummer sheaf of g_A on U'. $\mathcal{G}'$ is lisse of rank $h(s) \geq 2$, pure of weight 1, tame, and geometrically irreducible (as in Step 2, by Lemma 15.14(i)–(iii) and (E1)–(E4)), so Lemma 15.15 applies with $|S \cup A|$ finite punctures and gives $|\sum_{x \in U'(\mathbb{F}_p)} \mathrm{Tr}(\mathrm{Frob}_x \mid \mathcal{G}')| \leq h(s)(|S \cup A| - 1)p \leq (s+u-1)h(s)p$. For $x \in U'(\mathbb{F}_p)$, $\chi_A(x)\tau_S(x) = -p^{-1}\mathrm{Tr}(\mathrm{Frob}_x \mid \mathcal{G}')$; χ_A vanishes on A; and the at most s points of $S \setminus A$ contribute at most $p^{-1} \cdot s \cdot sr = s^2 r^3$ to $\mathbb{E}_x \chi_A \tau_S$. So $|\mathbb{E}_x \chi_A \tau_S| \leq r^2(s+u-1)h(s) + r^3(s+u)s$, which is (15.11). □

The sizes enter the bounds through the ranks $h(s)$, $h(t)$, the puncture bound $s+t+u$, and the term $r(s+t+u)st$ in (15.10). The root cuts use root sets of sizes 2 to 11 and weights with at most ten roots, and they use the rank-sensitive bounds themselves, through the pivot groups of Section 15.4.3. The product rows use root sets of sizes 2 to 5, weights with at most four roots, and the uniform constant 512 below; the constant 6000 is the corresponding uniform constant for the sizes of the root cuts.

Corollary 15.20 (Uniform constants)**.** *For every odd prime p and $S, T, A \subseteq \mathbb{F}_p$ with $2 \leq |S|, |T| \leq 11$ and $|A| \leq 10$,*

$$\Big| \mathbb{E}_x \chi_A \tau_S \tau_T - \frac{\mathbf{1}_{S=T,\ A=\emptyset}}{p} \Big| \leq 6000\, p^{-3/2}.$$

If $|S|, |T| \leq 5$ and $|A| \leq 4$, the constant 6000 can be replaced by 512.

Proof. The bracket of (15.10) is nondecreasing in s, t and u. At $(s,t,u) = (11,11,10)$ it is $31 \cdot 100 + 1 + 32 \cdot 121r = 3101 + 3872r$ (rank 100, at most 32 finite punctures); at $(5,5,4)$ it is $13 \cdot 16 + 1 + 14 \cdot 25r = 209 + 350r$ (rank 16, at most 14 finite punctures). Since $p \geq 3$, $r \leq 1/\sqrt{3}$, and $3101 + 3872/\sqrt{3} < 6000$, $209 + 350/\sqrt{3} < 512$. □

Corollary 15.21 (Conditional L^2 bound)**.** *Let $y_1, \dots, y_d \in \mathbb{F}_p$ be distinct, $\varsigma_1, \dots, \varsigma_d \in \{\pm 1\}$, $E \subseteq \mathbb{F}_p$ a set containing $y_1, \dots, y_d$, and $\Phi = \{x \in \mathbb{F}_p \setminus E : \varsigma_j(x - y_j) \in Q_p^* \text{ for } j \leq d\}$; let ν be the uniform law on Φ, and suppose $|\Phi| \geq (1-e)p2^{-d}$ with $e < 1$, so that $\gamma = p2^{-d}/|\Phi| \leq (1-e)^{-1}$. Let $S_1, \dots, S_m$ be distinct subsets of $\mathbb{F}_p$ of sizes in $[2, \bar{s}]$ and $b_1, \dots, b_m \in \mathbb{R}$. Then*

$$\Big\| \sum_l b_l \tau_{S_l} \Big\|_{L^2(\nu)} \leq \sqrt{\gamma}\, r\Big[\sum_l b_l^2 + 2^d C\, r\Big(\sum_l |b_l| \Big)^2 \Big]^{1/2},$$

with $C = 6000$ if $\bar{s} \leq 11$ and $d \leq 10$, and $C = 512$ if $\bar{s} \leq 5$ and $d \leq 4$. In particular, for at most four pinned labels $y_1, \dots, y_d$ ($d \leq 4$) and root sets of sizes 2 to 5, the factor $2^d C$ may be taken equal to 8192.

Proof. For $x \in \Phi$ each factor $1 + \chi(\varsigma_j(x - y_j))$ equals 2, and every factor is nonnegative; so the density of ν relative to uniform probability is at most $\gamma \prod_j (1 + \chi(\varsigma_j(x - y_j)))$, and, the function $F = \sum_l b_l \tau_{S_l}$ being real, $\mathbb{E}_\nu F^2 \leq \gamma\, \mathbb{E}_x\big[\prod_j (1 + \chi(\varsigma_j(x - y_j)))F^2\big]$: dropping the factor $\mathbf{1}[x \notin E] \leq 1$

from the nonnegative integrand only increases the right side. Expanding the product over the subsets $P \subseteq \{1,\dots,d\}$, with $y(P) = \{y_j : j \in P\}$,

$$\mathbb{E}_x\Big[\prod_j (1+\chi(\varsigma_j(x-y_j)))F^2\Big] = \sum_P \chi\Big(\prod_{j\in P}\varsigma_j\Big)\sum_{l,l'} b_l b_{l'}\mathbb{E}_x\chi_{y(P)}\tau_{S_l}\tau_{S_{l'}}.$$

By Corollary 15.20 (arbitrary overlaps of S_l, $S_{l'}$ and the pins are allowed), the main terms with $P=\emptyset$ and $l=l'$ give $p^{-1}\sum_l b_l^2$, and all error terms, including the diagonal errors, together are at most $2^d C\, r^3(\sum_l |b_l|)^2$ in modulus, since the root sets S_l are distinct. For $d \le 4$ and $C = 512$, $2^d C \le 16\cdot 512 = 8192$. □

Remark 15.22 (One family of moments). Theorem 15.13 and (15.11) are first moments, and (15.10) is a second moment, of the same normalized traces: $\Sigma = p^2\mathbb{E}_z\chi_{Z(g)}\tau_f - [m \text{ odd}]\sum_z \chi_{Z(g)}(z)$ when f and g are monic. Their common cohomological step is Lemma 15.15. The output statements keep the data needed by their different applications: Theorem 15.13 keeps the exact number of common roots and the symmetric variant, which the product rows and the completion variance use; Lemma 15.19 keeps the ranks and the puncture counts, which the pivot groups below need.

15.4.3. *Grouping stars by a distinguished free label.* The conditional laws to which the moments are applied are those of a half law given all labels but one.

Definition 15.23 (Pinned half laws)**.** Let U be a finite set of n vertices, and for each $i \in U$ let $N^0(i) \subseteq U\setminus\{i\}$, $d_i = |N^0(i)|$ (the degree inside the half; d_i^0 in the notation of Section 14), and signs $\varsigma_{ij}^0 \in \{\pm 1\}$ $(j \in N^0(i))$. A probability law μ on the injective maps $x : U \to \mathbb{F}_p$ is a *pinned half law* with count errors $e_i \in [0,1)$ if, for every $i \in U$ and every x in the support of μ, the conditional law of x_i given $x_{-i} = (x_j)_{j\ne i}$ is the uniform law on

$$\Phi_i(x_{-i}) = \{y \in \mathbb{F}_p \setminus x(U\setminus\{i\}) :\ \varsigma_{ij}^0(y-x_j) \in Q_p^* \text{ for all } j \in N^0(i)\},$$

and $|\Phi_i(x_{-i})| \ge (1-e_i)p2^{-d_i}$.

For an oriented graph G^+ on U, the uniform law $\lambda_p(G^+)$ on $\mathrm{Inj}_p(G^+)$ (Definition 12.1) is a pinned half law, with $N^0(i)$ the neighbours of i, $\varsigma_{ij}^0 = +1$ for an edge (j,i) and -1 for an edge (i,j): given x_{-i}, the admissible values of x_i are exactly $\Phi_i(x_{-i})$. By Lemma 12.2, with the at most $n-1$ other labels excluded, one may take $e_i = e_{d_i}(p,n)$ whenever this is below one.

Lemma 15.24 (Ordered almost orthogonality)**.** *Let $x = (x_1,\dots,x_n)$ be a random vector with values in a finite set, with an arbitrary joint law, and let $F = \sum_{i=1}^n F_i$, where each F_i is a real function of $(x_j)_{j\le i}$. Suppose that for every i, almost surely, $\mathbb{E}[F_i^2 \mid x_{-i}] \le \alpha_i^2$ and $|\mathbb{E}[F_i \mid x_{-i}]| \le \beta_i$. Then*

$$\mathbb{E}F^2 \le \sum_i \alpha_i^2 + 2\sum_i \beta_i \sum_{j<i}\alpha_j.$$

Proof. $\mathbb{E}F^2 = \sum_i \mathbb{E}F_i^2 + 2\sum_{j<i}\mathbb{E}F_iF_j$, and $\mathbb{E}F_i^2 \le \alpha_i^2$. For $j<i$ the function F_j does not involve x_i, so it is a function of x_{-i} and $\mathbb{E}F_iF_j = \mathbb{E}\big[F_j\,\mathbb{E}[F_i \mid x_{-i}]\big]$, of modulus at most $\beta_i\mathbb{E}|F_j| \le \beta_i(\mathbb{E}F_j^2)^{1/2} \le \beta_i\alpha_j$. □

No independence between the F_i is assumed: the cross terms are controlled by conditioning on all variables but the later one, which turns the later function into a combination of traces with fixed roots and the earlier one into a constant.

Pivot groups. Let μ be a pinned half law on U, fix a total order of U, and let $a_T \in \mathbb{R}$ be coefficients indexed by the subsets $T \subseteq U$ with $|T| \geq 3$. Each such T is assigned to its greatest vertex $i(T)$, its *pivot*; put $s_T = |T| - 1$ and $F_i(x) = \sum_{T:\, i(T)=i} a_T\, \tau_{x(T\setminus\{i\})}(x_i)$. For $i \in U$, with all sums over the T with $i(T) = i$ and with $d = d_i$, put

$$\begin{gathered} A_i = \sum_T a_T^2, \qquad L_i = \sum_T |a_T|, \qquad U_i = \sum_T |a_T| h(s_T), \\ V_i = \sum_T |a_T|(s_T - 1)h(s_T), \qquad W_i = \sum_T |a_T| s_T, \qquad X_i = \sum_T |a_T| s_T^2, \\ E_i = 2^d\Big[2U_iV_i + (\tfrac{d}{2}+1)U_i^2 + L_i^2 + r\big\{2W_iX_i + \tfrac{d}{2}W_i^2\big\}\Big], \qquad D_i^2 = \frac{A_i + rE_i}{1-e_i}, \\ K_i = \frac{2^d}{1-e_i}\Big[V_i + \tfrac{d}{2}U_i + r\big\{X_i + (\tfrac{d}{2}+n-1)W_i\big\}\Big]. \end{gathered} \tag{15.13}$$

These nonnegative expressions also apply when $d_i = 0$, and they vanish for a vertex that is the pivot of no support.

Lemma 15.25 (Conditional moments of a pivot group)**.** *In this setting, for every $i \in U$ and every x in the support of μ,*

(a) $\mathbb{E}_\mu[F_i^2 \mid x_{-i}] \leq r^2 D_i^2$;
(b) $|\mathbb{E}_\mu[F_i \mid x_{-i}]| \leq r^2 K_i$.

Proof. Fix i and x_{-i}, and write $d = d_i$, $X_{-i} = x(U \setminus \{i\})$ (a set of $n-1$ labels), $\varsigma_j = \varsigma_{ij}^0$ and $y_j = x_j$ for $j \in N^0(i)$. Given x_{-i}, $F_i = \sum_T a_T \tau_{S_T}$ is a combination of traces with the fixed root sets $S_T = x(T \setminus \{i\})$ of sizes $s_T \geq 2$, and these root sets are distinct, since x is injective and distinct T with the same pivot have distinct sets $T \setminus \{i\}$. The conditional law of x_i has density $\gamma_i \prod_{j\in N^0(i)} (1 + \chi(\varsigma_j(y - x_j)))\mathbf{1}[y \notin X_{-i}]$ relative to uniform probability on $\mathbb{F}_p$, with $\gamma_i = p2^{-d}/|\Phi_i(x_{-i})| \leq (1-e_i)^{-1}$. For $P \subseteq N^0(i)$ write $x(P) = \{x_j : j \in P\}$ and $\varsigma_P = \prod_{j\in P} \varsigma_j$; then $\prod_{j\in N^0(i)}(1 + \chi(\varsigma_j(y - x_j))) = \sum_P \chi(\varsigma_P)\chi_{x(P)}(y)$. Over the 2^d subsets, $\sum_P 1 = 2^d$ and $\sum_P |P| = \frac{d}{2}2^d$.

(a) As in the proof of Corollary 15.21, discarding the restriction $y \notin X_{-i}$ from a square, $\mathbb{E}_\mu[F_i^2 \mid x_{-i}] \leq \gamma_i \sum_P \chi(\varsigma_P) \sum_{T,T'} a_T a_{T'} \mathbb{E}_y \chi_{x(P)} \tau_{S_T} \tau_{S_{T'}}$. By (15.10) with $(s,t,u) = (s_T, s_{T'}, |P|)$, the terms with $P = \emptyset$ and $T = T'$ give $p^{-1}\sum_T a_T^2 = r^2 A_i$, and the error is at most

$$r^3 \sum_P \sum_{T,T'} |a_T||a_{T'}|\Big[(s_T + s_{T'} + |P| - 1)h(s_T)h(s_{T'}) + 1 + r(s_T + s_{T'} + |P|)s_T s_{T'}\Big].$$

Summing over P replaces $|P|$ by $\frac{d}{2}$ and multiplies by 2^d. Writing $s_T + s_{T'} - 1 + \frac{d}{2} = (s_T - 1) + (s_{T'} - 1) + (1 + \frac{d}{2})$, the rank part sums to $2U_iV_i + (\frac{d}{2}+1)U_i^2 + L_i^2$ and the restored-point part to $2W_iX_i + \frac{d}{2}W_i^2$. So the error is at most r^3E_i, and $\mathbb{E}_\mu[F_i^2 \mid x_{-i}] \leq \gamma_i(r^2A_i + r^3E_i) \leq r^2D_i^2$.

(b) Unlike the second moment, the first moment cannot drop the excluded labels. Write $\mathbf{1}[y \notin X_{-i}] = 1 - \mathbf{1}[y \in X_{-i}]$:

$$\mathbb{E}_\mu[F_i \mid x_{-i}] = \gamma_i \sum_P \chi(\varsigma_P) \sum_T a_T \mathbb{E}_y \chi_{x(P)} \tau_{S_T} - \frac{\gamma_i}{p} \sum_{y\in X_{-i}} \prod_{j\in N^0(i)} (1 + \chi(\varsigma_j(y - x_j)))F_i(y).$$

By (15.11), the first part is at most $\gamma_i r^2 \sum_P \sum_T |a_T|\big[(s_T + |P| - 1)h(s_T) + r(s_T + |P|)s_T\big] = \gamma_i r^2 2^d\big[V_i + \frac{d}{2}U_i + r(X_i + \frac{d}{2}W_i)\big]$ in modulus. In the second part there are $n-1$ points, the pin product is at most 2^d, and $|F_i(y)| \leq \sum_T |a_T| s_T r = rW_i$ by Lemma 15.17(b); so it is at most $\gamma_i p^{-1}(n-1)2^d r W_i = \gamma_i r^3 2^d (n-1)W_i$. This $(n-1)W_i$ term covers the excluded labels exactly, and the sum is at most r^2K_i. □

Every vertex of a support with pivot j precedes or equals j, so F_j is a function of $(x_k)_{k\le j}$, as Lemma 15.24 requires.

Proposition 15.26 (Norm of a sum of higher star functions in a fixed vertex order)**.** *Let μ be a pinned half law on U, fix a total order of U and real coefficients a_T ($T\subseteq U$, $|T|\ge 3$), and let F_i, D_i, K_i be as in* (15.13)*. Then*

$$\Big\|\sum_{i\in U}F_i\Big\|_{L^2(\mu)}\le M_\star:=r\Big[\sum_i D_i^2+2r\sum_i K_i\sum_{j<i}D_j\Big]^{1/2},\qquad |\mathbb{E}_\mu F_i|\le r^2K_i\quad(i\in U).$$

In particular $\|P_0\sum_i F_i\|_\mu\le M_\star$, where P_0 is the projection off constants. No bound on n, on the degrees or on $|T|$ is needed; the root cuts of the upper range have $n=36$, $d_i\le 10$ and $|T|\le 12$.

Proof. Apply Lemma 15.24, with the vertices listed in the fixed order, $\alpha_i=rD_i$ and $\beta_i=r^2K_i$ (Lemma 15.25(a),(b)): $\mathbb{E}_\mu(\sum_i F_i)^2\le r^2\sum_i D_i^2+2r^3\sum_i K_i\sum_{j<i}D_j=M_\star^2$. Integrating Lemma 15.25(b) gives the bound for $\mathbb{E}_\mu F_i$, and P_0 is a contraction. □

Remark 15.27 (The order, and monotonicity in r). The order is only a choice of bound: the proposition holds for every total order. The certificates place the vertices in decreasing internal degree, breaking ties by their original indices, so every support uses one of its vertices of smallest internal degree as its pivot. If each e_i is replaced by a bound $\bar e_i(r)<1$ that is a polynomial in r with nonnegative coefficients (as $e_t(p,n)$ of Lemma 12.2 is), then D_i, K_i, and hence $M_\star/r$ and the quantity $z_\star/r^2$ of Lemma 15.29, are nondecreasing functions of r; so evaluation at the largest r in the range (the smallest prime) bounds every smaller r, hence every larger prime (Lemma 4.18).

Diagonal and pivot bounds. The next lemma bounds the same kind of star marginal for a graph law, with arbitrary real coefficients, by the $T=T'$ density bounds of part (a) and the conditional-mean bounds of part (b) computed with Theorem 15.13 instead of conditional moments. It needs no order of the vertices: each cross term is conditioned on all labels but that of one vertex outside the other support.

Lemma 15.28 (Weighted star marginals in L^2)**.** *Let G^+ be an oriented graph on a set U of n vertices with degrees d_i and maximum degree Δ^+, let p be an odd prime with $e_{\Delta^+}(p,n)<1$, put $q=1/p$, and let $\mu=\lambda_p(G^+)$, a pinned half law whose neighbour sets $N^0(i)$ are the sets of neighbours in G^+ (Definition 15.23). For $T\subseteq U$ let $e(T)$ be the number of edges of G^+ inside T and $w_T(x)=\mathbb{E}_z\prod_{j\in T}\chi(z-x_j)$ as in Lemma 15.18, and put $R_U=\prod_{i\in U}\frac{1+e_{d_i}(p,n)}{1-e_{d_i}(p,n)}$ (the quantity R_{full} of Lemma 14.31 when $U=U^+$). For $|T|\ge 3$ and $j\in T$ put $s_T=|T|-1$, $d=d_j$, $o=|N^0(j)\cap T|$,*

$$D_T^{\mathrm{st}}=\min\Big(2^{e(T)}R_U,\ \min_{i\in T}\frac{2^{d_i}}{1-e_{d_i}(p,n)},\ (|T|-1)^2\Big),\tag{15.14}$$

(15.15)

$$\beta_{T,j}=\frac{q}{1-e_d^-(n)}\Big[(d-o)+d(s_T-1)r+\sum_{a=2}^{d}\sum_{u\ge 0}\binom{o}{u}\binom{d-o}{a-u}\mathcal{B}_{\min}(s_T,a,u)+2^d(n-1)s_Tr\Big],$$

with the graded count error $e_d^-(n)$ of (12.2)*, $\mathcal{B}_{\min}(m,a,u)=\min\big(\mathcal{B}(m,a,u),\mathcal{B}(a,m,u)\big)$ with $\mathcal{B}$ of* (15.7)*, and binomial coefficients outside their range zero. For $T\ne T'$ of size at least three let $\Xi_{TT'}$ be the least of the numbers $\beta_{T,j}(D_{T'}^{\mathrm{st}})^{1/2}r$ ($j\in T\setminus T'$) and $\beta_{T',j}(D_T^{\mathrm{st}})^{1/2}r$ ($j\in T'\setminus T$). Then for all $T,T'\subseteq U$ of size at least three:*

(a) $\mathbb{E}_\mu w_T^2\le qD_T^{\mathrm{st}}$;
(b) $|\mathbb{E}_\mu[w_T\mid x_{U\setminus\{j\}}]|\le\beta_{T,j}$ *for $j\in T$ and every feasible $x_{U\setminus\{j\}}$;*
(c) $|\mathbb{E}_\mu w_Tw_{T'}|\le\Xi_{TT'}$ *if $T\ne T'$.*

Consequently, for real coefficients c_T,

$$\Big\| \sum_{|T|\ge 3} c_T w_T \Big\|_\mu^2 \le q \sum_T c_T^2 D_T^{\mathrm{st}} + \sum_{T\ne T'} |c_T c_{T'}|\, \Xi_{TT'}.$$

Proof. By Lemma 12.5(a),(d), $\mathrm{Inj}_p(G^+) \ne \emptyset$, so μ is defined, and it is supported on injective labellings. Write $x_{-j} = x_{U\setminus\{j\}}$ and $X_{-j} = \{x_k : k \ne j\}$. Since $e_t(p,n)$ is nondecreasing in t ((E3)), $e_{d_i}(p,n) \le e_{\Delta^+}(p,n) < 1$ for every i, and $e_d^-(n) \le e_d(p,n) < 1$ ((E2)).

(a) Any one of three bounds suffices. *Weil.* On injective labellings $\prod_{j\in T}(z - x_j)$ has $|T|$ distinct roots, so $|w_T| \le (|T|-1)r$ by Theorem 4.7(a), and $\mathbb{E}_\mu w_T^2 \le (|T|-1)^2 q$. *Density.* By Lemma 12.6(e) with $V = T$, $C = \emptyset$ and any ordering of T, the law of x_T under μ has density at most $2^{e(T)} R_{U\setminus T} \prod_{i\in T}(1 - e_{t_i}(p,n))^{-1}$ with respect to the uniform probability on $\mathbb{F}_p^T$, where $t_i \le d_i$ is the number of earlier neighbours of i in T, so that $\sum_i t_i = e(T)$; as $(1-e_{t_i})^{-1} \le (1-e_{d_i})^{-1} \le (1+e_{d_i})/(1-e_{d_i})$, the density is at most $2^{e(T)} R_U$. Extend w_T to all of $\mathbb{F}_p^T$ by its defining formula. By (4.2), with $s = |T| \ge 3$,

$$\begin{aligned}\mathbb{E}_{v\in\mathbb{F}_p^T} w_T(v)^2 &= \mathbb{E}_{z,z'} \prod_{j\in T} \mathbb{E}_{v_j} \chi(z - v_j)\chi(z' - v_j) = \mathbb{E}_{z,z'}(\mathbf{1}_{z=z'} - q)^s \\ &= q\big[(1-q)^s + (p-1)(-q)^s\big] \le q(1-q),\end{aligned}$$

since $(1-q)^s \le (1-q)^2$ and $(p-1)q^s \le (p-1)q^2$; so $\mathbb{E}_\mu w_T^2 \le 2^{e(T)} R_U\, q$. *Conditional.* Fix $i \in T$, let $f = \prod_{l\in T\setminus\{i\}}(X - x_l)$ and $\mathcal{G}(\xi) = p^{-1}\sum_{y\in\mathbb{F}_p} \chi(f(y)(y-\xi))$, so that $w_T = \mathcal{G}(x_i)$. Since $\sum_\xi \chi((y-\xi)(y'-\xi)) = p\mathbf{1}_{y=y'} - 1$ by (4.2),

$$\sum_{\xi\in\mathbb{F}_p} \mathcal{G}(\xi)^2 = p^{-2}\sum_{y,y'} \chi(f(y)f(y'))(p\mathbf{1}_{y=y'} - 1) = p^{-2}\Big[p(p - |T| + 1) - \Big(\sum_y \chi(f(y))\Big)^2\Big] \le 1.$$

By Lemma 12.6(b), given a feasible x_{-i} the label x_i is uniform on a set Φ_i of at least $p2^{-d_i}(1 - e_{d_i}(p,n))$ elements, so $\mathbb{E}_\mu[w_T^2 \mid x_{-i}] \le |\Phi_i|^{-1}\sum_\xi \mathcal{G}(\xi)^2 \le q2^{d_i}/(1 - e_{d_i}(p,n))$. Averaging over x_{-i}, and taking the least of the three bounds and of the choices of i, gives (a).

(b) Fix $j \in T$ and a feasible x_{-j}; let d, o, s_T be as in (15.15), and f, $\mathcal{G}$ as in (a) with $i = j$, so that $w_T = \mathcal{G}(x_j)$ and $\deg f = s_T \ge 2$. Given x_{-j}, the label x_j is uniform on the admissible set $\Phi_j = \Phi_j(x_{-j})$ of Definition 15.23 (Lemma 12.6(b)). Put $\varsigma_k = \varsigma_{jk}^0$, $\varsigma_P = \prod_{k\in P}\varsigma_k$ and $x(P) = \{x_k : k \in P\}$, and use the character weights χ_A of Section 15.4.2. For $\xi \notin X_{-j}$ every $\xi - x_k$ is nonzero, and

$$\mathbf{1}_{\Phi_j}(\xi) = \prod_{k\in N^0(j)} \tfrac12(1 + \chi(\varsigma_k)\chi(\xi - x_k)) = 2^{-d} \sum_{P\subseteq N^0(j)} \chi(\varsigma_P)\chi_{x(P)}(\xi).$$

Hence, with $g_P = \prod_{k\in P}(X - x_k)$,

$$\sum_{\xi\in\Phi_j} \mathcal{G}(\xi) = 2^{-d} \sum_{P\subseteq N^0(j)} \chi(\varsigma_P)\Big[\frac{\Sigma_P}{p} - \sum_{\xi\in X_{-j}} \chi_{x(P)}(\xi)\mathcal{G}(\xi)\Big], \qquad \Sigma_P = \sum_{y,\xi\in\mathbb{F}_p} \chi\big(f(y)\, g_P(\xi)\,(y-\xi)\big).$$

$P = \emptyset$: $\Sigma_\emptyset = \sum_y \chi(f(y))\sum_\xi \chi(y-\xi) = 0$. *$P = \{k\}$:* $\sum_\xi \chi((\xi - x_k)(y-\xi)) = \chi(-1)(p\mathbf{1}_{y=x_k} - 1)$ by (4.2), so $\Sigma_P = \chi(-1)\big[p\chi(f(x_k)) - \sum_y \chi(f(y))\big]$ and $|\Sigma_P|/p \le |\chi(f(x_k))| + (s_T - 1)r$ by Theorem 4.7(a); as x is injective, $\chi(f(x_k)) = 0$ exactly when $k \in T$. So the d sets with $|P| = 1$ contribute at most $(d-o) + d(s_T-1)r$. *$|P| = a \ge 2$:* f and g_P are monic and split with distinct roots, of degrees $s_T \ge 2$ and $a \ge 2$, and by injectivity they have exactly $u = |P \cap T|$ common roots; Theorem 15.13 with $c = c' = 1$, in both of its forms, gives $|\Sigma_P| \le p\,\mathcal{B}_{\min}(s_T, a, u)$, and there are $\binom{o}{u}\binom{d-o}{a-u}$ such sets P for given a and u. *The excluded labels:* $|\mathcal{G}(\xi)| \le s_T r$ for every ξ. If ξ is not a root of f, $f(y)(y-\xi)$ has $s_T + 1$ distinct roots and Theorem 4.7(a) applies; if it is,

$f(y)(y-\xi)=(y-\xi)^2\tilde f(y)$ with $\tilde f$ of degree $s_T-1\ge 1$ with distinct roots, the term $y=\xi$ vanishes, and $|p\,\mathcal G(\xi)|\le|\sum_y\chi(\tilde f(y))|+1\le(s_T-2)\sqrt p+1\le s_T\sqrt p$. The 2^d sets P and the $n-1$ labels of X_{-j} therefore contribute at most $2^d(n-1)s_Tr$. Altogether $|\sum_{\xi\in\Phi_j}\mathcal G(\xi)|$ is at most 2^{-d} times the bracket of (15.15). Finally $\mathbb E_\mu[w_T\mid x_{-j}]=|\Phi_j|^{-1}\sum_{\xi\in\Phi_j}\mathcal G(\xi)$, and Lemma 12.2(c), with the d distinct roots x_k ($k\in N^0(j)$), the signs $\chi(\varsigma_k)$ and the excluded set X_{-j}, which has $n-1\le n$ elements and contains the roots, gives $|\Phi_j|\ge p2^{-d}(1-e_d^-(n))>0$. Dividing gives (b).

(c) Let $j\in T\setminus T'$. Then $w_{T'}$ is a function of $x_{T'}$ and $T'\subseteq U\setminus\{j\}$, so $\mathbb E_\mu w_Tw_{T'}=\mathbb E_\mu\big[w_{T'}\,\mathbb E_\mu[w_T\mid x_{-j}]\big]$, and by (b) and (a) $|\mathbb E_\mu w_Tw_{T'}|\le\beta_{T,j}\mathbb E_\mu|w_{T'}|\le\beta_{T,j}(qD^{\mathrm{st}}_{T'})^{1/2}=\beta_{T,j}(D^{\mathrm{st}}_{T'})^{1/2}r$. The case $j\in T'\setminus T$ is symmetric, and one of the two sets is nonempty as $T\ne T'$. For the last assertion expand $\|\sum_T c_Tw_T\|_\mu^2=\sum_{T,T'}c_Tc_{T'}\mathbb E_\mu w_Tw_{T'}$, and bound the diagonal terms by (a) and the others by (c). □

For a soft cross kernel the higher-star part of the marginal density of one half relative to μ is $\sum_{|T|\ge3}c_Tw_T$, where T runs over sets of leaves in that half and c_T is the sum, over the centres in the other half, of the star coefficients of support T (Corollary 14.5); for the cut kernel K_0 of Section 14 (all soft coefficients equal to 1 on $E_\times$) $c_T=a_T$ of (15.9). The coefficients are arbitrary, so the signs and sizes of the soft entries play no role in the lemma. The certificates of the dense comparisons and of the six-query mixture (Propositions 28.5 and 28.6) evaluate the bound with the count error $\big(\sum_{k\ge3}\binom dk(k-1)\big)r+\big(\binom d2+2^dn\big)q$ in place of $e_d^-(n)$. Since $\sum_{k\ge3}\binom dk(k-1)=a_d+b_d-\binom d2$ (proof of Lemma 12.2, and (E2)), this error exceeds $e_d^-(n)$ by $\big(b_d-\binom d2\big)(r-q)\ge0$ ($b_d=2^{d-1}-1\ge\binom d2$ for $d\ge1$); the certificates use this weaker form: $\beta_{T,j}$ increases with $e_d^-(n)$ through the factor $(1-e_d^-(n))^{-1}$ in (15.15), so replacing $e_d^-(n)$ by a larger error below 1 only enlarges the bound. For fixed coefficients, D^{st}_T, $\beta_{T,j}/q$ and $\Xi_{TT'}/(qr)$ are nondecreasing functions of r: they are built from the polynomials $e_t(p,n)$, $e_t^-(n)$ and $\mathcal B(m,a,u)$ in r with nonnegative coefficients ((E3), (15.7)) by sums, products, minima and the maps $x\mapsto(1-x)^{-1}$, $x\mapsto(1+x)/(1-x)$ on $[0,1)$. So the bound divided by q is nondecreasing in r, and a check at one prime covers every larger prime at which the coefficients are the same (Lemma 4.18).

15.4.4. *The complete-star mean.* The constant $Z_0=\mathbb E_{\mu\otimes\mu}K_0$ normalizing the cut kernel K_0 of Section 14 needs the mean of the star function, not only its centred part; here the affine constants of Lemma 15.18 must be kept.

Lemma 15.29 (The complete-star mean)**.** *Consider spectral data with the star function $h_\star$ and the coefficients a_T of* (15.9)*. Let $\mu=\lambda_p(G^+)$ be a pinned half law with count errors e_i, fix a total order of U^+, let F_i, K_i, $M_\star$ be the quantities of Section 15.4.3 for the coefficients a_T with $|T|\ge3$, and put*

$$B_{\mathrm{even}}=\sum_{T:\ |T|\ \mathit{even}\ \ge2}a_T=\sum_{k\in U^+}\Big(\frac12\Big[\prod_{j\in N(k)}(1+\chi(\varsigma_{jk}))+\prod_{j\in N(k)}(1-\chi(\varsigma_{jk}))\Big]-1\Big).$$

Then $h_\star=\sum_iF_i-B_{\mathrm{even}}/p$ on $\mathrm{Inj}_p(G^+)$. Consequently $h_\star-\mathbb E_\mu h_\star=P_0\sum_iF_i$, $\|h_\star-\mathbb E_\mu h_\star\|_\mu\le M_\star$, and

$$|\mathbb E_\mu h_\star|\le z_\star:=r^2\Big(\sum_iK_i+|B_{\mathrm{even}}|\Big).$$

Proof. By (15.9) and Lemma 15.18, the supports with $|T|=1$ contribute nothing, those with $|T|=2$ contribute $-a_T/p$, and those with $|T|\ge3$ contribute $a_T\big(\tau_{x(T\setminus\{i(T)\})}(x_{i(T)})-\mathbf1_{|T|\ \mathrm{even}}/p\big)$. Summing, $h_\star=\sum_iF_i-p^{-1}\big(\sum_{|T|=2}a_T+\sum_{|T|\ge4\ \mathrm{even}}a_T\big)=\sum_iF_i-B_{\mathrm{even}}/p$. For the second expression of B_{even}: for a centre k with signs $c_j=\chi(\varsigma_{jk})$, $\sum_{T\subseteq N(k),\ |T|\ \mathrm{even}}\prod_{j\in T}c_j=\frac12\big[\prod_j(1+c_j)+\prod_j(1-c_j)\big]$, and the empty support, which contributes 1, is subtracted. (For $|N(k)|\ge1$ a centre whose signs c_j are all equal contributes $2^{|N(k)|-1}-1$, and every other centre contributes -1.) Then

$\mathbb{E}_\mu h_\star = \sum_i \mathbb{E}_\mu F_i - B_{\mathrm{even}}/p$ and $h_\star - \mathbb{E}_\mu h_\star = P_0 \sum_i F_i$, and Proposition 15.26 gives both bounds, as $1/p = r^2$. □

Remark 15.30 (How the mean enters, and what is not done). The row stars have the same mean as the column stars, and together the stars contribute $2\mathbb{E}_\mu h_\star$ to the normalizer of the cut kernel (Lemma 14.19(c)). This is how $z_\star$ and $Z_\star \geq 2|\mathbb{E}_\mu h_\star|$ enter hypothesis (H3) of Theorem 14.27, and how the same bound enters the normalizer error of route A (Theorem 14.21). The number $z_\star$ depends on the class of p through the signs $\chi(\varsigma_{jk})$: the root caps proved by Route B, applied at $p \equiv 3 \pmod 4$, use its value for that class (Lemma 27.6(ii)), and the completed-root caps use the larger of its two values (Lemma 27.10). Two cautions apply. The affine constants are kept: the two-leaf stars and the constants $-\mathbf{1}_{|T|\ \mathrm{even}}/p$ make up B_{even}/p, which does not vanish in the mean. And the estimate bounds $\mathbb{E}_\mu h_\star$ itself; the bound for the centred part $P_0 h_\star = h_\star - \mathbb{E}_\mu h_\star$ is not used in its place.

For one half law the mean is known exactly. On the half reference of the middle root cut L20, the graph law on the six vertices $(a_0, a_1, b_0, b_1, c_0, c_1)$ whose only edges are the four arrows $a_i \to c_j$ $(i, j \in \{0, 1\})$ from the pair (a_0, a_1) to the pair (c_0, c_1), the remaining two vertices (b_0, b_1) being isolated, the two-leaf star means cancel, the three-leaf ones add up to zero at each centre, and the four-leaf ones are given exactly in terms of the graph moment M_{G_5} of the variance credits (Lemma 24.5, proved in Section 24 next to that moment). The normalizer of that cut uses this exact value instead of the general bound (Section 20.4).

15.5. **Where these estimates enter.** Table 6 lists every use in the final construction. The sheaves of Lemma 15.14 stand behind all rows except those of Theorem 15.8 and Corollaries 15.9 and 15.10, which rest on the representation theory of $\mathrm{PGL}_2(\mathbb{F}_p)$, and the last row, an external evaluation through a modular form (Section A.3).
Consequences and what to read next. This section used the following inputs; its outputs are the three second-order estimates of the opening, whose uses are listed in Table 6. From Part II: Weil's bound (Theorem 4.7), Jacobi sums (Theorem 4.10), the two-root identity (4.2), the Paley operator (Lemma 4.11(c)), and, in Remark 15.27 and in the discussion after Lemma 15.28 only (not in any proof), the admissible majorants of Lemma 4.18. From Part IV: strict-square labellings and the extension count (Definition 12.1 and Lemma 12.2), to see that half laws are pinned and, with the one-vertex conditional laws and tuple densities of graph laws (Lemma 12.6(b),(e)), the sequential counts (Lemma 12.5), the graded lower count Lemma 12.2(c) and the properties (E2), (E3) of the count errors, in Lemma 15.28; spectral data (Definition 13.1) for the cross signs of the star function; and, in remarks and discussion only (not in any proof), Lemmas 13.9 and 14.19 and Corollary 14.5. From Appendix A: the character table of $\mathrm{GL}_2(\mathbb{F}_p)$ with its class sizes (Theorem A.1); the ℓ-adic facts (E1)–(E4); the trace formula, the cohomology of open curves, Deligne's theorem and the Euler characteristic formula (Theorems A.2 to A.5); and the Katz–Sarnak family with its self-duality and its local monodromy at the finite roots (Theorem A.6).

16. The non-star part of the character expansion and coincident labels

What this section proves and how to read it. Expanding a cut kernel in characters gives terms indexed by sets of cross pairs. Terms whose pairs share an endpoint (*stars*) are handled in Section 14; this section bounds the remaining terms and the terms where labels coincide, showing both are second order with explicit constants. No new cancellation is proved here: the work is bookkeeping to keep constants small when many subsets are summed. *Extraction* means the following choice, made for each group of subsets (a *class* $\mathcal{C}(P, A)$: pairs in P present, pairs in A absent, others free): choose two disjoint present pairs (*witnesses*, certifying second order), choose sets of vertices meeting every other undecided pair (a *vertex cover*, whose labels are conditioned on), and choose labelling orders. An *extraction datum* records these choices (called a "record

TABLE 6. Where the second-order estimates enter.

estimate	enters as	in
Theorem 15.8, Corollary 15.9(a)	the norm $2r^2$ of a two-vertex strongly connected block of a witness kernel (Lemma 16.4), hence the complete-rectangle coefficient of the compressed partitions (Lemma 16.2)	the nonstar bounds of every spectral cap: the twelve-label grid, the caps of the root cut L20 (Sections 20.4 and 21), the ten nonroot upper cuts, N_{ns} of the direct-star root (Lemma 27.6(iii)), the pinned rows of Section 16.5
Corollary 15.9	the injective $K_{2,2}$ core $2r^2$ of a matched neighbourhood in the completion variance (Lemma 14.15, through Lemma 16.4(b))	the coupling block (Lemma 14.14) of the completed-root caps (Lemma 27.10) from 2^{52}
Corollary 15.10	the remainder $\|R\|^2 \le c_0 q^2$ of the raw pair maps (b), and the norms r and $\sqrt{q(1-q)}$ of the maps of (a)	the square identity, the linear trace and the layer squares of the positive roots (Lemmas 21.4 and 21.5 and Proposition 21.9)
Theorem 15.13	the pair term of the completion variance, through the coefficient triples of $\mathcal{B}(a,b,0)$ (Remark 15.16(ii))	the coupling block of the completed-root caps (Lemma 27.10) from 2^{52}
Theorem 15.13	the complete conditional means (PR1) of the product rows, with $p\,\mathcal{B}(4,4,0) < 40p$ (Remark 15.16(iii), Section 27.6.2)	the row inputs of the upper bands
Lemma 15.28 (through Theorem 15.13)	the L^2 norm of the weighted higher-star marginals of a soft cross kernel	the dense comparisons and the six-query mixture of the upper range (Propositions 28.5 and 28.6)
Corollaries 15.20 and 15.21	the weighted moments (PR7) and the conditional L^2 bounds (PR8) of the product rows	the row inputs of the upper bands
Proposition 15.26 and Lemma 15.29	$M_\star$, $z_\star$ and $Z_\star$ (Lemma 27.6(i),(ii))	the direct-star root on the upper bands from P_3 to 2^{52} (Theorem 14.27) and the completed-root caps from 2^{52} (Lemma 27.10)
Theorem A.7 (Appendix A)	the bounds $-(1-q)(2qr+q^2) \le M_{G_5} \le (1-q)(2qr-q^2)$ of the graph moment (24.3), in the second moment of a neighbourhood count (Lemma 24.3) and in the quartic star mean (Lemma 24.5)	the root-half variances χ_p (the χ^2-divergences of Section 24, not the character χ_p above) of the middle roots (Lemma 24.7) in the restorations and the completion of the root cut L20 (Section 20.4); the star mean of that cut; the linear prices and the normalization of the positive roots (Lemmas 21.5 and 21.6)

of a class" in the text below, not to be confused with records of Part III). Inputs: Sections 13 and 15. Output: the per-class bound (Section 16.3) used by the spectral criteria. A first reading should follow the class definition and the statement of the per-class bound; the counting details are reference material. Writing each cross constraint at $e = (j,k)$ as $\frac{1}{2}(1 + \chi(\varsigma_{jk}(x'_k - x_j)))$ gives

the expansion; the non-star part T_{ns} is second order because a non-star contains two disjoint pairs, and the coincidence part T_{coll} corrects for half labellings sharing a label. The difficulty is the constant at the first prime of a range, where $2^{m_\times}$ subsets are summed; it is kept small by stopping at the first two disjoint pairs, conditioning only on a vertex cover, counting constraints met by each vertex, and labelling the coincident vertices (those forced to carry equal labels) first in the coincidence count.

16.1. **The character expansion and the compressed partition.**
Setting. Throughout this section, except in Section 16.5, $(G, \vartheta, U^+, \mathfrak{o})$ are spectral data (Definition 13.1) with the notation fixed there: $n = |U^+|$, the half graph G^+, adjacency $j \sim k$ in G^+, the degrees d_k^0 and $\Delta^+ = \max_k d_k^0$, the cross signs ς_{jk}, the neighbourhoods $N(k) = \{j : \varsigma_{jk} \neq 0\}$ with $r_k = |N(k)|$, $d_k = d_k^0 + r_k$, $\Delta = \max_k d_k$, the cross pairs $E_\times$ and $m_\times = |E_\times|$. We write $\mathrm{Adj}(i) = \{j \in U^+ : j \sim i\}$, so that $|\mathrm{Adj}(i)| = d_i^0$, and $e(L)$, $e(L, L')$ for the numbers of edges of G^+ inside L and between disjoint sets L, L'. The prime p is odd, $r = p^{-1/2}$, $e_t(p, n') = 2^t(h_t r + n' r^2)$ is the count error of (12.1), $\mathcal{I}^+ = \mathrm{Inj}_p(G^+)$ and $\mu = \lambda_p(G^+)$ is the half law. For $Y \subseteq U^+$ put

$$R_Y = \prod_{i \in Y} \frac{1 + e_{d_i^0}(p, n)}{1 - e_{d_i^0}(p, n)}, \qquad R_{\mathrm{full}} = R_{U^+}, \qquad \hat{e} = e_{\Delta^+}(p, n),$$

the completion ratios of Lemma 12.6 for the graph G^+. Under (S1) at p, $e_t(p, n') < 1$ for $t \le \Delta$ and $n' \le 2n$; in particular $\hat{e} < 1$, and every factor of every R_Y is defined and at least one.

The two labellings of a pair $(x, x') \in \mathcal{I}^+ \times \mathcal{I}^+$ live on two copies X and Z of U^+; in a cross pair (j, k) the x-endpoint j is a vertex of X and the x'-endpoint k a vertex of Z. Graph quantities (Adj, d^0, N, r) of a vertex of X or Z are those of the corresponding vertex of U^+. The characters $\chi_e(x, x') = \chi(\varsigma_{jk}(x'_k - x_j))$ of the cross pairs $e = (j, k)$, their products χ_S, the operators T_S on $L^2(\mu)$ with kernel χ_S relative to $\mu \otimes \mu$, the stars and nonstars, $T_{\mathrm{ns}} = \sum_{S \text{ nonstar}} T_S$, and the collision operator T_{coll}, whose kernel K_{coll} is $\prod_{e \in E_\times}(1 + \chi_e)$ on the pairs (x, x') that share a label and 0 elsewhere, are those of Section 14.6. By Lemma 14.18, $2^{m_\times}$ times the cut kernel is $\mathbf{1} \otimes \mathbf{1} + \sum_{S \text{ star}} T_S + T_{\mathrm{ns}} - T_{\mathrm{coll}}$ (14.20), and K_{coll} is nonnegative and symmetric; the constants and the stars are treated in Section 14. By Lemma 13.2, $(j, k) \in E_\times$ iff $(k, j) \in E_\times$; so $j \in N(k)$ iff $k \in N(j)$, and r_j is also the number of cross pairs with x-endpoint j.
Graded errors. Every statement of this section outside Section 16.5, whose rows carry their own error e_d^{pin}, holds with the same proof when the coarse error $e_t(p, n')$ is replaced throughout (in R_Y, R_{full}, $\hat{e}$, the vertex and leaf factors, L_{int}, Υ and Υ_k) by the graded error $e_t^-(n')$ of (12.2). Indeed the proofs use three properties of $e_t(p, n')$: it bounds the relative error of a count of Lemma 12.2 from above whatever the excluded set, and from below when at most n' labels are excluded (Lemma 12.2(b),(d)); it is nondecreasing in t and n' and a polynomial in r with nonnegative coefficients ((E3)); and the densities and completion ratios of Lemma 12.6 are formed with it. The graded error has the first two properties by Lemma 12.2(b),(c) with (E2) and (E3). For the third, Lemma 12.6(b)–(e) hold with $e_t^-(n)$ in place of $e_t(p, n)$, under $e_\Delta^-(n) < 1$: in their proofs Lemma 12.2(d) enters only through lower counts, for which (c) with $z = n$ serves, and Lemma 12.5(a),(d) for the coarse pair may be replaced by (a) for the graded pair, whose ratio $\prod_i (1+e_{t_i}^+)/(1-e_{t_i}^-(n))$ is at most $\prod_i (1+e_{d_i}^-(n))/(1-e_{d_i}^-(n))$ by (E2) and (E3), since $t_i \le d_i$. Where a proof uses the explicit form of $e_t(p, n')$ (Corollary 16.14), it applies through $e_t^-(n') \le e_t(p, n')$ ((E2)); in particular, under (S1), $e_t^-(n') < 1$ for $t \le \Delta$ and $n' \le 2n$. The *error pair of the setting* is the coarse pair or the graded pair of Definition 12.4; one choice is made for all the errors of a bound, and the nonstar coefficients of the root cuts of the upper range below 2^{52} use the graded one.
Classes. For disjoint $P, A \subseteq E_\times$ the *class* $\mathcal{C}(P, A) = \{S : P \subseteq S \subseteq E_\times \setminus A\}$ consists of the sets in which the pairs of P are *present*, those of A *absent*, and those of $F = E_\times \setminus (P \cup A)$ *free*. A

pair is *decided* if it is present or absent; pairs of $X \times Z$ outside $E_\times$ count as absent. The *mass* of a class is $2^{|F|}$, and $T_{\mathcal{C}} = \sum_{S \in \mathcal{C}} T_S$. An *exact partition* of a family of subsets of $E_\times$ is a set of pairwise disjoint classes whose union is the family. If $\mathcal{C}$ is the disjoint union of $\mathcal{C}_1$ and $\mathcal{C}_2$, then $T_{\mathcal{C}} = T_{\mathcal{C}_1} + T_{\mathcal{C}_2}$. Hence bounds for the classes of an exact partition add up to a bound for the whole family, and for a class split in two the smaller of a bound for the class and the sum of bounds for its two parts is again a bound for the class.

Definition 16.1 (Witnesses and covers)**.** Let $\mathcal{C} = \mathcal{C}(P, A)$ be a class with free pairs F, and $k \geq 1$.

(a) A *witness of size* k consists of pairs $(j_1, k_1), \ldots, (j_k, k_k) \in P$ with distinct j_i and distinct k_i such that, with $V = \{j_1, \ldots, j_k\} \subseteq X$ and $W = \{k_1, \ldots, k_k\} \subseteq Z$, every pair of the *rectangle* $E_{VW} = (V \times W) \cap E_\times$ is decided; call this (W_k).

(b) The *witness digraph* $D_{\mathcal{C}}$ has the vertex set $\{1, \ldots, k\}$ and an arc $i \to l$ ($i \neq l$) whenever $(j_i, k_l) \in P$. If all its strongly connected components have at most two vertices, $\beta(\mathcal{C})$ denotes the number of components with two vertices.

(c) A *cover* is a pair $C_x \subseteq X \setminus V$, $C_z \subseteq Z \setminus W$ such that every pair of $P \cup F$ outside $V \times W$ has its x-endpoint in C_x or its x'-endpoint in C_z; call this (VC_k). The *integrated* vertices are $U_x = X \setminus (V \cup C_x)$ and $U_z = Z \setminus (W \cup C_z)$. The pair $(X \setminus V, Z \setminus W)$ is always a cover.

The partition lemma concerns only the combinatorics of the cross pairs, and we state it for an arbitrary finite set $E_\times \subseteq X \times Z$ of pairs, symmetric or not (the pinned rows of Section 16.5 need a set that is not symmetric). For such a set let r_j^X and r_k^Z be the numbers of its pairs with x-endpoint j and with x'-endpoint k; for spectral data both are the cross degrees r_j, r_k. Two pairs are *disjoint* if their x-endpoints differ and their x'-endpoints differ.

Lemma 16.2 (The compressed partition)**.** *Let $E_\times \subseteq X \times Z$ be finite, $m_\times = |E_\times|$, with a total order $\prec$.*

(a) *A subset of $E_\times$ contains no two disjoint pairs if and only if it is empty or a star. The empty set and the stars are*

$$1 + m_\times + \sum_j (2^{r_j^X} - 1 - r_j^X) + \sum_k (2^{r_k^Z} - 1 - r_k^Z)$$

subsets; so the masses of the classes of an exact partition of the nonstar subsets add up to $2^{m_\times}$ minus this number.

(b) *For a nonstar S let $e = (a, b)$ be its first pair. Exactly one of the following holds, and it defines a class $\mathcal{C}(S) = \mathcal{C}(P, A)$.*

(A) *Some pair of S is disjoint from e. Let $f = (a', b')$ be the first one, let* Con $= \{(a, b'), (a', b)\} \cap E_\times$ *be the* connectors*, and*

$$P = \{e, f\} \cup (S \cap \mathrm{Con}), \qquad A = \{g : g \prec e\} \cup \{g \prec f : g \text{ disjoint from } e\} \cup (\mathrm{Con} \setminus S).$$

(B) *No pair of S is disjoint from e. Then S contains, besides e, a pair with x-endpoint a and a pair with x'-endpoint b (the* arms *of e). Let $g_1 = (a, b'')$ and $g_2 = (a'', b)$ be the first arms of the two kinds, let A_1 (A_2) be the set of pairs $g \neq e$ with x-endpoint a (x'-endpoint b) and $g \prec g_1$ ($g \prec g_2$), and*

$$P = \{e, g_1, g_2\}, \qquad A = \{g : g \prec e\} \cup A_1 \cup A_2 \cup \{g : g \text{ disjoint from } e\}.$$

Every element of $\mathcal{C}(S)$ is a nonstar whose class is $\mathcal{C}(S)$. So the distinct classes $\mathcal{C}(S)$ form an exact partition of the nonstar subsets, the compressed partition*. In case* (A) *the pairs e, f, and in case* (B) *the arms g_1, g_2, form a witness of size two of $\mathcal{C}(S)$. Its witness digraph is a two-cycle if, in case* (A)*, both connectors are present, and it is acyclic otherwise.*

(c) *Let $\mathcal{C}(P, A)$ be a class and g a free pair. The* children *$\mathcal{C}(P \cup \{g\}, A)$ and $\mathcal{C}(P, A \cup \{g\})$ partition it; so replacing a class of an exact partition by its children, any number of times,*

gives an exact partition. A witness of $\mathcal{C}(P, A)$ is a witness of each child, with the same witness digraph, and a cover of $\mathcal{C}(P, A)$ is a cover of each child. If k pairs of P have distinct x-endpoints and distinct x'-endpoints, they form a witness of size k of every class obtained from $\mathcal{C}(P, A)$ by deciding, one at a time, the undecided pairs of their rectangle.

Proof. (a) Let S be nonempty without two disjoint pairs, and fix $e = (a, b) \in S$. Every other pair of S shares a or b with e. If S contained a pair (a, b_1) with $b_1 \neq b$ and a pair (a_1, b) with $a_1 \neq a$, these two would be disjoint. So all pairs of S have the x-endpoint a, or all have the x'-endpoint b. Conversely any two pairs of a star share an endpoint. A set of at least two pairs is not both a row star and a column star, since two pairs with the same two endpoints coincide, while a single pair is both. Counting the empty set, the $m_\times$ single pairs, the row stars of at least two pairs inside each row and the column stars of at least two pairs inside each column gives the number.

(b) By (a) a nonstar S contains two disjoint pairs, and so it has a first pair e and falls in exactly one of the two cases. In case (B) every pair of S shares an endpoint with e, and S is neither contained in the row of e nor in its column, which gives the arms; g_1 and g_2 are disjoint, and both follow e. In both cases $P \cap A = \emptyset$, $P \subseteq S$, and no pair of A lies in S: the pairs before e by the choice of e, the pairs disjoint from e before f by the choice of f, the arms before g_1, g_2 by the choice of the arms, and in case (B) the pairs disjoint from e by assumption. So $S \in \mathcal{C}(S)$. Let $S' \in \mathcal{C}(S)$. In case (A), $e \in S'$ and no pair before e lies in S', so e is the first pair of S'; $f \in S'$ is disjoint from e and no pair disjoint from e precedes it in S'; and $S' \cap \mathrm{Con} = S \cap \mathrm{Con}$. Hence S' is a nonstar in case (A) with the same e, f and connectors, and $\mathcal{C}(S') = \mathcal{C}(S)$. In case (B), e is again the first pair of S', no pair of S' is disjoint from e, and g_1, g_2 are the first arms of S'; S' is a nonstar because it contains the disjoint pairs g_1, g_2, and $\mathcal{C}(S') = \mathcal{C}(S)$. So two classes $\mathcal{C}(S_1)$, $\mathcal{C}(S_2)$ with a common element coincide, and the classes cover the nonstar subsets exactly once.

In case (A) take $(j_1, k_1) = e$ and $(j_2, k_2) = f$, so $V = \{a, a'\}$ and $W = \{b, b'\}$. The rectangle consists of e, f and the connectors, all decided; $D_\mathcal{C}$ has the arc $1 \to 2$ iff $(a, b') \in P$ and the arc $2 \to 1$ iff $(a', b) \in P$. In case (B) take $(j_1, k_1) = g_1 = (a, b'')$ and $(j_2, k_2) = g_2 = (a'', b)$, so $V = \{a, a''\}$ and $W = \{b'', b\}$. The rectangle consists of g_1, g_2, the present pair $e = (a, b)$, which gives the arc $1 \to 2$, and the pair (a'', b''), which is disjoint from e and therefore absent; there is no arc $2 \to 1$.

(c) The children are the elements of $\mathcal{C}(P, A)$ that contain g and those that do not. A child has more present and more absent pairs; the witness pairs stay present, the rectangle stays decided, and $P \cap (V \times W)$ does not change, because g was free and so lies outside the rectangle. The non-absent pairs of a child are non-absent in the parent, so (VC_k) passes to the child. Deciding the undecided pairs of a rectangle produces classes in which the rectangle is decided, which is (W_k). □

Remark 16.3 (Comments on the partition). (1) The arm case cannot be omitted: a three-edge path whose middle pair comes first contains no pair disjoint from that pair. In the arm case the first pair e is a present connector of the witness, not a witness pair. (2) A set $J \times K$ with $|J|, |K| \geq 2$ contains two disjoint pairs, so complete bipartite subsets lie in classes and need no separate estimate. (3) Refinement does not worsen the available bound: by (c) the witness and cover of a class remain valid at all its descendants, so the size-two bound is available for every class obtained by repeated refinement, and such a class may keep the smaller of its own bound and the sum of the bounds of its children. Larger witnesses are found along augmenting paths: if the present pairs contain a matching of size $k + 1$, the symmetric difference with a matching M of size k contains an augmenting path for M (Berge's lemma), of length one, three or five when $k = 2$. The search for witnesses is heuristic and lies outside the proof; a proposed witness is used only after (W_k) and the digraph condition have been verified. (4) Summed over a refinement of the compressed partition, the bounds for the classes at which refinement stops (the leaves) give the nonstar coefficients $C_k(Q)$ of Corollary 16.11(b). Every choice (the order, the pair to decide, the

witness, when to stop, the cover and the orders) affects only constants. The certificates make the choices once, in floating point at one prime, and fix them for all primes; their builders also check that the masses of the classes and the number in (a) add up to $2^{m\times}$, and that a refined union of classes carries the mass it replaces. These checks accompany the proof; they do not replace it. (In the data files, whose names are reproduced in Appendix D, the classes of case (B) are called type A and those of case (A) type B — the reverse of the letters in this text; the cases here are always referred to as (A) and (B).)

16.2. **Bounding a class by two designated pairs.** Fix a class $\mathcal{C}(P,A)$ and its designated witness pairs (Definition 16.1). After conditioning on all other labels, the kernel that remains on the witness labels is a product of characters $\chi(\pm(y_l-x_i))$ over a set $\mathcal{K}$ of pairs. The lemma bounds such a kernel: its compressed norm is at most $2^\beta r^k$ (k the number of witness labels, so second order as soon as $k\ge 2$), provided the directed graph of the pairs has no strongly connected component larger than a 2×2 rectangle, with β the number of two-vertex components. These components use the four-cycle bound of Section 15; larger components require a separate estimate.

Lemma 16.4 (Witness kernels)**.** *Let $k\ge 1$, let $\mathcal{K}\subseteq\{1,\dots,k\}^2$ contain the diagonal pairs (i,i), let $\varsigma_{il}\in\{\pm1\}$ for $(i,l)\in\mathcal{K}$, and let $T_\mathcal{K}$ be the operator on $L^2(\mathbb{F}_p^k)$, with uniform probability, given by*

$$(T_\mathcal{K}f)(x)=\mathbb{E}_{y\in\mathbb{F}_p^k}\Big[\prod_{(i,l)\in\mathcal{K}}\chi\big(\varsigma_{il}(y_l-x_i)\big)\,f(y)\Big].$$

Let $D_\mathcal{K}$ be the digraph on $\{1,\dots,k\}$ with an arc $i\to l$ for every $(i,l)\in\mathcal{K}$ with $i\ne l$.

(a) *If $D_\mathcal{K}$ is acyclic, then $\|T_\mathcal{K}\|\le r^k$.*

(b) *Suppose that every strongly connected component of $D_\mathcal{K}$ has at most two vertices and that β of them have two. Let P_{in} and P_{out} be the multiplications by the indicators of sets of tuples y, respectively x, with $y_i\ne y_l$, respectively $x_i\ne x_l$, for every two-vertex component $\{i,l\}$ (for instance the injective tuples). Then $\|P_{\mathrm{out}}T_\mathcal{K}P_{\mathrm{in}}\|\le 2^\beta r^k$.*

(c) *$D_\mathcal{K}$ is acyclic if and only if the identity is the only permutation π of $\{1,\dots,k\}$ with $(i,\pi(i))\in\mathcal{K}$ for all i, that is, if and only if the diagonal is the only perfect matching contained in $\mathcal{K}$.*

(d) Inserted multipliers. *Let $k=2$ and $\mathcal{K}=\{(1,1),(2,2)\}$, and let $F_{\mathrm{out}}(x_1,x_2)$, $F_{\mathrm{mid}}(x_1,y_2)$ and $F_{\mathrm{in}}(y_1,y_2)$ be bounded functions. The operator on $L^2(\mathbb{F}_p^2)$ with kernel*

$$\chi\big(\varsigma_{11}(y_1-x_1)\big)\,\chi\big(\varsigma_{22}(y_2-x_2)\big)\,F_{\mathrm{out}}(x_1,x_2)\,F_{\mathrm{mid}}(x_1,y_2)\,F_{\mathrm{in}}(y_1,y_2)$$

has norm at most $r^2\,\|F_{\mathrm{out}}\|_\infty\,\|F_{\mathrm{mid}}\|_\infty\,\|F_{\mathrm{in}}\|_\infty$, and so has the operator with $F_{\mathrm{mid}}(y_1,x_2)$ in place of $F_{\mathrm{mid}}(x_1,y_2)$.

Proof. Let $C_1,\dots,C_{n_c}$ be the strongly connected components of $D_\mathcal{K}$, ordered so that every arc between two components goes from an earlier to a later one. Consider functions of mixed coordinates: x_i for i in the components already treated and y_i for the others, with uniform probability. Step s replaces the coordinates y_{C_s} by x_{C_s}:

$$g\longmapsto M_s\cdot B_sg,\qquad B_sg=\mathbb{E}_{y_{C_s}}\Big[\prod_{(i,l)\in\mathcal{K}\cap C_s^2}\chi\big(\varsigma_{il}(y_l-x_i)\big)\,g\Big],\qquad M_s=\prod_{(i,l)\in\mathcal{K}_s}\chi\big(\varsigma_{il}(y_l-x_i)\big),$$

where $\mathcal{K}_s$ is the set of $(i,l)\in\mathcal{K}$ with $i\in C_s$ and l in a later component. When step s is performed the coordinates x_{C_s} have just been created and every y_l with l in a later component is still present, so M_s is a function of the current coordinates. Every pair of $\mathcal{K}$ is used exactly once: a pair inside a component in its block B_s, and a pair between two components in M_s for the component C_s of its first index, since its second index lies in a later component. Hence $T_\mathcal{K}=M_{n_c}B_{n_c}\cdots M_1B_1$. Each M_s is a multiplication by a function of modulus at most one, and B_s is the operator of the block C_s tensored with the identity in the other coordinates, which has the same norm.

(a) If $D_{\mathcal{K}}$ is acyclic, every component is a single vertex i, and B_s is $h \mapsto \mathbb{E}_y \chi(\varsigma_{ii}(y-x))h(y) = \chi(\varsigma_{ii})Ch$ in the coordinate i, with the Gauss operator C of Lemma 14.1, of norm r. So $\|T_{\mathcal{K}}\| \le r^k$.

(b) For a two-vertex component $C_s = \{i,l\}$, strong connectivity gives both arcs, so the kernel of B_s is the complete rectangle $\prod_{a,b\in\{i,l\}} \chi(\varsigma_{ab}(y_b - x_a))$. Let Q_{in} and Q_{out} be the multiplications by $\mathbf{1}[y_i \ne y_l]$ and by $\mathbf{1}[x_i \ne x_l]$. The steps before s act on other coordinates or multiply pointwise, so they commute with Q_{in}, and $Q_{\mathrm{in}}P_{\mathrm{in}} = P_{\mathrm{in}}$. The steps after s, and M_s, act on other coordinates or multiply pointwise, so they commute with Q_{out}, and $P_{\mathrm{out}}Q_{\mathrm{out}} = P_{\mathrm{out}}$. Hence Q_{in} may be inserted immediately before B_s and Q_{out} immediately after it without changing $P_{\mathrm{out}}T_{\mathcal{K}}P_{\mathrm{in}}$, for every two-vertex component. By Corollary 15.9(a) with $a = b = 2$, $\|Q_{\mathrm{out}}B_sQ_{\mathrm{in}}\| \le 2/p = 2r^2$, and the singleton blocks have norm r as in (a). Multiplying, $\|P_{\mathrm{out}}T_{\mathcal{K}}P_{\mathrm{in}}\| \le r^{k-2\beta}(2r^2)^\beta = 2^\beta r^k$.

(c) If $i_1 \to i_2 \to \cdots \to i_\ell \to i_1$ is a directed cycle with distinct vertices, the permutation with $\pi(i_s) = i_{s+1}$ (indices modulo ℓ) and $\pi(i) = i$ otherwise is a second perfect matching. Conversely, if $\pi \ne \mathrm{id}$ is a perfect matching in $\mathcal{K}$, every i with $\pi(i) \ne i$ gives an arc $i \to \pi(i)$, and following a cycle of π gives a directed cycle.

(d) Let M_F be the multiplication by F, and let $C_\varsigma \otimes I$ and $I \otimes C_\varsigma$ be the Gauss operator $(C_\varsigma h)(a) = \mathbb{E}_b \chi(\varsigma(b-a))h(b)$ of Lemma 14.1 acting in the first and in the second coordinate. Applied to $f(y_1, y_2)$, the operator multiplies by F_{in}, replaces y_1 by x_1 through the kernel $\chi(\varsigma_{11}(y_1 - x_1))$, multiplies by $F_{\mathrm{mid}}(x_1, y_2)$, replaces y_2 by x_2 through $\chi(\varsigma_{22}(y_2 - x_2))$, and multiplies by F_{out}: it is $M_{F_{\mathrm{out}}}(I \otimes C_{\varsigma_{22}})M_{F_{\mathrm{mid}}}(C_{\varsigma_{11}} \otimes I)M_{F_{\mathrm{in}}}$. With $F_{\mathrm{mid}}(y_1, x_2)$ the two transforms are applied in the other order. Each Gauss operator has norm r, and each multiplication has norm at most the supremum of its function. □

Remark 16.5 (The witness kernels that occur). For a witness of size two the rectangle is a matching, a three-edge path (one connector present) or a complete rectangle (both present). The first two are acyclic, so their kernels have norm at most $r^2 = 1/p$ for independent uniform labels, with no injectivity condition (the *incomplete-rectangle bound*). The complete rectangle has norm at most $2r^2$ on distinct pairs; numerical evidence for the sharpness of 2 is discussed in Remark 15.11. A witness of size three or four found along an augmenting path is used only if its digraph has no strongly connected component of three or more vertices in the selected partition. A complete three-vertex component may instead be priced at $6r^3$ by Theorem 15.12, with internal distinctness on both sides. This extension does not apply to an incomplete three-vertex component. The original criterion holds, for instance, when the connectors between a new witness pair and the others all point one way: the new vertex then lies on no directed cycle, and the kernel factors through one more Gauss operator. Part (d) keeps inside the norm a bounded multiplier that depends on one endpoint of each of the two pairs of a matching; a third character $\chi(\varsigma_{12}(y_2 - x_1))$ or $\chi(\varsigma_{21}(y_1 - x_2))$ has modulus at most one and may be absorbed into F_{mid}, so the three-edge paths are covered too. In the applications the other coordinates of a larger witness are fixed first and the multipliers may depend on them; the score multipliers of the positive roots are of this form (Section 21). No Schur-multiplier bound for a signed kernel is used.

16.3. **A bound for each class $\mathcal{C}(P, A)$.** A class is bounded in four steps. (1) Its character sum is written as the witness kernel times the nonnegative factors $1 + \chi_e$ of the free pairs. (2) One conditions on the labels of a vertex cover. (3) The witness labels then have a bounded density, and their kernel is bounded by Lemma 16.4. (4) The remaining labels are integrated one vertex at a time, each contributing its vertex factor F_v of Definition 16.6, with an additional second-order term when a label coincides with a fixed label of the other row. The definitions below record exactly the choices (witness, cover, orders) and the factor contributed by each non-witness vertex.

Definition 16.6 (Records of a class (extraction data)). Let $\mathcal{C} = \mathcal{C}(P, A)$ be a class with free pairs F, a witness of size k with sets $V \subseteq X$, $W \subseteq Z$, and a cover (C_x, C_z) with integrated sets U_x, U_z (Definition 16.1). Fix orders of the four sets U_x, U_z, C_x, C_z. Witness, cover and orders form a

record of $\mathcal{C}$ (extraction data, not a record in the sense of Section 1.8). For a *non-witness vertex* $v \in (X \setminus V) \cup (Z \setminus W)$ define t_v, the set H_v of its *free partners*, $h_v = |H_v|$ and $s_v = t_v + h_v$:

- $u \in U_x$: t_u is the number of G^+-neighbours of u in $V \cup C_x$ or earlier in U_x, and $H_u = \{b : (u,b) \in F\}$;
- $u \in U_z$: t_u is the number of G^+-neighbours of u in $W \cup C_z$ or earlier in U_z, and $H_u = \{a : (a,u) \in F\}$;
- $b \in C_z$: t_b is the number of G^+-neighbours of b earlier in C_z, and $H_b = \{a : (a,b) \in F\}$ (all free pairs at b, also those from V);
- $a \in C_x$: t_a is the number of G^+-neighbours of a earlier in C_x, and $H_a = \{b : (a,b) \in F\}$.

Viewed in U^+, $H_v \subseteq N(v)$, by the symmetry of N; and by (VC_k), $H_u \subseteq C_z$ for $u \in U_x$ and $H_u \subseteq C_x$ for $u \in U_z$. Thus $t_v \le d_v^0$, $h_v \le r_v$ and $s_v \le d_v \le \Delta$. Put $t_V = e(V) + e(V, C_x)$ and $t_W = e(W) + e(W, C_z)$, let f_V and f_W be the numbers of free pairs with x-endpoint in V and with x'-endpoint in W, and let $\mathrm{ex}(\mathcal{C}) = t_V + t_W + f_V + f_W$ be the *exponent* of the record. With the *merge counts* (for the merged roots of Lemma 16.7 below, not the merges of Section 17)

$$\mathrm{mg}(j,i) = \#\{i' \in \mathrm{Adj}(i) : j \in N(i')\}, \qquad c^U(j,i) = \#\{u \in \mathrm{Adj}(j) : u \in N(i)\} \qquad (i,j \in U^+),$$

the *vertex factors* of the record are, with Iverson brackets,

$$F_u = \frac{1 + e_{s_u}(p,n) + r^2 2^{s_u} \sum_{b \in C_z} 2^{c^U(u,b) - [b \in H_u]}}{1 - e_{t_u}(p,n)} \qquad (u \in U_x),$$

$$F_u = \frac{1 + e_{s_u}(p,n) + r^2 2^{s_u} \sum_{a \in C_x} 2^{c^U(u,a) - [a \in H_u]}}{1 - e_{t_u}(p,n)} \qquad (u \in U_z),$$

$$F_b = \frac{1 + e_{s_b}(p,2n) + r^2 2^{s_b} \sum_{j \in U^+} 2^{\mathrm{mg}(j,b) + c^U(j,b) - [j \in H_b]}}{1 - e_{t_b}(p,n)} \qquad (b \in C_x \cup C_z).$$

The brackets record that the free factor of a pair whose two labels coincide is $1 + \chi(0) = 1$. There are exactly $2(n-k)$ non-witness vertices, hence $2(n-k)$ vertex factors. A class may carry several records; each choice is made once, independently of p.

Lemma 16.7 (Counts with merged roots)**.** *Let $\beta_1, \dots, \beta_t \in \mathbb{F}_p$ be distinct, let $\gamma_1, \dots, \gamma_h \in \mathbb{F}_p$ be distinct, let $s_1, \dots, s_t, s'_1, \dots, s'_h \in \{\pm 1\}$, and let $\mathcal{X} \subseteq \mathbb{F}_p$ contain all β_i and γ_l. Let $\mu_{\mathrm{m}} = \#\{(i,l) : \beta_i = \gamma_l\}$, the number of* merged *roots. Then for every $n' \ge 0$*

$$\sum_{\substack{y \in \mathbb{F}_p \setminus \mathcal{X} \\ \chi(y - \beta_i) = s_i \ \forall i}} \prod_{l=1}^{h} (1 + s'_l \chi(y - \gamma_l)) \ \le \ p\, 2^{-t} 2^{\mu_{\mathrm{m}}} (1 + e_{t+h-\mu_{\mathrm{m}}}(p,n')) \ \le \ p\, 2^{-t} 2^{\mu_{\mathrm{m}}} (1 + e_{t+h}(p,n')).$$

Proof. For $y \notin \mathcal{X}$ every factor $1 + s'_l \chi(y - \gamma_l)$ is 0 or 2, so the sum is 2^h times the number of $y \notin \mathcal{X}$ satisfying the $t + h$ conditions $\chi(y - \beta_i) = s_i$ and $\chi(y - \gamma_l) = s'_l$. Since the β_i are distinct and the γ_l are distinct, the merged roots pair one β_i with one γ_l. Two conditions at the same root are identical or incompatible; in the second case the sum is 0. Otherwise the conditions are $t + h - \mu_{\mathrm{m}}$ conditions with distinct roots, all in $\mathcal{X}$, and by Lemma 12.2(b) and (E2) their solution set has at most $p 2^{-(t+h-\mu_{\mathrm{m}})}(1 + e_{t+h-\mu_{\mathrm{m}}}(p,n'))$ elements. Multiply by 2^h; the second inequality holds because $e_t(p,n')$ is nondecreasing in t (E3). □

Lemma 16.8 (Cover rows)**.** *Assume* (S1) *at p. Let $C \subseteq U^+$ be a set of x'-vertices with an order, let $x \in \mathcal{I}^+$ and $H \subseteq \{(a,b) \in E_\times : b \in C\}$. For $b \in C$ let t_b be the number of G^+-neighbours of b earlier in C, $H_b = \{a : (a,b) \in H\}$, $h_b = |H_b|$, $s_b = t_b + h_b$, and let $F_b(H)$ be the expression of F_b in Definition 16.6 with these counts and these sets H_b. Let μ_C be the law of x'_C for $x' \sim \mu$, and*

$\kappa_C(x,c') = \sum_{b\in C}\sum_{i\in U^+} c^U(i,b)\mathbf{1}[c'_b = x_i]$. *Then*

$$\mathbb{E}_{c'\sim\mu_C}\Big[\prod_{(a,b)\in H}\big(1+\chi(\varsigma_{ab}(c'_b-x_a))\big)\,2^{\kappa_C(x,c')}\Big] \;\le\; R_{U^+\setminus C}\prod_{b\in C}F_b(H).$$

Proof. Call the integrand $\Pi(c') \ge 0$.

Step 1 (reference law). Let $\lambda_C = \lambda_p(G^+[C])$. By Lemma 12.6(d) the density of μ_C with respect to λ_C takes values in an interval $[c_0, R_{U^+\setminus C}c_0]$; since it has mean one, $c_0 \le 1$ and the density is at most $R_{U^+\setminus C}$. So it suffices to show $\mathbb{E}_{\lambda_C}\Pi \le \prod_{b\in C}F_b(H)$.

Step 2 (lower count). By Lemma 12.5(a) with the error pair of the setting, no anchors and the given order,

$$|\operatorname{Inj}_p(G^+[C])| \ge \prod_{b\in C} p2^{-t_b}\big(1-e_{t_b}(p,n)\big).$$

Write $\Pi(c') = \prod_b \omega_b(c'_b)$ with

$$\omega_b(y) = \prod_{a\in H_b}\big(1+\chi(\varsigma_{ab}(y-x_a))\big)\cdot 2^{\sum_i c^U(i,b)\mathbf{1}[y=x_i]}.$$

Step 3 (merge weights). Call b *coincident* if $c'_b = x_{j(b)}$ for some (unique) $j(b)\in U^+$. A *merge at* b is a pair (b',a) with b' earlier than b, $b'\sim b$, $a\in H_b$ and $c'_{b'} = x_a$; let $\mu_b = \mu_b(c'_{<b})$ be their number. If (b',a) is a merge at b, then b' is coincident with $j(b') = a$, and $b\in\operatorname{Adj}(b')$ with $j(b')\in N(b)$. So a fixed b' causes at most $\operatorname{mg}(j(b'),b')$ merges, and $\sum_b\mu_b \le \sum_{b'\text{ coincident}}\operatorname{mg}(j(b'),b')$. With $\hat\omega_b(y) = \omega_b(y)2^{\operatorname{mg}(j,b)}$ if $y = x_j$ for some $j\in U^+$, and $\hat\omega_b(y) = \omega_b(y)$ otherwise,

$$\Pi(c') \le \prod_b \hat\omega_b(c'_b)\,2^{-\mu_b(c'_{<b})}.$$

Step 4 (one vertex). Fix b and the earlier labels $c'_{<b}$, and let Adm_b be the set of y distinct from the earlier labels that satisfy the t_b constraints of G^+ with the earlier neighbours of b. We claim

$$\sum_{y\in\operatorname{Adm}_b}\hat\omega_b(y)\,2^{-\mu_b} \le p2^{-t_b}\Big(1+e_{s_b}(p,2n)+r^2 2^{s_b}\sum_{j\in U^+}2^{\operatorname{mg}(j,b)+c^U(j,b)-[j\in H_b]}\Big). \tag{16.1}$$

For $y\notin x(U^+)$, $\hat\omega_b(y) = \prod_{a\in H_b}(1+\chi(\varsigma_{ab}(y-x_a)))$. The internal roots (the labels of the earlier neighbours) are distinct, the cross roots x_a ($a\in H_b$) are distinct, and they coincide exactly at the μ_b merges. With $\mathcal{X} = x(U^+)\cup\{c'_{b'} : b' \text{ earlier}\}$, Lemma 16.7 bounds the sum of $\hat\omega_b(y)$ over these values by $p2^{-t_b}2^{\mu_b}(1+e_{s_b}(p,2n))$, so their contribution to the left side of (16.1) is at most $p2^{-t_b}(1+e_{s_b}(p,2n))$. For $y = x_j$ the factor of $a = j$, if $j\in H_b$, is $1+\chi(0) = 1$, the other factors are at most 2, and x is injective; so $\hat\omega_b(x_j)2^{-\mu_b} \le 2^{h_b-[j\in H_b]+c^U(j,b)+\operatorname{mg}(j,b)} = p2^{-t_b}r^2 2^{s_b+\operatorname{mg}(j,b)+c^U(j,b)-[j\in H_b]}$, because $pr^2 = 1$. Summing over $j\in U^+$ proves (16.1).

Step 5 (product). The right side of (16.1) does not depend on $c'_{<b}$, and μ_b depends only on $c'_{<b}$. Summing the bound of Step 3 over $c'\in\operatorname{Inj}_p(G^+[C])$, successively over the labels in reverse order and using the uniform bound (16.1) for each label given the earlier ones, gives $\sum_{c'}\Pi(c') \le \prod_b p2^{-t_b}(1+e_{s_b}(p,2n)+r^2 2^{s_b}\sum_j 2^{\operatorname{mg}(j,b)+c^U(j,b)-[j\in H_b]})$, and dividing by the count of Step 2 gives $\mathbb{E}_{\lambda_C}\Pi \le \prod_b F_b(H)$. □

Theorem 16.9 (Grouped extraction with a witness of size k)**.** *Assume* (S1) *at* p. *Let* $\mathcal{C} = \mathcal{C}(P,A)$ *be a class with a record (Definition 16.6) whose witness has size* k, *and suppose that the kernel*

$$\mathsf{W}(v,w) = \prod_{(a,b)\in P\cap(V\times W)}\chi\big(\varsigma_{ab}(w_b-v_a)\big)\qquad (v\in\mathbb{F}_p^V,\ w\in\mathbb{F}_p^W)$$

of its present rectangle pairs, compressed to injective tuples, has norm at most $c_{\ker}r^k$ *for uniform probability; by Lemma 16.4,* $c_{\ker} = 2^{\beta(\mathcal{C})}$ *when every strongly connected component of* $D_{\mathcal{C}}$ *has at*

most two vertices. For either integrated set put $\widetilde{R}_U = \prod_{u\in U}(1+e_{t_u}(p,n))/(1-e_{t_u}(p,n))$*, where* t_u *is its actual internal insertion degree. Then for all* $f,h\in L^2(\mu)$

$$\Big|\sum_{S\in\mathcal{C}}\langle f, T_S h\rangle_\mu\Big| \le c_{\ker}\, r^k\, 2^{\mathrm{ex}(\mathcal{C})/2}\,\frac{\sqrt{\widetilde{R}_{U_x}\widetilde{R}_{U_z}}}{(1-\hat{e})^k}\prod_{v\in(X\setminus V)\cup(Z\setminus W)} F_v^{1/2}\ \|f\|_\mu\,\|h\|_\mu .$$

Proof. Let E_{VW} be the rectangle and F the free pairs.

Step 1 (grouping). The sets of $\mathcal{C}$ are the sets $P\cup S'$ with $S'\subseteq F$, so $\sum_{S\in\mathcal{C}}\chi_S = \prod_{e\in P}\chi_e\prod_{e\in F}(1+\chi_e) = \prod_{e\in P\cup F}\psi_e$, with $\psi_e=\chi_e$, $|\psi_e|\le 1$, for $e\in P$ and $\psi_e = 1+\chi_e\in[0,2]$ for $e\in F$. By (W_k) no pair of E_{VW} is free. Let (a,b) be a non-absent pair outside E_{VW}. If $a\in V\cup U_x$, then $a\notin C_x$ and $(a,b)\notin V\times W$, so $b\in C_z$ by (VC_k). Hence every such pair lies in exactly one of the groups

$$\mathcal{G}_1:\ a\in V\cup U_x,\ b\in C_z;\qquad \mathcal{G}_2:\ a\in C_x,\ b\in W\cup U_z;\qquad \mathcal{G}_\Gamma:\ a\in C_x,\ b\in C_z,$$

and $\sum_{S\in\mathcal{C}}\chi_S = \mathsf{W}(v,w)\,\Theta_1\Theta_2\Gamma$ with $v=x_V$, $w=x'_W$ and Θ_1,Θ_2,Γ the products of ψ_e over $\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_\Gamma$. Here Θ_1 depends on $x_{V\cup U_x}$ and x'_{C_z}, Θ_2 on x_{C_x} and $x'_{W\cup U_z}$, and Γ on x_{C_x} and x'_{C_z}. Let Θ_1^+ and Γ^+ be the products of $1+\chi_e$ over the free pairs of $\mathcal{G}_1$ and of $\mathcal{G}_\Gamma$; then $|\Theta_1|\le\Theta_1^+$ and $|\Gamma|\le\Gamma^+$. A free pair with x-endpoint in V lies in $\mathcal{G}_1$, so $\Theta_1^+ = \Theta_V^+\Theta_U^+$ with $\Theta_V^+ = \prod_{(a,b)\in F,\,a\in V}(1+\chi_{ab})\le 2^{f_V}$ and $\Theta_U^+ = \prod_{(u,b)\in F,\,u\in U_x}(1+\chi_{ub})$.

Step 2 (conditioning). Put $c=x_{C_x}$ and $c'=x'_{C_z}$. Under $\mu\otimes\mu$ the labellings x and x' are independent, so given (c,c') the pairs (x_V,x_{U_x}) and (x'_W,x'_{U_z}) are independent. Let ν_V and ν_W be the conditional laws of x_V given c and of x'_W given c' under μ, and put $\mathsf{a}(v)=\mathbb{E}[f(x)\Theta_1\mid x_V=v,c,c']$ and $\mathsf{b}(w)=\mathbb{E}[\Theta_2 h(x')\mid x'_W=w,c,c']$. The characters being real,

$$\sum_{S\in\mathcal{C}}\langle f,T_S h\rangle_\mu = \mathbb{E}_{\mu\otimes\mu}\Big[\overline{f(x)}\sum_{S\in\mathcal{C}}\chi_S\,h(x')\Big] = \mathbb{E}_{c,c'}\Big[\Gamma\,\mathbb{E}_{v\sim\nu_V,\,w\sim\nu_W}\big[\overline{\mathsf{a}(v)}\,\mathsf{W}(v,w)\,\mathsf{b}(w)\big]\Big].$$

Step 3 (density together with its cover marginal). Keep the actual conditional density supremum $D_V(c)=\sup_v\tilde{\nu}_V(v\mid c)$ as a function of the cover labels. Write $N_C(c)$ for the number of completions on $V\sqcup U_x$, $N_U(v,c)$ for the number on U_x, and λ_C for the uniform strict labelling law on C_x. Exact counting gives

$$\frac{D_V(c)\mu_{C_x}(c)}{\lambda_C(c)} = \frac{p^k|\mathcal{I}_{C_x}|\max_v N_U(v,c)}{|\mathcal{I}^+|}.$$

All cover labellings extend by (S1). Inserting V after C_x gives at least $p^k 2^{-t_V}(1-\hat{e})^k$ choices. The subsequent insertion of U_x gives lower and upper products $\prod_u p2^{-t_u}(1-e_{t_u})$ and $\prod_u p2^{-t_u}(1+e_{t_u})$. Thus, as measures,

$$D_V\,d\mu_{C_x}\le\frac{2^{t_V}\widetilde{R}_{U_x}}{(1-\hat{e})^k}\,d\lambda_{C_x}. \tag{16.2}$$

The same assertion holds for $D_W(c')$ and C_z.

Step 4 (the witness kernel). The functions $\mathsf{a}\tilde{\nu}_V$ and $\mathsf{b}\tilde{\nu}_W$ vanish off injective tuples, so the kernel hypothesis and $\mathbb{E}|\mathsf{a}|^2\tilde{\nu}_V^2\le D_V(c)\mathbb{E}|\mathsf{a}|^2\tilde{\nu}_V$ (expectations for uniform probability) give $|\mathbb{E}_{\nu_V\otimes\nu_W}[\bar{\mathsf{a}}\mathsf{W}\mathsf{b}]| = |\,\langle \mathsf{a}\tilde{\nu}_V, T_{\mathsf{W}}(\mathsf{b}\tilde{\nu}_W)\rangle\,|\le c_{\ker}r^k\sqrt{D_V(c)D_W(c')}\,\|\mathsf{a}\|_{\nu_V}\,\|\mathsf{b}\|_{\nu_W}$. Hence

$$\Big|\sum_{S\in\mathcal{C}}\langle f,T_S h\rangle_\mu\Big|\le c_{\ker}r^k\mathbb{E}_{c,c'}\big[|\Gamma|\sqrt{D_V(c)D_W(c')}\ \|\mathsf{a}\|_{\nu_V}\,\|\mathsf{b}\|_{\nu_W}\big]. \tag{16.3}$$

Step 5 (Cauchy–Schwarz in the integrated variables). With the weight $|\Theta_1|\le\Theta_1^+$, $|\mathsf{a}(v)|^2\le\mathbb{E}[|f|^2\Theta_1^+\mid v,c,c']\,\mathbb{E}[\Theta_1^+\mid v,c,c']$. Given (v,c) the labels x_{U_x} are uniform on the set $\mathcal{Y}(v,c)$ of completions of (v,c) to an element of $\mathcal{I}^+$, so

$$\mathbb{E}[\Theta_1^+\mid v,c,c'] = \Theta_V^+\,\mathsf{m}(v,c,c')\le 2^{f_V}\mathsf{m}(v,c,c'),\qquad \mathsf{m}(v,c,c')=\frac{1}{|\mathcal{Y}|}\sum_{y\in\mathcal{Y}}\Theta_U^+(y,c').$$

Step 6 (the integrated vertices). Label U_x in its order, writing y_u for the label of u. For $u \in U_x$ the free partners are $H_u = \{b : (u,b) \in F\}$; a free pair (u,b) is not in E_{VW} and $u \notin C_x$, so $H_u \subseteq C_z$ by (VC_k), and $\Theta_U^+ = \prod_u w_u(y_u)$ with $w_u(y) = \prod_{b\in H_u}(1+\chi(\varsigma_{ub}(c'_b - y)))$. A *merge at* u is a pair (i,b) with $i \in \mathrm{Adj}(u)$ labelled before u (so $i \in V \cup C_x$, or $i \in U_x$ earlier than u), $b \in H_u$, and label of i equal to c'_b. Let κ_u^f count the merges at u with $i \in V \cup C_x$, which are determined by (v,c,c'), let $\kappa_u^r(y_{<u})$ count those with $i \in U_x$, and $\kappa^f = \sum_u \kappa_u^f$. Put $\hat w_u(y) = w_u(y)2^{c^U(u,b)}$ if $y = c'_b$ for some $b \in C_z$ (unique, as c' is injective) and $\hat w_u(y) = w_u(y)$ otherwise. If (i,b) is a merge at u with $i \in U_x$, then $y_i = c'_b$, $u \in \mathrm{Adj}(i)$ and $u \in N(b)$; so an $i \in U_x$ with $y_i = c'_b$ causes at most $c^U(i,b)$ merges, and $\prod_u w_u(y_u) \le \prod_u \hat w_u(y_u) 2^{-\kappa_u^r(y_{<u})}$.

Fix u and the earlier labels, and sum over the admissible values y of y_u (distinct from the labels already used, satisfying the t_u constraints with the labelled neighbours of u). For the *generic values* $y \notin \{c'_b : b \in C_z\}$, $\hat w_u(y) = w_u(y)$; the internal roots (labels of the labelled neighbours) and the cross roots c'_b ($b \in H_u$) merge exactly at the $\kappa_u^f + \kappa_u^r$ merges at u. With $\mathcal{X}$ the set of labels already used together with $\{c'_b : b \in C_z\}$, Lemma 16.7 bounds the sum of $\hat w_u(y)$ over these values by $p2^{-t_u}2^{\kappa_u^f+\kappa_u^r}(1+e_{s_u}(p,n))$. For the *special values* $y = c'_b$, $b \in C_z$, every factor of $w_u(c'_b)$ is at most 2, and that of b itself, if $b \in H_u$, is $1+\chi(0) = 1$; so $\hat w_u(c'_b) \le 2^{h_u - [b\in H_u] + c^U(u,b)} = p2^{-t_u}r^2 2^{s_u + c^U(u,b) - [b\in H_u]}$; distinct b give distinct values. Hence, uniformly in the earlier labels,

$$\sum_y \hat w_u(y)\, 2^{-\kappa_u^r} \le p2^{-t_u}2^{\kappa_u^f}\Big(1 + e_{s_u}(p,n) + r^2 2^{s_u}\sum_{b\in C_z} 2^{c^U(u,b)-[b\in H_u]}\Big).$$

Summing over $y \in \mathcal{Y}$ with the last vertex first bounds $\sum_{y\in\mathcal{Y}}\Theta_U^+(y,c')$ by the product of these bounds. Since (v,c) is feasible, Lemma 12.5(a) with the error pair of the setting, the anchors $V \cup C_x$ and the order of U_x gives $|\mathcal{Y}| \ge \prod_{u\in U_x} p2^{-t_u}(1-e_{t_u}(p,n))$. Dividing,

$$\mathsf{m}(v,c,c') \le 2^{\kappa^f} \prod_{u\in U_x} F_u. \tag{16.4}$$

Finally $\kappa^f \le \kappa_{C_z}(x,c')$ for every completion x of (v,c), with κ_{C_z} of Lemma 16.8: a pair (i,b) with $x_i = c'_b$ is a merge at most at the $c^U(i,b)$ vertices of $\mathrm{Adj}(i) \cap N(b)$.

Step 7 (the cover rows). Apply Cauchy–Schwarz to (16.3), placing $D_W(c')$ with $\|\mathsf{a}\|_{\nu_V}^2$ and $D_V(c)$ with $\|\mathsf{b}\|_{\nu_W}^2$. By Steps 5 and 6, since Γ^+ depends only on (c,c'), $|\Gamma|\,|\mathsf{a}(v)|^2 \le 2^{f_V}\prod_{u\in U_x} F_u\,\mathbb{E}[|f(x)|^2\Theta_1^+\Gamma^+ 2^{\kappa_{C_z}(x,c')} \mid v,c,c']$. Integrating over $v \sim \nu_V$ and over $(c,c') \sim \mu_{C_x}\otimes\mu_{C_z}$, under which $x \sim \mu$ and $c' \sim \mu_{C_z}$ are independent,

$$\mathbb{E}_{c,c'}\big[|\Gamma| D_W(c')\,\|\mathsf{a}\|_{\nu_V}^2\big] \le 2^{f_V}\prod_{u\in U_x} F_u\,\mathbb{E}_{x\sim\mu}\Big[|f(x)|^2\,\mathbb{E}_{c'\sim\mu_{C_z}}\big[D_W(c')\Theta_1^+\Gamma^+ 2^{\kappa_{C_z}(x,c')}\big]\Big].$$

The product $\Theta_1^+\Gamma^+ = \Theta_V^+\Theta_U^+\Gamma^+$ runs over exactly the free pairs with x'-endpoint in C_z, their x-endpoints lying in V, U_x or C_x. Lemma 16.8 with $C = C_z$ in its order and $H = \{(a,b) \in F : b \in C_z\}$, whose counts and sets H_b are those of Definition 16.6, is applied after (16.2) has replaced $D_W\,d\mu_{C_z}$ by its multiple of $d\lambda_{C_z}$. Only Steps 2–5 of that row lemma are then needed; its separate marginal comparison is omitted. The inner expectation is at most $2^{t_W}\widetilde R_{U_z}(1-\hat e)^{-k}\prod_{b\in C_z} F_b$. Consequently

$$\mathbb{E}_{c,c'}[|\Gamma| D_W\,\|\mathsf{a}\|_{\nu_V}^2] \le \frac{2^{f_V+t_W}\widetilde R_{U_z}}{(1-\hat e)^k}\prod_{u\in U_x} F_u \prod_{b\in C_z} F_b\,\|f\|_\mu^2.$$

Step 8 (the other side, and conclusion). The map $(x,x') \mapsto (x',x)$ preserves $\mu\otimes\mu$, and $\chi_{(a,b)}(x,x') = \chi(\varsigma_{ab}(x'_b - x_a)) = \chi(\varsigma_{ba}(x_a - x'_b)) = \chi_{(b,a)}(x',x)$ by Lemma 13.2. With the two labellings exchanged, b is therefore the function a of the transposed class $\mathcal{C}(P^{\mathsf T}, A^{\mathsf T})$, $P^{\mathsf T} = \{(b,a) : (a,b) \in P\}$, of the same spectral data, with h in place of f; its record is the transposed witness with the same cover and orders, (W, C_z, U_z) playing the role of (V, C_x, U_x). Since both labellings carry

the half graph G^+ and N is symmetric, the counts and factors of the transposed record are those defined in Definition 16.6 for U_z and C_x. Steps 5–7 for the transposed class give $\mathbb{E}_{c,c'}[|\Gamma|D_V\,\|\mathsf{b}\|_{\nu_W}^2]$ bounded by $2^{f_W+t_V}\widetilde{R}_{U_x}(1-\hat{e})^{-k}\prod_{u\in U_z}F_u\prod_{a\in C_x}F_a\;\|h\|_\mu^2$. The cross-weighted Cauchy–Schwarz inequality of Step 7 gives

$$\Big|\sum_{S\in\mathcal{C}}\langle f,T_Sh\rangle_\mu\Big|\le c_{\ker}r^k\,2^{\mathrm{ex}(\mathcal{C})/2}\frac{\sqrt{\widetilde{R}_{U_x}\widetilde{R}_{U_z}}}{(1-\hat{e})^k}\prod_{v\in(X\setminus V)\sqcup(Z\setminus W)}F_v^{1/2}\,\|f\|_\mu\,\|h\|_\mu\,.$$

Here $\mathrm{ex}=t_V+t_W+f_V+f_W$. The vertex factors left are those of $U_x\sqcup C_z\sqcup U_z\sqcup C_x=(X\setminus V)\sqcup(Z\setminus W)$. This is the claim. Since each insertion degree is at most its full internal degree, $\sqrt{\widetilde{R}_{U_x}\widetilde{R}_{U_z}}\le R_{\mathrm{full}}$. In particular the earlier, coarser factor R_{full}^2 remains valid for the retained leaf and histogram certificates below. □

Remark 16.10 (What the proof uses). (1) Only Steps 3 and 4 see the size k of the witness, through the density of the witness labels and the norm of the witness kernel; the other steps use only the cover condition and the counts $s_v=t_v+h_v$. So witnesses of size three and four need no hypothesis beyond Lemma 16.4. (2) The cover may be chosen for each class. With $C_x=X\setminus V$ and $C_z=Z\setminus W$ no vertex is integrated. Integrated vertices may carry free pairs; only their coincidences with fixed labels of the other row produce the weight 2^{κ^f}, which the cover rows absorb at the second-order cost $r^2 2^{s_b}\sum_j 2^{\mathrm{mg}(j,b)+c^U(j,b)-[j\in H_b]}$. Each merge costs one factor 2, moved to the coincident vertex that caused it, so the bound factorizes over the coincidence patterns. (3) The count h_b of a cover vertex must include its free pairs to the witness set V: their factors occur in Step 7 a second time, inside the product integrated by Lemma 16.8, after the first copy was bounded by 2^{f_V} in Step 5. A vertex $u\in U_x$ has no free pair to W, by (VC_k), so h_u is its full free degree. (4) Step 8 is the only use of the symmetry of N. Without it the transposed class has its own counts, taken in the column neighbourhoods (Section 16.5).

The certificates evaluate two consequences of Theorem 16.9. The first keeps the leaf-specific counts s_v, bounds each special sum by its largest value over the vertices and the free partners that the count s_v allows, and bounds the denominators uniformly; the second averages the labelling orders and is linear in the logarithm. For $0\le s\le\Delta$ and $i\in U^+$ put $h_i^0(s)=\max(0,s-d_i^0)$, $w_{ij}^{\mathrm{m}}=2^{c^U(i,j)}$ and $w_{ij}^{\varrho}=2^{\mathrm{mg}(j,i)+c^U(j,i)}$ $(j\in U^+)$, and let $\sigma_i^{\mathrm{m}}(s)$ and $\sigma_i^{\varrho}(s)$ be the sums of the $h_i^0(s)$ smallest of the numbers w_{ij}^{m}, respectively w_{ij}^{ϱ}, over $j\in N(i)$; if $d_i\ge s$, then $h_i^0(s)\le r_i$. Put

$$\begin{aligned}\mathsf{S}_{\mathrm{m}}(s)&=\max_{i\in U^+,\ d_i\ge s}\Big(\sum_{j\in U^+}w_{ij}^{\mathrm{m}}-\tfrac12\sigma_i^{\mathrm{m}}(s)\Big),\\ \mathsf{S}_{\varrho}(s)&=\max_{i\in U^+,\ d_i\ge s}\Big(\sum_{j\in U^+}w_{ij}^{\varrho}-\tfrac12\sigma_i^{\varrho}(s)\Big),\end{aligned}\tag{16.5}$$

$$\mathsf{f}_{\mathrm{m}}(s)=\frac{1+e_s(p,n)+r^2 2^s\mathsf{S}_{\mathrm{m}}(s)}{1-\hat{e}},\qquad \mathsf{f}_{\varrho}(s)=\frac{1+e_s(p,2n)+r^2 2^s\mathsf{S}_{\varrho}(s)}{1-\hat{e}}.$$

The maxima are over nonempty sets, since $\Delta=\max_i d_i$. The constants $\mathsf{S}_{\mathrm{m}}(s)$ and $\mathsf{S}_{\varrho}(s)$ are at most $\max_i\sum_j w_{ij}^{\mathrm{m}}$ and $\max_i\sum_j w_{ij}^{\varrho}$, the values that bound the special sums when every factor at a special value is bounded by 2. The subtracted terms are exact maxima: for a set $H\subseteq N(i)$ of at least $h_i^0(s)$ free partners, $\sum_j w_{ij}2^{-[j\in H]}=\sum_j w_{ij}-\frac12\sum_{j\in H}w_{ij}$, which is largest when H consists of $h_i^0(s)$ elements of smallest weight.

Corollary 16.11 (Leaf bound and nonstar coefficients)**.**

(a) *In Theorem 16.9 with* $c_{\ker} = 2^{\beta(\mathcal{C})}$ *put* $\mathrm{ex}^\star(\mathcal{C}) = \mathrm{ex}(\mathcal{C}) + 2\beta(\mathcal{C})$. *Then*

$$\|T_{\mathcal{C}}\| \le r^k\, 2^{\mathrm{ex}^\star(\mathcal{C})/2}\, \frac{R_{\mathrm{full}}^2}{(1-\hat{e})^k} \prod_{u \in U_x \cup U_z} \mathsf{f}_{\mathrm{m}}(s_u)^{1/2} \prod_{b \in C_x \cup C_z} \mathsf{f}_{\varrho}(s_b)^{1/2}.$$

If moreover the monotone-factor condition *(on the factor functions* $\mathsf{S}_{\mathrm{m}}, \mathsf{S}_{\varrho}$ *of* (16.5)*)*

$$2^{s+1}\mathsf{S}_{\mathrm{m}}(s+1) \ge 2^s \mathsf{S}_{\mathrm{m}}(s), \qquad 2^{s+1}\mathsf{S}_{\varrho}(s+1) \ge 2^s \mathsf{S}_{\varrho}(s) \qquad (0 \le s < \Delta) \tag{16.6}$$

holds, then f_{m} *and* f_{ϱ} *are nondecreasing on* $\{0, \dots, \Delta\}$, *and the bound holds with each* s_v *replaced by any* s'_v *with* $s_v \le s'_v \le \Delta$.

(b) *Let* $\mathfrak{P}$ *be an exact partition of the nonstar subsets of* $E_\times$ *into classes with fixed records whose witness digraphs have strongly connected components of at most two vertices, the witness sizes lying in a finite set* $\mathcal{K}_{\mathrm{w}}$*; for instance a refinement of the compressed partition (Lemma 16.2). Let* (S1) *hold at a real* $Q > 1$, *and let* $C_k(Q)$ *be the sum over the classes of* $\mathfrak{P}$ *with witness size* k *of* $2^{\mathrm{ex}^\star(\mathcal{C})/2} R_{\mathrm{full}}^2 (1-\hat{e})^{-k} \prod_u \mathsf{f}_{\mathrm{m}}(s_u)^{1/2} \prod_b \mathsf{f}_{\varrho}(s_b)^{1/2}$, *evaluated at* $r = Q^{-1/2}$. *Then for every prime* $p \ge Q$

$$\|T_{\mathrm{ns}}\|_{L^2(\mu)} \le \sum_{k \in \mathcal{K}_{\mathrm{w}}} C_k(Q)\, p^{-k/2}.$$

Under the condition (16.6) *the same holds when* $C_k(Q)$ *is computed from stored numbers* s'_v *with* $s_v \le s'_v \le \Delta$.

Proof. (a) Since $t_v \le d_v^0 \le \Delta^+$ and $e_t(p,n)$ is nondecreasing in t, $1 - e_{t_v}(p,n) \ge 1 - \hat{e}$. Let $u \in U_x$. As $H_u \subseteq C_z$, the special sum of F_u is

$$\sum_{b \in C_z} 2^{c^U(u,b) - [b \in H_u]} = \sum_{b \in C_z} w_{ub}^{\mathrm{m}} - \frac{1}{2} \sum_{b \in H_u} w_{ub}^{\mathrm{m}} \le \sum_{b \in U^+} w_{ub}^{\mathrm{m}} - \frac{1}{2} \sum_{b \in H_u} w_{ub}^{\mathrm{m}}.$$

The set $H_u \subseteq N(u)$ has $h_u = s_u - t_u \ge s_u - d_u^0$ elements, hence at least $h_u^0(s_u)$, so its weights add up to at least $\sigma_u^{\mathrm{m}}(s_u)$; and $d_u \ge s_u$. So the special sum is at most $\mathsf{S}_{\mathrm{m}}(s_u)$, and $F_u \le \mathsf{f}_{\mathrm{m}}(s_u)$. For $u \in U_z$ the same argument applies with C_x, and for a cover vertex b with the weights w_{bj}^{ϱ} and $H_b \subseteq N(b)$, $|H_b| = s_b - t_b \ge h_b^0(s_b)$; this gives $F_b \le \mathsf{f}_{\varrho}(s_b)$. Finally $2^\beta = 2^{2\beta/2}$. Under this condition the numerators of $\mathsf{f}_{\mathrm{m}}(s)$ and $\mathsf{f}_{\varrho}(s)$ are nondecreasing in s, since $e_s(p,n)$ and $e_s(p,2n)$ are ((E3)) and $r^2 2^s \mathsf{S}(s)$ is by (16.6), while the denominator does not depend on s; so $\mathsf{f}(s_v) \le \mathsf{f}(s'_v)$.

(b) (S1) at Q implies (S1) at every prime $p \ge Q$ (Definition 13.1). By the triangle inequality over $\mathfrak{P}$ and (a), $\|T_{\mathrm{ns}}\| \le \sum_{\mathcal{C} \in \mathfrak{P}} r^{k_{\mathcal{C}}} \varphi_{\mathcal{C}}(r)$, where $\varphi_{\mathcal{C}}(r)$ is the coefficient of $r^{k_{\mathcal{C}}}$ in (a) as a function of r. The count errors $e_s(p,n)$, $e_s(p,2n)$ and r^2 are polynomials in r with nonnegative coefficients, and on $(0, Q^{-1/2}]$ every $e_t(p,n')$ with $t \le \Delta$, $n' \le 2n$ is below one; so $\varphi_{\mathcal{C}}$ is admissible by Lemma 4.18(i),(ii) (quotients $1/(1-\varphi)$ and $(1+\varphi)/(1-\varphi)$, products and square roots). Since the records are fixed, $\varphi_{\mathcal{C}}(p^{-1/2}) \le \varphi_{\mathcal{C}}(Q^{-1/2})$ by Lemma 4.18(a), and summing over the classes of each witness size gives the claim. With stored numbers s'_v every class bound of (a) only grows, and the same argument applies. □

For a fixed partition with fixed records, the coefficients $C_k(P)$ between two evaluation points $P_0 < P_1$ are bounded by log-convexity in $\log P$. The leaf factors have a common factor; after it is divided out, the rest is a positive combination of square roots of positive sums of exponentials in $\log p$.

Lemma 16.12 (Log-convex nonstar coefficients)**.** *Fix* $\mathfrak{P}$ *and its records as in Corollary 16.11(b), and one error pair of the setting. For real* $P > 1$ *at which* (S1) *holds let* $C_k(P)$ *be the coefficients of Corollary 16.11(b), and put* $M(P) = R_{\mathrm{full}}(P)^2 (1 - \hat{e}(P))^{-n}$, *both evaluated at* $r = P^{-1/2}$. *Then* $C_k = M D_k$, *where* D_k *is a finite positive sum of products of square roots of the numerators of* f_{m} *and* f_{ϱ}, *and* M, D_k *and* C_k *are nonincreasing functions of* P *and log-convex functions of*

$\log P$. *Consequently, if* $P_0 < P_1$, (S1) *holds at* P_0, $c_i \ge C_k(P_i)$ *for* $i = 0, 1$, $P_0 \le Q \le P_1$ *and* $\theta = \log(Q/P_0)/\log(P_1/P_0)$, *then*

$$C_k(Q) \le M(Q)\Big(\frac{c_0}{M(P_0)}\Big)^{1-\theta}\Big(\frac{c_1}{M(P_1)}\Big)^{\theta} \le c_0^{1-\theta}c_1^{\theta}.$$

Proof. A class of witness size k contributes to C_k the term $2^{\mathrm{ex}^\star/2}R_{\mathrm{full}}^2(1-\hat e)^{-k}\prod_v \mathsf{f}(s_v)^{1/2}$ over its $2(n-k)$ non-witness vertices, and every $\mathsf{f}(s_v)$ is a numerator divided by $1-\hat e$. So the powers of $1-\hat e$ add up to $-k-(n-k) = -n$, and the term is M times $2^{\mathrm{ex}^\star/2}$ times the product of the square roots of the numerators; this gives $C_k = MD_k$.

Put $u = \log P$, so that $r = e^{-u/2}$. Every count error of (12.1) or (12.2), and every $r^2 2^s \mathsf{S}(s)$, is a nonnegative combination of $e^{-u/2}$ and e^{-u}. A positive function of u is log-convex if its logarithm is convex; the functions e^{cu} are log-convex, and products, positive powers and positive sums of log-convex functions are log-convex, the last by Hölder's inequality. So every numerator $1 + e_s + r^2 2^s \mathsf{S}(s)$, every factor $1 + e_{d_i^0}$ and D_k are log-convex. If ε is a nonnegative combination of decreasing exponentials of u with $\varepsilon < 1$, then $(1-\varepsilon)^{-1}$ is log-convex, since

$$\frac{d^2}{du^2}\big(-\log(1-\varepsilon)\big) = \frac{\varepsilon''}{1-\varepsilon} + \frac{(\varepsilon')^2}{(1-\varepsilon)^2} \ge 0,$$

as $\varepsilon'' \ge 0$. Hence M and $C_k = MD_k$ are log-convex. Every exponential $e^{-u/2}$, e^{-u} decreases in u, so all these functions are nonincreasing in P. On $[P_0, P_1]$ all denominators are positive, as (S1) at P_0 gives (S1) at every $P \ge P_0$ (Definition 13.1).

Write $\log Q = (1-\theta)\log P_0 + \theta \log P_1$. Log-convexity of D_k gives $D_k(Q) \le D_k(P_0)^{1-\theta} D_k(P_1)^{\theta}$, and $D_k(P_i) = C_k(P_i)/M(P_i) \le c_i/M(P_i)$; multiplying by $M(Q)$ gives the first inequality. The second is the log-convexity of M, $M(Q) \le M(P_0)^{1-\theta} M(P_1)^{\theta}$. □

At an evaluation point the certificates bound $C_k(P_i)$ by rounding every square root of a vertex factor upward to a multiple of 2^{-48} and summing the stored histogram exactly. The lemma is applied to the unrounded functions C_k, with these upper bounds as c_i, so the rounded values need not be log-convex.

Lemma 16.13 (An order with a bounded weighted extension sum)**.** *For an integer* $g \ge 0$ *put* $\mathrm{av}(g) = (2^{g+1}-1)/(g+1)$. *If a vertex has* g *neighbours in a uniformly ordered vertex set containing it, then the expected value of* 2^t, *where* t *counts its earlier neighbours, is* $\mathrm{av}(g)$. *Consequently, for any fixed nonnegative weights* ω_v *on the vertices, some deterministic order has* $\sum_v \omega_v 2^{t_v} \le \sum_v \omega_v \mathrm{av}(g_v)$, *where* g_v *is the number of neighbours of* v *in the set.*

Proof. The vertex has uniform rank among itself and its g neighbours, so t is uniform on $\{0, \dots, g\}$, and $\mathbb{E}2^t = \frac{1}{g+1}\sum_{t=0}^{g} 2^t = \mathrm{av}(g)$. By linearity the weighted sum has expectation $\sum_v \omega_v \mathrm{av}(g_v)$, and a finite average is at least its minimum. □

Corollary 16.14 (Compressed classes, log-linear form)**.** *Assume* (S1) *at* p. *Let* $\mathcal{C}$ *be a class with a witness of size two and a cover, and let* $c_{\ker}(\mathcal{C}) \in \{1, 2\}$ *be its kernel constant (*2 *for the complete rectangle). For a non-witness vertex* i *let* h_i *be the number of free pairs at* i, *and put*

$$\begin{aligned}\mathrm{oc}(\mathcal{C}) = {} & \sum_{i \in C_x} 2^{h_i}\,\mathrm{av}(|\operatorname{Adj}(i) \cap C_x|) + \sum_{i \in U_x} 2^{h_i + d_i^0 - |\operatorname{Adj}(i)\cap U_x|}\,\mathrm{av}(|\operatorname{Adj}(i) \cap U_x|) \\ & + \sum_{i \in C_z} 2^{h_i}\,\mathrm{av}(|\operatorname{Adj}(i) \cap C_z|) \\ & + \sum_{i \in U_z} 2^{h_i + d_i^0 - |\operatorname{Adj}(i)\cap U_z|}\,\mathrm{av}(|\operatorname{Adj}(i) \cap U_z|),\end{aligned}$$

$\gamma_{\mathrm{cmp}} = nr^2 + \frac{\Delta}{4}r$, $J_{\mathrm{sp}} = \sum_{i,j\in U^+} 2^{d_i + |\operatorname{Adj}(i)\cap N(j)| + |\operatorname{Adj}(j)\cap N(i)|}$ *and*

$$\Lambda_0 = \frac{R_{\mathrm{full}}^2 \exp(r^2 J_{\mathrm{sp}})}{(1-\hat{e})^2 \prod_{i\in U^+}(1 - e_{d_i^0}(p,n))}.$$

Then $\|T_{\mathcal{C}}\| \le c_{\ker}(\mathcal{C})\, r^2\, 2^{\mathrm{ex}(\mathcal{C})/2} \Lambda_0 \exp(\gamma_{\mathrm{cmp}} \operatorname{oc}(\mathcal{C}))$. *Consequently, group the classes into* buckets *indexed by integers* $\mathfrak{b}$*: if every class of an exact partition of the nonstar subsets is assigned a bucket index* $\mathfrak{b} \ge \operatorname{oc}(\mathcal{C}) 2^{-\mathfrak{q}}$*, for a fixed integer* $\mathfrak{q} \ge 0$*, and* $\mathfrak{a}_{\mathfrak{b}} + \mathfrak{b}_{\mathfrak{b}}\sqrt{2} = \sum_{\mathcal{C}\mapsto\mathfrak{b}} c_{\ker}(\mathcal{C}) 2^{\mathrm{ex}(\mathcal{C})/2}$ *with* $\mathfrak{a}_{\mathfrak{b}}, \mathfrak{b}_{\mathfrak{b}} \in \mathbb{Z}_{\ge 0}$*, then*

$$\|T_{\mathrm{ns}}\| \le r^2 \Lambda_0 \sum_{\mathfrak{b}} (\mathfrak{a}_{\mathfrak{b}} + \mathfrak{b}_{\mathfrak{b}}\sqrt{2}) \exp(\gamma_{\mathrm{cmp}}\, \mathfrak{b}\, 2^{\mathfrak{q}}).$$

For fixed classes and a fixed assignment of each class to an integer $\mathfrak{b}$ *(its bucket) as above, this right side divided by* r^2 *is admissible (Lemma 4.18) as a function of* r *on the range where* (S1) *holds.*

Proof. We choose the orders of the record and apply Theorem 16.9 with $k = 2$. For $i \in C_x$ the number t_i counts the neighbours of i earlier in C_x; for $i \in U_x$ it counts the $d_i^0 - |\operatorname{Adj}(i) \cap U_x|$ neighbours outside U_x and the neighbours earlier in U_x. By Lemma 16.13, applied to each of the four sets with the weights 2^{h_i} (multiplied by $2^{d_i^0 - |\operatorname{Adj}(i)\cap U_x|}$ on U_x, and likewise on U_z), there are orders with $\sum_v 2^{s_v} \le \operatorname{oc}(\mathcal{C})$. Since $s \le \Delta$, (E1) gives $e_s(p,n) \le e_s(p,2n) \le 2^s(\frac{\Delta}{2}r + 2nr^2) = 2^s \cdot 2\gamma_{\mathrm{cmp}}$. By the symmetry of N, $\operatorname{mg}(j,i) = |\operatorname{Adj}(i)\cap N(j)|$ and $c^U(j,i) = |\operatorname{Adj}(j)\cap N(i)|$, so $c^U(u,b) = \operatorname{mg}(b,u)$. Put $\mathrm{sp}_v = \sum_{j\in U^+} 2^{d_v + \operatorname{mg}(j,v) + c^U(j,v)}$. As $s_v \le d_v$, the special sum of an integrated vertex u is at most $r^2 \sum_b 2^{s_u + \operatorname{mg}(b,u)} \le r^2 \mathrm{sp}_u$, and that of a cover vertex is at most $r^2 \mathrm{sp}_b$; with $t_v \le d_v^0$, every vertex factor satisfies

$$F_v \le \frac{1 + 2\gamma_{\mathrm{cmp}} 2^{s_v} + r^2 \mathrm{sp}_v}{1 - e_{d_v^0}(p,n)} \le \frac{\exp(2\gamma_{\mathrm{cmp}} 2^{s_v} + r^2 \mathrm{sp}_v)}{1 - e_{d_v^0}(p,n)}.$$

Each $i \in U^+$ occurs at most twice among the $2(n-2)$ non-witness vertices (once in $X \setminus V$, once in $Z \setminus W$), and $J_{\mathrm{sp}} = \sum_i \mathrm{sp}_i$. Hence $\prod_v F_v^{1/2} \le \exp(\gamma_{\mathrm{cmp}} \operatorname{oc}(\mathcal{C}) + r^2 J_{\mathrm{sp}}) \prod_{i\in U^+}(1 - e_{d_i^0}(p,n))^{-1}$, every factor $(1-e)^{-1}$ being at least one, and Theorem 16.9 gives the class bound. The sum follows from the triangle inequality, since $\operatorname{oc}(\mathcal{C}) \le \mathfrak{b} 2^{\mathfrak{q}}$. Finally γ_{cmp} and $r^2 J_{\mathrm{sp}}$ are polynomials in r with nonnegative coefficients, and Λ_0 is built from them and from the count errors, which are such polynomials and are below one where (S1) holds, by products, exponentials and the quotients $1/(1-\varphi)$ and $(1+\varphi)/(1-\varphi)$; so Lemma 4.18(i),(ii) give the admissibility. □

16.4. **Ordered coincidences.** The collision part T_{coll} of (14.20) lives on the pairs of half labellings that share a label. Such a pair has an *exact coincidence matching*: the set I of the x'-vertices whose labels are labels of x, and the injective map $\phi\colon I \to U^+$ with $x'_i = x_{\phi(i)}$. For fixed x and (I,ϕ) the set of x' with exact coincidence matching (I,ϕ), called a *stratum* below, is counted by labelling the coincident vertices first; a coincident vertex costs a factor $r^2 = 1/p$, and the remaining vertices are counted sequentially in an order chosen for the stratum type.

Lemma 16.15 (Ordered coincidence bound)**.** *Let spectral data satisfy* $e_\Delta(p,2n) < 1$*. Put*

$$L_{\mathrm{int}} = \prod_{i\in U^+}(1 - e_{d_i^0}(p,n)), \qquad \Upsilon = \prod_{i\in U^+}(1 + e_{d_i}(p,2n)),$$
$$a_{ij} = 2^{r_i - [j\in N(i)] + \operatorname{mg}(j,i)} \quad (i,j \in U^+), \qquad \Xi = 2^{\Delta^+/2} p^{-1} \sum_{i,j\in U^+} a_{ij},$$

and write $T_{\mathrm{coll}} = T_{\mathrm{coll},1} + T_{\mathrm{coll},\ge 2}$ *for the parts of* T_{coll} *on the pairs with exactly one and with at least two common labels.*

(a) Exactly one coincidence. *For each $(i,j) \in U^+ \times U^+$ choose an order of $U^+ \setminus \{i\}$, and for $\ell \neq i$ let*

$$t_\ell^{ij} = r_\ell + [\ell \sim i] - [\ell \sim i \text{ and } j \in N(\ell)] + |\operatorname{Adj}(\ell) \cap \{\text{vertices before } \ell\}|.$$

Then $\|T_{\mathrm{coll},1}\| \le D_1 := \dfrac{1}{pL_{\mathrm{int}}} \sum_{i,j\in U^+} a_{ij} \prod_{\ell\neq i}(1 + e_{t_\ell^{ij}}(p,2n))$.

(b) At least two coincidences. *For each $k \ge 2$ choose an order π_k of U^+, independently of the coincidence matching; let $b_i(\pi_k)$ be the number of G^+-neighbours of i preceding i in π_k, $t_i^{(k)} = \min\bigl(r_i + d_i^0,\ r_i + b_i(\pi_k) + k\bigr)$ and $\Upsilon_k = \prod_i (1 + e_{t_i^{(k)}}(p,2n))$. Then*

$$\|T_{\mathrm{coll},\ge 2}\| \le D_{\ge 2} := L_{\mathrm{int}}^{-1} \sum_{k=2}^{n} \Upsilon_k \frac{\Xi^k}{k!}. \tag{16.7}$$

(c) Series tail in k. *$\Upsilon_k = \Upsilon$ for $k \ge \Delta^+$. Put $k_0 = \max(2, \Delta^+)$. If $\Xi < k_0 + 1$, then*

$$\frac{1}{L_{\mathrm{int}}} \sum_{k\ge k_0} \Upsilon_k \frac{\Xi^k}{k!} \le \frac{\Upsilon}{L_{\mathrm{int}}} \frac{\Xi^{k_0}}{k_0!} \Bigl(1 - \frac{\Xi}{k_0+1}\Bigr)^{-1}.$$

(d) Coarse forms. *For all orders, $D_1 \le \Upsilon L_{\mathrm{int}}^{-1} p^{-1} \sum_{i,j} a_{ij}$ and $D_{\ge 2} \le \Upsilon L_{\mathrm{int}}^{-1}(e^\Xi - 1 - \Xi)$. In particular*

$$\|T_{\mathrm{coll}}\| \le D_{\mathrm{nat}} := \frac{\Upsilon}{L_{\mathrm{int}}} \Bigl[p^{-1} \sum_{i,j\in U^+} a_{ij} + e^\Xi - 1 - \Xi\Bigr], \tag{16.8}$$

the natural bound.

(e) Minima and endpoint monotonicity. *Put $D_{\mathrm{ord}} = D_1 + \Upsilon L_{\mathrm{int}}^{-1}(e^\Xi - 1 - \Xi)$, the ordered single-coincidence bound. The minimum of any finitely many valid bounds for $\|T_{\mathrm{coll}}\|$ — in particular of $D_1 + D_{\ge 2}$ (with the tail of* (c) *for $k \ge k_0$), of D_{ord} and of the natural bound D_{nat}, for any choices of the orders — is again a bound for $\|T_{\mathrm{coll}}\|$. If the orders are fixed and $e_\Delta(P,2n) < 1$ at a real $P > 1$, then at every prime $p \ge P$ the quantities D_1/r^2, $D_{\ge 2}/r^4$, D_{ord}/r^2 and D_{nat}/r^2 are at most their values at P, and so is the tail of* (c) *divided by r^4 if $\Xi < k_0 + 1$ at P.*

Proof. The kernel K_{coll} is nonnegative and symmetric (Lemma 14.18), and the number of common labels is symmetric too; so both parts have nonnegative symmetric kernels, and by Schur's test their norms are at most the supremum over x of their row sums $\mathbb{E}_{x'\sim\mu}[\cdot]$. Fix $x \in \mathcal{I}^+$. By Lemma 12.5(a) with the error pair of the setting and any order, $|\mathcal{I}^+| \ge p^n 2^{-|E(G^+)|} L_{\mathrm{int}}$, using $1 - e_t(p,n) \ge 1 - e_{d_i^0}(p,n)$ for $t \le d_i^0$. In a stratum (I,ϕ) the cross factors at an x'-vertex i are $w_i(x'_i)$ with $w_i(y) = \prod_{j\in N(i)}(1 + \chi(\varsigma_{ji}(y - x_j)))$; at a coincident vertex, $x'_i = x_{\phi(i)}$, the factor of $j = \phi(i)$, if $\phi(i) \in N(i)$, is $1 + \chi(0) = 1$ and the others are at most 2, so $w_i(x_{\phi(i)}) \le 2^{r_i - [\phi(i)\in N(i)]}$. We drop the internal constraints among coincident vertices, which only increases the sums. The stratum is summed by labelling the vertices of x' one at a time; the sum of w_i over the admissible labels of i, given the earlier labels, is the *one-step sum* at i, and the stratum sum is at most the product of bounds for the one-step sums that are uniform in the earlier labels (sum the last vertex first).

(a) Let $I = \{i\}$ and $\phi(i) = j$. Label i first and the other vertices in the chosen order. A vertex $\ell \neq i$ has label outside $x(U^+)$, so its cross roots $x_{j'}$ ($j' \in N(\ell)$) and its internal roots (the labels of its labelled neighbours: x_j if $\ell \sim i$, and labels outside $x(U^+)$ otherwise) merge exactly when $\ell \sim i$ and $j \in N(\ell)$. With $\mathcal{X} = x(U^+) \cup \{\text{labels of } x' \text{ already chosen}\}$, Lemma 16.7 bounds its one-step sum by $p 2^{-t_\ell^0} 2^{\mu_\ell}(1 + e_{t_\ell^{ij}}(p,2n))$, where t_ℓ^0 is its number of labelled neighbours, $\mu_\ell = [\ell \sim i \text{ and } j \in N(\ell)]$, and $t_\ell^{ij} = r_\ell + t_\ell^0 - \mu_\ell$ is its number of distinct roots. Every edge of G^+

is counted in t^0_ℓ at its later endpoint, so $\sum_{\ell\neq i} t^0_\ell = |E(G^+)|$, and $\sum_{\ell\neq i}\mu_\ell = \mathrm{mg}(j,i)$. The stratum sum is therefore at most $p^{n-1}2^{-|E(G^+)|}a_{ij}\prod_{\ell\neq i}(1+e_{t^{ij}_\ell}(p,2n))$, and dividing by $|\mathcal{I}^+|$ and summing over (i,j) bounds the row sum by D_1.

(b) Fix a stratum (I,ϕ) with $|I|=k\geq 2$, label I first and the other vertices in the restriction of π_k. A vertex $i\notin I$ has r_i distinct cross roots and t^0_i internal roots, the labels of its labelled neighbours, where $t^0_i\leq \min(d^0_i, b_i(\pi_k)+k)$: the labelled neighbours are at most k coincident ones and the neighbours preceding i in π_k. A cross root x_j equals an internal root exactly when a coincident neighbour $l\in I$ has $\phi(l)=j\in N(i)$; let μ_i be the number of such l. As $r_i+t^0_i-\mu_i\leq t^{(k)}_i$, Lemma 16.7 bounds the one-step sum at i by $p2^{-t^0_i}2^{\mu_i}(1+e_{t^{(k)}_i}(p,2n))$, whatever the earlier labels. Each merge counted in μ_i is caused by some $l\in I$ with $i\in\mathrm{Adj}(l)$ and $\phi(l)\in N(i)$, so $\sum_{i\notin I}\mu_i\leq\sum_{l\in I}\mathrm{mg}(\phi(l),l)$. Every edge of G^+ not inside I is counted in t^0 of its later endpoint, which lies outside I; so $\sum_{i\notin I}t^0_i = |E(G^+)|-e(I)$. Inserting the factors of Υ_k at the coincident vertices, each at least one, the stratum sum is at most $p^{n-k}2^{-|E(G^+)|+e(I)}\prod_{l\in I}a_{l,\phi(l)}\Upsilon_k$, and after division by $|\mathcal{I}^+|$ at most $L^{-1}_{\mathrm{int}}\Upsilon_k p^{-k}2^{e(I)}\prod_{l\in I}a_{l,\phi(l)}$. Now $2^{e(I)}\leq 2^{k\Delta^+/2}$, and expanding $(\sum_{i,j}a_{ij})^k$ counts every pair (I,ϕ) with $|I|=k$ exactly $k!$ times, besides other nonnegative terms. Summing over the strata with $|I|=k$ and over $k\geq 2$ bounds the row sum by $D_{\geq 2}$.

(c) For $k\geq\Delta^+$, $b_i(\pi_k)+k\geq d^0_i$, so $t^{(k)}_i = r_i+d^0_i = d_i$. The ratio of consecutive terms of $\sum_{k\geq k_0}\Xi^k/k!$ is at most $\Xi/(k_0+1)<1$, which gives the geometric bound.

(d) Always $t^{ij}_\ell\leq r_\ell+d^0_\ell = d_\ell$ and $t^{(k)}_i\leq d_i$, so every product is at most Υ, and $\sum_{k\geq 2}\Xi^k/k!\leq e^\Xi-1-\Xi$.

(e) Each listed quantity bounds the same norm, so their minimum does; no error bound is subtracted. For the endpoints write $r=p^{-1/2}$. Every count error is a polynomial in r with nonnegative coefficients, L^{-1}_{int} is a product of factors $1/(1-\varphi)$ with $\varphi<1$ on $(0,P^{-1/2}]$, and $\Xi = r^2\Xi_0$ with a constant Ξ_0. Hence D_1/r^2, $D_{\geq 2}/r^4 = L^{-1}_{\mathrm{int}}\sum_k\Upsilon_k\Xi_0^k r^{2k-4}/k!$, $(e^\Xi-1-\Xi)/r^2 = \sum_{k\geq 2}\Xi_0^k r^{2k-2}/k!$, and so D_{ord}/r^2 and D_{nat}/r^2, and the tail divided by r^4 are admissible in the sense of Lemma 4.18, and Lemma 4.18(a) gives the claim. □

Remark 16.16 (Why the coincident vertices are labelled first). With one fixed order of all vertices for every stratum, a coincident vertex i pays $2^{t^0_i}\leq 2^{d^0_i}$ for its earlier neighbours, that is, the charge $r^2 2^{d_i-[j\in N(i)]+\mathrm{mg}(j,i)} = 2^{d^0_i}r^2a_{ij}$ instead of r^2a_{ij}. Labelling the coincident vertices first moves these factors into the single factor $2^{e(I)}$, which is 1 for one coincidence. This factor cannot be dropped: the internal constraint between two coincident vertices is not counted, although the normalization $|\mathcal{I}^+|$ contains a factor $\frac12$ for it. The cardinality-sensitive orders of (b) refine the count of the vertices outside I once more: a vertex has at most k coincident neighbours, so in a stratum with few coincidences its internal degree is effectively that of the order π_k.

Soft factors and incident weights. The dense comparisons of the upper range (Proposition 28.5) use kernels whose cross factors are $1+D_e\chi_e$ with $|D_e|\leq 1$, and the L20 cut of the middle root (Proposition 20.21) needs the coincidence bound with the actual cross factors of the coincident vertices. Both follow from the proofs above.

Corollary 16.17 (Soft cross factors)**.** *Let numbers $D_e\in[-1,1]$ be given for $e\in E_\times$, put $D_S = \prod_{e\in S}D_e$, and consider the* soft kernel $K^D_0(x,x') = \mathbf{1}\{x(U^+)\cap x'(U^+)=\emptyset\}\prod_{e\in E_\times}(1+D_e\chi_e(x,x'))$, *its nonstar part $T^D_{\mathrm{ns}} = \sum_{S\ \mathrm{nonstar}} D_S T_S$, and its collision part T^D_{coll}, the operator with kernel $\mathbf{1}\{x(U^+)\cap x'(U^+)\neq\emptyset\}\prod_{e\in E_\times}(1+D_e\chi_e)$; as in Lemma 14.18, $K^D_0 = \mathbf{1}\otimes\mathbf{1}+\sum_{S\ \mathrm{star}}D_S T_S + T^D_{\mathrm{ns}} - T^D_{\mathrm{coll}}$ on $\mathcal{I}^+\times\mathcal{I}^+$.*

(a) *For a class $\mathcal{C}$ with a record, $T^D_{\mathcal{C}} = \sum_{S\in\mathcal{C}} D_S T_S$ satisfies the bounds of Theorem 16.9, Corollary 16.11(a) and Corollary 16.14; hence T^D_{ns} satisfies those of Corollary 16.11(b) and Corollary 16.14, with the partitions, records and constants computed for $E_\times$.*
(b) *T^D_{coll}, and its parts on the pairs with exactly one and with at least two common labels, satisfy the bounds of Lemma 16.15.*
(c) *The same holds for the pinned rows of Section 16.5: the bounds of Propositions 16.20 and 16.21 hold when the cross factors are $1 + D_{ij}\chi_{ij}$ with $|D_{ij}| \le 1$.*

In particular, let $E' \supseteq \{e \in E_\times : D_e \neq 0\}$ be the set of cross pairs of spectral data with the same half graph G^+. As $D_e\chi(\varsigma_e y) = D'_e\chi(\varsigma'_e y)$ with $D'_e = \pm D_e$ for any two signs $\varsigma_e, \varsigma'_e$, the soft kernel is a soft kernel of E' (with the signs of E' and $D'_e = 0$ off the support of D). So the bounds computed for the Boolean kernel of E', all of whose coefficients are one, bound its nonstar and collision parts, and bounds certified at a threshold p_0 hold at every prime $p \ge p_0$ (Corollary 16.11(b), Lemma 16.15(e)).

Proof. The expansion is $\prod_e(1 + D_e\chi_e) = \sum_S D_S\chi_S$, and the identity follows as in Lemma 14.18. The proofs of the cited statements use the cross factors only in four ways, and each survives the soft factors. (1) The class sum is $\sum_{S\in\mathcal{C}} D_S\chi_S = \prod_{e\in P} D_e\chi_e \prod_{e\in F}(1+D_e\chi_e)$: a present pair contributes a factor of modulus at most one, so the witness kernel is multiplied by a constant of modulus at most one, and a free pair contributes a factor in $[0, 2]$. (2) Every bound for a sum over one label of a product of free factors is uniform in the prescribed signs: $1 + b\chi = \frac{1+b}{2}(1 + \chi) + \frac{1-b}{2}(1 - \chi)$ is a convex combination of the two orientations, so a product of such factors is a convex combination of products of factors $1 \pm \chi(\cdot)$, each of which Lemma 16.7 bounds by the same quantity, whatever the signs. No free factor is squared: the Cauchy–Schwarz steps use the free factors as weights (Step 5 of Theorem 16.9). (3) At a special value or at a coincidence the factor of the coincident pair is $1 + b\chi(0) = 1$, and every other factor is at most $1 + |b| \le 2$. (4) The collision kernel is nonnegative. It need not be symmetric, and Schur's test is then applied with its row sums and its column sums; the column sums are the row sums of the soft kernel with the coefficient $D_{(j,k)}$ at the pair (k, j), which is a soft kernel of the same spectral data (as in the proof of Lemma 14.18), so the same bound holds for both. The pinned rows are covered by the same four remarks; their collision bound already uses row and column sums. Only absolute values enter: every term of the expansion is bounded, and no monotonicity of a signed norm under the deletion of pairs is used. □

Corollary 16.18 (Incident weights at coincident vertices)**.** *In Lemma 16.15, with hard or soft cross factors (Corollary 16.17), fix $x \in \mathcal{I}^+$. For an x'-vertex i let $w_i(y) = \prod_{l\in N(i)}(1+D_{(l,i)}\chi(\varsigma_{li}(y-x_l)))$ be its cross factor at the label y ($D \equiv 1$ for the hard kernel), and put $\alpha_{ij}(x) = w_i(x_j)2^{\mathrm{mg}(j,i)}$ and $\rho_i(x) = r^2\sum_{j\in U^+}\alpha_{ij}(x)$. Then $\alpha_{ij}(x) \le a_{ij}$, and with the orders of Lemma 16.15(a),(b) the row sums at x of $T_{\mathrm{coll},1}$ and of $T_{\mathrm{coll},\ge 2}$ are at most*

$$\frac{1}{L_{\mathrm{int}}}\sum_{i,j\in U^+} r^2\alpha_{ij}(x)\prod_{\ell\neq i}(1 + e_{t^{ij}_\ell}(p, 2n)) \quad \text{and} \quad \frac{1}{L_{\mathrm{int}}}\sum_{k=2}^{n}\Upsilon_k \sum_{I\subseteq U^+,\,|I|=k} 2^{e(I)}\prod_{i\in I}\rho_i(x).$$

If the kernel is symmetric, the norms of $T_{\mathrm{coll},1}$ and $T_{\mathrm{coll},\ge 2}$ are at most the suprema over x of these bounds; in general they are at most the geometric means of these suprema and of the same suprema for the transposed coefficients. A stratum (I, ϕ) with $\alpha_{i\phi(i)}(x) = 0$ for some $i \in I$ contributes nothing, and any relation among the numbers $\alpha_{ij}(x)$ that holds for every x (vanishing weights, disjoint supports, a bound for the number of nonzero ones) may be used to bound these sums before the supremum is taken.

Proof. In the proof of Lemma 16.15 keep, at a coincident vertex i with label $x_{\phi(i)}$, its cross factor $w_i(x_{\phi(i)})$ instead of its bound $2^{r_i-[\phi(i)\in N(i)]}$; that bound is $w_i(x_j) \le 2^{r_i-[j\in N(i)]}$, the factor of $l = j$

being $1 + D_{(j,i)}\chi(0) = 1$ and every other factor at most 2, which gives $\alpha_{ij} \le a_{ij}$. The one-step sums at the vertices outside I are bounded as there (Corollary 16.17). In (a) the stratum $(\{i\}, j)$ then has sum at most $p^{n-1}2^{-|E(G^+)|}\alpha_{ij}(x)\prod_{\ell\neq i}(1 + e_{t_\ell^{ij}}(p, 2n))$. In (b) the stratum (I, ϕ) has, after division by $|\mathcal{I}^+|$, sum at most $L_{\mathrm{int}}^{-1}\Upsilon_k 2^{e(I)}\prod_{l\in I} r^2\alpha_{l\phi(l)}(x)$, and the sum over the injective maps $\phi\colon I \to U^+$ is at most the sum over all maps, which is $\prod_{l\in I}\rho_l(x)$. These bounds hold pointwise in x, which gives the last assertion, and Schur's test is applied as in Lemma 16.15, with the column sums of Corollary 16.17 for a nonsymmetric kernel. The balanced coincidence bound of the L20 cut is this corollary (Proposition 20.21). □

Lemma 16.19 (Sign-incompatible collision strata)**.** *Suppose $p \equiv 3 \pmod 4$ and the two half graphs have the same signed adjacency $A_{ij} = \mathbf{1}[i \to j] - \mathbf{1}[j \to i]$. Write the hard cross kernel as $\prod_{k,i}(1 + D_{ki}\chi(y_i - x_k))$, omitting zero coefficients, with $D_{ki} \in \{-1, 0, 1\}$. Define*

$$\gamma_{ij} = \mathbf{1}[A_{kj}D_{ki} \neq -1 \text{ for all } k, \quad A_{i\ell}D_{j\ell} \neq -1 \text{ for all } \ell].$$

Every collision stratum containing $y_i = x_j$ vanishes when $\gamma_{ij} = 0$. Consequently in the nonnegative collision sums of Lemma 16.15, with the notation of that lemma, one may replace the old pin weights by

$$\widetilde{a}_{ij} = \gamma_{ij}2^{r_i - \mathbf{1}[j\in N(i)] + \mathrm{mg}(j,i)}, \qquad s_i = \sum_j \widetilde{a}_{ij}.$$

In particular, for the same fixed insertion orders, the single- and multiple-collision row bounds become

$$D_1^{\#} = \frac{q}{L_{\mathrm{int}}}\sum_{i,j}\widetilde{a}_{ij}\prod_{\ell\neq i}(1 + e_{t_\ell^{ij}}(p, 2n)),$$

$$D_{\ge 2}^{\#} = \frac{1}{L_{\mathrm{int}}}\sum_{k=2}^{n} q^k\Upsilon_k \sum_{|I|=k} 2^{e(I)}\prod_{i\in I} s_i.$$

The transposed argument supplies the column bounds.

Proof. On the pin $y_i = x_j$, an internal plus constraint fixes $\chi(x_j - x_k) = A_{kj}$. If its product with D_{ki} is -1, the corresponding cross factor is zero. The second test uses the internal minus constraint $\chi(y_\ell - y_i) = A_{i\ell}$ in exactly the same way. Within-half injectivity guarantees that these differences are nonzero. Adding further cross coincidences cannot remove the vanishing factor. Delete these zero strata before applying the positive extension bounds. For multiple collisions, sum first over injective compatible target maps, then enlarge to all compatible maps; the latter sum factors as $\prod_{i\in I} s_i$. The internal pinned-edge cost $2^{e(I)}$ remains. This proves both bounds. □

The second sign test is a property of the entire stratum; it is not an upper bound for an isolated incident factor. Nor does it apply to a softened factor $1 + b\chi$ with $|b| < 1$. These distinctions matter when the same graph is used for different comparison laws.

16.5. **Conditional laws given fixed labels.** The conditional rows of the upper range (Section 27.6.2) are labellings of n labels that avoid z fixed labels and satisfy a few square-class constraints against fixed labels; there are no edges inside a row, and the cross constraints between two rows need not be symmetric. The pins play the role of the half graph: a fixed root never coincides with a free root, so there are no merge charges, while every count gains the pins and the fixed labels as exclusions. Before the rows are made injective their labels are independent, each uniform on its allowed set; so the star part of a row factors through one coefficient matrix, as the stars of an invariant law do in Corollary 14.5 (Proposition 16.22), and the fold and matching rows of one coordinate, which have the same allowed sets, are bounded jointly on one chart (Lemma 16.23).

Setting. Let $n \ge 2$, $z \ge 0$, and let $\mathcal{M} \subset \mathbb{F}_p$ be a set of z fixed labels. Every coordinate $i \in [n]$ carries a set of $\varpi_i \le \varpi$ *pins* $(\beta, \epsilon) \in \mathcal{M} \times \{\pm 1\}$ with distinct roots β, the same for both copies. Let $\mathcal{I}_{\mathrm{pin}}$ be the set of injective $x \in (\mathbb{F}_p \setminus \mathcal{M})^n$ with $\epsilon(x_i - \beta) \in Q_p^*$ for every pin (β, ϵ) of every i, and let μ be the uniform law on $\mathcal{I}_{\mathrm{pin}}$. The *allowed set* of a coordinate i consists of the $t \in \mathbb{F}_p \setminus \mathcal{M}$ with $\epsilon(t - \beta) \in Q_p^*$ for every pin (β, ϵ) of i; with λ_i the uniform law on it and $\lambda = \bigotimes_i \lambda_i$, the law μ is λ conditioned on injectivity. Let $E_\times \subseteq [n] \times [n]$ carry signs $\varsigma_{ij} \in \{\pm 1\}$, and put $N_X(i) = \{j : (i,j) \in E_\times\}$ and $N_Y(j) = \{i : (i,j) \in E_\times\}$; no symmetry is assumed. The rows are the copies X (labels x) and Y (labels y) of $[n]$; $\chi_{ij}(x,y) = \chi(\varsigma_{ij}(y_j - x_i))$, and T_S, classes, witnesses and covers are as in Section 16.1, with y in place of x'. For a non-witness vertex v let h_v be its number of free pairs, counted in N_X on the row side and in N_Y on the column side. Put

$$e_d^{\mathrm{pin}} = 2^d\Big((2n+z)r^2 + \frac{d}{2}\,r\Big), \qquad \Delta_{\mathrm{pin}} = \varpi + \max_{i \in [n]} \max\big(|N_X(i)|, |N_Y(i)|\big),$$

$$\gamma_{\mathrm{pin}} = \frac{2n+z}{2}\,r^2 + \frac{\Delta_{\mathrm{pin}}}{4}\,r, \qquad R_{\mathrm{pin}} = \Big(\frac{1 + e_\varpi^{\mathrm{pin}}}{1 - e_\varpi^{\mathrm{pin}}}\Big)^n, \qquad \Lambda_0^{\mathrm{pin}} = \frac{R_{\mathrm{pin}}^2 \exp(r^2 n^2 2^{\Delta_{\mathrm{pin}}})}{(1 - e_\varpi^{\mathrm{pin}})^{n+2}}.$$

By (E1), $e_d(p, n') \le e_d^{\mathrm{pin}}$ for $n' \le 2n + z$; so by Lemma 12.2(b),(d) every set Φ of Lemma 12.2 with d roots and at most $2n + z$ excluded labels has between $p2^{-d}(1 - e_d^{\mathrm{pin}})$ and $p2^{-d}(1 + e_d^{\mathrm{pin}})$ elements, and the upper bound needs no condition on the exclusions.

Proposition 16.20 (Pinned compressed classes)**.** *Assume $e_\varpi^{\mathrm{pin}} < \frac{1}{2}$. Let $\mathcal{C}$ be a class of nonstar subsets of $E_\times$ with a witness of size two whose kernel, compressed to injective pairs, has norm at most $c_{\ker}(\mathcal{C}) r^2$, $c_{\ker}(\mathcal{C}) \in \{1, 2\}$, and with a cover; put* $\mathrm{ex}^{\mathrm{pin}}(\mathcal{C}) = f_V + f_W$ *(the pinned analogues of* ex *and* oc *in Corollary 16.14, with different definitions) and* $\mathrm{oc}^{\mathrm{pin}}(\mathcal{C}) = \sum_v 2^{h_v}$ *over the $2(n-2)$ non-witness vertices. Then*

$$\|T_{\mathcal{C}}\|_{L^2(\mu) \to L^2(\mu)} \le c_{\ker}(\mathcal{C})\, r^2\, 2^{\mathrm{ex}^{\mathrm{pin}}(\mathcal{C})/2 + 2\varpi}\, \Lambda_0^{\mathrm{pin}} \exp\big(\gamma_{\mathrm{pin}} 2^\varpi\, \mathrm{oc}^{\mathrm{pin}}(\mathcal{C})\big). \tag{16.9}$$

Consequently, for an exact partition of the nonstar subsets of $E_\times$ into such classes (for instance the compressed partition of Lemma 16.2(b), with $c_{\ker}$ from Lemma 16.4), grouped by integers $\mathfrak{b} \ge \mathrm{oc}^{\mathrm{pin}}(\mathcal{C}) 2^{-\mathfrak{q}}$ with $\mathsf{a}_\mathfrak{b} + \mathsf{b}_\mathfrak{b}\sqrt{2} = \sum_{\mathcal{C} \mapsto \mathfrak{b}} c_{\ker}(\mathcal{C}) 2^{\mathrm{ex}^{\mathrm{pin}}(\mathcal{C})/2}$, the nonstar part is bounded by

$$N^{\mathrm{pin}} = r^2 \Lambda_0^{\mathrm{pin}}\, 2^{2\varpi} \sum_{\mathfrak{b}} \big(\mathsf{a}_\mathfrak{b} + \mathsf{b}_\mathfrak{b}\sqrt{2}\big) \exp\big(\gamma_{\mathrm{pin}} 2^\varpi\, \mathfrak{b}\, 2^{\mathfrak{q}}\big).$$

Proof. We follow the proof of Theorem 16.9 for $\mathcal{C}$, with $k = 2$, the pinned law μ on both rows, and any orders; only the counts change. Every count below is that of a set Φ of Lemma 12.2 with at most $2n + z$ excluded labels.

Steps 1, 2 and 5 use only the product form of the weights, the cover condition and the independence of the two rows under $\mu \otimes \mu$, and hold verbatim.

Step 3. Given the cover labels $c = x_{C_x}$, a label of U_x meets only its $\varpi_u \le \varpi$ pins, whose roots lie in $\mathcal{M}$ and so are distinct, and avoids $\mathcal{M}$ and the labels already used. So the number of completions of (c, v) on U_x lies in $\prod_{u \in U_x} p2^{-\varpi_u}[1 - e_\varpi^{\mathrm{pin}}, 1 + e_\varpi^{\mathrm{pin}}]$, a ratio at most R_{pin}; in particular every admissible (c, v) has a completion, as $e_\varpi^{\mathrm{pin}} < 1$. Labelling the two witness labels in turn, there are at least $p^2 2^{-\varpi_{j_1} - \varpi_{j_2}}(1 - e_\varpi^{\mathrm{pin}})^2$ admissible pairs v. Hence ν_V is supported on injective pairs and has density at most $D_V = 2^{2\varpi} R_{\mathrm{pin}}(1 - e_\varpi^{\mathrm{pin}})^{-2}$ with respect to uniform probability on $\mathbb{F}_p^2$, and likewise D_W. This replaces $2^{t_V} R_{U_x}(1 - \hat{e})^{-2}$: each witness label contributes the factor 2^ϖ of its pins.

Step 4 holds verbatim: the kernel bound for uniform probability, compressed to injective pairs, applies to the zero extensions of $\mathsf{a}\tilde{\nu}_V$ and $\mathsf{b}\tilde{\nu}_W$.

Step 6. An integrated vertex $u \in U_x$ meets ϖ_u pins and the h_u cross roots c'_b ($b \in H_u \subseteq C_z$). The cross roots are labels of the other row and avoid $\mathcal{M}$, so no root merges, and there are no

internal edges. For the generic values $y \notin c'(C_z)$, Lemma 16.7 with the pins as the conditions $\chi(y-\beta_i)=s_i$ bounds the contribution by $p2^{-\varpi_u}(1+e^{\mathrm{pin}}_{\varpi_u+h_u})$; the at most n special values $y=c'_b$ contribute at most $2^{h_u}=p2^{-\varpi_u}r^2 2^{\varpi_u+h_u}$ each. The lower count is $|\mathcal{Y}|\ge\prod_u p2^{-\varpi_u}(1-e^{\mathrm{pin}}_{\varpi})$. So $\mathsf{m}\le\prod_{u\in U_x}F^{\mathrm{pin}}_u$ with $F^{\mathrm{pin}}_v=(1+e^{\mathrm{pin}}_{\varpi_v+h_v}+r^2n2^{\varpi_v+h_v})/(1-e^{\mathrm{pin}}_{\varpi})$.

Step 7. The cover rows are bounded as in Lemma 16.8, without merge weights: the marginal of μ on C_z has density at most R_{pin} with respect to the uniform law on the pinned injective labellings of C_z, which has at least $\prod_{b\in C_z}p2^{-\varpi_b}(1-e^{\mathrm{pin}}_{\varpi})$ elements; a value $y\notin x([n])$ of b contributes as a generic value above, and the at most n values $y=x_j$ at most $2^{h_b}=p2^{-\varpi_b}r^2 2^{\varpi_b+h_b}$ each. So the cover row of b costs F^{pin}_b, with h_b the number of free pairs at b (in N_Y).

Step 8. The transposed class is a class of the same kind for the support $\{(j,i):(i,j)\in E_\times\}$, whose row and column neighbourhoods are N_Y and N_X, with the same pinned law on both rows. Steps 5–7 apply to it with its own counts, which are the counts h_v of the statement on U_z and C_x. The symmetry of the support is never used.

Conclusion. As in Step 8 of Theorem 16.9, $\|T_{\mathcal{C}}\|\le c_{\ker}r^2\sqrt{D_VD_W}\,2^{(f_V+f_W)/2}R_{\mathrm{pin}}\prod_v F_v^{\mathrm{pin}1/2}$. Since $\varpi_v+h_v\le\Delta_{\mathrm{pin}}$ and e^{pin}_d is nondecreasing in d, $e^{\mathrm{pin}}_{\varpi_v+h_v}\le e^{\mathrm{pin}}_{\varpi+h_v}\le 2^{\varpi}2^{h_v}\cdot 2\gamma_{\mathrm{pin}}$ and $r^2n2^{\varpi_v+h_v}\le r^2n2^{\Delta_{\mathrm{pin}}}$. With $1+x\le e^x$ and the $2(n-2)\le 2n$ factors, $\prod_v F_v^{\mathrm{pin}1/2}\le\exp(\gamma_{\mathrm{pin}}2^{\varpi}\,\mathrm{oc}^{\mathrm{pin}}(\mathcal{C})+r^2n^22^{\Delta_{\mathrm{pin}}})(1-e^{\mathrm{pin}}_{\varpi})^{-n}$, and $\sqrt{D_VD_W}=2^{2\varpi}R_{\mathrm{pin}}(1-e^{\mathrm{pin}}_{\varpi})^{-2}$. Collecting the factors gives (16.9), and the sum follows as in Corollary 16.14. □

Proposition 16.21 (Pinned coincidences)**.** *In the setting of this subsection assume $e^{\mathrm{pin}}_{\varpi}<1$, and let $T^{\mathrm{pin}}_{\mathrm{coll}}$ be the operator on $L^2(\mu)$ with kernel $\mathbf{1}\{x([n])\cap y([n])\ne\emptyset\}\prod_{(i,j)\in E_\times}(1+\chi_{ij}(x,y))$ relative to $\mu\otimes\mu$. Put*

$$U_Y=r^2\sum_{j,i\in[n]}2^{\varpi+|N_Y(j)|-[i\in N_Y(j)]},\qquad \Upsilon_Y=\prod_{j\in[n]}(1+e^{\mathrm{pin}}_{\varpi+|N_Y(j)|}),$$
$$D_Y=\frac{\Upsilon_Y}{(1-e^{\mathrm{pin}}_{\varpi})^n}(e^{U_Y}-1),$$

and let U_X, Υ_X, D_X be the same with N_X in place of N_Y. Then $\left\|T^{\mathrm{pin}}_{\mathrm{coll}}\right\|\le(D_XD_Y)^{1/2}\le\max(D_X,D_Y)$. If the multisets $\{|N_X(i)|\}_i$ and $\{|N_Y(j)|\}_j$ coincide, then $D_X=D_Y$. For fixed data, D_X/r^2, D_Y/r^2 and N^{pin}/r^2 of Proposition 16.20 are admissible (Lemma 4.18) as functions of r on the range where $e^{\mathrm{pin}}_{\varpi}<1$.

Proof. The kernel is nonnegative, so by Schur's test its norm is at most the geometric mean of the suprema of its row sums $\mathbb{E}_{y\sim\mu}[\cdot]$ and its column sums $\mathbb{E}_{x\sim\mu}[\cdot]$. Fix $x\in\mathcal{I}_{\mathrm{pin}}$ and a coincidence pattern: a nonempty set I of y-vertices and a map $\phi\colon I\to[n]$ with $y_j=x_{\phi(j)}$ for $j\in I$ and $y_j\notin x([n])$ otherwise. The cross factors at j are $\prod_{i\in N_Y(j)}(1+\chi(\varsigma_{ij}(y_j-x_i)))$. A coincident vertex j has weight at most $2^{|N_Y(j)|-[\phi(j)\in N_Y(j)]}=p2^{-\varpi_j}\,r^2 2^{\varpi_j+|N_Y(j)|-[\phi(j)\in N_Y(j)]}$. A free vertex j meets its pins and the $|N_Y(j)|$ cross roots x_i, which avoid $\mathcal{M}$; with no internal edges no root merges, and Lemma 16.7 bounds its weight by $p2^{-\varpi_j}(1+e^{\mathrm{pin}}_{\varpi+|N_Y(j)|})$, whatever the order. Dividing by $|\mathcal{I}_{\mathrm{pin}}|\ge\prod_j p2^{-\varpi_j}(1-e^{\mathrm{pin}}_{\varpi})$, the stratum contributes at most $(1-e^{\mathrm{pin}}_{\varpi})^{-n}\prod_{j\in I}u_{j,\phi(j)}\prod_{j\notin I}(1+e^{\mathrm{pin}}_{\varpi+|N_Y(j)|})$, where $u_{j,i}=r^2 2^{\varpi+|N_Y(j)|-[i\in N_Y(j)]}$. Summing over all nonempty I and all maps ϕ (an upper bound), and with $u_j=\sum_i u_{j,i}$, the row sum is at most $\Upsilon_Y(1-e^{\mathrm{pin}}_{\varpi})^{-n}(\prod_j(1+u_j)-1)$, and $\prod_j(1+u_j)-1\le e^{U_Y}-1$. So every row sum is at most D_Y. Exchanging the two rows exchanges N_X and N_Y and leaves the pinned law unchanged, so every column sum is at most D_X. Next, Υ_Y and $U_Y=r^2\sum_j 2^{\varpi+|N_Y(j)|}(n-\frac12|N_Y(j)|)$ depend only on the multiset of the column degrees, and Υ_X, U_X in the same way on the row degrees. Finally U_Y/r^2 is a constant, so $(e^{U_Y}-1)/r^2$ is a power series in r with nonnegative coefficients; e^{pin}_d, γ_{pin} and $r^2n^22^{\Delta_{\mathrm{pin}}}$ are polynomials in

r with nonnegative coefficients; and $\Upsilon_Y(1-e_{\varpi}^{\mathrm{pin}})^{-n}$ and Λ_0^{pin} are built from them by products, exponentials and the quotients $1/(1-\varphi)$ and $(1+\varphi)/(1-\varphi)$. So Lemma 4.18(i),(ii) give the admissibility. □

Star matrices. The star part of a pinned row is of first order, and the extraction does not touch it. Under a product of coordinate laws the leaves supported on disjoint exact supports (a leaf has *exact support* S if it depends exactly on the coordinates in S and has mean zero in each of them) are orthogonal, and the complete star part factors through one coefficient matrix; the leaves are centred in their own laws, and their means, bounded by β_χ in Proposition 16.22(b), contribute the separate term T_{mean} there.

Proposition 16.22 (Star matrix under pinned product laws)**.** *Let $p\equiv 3 \pmod 4$, let $\mathcal{H}=L_0^2(\mathbb{F}_p)$ for the uniform law, and let $\hat P = C/r$ be the normalized Paley operator, so that $\hat P\mathbf{1}=0$, $\hat P^*=-\hat P$ and $\hat P^2=-I$ on $\mathcal{H}$ (Lemma 14.1). Let X and Y be finite sets, let $\nu_X=\bigotimes_{i\in X}\nu_i$ and $\nu_Y=\bigotimes_{j\in Y}\nu'_j$ be product laws on $\mathbb{F}_p^X$ and $\mathbb{F}_p^Y$ whose coordinate laws have densities at most κ relative to the uniform law and are supported in a set $\Omega\subseteq\mathbb{F}_p$, and let $D\in\{0,\pm1\}^{X\times Y}$, $N_X(i)=\{j: D_{ij}\neq 0\}$ and $N_Y(j)=\{i: D_{ij}\neq 0\}$. In the expansion of the kernel $\prod_{i,j}(1+D_{ij}\chi(y_j-x_i))$ the* star part *consists of the single pairs $D_{ij}\chi(y_j-x_i)$, the* row stars $\prod_{j\in T}D_{ij}\chi(y_j-x_i)$ *with centre i and leaves T, and the* column stars $\prod_{i\in S}D_{ij}\chi(y_j-x_i)$ *with centre j and leaves S ($|T|,|S|\geq 2$); let $T_\star\colon L_0^2(\nu_Y)\to L_0^2(\nu_X)$ be its operator relative to $\nu_X\otimes\nu_Y$, compressed to the mean-zero functions on both sides. (For a pinned row at $p\equiv 3 \pmod 4$, $D_{ij}=\varsigma_{ij}$ for $(i,j)\in E_\times$, since $\chi(\varsigma_{ij})=\varsigma_{ij}$, and $D_{ij}=0$ off $E_\times$.) For sets $T\subseteq Y$ and $S\subseteq X$ with at least two elements put $A_{iT}=(-1)^{|T|}\prod_{j\in T}D_{ij}$ and $B_{jS}=\prod_{i\in S}D_{ij}$, and let $M(\hat P)$ be the block operator matrix with rows indexed by X and the sets S and columns indexed by Y and the sets T, with the singleton block $D\otimes\hat P$, the blocks $A\otimes I_{\mathcal{H}}$ (rows i, columns T) and $B^*\otimes I_{\mathcal{H}}$ (rows S, columns j), and zero between sets, as in Corollary 14.5; let $M(i)=\left(\begin{smallmatrix} iD & A\\ B^* & 0\end{smallmatrix}\right)$.*

(a) *For $\mathsf{k}>0$, $\|M(i)\|\leq\mathsf{k}$ if and only if*

$$\begin{pmatrix}\mathsf{k}^2I-AA^* & \mathsf{k}D\\ \mathsf{k}D^* & \mathsf{k}^2I-BB^*\end{pmatrix}\succeq 0. \tag{16.10}$$

(b) *Let β_χ bound $|\mathbb{E}_{\nu_i}\chi(t-x_i)|$ and $|\mathbb{E}_{\nu'_j}\chi(t-y_j)|$ for all i, j and $t\in\Omega$. Let d_{L} and d_{R} be the square roots of the largest row sum times the largest column sum of the nonnegative $X\times Y$ matrices with the entries $(1+\beta_\chi)^{|N_X(i)|-1}-1$, respectively $(1+\beta_\chi)^{|N_Y(j)|-1}-1$, where $D_{ij}\neq 0$, and zero elsewhere, and $\mathsf{d}_1=\mathsf{d}_{\mathrm{L}}+\mathsf{d}_{\mathrm{R}}$; with $m_{ii'}=|N_X(i)\cap N_X(i')|$ put*

$$\mathsf{e}_{\mathrm{L}}^2=\max_{i\in X}\sum_{i'\in X}\sum_{s=2}^{m_{ii'}}\binom{m_{ii'}}{s}\big((1+\beta_\chi)^{|N_X(i)|-s}-1\big)\big((1+\beta_\chi)^{|N_X(i')|-s}-1\big),$$

let e_{R} be defined likewise with N_Y, and put $\mathsf{e}=\max(\mathsf{e}_{\mathrm{L}},\mathsf{e}_{\mathrm{R}})$ and $\mathsf{e}_\star=\frac12\big(\mathsf{d}_1+\sqrt{\mathsf{d}_1^2+4\mathsf{e}^2}\big)$. Then $T_\star=\kappa r\,\mathcal{F}_X M(\hat P)\mathcal{F}_Y^*+T_{\mathrm{mean}}$, with contractions $\mathcal{F}_X\colon\mathcal{H}^X\oplus\mathcal{H}^{\mathcal{S}_X}\to L_0^2(\nu_X)$, where $\mathcal{S}_X=\{S\subseteq X:|S|\geq 2\}$, and $\mathcal{F}_Y$ likewise with Y in place of X, built from ν_X, respectively ν_Y, alone (defined in the proof) and $\|T_{\mathrm{mean}}\|\leq\kappa r\mathsf{e}_\star$. Hence $\|T_\star\|\leq\kappa r(\|M(i)\|+\mathsf{e}_\star)$, and under (a) $\|T_\star\|\leq\kappa r(\mathsf{k}+\mathsf{e}_\star)$. No invariance of the coordinate laws is used.

(c) Decay. *Let $\kappa_0\geq\kappa$ and $0<\omega_s\leq 1$ ($s\geq 2$) satisfy $\kappa_0\omega_s^2\geq\min\{\kappa_0,1+\nu_\chi^s/r^2\}$, where ν_χ bounds $|\operatorname{Cov}_\nu(\chi(t-x),\chi(t'-x))|$ for every coordinate law ν and all distinct $t,t'\in\Omega$ (if $\omega_s=1$ for every $s\geq 2$ no bound on ν_χ is needed; an individual s with $\omega_s<1$ uses the bound at that s). Let A_ω, B_ω be A, B with every column of a set T, respectively S, multiplied by $\omega_{|T|}$, respectively $\omega_{|S|}$, and let M_ω be built from them. Then (b) holds with κ_0, M_ω and contractions $\mathcal{F}_X^\omega$, $\mathcal{F}_Y^\omega$ in place of κ, M, $\mathcal{F}_X$, $\mathcal{F}_Y$; so* (16.10) *for A_ω, B_ω gives*

$\|T_\star\| \le \kappa_0 r(\mathsf{k}+\mathsf{e}_\star)$. If κ and ν_χ/r are admissible functions of r (Lemma 4.18) and rational κ_0, ω_s satisfy these conditions at a threshold p_0, they satisfy them at every prime $p \ge p_0$.

Proof. Centred feature maps. (The maps U_ν and W_S below are the *feature maps*; $\mathcal{F}_X, \mathcal{F}_Y$ are assembled from them.) For a coordinate law ν on $\mathbb{F}_p$ with density $\rho \le \kappa$ let $\iota_\nu \colon L^2(\mathbb{F}_p) \to L^2(\nu)$ be the evaluation $f \mapsto f$; then $\|\iota_\nu f\|_\nu^2 = \mathbb{E}\rho|f|^2 \le \kappa \|f\|^2$ and $\iota_\nu^* g = \rho g$. The *centred evaluation* $U_\nu = P_0 \iota_\nu|_{\mathcal{H}} \colon \mathcal{H} \to L_0^2(\nu)$ has norm at most $\sqrt{\kappa}$; since $P_0\iota_\nu \mathbf{1} = 0$, $P_0\iota_\nu = U_\nu P_{\mathcal{H}}$, with $P_{\mathcal{H}}$ the projection of $L^2(\mathbb{F}_p)$ onto $\mathcal{H}$; and for $g \in L_0^2(\nu)$, $U_\nu^* g = \rho g$ is a mean-zero function supported in $\operatorname{supp}\nu \subseteq \Omega$. Put $m_\nu(t) = \mathbb{E}_{x\sim\nu}\chi(t-x)$ and $\phi_\nu(t,x) = \chi(t-x) - m_\nu(t)$, so that $\mathbb{E}_{x\sim\nu}\phi_\nu(t,x) = 0$. For $S \subseteq X$ with $|S| \ge 2$ let $W_S \colon L^2(\mathbb{F}_p) \to L^2(\nu_X)$, $(W_S f)(x) = \mathbb{E}_t f(t) \prod_{i\in S} \phi_{\nu_i}(t, x_i)$, and define W_T' for $T \subseteq Y$ likewise with the laws ν_j'. Write $U_i = U_{\nu_i}$ and $U_j' = U_{\nu_j'}$, their outputs read as functions of x_i and of y_j.

Step 1 (the contractions $\mathcal{F}_X, \mathcal{F}_Y$). The product law gives the orthogonal decomposition $L^2(\nu_X) = \bigoplus_{R\subseteq X} H_R$, where H_R consists of the functions of x_R whose conditional mean in every coordinate of R vanishes; U_i maps into $H_{\{i\}}$ and W_S into H_S. By independence, $\langle W_S f, W_S f'\rangle = \mathbb{E}_{t,t'} \overline{f(t)} f'(t') \prod_{i\in S} \Gamma_i(t,t')$ with $\Gamma_i(t,t') = \mathbb{E}_{x\sim\nu_i} \phi_{\nu_i}(t,x)\phi_{\nu_i}(t',x)$. Each Γ_i is a positive semidefinite kernel with $\Gamma_i(t,t) \le 1$, and as an operator relative to the uniform law it is $V_i^* V_i$ with $V_i f = \mathbb{E}_t \phi_{\nu_i}(t,\cdot) f(t) = P_0 \iota_{\nu_i} C f$, so $0 \preceq \Gamma_i \preceq \kappa r^2 I$. For $i_1 \in S$, the pointwise product of the Γ_i with $i \ne i_1$ is a positive semidefinite kernel with diagonal at most one (Schur product theorem), so by Lemma 13.9(b) the kernel $\prod_{i\in S}\Gamma_i$ defines an operator between 0 and $\kappa r^2 I$, and $\|W_S\| \le \sqrt{\kappa}\, r$ on all of $L^2(\mathbb{F}_p)$. As the ranges are orthogonal,

$$\mathcal{F}_X\big((f_i)_{i\in X}, (g_S)_S\big) = \sum_{i\in X} \frac{U_i f_i}{\sqrt{\kappa}} + \sum_S \frac{W_S g_S}{\sqrt{\kappa}\, r}$$

is a contraction from $\mathcal{H}^X \oplus \mathcal{H}^{\mathcal{S}_X}$ into $L_0^2(\nu_X)$, $\mathcal{S}_X$ the set of the $S \subseteq X$ with $|S| \ge 2$; $\mathcal{F}_Y$ is built in the same way from the U_j' and the W_T'.

Step 2 (exact supports). Let $g \in L_0^2(\nu_Y)$, and let $\bar g_j$ be its component in $H_{\{j\}}$. A kernel has *exact support* R in y if, for fixed x, it depends only on y_R and has conditional mean zero in each coordinate of R. Such a kernel sees only the component of g in H_R. *Single pairs:* $\mathbb{E}_y \chi(y_j - x_i) g(y) = \mathbb{E}_b \chi(b - x_i)\rho_j'(b)\bar g_j(b) = (C U_j'^* \bar g_j)(x_i)$, with $U_j'^* \bar g_j \in \mathcal{H}$; after the output is centred, the pair (i,j) gives $r D_{ij} U_i \hat P U_j'^*$. *Row stars:* since $\chi(-1) = -1$, $\prod_{j\in T} D_{ij}\chi(y_j - x_i) = A_{iT} \prod_{j\in T}\big(m_{\nu_j'}(x_i) + \phi_{\nu_j'}(x_i, y_j)\big)$. In the expansion, the term with all factors ϕ has exact support T and gives, after centring, $A_{iT} P_0 \iota_{\nu_i} W_T'^* = A_{iT} U_i P_{\mathcal{H}} W_T'^* = A_{iT} U_i (W_T'|_{\mathcal{H}})^*$; the term without a factor ϕ is a function of x_i and vanishes on mean-zero g; the others, with exact supports $\emptyset \ne T' \subsetneq T$, carry the *complementary means* $\prod_{j\in T\setminus T'} m_{\nu_j'}(x_i)$. *Column stars:* $\prod_{i\in S} D_{ij}\chi(y_j - x_i) = B_{jS} \prod_{i\in S}\big(m_{\nu_i}(y_j) + \phi_{\nu_i}(y_j, x_i)\big)$; the term with all factors ϕ gives $B_{jS} W_S U_j'^*$, the term without a factor ϕ has a constant image, which the centring removes, and the others carry complementary means. So $T_\star = T_{\mathrm{pure}} + T_{\mathrm{mean}}$, where T_{mean} collects the terms with complementary means and

$$T_{\mathrm{pure}} = \sum_{i,j} r D_{ij} U_i \hat P U_j'^* + \sum_{i,T} A_{iT} U_i (W_T'|_{\mathcal{H}})^* + \sum_{j,S} B_{jS} W_S U_j'^* = \kappa r\, \mathcal{F}_X M(\hat P) \mathcal{F}_Y^*,$$

as one checks block by block.

Step 3 (the pure part). By Lemma 14.1, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with $\hat P = \pm i I$ on $\mathcal{H}_\pm$, and $M(\hat P)$ acts on the two parts as $M(i) \otimes I$ and as $\overline{M(i)} \otimes I$; so $\big\|M(\hat P)\big\| = \|M(i)\|$ and $\|T_{\mathrm{pure}}\| \le \kappa r \|M(i)\|$. The matrix is decomposed before the contractions are applied, so no commutation of $\hat P$ with the features is used.

Step 4 (the gate). For $\mathsf{k} > 0$, $\|M(i)\| \le \mathsf{k}$ if and only if the Hermitian dilation $\left(\begin{smallmatrix} \mathsf{k} I & M(i) \\ M(i)^* & \mathsf{k} I \end{smallmatrix}\right)$ is positive semidefinite. Order its index set as $(X, \mathcal{S}_X, Y, \mathcal{T}_Y)$, $\mathcal{T}_Y$ the sets T, and take the Schur

complement of its diagonal block $\mathsf{k}I$ on $(\mathcal{S}_X, \mathcal{T}_Y)$: the dilation is positive semidefinite if and only if, by Lemma 13.5(a),

$$\begin{pmatrix} \mathsf{k}I - \mathsf{k}^{-1}AA^* & iD \\ -iD^* & \mathsf{k}I - \mathsf{k}^{-1}BB^* \end{pmatrix} \succeq 0.$$

Multiplying by k and conjugating with the unitary $\mathrm{diag}(I, -iI)$ gives (16.10). This proves (a).

Step 5 (complementary means). Fix a row centre i and an exact support $T' \neq \emptyset$. The terms of the row stars at i with this support add up to $(-1)^{|T'|}\prod_{j\in T'} D_{ij}\, c(x_i)\prod_{j\in T'}\phi_{\nu'_j}(x_i, y_j)$, where $c(t) = \prod_{l\in N_X(i)\setminus T'}(1 - D_{il}m_{\nu'_l}(t)) - 1$ satisfies $|c| \le (1+\beta_\chi)^{|N_X(i)|-|T'|} - 1$ on Ω. Their operator maps $H_{T'}(\nu_Y)$ into $H_{\{i\}}(\nu_X)$: for $|T'| \ge 2$ it is $g \mapsto \pm P_0[c\cdot \iota_{\nu_i} W'^*_{T'} g]$, of norm at most $\kappa r\,\|c\|_\infty$ by Step 1, and for $T' = \{j\}$ it is $g \mapsto \pm P_0[c\cdot \iota_{\nu_i} C^* U'^*_j \bar g_j]$, with the same bound, since $\mathbb{E}_{y_j}\phi_{\nu'_j}(t, y_j)\bar g_j(y_j) = (C^*U'^*_j\bar g_j)(t)$. Likewise the column stars at j with exact support S' give an operator from $H_{\{j\}}(\nu_Y)$ into $H_{S'}(\nu_X)$, namely $g \mapsto W_{S'}\iota^*_{\nu'_j}(b\,\bar g_j)$ with $|b| \le (1+\beta_\chi)^{|N_Y(j)|-|S'|} - 1$ on Ω, and for $S' = \{i\}$ the same with $P_0\iota_{\nu_i}C$ in place of $W_{S'}$; its norm is at most $\kappa r\,\|b\|_\infty$. Write $L^2_0(\nu_X) = \mathcal{V}_X \oplus \mathcal{W}_X$ with $\mathcal{V}_X = \bigoplus_i H_{\{i\}}$ and $\mathcal{W}_X = \bigoplus_{|S|\ge 2} H_S$, and likewise for Y. The norm of an operator matrix between orthogonal sums is at most the norm of the matrix of the norms of its entries. The block $\mathcal{V}_X \leftarrow \mathcal{V}_Y$ of T_{mean} has entries $H_{\{i\}} \leftarrow H_{\{j\}}$ of norm at most κr times the sum of the entries of the two matrices of (b), so its norm is at most $\kappa r(\mathsf{d}_{\mathrm{L}} + \mathsf{d}_{\mathrm{R}})$ by Schur's test. The block $\mathcal{V}_X \leftarrow \mathcal{W}_Y$ has entries $H_{\{i\}} \leftarrow H_{T'}$ of norm at most $\kappa r e_{iT'}$, with $e_{iT'} = (1+\beta_\chi)^{|N_X(i)|-|T'|} - 1$ for $T' \subseteq N_X(i)$ and zero otherwise; the squared norm of the matrix $(e_{iT'})$ is the norm of its Gram matrix $(\sum_{T'} e_{iT'}e_{i'T'})_{i,i'}$, at most its largest row sum $\mathsf{e}^2_{\mathrm{L}}$. Likewise the block $\mathcal{W}_X \leftarrow \mathcal{V}_Y$ has norm at most $\kappa r\mathsf{e}_{\mathrm{R}}$, and the block $\mathcal{W}_X \leftarrow \mathcal{W}_Y$ vanishes, since every star has a singleton centre. So $\|T_{\mathrm{mean}}\| \le \kappa r \left\|\left(\begin{smallmatrix} \mathsf{d}_1 & \mathsf{e} \\ \mathsf{e} & 0 \end{smallmatrix}\right)\right\| = \kappa r \mathsf{e}_\star$, which with Step 3 proves (b).

Step 6 (decay). The Gram kernel of $W_S\mathbf{1}_\Omega$ is $\mathbf{1}_\Omega(t)\mathbf{1}_\Omega(t')\prod_{i\in S}\Gamma_i(t,t')$, with diagonal at most one and off-diagonal entries of modulus at most $\nu_\chi^{|S|}$; its absolute row sums relative to the uniform law are at most $r^2 + \nu_\chi^{|S|}$, and Schur's test gives $\|W_S\mathbf{1}_\Omega\|^2 \le r^2 + \nu_\chi^{|S|}$. Since U'^*_j has mean-zero range supported in Ω, $W_S U'^*_j = W_S\mathbf{1}_\Omega P_{\mathcal{H}} U'^*_j$, the two projections being applied in this order; and since ι_{ν_i} sees only Ω, $U_i P_{\mathcal{H}} W'^*_T = U_i P_{\mathcal{H}}\mathbf{1}_\Omega W'^*_T = U_i(W'_T\mathbf{1}_\Omega P_{\mathcal{H}})^*$. By the choice of ω_s, $\|W_S\mathbf{1}_\Omega P_{\mathcal{H}}\| \le \sqrt{\kappa_0}\, r\omega_{|S|}$: if $\omega_{|S|} < 1$, then $\kappa_0\omega^2_{|S|} \ge 1 + \nu_\chi^{|S|}/r^2$, and otherwise Step 1 applies. So the maps of Step 1 with $\sqrt{\kappa_0}$ in place of $\sqrt{\kappa}$ and $W_S\mathbf{1}_\Omega P_{\mathcal{H}}/(\sqrt{\kappa_0}\, r\omega_{|S|})$ in place of $W_S/(\sqrt{\kappa}\, r)$, and likewise for Y, are contractions $\mathcal{F}^\omega_X$, $\mathcal{F}^\omega_Y$, and $T_{\mathrm{pure}} = \kappa_0 r\,\mathcal{F}^\omega_X M_\omega(\hat P)\mathcal{F}^{\omega *}_Y$, while $\|T_{\mathrm{mean}}\| \le \kappa r\mathsf{e}_\star \le \kappa_0 r\mathsf{e}_\star$. At a prime $p \ge p_0$, $\kappa(p) \le \kappa(p_0) \le \kappa_0$, and $1 + \nu_\chi^s/r^2 = 1 + (\nu_\chi/r)^s r^{s-2}$ ($s \ge 2$) is at most its value at p_0, by admissibility; the gate does not depend on p. □

For an invariant law the same matrix appears in Corollary 14.5; under a product law the leaves are centred in their own laws and their means are paid by $\mathsf{e}_\star$. In the conditional rows of the upper range (Sections 27.6.1 and 27.6.2) the coordinate laws are the allowed-set laws, Ω is the complement of the middle labels, $\kappa = 2^{\varpi}/(1 - e^-_\varpi(z))$, β_χ is the one-character mean bound and ν_χ the covariance bound (display (PR3) in Section 27.6.2); in the bands the bound $\min\{C_\star, \kappa_0 r(\mathsf{k} + \mathsf{e}_\star)\}$ takes the place of the scalar majorant $C_\star$ of (27.5) in the correlation bound (27.6).

Joint fold and matching rows. The completion of an upper coordinate bounds its fold row and its matching row jointly, relative to one chart (Lemma 17.5(f)). The two rows have the same allowed sets, so their star parts factor through the same contractions, and the joint bound is one more matrix test.

Lemma 16.23 (Joint fold and matching rows)**.** *Let $p \equiv 3 \pmod 4$, and in the setting of this subsection let κ bound the densities of the allowed-set laws λ_i relative to the uniform law and $I_0 \le \lambda(\mathcal{I}_{\mathrm{pin}})$. Let two sets of signed cross pairs be given, the* fold *row* F *and the* matching *row* E,

with signed incidence matrices $D_{\mathrm{F}}, D_{\mathrm{E}} \in \{0,\pm1\}^{n\times n}$ ($D_\bullet(i,j) = \varsigma_{ij}$ on the pairs of the row and zero elsewhere) and $D_{\mathrm{E}}^{\mathsf{T}} = -D_{\mathrm{E}}$. For $\bullet \in \{\mathrm{F},\mathrm{E}\}$ let $K_\bullet$ be the raw kernel relative to $\mu\otimes\mu$ (the indicator that $x([n])\cap y([n]) = \emptyset$, times the product over the pairs of $1+\chi_{ij}$), $Z_\bullet = \langle \mathbf{1}, K_\bullet \mathbf{1}\rangle_\mu$ and $\pi_\bullet = K_\bullet\,\mu\otimes\mu/Z_\bullet$. Let ν be the marginal of π_{E}, and suppose $d\nu/d\mu \geq e^{-\ell_{\mathrm{E}}}$. Relative to $\nu\otimes\nu$ let T_{F} and T_{E} be the densities of π_{F} and π_{E}, and u_{L}, u_{R} the densities of the two marginals of π_{F} relative to ν, and put $C_{\mathrm{F}} = T_{\mathrm{F}} - u_{\mathrm{L}}\otimes u_{\mathrm{R}}$ and $C_{\mathrm{E}} = T_{\mathrm{E}} - \mathbb{J}$, as operators on $L^2(\nu)$. With κ_0 and ω_s as in Proposition 16.22(c) for the laws λ_i and $\Omega = \mathbb{F}_p \setminus \mathcal{M}$, put, for $D = D_{\mathrm{F}}, D_{\mathrm{E}}$ and all $S\subseteq[n]$ with $|S|\geq 2$,

$$A_D(i,S) = (-1)^{|S|}\omega_{|S|}\prod_{j\in S} D(i,j), \qquad B_D(i,S) = \omega_{|S|}\prod_{j\in S} D(j,i),$$

and

$$\mathcal{Z} = \begin{pmatrix} D_{\mathrm{E}} & D_{\mathrm{F}} \\ -D_{\mathrm{F}}^{\mathsf{T}} & D_{\mathrm{E}}\end{pmatrix}, \qquad \mathcal{A} = \begin{pmatrix} A_{D_{\mathrm{E}}} & A_{D_{\mathrm{F}}} \\ B_{D_{\mathrm{F}}} & A_{D_{\mathrm{E}}}\end{pmatrix}, \qquad \mathcal{G} = \mathcal{A}\mathcal{A}^*.$$

Then C_{F} and C_{E} annihilate the constants on both sides, $C_{\mathrm{E}} = C_{\mathrm{E}}^*$, and $A_{D_{\mathrm{E}}} = B_{D_{\mathrm{E}}}$.

(a) *Let $\mathsf{k} > 0$ satisfy*

$$\mathsf{k}^2 I + i\mathsf{k}\mathcal{Z} - \mathcal{G} \succeq 0, \tag{16.11}$$

a Hermitian test of size $2n$, rational of size $4n$ in real form (Remark 14.7). For each row $\bullet$ let $e_\bullet$ bound the norm of the complementary-mean operator $T^\bullet_{\mathrm{mean}}$ of Proposition 16.22(b) for $D_\bullet$ on $L^2_0(\lambda)$, where λ is the product of the allowed-set laws, divided by $\kappa_0 r$ (for instance $\mathsf{e}_\star$ of Proposition 16.22(b) for $D_\bullet$); let $S^\star_\bullet \geq \|P_0 K^{\mathrm{st}}_\bullet P_0\|$ for the star part $K^{\mathrm{st}}_\bullet$ of $K_\bullet$, $N_\bullet \geq \|T^\bullet_{\mathrm{ns}}\| + \|T^\bullet_{\mathrm{coll}}\|$ for its nonstar and collision parts, $\varepsilon_\bullet \geq |Z_\bullet - 1|$ with $\varepsilon_\bullet < 1$, and $m_\bullet \geq \|P_0 K_\bullet \mathbf{1}\|, \|P_0 K_\bullet^* \mathbf{1}\|$, all on $L^2(\mu)$. Then

$$\begin{pmatrix} C_{\mathrm{E}} & C_{\mathrm{F}} \\ C_{\mathrm{F}}^* & C_{\mathrm{E}}\end{pmatrix} \succeq -\mathfrak{k}_p I \quad \text{on } L^2(\nu)\oplus L^2(\nu), \qquad \mathfrak{k}_p = e^{\ell_{\mathrm{E}}}\Big[\frac{\kappa_0 r(\mathsf{k}+e_{\mathrm{F}}+e_{\mathrm{E}})}{I_0} + \mathfrak{c}_{\mathrm{F}} + \mathfrak{c}_{\mathrm{E}}\Big],$$

where $\mathfrak{c}_\bullet = (\varepsilon_\bullet S^\star_\bullet + N_\bullet)/(1-\varepsilon_\bullet) + m_\bullet^2/(1-\varepsilon_\bullet)^2$.

(b) *Let $\mathcal{B}$ be a finite set of primes and, for $p\in\mathcal{B}$, let $C_{\mathrm{F},p}$ and $C_{\mathrm{E},p} = C_{\mathrm{E},p}^*$ be operators on $L^2(\nu_p)$ that annihilate the constants on both sides, for instance those of (a) at p. Put $C_{\mathcal{B}} = \bigotimes_p(\mathbb{J}_p + C_{\mathrm{F},p}) - \mathbb{J}_{\mathcal{B}}$ and $E_{\mathcal{B}} = \bigotimes_p(\mathbb{J}_p + C_{\mathrm{E},p}) - \mathbb{J}_{\mathcal{B}}$ on $L^2(\bigotimes_p \nu_p)$. They act on each exact prime-support sector $\bigotimes_{p\in S} L^2_0(\nu_p)\otimes\mathbf{1}$ ($\emptyset\neq S\subseteq\mathcal{B}$) as $\bigotimes_{p\in S} C_{\mathrm{F},p}$ and $\bigotimes_{p\in S} C_{\mathrm{E},p}$, so the joint block of $(E_{\mathcal{B}}, C_{\mathcal{B}})$ is the direct sum of the joint blocks of the sectors. If $\|C_{\mathrm{F},p}\| \leq c < 1$ and $\|C_{\mathrm{E},p}\| \leq h < 1$ for every p, the joint block of every sector with at least two primes is $\succeq -(c^2+h^2)I$; so the joint block of $(E_{\mathcal{B}}, C_{\mathcal{B}})$ is $\succeq -\max(c^2+h^2, \max_p \mathfrak{k}'_p)I$ whenever the joint block at each prime p is $\succeq -\mathfrak{k}'_p I$.*

(c) *If moreover $C_{\mathrm{F},p} = T_{\mathrm{F},p} - u_{\mathrm{L},p}\otimes u_{\mathrm{R},p}$ for a kernel $T_{\mathrm{F},p}$ of mass one with marginal densities $u_{\mathrm{L},p}$, $u_{\mathrm{R},p}$ relative to ν_p, and $U_{\mathrm{L}} = \bigotimes_p u_{\mathrm{L},p}$, $U_{\mathrm{R}} = \bigotimes_p u_{\mathrm{R},p}$ satisfy $\|U_{\mathrm{L}} - \mathbf{1}\|, \|U_{\mathrm{R}} - \mathbf{1}\| \leq \upsilon$, then $T_{\mathrm{F},\mathcal{B}} = \bigotimes_p T_{\mathrm{F},p}$ satisfies*

$$\|T_{\mathrm{F},\mathcal{B}} - U_{\mathrm{L}}\otimes U_{\mathrm{R}} - C_{\mathcal{B}}\| \leq (2\upsilon+\upsilon^2)\Big[\prod_{p\in\mathcal{B}}(1+\|C_{\mathrm{F},p}\|) - 1\Big],$$

which is at most $(2\upsilon+\upsilon^2)(e^{Nc}-1)$ if $\mathcal{B}$ has at most N primes, for an integer $N\geq|\mathcal{B}|$, and $\|C_{\mathrm{F},p}\|\leq c$ for every $p\in\mathcal{B}$.

Proof. Preliminaries. At $p\equiv 3 \pmod 4$ the factor of a pair is $1+\varsigma_{ij}\chi(y_j - x_i)$. The factor of (i,j) in $K_{\mathrm{E}}(y,x)$ is $1+\varsigma_{ij}\chi(x_j-y_i) = 1 + D_{\mathrm{E}}(j,i)\chi(y_i - x_j)$, the factor of (j,i) in $K_{\mathrm{E}}(x,y)$; so K_{E} is symmetric, π_{E} has two equal marginals ν, and T_{E} is a symmetric kernel with marginal densities one; hence $C_{\mathrm{E}} = C_{\mathrm{E}}^*$ annihilates the constants. As $T_{\mathrm{F}}\mathbf{1} = u_{\mathrm{L}}$ and $\mathbb{E}_\nu u_{\mathrm{R}} = 1$, $C_{\mathrm{F}}\mathbf{1} = 0$, and likewise $C_{\mathrm{F}}^*\mathbf{1} = 0$. Finally $B_{D_{\mathrm{E}}}(i,S) = \omega_{|S|}\prod_{j\in S}(-D_{\mathrm{E}}(i,j)) = A_{D_{\mathrm{E}}}(i,S)$.

(a) *The actual chart.* Let $g = d\nu/d\mu = K_{\mathrm{E}}\mathbf{1}/Z_{\mathrm{E}}$, $g_{\mathrm{L}} = K_{\mathrm{F}}\mathbf{1}/Z_{\mathrm{F}}$ and $g_{\mathrm{R}} = K_{\mathrm{F}}^{*}\mathbf{1}/Z_{\mathrm{F}}$. Then $T_{\mathrm{F}}(x,y) = K_{\mathrm{F}}(x,y)/(Z_{\mathrm{F}}g(x)g(y))$, $u_{\mathrm{L}} = g_{\mathrm{L}}/g$ and $u_{\mathrm{R}} = g_{\mathrm{R}}/g$, so for $f, h \in L^2(\nu)$

$$\langle f, C_{\mathrm{F}}h\rangle_{\nu} = \mathbb{E}_{\pi_{\mathrm{F}}}[\overline{f(x)}h(y)] - \mathbb{E}_{\nu}[\overline{f}u_{\mathrm{L}}]\,\mathbb{E}_{\nu}[u_{\mathrm{R}}h] = \langle f, (K_{\mathrm{F}}/Z_{\mathrm{F}} - g_{\mathrm{L}}\otimes g_{\mathrm{R}})h\rangle_{\mu},$$

and likewise $\langle f, C_{\mathrm{E}}h\rangle_{\nu} = \langle f, (K_{\mathrm{E}}/Z_{\mathrm{E}} - g\otimes g)h\rangle_{\mu}$. So the quadratic form of the joint block at (f,h) is that of the self-adjoint operator $\mathbb{K}$ on $L^2(\mu)^2$ with the diagonal entries $K_{\mathrm{E}}/Z_{\mathrm{E}} - g\otimes g$ and the off-diagonal entries $K_{\mathrm{F}}/Z_{\mathrm{F}} - g_{\mathrm{L}}\otimes g_{\mathrm{R}}$ and its adjoint, and $\|f\|_{\mu}^2 = \mathbb{E}_{\nu}[|f|^2/g] \le e^{\ell_{\mathrm{E}}}\|f\|_{\nu}^2$. Hence $\mathbb{K} \succeq -cI$ with $c \ge 0$ gives the joint block $\succeq -ce^{\ell_{\mathrm{E}}}I$. A block operator $\left(\begin{smallmatrix} X & Y \\ Y^{*} & X \end{smallmatrix}\right)$ has norm at most $\|X\| + \|Y\|$, which bounds every error term below.

Centring and normalization. For a kernel K with $Z = \langle \mathbf{1}, K\mathbf{1}\rangle$ and marginal densities $g_1 = K\mathbf{1}/Z$, $g_2 = K^{*}\mathbf{1}/Z$ the exact identity $K/Z - g_1\otimes g_2 = P_0KP_0/Z - (g_1-\mathbf{1})\otimes(g_2-\mathbf{1})$ holds, and for $K = K_{\bullet}$ the rank-one term has norm at most $m_{\bullet}^2/(1-\varepsilon_{\bullet})^2$. On $\mathcal{I}_{\mathrm{pin}}\times\mathcal{I}_{\mathrm{pin}}$, $K_{\bullet} = \mathbf{1}\otimes\mathbf{1} + K_{\bullet}^{\mathrm{st}} + T_{\mathrm{ns}}^{\bullet} - T_{\mathrm{coll}}^{\bullet}$, the expansion of Lemma 14.18 with the collision part of Proposition 16.21; so $P_0K_{\bullet}P_0/Z_{\bullet} = P_0K_{\bullet}^{\mathrm{st}}P_0 + (Z_{\bullet}^{-1}-1)P_0K_{\bullet}^{\mathrm{st}}P_0 + Z_{\bullet}^{-1}P_0(T_{\mathrm{ns}}^{\bullet} - T_{\mathrm{coll}}^{\bullet})P_0$, and the last two terms have norm at most $(\varepsilon_{\bullet}S_{\bullet}^{\star} + N_{\bullet})/(1-\varepsilon_{\bullet})$. Hence $\mathbb{K} \succeq \mathbb{K}^{\mathrm{st}} - (\mathfrak{c}_{\mathrm{F}} + \mathfrak{c}_{\mathrm{E}})I$, where $\mathbb{K}^{\mathrm{st}}$ is the joint block of the operators $P_0K_{\bullet}^{\mathrm{st}}P_0$.

Restriction. Both rows have the law $\mu = \lambda(\cdot \mid \mathcal{I}_{\mathrm{pin}})$, and $P = \lambda(\mathcal{I}_{\mathrm{pin}}) \ge I_0$. By Lemma 13.23(a), with the isometry $\iota f = \mathbf{1}_{\mathcal{I}_{\mathrm{pin}}}f/\sqrt{P}$, which preserves centring, $\langle F, \mathbb{K}^{\mathrm{st}}F\rangle = \langle \iota F, \mathbb{T}\iota F\rangle / P$ for $F \in L_0^2(\mu)^2$, where $\mathbb{T}$ is the joint block, on $L_0^2(\lambda)^2$, of the operators $T_{\star}^{\bullet}$ of Proposition 16.22 for $\nu_X = \nu_Y = \lambda$ and $D = D_{\bullet}$; and $\mathbb{K}^{\mathrm{st}}$ vanishes on the constants. So $\mathbb{T} \succeq -c'I$ gives $\mathbb{K}^{\mathrm{st}} \succeq -(c'/I_0)I$.

Pure stars. By Proposition 16.22(b),(c), $T_{\star}^{\bullet} = \kappa_0 r\,\mathcal{F}M_{\bullet}(\hat{P})\mathcal{F}^{*} + T_{\mathrm{mean}}^{\bullet}$, with one contraction $\mathcal{F} = \mathcal{F}_X^{\omega} = \mathcal{F}_Y^{\omega}$ for both rows and both laws (it depends only on λ; take all sets $S \subseteq [n]$, $|S| \ge 2$), with $M_{\bullet}$ built from $D_{\bullet}$, $A_{D_{\bullet}}$ and $B_{D_{\bullet}}$, and with $\|T_{\mathrm{mean}}^{\bullet}\| \le \kappa_0 r e_{\bullet}$. So $\mathbb{T} = \kappa_0 r(\mathcal{F}\oplus\mathcal{F})\mathbb{M}(\mathcal{F}\oplus\mathcal{F})^{*} + \mathbb{T}_{\mathrm{mean}}$ with $\|\mathbb{T}_{\mathrm{mean}}\| \le \kappa_0 r(e_{\mathrm{E}} + e_{\mathrm{F}})$, where $\mathbb{M}$ has the diagonal entries $M_{\mathrm{E}}(\hat{P})$ and the off-diagonal entries $M_{\mathrm{F}}(\hat{P})$ and $M_{\mathrm{F}}(\hat{P})^{*}$. As $\hat{P}^{*} = -\hat{P}$, $M_{\mathrm{F}}(\hat{P})^{*}$ has the singleton block $-D_{\mathrm{F}}^{\mathsf{T}}\otimes\hat{P}$, the block $B_{D_{\mathrm{F}}}\otimes I$ from the sets to the singletons and $A_{D_{\mathrm{F}}}^{*}\otimes I$ from the singletons to the sets; and $D_{\mathrm{E}}^{\mathsf{T}} = -D_{\mathrm{E}}$, $A_{D_{\mathrm{E}}} = B_{D_{\mathrm{E}}}$ give $M_{\mathrm{E}}(\hat{P})^{*} = M_{\mathrm{E}}(\hat{P})$. Grouping the singletons of the two copies and the sets of the two copies,

$$\mathbb{M} = \begin{pmatrix} \mathcal{Z}\otimes\hat{P} & \mathcal{A}\otimes I \\ \mathcal{A}^{*}\otimes I & 0 \end{pmatrix}.$$

On $\mathcal{H}_{\pm}$ it acts as $\left(\begin{smallmatrix} \pm i\mathcal{Z} & \mathcal{A} \\ \mathcal{A}^{*} & 0 \end{smallmatrix}\right)\otimes I$; these are complex conjugate Hermitian matrices, $\mathcal{Z}$ being real and antisymmetric. By Lemma 13.5(a), eliminating the block $\mathsf{k}I$, the one with $+i$ is $\succeq -\mathsf{k}I$ if and only if $\mathsf{k}I + i\mathcal{Z} - \mathsf{k}^{-1}\mathcal{G} \succeq 0$, which is (16.11). So $\mathbb{M} \succeq -\mathsf{k}I$ and $\mathbb{T} \succeq -\kappa_0 r(\mathsf{k} + e_{\mathrm{F}} + e_{\mathrm{E}})I$, and the three steps give $\mathfrak{k}_p$.

(b) As $C_{\mathrm{F},p}$ annihilates the constants on both sides, $\bigotimes_p(\mathbb{J}_p + C_{\mathrm{F},p}) = \sum_{S\subseteq\mathcal{B}}\bigotimes_{p\in S}C_{\mathrm{F},p}\otimes\bigotimes_{p\notin S}\mathbb{J}_p$, the term of S acting on the sector of S and vanishing on the others; the same holds for $E_{\mathcal{B}}$. On a sector with $|S| \ge 2$ the joint block is $\succeq -(\left\|\bigotimes_{p\in S}C_{\mathrm{E},p}\right\| + \left\|\bigotimes_{p\in S}C_{\mathrm{F},p}\right\|)I \succeq -(h^{|S|} + c^{|S|})I \succeq -(h^2+c^2)I$; on a singleton sector it is the joint block at the prime.

(c) Expand $T_{\mathrm{F},\mathcal{B}} = \bigotimes_p(u_{\mathrm{L},p}\otimes u_{\mathrm{R},p} + C_{\mathrm{F},p})$ over the sets $S \subseteq \mathcal{B}$ of the primes at which the factor $C_{\mathrm{F},p}$ is taken. The term of $S = \emptyset$ is $U_{\mathrm{L}}\otimes U_{\mathrm{R}}$, and $C_{\mathcal{B}}$ is the sum of the other terms with the marginal rank $\bigotimes_{p\notin S}u_{\mathrm{L},p}\otimes u_{\mathrm{R},p} = a\otimes b$ of the inactive primes replaced by $\mathbb{J} = \mathbf{1}\otimes\mathbf{1}$. Since $\mathbb{E}u_{\mathrm{L},p} = 1$, $\left\|\bigotimes_{p\in A}u_{\mathrm{L},p} - \mathbf{1}\right\|^2 = \prod_{p\in A}\|u_{\mathrm{L},p}\|^2 - 1$ is nondecreasing in A, so $\|a-\mathbf{1}\| \le \upsilon$, and likewise $\|b-\mathbf{1}\| \le \upsilon$; hence $a\otimes b - \mathbf{1}\otimes\mathbf{1} = (a-\mathbf{1})\otimes b + \mathbf{1}\otimes(b-\mathbf{1})$ has norm at most $\upsilon(1+\upsilon)+\upsilon$. Summing $\prod_{p\in S}\|C_{\mathrm{F},p}\|$ times this over $S \ne \emptyset$ gives the bound, and $1 + x \le e^x$ the last one. □

Corollary 16.24 (The joint fluctuation in a positive row)**.** *In Lemma 16.23(b) let $h \ge 0$ and $\mathsf{k}_{\mathcal{B}} \ge h$ satisfy $E_{\mathcal{B}} \succeq -hI$ and $\left(\begin{smallmatrix} E_{\mathcal{B}} & C_{\mathcal{B}} \\ C_{\mathcal{B}}^{*} & E_{\mathcal{B}} \end{smallmatrix}\right) \succeq -\mathsf{k}_{\mathcal{B}}I$ (a bound k of the joint block with $\mathsf{k} < h$ is replaced*

by h). If $0 \le m \le H$ and $a_{\mathrm{S}} \ge H a_{\mathrm{D}} \ge 0$, then

$$\begin{pmatrix} a_{\mathrm{S}} E_{\mathcal{B}} & m a_{\mathrm{D}} C_{\mathcal{B}} \\ m a_{\mathrm{D}} C_{\mathcal{B}}^{*} & a_{\mathrm{S}} E_{\mathcal{B}} \end{pmatrix} \succeq -(h a_{\mathrm{S}} + H a_{\mathrm{D}}(\mathsf{k}_{\mathcal{B}} - h)) I.$$

With $T_{\mathrm{F},\mathcal{B}} - U_{\mathrm{L}} \otimes U_{\mathrm{R}}$ in place of $C_{\mathcal{B}}$ the bound decreases by at most $H a_{\mathrm{D}}(2\upsilon + \upsilon^2)[\prod_p (1 + \|C_{\mathrm{F},p}\|) - 1]$, by Lemma 16.23(c); the $S = \emptyset$ term $U_{\mathrm{L}} \otimes U_{\mathrm{R}}$ of Lemma 16.23(c) is not counted in this fluctuation bound; it is instead absorbed by the mean allowance term in Lemma 17.5(f), where m is the fold mass m_b and a_{S}, a_{D} are the five-type weights.

Proof. The matrix is the sum of $m a_{\mathrm{D}}$ copies of the joint block and of $a_{\mathrm{S}} - m a_{\mathrm{D}}$ copies of $\operatorname{diag}(E_{\mathcal{B}}, E_{\mathcal{B}})$, both coefficients being nonnegative as $m \le H$ and $a_{\mathrm{S}} \ge H a_{\mathrm{D}}$. Its lower bound is $-(m a_{\mathrm{D}} \mathsf{k}_{\mathcal{B}} + (a_{\mathrm{S}} - m a_{\mathrm{D}}) h) = -(h a_{\mathrm{S}} + m a_{\mathrm{D}}(\mathsf{k}_{\mathcal{B}} - h))$, and $m \le H$ is used only after $\mathsf{k}_{\mathcal{B}} - h \ge 0$. The last assertion is Lemma 16.23(c), multiplied by $m a_{\mathrm{D}} \le H a_{\mathrm{D}}$ in the off-diagonal blocks. □

When the pins of a row make some allowed sets disjoint, the coordinates whose allowed sets contain a given label lie in one member of the family $\mathcal{S}$ of the lemma, and the density bound κ_0 then enters through the principal submatrices on the members of $\mathcal{S}$.

Lemma 16.25 (Positive potentials)**.** *In Lemma 16.23(a) let ρ_i be the density of λ_i relative to the uniform law, and let $\mathcal{S}$ be a family of sets of coordinates (c, i) of $\mathcal{Z}$ ($c \in \{1, 2\}$ the copy, $i \in [n]$) such that for every $t \in \mathbb{F}_p$ the coordinates with $\rho_i(t) > 0$ lie in one member of $\mathcal{S}$. Let T be a real symmetric $2n \times 2n$ matrix with $T \pm i\mathcal{Z} \succeq 0$, for instance $T = |\mathcal{Z}| = (-\mathcal{Z}^2)^{1/2}$. If $\mathsf{k} > 0$ and*

$$\mathsf{k}^2 I_S - \mathsf{k}\kappa_0 T_S - \kappa_0^2 \mathcal{G}_S \succeq 0 \qquad (S \in \mathcal{S}), \tag{16.12}$$

where X_S denotes the principal submatrix of X on S, then the conclusion of Lemma 16.23(a) holds without (16.11) *and with $r\mathsf{k}/I_0$ in place of $\kappa_0 r\mathsf{k}/I_0$ in $\mathfrak{k}_p$; all other terms are kept. For rational T and rational gate parameters, both $T \pm i\mathcal{Z} \succeq 0$ and* (16.12) *are rational tests in real form (Remark 14.7).*

Proof. In the proof of Lemma 16.23(a) only the bound of the pure part $\mathbb{P} = \kappa_0 r(\mathcal{F} \oplus \mathcal{F})\mathbb{M}(\mathcal{F} \oplus \mathcal{F})^*$ changes. Let $\Psi_i \colon L^2(\lambda_i) \to L^2(\mathbb{F}_p)$, $g \mapsto \sqrt{\rho_i}\, g$, an isometry onto the functions supported in the allowed set of i; then $\Psi_i \iota_{\lambda_i}$ is the multiplication by $\sqrt{\rho_i}$, and $U_{\lambda_i} = \Psi_i^* \Pi_i \sqrt{\rho_i}\,|_{\mathcal{H}}$, where $\Pi_i = \Psi_i P_0 \Psi_i^*$ is an orthogonal projection. Let D_ρ be the multiplication by $\sqrt{\rho_i}$ on the copy of $L^2(\mathbb{F}_p)$ of the coordinate (c, i), extend $\hat{P}$ by zero to the constants, and let $\mathcal{W}$ be the higher part of $\mathcal{F} \oplus \mathcal{F}$, a contraction. Since the singleton part of $\mathcal{F}$ is $U_{\lambda_i}/\sqrt{\kappa_0}$, $\mathbb{P}$ is the compression, by Π on the singleton coordinates (followed by Ψ^*), of

$$\mathbb{S} = \begin{pmatrix} r D_\rho (\mathcal{Z} \otimes \hat{P}) D_\rho & \sqrt{\kappa_0}\, r D_\rho (\mathcal{A} \otimes I) \mathcal{W}^* \\ \sqrt{\kappa_0}\, r \mathcal{W} (\mathcal{A}^* \otimes I) D_\rho & 0 \end{pmatrix},$$

and compressions keep lower bounds; so it suffices that $\mathbb{S} \succeq -r\mathsf{k} I$. By Lemma 13.5(a) on the block $r\mathsf{k}I$ and $\mathcal{W}^*\mathcal{W} \preceq I$, this follows from $r\mathsf{k}I + r D_\rho (\mathcal{Z} \otimes \hat{P}) D_\rho - (\kappa_0 r/\mathsf{k}) D_\rho (\mathcal{G} \otimes I) D_\rho \succeq 0$. On the eigenspaces of $\hat{P}$ for i, $-i$ and 0 the operator $T \otimes I + \mathcal{Z} \otimes \hat{P}$ is $T + i\mathcal{Z}$, $T - i\mathcal{Z}$ and T, all positive semidefinite (the last by averaging the first two); so $D_\rho(\mathcal{Z} \otimes \hat{P}) D_\rho \succeq -D_\rho (T \otimes I) D_\rho$, and it suffices that $\mathsf{k}^2 I - D_\rho((\mathsf{k}T + \kappa_0 \mathcal{G}) \otimes I) D_\rho \succeq 0$. This operator is the multiplication by the matrix function $t \mapsto \mathsf{k}^2 I - D(t)(\mathsf{k}T + \kappa_0 \mathcal{G}) D(t)$, $D(t) = \operatorname{diag}(\sqrt{\rho_i(t)})$. At each t, $D(t)$ vanishes outside a set $S \in \mathcal{S}$ and $0 \preceq D(t) \preceq \sqrt{\kappa_0} I$; as $\mathsf{k}T + \kappa_0 \mathcal{G} \succeq 0$, the matrix $D(t)(\mathsf{k}T + \kappa_0 \mathcal{G}) D(t)$ is at most $\kappa_0 \lambda_{\max}((\mathsf{k}T + \kappa_0 \mathcal{G})_S) I \preceq \mathsf{k}^2 I$, by (16.12). For $T = |\mathcal{Z}|$, $T \pm i\mathcal{Z} \succeq 0$ because $i\mathcal{Z}$ is Hermitian with $|i\mathcal{Z}| = |\mathcal{Z}|$. □

Remark 16.26 (The pinned rows of the construction). The conditional rows of Section 27.6.2 come in six geometries (the other type L, F or E, with one or two layers), each for the two laws F and E. In all twelve there are no edges inside a row, $n = z \in \{3, 6, 12\}$, and the pin bound is $\varpi \in \{0, 1, 2, 4\}$; in

the two geometries with other type F the actual pin numbers are mixed ($\{0,1\}$ and $\{0,2\}$) and are majorized by ϖ. The classes are those of the compressed partition, grouped with $\mathfrak{q}=0$: the integer $\mathfrak{b}=\mathrm{oc}(\mathcal{C})$, at most 1280, is used exactly, and for the classes of case (A) of Lemma 16.2(b) the better of two covers is taken class by class (a class may carry several records, Definition 16.6). The E law of the E geometries has no cross pairs, so its nonstar part vanishes; the other ten rows have 7164 classes, counted with their kernel constants $c_{\ker}(\mathcal{C})$. In all twelve the multisets of row and column degrees coincide. Hence $\Delta_{\mathrm{pin}}=\varpi+\max_i|N_X(i)|$ and $D_X=D_Y$, and the evaluator, which computes $\Upsilon_X(1-e_{\varpi}^{\mathrm{pin}})^{-n}(e^{U_Y}-1)$, returns this common value. These facts are read off the row geometry and the stored partitions (`research/advance_2026_10_17/sparse/product_rows_all.py` and `product_rows_all_parts.json`; the exact grouping is `research/advance_2026_10_24/upper/conditional_fine_parts.json`, checked by `check_conditional_fine_buckets.py` of the same directory into `conditional_fine_parts_review.json`). Rows with internal edges, which would need the merge charges of Theorem 16.9, do not occur. The fold and matching rows of one geometry have the same pins, which is the hypothesis of Lemma 16.23.

The forms that the certificates evaluate. *Nonstar bounds.* For the two root cuts of the upper range ($n=36$, $\Delta^+=10$, $\Delta=20$) the partition is a refinement of the compressed partition with witnesses of sizes 2, 3 and 4, and the nonstar bound is Corollary 16.11(b) with the graded errors, the free partners of the records (Definition 16.6) and the constants of (16.5), under the monotone-factor condition (16.6). For the direct Route B criterion (Lemma 27.6), from P_3 to 2^{52}, the coefficients are evaluated at the nineteen points $2^{47},\dots,2^{52}$, $167\cdot 2^{40}$ and $j2^{42}$, $j\in\{40,42,43,44,46,48,50,52,56,60,80,96\}$, and between consecutive ones bounded by Lemma 16.12. The Route A criterion from 2^{52} on uses the tables $C_k(Q)$ at $Q=2^{50},2^{52},\dots,2^{60}$, each prime using the table of the largest $Q\le p$; these tables are computed with the coarse errors and the constants $\max_i\sum_j w_{ij}^{\mathrm{m}}$, $\max_i\sum_j w_{ij}^{\varrho}$, which dominate the factors of (16.5). The classes are stored as a histogram of signatures (the witness size, $\mathrm{ex}^\star$, and the multiset of the pairs (kind, s_v), the kind being f_{m} on $U_x\cup U_z$ and f_{ϱ} on $C_x\cup C_z$) with integer multiplicities, 30 645 955 distinct signature rows in all (17 286 284 for LL20 and 13 359 671 for LL22); every square root of a vertex factor is rounded upward to a multiple of 2^{-48}, and the integer sums over the histogram are exact. The nonroot caps (F6) and the route-A twelve-label caps (Section 20) use Corollary 16.14 with $\mathfrak{q}=0$ on the compressed partition, each vertex summand of $\mathrm{oc}(\mathcal{C})$ rounded upward at 2^{-20}. The conditional rows (F3) of the bands use Proposition 16.20 with $\mathfrak{q}=0$ (Remark 16.26), Proposition 16.21 and the star matrices of Proposition 16.22(a)–(c), and their joint fold and matching bounds Lemma 16.23 (for the L rows with Lemma 16.25); those of the tail datum (Definition 27.1) are computed with $\mathfrak{q}=20$ and the scalar majorant (27.5). Soft kernels (Corollary 16.17) are evaluated with every coefficient D_e replaced by $|D_e|$ on its support $\{e:D_e\neq 0\}$ (its *Boolean support*), or on a larger set of pairs containing that support whose admissibility is checked in Appendix D (a *certified Boolean supergraph*). *Coincidence bounds.* For the root cuts the Route B criterion takes the minimum in Lemma 16.15(e) of $D_1+D_{\ge 2}$ (with one order for each pair (i,j), one order π_k for each $k=2,\dots,9$ chosen at each of the points $2^{47},\dots,2^{52}$, and the tail (c) from $k_0=10$), D_{ord} and D_{nat}; a band evaluates this minimum at its start X, with the orders chosen at the evaluation point of its band (Certificate 27.9). The Route A criterion takes $\min(D_{\mathrm{ord}},D_{\mathrm{nat}})$. The nonroot caps and the route-A twelve-label caps use D_{nat} of (16.8), the L20 cut of the middle root the balanced bound of Proposition 20.21 (Corollary 16.18), sharpened, for the rows in which the principal (root) law is the uniform law, by the protected-subspace compression of Proposition 21.11, and the product rows Proposition 16.21.

Consequences and what to read next. This section uses the labellings and the extension count with its coarse and graded errors (Lemma 12.2, (12.1), (12.2), (E1)–(E3)); one-step error pairs, sequential counts, vertex densities and marginal ratios (Definition 12.4 and Lemmas 12.5 and 12.6); spectral data, their elementary symmetries and the cut kernel (Definition 13.1 and Lemmas 13.2

and 13.3); the positive Schur order (Lemma 13.9(b)) and the restriction to an injectivity event (Lemma 13.23(a)); the character expansion, the Gauss operator and the real form of the phase gates (Lemmas 14.1 and 14.18 and Remark 14.7); the four-cycle kernel for uniform probability (Corollary 15.9); and admissible majorants (Lemma 4.18). Three standard facts are used without statement: the Schur product theorem, Schur's test for the norm of a kernel or a matrix, and Hölder's inequality.

17. Making a graph law reflection positive, and replacing a law at fixed marginals

What this section proves and how to read it. A graph law need not be reflection positive: its cut operator can have a negative part of size about $gp^{-1/2}$. This section proves three constructions that, with the spectral caps, produce the objects Theorem C needs. A *completion* adds four smaller summand types (folds, matchings, merges, constants, with weights in powers of $r = p^{-1/2}$) to make the law reflection positive at every cut. *Polarization* dominates two summands by two others in the 2×2 block order. A *replacement at fixed marginals* (called a *repair* in Part V) reweights a coupling by $1 + f(x) + g(y)$ to prescribed one-half marginals with the same support; "repair" never means a small perturbation, and a repaired record keeps the actual half marginals of the record it replaces, so differences annihilate constants. Inputs: spectral caps (Section 14). Outputs: the completion, polarization and repair theorems applied in Part V. A first reading should follow the chamber geometry and the three statements below; weight calculations are reference material.

(1) *The five-type completion* (Lemma 17.5). Adding to a root law its folds, matchings, merges and constants, with weights in powers of $r = p^{-1/2}$ fixed by four scalar inequalities, gives a law that is reflection positive at every chamber cut; adding one-layer diagonals restores the reserved cut ι. This is (CP1) of Definition 9.10 for the middle atoms (Lemma 22.1), the upper atoms (Lemma 27.17) and the tail atoms (Section 29). Its joint form (f) completes rows whose fold and matching kernels are bounded together, in one Gram matrix, with a mean allowance $t_Y > 0$ for each row subject to $t_Y \le H^{-1}$ and $M \ge H + t_Y$ (Corollary 17.6); the bands of the upper range use it.

(2) *Polarization* (Lemma 17.11). At the chamber cut ϑ_0 the two nonaligned fold records are dominated by the two matching records plus a merge correction of one more order in r. This is the first-stage order (CP2)(a) of Definition 9.11 (Section 22), the matching bridge of the variance credits (Section 24) and, in all degrees, the fold coupon of the mean credit (Proposition 26.7).

(3) *Additive marginal repair* (Lemma 17.14) with the centred norm of an additive reweighting (Lemma 17.13). A coupling with the required support whose prescribed target marginals are $\mu_X = (1 + u)\nu_X$ and $\mu_Y = (1 + v)\nu_Y$ with $|u| \le U_X$, $|v| \le U_Y$ (and the L^2 bounds of Lemma 17.14) is reweighted by $1 + f(x) + g(y)$ to a coupling with exactly those marginals and the same support, and its centred conditional norm stays of the same order; the centred correction is bounded in the actual marginal space, with separate symmetric and skew parts and, against a law with the same marginals, without a marginal-variance term (Lemma 17.16). If a coupling with the prescribed marginals and the required support is at hand, a positive kernel is completed by mixing it in (Corollary 17.17). Apart from the matching records of the middle range, whose law keeps the half marginals of the records it replaces without a repair (Lemma 23.15), the comparison records of (CP4) of Definition 9.14 are such repaired records or finite combinations of them: the signed fold law of the middle range combines three positive components completed through an accepting coupling (Section 23); the middle merge repairs use Lemma 17.14 (Lemma 23.20); the upper range uses one-way repairs (imposing edges in one direction only across the reserved

cut), dense comparisons repaired by the initial-segment bound of Lemma 17.14(g) and a convex mixture of such repairs (Section 28); and the tail uses one-way repairs (Section 29).

A repaired record keeps the actual half marginals of the record it replaces, and so does a signed combination of such records with total coefficient one; so every record difference annihilates the constants on both sides of the reserved cut. This is what makes the sharp prices, the variance credits and the common mean of Part V (the empty-support component retained in Lemma 10.1(a)) possible. Section 17.1 fixes the chamber geometry of the records, which all three statements use.

17.1. **How records sit inside chambers (supports and symmetries).** Throughout this section the patterns below act in one chamber coordinate $\mathbb{Z}/6\mathbb{Z}$ (two in Lemma 17.5(d)); all remaining slot data of a configuration are carried by the objects in $\mathfrak{X}$. We use the rotation c, the chamber reflections $\vartheta_i(j) = 2i - 1 - j$ with plus halves $H_i = \{i, i+1, i+2\}$ ($\vartheta_{i+3} = \vartheta_i$), and the group $\mathcal{G}_6$ of Definition 5.12 and Lemma 5.13(i): the translations $u \mapsto u + 2k$ and the reflections $\vartheta_0, \vartheta_1, \vartheta_2$ ($u \mapsto 5 - u$, $1 - u$, $3 - u$).

An *object* carries the labels of a finite set $\mathfrak{X}$, and a *configuration* is $\Psi = (\Psi_u)_{u \in \mathbb{Z}/6\mathbb{Z}} \in \mathfrak{X}^{\mathbb{Z}/6\mathbb{Z}}$. For a cut $\vartheta = \vartheta_i$ we use the cut matrix C_ϑ, the order $\succeq_\vartheta$ and the cut form of Definition 5.2 on configurations, and write the diagonal cut form at a real function g on $\mathfrak{X}^{H_i}$ as

$$Q_\kappa^\vartheta(g) = \int g(\Psi|_{H_i})\, g((\Psi \circ \vartheta)|_{H_i})\, d\kappa(\Psi) = \langle g, C_\vartheta(\kappa) g \rangle\,, \qquad Q_\kappa = Q_\kappa^{\vartheta_0}.$$

For a law λ on $\mathfrak{X}^{H_i}$ the diagonal copy $\mathrm{Diag}_\vartheta(\lambda)$ (Lemma 5.4(i)) is fixed *samplewise* by ϑ, meaning $\Psi \circ \vartheta = \Psi$ for every configuration Ψ in its support, and its cut form is $\mathbb{E}_\lambda g^2$.

Definition 17.1 (Patterns and types)**.** For a map $\phi\colon \mathbb{Z}/6\mathbb{Z} \to O$ into a finite set of objects and a law λ on $\mathfrak{X}^O$, $\Pi_\phi(\lambda)$ is the law of the configuration $\omega \circ \phi = (\omega_{\phi(u)})_u$, $\omega \sim \lambda$; we write Π_ϕ when λ is clear. The *patterns*, listed as $(\phi(0), \dots, \phi(5))$, are:

- the identity of $\mathbb{Z}/6\mathbb{Z}$ (type L, the *root*);
- the three *folds* $F_0 = (0,1,2,2,1,0)$, $F_1 = (1,0,0,1,2,2)$, $F_2 = (2,2,1,0,0,1)$ (type D), with $O = \{0,1,2\}$;
- the two *matchings* $E_0 = (0,1,1,2,2,0)$ and $E_1 = (0,0,1,1,2,2)$ (type S), constant on the blocks of the pair partitions $\mathcal{P}_e = \{\{0,5\},\{1,2\},\{3,4\}\}$ and $\mathcal{P}_o = \{\{0,1\},\{2,3\},\{4,5\}\}$, whose kth blocks are $E_0^{-1}(k)$ and $E_1^{-1}(k)$;
- the six *merges* $G_{sk} = \mathbf{1}[E_s(\cdot) = k]$, $s \in \{0,1\}$, $k \in \{0,1,2\}$ (type R), with values in $\{0,1\}$: $G_{00} = (1,0,0,0,0,1)$, $G_{01} = (0,1,1,0,0,0)$, $G_{02} = (0,0,0,1,1,0)$, $G_{10} = (1,1,0,0,0,0)$, $G_{11} = (0,0,1,1,0,0)$, $G_{12} = (0,0,0,0,1,1)$; we write $G_e = G_{00}$ and $G_o = G_{11}$;
- the constant map 0, counted twice as C_0 and C_1 (type C), so that π_{C} in Definition 17.3 contains two copies of $\Pi_0(\nu)$ and contributes the term $2a_{\mathrm{C}} z^2$ in Lemma 17.5(a).

The types $\mathrm{L}, \mathrm{D}, \mathrm{S}, \mathrm{R}, \mathrm{C}$ have the *multiplicities* $1, 3, 2, 6, 2$ and the *formal degrees* $0, 1, 1, 2, 3$, where the formal degree is the exponent of r in the graded weight of Lemma 17.5.

The geometry of these patterns is the following finite check. The root graph of the middle reference, with one or two layers, is the *rotation lift* G_{L} below; with two layers its uniform law is the twelve-label root of Section 20, and on the three late middle pieces the root is a positive reweighting of this law, with the same support and the same symmetries (Section 21).

Lemma 17.2 (Chamber geometry of the records)**.** *Let $\mathcal{B}$ be a finite set of primes and Λ a finite set of layers. Let G_{L} be the oriented graph on $\mathbb{Z}/6\mathbb{Z} \times \Lambda$ with the edges $(u,\ell) \to (\mathsf{c}u, \ell')$ for all u and all $\ell, \ell' \in \Lambda$ (all layer lifts of the six rotation edges $0 \to 2 \to 4 \to 0$ and $1 \to 5 \to 3 \to 1$); let G_{F} be the graph on $\{0,1,2\} \times \Lambda$ with the edges $(0,\ell) \to (2,\ell')$ for all $\ell, \ell' \in \Lambda$, and G_{E} the graph on $\{0,1,2\} \times \Lambda$ without edges. An object carries one label of $X_{\mathcal{B}} = \prod_{p \in \mathcal{B}} \mathbb{F}_p$ per layer, so $\mathfrak{X} = X_{\mathcal{B}}^{\Lambda}$. The graph law $\lambda_{\mathcal{B}}(G_{\mathrm{L}})$ is a law on $\mathfrak{X}^{\mathbb{Z}/6\mathbb{Z}}$; put $\lambda_{\mathrm{F}} = \lambda_{\mathcal{B}}(G_{\mathrm{F}})$ and $\lambda_{\mathrm{E}} = \lambda_{\mathcal{B}}(G_{\mathrm{E}})$.*

pattern	plus half (Ψ_0,Ψ_1,Ψ_2)	reflected half (Ψ_5,Ψ_4,Ψ_3)	position
F_0	$(\omega_0,\omega_1,\omega_2)$	$(\omega_0,\omega_1,\omega_2)$	diagonal
F_1	$(\omega_1,\omega_0,\omega_0)$	$(\omega_2,\omega_2,\omega_1)$	$\mathcal{U}\to\mathcal{V}$
F_2	$(\omega_2,\omega_2,\omega_1)$	$(\omega_1,\omega_0,\omega_0)$	$\mathcal{V}\to\mathcal{U}$
E_0	$(\omega_0,\omega_1,\omega_1)$	$(\omega_0,\omega_2,\omega_2)$	$\mathcal{U}\to\mathcal{U}$
E_1	$(\omega_0,\omega_0,\omega_1)$	$(\omega_2,\omega_2,\omega_1)$	$\mathcal{V}\to\mathcal{V}$
$G_{00}=G_e$	$(\omega_1,\omega_0,\omega_0)$	$(\omega_1,\omega_0,\omega_0)$	diagonal on $\mathcal{U}$
G_{01}	$(\omega_0,\omega_1,\omega_1)$	$(\omega_0,\omega_0,\omega_0)$	$\mathcal{U}\to\mathcal{Z}$
G_{02}	$(\omega_0,\omega_0,\omega_0)$	$(\omega_0,\omega_1,\omega_1)$	$\mathcal{Z}\to\mathcal{U}$
G_{10}	$(\omega_1,\omega_1,\omega_0)$	$(\omega_0,\omega_0,\omega_0)$	$\mathcal{V}\to\mathcal{Z}$
$G_{11}=G_o$	$(\omega_0,\omega_0,\omega_1)$	$(\omega_0,\omega_0,\omega_1)$	diagonal on $\mathcal{V}$
G_{12}	$(\omega_0,\omega_0,\omega_0)$	$(\omega_1,\omega_1,\omega_0)$	$\mathcal{Z}\to\mathcal{V}$
C_0,C_1	$(\omega_0,\omega_0,\omega_0)$	$(\omega_0,\omega_0,\omega_0)$	diagonal on $\mathcal{Z}$

TABLE 7. The halves of $\omega\circ\phi$ at ϑ_0, for $\omega=(\omega_0,\omega_1,\omega_2)$, and the corners $\mathcal{U}=\{(b,y,y)\colon y\neq b\}$, $\mathcal{V}=\{(y,y,b)\colon y\neq b\}$, $\mathcal{Z}=\{(b,b,b)\}$ of $\mathfrak{X}^{H_0}$ in which they lie when the labels ω_o that occur are distinct. In the position column $\mathcal{A}\to\mathcal{B}$ means that the plus half lies in $\mathcal{A}$ and the reflected half in $\mathcal{B}$.

α	$(F_0\circ\alpha,F_1\circ\alpha,F_2\circ\alpha)$	$\psi_{0,\alpha}$	$\psi_{1,\alpha}$
$\vartheta_1\colon u\mapsto 1-u$	(F_1,F_0,F_2)	(0 1)	(1 2)
$\vartheta_2\colon u\mapsto 3-u$	(F_2,F_1,F_0)	(0 2)	(0 1)
$\vartheta_0\colon u\mapsto 5-u$	(F_0,F_2,F_1)	(1 2)	(0 2)
$u\mapsto u+2$	(F_2,F_0,F_1)	(0 1 2)	(0 1 2)

TABLE 8. Precomposition with the generators of $\mathcal{G}_6$: $E_s\circ\alpha=\psi_{s,\alpha}\circ E_s$, where (0 1 2) is the cycle $0\mapsto1\mapsto2\mapsto0$.

(a) *The reflections $u\mapsto k-u$ ($k\in\{1,3,5\}$) and the translation $u\mapsto u+2$ map the edge set of G_{L} onto itself. Hence $\lambda_{\mathcal{B}}(G_{\mathrm{L}})$ is invariant under $\mathcal{G}_6$, and at each ϑ_i it is a coupling of two copies of one half law.*

(b) *For $j\in\{0,1,2\}$, $F_j\circ\vartheta_{2j}=F_j$, and F_j maps the points $2j,2j+1,2j+2$ of H_{2j} (indices mod 6) to $0,1,2$. Hence for every law λ on $\mathfrak{X}^{\{0,1,2\}}$, $\Pi_{F_j}(\lambda)=\mathrm{Diag}_{\vartheta_{2j}}(\lambda^{(j)})$, where $\lambda^{(j)}$ is the law on $\mathfrak{X}^{H_{2j}}$ which puts the labels of the object o at $2j+o$, $o\in\{0,1,2\}$. We call ϑ_{2j} the* fold chamber *of F_j.*

(c) *The subgraph of G_{L} induced on $H_{2j}\times\Lambda$, read in the order $(2j,2j+1,2j+2)$, is G_{F}. Hence the half marginal of $\lambda_{\mathcal{B}}(G_{\mathrm{L}})$ on H_{2j}, read in this order, is a law on the support of λ_{F}, and it does not depend on j.*

(d) *Let $\Lambda=\{0,1\}$ and let ι be the layer swap, with plus half the layer 0. If a law λ on $\mathfrak{X}^O$ is invariant under the swap of the two layer labels of every object, then every image $\Pi_\phi(\lambda)$ is symmetric at ι, and its actual half marginal at ι is the image under ϕ of the layer-zero marginal of λ. In particular $\Pi_{F_j}(\lambda_{\mathrm{F}})$ and $\Pi_{E_s}(\lambda_{\mathrm{E}})$ are symmetric at ι; the actual half marginal of the first is $\Pi_{F_j}(m_0)$, where m_0 is the law of the layer-zero triple under λ_{F}, that of the second the uniform law on the ordered triples of labels distinct at every prime of $\mathcal{B}$, placed by E_s.*

(e) *Precomposition with $\alpha\in\mathcal{G}_6$ permutes the folds, maps E_s to $\psi\circ E_s$ for a permutation $\psi=\psi_{s,\alpha}$ of $\{0,1,2\}$, maps G_{sk} to $G_{s,\psi^{-1}(k)}$, and fixes the constant maps; for the generators the action is that of Table 8. The fold F_j and the merges G_e, G_o are fixed by the following chamber reflections: F_0, G_e and G_o by ϑ_0, F_1 by ϑ_2, F_2 by $\vartheta_4=\vartheta_1$.*

(f) *At ϑ_0 the halves of $\omega \circ \phi$ are those of Table 7. The corners $\mathcal{U}$, $\mathcal{V}$, $\mathcal{Z}$ are pairwise disjoint. Parts (b), (e) and (f) use only the patterns, and hold for objects with labels in any finite set $\mathfrak{X}$.*

Proof. (a) Under $u \mapsto 3-u$ the edges $0\to 2$, $2\to 4$, $4\to 0$ go to $3\to 1$, $1\to 5$, $5\to 3$, and conversely; under $u\mapsto 1-u$ they go to $1\to 5$, $5\to 3$, $3\to 1$, and under $u\mapsto 5-u$ to $5\to 3$, $3\to 1$, $1\to 5$; the translation by two permutes each of the two directed triangles. Since all layer lifts of every rotation edge are edges, these maps, acting on the first factor, are automorphisms of G_{L}, and they generate $\mathcal{G}_6$. A graph automorphism permutes the injective strict-square labellings at every prime, so it preserves $\lambda_{\mathcal{B}}(G_{\mathrm{L}})$. A law invariant under ϑ_i has the same law on its two halves.

(b) At the points $2j, 2j+1, 2j+2$ the fold F_j takes the values $0,1,2$ (for F_2 the points are $4,5,0$), and $\vartheta_{2j}(u) = 4j-1-u$, that is $5-u$, $3-u$ and $1-u$ for $j=0,1,2$; substitution in the tuples gives $F_j(\vartheta_{2j}u) = F_j(u)$. So every configuration $\omega\circ F_j$ is fixed by ϑ_{2j}, and its plus half carries ω_o at $2j+o$.

(c) Inside H_{2j} the only rotation edge is $2j\to 2j+2$: $\mathsf{c}(2j) = 2j+2$, while $\mathsf{c}(2j+1) = 2j-1$ and $\mathsf{c}(2j+2) = 2j+4$ lie outside H_{2j}, and the odd position $2j+1$ is joined in G_{L} only to odd positions. With all layer lifts the induced graph is G_{F} in the stated order. The restriction of an injective strict-square labelling of G_{L} to $H_{2j}\times\Lambda$ is one of the induced graph. The translation $u\mapsto u+2j$ maps H_0 onto H_{2j} in order and preserves G_{L}, so by (a) the half marginal does not depend on j.

(d) The layer swap commutes with $\omega\mapsto\omega\circ\phi$, so the image of a swap-invariant law is ι-invariant, which is the symmetry of its cut matrix at ι; its plus half is the layer-zero part. The swap is an automorphism of G_{F} (all layer lifts of $0\to 2$ are edges) and of G_{E}, so λ_{F} and λ_{E} are swap invariant. Under λ_{E} the six labels are uniform among those distinct at every prime, so the layer-zero triple is uniform on the ordered triples of distinct labels.

(e) Substitution in the tuples gives Table 8; for instance $E_0(1-u) = (1,0,0,2,2,1)$, which is E_0 with the names $0,1$ exchanged, and $F_1(3-u) = F_1(u)$. Since the generators map each block of $\mathcal{P}_e$ and of $\mathcal{P}_o$ onto a block of the same partition, so does every $\alpha\in\mathcal{G}_6$, and $E_s\circ\alpha = \psi\circ E_s$ for the induced permutation ψ of the block names. Then $G_{sk}(\alpha u) = \mathbf{1}[\psi(E_s(u)) = k] = G_{s,\psi^{-1}(k)}(u)$. The last sentence is (b) together with $G_{00}(5-u) = G_{00}(u)$ and $G_{11}(5-u) = G_{11}(u)$.

(f) Substitution gives Table 7. If $(b,y,y) = (y',y',b')$ with $y\neq b$ and $y'\neq b'$, then $y' = b$ and $y' = y$, contradicting $y\neq b$; a point of $\mathcal{Z}$ has three equal entries, so in either form above it has $y = b$ and lies in neither $\mathcal{U}$ nor $\mathcal{V}$. Hence the corners are pairwise disjoint. □

The cut forms of all components at ϑ_0 are read off Table 7: the cut form of every component other than the root and the aligned fold F_0 depends on a half-function g only through its restrictions to the corners $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{Z}$. In Part V the two facts used most are (b), which says that the two-layer fold record is the diagonal copy, at its own chamber, of its induced law, and (d), which identifies its actual half marginal at the reserved cut.

17.2. **The five-type completion.** A labelling law is made positive at a chamber cut in steps of one order of $r = p^{-1/2}$ each. Relative to the diagonal of its own half law, its cut operator is bounded below by $-\delta$ with δ of order r (the *root envelope*, from the spectral caps). In the graph-law applications that diagonal is supplied by the aligned fold (Lemma 17.2(b),(c)): a copy of the half law placed symmetrically, which is fixed by the cut and positive there. In Lemma 17.5 this is the general root-envelope hypothesis $\Pi_{\mathrm{L}} \succeq_{\vartheta_0} -\delta\,\Pi_{F_0}$, with Π_{F_0} built from π_{F}. The aligned fold at one cut is a nonaligned fold at the others, where its form couples the two corners $\mathcal{U},\mathcal{V}$; this coupling is dominated by the matchings up to the kernel errors $e_{\mathrm{F}}, e_{\mathrm{E}}$ of Lemma 17.5(b); those errors, and the mean-zero parts U°, V° of (c), are dominated by the merges, and the linear terms of the merges by the constants. So the five types enter with the formal degrees $0,1,1,2,3$, and the constants may carry degree three although the merges have degree two, because the last condition $a_{\mathrm{C}}a_{\mathrm{S}} \geq a_{\mathrm{R}}^2$ of Lemma 17.5(b) has total degree $3+1 = 2+2$ in the graded weights.

Definition 17.3 (Five-type datum)**.** A *five-type datum* on a finite set $\mathfrak{X}$ consists of

(i) a probability measure ν on $\mathfrak{X}$, the law of the middle object ω_1 in (iii) and (iv) below (the *middle law*), and, for each $b \in \operatorname{supp}\nu$, a probability measure ν^b on $\mathfrak{X}$ (the *conditional reference*);

(ii) a symmetric probability measure Γ on $\mathfrak{X}^2$ with $\Gamma(dx\,db) = \nu(db)\nu^b(dx)$;

(iii) an exchangeable probability measure π_{E} on $\mathfrak{X}^{\{0,1,2\}}$ with

$$\pi_{\mathrm{E}}(d\omega_0\,d\omega_1\,d\omega_2) = \nu(d\omega_1)\,\nu^{\omega_1}(d\omega_0)\,\nu^{\omega_1}(d\omega_2)\,T_{\mathrm{E}}^{\omega_1}(\omega_0,\omega_2);$$

(iv) a probability measure π_{F} on $\mathfrak{X}^{\{0,1,2\}}$ with

$$\pi_{\mathrm{F}}(d\omega_0\,d\omega_1\,d\omega_2) = \nu(d\omega_1)\,\nu^{\omega_1}(d\omega_0)\,\nu^{\omega_1}(d\omega_2)\,T_{\mathrm{F}}^{\omega_1}(\omega_0,\omega_2);$$

(v) a probability measure π_{L} on $\mathfrak{X}^{\mathbb{Z}/6\mathbb{Z}}$ invariant under $\mathcal{G}_6$;

with nonnegative kernels $T_{\mathrm{E}}^b, T_{\mathrm{F}}^b$. A kernel T is the operator $Th(x) = \int T(x,y)h(y)\,\nu^b(dy)$ on $L^2(\nu^b)$; norms, inner products, means and the projection $\mathbb{J}$ onto the constants refer to $L^2(\nu^b)$. The *components* of the datum are $\Pi_{\mathrm{L}} = \pi_{\mathrm{L}}$, $\Pi_{F_j} = \Pi_{F_j}(\pi_{\mathrm{F}})$, $\Pi_{E_s} = \Pi_{E_s}(\pi_{\mathrm{E}})$, $\Pi_{G_{sk}} = \Pi_{G_{sk}}(\Gamma)$ (the objects $0,1$ of Γ) and $\Pi_{C_0} = \Pi_{C_1} = \Pi_0(\nu)$; π_t is the sum of the components of the type t, and for weights $a = (a_{\mathrm{L}}, a_{\mathrm{D}}, a_{\mathrm{S}}, a_{\mathrm{R}}, a_{\mathrm{C}}) \geq 0$ the *five-type measure* is $\mathcal{Q}_a = \sum_t a_t\pi_t$.

In the applications Γ is the two-object marginal of π_{E} and ν, ν^b are its one-object marginal and conditional law (the merges and constants use only some objects of the matching law); then (ii) holds, and T_{E}^b, the density of the conditional law of the objects $0,2$ given $\omega_1 = b$, is a coupling of ν^b with itself, $T_{\mathrm{E}}^b\mathbf{1} = (T_{\mathrm{E}}^b)^*\mathbf{1} = \mathbf{1}$. The fold kernel T_{F}^b contains the density of the middle law of π_{F} relative to ν and those of its conditional outer marginals relative to ν^b, so no equality of the middle laws of the fold and matching laws is assumed.

Definition 17.4 (Two coordinates)**.** Let X be a finite set of labels, Λ a finite set of layers and $W = \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \Lambda$. Put $O_{\mathrm{L}} = \mathbb{Z}/6\mathbb{Z}$ and $O_{\mathrm{F}} = O_{\mathrm{E}} = \{0,1,2\}$. The types $\mathrm{L}, \mathrm{D}, \mathrm{S}, \mathrm{R}, \mathrm{C}$ have the *field types* (the named graph $G_{\mathrm{L}}, G_{\mathrm{F}}$ or G_{E} from which the coordinate law is taken; "field" here does not refer to $\mathbb{F}_p$) $\mathfrak{f}(\mathrm{L}) = \mathrm{L}$, $\mathfrak{f}(\mathrm{D}) = \mathrm{F}$, $\mathfrak{f}(\mathrm{S}) = \mathfrak{f}(\mathrm{R}) = \mathfrak{f}(\mathrm{C}) = \mathrm{E}$ and the pattern sets $P_{\mathrm{L}} = \{\mathrm{id}\}$, $P_{\mathrm{D}} = \{F_0, F_1, F_2\}$, $P_{\mathrm{S}} = \{E_0, E_1\}$, $P_{\mathrm{R}} = \{G_{sk}\}$, $P_{\mathrm{C}} = \{C_0, C_1\}$. Suppose given, for each pair of field types, a probability law $\lambda_{\mathfrak{f}_0\mathfrak{f}_1}$ on $X^{O_{\mathfrak{f}_0}\times O_{\mathfrak{f}_1}\times\Lambda}$. A *state* is $t = (t_0, t_1)$, of degree $\deg t_0 + \deg t_1$. For patterns $\phi = (\phi_0, \phi_1) \in P_{t_0} \times P_{t_1}$ the component $\Pi_{t,\phi}$ is the law of $e_\phi(y)$, where $e_\phi(y)_{(u_0,u_1,\ell)} = y_{\phi_0(u_0),\phi_1(u_1),\ell}$ and $y \sim \lambda_{\mathfrak{f}(t_0)\mathfrak{f}(t_1)}$; $\pi_t = \sum_\phi \Pi_{t,\phi}$, and for weights $a_t \geq 0$ we put $\mathcal{Q} = \sum_t a_t\pi_t$. The *row data of the coordinate* 0 *at the field type* $\mathfrak{f}'$ are the laws, on the row space $\mathfrak{X} = X^{O_{\mathfrak{f}'}\times\Lambda}$, of the families of rows $(y_{o,\cdot,\cdot})_o$ of $y \sim \lambda_{\mathrm{L}\mathfrak{f}'}$ (this is π_{L}), $y \sim \lambda_{\mathrm{F}\mathfrak{f}'}$ (π_{F}) and $y \sim \lambda_{\mathrm{E}\mathfrak{f}'}$ (π_{E}), together with the law Γ of the rows $0,1$ and the law ν of the row 0 of $\lambda_{\mathrm{E}\mathfrak{f}'}$, and the conditional law ν^b of the row 0 given that the row 1 is b. The row data of the coordinate 1 are defined in the same way with the two coordinates exchanged. We write $\vartheta_{c,i}$ for the reflection ϑ_i in the coordinate c.

Lemma 17.5 (Five-type completion)**.** *Let* $(\nu, \nu^b, \Gamma, \pi_{\mathrm{E}}, \pi_{\mathrm{F}}, \pi_{\mathrm{L}})$ *be a five-type datum on* $\mathfrak{X}$*, and* $a \geq 0$ *weights.*

(a) Invariance and cut forms. $\mathcal{Q}_a$ *is invariant under* $\mathcal{G}_6$*. For a real function* g *on* $\mathfrak{X}^{H_0}$ *put* $U_b(y) = g(b,y,y)$*,* $V_b(y) = g(y,y,b)$*,* $z_b = g(b,b,b)$*,* $u_b = \mathbb{E}_{\nu^b}U_b$ *and* $v_b = \mathbb{E}_{\nu^b}V_b$*, and let* $\mathbb{E}_b$ *be the average over* $b \sim \nu$*. Then*

$$Q_{\Pi_{F_0}}(g) = \mathbb{E}_{\pi_{\mathrm{F}}}g(\omega)^2, \qquad Q_{\Pi_{F_1}}(g) = Q_{\Pi_{F_2}}(g) = \mathbb{E}_b\left\langle U_b, T_{\mathrm{F}}^bV_b\right\rangle,$$

$$Q_{\Pi_{E_0}}(g) = \mathbb{E}_b\left\langle U_b, T_{\mathrm{E}}^bU_b\right\rangle, \qquad Q_{\Pi_{E_1}}(g) = \mathbb{E}_b\left\langle V_b, T_{\mathrm{E}}^bV_b\right\rangle,$$

$$Q_{\Pi_{G_e}}(g) = \mathbb{E}_b\|U_b\|^2, \qquad Q_{\Pi_{G_o}}(g) = \mathbb{E}_b\|V_b\|^2,$$

$$Q_{\Pi_{G_{01}}}(g) = Q_{\Pi_{G_{02}}}(g) = \mathbb{E}_b[z_bu_b], \qquad Q_{\Pi_{G_{10}}}(g) = Q_{\Pi_{G_{12}}}(g) = \mathbb{E}_b[z_bv_b], \qquad Q_{\Pi_{C_k}}(g) = \mathbb{E}_bz_b^2.$$

For every $b \in \operatorname{supp}\nu$, every linear operator T on $L^2(\nu^b)$, every $e \ge 0$ with $\|T - \mathbb{J}\| \le e$ and all real $\Phi, \Psi \in L^2(\nu^b)$, $\langle \Phi, T\Psi\rangle \ge \mathbb{E}_{\nu^b}\Phi\, \mathbb{E}_{\nu^b}\Psi - e\|\Phi\|\|\Psi\|$.

(b) Full-norm criterion. *Suppose that $\|T^b_{\mathrm{F}} - \mathbb{J}\| \le e_{\mathrm{F}}$ and $\|T^b_{\mathrm{E}} - \mathbb{J}\| \le e_{\mathrm{E}}$ for every $b \in \operatorname{supp}\nu$, and*

$$a_{\mathrm{S}} \ge a_{\mathrm{D}}, \qquad a_{\mathrm{R}} \ge e_{\mathrm{F}} a_{\mathrm{D}} + e_{\mathrm{E}} a_{\mathrm{S}}, \qquad a_{\mathrm{C}} a_{\mathrm{S}} \ge a_{\mathrm{R}}^2.$$

Then $\mathcal{Q}_a - a_{\mathrm{L}}\Pi_{\mathrm{L}} \succeq_{\vartheta_0} a_{\mathrm{D}}\Pi_{F_0}$. If moreover the root envelope $\Pi_{\mathrm{L}} \succeq_{\vartheta_0} -\delta\,\Pi_{F_0}$ holds and $a_{\mathrm{D}} \ge \delta a_{\mathrm{L}}$ (no condition if $a_{\mathrm{L}} = 0$), then $\mathcal{Q}_a$ is reflection positive at every chamber cut ϑ_i, $i \in \mathbb{Z}/6\mathbb{Z}$. For $r > 0$ and the graded weights $(w_{\mathrm{L}}, w_{\mathrm{D}} r, w_{\mathrm{S}} r, w_{\mathrm{R}} r^2, w_{\mathrm{C}} r^3)$*, with $e_{\mathrm{F}} = fr$, $e_{\mathrm{E}} = hr$ and $\delta = dr$, the conditions read*

$$w_{\mathrm{D}} \ge d\, w_{\mathrm{L}}, \qquad w_{\mathrm{S}} \ge w_{\mathrm{D}}, \qquad w_{\mathrm{R}} \ge f w_{\mathrm{D}} + h w_{\mathrm{S}}, \qquad w_{\mathrm{C}} w_{\mathrm{S}} \ge w_{\mathrm{R}}^2.$$

(c) Schur criterion. *Suppose that for every $b \in \operatorname{supp}\nu$ the matching kernel has exact marginals, $T^b_{\mathrm{E}}\mathbf{1} = (T^b_{\mathrm{E}})^*\mathbf{1} = \mathbf{1}$, with $\|T^b_{\mathrm{E}} - \mathbb{J}\| \le \rho_{\mathrm{E}}$, and that the fold kernel $K_b = T^b_{\mathrm{F}}$ has the* fold data

$$m_b := \langle \mathbf{1}, K_b\mathbf{1}\rangle \in [H^{-1}, H], \qquad \|(I-\mathbb{J})K_b\mathbf{1}\|,\ \|(I-\mathbb{J})K_b^*\mathbf{1}\| \le \beta, \qquad \|(I-\mathbb{J})K_b(I-\mathbb{J})\| \le q_\perp.$$

Let $a_{\mathrm{D}} > 0$, $M = a_{\mathrm{S}}/a_{\mathrm{D}}$, $s = a_{\mathrm{R}}/a_{\mathrm{D}}$, $k = a_{\mathrm{C}}/a_{\mathrm{D}}$, and let $t_Y > 0$ (the mean allowance*) satisfy*

$$M \ge H + t_Y, \qquad t_Y \le H^{-1}, \qquad s \ge q_\perp + M\rho_{\mathrm{E}} + \beta^2/t_Y, \qquad kM \ge s^2.$$

Then $\mathcal{Q}_a - a_{\mathrm{L}}\Pi_{\mathrm{L}} \succeq_{\vartheta_0} a_{\mathrm{D}}\Pi_{F_0}$, and under the root envelope and $a_{\mathrm{D}} \ge \delta a_{\mathrm{L}}$ of (b) the measure $\mathcal{Q}_a$ is reflection positive at every chamber cut. For $t_Y = \frac{1}{2}$ the conditions follow from $H \le \frac{3}{2}$, $M \ge 2$, $s \ge q_\perp + M\rho_{\mathrm{E}} + 2\beta^2$ and $kM \ge s^2$. For graded weights, with $h = \rho_{\mathrm{E}}/r$ and $f_{\mathrm{al}} = (q_\perp + \beta^2/t_Y)/r$, the conditions read $w_{\mathrm{D}} \ge d\, w_{\mathrm{L}}$, $w_{\mathrm{S}} \ge (H + t_Y) w_{\mathrm{D}}$, $w_{\mathrm{R}} \ge f_{\mathrm{al}} w_{\mathrm{D}} + h w_{\mathrm{S}}$ and $w_{\mathrm{C}} w_{\mathrm{S}} \ge w_{\mathrm{R}}^2$, together with $t_Y \le H^{-1}$.

(d) Two coordinates (slice principle). *In the setting of Definition 17.4, suppose that for each coordinate c and each field type $\mathfrak{f}'$ the row data of c at $\mathfrak{f}'$ form a five-type datum, and that for each type t' the five weights a_t of the states t with $t_{1-c} = t'$, as t_c ranges over the five types, satisfy the hypotheses of (b), of (c) or of (f) for the row data of c at $\mathfrak{f}(t')$, these five weights playing the roles of $a_{\mathrm{L}}, a_{\mathrm{D}}, a_{\mathrm{S}}, a_{\mathrm{R}}, a_{\mathrm{C}}$ there respectively, in each case together with the root envelope and the inequality $a_{\mathrm{D}} \ge \delta a_{\mathrm{L}}$ of (b) in these roles. Then $\mathcal{Q}$ is reflection positive at every chamber cut $\vartheta_{c,i}$ of both coordinates. More precisely, for fixed c, t' and $\phi' \in P_{t'}$, the sum of the components $a_t\Pi_{t,\phi}$ with $t_{1-c} = t'$ and $\phi_{1-c} = \phi'$ is the image of the five-type measure of these row data under the emission (placement map) $e_{\phi'} \colon \mathfrak{X}^{\mathbb{Z}/6\mathbb{Z}} \to X^W$, $e_{\phi'}(\Psi)_{(u_0,u_1,\ell)} = \Psi_{u_c}(\phi'(u_{1-c}), \ell)$.*

(e) The reserved cut (fold pairing). *Let $\Lambda = \{0,1\}$, let ι be the layer swap, and for a one-layer measure κ let $\mathrm{Diag}_\iota(\kappa)$ be the law that carries the same one-layer configuration on both layers. Consider the components of one coordinate (objects carrying one label per layer) or of two coordinates as in (d), and the one-layer components $\Pi^{(1)}$ built in the same way from one-layer laws; let $\pi_{t,j}$ be the sum of the j-layer components of the type or state t. Suppose that every two-layer component Π of t satisfies $\Pi \succeq_\iota -k_t r\,\mathrm{Diag}_\iota(\Pi^{(1)})$, where $\Pi^{(1)}$ is a fixed one-layer component with the same patterns as Π (one such component being fixed for each two-layer component), and let $w_t, v_t \ge 0$ with $v_t \ge k_t w_t$ for every t. Then*

$$\widetilde{\mu} = \sum_t w_t r^{\deg t}\pi_{t,2} + r\sum_t v_t r^{\deg t}\,\mathrm{Diag}_\iota(\pi_{t,1}) \succeq_\iota 0.$$

Diag_ι *preserves the order $\succeq_\vartheta$ at every chamber cut ϑ; in particular it maps a one-layer measure that is reflection positive at a chamber cut to a measure reflection positive there. Hence if the two-layer data with the weights $w_t r^{\deg t}$ and the one-layer data with the weights $v_t r^{\deg t}$ satisfy the hypotheses of (b), (c), (d) or (f), root envelopes included (for (b) in graded form: $v_{\mathrm{D}} \ge d_1 v_{\mathrm{L}}$, $v_{\mathrm{S}} \ge v_{\mathrm{D}}$, $v_{\mathrm{R}} \ge f_1 v_{\mathrm{D}} + h_1 v_{\mathrm{S}}$, $v_{\mathrm{C}} v_{\mathrm{S}} \ge v_{\mathrm{R}}^2$ with the one-layer inputs;*

all these conditions are homogeneous in υ), then $\widetilde{\mu}$ is reflection positive at every chamber cut of the active coordinates and at ι. Finally, if a field law λ and its one-layer law $\lambda^{(1)}$ satisfy $\lambda \succeq_\iota -kr\,\mathrm{Diag}_\iota(\lambda^{(1)})$, then so do their images under every emission e_ϕ (in one coordinate, the map $\omega \mapsto \omega\circ\phi$); so components that use only some objects of a field law, the merges and the constants, inherit its bound. A component may also be used with a smaller proved coefficient than k_t in its own order at ι, and $\Pi^{(1)}$ need not be the one-layer marginal of Π: only the order $\Pi \succeq_\iota -k_t r\,\mathrm{Diag}_\iota(\Pi^{(1)})$ is used.

(f) Joint Schur criterion. *Suppose that for every $b \in \operatorname{supp}\nu$ the matching kernel has exact marginals, as in (c), and the fold kernel has the* joint data

$$K_b = T_{\mathrm{F}}^b = m_b(\varphi_0\otimes\varphi_2 + C_b + R_b), \qquad m_b \in [H^{-1}, H],$$

with $\mathbb{E}\varphi_0 = \mathbb{E}\varphi_2 = 1$ (both φ_0 and φ_2 depend on b, the index being suppressed) and $\|\varphi_o - \mathbf{1}\| \le \upsilon$ ($o = 0, 2$), where $\varphi_0\otimes\varphi_2$ is the kernel $\varphi_0(x)\varphi_2(y)$ and the kernels C_b, R_b annihilate the constants on both sides, with $\|R_b\| \le \varepsilon_{\mathrm{T}}$; and that, with $E_b = \frac{1}{2}(T_{\mathrm{E}}^b + (T_{\mathrm{E}}^b)^*) - \mathbb{J}$,

$$E_b \succeq -\varepsilon_{\mathrm{E}} I, \qquad \begin{pmatrix} E_b & C_b \\ C_b^* & E_b \end{pmatrix} \succeq -\kappa_{\mathrm{J}} I \quad \text{on } L^2(\nu^b)\oplus L^2(\nu^b).$$

Let a_{D}, M, s and k be as in (c), and let $t_Y > 0$ satisfy

$$t_Y \le H^{-1}, \qquad M \ge H + t_Y, \qquad s \ge s_{\mathrm{J}}, \qquad kM \ge s^2,$$

where

$$s_{\mathrm{J}} = M\varepsilon_{\mathrm{E}} + H(\kappa_{\mathrm{J}} - \varepsilon_{\mathrm{E}})_+ + H\varepsilon_{\mathrm{T}} + \Big(H + \frac{H^2}{t_Y}\Big)\upsilon^2.$$

Then the conclusions of (c) hold. For graded weights, with $h = \varepsilon_{\mathrm{E}}/r$, $k_{\mathrm{J}} = \kappa_{\mathrm{J}}/r$ and

$$f_{\mathrm{J}} = \frac{H\{(k_{\mathrm{J}} - h)_+ r + \varepsilon_{\mathrm{T}}\} + (H + H^2/t_Y)\upsilon^2}{r},$$

the conditions read $w_{\mathrm{D}} \ge d\,w_{\mathrm{L}}$, $w_{\mathrm{S}} \ge (H + t_Y)w_{\mathrm{D}}$, $w_{\mathrm{R}} \ge f_{\mathrm{J}} w_{\mathrm{D}} + h w_{\mathrm{S}}$ and $w_{\mathrm{C}} w_{\mathrm{S}} \ge w_{\mathrm{R}}^2$, together with $t_Y \le H^{-1}$.

Proof. (a) By Lemma 17.2(e), precomposition with $\alpha \in \mathcal{G}_6$ permutes the fold patterns, maps E_s to $\psi\circ E_s$, permutes the merges of each partition and fixes the constant maps. The law of $\Psi\circ\alpha$ under $\Pi_\phi(\lambda)$ is $\Pi_{\phi\circ\alpha}(\lambda)$, and $\Pi_{\psi\circ E_s}(\pi_{\mathrm{E}}) = \Pi_{E_s}(\pi_{\mathrm{E}})$ because π_{E} is exchangeable. Hence every sum π_t with $t \ne \mathrm{L}$ is invariant, and Π_{L} is invariant by (v).

The cut forms are read off Table 7. The halves of Π_{F_0} coincide. For F_1 the halves are $(\omega_1, \omega_0, \omega_0)$ and $(\omega_2, \omega_2, \omega_1)$, so its form is $\mathbb{E}_{\pi_{\mathrm{F}}}[U_{\omega_1}(\omega_0)V_{\omega_1}(\omega_2)] = \mathbb{E}_b\left\langle U_b, T_{\mathrm{F}}^b V_b\right\rangle$ by (iv); F_2 exchanges the two halves and gives the same value. For E_1 the form is $\mathbb{E}_{\pi_{\mathrm{E}}}[V_{\omega_1}(\omega_0)V_{\omega_1}(\omega_2)] = \mathbb{E}_b\left\langle V_b, T_{\mathrm{E}}^b V_b\right\rangle$ by (iii). For E_0 the object on both halves is ω_0, and the form is $\mathbb{E}_{\pi_{\mathrm{E}}}[U_{\omega_0}(\omega_1)U_{\omega_0}(\omega_2)] = \mathbb{E}_b\left\langle U_b, T_{\mathrm{E}}^b U_b\right\rangle$, because $(\omega_1, \omega_0, \omega_2)$ again has the law π_{E}. For the merges $(\omega_0, \omega_1) \sim \Gamma$: G_e gives $\mathbb{E}_\Gamma[U_{\omega_1}(\omega_0)^2] = \mathbb{E}_b\|U_b\|^2$ by (ii), and G_{01}, G_{02} give $\mathbb{E}_\Gamma[z_{\omega_0}U_{\omega_0}(\omega_1)] = \mathbb{E}_b[z_b u_b]$ by (ii) and the symmetry of Γ; G_o, G_{10} and G_{12} are the same with V in place of U. The constants give $\mathbb{E}_\nu z_b^2$. Finally $\langle\Phi, T\Psi\rangle = \langle\Phi, \mathbb{J}\Psi\rangle + \langle\Phi, (T - \mathbb{J})\Psi\rangle$ and $\langle\Phi, \mathbb{J}\Psi\rangle = \mathbb{E}\Phi\,\mathbb{E}\Psi$.

(b) Fix $b \in \operatorname{supp}\nu$ and omit it from the notation. Let Φ_b be the sum of the terms of the integrand of $Q_{\mathcal{Q}_a - a_{\mathrm{L}}\Pi_{\mathrm{L}} - a_{\mathrm{D}}\Pi_{F_0}}(g)$ given by (a):

$$\Phi_b = 2a_{\mathrm{D}}\,\langle U, T_{\mathrm{F}}V\rangle + a_{\mathrm{S}}\big(\langle U, T_{\mathrm{E}}U\rangle + \langle V, T_{\mathrm{E}}V\rangle\big) + a_{\mathrm{R}}\big(\|U\|^2 + \|V\|^2 + 2z(u+v)\big) + 2a_{\mathrm{C}}z^2.$$

By (a) and $2\|U\|\|V\| \le \|U\|^2 + \|V\|^2$,

$$\Phi_b \ge (a_{\mathrm{R}} - e_{\mathrm{F}}a_{\mathrm{D}} - e_{\mathrm{E}}a_{\mathrm{S}})\big(\|U\|^2 + \|V\|^2\big) + a_{\mathrm{S}}(u^2 + v^2) + 2a_{\mathrm{D}}uv + 2a_{\mathrm{R}}z(u+v) + 2a_{\mathrm{C}}z^2.$$

The first term is nonnegative. With $w = u + v$, $a_{\mathrm{S}}(u^2 + v^2) + 2a_{\mathrm{D}}uv = \frac{1}{2}(a_{\mathrm{S}} + a_{\mathrm{D}})w^2 + \frac{1}{2}(a_{\mathrm{S}} - a_{\mathrm{D}})(u - v)^2 \geq \frac{1}{2}a_{\mathrm{S}}w^2$, because $a_{\mathrm{S}} \geq a_{\mathrm{D}} \geq 0$. The binary form $\frac{1}{2}a_{\mathrm{S}}w^2 + 2a_{\mathrm{R}}zw + 2a_{\mathrm{C}}z^2$ has nonnegative diagonal coefficients and determinant $a_{\mathrm{S}}a_{\mathrm{C}} - a_{\mathrm{R}}^2 \geq 0$, so $\Phi_b \geq 0$. The cut form at ϑ_0 of $\mathcal{Q}_a - a_{\mathrm{L}}\Pi_{\mathrm{L}} - a_{\mathrm{D}}\Pi_{F_0}$ at g is $\mathbb{E}_b\Phi_b \geq 0$, and its cut matrix is symmetric, because $\mathcal{Q}_a$ and Π_{L} are ϑ_0-invariant by (a) and Π_{F_0} is fixed samplewise by ϑ_0 (Lemma 17.2(b)). This is the first claim. Under the root envelope and $a_{\mathrm{D}} \geq \delta a_{\mathrm{L}}$ (for $a_{\mathrm{L}} = 0$ the term $a_{\mathrm{L}}\Pi_{\mathrm{L}}$ vanishes, and the chain holds with $\delta = 0$),

$$\mathcal{Q}_a \succeq_{\vartheta_0} a_{\mathrm{L}}\Pi_{\mathrm{L}} + a_{\mathrm{D}}\Pi_{F_0} \succeq_{\vartheta_0} (a_{\mathrm{D}} - \delta a_{\mathrm{L}})\Pi_{F_0} \succeq_{\vartheta_0} 0,$$

the last step by Lemma 5.4(i). Since $\mathcal{Q}_a$ is $\mathcal{G}_6$-invariant, chamber transport (Lemma 5.13(i)) gives every cut ϑ_i. For the graded weights, divide $a_{\mathrm{D}} \geq \delta a_{\mathrm{L}}$ and the first displayed condition by r and the second by r^2; in the third both sides carry r^4, since $3 + 1 = 2 + 2$.

(c) Fix b and omit it. Write $U = u + U^\circ$, $V = v + V^\circ$ with $U^\circ, V^\circ \perp \mathbf{1}$, $\sigma_U = \|U^\circ\|$, $\sigma_V = \|V^\circ\|$, and $K = K_b$; only m_b keeps its index. Then $\langle U, KV\rangle = m_b uv + u\,\langle (I - \mathbb{J})K^*\mathbf{1}, V^\circ\rangle + v\,\langle U^\circ, (I - \mathbb{J})K\mathbf{1}\rangle + \langle U^\circ, (I - \mathbb{J})K(I - \mathbb{J})V^\circ\rangle$, so the fold data give

$$2\,\langle U, KV\rangle \geq 2m_b uv - 2\beta\big(|u|\sigma_V + |v|\sigma_U\big) - q_\perp(\sigma_U^2 + \sigma_V^2).$$

Since T_{E} has exact marginals, $\langle U, T_{\mathrm{E}}U\rangle = u^2 + \langle U^\circ, (T_{\mathrm{E}} - \mathbb{J})U^\circ\rangle \geq u^2 - \rho_{\mathrm{E}}\sigma_U^2$, and likewise for V; and $\|U\|^2 = u^2 + \sigma_U^2$. Hence Φ_b/a_{D}, with Φ_b as in (b), is at least

$$(s - q_\perp - M\rho_{\mathrm{E}})(\sigma_U^2 + \sigma_V^2) + (M + s)(u^2 + v^2) + 2m_b uv - 2\beta\big(|u|\sigma_V + |v|\sigma_U\big) + 2sz(u + v) + 2kz^2.$$

By $2\beta|u|\sigma_V \leq t_Y u^2 + \beta^2\sigma_V^2/t_Y$ and the same for (v, σ_U), this is at least

$$\Big(s - q_\perp - M\rho_{\mathrm{E}} - \frac{\beta^2}{t_Y}\Big)(\sigma_U^2 + \sigma_V^2) + A(u^2 + v^2) + 2m_b uv + 2sz(u + v) + 2kz^2, \qquad A = M + s - t_Y.$$

The fluctuation term is nonnegative. With $w_\pm = (u \pm v)/\sqrt{2}$ the rest is $(A - m_b)w_-^2 + (A + m_b)w_+^2 + 2\sqrt{2}\,szw_+ + 2kz^2$. Here $A - m_b \geq M - t_Y - H + s \geq s \geq 0$, and $A + m_b \geq M + s - t_Y + H^{-1} \geq M$, so the binary form in (w_+, z) has nonnegative diagonal coefficients and determinant $2k(A + m_b) - 2s^2 \geq 2(kM - s^2) \geq 0$. Thus $\Phi_b \geq 0$, and the rest of the proof is that of (b). For $t_Y = \frac{1}{2}$: $M \geq 2 \geq H + \frac{1}{2}$ and $\frac{1}{2} < \frac{2}{3} \leq H^{-1}$. The graded form follows as in (b), since $s = w_{\mathrm{R}}r/w_{\mathrm{D}}$, $M = w_{\mathrm{S}}/w_{\mathrm{D}}$ and $k = w_{\mathrm{C}}r^2/w_{\mathrm{D}}$.

(d) Fix $c = 0$; the coordinate 1 is the same with the coordinates exchanged. Fix a type t' of the coordinate 1 and $\phi_1 \in P_{t'}$, and let $\mathcal{Q}^{\mathrm{row}}$ be the five-type measure of the row data of 0 at $\mathfrak{f}(t')$ with the weights $a_{(t_0,t')}$. For a type t_0 and $\phi_0 \in P_{t_0}$, the configuration $\Psi_{u_0} = y_{\phi_0(u_0),\cdot,\cdot}$ of rows of $y \sim \lambda_{\mathfrak{f}(t_0)\mathfrak{f}(t')}$ has the law of the component ϕ_0 of the row data: the rows of $\lambda_{\mathrm{L}\mathfrak{f}(t')}$ are π_{L}, the three rows of $\lambda_{\mathrm{F}\mathfrak{f}(t')}$ are π_{F}, those of $\lambda_{\mathrm{E}\mathfrak{f}(t')}$ are π_{E}, and a merge or a constant reads only the rows $0, 1$ or the row 0, whose laws are Γ and ν. Moreover $e_{\phi_1}(\Psi)_{(u_0,u_1,\ell)} = y_{\phi_0(u_0),\phi_1(u_1),\ell}$. So the sum of $a_t\Pi_{t,\phi}$ over the states $t = (t_0, t')$ and the patterns (ϕ_0, ϕ_1) is $(e_{\phi_1})_*\mathcal{Q}^{\mathrm{row}}$, which is the slice decomposition. Let $\vartheta = \vartheta_{0,i}$, with plus half $W_+ = H_i \times \mathbb{Z}/6\mathbb{Z} \times \Lambda$. The emission is half-local for the cuts ϑ_i of $\mathbb{Z}/6\mathbb{Z}$ and ϑ of W in the sense of Lemma 5.4(v): $e_{\phi_1}(\Psi)|_{W_+} = \varphi(\Psi|_{H_i})$ and the reflected half of $e_{\phi_1}(\Psi)$ is φ of the reflected half of Ψ, with $\varphi(x) = (x_{u_0}(\phi_1(u_1), \ell))_{(u_0,u_1,\ell)\in W_+}$ for $x \in \mathfrak{X}^{H_i}$; so the image of a measure reflection positive at ϑ_i is reflection positive at ϑ. By (b), (c) or (f), $\mathcal{Q}^{\mathrm{row}}$ is reflection positive at every ϑ_i, and so is each image. Summing over t' and ϕ_1 proves the claim.

(e) A component Π of t, with the weight $w_t r^{\deg t}$, together with its one-layer diagonal, with the weight $v_t r^{\deg t+1}$, satisfies

$$w_t r^{\deg t}\Pi + v_t r^{\deg t+1}\,\mathrm{Diag}_\iota(\Pi^{(1)}) \succeq_\iota (v_t - k_t w_t)\,r^{\deg t+1}\,\mathrm{Diag}_\iota(\Pi^{(1)}) \succeq_\iota 0,$$

by the hypothesis and because a diagonal copy is fixed samplewise by ι (Lemma 5.4(i)); summing over the components gives $\widetilde{\mu} \succeq_\iota 0$. Let ϑ be a chamber cut, acting on the chamber coordinates only. The diagonal embedding $z' \mapsto z$, $z_{(u,\ell)} = z'_u$, is half-local for ϑ on the one-layer and on the

two-layer slots, so by Lemma 5.4(v) it preserves the order $\succeq_\vartheta$; this is the statement about Diag_ι, and the two sums of $\widetilde{\mu}$ are then reflection positive at the chamber cuts by (b), (c), (d) or (f). Finally e_ϕ is half-local for the layer swaps of the field labels and of the slots, so by Lemma 5.4(v) the image of $\lambda + kr\,\mathrm{Diag}_\iota(\lambda^{(1)}) \succeq_\iota 0$ is $(e_\phi)_*\lambda + kr\,\mathrm{Diag}_\iota((e_\phi)_*\lambda^{(1)}) \succeq_\iota 0$. The argument uses each component only through its own order at ι and the one-layer sums only through their own completion inequalities; this is the last sentence of (e).

(f) Fix b and omit it; write $U = u + U^\circ$, $V = v + V^\circ$, σ_U, σ_V as in (c), and $\alpha = \langle U^\circ, \varphi_0 - \mathbf{1}\rangle$, $\alpha' = \langle \varphi_2 - \mathbf{1}, V^\circ\rangle$, so that $|\alpha| \le \upsilon\sigma_U$ and $|\alpha'| \le \upsilon\sigma_V$. Since $\mathbb{E}\varphi_0 = \mathbb{E}\varphi_2 = 1$ and C_b, R_b annihilate the constants on both sides,

$$\langle U, K_b V\rangle = m_b(u+\alpha)(v+\alpha') + m_b\,\langle U^\circ, (C_b + R_b)V^\circ\rangle\,.$$

By $0 \le m_b \le H$ and Young's inequality with the allowance t_Y,

$$\begin{aligned} 2m_b(u\alpha' + v\alpha) &\ge -t_Y(u^2+v^2) - \frac{H^2\upsilon^2}{t_Y}(\sigma_U^2+\sigma_V^2),\\ 2m_b(\alpha\alpha' + \langle U^\circ, R_bV^\circ\rangle) &\ge -H(\upsilon^2+\varepsilon_{\mathrm{T}})(\sigma_U^2+\sigma_V^2). \end{aligned}$$

The matching kernel has exact marginals, so $\langle U, T_{\mathrm{E}}U\rangle = u^2 + \langle U^\circ, E_bU^\circ\rangle$, and likewise for V. The remaining terms $a_{\mathrm{S}}(\langle U^\circ, E_bU^\circ\rangle + \langle V^\circ, E_bV^\circ\rangle) + 2m_ba_{\mathrm{D}}\,\langle U^\circ, C_bV^\circ\rangle$ are the form of the block operator $\left(\begin{smallmatrix} a_{\mathrm{S}}E_b & m_ba_{\mathrm{D}}C_b \\ m_ba_{\mathrm{D}}C_b^* & a_{\mathrm{S}}E_b\end{smallmatrix}\right)$ at (U°, V°). Apply Corollary 16.24 with $h = \varepsilon_{\mathrm{E}}$, $\mathsf{k}_{\mathcal{B}} = \max(\kappa_{\mathrm{J}}, \varepsilon_{\mathrm{E}})$ and $m = m_b$. Its hypotheses hold because $a_{\mathrm{S}} = Ma_{\mathrm{D}} \ge Ha_{\mathrm{D}}$ and $m_b \le H$; the splitting uses only these block inequalities, not the prime-product origin of the operators. Thus the remaining terms are at least

$$-\big(a_{\mathrm{S}}\varepsilon_{\mathrm{E}} + Ha_{\mathrm{D}}(\kappa_{\mathrm{J}} - \varepsilon_{\mathrm{E}})_+\big)(\sigma_U^2+\sigma_V^2).$$

Collecting, Φ_b/a_{D}, with Φ_b as in (b), is at least

$$(s - s_{\mathrm{J}})(\sigma_U^2+\sigma_V^2) + A(u^2+v^2) + 2m_buv + 2sz(u+v) + 2kz^2, \qquad A = M + s - t_Y,$$

which is the form of the proof of (c) with the fluctuation coefficient $s - s_{\mathrm{J}} \ge 0$. Its mean part is nonnegative exactly as there, because $M \ge H + t_Y$, $t_Y \le H^{-1}$ and $kM \ge s^2$; so $\Phi_b \ge 0$, and the rest of the proof is that of (b). In graded form $\varepsilon_{\mathrm{E}} = hr$ and $\kappa_{\mathrm{J}} = k_{\mathrm{J}}r$, so $s_{\mathrm{J}} = (w_{\mathrm{S}}/w_{\mathrm{D}})hr + f_{\mathrm{J}}r$ and $s \ge s_{\mathrm{J}}$ reads $w_{\mathrm{R}} \ge f_{\mathrm{J}}w_{\mathrm{D}} + hw_{\mathrm{S}}$; the other conditions read as in (c). □

Corollary 17.6 (Joint rows of a block)**.** *Let $\mathcal{B}$ be a finite set of primes with least prime P, $r = P^{-1/2}$ and $r_p = p^{-1/2}$. Fix a value b of the middle object of a five-type datum whose conditional kernels at b are products over the primes of $\mathcal{B}$: $\nu^b = \bigotimes_p \nu_p$, $T_{\mathrm{E}}^b = \bigotimes_p T_{\mathrm{E},p}$ with $T_{\mathrm{E},p}$ a self-adjoint coupling of ν_p with itself, and $T_{\mathrm{F}}^b = m_bT_{\mathrm{F},\mathcal{B}}$, $T_{\mathrm{F},\mathcal{B}} = \bigotimes_p T_{\mathrm{F},p}$, with $T_{\mathrm{F},p}$ of mass one and marginal densities $u_{\mathrm{L},p}$, $u_{\mathrm{R},p}$ relative to ν_p. Let $C_{\mathrm{F},p} = T_{\mathrm{F},p} - u_{\mathrm{L},p}\otimes u_{\mathrm{R},p}$ and $C_{\mathrm{E},p} = T_{\mathrm{E},p} - \mathbb{J}_p$, as in Lemma 16.23, and suppose that for every $p \in \mathcal{B}$*

$$\|C_{\mathrm{F},p}\| \le c_{\mathrm{F}}r_p, \qquad \|C_{\mathrm{E},p}\| \le c_{\mathrm{E}}r_p, \qquad \begin{pmatrix} C_{\mathrm{E},p} & C_{\mathrm{F},p} \\ C_{\mathrm{F},p}^* & C_{\mathrm{E},p}\end{pmatrix} \succeq -k_{\mathrm{J}}r_pI,$$

with $c_{\mathrm{F}}r, c_{\mathrm{E}}r < 1$ and $(c_{\mathrm{F}}^2 + c_{\mathrm{E}}^2)r \le k_{\mathrm{J}}$. Let $m_b \in [H^{-1}, H]$, let the block marginal densities $U_{\mathrm{L}} = \bigotimes_p u_{\mathrm{L},p}$ and $U_{\mathrm{R}} = \bigotimes_p u_{\mathrm{R},p}$ satisfy $\|U_{\mathrm{L}} - \mathbf{1}\|, \|U_{\mathrm{R}} - \mathbf{1}\| \le \upsilon$, let $E_{\mathcal{B}} = T_{\mathrm{E}}^b - \mathbb{J} \succeq -hrI$, and put

$$E_{\mathrm{T}} = (2\upsilon + \upsilon^2)\Big(\prod_{p\in\mathcal{B}}(1 + c_{\mathrm{F}}r_p) - 1\Big),$$

which is at most $(2\upsilon+\upsilon^2)(e^{nc_{\mathrm{F}}r} - 1)$ if $\mathcal{B}$ has at most n primes. Then the joint data of Lemma 17.5(f) hold at b with $\varphi_0 = U_{\mathrm{L}}$, $\varphi_2 = U_{\mathrm{R}}$, $\varepsilon_{\mathrm{E}} = hr$, $\kappa_{\mathrm{J}} = k_{\mathrm{J}}r$ and $\varepsilon_{\mathrm{T}} = E_{\mathrm{T}}$. Consequently, if this holds at

every $b \in \operatorname{supp}\nu$ and $0 < t_Y \le H^{-1}$, then for row parameters

$$M \ge H + t_Y, \qquad f \ge \frac{H\{(k_{\mathrm{J}} - h)_+ r + E_{\mathrm{T}}\} + (H + H^2/t_Y)\upsilon^2}{r},$$

and h, the completion inequalities $w_{\mathrm{D}} \ge d\, w_{\mathrm{L}}$, $w_{\mathrm{S}} \ge M w_{\mathrm{D}}$, $w_{\mathrm{R}} \ge f w_{\mathrm{D}} + h w_{\mathrm{S}}$, $w_{\mathrm{C}} w_{\mathrm{S}} \ge w_{\mathrm{R}}^2$ imply the graded conditions of Lemma 17.5(f). This holds for the two-layer and for the one-layer rows alike, so the corollary serves the ordinary completion, with the two-layer rows, and the reserved one, with the one-layer rows (Lemma 17.5(e)).

Proof. The kernels $C_{\mathrm{F},p}$ and $C_{\mathrm{E},p}$ annihilate the constants on both sides. Hence $C_{\mathcal{B}} = \bigotimes_p(\mathbb{J}_p + C_{\mathrm{F},p}) - \mathbb{J}_{\mathcal{B}}$ and $E_{\mathcal{B}} = \bigotimes_p(\mathbb{J}_p + C_{\mathrm{E},p}) - \mathbb{J}_{\mathcal{B}}$ annihilate them too, and their joint block is bounded by Lemma 16.23(b). Indeed on its singleton sectors the joint block is $\succeq -k_{\mathrm{J}} r I$ (using $r_p \le r$), and on sectors with at least two primes it is $\succeq -(c_{\mathrm{F}}^2 + c_{\mathrm{E}}^2) r^2 I \succeq -k_{\mathrm{J}} r I$. Hence the joint block of $(E_{\mathcal{B}}, C_{\mathcal{B}})$ is $\succeq -k_{\mathrm{J}} r I$. Write $T_{\mathrm{F},\mathcal{B}} = U_{\mathrm{L}} \otimes U_{\mathrm{R}} + C_{\mathcal{B}} + R_{\mathcal{B}}$. Since $T_{\mathrm{F},\mathcal{B}}\mathbf{1} = U_{\mathrm{L}}$, $T_{\mathrm{F},\mathcal{B}}^{*}\mathbf{1} = U_{\mathrm{R}}$ and $C_{\mathcal{B}}$ annihilates the constants, so does $R_{\mathcal{B}}$; and $\|R_{\mathcal{B}}\| \le (2\upsilon + \upsilon^2)(\prod_p(1 + \|C_{\mathrm{F},p}\|) - 1) \le E_{\mathrm{T}}$ by Lemma 16.23(c). With $C_b = C_{\mathcal{B}}$ and $R_b = R_{\mathcal{B}}$ these are the joint data of Lemma 17.5(f); the bound by $e^{n c_{\mathrm{F}} r}$ is $1 + x \le e^x$. Finally $f \ge f_{\mathrm{J}}$ and $M \ge H + t_Y$, so the four completion inequalities with (d, M, f, h) imply those of Lemma 17.5(f). Nothing in the argument depends on the number of layers. □

Remark 17.7 (Fold data from marginal densities). Suppose that, for every $b \in \operatorname{supp}\nu$, the middle density m_b — the density at b of the ω_1-marginal of π_{F} relative to ν, equal to $\langle \mathbf{1}, K_b \mathbf{1}\rangle$ for $K_b = T_{\mathrm{F}}^b$ by the factorisation in Definition 17.3(iv) — lies in $[H^{-1}, H]$, and the conditional law of (ω_0, ω_2) given $\omega_1 = b$ under π_{F} has marginal densities φ_0, φ_2 relative to ν^b, with $\varphi_o \le R_{\mathrm{m}}$ and $\|\varphi_o - \mathbf{1}\| \le \upsilon$, and maximal correlation at most ρ_{F} relative to its own marginals. Then the fold data of (c) hold with $\beta = H\upsilon$ and $q_\perp = H(R_{\mathrm{m}}\rho_{\mathrm{F}} + \upsilon^2)$, and

$$f_{\mathrm{al}} = \frac{H R_{\mathrm{m}} \rho_{\mathrm{F}} + (H + H^2/t_Y)\upsilon^2}{r}.$$

Indeed $K_b = m_b(\varphi_0 \otimes \varphi_2 + c_b)$, where the kernel c_b represents the covariance, $\langle h, c_b h'\rangle = \operatorname{Cov}(h(\omega_0), h'(\omega_2) \mid \omega_1 = b)$; so $c_b\mathbf{1} = c_b^{*}\mathbf{1} = 0$ and $\|c_b\| \le R_{\mathrm{m}}\rho_{\mathrm{F}}$, because $\|h\|_{\varphi_o \nu^b}^2 \le R_{\mathrm{m}}\|h\|^2$. Hence $\langle \mathbf{1}, K_b\mathbf{1}\rangle = m_b$, $(I - \mathbb{J})K_b\mathbf{1} = m_b(\varphi_0 - \mathbf{1})$, and the perpendicular block is $m_b((\varphi_0 - \mathbf{1}) \otimes (\varphi_2 - \mathbf{1}) + (I - \mathbb{J})c_b(I - \mathbb{J}))$. With $t_Y = 1/50$ this is the Schur row coefficient $f_{\mathrm{Schur}} = (H R_{\mathrm{m}}\rho_{\mathrm{F}} + (H + 50H^2)\upsilon^2)/r$ of the tail datum of the upper range ((F5)). The joint form (f) splits the covariance kernel as $c_b = C_b + R_b$ and bounds C_b together with the matching row, in one Gram matrix, instead of adding the two norms: the term $H R_{\mathrm{m}}\rho_{\mathrm{F}}$ becomes $H\{(k_{\mathrm{J}} - h)_+ r + \varepsilon_{\mathrm{T}}\}$, and on a block with the data of Corollary 17.6

$$f_{\mathrm{J}} = \frac{H\{(k_{\mathrm{J}} - h)_+ r + E_{\mathrm{T}}\} + (H + H^2/t_Y)\upsilon^2}{r},$$

with a rational t_Y chosen for each row; this is the row coefficient of the bands of the upper range ((F5)). Both criteria keep a change of marginals and a centred correlation apart: in the constant direction the matching contribution dominates the fold term because $M \ge H + t_Y$ while $m_b \le H$, and Young's inequality with the allowance t_Y absorbs the constant–perpendicular coupling of the fold, so a linear marginal error enters only through its square. The allowance enters only through $M \ge H + t_Y$, $t_Y \le H^{-1}$ and the term $H^2\upsilon^2/t_Y$, so it may be chosen freely, row by row.

Remark 17.8 (The model computation and its ancestors). The proof of (b), (c) and (f) is one computation: the cut forms of the twelve components other than the root and the aligned fold are explicit in the conditional means u_b, v_b, the fluctuations and the constant value z_b, and the four inequalities are the conditions for the resulting form in (u, v, z) to be positive after the rotation $w_\pm = (u \pm v)/\sqrt{2}$. Only the inequalities matter: any nonnegative weights satisfying them give a positive law, so the certificates may choose rational weights by a sequential recipe or jointly,

and then check the inequalities exactly. Without a root ($a_{\mathrm{L}} = 0$) and with $a_{\mathrm{D}} = 1$, $a_{\mathrm{S}} = 1 + a_{\mathrm{R}}$ and $a_{\mathrm{C}} = a_{\mathrm{R}}$, (b) is the array completion of the middle range of the 216-slot design: its condition $a_{\mathrm{R}}(1 - e_{\mathrm{E}}) \geq e_{\mathrm{F}} + e_{\mathrm{E}}$ is the second inequality of (b), and the other two hold for these weights. Its later row completions, with a separate bound for the middle density and with split marginal and fluctuation bounds, are described in Section 34.

Remark 17.9 (How the completions are certified). Every certified completion inequality is an instance of (b)–(f) with the stored inputs. A middle reference (Lemma 22.1) uses (b) with the full-norm row coefficients f_2, h_2, (b) for the one-layer diagonals with f_1, h_1, and (e): $4 + 4 + 5 = 13$ inequalities per piece, 429 for the 33 pieces (Certificate D.3(c)); the type-S inequality of (e) (the matching type, of field type E) is checked in the stronger form $v_{\mathrm{S}} \geq 2k_{\mathrm{E}}w_{\mathrm{S}}$, where $k_{\mathrm{E}} = k_{\mathrm{S}}$ is the coefficient k_t of (e) for $t = \mathrm{S}$, and the protection inequality $v_{\mathrm{D}} \geq 2k_{\mathrm{D}}w_{\mathrm{D}}$ of Lemma 22.1(iv), which implies the type-D inequality of (e), is checked with them. An upper band uses (d) in both coordinates and for both layer counts, with (e): 40 slice inequalities for w, 40 for v and the 25 domination inequalities $v_t \geq k_t w_t$ of (e), one for each of the 25 states, 105 per band and 32550 for the 310 bands (Certificate 27.18). Its slices use (f), through Corollary 17.6, with $h = \varepsilon_{\mathrm{E}}/r$ and a rational mean allowance t_Y for each row, with $t_Y H < 1$ and $M \geq H + t_Y$. The tail datum of the upper range (Definition 27.1(e)) has its own 105 inequalities, and its slices use (c) with $t_Y = 1/50$ and $h = \rho_{\mathrm{E}}/r$: in all six rows of the tail datum (three field types, two layer counts) the coefficient f_{Schur} of Remark 17.7 is smaller than the full-norm one and $H < 103/100$, and the floor $w_{\mathrm{S}} \geq \frac{21}{20}w_{\mathrm{D}}$ of its slices gives $M \geq H + t_Y$ and $t_Y \leq H^{-1}$ ((F5)).

Remark 17.10 (Fold pairing). Part (e) pairs each component with the diagonal copy of a one-layer law, one power of r higher: the copy is fixed samplewise by the cut, hence positive there, and it is positive at every other cut at which that law is. The reserved cut is cheap because the half graph of a two-layer labelling at ι is its one-layer graph, so the copy that cancels a component's negative envelope is the one-layer version of the same component, and the one-layer versions are made positive at the chamber cuts by their own completion inequalities. The paired law need not be the half law of the component. On the three late middle pieces the two-layer root is a positive reweighting of the uniform root, and its layer cut is capped against the diagonal copy of the uniform one-layer root, which the completion supplies (Section 21); the reserved merges and constants of the upper range carry smaller caps, proved for the marginals they observe (Proposition 27.14). The same pairing, applied in every chamber coordinate at once, made the high range of the 216-slot design positive (Section 34).

17.3. **Polarization: dominating two summands by two others.** The first stage λ must drop the fold records F_1, F_2, which do not accept the queries, without losing positivity at the chamber cut ϑ_0 used for the first stage. At ϑ_0 these two folds are nonaligned: their cut forms couple the two corners $\mathcal{U}$ and $\mathcal{V}$ of the half space, while the matching records E_0, E_1 act inside each corner separately (Table 7). Polarization is the inequality that lets the matchings replace the folds: the sum of the two matching forms exceeds twice the fold form by the square $(u - v)^2$ of the difference of the conditional means on the corners (a nonnegative deficit term, favourable to us), less the explicit error term of (17.1) (the row errors), and those errors are in turn dominated by the two merge records G_e, G_o, which are diagonal on the corners. When the domination holds with both signs in front of the fold terms, it also holds after taking tensor products over any set of blocks (part (d)), which the mean credit of (CP7) uses.

Lemma 17.11 (Polarization). *Let ν^b be a probability measure on a finite set, and T_{F}, T_{E}, $T_{\mathrm{E}'}$ kernels on $L^2(\nu^b)$ with $\|T_{\mathrm{F}} - \mathbb{J}\| \leq e_{\mathrm{F}}$ and $\|T_{\mathrm{E}} - \mathbb{J}\|, \|T_{\mathrm{E}'} - \mathbb{J}\| \leq e_{\mathrm{E}}$. For real $U, V \in L^2(\nu^b)$ with means u, v,*

$$\langle U, T_{\mathrm{E}}U\rangle + \langle V, T_{\mathrm{E}'}V\rangle - 2\langle U, T_{\mathrm{F}}V\rangle \geq (u - v)^2 - (e_{\mathrm{F}} + e_{\mathrm{E}})(\|U\|^2 + \|V\|^2). \tag{17.1}$$

Consequently, for the components of a five-type datum with $\|T_{\mathrm{F}}^b - \mathbb{J}\| \le e_{\mathrm{F}}$ *and* $\|T_{\mathrm{E}}^b - \mathbb{J}\| \le e_{\mathrm{E}}$ *for every* $b \in \operatorname{supp}\nu$*, and every* $\varepsilon \ge e_{\mathrm{F}} + e_{\mathrm{E}}$*:*

(a) $\Pi_{E_0} + \Pi_{E_1} + \varepsilon(\Pi_{G_e} + \Pi_{G_o}) \succeq_{\vartheta_0} \Pi_{F_1} + \Pi_{F_2}$*;*

(b) *if* $\nu^b(\{b\}) = 0$ *for every* $b \in \operatorname{supp}\nu$*, so that the conditional law* ν^b *gives mass* 0 *to the value* b *itself and the middle object differs from each outer object almost surely under the fold law, then both orders* $\Pi_{E_0} + \Pi_{E_1} + \varepsilon(\Pi_{G_e} + \Pi_{G_o}) \succeq_{\vartheta_0} \pm(\Pi_{F_1} + \Pi_{F_2})$ *hold. Equivalently, with* $A = C_{\vartheta_0}(\Pi_{E_0} + \varepsilon\Pi_{G_e})$*,* $D = C_{\vartheta_0}(\Pi_{E_1} + \varepsilon\Pi_{G_o})$ *and* $K = C_{\vartheta_0}(\Pi_{F_1})$*, viewed as matrices on* $\mathbb{R}^{\mathfrak{X}^{H_0}}$ *supported on the blocks* $\mathcal{U}\times\mathcal{U}$*,* $\mathcal{V}\times\mathcal{V}$ *and* $\mathcal{U}\times\mathcal{V}$ *respectively and zero elsewhere, both matrices* $\left(\begin{smallmatrix} A & \pm K \\ \pm K^{\mathsf{T}} & D \end{smallmatrix}\right)$ *on* $\mathbb{R}^{\mathcal{U}} \oplus \mathbb{R}^{\mathcal{V}}$ *are positive semidefinite; in particular* $A, D \succeq 0$*;*

(c) *let* X *be a finite nonempty set of blocks, and for* $\mathcal{B} \in X$ *let a five-type datum on* $\mathfrak{X}_{\mathcal{B}}$ *satisfy the hypotheses of (b) with constants* $\varepsilon_{\mathcal{B}} \ge e_{\mathrm{F},\mathcal{B}} + e_{\mathrm{E},\mathcal{B}}$*. Then, as measures on* $(\prod_{\mathcal{B}} \mathfrak{X}_{\mathcal{B}})^{\mathbb{Z}/6\mathbb{Z}}$ *and for both signs,*

$$\bigotimes_{\mathcal{B}}(\Pi_{E_0,\mathcal{B}} + \varepsilon_{\mathcal{B}}\Pi_{G_e,\mathcal{B}}) + \bigotimes_{\mathcal{B}}(\Pi_{E_1,\mathcal{B}} + \varepsilon_{\mathcal{B}}\Pi_{G_o,\mathcal{B}}) \pm \Big(\bigotimes_{\mathcal{B}} \Pi_{F_1,\mathcal{B}} + \bigotimes_{\mathcal{B}} \Pi_{F_2,\mathcal{B}}\Big) \succeq_{\vartheta_0} 0,$$

and $\bigotimes_{\mathcal{B}}(\Pi_{E_0,\mathcal{B}} + \Pi_{E_1,\mathcal{B}} + \varepsilon_{\mathcal{B}}(\Pi_{G_e,\mathcal{B}} + \Pi_{G_o,\mathcal{B}})) \succeq_{\vartheta_0} \bigotimes_{\mathcal{B}} \Pi_{F_1,\mathcal{B}} + \bigotimes_{\mathcal{B}} \Pi_{F_2,\mathcal{B}}$*;*

(d) *for every number* k *of tensor factors: if* ϑ *is a cut of a slot set* V *and, for* $j = 1, \dots, k$*,* κ_j *and* ζ_j *are finite signed measures on* $\mathfrak{X}_j^V$ *with* $\kappa_j \pm \zeta_j \succeq_{\vartheta} 0$ *for both signs, then*

$$\bigotimes_{j=1}^{k} \kappa_j - \bigotimes_{j=1}^{k} \zeta_j = 2^{-k} \sum_{\sigma\in\{\pm1\}^k} (1 - \sigma_1 \cdots \sigma_k) \bigotimes_{j=1}^{k} (\kappa_j + \sigma_j\zeta_j),$$

every product on the right is $\succeq_{\vartheta} 0$*, and* $\bigotimes_j \kappa_j \pm \bigotimes_j \zeta_j \succeq_{\vartheta} 0$ *for both signs. In particular, under the hypotheses of (c), with* $\mathcal{Q}_{\mathcal{B}}^{\mathrm{E}} = \Pi_{E_0,\mathcal{B}} + \Pi_{E_1,\mathcal{B}} + \varepsilon_{\mathcal{B}}(\Pi_{G_e,\mathcal{B}} + \Pi_{G_o,\mathcal{B}})$ *and* $\mathcal{Q}_{\mathcal{B}}^{\mathrm{F}} = \Pi_{F_1,\mathcal{B}} + \Pi_{F_2,\mathcal{B}}$*,*

$$\bigotimes_{\mathcal{B}\in X} \mathcal{Q}_{\mathcal{B}}^{\mathrm{E}} \pm \bigotimes_{\mathcal{B}\in X} \mathcal{Q}_{\mathcal{B}}^{\mathrm{F}} \succeq_{\vartheta_0} 0 \qquad \textit{for both signs.}$$

Proof. By Lemma 17.5(a), $\langle U, T_{\mathrm{E}}U\rangle \ge u^2 - e_{\mathrm{E}}\|U\|^2$, $\langle V, T_{\mathrm{E}'}V\rangle \ge v^2 - e_{\mathrm{E}}\|V\|^2$ and $2\langle U, T_{\mathrm{F}}V\rangle \le 2uv + 2e_{\mathrm{F}}\|U\|\|V\| \le 2uv + e_{\mathrm{F}}(\|U\|^2 + \|V\|^2)$, which gives (17.1).

(a) By Lemma 17.5(a), the cut form at ϑ_0 of the difference of the two sides at g is $\mathbb{E}_b[\langle U_b, T_{\mathrm{E}}^b U_b\rangle + \langle V_b, T_{\mathrm{E}}^b V_b\rangle + \varepsilon(\|U_b\|^2 + \|V_b\|^2) - 2\langle U_b, T_{\mathrm{F}}^b V_b\rangle]$, which is at least $\mathbb{E}_b(u_b - v_b)^2 \ge 0$ by (17.1). The cut matrix is symmetric: Π_{E_s} and Π_{G_e}, Π_{G_o} are ϑ_0-invariant, and ϑ_0 exchanges Π_{F_1} and Π_{F_2} (Lemma 17.2(e)).

(b) If $\nu^b(\{b\}) = 0$, the fold law has $\omega_1 \ne \omega_0, \omega_2$ almost surely. The matching law has all three objects distinct by exchangeability, and the merge law has its two objects distinct. In particular, for E_0, $\omega_0 \ne \omega_2$ because (ω_0, ω_2) has the law of (ω_0, ω_1) by exchangeability. Thus the cut matrices of Π_{E_0} and Π_{G_e} live on $\mathcal{U}\times\mathcal{U}$, those of Π_{E_1} and Π_{G_o} on $\mathcal{V}\times\mathcal{V}$, that of Π_{F_1} on $\mathcal{U}\times\mathcal{V}$ and that of Π_{F_2}, which is K^{T} because Π_{F_2} is the image of Π_{F_1} under ϑ_0 (Remark 5.3(a)), on $\mathcal{V}\times\mathcal{U}$. Given g, let $\tilde{g}$ equal g on $\mathcal{U}$ and $-g$ on $\mathcal{V}$; the corners are disjoint (Lemma 17.2(f)). The forms of Π_{E_s}, Π_{G_e}, Π_{G_o} at $\tilde{g}$ equal those at g, and the fold forms change sign, so (a) at $\tilde{g}$ is the order with the lower sign at g. The two orders are the two block matrices evaluated at $(x|_{\mathcal{U}}, x|_{\mathcal{V}})$, and a diagonal block of a positive semidefinite matrix is positive semidefinite.

(c) Let $\mathcal{H}_{\mathcal{B}} = \mathbb{R}^{\mathfrak{X}_{\mathcal{B}}^{H_0}}$, and extend $A_{\mathcal{B}}$, $D_{\mathcal{B}}$, $K_{\mathcal{B}}$ of (b) by zero to $\mathcal{H}_{\mathcal{B}}$. The block matrices $\left(\begin{smallmatrix} A_{\mathcal{B}} & \pm K_{\mathcal{B}} \\ \pm K_{\mathcal{B}}^{\mathsf{T}} & D_{\mathcal{B}} \end{smallmatrix}\right)$ remain positive semidefinite on $\mathcal{H}_{\mathcal{B}} \oplus \mathcal{H}_{\mathcal{B}}$, because their form at (x, y) is the form of (b) at $(x|_{\mathcal{U}}, y|_{\mathcal{V}})$. Choose the sign $-$ at one block and $+$ at the others, or $+$ at every block. By Lemma 5.6, $\left(\begin{smallmatrix} \bigotimes A_{\mathcal{B}} & \pm\bigotimes K_{\mathcal{B}} \\ \pm\bigotimes K_{\mathcal{B}}^{\mathsf{T}} & \bigotimes D_{\mathcal{B}} \end{smallmatrix}\right) \succeq 0$ for both signs, and its form at (x, x) is

$\big\langle x, (\bigotimes A_{\mathcal{B}} + \bigotimes D_{\mathcal{B}} \pm (\bigotimes K_{\mathcal{B}} + \bigotimes K_{\mathcal{B}}^{\mathsf{T}}))x \big\rangle$. By (5.2) these four Kronecker products are the cut matrices at ϑ_0 of the four tensor products in the first claim, which is therefore proved; the cut matrices are symmetric, each factor being ϑ_0-invariant or exchanged with its partner. For the second claim expand the left side into the $2^{|X|}$ products $\bigotimes_{\mathcal{B}}(\Pi_{E_{s_{\mathcal{B}}},\mathcal{B}} + \varepsilon_{\mathcal{B}}\Pi_{G_{s_{\mathcal{B}}s_{\mathcal{B}}},\mathcal{B}})$, $s \in \{0,1\}^X$. The two constant choices of s give the first claim with the sign $-$; every other product has the cut matrix $\bigotimes_{\mathcal{B}} M_{\mathcal{B}}$ with $M_{\mathcal{B}} \in \{A_{\mathcal{B}}, D_{\mathcal{B}}\}$, a Kronecker product of positive semidefinite matrices.

(d) Expand the products on the right. The product with ζ_j at the positions $j \in S$ and κ_j elsewhere has the coefficient $2^{-k}\sum_\sigma(1 - \prod_j \sigma_j)\prod_{j\in S}\sigma_j = [S = \emptyset] - [S = \{1,\dots,k\}]$, because $\sum_\sigma \prod_{j\in T}\sigma_j = 2^k[T = \emptyset]$ for every $T \subseteq \{1,\dots,k\}$ and $\prod_j \sigma_j \prod_{j\in S}\sigma_j = \prod_{j\notin S}\sigma_j$; this is the identity. Each factor $\kappa_j + \sigma_j\zeta_j$ is $\succeq_\vartheta 0$, so each product is $\succeq_\vartheta 0$ by (5.2), and the coefficients $1 - \sigma_1\cdots\sigma_k \in \{0,2\}$ are nonnegative; hence $\bigotimes_j \kappa_j - \bigotimes_j \zeta_j \succeq_\vartheta 0$. Replacing ζ_1 by $-\zeta_1$, which satisfies the same hypothesis, gives the other sign. For cut matrices this is Lemma 5.6(d); the identity moreover writes the difference as a nonnegative combination of tensor products of the $\succeq_\vartheta 0$ signed measures $\kappa_j + \sigma_j\zeta_j$, so that any of them may be discarded. For the last claim, (b) gives $\mathcal{Q}^{\mathrm{E}}_{\mathcal{B}} \pm \mathcal{Q}^{\mathrm{F}}_{\mathcal{B}} \succeq_{\vartheta_0} 0$ for every $\mathcal{B} \in X$. □

Remark 17.12 (Where polarization is used). (1) In the middle frame (Definition 22.4) (a) is applied to the one-layer datum with $\varepsilon = (f_1 + h_1)r$; multiplied by the record weight $v_{\mathrm{D}}r^2$ and copied to the reserved diagonal, which preserves the order at ϑ_0 (Lemma 17.5(e)), it is the first-stage order $E_{\mathrm{add}} + G \succeq_{\vartheta_0} F_{\mathrm{fold}}$ of (CP2)(a). (2) In the variance credits (a) is applied to the two-layer datum with $\varepsilon_{\mathcal{B}} = (f_2 + h_2)r_{\mathcal{B}}$ (Lemma 24.10(a)); this datum also satisfies the hypothesis $\nu^b(\{b\}) = 0$ of (b), and (c) gives the aligned bridge of several blocks (Lemma 24.11(a)). (3) The one-layer datum of (1) also satisfies the hypothesis of (b), whose two orders, copied to the reserved diagonal as in (1), give $E_{\mathrm{add}} + G \pm F_{\mathrm{fold}} \succeq_{\vartheta_0} 0$ on every block; (d) carries both orders to the products over any set of blocks, in every degree. This is the fold coupon of the mean credit (Proposition 26.7). In all three the order is an order of cut matrices, not of measures; only actual components of the reference are subtracted, so the laws built from it stay nonnegative. In the middle range T^b_{F} carries no middle density, the F and E laws having the same middle law at class-three primes (Lemma 20.9).

17.4. **Reweighting to prescribed marginals.** A comparison record must accept the queries, which is a condition on its support, and it must keep the actual half marginals at ι of the record it replaces, so that the difference annihilates the constants. A coupling with the right support and slightly wrong marginals is corrected by the additive reweighting $1 + f(x) + g(y)$: the marginal equations are linear in (f, g), and a positive reweighting keeps the support. The next lemma bounds the centred norm of such a reweighting; it holds for any additive perturbation with centred f, g, not only for the solution of the marginal equations.

Lemma 17.13 (Centred norm of an additive reweighting)**.** *Let ν be a probability measure on $X \times Y$, finite sets, with marginals ν_X, ν_Y of full support, and let its centred conditional operator $A\colon L^2_0(\nu_Y) \to L^2_0(\nu_X)$, $(Ag)(x) = \mathbb{E}_\nu[g(y) \mid x]$, have norm at most ρ. Let $d\tilde\nu = (1+f(x)+g(y))\,d\nu \geq 0$ with $\mathbb{E}_{\nu_X} f = \mathbb{E}_{\nu_Y} g = 0$, and let $1 + u$, $1 + v$ be the densities of its marginals relative to ν_X, ν_Y. If $u \geq -U_X$, $v \geq -U_Y$, $\|u\|_2 \leq e_X$, $\|v\|_2 \leq e_Y$, $\|f\|_\infty \leq F_X$ and $\|g\|_\infty \leq F_Y$, with $U_X, U_Y < 1$, then the centred conditional operator of $\tilde\nu$, relative to its own marginals, has norm at most*

$$\frac{\rho(1 + F_X + F_Y) + e_X e_Y}{\sqrt{(1 - U_X)(1 - U_Y)}}.$$

The bounds $\|u\|_2 \leq e_X$, $\|v\|_2 \leq e_Y$ enter only through $\|u\|_2\|v\|_2 \leq e_X e_Y$, so the conclusion holds whenever $e_X e_Y \geq \|u\|_2\|v\|_2$.

Proof. Apply Lemma 13.19(a), with the references ν_X, ν_Y, to the form $\mathfrak{b}(h,k) = \int hk\,d\tilde\nu$ (Remark 13.20(3)): its densities are $\varrho_L = 1 + u$ and $\varrho_R = 1 + v$, so for h, k centred under the marginals

of $\tilde{\nu}$, with $h_0 = h - \mathbb{E}_{\nu_X} h$ and $k_0 = k - \mathbb{E}_{\nu_Y} k$,

$$\int hk \, d\tilde{\nu} = \int h_0 k_0 (1 + f + g) \, d\nu - \langle u, h_0 \rangle \langle v, k_0 \rangle .$$

Since k_0 is centred, $\|Ak_0\| \le \rho \|k_0\|$. Hence

$$\left| \int h_0 k_0 \, d\nu \right| = |\langle h_0, Ak_0 \rangle| \le \rho \|h_0\| \|k_0\|, \qquad \left| \int h_0 f k_0 \, d\nu \right| = |\langle h_0 f, Ak_0 \rangle| \le F_X \rho \|h_0\| \|k_0\|,$$

and the term with g is bounded in the same way through A^*. By Cauchy–Schwarz, $|\langle u, h_0 \rangle \langle v, k_0 \rangle| \le e_X e_Y \|h_0\| \|k_0\|$. Finally $1 + u \ge 1 - U_X$ gives $\|h\|_{\tilde{\nu}_X}^2 \ge (1 - U_X) \mathbb{E}_{\nu_X} h^2 \ge (1 - U_X) \|h_0\|^2$, and likewise for k. □

Lemma 17.14 (Additive marginal repair)**.** *Let ν, A and $\rho < 1$ be as in Lemma 17.13, and let K_∞ bound A and A^* as operators from L^2 to L^∞, $\|Ag\|_\infty \le K_\infty \|g\|_{L^2(\nu_Y)}$ and $\|A^* h\|_\infty \le K_\infty \|h\|_{L^2(\nu_X)}$; one may take $K_\infty = (\sup d\nu / d(\nu_X \otimes \nu_Y))^{1/2}$. Let $\mu_X = (1+u)\nu_X$ and $\mu_Y = (1+v)\nu_Y$ be probability measures with $|u| \le U_X$, $|v| \le U_Y$, $\|u\|_2 \le e_X$ and $\|v\|_2 \le e_Y$.*

(a) *The equations $f + Ag = u$, $g + A^* f = v$ have exactly one solution with $f \in L_0^2(\nu_X)$, $g \in L_0^2(\nu_Y)$, namely $f = (I - AA^*)^{-1}(u - Av)$, $g = v - A^* f$. The signed measure $\nu' = (1 + f(x) + g(y)) \, \nu$ has the marginals μ_X, μ_Y. The norms are bounded equation by equation: $\|f\|_2 \le (e_X + \rho e_Y)/(1 - \rho^2)$ and $\|g\|_2 \le (e_Y + \rho e_X)/(1 - \rho^2)$. If a group acts on X and on Y, the diagonal action $(x, y) \mapsto (gx, gy)$ preserves ν, and the two actions preserve μ_X and μ_Y respectively, then f, g and ν' are invariant under these actions.*

(b) *$\|f\|_\infty \le U_X + K_\infty \|g\|_2 \le F_X := U_X + K_\infty (e_Y + \rho e_X)/(1 - \rho^2)$, and likewise $\|g\|_\infty \le F_Y$.*

(c) *If $\|f\|_\infty + \|g\|_\infty < 1$, for instance if $F_X + F_Y < 1$, then ν' is a probability measure with the marginals μ_X, μ_Y, and $1 + f + g \ge 1 - \|f\|_\infty - \|g\|_\infty > 0$; so ν' has the support of ν and satisfies every constraint that holds ν-almost surely.*

(d) (One-way repair.) *If $|u|, |v| \le U < 1$, then $\|f\|_2, \|g\|_2 \le U/(1-\rho)$ and $\|f\|_\infty, \|g\|_\infty \le F := U(1 + K_\infty/(1-\rho))$; if $2F < 1$, (c) applies.*

(e) (Swap symmetry.) *If $X = Y$, ν is invariant under $(x, y) \mapsto (y, x)$ and $\mu_X = \mu_Y$, then $A^* = A$ and $f = g = s$, where s is the centred solution of $(I + A)s = u$; ν' is swap invariant.*

(f) (Centred norm and difference.) *Under (c) and $U_X, U_Y < 1$, ν' satisfies the hypotheses of Lemma 17.13 with the bounds U_X, U_Y, e_X, e_Y above (or any e_X, e_Y with $e_X e_Y \ge \|u\|_2 \|v\|_2$) and any $F_X \ge \|f\|_\infty$, $F_Y \ge \|g\|_\infty$; so its centred conditional operator, relative to μ_X, μ_Y, has norm at most*

$$\rho' = \frac{\rho(1 + F_X + F_Y) + e_X e_Y}{\sqrt{(1 - U_X)(1 - U_Y)}};$$

in the case of (d),

$$\rho' \le \frac{\rho(1 + 2F) + U^2}{1 - U} \le \frac{\rho(1 + 2F) + U^2 + 2FU}{1 - U}.$$

If π is a coupling of μ_X and μ_Y whose centred conditional operator has norm at most $\rho(\pi)$, then $\nu' - \pi$ has zero marginals and, as an operator from $L^2(\mu_Y)$ to $L^2(\mu_X)$, norm at most $\rho' + \rho(\pi)$.

(g) (Neumann series split after N terms.) *For this part $\rho < 1$ is not assumed. Let $\mathcal{V}_X \subseteq L_0^2(\nu_X)$ and $\mathcal{V}_Y \subseteq L_0^2(\nu_Y)$ be subspaces with $A\mathcal{V}_Y \subseteq \mathcal{V}_X$, $A^*\mathcal{V}_X \subseteq \mathcal{V}_Y$ and $\|A|_{\mathcal{V}_Y}\| \le \rho_I < 1$. If $u \in \mathcal{V}_X$, $v \in \mathcal{V}_Y$, $|u|, |v| \le U$ and $\|u - Av\|_2, \|v - A^* u\|_2 \le e_I$, then*

$$f = \sum_{j \ge 0} (AA^*)^j (u - Av) \in \mathcal{V}_X, \qquad g = \sum_{j \ge 0} (A^* A)^j (v - A^* u) \in \mathcal{V}_Y$$

solve the equations of (a), and for every integer $N \geq 1$

$$\|f\|_\infty,\ \|g\|_\infty \leq F_N := 2NU + \frac{K_\infty \rho_I^{2N-1} e_I}{1-\rho_I^2}.$$

If $2F_N < 1$, the conclusions of (c) hold for this solution. If a group acts on X and on Y, the diagonal action $(x,y) \mapsto (gx, gy)$ preserves ν, and $\mathcal{V}_X$, $\mathcal{V}_Y$ consist of invariant functions, then f, g and ν' are invariant.

Proof. (a) A maps centred functions to centred functions, since $\mathbb{E}_{\nu_X} Ag = \mathbb{E}_{\nu_Y} g$. The positive operator AA^* on $L_0^2(\nu_X)$ has norm at most $\rho^2 < 1$, so $I - AA^*$ is invertible, and substitution gives $f + Ag = (I - AA^*)f + Av = u$ and $g + A^* f = v$; the targets are centred because μ_X and μ_Y are probability measures. If (f, g) solves the equations with $u = v = 0$, then $f = AA^* f$, so $f = 0$ and $g = 0$: the solution is unique. For centred f, g the marginal densities of ν' are $1 + f + \mathbb{E}_\nu[g(y) \mid x] = 1 + f + Ag = 1 + u$ and $1 + v$. Taking norms in each equation separately, $\|f\| \leq e_X + \rho\|g\|$ and $\|g\| \leq e_Y + \rho\|f\|$, so $\|f\| \leq e_X + \rho e_Y + \rho^2\|f\|$, which is the first bound; likewise for g. (The norm ρ of the off-diagonal operator on $L_0^2(\nu_X) \oplus L_0^2(\nu_Y)$ alone would give only $\|(f,g)\| \leq \|(u,v)\|/(1-\rho)$, a loss of $\sqrt{2}$ in the one-way case.) A symmetry of the data commutes with A and A^* and maps a centred solution to a centred solution, which is therefore the same one.

(b) $f = u - Ag$, so $\|f\|_\infty \leq \|u\|_\infty + K_\infty \|g\|_2$, and (a) bounds $\|g\|_2$. If $k = d\nu/d(\nu_X \otimes \nu_Y)$, then $|Ag(x)|^2 \leq \int k(x,y)^2\, \nu_Y(dy)\, \|g\|_2^2 \leq \sup k\, \|g\|_2^2$, because $\int k(x,y)\, \nu_Y(dy) = 1$.

(c) By (a) ν' has mass one and the marginals μ_X, μ_Y, and $1 + f + g \geq 1 - \|f\|_\infty - \|g\|_\infty > 0$ on the support of ν.

(d) With $U_X = U_Y = e_X = e_Y = U$, (a) gives $U(1+\rho)/(1-\rho^2) = U/(1-\rho)$ and (b) gives $U + K_\infty U/(1-\rho) = F$.

(e) Under the swap the system for (f, g) becomes the system for (g, f); by uniqueness $f = g$, and then $f + Af = u$.

(f) $\nu' \geq 0$ by (c), f, g are centred and $1 + u, 1 + v$ are the densities of the marginals of ν', with $u \geq -U_X$ and $v \geq -U_Y$; Lemma 17.13 gives ρ', and in the case of (d) $F_X = F_Y = F$, $e_X e_Y \leq U^2$ and $\sqrt{(1-U)^2} = 1 - U$. Finally ν' and π have the same marginals; their operators relative to μ_X, μ_Y agree on the constants, both fix them on both sides, and both map centred functions to centred functions, so the difference is the difference of the centred parts, of norm at most $\rho' + \rho(\pi)$.

(g) Since $A^* \mathcal{V}_X \subseteq \mathcal{V}_Y$, the restriction of A^* to $\mathcal{V}_X$ is the adjoint of $A|_{\mathcal{V}_Y} : \mathcal{V}_Y \to \mathcal{V}_X$; so AA^* maps $\mathcal{V}_X$ into itself with norm at most ρ_I^2, and A^*A maps $\mathcal{V}_Y$ into itself with norm at most ρ_I^2. Hence both series converge, $f \in \mathcal{V}_X$, $g \in \mathcal{V}_Y$, $(I - AA^*)f = u - Av$, and $A^*(AA^*)^j = (A^*A)^j A^*$ gives $g = \sum_j (A^*A)^j v - \sum_j (A^*A)^j A^* u = v - A^* f$. Then $g + A^* f = v$ and $f + Ag = (I - AA^*)f + Av = u$. The operators A and A^* are conditional expectations, hence contractions of L^∞, so each of the first N terms of the series for f has sup norm at most $\|u - Av\|_\infty \leq 2U$. For $j \geq N$ write $(AA^*)^j(u - Av) = A\big[A^*(AA^*)^{j-1}(u - Av)\big]$; its sup norm is at most $K_\infty \|A^*(AA^*)^{j-1}(u - Av)\|_2 \leq K_\infty \rho_I^{2j-1} e_I$. Summing over $j \geq N$ gives F_N, and the same argument with A and A^* exchanged bounds g. The marginal densities of ν' are $1 + f + Ag = 1 + u$ and $1 + g + A^* f = 1 + v$, as in (a), and the proof of (c) applies. In the last claim $f \in \mathcal{V}_X$ and $g \in \mathcal{V}_Y$ are invariant, and so is $\nu' = (1 + f + g)\nu$, since ν is. □

Remark 17.15 (Blocks and the certified forms). (1) On a block the replaced and the repaired records are tensor products over the primes whose prime operators fix the constants of the same marginal spaces; if every prime centred norm of both is at most $\frac{1}{2}$, Lemma 6.10(b) bounds the block difference by the largest prime difference: on functions with exact support a set S of k primes the difference is a sum of k terms, each the prime difference at one prime of S times centred factors of norm at most $\frac{1}{2}$ at the other $k - 1$ primes, hence at most $k 2^{1-k} \leq 1$ times the largest prime difference, and on the empty support the difference vanishes. (2) The certificates evaluate

(f) in two forms. The middle merge repairs (Lemma 23.20) use the weaker form with $+2FU$, which the stronger one implies, with the influence bound for $\rho(\pi)$; the one-way repairs of the upper range (Section 28) use $(\rho(1+2F)+U^2)/(1-U)$, with the smaller of the influence bound and the spectral bound of the replaced record for $\rho(\pi)$, and in their joint-star refinement a smaller bound for $\|u\|_2\|v\|_2$ in place of U^2, which (f) allows. In every case the targets are the actual half marginals of the replaced record, the solution of (a) keeps the support of the new graph law, and (a) makes it invariant under the square-affine maps when the data are. (3) The dense comparisons of the upper range and the components of its six-query mixture (Propositions 28.5 and 28.6) are repaired by (g) on the functions invariant under the square-affine maps, where the centred leading stars vanish; on these subspaces $\|A\| \le \rho_I$ with the certified $\rho_I < 1$ of the applications, and the tail of the series in (g) beyond its first N terms contributes at most $K_\infty \rho_I^{2N-1} e_I/(1-\rho_I^2)$ to the L^∞ bound F_N there; the certificates record N for each endpoint ($N = 2$ for the dense records, $N = 1$ for the components of the mixture). The changes of these records, and those of the forest records (Proposition 28.3), from the records they replace are bounded in the actual marginal space by Lemma 17.16: for the dense comparisons and the mixture by its part (c), without a marginal-variance term; for the forest records by its part (b), which also gives the full operator norms on $L^2(m)$ entering the block-gate inequalities (28.3) for the forest records, and for the field pairs FE and EF the symmetric part by its part (c).

A repaired law is compared with the record it replaces in the space of its actual marginals, while its bounds are proved in one common fixed half chart. The next lemma carries out this change of space for a reweighting $(1+f(x)+g(y))K$. It bounds separately the norm of the centred operator and the norms of its symmetric and skew parts; and against a second law with the same marginals, the rank-one terms produced by centring on the left and on the right cancel, so that no marginal-variance term remains.

Lemma 17.16 (Centred additive correction)**.** *Let ν be a probability measure on a finite set $\mathcal{H}$, $\Pi_\perp = I - \mathbb{J}$ on $L^2(\nu)$, and K a real kernel on $\mathcal{H}\times\mathcal{H}$ with $\int K\, d(\nu\otimes\nu) = 1$ (in the applications a probability density relative to $\nu\otimes\nu$). Write $h\otimes k$ for the kernel $h(x)k(y)$, and put $\delta_L = K\mathbf{1} - \mathbf{1}$, $\delta_R = K^*\mathbf{1} - \mathbf{1}$ and $K_c = \Pi_\perp K \Pi_\perp$, so that*

$$K = \mathbf{1}\otimes\mathbf{1} + \delta_L\otimes\mathbf{1} + \mathbf{1}\otimes\delta_R + K_c,$$

with δ_L, δ_R centred. Let $\|K_c\| \le c$, $\|\operatorname{Sym} K_c\| \le s$, $\|\operatorname{Skew} K_c\| \le a$ and $\|\delta_L\|_\nu, \|\delta_R\|_\nu \le e$, all on $L^2(\nu)$. Let f, g be real functions on $\mathcal{H}$ with $\|f\|_\infty, \|g\|_\infty \le F$, and $K' = (1+f(x)+g(y))K$. Here a kernel κ denotes both the measure $\kappa\, d(\nu\otimes\nu)$ and the operator $h \mapsto \int \kappa(\cdot,y)h(y)\,\nu(dy)$ on $L^2(\nu)$; its half marginals are the marginals of that measure. For a kernel κ of mass one with both half marginals equal to a probability measure m, the centred operator $\mathcal{C}_\kappa$ *on $L^2(m)$ is defined by $\langle\phi, \mathcal{C}_\kappa\psi\rangle_m = \int \phi(x)\psi(y)\kappa(x,y)\,\nu(dx)\,\nu(dy) - \mathbb{E}_m\phi\,\mathbb{E}_m\psi$; for a probability density it is the centred conditional operator relative to the marginals.*

(a) *The centred correction $D = \Pi_\perp(K'-K)\Pi_\perp$ is*

$$D = (\Pi_\perp f)\otimes\delta_R + \delta_L\otimes(\Pi_\perp g) + \Pi_\perp(fK_c)\Pi_\perp + \Pi_\perp(K_c g)\Pi_\perp,$$

where fK_c and $K_c g$ are the kernels $f(x)K_c(x,y)$ and $K_c(x,y)g(y)$, and

$$\|D\| \le \|f\|_\nu\|\delta_R\|_\nu + \|\delta_L\|_\nu\|g\|_\nu + (\|f\|_\infty + \|g\|_\infty)\|K_c\| \le 2F(c+e).$$

(b) *Suppose that K' has mass one and both half marginals $m = (1+u_0)\nu$, with $1+u_0 \ge e^{-\ell_m}$ and $\|u_0\|_\nu^2 \le b_0$. Then*

$$K' - (1+u_0)\otimes(1+u_0) = K_c + D - u_0\otimes u_0,$$

and the norms of $\mathcal{C}_{K'}$, of its symmetric part and of its skew part are at most

$$e^{\ell_m}\big(c + 2F(c+e) + b_0\big), \qquad e^{\ell_m}\big(s + 2F(c+e) + b_0\big), \qquad e^{\ell_m}\big(a + 2F(c+e)\big).$$

(c) *Suppose moreover that O is a kernel of mass one with the same half marginals m (in the applications the record that is replaced). Then*

$$K' - O = \Pi_\perp (K - O)\Pi_\perp + D,$$

and the symmetric and skew parts of $\mathcal{C}_{K'} - \mathcal{C}_O$ have norms at most $e^{\ell_m}(\|\operatorname{Sym}\Pi_\perp(K - O)\Pi_\perp\| + 2F(c+e))$ and $e^{\ell_m}(\|\operatorname{Skew}\Pi_\perp(K-O)\Pi_\perp\| + 2F(c+e))$: there is no marginal-variance term.

Proof. The kernel $h(x)k(y)$ has the compression $(\Pi_\perp h)(x)(\Pi_\perp k)(y)$, so a term that is constant in x or in y is killed by $\Pi_\perp \cdot \Pi_\perp$. The four terms of the decomposition of K are its four compressions by $\mathbb{J}$ and $\Pi_\perp$: $\mathbb{J}K\mathbb{J} = \mathbf{1}\otimes\mathbf{1}$ because K has mass one, $\Pi_\perp K\mathbb{J} = (K\mathbf{1}-\mathbf{1})\otimes\mathbf{1}$, $\mathbb{J}K\Pi_\perp = \mathbf{1}\otimes(K^*\mathbf{1}-\mathbf{1})$, and δ_L, δ_R are centred for the same reason.

(a) Expand $(f(x)+g(y))K$ with this decomposition and compress. In $f(x)K(x,y)$ the terms $f\otimes\mathbf{1}$ and $(f\delta_L)\otimes\mathbf{1}$ are killed, and $f\otimes\delta_R$ and fK_c give $(\Pi_\perp f)\otimes\delta_R$ and $\Pi_\perp(fK_c)\Pi_\perp$, since $\Pi_\perp\delta_R = \delta_R$; in $g(y)K(x,y)$ the terms $\mathbf{1}\otimes g$ and $\mathbf{1}\otimes(g\delta_R)$ are killed, and $\delta_L\otimes g$ and $K_c g$ give the other two terms. The rank-one terms have norms $\|\Pi_\perp f\|_\nu\|\delta_R\|_\nu \le \|f\|_\nu\|\delta_R\|_\nu$ and $\|\delta_L\|_\nu\|\Pi_\perp g\|_\nu \le \|\delta_L\|_\nu\|g\|_\nu$; the last two are compressions of the products of K_c with the multiplications by f and by g, of norms at most $\|f\|_\infty\|K_c\|$ and $\|g\|_\infty\|K_c\|$. With $\|f\|_\nu \le \|f\|_\infty \le F$ and $\|g\|_\nu \le F$ this is at most $2F(c+e)$. The bound uses only the sup norms of f, g, and no centring of them.

(b) Since $K'\mathbf{1} = K'^*\mathbf{1} = 1+u_0$ and $\mathbb{E}_\nu u_0 = 0$, the kernel $X = K' - (1+u_0)\otimes(1+u_0)$ annihilates the constants on both sides, so it equals its compression $\Pi_\perp K'\Pi_\perp - u_0\otimes u_0 = K_c + D - u_0\otimes u_0$. By the density transport of Lemma 13.21(b), with $\psi = 1+u_0$ and $W\phi = (1+u_0)^{1/2}\phi$,

$$\mathcal{C}_{K'} = W^* M_{1+u_0}^{-1/2} X M_{1+u_0}^{-1/2} W.$$

Here M_{1+u_0} denotes multiplication by $1+u_0$. The transport multiplies norm bounds by at most e^{ℓ_m} and preserves symmetric and skew parts. The rank-one term $u_0\otimes u_0$ is self-adjoint, of norm $\|u_0\|_\nu^2 \le b_0$, so it enters the full and the symmetric norms and not the skew one; and the symmetric and skew parts of D have norms at most $\|D\|$.

(c) K' and O have mass one and the same half marginals, so $K'-O$ annihilates the constants on both sides and equals its compression $\Pi_\perp(K'-O)\Pi_\perp = \Pi_\perp(K-O)\Pi_\perp + D$. As in (b), $\mathcal{C}_{K'} - \mathcal{C}_O = W^* M_{1+u_0}^{-1/2}(K'-O)M_{1+u_0}^{-1/2}W$: the two subtractions $(1+u_0)\otimes(1+u_0)$ cancel. Take symmetric and skew parts. □

A second way to correct the marginals of a positive kernel adds instead of reweighting. If a coupling J with the prescribed (target) marginals and the required support is at hand, a kernel with marginal defects α, β bounded in absolute value by M and satisfying $J\beta = 0$ and $J^*\alpha = 0$ is corrected by the added term $J(\alpha+\beta+2M)$ of Corollary 17.17. This uses neither the inversion of $I - AA^*$ nor a hypothesis $\rho < 1$, and the centred change is at most $2M(\|QRQ\| + 2\|QJQ\|)$ there.

Corollary 17.17 (Completion by mixing in a coupling of the target marginals)**.** *Let m_X, m_Y be probability measures on finite sets X, Y, and let $R \ge 0$ be a density relative to $m_X\otimes m_Y$ of mass one, with marginal defects $\alpha = \mathbf{1} - R\mathbf{1}$ and $\beta = \mathbf{1} - R^*\mathbf{1}$, $|\alpha|, |\beta| \le M$. Let $J \ge 0$ be a density relative to $m_X\otimes m_Y$ with $J\mathbf{1} = \mathbf{1}$ and $J^*\mathbf{1} = \mathbf{1}$ (a coupling of m_X and m_Y) such that $J\beta = 0$ and $J^*\alpha = 0$. Then*

$$P = \frac{R + J\big(\alpha(x) + \beta(y)\big) + 2MJ}{1+2M},$$

where $J(\alpha(x)+\beta(y))$ is the kernel $J(x,y)(\alpha(x)+\beta(y))$, is a probability density relative to $m_X\otimes m_Y$ with marginals m_X and m_Y, it vanishes wherever R and J both vanish, and, with Q the projections off the constants,

$$\|Q(P-R)Q\| \le 2M\big(\|QRQ\| + 2\|QJQ\|\big),$$

the norms being those of operators from $L^2(m_Y)$ to $L^2(m_X)$. If a group acts on X and on Y preserving m_X and m_Y, and the diagonal action on $X \times Y$ preserves the densities R and J, then P is invariant under the diagonal action.

Proof. $J(x,y)(\alpha(x)+\beta(y)+2M) \geq 0$ because $\alpha+\beta \geq -2M$, so $P \geq 0$. As an operator the kernel $J(x,y)(\alpha(x)+\beta(y))$ is $M_\alpha J + JM_\beta$, with M_α the multiplication by α. By $J\mathbf{1} = \mathbf{1}$ and $J\beta = 0$, $(1+2M)P\mathbf{1} = (\mathbf{1}-\alpha)+\alpha+0+2M\mathbf{1}$; by $J^*\mathbf{1} = \mathbf{1}$ and $J^*\alpha = 0$, $(1+2M)P^*\mathbf{1} = (\mathbf{1}-\beta)+0+\beta+2M\mathbf{1}$. So P has the marginals m_X and m_Y, and mass one; the support claim is clear. Next, J maps centred functions to centred functions, since $\langle \mathbf{1}, J\psi\rangle = \langle J^*\mathbf{1}, \psi\rangle$, and constants to constants; so $JQ = QJQ$ and $QJ\mathbb{J} = 0$. Hence $QM_\alpha JQ = QM_\alpha(QJQ)$ and $QJM_\beta Q = (QJQ)(QM_\beta Q)$, and $\|Q(M_\alpha J + JM_\beta)Q\| \leq 2M\|QJQ\|$. Therefore

$$Q(P-R)Q = \frac{Q(M_\alpha J + JM_\beta)Q + 2M\,QJQ - 2M\,QRQ}{1+2M}$$

has norm at most $2M(\|QRQ\| + 2\|QJQ\|)$. Finally α and β are invariant when R, m_X and m_Y are, and the formula for P is equivariant. □

Consequences and what to read next. From Section 5: cut matrices and the order $\succeq_\vartheta$ (Definition 5.2, (5.2), Remark 5.3), diagonal laws of samplewise fixed records and images under half-local maps (Lemma 5.4(i),(v)), block brackets and their tensorization (Lemma 5.6, with (d) for symmetric brackets), and the chamber group with chamber transport (Definition 5.12 and Lemma 5.13). From Section 6: the sector lemma (Lemma 6.10), for block differences. From Section 12: the laws $\lambda_{\mathcal{B}}(G)$ (Definition 12.1). From Section 13: the recentering identity (Lemma 13.19(a)) and density transport (Lemma 13.21(b)). From Section 16: the joint fold and matching rows (Lemma 16.23(b),(c), Corollary 16.24). Nothing from Part III or Part V is used; the references to Part V name applications.

18. Overview of the construction verifying (CP0)–(CP8)

What this section proves and how to read it. Part V exhibits laws satisfying (CP0)–(CP8) with baseline $\Theta < 0.99999999997327 < 1$. This section specifies the construction once: parameters, queries, prime ranges, and which graph law is used in each range. Later sections verify the hypotheses range by range; each ends by naming the rows of Table 5 it proves. Inputs: Theorem C and Part IV. Output: the standing data Definition 18.1 for the baseline. The improved ($\vartheta > 1$) parameters differ and are specified in Sections 31 to 33; do not use baseline numbers there. "Admissible" in Part V has three qualified senses, distinguished in Section 18.4: admissible for the energy estimates (Definition 8.9), admissible tuples (local degree conditions), and admissible price bounds $c_{\mathcal{C}}$ (allowed (CP5) scalars). Where a sum is computed at a smaller evaluation exponent and converted to a (repricing), Lemma 9.3 and Corollary 9.4 justify the step and require $a \geq a_0$.

The prime ranges are: prefix $p \equiv 1 \pmod 4$, $p < 2^{512}$ (Section 19); middle $p \equiv 3 \pmod 4$ in $[Q_3, P_3)$ (Sections 20 to 26); upper $p \equiv 3 \pmod 4$ in $[P_3, 2^{512})$ (Sections 27 and 28); large primes $p \geq 2^{512}$ and the external index (Section 29); closing (Section 30). Baseline totals are $L_0, S, A, D, \mathcal{E}, T, R_I, J, W, m$ as in Section 1.6 and Table 1.

18.1. **The baseline numerical parameters.**

The construction is now fixed completely: exponents, number of intervals, prime ranges, and which law is used in each range. These are the baseline values ($\theta = \vartheta = 1$); the improved closing changes only the values named in Section 33.

Definition 18.1 (Parameters)**.** The construction has the following data.

(a) *Exponents.* The price exponent $a = \frac{5422587}{6103515625}$, the density exponent $\theta = 1$, and the exponent of the baseline theorem $\varepsilon = a/(72-\theta) = a/71 = \frac{5422587}{433349609375}$.

(b) *Slots, queries and intervals.* The slot array $V = (\mathbb{Z}/6\mathbb{Z})^2 \times \{0,1\}$ with its family $\mathfrak{F}$ of cuts and its reserved cut ι (Definition 5.12); the six queries $\mathcal{E}_6$ (Section 18.2); and $K = 890666$ intervals.

(c) *Prime ranges.*

- *Prefix:* the primes $p \equiv 1 \pmod 4$ below $P_4 = 2^{512}$. Each is an atom $\{p\}$ with label group $\mathbb{Z}/p^{e_p}\mathbb{Z}$, $e_p = 2m_{\mathrm{cl}}$, $m_{\mathrm{cl}} = 8192$ digit pairs, carrying the square-clique law of Definition 19.1.
- *Small modulus:* $M_0 = 2^{2m_{\mathrm{ext}}+1} \prod_{3 \le p < Q_3,\ p \equiv 3\ (4)} p^{2m_{\mathrm{ext}}}$, $m_{\mathrm{ext}} = 64$, $Q_3 = 19951586$. Its primes are not coordinates of the label group; they enter only through the index (Definition 9.6).
- *Middle range:* the primes $p \equiv 3 \pmod 4$ in $[Q_3, P_3)$, $P_3 = 167 \cdot 2^{40}$, in 33 pieces $[X_i, X_{i+1})$ (Table 24), each partitioned into exact blocks (Definition 20.1); the seven tilted pieces start at 2^{30}, 2^{31}, 2^{32}, 2^{34}, 2^{36}, 2^{40} and 2^{45}.
- *Upper range:* the primes $p \equiv 3 \pmod 4$ in $[P_3, P_4)$, in 310 bands, each partitioned into blocks (Definition 27.1): 45 bands in $[P_3, 2^{48})$, namely $[167 \cdot 2^{40}, 168 \cdot 2^{40})$ and the 44 bands $[j2^{41}, (j+1)2^{41})$, $84 \le j < 128$; 64 equal bands in each octave $[2^b, 2^{b+1})$, $48 \le b \le 51$; and the nine bands $[2^b, 2^{b+1})$, $52 \le b \le 57$, $[2^{58}, 2^{60})$, $[2^{60}, 2^{64})$ and $[2^{64}, P_4)$.
- *Tail:* every prime $p \ge P_4$, an atom $\{p\}$ with $e_p = 1$; the threshold in the mixing condition (R4) (Lemma 29.3) is $P_5 = 2^{65536}$.

Every other prime (2, and the primes $p \equiv 3 \pmod 4$ below Q_3) divides M_0.

(d) *Block budgets.* Every middle block has $\log q_{\mathcal{B}} \le L = 570$, so that its selection factor is at most e^{2aL}, with $2aL = 1236349836/1220703125 < 1.013$. The local laws of the middle range are evaluated for blocks of logarithmic length at most the local cap $L_{\mathrm{loc}} = 585$, at which their constants are computed (Lemma 20.3). Every upper block has $\log q_{\mathcal{B}}$ at most the recorded budget L_{band} of its band, an integer between 550 and 886 (Definition 27.1), at which the local data of the band are evaluated; the completed-root caps of the bands from 2^{52} and the local data of the tail law (Lemma 29.2) are evaluated for the larger block length $L_{\mathrm{u}} = 50000/27$.

(e) *Fold records.* In the comparison law the two-layer fold records F_1, F_2 of the middle blocks are replaced by the signed whole-block fold records with parameter $\mathsf{t} = \frac{127}{500}$ (Definition 23.9).

(f) *Roots.* On a middle piece starting below 2^{30} the two-layer root is the uniform strict-square labelling law $\lambda_{\mathcal{B}}(G_{\mathrm{L},2})$. The other seven pieces use the positive reweightings proved in Section 21:

piece start	multiplier	u
2^{30}	$\frac{1}{3}\sum_j (1 + \mathsf{u}\mathsf{H}_j)^2$	711/4096
2^{31}	same	1021/4096
2^{32}	same	1455/4096
2^{34}	same	833/1024
2^{36}	same	3851/4096
2^{40}	same	4049/4096
2^{45}	$1/100 + (1 + \mathsf{u}H_{39})^2$	67/40

Each multiplier is normalized at every prime; $H_{39} = H_{\mathcal{D}_{39}}$ is the collected signed trace sum with coefficients $\pm 1/15$ specified in Section 21. The one-layer root remains $\lambda_{\mathcal{B}}(G_{\mathrm{L},1})$ (the graph $G_{\mathrm{L},1}$ of Section 20.3) on every piece.

(g) *First stage and bracket.* The one-layer folded copies of the fold records are fully polarized (polarized fraction one, i.e. the whole of F_{fold} is subtracted in forming ρ; Definition 18.3(c)); the folded matching records use the one-stage repair of Lemma 22.6, whose record differences,

of the two types E_0 and E_1, form the two tags of (CP2)(b) (Definition 22.4(e)); the dominator N of (CP3) carries one folded copy of each fold record (Definition 22.4(d)).

(h) *Closing constants.* The common-mean coefficient $63/64 = 1 - 2^{-6}$ (Theorem 9.21 with $k = 6$), and the side weight $\tau_{\mathrm{side}} = 31/32$ for the auxiliary sides of the joint Gram block (Definition 26.2); every numerical side bound of the matching envelopes is at least the corresponding auxiliary side bound (Proposition 26.4).

(i) *Credits.* Variance credits of degrees one to four, with positive backgrounds of degrees one and two (Section 24); mean savings of all degrees up to four (Proposition 26.7); on the pieces starting at or above 2^{31}, pooling over those pieces adds background degrees three and four and mean degrees five and six (Lemma 24.16); every restoration, i.e. every recovery of product energy as in Section 24.3, is formed at the exponent a with the piece product caps of Lemma 20.7.

(j) *Analytic stage* (Definition 8.1). $\beta = 1/20$, the major-arc exponent $\eta = 17/100$, $\gamma = 255/1024$, $\tau = 1/201$, $t_{\mathrm{an}} = 1/60$, $\sigma = 1/825$, and the moment losses $\delta = 10^{-10}$ and $\ell_0 = 10^{-8}$.

The completion weights of the 33 middle pieces and of the 310 upper bands, and the other finite data of the construction, are listed or referenced in Appendix E; the finite consequences of these choices that the frame uses (the tiling of $[Q_3, P_3)$ by the pieces, the completion inequalities and the protection inequality (22.1) in every piece, the one-stage gates, the band starts and the prefix gate) are collected in Certificate D.3 (Appendix D). The price exponent is below the error exponent of the analytic stage, $\sigma - a = 65195254/201416015625 > 0$ (Certificate D.3(a)); the strict inequalities (S1)–(S7) of the analytic stage hold (Certificate D.2), and the gates of the analytic transfer that depend on a hold on the whole interval $0 < a \le 9/10000$, for every fixed $K \ge 2$ (`research/advance_2026_10_24/global/analytic_domain.py`; the baseline verifier asserts $0 < a \le 9/10000$). The larger parameter domain used for the main theorem is verified in Section 33. All data are fixed integers or rationals, independent of the scale; the analysis of Part II applies with constants that depend on them (Remark 8.13). The certificate computes the prices of the middle range at the anchor $a_0 = \frac{4442183}{5000000000} < a$ and reprices them by the factor $t = e^{2(a-a_0)L}$, about 1.0000000616512; it computes the prices of the upper bands at the anchor $a_{\mathrm{u}} = 883/10^6 < a$ and reprices them band by band with the logarithmic-product cap of each band (Lemma 9.3 and Corollary 9.4). The numbers a_0 and a_{u} are points of evaluation of the certificate, not parameters of the construction.

18.2. **The six queries $\mathcal{E}_6$ and the twelve pairs in projections.** The comparison law accepts the six queries $e^{(j)} = ((0, j, 0), (\mathsf{c}0, \mathsf{c}j, 1)) = ((0, j, 0), (2, \mathsf{c}j, 1))$, $j \in \mathbb{Z}/6\mathbb{Z}$, of Definition 5.12(e), the family $\mathcal{E}_6$. By Lemma 5.13(iv) their twelve endpoints are distinct, every source lies in $V_+ = \{\ell = 0\}$ and every reflected target $\iota(2, \mathsf{c}j, 1) = (2, \mathsf{c}j, 0)$ lies in V_+; so $\mathcal{E}_6$ is a family of $k = 6$ queries in the sense of Definition 9.1(c). Each query crosses the reserved cut from layer 0 to layer 1 and applies c to both chamber coordinates.

All six queries have the same first-coordinate pair $(0, \mathsf{c}0) = (0, 2)$; they differ only in the second chamber coordinate, on which they act by the permutation c. A law that ignores the second coordinate therefore sees one query. The middle laws are such laws: they live on one chamber coordinate and the layer, and the pullback reads every query of $\mathcal{E}_6$ through the single pair $((0, 0), (2, 1))$ (Lemma 5.13(v)). In the middle records the six queries thus impose one directed edge between the layers: the two-layer root and the aligned fold F_0 accept it (on the pieces from 2^{30} the root is the positive root of Definition 21.1, which has the accepting support of the uniform root), the two-layer fold records F_1, F_2 and matching records E_0, E_1 need one added cross edge each, and the folded matching records one edge between their two private labels (the two labels not shared between its halves; Certificate 23.10). The square-clique law of the prefix ignores the

layer as well, and sees the six queries as one coincidence event of two labels (Lemma 19.2(v)). This is why the six queries impose on the middle range the same single edge condition as one query, while the matching term of the closing carries the sixth power $(R_I/2)^6$ (Theorem 9.21).

Remark 18.2 (The twelve-query superset). The family

$$\mathcal{E}_{12} = \{((u, j, 0), (\mathsf{c}u, \mathsf{c}j, 1)) : u \in \{0, 5\},\ j \in \mathbb{Z}/6\mathbb{Z}\}$$

consists of twelve queries with twenty-four distinct endpoints, each running from layer 0 to layer 1. Its first-coordinate pairs are $(0, 2)$ and $(5, \mathsf{c}5) = (5, 3)$, and $\mathcal{E}_6$ is its part with $u = 0$. Some parts of the construction accept all twelve queries: the one-way repairs of the upper range, whose ledger was built for $\mathcal{E}_{12}$ (Section 28.1), the dense comparison of the two EL records (field type E in one chamber coordinate and L in the other; Definition 28.4, and Section 27.2.1 for the codes), and the tail law with its one-way repairs (Lemma 29.5). Others accept $\mathcal{E}_6$ but not all of $\mathcal{E}_{12}$: the mirror merge repairs and accepting two-layer constants (Lemmas 23.20 and 23.23), the signed fold records of the middle (Definition 23.9), the positive mixture of the six LE records (the field types L and E exchanged; Proposition 28.6), and the upper records $G_{01}L$, $G_{10}L$ with their reserved one-layer companions, which accept $\mathcal{E}_6$ and are retained (Lemma 28.1). So $\mathcal{E}_{12}$ is only an acceptance set of parts of the construction: the comparison theorem is applied with $\mathcal{E}_6$, $k = 6$ (what twelve queries would need is in Section 35.7). In the density-linear baseline (density exponent $\theta = 1$), the exception of the clique prefix is charged with the two-pair mass $\eta_{m_{\mathrm{cl}}} = 2^{1-m_{\mathrm{cl}}} - 2^{-2m_{\mathrm{cl}}}$, its exact mass for $\mathcal{E}_{12}$ and an upper bound for $\mathcal{E}_6$ (Lemma 19.2(v)).

18.3. **Which graph law is used in each prime range.** The construction is one set of comparison data in the sense of Part III. Its objects are built in the sections named below; here they are listed together.

Definition 18.3 (The construction)**.** The comparison data of Part V, at the parameters of Definition 18.1, are the following.

(a) *Frame data* (Definition 9.1). The slot array V with $\mathfrak{F}$ and ι. At scale N the atoms of Definition 6.1: the prefix atoms, the blocks of the 33 middle pieces and of the 310 upper bands, and a tail atom $\{p\}$ for every prime $P_4 \le p \le N^{\beta}$. The queries $\mathcal{E}_6$. The index law: independent indices $\omega_e = (b_e, s_e) \in \mathbb{Z}/K\mathbb{Z} \times \mathbb{Z}/M_0\mathbb{Z}$, $e \in \mathcal{E}_6$, with coupled index $(b_e + 1, s_e + r_e)$, where each prime-power component of r_e is the square of a uniform residue conditioned so that the residue itself has p-adic valuation less than 64. The independent depths J_p have tails $\mathbb{P}(J_p \ge j) = \delta_{p,j}$ of Lemma 29.6; the interval coordinate b_e is always revealed (conditioned on, not averaged over; Definition 9.6 and Proposition 29.7).

(b) *Reference* $\mu = \bigotimes_{\mathcal{B}} \mu_{\mathcal{B}}$ (Definition 9.10): at a prefix atom the square-clique law (Definition 19.1); on a middle block the twelve-label five-type completion of Lemma 22.1, a law on one chamber coordinate and the layer pulled back to V, whose two-layer root is the uniform labelling law on the pieces below 2^{30} and the positive root of Definition 21.1 on the seven pieces from 2^{30}; on an upper block the 25-state five-type completion of the two-layer graph law, with independent one-layer fold weights (Definition 27.1, Lemma 27.17); at a tail atom the tail law of Definition 29.1 at the single prime.

(c) *First stage and bracket* (Definitions 9.11 and 9.13). The first-stage atoms are the middle blocks. There the records F_{fold} (the two nonaccepting one-layer fold components on the reserved diagonal), E_{fold} (the two reserved-folded matching components), E_{add}, the polarization correction G, the reserved half diagonals H, and the one-stage repair E_{new} of $E_{\mathrm{fold}} + E_{\mathrm{add}}$ give, with ρ the intermediate law obtained from μ by polarization before the

one-stage repair, and N_0 the auxiliary dominator of Definition 9.13,

$$\rho = \mu - F_{\mathrm{fold}} + E_{\mathrm{add}} + G, \qquad \lambda = \rho + \delta_E, \qquad \delta_E = E_{\mathrm{new}} - E_{\mathrm{fold}} - E_{\mathrm{add}},$$
$$\mathsf{N} = \lambda + F_{\mathrm{fold}} + H, \qquad \mathsf{N}_0 = \mu + E_{\mathrm{add}} + G + H$$

(Definition 22.4). The record differences of δ_E of type E_0, whose sharp (O2) bound is at their first-coordinate cut ϑ_1, form one of the two tags of (CP2)(b), and those of type E_1, whose sharp bound is at ϑ_2, the other. Off the middle blocks, $\lambda_{\mathcal{B}} = \mathsf{N}_{\mathcal{B}} = \mu_{\mathcal{B}}$.

(d) *Comparison law* $\nu = \bigotimes_{\mathcal{B}} \nu_{\mathcal{B}}$ (Definition 9.14), with the normalizing masses of μ, as described in the next paragraph. On a middle block $\nu_{\mathcal{B}}$ is a finite real combination of nonnegative measures that accept the queries, each of the two signed fold records contributing three product components (Definition 23.9 and Lemma 23.24); every other $\nu_{\mathcal{B}}$ is nonnegative. As the middle blocks are finitely many, the variation factor of Definition 9.14 is a fixed finite number.

(e) *Classes, envelopes and the mean credit* (Definitions 9.16 to 9.18). The variance classes of degrees one to four with their gains and positive backgrounds (Section 24); the positive joint envelope classes with $n \geq 1$ fold factors and m matching factors in the tensor product over distinct middle blocks defining the class, $n + m \in \{2, 4, 6\}$, the aligned and crossed matching pairs and the matching classes of degrees four and six, with the class bounds of Corollary 25.10 in the auxiliary prices of Lemma 23.24, the invariant background at its assigned tag price U^{inv} (Lemma 25.6) and the envelope portfolio of Table 33 (Theorem 25.8 and Lemmas 25.7 and 25.9); matching envelopes whose sides are the larger of the signed direct-Gram side and the auxiliary side, rounded up, with chosen side diagonals (Propositions 26.4 and 26.5); and the mean credit (Proposition 26.7).

The comparison law in words. On a middle block the comparison law keeps the two-layer root and the aligned fold F_0, which accept the queries. It replaces the two other two-layer fold records F_1, F_2 by the signed whole-block law: on the block, the combination $\bigotimes_p P_{++,p} + (\mathsf{t}/2) \bigotimes_p P_{-+,p} - (\mathsf{t}/2) \bigotimes_p P_{--,p}$ of three positive accepting product laws, each of which is at every prime the joint law (Markov coupling) of two nonnegative kernels, both forcing the extra cross edge, whose two marginals are the actual half marginals of the reference fold record (an *accepting Markov coupling*) (Definition 23.9 and Lemma 23.7). The combination is taken after the block products, so its total variation is at most $1 + \mathsf{t}$ whatever the number of primes; F_2 is the reflected transpose of F_1, and the case $\mathsf{t} = 0$ is the mirror pairing of Lemma 23.5. It replaces the two two-layer matching records by the exact-marginal law K_1 (Lemma 23.17), repairs every two-layer merge record by its actual-marginal mirror law (Lemma 23.20), replaces the two-layer constants by the accepting edge law (Lemma 23.23), and, relative to the first-stage law λ, deletes the folded merge and constant records and the polarization correction G, and keeps E_{new} unchanged. On an upper block it keeps every record that accepts the six queries, among them the records $G_{01}L$, $G_{10}L$ (the two EL records of state RL) and their reserved one-layer companions (Lemma 28.1); it replaces the six degree-two LE records by the positive mixture of four separately repaired six-query laws (Proposition 28.6), the two remaining degree-two EL records by the dense positive comparison (Definition 28.4 and Proposition 28.5), each of the other 84 nonaccepting two-layer records of degrees two and three by its one-way repair, and deletes every other nonaccepting record (Definition 28.13); the tail atoms use the one-way replacements of the tail law at single primes, with the all-prime caps (Lemma 29.5). At each prefix prime it deletes the exception of Lemma 19.2. Every change is either a repair with the same actual half marginals at ι as the record it replaces, or the deletion of a positive record. Every component of the comparison law is nonnegative and accepts the six queries, and only the middle blocks carry signed combinations. The comparison law is not reflection positive; it is compared with the first stage only at ι, through the bracket N and the norm envelopes of its changes, with the symmetric and skew parts priced separately.

One recipe per range. Table 9 lists, for each range of primes, the reference law and the reason for its positivity, the change made by the comparison law, and where the range enters the closing number Θ.

Table 9: The ranges of primes.

Range	Reference law and its positivity	Comparison change	Contribution to Θ
Small modulus: 2^{129} and p^{128} for $3 \le p < Q_3$, $p \equiv 3$ (4) (not atoms)	no law; an external index per query, with coupled index $(b+1, s+r)$; doubly stochastic kernels, no positivity needed	none	the conductor moment $R_I = M_{\mathrm{pad}}(K)$ in J and W, at $K = 890666$ (Proposition 29.7 and Lemma 29.9)
Prefix: $p \equiv 1$ (4), $p < P_4$; atoms $\{p\}$, $e_p = 2m_{\mathrm{cl}}$	independent draws from a random square clique; at the chamber cuts of the first coordinate the square of a conditional mean, the other cuts fix it samplewise (Lemma 19.2)	delete the zero-difference exception	mass at most $\eta_{m_{\mathrm{cl}}}$; total price below $2^{513-m_{\mathrm{cl}}+1024m_{\mathrm{cl}}a}$, in S and T only, inside the allowance 10^{-12} (Lemma 19.3)
Middle: $p \equiv 3$ (4) in $[Q_3, P_3)$; 33 pieces, blocks with $\log q_{\mathcal{B}} \le L$	twelve-label five-type completion on one chamber coordinate and the layer (Lemma 22.1), its root uniform, or positive on the seven pieces from 2^{30} (Definition 21.1); chamber cuts by the completion lemma with the caps of the root (Lemma 20.12 and Propositions 20.21, 21.7 and 21.9) and the rows, the layer flip by fold pairing (Lemma 17.5)	first stage: polarized folds, one-stage repair E_{new} in two tags; comparison: signed whole-block F_1, F_2, exact-marginal E, mirror merges, accepting two-layer constants, folded deletions	L_0 and the energy factor B through the tagged series (Section 22); the direct prices of the middle records in S, A, and the auxiliary prices of the fold and matching records in the class bounds (Lemma 23.24); variance and envelope credits in $\mathcal{E}$ (Section 24, Corollary 25.10); the prices of the matching and other records and the side excess T_K (Proposition 26.4(iii)) of the matching envelopes in the total T, the mean savings in m (Section 26)
Upper: $p \equiv 3$ (4) in $[P_3, P_4)$; 310 bands	25-state two-layer five-type completion with independent reserved fold weights and observed reserved caps (Lemma 27.17 and Proposition 27.14); root caps by the direct Route B criterion below 2^{52}, with graded counts and interpolated coefficients, and by the Route A criterion from 2^{52} (Theorems 14.21 and 14.27); joint fold and matching rows (Lemma 17.5)	one-way repairs of 84 records, a dense comparison of two EL records, a positive mixture of the six LE records; deletion of the other nonaccepting records	$S_{\mathrm{u}}(a)$, $A_{\mathrm{u}}(a)$ in S, A and T, the band ledger repriced band by band (Proposition 28.14)

Range	Reference law and its positivity	Comparison change	Contribution to Θ
Tail: every $p \geq P_4$; atoms $\{p\}$, $e_p = 1$	the tail law at one prime (Definition 29.1), at class-one primes with the all-prime caps; mixing (R4) for $p \geq P_5$ (Lemmas 29.2 and 29.3)	one-way repairs of the 94 records that do not accept the twelve queries, with the all-prime caps; deletion of the other nonaccepting records	per prime at most $Cp^{-3/2+2a}$ with the absolute constant C of Lemma 29.5, inside $S_{\mathrm{u}}(a)$ and $A_{\mathrm{u}}(a)$ (Lemma 29.5)

18.4. **Three uses of "admissible", and why each holds.** The energy estimates of Part II apply to the energies of the comparison when the laws and the reductions are admissible (Definition 8.9), and (CP8)(a) asks this of every law of the construction. Each construction lemma of Section 19 and of the sections after it states the admissibility of its laws; Table 10 collects them, with the two facts that give (O1) and (O2) for each.

Table 10: The laws of the construction and their admissibility.

Law	Sign; one-slot marginals (O1)	Sharp bounds (O2)	Proved in
prefix: λ_p, the exception λ_p^{ex}, $\lambda_p - \lambda_p^{\mathrm{ex}}$	nonnegative; invariant under $\mathfrak{G}_p$	none; the exception is charged by its mass	Lemma 19.2(vi)
middle and upper references $\mu_{\mathcal{B}}$	nonnegative; invariant under the maps of (R3)	none	Lemma 22.1, Lemmas 21.2 and 21.17, Lemma 27.17
first-stage and bracket records F_{fold}, E_{fold}, E_{add}, G, H; $\mathsf{N}_0 - \mu = E_{\mathrm{add}} + G + H$	nonnegative, reserved diagonal; images of invariant laws under vertex maps, then diagonal copies	none; variation and mass prices	Definition 22.4, Lemma 22.7
one-stage repair E_{new}	nonnegative, reserved diagonal; invariant	$k_{\mathrm{new}}(X)b^{-1/2}$, and $s_{\mathrm{fold}}(X)/b$ for the symmetric part, for every value of the shared label, where X is the piece start and b the least prime of the block	Lemma 22.6
middle comparison records: the three product components of each signed fold record, the law K_1, merge and two-layer constant repairs	nonnegative; invariant under the maps of (R3), independently at the primes of the block; the actual half marginals of the replaced record	operator norms between actual half spaces at ι, nothing conditioned	Lemma 23.7, Proposition 23.13, Lemma 23.17, Lemmas 23.20 and 23.23; Lemma 23.24
upper and tail comparison laws: one-way repairs, the dense EL comparison, the components of the LE mixture	nonnegative; the actual half marginals of the replaced record, each mixture component repaired to them before mixing	as for the middle records	Proposition 28.14, Proposition 28.5, Proposition 28.6, Lemma 29.5

Law	Sign; one-slot marginals (O1)	Sharp bounds (O2)	Proved in
deleted records (middle, upper, tail)	records of the reference	none; charged by mass (Remark 9.15)	Lemma 23.24, Proposition 28.14, Lemma 29.5
tail law μ_p, $p \geq P_4$	as the upper references, at one prime	none	Lemma 29.2
selected laws of the variance credits	positive; invariant under independent translations at the primes of their blocks	$d = 1$ for a probability coupling; centred norms of a record's own operator at ι	Definition 24.1, Lemma 24.12
couplings of the positive joint envelopes	normalized positive couplings, among them the singleton couplings $M_{ij} = V_i V_j^*$ (a uniform label and independent conditioned halves); invariant under translations at each prime	none (the envelope bounds are brackets at ι between actual half spaces)	Theorem 25.8(iv), Lemma 25.3
matching-envelope diagonals $M_{s,\mathcal{B}} - \mathsf{N}_{\mathcal{B}}$	positive, reserved diagonal (norm envelopes and chosen side diagonals); normalizations with uniform one-slot marginals	block Grams at ι between actual half spaces	Proposition 26.5, Proposition 26.4
companions of the mean credit	positive normalized laws fixed by ι, uniform one-slot marginals	none	Lemma 26.6

How the table is read. For (O1), every law of the table is nonnegative (a signed fold record of the middle enters only through its three nonnegative product components, each of which is such a law), and its one-slot marginals are uniform for one of two reasons named in the table: it is invariant under the translations, or all square-affine maps, at the primes of its atom, so that Lemma 5.16(2) applies; or it is nonnegative with the actual half marginals of such an invariant law, whose one-slot marginals it therefore shares. Invariance passes to every law that the reductions of Part III produce from these: to variations, normalizations, restrictions to invariant events, half marginals and their diagonal copies, images under folds and pullbacks, products over disjoint atoms and mixtures (Lemma 5.16(1)). Every law of the table is a law of the field labels of its atoms, and the index is external (Definition 9.1(d)); so no law depends on the interval index b_e, the residue s_e, or the inclusion indices of Definition 9.6, which is the last clause of (O1). For (O2), the third column says why every sharp bound applied to the law holds uniformly in the shared labels: either it is an operator norm between actual half spaces with nothing conditioned, or it is proved for every value of the shared label. The remaining two obligations concern the reductions, not the laws. (O3): the index-support projections are taken before the field folds and the inclusion conductors are averaged last, in the reductions of Part III (Lemmas 10.1, 10.6 and 10.17), and no section of Part V changes this order. (O4): every chamber fold commutes with ι on the slot array (Lemma 5.13(iii)).

18.5. **Where the hypotheses are verified (guide to Part V).** Table 5 lists each hypothesis of Theorem 9.21, what Part V shows for the construction of Definition 18.3, and the statements that show it. The frame data (CP0) are verified by the parameters above, by the slot array (Lemma 5.13) and by the index law (Proposition 29.7 and Lemma 29.9); the hypotheses (CP1)–(CP7) range by range in the following sections; the admissibility clause (CP8)(a) by Table 10; and the summability clause (CP8)(c) because only the tail atoms are infinitely many and their prices are summable

(Lemma 29.5(iv)), while the auxiliary prices of (CP5) are nonzero only at the finitely many middle atoms. The robustness clause (CP8)(b) follows from Lemma 11.4(i) (Remark 11.5): under the inflation of Definition 9.19 (multiplying structured energy bounds by $1+z$ and adding remainders at most zY), every certified inequality (gate) whose left side changes is strict, except two: the orbit saving of Lemma 25.7, which is a nonnegativity condition on a coefficient that only grows under inflation, and the hypothesis $\chi<1$ selecting the branch of the restoration function (32.5), which inflation preserves. The remaining non-strict gates concern fixed numbers of the construction. Only the finitely many middle blocks carry signed comparison laws (Definition 18.3(d)), so the variation factor of Definition 9.14 is finite and independent of the scale.

Consequences and what to read next. This section uses the slot array and its queries (Definition 5.12, Lemma 5.13(iii), (iv), (v)); the comparison data and the hypotheses (CP0)–(CP8) (Definitions 9.1, 9.10, 9.11, 9.13, 9.14, 9.16 to 9.18 and 9.20); admissibility and its closure under the operations of the reductions (Definition 8.9, Lemma 5.16); the functional, the analytic stage and the repricing (Definition 6.1, Definition 8.1, Certificate D.2, Lemma 9.3, Corollary 9.4); and, as pointers, the construction lemmas named in Tables 9 and 10.

Hypotheses verified in this section (Table 5). (CP0)(b) for the construction: the atoms and exponents of Definition 18.1, with $a<\sigma$, the stage inequalities (S1)–(S7) and the analytic gates on $0<a\le 9/10000$ (Certificate D.2); (CP0)(c): the six queries have disjoint endpoints, with sources and reflected targets in V_+ (Section 18.2, Lemma 5.13(iv)); for (CP4), that only the finitely many middle blocks carry signed combinations, so that the variation factor is finite and independent of the scale (Definition 18.3(d)); and the summary of (CP8)(a) in Table 10. These are the rows "(CP0)(b)", "(CP0)(c)" and "(CP8)(a)" of Table 5, and the finiteness of the variation factor in the row "(CP4) middle F", whose variation $1+\mathtt{t}$ per block is that of the signed fold record (Definition 23.9(c)).

19. Primes $p\equiv 1 \pmod 4$: independent labels from a random square clique

What this section proves and how to read it. At $p\equiv 1 \pmod 4$, -1 is a square, so "differs by a nonzero square" is symmetric and independent labels can force it. This section constructs, for $p\equiv 1 \pmod 4$ below 2^{512}, a reflection-positive law (CP1) with negligible comparison price: draw labels independently from a random *square clique* (a set of residues pairwise differing by a square, defined below), so every query difference is a square except on the probability-$2^{-m_{\mathrm{cl}}}$ event that its ends coincide (the *deleted coincidence event*, called "the exception" below); at the baseline exponent, deleting that event gives total price below $8\cdot 10^{-69}$. At $p\equiv 3 \pmod 4$ no exchangeable law (one invariant under permuting the slots) can force a directed square difference, which is why middle and upper ranges need oriented graph laws. Output: (CP1), (R5) and the prefix parts of (CP4),(CP6). Large primes $p\ge 2^{512}$ of both classes are in Section 29.

19.1. **The square-clique law.** At primes $p\equiv 1 \pmod 4$ the relation $x\to y$ defined by $\chi(y-x)=1$ is symmetric, not a tournament as in Section 20.2, so the reference law is built differently: labels are drawn independently from one random set of residues modulo $p^{2m_{\mathrm{cl}}}$, the square clique, which is built digit by digit from strict-square pairs so that its elements have square differences. The definition makes this precise.

Definition 19.1 (The square-clique law)**.** Let $p\equiv 1 \pmod 4$ be prime and $m_{\mathrm{cl}}\ge 1$ an integer. Represent $\mathbb{F}_p$ by $\{0,\dots,p-1\}$. Choose independently, for $0\le j<m_{\mathrm{cl}}$, a uniform ordered strict-square pair $(a_j,b_j)\in\Omega_p:=\{(a,b)\in\mathbb{F}_p^2:\chi(b-a)=1\}$ (Definition 4.5) and two uniform digits $k_j,k_j'\in\{0,\dots,p-1\}$, and put

$$A_j=a_j+pk_j,\qquad B_j=b_j+pk_j',\qquad \mathcal{C}_p=\Big\{\sum_{j=0}^{m_{\mathrm{cl}}-1}p^{2j}x_j:\ x_j\in\{A_j,B_j\}\Big\}\subseteq\mathbb{Z}/p^{2m_{\mathrm{cl}}}\mathbb{Z}.$$

Conditionally on these data, choose six labels $y_0, \dots, y_5$, one for each chamber of the first coordinate, independently and uniformly from $\mathcal{C}_p$, and a uniform square-affine map $g(x) = sx + c$ of $\mathbb{Z}/p^{2m_{\mathrm{cl}}}\mathbb{Z}$ (s a uniform unit square, c uniform), independent of the clique data and of $y_0, \dots, y_5$. The *square-clique law* λ_p is the law of the labelling

$$z_{(u_0,u_1,\ell)} = g(y_{u_0}) \qquad ((u_0, u_1, \ell) \in V)$$

of the slot array by $\mathbb{Z}/p^{2m_{\mathrm{cl}}}\mathbb{Z}$: the pullback of the six labels $(g(y_i))_{i\in\mathbb{Z}/6\mathbb{Z}}$ along the first chamber coordinate (Lemma 5.13(v)). We call $\mathcal{C}_p$ the *random square clique*.

In the construction (Definition 18.1(c)) every prime $p \equiv 1 \pmod 4$ below P_4 is an atom $\{p\}$ with label group $\mathbb{Z}/p^{e_p}\mathbb{Z}$, $e_p = 2m_{\mathrm{cl}} = 16384$, and reference law $\mu_p = \lambda_p$ with $m_{\mathrm{cl}} = 8192$. The prefix is a fixed finite modulus $Q_{\mathrm{cl}} = \prod_{p<P_4,\, p\equiv 1 \ (4)} p^{e_p}$, and its analytic interface is in Part II (Remark 8.13 lists where it enters: the lift retains every prefix character below the conductor cutoff, and the actual-square kernel divides out an even p-adic valuation with square unit part and keeps every prefix prime in the modulus, Proposition 7.3(ii)). Here only the local law is needed.

19.2. **Reflection positivity and forcing except for coincidences.** For a pair of slots (u, v), its *first-coordinate pair* is (u_0, v_0), the pair of their first chamber coordinates.

Lemma 19.2 (The square-clique law)**.** *Let $p \equiv 1 \pmod 4$ be prime, $m_{\mathrm{cl}} \geq 1$, λ_p the law of Definition 19.1, and $\mathfrak{G}_p$ the group of the maps $x \mapsto sx + c$ of $\mathbb{Z}/p^{2m_{\mathrm{cl}}}\mathbb{Z}$ with s a unit square (Definition 5.15).*

- *(i) λ_p is a probability measure on $(\mathbb{Z}/p^{2m_{\mathrm{cl}}}\mathbb{Z})^V$, reflection positive at every cut of the family $\mathfrak{F}$: at the chamber cuts of both chamber coordinates and at ι.*
- *(ii) λ_p is invariant under the simultaneous maps $z \mapsto (h(z_v))_v$, $h \in \mathfrak{G}_p$; in particular its one-slot marginals are uniform.*
- *(iii) The random square clique has exactly $2^{m_{\mathrm{cl}}}$ elements, and every nonzero difference $z_v - z_w$ of two labels of a labelling in the support of λ_p is $p^{2j}u$ with $0 \leq j < m_{\mathrm{cl}}$ and u a unit square modulo $p^{2m_{\mathrm{cl}}-2j}$.*
- *(iv) Let $\mathcal{Q}$ be a family of ordered pairs of slots with first-coordinate pairs (u_0, v_0), $u_0 \neq v_0$, and let λ_p^{ex}, the* exception *of $\mathcal{Q}$, be the restriction of λ_p to the event E that $z_u = z_v$ for some $(u, v) \in \mathcal{Q}$. Then λ_p^{ex} is a nonnegative measure, supported on labellings z with $z_{\iota v} = z_v$ for all v (fixed samplewise by ι) and invariant as in (ii), so that its one-slot marginals are $\lambda_p^{\mathrm{ex}}(1)$ times the uniform law; and $\lambda_p - \lambda_p^{\mathrm{ex}}$ accepts every pair of $\mathcal{Q}$: its difference $z_v - z_u$ is nonzero, of even valuation $2j < 2m_{\mathrm{cl}}$, with a square unit part. If the first-coordinate pairs of $\mathcal{Q}$ form n distinct unordered pairs, then $\lambda_p^{\mathrm{ex}}(1) \leq n2^{-m_{\mathrm{cl}}}$, and $\lambda_p^{\mathrm{ex}}(1) = 1 - (1 - 2^{-m_{\mathrm{cl}}})^n$ if these pairs are pairwise disjoint.*
- *(v) For the six queries $\mathcal{E}_6$, $\lambda_p^{\mathrm{ex}}(1) = 2^{-m_{\mathrm{cl}}}$, so $\lambda_p - \lambda_p^{\mathrm{ex}}$ has mass $1 - 2^{-m_{\mathrm{cl}}} > 0$; for the twelve queries of Remark 18.2, $\lambda_p^{\mathrm{ex}}(1) = \eta_{m_{\mathrm{cl}}} = 2^{1-m_{\mathrm{cl}}} - 2^{-2m_{\mathrm{cl}}}$. In both cases $\lambda_p^{\mathrm{ex}}(1) \leq \eta_{m_{\mathrm{cl}}}$.*
- *(vi) The measures λ_p, λ_p^{ex} and $\lambda_p - \lambda_p^{\mathrm{ex}}$ are admissible (Definition 8.9). Thus $\mu_p = \lambda_p$ satisfies (R1)–(R3) of Hypothesis 6.2, with (R2) at every cut of $\mathfrak{F}$, and the accepted part of $\lambda_p - \lambda_p^{\mathrm{ex}}$ for $\mathcal{Q} = \mathcal{E}_6$ satisfies (R5).*

Proof. The clique. Since $A_j \equiv a_j$ and $B_j \equiv b_j \pmod p$ and $b_j - a_j$ is a nonzero square in $\mathbb{F}_p$ by the definition of Ω_p, $A_j \neq B_j$. Let $x = \sum_j p^{2j}x_j$ and $x' = \sum_j p^{2j}x'_j$ be given by two different choices, and let j be the least index with $x_j \neq x'_j$, so that $\{x_j, x'_j\} = \{A_j, B_j\}$. Then $x - x' = p^{2j}u$ with $u = (x_j - x'_j) + p^2 w$ for an integer w, and $u \equiv \pm(b_j - a_j) \pmod p$. As $p \equiv 1 \pmod 4$, -1 is a square modulo p, so u is a nonzero square modulo p, and since p is odd, Hensel's lemma makes u the square of a unit modulo $p^{2m_{\mathrm{cl}}-2j}$ (the remark after Definition 4.5). As $2j < 2m_{\mathrm{cl}}$, $x \neq x'$: the $2^{m_{\mathrm{cl}}}$ choices give $2^{m_{\mathrm{cl}}}$ distinct elements, any two of which differ by p^{2j} times a unit square. A

map $g \in \mathfrak{G}_p$ multiplies differences by the unit square s, so the same holds in $g(\mathcal{C}_p)$; and two labels $z_v = g(y_{v_0})$, $z_w = g(y_{w_0})$ are equal or two distinct elements of $g(\mathcal{C}_p)$. This is (iii).

Invariance. The map g is uniform on the group $\mathfrak{G}_p$ and independent of the clique and of the y_i. For fixed $h \in \mathfrak{G}_p$, $h \circ g$ has the same properties, so $(h(g(y_i)))_i$ has the law of $(g(y_i))_i$, and the pullback λ_p is invariant under $z \mapsto (h(z_v))_v$ (Lemma 5.13(v)(d)). The translations act transitively, so the one-slot marginals are uniform (Lemma 5.16(2)). This is (ii).

Positivity. Let ω be the clique data together with g. Conditionally on ω the six labels $Y_i = g(y_i)$, $i \in \mathbb{Z}/6\mathbb{Z}$, are independent and uniform on $g(\mathcal{C}_p)$, and $z_v = Y_{v_0}$. A chamber cut $\vartheta_{1,a}$ of the second coordinate and the cut ι do not change the first coordinate, so they fix every labelling in the support: the cut matrix of λ_p there is nonnegative and diagonal, and λ_p is reflection positive (Lemma 5.13(v)(a), Lemma 5.4(i)). Consider a chamber cut $\vartheta_{0,a}$ of the first coordinate, with reflection ϑ_a and plus half $W_a = \{v : v_0 \in H_a\}$. The family $(Y_i)_i$ is conditionally exchangeable, so λ_p is invariant under the reflection. A real function F of the labels on W_a has the form $F(z_{W_a}) = \Phi((Y_i)_{i\in H_a})$, and then $F((z_{\vartheta_{0,a}v})_{v\in W_a}) = \Phi((Y_{\vartheta_a i})_{i\in H_a})$. As ϑ_a maps H_a onto $\mathbb{Z}/6\mathbb{Z}\setminus H_a$, the families $(Y_i)_{i\in H_a}$ and $(Y_{\vartheta_a i})_{i\in H_a}$ are conditionally independent and identically distributed, so

$$\int F(z_{W_a})\, F\big((z_{\vartheta_{0,a}v})_{v\in W_a}\big)\, d\lambda_p(z) = \mathbb{E}_\omega\Big[\Big(\mathbb{E}\big[\Phi\big((Y_i)_{i\in H_a}\big) \mid \omega\big]\Big)^2\Big] \geq 0,$$

which is reflection positivity at $\vartheta_{0,a}$ (Definition 5.1). This is (i).

The exception. The event E depends only on the first-coordinate labels and is preserved by every bijection $h \in \mathfrak{G}_p$; so $\lambda_p^{\mathrm{ex}} = \mathbf{1}_{\mathsf{E}}\lambda_p$ is invariant and its one-slot marginals are $\lambda_p^{\mathrm{ex}}(1)$ times the uniform law (Lemma 5.16(1), (2)). Like λ_p, it is supported on labellings fixed by ι. Off E every pair of $\mathcal{Q}$ has a nonzero difference, of the form stated by (iii). Conditionally on ω the Y_i are independent and uniform on $2^{m_{\mathrm{cl}}}$ elements, so $\mathbb{P}(Y_i = Y_{i'} \mid \omega) = 2^{-m_{\mathrm{cl}}}$ for $i \neq i'$; the union bound gives $\lambda_p^{\mathrm{ex}}(1) \leq n2^{-m_{\mathrm{cl}}}$, and if the n pairs are disjoint the n coincidence events involve disjoint sets of conditionally independent labels, so that $\mathbb{P}(\mathsf{E}^c \mid \omega) = (1 - 2^{-m_{\mathrm{cl}}})^n$.

(v) Every query of $\mathcal{E}_6$ has the first-coordinate pair $(0, \mathsf{c}0) = (0, 2)$ (Definition 5.12(e)), so $n = 1$. The twelve queries have the disjoint first-coordinate pairs $\{0, 2\}$ and $\{5, \mathsf{c}5\} = \{5, 3\}$, so $\lambda_p^{\mathrm{ex}}(1) = 1 - (1 - 2^{-m_{\mathrm{cl}}})^2 = \eta_{m_{\mathrm{cl}}}$; and $\eta_{m_{\mathrm{cl}}} - 2^{-m_{\mathrm{cl}}} = 2^{-m_{\mathrm{cl}}} - 2^{-2m_{\mathrm{cl}}} \geq 0$.

(vi) The three measures are nonnegative, invariant under $\mathfrak{G}_p$, so that their one-slot marginals and those of their variations are multiples of the uniform law (Lemma 5.16), and they are measures of the field labels of $\{p\}$ only, so they do not depend on the interval, residue or inclusion indices: this is (O1). No sharp operator bound is applied to them (the exception is charged by its mass, Lemma 19.3), so (O2) holds. The law does not depend on the scale; with (i) and (ii) this gives (R1)–(R3), and (R5) at a clique prime is (iv) for $\mathcal{E}_6$, with $2m_{\mathrm{cl}} = e_p$. □

Outside its own atoms, the prefix enters the estimates of Part V only through parts (i) and (ii) of Lemma 19.2, as an unchanged tensor factor (a spectator) at other atoms.

19.3. **Price of deleting coincident labels.** Selecting a prefix atom fixes the full group $\mathbb{Z}/p^{e_p}\mathbb{Z}$ at every slot, so the price of a deletion is its mass times $p^{e_p a}$. With m_{cl} digit pairs the mass is $2^{-m_{\mathrm{cl}}}$, while the price factor grows only like $p^{2m_{\mathrm{cl}}a}$.

Lemma 19.3 (The price of the prefix)**.** *Let $a > 0$, $m_{\mathrm{cl}} \geq 1$ and $P \geq 2$. Let every prime $p \equiv 1 \pmod 4$ below P be a prefix atom $\{p\}$ with $e_p = 2m_{\mathrm{cl}}$ and reference law λ_p, and let the comparison law at p be $\nu_p = \lambda_p - \lambda_p^{\mathrm{ex}}$, where λ_p^{ex} is the exception of a family of queries with $\lambda_p^{\mathrm{ex}}(1) \leq \eta_{m_{\mathrm{cl}}}$ (Lemma 19.2(iv),(v)).*

(i) The selection factor of $\{p\}$ is $p^{e_p a} = p^{2m_{\mathrm{cl}}a}$: in every structured energy in which $\{p\}$ is selected it contributes the factor $p^{2m_{\mathrm{cl}}a}$ to the price, density factors included; an unselected prefix atom contributes nothing.

(ii) The change $\nu_p - \lambda_p = -\lambda_p^{\mathrm{ex}}$ is symmetric at ι, and (CP4)(b) holds for it with $\zeta^{\mathrm{s}} = -\lambda_p^{\mathrm{ex}}$, $\zeta^{\mathrm{a}} = 0$, $D^{\mathrm{s}} = \lambda_p^{\mathrm{ex}}/\lambda_p^{\mathrm{ex}}(1)$ and $\tilde{\pi}^{\mathrm{s}} = \lambda_p^{\mathrm{ex}}(1)$: the deletion is charged to the symmetric part only, at the price $\lambda_p^{\mathrm{ex}}(1)p^{2m_{\mathrm{cl}}a} \le \eta_{m_{\mathrm{cl}}}p^{2m_{\mathrm{cl}}a}$. Its norm envelope is λ_p^{ex}, and (CP6) holds at p with $M_{\mathrm{L}} = M_{\mathrm{R}} = \lambda_p + \lambda_p^{\mathrm{ex}}$, at the same price. If $\lambda_p^{\mathrm{ex}}(1) = 0$, omit the normalized record and take $\zeta^{\mathrm{s}} = \zeta^{\mathrm{a}} = 0$ and price zero.

(iii) The total price of the deletions is at most

$$H_{\mathrm{iid}} = \eta_{m_{\mathrm{cl}}} \sum_{\substack{p<P \\ p\equiv 1 \ (4)}} p^{2m_{\mathrm{cl}}a} \le \eta_{m_{\mathrm{cl}}} P^{1+2m_{\mathrm{cl}}a} < 2P\Big(\frac{P^{2a}}{2}\Big)^{m_{\mathrm{cl}}}.$$

(iv) For the baseline construction, $P = P_4 = 2^{512}$, $m_{\mathrm{cl}} = 8192$ and $a = \frac{5422587}{6103515625}$:

$$H_{\mathrm{iid}} < 2^{513-m_{\mathrm{cl}}+1024m_{\mathrm{cl}}a} = 2^{-1380939795479/6103515625} < 8\cdot 10^{-69}.$$

Proof. (i) Selecting $\{p\}$ fixes the residues modulo p^{e_p} of the inputs at every slot and multiplies the square step of every slot by p^{e_p}, a square since e_p is even ((8.1), Lemma 8.2). So each slot price ξ_v of Definition 8.5 contains p^{e_p}, and the main price $\prod_{v\in V}\xi_v^{a/72}$ contains $(p^{e_p})^{72\cdot a/72} = p^{e_p a}$, the selection factor of Definition 9.2; the density factors depend on the index law only. The structured energies use two properties of the prefix laws, both from Lemma 19.2(ii): a selected law at p (here λ_p^{ex} and its normalization) has a variation with uniform one-slot marginals, as Definition 8.5(b) requires; and an unselected prefix law is preserved by the square-unit dilation of the rescaling ((R3)).

(ii) If the exception has mass zero it is the zero measure and the claims follow directly. Otherwise λ_p^{ex} is nonnegative and fixed samplewise by ι, so its cut matrix at ι is nonnegative and diagonal (Lemma 5.4(i)), in particular symmetric. Its normalization $\kappa = \lambda_p^{\mathrm{ex}}/\lambda_p^{\mathrm{ex}}(1)$ is a probability measure on labellings fixed by ι, hence $\kappa = \mathrm{Diag}_\iota(m_\kappa)$ for its half marginal m_κ, with uniform one-slot marginals; so Remark 9.15(i), applied to the deleted record $\lambda_p^{\mathrm{ex}} = \lambda_p^{\mathrm{ex}}(1)\kappa$, gives (CP4)(b) with the stated data, and the price is $\tilde{\pi}^{\mathrm{s}}$ times the factor of (i). For (CP6): $\mathsf{N}_p = \lambda_p$, a prefix atom not being a first-stage atom, and ν_p is symmetric at ι, so it suffices that $M \pm \nu_p \succeq_\iota 0$ for $M = \lambda_p + \lambda_p^{\mathrm{ex}}$ (Definition 9.9); indeed $M + \nu_p = 2\lambda_p$ and $M - \nu_p = 2\lambda_p^{\mathrm{ex}}$ are nonnegative measures on labellings fixed by ι (Lemma 5.4(i)). The difference $M - \mathsf{N}_p = \lambda_p^{\mathrm{ex}}$ is a positive reserved-diagonal record with uniform one-slot marginals that does not depend on the index (Lemma 19.2(vi)); this is the matching envelope of Proposition 26.5(i) at a prefix atom.

(iii) There are fewer than P primes below P, and $p^{2m_{\mathrm{cl}}a} < P^{2m_{\mathrm{cl}}a}$ for each; and $\eta_{m_{\mathrm{cl}}} < 2^{1-m_{\mathrm{cl}}}$, with $2^{1-m_{\mathrm{cl}}}P^{1+2m_{\mathrm{cl}}a} = 2P(P^{2a}/2)^{m_{\mathrm{cl}}}$.

(iv) By (iii), $H_{\mathrm{iid}} < 2^{1-m_{\mathrm{cl}}}2^{512(1+2m_{\mathrm{cl}}a)} = 2^{513-m_{\mathrm{cl}}+1024m_{\mathrm{cl}}a}$. At $m_{\mathrm{cl}} = 8192$ and $a = \frac{5422587}{6103515625}$ the exponent is $-1380939795479/6103515625 = -226.25\ldots$. The decimal bound follows from $513 - m_{\mathrm{cl}} + 1024m_{\mathrm{cl}}a < -905/4$ and the integer inequality $10^{276} < 2^{917}$. The broader domain check is Certificate D.3(f). □

Remark 19.4 (The allowance and the number of digit pairs). The finite check behind (iv) is item (f) of Certificate D.3: the domain audit `research/advance_2026_10_24/global/analytic_domain.py` bounds the price of the deleted prefix exception together with the tail prices beyond P_4, which the upper totals $S_{\mathrm{u}}(a)$ and $A_{\mathrm{u}}(a)$ of Proposition 28.14 also contain, by less than $1.2332\cdot 10^{-39}$ at $a = 9/10000$ (prefix exponent $-80783/625$), hence at every $0 < a \le 9/10000$, both prices increasing with a; this is inside the allowance 10^{-12} that the closing adds to S, to A and to the matching exponent T (Definition 30.1). By Lemma 19.3(ii) the prefix enters only S and T, so this is conservative, and so is the use of the two-pair mass $\eta_{m_{\mathrm{cl}}}$, exact for the twelve queries, in place of the mass $2^{-m_{\mathrm{cl}}}$ of the deletion for $\mathcal{E}_6$ (which would replace 513 by 512 in the exponent). At the baseline exponent the exponent $513 - m_{\mathrm{cl}}(1 - 1024a)$ of (iv) is negative only for $m_{\mathrm{cl}} \ge 5685$; with

$m_{\mathrm{cl}} = 8192$ it is negative for every $a < 7679/8388608$, about 0.000915, an interval that contains the analytic baseline domain $a \le 9/10000$, at the cost of a larger fixed modulus Q_{cl}, which affects only constants and the starting scale (Remark 8.13). Why this layout of the prefix needs many digit pairs, and where it stops working, is the subject of Remark 35.6.

Consequences and what to read next. The slot array and the cylinder pullback (Definition 5.12, Lemma 5.13(v)); reflection positivity and diagonal measures (Definition 5.1, Lemma 5.4); square-affine invariance (Definition 5.15, Lemma 5.16); Hensel's lemma (Definition 4.5); structured energies, the square step and prices (Definition 8.5, (8.1), Lemma 8.2, Definition 9.2); the charging of deleted records and the brackets of (CP4) and (CP6) (Remark 9.15, Definition 9.9, Definition 9.14, Definition 9.17); admissibility (Definition 8.9); and Certificate D.3(f).
Hypotheses verified in this section (Table 5). At the prefix atoms: (CP1), that is, (R1)–(R3) with (R2) at every cut of $\mathfrak{F}$, by parts (i), (ii) and (vi) of Lemma 19.2; (R5) and the acceptance part of (CP4)(a), by its parts (iii)–(v); the prefix parts of (CP4)(b) and of (CP6), the deletion of the exception charged to S only, by part (ii) of Lemma 19.3, with baseline total price below $8 \cdot 10^{-69}$ by its part (iv) and Certificate D.3(f); and (CP8)(a) for the prefix laws, by part (vi) of Lemma 19.2. The prefix atoms are finitely many, so they do not affect the summability (CP8)(c). These are the rows "(CP1) prefix", "(R5), (CP4)(a)" (at the prefix) and "(CP4) prefix" of Table 5.

20. Middle primes I: blocks and spectral bounds for the twelve-label law

What this section proves and how to read it. The middle range is $p \equiv 3 \pmod 4$ in $[Q_3, P_3) = [19\,951\,586, 167 \cdot 2^{40})$. This section partitions it into prime blocks $\mathcal{B}$ with $\sum_{p\in\mathcal{B}} \log p \le 570$ (grouped into 33 consecutive intervals, called *pieces* in displays), proves degree-sum bounds converting per-prime constants into block prices (sums $\sum q_{\mathcal{B}}^{2a} b_{\mathcal{B}}^{-d/2}$; "block charge" below means such a sum), and proves spectral bounds $(-gp^{-1/2}I \preceq A_p \preceq (1 + c^+/p)I$, called *caps*) for the twelve-label graph laws. The summand of weight 1 in the five-type completion of Lemma 22.1 is called the *principal law* (*root* in this part; the word has nothing to do with square roots). Cut codes name mirror symmetries: L20 is one chamber symmetry, expanded at first use. Inputs: Part IV. Outputs: blocks and caps used to build μ, λ in Section 22. On seven intervals from 2^{30} the principal law is reweighted; Section 21 bounds it. A first reading should follow the block definition and cap statements; threshold arithmetic is reference material.

Conventions. In this section ϑ_a is the chamber cut $\vartheta_{0,a}$ of the first coordinate: the reflection $u_0 \mapsto 2a - 1 - u_0$ with plus half $H_a = \{a, a+1, a+2\}$ (indices mod 6), $a \in \mathbb{Z}/6\mathbb{Z}$. The reserved cut ι flips the layer ℓ and has plus half $V_+ = \{\ell = 0\}$. For a prime p we write $r = p^{-1/2}$ and $q = 1/p$; on a block $\mathcal{B}$ with least prime $b = b_{\mathcal{B}}$, $r = b^{-1/2}$ and $q = 1/b$, while $q_{\mathcal{B}} = \prod_{p\in\mathcal{B}} p$. The order $\succeq_c$ of cut matrices is that of Definition 5.2. Every middle block has $\log q_{\mathcal{B}} \le L = 570$; the local data of a piece $[X, Y)$ are evaluated at its start X and at the local cap $L_{\mathrm{loc}} = 585$ (Lemma 20.3).

20.1. Middle blocks with $\sum \log p \le 570$ and their degree sums.

This subsection fixes the partition of the middle primes $p \equiv 3 \pmod 4$ (called class-three primes below; $p \equiv 1 \pmod 4$ are class-one) into blocks and bounds the three sums

$$b_d(X, Y) = \sum_{\mathcal{B}\subseteq[X,Y)} q_{\mathcal{B}}^{2a}\, b_{\mathcal{B}}^{-d/2} \qquad (d = 2, 3, 4)$$

over the blocks $\mathcal{B}$ of a middle piece $[X, Y)$. Multiplied by the local coefficients evaluated at X, these sums give every middle price (Lemma 22.9). The subsection also gives the lower bounds for $\sum_{\mathcal{B}} b_{\mathcal{B}}^{-1}$ used by the mean credit (Lemma 20.6) and the caps of the selection factors of the pieces used by the restorations (Lemma 20.7).

Below the explicit Chebyshev threshold the partition consists of exact blocks built from an authenticated prime histogram; above it, of geometrically spaced cells partitioned into blocks

of fixed cardinality n_j. Every block has logarithmic length at most the budget $L = 570$, so its selection factor is at most e^{2aL}; the local laws of a piece are evaluated at the larger local cap $L_{\mathrm{loc}} = 585$. The partition depends on no parameter of the induction and is fixed before the scale N.

Throughout, $\theta(t;4,3) = \sum_{p\le t,\ p\equiv 3\ (4)} \log p$, and $\mathsf{e}_X = 2411/(5000000\log X)$ is the relative error of Theorem 4.12. We put $X_{\mathrm{Ch}} = 4\,800\,162\,890$; thus $X_{\mathrm{Ch}} - 1$ is the threshold of that theorem, and $|\theta(t;4,3) - t/2| \le \mathsf{e}_X t$ for all $t \ge X \ge X_{\mathrm{Ch}}$. The threshold has no slack: an analytic cell may not start below X_{Ch}.

Definition 20.1 (Exact log-budget blocks)**.** Let $L = 570$ and let

$$\mathfrak{u}_* = 2^{40}L = 626\,721\,627\,832\,320$$

be the integer budget.

(a) *Pieces.* The middle range $[Q_3, P_3) = [19\,951\,586,\ 167\cdot 2^{40})$ is the union of the 33 pieces $[X_i, Y_i) = [X_i, X_{i+1})$, $1 \le i \le 33$, whose endpoints $X_1 < X_2 < \cdots < X_{34}$ are

$$\begin{gathered} 19951586,\ 21757358,\ 23726568,\ 25429506,\ 25874006,\ 28215802,\\ 2^{25},\ 39903170,\ 47453134,\ 56431604,\ 2^{26},\ 79806340,\ 94906266,\\ 112863208,\ 2^{27},\ 159612678,\ 189812532,\ 225726414,\ 2^{28},\\ 319225356,\ 379625064,\ 451452826,\ 2^{29},\ 638450710,\ 759250126,\\ 902905652,\ 2^{30},\ 2^{31},\ 2^{32},\ 2^{34},\ 2^{36},\ 2^{40},\ 2^{45},\ 167\cdot 2^{40}; \end{gathered}$$

thus $X_1 = Q_3$, $X_{33} = 2^{45}$ and $Y_{33} = X_{34} = P_3$, the numbering of Table 24. A block of a piece contains only primes $p \equiv 3 \pmod 4$ of that piece.

(b) *Mandatory threshold points.* For $97 \le j \le 128$ let $c_j = 2\lceil 2^{j/4}/2\rceil$ be the least even integer $\ge 2^{j/4}$; let $c'_1 = 21\,757\,358$, $c'_2 = 25\,429\,506$ and $c'_3 = 25\,874\,006$, the least even integers $\ge 2^{195/8}$, $2^{123/5}$ and $2^{197/8}$; and put $c_{129} = X_{\mathrm{Ch}}$. These 36 numbers are the *threshold points* (called cuts in the data; they are not mirror cuts), and the 35 intervals between consecutive threshold points are the *parts*. Every endpoint of (a) below 2^{32} is a threshold point ($Q_3 = c_{97}$), so each piece below 2^{32} is a union of parts, and the piece $[2^{32}, 2^{34})$ contains the part $[c_{128}, c_{129})$.

(c) *Exact blocks below X_{Ch}.* Every prime $p \in [2^{24}, X_{\mathrm{Ch}})$ carries an integer upper logarithm $\mathfrak{u}(p)$: the interval $[2^{24}, X_{\mathrm{Ch}})$ is covered by 5668 fixed consecutive count bins $[\mathrm{lo}, \mathrm{hi})$, and $\mathfrak{u}(p)$ is the least integer $\ge 2^{40}\log \mathrm{hi}$ for the bin containing p, so that $\log p < \mathfrak{u}(p)/2^{40}$ (the bins and the authenticated histogram are in Section E.2). In each part list its primes $p \equiv 3 \pmod 4$ in increasing order. A prime joins the current block if the sum of $\mathfrak{u}$ over the block stays at most $\mathfrak{u}_*$; otherwise it starts a new block. The last block of a part is closed at the end of the part.

(d) *Fixed-cardinality cells above X_{Ch}.* In a piece $[X, Y)$ with $Y > X_{\mathrm{Ch}}$ put $x = \max(X, X_{\mathrm{Ch}})$, $\mathsf{n} = \lceil 128\log_2(Y/x)\rceil$ and

$$x_j = x^{1-j/\mathsf{n}}Y^{j/\mathsf{n}}, \qquad n_j = \Big\lfloor \frac{L}{\log x_{j+1}} \Big\rfloor \qquad (0 \le j < \mathsf{n}).$$

In the cell $[x_j, x_{j+1})$ group its primes $p \equiv 3 \pmod 4$, in increasing order, into consecutive blocks of n_j primes; the last block of the cell may have fewer. The five pieces above X_{Ch} have $\mathsf{n} = 236, 256, 512, 640, 306$; their integers n_j are recorded in Certificate D.4.

The *middle blocks* are the blocks of (c) and (d). Each lies in one part or one cell, and therefore in one piece.

No endpoint used in (c) or (d) is a prime: X_{Ch}, the threshold points and every Y are even, and an interior x_j cannot be an odd prime p, because $p^{\mathsf{n}} = x^{\mathsf{n}-j}Y^j$ would be even. The cells and the cardinalities are deterministic functions of L and of the endpoints; the bin integers are fixed rational upper logarithms. No numerical choice depends on a or on N.

Remark 20.2 (Restarts and fixed cardinality). A block never straddles a part or a cell: the partition restarts at every cut. This costs one restart term per cell in Lemma 20.4, but it removes every analysis of blocks that cross a boundary between pieces with different local coefficients, and it lets each piece carry its own local law. Inside a cell all blocks except the last have exactly n_j primes, so the start of every later block is controlled by the full block that precedes it; this turns the sum over block starts into a sum over primes, to which the explicit Chebyshev bound is applied once, on the whole interval. The pieces need not be equally spaced: the threshold points c_1', c_2', c_3', together with c_{98}, are endpoints X_i, so $[c_{97}, c_{99})$ is the union of five consecutive pieces, each a union of parts. Each piece carries its own prime counts, product budget and restart, endpoint and jump terms; the pieces are disjoint and cover the range, so adding their charges drops no boundary term.

Lemma 20.3 (Domain of the local laws)**.** *Let $\mathcal{B}$ be a middle block in the piece $[X, Y)$. Then:*

- *(i) every prime of $\mathcal{B}$ satisfies $p \equiv 3 \pmod 4$ and $X \le b_{\mathcal{B}} \le p < Y$;*
- *(ii) $\log q_{\mathcal{B}} \le L$; hence $|\mathcal{B}| \le \lfloor L/\log X \rfloor \le n_X := \lfloor L_{\mathrm{loc}}/\log X \rfloor$ with $L_{\mathrm{loc}} = 585$, $\sum_{p\in\mathcal{B}} 1/p \le L/(X \log X)$ and $q_{\mathcal{B}}^{2a} \le e^{2aL}$;*
- *(iii) each bound listed below, computed with threshold $Q = X$ and logarithmic block length L_{loc}, holds for every prime and block as stated in its cited result, hence for $\mathcal{B}$; these are the caps of the cuts of the root and of the twelve-label graphs with their block inflation (Lemma 20.12; Proposition 21.11 on the pieces below 2^{30}; Certificate 21.25 on the pieces from 2^{30}), the cap k_{D} of the reference fold record (Lemma 23.2), the row constants f_j, h_j, the completion weights, the one-stage coefficients k_{new} and s_{fold} and the coefficients of the middle prices (Lemma 22.1, Lemma 22.6, Lemma 22.9 and Lemma 23.24).*

Proof. (i) holds by construction. For (ii), a block of Definition 20.1(c) satisfies

$$\log q_{\mathcal{B}} < \sum_{p\in\mathcal{B}} \mathfrak{u}(p)/2^{40} \le \mathfrak{u}_*/2^{40} = L.$$

A block of (d) in the cell $[x_j, x_{j+1})$ has at most n_j primes, all below x_{j+1}, so $\log q_{\mathcal{B}} < n_j \log x_{j+1} \le L$ by the definition of n_j. Every prime of $\mathcal{B}$ is at least X, so $|\mathcal{B}| \log X \le \log q_{\mathcal{B}} \le L \le L_{\mathrm{loc}}$, $\sum_{p\in\mathcal{B}} 1/p \le |\mathcal{B}|/X$, and $q_{\mathcal{B}}^{2a} = e^{2a \log q_{\mathcal{B}}} \le e^{2aL}$.

(iii) The row constants, k_{new}, s_{fold} and the coefficients of Lemma 22.9 are admissible functions in the sense of Lemma 4.18, of $r = p^{-1/2}$ and of the block count, with their powers of r cancelled (for the row constants see the proof of Lemma 22.1(i), for k_{new} and s_{fold} their formulas). By Lemma 4.18(b) the value of such a coefficient at X, with the count $n_X \ge |\mathcal{B}|$ of (ii), bounds its value at $b_{\mathcal{B}} \ge X$. The coefficients of Lemma 23.24 combine the weights and f_1, h_1 with the constants s_F, a_F and $\ell_{\mathfrak{t}}$ of the signed fold records and the coefficients s_E, a_E and $g_\Sigma(X)$, which are valid at every block of the piece by (23.5) with Proposition 23.13, by Lemma 23.17(ii) and by Corollary 23.21. The caps are constants valid for every prime at least the threshold at which they are computed, and this threshold is the piece start X itself: every piece start is a threshold of Certificate 20.13 and an endpoint of Certificate 20.22, and the caps of the positive roots are evaluated at X (Certificate 21.25). The block caps hold for every block of the piece by (i) and (ii), since $L \le L_{\mathrm{loc}}$; the grid domain $Q \ge 2^{24}$, length at most 10000, contains (X, L_{loc}). The cap k_{D} holds at every block of the piece by Lemma 23.2. The completion weights are fixed numbers whose inequalities involve only these inputs. That the local data of each of the 33 pieces were computed at (X, L_{loc}) is part of Certificate D.4. □

The selection exponent $2aL$ exceeds one: it is $1236349836/1220703125 < 1.013$ at the baseline exponent a, and $253204431/250000000$ at the reference exponent $a_0 = \frac{4442183}{5000000000}$ of the tabulated middle prices. No statement of the paper requires $2aL \le 1$: the restorations and variance credits take an arbitrary upper bound for the block selection factor, and every block is priced at its own $q_{\mathcal B}^{2a}$. A middle price at $a \ge a_0$ is at most $t = e^{2(a-a_0)L}$ times its value at a_0 (Lemma 9.3), and the selection factor of one block is at most $C = e^{2aL}$, which is below 2.753348440327 at the baseline exponent (Definition 30.1, Table 12). The restorations use the sharper caps of the individual pieces (Lemma 20.7).

Lemma 20.4 (Block charges)**.** *Let $a > 0$ and $s > 0$.*

(i) Cells. *Let $[x, Y)$ be an interval with $x \ge X_{\mathrm{Ch}}$ and x, Y even, $x_j = x^{1-j/\mathsf n} Y^{j/\mathsf n}$ for $0 \le j \le \mathsf n$, and let $n_0, \dots, n_{\mathsf n-1} \ge 1$ be integers. Partition the primes $p \equiv 3 \pmod 4$ of each cell $[x_j, x_{j+1})$ into consecutive blocks of n_j primes, the last possibly shorter. Put*

$$\Pi_j = \exp(2an_j \log x_{j+1}), \qquad g_j = \Pi_j / n_j, \qquad I_j(s) = \int_{x_j}^{x_{j+1}} \frac{t^{-s}}{\log t}\, dt$$

(numbers attached to the cells). Then, with $\mathsf e = \mathsf e_x := 2411/(5000000 \log x)$,

$$\begin{aligned} \sum_{\mathcal B} q_{\mathcal B}^{2a} b_{\mathcal B}^{-s} \le{}& \sum_{j<\mathsf n} \Pi_j x_j^{-s} + \Big(\frac12 + \mathsf e\Big) \sum_{j<\mathsf n} g_j I_j(s) \\ &+ 2\mathsf e\, \frac{g_0 x^{1-s}}{\log x} + 2\mathsf e \sum_{0<j<\mathsf n} (g_j - g_{j-1})_+ \frac{x_j^{1-s}}{\log x_j}. \end{aligned} \tag{20.1}$$

(ii) Exact blocks. *For a block $\mathcal B$ of Definition 20.1(c) put $\mathfrak u(\mathcal B) = \sum_{p\in\mathcal B} \mathfrak u(p)$, $k_{\mathcal B} = \lceil 2^{16}\mathfrak u(\mathcal B)/\mathfrak u_*\rceil$ and*

$$w_2(\mathcal B) = \Big\lceil \frac{2^{95}}{b_{\mathcal B}} \Big\rceil, \quad w_3(\mathcal B) = \Big\lceil \frac{2^{127}}{b_{\mathcal B} \lfloor (2^{64} b_{\mathcal B})^{1/2} \rfloor} \Big\rceil, \quad w_4(\mathcal B) = \Big\lceil \frac{2^{95}}{b_{\mathcal B}^2} \Big\rceil.$$

Then $1 \le k_{\mathcal B} \le 2^{16}$, $q_{\mathcal B}^{2a} \le \exp(2a\mathfrak u_* k_{\mathcal B}/2^{56}) \le e^{2aL}$ and $b_{\mathcal B}^{-d/2} \le w_d(\mathcal B)/2^{95}$. Consequently, for every part $\mathfrak p$,

$$\sum_{\mathcal B\subseteq\mathfrak p} q_{\mathcal B}^{2a} b_{\mathcal B}^{-d/2} \le \sum_{k=1}^{2^{16}} \exp\Big(\frac{2a\mathfrak u_* k}{2^{56}}\Big) \frac{w_d(\mathfrak p, k)}{2^{95}}, \qquad w_d(\mathfrak p, k) = \sum_{\mathcal B\subseteq\mathfrak p,\ k_{\mathcal B}=k} w_d(\mathcal B).$$

In particular, for every middle piece $[X, Y)$ and $d \in \{2, 3, 4\}$, $b_d(X, Y)$ is at most the sum of (ii) over the parts of the piece below X_{Ch} and of (20.1) with $s = d/2$ over its cells above X_{Ch}.

Proof. (i) *Block starts.* Fix a cell j. Its first block starts at a prime $b \ge x_j$, so it contributes at most $\Pi_j x_j^{-s}$: its primes are below x_{j+1}, and it has at most n_j of them, so $q_{\mathcal B} < x_{j+1}^{n_j}$. Every later block $\mathcal B'$ of the cell is preceded by a block $\mathcal B$ of exactly n_j primes, all smaller than $b_{\mathcal B'}$. Since t^{-s} decreases, $b_{\mathcal B'}^{-s} \le n_j^{-1} \sum_{p\in\mathcal B} p^{-s}$. Distinct blocks $\mathcal B'$ give distinct blocks $\mathcal B$, so

$$\sum_{\mathcal B\subseteq \text{cell } j} q_{\mathcal B}^{2a} b_{\mathcal B}^{-s} \le \Pi_j x_j^{-s} + g_j \sum_{\substack{x_j \le p < x_{j+1} \\ p\equiv 3\ (4)}} p^{-s}.$$

Empty cells and short last blocks are covered, and every cell keeps its own restart term $\Pi_j x_j^{-s}$.

Partial summation. Put $f_s(t) = t^{-s}/\log t$ and let $\phi(t) = g_j f_s(t)$ for $x_j \le t < x_{j+1}$. Then ϕ is right-continuous and positive, it decreases on each open cell (as $s > 0$ and $t > 1$), and its jump at

an interior endpoint is $\Delta_j = (g_j - g_{j-1}) f_s(x_j)$. With $\eta(t) = \theta(t;4,3) - t/2$,

$$\sum_j g_j \sum_{\substack{x_j \le p < x_{j+1} \\ p \equiv 3 \ (4)}} p^{-s} = \int_{[x,Y)} \phi \, d\theta(\cdot;4,3) = \frac{1}{2}\int_x^Y \phi(t)\,dt + \int_{[x,Y)} \phi \, d\eta.$$

No endpoint is a prime, so η has no atom at any x_j or at Y, and Stieltjes integration by parts with the jumps of ϕ gives

$$\int_{[x,Y)} \phi \, d\eta = \eta(Y)\phi(Y^-) - \eta(x)\phi(x) - \int_x^Y \eta(t)\phi'(t)\,dt - \sum_{0<j<\mathsf{n}} \eta(x_j)\Delta_j,$$

the integral being over the open cells. By Theorem 4.12, $|\eta(t)| \le \mathsf{e}t$ on $[x,Y]$. Since $\phi \ge 0$ and $\phi' < 0$ on the open cells, the modulus of the right side is at most

$$\mathsf{e}\Big[Y\phi(Y^-) + x\phi(x) + \int_x^Y t(-\phi'(t))\,dt + \sum_j x_j|\Delta_j|\Big].$$

Integrating by parts on each open cell and summing, $\int_x^Y t(-\phi')\,dt = x\phi(x) - Y\phi(Y^-) + \int_x^Y \phi + \sum_j x_j\Delta_j$. Since $\Delta_j + |\Delta_j| = 2(\Delta_j)_+$, the error is at most $\mathsf{e}[2x\phi(x) + \int_x^Y \phi + 2\sum_j x_j(\Delta_j)_+]$. Adding the main term $\frac{1}{2}\int\phi = \frac{1}{2}\sum_j g_j I_j(s)$ and the restart terms gives (20.1). The argument does not use monotonicity of g_j, and it applies a single Chebyshev error to the whole interval; no additional error term arises at interior cell endpoints beyond the jump terms displayed in (20.1).

(ii) The bin integers satisfy $\log p < \mathfrak{u}(p)/2^{40}$, so $\log q_{\mathcal{B}} < \mathfrak{u}(\mathcal{B})/2^{40}$. A single prime has $\mathfrak{u}(p) \le \lceil 2^{40}\log X_{\mathrm{Ch}}\rceil < 2.46\cdot 10^{13} < \mathfrak{u}_*$, so a block started by one prime is within the budget, and the greedy rule gives $1 \le \mathfrak{u}(\mathcal{B}) \le \mathfrak{u}_*$, hence $1 \le k_{\mathcal{B}} \le 2^{16}$ and $\mathfrak{u}(\mathcal{B}) \le \mathfrak{u}_* k_{\mathcal{B}}/2^{16}$, which gives the price bound and $\exp(2a\mathfrak{u}_* k_{\mathcal{B}}/2^{56}) \le \exp(2a\mathfrak{u}_*/2^{40}) \le e^{2aL}$. For $d = 2, 4$ the weight bound is immediate. For $d = 3$ put $s_0 = \lfloor (2^{64}b)^{1/2}\rfloor \le 2^{32}b^{1/2}$; then $2^{127}/(bs_0) \ge 2^{95}b^{-3/2}$. Group the blocks of the part by $k_{\mathcal{B}}$. Every block, including the first block after each cut, is an actual block of the partition, so no restart term is needed. □

Corollary 20.5 (One positive functional for all charge degrees)**.** *For fixed cells and fixed a, the right side of* (20.1) *is $\Phi(t^{-s})$, where*

$$\Phi(f) = \sum_j \Pi_j f(x_j) + (\tfrac{1}{2} + \mathsf{e})\sum_j g_j \int_{x_j}^{x_{j+1}} \frac{f(t)}{\log t}\,dt + 2\mathsf{e}\frac{g_0 x}{\log x} f(x) + 2\mathsf{e}\sum_{j>0}(g_j - g_{j-1})_+ \frac{x_j}{\log x_j} f(x_j).$$

This functional is positive and independent of the degree s. Hence any pointwise polynomial inequality in $r = t^{-1/2}$ on $[x,Y]$ may be integrated against Φ, including an inequality with signed coefficients. A common positive repricing factor preserves this property.

Proof. Every displayed weight is nonnegative, and substitution of $f(t) = t^{-s}$ gives (20.1). Apply positivity to the difference of the two sides of the pointwise inequality. □

The inequality is applied before rounding. Independently rounded upper bounds for different monomials do not define the same positive functional and cannot be substituted into its negative coefficients. For the exact blocks the same principle applies to the actual common block sum; their independently rounded degree histograms supply upper bounds, not signed functional identities.

The same partial summation, with the error subtracted, gives lower bounds for the unpriced start sums.

Lemma 20.6 (Lower start sums)**.**

(i) In the setting of Lemma 20.4(i),

$$\sum_{\mathcal{B}} b_{\mathcal{B}}^{-1} \;\ge\; \Big(\frac{1}{2}-\mathsf{e}\Big)\sum_{j<\mathsf{n}} \frac{1}{n_j}\log\frac{\log x_{j+1}}{\log x_j} - 2\mathsf{e}\Big(\frac{1}{n_0\log x} + \sum_{0<j<\mathsf{n}} \frac{(1/n_j - 1/n_{j-1})_+}{\log x_j}\Big).$$

(ii) For every part $\mathfrak{p}$ of Definition 20.1(c),

$$\sum_{\mathcal{B}\subseteq\mathfrak{p}} b_{\mathcal{B}}^{-1} \;\ge\; \frac{\sum_{\mathcal{B}\subseteq\mathfrak{p}} w_2(\mathcal{B}) - \#\{\mathcal{B}\subseteq\mathfrak{p}\}}{2^{95}}.$$

Proof. (i) A block of cell j has at most n_j primes, all at least its first prime b, so $\sum_{p\in\mathcal{B}} p^{-1} \le n_j b^{-1}$. Summing over the blocks, $\sum_{\mathcal{B}} b_{\mathcal{B}}^{-1} \ge \sum_j n_j^{-1} \sum_{p\in\text{cell } j} p^{-1}$. Apply the partial summation of the proof of Lemma 20.4(i) with $s=1$ and g_j replaced by $1/n_j$: the main term is $\frac{1}{2}\sum_j n_j^{-1} I_j(1)$ with $I_j(1) = \log(\log x_{j+1}/\log x_j)$, and the same error bound, now subtracted, gives (i). No first-block term is subtracted, and the bound holds for arbitrary n_j. (ii) $w_2(\mathcal{B}) < 2^{95}/b_{\mathcal{B}} + 1$. □

The lower sum of (i) is a bound for a whole interval: it does not assume that a single fine cell has relative prime density close to one half.

One budget L bounds every block, but the largest block product of a piece may be smaller; the restorations of Section 24 use the following caps, one for each piece.

Lemma 20.7 (Piece caps)**.** *For a middle piece $[X_i, Y_i)$ let L_i be the largest of the numbers $k_{\mathcal{B}} L/2^{16}$ over its blocks $\mathcal{B}$ of Definition 20.1(c) and of the recorded upper bounds for $n_j \log x_{j+1}$ over its cells of Definition 20.1(d). Then every block of the piece has $\log q_{\mathcal{B}} \le L_i \le L$. Consequently, for $0 < a \le a_*$ and $C_i = e^{2a_* L_i}$, every block of piece i has $q_{\mathcal{B}}^{2a} \le C_i$, and distinct blocks $\mathcal{B}_1, \dots, \mathcal{B}_k$ lying in pieces $i_1, \dots, i_k$ have $\prod_l q_{\mathcal{B}_l}^{2a} \le C_m := \prod_l C_{i_l}$ (m the multiset of the pieces $i_1, \dots, i_k$).*

Proof. For a block of (c), $\log q_{\mathcal{B}} < \mathfrak{u}(\mathcal{B})/2^{40} \le k_{\mathcal{B}}\mathfrak{u}_*/2^{56} = k_{\mathcal{B}} L/2^{16}$ by Lemma 20.4(ii), as $\mathfrak{u}_* = 2^{40}L$; and $k_{\mathcal{B}} \le 2^{16}$. A block of (d) in the cell $[x_j, x_{j+1})$ has at most n_j primes, all below x_{j+1}, so $\log q_{\mathcal{B}} < n_j \log x_{j+1}$, and $n_j \log x_{j+1} \le L$ by the definition of n_j. The authenticated histogram stores $k_{\mathcal{B}}$ for every exact block, so the maximum over the piece bounds every block of it; it is not an average. The products follow from $q_{\mathcal{B}}^{2a} \le e^{2aL_i} \le e^{2a_* L_i}$. □

The numbers L_i lie between 556.79 and 570 (Table 27). They are recorded in `research/push_2026_10_25/artifacts/author.json` (key `restoration_product_caps`; producer `research/advance_2026_10_24/envelopes/piece_product_caps.py`, which authenticates the histogram and the partition run). The restorations (Section 24.8) use them at $a_* = a$.

Certificate D.4 (Appendix D) records the finite facts about this partition (the tiling, the bins and the histogram of the exact blocks, the cells and their integers n_j, the piece log caps L_i and the lower start sums of Lemma 20.6) and, at the reference exponent a_0, the four totals of Section 22.4. The degree charges of each piece are listed in Table 24, and its four sums in Table 25.

Remark 20.8 (What rests on the explicit Chebyshev bound). In the middle range Theorem 4.12 enters only through (20.1) and Lemma 20.6(i): it prices the five analytic intervals above X_{Ch}, which carry about two thirds of the total degree-two charge at the reference exponent a_0 (pieces 29–33 of Table 24), and it gives the analytic lower start sums of the mean credit. Below X_{Ch} the blocks are built from exact prime counts, and no Chebyshev estimate is used.

20.2. **Ordered pairs at $p \equiv 3 \pmod 4$.** Let $p \equiv 3 \pmod 4$ be prime and χ its quadratic character, and write $x \to y$ if $\chi(y-x) = 1$. Since $\chi(-1) = -1$ this is a tournament on $\mathbb{F}_p$ (the Paley tournament).

Lemma 20.9 (Class-three middle pairs)**.** *Let $p \equiv 3 \pmod 4$ be prime.*

(i) *For distinct $x, y \in \mathbb{F}_p$ there are exactly $(p-3)/4$ labels z with $x \to z$ and $y \to z$, and exactly $(p-3)/4$ with $z \to x$ and $z \to y$.*

(ii) *Call $(x_0, x_0', x_2, x_2', u, u')$, six distinct labels, an F-*configuration *if $y \to z$ for all $y \in \{x_0, x_0'\}$ and $z \in \{x_2, x_2'\}$. For $p \geq 11$ every ordered pair (u, u') of distinct labels is the middle pair of exactly $(p-3)(p-7)(p-4)(p-5)/16$ F-configurations.*

(iii) *Let $p > 15$. Fix distinct u, u' and an ordered pair (x_0, x_0') of distinct labels outside $\{u, u'\}$. The number of ordered pairs (x_2, x_2') completing an F-configuration lies between $(p-11)(p-15)/16$ and $(p-3)(p-7)/16$. Consequently the conditional laws of (x_0, x_0') and of (x_2, x_2') given (u, u') under $\lambda_p(G_{\mathrm{F},2})$ (Section 20.3), relative to the uniform distinct-pair law avoiding the middle pair, have densities in $[R_2^{-1}, R_2]$ with $R_2 = (p-3)(p-7)/((p-11)(p-15))$. For one layer the same holds with $R_1 = (p-1)/(p-3)$.*

Proof. (i) is part of Lemma 12.10.

(ii) By (i) there are $p(p-1)\lambda_0(\lambda_0-1)$ ordered outer $K_{2,2}$ embeddings (x_0, x_0', x_2, x_2'), with $\lambda_0 = (p-3)/4$; each allows $(p-4)(p-5)$ ordered middle pairs. The maps $z \mapsto \alpha z + \beta$ with $\chi(\alpha) = 1$ preserve F-configurations and act transitively on the ordered distinct pairs (u, u') with fixed $\chi(u'-u)$. The middle labels are unconstrained, so (u, u') and (u', u) have the same number of completions; since $\chi(-1) = -1$, this exchange interchanges the two orbits. Hence the count is constant, equal to $\lambda_0(\lambda_0-1)(p-4)(p-5) = (p-3)(p-7)(p-4)(p-5)/16$.

(iii) By (i), the common successors of x_0, x_0' number λ_0; removing those among u, u' leaves $\lambda_0 - h$ with $0 \leq h \leq 2$. Ordered pairs of distinct ones number $(\lambda_0-h)(\lambda_0-h-1)$, which lies between $(\lambda_0-2)(\lambda_0-3)$ and $\lambda_0(\lambda_0-1)$. The conditional density of (x_0, x_0') is its count divided by the mean count, so it lies in $[R_2^{-1}, R_2]$. The same argument with common predecessors gives the opposite outer marginal. For one layer the successors of x_0 other than the middle label number $(p-1)/2$ or $(p-3)/2$, whence R_1. □

Remark 20.10 (Why the middle range is class three). Part (ii) fails for $p \equiv 1 \pmod 4$: at $p = 13$ the two orbits of ordered middle pairs have 312 and 264 completions. Part (ii) gives the F and E laws the same middle law, so that in the completion and in the first stage the F and E kernels are densities against the one conditional reference of part (iii) (the uniform distinct-pair law given the middle pair) and the fold kernel has middle-pair density identically 1 (Remark 17.12); this is why the middle range consists of class-three primes, every class-one prime below P_4 being a prefix prime (Lemma 19.2).

20.3. **The twelve-label graphs and bounds $-gp^{-1/2}I \preceq A_p \preceq (1 + c^+/p)I$.** On a block $\mathcal{B}$ of class-three primes we write $x \to y$ if $\chi_p(y-x) = 1$ at every prime p of $\mathcal{B}$. The *twelve-label root* $G_{\mathrm{L},2}$, the *F graph* $G_{\mathrm{F},2}$ and the *E graph* $G_{\mathrm{E},2}$ are the graphs G_{L}, G_{F}, G_{E} of Lemma 17.2 with the layer set $\{0,1\}$, and $G_{\mathrm{L},1}$, $G_{\mathrm{F},1}$, $G_{\mathrm{E},1}$ are their one-layer versions (layer set $\{0\}$). Thus $G_{\mathrm{L},2}$ has the vertices $(u, \ell) \in \mathbb{Z}/6\mathbb{Z} \times \{0,1\}$ and the edges $(u, \ell) \to (\mathsf{c}u, \ell')$ for all u, ℓ, ℓ', the four layer lifts of each of the six rotation edges $0 \to 2 \to 4 \to 0$ and $1 \to 5 \to 3 \to 1$: 12 vertices, 24 edges, every degree 4. The F and E graphs have three *objects* $0, 1, 2$, each carrying one label per layer; the F graphs have all the edges from the labels of object 0 to those of object 2 (four, respectively one), the E graphs no edges. For an oriented graph G, $\lambda_{\mathcal{B}}(G) = \bigotimes_{p \in \mathcal{B}} \lambda_p(G)$ is the product of the uniform laws on the injective strict-square labellings (Definition 12.1).

Lemma 20.11 (Twelve-label geometry)**.**

(a) *The chamber reflections $u \mapsto k - u$ ($k \in \{1,3,5\}$), the translation $u \mapsto u + 2$ and the layer swap $\ell \mapsto 1 - \ell$ map the edge set of $G_{\mathrm{L},2}$ onto itself; the first two do the same for $G_{\mathrm{L},1}$; the layer swap is an automorphism of $G_{\mathrm{F},2}$ and $G_{\mathrm{E},2}$.*

(b) *The subgraph of $G_{\mathrm{L},j}$ induced on the objects $0, 1, 2$ of the plus half H_0 is $G_{\mathrm{F},j}$, $j = 1, 2$.*

(c) *The subgraph of $G_{\mathfrak{f},2}$ induced on the layer $\ell = 0$ is $G_{\mathfrak{f},1}$, $\mathfrak{f} \in \{\mathrm{L}, \mathrm{F}, \mathrm{E}\}$.*

(d) $G_{\mathrm{F},j}$ and $G_{\mathrm{E},j}$ induce the same graph, without edges, on the object pairs $\{0,1\}$ and $\{1,2\}$.

Proof. The chamber maps of (a) and part (b) are Lemma 17.2(a),(c); the layer swap permutes the four layer lifts of every edge. (c) and (d) are read off from the definitions. The script `research/push_2026_10_17/sparse/layer12.py` reconstructs these facts. □

The spectral caps concern five cuts: the first-coordinate cut ϑ_0 of $G_{\mathrm{L},2}$ and of $G_{\mathrm{L},1}$ (named L20 and L10), and the layer cut ι of $G_{\mathrm{L},2}$, $G_{\mathrm{F},2}$ and $G_{\mathrm{E},2}$ (named L21, F21 and E21). Each is spectral data in the sense of Definition 13.1, with the deletion order $\mathfrak{o}$ of that definition fixed as in the data of Certificate 20.13; by Lemma 20.11(b),(c) its half graph G^+ is $G_{\mathrm{F},j}$ for L20 and L10 and the one-layer graph for the three layer cuts. For a prime p let A_p be the normalized cut operator of Lemma 13.3 on $L^2(\lambda_p(G^+))$, so that $\mathbb{E}_{\lambda_p(G)}[\Phi(x)\Phi(x')] = \langle \Phi, \mathsf{A}_p\Phi\rangle$ for every real function Φ on the half, x and x' being the half and the reflected half of a labelling; on a block, $\mathsf{A}_{\mathcal{B}} = \bigotimes_{p\in\mathcal{B}} \mathsf{A}_p$, and the same identity holds for $\lambda_{\mathcal{B}}(G)$ (Lemma 13.3(e)).

Route A (Theorem 14.21) gives the caps of L21, E21, L10 and F21 (Lemma 20.12). The cap of L20 of the uniform root is proved directly from complete actual stars (Section 20.4); on the seven pieces from 2^{30} the caps of L20 and L21 of the positive root are those of Section 21. For the folds the reference uses the cap k_{D} of Lemma 23.2, not the F21 cap.

Lemma 20.12 (Twelve-label caps)**.** *Let X be the start of a middle piece, and for one of the cuts L21, E21, L10 and F21 let g_X be the class-three negative constant and c_X^+ the positive constant recorded at the threshold X in Certificate 20.13.*

(a) For every prime $p \ge X$ with $p \equiv 3 \pmod 4$, $-g_X p^{-1/2} I \preceq \mathsf{A}_p \preceq (1 + c_X^+/p) I$.

(b) Let $\mathcal{B}$ be a middle block, with least prime b, of the piece starting at X. Then $\mathsf{A}_{\mathcal{B}} \succeq -g_X \exp\bigl(c_X^+ L_{\mathrm{loc}}/(X \log X)\bigr)\, b^{-1/2}$. The coefficient $g_X \exp(c_X^+ L_{\mathrm{loc}}/(X\log X))$ is the block cap *of the cut for the piece.*

The reference uses the block caps of L21 on the pieces below 2^{30} and those of E21 and L10 on every piece; the cap of F21 is recorded but not used.

Proof. (a) This is route A, Theorem 14.21(a),(d) for the class $\mathfrak{c} = 3$, with $p_0 = X$ and c_X^+ the constant B^+ of (d). The spectral data satisfy (S1) at X: the graphs have at most twelve vertices (half-graphs on $n \le 6$ vertices, in the notation of Theorem 14.21) and maximum degree at most four, and $X \ge Q_3 > 2^{24}$, so $e_\Delta(X, 2n) \le e_4(2^{24}, 12) < 5\cdot 10^{-3}$ and $X > 4n^2$. The inputs (R1)–(R3) are the standard ones of Remark 14.26(i): the nonstar bound of the compressed matching partition for these spectral data (Corollary 16.14), the natural coincidence bound (16.8) of Lemma 16.15, and the standard star and coupling inputs. The evaluator verifies the test (14.21) in exact rational arithmetic at X for both class-three signs, with one common value of $\tau^{\mathrm{sp}}_{3,\pm}$, and records g_X, rounded upward, as the class-three constant g_3 of the theorem; it computes c_X^+ with the largest g of all four tests, which satisfies $gX^{-1/2} < 1$ and is at least g_3, as (d) allows (Remark 14.30). (b) By Lemma 20.3(i),(ii), the primes of $\mathcal{B}$ satisfy $p \equiv 3 \pmod 4$ and $p \ge X$, and $\sum_{p\in\mathcal{B}} \log p \le L \le L_{\mathrm{loc}}$. Apply Corollary 14.22 with its P equal to X and its length L equal to L_{loc}. □

Certificate 20.13 (The twelve-label grids)**.** For each of the five cuts of the twelve-label law and each of the 35 thresholds listed below, the constants of the Route A criterion (Theorem 14.21) have been evaluated and its four exact positive-semidefiniteness tests (14.21) pass; every middle piece start is one of the thresholds, so these give the caps at every piece. The values are stored in two data files, built with the spectral evaluator `research/push_2026_10_17/sparse/layer72_spectral.py` applied to the twelve-label graphs (`research/push_2026_10_17/sparse/layer12_spectral.py`). The record `research/frontier_2026_10_17/root/middle12_grid.json`, built by `research/frontier_2026_10_17/root/build_middle12_grid.py`, contains 26 thresholds: 2^{24}, 2^{25}, 2^{26}, the sixteen piece starts from 79 806 340 to 2^{30}, and 2^{31}, 2^{32}, 2^{34}, 2^{36}, 2^{40}, 2^{45}, 2^{50}; the replay record

`research/frontier_2026_10_17/root/middle12_grid_replay.json` repeats its 520 tests with fresh partitions. The record `research/continuation_2026_10_24/global/middle_grid_new.json`, built by `research/continuation_2026_10_24/global/middle_grid.py`, contains the nine piece starts 19 951 586, 21 757 358, 23 726 568, 25 429 506, 25 874 006, 28 215 802, 39 903 170, 47 453 134, 56 431 604; the review record `research/continuation_2026_10_24/global/middle_grid_review.json` repeats its 180 tests with fresh partitions. The two records contain the same compressed matching partition of each cut, which each review rebuilds, and, for each threshold and cut, the constants of Theorem 14.21 and four exact rational positive semidefiniteness tests of (14.21) (both real class-one signs and both Hermitian class-three signs); in every record the two class-three tests use the same τ^{sp}. Every piece start is a threshold of one of the records: pieces 1–6 and 8–10 read the second, pieces 7 and 11–33 the first. The final source stores for each piece and cut the threshold X, the length 585, the class-three g, c^+ and the block cap; the native check (`research/advance_2026_10_24/global/check_low_reference_native.py`) binds every stored route-A spectrum to the record entry at X, and the L20 spectrum to the endpoint X of Certificate 20.22, and recomputes the block caps; the native check of the final source verifies that its spectra are those of the source so checked, with L20 replaced by the protected collision cap at every start and with both L20 and L21 replaced on the seven middle pieces from 2^{30} by Certificate 21.25. At Q_3 the class-three values of g are 2.57407111 (L21), 0.00454092 (E21) and 1.75129044 (L10), with block caps below 2.574071293695, 0.004540920001 and 1.751290440133. The records also contain route-A constants of L20 and F21, which the reference does not use (Section 20.4; for the folds Lemma 23.2, Certificate D.6), and of L21 at the starts from 2^{30}, where the root is a positive tilt (Section 21). The finite checks are inputs of the spectral proof, not substitutes for its graph decomposition. The data are listed in Table 34.

20.4. **The chamber symmetry L20 for the uniform principal law.** One cut governs the middle range: the chamber reflection in the first coordinate, called L20 in the tables and certificates. The lower spectral bound (the *cap*) of its operator for the twelve-label principal law is the most negative in the middle range, so it is the bound the completion has to compensate. On the middle intervals ("pieces") starting below 2^{30} the principal law is the uniform law $\lambda_{\mathcal{B}}(G_{\mathrm{L},2})$; the tilted laws of Section 21 start at 2^{30}. This subsection bounds the L20 operator for the uniform law directly, by computing its star part exactly in its own half law (Section 14.2) instead of using the general criteria of Section 14. Fix a prime $p \equiv 3 \pmod 4$. By Lemma 20.11(b) the half graph of L20 is $G_{\mathrm{F},2}$, so the half reference $\mu_p = \lambda_p(G_{\mathrm{F},2})$ is the uniform law on injective tuples $x = (a_0, a_1, b_0, b_1, c_0, c_1)$ with $a_i \to c_j$ for all i, j (the labels of the objects $0, 1, 2$ in the layers $0, 1$; coordinates numbered $0, \dots, 5$ in this order). Let $T = \mathsf{A}_p$ be the L20 cut operator on $L^2(\mu_p)$, $\mathcal{I}$ the subspace of functions invariant under the simultaneous translations $x_k \mapsto x_k + y$ ($y \in \mathbb{F}_p$), and $\chi_p^+ = D_{\mathrm{full}}^2 / L_{\mathrm{low}}^2$ the second bound of Lemma 24.7(ii), with $D_{\mathrm{full}}, L_{\mathrm{low}}$ as defined there. As $T\mathbf{1}$ is the density of the half law of the root relative to $\mu_p = \lambda_{F,p}$ (Lemma 13.3(b)), $\|T\mathbf{1} - \mathbf{1}\|^2 = \chi^2(\nu_{\mathrm{L},p} \,\|\, \lambda_{F,p}) \le \chi_p^+$, where $\nu_{\mathrm{L},p}$ is the ϑ_0-half law of $\lambda_p(G_{\mathrm{L},2})$.

The cross pairs of the cut are the sixteen layer lifts of the four root edges $b' \to a$, $c \to b'$, $b \to a'$ and $c' \to b$ between the half x and the reflected half x'. The *soft L20 kernel* is the product over the sixteen cross pairs of the factors $1 + (D_{20})_{ij}\chi(x'_j - x_i)$, without the indicator that x and x' have no common label (its factors are the signed factors of the cut, not the soft factors of Corollary 16.17),

with the signed cross matrix

$$D_{20} = \begin{pmatrix} 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \tag{20.2}$$

(rows indexed by the coordinates of x, columns by those of x'; an edge $y \to y'$ means $\chi(y'-y)=1$, and $\chi(-1)=-1$). The matrix D_{20} is antisymmetric. In the notation of Lemma 14.18 the cross signs are $\varsigma_{ij} = (D_{20})_{ij}$, so $T = K_0/Z_0$ and $K_0 = K_{\mathrm{soft}} - T_{\mathrm{coll}}$, where $K_{\mathrm{soft}} = \mathbf{1}\otimes\mathbf{1} + \sum_{S\ \mathrm{star}} T_S + T_{\mathrm{ns}}$ is the soft kernel as an operator on $L^2(\mu_p)$. We call $C = (a_0, a_1, c_0, c_1)$ (coordinates $0,1,4,5$) the *core* of x and $b = (b_0, b_1)$ (coordinates $2,3$; not the least prime $b_{\mathcal{B}}$) its *pair*, write $\mathcal{H} = L^2_0(\mathbb{F}_p)$ and $\hat{P} = P/r$ for the normalized Paley operator, where P is the Paley operator of Lemma 4.11, and use the features V_k, W_S of Section 14.2 relative to μ_p and their synthesis $\mathcal{F} = (V_k, (W_S/r)_S)$. The signs of the core coordinates are $s = (s_0, s_1, s_4, s_5) = (-1,-1,1,1)$, as in Lemma 24.3.

Lemma 20.14 (Complete stars of L20)**.** *Let $p \equiv 3 \pmod 4$ and $p > 15$.*

(a) *The leaf supports with at least two elements of the stars of K_{soft} are the twelve sets* 01, 014, 0145, 015, 04, 045, 05, 14, 145, 15, 23, 45, *the subsets with at least two elements of the row supports $\{2,3\}$ and $\{0,1,4,5\}$ of D_{20}. With the six singletons, the Hermitian matrix $M(\hat{P})$ of Corollary 14.5 satisfies*

$$\det(\lambda I - M(i)) = \lambda^{14}(\lambda^2 - 4)(\lambda^2 - 30);$$

in particular $\|M(i)\| = \sqrt{30}$.

(b) *Under μ_p the conditional probability of each value of a core coordinate given the other five coordinates is at most D_j/p with $D_j = 4/(1-15q)$, and that of a pair coordinate is D_j/p with $D_j = 1/(1-5q)$. Let Γ_{20} be one plus the largest row sum of the symmetric 18×18 matrix, indexed by the six singletons V_k and the twelve supports S of part (a) (the features W_S/r), whose diagonal entries are 0 and whose off-diagonal entries are: for two supports $S \neq T$, the bound of Lemma 14.4(c) with the pivot $j \in S\triangle T$ of least D_j; for a singleton k and a support S, the bound of Lemma 14.4(d); and for two singletons, the bound of Lemma 14.4(c) applied to their pair. Then $\|\mathcal{F}\|^2 \le \Gamma_{20}$; Γ_{20} is a nondecreasing function of r (with $q = r^2$), and $\Gamma_{20} \to 1$ as $p \to \infty$.*

(c) *For a label z put*

$$\mathsf{K}(z, C) = \prod_{i=0,1} (1 - \chi(z - a_i)) \prod_{j=0,1} (1 + \chi(z - c_j)),$$

$$\ell(z, C) = -\chi(z-a_0) - \chi(z-a_1) + \chi(z-c_0) + \chi(z-c_1),$$

$\mathsf{K}_\star = \mathsf{K} - 1 - \ell$ and $(W_\star h)(x) = \mathbb{E}_z \mathsf{K}_\star(z, C) h(z)$ for $h \in \mathcal{H}$, and put

$$\mathbf{u} = \tfrac{1}{2}(V_0 + V_1 - V_4 - V_5), \qquad \mathbf{v} = \tfrac{1}{\sqrt{2}}(V_2 + V_3), \qquad \mathbf{w} = \tfrac{1}{2}(V_0 + V_1 + V_4 + V_5),$$

$$\mathbf{e} = W_{23}/r, \qquad \mathbf{z} = \tfrac{1}{\sqrt{15}}(W_\star/r - 2\mathbf{u}\hat{P}).$$

Then $\mathsf{K} = 16 \cdot \mathbf{1}\{c_0, c_1 \to z \to a_0, a_1\}$; in particular K vanishes at the core labels, $\mathsf{K}^2 = 16\mathsf{K}$, $\mathsf{K}\ell = 4\mathsf{K}$ and $\mathbb{E}_z \ell^2 = 4$. The function $\mathsf{K}_\star$ is the signed sum $\sum_S \sigma_S \psi_S$ of the eleven higher stars on the core, $\sigma_S = \prod_{k\in S} s_k$, $\psi_S(z, x) = \prod_{k\in S} \chi(z - x_k)$, and the doubly centred complete star part of K_{soft} is

$$T_\star = r\{\sqrt{30}(\mathbf{v}\mathbf{z}^* + \mathbf{z}\mathbf{v}^*) + 2(\mathbf{w}\mathbf{e}^* + \mathbf{e}\mathbf{w}^*)\}. \tag{20.3}$$

Proof. (c) For z outside the core each factor of K is 0 or 2, and all four are 2 exactly when $\chi(z-a_i)=-1$ and $\chi(z-c_j)=1$ for all i,j, that is, when $z\to a_i$ and $c_j\to z$. At $z=a_i$ the factor $1+\chi(a_i-c_j)$ vanishes, because $a_i\to c_j$ and $\chi(-1)=-1$; at $z=c_j$ the factor $1-\chi(c_j-a_i)$ vanishes. Hence $\mathsf{K}\in\{0,16\}$, $\mathsf{K}^2=16\mathsf{K}$, and $\ell=4$ where $\mathsf{K}\neq 0$. Since $\mathbb{E}_z\chi(z-y)\chi(z-y')=\mathbf{1}_{y=y'}-q$ (4.2) and $\sum_k s_k=0$, $\mathbb{E}_z\ell^2=4(1-q)+4q=4$. Expanding the product, $\mathsf{K}=\sum_S\sigma_S\psi_S$ over all $S\subseteq\{0,1,4,5\}$; the empty set gives 1 and the singletons give ℓ, so $\mathsf{K}_\star$ is the sum over the eleven sets with at least two elements.

(a) and (20.3). By Corollary 14.5, $M(\hat P)$ has the singleton block $D_{20}\otimes\hat P$, the entry $(-1)^{|S|}\prod_{j\in S}(D_{20})_{kj}$ at (singleton k, support S) and $\prod_{i\in S}(D_{20})_{ij}$ at (support S, singleton j). The leaf sets of the left stars are the subsets of the row supports of D_{20}, those of the right stars the subsets of its column supports, and these are the same because D_{20} is antisymmetric; this gives the twelve supports. In $\mathbb{C}^{18}$, with the coordinate vectors e_k $(0\le k\le 5)$ and e_S, put, at the phase $\hat P=i$,

$$\hat{\mathbf{u}}=\tfrac12(e_0+e_1-e_4-e_5),\quad \hat{\mathbf{v}}=\tfrac{1}{\sqrt2}(e_2+e_3),\quad \hat{\mathbf{w}}=\tfrac12(e_0+e_1+e_4+e_5),\quad \hat{\mathbf{e}}=e_{23},$$
$$\hat{\mathbf{z}}=\tfrac{1}{\sqrt{15}}\Big(\sum_S\sigma_S e_S-2i\,\hat{\mathbf{u}}\Big),$$

the sum over the eleven supports contained in $\{0,1,4,5\}$. The four vectors $\hat{\mathbf{v}},\hat{\mathbf{z}},\hat{\mathbf{w}},\hat{\mathbf{e}}$ are orthonormal ($|\hat{\mathbf{z}}|^2=(11+4)/15$), and

$$M(i)=\sqrt{30}\big(\hat{\mathbf{v}}\hat{\mathbf{z}}^*+\hat{\mathbf{z}}\hat{\mathbf{v}}^*\big)+2\big(\hat{\mathbf{w}}\hat{\mathbf{e}}^*+\hat{\mathbf{e}}\hat{\mathbf{w}}^*\big),$$

as a comparison of entries shows: on the right the singleton entries are i at (b_l,a_k) and (c_k,b_l) and $-i$ at (b_l,c_k) and (a_k,b_l), which are those of iD_{20}; the entries at (b_l,S) and (S,b_l), $S\subseteq\{0,1,4,5\}$, are $\sigma_S=(-1)^{|S|}\prod_{j\in S}(D_{20})_{2j}=\prod_{i\in S}(D_{20})_{i2}$; the entries at $(k,23)$ and $(23,k)$, k a core coordinate, are $1=(D_{20})_{k2}(D_{20})_{k3}$; and all other entries vanish on both sides, since a core row of D_{20} has the support $\{2,3\}$ and a pair row the support $\{0,1,4,5\}$. So $M(i)$ has the eigenvalues $\pm\sqrt{30}$ and ±2 on the span of the four vectors and 0 on its orthogonal complement, which is (a). The synthesis maps e_k to V_k and e_S to W_S/r, the phase acting as $\hat P$ on the label input; it maps $\hat{\mathbf{v}}$, $\hat{\mathbf{w}}$, $\hat{\mathbf{e}}$ to $\mathbf{v}$, $\mathbf{w}$, $\mathbf{e}$, and $\hat{\mathbf{z}}$ to $\mathbf{z}$, since $\sum_S\sigma_S W_S=W_\star$ by (c). The identity $T_\star=r\mathcal{F}M(\hat P)\mathcal{F}^*$ of Corollary 14.5 is therefore (20.3).

(b) The coordinate a_0 is a common predecessor of c_0,c_1 distinct from a_1,b_0,b_1. By Lemma 20.9(i) two distinct labels have exactly $(p-3)/4$ common predecessors, so given the other coordinates a_0 is uniform on at least $(p-3)/4-3=(p-15)/4$ labels; the same holds for a_1 and, with common successors of a_0,a_1, for c_0,c_1. A pair coordinate is uniform on the $p-5$ labels outside the other five. This gives the D_j. The off-diagonal Grams of $\mathcal{F}$ are bounded by the entries of the matrix (Lemma 14.4(c),(d)), the diagonal ones are at most one (Lemma 14.4(a)), and Lemma 14.4(f) gives $\|\mathcal{F}\|^2\le\Gamma_{20}$. Every entry is built from q, r and $D_j-1\in\{(3+15q)/(1-15q),\,5q/(1-5q)\}$ by sums, products, square roots and division by $1-q$, so it is admissible (Lemma 4.18) and tends to 0. □

The synthesis bound $\|\mathcal{F}\|^2\le\Gamma_{20}$ alone gives the coefficient $\sqrt{30}\,\Gamma_{20}$ of r in the negative cap; its excess over $\sqrt{30}$ is of order r, with a large coefficient coming from the overlaps of the eighteen features. The following propositions separate the translation-invariant part of the kernel (Proposition 20.15), replace this synthesis bound by the exact Gram matrices of the four features of (20.3), so that the leading coefficient becomes $\sqrt{30}+O(q)$ (Proposition 20.16), and factor the root through a rectangular kernel whose nonstar part is priced quadratically (Propositions 20.17 and 20.20). What remains of order r is the collision cap (Proposition 20.21 and the remark after it).

Proposition 20.15 (Translation sectors)**.** *Let $p \equiv 3 \pmod 4$, $p \geq 2^{20}$, let N_{ns} and D_{col} be raw nonstar and collision caps of the soft L20 kernel ("raw": bounds for the unnormalized kernel K_0, before division by Z_0), $\|T_{\mathrm{ns}}\| \leq N_{\mathrm{ns}}$ and $\|T_{\mathrm{coll}}\| \leq D_{\mathrm{col}}$ (Corollary 16.14 and Lemma 16.15), let $m_\star$ be the averaged-star mean bound of Lemma 24.5, and put $\varepsilon_Z = N_{\mathrm{ns}} + D_{\mathrm{col}} + m_\star$ and $H_{20} = (N_{\mathrm{ns}} + D_{\mathrm{col}})/(1 - \varepsilon_Z)$, assuming $\varepsilon_Z < 1$. Then T preserves $\mathcal{I}$ and $\mathcal{I}^\perp$, the ranges of the features V_k, W_S lie in $\mathcal{I}^\perp$, and the centred block of T on $\mathcal{I} \cap \mathbf{1}^\perp$ has norm at most H_{20}. If moreover $H_{20} < 1$ and $C_\perp < 1$ satisfy $T \succeq -C_\perp I$ and $\|T|_{\mathcal{I}^\perp}\| \leq C_\perp$, then*

$$-g\,rI \preceq T \preceq (1 + c^+/p)I, \qquad g = C_\perp/r, \qquad c^+ = \frac{p\chi_p^+}{1 - H_{20}}. \tag{20.4}$$

Proof. By Lemma 14.18, $T = K_0/Z_0$ with $Z_0 = \langle \mathbf{1}, K_0\mathbf{1}\rangle_{\mu_p}$, and by Lemma 24.5 the star means of K_{soft} sum to at most $m_\star$ in modulus, so $Z_0 \geq 1 - \varepsilon_Z > 0$. The simultaneous translations preserve μ_p and K_0, so T commutes with them and preserves $\mathcal{I}$, and $\mathcal{I}^\perp$ as it is self-adjoint. The ranges of the features lie in $\mathcal{I}^\perp$ by Lemma 14.4. Let $h, h' \in \mathcal{I} \cap \mathbf{1}^\perp$. Then $\mathbb{E}_{\mu_p}[h \mid x_k] = 0$ for every coordinate k (Lemma 14.2(d)), so $T_S h = 0$ for every column star S (Lemma 14.19(a)) and $\langle h', T_S h\rangle = \langle T_{S^{\mathrm{t}}} h', h\rangle = 0$ for every row star (Lemma 14.19(b)); also $(\mathbf{1} \otimes \mathbf{1})h = 0$. By (14.20), $\langle h', K_0 h\rangle = \langle h', (T_{\mathrm{ns}} - T_{\mathrm{coll}})h\rangle$, of modulus at most $(N_{\mathrm{ns}} + D_{\mathrm{col}})\,\|h\|\,\|h'\|$, and division by Z_0 bounds the centred block by H_{20}. The vector $T\mathbf{1} - \mathbf{1}$ is translation invariant and orthogonal to $\mathbf{1}$, as $\langle \mathbf{1}, T\mathbf{1}\rangle = 1$, and its squared norm is at most χ_p^+. Now Proposition 13.6(a″), for the group of translations, gives $T \preceq \max\bigl(C_\perp, 1 + \chi_p^+/(1 - H_{20})\bigr)I = (1 + c^+/p)I$, as $C_\perp < 1$. The lower bound is the hypothesis. □

The lower bound is required on the whole space; Proposition 20.17 supplies it.

Proposition 20.16 (Gram bounds for the four L20 star features)**.** *Let $p \equiv 3 \pmod 4$, $p \geq 2^{20}$, let $\bar{\mathsf{K}}(C) = \mathbb{E}_z \mathsf{K}(z, C)$, $m_{\mathsf{K}} = \mathbb{E}_{\mu_p}\bar{\mathsf{K}} - 1$ and $v_{\mathsf{K}} = \mathbb{E}_{\mu_p}(\bar{\mathsf{K}} - 1)^2$, let $m_\star$ be the bound of Lemma 24.5, and put*

$$\mathfrak{m}_+ = 2q + \tfrac{1}{4}m_\star, \qquad g_{\mathbf{z}} = \frac{1 + \frac{14}{15}\mathfrak{m}_+}{1 - q}, \qquad \rho = \frac{(12 + 8r)q^2}{(1 - 4q)(1 - 5q)},$$

$$\rho_{\mathbf{vz}} = \frac{4\sqrt{2/15}\,qr(1 + \mathfrak{m}_+)}{(1 - 4q)(1 - q)}, \qquad \rho_{\mathbf{vw}} = \frac{2\sqrt{2}\,q}{1 - q}, \qquad \rho_{\mathbf{ve}} = \frac{\sqrt{2}\,qr}{1 - q},$$

$$\rho_{\mathbf{zw}} = \frac{2r(1 + \mathfrak{m}_+)}{\sqrt{15}(1 - q)}, \qquad \rho_{\mathbf{ze}} = \sqrt{g_{\mathbf{z}}}\,\rho, \qquad k_{\mathrm{top}} = \sqrt{30}\,(g_{\mathbf{z}} + \rho_{\mathbf{vz}}), \qquad k_{\mathrm{low}} = 2(1 + \rho),$$

$$k_\times^2 = 2\sqrt{30}(\rho_{\mathbf{vw}}^2 + \rho_{\mathbf{ve}}^2 + \rho_{\mathbf{zw}}^2 + \rho_{\mathbf{ze}}^2), \qquad k_4 = k_{\mathrm{top}} + \frac{k_\times^2}{\sqrt{30} - k_{\mathrm{low}}}.$$

Then $\sqrt{30} - k_{\mathrm{low}} > 0$, $|m_{\mathsf{K}}| \leq \mathfrak{m}_+$, and on $\mathcal{H}$:

(a) $\mathbf{u}^*\mathbf{u} = \frac{1}{1-q}I$, $\mathbf{w}^*\mathbf{w} = \frac{1-4q}{1-q}I$, $\mathbf{u}^*\mathbf{w} = \frac{2r}{1-q}\hat{P}$ *and* $\mathbf{v}^*\mathbf{v} = \frac{1-2q}{1-q}I$;

(b) $W_\star^* W_\star = \frac{q(11+6m_{\mathsf{K}}-v_{\mathsf{K}})}{1-q}I$, $\mathbf{u}^* W_\star = -\frac{2m_{\mathsf{K}}r}{1-q}\hat{P}$ *and* $\mathbf{w}^* W_\star = \frac{2q(1-m_{\mathsf{K}})}{1-q}I$;

(c) $\mathbf{z}^*\mathbf{z} = \frac{15+14m_{\mathsf{K}}-v_{\mathsf{K}}}{15(1-q)}I \preceq g_{\mathbf{z}}I$, $\mathbf{w}^*\mathbf{z} = -\frac{2r(1+m_{\mathsf{K}})}{\sqrt{15}(1-q)}I$, $\mathbf{v}^*\mathbf{z} = \frac{4\sqrt{2/15}\,qr(1+m_{\mathsf{K}})}{(1-4q)(1-q)}I$ *and* $\mathbf{v}^*\mathbf{w} = -\frac{2\sqrt{2}\,q}{1-q}I$;

(d) $\mathbf{e}^*\mathbf{e} = \frac{1-2q-q^2}{1-q}I$, $\mathbf{v}^*\mathbf{e} = \frac{\sqrt{2}\,qr}{1-q}I$, $\|\mathbf{w}^*\mathbf{e}\| \leq \rho$ *and* $\|\mathbf{z}^*\mathbf{e}\| \leq \rho_{\mathbf{ze}}$.

Consequently, if $\mathfrak{F} = (\mathbf{v}, \mathbf{z}, \mathbf{w}, \mathbf{e})$ and Ξ is a real symmetric 4×4 matrix with $-\Delta_\star \preceq \Xi \preceq \Delta_\star$, $\Delta_\star = \mathrm{diag}(\sqrt{30}, \sqrt{30}, 2, 2)$, then $\|\mathfrak{F}\Xi\mathfrak{F}^*\| \leq k_4$. In particular $\|T_\star\| \leq k_4 r$.

Proof. For $p \geq 2^{20}$, $\rho < 1/2$, so $k_{\mathrm{low}} < 3 < \sqrt{30}$.

The mean. By Lemma 20.14(c), $\bar{\mathsf{K}} - 1 = \sum_S \sigma_S w_S$ over the eleven higher supports S on the core, with $w_S(C) = \mathbb{E}_z \psi_S(z, x)$. The six two-element supports have $w_S = -q$ (4.2) and $\sum \sigma_S = \frac{1}{2}((\sum_k s_k)^2 - 4) = -2$, which contributes $2q$. Under μ_p the core is uniform on the admissible quadruples of Lemma 24.3 (each has exactly $(p-4)(p-5)$ completions by the pair), whose probability under independent uniform labels is $Z_4/16$ with $Z_4 = (1-q)(1-3q)(1-7q)$, while $P_{\mathrm{adm}} D_4 = 16$ on them (D_4 the indicator that the four labels are distinct, and P_{adm} the product of Lemma 24.3, 16 on admissible quadruples and 0 on the other distinct quadruples). In the notation of Lemma 24.5 the three- and four-element supports therefore contribute $\mathbb{E}_{\mu_p}[\mathcal{W}_3/p] = 0$ and $\mathbb{E}_{\mu_p}[\mathcal{W}_4/p] = (M_{G_5} + 8q^2 - 33q^3 + 25q^4)/Z_4$ (24.4), and $|M_{G_5}| \le (1-q)(q^2 + 2qr)$ (Lemma 24.3(b)) gives $|m_{\mathsf{K}} - 2q| \le m_\star/4$.

The Grams. Each operator below commutes with the square-affine maps acting on the label input and on all coordinates, so on $\mathcal{H}$ it is $\alpha I + \gamma \hat{P}$, where α is its trace divided by $p-1$ and $\gamma(p-1)$ is the trace of its product with $\hat{P}^*$; if it is self-adjoint, $\gamma = 0$ (Lemma 14.2(d)). For an orthonormal basis (h) of $\mathcal{H}$, $\sum_h h(y)h(y') = p\mathbf{1}_{y=y'} - 1$, which gives the traces.

(a) Two core coordinates on the same side form a uniform distinct pair, and (a_i, c_j) a uniform directed pair, so $V_k^* V_{k'} = -\frac{q}{1-q} I$ for a same-side pair and $V_{a_i}^* V_{c_j} = -\frac{q}{1-q} I + \frac{r}{1-q}\hat{P}$. The coefficients of $\mathbf{u}$ sum to zero, and the Paley parts of $V_{a_i}^* V_{c_j}$ and $V_{c_j}^* V_{a_i}$ cancel in $\mathbf{u}^*\mathbf{u}$ and $\mathbf{w}^*\mathbf{w}$ and add up in $\mathbf{u}^*\mathbf{w}$; this gives (a) for $\mathbf{u}$ and $\mathbf{w}$. The pair (b_0, b_1) is a uniform distinct pair, which gives $\mathbf{v}^*\mathbf{v}$.

(b) By Lemma 20.14(c), $\mathbb{E}_z \mathsf{K}_\star^2 = 16\bar{\mathsf{K}} + 1 + 4 - 2\bar{\mathsf{K}} - 8\bar{\mathsf{K}} = 6\bar{\mathsf{K}} + 5$ and $\mathbb{E}_z \mathsf{K}_\star = \bar{\mathsf{K}} - 1$, so the trace of $W_\star^* W_\star$ is $\mathbb{E}_{\mu_p} \operatorname{Var}_z \mathsf{K}_\star = 11 + 6m_{\mathsf{K}} - v_{\mathsf{K}}$. For a core coordinate k, $\mathsf{K}(x_k, C) = 0$ and $\ell(x_k, C) = -2 \pm \chi(x_k - x_{k'})$, k' the other coordinate on the same side, so $\mathbb{E}_{\mu_p} \mathsf{K}_\star(x_k, C) = 1$ by the exchange of x_k and $x_{k'}$, and the trace of $V_k^* W_\star$ is $1 - m_{\mathsf{K}}$. Where $\mathsf{K} \ne 0$, $\chi(z - x_k) = s_k$, and $\mathbb{E}_z \chi(z - x_k)\ell = s_k$; so $\mathbb{E}_z \chi(z - x_k)\mathsf{K}_\star = s_k(\bar{\mathsf{K}} - 1)$ and $V_k^* W_\star = \frac{q(1-m_{\mathsf{K}})}{1-q} I + \frac{s_k m_{\mathsf{K}} r}{1-q}\hat{P}$. Summing with the coefficients of $\mathbf{u}$ and of $\mathbf{w}$ gives (b).

(c) By (a), (b), $\hat{P}^2 = -I$ and $\hat{P}^* = -\hat{P}$ on $\mathcal{H}$, $15\,\mathbf{z}^*\mathbf{z} = W_\star^* W_\star/q + 8m_{\mathsf{K}}(1-q)^{-1} I + 4\,\mathbf{u}^*\mathbf{u}$ and $\sqrt{15}\,\mathbf{w}^*\mathbf{z} = \mathbf{w}^* W_\star/r - 2\mathbf{w}^*\mathbf{u}\hat{P}$; this gives $\mathbf{z}^*\mathbf{z}$ and $\mathbf{w}^*\mathbf{z}$, and $\mathbf{z}^*\mathbf{z} \preceq g_{\mathbf{z}} I$ as $v_{\mathsf{K}} \ge 0$. Given the core, each pair coordinate is uniform on the $p-4$ labels outside it, so $\mathbb{E}_{\mu_p}[h(b_l) \mid C] = -2(\mathbf{w}h)(C)/(p-4)$ for $h \in \mathcal{H}$; hence $\mathbf{v}^* Y = -\frac{2\sqrt{2}\,q}{1-4q}\mathbf{w}^* Y$ for every operator Y with values in the functions of the core, and $Y = \mathbf{w}, \mathbf{z}$ give $\mathbf{v}^*\mathbf{w}$ and $\mathbf{v}^*\mathbf{z}$.

(d) Lemma 14.4(a) with $w_{23} = -q$ gives $\mathbf{e}^*\mathbf{e}$. The trace of $V_2^* W_{23}$ is $\mathbb{E}[\chi(0)\chi(b_0 - b_1)] + q = q$, and the trace of its product with $\hat{P}^*$ is $r^{-1}\mathbb{E}\,\mathbb{E}_z \chi(z - b_0)^2 \chi(z - b_1) = -qr^{-1}\mathbb{E}\chi(b_0 - b_1) = 0$; so $V_2^* W_{23} = V_3^* W_{23} = \frac{q^2}{1-q} I$, which gives $\mathbf{v}^*\mathbf{e}$. Given the core, (b_0, b_1) is uniform on the ordered distinct pairs outside it, and expanding $\sum_{y \ne y'} \chi(z - y)\chi(z - y')$ over such pairs gives, on $\mathcal{H}$,

$$\mathbb{E}_{\mu_p}[W_{23} \mid C] = \frac{-2q \sum_k V_k + 2 \sum_{|S|=2} W_S}{(p-4)(p-5)},$$

the sums over the core coordinates and over the six two-element subsets of the core. As $\|V_k\| = 1$ and $\|W_S\| \le r$ (Lemma 14.4(a)), the conditional expectation $\Pi_C \mathbf{e}$ of $\mathbf{e}$ onto the functions of the core has norm at most ρ. For $Y = \mathbf{w}, \mathbf{z}$, which take values in the functions of the core, $\mathbf{e}^* Y = (\Pi_C \mathbf{e})^* Y$, and $\|\mathbf{w}\| \le 1$, $\|\mathbf{z}\| \le \sqrt{g_{\mathbf{z}}}$.

The consequence. As $-\Delta_\star \preceq \Xi \preceq \Delta_\star$, $\|\mathfrak{F}\Xi\mathfrak{F}^*\| \le \|\mathfrak{F}\Delta_\star\mathfrak{F}^*\| = \left\|\Delta_\star^{1/2}\mathfrak{G}\Delta_\star^{1/2}\right\|$, where $\mathfrak{G} = \mathfrak{F}^*\mathfrak{F}$ is the Gram of the four features. In the blocks $(\mathbf{v}, \mathbf{z})$ and $(\mathbf{w}, \mathbf{e})$ this matrix has diagonal blocks of norms at most k_{top} and k_{low}, by (a)–(d), and an off-diagonal block of squared norm at most $k_\times^2$, the weighted sum of the squared norms of its four entries. Since $k_{\mathrm{top}} \ge \sqrt{30} > k_{\mathrm{low}}$, the largest eigenvalue of $\left(\begin{smallmatrix} k_{\mathrm{top}} & k_\times \\ k_\times & k_{\mathrm{low}} \end{smallmatrix}\right)$, at most $k_{\mathrm{top}} + k_\times^2/(k_{\mathrm{top}} - k_{\mathrm{low}}) \le k_4$, bounds the norm. By (20.3), $T_\star/r = \mathfrak{F}\Xi_\star\mathfrak{F}^*$ with $\Xi_\star = \left(\begin{smallmatrix} 0 & \sqrt{30} \\ \sqrt{30} & 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$, and $-\Delta_\star \preceq \Xi_\star \preceq \Delta_\star$. □

The coefficient k_4 is $\sqrt{30}+O(q)$: the only overlap of order r, the entry $\rho_{\mathbf{zw}}$ between the two blocks, enters quadratically across the gap $\sqrt{30}-2$.

Proposition 20.17 (The soft L20 kernel as a swapped rectangle product)**.** *Let $p\equiv 3 \pmod 4$, $p\ge 2^{20}$. Let μ^{core} be the uniform law on the admissible cores C (distinct labels with $a_i\to c_j$) and μ^{pair} the uniform law on the ordered distinct pairs b, and let $\mathsf{R}\colon L^2(\mu^{\mathrm{pair}})\to L^2(\mu^{\mathrm{core}})$ have the kernel $\mathsf{R}(C;b)=\mathsf{K}(b_0,C)\mathsf{K}(b_1,C)$; the features $\mathbf{z},\mathbf{w}$ of the core and $\mathbf{v},\mathbf{e}$ of the pair are those of Lemma 20.14, read in these laws.*

(a) *R has the row mass $\mathfrak{r}(C)=\mathsf{R}\mathbf{1}(C)=(\bar{\mathsf{K}}(C)^2-16q\bar{\mathsf{K}}(C))/(1-q)$ and a constant column mass, and $\mathsf{R}_c=\mathsf{R}-\mathfrak{r}\otimes\mathbf{1}$ (where $(\mathfrak{r}\otimes\mathbf{1})h=(\mathbb{E}h)\,\mathfrak{r}$) satisfies $\mathsf{R}_c\mathbf{1}=\mathsf{R}_c^*\mathbf{1}=0$ and $\mathsf{R}_c^*\mathfrak{r}=0$.*

(b) *Let N_A be an upper bound for the norm of the double-centred nonstar remainder $P_0(\text{nonstars})P_0$. A raw cap from Corollary 16.17(a) suffices; the sharper bound of Theorem 20.19 also applies. Then*

$$\mathsf{R}=\mathfrak{r}\otimes\mathbf{1}+\mathsf{R}_\star+\mathsf{R}_\circ,\qquad \mathsf{R}_\star=r\big(\sqrt{30}\,\mathbf{z}\mathbf{v}^*+2\mathbf{w}\mathbf{e}^*\big),\qquad \|\mathsf{R}_\circ\|\le N_A, \tag{20.5}$$

and $\|\mathsf{R}_\star\|\le k_4 r$; hence $\|\mathsf{R}_c\|\le k_4 r+N_A$.

(c) *On $L^2(\mu^{\mathrm{core}}\otimes\mu^{\mathrm{pair}})$ the soft L20 kernel is*

$$K_{\mathrm{soft}}\big((C,b),(C',b')\big)=\mathsf{R}(C;b')\,\mathsf{R}(C';b),\qquad K_{\mathrm{soft}}=T[\mathsf{R},\mathsf{R}]=(\mathsf{R}\otimes\mathsf{R}^*)\,\mathrm{Swap}\,. \tag{20.6}$$

(d) *Put $\eta=2q+11r$, $h_p=(1-4q)(1-5q)/(1-q)$ and*

$$s_p=\frac{\big[1+4\mathfrak{m}_+ +(6+4\eta+\eta^2)\eta^2+256q^2(1+2\mathfrak{m}_+ +\eta^2)\big]^{1/2}}{1-q}.$$

Then $\|\mathfrak{r}\|\le s_p$. If $\|\mathsf{R}_c\|\le c_{\mathrm{R}}<1$, then for every collision cap D_{col} with $\varepsilon_Z=N_{\mathrm{ns}}+D_{\mathrm{col}}+m_\star<1$ as in Proposition 20.15,

$$T\succeq -C'I,\qquad \|T|_{\mathcal{I}^\perp}\|\le C',\qquad C'=\frac{s_p c_{\mathrm{R}}/h_p+D_{\mathrm{col}}}{1-\varepsilon_Z}.$$

Proof. (a) Given C, under μ^{pair}, $\mathfrak{r}(C)=\frac{1}{p(p-1)}\sum_{y\neq y'}\mathsf{K}(y,C)\mathsf{K}(y',C)=\frac{p^2\bar{\mathsf{K}}^2-\sum_y\mathsf{K}(y,C)^2}{p(p-1)}$, which is the stated formula by $\mathsf{K}^2=16\mathsf{K}$. If φ is a function of the core invariant under the square-affine maps, then $\mathsf{R}^*\varphi(b)=\mathbb{E}_{\mu^{\mathrm{core}}}[\mathsf{R}(C;b)\varphi(C)]$ is invariant under these maps acting on b, hence constant on each of the two orbits of ordered distinct pairs (the sign of $\chi(b_1-b_0)$); R is symmetric in b_0,b_1, and the exchange maps one orbit onto the other because $\chi(-1)=-1$, so $\mathsf{R}^*\varphi$ is constant. With $\varphi=\mathbf{1}$ and $\varphi=\mathfrak{r}$ this gives the constant column mass $\mathbb{E}\mathfrak{r}$ and $\mathsf{R}^*\mathfrak{r}=\|\mathfrak{r}\|^2\,\mathbf{1}$; hence $\mathsf{R}_c\mathbf{1}=0$, $\mathsf{R}_c^*\mathbf{1}=0$ and $\mathsf{R}_c^*\mathfrak{r}=0$.

(b) By (a), $\mathsf{R}_c=P_0\mathsf{R}P_0$, with P_0 the projections onto the mean-zero functions. Expand $\mathsf{R}=\prod(1+s_k\chi(b_l-x_k))$ over the eight cross pairs (core coordinate k, pair coordinate b_l) into its 256 subset terms: the constant, 34 stars (8 single pairs, 22 sets of at least two pairs at one pair coordinate and 4 pairs $\{(k,b_0),(k,b_1)\}$ at one core coordinate) and 221 nonstars. The double centring removes the constant. The stars at b_l add up to $(\ell+\mathsf{K}_\star)(b_l,C)$, and $\mathbb{E}_z\ell(z,C)h(z)=\sum_k s_k(Ph)(x_k)=-2r(\mathbf{u}\hat{P}h)(C)$; so after the double centring they give $(W_\star-2r\mathbf{u}\hat{P})V_{b_l}^*=\sqrt{15}\,r\,\mathbf{z}V_{b_l}^*$, with $V_{b_l}h=h(b_l)$; the stars at the core coordinates give $\sum_k V_kW_{23}^*=2r\mathbf{w}\mathbf{e}^*$, their rank-one marginal terms being exactly those removed by the double centring (Corollary 14.5). Hence the centred stars give $\sqrt{15}\,r\,\mathbf{z}(V_{b_0}^*+V_{b_1}^*)+2r\mathbf{w}\mathbf{e}^*=\mathsf{R}_\star$, and $\mathsf{R}_\circ=P_0(\text{nonstars})P_0$ has norm at most that of the sum of the nonstar terms. The raw extraction bound of Corollary 16.17(a) also bounds this compression: the core and pair coordinate embeddings are isometries, and the eight-edge kernel uses the same 221 subsets. The sharper direct calculation is Theorem 20.19. Finally $\mathsf{R}_\star$ maps the mean-zero functions of the pair to those of the core. Its Hermitian dilation on $L^2(\mu^{\mathrm{core}})\oplus L^2(\mu^{\mathrm{pair}})$ is $r\mathfrak{F}'\Xi_\star\mathfrak{F}'^*$ for the four features placed in their summands; their Gram has the blocks of Proposition 20.16

inside each summand ($\mathbf{z}, \mathbf{w}$ depend only on the core and $\mathbf{v}, \mathbf{e}$ only on the pair, whose laws under μ_p are μ^{core} and μ^{pair}) and zero blocks across them. The argument of Proposition 20.16 gives $\|\mathsf{R}_\star\| \le k_4 r$.

(c) The cross factors between the core of x and the pair of x' are $1-\chi(b_l'-a_i)$ and $1+\chi(b_l'-c_j)$, the entries -1 and 1 of D_{20} in the rows of a_i and c_j; their product is $\mathsf{R}(C; b')$. By the antisymmetry of D_{20} the factors between the pair of x and the core of x' are $1+\chi(a_i'-b_l) = 1-\chi(b_l-a_i')$ and $1-\chi(c_j'-b_l) = 1+\chi(b_l-c_j')$, with product $\mathsf{R}(C'; b)$, and D_{20} has no other nonzero entries. The second form is Lemma 13.9(a).

(d) Write $\bar{\mathsf{K}} = 1+y$. By Lemma 20.14(c) and Theorem 4.7, $|y| \le 2q+4\cdot 2r+3r = \eta$ (six two-root, four three-root and one four-root sum), and $\mathbb{E}_{\mu^{\mathrm{core}}} y = m_{\mathsf{K}} \le \mathfrak{m}_+$ (Proposition 20.16). Hence $\mathbb{E}\bar{\mathsf{K}}^4 \le 1+4\mathfrak{m}_+ + (6+4\eta+\eta^2)\eta^2$ and $\mathbb{E}\bar{\mathsf{K}}^2 \le 1+2\mathfrak{m}_+ + \eta^2$, and dropping the term $-32q\bar{\mathsf{K}}^3 \le 0$ in $(1-q)^2\mathfrak{r}^2$ gives $\|\mathfrak{r}\| \le s_p$; note $s_p \ge 1$. By (a), R has the singular pair $(\mathfrak{r}/\|\mathfrak{r}\|, \mathbf{1})$ with the value $\|\mathfrak{r}\|$, and all its other singular values are at most c_{R}. By Lemma 13.9(a) the eigenvalues of $T[\mathsf{R}, \mathsf{R}]$ are 0, the squares of the singular values and their products $\pm\sigma_i\sigma_j$ $(i \ne j)$; as $c_{\mathrm{R}} < 1 \le s_p$, all negative eigenvalues are at least $-s_p c_{\mathrm{R}}$, and all eigenvalues except the one at the translation-invariant vector $(\mathfrak{r}/\|\mathfrak{r}\|) \otimes \mathbf{1}$ have modulus at most $s_p c_{\mathrm{R}}$. The law μ_p is $\mu^{\mathrm{core}} \otimes \mu^{\mathrm{pair}}$ conditioned on the translation-invariant event $\mathcal{E}$ that the pair avoids the core, which has probability h_p for every core. With the isometry $Jf = \mathbf{1}_{\mathcal{E}} f/\sqrt{h_p}$ of $L^2(\mu_p)$ into $L^2(\mu^{\mathrm{core}} \otimes \mu^{\mathrm{pair}})$, $K_{\mathrm{soft}} = h_p^{-1} J^* T[\mathsf{R}, \mathsf{R}] J$ on $L^2(\mu_p)$, and J maps $\mathcal{I}^\perp$ into the orthogonal complement of the translation-invariant functions. So $K_{\mathrm{soft}} \succeq -(s_p c_{\mathrm{R}}/h_p) I$ and $\|K_{\mathrm{soft}}|_{\mathcal{I}^\perp}\| \le s_p c_{\mathrm{R}}/h_p$. Finally $T = (K_{\mathrm{soft}} - T_{\mathrm{coll}})/Z_0$ with $\|T_{\mathrm{coll}}\| \le D_{\mathrm{col}}$ and $Z_0 \ge 1-\varepsilon_Z$ (proof of Proposition 20.15). □

20.4.1. *Bounding the centred remainder through its Gram.* The 221 nonstar terms of the rectangle share their four core variables. Estimating the Gram of their sum retains cancellations that are lost when the 221 terms are bounded separately and their norms are then added. The following estimate is for precisely $\mathsf{R}_\circ$ in (20.5).

Lemma 20.18 (Two-sided bounds for the squared cubic trace H_u)**.** *For $u \in \mathbb{F}_p$, on counting measure on $\mathbb{F}_p \setminus \{u\}$, put*

$$T_u(v,w) = \mathbb{E}_y \chi(u-y)\chi(v-y)\chi(w-y), \qquad H_u = T_u \circ T_u.$$

Then $\|T_u\| \le 1$, $\|H_u\| \le 1-2q-2q^2$, and

$$H_u \succeq -(3r+q-q^2) I.$$

Proof. The full character matrix C satisfies $CC^* = pI - J$. The matrix T_u is a compression of $qC\,\mathrm{diag}(\chi(u-y))C^*$, so has norm at most 1. Character orthogonality gives each row sum of the nonnegative matrix H_u as $1-2q-2q^2$, proving its absolute norm bound. For the lower bound translate u to zero. Dilation gives $T_0(v,w) = \chi(v)T_0(1, w/v)$, so H_0 is multiplicative convolution. Its trivial-character eigenvalue is its positive row sum. For a nontrivial multiplicative character ρ, the eigenvalue is

$$-q+q^2+p^{-2}\sum_{\lambda \ne 0,1} \rho(\lambda)\,\mathrm{tr}(\mathrm{Frob}_\lambda \mid \mathrm{Sym}^2 \mathcal{F}),$$

where $\mathcal{F}$ is the Legendre sheaf of Theorem A.6 with $h(X) = X(X-1)$. Indeed its determinant is $\overline{\mathbb{Q}}_\ell(-1)$, so the symmetric-square trace is the square of the cubic character sum minus p; the missing value at 1 is q^2. The Kummer sheaf for ρ is the corresponding eigensheaf of $z \mapsto z^{p-1}$ on $\mathbb{G}_m$. It has weight zero and is tame. The twist of $\mathrm{Sym}^2 \mathcal{F}$ has rank 3, weight 2, and is tame at $0, 1, \infty$. Its geometric irreducibility follows from the SL_2 monodromy of $\mathcal{F}$ and irreducibility of the symmetric square. Thus its invariants and coinvariants vanish. By Theorems A.3 and A.5, $H_c^0 = H_c^2 = 0$ and $\dim H_c^1 = 3$. The trace formula and Deligne's bound (Theorems A.2 and A.4)

bound the sum by $3p^{3/2}$. All eigenvalues are real because H_0 is symmetric. This proves the stated lower bound. □

Theorem 20.19 (The centred accepting rectangle)**.** *For $p \equiv 3 \pmod 4$, $p \geq 2^{20}$, let $Z_4 = (1-q)(1-3q)(1-7q)$. Then*

$$\|\mathsf{R}_\circ\| \leq 4q\sqrt{\frac{920 + 4012r + 48641q + 3008q^2 + 320q^3}{(1-q)Z_4}} \leq 121.588q. \tag{20.7}$$

For $p \geq 2^{36}$ the last bound improves to $121.328q$. Without Lemma 20.18, the same argument gives $128q$. These are centred bounds; they do not replace a raw nonstar or collision bound elsewhere in the proof.

Proof. Put $h = 1 - q$ for this proof. Initially give the four core coordinates independent uniform laws and retain the uniform distinct private pair. With $s = (-1, -1, 1, 1)$, set

$$k_x(t) = \prod_{i=1}^{4}(1 + s_i\chi(t - x_i)), \qquad N(x; b_0, b_1) = (k_x(b_0) - 1)(k_x(b_1) - 1) - \sum_i \chi(b_0 - x_i)\chi(b_1 - x_i).$$

This is exactly the sum of the nonstar terms. The accepting-core event has probability $Z_4/16$: each distinct ordered c-pair has $(p-3)/4$ common predecessors, from which the ordered a-pair is chosen distinctly. Restriction therefore costs $4/\sqrt{Z_4}$; output centring is a contraction. We bound NP_0 on the full core cube.

The Gram away from private overlaps. For four distinct private labels z_1, z_2, z_3, z_4, write

$$v = \mathbb{E}_y \prod_i \chi(z_i - y), \qquad t_i = \mathbb{E}_y \prod_{j \neq i} \chi(z_j - y), \qquad T = \sum_i t_i.$$

Independent integration in the four core variables gives

$$\begin{aligned} G = {} & [(1 - 6q + v)^2 - T^2]^2 - \sum_i [(1 - 3q)^2 - t_i^2]^2 + 6h^4 - 3 \\ & - 8[h^3(v - q) + q] + 4v + 12q^2. \end{aligned} \tag{20.8}$$

Denote the same polynomial on all private pair pairs by $\widetilde{G}$. With $a = 1 - 6q$, $b = 1 - 3q$, its nonconstant part is

$$\alpha v + 6a^2v^2 + (12q - 54q^2)\sum_i t_i^2 - 4a^2\sum_{i<j} t_it_j + 4a(v^3 - vT^2) + Q_4,$$

$$\alpha = -48q + 408q^2 - 856q^3 < 0, \qquad Q_4 = v^4 - 2v^2T^2 + T^4 - \sum_i t_i^4.$$

Powers denote pointwise products. The full kernel v is a positive Gram, so the term with negative coefficient is $\preceq 0$ in Loewner order. The constant kernel vanishes after compression by P_0.

Quadratic terms. Put $g = (1 - 2q)/(1 - q)$. Applying the four-cycle estimate to the off-diagonal output of the pair-to-field-square map, and the singleton incidence map to its diagonal, gives

$$\|P_0v^2P_0\| \leq q^2\left(\frac{4}{h} + 2gq\right).$$

For t_it_j with both omissions on the same private half, two outer Paley transforms give $q^2\sqrt{4/h^2 + 2gq/h}$; there are two such pairs. For each of the four opposite-half pairs, the middle kernel is a two-edge matching, giving q^2/h. These bounds follow from Corollaries 15.9 and 15.10 and Lemma 16.4 by keeping the common auxiliary labels. Together they bound the quadratic cross terms by $(56/h + 16q)q^2$. For t_i^2, split its two auxiliary labels into distinct and equal ones. The distinct part factors through the raw pair-to-singleton map and the four-cycle map and has norm

at most $2q^{3/2}$. The equal part is $q(1-3q)J + q^2(\mathbf{1}[c=d]+\mathbf{1}[c=e])$ on the singleton/pair chart. After centring its norm is at most $\sqrt{2g}q^3$. Thus the four square terms cost at most $(96r+96q^2)q^2$.

Cubic auxiliary collisions. For four distinct private labels put $A(y)=\prod_i \chi(z_i-y)$ and $B(y)=\sum_i \prod_{j\neq i}\chi(z_j-y)$. Let D_3 be the unrenormalized average of $A(u)A(v')A(w)-A(u)B(v')B(w)$ over three distinct auxiliary labels. The equality-partition formula for three summation labels gives

$$v^3 - vT^2 = D_3 - qv + q^2(6+L)v - 2q^3L, \qquad L=\sum_i\left(\sum_{j\neq i}\chi(z_j-z_i)\right)^2 \le 20.$$

For verification, the merged moments are $\mathbb{E}A^2 = 1-4q$, $\mathbb{E}A^3 = v$, $\mathbb{E}AB = 0$, $\mathbb{E}B^2 = 4(1-3q)-qL$, and $\mathbb{E}AB^2 = 4v - qL$. The bound on L is the score bound for a four-vertex tournament. Since $|v| \le 2r+q$, the error away from overlaps is at most $52q^2r + 66q^3$. On the private diagonal the error after $-qv$ is $-4q+40q^2-80q^3$, on a row fraction $2q^2/h$. On exactly one overlap it is at most $73q^2$, on a row fraction at most $4q/h$. These follow by the same three-label partition: there $|v|\le 2q$, $|T|\le 4r+2q$, $|\mathbb{E}AB|\le 2q$, $\mathbb{E}A^2\le 1$, $|\mathbb{E}B^2|, |\mathbb{E}AB^2| \le 16$. The absolute row and column bounds therefore give

$$v^3 - vT^2 = D_3 - qv + E_3, \qquad \|E_3\| \le 52q^2r + 400q^3/h.$$

The term D_3 has two-label witness mass 28: 4 for v^3, $8\cdot 2$ for the ordered same-half omissions, and 8 for the opposite-half omissions. Matching, path and four-cycle bounds give $\|D_3\| \le 28q^2/h$. Auxiliary distinctness is an output projection and is not renormalized; each private-half restriction costs $h^{-1/2}$. Dropping the negative semidefinite term $-qv$, the cubic contribution is at most $(112/h + 208r + 1600q/h)q^2$.

Quartic auxiliary collisions. For four support characters ψ_i, write $\tau_i = \mathbb{E}\psi_i$ and $m_I = \mathbb{E}\prod_{i\in I}\psi_i$. The exact partition identity is

$$\begin{aligned}\tau_1\tau_2\tau_3\tau_4 = D_4 &+ q\sum_{ij} m_{ij}\tau_k\tau_l - q^2\sum_{ij|kl} m_{ij}m_{kl}\\ &- 2q^2\sum_{ijk} m_{ijk}\tau_l + 6q^3 m_{1234},\end{aligned}$$

where the multiplicities are $6,3,4,1$ and D_4 retains four distinct auxiliary labels. This identity includes every root-hit hole. For any of the five traces $v, t_1, \dots, t_4$, and any nonempty XOR of at most three of their supports, the row and column second moments are at most q, by Paley orthogonality and the raw pair Gram. Hence a product of two such traces has absolute Schur norm at most q. Also

$$\|P_0(\tau\tau')P_0\| \le 2q^{3/2}/h + 16q^2/h.$$

The square-cubic case was proved above; different cubics use the quadratic cross bounds. For v times a cubic the distinct term has a path and a rectangle witness, and the equal term has a singleton trace with at most three holes, giving $5q^2/h$. Each m_I differs from its XOR trace by at most $4q$. These observations give the following bounds after centring:

one partition term	upper bound
$qm_{ij}\tau_k\tau_l$	$2q^{5/2}/h + 20q^3/h$
$q^2 m_{ij}m_{kl}$	$9q^3$
$2q^2 m_{ijk}\tau_l$	$3q^3$
$6q^3 m_{1234}$	$6q^3$

For the first row, identical supports use the preceding centred bound, with $4q^3$ for holes. Non-identical supports have singleton or two-root XOR. The latter is $-q$ plus a possible cross-half equality; on that equality the product of traces is at most $4q$, apart from a diagonal of row mass

q^2/h. This gives at most $10q^3/h$ including holes. For the second row two identical blocks give $q^2((1-qu_i)(1-qu_j)-1)$, $0 \le u_i, u_j \le 4$; one or no identical blocks give respectively $6q^3$ or $2q^3$. The third row has nonempty XOR and costs $2(1+4r)q^3$ by row and column Cauchy–Schwarz. The last row uses $|m_I| \le 1$.

The absolute word mass in Q_4 is $1+32+252=285$. Its distinct-label witness mass is 332: 4 from v^4, $2(8\cdot 2+8)=48$ from $-2v^2T^2$, and $28\cdot 2+224=280$ from $T^4-\sum t_i^4$. The same two-label witnesses as above therefore give

$$\|P_0Q_4P_0\| \le (332+3420r+47025q)q^2/h.$$

Here $3420 = 285\cdot 6\cdot 2$ and $47025 = 285(6\cdot 20+3\cdot 9+4\cdot 3+6)$.

Private overlaps. It remains to bound $\Delta = N^*N - \widetilde{G}$. Use unordered private pairs, of cardinality $n=\binom{p}{2}$; symmetry makes this equivalent to the ordered-pair operator. A universal expression, valid also at repeated roots, is obtained by setting

$$m_{\epsilon,J} = \mathbb{E}_y \prod_{j\in J}(1+\epsilon\chi(z_j-y)), \qquad F(z) = \sum_{J\subseteq[4]} (-1)^{4-|J|} m_{-,J}^2 m_{+,J}^2.$$

The true Gram is $F(z) - 8[h^3(v-q)+q] + 4v + 12q^2$. Subtracting (20.8) gives

$$\Delta_{A,A} = 210 - 1512q + 4020q^2 - 3720q^3.$$

For $A=\{u,v_0\}$, $B=\{u,w\}$ put $e=\chi(u-v_0)$, $f=\chi(u-w)$, $s=e+f$, $\eta=ef$, and $t=T_u(v_0,w)$. The other nonzero entries are

$$\begin{aligned}\Delta_{A,B} = {} & -28(2+\eta)q - 24t^2 + (594+240\eta)q^2 - (1204+528\eta)q^3 \\ & + (112+16\eta)qt^2 + s(-24q+128q^2)t.\end{aligned}$$

These formulas follow by expanding the displayed $m_{\epsilon,J}$; no distinct-root formula has been applied on an overlap.

For clarity, the one-sided operator bound can be checked with incidence matrices. On counting measure let $U(A,u) = \mathbf{1}[u\in A]$ and $W(\{u,v_0\},u) = \chi(u-v_0)$, zero otherwise. Their Grams have diagonal 2 and off-diagonal entries 1 and η on a shared endpoint. Their coefficients in Δ are $-56q+594q^2-1204q^3$ and $-28q+240q^2-528q^3$, both negative. After their diagonal terms are removed, their sum is at most $(168q-1668q^2+3464q^3)I$. Let R_u restrict to pairs containing u, and let $E_u = \operatorname{diag}_{v_0\ne u}\chi(u-v_0)$. Since $\sum_u R_u^*R_u = 2I$, the remaining trace-square and trace-linear terms are

$$\begin{aligned}&(-24+112q)(H^{(0)}-2q^2I) + 16q(H^{(1)}-2q^2I) \\ &\qquad + (-24q+128q^2)(K+4qI),\end{aligned}$$

$$H^{(0)} = \sum_u R_u^*H_uR_u, \quad H^{(1)} = \sum_u R_u^*E_uH_uE_uR_u, \quad K = \sum_u R_u^*(E_uT_u+T_uE_u)R_u.$$

By Lemma 20.18, $H^{(0)} \succeq -2(3r+q-q^2)I$, $\|H^{(1)}\| \le 2(1-2q-2q^2)$, and $\|K\| \le 4$. Adding the diagonal and incidence contributions, and dividing by n, gives the probability-normalized bound

$$\Delta \preceq \frac{q^2}{h}[420+(288-1344q)r-2336q+2912q^2+320q^3]I.$$

Using only $\|H_u\| \le 1-2q-2q^2$ instead gives $\Delta \preceq 516q^2I$ for $q \le 1/16$.

Assembly. Add the quadratic, cubic, quartic and overlap estimates, discarding only negative terms. On mean-zero private inputs this yields

$$\|NP_0\|^2 \le q^2\frac{920+4012r+48641q+3008q^2+320q^3}{h}.$$

The accepting-core restriction gives (20.7). The factor multiplying q in the first bound is increasing in q and r while $Z_4 > 0$. Exact evaluation at $(q, r) = (2^{-20}, 2^{-10})$ and $(2^{-36}, 2^{-18})$ gives respectively coefficients less than 121.588 and 121.328. The elementary overlap bound gives instead

$$\|NP_0\|^2 \le q^2 \left(516 + \frac{500 + 3724r + 48641q + 96q^2}{h}\right),$$

which proves the stated $128q$ alternative on the same domain. □

The coefficient in (20.7) may be used as N_A/q in the gap estimate below. The selected completion weights and their certificates retain the larger, previously certified rectangle prices. They remain valid upper assignments; no global saving is claimed merely by reducing this local norm bound.

Proposition 20.20 (Sharper bound for the centred rectangle, using the gap $30q - \ell_g^2$)**.** *In the setting of Proposition 20.17 put*

$$g_{\mathbf{v}} = \frac{1-2q}{1-q}, \qquad M_g = 15 + 44\mathfrak{m}_+ + (42 + 12\eta)\eta^2, \qquad \tau_g^2 = \frac{2qM_g}{(1-q)^2(1-2q)},$$
$$e_g = \frac{\sqrt{30}\, r(\eta + 15q)\sqrt{g_{\mathbf{z}}}}{(1-q)\sqrt{g_{\mathbf{v}}}}, \qquad w_g = \frac{\sqrt{30}\, r\rho_{\mathbf{zw}}}{\sqrt{g_{\mathbf{v}}}} + e_g, \qquad l_g = 2r + N_A,$$
$$x_g = 2rw_g + \tau_g N_A, \qquad \Delta_g = 30q - l_g^2.$$

If $\Delta_g > 0$, then $\|\mathsf{R}_c\| \le c_{\mathrm{gap}} := \left(\tau_g^2 + x_g^2/\Delta_g\right)^{1/2}$.

Proof. On $\mathcal{H}$, $\mathbf{v}^*\mathbf{v} = g_{\mathbf{v}} I$, so $\mathbf{v}' = \mathbf{v}/\sqrt{g_{\mathbf{v}}}$ is an isometry into the mean-zero functions of the pair. As $\mathbb{E}_z \ell(z, C)h(z) = -2r(\mathbf{u}\hat{P}h)(C)$ (proof of Proposition 20.17(b)), $\mathbf{z}h = \mathbb{E}_z \mathsf{K}(z, C)h(z)/(\sqrt{15}\, r)$ for $h \in \mathcal{H}$. Given $b_0 = y$, the label b_1 is uniform on the other $p - 1$ labels and $\sum_{y' \ne y} \mathsf{K}(y', C) = p\bar{\mathsf{K}} - \mathsf{K}(y, C)$; with $\mathsf{K}^2 = 16\mathsf{K}$ this gives the exact action

$$\mathsf{R}\mathbf{v} = \sqrt{30}\, r\, \frac{\bar{\mathsf{K}} - 16q}{1-q}\, \mathbf{z}.$$

The Gram $(\mathsf{R}\mathbf{v}')^*\mathsf{R}\mathbf{v}'$ is self-adjoint and commutes with the square-affine maps, so it is a scalar (Lemma 14.2(d)); its trace is $30q\,\mathbb{E}[(\bar{\mathsf{K}} - 16q)^2(1-q)^{-2}\,\|P_0\mathsf{K}(\cdot, C)\|^2]/(15q\, g_{\mathbf{v}})$ with $\|P_0\mathsf{K}(\cdot, C)\|^2 = 16\bar{\mathsf{K}} - \bar{\mathsf{K}}^2$, and division by $p - 1$ gives

$$(\mathsf{R}\mathbf{v}')^*\mathsf{R}\mathbf{v}' = \frac{2q\,\mathbb{E}\big[(\bar{\mathsf{K}} - 16q)^2(16\bar{\mathsf{K}} - \bar{\mathsf{K}}^2)\big]}{(1-q)^2(1-2q)} I.$$

Here $\bar{\mathsf{K}} = 16n/p = 16nq$, where $n = \#\{z : \mathsf{K}(z, C) = 16\}$; the weight vanishes at $n = 0$, and for $n \ge 1$, $0 \le \bar{\mathsf{K}} - 16q \le \bar{\mathsf{K}} \le 16$, so the expectation is at most $\mathbb{E}(16\bar{\mathsf{K}}^3 - \bar{\mathsf{K}}^4)$; expanding at $\bar{\mathsf{K}} = 1 + y$, $16\bar{\mathsf{K}}^3 - \bar{\mathsf{K}}^4 = 15 + 44y + 42y^2 + 12y^3 - y^4$, and $|y| \le \eta$, $\mathbb{E}y \le \mathfrak{m}_+$ bound it by M_g. Thus $\|\mathsf{R}\mathbf{v}'\| \le \tau_g$, and $\tau_g^2 \ge 30q$ as $M_g \ge 15$. On the orthogonal complement of the range of $\mathbf{v}'$ in the mean-zero functions of the pair, $\mathbf{v}^* = 0$, so there $\mathsf{R}_c = 2r\mathbf{w}\mathbf{e}^* + \mathsf{R}_\circ$ has norm at most l_g, as $\|\mathbf{w}\|, \|\mathbf{e}\| \le 1$. Moreover $|(\bar{\mathsf{K}} - 16q)/(1-q) - 1| \le (\eta + 15q)/(1-q)$ gives $\left\|\mathsf{R}\mathbf{v}' - \sqrt{30}\, r\mathbf{z}/\sqrt{g_{\mathbf{v}}}\right\| \le e_g$ and $\|(\mathsf{R}\mathbf{v}')^*\mathbf{w}\| \le w_g$, so the off-diagonal block of $\mathsf{R}_c^*\mathsf{R}_c$ in this decomposition has norm at most $2rw_g + \tau_g N_A = x_g$. The diagonal blocks have norms at most τ_g^2 and l_g^2, and $\tau_g^2 - l_g^2 \ge \Delta_g > 0$, so the largest eigenvalue of the scalar comparison matrix, and with it $\|\mathsf{R}_c\|^2$, is at most $\tau_g^2 + x_g^2/\Delta_g$. □

The nonstar term N_A enters c_{gap}^2 through x_g, of order qr, across the gap Δ_g of order q, and through l_g in the gap itself, at relative order r; so it changes $(c_{\mathrm{gap}}/r)^2$ by $O(q)$, whereas in $k_4 r + N_A$ it enters c/r linearly, at order r.

Proposition 20.21 (Collisions with the core–pair weights of the proof; the caps of L20)**.** *For $p \equiv 3 \pmod 4$, $p \ge 2^{20}$, put $E_{c,j} = e_j(p,12) = ((j/2-1)2^j+1)r + 12\cdot 2^j q$ $(j=2,3,4)$, $\Lambda_c = (1-e_2(p,6))^4(1-e_0(p,6))^2 = (1-r-24q)^4(1-6q)^2$, $U_c = (1+E_{c,4})^6$,*

$$P_c(q) = 96q + 3712q^2 + 81920q^3 + 1245184q^4 + 12582912q^5 + 67108864q^6,$$
$$S_c = 32q(1+E_{c,3})^2(1+E_{c,4})^3 + 64q(1+E_{c,2})(1+E_{c,3})^2(1+E_{c,4})^2,$$
$$D_{\mathrm{bal}} = \frac{S_c + U_c(P_c - 96q)}{\Lambda_c}.$$

(a) *$\|T_{\mathrm{coll}}\| \le D_{\mathrm{bal}}$ for the soft L20 kernel.*

(b) *Put $\varepsilon_Z = N_{\mathrm{ns}} + D_{\mathrm{bal}} + m_\star$, $C_{20} = (s_p c_{\mathrm{gap}}/h_p + D_{\mathrm{bal}})/(1-\varepsilon_Z)$, $H_{20} = (N_{\mathrm{ns}} + D_{\mathrm{bal}})/(1-\varepsilon_Z)$, $g_{20} = C_{20}/r$ and $c_{20}^+ = p\chi_p^+/(1-H_{20})$. If $\varepsilon_Z < 1/2$, $\Delta_g > 0$, $c_{\mathrm{gap}} < 1$, $C_{20} < 1$ and $H_{20} < 1$ at an endpoint $P \ge 2^{20}$ (that is, with $q = 1/P$ and $r = P^{-1/2}$), then $-g_{20}(P)p^{-1/2} \le \mathsf{A}_p \le 1 + c_{20}^+(P)/p$ for every prime $p \ge P$ with $p \equiv 3 \pmod 4$, and a middle block $\mathcal{B}$ of a piece starting at $X \ge P$, with least prime b, has*

$$\mathsf{A}_{\mathcal{B}} \ge -g_{20}(P)\exp\bigl(c_{20}^+(P)L_{\mathrm{loc}}/(X\log X)\bigr)\, b^{-1/2}.$$

Proof. (a) The soft kernel contains the factors $\mathsf{K}(b_l', C)$ and $\mathsf{K}(b_l, C')$ (Proposition 20.17(c)), which vanish at the core labels (Lemma 20.14(c)). So the strata of T_{coll} with a coincidence between a pair label of one half and a core label of the other contribute nothing, and in the bounds of Corollary 16.18 the weights $\alpha_{ij}(x)$ of the 16 pairs (i,j) joining a core coordinate and a pair coordinate may be replaced by zero: a stratum containing such a pair is bounded by zero, and the bounds of the other strata do not change. The remaining coincidences $x_i' = x_j$ join two core coordinates (16 choices of (i,j)) or two pair coordinates (4 choices); for these, $j \notin N(i)$ and $\mathrm{mg}(j,i) = 0$ in Lemma 16.15. Fix x and apply Corollary 16.18 with the actual incident weights $\alpha_{ij}(x) = w_i(x_j)$, where w_i is the cross factor at vertex i of that corollary. For an a-coordinate i of x' and a core coordinate j, $w_i(x_j) = \prod_{l=0,1}(1-\chi(b_l - x_j))$; for a c-coordinate, $w_i(x_j) = \prod_{l=0,1}(1+\chi(b_l - x_j))$; for a pair coordinate i and a pair coordinate j, $w_i(x_j) = \mathsf{K}(x_j, C)$. At a core label y of x the labels y, b_0, b_1 are distinct, so the two products are 0 or 4, and not both are 4. Let n_- and n_+ count the core labels of x at which the first, respectively the second, product is 4; then $n_- + n_+ \le 4$, and the row charges $\rho_i(x) = q\sum_j \alpha_{ij}(x)$ are $\rho_a = 4n_-q$ for the two a-coordinates, $\rho_c = 4n_+q$ for the two c-coordinates and $\rho_{b_l} = q(\mathsf{K}(b_0,C) + \mathsf{K}(b_1,C)) \le 32q$ for the two pair coordinates.

One coincidence. Order the other vertices so that, for a coincident core coordinate, its two neighbours on the opposite side of the core come before the other coordinate on its side, and, for a coincident pair coordinate, the core alternates between the sides. The five remaining vertices then have t_ℓ^{ij} equal to $3,3,4,4,4$, respectively $2,3,3,4,4$, and Corollary 16.18 bounds the row sum of $T_{\mathrm{coll},1}$ by Λ_c^{-1} times

$$\sum_{i\ \mathrm{core}} \rho_i(x)(1+E_{c,3})^2(1+E_{c,4})^3 + \sum_{i\ \mathrm{pair}} \rho_i(x)(1+E_{c,2})(1+E_{c,3})^2(1+E_{c,4})^2 \le S_c,$$

since the core charges add up to $8(n_- + n_+)q \le 32q$ and the pair charges to at most $64q$; here $\Lambda_c = L_{\mathrm{int}}$ and $e_t(p,2n) = E_{c,t}$.

Two or more coincidences. Every vertex has degree $d_i = 4$, so $\Upsilon_k \le (1+E_{c,4})^6 = U_c$, and the row sum of $T_{\mathrm{coll},\ge 2}$ is at most $U_c\Lambda_c^{-1}$ times the part of degree at least two of $\sum_I 2^{e(I)}\prod_{i\in I}\rho_i(x)$. The internal edges join the a-coordinates to the c-coordinates, so this sum is $F(\rho_a,\rho_c)(1+\rho_{b_0})(1+\rho_{b_1})$

with

$$F(\rho_a,\rho_c)=\sum_{k,l=0}^{2}\tbinom{2}{k}\tbinom{2}{l}2^{kl}\rho_a^k\rho_c^l$$
$$=1+2(\rho_a+\rho_c)+(\rho_a+\rho_c)^2+6\rho_a\rho_c+8(\rho_a+\rho_c)\rho_a\rho_c+16\rho_a^2\rho_c^2.$$

As a polynomial in $\rho_a+\rho_c$, $\rho_a\rho_c$, ρ_{b_0}, ρ_{b_1} its part of degree at least two has nonnegative coefficients, and $\rho_a+\rho_c\le 16q$, $\rho_a\rho_c\le(\rho_a+\rho_c)^2/4$, $\rho_{b_l}\le 32q$; so it is at most its value at $\rho_a=\rho_c=8q$, $\rho_{b_l}=32q$, which is $(1+32q)^2F(8q,8q)-1-96q=P_c-96q$. All these bounds are pointwise in x, and Corollary 16.18 turns them into the norm bound $\|T_{\mathrm{coll}}\|\le D_{\mathrm{bal}}$.

(b) *Endpoint monotonicity.* Write every quantity as a function of $r=p^{-1/2}$, with $q=r^2$, on $(0,P^{-1/2}]$. The functions N_{ns}/q and N_A/q are admissible in the sense of Lemma 4.18 (Corollaries 16.11 and 16.14), and so is D_{bal}/q: its numerator divided by q is a polynomial with nonnegative coefficients in r and the count errors, and Λ_c is a product of factors $1-\varphi$ with admissible $\varphi<1$. The functions $m_\star/q$, $\mathfrak{m}_+/q$ and χ_p^+/q are nondecreasing: written in r, the numerator of $m_\star/q$ is nondecreasing on $(0,1]$ and its denominator Z_4 is positive and nonincreasing, and $\chi_p^+/q=(D_{\mathrm{full}}/r)^2/L_{\mathrm{low}}^2$ with L_{low} positive and nonincreasing (Lemma 24.7(iii)) and D_{full}/r nondecreasing, since each term of D_{full} carries one of the factors Δ_2, q, $\mathfrak{a}_p$, and Δ_2/r, q/r and $\mathfrak{a}_p/r$ are nondecreasing. The quantities $g_{\mathbf{z}}$, $1/g_{\mathbf{v}}=1+q/(1-2q)$, $1/h_p=(1+3q/(1-4q))/(1-5q)$, ρ/q, η/r, s_p and k_4 are built from these by sums, products, square roots and quotients by positive nonincreasing denominators. Hence ε_Z, H_{20}, τ_g/r, w_g/q, $x_g/q=2r\,w_g/q+r(\tau_g/r)(N_A/q)$ and $c_{20}^+=(\chi_p^+/q)/(1-H_{20})$ are nondecreasing in r while ε_Z and H_{20} are below one; $\Delta_g/q=30-(l_g/r)^2$, with $l_g/r=2+r\,N_A/q$, is nonincreasing; and where $\Delta_g>0$ the functions $(c_{\mathrm{gap}}/r)^2=(\tau_g/r)^2+(x_g/q)^2/(\Delta_g/q)$ and $C_{20}/r=(s_p(c_{\mathrm{gap}}/r)/h_p+D_{\mathrm{bal}}/r)/(1-\varepsilon_Z)$ are nondecreasing. If the gates hold at P, they therefore hold at every prime $p\ge P$, with $g_{20}(p)\le g_{20}(P)$ and $c_{20}^+(p)\le c_{20}^+(P)$.

The caps. At such a prime, Proposition 20.20 gives $\|\mathsf{R}_c\|\le c_{\mathrm{gap}}<1$, Proposition 20.17(d) with $c_{\mathrm{R}}=c_{\mathrm{gap}}$ and $D_{\mathrm{col}}=D_{\mathrm{bal}}$ from (a) gives $T\succeq -C_{20}I$ and $\|T|_{\mathcal{I}^\perp}\|\le C_{20}$, and Proposition 20.15 with $C_\perp=C_{20}$ gives $-g_{20}(p)rI\preceq\mathsf{A}_p\preceq(1+c_{20}^+(p)/p)I$. The block bound follows from Lemma 13.7(b) with $g=g_{20}(P)$ and $c^+=c_{20}^+(P)$, where $g_{20}(P)p^{-1/2}\le C_{20}(P)<1$, and from $\sum_{p\in\mathcal{B}}1/p\le L/(X\log X)\le L_{\mathrm{loc}}/(X\log X)$ (Lemma 20.3(ii)). □

At the endpoint $p=Q_3$ (just below the middle primes) the main terms of C_{20}/r are $s_pc_{\mathrm{gap}}/(h_pr)=5.47787\ldots$, against their limit $\sqrt{30}=5.47723\ldots$ as $p\to\infty$, and $D_{\mathrm{bal}}/r=0.02176\ldots$ ($D_{\mathrm{bal}}/q=97.18\ldots$, close to its leading coefficient 96); the normalizer h_p^{-1} adds about $4.01\cdot10^{-7}$ as a relative correction. So the collision bound D_{bal} is the largest correction, and $g_{20}-\sqrt{30}$ is asymptotic to $96r$ as $p\to\infty$. The other inputs at Q_3 are $N_{\mathrm{ns}}=7.336\cdot10^{-5}$ up to the displayed digits, $N_A/q=281.37$, $k_4=5.4772264$, $c_{\mathrm{gap}}=1.226351\cdot10^{-3}$, $\chi_p^+=1.0341\cdot10^{-6}$ and $H_{20}=7.8240\cdot10^{-5}$.

Certificate 20.22 (The L20 grid)**.** At each of the 45 endpoints below, the quantities N_{ns}, N_A, D_{bal}, $m_\star$, ε_Z, k_4, s_P, h_P, c_{gap}, C_{20}, H_{20} and χ_P^+ of the L20 estimates satisfy $\varepsilon_Z<1/2$, $\Delta_g>0$, $c_{\mathrm{gap}}<1$, $C_{20}<1$ and $H_{20}<1$ in outward rational interval arithmetic, with the recorded values of g_{20} and c_{20}^+ stated below. The scripts and records are in the directory `research/advance_2026_10_24/local/`. The record `l20_refined_Q_grid.json`, built by `l20_collision_balanced.py` on the rank-four, rectangular and gap evaluators `l20_rank4.py`, `l20_rectangular.py` and `l20_rectangular_gap.py` (and on the star matrix and the frame cap of `l20_full_star.py`, which asserts the characteristic polynomial of Lemma 20.14(a) in exact symbolic arithmetic), contains 45 endpoints from 2^{20} to 2^{50}, among them the 33 piece starts. The nonstar ledger of N_A (Proposition 20.17(b)) is the histogram `middle_factor_probe/rectangular_A.hist.json.gz` of the same directory. At each endpoint P it evaluates in outward rational interval arithmetic

N_{ns}, N_A, D_{bal}, $m_\star$, ε_Z, k_4, s_P, h_P, c_{gap}, C_{20}, H_{20} and χ_P^+, checks $\varepsilon_Z < 1/2$, $\Delta_g > 0$, $c_{\mathrm{gap}} < 1$, $C_{20} < 1$ and $H_{20} < 1$, and records $g_{20}(P)$, rounded up to 10^{-12}, and $c_{20}^+(P)$. At Q_3 it gives $g_{20} = 5.500057067029$ and $c_{20}^+ < 20.633953751661$, hence a block cap below 5.500255036718 at L_{loc}; at 2^{20}, $g_{20} = 5.597878403332$; the limit is $\sqrt{30}$. These are the inherited uniform-root prices, preserved as a baseline in Table 35. The selected native reconstruction recomputes the protected collision price of Proposition 21.11 at all 33 piece starts, with the same positive coefficient and marginal data. It then replaces L20 and L21 on the seven middle pieces from 2^{30} by the prices of Certificate 21.25. The reference uses the selected value at Q_3, $g_{20} = 5.481634403216$, proved through Proposition 21.11; the larger value 5.500057067029 recorded above is the weaker bound of Proposition 20.21, retained only as a comparison. At Q_3 the selected uniform root also has $c_{20}^+ = 20.633953751661$ and block cap below 5.481831709798 at budget 585. The exact selected spectra and weights are in `research/push_2026_10_25/global/refined_L570_source.json`; `research/improvements_2026_10_25/verify.py` independently checks the collision, endpoint and native-completion interfaces, and `research/push_2026_10_25/verify.py` authenticates their unchanged identity after scalar reanchoring.

Consequences and what to read next. Theorem 4.12 (the cell charges above X_{Ch}); Lemma 12.10 (class-three counts); Lemma 17.2, Definition 12.1 and Definition 13.1 (the twelve-label graphs and their cuts); Theorem 14.21 and Corollary 14.22, with the inputs of Corollary 16.14 and Lemma 16.15 (the route-A caps); Lemma 4.18 (evaluation at the piece start and at the endpoints); Lemmas 14.18 and 14.19, Lemma 14.4, Corollary 14.5, Lemma 14.2(d), Lemma 24.5 and Lemma 24.3, Lemma 13.9, Proposition 13.6(a″), Theorem 4.7, (4.2), Corollary 16.17(a), Corollary 16.11, Corollary 16.18 and Lemma 24.7(ii),(iii) (the cap of L20); Lemma 13.3 and Lemma 13.7(b) (block caps); Lemma 9.3 and Definition 30.1 (prices and the selection factor at a). Certificates: Certificates D.4, 20.13 and 20.22.
Hypotheses verified in this section (Table 5). the inputs of the rows "(CP1) middle" and "P, V, D, U_0, tags" that do not depend on the reference: the middle partition and its charges (Definition 20.1 and Lemmas 20.4 and 20.6, Certificate D.4) and the caps of the twelve-label cuts L21, E21 and L10 and of the cut L20 of the uniform root (Lemma 20.12 and Proposition 20.21; Certificates 20.13 and 20.22); and the piece caps of the restorations (Lemma 20.7), an input of the row "(CP5) variance classes". The first two rows are completed in Section 22.

21. Middle primes II: reweighting the principal law on seven intervals

What this section proves and how to read it. On seven middle intervals from 2^{30} onward, the uniform twelve-label law has spectral bound $g \to \sqrt{30}$, which forces completion weights too large for $\Theta < 1$. This section reweights it by an explicit nonnegative function (a *tilted law*, called a *positive root* in the text) that preserves support and one-slot marginals and improves the bound to $g(\lambda) < \sqrt{30}$ (where $\lambda = 2\mathsf{u}/(1+\mathsf{u}^2)$ in terms of the tilt parameter u of Definition 21.1, and $g(\lambda)$ is the leading coefficient defined there), at the price of a larger bound at a second cut and a changed one-half marginal, both bounded here. The other 26 intervals use the uniform law of Section 20; the choice of which intervals are reweighted is part of the construction (Section 18). Inputs: Section 20 and Part IV trace estimates. Outputs: the two cut bounds and marginal controls used in Section 22. A first reading should follow the definition and the two bound statements; the pair-score calculations are reference material. On seven pieces of the middle range, from 2^{30} onward, the two-layer root is a positive reweighting of the uniform twelve-label law. Six pieces use individual squares of the three quartic pair scores; the last uses 39 coherent traces, with a positive floor. The support, invariance under the chamber reflections and layer swap of Lemma 20.11(a) and uniform one-slot marginals are preserved. We bound the chamber and layer operators in the fixed uniform induced-half charts used by Section 22, and give the actual marginal variance and

range needed by Lemma 24.7(iv). The pair-score toolkit is proved first; then come the sharper collision bound on the two protected subspaces, the interpolation between the pair bounds for the actual (physical) kernel (Corollary 21.14), the layer remainder summed without expanding all monomials (Proposition 21.15), and the finite bounds for the coherent law. The seven endpoint caps g, c^+ are in Certificate 21.25. These caps determine the completion weights w_{D} and v_{L}.

The reason for the tilt is visible in the leading matrices. The negative cap of L20 of the uniform root tends to $\sqrt{30}$, the norm of the complete star matrix of its soft kernel (Lemma 20.14), and through the completion inequality $w_{\mathrm{D}} \geq d$ of Lemma 22.1(iii), where d is the block cap of L20, this cap fixes the first-order weights $w_{\mathrm{D}} = w_{\mathrm{S}}$ of the reference. The linear part $2\mathsf{u}\bar{\mathsf{H}}$ of a tilt adds to that matrix a coupling of the pairs of opposite base labels, and the eigenvector of the least eigenvalue has a component along this coupling; the least eigenvalue rises to $-g(\lambda)$ with $g(\lambda) < \sqrt{30}$ (Lemma 21.3). This gain is at order one in the coefficient of r. The costs are of order r: the layer cut of the tilted root has a larger cap, which enters the reference only through the larger one-layer weight v_{L} it forces (Remark 22.2), and the half marginal of the root changes, which the restorations pay through its range ratio and its chi-square.

Conventions. Those of Section 20. Throughout $p \equiv 3 \pmod 4$, $q = 1/p$, $r = p^{-1/2}$, every space carries a probability measure, $\mathsf{u} \geq 0$ is rational, and P_0 is the projection onto mean-zero functions. In Proposition 21.11 the same centering P_0 is written Q, with protected subspaces $P_{\mathrm{c}}, P_{\mathrm{p}}$ defined there. For an operator T with a kernel on a product of two spaces and a function f on that product, $T \circ f$ is the operator whose kernel is the pointwise product of the kernel of T with f, as in Lemmas 13.9 and 13.22.

21.1. **The tilted law $\Gamma_{p,\mathsf{u}}$.** On seven middle intervals the uniform twelve-label law is reweighted (tilted) by an explicit sum of squares, hence by a nonnegative weight with the same support. This one-parameter family is the positive root; the definition fixes the weight and the parameter value used on each interval.

Definition 21.1 (Positive roots)**.** Let x be a labelling of the twelve vertices (u, ℓ) of $G_{\mathrm{L},2}$ at the prime p. For $j \in \{0, 1, 2\}$ let

$$\mathsf{H}_j(x) = r \sum_{z \in \mathbb{F}_p} \chi\Big(\prod_{u \in \{j, j+3\}} \prod_{\ell \in \{0,1\}} (z - x_{u,\ell}) \Big)$$

be the normalized trace score of the opposite base pair $\{j, j+3\}$, and $\bar{\mathsf{H}} = (\mathsf{H}_0 + \mathsf{H}_1 + \mathsf{H}_2)/3$. For rational $\mathsf{u} \geq 0$ the *individual-square positive root* is

$$\Gamma_{p,\mathsf{u}} = \frac{w_{\mathsf{u}}}{\mathbb{E} w_{\mathsf{u}}} \lambda_p(G_{\mathrm{L},2}), \qquad w_{\mathsf{u}} = \frac{1}{3} \sum_{j=0}^{2} (1 + \mathsf{u}\mathsf{H}_j)^2, \tag{21.1}$$

the expectation being under $\lambda_p(G_{\mathrm{L},2}) = \Gamma_{p,0}$. Thus

$$w_{\mathsf{u}} = 1 + 2\mathsf{u}\bar{\mathsf{H}} + \mathsf{u}^2 V, \qquad V = \frac{1}{3} \sum_j \mathsf{H}_j^2.$$

We put $\gamma = \mathsf{u}^2$, $s = 1 + \gamma$ and $\lambda = 2\mathsf{u}/s$, the *strength* of the tilt. On a block, $\Gamma_{\mathcal{B},\mathsf{u}} = \bigotimes_{p \in \mathcal{B}} \Gamma_{p,\mathsf{u}}$. The selected construction uses individual squares on the six successive middle pieces starting at $2^{30}, 2^{31}, 2^{32}, 2^{34}, 2^{36}, 2^{40}$, with parameters 711/4096, 1021/4096, 1455/4096, 833/1024, 3851/4096, 4049/4096, respectively. On $[2^{45}, P_3)$ it uses instead

$$\Gamma_{p,\mathsf{u}} = \frac{1/100 + (1 + \mathsf{u}H_{\mathcal{D}_{39}})^2}{\mathbb{E}[1/100 + (1 + \mathsf{u}H_{\mathcal{D}_{39}})^2]} \lambda_p(G_{\mathrm{L},2}), \qquad \mathsf{u} = 67/40,$$

where $H_{\mathcal{D}_{39}} = \sum_{S \in \mathcal{D}_{39}} c_S H_S$ with coefficients $c_S \in \{\pm 1/15\}$ and H_S the trace function of the support S; the list $\mathcal{D}_{39}$ of 39 subsets ("supports") of the twelve label positions, with the signs c_S,

is defined in Section 21.9. All expectations normalising these weights are under the uniform law $\lambda_p(G_{\mathrm{L},2})$. Below 2^{30} the principal law remains uniform.

Why squares of the individual scores, rather than the square $(1+\mathsf{u}\bar{\mathsf{H}})^2$ of their average? The two weights have the same linear term $2\mathsf{u}\bar{\mathsf{H}}$, but the square of the average also contains the mixed products $\mathsf{H}_j\mathsf{H}_k$ $(j\neq k)$, whose cut operators are harder to bound; the individual-square weight avoids them. On the last interval such products are instead included in a controlled form: the score $H_{\mathcal{D}_{39}}$ is one symmetrised list of trace functions, invariant under all chamber reflections and the layer swap, so that the same tilted law can be bounded at every sorting cut. It is called the *coherent* law for this reason, and the list of supports with their coefficients is called its *dictionary*.

The three base pairs are $\{0,3\}$, $\{1,4\}$ and $\{2,5\}$: each consists of a base of the even component $\{0,2,4\}$ of $G_{\mathrm{L},2}$ and a base of the odd component $\{1,3,5\}$, and a score is a quartic character sum over the four labels of the pair, two in each layer.

Lemma 21.2 (Properties of the positive roots)**.** *Let $\Gamma_{p,\mathsf{u}}$ be the individual-square root of* (21.1)*. The coherent root has the separate admissibility statement of Lemma 21.17.*

- *(a) On the support of $\Gamma_{p,0}$ every score lies in $r\mathbb{Z}$ and satisfies $|\mathsf{H}_j|\le 2+r$, and $w_{\mathsf{u}}>0$. Hence $\Gamma_{p,\mathsf{u}}$ is a probability law with the support of $\Gamma_{p,0}$, and it accepts the six queries.*
- *(b) The chamber reflections $u\mapsto k-u$ $(k\in\{1,3,5\})$ and the translation $u\mapsto u+2$ permute the three scores, and the layer swap $\ell\mapsto 1-\ell$ and every map $z\mapsto \alpha z+\beta$ of the labels with $\alpha\neq 0$ fix each of them. Hence $\Gamma_{p,\mathsf{u}}$, and $\Gamma_{\mathcal{B},\mathsf{u}}$ on a block, is invariant under the maps of Lemma 20.11(a) and under the square-affine maps, and every slot is uniform.*
- *(c) At the chamber cut ϑ_0, with plus half (a,b,c) on the objects $0,1,2$ and reflected half (a',b',c') on the objects $5,4,3$, the scores pair a with c', b with b' and c with a'; the reflection exchanges H_0 and H_2 and fixes H_1. At the layer cut ι each score selects in each layer one label of each of the two components of $G_{\mathrm{L},2}$, at the same bases.*

The half marginals of $\Gamma_{p,\mathsf{u}}$ differ from the uniform induced half laws at order r, as quantified in Propositions 21.7 and 21.9; all caps below are proved in those fixed uniform half charts, which are the diagonal measures of the completion.

Proof. (a) On the support all twelve labels are distinct, so the four labels of a base pair are distinct. The sum in H_j is an integer, and Theorem 4.7(b) with $t=4$ bounds its modulus by $2\sqrt{p}+1$; hence $\mathsf{H}_j\in r\mathbb{Z}$ and $|\mathsf{H}_j|\le 2+r$. Let $\mathsf{u}>0$. If $1+\mathsf{u}\mathsf{H}_j=0$ with $\mathsf{H}_j=rn$, $n\in\mathbb{Z}$, then $\sqrt{p}=-\mathsf{u}n$ would be rational; so every square in w_{u} is strictly positive on the support (and $w_0=1$), $\mathbb{E}w_{\mathsf{u}}>0$, and $\Gamma_{p,\mathsf{u}}$ has the support of $\Gamma_{p,0}$. Acceptance of a query is a property of the support, and $\Gamma_{p,0}$ accepts the six queries.

(b) H_j depends only on the four labels on the vertices (u,ℓ) with $u\in\{j,j+3\}$, symmetrically. The map $u\mapsto k-u$ sends $\{j,j+3\}$ to $\{k-j,k-j+3\}$ and $u\mapsto u+2$ sends it to $\{j+2,j+5\}$ (indices mod 6), both base pairs; the layer swap fixes the vertex set of each base pair. Under $z\mapsto\alpha z+\beta$ the substitution $z=\alpha z'+\beta$ in the defining sum multiplies the product by α^4, and $\chi(\alpha^4)=1$. The weight w_{u} is a symmetric function of $(\mathsf{H}_0,\mathsf{H}_1,\mathsf{H}_2)$, so every map that preserves $\Gamma_{p,0}$ and permutes the scores preserves w_{u} and $\Gamma_{p,\mathsf{u}}$. The maps of Lemma 20.11(a) are automorphisms of $G_{\mathrm{L},2}$ and preserve its uniform labelling law, and so do the square-affine maps, which preserve strict squareness of differences; a block law is a product. Translations act transitively on the labels, so every slot is uniform (Lemma 5.16).

(c) At ϑ_0 the reflection is $u\mapsto 5-u$, so the objects $0,1,2$ of the plus half face $5,4,3$. The base pairs $\{0,3\}$, $\{1,4\}$, $\{2,5\}$ therefore join a with c', b with b' and c with a', and $u\mapsto 5-u$ maps $\{0,3\}$ to $\{2,5\}$ and $\{1,4\}$ to itself. At ι the plus half is the layer $\ell=0$, and the base pair $\{j,j+3\}$ has one base in each component. □

21.2. **The chamber symmetry for the tilted law.** This subsection proves the cap of the chamber cut L20 of a positive root. The proof follows that of Section 20.4 for the uniform root: the soft kernel is a swapped rectangle, its leading part is a complete star matrix synthesized by the actual frame, and everything else is a remainder of order q. The tilt adds to the leading part one exact matrix term, the coupling of the pair features by the linear term $2\mathsf{u}\bar{\mathsf{H}}$ (Lemma 21.3); its square terms are, up to order q, multiples of the untilted kernel (Lemma 21.4); the linear term has an explicit remainder (Lemma 21.5); and the normalization, the half marginal and the pointwise conditional score mean are bounded in Lemma 21.6. All estimates are written so that they can be evaluated at a prime endpoint and hold for every larger prime.

The chart. At ϑ_0 the half graph of the root is $G_{\mathrm{F},2}$ (Lemma 20.11(b)), and the half reference is $\mu_p = \lambda_p(G_{\mathrm{F},2}) = \lambda_{F,p}$, the uniform law on injective tuples $x = (a_0, a_1, b_0, b_1, c_0, c_1)$ with $a_i \to c_j$. We use the notation of Section 20.4: the *core* $C = (a_0, a_1, c_0, c_1)$ and the *pair* $b = (b_0, b_1)$ of x; the independent laws μ^{core} (the uniform law of the admissible cores, injective directed copies of $K_{2,2}$) and μ^{pair} (the uniform law of the ordered distinct pairs) of Proposition 20.17; and the features V_k, W_S, $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}, \mathbf{e}$ of Lemma 20.14, the same formulas defining them on μ_p and on $\mu^{\mathrm{core}} \otimes \mu^{\mathrm{pair}}$. The law μ_p is $\mu^{\mathrm{core}} \otimes \mu^{\mathrm{pair}}$ conditioned on $b \cap C = \emptyset$, an event of probability

$$h_p = \frac{(1-4q)(1-5q)}{1-q} \;\geq\; h_p' = (1-4q)(1-5q),$$

and $J_\mu \colon L^2(\mu_p) \to L^2(\mu^{\mathrm{core}} \otimes \mu^{\mathrm{pair}})$ denotes its isometric embedding (Lemma 13.23(a)). The rectangle of Proposition 20.17 is built from $\mathsf{K}(z, C)$, $\bar{\mathsf{K}}(C) = \mathbb{E}_z \mathsf{K}(z, C)$, $\mathsf{R}(C; b) = \mathsf{K}(b_0, C)\mathsf{K}(b_1, C)$ and $\mathfrak{r} = \mathsf{R}\mathbf{1}$:

$$\mathsf{R} = \mathfrak{r} \otimes \mathbf{1} + \mathsf{R}_\star + \mathsf{R}_\circ, \qquad \mathsf{R}_\star = r\big(\sqrt{30}\,\mathbf{z}\mathbf{v}^* + 2\mathbf{w}\mathbf{e}^*\big), \qquad \|\mathsf{R}_\circ\| \leq N_A, \qquad \mathsf{R}_\circ \mathbf{1} = \mathsf{R}_\circ^* \mathbf{1} = 0$$

(20.5), with $\|\mathbf{z}\|^2 \leq g_{\mathbf{z}}$, $\|\mathbf{w}\|, \|\mathbf{v}\|, \|\mathbf{e}\| \leq 1$ (Proposition 20.16) and $\|\mathsf{R}_\star\| \leq k_4 r$ (Proposition 20.17(b)), and $\mathsf{R}^*\mathbf{1} = \mathbb{E}\mathfrak{r}$ is constant. The soft L20 kernel on $L^2(\mu^{\mathrm{core}} \otimes \mu^{\mathrm{pair}})$ is $K_{\mathrm{soft}} = T[\mathsf{R}, \mathsf{R}] = (\mathsf{R} \otimes \mathsf{R}^*)\,\mathrm{Swap}$ (20.6), with kernel $\mathsf{R}(C; b')\mathsf{R}(C'; b)$ at a pair of halves $x = (C, b)$, $x' = (C', b')$. Further

$$Z_4 = (1-q)(1-3q)(1-7q), \qquad \eta = 11r + 2q, \qquad c_0 = \frac{4}{1-q} + 2q,$$

$$\mathfrak{r}_+ = \frac{(1+\eta)^2}{1-q}, \qquad \mathfrak{r}_- = (1-\eta)(1-\eta-16q),$$

so that $\mathfrak{r}_- \leq \mathfrak{r} \leq \mathfrak{r}_+$, as $|\bar{\mathsf{K}} - 1| \leq \eta$ (Proposition 20.17(d)). We write $\mathfrak{r}_2 = s_p \geq \|\mathfrak{r}\|_2$ for the bound of Proposition 20.17(d), $\mathfrak{m}_+ \geq |\mathbb{E}\bar{\mathsf{K}} - 1|$ for that of Proposition 20.16, $\Gamma_{20} \geq \|\mathcal{F}\|^2$ for the frame cap of Lemma 20.14(b) ($\mathcal{F}$ the synthesis of the eighteen features on $L^2(\mu_p)$, in the coefficient space of Lemma 20.14), N_A, D_{bal} and ε_Z for the caps and the normalizer error of Propositions 20.17 and 20.21, $\chi_p^+ \geq \|T_0\mathbf{1} - \mathbf{1}\|^2$ for the root-half chi-square bound of Section 20.4 (Lemma 24.7(ii)), and R_p for that of Lemma 24.7(i). For a feature map Y on $\mu^{\mathrm{core}} \otimes \mu^{\mathrm{pair}}$ and the same map Y_μ on μ_p, $J_\mu^* Y = \sqrt{h_p}\, Y_\mu$ (Lemma 13.23(a)); hence a leading operator YMY'^* of the soft kernel becomes $Y_\mu M Y_\mu'^*$ under the compression $h_p^{-1} J_\mu^*(\cdot) J_\mu$, with no factor h_p^{-1}, which only norm remainders pay.

The cut operator of the uniform root on $L^2(\mu_p)$ is the operator T of Section 20.4, which we write T_0:

$$T_0 = K_0/Z_0, \qquad K_0 = K_{\mathrm{soft}}^{\mu} - T_{\mathrm{coll}}, \qquad K_{\mathrm{soft}}^{\mu} = h_p^{-1} J_\mu^* K_{\mathrm{soft}} J_\mu,$$

where K_{soft}^{μ} is the soft kernel as an operator on $L^2(\mu_p)$, T_{coll} is its restriction to the pairs of halves with a common label, a nonnegative kernel with $\|T_{\mathrm{coll}}\| \leq D_{\mathrm{bal}}$ (Proposition 20.21(a)), and $Z_0 = \langle \mathbf{1}, K_0 \mathbf{1} \rangle \geq 1 - \varepsilon_Z$ (Propositions 20.15 and 20.17); on cross-injective pairs every factor of the

soft kernel is 0 or 2, so the *raw kernel* K_0 is 2^{16} times the indicator that (x,x') join to a labelling of $G_{\mathrm{L},2}$. By Lemma 13.3(f) the cut operator of $\Gamma_{p,\mathsf{u}}$ on $L^2(\mu_p)$ is

$$T_{\mathsf{u}} = \frac{T_0 \circ w_{\mathsf{u}}}{\mathbb{E}_{\Gamma_{p,0}} w_{\mathsf{u}}} = \frac{K_0 \circ w_{\mathsf{u}}}{\langle \mathbf{1}, (K_0 \circ w_{\mathsf{u}})\mathbf{1}\rangle},$$

self-adjoint because w_{u} is invariant under the reflection (Lemma 21.2(c)).

The scores as kernels. For a pair $y=(y_0,y_1)$ of labels put $\psi_y(z)=\chi(z-y_0)\chi(z-y_1)$. By Lemma 21.2(c), at ϑ_0

$$\mathsf{H}_0(x,x') = \frac{1}{r}\mathbb{E}_z \psi_a(z)\psi_{c'}(z), \qquad \mathsf{H}_1(x,x') = \frac{1}{r}\mathbb{E}_z \psi_b(z)\psi_{b'}(z), \qquad \mathsf{H}_2(x,x') = \frac{1}{r}\mathbb{E}_z \psi_c(z)\psi_{a'}(z),$$

where a,b,c are the pairs of x and a',b',c' those of x'; so the multiplier of a score is a scalar product of rows of raw pair maps, the setting of Lemma 13.9(d), and we call $\mathsf{H}_0,\mathsf{H}_2$ the *core scores* and H_1 the *private score*. Let $\mathcal{U}_a = W_{01}/r$ and $\mathcal{U}_c = W_{45}/r$ be the normalized pair features of the core and $\mathcal{U}_b = W_{23}/r = \mathbf{e}$ that of the private pair. Since $\mathbb{E}_z\psi_y(z) = -q$ for distinct labels, W_{01} has the kernel $\psi_a(z)+q$ on mean-zero label functions, and

$$\mathsf{H}_0 = r\,\mathcal{U}_a\mathcal{U}_c^* + qr, \qquad \mathsf{H}_1 = r\,\mathcal{U}_b\mathcal{U}_b^* + qr, \qquad \mathsf{H}_2 = r\,\mathcal{U}_c\mathcal{U}_a^* + qr \tag{21.2}$$

as kernels on the half, the term qr denoting the constant kernel $qr\,\mathbf{1}\otimes\mathbf{1}$. In the coefficient space of Lemma 20.14 the three couplings are $e_{01}e_{45}^*$, $e_{23}e_{23}^*$ and $e_{45}e_{01}^*$.

Lemma 21.3 (The pair-feature matrix of a tilt)**.** *In the coefficient space of Lemma 20.14, with orthonormal coordinates e_i ($0\le i\le 5$) and e_S (the twelve supports) and $\hat P$ specialized to i (Remark 14.7), let M be the star matrix of the soft L20 kernel and, for $\lambda\ge 0$,*

$$M_\lambda = M + \tfrac{\lambda}{3}(e_{01}e_{45}^* + e_{45}e_{01}^* + e_{23}e_{23}^*).$$

Then M_λ is unitarily equivalent to the direct sum of $\begin{pmatrix} 0 & 2 & \sqrt{26} \\ 2 & \lambda/3 & 0 \\ \sqrt{26} & 0 & 0 \end{pmatrix}$, $\left(\begin{smallmatrix} 0 & 2 \\ 2 & \lambda/3 \end{smallmatrix}\right)$, $(-\lambda/3)$ and the zero matrix of size twelve; its characteristic polynomial is $x^{12}(x+\frac{\lambda}{3})(x^2-\frac{\lambda}{3}x-4)(x^3-\frac{\lambda}{3}x^2-30x+\frac{26\lambda}{3})$. Consequently, for $0\le\lambda\le 3\sqrt{26}$ the least eigenvalue of M_λ is $-g(\lambda)$, where $g(\lambda)\in(\sqrt{26},\sqrt{30}]$ is the root of

$$3g^3 + \lambda g^2 - 90g - 26\lambda = 0, \tag{21.3}$$

decreasing in λ, $g(0)=\sqrt{30}$; every other negative eigenvalue is at least $-\max(2,\lambda/3)$; and $\|M_\lambda\| \le \sqrt{30}+\lambda/3$. The same holds at $\hat P = -i$.

Proof. (1) As in the proof of Lemma 20.14(a) put $\hat{\mathbf{v}} = (e_2+e_3)/\sqrt{2}$, $\hat{\mathbf{w}} = (e_0+e_1+e_4+e_5)/2$, $\hat{\mathbf{e}} = e_{23}$, $\hat{\mathbf{u}} = (e_0+e_1-e_4-e_5)/2$ and

$$\hat{\mathbf{z}} = \frac{1}{\sqrt{15}}\Big(\sum_S \sigma_S e_S - 2i\,\hat{\mathbf{u}}\Big), \qquad \sigma_S = (-1)^{|S\cap\{0,1\}|},$$

the sum running over the eleven supports $S\subseteq\{0,1,4,5\}$ of the core stars, and σ_S being the sign of the star S in the expansion of K. The vectors $\hat{\mathbf{v}},\hat{\mathbf{z}},\hat{\mathbf{w}},\hat{\mathbf{e}}$ are orthonormal ($|\hat{\mathbf{z}}|^2 = (11+4)/15$), and a comparison of entries gives

$$M = \sqrt{30}\,(\hat{\mathbf{v}}\hat{\mathbf{z}}^* + \hat{\mathbf{z}}\hat{\mathbf{v}}^*) + 2(\hat{\mathbf{w}}\hat{\mathbf{e}}^* + \hat{\mathbf{e}}\hat{\mathbf{w}}^*).$$

(2) $\sigma_{01}=\sigma_{45}=1$, so the coordinates of $\hat{\mathbf{z}}$ along e_{01} and e_{45} are both $1/\sqrt{15}$. With $\hat{\mathbf{u}}_\pm = (e_{01}\pm e_{45})/\sqrt{2}$ this gives

$$\hat{\mathbf{z}} = \sqrt{2/15}\;\hat{\mathbf{u}}_+ + \sqrt{13/15}\;\hat{\mathbf{z}}_\perp$$

with a unit vector $\hat{\mathbf{z}}_\perp$ orthogonal to $\hat{\mathbf{v}}$, $\hat{\mathbf{w}}$, $\hat{\mathbf{e}}$, $\hat{\mathbf{u}}_+$ and $\hat{\mathbf{u}}_-$; the six vectors $\hat{\mathbf{v}},\hat{\mathbf{u}}_+,\hat{\mathbf{z}}_\perp,\hat{\mathbf{w}},\hat{\mathbf{e}},\hat{\mathbf{u}}_-$ are orthonormal.

(3) $e_{01}e_{45}^* + e_{45}e_{01}^* = \hat{\mathbf{u}}_+\hat{\mathbf{u}}_+^* - \hat{\mathbf{u}}_-\hat{\mathbf{u}}_-^*$, so the pair term is $\frac{\lambda}{3}(\hat{\mathbf{u}}_+\hat{\mathbf{u}}_+^* - \hat{\mathbf{u}}_-\hat{\mathbf{u}}_-^* + \hat{\mathbf{e}}\hat{\mathbf{e}}^*)$.
(4) As $\sqrt{30}\sqrt{2/15} = 2$ and $\sqrt{30}\sqrt{13/15} = \sqrt{26}$,

$$M_\lambda = 2(\hat{\mathbf{v}}\hat{\mathbf{u}}_+^* + \hat{\mathbf{u}}_+\hat{\mathbf{v}}^*) + \sqrt{26}\,(\hat{\mathbf{v}}\hat{\mathbf{z}}_\perp^* + \hat{\mathbf{z}}_\perp\hat{\mathbf{v}}^*) + \tfrac{\lambda}{3}\hat{\mathbf{u}}_+\hat{\mathbf{u}}_+^* + 2(\hat{\mathbf{w}}\hat{\mathbf{e}}^* + \hat{\mathbf{e}}\hat{\mathbf{w}}^*) + \tfrac{\lambda}{3}\hat{\mathbf{e}}\hat{\mathbf{e}}^* - \tfrac{\lambda}{3}\hat{\mathbf{u}}_-\hat{\mathbf{u}}_-^*.$$

In the orthonormal bases $(\hat{\mathbf{v}}, \hat{\mathbf{u}}_+, \hat{\mathbf{z}}_\perp)$, $(\hat{\mathbf{w}}, \hat{\mathbf{e}})$, $\hat{\mathbf{u}}_-$, completed by an orthonormal basis of the complement, on which M_λ vanishes, this is the displayed direct sum. Its blocks have the characteristic polynomials $x^3 - \frac{\lambda}{3}x^2 - 30x + \frac{26\lambda}{3}$, $x^2 - \frac{\lambda}{3}x - 4$ and $x + \frac{\lambda}{3}$.

(5) For $g > 0$ the value of the cubic at $x = -g$ is $-f(g)/3$ with $f(g) = 3g^3 + \lambda g^2 - 90g - 26\lambda$. Since $f(0) = -26\lambda \le 0$ and $f'' \ge 0$ on $[0, \infty)$, f has exactly one positive root $g(\lambda)$ (for $\lambda = 0$ it is $\sqrt{30}$); as $f(\sqrt{26}) = -12\sqrt{26} < 0$ and $f(\sqrt{30}) = 4\lambda \ge 0$, it lies in $(\sqrt{26}, \sqrt{30}]$. At the root $\partial_g f = 9g^2 + 2\lambda g - 90 > 0$ and $\partial_\lambda f = g^2 - 26 > 0$, so g decreases in λ. So the 3×3 block has the single negative eigenvalue $-g(\lambda)$. The 2×2 block has the negative eigenvalue $\lambda/6 - (\lambda^2/36 + 4)^{1/2} \in [-2, 0)$, and the last block $-\lambda/3$; for $\lambda \le 3\sqrt{26} < 3g(\lambda)$ both lie above $-g(\lambda)$. The 3×3 block is $B_0 + \frac{\lambda}{3}\hat{\mathbf{u}}_+\hat{\mathbf{u}}_+^*$, where B_0 has the characteristic polynomial $x^3 - 30x$, so its norm is at most $\sqrt{30} + \lambda/3$; the 2×2 block has norm $\lambda/6 + (\lambda^2/36 + 4)^{1/2} \le 2 + \lambda/3$. At $\hat{P} = -i$ the matrix is the complex conjugate (Remark 14.7), with the same spectrum. □

The orientation of the pair term matters. The scores couple the core pair a of one half with the core pair c' of the other, and the leading vector $\hat{\mathbf{z}}$ has a component along $\hat{\mathbf{u}}_+$, on which this coupling is positive. With the opposite sign on the two core couplings, $\hat{\mathbf{u}}_+$ and $\hat{\mathbf{u}}_-$ exchange their roles, the cubic becomes $3g^3 - \lambda g^2 - 90g + 26\lambda = 0$, and its root exceeds $\sqrt{30}$.

Lemma 21.4 (Square of a score on the rectangle)**.** *Suppose $k_4 r + N_A < 1$. Then for each score, on $L^2(\mu^{\mathrm{core}} \otimes \mu^{\mathrm{pair}})$,*

$$\left\|K_{\mathrm{soft}} \circ \mathsf{H}_j^2 - (1-2q)^2\, K_{\mathrm{soft}}\right\| \le 256 c_0 \mathfrak{r}_+ q, \qquad \left\|K_{\mathrm{soft}} \circ \mathsf{H}_j^2 - K_{\mathrm{soft}}\right\| \le 4q\mathfrak{r}_2^2 + 256 c_0 \mathfrak{r}_+ q.$$

Proof. Take the core score H_0; H_2 is its transpose. For a pair y of distinct labels let $\Psi_y(z, z') = \psi_y(z)\psi_y(z')$ in $L^2(\mathbb{F}_p^2)$, so that $\mathsf{H}_0^2(x, x') = q^{-1}\langle \Psi_a, \Psi_{c'}\rangle$. By Corollary 15.10(b), $\Psi_y = r(1-2q)d + \rho_y$, where $d = \sqrt{p}\,\mathbf{1}_{z=z'}$ is a unit vector, $\rho_y \perp d$ is the row of the remainder of (15.5), and the map $F \mapsto \langle \rho_y, F\rangle$ has norm at most $\sqrt{c_0}\,q$ into L^2 of a uniform distinct pair y. Under μ^{core} the pairs a and c of a core are uniform distinct pairs, so the same bound holds into $L^2(\mu^{\mathrm{core}})$. Since $\rho \perp d$, the cross terms vanish and

$$\mathsf{H}_0^2 = (1-2q)^2 + q^{-1}\langle \rho_a, \rho_{c'}\rangle, \qquad K_{\mathrm{soft}} \circ \mathsf{H}_0^2 - (1-2q)^2 K_{\mathrm{soft}} = q^{-1}\, T[\mathsf{R}, \mathsf{R}] \circ \langle \rho_a, \rho_{c'}\rangle.$$

By Lemma 13.9(d) the right side is $q^{-1}(\mathcal{Y}_a \otimes I)\,\mathrm{Swap}(\mathcal{Y}_c^* \otimes I)$, the auxiliary label being shared and the private variables exchanged, where $\mathcal{Y}_a$ has the kernel $\mathsf{R}(C; b)\rho_{a(C)}$ from $L^2(\mu^{\mathrm{pair}}) \otimes L^2(\mathbb{F}_p^2)$ to $L^2(\mu^{\mathrm{core}})$, and $\mathcal{Y}_c$ likewise. The Gram $\mathcal{Y}_a\mathcal{Y}_a^*$ is the pointwise product of the positive semidefinite kernel $\mathsf{R}\mathsf{R}^*$, whose diagonal is $\mathbb{E}_b \mathsf{R}(C; b)^2 = 256\,\mathfrak{r}(C) \le 256\,\mathfrak{r}_+$ (R takes the values 0 and 256), with the Gram of the residual pair map, which lies between 0 and $c_0 q^2$. By the positive Schur order of Lemma 13.9(b), $\|\mathcal{Y}_a\|^2 \le 256 c_0 \mathfrak{r}_+ q^2$, and likewise for $\mathcal{Y}_c$; this proves the first bound for the core scores. For the private score the pairs b and b' are the input variables of R: in the variables (b, C) the soft kernel is $T[\mathsf{R}^*, \mathsf{R}^*]$ (the last sentence of Lemma 13.9), and the same argument applies with the rows of R^*, whose square norm is the constant $256\,\mathbb{E}\mathfrak{r} \le 256\,\mathfrak{r}_+$.

For the second bound, $\mathsf{R} = \mathfrak{r} \otimes \mathbf{1} + \mathsf{R}_c$ with $\mathsf{R}_c\mathbf{1} = 0$ and $\mathsf{R}_c^*\mathfrak{r} = 0$ (Proposition 20.17(a)), so R maps $\mathbf{1}$ to $\mathfrak{r}$ and $\mathbf{1}^\perp$ into $\mathfrak{r}^\perp$, and its norm is the larger of $\|\mathfrak{r}\|_2 \le \mathfrak{r}_2$ and $\|\mathsf{R}_c\| \le k_4 r + N_A < 1$ (Proposition 20.17(b),(d)). As $\mathfrak{r}_2 = s_p \ge 1/(1-q) > 1$, $\|K_{\mathrm{soft}}\| = \|\mathsf{R}\|^2 \le \mathfrak{r}_2^2$ (Lemma 13.9(a)), and $1 - (1-2q)^2 \le 4q$. □

The estimate bounds the square of a score on the whole rectangle at once, with the coefficient $4\mathfrak{r}_2^2 + 256c_0\mathfrak{r}_+$ of q, which tends to 1028 as $p \to \infty$. Bounded class by class instead, the remainder class of the rectangle alone would give about N_A^2/q, about $8 \cdot 10^4 q$.

Linear prices. The linear part of the tilt is $K_{\mathrm{soft}} \circ \bar{\mathsf{H}}$. Its finite price uses the following constants. Put

$$d_{\mathsf{K}} = \Big(\frac{5q + 34qr + 11q^3}{Z_4}\Big)^{1/2} + 2q, \quad d_{\mathfrak{r}} = \frac{(2 + \eta + 16q)d_{\mathsf{K}} + 15q}{1 - q}, \quad \delta_{\mathfrak{r}} = \frac{d_{\mathfrak{r}}}{\sqrt{1-q}}, \quad A_{\mathfrak{r}} = 1 + \delta_{\mathfrak{r}}.$$

In the expansion of $\bar{\mathsf{K}} - 1$ the single terms vanish and the pair terms sum to exactly $2q$; the higher terms are $\mathcal{W}/p$ in the notation of Lemma 24.3 (signs $(-1,-1,1,1)$ on (a_0, a_1, c_0, c_1)), and the core law μ^{core} has the density $P_{\mathrm{adm}} D_4/Z_4$ relative to independent uniform labels (D_4 is the indicator that the four labels are distinct, and P_{adm} is 16 on the admissible cores, of total probability $Z_4/16$, and 0 on the other distinct quadruples), so $\mathbb{E}_{\mu^{\mathrm{core}}}(\mathcal{W}/p)^2 \le \mathbb{E}[P_{\mathrm{adm}}\mathcal{W}^2]/(p^2 Z_4)$. Hence Lemma 24.3(b) gives $\|\bar{\mathsf{K}} - 1\|_2 \le d_{\mathsf{K}}$, and since $(1-q)(\mathfrak{r} - 1) = (\bar{\mathsf{K}} - 1)(\bar{\mathsf{K}} + 1 - 16q) - 15q$, $\|\mathfrak{r} - 1\|_2 \le d_{\mathfrak{r}}$. For a core function f invariant under the square-affine maps and a normalized core star $\mathcal{U}_S = W_S/r$, the Gram of $f\mathcal{U}_S$ commutes with the affine group and is self-adjoint, hence scalar (Lemma 14.2(d)), and its trace gives $\|f\mathcal{U}_S\| \le \|f\|_2/\sqrt{1-q}$ (as in Lemma 14.4(a)). So multiplying a core star by $\mathfrak{r}$ changes it by at most $\delta_{\mathfrak{r}}$ and enlarges its norm by at most the factor $A_{\mathfrak{r}}$; for the private features, which are independent of the core, the change is at most $d_{\mathfrak{r}} \le \delta_{\mathfrak{r}}$.

Product maps. Fix a same-side core pair A_0 (the pair a or the pair c), with normalized pair feature $\mathcal{U}_{A_0}$. For a core support S with $2 \le |S| \le 4$ the product map $\mathcal{U}_S \times \mathcal{U}_{A_0} \colon \mathcal{H} \otimes \mathcal{H} \to L^2(\mu^{\mathrm{core}})$, $F \mapsto \mathbb{E}_{t,z}\mathcal{U}_S(\cdot;t)\mathcal{U}_{A_0}(\cdot;z)F(t,z)$ ($\mathcal{H} = L_0^2(\mathbb{F}_p)$), is bounded on three orthogonal pieces of $\mathcal{H} \otimes \mathcal{H}$: the functions vanishing on the diagonal $t = z$, the mean-zero diagonal functions $D_{\mathcal{H}} f = (P_0 \otimes P_0)\sqrt{p}\,\mathbf{1}_{t=z} f(t)$, $f \in \mathcal{H}$, whose Gram is $(1-2q)I$, and the constant diagonal vector $e_0 = (p\mathbf{1}_{t=z} - 1)/\sqrt{p-1}$ of Lemma 13.9(c). With $d = |S \triangle A_0|$, $c = |S \cap A_0|$, $u = |S \cup A_0|$, the densities $\mathfrak{d}_2 = 1/(1-q)$, $\mathfrak{d}_3 = 4/((1-q)(1-3q))$, $\mathfrak{d}_4 = 16/Z_4$ of a same-side pair, a triple of core labels consisting of one same-side pair and one further core label, and the whole core relative to independent uniform labels, and the bounds $\omega_1 = 0$, $\omega_2 = q$, $\omega_3 = 2r$, $\omega_4 = 2r + q$ for the complete character mean $|\mathbb{E}_t \psi_S(t)|$ of $1, \dots, 4$ distinct labels, put

$$\varrho_S^o = \begin{cases} 2\sqrt{\mathfrak{d}_2}, & S = A_0, \\ \sqrt{\mathfrak{d}_u}, & S \ne A_0, \end{cases} \qquad \varrho_S^d = \frac{\mathbf{1}_{d \ne 0} + cr + q + \omega_{|S|}}{\sqrt{1-2q}}, \qquad \varrho_S^c = \begin{cases} 0, & S = A_0, \\ \dfrac{\omega_d + cq + \omega_{|S|}q}{r\sqrt{1-q}}, & S \ne A_0, \end{cases}$$

$$R_S = \big((\varrho_S^o)^2 + (\varrho_S^d)^2 + (\varrho_S^c)^2\big)^{1/2}.$$

Then $\mathcal{U}_S \times \mathcal{U}_{A_0}$ has norm at most R_S, except that for $S = A_0$ the vector e_0 has the constant image $h_{A_0}\mathbf{1}$, $h_{A_0} = (1 - 2q - q^2)/(r\sqrt{1-q})$, which is set apart. Indeed, on the off-diagonal piece choose an output label in $S \setminus A_0$ and one in A_0: with the two auxiliary labels they span a matching or a three-edge path, of norm q by two Paley transforms (Lemma 16.4(a)), the other labels entering as bounded multipliers and the density of the u labels costing $\sqrt{\mathfrak{d}_u}$; for $S = A_0$ the incidence is a $K_{2,2}$, of counting norm $2p$ (Theorem 15.8), whence $2q$; the separate distinctness compressions stay in place. On $D_{\mathcal{H}}\mathcal{H}$ the image is $r^{-1}(W_{S \triangle A_0} - c \text{ holes} - w_{A_0}W_S - w_S W_{A_0})$, with $w_{A_0} = -q$, each hole (evaluation at a common label) costing q and every nonempty star having norm at most r. On e_0 the image is $(\mathbb{E}_t \psi_S \psi_{A_0} - w_S w_{A_0})/(r\sqrt{1-q})$, bounded by ϱ_S^c for $S \ne A_0$ and equal to h_{A_0} for $S = A_0$. A singleton inside A_0 times $\mathcal{U}_{A_0}$ costs $1/\sqrt{1-q}$ and one outside A_0 costs $\sqrt{\mathfrak{d}_3}$; so the products of $\mathbf{w}$, and of $\mathbf{u}$ and $\mathbf{u}\hat{P}$ ($\hat{P}$ is unitary on $\mathcal{H}$), with $\mathcal{U}_{A_0}$ have norm at most

$R_{\mathbf{w}} = 1/\sqrt{1-q} + \sqrt{\mathfrak{d}_3}$. With $\mathbf{z} = (\sum_S \sigma_S \mathcal{U}_S - 2\mathbf{u}\hat{P})/\sqrt{15}$ and $\sigma_{A_0} = 1$,

$$\mathbf{z} \times \mathcal{U}_{A_0} = h'_{\mathbf{z}}\, \mathbf{1} e_0^* + (\text{residual}), \qquad 0 \le h'_{\mathbf{z}} \le h_{\mathbf{z}} = \frac{1}{r\sqrt{15(1-q)}},$$

$$\|\text{residual}\| \le R_{\mathbf{z}} = \frac{\sum_S R_S + 2R_{\mathbf{w}}}{\sqrt{15}},$$

the sum running over the eleven core supports. For the private pair, $\mathbf{v} \times \mathbf{e}$ has norm at most $R_{\mathbf{v}} = (2/(1-q))^{1/2}$, and $\mathbf{e} \times \mathbf{e} = h'_{\mathbf{e}} \mathbf{1} e_0^* + (\text{residual})$ with residual norm at most $R_{\mathbf{e}} = (4/(1-q) + (2r+2q)^2/(1-2q))^{1/2}$ and $0 \le h'_{\mathbf{e}} \le h_{\mathbf{e}} = 1/(r\sqrt{1-q})$, by the computation for $S = A_0$, the private pair being a uniform distinct pair. As $p \to \infty$, $R_{\mathbf{z}}$ tends to about 10.14, $R_{\mathbf{w}}$ to 3, $R_{\mathbf{v}}$ to $\sqrt{2}$ and $R_{\mathbf{e}}$ to 2. The class prices are

$$\begin{aligned}
\varepsilon_{00} &= r(2\delta_{\mathfrak{r}} + \delta_{\mathfrak{r}}^2), \qquad \varepsilon_{01,c} = 2qA_{\mathfrak{r}}(\sqrt{30}R_{\mathbf{z}} + 2R_{\mathbf{w}}),\\
\varepsilon_{01,b} &= 2q\{\sqrt{30}\sqrt{g_{\mathbf{z}}}\,\mathfrak{r}_2 R_{\mathbf{v}} + 2(\mathfrak{r}_2 R_{\mathbf{e}} + h_{\mathbf{e}} d_{\mathfrak{r}})\},\\
\varepsilon_{11,c} &= r^3\{30(R_{\mathbf{z}}^2 + 2h_{\mathbf{z}}R_{\mathbf{z}} + \tfrac{h_{\mathbf{z}}^2}{p-1}) + 4\sqrt{30}(R_{\mathbf{z}} + h_{\mathbf{z}})R_{\mathbf{w}} + 4R_{\mathbf{w}}^2\},\\
\varepsilon_{11,b} &= r^3\{30 g_{\mathbf{z}} R_{\mathbf{v}}^2 + 4\sqrt{30}\sqrt{g_{\mathbf{z}}}\,R_{\mathbf{v}}(R_{\mathbf{e}} + h_{\mathbf{e}}) + 4(R_{\mathbf{e}}^2 + 2h_{\mathbf{e}}R_{\mathbf{e}} + \tfrac{h_{\mathbf{e}}^2}{p-1})\},\\
\varepsilon_{02} &= 2A_{\mathfrak{r}}N_A, \qquad \varepsilon_{12,c} = 2N_A\{\sqrt{30}(rh_{\mathbf{z}} + rR_{\mathbf{z}}) + 2rR_{\mathbf{w}}\},\\
\varepsilon_{12,b} &= 2N_A\{\sqrt{30}\sqrt{g_{\mathbf{z}}}\,rR_{\mathbf{v}} + 2(rh_{\mathbf{e}} + rR_{\mathbf{e}})\}, \qquad \varepsilon_{22} = N_A^2/r,
\end{aligned}$$

and, a core value counted twice and the private value once,

$$\varepsilon_{\text{lin}} = \varepsilon_{00} + \frac{2\varepsilon_{01,c} + \varepsilon_{01,b}}{3} + \frac{2\varepsilon_{11,c} + \varepsilon_{11,b}}{3} + \varepsilon_{02} + \frac{2\varepsilon_{12,c} + \varepsilon_{12,b}}{3} + \varepsilon_{22} + qr\mathfrak{r}_2^2,$$

$$\mathfrak{l}_0 = \frac{2\sqrt{30}\,rh_{\mathbf{z}}A_{\mathfrak{r}} + 2rh_{\mathbf{e}}\mathfrak{r}_2}{3}.$$

Lemma 21.5 (Linear root trace)**.** *Suppose $k_4 r + N_A < 1$. On $L^2(\mu^{\text{core}} \otimes \mu^{\text{pair}})$,*

$$P_0(K_{\text{soft}} \circ \bar{\mathsf{H}})P_0 = \frac{r}{3}(\mathcal{U}_a\mathcal{U}_c^* + \mathcal{U}_c\mathcal{U}_a^* + \mathcal{U}_b\mathcal{U}_b^*) + \mathcal{R}_{\text{lin}}, \qquad \|\mathcal{R}_{\text{lin}}\| \le \varepsilon_{\text{lin}},$$

the leading term being the pair term of Lemma 21.3 with coefficient $r/3$ per score, synthesized by the features. Moreover $\|P_0(K_{\text{soft}} \circ \bar{\mathsf{H}})\mathbf{1}\| \le \mathfrak{l}_0 r + \varepsilon_{\text{lin}}$ and $\|P_0(K_{\text{soft}} \circ \bar{\mathsf{H}})\| \le r(1 + 2\mathfrak{l}_0) + \varepsilon_{\text{lin}}$, and $\mathfrak{l}_0 \to (2\sqrt{2}+2)/3$.

Proof. By (21.2), $K_{\text{soft}} \circ \mathsf{H}_0 = r\, K_{\text{soft}} \circ \mathcal{U}_a\mathcal{U}_c^* + qr\, K_{\text{soft}}$, and the last term has norm at most $qr\mathfrak{r}_2^2$ (proof of Lemma 21.4); the same holds for the other scores. Expand both factors of $K_{\text{soft}} = T[\mathsf{R}, \mathsf{R}]$ by (20.5): K_{soft} is the sum of the operators $T[X, Y]$ with X, Y the rank $\mathfrak{r} \otimes \mathbf{1}$, the star $\mathsf{R}_\star$ or the remainder $\mathsf{R}_\circ$, six classes up to transposition. For a core score write the kernel of $T[X, Y] \circ \mathcal{U}_a\mathcal{U}_c^*$ at (x, x') as $\mathbb{E}_z[X(C; b')\mathcal{U}_a(C; z)]\,[Y(C'; b)\mathcal{U}_c(C'; z)]$: it factors through the shared auxiliary label z and the exchange of the private variables (Lemma 13.9(d)), and its norm is at most the product of the norms of the two factors, each a product map of the arms (core or pair feature factors) of X or Y with a pair feature. The star $\mathsf{R}_\star$ has the two arms $(\mathbf{z}, \mathbf{v})$ and $(\mathbf{w}, \mathbf{e})$ with the coefficients $\sqrt{30}\,r$ and $2r$.

Rank/rank. The factors are $\mathfrak{r}\mathcal{U}_a$ and $\mathfrak{r}\mathcal{U}_c$, so the class is $r\,(\mathfrak{r}\mathcal{U}_a)(\mathfrak{r}\mathcal{U}_c)^*$. Since $P_0(\mathfrak{r}\mathcal{U}_a) - \mathcal{U}_a = P_0((\mathfrak{r}-1)\mathcal{U}_a)$ has norm at most $\delta_{\mathfrak{r}}$, it is the leading term $r\,\mathcal{U}_a\mathcal{U}_c^*$ up to ε_{00} after centring on both sides.

Rank/star. The factors are $\mathfrak{r}\mathcal{U}_a$ (norm at most $A_{\mathfrak{r}}$) tensored with the private arm, and the product of the core arm with $\mathcal{U}_c$, with the coefficient $r \cdot r = q$ of the star and the score. By the product maps the second factor is bounded by $R_{\mathbf{z}}$ or $R_{\mathbf{w}}$ except for the large constant $h'_{\mathbf{z}}\mathbf{1}e_0^*$ of $\mathbf{z} \times \mathcal{U}_c$, which yields an operator whose input is the constant function of x': a constant-to-feature

rank, killed by P_0 on the right. The two placements give $\varepsilon_{01,c}$. For the private score the large constant is that of $\mathbf{e}\times\mathbf{e}$; its output is $\mathfrak{r}$ times a constant, and $P_0(\mathfrak{r}h)=hP_0\mathfrak{r}$ is paid by $h_{\mathbf{e}}d_{\mathfrak{r}}$ in $\varepsilon_{01,b}$.

Star/star. The coefficient is $r\cdot r\cdot r$ and the four arm pairs give the products of the bounds above. The only term with a large constant on both sides is $\mathbf{z}\times\mathcal{U}_a$ against $\mathbf{z}\times\mathcal{U}_c$ (for the private score $\mathbf{e}\times\mathbf{e}$ against itself); there the two vectors e_0 meet through the exchange of the auxiliary labels, and the entangled swap of Lemma 13.9(c) reduces the term to $I/(p-1)$. This gives $\varepsilon_{11,c}$ and $\varepsilon_{11,b}$.

Classes with the remainder. Apply $\mathsf{R}_\circ$ fibrewise in its own core and private variables, keeping the shared auxiliary label (it is not duplicated), then the free feature on the other side: rank/remainder costs $2A_{\mathfrak{r}}N_A$, star/remainder $\varepsilon_{12,c}$ or $\varepsilon_{12,b}$ (here the large constant costs $rh_{\mathbf{z}}$ or $rh_{\mathbf{e}}$), and remainder/remainder N_A^2/r, the raw pair rows having norm at most one.

Averaging the three scores, a core value twice and the private value once, gives the first assertion. The constant-to-feature ranks are explicit: at a core score the vector obtained by contracting e_0 with $\mathfrak{r}\mathcal{U}\otimes\mathbf{v}$ has norm at most $A_{\mathfrak{r}}$ and mean zero, by translation invariance and independence, with the coefficient $\sqrt{30}\,rh_{\mathbf{z}}\cdot r$; at the private score the vector from $\mathbf{w}\otimes\mathbf{e}$ has norm at most $\mathbb{E}\mathfrak{r}\le\mathfrak{r}_2$, with the coefficient $2rh_{\mathbf{e}}\cdot r$. Exactly one of the two orientations survives in $P_0(K_{\mathrm{soft}}\circ\bar{\mathsf{H}})\mathbf{1}$, which gives $\mathfrak{l}_0 r$ rather than twice that number; the other parts of $P_0(K_{\mathrm{soft}}\circ\bar{\mathsf{H}})\mathbf{1}$ are bounded by their norms. The leading pair term has norm at most r, which gives the bound for $\|P_0(K_{\mathrm{soft}}\circ\bar{\mathsf{H}})\|$. As $p\to\infty$, $A_{\mathfrak{r}},\mathfrak{r}_2\to1$ and $\mathfrak{l}_0\to(2\sqrt{30}/\sqrt{15}+2)/3$. □

Lemma 21.6 (Normalization, marginal and conditional prices)**.** *Let $p\ge2^{20}$. Put*

$$\bar{\mathfrak{r}}=\frac{1+2\mathfrak{m}_++d_{\mathsf{K}}^2}{1-q},\qquad \mathfrak{s}_b=\frac{512\,\bar{\mathfrak{r}}q^2}{1-q},\qquad \mathfrak{s}_c=\frac{32\,\mathfrak{r}_+(1+\eta)^2q^2}{Z_4},\qquad \mathfrak{s}=\frac{2\mathfrak{s}_c+\mathfrak{s}_b}{3},$$

$$c_{\mathrm{lin}}=(2+r)D_{\mathrm{bal}}+\frac{\mathfrak{s}}{h'_pr},\qquad \varepsilon_{\mathrm{lin}}^{\mathrm{raw}}=\frac{\varepsilon_{\mathrm{lin}}}{h'_p}+c_{\mathrm{lin}},$$

$$\varepsilon_{\mathrm{sq}}^{\mathrm{raw}}=\frac{4q\mathfrak{r}_2^2+256c_0\mathfrak{r}_+q}{h'_p}+((2+r)^2-1)D_{\mathrm{bal}}+\frac{\mathfrak{s}}{h'_pq},\qquad \varepsilon_{\mathrm{sq}}=\frac{\varepsilon_{\mathrm{sq}}^{\mathrm{raw}}}{1-\varepsilon_Z},$$

$$\mathfrak{l}=\frac{1}{1-\varepsilon_Z}\Big\{\frac{\mathfrak{l}_0r+\varepsilon_{\mathrm{lin}}}{\sqrt{h'_p}}+\frac{[r(1+2\mathfrak{l}_0)+\varepsilon_{\mathrm{lin}}]\sqrt{8q/(1-q)}}{h'_p}+c_{\mathrm{lin}}\Big\},$$

$$h_0=\frac{72q+37qr+2qr/(1-q)}{1-36q}.$$

(a) If $k_4r+N_A<1$, then in the fixed half chart $L^2(\mu_p)$, $\big\|K_0\circ\frac13\sum_j\mathsf{H}_j^2-K_0\big\|\le\varepsilon_{\mathrm{sq}}^{\mathrm{raw}}$ and

$$\Big\|P_0(K_0\circ\bar{\mathsf{H}})P_0-\frac{r}{3}\,\mathcal{F}\big(e_{01}e_{45}^*+e_{45}e_{01}^*+e_{23}e_{23}^*\big)\mathcal{F}^*\Big\|\le\varepsilon_{\mathrm{lin}}^{\mathrm{raw}};$$

consequently $\big\|T_0\circ\frac13\sum_j\mathsf{H}_j^2-T_0\big\|\le\varepsilon_{\mathrm{sq}}$ and $\|P_0(T_0\circ\bar{\mathsf{H}})\mathbf{1}\|\le\mathfrak{l}$.

(b) For each score, $|\mathbb{E}_{\Gamma_{p,0}}\mathsf{H}_j|\le h_0$ and $\mathbb{E}_{\Gamma_{p,0}}\mathsf{H}_j^2\ge1-4q-36q(2+r)^2-2q/(1-q)$; in particular $|\mathbb{E}_{\Gamma_{p,0}}\bar{\mathsf{H}}|\le h_0$.

(c) Put $\Delta_{\mathrm{core}}=(1-q)^4-16r-144q-512qr-512q^2$, positive for $p\ge2^{20}$, and

$$c_{\mathrm{core}}=\frac{148q+616qr+580q^2}{r\Delta_{\mathrm{core}}}+2r,\qquad \varphi_b=\frac{24r+8q}{1-\eta},\qquad d_b=\frac{16q}{1-\eta-16q},$$

$$c_b=\frac{4q+(1+d_b)\varphi_b^2+d_b}{r},\qquad c_{\mathrm{avg}}=\frac{2c_{\mathrm{core}}+c_b}{3},\qquad I_{\mathrm{in}}=\frac{128\,q\,\mathfrak{r}_+\bar{\mathfrak{r}}}{1-\eta},\qquad \rho_{\mathrm{in}}=\mathfrak{r}_-^2-I_{\mathrm{in}}-D_{\mathrm{bal}}.$$

If $\rho_{\mathrm{in}} > 0$, then for every half x

$$\left|\mathbb{E}_{\Gamma_{p,0}}[\bar{\mathsf{H}} \mid x]\right| \le c_{\mathrm{cond}} := \frac{\mathfrak{r}_+\bar{\mathfrak{r}}\, c_{\mathrm{avg}} + (2+r)(I_{\mathrm{in}} + D_{\mathrm{bal}}) + \mathfrak{s}/r}{\rho_{\mathrm{in}}}.$$

Proof. (a) *Spikes.* $\bar{\mathfrak{r}}$ bounds $\mathbb{E}\mathfrak{r} \le \mathbb{E}\bar{\mathsf{K}}^2/(1-q)$, as $\mathbb{E}\bar{\mathsf{K}}^2 \le 1 + 2\mathfrak{m}_+ + d_{\mathsf{K}}^2$. When the two halves share a label, the only large values of a score occur when the two selected pairs are equal as sets; there $\mathsf{H}_j = (1-2q)/r$, and otherwise $|\mathsf{H}_j| \le 2 + r$. For the private score, $\{b'\} = \{b\}$ has probability $2q^2/(1-q)$ and $\mathsf{R} \le 256$; integrating the core of the other half gives the column mass $\mathbb{E}\mathfrak{r} \le \bar{\mathfrak{r}}$, so the rows and columns of the soft kernel on this event are at most $\mathfrak{s}_b$. For a core score, $\{c'\} = \{a\}$ has the same probability; given it, the labels a' are a uniform ordered distinct pair among the $(p-3)/4$ common predecessors of the pinned pair (Lemma 12.10(a)), and both must lie in the set of labels that extend (b, c') cyclically, of at most $p(1+\eta)/16$ points, so the conditional mean of $\mathsf{R}(C'; b)$ is at most $16(1+\eta)^2/((1-3q)(1-7q))$; integrating the pair of the other half gives $\mathfrak{r}(C) \le \mathfrak{r}_+$, and this gives $\mathfrak{s}_c$. Away from these events a score has $|\mathsf{H}_j| \le 2 + r$, as a single repeated label leaves a character sum of degree two, so $|\mathsf{H}_j^2 - 1| \le (2+r)^2 - 1$, and the nonnegative kernel T_{coll} contributes in the displayed directions. The spike prices are row and column bounds of nonnegative kernels (Schur's test), compressed with the factor $1/h_p \le 1/h'_p$.

Squares. For $V = \frac{1}{3}\sum_j \mathsf{H}_j^2$, $K_0 \circ V - K_0 = h_p^{-1} J_\mu^*(K_{\mathrm{soft}} \circ V - K_{\mathrm{soft}})J_\mu - T_{\mathrm{coll}} \circ (V - 1)$, and Lemmas 13.23 and 21.4(b) bound the first term by $(4q\mathfrak{r}_2^2 + 256 c_0 \mathfrak{r}_+ q)/h'_p$; with the collision bounds this is $\varepsilon_{\mathrm{sq}}^{\mathrm{raw}}$, and division by $Z_0 \ge 1 - \varepsilon_Z$ gives $\varepsilon_{\mathrm{sq}}$.

Linear term. By Lemma 13.23(a) the embedding maps centred functions to centred functions and $J_\mu^*\mathbf{1} = \sqrt{h_p}\mathbf{1}$, so $P_0(K_{\mathrm{soft}}^{\mu} \circ \bar{\mathsf{H}})P_0 = h_p^{-1} P_0 J_\mu^*[P_0(K_{\mathrm{soft}} \circ \bar{\mathsf{H}})P_0] J_\mu P_0$. By Lemma 21.5 and the identity $J_\mu^* Y = \sqrt{h_p} Y_\mu$ for the features, the leading term becomes $\frac{r}{3}\mathcal{F}(e_{01}e_{45}^* + e_{45}e_{01}^* + e_{23}e_{23}^*)\mathcal{F}^*$ and the remainder has norm at most $\varepsilon_{\mathrm{lin}}/h'_p$; the collision part $T_{\mathrm{coll}} \circ \bar{\mathsf{H}}$ has norm at most c_{lin}. For the marginal, Lemma 13.23(b) with both events of probability h_p gives

$$\|P_0(K_{\mathrm{soft}}^{\mu} \circ \bar{\mathsf{H}})\mathbf{1}\| \le \frac{\|P_0(K_{\mathrm{soft}} \circ \bar{\mathsf{H}})\mathbf{1}\|}{\sqrt{h_p}} + \|P_0(K_{\mathrm{soft}} \circ \bar{\mathsf{H}})P_0\| \, \frac{(1-h_p)^{1/2}}{h_p},$$

where $1 - h_p = (8q - 20q^2)/(1-q) \le 8q/(1-q)$ and Lemma 21.5 bounds the two norms. Adding c_{lin} and dividing by Z_0 gives $\mathfrak{l}$.

(b) Before global injectivity the root is the product of the uniform laws of its two components, each a cyclic $K_{2,2,2}$ on the bases of one parity (the parts being the two layer labels at each of the three bases, with all edges directed cyclically to the next base), and each score pairs the two labels of one base of one component with those of a base of the other. Each such pair is a uniform ordered distinct pair: square-affine maps act transitively on each of the two orbits of ordered distinct pairs and the layer swap exchanges them. Hence, with $\mathbb{E}\psi_y(z) = -q$, $\mathbb{E}\mathsf{H}_j = r^{-1}\mathbb{E}_z(\mathbb{E}\psi_y(z))^2 = qr$, and

$$\mathbb{E}\mathsf{H}_j^2 = q^{-1}\mathbb{E}_{z,z'}\big(\mathbb{E}_y \psi_y(z)\psi_y(z')\big)^2 = (1-2q)^2 + \frac{q(1-3q)^2}{1-q} \;\ge\; 1 - 4q,$$

as $\mathbb{E}_y\psi_y(z)^2 = 1 - 2q$ and $\mathbb{E}_y\psi_y(z)\psi_y(z') = -q(1-3q)/(1-q)$ for $z \ne z'$. The thirty-six coincidences between the labels of the two components have total probability at most $36q$; on them $|\mathsf{H}_j| \le 2 + r$, except on the events where the two selected pairs are equal as sets, of probability at most $2q^2/(1-q)$, where $|\mathsf{H}_j| \le 1/r$. Conditioning on injectivity therefore gives $|\mathbb{E}_{\Gamma_{p,0}}\mathsf{H}_j| \le (qr + 36q(2+r) + 2qr/(1-q))/(1-36q) = h_0$ and $\mathbb{E}_{\Gamma_{p,0}}\mathsf{H}_j^2 \ge \mathbb{E}\mathsf{H}_j^2 - 36q(2+r)^2 - 2q(1-2q)^2/(1-q)$, whence the bounds.

(c) Fix $x = (C, b)$. On $\mu^{\mathrm{core}} \otimes \mu^{\mathrm{pair}}$ the soft conditional law of $x' = (C', b')$ given x is proportional to $\mathsf{R}(C; b')\mathsf{R}(C'; b)$; it is the product of a law of C' that depends on b and a law of b' that depends on C, and $\mathbb{E}_{x'} K_{\mathrm{soft}}(x, x') = \mathfrak{r}(C)\mathbb{E}\mathfrak{r}$.

A core score. Take H_0, which tests the pair c' of C': $\mathbb{E}[\mathsf{H}_0 \mid x] = r^{-1}\mathbb{E}_z\psi_a(z)\,\mathbb{E}[\psi_{c'}(z) \mid b]$. In independent uniform labels (a'_0, a'_1, c'_0, c'_1) the law of C' given b has a density proportional to the product of the four internal edge factors $1+\chi(c'_j - a'_i)$, of the vertex weights $w_A(a'_i)$ and $w_C(c'_j)$, with $w_A(y) = (1-\chi(b_0-y))(1-\chi(b_1-y))$ and $w_C(y) = (1+\chi(b_0-y))(1+\chi(b_1-y))$, and of the indicator that the four labels are distinct. Each weight has mean $1-q$, L^2 norm at most 2 and supremum at most 4; testing a vertex by $\chi(z-y)$, for $z \notin \{b_0, b_1\}$, gives a weight of mean at most $2r+2q$ in modulus (two quadratic character sums contribute q each and the cubic one $2r$), with the same norm bounds. Expand the four internal edge factors into the 16 subsets of edges. A single edge, a Paley operator of norm r between two weights of norm 2, costs $4r$; each of the two matchings costs $16q$; each of the four adjacent two-edge paths costs $16q$ (integrate the two leaf Paley transforms first and use the supremum of the internal weight); each of the four three-edge paths costs $64qr$; and the cycle costs $16q$, as a product of two operators $M_{w_1}PM_{w_2}P$, each of Hilbert–Schmidt norm at most $4r$. The distinctness costs only the same-side coincidences: $w_A(y)w_C(y) = (1-\chi(b_0-y)^2)(1-\chi(b_1-y)^2) = 0$ kills the four cross-side ones, and on a same-side coincidence $w^2 \le 4w$ and every doubled edge factor is at most twice the original, so its mass is at most $16q\,\mathbb{E}_a[w_A(a)(\mathbb{E}_c w_C(c)(1+\chi(c-a)))^2] \le 16q(1+8r+16q)$; the two such events cost $C_{\mathrm{coll}} = 32q + 256qr + 512q^2$. Without tests this bounds the normalizing mass below by $(1-q)^4 - 16r - 112q - 256qr - C_{\mathrm{coll}} = \Delta_{\mathrm{core}}$. With the two vertices of c' tested, the empty graph gives m^2, the four single edges $16rm$, the matchings $32q$, the two paths centred on the untested side $32q$, the two paths centred at a tested vertex $32qm$, the three-edge paths $256qr$, the cycle $16q$ and the coincidences C_{coll}; the sum is

$$m^2 + 16rm + 80q + 32qm + 256qr + C_{\mathrm{coll}} = 148q + 616qr + 580q^2.$$

Hence $|\mathbb{E}[\psi_{c'}(z) \mid b]| \le (148q + 616qr + 580q^2)/\Delta_{\mathrm{core}}$ for $z \notin \{b_0, b_1\}$, and the two exceptional labels cost $2q$, which gives the soft bound c_{core}. The lower bound Δ_{core} increases with p and is positive at $p = 2^{20}$, hence for $p \ge 2^{20}$.

The private score. $\mathbb{E}[\mathsf{H}_1 \mid x] = r^{-1}\mathbb{E}_z\psi_b(z)\mathbb{E}[\psi_{b'}(z) \mid C]$, and given C the pair b' is a uniform ordered distinct pair from the set L_C of cyclic extensions of C, of $n = p\bar{\mathsf{K}}(C)/16 \ge p(1-\eta)/16$ points. For $z \notin C$, expanding $\mathsf{K}(t, C)\chi(z-t)$ in t gives $|\mathbb{E}_{t\in L_C}\chi(z-t)| \le \varphi_b$: the four terms with one further character cost q each, the six cubic terms $2r$ each, the four quartic terms $2r+q$ each (Theorem 4.7(b)) and the quintic term $4r$, and the mass is $\bar{\mathsf{K}} \ge 1-\eta$. Sampling without replacement gives $\mathbb{E}\chi(z-b'_0)\chi(z-b'_1) = (n\gamma_C^2 - \mathbb{E}_{t\in L_C}\chi(z-t)^2)/(n-1)$ with $\gamma_C = \mathbb{E}_{t\in L_C}\chi(z-t)$, of modulus at most $(1+d_b)\varphi_b^2 + d_b$, as $1/(n-1) \le d_b$; the four labels $z \in C$ cost $4q$. This gives the soft bound c_b, and c_{avg} for $\bar{\mathsf{H}}$.

The row of the actual kernel. ("Physical" in this proof means: of the actual kernel K_0, as opposed to the soft kernel K_{soft}.) The soft row of $K_{\mathrm{soft}} \circ \bar{\mathsf{H}}$ at x is at most $c_{\mathrm{avg}}\,\mathfrak{r}(C)\mathbb{E}\mathfrak{r} \le \mathfrak{r}_+\bar{\mathfrak{r}}\,c_{\mathrm{avg}}$ in modulus, and the soft row of K_{soft} is at least $\mathfrak{r}_-^2$. Passing to the physical conditional law removes, first, the eight coincidences of b' with the core C', which are excluded from the actual half: given C the pair b' is uniform in L_C, so each costs at most $16q/(1-\eta)$ of the row mass $\mathfrak{r}(C)\mathbb{E}\mathfrak{r}$, together I_{in}; second, the coincidences across the cut, of row mass at most D_{bal} (Proposition 20.21(a), the compression factor $h_p \le 1$ being used in the conservative direction); third, the equal-pair spikes, of score at most $1/r$ and row mass at most $\mathfrak{s}$. On the first two kinds $|\bar{\mathsf{H}}| \le 2 + r$. The ratio of the remaining numerator and denominator is at most c_{cond}. □

The conditional bound c_{cond} is about $299r$; it is used only for the range of the tilted marginal, by Jensen's inequality in Lemma 13.22, while the much smaller L^2 bound $\mathfrak{l}$ prices its variance.

Proposition 21.7 (The chamber cut of a positive root)**.** *Let $p \ge 2^{30}$, suppose $k_4 r + N_A < 1$ and $\rho_{\mathrm{in}} > 0$, and let $\Gamma_{p,\mathsf{u}}$ be the individual-square root, with $s = 1+\mathsf{u}^2$, $\lambda = 2\mathsf{u}/s$, and $g = g(\lambda)$ of*

(21.3). *Put*

$$\varepsilon_0 = \frac{2rd_{\mathfrak{r}}(\sqrt{30}\sqrt{g_{\mathbf{z}}}+2) + k_4^2 q + 2\mathfrak{r}_2 N_A + 2k_4 r N_A + N_A^2}{h'_p} + D_{\mathrm{bal}}, \qquad \varepsilon_{\mathsf{u}} = s\varepsilon_0 + 2\mathsf{u}\varepsilon_{\mathrm{lin}}^{\mathrm{raw}} + \mathsf{u}^2\varepsilon_{\mathrm{sq}}^{\mathrm{raw}},$$

$$z_- = s - 2\mathsf{u}h_0 - \mathsf{u}^2\varepsilon_{\mathrm{sq}}, \qquad z_{\mathrm{raw}} = (1-\varepsilon_Z)z_-, \qquad g_{\mathrm{ch}} = \frac{sg\, r\Gamma_{20} + \varepsilon_{\mathsf{u}}}{z_{\mathrm{raw}}},$$

$$c_{\mathrm{ch}} = \frac{s(\sqrt{30}+\lambda/3)r\Gamma_{20} + \varepsilon_{\mathsf{u}} + sd_{\mathfrak{r}}^2/h'_p}{z_{\mathrm{raw}}}, \qquad d_{\mathsf{u}} = \frac{s\sqrt{\chi_p^+} + 2\mathsf{u}\mathfrak{l} + \mathsf{u}^2\varepsilon_{\mathrm{sq}}}{z_-}.$$

If $z_- > 0$, $\mathsf{u}c_{\mathrm{cond}} < 1$, $c_{\mathrm{ch}} < 1$ and $d_{\mathsf{u}} < 1$, the cut operator T_{u} of $\Gamma_{p,\mathsf{u}}$ at ϑ_0 in the fixed uniform actual half chart $L^2(\mu_p)$ satisfies

$$-\Big(g_{\mathrm{ch}} + \frac{d_{\mathsf{u}}^2}{1+g_{\mathrm{ch}}}\Big)I \preceq T_{\mathsf{u}} \preceq \Big(1 + \frac{d_{\mathsf{u}}^2}{1-c_{\mathrm{ch}}}\Big)I, \qquad \|T_{\mathsf{u}}\mathbf{1} - \mathbf{1}\|^2 \le d_{\mathsf{u}}^2,$$

$$\frac{\sup T_{\mathsf{u}}\mathbf{1}}{\inf T_{\mathsf{u}}\mathbf{1}} \le R_p \frac{(1+\mathsf{u}(2+r))^2}{(1-\mathsf{u}c_{\mathrm{cond}})^2}.$$

With $g_{20}^{\mathsf{u}} = (g_{\mathrm{ch}} + d_{\mathsf{u}}^2/(1+g_{\mathrm{ch}}))/r$ and $c_{20}^{+,\mathsf{u}} = d_{\mathsf{u}}^2/(q(1-c_{\mathrm{ch}}))$ evaluated at an endpoint P, the same bounds, $-g_{20}^{\mathsf{u}}p^{-1/2} \le T_{\mathsf{u}} \le 1 + c_{20}^{+,\mathsf{u}}/p$, the variance d_{u}^2 and the range ratio hold for every prime $p \ge P$ with the same u. The block caps follow from Lemma 13.7(b): a middle block $\mathcal{B}$ of a piece starting at $X \ge P$, with least prime b and $\sum_{p\in\mathcal{B}} \log p \le L_{\mathrm{loc}}$, has $T_{\mathcal{B},\mathsf{u}} \succeq -g_{20}^{\mathsf{u}} \exp(c_{20}^{+,\mathsf{u}} L_{\mathrm{loc}}/(X\log X)) b^{-1/2}$ if $g_{20}^{\mathsf{u}} X^{-1/2} < 1$.

Proof. The base kernel. Expand $K_{\mathrm{soft}} = T[\mathsf{R},\mathsf{R}]$ by (20.5) and keep the positive rank: $T[\mathfrak{r}\otimes\mathbf{1}, \mathfrak{r}\otimes\mathbf{1}]$ is the rank-one operator $(\mathfrak{r}\otimes\mathbf{1})\langle\mathfrak{r}\otimes\mathbf{1},\cdot\rangle$. The two rank/star classes are the star part $T_\star = r\{\sqrt{30}(\mathbf{v}\mathbf{z}^* + \mathbf{z}\mathbf{v}^*) + 2(\mathbf{w}\mathbf{e}^* + \mathbf{e}\mathbf{w}^*)\}$ of (20.3), up to the factor $\mathfrak{r}$ on one side, which costs $2rd_{\mathfrak{r}}(\sqrt{30}\sqrt{g_{\mathbf{z}}}+2)$; star/star costs $k_4^2 q$, rank/remainder $2\mathfrak{r}_2 N_A$, star/remainder $2k_4 r N_A$ and remainder/remainder N_A^2, as $\|T[X,Y]\| \le \|X\|\|Y\|$. The compression to μ_p (Lemma 13.23) and the collisions give, in the coefficient space,

$$P_0 K_0 P_0 = r\,\mathcal{F}M\mathcal{F}^* + P_0\mathfrak{r}_\mu\mathfrak{r}_\mu^* P_0 + \mathcal{R}_0, \qquad \|\mathcal{R}_0\| \le \varepsilon_0,$$

where $\mathfrak{r}_\mu$ is the function $\mathfrak{r}(C)$ on the actual half; the centred rank $P_0\mathfrak{r}_\mu\mathfrak{r}_\mu^* P_0$ is positive semidefinite, of norm at most $\|\mathfrak{r}-1\|_2^2/h_p \le d_{\mathfrak{r}}^2/h'_p$, and enters only the absolute norm.

The tilt. By Definition 21.1, $K_0 \circ w_{\mathsf{u}} = K_0 + 2\mathsf{u}\,K_0\circ\bar{\mathsf{H}} + \mathsf{u}^2\,K_0\circ V$ with $V = \frac{1}{3}\sum_j \mathsf{H}_j^2$. By Lemma 21.6(a), $K_0\circ V = K_0 + \mathcal{R}_V$ with $\|\mathcal{R}_V\| \le \varepsilon_{\mathrm{sq}}^{\mathrm{raw}}$, and $P_0(K_0\circ\bar{\mathsf{H}})P_0$ is the pair term up to $\varepsilon_{\mathrm{lin}}^{\mathrm{raw}}$. As $1+\mathsf{u}^2 = s$ and $2\mathsf{u}/s = \lambda$,

$$P_0(K_0\circ w_{\mathsf{u}})P_0 = s\,r\,\mathcal{F}M_\lambda\mathcal{F}^* + sP_0\mathfrak{r}_\mu\mathfrak{r}_\mu^*P_0 + \mathcal{R}_{\mathsf{u}}, \qquad \|\mathcal{R}_{\mathsf{u}}\| \le \varepsilon_{\mathsf{u}}.$$

By Lemma 21.3, applicable since $\lambda \le 1$, $M_\lambda \succeq -gI$ and $\|M_\lambda\| \le \sqrt{30} + \lambda/3$, so $\mathcal{F}M_\lambda\mathcal{F}^* \succeq -g\Gamma_{20}$ and $\|\mathcal{F}M_\lambda\mathcal{F}^*\| \le (\sqrt{30}+\lambda/3)\Gamma_{20}$, the features mapping into mean-zero functions.

Mass, marginal and range. Apply Lemma 13.22 to the kernel of T_0, which has mass one, $\|T_0\mathbf{1}-\mathbf{1}\| \le (\chi_p^+)^{1/2}$ (Lemma 24.7(ii)) and range ratio $\sup T_0\mathbf{1}/\inf T_0\mathbf{1} \le R_p$: by (24.6) the density is a ratio of completion counts between two bounds whose quotient is R_p. Take the scores $\eta_j = \mathsf{H}_j$, which the transposition permutes (Lemma 21.2(c)), and the constants

$$(c_V, \delta_0, e, \ell, h, c, b) = (1, (\chi_p^+)^{1/2}, \varepsilon_{\mathrm{sq}}, \mathfrak{l}, h_0, c_{\mathrm{cond}}, 2+r)$$

of Lemma 21.6 and Lemma 21.2(a). It gives $\mathbb{E}_{\Gamma_{p,0}} w_{\mathsf{u}} \ge z_-$, $\|T_{\mathsf{u}}\mathbf{1}-\mathbf{1}\| \le d_{\mathsf{u}}$ and the range ratio. Since the raw mass of $K_0\circ w_{\mathsf{u}}$ is $Z_0\,\mathbb{E}_{\Gamma_{p,0}} w_{\mathsf{u}} \ge z_{\mathrm{raw}}$, $P_0T_{\mathsf{u}}P_0 \succeq -g_{\mathrm{ch}}P_0$ and $\|P_0T_{\mathsf{u}}P_0\| \le c_{\mathrm{ch}}$.

The constant block. $T_{\mathfrak{u}}$ is self-adjoint with $\langle \mathbf{1}, T_{\mathfrak{u}}\mathbf{1}\rangle = 1$ and $\|P_0 T_{\mathfrak{u}}\mathbf{1}\| = \|T_{\mathfrak{u}}\mathbf{1} - \mathbf{1}\| \le d_{\mathfrak{u}}$, so Proposition 13.6(a) and (a′) give the two full bounds; no alignment of spectral directions is assumed.

Uniformity. The quantities $\mathfrak{r}_-$, h'_p, the square term $4q\mathfrak{r}_2^2$ and the conditional ratio were coarsened so that every normalized remainder has a positive increasing numerator and a positive decreasing denominator in q, r; the retained N_A/q, D_{bal}/q, frame cap, normalizer error, χ_p^+ and R_p have this property (Proposition 20.21, Lemma 24.7). Thus $G = g_{\mathrm{ch}}/r$, c_{ch}/r and $D = d_{\mathfrak{u}}/r$ increase toward a smaller prime endpoint. The negative coefficient is $G + D^2 r/(1+Gr)$; its derivative in G is $1 - D^2r^2/(1+Gr)^2 > 0$, as $Dr < 1$, and its derivatives in D and r are nonnegative (Lemma 4.18(ii)). The positive coefficient $D^2/(1-(c_{\mathrm{ch}}/r)\,r)$, the variance divided by q and the range multiplier increase as well on the gated domain, so the values at P bound every prime $p \ge P$. For the block caps, the primes of $\mathcal{B}$ satisfy the prime bounds and $\sum_{p\in\mathcal{B}} 1/p \le L_{\mathrm{loc}}/(X\log X)$ (Lemma 20.3), the cut operator of $\Gamma_{\mathcal{B},\mathfrak{u}}$ is the tensor product of the prime operators (Lemma 13.3(f)), and Lemma 13.7(b) applies. □

21.3. **The layer-flip symmetry ι for the tilted law.** This subsection bounds the reserved cut ι of a positive root, the cut L21. Here the scores are positive Gram kernels, so the tilt costs only at order r; the bounds are proved on the fixed uniform induced layer half, the half law of the one-layer root, which is not tilted.

The two components. At ι the half graph of $G_{\mathrm{L},2}$ is $G_{\mathrm{L},1}$, two directed triangles on the bases $\{0,2,4\}$ and $\{1,5,3\}$ (Lemma 20.11(c)). Let $\mu_\triangle$ be the uniform law of a directed cyclic triangle $x_0 \to x_1 \to x_2 \to x_0$ of $\mathbb{F}_p$, let $\nu_\triangle = \mu_\triangle \otimes \mu_\triangle$ (one triangle for each component), and let $\nu = \lambda_p(G_{\mathrm{L},1})$ be $\nu_\triangle$ conditioned on its six labels being distinct, an event of probability $h \ge h_- = 1 - 9q$. In a component, with the labels x of the layer 0 and x' of the layer 1 indexed along the triangle, the cross edges are $x_i \to x'_{i+1}$ and $x'_i \to x_{i+1}$: a six-cycle without an edge between the two labels of one base. The soft kernel of a component is

$$\Phi(x,x') = \prod_{i\ne j}\bigl(1 + (D_\triangle)_{ij}\chi(x'_j - x_i)\bigr), \qquad N_\Phi = \Phi - 1,$$

$D_\triangle$ being the skew adjacency matrix of the directed triangle ($(D_\triangle)_{i,i+1} = 1$, $(D_\triangle)_{i+1,i} = -1$ with indices mod 3, and 0 on the diagonal), and the soft layer kernel on $\nu_\triangle \otimes \nu_\triangle$ is $\Phi \otimes \Phi$. A score H_j multiplies the kernel by $r^{-1}\langle f_j(x), f_j(x')\rangle$, $f_j(x)(z) = \chi(z - x_j)\chi(z - x_{j+3})$, the product of the raw pair rows of the two selected labels of a half, one from each component. Put

$$d_2 = \frac{2}{1-q}, \quad d_\mu = \frac{8}{1-q^2}, \quad \kappa = \frac{\sqrt{3}+\sqrt{7}}{2},$$
$$\mathfrak{b} = 6 + 15d_2 + 18\sqrt{d_2 d_\mu} + 4d_\mu + 14 d_\mu r, \quad \mathfrak{n} = \kappa(1+5r)r + \mathfrak{b}q, \quad M_\Phi = d_\mu(1+q)^3,$$
$$Z_- = (1-\mathfrak{b}q)^2 - q\bigl[18M_\Phi^2 + 384 M_\Phi + 72 d_\mu^2 (1+q)^5\bigr],$$

and let $p \ge 2^{30}$. The following facts hold for every score.

(L1) $\|N_\Phi\| \le \mathfrak{n}$ and $\|N_\Phi \mathbf{1}\| \le \mathfrak{b}q$ on $L^2(\mu_\triangle)$. Indeed the six single edges and the six two-edge stars have the coefficient matrix $\left(\begin{smallmatrix} iD_\triangle & -I_3 \\ -I_3 & 0\end{smallmatrix}\right)$ (Corollary 14.5), whose characteristic polynomial is $(z^2-1)(z^4 - 5z^2 + 1)$ and whose norm is κ. The six features (the coordinate isometries V_i and the pair stars $W_{\{j,k\}}/r$) have actual Gram at most $(1+5r)I$ for $p \ge 2^{20}$ (Lemma 14.4): the off-diagonal norms are at most $r\sqrt{1+q}/(1-q)$ (singleton/singleton), $2q/(1-q)$ (incident singleton/pair), exactly r (opposite singleton/pair: condition on that coordinate and negate while swapping the other two) and $(3q + \sqrt{3}r)/(1-q)$ (pair/pair), with row sums at most $5r$. The uncentred stars leave $+6q$ times the constant kernel, each pair mean being $-q$ and each star coefficient -1. The remaining 51 masks can be priced

using just their active coordinates. The one-, two-, and three-coordinate densities of a directed triangle are $1, d_2, d_\mu$. A kernel using u output and v input coordinates therefore incurs only $\sqrt{d_u d_v}$. The exact mask partition is

mask type	(u,v)	number	bound per mask
two-edge matching or forest	$(2,2)$	15	$d_2 q$
matching number two forest	$(2,3),(3,2)$	$9+9$	$\sqrt{d_2 d_\mu}\,q$
two disjoint two-edge stars	$(3,3)$	3	$d_\mu q$
matching number three forest	$(3,3)$	14	$d_\mu r q$
full six-cycle	$(3,3)$	1	$d_\mu q$

A path of $k \le 5$ edges has full-cube norm at most $r^{\lceil k/2 \rceil}$: the two-edge star has squared norm $q(1-q)$, the three- and five-edge paths factor into Paley transforms with intervening contractions, and the four-edge path is bounded by fixing its middle vertex and applying two transforms. Disconnected components tensor. For the full cycle, fix one coordinate on each side and apply the two-edge matching/path bound, at cost q; no r^3 estimate for that cycle is used. Summing the table and the six means gives $\mathfrak{b}$. In particular

$$\mathfrak{b} \le 6 + \frac{102}{1-q} + \frac{32+112r}{1-q^2} < 140.004 \qquad (p \ge 2^{30}),$$

with limit 140. All these majorants are increasing in q, r.

(L2) The passive selected-coordinate maps $h \mapsto \mathbb{E}_z \chi(z-y)h(z)$ and $F \mapsto \mathbb{E}_{z,z'} \chi(z-y)\chi(z'-y)F(z,z')$ have norms r and $(q(1-q))^{1/2}$ (Corollary 15.10(a)).

(L3) *Insertion of a score multiplier in a monomial of $N_\Phi \otimes N_\Phi$.* In a monomial of $N_\Phi \otimes N_\Phi$ choose one cross edge in each component; it has at most one endpoint among the selected labels of a score, as no cross edge joins the two labels of a base. With the other vertices pinned, a bounded multiplier depending on at most one endpoint of each chosen edge can be placed between the two Paley transforms, or before or after their tensor product, so the kernel has norm at most q times the supremum of the multiplier (Lemma 16.4(d)). Given the other two vertices of a triangle, the active vertex has exactly $(p+1)/4$ completions (Lemma 12.10(a)), a conditional density at most $4/(1+q) \le 4$; Cauchy–Schwarz over the pinned vertices recovers the triangle energies, at the cost 16.

(L4) *Rows and coincidences.* Each two-edge column of Φ has uniform mean $1+q$, so the rows of Φ are at most M_Φ. A cross-component equality across the cut costs at most $4d_\mu^2(1+q)^5 q$ of row mass (given a coordinate, the other two of its triangle have density d_μ), a twin equality $x_u = x'_u$ at most $64 M_\Phi q$, an equality between the two components inside one half at most $M_\Phi^2 q$ (integrate the other half against the two row bounds), and a same-component equality at different bases makes Φ vanish, the third base supplying a contradictory edge. The soft mass is at least $(1-\mathfrak{b}q)^2$ by (L1); counting the eighteen cross-component coincidences inside the two halves, the six twins and the eighteen cross-component coincidences across the cut, the raw mass of the uniform root relative to $\nu \otimes \nu$ is at least Z_-; division by the half probability squared only increases it.

Lemma 21.8 (Inserting an opposite-base-pair score into the soft kernel)**.** *Fix one of the opposite-base-pair scores $\mathsf{H}_0, \mathsf{H}_1, \mathsf{H}_2$ introduced above and denote it by H. On the two independent cyclic triangles put $K = \Phi \otimes \Phi$, $e = \mathfrak{b} - 6$. Then, for $p \ge 2^{30}$,*

$$\|K \circ \mathsf{H} - \mathsf{H}\| \le C_{\mathrm{pair}} q, \qquad C_{\mathrm{pair}} = 2304 + 2\kappa(1+5r)(1+e) + r(12+14e+e^2) < 2897.$$

Restriction to the actual six-distinct half costs at most $(1-9q)^{-1}$. Cross-cut collisions must still be charged separately.

Proof. Keep the same auxiliary label in the two raw pair rows. For an operator A on the first triangle, let

$$(VF)(x,y)=\mathbb{E}_s\chi(s-x_i)\chi(s-y_j)F(x,s).$$

Uniformity of y_j and the Paley bound give $\|V\|\le r$, while direct composition gives $(A\otimes J)\circ\mathsf{H}=r^{-1}V(A\otimes I)V^*$. Thus this term has norm at most $r\|A\|$. Write $\Phi=J+S+R$, with S the twelve raw star monomials. The preceding bounds give $\|S\|\le\kappa(1+5r)r+6q$ and $\|R\|\le eq$. Each nonempty old-edge monomial times H costs at most $16q$: select one old edge and the passive score coordinate in the other triangle. The passive Paley map and the two-edge matching/path bound leave r^3 before the raw factor r^{-1}; the conditional triangle densities cost 16. The absence of an old edge at the selected base is used here. Apply this bound to the $12^2=144$ terms of $S\otimes S$. The one-sided S and R terms use the displayed sandwich. The remaining $R\otimes S$, $S\otimes R$, $R\otimes R$ terms use the Schur multiplier norm at most r^{-1} of the positive raw pair Gram. Summing gives the stated bound $C_{\mathrm{pair}}q$. The endpoint inequality uses $r\le 2^{-15}$ and the rational upper bound for $\mathfrak{b}$ above. Conditioning both half laws on six distinct labels gives the single factor $(1-9q)^{-1}$. □

Proposition 21.9 (The layer cut of a positive root)**.** *Let $p\ge 2^{30}$ and let $\Gamma_{p,\mathsf{u}}$ be the individual-square root, with $\gamma=\mathsf{u}^2$. Let ν be the fixed uniform induced layer half. With $d_\mu,\mathfrak{b},\mathfrak{n},M_\Phi,Z_-$ as above and h_0 of Lemma 21.6, put $B_H=2+r$ and*

$$m_{\mathsf{u}}=1+2\mathsf{u}r+\gamma,\qquad F_{\mathsf{u}}=1+2\mathsf{u}B_H+\gamma B_H^2,\qquad C_{\mathrm{nn},\mathsf{u}}=3969\cdot 16\,F_{\mathsf{u}},$$
$$C_{\mathrm{sc},\mathsf{u}}=127\big[4(1+4\mathsf{u}r+4\gamma q)+2(q+2\mathsf{u}r+\gamma)\big],\qquad C_{\mathrm{cr},\mathsf{u}}=\big[256M_\Phi+64d_\mu^2(1+q)^5\big]F_{\mathsf{u}},$$
$$C_{\mathsf{u}}=C_{\mathrm{nn},\mathsf{u}}+C_{\mathrm{sc},\mathsf{u}}+C_{\mathrm{cr},\mathsf{u}},\qquad C_{\mathrm{in},\mathsf{u}}=127\big[9F_{\mathsf{u}}+2(q+2\mathsf{u}r+\gamma)\big],$$
$$z_{\mathsf{u}}^{\iota}=1+\gamma-\gamma\Big[4q+36qB_H^2+\frac{2q}{1-q}\Big]-2\mathsf{u}h_0,$$

and

$$g_{\mathsf{u}}^{\iota}=\frac{2\mathfrak{n}m_{\mathsf{u}}+C_{\mathsf{u}}q}{rz_{\mathsf{u}}^{\iota}h_-Z_-},$$
$$c_{\mathsf{u}}^{\iota}=\frac{2\mathsf{u}r+\gamma c_0q+(2\mathfrak{n}m_{\mathsf{u}}+C_{\mathsf{u}}q)/h_-}{z_{\mathsf{u}}^{\iota}Z_-},$$
$$A_{\mathsf{u}}^{\iota}=\frac{C_{\mathsf{u}}+C_{\mathrm{in},\mathsf{u}}+910\gamma+2(1+\gamma)\mathfrak{b}/h_-}{z_{\mathsf{u}}^{\iota}h_-Z_-},\qquad c_{\mathsf{u}}^{+,\iota}=\frac{q(A_{\mathsf{u}}^{\iota})^2}{1-c_{\mathsf{u}}^{\iota}}.$$

Here $C_{\mathrm{nn},\mathsf{u}}$ bounds the $N_\Phi\otimes N_\Phi$ monomials, $C_{\mathrm{sc},\mathsf{u}}$ the selected-label coincidences, $C_{\mathrm{cr},\mathsf{u}}$ the other cross coincidences, and $C_{\mathrm{in},\mathsf{u}}$ the input-half coincidences. If $z_{\mathsf{u}}^{\iota}>0$ and $c_{\mathsf{u}}^{\iota}<1$, the cut operator T_{u}^{ι} of $\Gamma_{p,\mathsf{u}}$ at ι on $L^2(\nu)$ satisfies

$$-g_{\mathsf{u}}^{\iota}rI\preceq T_{\mathsf{u}}^{\iota}\preceq(1+c_{\mathsf{u}}^{+,\iota}q)I,\qquad \|P_0T_{\mathsf{u}}^{\iota}\mathbf{1}\|\le qA_{\mathsf{u}}^{\iota},\qquad \|P_0T_{\mathsf{u}}^{\iota}P_0\|\le c_{\mathsf{u}}^{\iota}.$$

The bounds at an endpoint hold for all larger primes with the same u, and $g_{\mathsf{u}}^{\iota}\to\sqrt{3}+\sqrt{7}$. The block caps follow from Lemma 13.7(b), as in Proposition 21.7.

Proof. Write $w_{\mathsf{u}}=\frac{1}{3}\sum_j\Upsilon_j$, with $\Upsilon_j=1+2\mathsf{u}\mathsf{H}_j+\gamma\mathsf{H}_j^2$. It suffices to bound the soft operator of one polynomial $\Upsilon=1+2\mathsf{u}\mathsf{H}+\gamma\mathsf{H}^2$ and then average and normalize. By Lemma 13.3(f), $T_{\mathsf{u}}^{\iota}=(T_0^{\iota}\circ w_{\mathsf{u}})/\mathbb{E}_{\Gamma_{p,0}}w_{\mathsf{u}}$.

Soft expansion. $\Phi\otimes\Phi=1\otimes1+N_\Phi\otimes1+1\otimes N_\Phi+N_\Phi\otimes N_\Phi$. The base term $(1\otimes1)\circ\Upsilon$ is positive semidefinite: H is a positive Gram kernel, so is its pointwise square, and the coefficients of Υ are nonnegative. In $(N_\Phi\otimes1)\circ\Upsilon$ the passive component carries the constant kernel and its selected labels; retaining the auxiliary variables, (L1) and (L2) give the absolute price $\mathfrak{n}[1+2\mathsf{u}r+\gamma(1-q)]\le\mathfrak{n}m_{\mathsf{u}}$ in both orders, and the other component the same. For the 63^2 monomials of $N_\Phi\otimes N_\Phi$, (L3) with the multiplier Υ on four distinct selected labels, where $|\mathsf{H}|\le B_H$ (Theorem 4.7(b)) and

$|\Upsilon| \le F_{\mathsf{u}}$, gives $16F_{\mathsf{u}}q$ each, together $C_{\mathrm{nn},\mathsf{u}}q$. The strict score mask is part of this multiplier; no other cross-injectivity mask is removed inside a norm.

Coincidences. The four single coincidences between the selected labels of the two halves have, before the input half is conditioned, total row and column mass at most $4q$, and there $|\mathsf{H}| \le 2r$, so $|\Upsilon| \le 1 + 4\mathsf{u}r + 4\gamma q$; the two equal-pair events have mass at most $2q^2$ and $\mathsf{H} = (1-2q)/r$, so $|\Upsilon| \le 1 + 2\mathsf{u}/r + \gamma/q$. They are deleted from the unmasked part $1 + N_\Phi \otimes 1 + 1 \otimes N_\Phi$, whose kernel is pointwise at most $1 + 2 \cdot 63 = 127$ in absolute value (N_Φ being a sum of 63 character monomials of modulus at most 1), which costs $C_{\mathrm{sc},\mathsf{u}}q$. After these four, the coincidences across the cut are four twins and sixteen cross-component equalities, on which $|\Upsilon| \le F_{\mathsf{u}}$, and (L4) gives $C_{\mathrm{cr},\mathsf{u}}q$. These are positive row and column bounds of the deleted kernels, not Schur monotonicity of signed kernels.

Mass. The raw mass of the uniform root is at least Z_- (L4). Its scores have $|\mathbb{E}_{\Gamma_{p,0}}\mathsf{H}_j| \le h_0$ and $\mathbb{E}_{\Gamma_{p,0}}\mathsf{H}_j^2 \ge 1 - 4q - 36qB_H^2 - 2q/(1-q)$ (Lemma 21.6(b)); so $\mathbb{E}_{\Gamma_{p,0}} w_{\mathsf{u}} \ge z_{\mathsf{u}}^{\iota}$.

Lower bound. The restriction to ν costs h_-^{-1} (Lemma 13.23) and the normalization Z_-^{-1} of the uniform root. Dividing by the tilt mass gives $T_{\mathsf{u}}^{\iota} \succeq -g_{\mathsf{u}}^{\iota} r$.

Centred norm. The selected pair of a half is a uniform distinct pair under ν, by square-affine invariance and the interchange of the two components, so the centred base is at most $2\mathsf{u}r + \gamma c_0 q$: $P_0 \mathsf{H} P_0 = r\,\mathcal{U}\mathcal{U}^*$ with $\|\mathcal{U}\| \le 1$, and Corollary 15.10(b) gives $\|P_0\mathsf{H}^2P_0\| \le c_0 q$. The other terms are those of the lower bound, which gives c_{u}^{ι}.

Row. On a six-distinct output half the unconditioned base row is constant, as H has mean 0 and H^2 has mean $(1-2q)^2 + q^3$ when the other half is unconditioned. In a one-component term the linear row vanishes after the passive coordinate is integrated, and the square row multiplies $N_\Phi(x, x')$ by

$$\mathbb{E}_z[\chi(z-y)^2\chi(z-y')^2\chi(z-y'')^2] - \big(\mathbb{E}_z\chi(z-y)\chi(z-y')\chi(z-y'')\big)^2, \qquad y \ne y'',$$

where y, y'' are the two selected labels of the output half and y' that of the input half in the active component; this differs from one by at most $7q$, also when $y' \in \{y, y''\}$. With $|N_\Phi| \le 65$ this costs $455q$. The two components therefore have row price $2(1+\gamma)\mathfrak{b}q/\sqrt{h} + 910\gamma q$. The nine input-half coincidences and their pair spikes cost $C_{\mathrm{in},\mathsf{u}}q$, by the same ordinary/equal-pair split; all other errors cost $C_{\mathsf{u}}q$. With $h^{-1/2} \le h_-^{-1}$, the half compression, the normalization of the uniform root and the tilt mass this gives $\|P_0 T_{\mathsf{u}}^{\iota} \mathbf{1}\| \le q A_{\mathsf{u}}^{\iota}$.

Upper bound. T_{u}^{ι} is self-adjoint of mass one, so Proposition 13.6(a) gives $T_{\mathsf{u}}^{\iota} \preceq 1 + q^2(A_{\mathsf{u}}^{\iota})^2/(1-c_{\mathsf{u}}^{\iota})$.

Uniformity. After the explicit factor r in g_{u}^{ι} is cancelled, all numerators are positive increasing in q, r and all lower denominators positive decreasing, so evaluation at $q = 1/P$, $r = P^{-1/2}$ is uniform for $p \ge P$. As $q \to 0$, $\mathfrak{n}/r \to \kappa$ and m_{u} and z_{u}^{ι} tend to $1+\gamma$, so $g_{\mathsf{u}}^{\iota} \to 2\kappa = \sqrt{3} + \sqrt{7}$. □

For $\gamma = 1$, the individual squares at $\mathsf{u} = 1$, the proposition is the individual-square layer bound at that parameter; then $z_1^{\iota} = 2 - 4q - 36qB_H^2 - 2q/(1-q) - 2h_0$, $2\mathfrak{n}m_1/r = 2[\kappa(1+5r) + \mathfrak{b}r](2+2r)$ and $F_1 = (3+r)^2$. The proved coefficient tends to 2κ for every $\mathsf{u} > 0$, because the two one-component terms are priced separately; the larger layer cap costs only through the weight v_{L} of the completion (Remark 22.2).

Sharper bounds for the tilted law (next nine subsections). The following subsections sharpen the two cut bounds above at the seven interval starts. In order they: isolate collision terms on two explicit subspaces; give a finite gap for the uniform law; split a remainder across that gap; retain a constant term at the layer-flip cut; list supports for the 39-trace law; extend it to all sorting cuts; use one parameter for both cuts; evaluate an explicit layer price; and evaluate a finite chamber price. A first reading can skip to Section 21.13 and use the stated bounds.

21.4. **Isolating collision terms on two explicit subspaces.** The remaining subsections of this section are finite calculations at one prime $p \equiv 3 \pmod 4$, used only to evaluate the caps tabulated

in Section 21.13. They do two things. First, they sharpen the collision part of the L20 bound for the uniform principal law, by computing it exactly on two explicit subspaces. Second, they construct and bound the coherent tilted law of the last interval $[2^{45}, P_3)$: its score is an explicit linear combination of the trace functions $\mathbb{E}_s \prod_{i\in D} \chi(s - x_i)$ over a list of 39 supports D (subsets of the twelve label positions), symmetrised so that one law serves at every sorting cut. Its matrix bounds are first proved at the endpoint parameter $t_* = 12/5$ and then transferred by convexity to the parameter $t = 67/40$ used in the construction. In this part of the paper a support or a law is called *physical* when it is described in the original twelve labels, as opposed to coordinates of a chosen half chart; a *gate* is one finite matrix or positivity inequality checked exactly (Appendix D). A reader who accepts the caps of Certificate 21.25 can skip to Section 22. Local words in this part: a *pin* is a label fixed by conditioning (core pins C_i, private pins b_l); a *mask* is a subset of the edge set of the small graph under discussion, determining one character product; an *arm* is one synthesis factor (feature map) in a product of two stars, and a *free leaf* is a label coordinate occurring in no other factor; a *spike* is an exceptional large value of a trace or score at special configurations (in particular at equal score pairs), always bounded separately; a *hole term* is a term in which the common auxiliary label equals one of the fixed labels; an argument is *rooted* when it keeps the common auxiliary summation label s unintegrated, and *unrooted* when s is integrated out first; a star factor is *passive* when it carries none of the displayed characters; the *realification* of a complex feature space is the real span of the real and imaginary parts of its functions; an *endpoint* X is the lower endpoint of an interval, at which bounds are evaluated; and *physical* (as defined above) is opposed to *soft*: the physical kernel is the actual kernel K_0 in the original labels, the soft kernel its expansion without distinctness.

We begin with the uniform law, before any tilt. The collision part of its L20 kernel—the terms where labels of the two halves coincide—is split according as the test function lies in one of two explicit subspaces or in their orthogonal complement; on the two subspaces it is computed exactly below. In the following local calculations write $A = \mathsf{R}$, $a = \mathfrak{r}$ for the rectangle and its private mean in Proposition 20.17, $C_0 = A - a \otimes 1$, and $Z = \boldsymbol{z}$, with the other normalized star features v, w, e as in Proposition 20.16. All spaces have probability normalization, $q = p^{-1}$ and $r = \sqrt{q}$. The letter a in these local formulas denotes a function on the core; the scalar exponent does not enter this calculation. Likewise t below denotes the root parameter u, not scalar repricing. Write $Q = I - J$ for ordinary centering (subtracting the mean). Put $a_2 = \|a\|_2$, $P_a = aa^*/a_2^2$ and

$$P_{\mathrm{c}} = I_{\mathrm{core}} \otimes J_{\mathrm{pair}}, \qquad P_{\mathrm{p}} = P_a \otimes Q_{\mathrm{pair}}, \qquad P = P_{\mathrm{c}} + P_{\mathrm{p}}, \qquad R = Q_a \otimes Q_{\mathrm{pair}}.$$

The core labels are $C = (a_0, a_1, c_0, c_1) = (C_0, C_1, C_2, C_3)$, with $a_i \to c_j$, and the private pair is $b = (b_0, b_1)$. Set

$$K_C(s) = \prod_i (1 - \chi(s - a_i)) \prod_j (1 + \chi(s - c_j)), \qquad \kappa_C = \mathbb{E}_s K_C(s), \qquad Z_C = (K_C - \kappa_C)/(\sqrt{15} r).$$

The six-distinct induced half is obtained by conditioning the independent core and distinct private pair on the six labels being pairwise distinct.

Lemma 21.10 (Compression of the constant kernels at pinned labels)**.** *For $h \in H = L_0^2(\mathbb{F}_p)$, let $\sigma = (-1, -1, 1, 1)$, $V_i h = h(C_i)$, $B_l h = h(b_l)$, $w = \sum_i V_i/2$, $u = (V_0 + V_1 - V_2 - V_3)/2$, $v = (B_0 + B_1)/\sqrt{2}$, and*

$$Wh(b) = \mathbb{E}_s \chi(s - b_0)\chi(s - b_1)h(s), \quad e = W/r, \quad R_l h(b) = \mathbb{E}_s \chi(b_l - s)h(s).$$

On H, for $\epsilon \in \{-1, 1\}$, define

$$F_i^{\epsilon} h = \prod_{l=0}^{1} (1 + \epsilon \chi(b_l - C_i))h(C_i), \qquad G_l h = K_C(b_l)h(b_l),$$

and set $k = \mathbb{E}a/a_2$, $k' = \mathbb{E}(a\kappa)/a_2$, where κ denotes the function $C \mapsto \kappa_C$ and the expectations are under the core law (uniform on core configurations). The sixteen core-pin constant kernels and the four private-pin constant kernels have auxiliary-mean-zero parts

$$K_c = q\sum_{i,j} F_i^{\sigma_j}(F_j^{\sigma_i})^*, \qquad K_b = q(G_0+G_1)(G_0+G_1)^*.$$

Their protected blocks are

(21.4)

	$P_{\mathrm{c}}KP_{\mathrm{c}}$	$P_{\mathrm{p}}KP_{\mathrm{p}}$	$P_{\mathrm{c}}KP_{\mathrm{p}}$
K_c	$4q(1-q)^2ww^*$	$16q^2k^2ee^*$	$4q(1-q)k[-2u(R_0+R_1)^* + 2wW^*]$
K_b	$60q^2ZZ^*$	$2q(k')^2vv^*$	$2\sqrt{30}qrk'Zv^*$

The full auxiliary constant-label pieces contribute, respectively, $16q(1-q)^2J$ and $4q\kappa\otimes\kappa$ on P_{c}, and zero other protected blocks.

Proof. Uniformity of the private pair and affine invariance of the actual core give exactly

$$P_{\mathrm{c}}F_i^{\epsilon} = (1-q)V_i, \quad P_{\mathrm{p}}F_i^{\epsilon} = k[\epsilon(R_0+R_1)+W], \quad P_{\mathrm{c}}G_l = \sqrt{15}rZ, \quad P_{\mathrm{p}}G_l = k'B_l.$$

Indeed $\mathbb{E}[a \mid C_i = s] = \mathbb{E}a$ and $\mathbb{E}_C[aK_C(s)] = \mathbb{E}(a\kappa)$, independently of s. Summing uses $\sum\sigma_i = 0$, $\sum V_i = 2w$, and $\sum\sigma_i V_i = -2u$, proving (21.4). On the constant label, $P_{\mathrm{c}}F_i^{\epsilon}1 = (1-q)1$, $P_{\mathrm{p}}F_i^{\epsilon}1 = 0$, $P_{\mathrm{c}}G_l1 = \kappa$, and $P_{\mathrm{p}}G_l1 = 0$, which proves the last assertion. □

21.5. **A finite collision bound for the uniform law.** This result is a finite bound for the uniform (untilted) L20 root in the original twelve labels. Its positive coefficient, marginal range, variance and layer data are consequently unchanged. Retain the independent-half rectangle $A = a\otimes 1 + C_0$ and the certified bounds for the rectangle and its domain gap from Proposition 20.17 and Proposition 20.20: numbers again called a_2, c with $a_2 \ge \|a\|_2$, $c \ge \|C_0\|$, $z_2 \ge \|Z\|^2$ (here a_2, c are upper bounds, although the same letters denoted the exact norms above). Put

$$\theta = (1-q)(1-3q)(1-7q), \quad h_- = (1-4q)(1-5q), \quad \eta = 11r+2q,$$

(21.5)
$$d_\kappa = \sqrt{\frac{5q+34qr+11q^3}{\theta}} + 2q, \qquad d_a = \frac{(2+\eta+16q)d_\kappa + 15q}{1-q},$$
$$a_- = (1-\eta)(1-\eta-16q), \qquad \omega = d_a/a_-.$$

From the rectangle estimates of Section 20 we retain the bounds $|\kappa - 1| \le \eta$, $\|a-1\|_2 \le d_a$ and $a \ge a_-$.

Proposition 21.11 (Finite bound for the collision term on the two protected subspaces)**.** *Define*

$$\delta_c = \frac{4(4r+92q)}{\sqrt{\theta(1-q)}}, \qquad \delta_b = \frac{4\sqrt{15}\,r}{\sqrt{\theta(1-q)}},$$

(21.6)
$$E_{\mathrm{pin}} = 256q(2\delta_c+\delta_c^2) + 1024q(2\delta_b+\delta_b^2), \qquad E_{\mathrm{mult}} = \frac{24903680q^2}{1-q},$$

and

(21.7)
$$\begin{aligned} b_B &= 64q(2+\eta),\\ A_0 &= 4q + 60q^2z_2, \qquad A_1 = 2q(1+\eta)^2 + 16q^2,\\ X &= 8qr(\sqrt{2/(1-q)}+1) + 2\sqrt{30}qr(1+\eta)\sqrt{z_2},\\ \alpha &= \max(A_0,A_1) + X + 16q\omega^2 + 4q(\eta+\omega)^2,\\ E_S &= \frac{b_B^2}{a_-^2 - b_B}, \qquad G = a_2c + \alpha, \qquad D = c^2 + b_B. \end{aligned}$$

For $p \equiv 3 \pmod 4$ satisfying $a_-^2 > b_B$ and $G > D$, the old mass-one root operator on its fixed induced half satisfies

$$T_{\mathrm{old}} \succeq -g_{\mathrm{pin}} r I, \qquad g_{\mathrm{pin}} = \frac{E_S + G + b_B^2/(G-D) + E_{\mathrm{pin}} + E_{\mathrm{mult}}}{h_-(1-\varepsilon_Z)r}. \tag{21.8}$$

Here ε_Z is the normalizer error of the uniform (untilted) law from Proposition 20.15. Exact rational endpoint evaluation is valid uniformly for every prime above the endpoint. For example, the endpoint 19951586 *gives* $g_{\mathrm{pin}} < 5.481634404$*, compared with the retained* 5.500057067029*.*

Proof. Let B be the sum of the twenty full-label constant pin-incidence kernels in Lemma 21.10. For a core pin $C_i = C'_j = t$, where C' denotes the core labels of the second half, its incident factors are two half multipliers bounded by four. After removing them, the cross polynomial is a swapped product of two 3-by-2 rectangles. Given one core coordinate, the remaining three have density $16/\theta$ relative to the three-label cube; the private distinct-pair density is $1/(1-q)$.

The six-edge rectangle has seventeen nonempty stars and forty-six other masks (edge subsets that are not stars). Group the stars on the full independent cube. Every raw star has norm at most r, because its Gram is a Schur product of Paley Grams with diagonal at most one and norm at most q. Distinct output supports are orthogonal ANOVA spaces. On nonzero translation sectors, the three singleton and four higher output supports, and the two singleton and one pair input supports, give a seven-by-three matrix. Collapsing the two equal singleton columns gives Gram

$$\begin{pmatrix} 14 & \pm i\sqrt{2} \\ \mp i\sqrt{2} & 3 \end{pmatrix},$$

whose largest eigenvalue is below sixteen: the three retained core signs sum to ± 1. On zero translation frequency the blocks are $2\sum_{|S|\ge 2}\sigma_S m_S \otimes 1$, of norm at most $4r$, and $3\,1 \otimes m_{\mathrm{pair}}$, of norm at most $3r$. Their input and output spaces are orthogonal. This retains the full auxiliary constant sector and proves that the seventeen stars together have norm at most $4r$. Every other mask contains two disjoint edges whose induced subkernel is a matching, a path, or a restricted $K_{2,2}$, of norm at most $2q$. Fixing the third output coordinate inserts only one-variable input multipliers and separate output exclusions. Distinctness of the selected input and output pairs stays inside the restricted kernel. Density congruence gives $\|A_i - J\| \le \delta_c$.

For a private pin the two incident factors are bounded by sixteen. The remaining rectangle is $K_C(s)$ with s uniform outside the pinned label. Its fifteen nonempty output supports are orthogonal on the full four-label cube, giving norm $\sqrt{15}r$ before the same density congruence. Its defect from J is at most δ_b. A swapped product of two J+defect rectangles differs from J by at most $2\delta + \delta^2$. Equality of two uniform coordinates disintegrates exactly as $q\mathbb{E}_t$; conditional Cauchy–Schwarz recovers the original half norms. The sixteen core pins and four private pins therefore give $\|D_{\mathrm{sum}} - B\| \le E_{\mathrm{pin}}$, where D_{sum} is the sum of those twenty pin terms, with multipliers $16 \cdot 4^2$ and $4 \cdot 16^2$. The condition that the labels involved are distinct is retained inside every norm estimated above.

The sixteen core/private equalities vanish because K_C is zero at every core label. Of the other twenty events, compatible double equalities form matchings on two six-distinct halves. Every two-coordinate input marginal has point mass at most $2q^2/(1-q)$: this is attained by a directed core pair; same-side and private pairs have mass $q^2/(1-q)$, and core/private pairs have mass q^2. The soft density is at most 65536. For multiplicity m, $m - 1 \le \binom{m}{2}$. Counting even all $\binom{20}{2} = 190$ pairs bounds the positive overlap correction by E_{mult}. Conditioning on the input half contributes the single factor h_-^{-1} in (21.8), which also covers every higher coincidence.

With $f_\epsilon(s) = \prod_l (1 + \epsilon\chi(b_l - s))$ the private factor in F_i^ϵ, the exact second moment $\mathbb{E} f_\epsilon^2 = 4(1-2q)$ gives $\|F_i^\epsilon\|^2 = 4(1-2q)$, while $\|G_l\|^2 = 16\mathbb{E}\kappa \le 16(1+\eta)$. Hence $\|B\| \le b_B$. In (21.4), use $\|w\|, \|v\|, \|e\| \le 1$, $\|u\| \le (1-q)^{-1/2}$, $k \le 1$ and $k' \le 1 + \eta$. The nonconstant protected blocks

are bounded by A_0, A_1, X. On the core space orthogonal to a, the constant-label blocks cost at most $16q\omega^2$ and $4q(\eta+\omega)^2$. Thus $P_- B P_-$ (with P_- as defined in the next paragraph) has norm at most α; no auxiliary constant mode is omitted.

Let $S = \operatorname{span}(a \otimes 1)$ and $P_- = P \ominus S$. The old rectangle reduces S, P_- and R. Its lower diagonal bounds are $\|a\|_2^2$, $-\|a\|_2\|C_0\|$, and $-\|C_0\|^2$. After subtracting B, the S block is at least $a_-^2 - b_B$; its off-diagonal norm is at most b_B. The positive-block elimination in Lemma 13.5(a) costs at most E_S times the remaining identity. The other two blocks have lower bounds $-G, -D$ and off-diagonal norm at most b_B. Since $G > D$, Lemma 13.5(b) gives their lower bound $-G - b_B^2/(G-D)$. Adding the two pin errors, compressing to the physical half and dividing by the old normalizer proves (21.8).

For uniform endpoint evaluation, retain certified endpoint upper bounds on $a_2, c/r, z_2, \varepsilon_Z$. All normalized numerators are positive increasing in q, r, and the lower denominators are positive decreasing on the gates. Before the last Schur step replace $G/r, D/r, b_B/r$ by endpoint majorants G_0, D_0, B_0. The lower block matrix with these fixed entries is valid at every larger prime; use $G_0 + B_0^2/(G_0 - D_0)$ after checking $G_0 > D_0$. No monotonicity of a moving gap denominator is assumed. □

21.6. **Allocating the pair-square remainder across a spectral gap.** All spaces in this subsection carry probability measures; its C is the rectangle C_0 above. Let U be the uniform directed $K_{2,2}$ core, with ordered side pairs $a(u), c(u)$, and let V be the uniform ordered distinct-pair space. Put

$$A(u;b) = K_u(b_0)K_u(b_1), \quad a = A1, \quad C = A - a \otimes 1,$$
$$a_2 = \|a\|_2, \quad a_\infty = \|a\|_\infty, \quad c = \|C\|.$$

Here K_u is the function K_C above with core $C = u$. The rectangle identities used in Proposition 21.7 give $C1 = C^*1 = C^*a = 0$. Write $M^2 = \sup_u \int |C(u;b)|^2\, db$; the exact row square $256a(u) - a(u)^2$ permits $M^2 \le 256a_\infty$. The column square is constant. Indeed square-affine changes and private-pair exchange act transitively on V and preserve C. If $F_a, F_c : L^2(V) \to L^2(U)$ are the side-pair pullbacks, the same symmetry proves

$$F_a^* M_{a^2} F_a = F_c^* M_{a^2} F_c = a_2^2 I. \tag{21.9}$$

For real kernels $X, Y : V \to U$, define

$$T[X,Y]((u,b),(u',b')) = X(u;b')Y(u';b).$$

Set

$$P_{\mathrm{c}} = I_U \otimes J_V, \quad P_{\mathrm{p}} = P_a \otimes Q_V, \quad P = P_{\mathrm{c}} + P_{\mathrm{p}}, \quad R = Q_a \otimes Q_V, \quad P_a = aa^*/a_2^2.$$

Put $K = T[A,A]$; expanding $A = a \otimes 1 + C$ shows that P, R reduce K exactly. The R block is $T[C,C]$, of norm at most c^2. On P, the positive rank-one term $aa^* \otimes J$ and the two adjoint blocks $a_2 C, C^* a_2$ give a lower bound $-a_2 c$.

Lemma 21.12 (The pair-square residual)**.** *Put $q = p^{-1}$, $r = \sqrt{q}$, and $H(b,d) = r^{-1}\mathbb{E}_s \chi(s - b_0)\chi(s - b_1)\chi(s - d_0)\chi(s - d_1)$. Then*

$$\sigma = (1-2q)^2 + \frac{q(1-3q)^2}{1-q}, \qquad L = H^{\circ 2} - \sigma J,$$
$$0 \preceq L \preceq \rho I, \quad L1 = 0, \quad \rho = q\left(\frac{4}{1-q} + 2q\right).$$

For

$$E_c = \tfrac{1}{2} K \circ \{L(a(u), c(u')) + L(c(u), a(u'))\},$$
$$E_b = K \circ L(b, b'), \qquad E = (2E_c + E_b)/3,$$

one has

$$PEP \succeq -\alpha P, \quad \|REP\| \le b, \quad \|RER\| \le \gamma,$$

where

$$\alpha = \frac{2\rho}{3}\max(a_2^2, c^2), \quad \gamma = \rho M^2, \quad b = \frac{\rho M}{3}\sqrt{(2a_2 + c)^2 + (a_2 + 2a_\infty c/a_2)^2}.$$

The separate $R \leftarrow P_{\mathrm{c}}$ bound (the norm of the block REP_{c}) is $b_0 = \rho M(2a_2 + c)/3$.

Proof. For the raw pair map W with kernel $\chi(s - b_0)\chi(s - b_1)$, the exact Paley Gram is $W^*W = qg_eQ + q^2J$, where $g_e = (1 - 2q - q^2)/(1 - q)$. This gives the constant row sum $\sigma = (1 - q)g_e^2 + q^3$. The auxiliary constant diagonal contributes $(1 - 2q)^2 J$ to $H^{\circ 2}$; the part of $H^{\circ 2}$ orthogonal to that contribution is positive with norm at most ρ, by the distinct-pair $K_{2,2}$ bound and the mean-zero diagonal bound. Removing its constant eigenvalue proves the asserted bounds on L.

Consider first the unaveraged core placement $L(a(u), c(u'))$. Its rank/rank term (the part $a \otimes 1$ in both factors; the rank/centered and centered/centered terms below are analogous) is supported on P_{c} and costs at most ρa_2^2, by (21.9). The rank/centered terms have zero P compression: use $C1 = 0$ on P_{c}, and (21.9) followed by $L1 = 0$ on P_{p}. Their $R \leftrightarrow P_{\mathrm{c}}$ norm is at most $a_2\rho M$. To see the normalization explicitly, their squared Gram is bounded by $a_2^2\rho^2(CC^*) \circ (F_aF_a^*)$; the last operator is a direct integral over pair fibers, each with trace at most M^2.

The centered/centered term kills P_{c}. Its P_{p} compression is $B_a^* L B_c/a_2^2$, where M_a is multiplication by the function a, $B_a = F_a^* M_a C$, and $B_c = F_c^* M_a C$. Equation (21.9) gives $\|B_a\|, \|B_c\| \le a_2 c$, so this compression costs ρc^2. The full image of $a \otimes g/a_2$ is

$$(Cg)(u)(LF_c^* M_a C)(a(u); b)/a_2.$$

The second factor has constant squared row norm, at most $\rho^2 a_\infty^2 M^2$, giving the $R \leftarrow P_{\mathrm{p}}$ bound $\rho a_\infty M c/a_2$. Factoring L into two square roots, retaining the same auxiliary label in the two product maps and the private-coordinate swap, gives the whole centered/centered norm at most ρM^2.

For the private placement, the rank/rank term is $aa^* \otimes L$ and is positive on P_{p}. The rank/centered term couples P_{p}, R with norm at most $a_2\rho M$: its squared Gram is bounded by $\rho^2(C^*C) \circ I$, whose diagonal is the constant column square. The centered/centered term kills P_{p}, has positive P_{c} compression CLC^*, and has $R \leftarrow P_{\mathrm{c}}$ norm at most $\rho M c$. The same square-root factorization bounds its full norm by ρM^2. Thus the two columns of REP cost respectively $\rho M(2a_2 + c)/3$ and $\rho M(a_2 + 2a_\infty c/a_2)/3$. Their squared sum proves the result. These are product-operator bounds, not an arbitrary-kernel Schur-multiplier assertion. □

Proposition 21.13 (Finite chamber-cut price for the pair-square law, with the gap bound)**.** *Use the retained finite constants of Proposition 21.7. In its notation (the following ten aliases are used only in this proposition; in particular $N, \Gamma, h, \varepsilon, e$ do not have their earlier meanings here), $N = N_A$, $\Gamma = \Gamma_{20}$, $E_L = \varepsilon_{\mathrm{lin}}$, $\ell_0 = \mathfrak{l}_0$, $h = h'_p$, $D_{\mathrm{old}} = D_{\mathrm{bal}}$, $P_{\mathrm{spike}} = \mathfrak{s}$, $\varepsilon = \varepsilon_Z$, $h_{\mathrm{mean}} = h_0$ and $e = \varepsilon_{\mathrm{sq}}$. Use $a_- \le a$, $a_+ \ge \|a\|_\infty$, $d \ge \|a - 1\|_2$, $\delta = d/\sqrt{1 - q}$, and*

$$\|QL_{\mathrm{soft}}Q - rM_{\mathrm{pair}}\| \le E_L, \qquad \|QL_{\mathrm{soft}}1\| \le \ell_0 r + E_L,$$

where $L_{\mathrm{soft}} = K \circ \bar{\mathsf{H}}$ is the soft kernel multiplied by the mean score and M_{pair} the leading pair matrix of Proposition 21.7. Let c be an upper bound for $\|C\|$, namely the retained rectangular-domain-gap bound, and assume $c < a_-$. Put $d_a = d/a_-$,

$$E_{0P} = 2rd(\sqrt{30\Gamma} + 2) + 2a_2 N, \quad E_{\mathrm{rank}} = r(2\delta + \delta^2),$$
$$s_+ = 1 + t^2 + 2tqr, \qquad s_- = 1 + t^2(1 - 4q).$$

For a fixed rational $0 < \Lambda \le 15$ with $k = 2t/\Lambda \le s_-$, let $g(\Lambda)$ be the unique root $> \sqrt{26}$ of $3g^3 + \Lambda g^2 - 90g - 26\Lambda = 0$ (the cubic is negative at $\sqrt{26}$ and strictly increasing beyond it). Define

$$\begin{aligned} G &= (s_+ - k)a_2 c + k\{g(\Lambda) r\Gamma + E_{0P}\} + 2tE_{\mathrm{rank}} + \tfrac{4t}{3} r c^2 + t^2\alpha, \\ d_L &= E_L + r d_a^2/3 + q r c^2, \\ b_L &= E_L + r d_a/3 + d_a(\ell_0 r + E_L), \\ h_L &= \ell_0 r + E_L(1 + d_a), \\ D &= s_+ c^2 + 2t d_L + t^2\gamma + \frac{(2t h_L + t^2 b_0)^2}{s_- a_-^2}, \qquad B = 2t b_L + t^2 b. \end{aligned}$$

Here the residual constants are those of Lemma 21.12, with $M = 16\sqrt{a_+}$ and a_2 in denominators replaced by the smaller bound a_-, which only enlarges the constants. If $G > D$, the independent-half weighted rectangle satisfies

$$K \circ \frac{1}{3}\sum_{j=0}^{2}(1 + t\mathsf{H}_j)^2 \succeq -\left(G + \frac{B^2}{G - D}\right) I.$$

For the actual normalized root, the following number g is valid in the bound $T \succeq -grI$ for its chamber operator T in the fixed half chart (a full fixed-half negative coefficient*):*

$$\begin{aligned} &\frac{h^{-1}[G + B^2/(G - D)] + C_w}{(1 - \varepsilon) z r}, \\ &C_w = (1 + 2tB_H + t^2 B_H^2) D_{\mathrm{old}} + \left(\frac{2t}{hr} + \frac{t^2}{hq}\right) P_{\mathrm{spike}}. \end{aligned} \tag{21.10}$$

Here $B_H = 2 + r$, h, D_{old}, P_{spike}, ε, and $z = 1 + t^2 - 2th_{\mathrm{mean}} - t^2 e$ are the retained conditioning, collision, spike, normalizer and moment bounds. The upper bounds for the norm, variance and range of that same law at the same parameter, proved in Proposition 21.7, remain valid.

Proof. Let S project onto $(a/a_2) \otimes 1$, and set $P_- = P - S$. The exact scalar pair mean is qr. With $L_0 = K \circ \frac{1}{3}\sum_j (H_j - qr)$, the weighted rectangle is $sK + 2tL_0 + t^2 E$, where $s = 1 + 2tqr + t^2\sigma \in [s_-, s_+]$. Conditional identity (21.9) and the centered pair kernels imply $SL_0S = SES = SL_0P_- = SEP_- = 0$. The positive S block is therefore sa_2^2.

On P_-, the exact old bound is $-a_2 c$. In comparing K with its leading stars, $T[C, C]$ has zero P compression, so only the rank/centered remainder E_{0P} is charged. The inherited eighteen-feature Gram bound Γ also applies on the independent half: cross-factor Grams vanish, and within-core and within-private conditional counts dominate the smaller counts used in its original proof. The finite leading matrix has negative norm $g(\Lambda)$; its remaining blocks cost at most $\max(2, \Lambda/3) \le 5 < g(\Lambda)$. The exact positive-coefficient splitting

$$sK + 2trM_{\mathrm{pair}} = (s - k)K + k(K + \Lambda r M_{\mathrm{pair}})$$

therefore incurs the error E_{0P} only in the second summand. The weighted rank error is E_{rank}. The other core P_{p} linear block costs at most $2rc^2/3$; the private P_{c} block is positive. This proves the stated G.

The angle of P_- to the ordinary constant is at most d_a. Splitting through Q gives b_L; the private leading pair term has coefficient $1/3$, while core terms have private-constant output and vanish against R. The same split gives d_L, with qrc^2 paying the removed raw mean. For the S column, the leading pair matrix sends $Q((a/a_2) \otimes 1)$ into $U \otimes 1$, which R annihilates; this gives the sharper h_L. Eliminating the positive S block costs only $(2th_L + t^2 b_0)^2/(s_- a_-^2)$ on R. The remaining two-block bound is $G + B^2/(G - D)$ by Lemma 13.5(b). It is already a full-space bound, so no second constant-block penalty is added after compression.

Actual-half compression costs h^{-1}. Outside the configurations where the two label pairs of a score are equal as unordered sets, the weight is at most $1 + 2tB_H + t^2B_H^2$; the equal-pair strata are paid by the two explicit spike terms in (21.10). The retained raw mass is at least $(1-\varepsilon)z$. These facts prove the bound in the original twelve labels, in the fixed half chart, without passing to the tilted half marginal.

For a uniform endpoint price, majorize $G/r, D/r, B/r$ first by fixed endpoint values G_0, D_0, B_0 and check $G_0 > D_0$. Use $G_0 + B_0^2/(G_0 - D_0)$ at every larger prime. All constituent normalized numerators increase in q, r, and all positive lower denominators decrease. This order of operations avoids assuming monotonicity of a quotient with a moving gap. The fixed Λ remains feasible: at the endpoint X (the lower endpoint of the interval) $k \le s_-(X)$, and $s_-(p)$ is nondecreasing in p, so $k \le s_-(p)$ for all $p \ge X$. □

Corollary 21.14 (Interpolation after physical normalization)**.** *Let $g_0 r$ be any full negative price for the mass-one unchanged physical root Γ_0. Fix $0 < \tau \le 1$. In Proposition 21.13, replace*

$$s_+ = \tau + t^2 + 2tqr, \qquad s_- = \tau + t^2(1 - 4q),$$

and replace the constant 1 in the collision coefficient $1 + 2tB_H + t^2B_H^2$ by τ. Keep all linear and square coefficients $2t, t^2$, and check the same positive-block and gap conditions. After these replacements, choose endpoint majorants G_0, D_0, B_0 for $G/r, D/r, B/r$, respectively, with $G_0 > D_0$, as in the final paragraph of the proposition's proof, and set

$$C_{\mathrm{soft}} = G_0 + \frac{B_0^2}{G_0 - D_0}.$$

This is the raw soft coefficient, before half-conditioning, scalar normalization and collision corrections. If $C_{w,\tau}$ is the corresponding collision price in (21.10)*, then*

$$g_{\mathrm{int}} = \frac{(1-\tau)g_0 + [C_{\mathrm{soft}}/h + C_{w,\tau}/r]/(1-\varepsilon)}{z}$$

is a full negative coefficient, in the sense above ($T \succeq -g_{\mathrm{int}} r I$), for the original individual-square (pair-score) law. Here h, ε and z are those of the proposition at the same t. Its marginal and layer data remain those at the original parameter t.

Proof. Before normalizing the new tilt, split its exact physical operator as

$$(1-\tau)\Gamma_0 + \Gamma_0 \left(\tau + 2t\,\frac{1}{3}\sum_j H_j + t^2\,\frac{1}{3}\sum_j H_j^2 \right).$$

The first summand has lower price $(1-\tau)g_0 r$ in the mass-one physical convention. The second summand can have signed entries; the three-block argument requires only its positive S block and the stated gap gates. Its raw scalar normalization alone costs $(1-\varepsilon)^{-1}$. Absolute collision weights are bounded by $\tau + 2tB_H + t^2B_H^2$, with the same exceptional-pair terms. Add these two lower prices and divide once by the common new mass lower bound z. No second old-normalizer or half-conditioning factor belongs to the first summand. □

21.7. **Retaining the constant coefficient at the layer cut.**

Proposition 21.15 (Layer-cut remainder, summed without expanding all monomials)**.** *For the individual-square law in Proposition 21.9, put*

$$d_\mu = \frac{8}{1-q^2}, \quad M_\Phi = d_\mu(1+q)^3, \quad b_* = 6 + 51 d_\mu,$$
$$n = r\{\kappa(1+5r) + b_* r\}, \quad \kappa = (\sqrt{3} + \sqrt{7})/2, \quad B_H = 2 + r.$$

Replace its remainder coefficients by

$$C_{\mathrm{nn}} = (n/r)^2 + 3969 \cdot 16(2tB_H + t^2 B_H^2),$$
$$C_0 = 128M_\Phi + 8d_\mu^2(1+q)^5,$$
$$C_{\mathrm{sc}} = C_0 + 127(20tr + 16t^2 q + 2t^2).$$

Keep $C_{\mathrm{cr}}, C_{\mathrm{in}}$ and both mass lower bounds unchanged, and substitute $C_{\mathrm{nn}} + C_{\mathrm{sc}} + C_{\mathrm{cr}}$ in its negative, centered and marginal formulas. The resulting prices hold for the same positive law, with no change to its support or parameter.

Proof. Write the probability-normalized triangle operator of Proposition 21.9 as $\Phi = J + N$, where $\|N\| \le n$. In the score-independent coefficient retain the full $N \otimes N$, of norm at most n^2, instead of expanding its 63^2 nonempty monomials. The score-containing terms retain their specified four-label mask, with all labels distinct, and the two-Paley price, giving C_{nn}.

Delete the removed constant mask from the whole positive $\Phi_1\Phi_2$ of the two triangle operators. Among its four specified equalities, two twins cost $64M_\Phi q$ each and two cross-component pins cost $4d_\mu^2(1+q)^5 q$ each. Their positive union bound gives C_0, including all intersections. For terms containing a score, single coincidences in the specified single strata have $|H| \le 2r$ for the pair score H of Definition 21.1, and equal unordered pairs have $|H| \le 1/r$; the unchanged mass bounds $4q, 2q^2$ and absolute base density 127 give the remaining terms of C_{sc}. In particular no mask is deleted through a signed $N \otimes N$ kernel. Its row contribution is bounded by its full norm, so the same replacement is valid in the marginal estimate. The input-half conditioning factor from Proposition 21.9 remains an upper bound here.

The saving in the remainder coefficient is exactly

$$3969 \cdot 16 - (n/r)^2 + 127(4 + 2q) - C_0,$$

whose limit is $62476 - \kappa^2 > 62471$. After cancellation of r, every displayed endpoint expression has positive increasing numerators and decreasing positive lower denominators. The explicit mass, centered-gap and tensor-domain gates (Appendix D) therefore certify uniform prices above the endpoint. They must still be combined with a chamber price for the same t, fresh native completion, and the corresponding marginal restoration. □

21.8. **Supports of the 39-trace law.** The tilted law of the last interval has weight $\eta + (1+tH)^2$, where each trace in H uses the same auxiliary summation label on its two halves. Squaring introduces two auxiliary labels; their diagonal must be retained. The following lemma bounds the product synthesis on one half, including this diagonal, before it is contracted with the other half.

Lemma 21.16 (Higher-star products)**.** *Let $H = L_0^2(\mathbb{F}_p)$ and $U_S = (\psi_S - \mathbb{E}\psi_S)/r : H \to L^2(C)$, where $\psi_S(s) = \prod_{i \in S} \chi(s - C_i)$. For subsets S, T of the four core coordinates with S nonempty and $|T| \ge 2$, let $U_S \cdot U_T : H \otimes H \to L^2(C)$ be the pointwise product synthesis, defined on simple tensors by $(U_S \cdot U_T)(f \otimes g) = (U_S f)(U_T g)$. It has the decomposition*

$$U_S \cdot U_T = he_0^* + R, \quad Re_0 = 0, \quad \|R\| = O(1),$$
$$h = 1_{S=T}/r + O(1) \text{ pointwise}, \qquad e_0(s,u) = \frac{p1_{s=u} - 1}{\sqrt{p-1}}. \tag{21.11}$$

Here h is a function on core configurations and the $O(1)$ constants are absolute, uniformly in p and in the labels. The same conclusion holds for fixed linear combinations, with the coefficient of $1/r$ equal to their coefficient-vector inner product. A coordinate evaluation times a higher star has product norm bounded by an absolute constant. The assertion does not include a shared singleton–singleton support.

Proof. For $Df(s,u) = \sqrt{p}1_{s=u}f(s)$, its projection D_H to $H \otimes H$ obeys

$$D_H^* D_H = (1-2q)P_H + (1-q)J. \tag{21.12}$$

On the orthogonal complement of its range, auxiliary functions vanish on $s = u$ and all auxiliary centering terms disappear. If $S \neq T$, choose two output coordinates whose incidence with s, u is a matching or a three-edge path. Such coordinates exist also when one support contains the other. Two Paley transforms give norm q, with a bounded middle multiplication for the path. If $S = T$, choose two common coordinates and use the restricted $K_{2,2}$ estimate $2q$, retaining distinctness of both output and auxiliary pairs. After fixing all other core coordinates, their characters are bounded auxiliary multipliers. Exclusions of fixed coordinates are separate output restrictions. The actual core density is $16/\theta$, $\theta = (1-q)(1-3q)(1-7q)$, so output compression costs at most $4/\sqrt{\theta}$. Division by $r^2 = q$ proves the off-diagonal bound without deleting a coupled injectivity mask.

On the mean-zero part of D_H, repeated-root cancellation gives $U_{S\triangle T}$, or zero if $S\triangle T$ is empty, plus at most four evaluation terms of size $q/r = r$. Terms containing an auxiliary mean are bounded by the degree-four character estimate (Theorem 4.7). Equation (21.12) gives a bounded diagonal inverse. Finally the e_0 image is

$$\frac{\mathbb{E}_s[(\psi_S - m_S)(\psi_T - m_T)]}{r\sqrt{1-q}}, \qquad m_S = \mathbb{E}_s \psi_S.$$

Its numerator is $1 + O(q)$ when $S = T$ and $O(r)$ otherwise. For evaluation times a higher star, fix the evaluated coordinate; one other leaf supplies a Paley factor r, canceling the star's normalization. Passing from the product law to the actual law changes the bound only by the density factor $\prod_j D_j$ of Lemma 20.14, an absolute constant. □

21.9. **The 39-trace law at every sorting cut.** We now define the list of supports. A symmetrised list $\mathcal{D}_{51}$ of 51 supports is defined first, and the action of its score and of its square on the soft kernel (the kernel without the distinctness indicator) is computed, keeping every mean term of a star, including the functions invariant under simultaneous translation of all labels. Removing the twelve supports with fewer than three labels in one layer then leaves the list $\mathcal{D}_{39}$ used in the law, and explicit finite matrix bounds at the endpoint parameter $t_* = 12/5$ follow.

The twelve vertices are grouped into six bases, numbered $0, \dots, 5$, each carrying two labels (j, ℓ), $\ell \in \{0, 1\}$, in the two layers (the "twins" $(j,0), (j,1)$). The directed edges between the bases are

$$0 \longrightarrow 2 \longrightarrow 4 \longrightarrow 0, \qquad 1 \longrightarrow 5 \longrightarrow 3 \longrightarrow 1. \tag{21.13}$$

At the first chamber cut, with the two halves identified by the reflection (the reflected chart), the halves are $(a,b,c) = (0,1,2)$ and $(a',b',c') = (5,4,3)$. On each four-label core retain the four cubic supports and the quartic support. Give a support S sign $(-1)^{|S\cap a|}$. Form the twenty-five products of pairs of supports, one from each core, divide their coefficients by five, and average over

$$j \mapsto j,\ j+2,\ j+4,\ 1-j,\ 3-j,\ 5-j \pmod 6. \tag{21.14}$$

After collecting equal physical supports this gives seventy-five nonzero coefficients, each $\pm 1/15$.

Retain a support if, at every chamber cut, either its intersections with the core labels of both halves have size at least two, or its intersections with the private labels of both halves have size two. Denote this fifty-one-element dictionary by $\mathcal{D}_{51}$, retaining the collected coefficients c_D. Put

$$H_{\mathcal{D}} = \frac{1}{r}\sum_{D\in\mathcal{D}} c_D\, \mathbb{E}_s \prod_{i\in D} \chi(s - x_i), \qquad \sigma_{\mathcal{D}} = \sum_D c_D^2. \tag{21.15}$$

Here $x = (x_i)$ is the configuration of the twelve labels. This definition uses physical labels and is independent of a chosen cut. In particular $\sigma_{51} = 17/75$ and $\sum |c_D| = 17/5$. We give the complete argument keeping the common auxiliary label s (the root) unintegrated, since integrating it out

first and using only moment orthogonality would not suffice. Write $K_{\mathrm{soft}} = T[A, A]$, with A the exact swapped rectangle of Proposition 20.17, and

$$A = a \otimes 1 + r(\sqrt{30}\, Zv^* + 2we^*) + E, \qquad \|E\| = O(q), \quad a = 1 + O(r). \tag{21.16}$$

Both row and column squared norms of A are at most $256a_\infty$ (with $a_\infty = \|a\|_\infty$). For two trace supports, retain their common auxiliary labels s, t and split $s = t$ from $s \neq t$. If two nonempty raw supports have union of size at least two, their off-diagonal two-auxiliary map has norm $O(q)$. Distinct supports use a matching or three-edge path; equal supports use the restricted $K_{2,2}$ estimate. Output and auxiliary distinctness are both retained.

If the two maps obtained from the core part of A have the property of the preceding paragraph (norm $O(r)$ after one free transform) on both halves, write the modulated rectangles (rectangles A with left and right multiplication operators M_f, M_g by the displayed private multiplier functions f, g and their reflected counterparts $\tilde{f}, \tilde{g}$) as $B_{st} = M_f A M_g$ and $D_{st} = M_{\tilde{f}} A M_{\tilde{g}}$. The private multipliers have modulus at most one, so positive Gram Schur order gives

$$\|\mathbb{E}_{s \neq t} B_{st} B_{st}^*\| \leq 256 a_\infty \|F_{\mathrm{core}}\|^2 = O(q^2), \tag{21.17}$$

where F_{core} is the core synthesis operator of (21.16); and the same for D. Operator Cauchy–Schwarz bounds $\mathbb{E} B_{st} \otimes D_{st}^*$ by $O(q^2)$, with the private swap unchanged. Division by the score-square normalization q gives $O(q)$. If both private maps instead have the property, use the input Grams and the column bound. Exact support enumeration covers all 51^2 ordered pairs: 2537 use core output Grams and 64 use private input Grams.

On $s = t$, cancel repeated roots. The identical-support term is $K_{\mathrm{soft}} + O(q)$. Other terms are raw traces on the symmetric difference, with at most twelve hole terms (terms where the common auxiliary label equals a fixed label) of size q. A raw trace with nonempty supports on both halves costs $O(q)$ after multiplication by K_{soft}. For core/core or private/private placement, use the one-auxiliary form of (21.17), with two raw norms $O(r)$. Pure core/private placement reduces to A times a raw-star cross-kernel; expand (21.16). Rank and remainder terms cost $O(q)$. Every leading edge has a coordinate evaluation at one endpoint. The higher/raw product has a bounded exceptional e_0 image and an $O(r)$ remainder, whereas the evaluation/raw image is

$$\frac{\psi_S(x_i) - \mathbb{E}\psi_S}{\sqrt{p-1}} = O(r). \tag{21.18}$$

The old factor r makes these terms $O(q)$. A shared singleton pair gives $1_{x_i = y_j} - q$, whose positive equality mass is $O(q)$. One-sided differences give invariant functions of size $O(r)$; their leading one-sided ranks vanish under ordinary centering, and multiplication by $K_{\mathrm{soft}} - J = O(r)$ costs $O(q)$. Thus

$$Q(K_{\mathrm{soft}} \odot H^2)Q = \sigma Q K_{\mathrm{soft}} Q + O(q). \tag{21.19}$$

The corresponding linear calculation gives

$$Q(K_{\mathrm{soft}} \odot H)Q = QHQ + O(q), \qquad \|K_{\mathrm{soft}} \odot H\| = O(r). \tag{21.20}$$

Here are the exceptional shared-label cases. Rank/star terms have a free raw core and private transform, each $O(r)$. For the product of two stars, the two arms (synthesis factors) from the untilted law are (Z, v) and (w, e). The cross pairing has a free leaf at its (w, v) endpoint, since each half score uses at least three labels. For equal (Z, v) pairings, two private leaves again give a free transform. Otherwise the private support is a singleton and the core support has at least two labels. Split its higher/raw product into the constant exceptional image plus an $O(r)$ remainder. The two exceptional images contract against the swapped private evaluations; the delta terms vanish by $\chi(0) = 0$ and the remaining probability operator is $qQ/(p-1)$. It is $O(q^2)$, not an omitted order-r block. The equal (w, e) case exchanges core and private. Every single remainder costs $O(q)$. An E/star term applies E fiberwise between two bounded product maps with the

shared label retained. The E/rank term has free raw transforms; E/E costs $O(\|E\|^2/r)$. These prove (21.20).

For a half support S (a support contained in one half), extending the notation of Lemma 21.16, let $U_S = (\psi_S - m_S)/r$ on mean-zero auxiliary labels and $M_S = m_S/r$. With C the symmetric matrix, indexed by the half supports, whose entries are the collected coefficients c_D of (21.15), and U the row of operators $(U_S)_S$,

$$H = rUCU^* + rMCM^*. \tag{21.21}$$

The two terms occupy respectively nontrivial translation sectors and the invariant sector. To specify the finite matrix gate without ambiguity, index all nonempty half supports by S. Let D_{ij} be $+1$ or -1 according to the directed cross edge between coordinate i and the reflected coordinate j, and zero if it is absent. The old matrix has singleton block iD, higher/singleton entry $\prod_{i\in S} D_{ij}$, its adjoint entry, and zero higher/higher block.

The Gram matrix of the core raw mean functions M_S is at most $16+O(r)$, by cube orthogonality and the core density. Group mixed means by their private singleton or pair support. Their raw Gram is at most $q[1+q/(1-q)+2qr/(1-q)]I$; for each fixed injective core the character supports have Gram $I+O(r)$. Conditional synthesis gives mixed mean Gram $I+O(r)$. The Gram matrix of the common actual stars is likewise $I+O(r)$ on nonconstant auxiliary labels. These estimates will price the invariant sector as well as the nonzero translation sectors.

Physical completion. Each half support has at least three labels. Complete character cancellation therefore requires at least three cross identifications, of positive row/column mass $O(q^3)$. Everywhere each single-support trace factor in (21.15), before its coefficient, is at most $1/r$ in absolute value, so even at the configurations where a trace takes an exceptional value (the spikes), its square contributes $O(q^2)$ after the normalizations. Other collisions have mass $O(q)$ and bounded score by the repeated-root Weil bound (Theorem 4.7). This controls all physical deletions and their intersections.

The positive constant $\eta = 1/100$ added in the weight (its floor) and the Weil bound on the strict support (distinct labels) give uniformly positive bounded new marginal densities. The full identities for the new half-marginal density g with respect to the fixed half law give $\|g-1\|_2 = O(r)$. For each normalized star, the weighted affine Gram with weight $(g^{-1/2}-1)^2$ is real symmetric and a scalar multiple of the identity on mean-zero labels, with scalar its trace divided by $p-1$; this shows that multiplying the finite feature synthesis by $g^{-1/2}-1$ changes its norm by $O(r)$. Thus the leading r operator changes by $O(q)$ under actual-marginal congruence. The finitely many invariant mean arms are pointwise bounded; their multiplication by $g^{-1/2}-1$ likewise has norm $O(r)$ by its L^2 bound, covering the second term of (21.21). Finally, writing T for the chamber operator of the tilted law on the induced half,

$$T = g \otimes g + QTQ - (g-1)\otimes(g-1) \tag{21.22}$$

keeps the constant rank positive and charges only $O(q)$. This proves that the leading bound transfers from the fixed half chart to the actual half marginal at every chamber cut.

Remove from $\mathcal{D}_{51}$ the twelve supports containing fewer than three labels in one layer. Call the remaining dictionary $\mathcal{D}_{39}$. It has twelve supports of degree six, twenty-four of degree seven, and three of degree eight. Consequently

$$\sigma_{39} = 13/75, \quad \sum |c_D| = 13/5, \quad |H_{\mathcal{D}_{39}}| \le 14 + r \quad \text{on strict physical support.} \tag{21.23}$$

The last bound uses $(k-2)r+q$ for even degree k and $(k-1)r$ for odd degree, before dividing by r.

Lemma 21.17 (Admissibility of the coherent root)**.** *For every rational $t \ge 0$ and every $\eta > 0$, normalize $\eta + (1 + tH_{\mathcal{D}_{39}})^2$ against the old physical root at the prime $p \equiv 3 \pmod 4$ fixed above.*

The resulting probability law has exactly the old root's support and accepts the six queries. It is invariant under the three chamber reflections $j \mapsto 2a-1-j$, $a=0,1,2$, the rotation $j \mapsto j+2$, the simultaneous layer interchange, and the square-affine maps of the labels. Every slot is uniform. These assertions also hold for the primewise product on a block. In particular they hold at the selected parameter $t = 67/40$ and floor $\eta = 1/100$.

Proof. The initial average in (21.14) makes the signed support dictionary invariant under the chamber reflections and the rotation. Simultaneous layer interchange permutes the four cubic supports and fixes the quartic support in each core; it preserves their signs and the twenty-five corresponding products. The filter defining $\mathcal{D}_{51}$ is imposed at every chamber and is unchanged by these permutations. The subsequent filter requiring at least three labels in each layer is also unchanged. Thus each of these permutations preserves the collected coefficients of $\mathcal{D}_{39}$ and fixes its score.

Under $x_i \mapsto \alpha x_i + \beta$, with $\chi(\alpha) = 1$, the substitution $s = \alpha s' + \beta$ fixes every character product in (21.15), in both odd and even degrees. All the stated maps preserve the old physical root, so they preserve its tilted weight and normalizer. Since the weight is at least $\eta > 0$, normalization is defined and the support is unchanged; query acceptance follows from that of the old root. Translations act transitively on the labels, so every slot is uniform. Tensor products give the block assertions. Only the simultaneous layer interchange is needed; independent flips of individual twins are not asserted. □

Proposition 21.18 (Thirty-nine coherent traces at every sorting cut: the leading coefficients)**.** *Let Γ be the coherent law at the endpoint parameter $t_* = 12/5$, that is, the uniform law reweighted by $1/100 + (1 + (12/5)H_{\mathcal{D}_{39}})^2$ and normalized. It is invariant under the chamber reflections $j \mapsto 2a-1-j$, $a=0,1,2$, and under the simultaneous interchange of the two layers $(j,\ell) \mapsto (j, 1-\ell)$; these are the reflections occurring in the sorting sequences. (Independent flips of single twins are not symmetries and are not used.) There are absolute constants C and P_0 such that for every prime $p \equiv 3 \pmod 4$ with $p \geq P_0$ the Markov operators of Γ at its cuts, on its actual marginals, obey*

$$T_{\text{chamber}} \succeq -(5.413r + Cq)I, \qquad T_{\text{layer}} \succeq -(2.19r + Cq)I, \tag{21.24}$$

the first at all three chamber cuts. This proposition identifies only the leading coefficients 5.413 *and* 2.19*; by itself it gives no bound at any specified prime. The explicit bounds used in the construction, with every remainder evaluated, are Propositions 21.20 and 21.23.*

Proof. The chamber calculation for $\mathcal{D}_{51}$ above applies to the retained dictionary $\mathcal{D}_{39}$. There are 1493 core-output and 28 private-input covariance cases. Its strength is $\lambda = 2t/(1+\eta+t^2\sigma) = 12000/5021$. Exact rational gates (Appendix D) give chamber coefficient $5413/1000$, core mean coefficient $7/100$ and mixed mean coefficient $21/100$. The invariant chamber price is below $\lambda(16 \cdot 7/100 + 21/100) < 16/5$.

At the layer cut a half is initially two independent directed cyclic triangles, on bases $\mathcal{A} = \{0,2,4\}$ and $\mathcal{B} = \{1,3,5\}$. The actual half conditions their six labels to be distinct, at cost $1 + O(q)$. Each component has

$$\Phi = J + rFM_0F^* + E, \quad \|E\| = O(q), \qquad M_0 = \begin{pmatrix} iD_0 & -I_3 \\ -I_3 & 0 \end{pmatrix}, \tag{21.25}$$

where F is the synthesis operator assembled from the six feature maps of the preceding paragraph. Its arms (factors) are the three evaluations and three pair stars. The six omitted pair means give $+6qJ$, included in E. There is no higher/higher edge.

For every retained trace each half support meets both triangles in one or two coordinates and has size at least three. In the quadratic product use $q^{-1}\mathbb{E}_{s \neq t} B_{st} \otimes D_{st}$. Operator Cauchy–Schwarz now uses BB^* and D^*D: the small raw maps must be output $\mathcal{A}$ and input $\mathcal{B}$, or input $\mathcal{A}$ and

output $\mathcal{B}$. The exact enumeration gives 1437 cases of the first kind and 84 of the second. Each raw map is $O(q)$; the component density $8/(1-q^2)$ and bounded row/column squares of Φ give the full $O(q)$ off-diagonal contribution.

On the auxiliary diagonal, supports in opposite components use the one-label crossed argument with two $O(r)$ maps. If both lie in the same component, expand (21.25). The rank and remainder terms cost $O(q)$; each star has an evaluation endpoint, so (21.18) controls the exceptional image. Two singleton supports instead give an equality kernel of bounded-density mass $O(q)$. For this particular dictionary, collecting all one-sided symmetric differences gives thirty pair supports of total absolute coefficient mass $8/25$. After conditioning the half to be six-distinct, every such raw trace is exactly $-q$. Therefore at either cut the full covariance, in the fixed induced-half chart, is, writing K for the tensor square of the layer-cut kernel,

$$K \odot H^2 = \sigma K + O(q). \tag{21.26}$$

This full-norm identity is not asserted on the independent triangle chart before conditioning.

For the layer linear term expand the two copies of (21.25). One star term and one rank term have, on both sides, free raw star factors carrying none of the displayed characters (passive factors). For two stars, the scalar is $r^2/r = r$. A coordinate/coordinate endpoint has a free leaf because the score uses at least three coordinates. Every higher/higher endpoint is opposite one of this type. The remaining coordinate/higher against higher/coordinate case either has a free leaf or has two identical pair-star exceptional images. The latter is the same shared-label $qQ/(p-1)$ contraction used above; all single remainders have norm $O(r)$. An E/star term is a fiberwise E between bounded product maps; E/rank has free raw transforms and E/E costs $O(\|E\|^2/r)$. Hence $K \odot H = H + O(q)$ on the induced half.

The real vector space spanned by the real and imaginary parts of the layer feature functions that occur (the active layer realification) has dimension 54; rational elimination proves coefficient $219/100$. Its raw-mean coefficient matrix is bounded below by $-21/100$. Group mean arms by the singleton or pair in one triangle. The normalized six-arm nonconstant Gram is $I + O(r)$; the three pair constant images add only $3q^2 J$ before normalization. Fixed opposite-triangle character Grams are $I + O(r)$, so the complete mean frame is $I + O(r)$. Its invariant negative coefficient is below $\lambda(21/100) < 1$. Thus no invariant sector is omitted. Physical collision deletion, positivity and actual-marginal transport follow exactly as in the physical-completion argument above: all retained half supports still have size at least three. The dictionary, all 1521 crossed cases, chamber and layer matrix gates and separate mean gates are reconstructed by `check_coherent_39_all_cuts.py` (Appendix D). □

21.10. **One parameter u for both cuts of the 39-trace law.** The matrix bounds above were proved at the endpoint parameter $t_* = 12/5$, because the finite checks are cleanest there. The law used in the construction has the smaller parameter $t = 67/40$. The following proposition transfers the endpoint bounds to every $0 \le t \le t_*$ by convexity—the leading matrix is affine in an explicit interpolation parameter $\alpha(t)$—at both the chamber cut and the layer cut simultaneously; all remaining terms are then evaluated directly at $t = 67/40$.

Proposition 21.19 (The common coherent parameter interval)**.** *Keep $\eta = 1/100$, $\sigma = 13/75$ and the 39-trace score, and let $0 \le t \le t_* = 12/5$. Set*

$$z(t) = 1 + \eta + \sigma t^2, \qquad \lambda(t) = \frac{2t}{z(t)}, \qquad \alpha(t) = \frac{\lambda(t)}{12000/5021}. \tag{21.27}$$

Then $0 \le \alpha(t) \le 1$. In the leading matrices of the normalized coherent law, valid negative coefficients are

$$g_{\mathrm{ch}}(t) = (1-\alpha(t))\sqrt{30} + \alpha(t)\frac{5413}{1000}, \qquad g_{\mathrm{lay}}(t) = \frac{219}{100}.$$

The invariant mean branches obey the same displayed bounds. Each estimate in Propositions 21.20 and 21.23 below applies to the physical weight $\eta + (1 + tH)^2$ by using this t, its scalar leading mass $z(t)$, and the corresponding leading coefficient. The error terms in the linear and covariance identities for the score itself do not change with t. A separately certified matrix bound at a specified parameter may improve its leading coefficient, provided the separate invariant-mean price remains below it. This changes neither the law nor its other prices.

Proof. The exact factorization

$$t_* z(t) - t z(t_*) = (t_* - t)(1 + \eta - \sigma t t_*), \qquad 1 + \eta - \sigma t_*^2 = 29/2500 \tag{21.28}$$

gives $0 \le \alpha \le 1$. At either cut, with M_{old} and C denoting the untilted leading matrix and the score coefficient matrix at that cut, the leading coefficient matrix is

$$M_{\mathrm{old}} + \lambda(t) C = (1 - \alpha) M_{\mathrm{old}} + \alpha\big(M_{\mathrm{old}} + (12000/5021) C\big).$$

The endpoint orders are those proved in Proposition 21.18. The old chamber coefficient is $\sqrt{30}$; the old layer coefficient $(\sqrt{3} + \sqrt{7})/2$ is less than $219/100$. Convexity gives the stated bounds. The old invariant centered leading operator is zero, so the separate invariant branches interpolate as well.

For the finite calculation, expand the raw weight before normalizing. The old-root, linear and covariance terms have coefficients $z(t), 2t, t^2$, respectively; the regular and exceptional collision weights are their explicit positive quadratic polynomials. Each primitive error is bounded separately below. Substituting these coefficients therefore prices the same physical law at both cuts. The $10q$ allowance for centering the layer mean arms is bounded at t_* and remains valid for smaller t (see the proof of Proposition 21.20). For each fixed t, the positive endpoint majorants give uniformity in p; no monotonicity of the final price in t is needed. □

21.11. **An explicit layer price for the coherent law.** We now evaluate the layer-cut bound for the same 39-trace law numerically. All norms in this subsection are taken in L^2 of the fixed uniform half law (the fixed chart), not in the tilted law's own marginal; the passage from the chart to the actual marginal was already bounded in the transport estimates above, so no extra factor for that change of measure is charged here. Fix $0 \le t \le 12/5$ and keep $\eta = 1/100$, $L = 13/5$, $\sigma = 13/75$ and $z = z(t) = 1 + \eta + t^2\sigma$ from Proposition 21.19. For $p \ge 2^{32}$ set

$$\begin{aligned} h &= 1 - 9q, \quad d = \frac{8}{1 - q^2}, \qquad M = d(1 + q)^3, \quad \Gamma = 1 + 1000r, \\ B_H &= 14 + r, \quad W = \eta + (1 + tB_H)^2. \end{aligned} \tag{21.29}$$

Define positive coefficients

$$\begin{aligned} C_{\mathrm{soft}} &= \frac{1100z + 2tL \cdot 100000 + t^2 \cdot 200000 + 10}{h}, \\ C_{\times} &= \frac{W}{h}[384M + 72d^2(1 + q)^5], \\ C_{\mathrm{exc}} &= \frac{9830400 d^2}{h}[(1 + \eta)q^2 + 2tLqr + t^2 L^2 q], \qquad C = C_{\mathrm{soft}} + C_{\times} + C_{\mathrm{exc}}. \end{aligned} \tag{21.30}$$

Here C is a finite coefficient, not an unspecified asymptotic constant. Put

$$m = z - Cq, \qquad \delta = \frac{Cq}{m}, \qquad c = \frac{4z\Gamma r + Cq}{m}. \tag{21.31}$$

Proposition 21.20 (Finite layer-cut bound for the coherent law in the fixed half chart)**.** *With $p \ge 2^{32}$ and $0 \le t \le 12/5$ as fixed above, on the explicit gates $m > 0$ and $c < 1$, the normalized*

physical layer operator of the law in Lemma 21.17, represented on the fixed uniform six-distinct induced half and written T, satisfies

$$-g_{21} r I \preceq T \preceq (1 + B_{21} q) I, \tag{21.32}$$

where

$$g_{21} = \frac{z(219/100)\Gamma + Cr}{m} + \frac{C^2 q r}{m^2}, \qquad B_{21} = \frac{C^2 q}{m^2(1-c)}. \tag{21.33}$$

Its marginal defect (the difference between its actual half marginal and the fixed chart marginal, as a density minus 1) has L^2 norm at most δ, hence variance at most δ^2. Its pointwise marginal range ratio is at most the old range ratio multiplied by W/η. Evaluation of these positive majorants at the endpoint $p = X$ (the lower endpoint of the interval) is valid uniformly for every prime $p \geq X$. The separate chamber inequalities and the completion inequalities for the law itself (not a converted one) remain to be checked.

Proof. We first justify every finite constant. A cyclic triangle has density $d = 8/(1-q^2)$ relative to its three-label cube and exactly $(p+1)/4$ completions of a directed pair. The inherited six-cycle expansion gives (21.25) with

$$\|E\| \leq (6 + 51d)q < 500q, \qquad \|rFM_0F^*\| \leq \kappa(1+5r)r, \quad \kappa = (\sqrt{3}+\sqrt{7})/2. \tag{21.34}$$

The twelve star entries are six evaluation/evaluation and six evaluation/pair entries. The positive row and column bound is M, and $0 \leq \Phi \leq 64$, so their squared norms are at most $64M < 600$; also $\|\Phi\| < 2$. Consequently the old two-component root equals its constant plus leading star operator with remainder at most $1100q$, since $1000 + [\kappa(1+5r) + 500r]^2 < 1100$.

For completeness the finite product bounds used next hold also when a raw symmetric-difference support is the full triangle. If $|S| = 2$ and $1 \leq |T| \leq 3$, the normalized pair star times ψ_T has decomposition $h_{ST} e_0^* + R_{ST}$. The exact diagonal image is $(\mathbb{E}\psi_S\psi_T - m_S m_T)/\sqrt{1-q}$. Off the auxiliary diagonal, a matching/path or restricted $K_{2,2}$ map costs at most $2\sqrt{d}\, r < 6r$. The mean-zero diagonal part is below $2r$, using $D_H^* D_H = (1-2q)P_H + (1-q)J$ and the at-most-three removed roots. The constant-input part is at most $2r$. Thus $\|R_{ST}\| \leq 10r$, $|h_{ST}| \leq 2$, and $S \neq T$ gives $|h_{ST}| \leq 3r$. A raw singleton gives a whole product at most $13r$. A coordinate times a raw pair or triple has norm at most $\sqrt{d}\, r < 3r$, by fixing that coordinate and retaining a free Paley transform. Every whole product has norm at most two. The coarser constants $100r, 4r, 2$ will suffice.

There are 24 one-component leading terms in the linear rooted expansion. Each costs at most $4q$. Of the 144 two-component terms, 126 have a free evaluated leaf and cost at most $10q$ each; fixing that coordinate leaves conditional density at most $4/(1+q) < 4$. The 18 exceptional evaluation/pair against pair/evaluation terms have residual at most $400r + 10000q + 8q^2 < 500r$ before their old factor r. This includes the exact double-e_0 contraction. The 24 remainder/star placements cost at most $24 \cdot 4 \cdot 500q$; the two remainder/rank terms cost $1000rq$, and the remainder/remainder term costs $250000rq$. Their sum is below $100000q$ per trace. Summing the absolute coefficient mass L proves the linear price used in (21.30).

For the quadratic term, each of the 1521 valid crossed factorizations has two raw two-label maps at most $6q$. Its positive Gram CS bound is at most $600 \cdot 9 \cdot 4q = 21600q$. The diagonal two-sided raw-trace bound is at most $6100q$: opposite components use two raw transforms of norm at most $2r$, giving $2400q$, while a common component uses its rank, twelve star entries and $500q$ remainder, followed by the other component of norm less than two. The evaluation endpoint controls its exceptional diagonal image. Holes cost at most $32L^2q$. Collect the one-sided pair terms only after conditioning both halves to be distinct; then they are exactly $-q$ and have absolute coefficient

mass $8/25$. Thus the full induced-half covariance error is at most

$$\frac{q}{h}\{(21600+6100+32)L^2+4(8/25)\}<200000q/h. \tag{21.35}$$

Before conditioning a pair spanning the two triangles may have trace $1_{x_i=x_j}-q$, so no unconditioned full-norm claim is used here.

For collisions in the actual kernel, twelve equalities between labels at different bases make the untilted (old) root kernel vanish. The six remaining twin events have positive total price $384Mq/h$; the eighteen cross-component events cost $72d^2(1+q)^5q/h$. Multiply their union by W. Partial root repetitions still satisfy the even/odd degree bound in (21.23). Complete cancellation requires a three-matching between the halves. There are at most $\binom{6}{3}^2 3!=2400$ such matchings. The input-half cube density is at most d^2/h, the root kernel at most 4096, and each matching has mass q^3. The unrestricted score bound L/r therefore gives exactly qC_{exc}. Positive unions include all further collisions and intersections; no coupled mask was removed inside a norm.

The finite actual-star Gram is at most $1+182r$. Indeed every conditional vertex has at least $(p-11)/4$ choices, and the common actual-star identities give off-diagonal Gram entries at most $7r$ among 27 active arms. The fifteen mean arms have Gram at most

$$\frac{(1+5r)(1+10r+5q)}{1-9q}\le 1+16r: \tag{21.36}$$

grouping by singleton/pair support in one triangle, using its raw six-arm frame, and then using the pointwise character Gram of the other triangle. Both are bounded by Γ.

Each mean arm has expectation at most $100q/h$. Before half conditioning its raw mean is zero when the support meets the two triangles in 1 and 2 labels, and q^2 when it meets them in 2 and 2 labels. Removing nine equalities costs $27qr$, except complete two-pair cancellation of probability at most $18q^2$; division by rh gives $(27+19r)q/h<100q/h$. With mean synthesis norm at most two, coefficient norm at most $\sqrt{\sigma}<1/2$, and fifteen arms, centering the raw-mean contribution to $2tH$ costs less than $10q$. Nonconstant auxiliary arms already kill constants by translation invariance.

We have therefore proved the exact decomposition $T_{\mathrm{raw}}=zJ+L_{\mathrm{cen}}+R$ with

$$L_{\mathrm{cen}}1=0,\quad \|R\|\le Cq,\quad L_{\mathrm{cen}}\succeq -z(219/100)\Gamma rI,\quad \|L_{\mathrm{cen}}\|\le 4z\Gamma r. \tag{21.37}$$

The lower order uses the 54-dimensional rational gate, the separate invariant mean gate and Proposition 21.19; the upper norm follows from $\kappa+\lambda\sqrt{\sigma}<4$. The actual raw scalar mass is at least m. After normalization the constant entry is exactly one, the off-diagonal vector has norm at most δ, and the centered block has norm at most c. By Lemma 13.5, the positive constant square costs at most δ^2 below, and the separated upper blocks cost $\delta^2/(1-c)$. These are precisely (21.33). The pointwise weight bounds $\eta\le w\le W$ prove the range assertion without a pointwise closeness claim.

Finally every numerator after its indicated power of q or r is removed has nonnegative coefficients and increases with q,r; $h,m,1-c$ decrease on their positive gates. Rational outward endpoint evaluation is uniform. The independent checker rebuilds the 39-support dictionary, both collected diagonals and 136 scalar quantities at eight endpoints, authenticating the separate matrix and triangle-product certificates. Its parameter replay checks 72 endpoint/parameter cases and 1296 rational quantities, recovering the values at the fixed parameter $t=t_*$. At $X=2^{45}$ the selected $t=67/40$ gives $g_{21}<2.869198$ and $B_{21}<0.460808$, compared with $g_{21}<3.163793$ and $B_{21}<0.947552$ at $t=t_*$. □

At the selected exact parameter $t=67/40$, the chamber strength is $\lambda=402000/179557$ and a rational matrix gate improves the convex chamber coefficient to 5.41674864. The checker `check_coherent_39_selected_matrix.py` reconstructs the physical dictionary and directed reflected matrix, proves all 56 rational LDL pivots positive, and verifies all 1596 exact reconstruction

identities. The invariant mean order remains below 3.2. The parameter interval above is used for the finite remainder and layer estimates; this stronger matrix gate is used only at its stated exact parameter.

21.12. **A finite chamber price for the coherent law.** This is the corresponding numerical evaluation at the chamber cut, for the same law Lemma 21.17 and with the same convention: all norms are in L^2 of the fixed uniform chamber half law, and no extra change-of-measure factor is charged here. Throughout this subsection

$$q = p^{-1}, \quad r = \sqrt{q}, \quad p \equiv 3 \pmod 4, \quad p \geq 2^{40},$$
$$0 \leq t \leq \frac{12}{5}, \quad \eta = \frac{1}{100}, \quad \sigma = \frac{13}{75}, \quad s = z(t) = 1 + \eta + t^2\sigma.$$

The weight of the law (before normalization) is $w = \eta + (1 + tH)^2$. Recall that the 39 coefficients in H are $\pm 1/15$, of total absolute mass $\ell = 13/5$ (denoted L in the preceding subsection). The degree count and Weil estimate in (21.23) give

$$|H| \leq B_H := 14 + r \tag{21.38}$$

on the support of the uniform law.

Use the exact rectangle of the uniform law (Proposition 20.17) and its probability-normalized decomposition

$$A = a \otimes 1 + r(\sqrt{30} Z v^* + 2 w e^*) + E, \qquad \|E\| \leq N_0,$$

with N_0 the nonstar bound of Proposition 21.23, and where the repeated letter w in this rectangle denotes its four-coordinate feature function, not the weight w above. Write a_+ for a pointwise upper bound on a, a_2 for its L^2 bound, and put

$$\theta = (1-q)(1-3q)(1-7q), \qquad h = (1-4q)(1-5q), \qquad D_6 = \frac{16}{\theta h}.$$

The row and column squared norms of A are at most $256 a_+$. Let K_ν be its swapped product after compression to the old six-distinct induced half. A norm-bounded independent-half remainder costs at most $1/h$; an unchanged leading feature has its original coefficient after compression.

Lemma 21.21 (Full coherent covariance)**.** *One has*

$$\|K_\nu \odot H^2 - \sigma K_\nu\| \leq E_2/h,$$

where

$$E_2 = q\left[256 a_+ \left\{ \frac{20992}{225\theta} + \frac{2108}{225(1-q)} \right\} + \left(\frac{32}{225} + \frac{8}{25} + \frac{2028}{25} \right) a_2^2 \right] < 26400 q. \tag{21.39}$$

This is a full operator estimate, including its constant sector.

Proof. For each ordered pair of score supports, keep their common auxiliary labels and split the diagonal from its complement. On the complement, use the cheaper of the two valid whole-rectangle Cauchy–Schwarz factorizations. A different-support core product costs $4q/\sqrt{\theta}$ and an equal-support product twice that; the private costs are $q/\sqrt{1-q}$ and twice that. The exact ordered counts of support pairs, by core and private product coefficient, are

core product coefficient	16	32	64
number	804	56	41
private product coefficient	1	2	4
number	184	240	196.

Multiplying by the rectangle row or column square and by $1/225$ gives

$$256 a_+ q \left\{ \frac{17280}{225\theta} + \frac{1448}{225(1-q)} \right\}.$$

The operator Cauchy–Schwarz uses the two output Grams for the swapped rectangle, or the two input Grams in its private version. It does not introduce a second copy of either auxiliary label.

On the auxiliary diagonal, collect equal symmetric differences before taking absolute values. Every nonzero one-sided difference has size two. Its trace in the induced half is exactly $-q$; the absolute coefficient mass is $4/25$ on each side. For the two-sided differences, the private raw-star maps give weighted cost $44/15$, and the remaining core maps give $3712/225$. A core singleton has norm r; a higher raw core star has norm at most $4r/\sqrt{\theta}$. The only unmatched terms are sixteen core-singleton/private-singleton terms of total mass $32/225$. Their raw trace is $1_{x=y} - q$, and the accepting rectangle vanishes at $x = y$, so their operator is exactly $-qK_\nu$. Finally the repeated-root holes cost at most $12\ell^2 q\|K_\nu\|$. These are precisely the remaining terms in (21.39). The one-sided identities are asserted after induced-half conditioning; before conditioning a core/private pair may coincide. The other remainder estimates are compressed with the single factor $1/h$. □

Lemma 21.22 (Finite coherent linear error)**.** *With* $E_1 = 12500q$*,*

$$\|Q(K_\nu \odot H - H)Q\| \le E_1/h, \qquad \|K_\nu \odot H\| \le (12r + E_1)/h.$$

Proof. Here are explicit rounded product bounds sufficient for the assertion. The inherited positive endpoint majorants give

$$N \le 282q, \qquad \|a - 1\|_2 \le 4.5r, \qquad a_+, a_2, \|Z\| < 1.001, \qquad \|K_\nu - J\| < 15r.$$

For higher core supports S, T, split the normalized product map by the constant diagonal vector $e_0 = (p1_{u=v} - 1)/\sqrt{p-1}$, its mean-zero diagonal, and its off-diagonal complement. Set

$$b_1 = 0, \quad b_2 = q, \quad b_3 = 2r, \quad b_4 = 2r + q,$$
$$D_2 = \frac{2}{1-q}, \quad D_3 = \frac{4}{(1-q)(1-3q)}, \quad D_4 = \frac{16}{\theta}.$$

For $S \ne T$, writing $c = |S \cap T|$ and $d = |S \triangle T|$, the three pieces are bounded respectively by

$$\sqrt{D_{|S\cup T|}}, \qquad \frac{1 + cr + b_{|S|} + b_{|T|}}{\sqrt{1-2q}}, \qquad \frac{b_d + cq + b_{|S|}b_{|T|}}{r\sqrt{1-q}}.$$

For $S = T$ double the first bound, omit the last, and remove the constant image $1/[r\sqrt{1-q}]$. Its remaining variation is at most $(|T|q + b_{|T|}^2)/[r\sqrt{1-q}]$. These bounds follow from the two-Paley matching/path extraction or the restricted $K_{2,2}$ bound, and the exact Gram of the projected auxiliary diagonal. Their orthogonal-input square sum, the eleven-star expression for Z, and its singleton correction give residual bound 22 and exceptional coefficient at most $.259/r$. The four-singleton aggregate has product bound 9. Each expression is a positive endpoint majorant; the displayed rational constants hold already at $p = 2^{40}$.

There are thirteen pure-core terms (whose supports at this cut lie in the core labels of the halves). For each, the rectangle expansion prices the three rank and star classes, the three classes containing a nonstar remainder, and the raw invariant means by the following coefficients of q:

$$9.1, \quad 278, \quad 400, \quad 565, \quad 801, \quad 1, \quad 61.$$

For clarity the nontrivial rounded expressions are

$$278 > 2(1 + 4.5r)(22\sqrt{30} + 18),$$
$$400 > 30r\,22^2 + 60(.259)22 + \frac{30(.259)^2 r}{1-q} + 4\sqrt{30}r\,22 \cdot 9 + 4\sqrt{30}(.259)9 + 4r\,9^2,$$
$$801 > 2 \cdot 282\{\sqrt{30}(.259 + 22r) + 18r\}.$$

The factor $1 + 4.5r$ is the invariant weighted-star correction from $a - 1$. The large constant output of a pure-core rank/star term is killed by Q on that half; its nonconstant variation was included

in the residual bound 22. The retained uncentered rank/star norm, after weighting all thirteen terms, is below $6r$. The mean contribution uses $(2+r)^2\|K_\nu - J\|r < 61q$. The sum of these seven prices is below 2300.

The other twenty-six terms are mixed. Weighted raw-star synthesis gives a rank/rank cost of at most $91q$. The free core and private transforms give a rank/star cost of $61q$. For the two cross-arm star/star terms, the w, v triple map has a free Paley leaf and norm below $6r$, while Z, e has norm below 1.001; their price is 132. Equal Z, v arms have exceptional amplitude at most .259 and residual $24r$; equal w, e arms have amplitude at most .501 and residual $6r$. The double exceptional contractions are exactly $qQ/(p-1)$, not products of their separate norms. These prices are 375 and 26. The nonstar/star chain applies the input product, then E fiberwise, then the output product, keeping both shared labels. Their norms are at most 1.04 and 1.001, so both placements cost less than

$$2(\sqrt{30}+2)(1.04)(1.001)282q < 4500q.$$

Nonstar/rank and nonstar/nonstar cost 566 and 1. Thus the mixed price is below 6000. The total is bounded by $(13 \cdot 2300 + 26 \cdot 6000)q/15 < 12500q$.

For the full-norm assertion, the raw score coefficient matrix has absolute row sum at most $1/3$. Cube star ranges are orthogonal, so compression gives $\|H\| \le D_6 r/3 < 5.334r$. Adding the retained rank/star norm below $6r$ and the error just proved gives the stated $12r + E_1$ bound, with its compression price. □

We state the final finite scalar interface explicitly. Let N_0, D, ϵ, χ be the authenticated old raw nonstar, positive collision, scalar-normalizer and normalized marginal-variance bounds: $N_0 = N_{\mathrm{ns}}$, $D = D_{\mathrm{bal}}$, $\epsilon = \varepsilon_Z$ and $\chi = \chi_p^+$ in the notation of Propositions 20.15 and 20.21. Put

$$\begin{aligned} P_* &= 144 \cdot 65536 D_6 q^3, \\ C_w &= [\eta + (1+tB_H)^2]D + P_*[\eta + 1 + 2t\ell/r + t^2\ell^2/q], \\ L &= \frac{(12r+E_1)/h + B_H D + \ell P_*/r}{1-\epsilon}, \\ V &= \frac{E_2/h + (B_H^2+\sigma)D + (\ell^2/q+\sigma)P_*}{1-\epsilon}, \\ z &= s - 2tL - t^2V, \qquad \delta = \frac{s\sqrt{\chi} + 2tL + t^2V}{z}. \end{aligned} \tag{21.40}$$

(From here z denotes the proved lower bound $s - 2tL - t^2V$ for the scalar mass whose leading value was called $z(t)$ above; the letter L in z is the linear error term of the display, not the block budget.) Let $\Gamma = 1 + 434r$. Let g_* be any number such that both the nontrivial and invariant branches at the specified t have leading negative coefficient at most g_*; the value $g_{\mathrm{ch}}(t)$ of Proposition 21.19 may be used, or a sharper certified coefficient at the specified t. In particular the selected $t = 67/40$ uses $g_* = 5.41674864$. Define

$$\begin{aligned} E_* &= sN_0 + 2tE_1/h + t^2E_2/h + C_w, \\ G &= \frac{sg_*\Gamma r + E_*}{(1-\epsilon)z}, \qquad c = \frac{s\sqrt{30}\Gamma r + 2tD_6 r/3 + E_*}{(1-\epsilon)z}. \end{aligned} \tag{21.41}$$

Proposition 21.23 (Finite chamber-cut bound and marginal data for the coherent law)**.** *The explicit gates $z > 0$, $c < 1$, and $\delta < 1$ hold on the stated domain. The normalized physical operator of Lemma 21.17, on the fixed old induced half and written T, satisfies*

$$-(G+\delta^2)I \preceq T \preceq \left(1 + \frac{\delta^2}{1-c}\right) I.$$

Its marginal variance is at most δ^2, and its marginal range ratio is at most the old ratio times $[\eta + (1 + tB_H)^2]/\eta$. Every endpoint price is uniform for all primes $p \equiv 3 \pmod 4$ above the endpoint.

Proof. Complete character-root cancellation needs three cross identifications for each of the twelve degree-six traces, or four for each of the three degree-eight traces. There are at most $12 \cdot 3! + 3 \cdot 4! = 144$ matchings. The raw root is bounded by 65536 and the input half has cube density at most D_6, giving the positive exceptional row/column mass P_*. Outside it, cancellation only reduces the Weil price in (21.38); everywhere $|H| \le \ell/r$. Thus C_w bounds the difference between the soft kernel and the kernel with the distinctness conditions imposed, including all intersections. The same argument and the two lemmas give the full normalized-old prices L and V. The new scalar mass is at least z, and its centered row has norm at most δ.

The general actual-star Gram bound applies to all 63 nonempty supports. Every distinct-feature entry is at most $7r$, giving Γ. The exact matrix inequality of Proposition 21.19 (its certified gate) gives the nontrivial negative leading price $sg_*\Gamma r$. In the invariant sector the bound from the core mean features price is $16/(\theta h)$, and the mixed price is at most

$$\frac{[1 + q/(1-q) + 2qr/(1-q)](1 + 45r + 60q)}{h}.$$

The latter follows by enlarging the core character list to all 15 nonempty supports: each distinct-support entry is at most $3r + 4q$, so its row sum is at most $1 + 42r + 56q$. The respective negative coefficient orders $-7/100$ and $-21/100$ keep this invariant branch strictly below the nontrivial branch. The full positive centered price follows from $\|M_{\mathrm{old}}\| = \sqrt{30}$ and $\|H\| \le D_6 r/3$. This proves the centered bounds $-G$ and c. The constant-block bounds of Lemma 13.5 cost at most δ^2 below and $\delta^2/(1-c)$ above. The pointwise multiplier lies between η and $\eta + (1 + tB_H)^2$, proving the range statement.

After substituting $q = r^2$, all new normalized numerator majorants are positive increasing functions of r, while z and $1 - c$ decrease. For example the three exceptional terms in C_w/r have powers r^5, r^4, r^3. The inherited majorants have the same endpoint property. Thus the gates and orders extend uniformly above the endpoint. Exact coefficient collection, finite product gates and an independently assembled scalar replay accompany the proof (Appendix D). For the endpoint parameter $t = t_*$ at 2^{45}, with g_{20} the negative coefficient in $T \succeq -g_{20} r I$ and B_{20} the excess in $T \preceq (1 + B_{20} q) I$ (both formed from G, δ, c by the formulas above, as g_{21}, B_{21} are for the layer cut), the values are $g_{20} < 5.441282211737$ and $B_{20} < 1100.871034665$. These are local interfaces; the layer price, native completion and full scalar comparison are checked separately. □

Proposition 21.24 (The coherent score on a single collision)**.** *For the same* 39*-trace coherent law, put*

$$B_c = \frac{64}{5} + \frac{8}{5} r, \qquad B_s = 10 + \frac{11}{5} r, \qquad B_x = \frac{26}{3} + \frac{31}{15} r, \qquad B = 14 + r.$$

Let A_c, A_p, A_m be the positive collision bounds

$$\begin{aligned}
e_2 &= r + 48q, \quad e_3 = 5r + 96q, \quad e_4 = 17r + 192q, \\
\Lambda_c &= (1 - r - 24q)^4 (1 - 6q)^2, \\
A_c &= \frac{32q(1 + e_3)^2 (1 + e_4)^3}{\Lambda_c}, \\
A_p &= \frac{32q(1 + e_2)(1 + e_3)^2 (1 + e_4)^2}{\Lambda_c}, \\
A_m &= \frac{(1 + e_4)^6}{\Lambda_c} (3712q^2 + 81920q^3 + 1245184q^4 + 12582912q^5 + 67108864q^6).
\end{aligned}$$

With $w(x) = \eta + (1 + tx)^2$, replace only C_w in the negative price of Proposition 21.23 by

$$C_{\mathrm{pin}} = A_c w(B_c) + A_p[w(B_s) + w(B_x)] + A_m w(B) + P_*[\eta + 1 + 2t\ell/r + t^2\ell^2/q].$$

Retain the previous z, δ, c and all positive, variance and range prices. The resulting lower coefficient is uniformly valid for $p \geq 2^{40}$ and each fixed parameter t for which the leading matrix bound has been certified.

Proof. At the chosen chamber cut, numbering the label (j, ℓ) by $2j + \ell$, the coordinate sets are

$$\text{cores: } \{0, 1, 4, 5\},\ \{6, 7, 10, 11\}; \qquad \text{private pairs: } \{2, 3\},\ \{8, 9\}.$$

Only their sixteen core/core and four private/private coincidences survive the soft accepting kernel. For exactly one equality $x_a = x_b = u$, use $\chi(v - u)^2 = 1 - 1_{\{v=u\}}$ in every score monomial and collect the remaining supports. In the score divided by r, the deleted evaluation contributes at most r times its absolute coefficient. No remaining support has fewer than four labels. For a remaining support of degree d, the even-degree character price is $d - 2 + r$ and the odd-degree price is $d - 1$. Exact collection gives eight pins with bound B_c, six with B_s, and six with B_x.

The eight B_c pins join the output a pair to the input a pair, or output c to input c. The other two classes are

$$\begin{array}{c|l} B_s & (0,6), (1,7), (4,10), (5,11), (2,8), (3,9) \\ B_x & (0,7), (1,6), (4,11), (5,10), (2,9), (3,8). \end{array}$$

For $p \geq 2^{40}$, B_c dominates the other two prices, so the original mutually exclusive core orientations retain their total incident charge $32q$. For each output private coordinate, the two input private pins have the same incident cyclic-extension weight, at most 16, and one belongs to each of the B_s, B_x classes. Their row charges are therefore $32q$ each. The counts of extensions of a pinned pair to full labellings (the extension histograms) are unchanged within the two groups and give A_c, A_p; transposition gives the same column bounds.

The strata with exactly one coincidence are disjoint from the set with at least two coincidences, already bounded by A_m. Keep the original A_m with the generic score bound B, and the entire exceptional mass P_* with the worst score ℓ/r. Schur's test gives the displayed collision price. Recompute the numerator of G in (21.41), with C_{pin} in place of C_w in E_*, using only positive terms, retaining the old denominator and constant-block charge. This proves the same-law substitution without subtracting monotone majorants. All normalized numerator prices increase with q and r, and the previous denominators decrease on their gates, proving endpoint uniformity. The independent replay verifies the 20 support collections, 180 finite identities and all 35 endpoint/parameter cases. At $p = 2^{45}$ and the rational parameter $t = 3337/2048$ (a rational below the endpoint $t_* = 12/5$ at which this refined check is run; the selected construction uses $t = 67/40$ with the coefficient above), the refined coefficient is $g_{20} < 5.434524687965$, with the positive and marginal prices unchanged. □

21.13. **Values of the bounds at the seven interval starts.** The preceding subsections proved bounds for the tilted law in terms of quantities evaluated at the start of each interval. The following certificate records those values at the seven starts, checked by exact and outward-interval arithmetic; the table at its end is the form in which later sections use them.

Certificate 21.25 (The positive roots)**.** The chamber caps for the selected individual-square (pair-score) laws combine Propositions 21.11 and 21.13 and Corollary 21.14; their layer prices use Proposition 21.15 where it improves the retained finite price. The last band uses Propositions 21.19 and 21.24, with the exact matrix gate at $\mathsf{u} = 67/40$. All prices concern the same tilted law at both cuts, and that law's actual marginal enters the restoration. Every positivity or contractivity condition required above — in particular $z_- > 0$, $\Delta_g > 0$, $\rho_{\mathrm{in}} > 0$, $c_{\mathrm{ch}} < 1$, $d_{\mathsf{u}} < 1$ and $gX^{-1/2} < 1$

at each endpoint X — is checked in outward interval arithmetic, with the uniform endpoint majorants proved above.

The exact selected data are the `local.spectra` and `local.further_root_law` entries of `research/push_2026_10_25/global/refined_parent.json`. They agree exactly with the native data authenticated by `research/improvements_2026_10_25/verify.py`; only scalar charges are reanchored by `research/push_2026_10_25/verify.py`. The former reruns independent collision, pair-gap, interpolation, resummation, coherent covariance, selected matrix, finite chamber, finite layer and single-pin checkers. Its full tier regenerates the native completion. The latter checks exact equality of all native records and reruns the independent closing and restoration assemblies. At $L_{\mathrm{loc}} = 585$, the exported prices, rounded upward, are:

piece start	law, u	cut	g	c^+	block cap
2^{30}	individual squares, 711/4096	L20	5.476619678296	25.456659470084	5.476623331076
		L21	6.313078885225	3.988889092310	6.313079545011
2^{31}	individual squares, 1021/4096	L20	5.474335244881	27.623843030000	5.474337162024
		L21	6.315256713466	3.962812801799	6.315257030740
2^{32}	individual squares, 1455/4096	L20	5.471257426364	30.397461416086	5.471258447645
		L21	6.312562621834	3.927444212402	6.312562774077
2^{34}	individual squares, 833/1024	L20	5.464133834236	36.835698528597	5.464134125055
		L21	6.307298870667	3.877354549660	6.307298906003
2^{36}	individual squares, 3851/4096	L20	5.460775686025	37.085210712622	5.460775755114
		L21	5.426082230198	1.143781717023	5.426082232316
2^{40}	individual squares, 4049/4096	L20	5.457521105807	37.019255725148	5.457521109684
		L21	4.646587567237	0.075183415124	4.646587567244
2^{45}	coherent 39, 67/40	L20	5.434450251962	983.324753609660	5.434450254811
		L21	2.869197519095	0.460807795864	2.869197519096

The seven per-prime marginal variances and range ratios are recorded as exact rationals in the corresponding rows of the data file named above and displayed in Table 36. They are bounds in the fixed uniform chamber half, with its actual root marginal retained. No return to a uniform root is made in computing the restorations.

The finite tables retain the larger choice $\mathfrak{b}_{\mathrm{cert}} = 6 + 51d_\mu$ in the layer formulas and the corresponding certified remainder prices. Since $\mathfrak{b} \le \mathfrak{b}_{\mathrm{cert}}$, these remain valid majorants for the same laws. The sharper triangle estimate and Lemma 21.8 do not change the chosen weights or permit an uncharged change of the actual half marginals.

The block caps of L20, rounded upward, give $w_{\mathrm{D}} = w_{\mathrm{S}}$, and those of L21 give v_{L} (Remark 22.2). The restorations use actual budget 570 and the respective maximum block cardinalities $\lfloor L/\log X \rfloor$ ($L = 570$), namely $27, 26, 25, 24, 22, 20, 18$ at the seven starts; local operator tensorization uses budget 585.

Remark 21.26 (The family of tilts). The root $g(\lambda)$ of (21.3) is the limit of the chamber coefficient g_{20}^{u} as $p \to \infty$ (Proposition 21.7: $\Gamma_{20} \to 1$, $z_{\mathrm{raw}} \to s$). It decreases from $g(0) = \sqrt{30}$, the limit of the uniform root, to about 5.466411 at $\lambda = 1/2$, 5.456962 at $\lambda = 24/25$ ($\mathsf{u} = 3/4$), 5.456161 at $\lambda = 1$ ($\mathsf{u} = 1$, the largest value of $2\mathsf{u}/(1+\mathsf{u}^2)$). For comparison, the leading matrix of a square-of-the-average weight has strength $6\mathsf{u}/(3+\mathsf{u}^2)$: the same cubic gives 5.443543 at $\lambda = 4080/2467$ ($\mathsf{u} = 51/40$) and 5.442090 at its maximum $\lambda = \sqrt{3}$. These matrix values explain the historical interest in that family; a finite law must also pay its mixed-product errors. The gain is at order one; the costs are at order r: the layer coefficient tends to $\sqrt{3}+\sqrt{7}$ for every u, the half marginal moves, and the range ratio grows like $(1+\mathsf{u}(2+r))^2/(1-\mathsf{u}c_{\mathrm{cond}})^2$ and enters the restorations through the per-block product of the per-prime range ratios, bounded by $M_{\mathcal{B}} \le R_{\mathsf{u}}(X)^n$ in Lemma 24.7(iv). The selected seven middle pieces and their parameters are those of Definition 21.1. The coherent last-band law has its own full chamber and layer estimates above; the one-layer root is unchanged.

Consequences and what to read next. Theorem 4.7 (the scores); Lemma 20.11 and Lemma 5.16 (the symmetries); Lemma 20.14 and Propositions 20.15 to 20.17 and 20.21 (the star matrix and the frame cap, the normalizer, the rank-four Gram, the rectangle and the collision cap of the uniform root); Lemma 14.4, Corollary 14.5, and Remark 14.7 and Lemma 14.2(d) (stars); Corollary 15.10 and Lemmas 13.9 and 24.5 (pair maps, swapped products, star means), with Theorem 15.8, Lemma 16.4 and Lemma 12.10 (witness kernels and completion counts); Lemma 24.3 (the fluctuation of the extension count); Lemma 13.23 (the actual half); Lemma 13.22 and Proposition 13.6(a),(a′) (mass, marginal, range and the constant block); Lemma 24.7(i),(ii) and (24.6) (R_p, χ_p^+); Lemma 13.3(f) and Lemma 13.7(b) (cut operators of a reweighted law, block caps); Lemma 4.18 (evaluation at an endpoint); Lemma 20.3 (the block budget). Certificate: Certificate 21.25.

Hypotheses verified in this section (Table 5). the seven positive-root bands supply the root inputs of "(CP1) middle": positivity, support, native sorting symmetries, uniform single-slot marginals, and finite chamber and layer caps. Pair laws use Lemma 21.2; the coherent law uses the physical dictionary, positive floor and square-affine invariance of Lemma 21.17. Their common finite interfaces and actual marginal data are certified in Certificate 21.25. The completion is in Section 22; restorations use Lemma 24.7(iv).

22. Middle primes III: the reference law μ, first stage λ, and its dominator

What this section proves and how to read it. This section builds μ (five-type completion of the twelve-label principal law), λ (polarization and one-stage replacements), and the law N dominating λ at ι, and proves (CP1)–(CP3) for middle atoms with prices L_0, B. Inputs: caps of Sections 20 and 21. Outputs: μ, λ, N for Section 23. A first reading should follow the three constructions and the (CP1)–(CP3) statements; weight verification is reference material. "Frame" in labels below means the package of these three laws together with the records and prices defined with them, as defined in Definition 22.4. On a middle block the reference (Section 22.1) is the five-type completion of the twelve-label principal law, which is uniform on the intervals below 2^{30} and tilted (Section 21) on the seven intervals from 2^{30}; its completion weights use the caps of the principal law (Sections 20 and 21) and of the reference fold record (Lemma 23.2). The first stage (Section 22.2) is obtained by polarization at one chamber cut, which removes the two nonaccepting one-layer fold copies $\mathrm{Diag}_\iota(\Pi_{F_1}^{(1)})$, $\mathrm{Diag}_\iota(\Pi_{F_2}^{(1)})$, and by replacing the one-layer matching copies by accepting laws with the same actual conditional marginals; the two resulting record differences are the tags of (CP2)(b). The law N (Section 22.3) then dominates λ at ι in a 2×2 block, giving (CP3), and Section 22.4 computes the resulting prices L_0 and B. Later sections build the comparison records on this reference (Section 23) and compare them with μ, λ and N through the variance savings (Section 24), the dominating matrices (Section 25) and the matching envelopes (Section 26); the closing (Section 30) uses the prices through B, R_{err} and L_0.

The middle reference uses twelve labels, two layers over one chamber coordinate, and is pulled back to the slot array V by ignoring the second chamber coordinate. It is the five-type completion of Lemma 17.5 applied to the twelve-label root: folds, matchings, merges and constants are added with weights in powers of $r = b^{-1/2}$ (b the least prime of the block) until every chamber cut is positive, and one-layer copies placed on the reserved diagonal restore positivity at the layer flip ι. On the seven tilted pieces the positive root changes two inputs of the completion, its caps at the chamber cut L20 and at the layer cut L21, both proved in fixed half charts (Remark 22.3). (Section 34 explains how this completion developed from an earlier version on 216 slots; nothing here depends on that history.) The comparison law built on this reference need not be reflection positive; it is compared with the first stage λ only at ι, through the 2×2 block of Section 22.3, with symmetric and skew errors priced separately.

Conventions. In this section ϑ_a is the chamber cut $\vartheta_{0,a}$ of the first coordinate: the reflection $u_0 \mapsto 2a-1-u_0$ with plus half $H_a = \{a, a+1, a+2\}$ (indices mod 6), $a \in \mathbb{Z}/6\mathbb{Z}$. The reserved cut ι flips the layer ℓ and has plus half $V_+ = \{\ell = 0\}$. For a prime p we write $r = p^{-1/2}$ and $q = 1/p$; on a block $\mathcal{B}$ with least prime $b = b_{\mathcal{B}}$, $r = b^{-1/2}$ and $q = 1/b$, while $q_{\mathcal{B}} = \prod_{p\in\mathcal{B}} p$. The order $\succeq_c$ of cut matrices is that of Definition 5.2. The pieces, the blocks, the twelve-label graphs and the caps of their cuts are those of Sections 20 and 21. Every middle block has $\log q_{\mathcal{B}} \le L = 570$; the local data of a piece are evaluated at its start X and at the local cap $L_{\mathrm{loc}} = 585$ (Lemma 20.3).

22.1. **The reference law of a middle block.** Fix a middle block $\mathcal{B}$ of a piece starting at X, with least prime b, $r = b^{-1/2}$ and $q = 1/b$. An *object* carries one label of $X_{\mathcal{B}}$ per layer. Write $\lambda_{\mathrm{F}} = \lambda_{\mathcal{B}}(G_{\mathrm{F},2})$ and $\lambda_{\mathrm{E}} = \lambda_{\mathcal{B}}(G_{\mathrm{E},2})$ for the two-layer F and E laws of three objects (y_0, y_1, y_2), $y_o = (y_{o,0}, y_{o,1})$, and $\lambda_{\mathrm{F}}^{(1)}$, $\lambda_{\mathrm{E}}^{(1)}$ for their one-layer versions. The components of the reference are images of these laws under the patterns of Definition 17.1: the folds $F_0 = (0,1,2,2,1,0)$, $F_1 = (1,0,0,1,2,2)$ and $F_2 = (2,2,1,0,0,1)$ applied to the F law, and the matchings $E_0 = (0,1,1,2,2,0)$ and $E_1 = (0,0,1,1,2,2)$, the six merges G_{sk} (which use the objects $0,1$) and the two constants C_0, C_1 (which use the object 0) applied to the E law. Under a pattern ϕ, a configuration (y_0, y_1, y_2) of three objects gives the slot (u_0, u_1, ℓ) of V the label $y_{\phi(u_0),\ell}$, whatever u_1 is. We write Π_ϕ ($\Pi_{\phi,\mathcal{B}}$ when the block matters) for the resulting image on $X_{\mathcal{B}}^V$ of the two-layer law, $\Pi_\phi^{(1)}$ for that of the one-layer law, and Π_{L} for the two-layer root of the block pulled back in the same way: the uniform law $\lambda_{\mathcal{B}}(G_{\mathrm{L},2})$ on the pieces below 2^{30}, and on the pieces from 2^{30} the positive root $\Gamma_{\mathcal{B},\mathfrak{u}}$ of Definition 21.1 with the parameter of the piece; $\Pi_{\mathrm{L}}^{(1)}$ is the pullback of the one-layer root $\lambda_{\mathcal{B}}(G_{\mathrm{L},1})$, which is not tilted. The five types $\mathrm{L}, \mathrm{D}, \mathrm{S}, \mathrm{R}, \mathrm{C}$, with multiplicities $1, 3, 2, 6, 2$ and degrees $0, 1, 1, 2, 3$, are those of Definition 17.1; $\pi_{\mathfrak{t},j}$ is the sum of the j-layer components of type $\mathfrak{t}$, of mass equal to the multiplicity. For a law κ of one-layer configurations, $\mathrm{Diag}_\iota(\kappa)$ places the same configuration on both layers (the diagonal copy of Lemma 5.4(i)); it is fixed samplewise by ι. The halves of the components at ϑ_0 are listed in Table 7.

Rows. Let $j \in \{1,2\}$. Under the j-layer E law let ν be the law of one object (in this paragraph ν is this one-object law, not the comparison law) and ν^ξ the conditional law of an outer object given that the middle object y_1 equals ξ: ν is uniform on the j-tuples, indexed by the layers, of block labels whose labels at each prime are pairwise distinct, and ν^ξ is uniform on those whose labels avoid the labels of ξ at every prime. Let T_{F}^ξ and T_{E}^ξ be the densities, relative to $\nu^\xi \otimes \nu^\xi$, of the conditional law of (y_0, y_2) given $y_1 = \xi$ under the j-layer F and E laws, regarded as operators on $L^2(\nu^\xi)$. For a prime p let $\rho_{\mathrm{F},j}(p)$ and $\rho_{\mathrm{E},j}(p)$ be the Schur-test bounds of Corollary 13.17 for the conditional maximal correlation of the labels of objects 0 and 2, with the labels of object 1 pinned, under $\lambda_p(G_{\mathrm{F},j})$ and $\lambda_p(G_{\mathrm{E},j})$, in the form of Remark 13.18(1) (graded count errors; the two vertex sets of the corollary are the labels of the two outer objects); they are explicit functions of $r = p^{-1/2}$, and $\rho_{\mathrm{F},j}(X)$, $R_j(X)$ below denote their values at $r = X^{-1/2}$. With $n = L_{\mathrm{loc}}/\log X$, $R_1(p) = (p-1)/(p-3)$, $R_2(p) = (p-3)(p-7)/((p-11)(p-15))$ and $\upsilon_j = (e^{n(R_j(X)-1)^2} - 1)^{1/2}$, the *row constants* of the piece are

$$f_j = \frac{R_j(X)^n \rho_{\mathrm{F},j}(X) + 2\upsilon_j + \upsilon_j^2}{X^{-1/2}}, \qquad h_j = \frac{\rho_{\mathrm{E},j}(X)}{X^{-1/2}}.$$

These are the row constants that the certificate evaluates (Remark 13.26(1)).

Lemma 22.1 (The reference)**.** *Let $\mathcal{B}$ be a middle block of a piece starting at X, with least prime b and $r = b^{-1/2}$.*

(i) Rows. *For $j \in \{1,2\}$ the middle object has the law ν under the j-layer F law as well as under the j-layer E law, and for every value ξ,*

$$\|T_{\mathrm{F}}^\xi - \mathbb{J}\| \le f_j r, \qquad \|T_{\mathrm{E}}^\xi - \mathbb{J}\| \le h_j r$$

on $L^2(\nu^\xi)$, $\mathbb{J}$ being the projection onto constants.

(ii) Negative envelopes. *Let d, d_1, k_{L} and k_{E} be the block caps, at the piece start X and the local cap L_{loc}, of the root cuts L20 and L10 and of the layer cuts L21 and E21: for L10 and E21, and for L21 on the pieces below 2^{30}, those of Lemma 20.12(b); for L20 on the pieces below 2^{30} that of Proposition 21.11; for L20 and L21 on the pieces from 2^{30} those of the positive root (Certificate 21.25). Let $k_{\mathrm{D}} = k_{\mathrm{D}}(X, L_{\mathrm{loc}})$ be the cap of the reference fold record of Lemma 23.2(e), and put $k_{\mathrm{S}} = k_{\mathrm{R}} = k_{\mathrm{C}} = k_{\mathrm{E}}$. Then*

$$\Pi_{\mathrm{L}} \succeq_{\vartheta_0} -dr\,\Pi_{F_0}, \qquad \Pi_{\mathrm{L}}^{(1)} \succeq_{\vartheta_0} -d_1 r\,\Pi_{F_0}^{(1)},$$

and $\Pi_\phi \succeq_\iota -k_{\mathfrak{t}} r\,\mathrm{Diag}_\iota(\Pi_\phi^{(1)})$ for every two-layer component Π_ϕ of type $\mathfrak{t}$, the root included.

(iii) Completion. *Let $w = (w_{\mathrm{L}}, w_{\mathrm{D}}, w_{\mathrm{S}}, w_{\mathrm{R}}, w_{\mathrm{C}})$ with $w_{\mathrm{L}} = 1$ and $v = (v_{\mathrm{L}}, v_{\mathrm{D}}, v_{\mathrm{S}}, v_{\mathrm{R}}, v_{\mathrm{C}})$ be nonnegative weights satisfying the thirteen inequalities*

$$\begin{aligned}
&w_{\mathrm{D}} \ge d, && w_{\mathrm{S}} \ge w_{\mathrm{D}}, && w_{\mathrm{R}} \ge f_2 w_{\mathrm{D}} + h_2 w_{\mathrm{S}}, && w_{\mathrm{C}} w_{\mathrm{S}} \ge w_{\mathrm{R}}^2,\\
&v_{\mathrm{D}} \ge d_1 v_{\mathrm{L}}, && v_{\mathrm{S}} \ge v_{\mathrm{D}}, && v_{\mathrm{R}} \ge f_1 v_{\mathrm{D}} + h_1 v_{\mathrm{S}}, && v_{\mathrm{C}} v_{\mathrm{S}} \ge v_{\mathrm{R}}^2,\\
&v_{\mathfrak{t}} \ge k_{\mathfrak{t}} w_{\mathfrak{t}} \quad (\mathfrak{t} \in \{\mathrm{L}, \mathrm{D}, \mathrm{S}, \mathrm{R}, \mathrm{C}\}).
\end{aligned}$$

Then

$$\widetilde{\mu}_{\mathcal{B}} = \sum_{\mathfrak{t}} w_{\mathfrak{t}} r^{\deg \mathfrak{t}} \pi_{\mathfrak{t},2} + r \sum_{\mathfrak{t}} v_{\mathfrak{t}} r^{\deg \mathfrak{t}}\,\mathrm{Diag}_\iota(\pi_{\mathfrak{t},1})$$

is a nonnegative measure on $X_{\mathcal{B}}^V$ of mass at least one, reflection positive at every cut of the family (the chamber cuts of both coordinates and ι), and invariant under the layer swap and under the square-affine maps of (R3) of Hypothesis 6.2; in particular its one-slot marginals are its mass times the uniform law.

(iv) Protection. *If the weights satisfy moreover the fourteenth inequality*

$$v_{\mathrm{D}} \ge 2k_{\mathrm{D}} w_{\mathrm{D}}, \tag{22.1}$$

which implies $v_{\mathrm{D}} \ge k_{\mathrm{D}} w_{\mathrm{D}}$ of the third line, then for $s \in \{1, 2\}$

$$2w_{\mathrm{D}} r\,\Pi_{F_s} + v_{\mathrm{D}} r^2\,\mathrm{Diag}_\iota(\Pi_{F_s}^{(1)}) \succeq_\iota 0.$$

Parts (i)–(iii) do not use (22.1)*; the bracket of Section 22.3 does.*

The reference law *of $\mathcal{B}$ is $\mu_{\mathcal{B}} = \widetilde{\mu}_{\mathcal{B}}/\widetilde{\mu}_{\mathcal{B}}(1)$. It satisfies (R1)–(R3) of Hypothesis 6.2, with (R2) at every cut of the family, and it is admissible (Definition 8.9).*

Proof. (i) *One prime.* For one layer the middle law is uniform for both laws, by translation invariance; for two layers it is uniform on the distinct pairs for the E law by symmetry and for the F law by Lemma 20.9(ii), since $p \ge Q_3 > 11$. Under the E law the outer objects given $y_1 = \xi$ have exactly the marginal ν^ξ, so T_{E}^ξ is the density of a coupling with exact marginals and $\|T_{\mathrm{E}}^\xi - \mathbb{J}\|$ is its conditional maximal correlation, at most $\rho_{\mathrm{E},j}(p)$ by Corollary 13.17. Under the F law the actual outer marginals given $y_1 = \xi$ have densities in $[R_j(p)^{-1}, R_j(p)]$ relative to ν^ξ (Lemma 20.9(iii), $p > 15$), and the coupling relative to these actual marginals has maximal correlation at most $\rho_{\mathrm{F},j}(p)$; in the language of the four-count comparison (Lemma 12.7), these densities are ratios of completion counts.

The block. Fix $j \in \{1, 2\}$. Given $\xi = (\xi_p)_p$ the conditional laws are products over the primes of $\mathcal{B}$. By Lemma 13.25(c), the densities of the actual outer marginals multiply, so they are at most $R_{\mathcal{B}} = \prod_p R_j(p) \le R_j(b)^n$, because R_j decreases and $|\mathcal{B}| \le n_X \le n$ (Lemma 20.3(ii)); their chi-square defects multiply, and a density with values in $[R^{-1}, R]$ has defect at most $R - 1$; writing g for an outer marginal density and g_p for its factor at p,

$$1 + \|g - 1\|^2 = \prod_p (1 + \|g_p - 1\|^2) \le \exp\Big(\sum_p (R_j(p) - 1)^2\Big) \le e^{n(R_j(b)-1)^2}$$

for each such g; and the maximal correlation of the product coupling is the largest of the factors, at most $\max_p \rho_{\mathrm{F},j}(p)$, respectively $\max_p \rho_{\mathrm{E},j}(p)$. Hence Lemma 13.25(a), with $H_{\mathrm{L}}, H_{\mathrm{R}} \le R_{\mathcal{B}}$ and $\upsilon_{\mathrm{L}}, \upsilon_{\mathrm{R}} \le \upsilon$, $\upsilon^2 = e^{n(R_j(b)-1)^2} - 1$, gives $\|T_{\mathrm{F}}^{\xi} - \mathbb{J}\| \le R_{\mathcal{B}} \max_p \rho_{\mathrm{F},j}(p) + 2\upsilon + \upsilon^2$; the E kernel has exact marginals, so $\|T_{\mathrm{E}}^{\xi} - \mathbb{J}\| \le \max_p \rho_{\mathrm{E},j}(p)$.

Evaluation at the piece start. As functions of $r = p^{-1/2}$ (with n fixed), $(R_1 - 1)/r^2 = 2/(1 - 3r^2)$ and $(R_2 - 1)/r^2 = (16 - 144r^2)/((1 - 11r^2)(1 - 15r^2)) = \sum_{m \ge 0}(24 \cdot 15^m - 8 \cdot 11^m) r^{2m}$ are power series with nonnegative coefficients, so υ/r and $\upsilon^2/r = r(\upsilon/r)^2$ are admissible majorant functions in the sense of Lemma 4.18 (a different use of *admissible* from admissible laws), since $(e^x - 1)/x$ increases; R_j^n is nondecreasing in r; and $\rho_{\mathrm{F},j}/r$, $\rho_{\mathrm{E},j}/r$ are admissible, the bound of Corollary 13.17 being r times an admissible function of r (Remark 13.18(3)). Hence $\max_p \rho(p) \le \rho(b)$, and the bound of the previous paragraph, divided by r, is a majorant function of r, so the bound is at most r times that quotient at $r = b^{-1/2}$, which is at most its value at X by Lemma 4.18(b): $\|T_{\mathrm{F}}^{\xi} - \mathbb{J}\| \le f_j r$ and $\|T_{\mathrm{E}}^{\xi} - \mathbb{J}\| \le h_j r$.

(ii) The plus half of ϑ_0 carries the objects $0, 1, 2$ of H_0, and by Lemma 20.11(b) the half graph of the root at ϑ_0 is $G_{\mathrm{F},2}$, whose labelling law λ_{F} is the law of the plus half of Π_{F_0}. The configurations of Π_{F_0} are fixed by ϑ_0 (as $F_0(5 - u) = F_0(u)$), so the cut form of Π_{F_0} at a real half-function Φ is $\mathbb{E}_{\lambda_{\mathrm{F}}} \Phi^2$. For the uniform root the cut form of Π_{L} is $\langle \Phi, \mathsf{A}_{\mathcal{B}} \Phi \rangle$, where $\mathsf{A}_{\mathcal{B}}$ is its cut operator on $L^2(\lambda_{\mathrm{F}})$, and Proposition 21.11 for the cut L20 gives the first inequality. For the positive root of a piece from 2^{30} the cut form of Π_{L} at Φ is $\langle \Phi, T_{\mathcal{B}} \Phi \rangle$ on $L^2(\lambda_{\mathrm{F}})$, where $T_{\mathcal{B}}$ is the cut operator of the reweighted graph law relative to the fixed half law λ_{F} (Lemma 13.3(f)); Certificate 21.25 with Lemma 13.7(b) gives $T_{\mathcal{B}} \succeq -dr$, which is the first inequality with the same Π_{F_0} (Remark 22.3). The one-layer root is the uniform law on every piece, and Lemma 20.12(b) for the cut L10 gives the second inequality. At ι, by Lemma 20.11(c) the half graphs of $G_{\mathrm{L},2}$ and $G_{\mathrm{E},2}$ are their one-layer graphs, and the cut form of $\mathrm{Diag}_{\iota}(\Pi_{\phi}^{(1)})$ at Φ is the mean of Φ^2 under $\Pi_{\phi}^{(1)}$. So the caps L21 and E21 give $\lambda_{\mathcal{B}}(G_{\mathrm{L},2}) \succeq_{\iota} -k_{\mathrm{L}} r \, \mathrm{Diag}_{\iota}(\lambda_{\mathcal{B}}(G_{\mathrm{L},1}))$ for the uniform root and $\lambda_{\mathrm{E}} \succeq_{\iota} -k_{\mathrm{E}} r \, \mathrm{Diag}_{\iota}(\lambda_{\mathrm{E}}^{(1)})$. For the positive root, the layer bounds of Certificate 21.25 with Lemma 13.3(f) and Lemma 13.7(b) give the inequality relative to the half law of $\mathrm{Diag}_{\iota}(\Pi_{\mathrm{L}}^{(1)})$, the untilted one-layer root: $\Gamma_{\mathcal{B},\mathsf{u}} \succeq_{\iota} -k_{\mathrm{L}} r \, \mathrm{Diag}_{\iota}(\lambda_{\mathcal{B}}(G_{\mathrm{L},1}))$. A component is the image of its field law under an emission, and it inherits the bound of the field law (Lemma 17.5(e)); this gives the inequality for the root and, with k_{E}, for the matchings, the merges and the constants. For the folds it is Lemma 23.2(e), with the cap $k_{\mathrm{D}} = k_{\mathrm{D}}(X, L_{\mathrm{loc}})$: its values at the 33 piece starts, with the gates of that lemma, are in Certificate D.6, and its conditions on the block ($p \ge X \ge 2^{24}$, $\sum_{p \in \mathcal{B}} \log p \le L_{\mathrm{loc}}$) hold by Lemma 20.3(i),(ii).

(iii) *Positivity, mass and invariance.* All components are probability laws and all weights are nonnegative; the root has weight $w_{\mathrm{L}} = 1$, so $\widetilde{\mu}_{\mathcal{B}}(1) \ge 1$. Every graph law $\lambda_p(G)$ is invariant under the diagonal maps $z \mapsto \alpha z + \beta$ with α a nonzero square, which preserve injectivity and multiply differences by squares; the component maps act on the vertices only and commute with them, and on a block the maps act independently at each prime. The positive root is invariant under the same maps and under the layer swap (Lemmas 21.2 and 21.17). Translations act transitively, so every one-slot marginal is its mass times the uniform law (Lemma 5.16). The layer swap is an automorphism of the three two-layer graphs (Lemma 20.11(a)), and a diagonal copy is fixed by it.

First-coordinate cuts. For $j \in \{1, 2\}$, the middle law ν and the conditional references ν^{ξ} of the j-layer E law, the two-object marginal of that law, the j-layer E and F laws and the j-layer root form a five-type datum on the label set of one object (Definition 17.3): the E law is exchangeable; by (i) the middle law of the F law is ν, so that the F law has the form required there with the kernels T_{F}^{ξ}; and the root is invariant under $\mathcal{G}_6$ (Lemma 20.11(a) for the uniform root, Lemmas 21.2 and 21.17 for the positive root). For $j = 2$, (i) gives the full-norm hypotheses of Lemma 17.5(b)

with $e_{\mathrm{F}} = f_2 r$ and $e_{\mathrm{E}} = h_2 r$, and (ii) the root envelope with $\delta = dr$; for the graded weights $(1, w_{\mathrm{D}} r, w_{\mathrm{S}} r, w_{\mathrm{R}} r^2, w_{\mathrm{C}} r^3)$ the conditions of Lemma 17.5(b) are the four inequalities of the first line. So $\sum_{\mathfrak{t}} w_{\mathfrak{t}} r^{\deg \mathfrak{t}} \pi_{\mathfrak{t},2}$ is reflection positive at every chamber cut of the first coordinate. The same holds for $j = 1$ with the inputs d_1, f_1, h_1 and the weights v (the second line, homogeneous in v_{L}), and the diagonal copy preserves this positivity (Lemma 17.5(e)).

The reserved cut. By (ii) every two-layer component of type $\mathfrak{t}$ is bounded below at ι by $-k_{\mathfrak{t}} r$ times the diagonal copy of its one-layer component, and the third line is $v_{\mathfrak{t}} \geq k_{\mathfrak{t}} w_{\mathfrak{t}}$; so $\widetilde{\mu}_{\mathcal{B}} \succeq_{\iota} 0$ by the fold pairing of Lemma 17.5(e). The one-layer component of a type need not be the one-layer marginal of its two-layer component, only the order of (ii) being used (Lemma 17.5(e)); for the positive root it is the untilted $\Pi_{\mathrm{L}}^{(1)}$. The first coordinate and the layer are the active coordinates of the pullback to V, so both conclusions hold on V (Lemma 5.13(v)).

Second-coordinate cuts. $\widetilde{\mu}_{\mathcal{B}}$ is the pullback of a law on the labels indexed by (u_0, ℓ), so every configuration is fixed by every chamber cut of the second coordinate, and the cut matrix there is nonnegative diagonal (Lemma 5.13(v), Lemma 5.4).

Normalization preserves positivity and the marginal assertions, and (R1) holds because the block and its law do not depend on the scale. The reference is a nonnegative law depending only on the labels in the finite fields at the primes of the block, invariant under the maps of (R3); so it has uniform whole-slot marginals and does not depend on the interval, residue or inclusion indices, which is (O1) of Definition 8.9; no sharp bound is claimed for it, so clause (O2) is vacuous for it.

(iv) By (ii) for the type D, $2 w_{\mathrm{D}} r \, \Pi_{F_s} \succeq_{\iota} -2 k_{\mathrm{D}} w_{\mathrm{D}} r^2 \, \mathrm{Diag}_{\iota}(\Pi_{F_s}^{(1)})$, and $\mathrm{Diag}_{\iota}(\Pi_{F_s}^{(1)}) \succeq_{\iota} 0$, being fixed samplewise by ι (Lemma 5.4); so (22.1) gives the claim. □

Remark 22.2 (The certified weights). In each of the 33 pieces the weights w, v are fixed rationals (Table 26), chosen at (X, L_{loc}) with the caps of the piece, and the thirteen inequalities of Lemma 22.1(iii) and the protection (22.1) are checked exactly with the inputs of the piece, 462 inequalities in all (Certificate D.3). In every piece three of them fix the leading weights: $w_{\mathrm{D}} = w_{\mathrm{S}}$ is the block cap d of L20, rounded up; v_{L} is the block cap k_{L} of L21, rounded up; and $v_{\mathrm{D}} = v_{\mathrm{S}}$ is $2 k_{\mathrm{D}} w_{\mathrm{D}}$, rounded up to the tabulated rational, so that the protection inequality holds, with equality before rounding. The inequality $v_{\mathrm{D}} \geq d_1 v_{\mathrm{L}}$ is slack, also on the seven pieces from 2^{30}, where v_{L} is the layer cap of the same positive root. Its smallest slack on these seven pieces is greater than $5.7 \cdot 10^{-5}$; the exact rational check retains this gate along with protection. The certificate checks $v_{\mathrm{S}} \geq 2 k_{\mathrm{E}} w_{\mathrm{S}}$, which is stronger than required. Since every input of Lemma 22.1 is evaluated at the piece start and is valid for every block of the piece (Lemma 20.3), one set of weights serves all blocks of a piece.

Remark 22.3 (The fixed half charts). The caps of the positive root are proved in two fixed charts: at the chamber cut on $L^2(\lambda_{\mathrm{F}})$, the half law of Π_{F_0}, and at the layer cut on $L^2(\lambda_{\mathcal{B}}(G_{\mathrm{L},1}))$, the half law of $\mathrm{Diag}_{\iota}(\Pi_{\mathrm{L}}^{(1)})$ (Certificate 21.25). These are the diagonal measures against which Lemma 22.1(ii) is stated and which the completion of (iii) uses, so the actual half marginal of the positive root, whose density relative to the fixed half law differs from 1 by $O(r)$ in the norms used below, is not needed there. Its change is paid where the half marginal enters: in the restorations of the variance credits and of the mean savings (Lemma 24.7(iv)).

Together with the laws of the other ranges (Lemma 19.2, Lemma 27.17, Lemma 29.2), the reference laws of the middle blocks form the reference $\mu = \bigotimes_{\mathcal{B}} \mu_{\mathcal{B}}$ of (CP1). Of its components, the root and the aligned fold Π_{F_0} accept the six queries, both as two-layer components and as one-layer diagonals (the positive root has the support of the uniform root); every other record is removed or replaced, by the first stage below or by the comparison law (Certificate 23.10).

22.2. **The first stage λ (polarization and one-stage replacements).** The comparison law will not contain the two one-layer fold records $\mathrm{Diag}_{\iota}(\Pi_{F_1}^{(1)})$ and $\mathrm{Diag}_{\iota}(\Pi_{F_2}^{(1)})$, which do not accept

the queries, nor the folded matching records in their reference form. The first stage removes them at the price of a second-order correction at one chamber cut, and replaces the folded matching records by laws that accept the queries and keep their actual conditional marginals.

Definition 22.4 (The middle frame)**.** The frame consists of the following objects.

(a) *Reference.* $\mu = \bigotimes_{\mathcal{B}} \mu_{\mathcal{B}}$ is the reference of (CP1); on a middle block, $\mu_{\mathcal{B}}$ is the law of Lemma 22.1.

(b) *Records.* Let $\mathcal{B}$ be a middle block with least prime b, and put $r = b^{-1/2}$ and $q = r^2$. Before the common normalization, define the following measures (the first two are summands of $\widetilde{\mu}_{\mathcal{B}}$; the next three are added in (d) below):

- $F_{\mathrm{fold}} = v_{\mathrm{D}} r^2 (\mathrm{Diag}_\iota(\Pi^{(1)}_{F_1}) + \mathrm{Diag}_\iota(\Pi^{(1)}_{F_2}))$ is the sum of the two nonaccepting one-layer fold components on the reserved diagonal, of mass $2v_{\mathrm{D}} r^2$;
- $E_{\mathrm{fold}} = v_{\mathrm{S}} r^2 (\mathrm{Diag}_\iota(\Pi^{(1)}_{E_0}) + \mathrm{Diag}_\iota(\Pi^{(1)}_{E_1}))$ is the sum of the two reserved-folded matching components, of mass $2v_{\mathrm{S}} r^2$;
- $E_{\mathrm{add}} = v_{\mathrm{D}} r^2 (\mathrm{Diag}_\iota(\Pi^{(1)}_{E_0}) + \mathrm{Diag}_\iota(\Pi^{(1)}_{E_1}))$ is a pair of reserved-folded matching records of total mass $2v_{\mathrm{D}} r^2$;
- $G = (f_1 + h_1) r\,(G_e + G_o)$, with $G_e = v_{\mathrm{D}} r^2 \,\mathrm{Diag}_\iota(\Pi^{(1)}_{G_{00}})$ and $G_o = v_{\mathrm{D}} r^2 \,\mathrm{Diag}_\iota(\Pi^{(1)}_{G_{11}})$, is the *polarization correction*: two merge records of common record weight $v_{\mathrm{D}} r^2$, of total mass $2v_{\mathrm{D}}(f_1 + h_1) r^3$;
- H adds, for each of the two two-layer matching components Π_{E_0}, Π_{E_1} of weight $w_{\mathrm{S}} r$, its actual half diagonal $\mathrm{Diag}_\iota(\Pi^{(1)}_{E_s})$ with coefficient $2w_{\mathrm{S}} r \cdot 3/(b-3)$; thus H has mass $12 w_{\mathrm{S}} r^3/(1-3q)$.

All five measures are nonnegative and reserved diagonal, that is, fixed samplewise by ι. In the decompositions below all five measures are divided by $\widetilde{\mu}_{\mathcal{B}}(1)$; only the first two are records of $\mu_{\mathcal{B}}$ itself.

(c) *First stage.* E_{new} is the nonnegative reserved-diagonal replacement of $E_{\mathrm{fold}} + E_{\mathrm{add}}$ given by Lemma 22.6. It has the same mass, it accepts the six queries, and at its own first-coordinate cut (ϑ_1 for E_0, ϑ_2 for E_1) each record of E_{new} has the same actual conditional marginals as the record it replaces.

(d) *Decomposition.* Blockwise, and with all records zero outside the middle blocks,

$$\begin{aligned} \rho &= \mu - F_{\mathrm{fold}} + E_{\mathrm{add}} + G, & \lambda &= \mu - F_{\mathrm{fold}} - E_{\mathrm{fold}} + E_{\mathrm{new}} + G, \\ \delta_E &= E_{\mathrm{new}} - E_{\mathrm{fold}} - E_{\mathrm{add}}, & \mathsf{N} &= \lambda + F_{\mathrm{fold}} + H, \\ \mathsf{N}_0 &= \mu + E_{\mathrm{add}} + G + H. \end{aligned}$$

Hence $\lambda = \rho + \delta_E$, $\mathsf{N} = \mathsf{N}_0 + \delta_E$, $\mathsf{N} - \lambda = F_{\mathrm{fold}} + H$ and $\mathsf{N} + \lambda = 2\mu - F_{\mathrm{fold}} - 2E_{\mathrm{fold}} + H + 2E_{\mathrm{new}} + 2G$; in particular F_{fold} occurs with coefficient one in $\mathsf{N} - \lambda$. (The bracket measures N and N_0 are unrelated to the scale $N = K\mathsf{L}$.)

(e) *Prices and tags.* U_0 is the summed price of $\mathsf{N}_0 - \mu = E_{\mathrm{add}} + G + H$ and P that of $F_{\mathrm{fold}} + E_{\mathrm{add}} + G$. The record differences of δ_E form two tags (Definition 9.11(b)): the differences of type E_0, sharp at their first-coordinate cut ϑ_1, form the tag 0, and those of type E_1, sharp at ϑ_2, the tag 1. V and D are the summed variation and sharp prices of δ_E, the sharp price of a difference being its mass times its sharp bound at its own cut, relative to its actual half marginals, times $q_{\mathcal{B}}^{2a}$ (Definition 9.2(b)). The two tags have equal totals, the sharp total $D/2$ and the variation total $V/2$ each, because the two records of a block have the same mass and the same sharp bound (Lemma 22.9). With the two-tag series $\mathfrak{h}$ of Corollary 10.9,

$$R_{\mathrm{err}} = e^{U_0}\mathfrak{h}(D/2, V/2), \qquad B = e^{U_0} + R_{\mathrm{err}}, \qquad L_0 = e^{P}\mathfrak{h}(D/2, V/2).$$

The actual half marginal of Π_{E_s} at ι is the uniform law on ordered triples of distinct labels placed by E_s, which is the half law of $\Pi^{(1)}_{E_s}$ (Lemma 17.2(d)); this is the diagonal that H adds. All records in Definition 22.4(b) are images of invariant laws under vertex maps, followed by a diagonal copy, so they are positive records (Definition 9.9): their normalizations are admissible. Admissibility of the replacements in Definition 22.4(c) is verified after Lemma 22.6 below.

Lemma 22.5 (The polarized first stage)**.** *For every middle block, $E_{\mathrm{add}} + G - F_{\mathrm{fold}} \succeq_{\vartheta_0} 0$. Hence $\rho_{\mathcal{B}} \succeq_{\vartheta_0} \mu_{\mathcal{B}} \succeq_{\vartheta_0} 0$, and (CP2)(a) holds at the cut ϑ_0, with the variation price $p_{\mathcal{B}}$ of the positive records F_{fold}, E_{add} and G.*

Proof. The one-layer data of the proof of Lemma 22.1(iii) form a five-type datum whose rows satisfy $\|T^{\xi}_{\mathrm{F}} - \mathbb{J}\| \le f_1 r$ and $\|T^{\xi}_{\mathrm{E}} - \mathbb{J}\| \le h_1 r$ (Lemma 22.1(i) with $j = 1$). By Lemma 17.11(a) with $\varepsilon = (f_1 + h_1)r$,

$$\Pi^{(1)}_{E_0} + \Pi^{(1)}_{E_1} + (f_1 + h_1)r(\Pi^{(1)}_{G_{00}} + \Pi^{(1)}_{G_{11}}) \succeq_{\vartheta_0} \Pi^{(1)}_{F_1} + \Pi^{(1)}_{F_2}.$$

A diagonal copy preserves this order of cut matrices (Lemma 17.5(e)); multiplying by the record weight $v_{\mathrm{D}} r^2$ gives $E_{\mathrm{add}} + G \succeq_{\vartheta_0} F_{\mathrm{fold}}$ (Remark 17.12(1)). The reference is positive at ϑ_0 (Lemma 22.1), and $\rho_{\mathcal{B}} - \mu_{\mathcal{B}}$, which is $E_{\mathrm{add}} + G - F_{\mathrm{fold}}$ divided by $\widetilde{\mu}_{\mathcal{B}}(1)$, is a real combination of the three positive records, of variation at most their total mass. □

So the first stage costs only second order in r (the removed fold copies have mass $2v_{\mathrm{D}}r^2$): the nonaccepting one-layer fold copies are traded for matching records at the one cut ϑ_0 where positivity is needed, and the row errors are paid by merges of degree three. Since only actual components of the reference are subtracted, ρ and λ stay nonnegative. The tensor step, $Y_\rho \ge Y_\mu$, is Step 1 of Lemma 10.10.

The folded matching records. For $s \in \{0, 1\}$ the folded matching record of type E_s in $E_{\mathrm{fold}} + E_{\mathrm{add}}$ has mass $(v_{\mathrm{S}} + v_{\mathrm{D}})r^2$: the one-layer law $\Pi^{(1)}_{E_s}$, uniform on the halves, placed on the reserved diagonal. By Certificate 23.10, at its first-coordinate cut (ϑ_1 for E_0, ϑ_2 for E_1) the label of object 2 lies on both sides of the cut (it is *shared*), the labels of objects 0 and 1 lie on one side each (they are *private*), and the six queries need exactly the edge $y_0 \to y_1$. At one prime (the block statement then follows by tensoring), given the shared label c, the private pair is uniform on the ordered distinct pairs of $V_c = \mathbb{F}_p \setminus \{c\}$. Write $V_c^{\pm} = \{a \in V_c : \chi(a - c) = \pm 1\}$, each of size $(p - 1)/2$, let $\mathbf{M}_c$ be the 0/1 matrix of the relation $a \to a'$ on V_c, and $\mathbf{M}_c^{+-}$ its restriction to $V_c^+ \times V_c^-$, extended by zero.

Lemma 22.6 (One-stage replacement of the folded matching records)**.** *Let $p \ge 11$ with $p \equiv 3 \pmod 4$, $k = (p - 3)/4$, $\kappa_p^{\mathrm{fold}} = (1 + 3r)r/(1 - 3q) = (\sqrt{p} + 3)/(p - 3)$ and*

$$k_{\mathrm{new}}(p) = \frac{1 + 5q}{(1 - 3q)(1 + q)} + \frac{qr}{(1 - 3q)(1 - 2q)}, \qquad s_{\mathrm{fold}}(p) = \frac{(1 + r)^2 + 4q}{(1 - 3q)^2} + \frac{q}{(1 - 2q)^2}.$$

(a) The four blocks of $\mathbf{M}_c$, from V_c^+ to V_c^+, from V_c^+ to V_c^-, from V_c^- to V_c^+ and from V_c^- to V_c^-, have equal row and column degrees, namely k, $k + 1$, k and k.

(b) The matrix $\mathcal{K}_+ = \mathbf{M}_c/(2k) - \mathbf{M}_c^{+-}/(2k(k + 1))$ is doubly stochastic, supported on the pairs $a \to a'$, with entries $1/(2k)$ or $1/(2(k + 1))$ there; from each a it chooses each destination sector with probability $1/2$ and, within a sector, is uniform on the out-neighbours of a in that sector.

(c) Let $\mathcal{K}_0 = (J_{\mathrm{all}} - I)/(p - 2)$ on V_c, the conditional coupling of the record being replaced, and $\Delta = \mathcal{K}_+ - \mathcal{K}_0$. On L^2 of the uniform law of V_c both $\mathcal{K}_+$ and $\mathcal{K}_0$ fix constants and preserve mean-zero functions; there $\|\mathcal{K}_+\| \le \kappa_p^{\mathrm{fold}}$ and $\|\mathcal{K}_0\| = 1/(p - 2)$; the symmetric part of Δ has norm exactly $1/(p - 2)$, attained at the sector contrast $\mathbf{1}_{V_c^+} - \mathbf{1}_{V_c^-}$; and $\|\Delta\| \le k_{\mathrm{new}}(p)\, r$.

(d) Let $\mathcal{B}$ be a block of such primes $p \ge X \ge 2^{24}$ with least prime b, put $r_X = X^{-1/2}$, $q_X = 1/X$, $\alpha_X = (1+3r_X)/(1-3q_X)$ and $\beta_X = 1/(1-2q_X)$, write $k_{\mathrm{new}}(X)$ and $s_{\mathrm{fold}}(X)$ for the displayed formulas evaluated at q_X, r_X, and suppose $\alpha_X^2 r_X + \beta_X^2 q_X r_X < k_{\mathrm{new}}(X)$. For every shared label $c = (c_p)_{p\in\mathcal{B}}$ the difference $\Delta_c = \bigotimes_p \mathcal{K}_+^{(c_p)} - \bigotimes_p \mathcal{K}_0^{(c_p)}$ satisfies $\|\Delta_c\| \le k_{\mathrm{new}}(X)\, b^{-1/2}$ and $\|\operatorname{Sym}\Delta_c\| \le s_{\mathrm{fold}}(X)/b$.

(e) Replace each folded matching record of type E_s by the measure of the same mass under which the shared label is uniform and, given it, the private pair is distributed by $\bigotimes_p \mathcal{K}_+^{(c_p)}$, placed on the reserved diagonal; let E_{new} be the sum of the two replacements. Then E_{new} is a nonnegative reserved-diagonal measure of the mass of $E_{\mathrm{fold}} + E_{\mathrm{add}}$; it accepts the six queries; it is invariant under the groups $\mathcal{G}_p$ of the maps $z \mapsto \alpha z + \beta$ (α a nonzero square) acting independently at the primes of $\mathcal{B}$, so it has uniform one-slot marginals and its queried pair is uniform on the strict-square pairs; and at its own first-coordinate cut each of its records has the same actual conditional marginals as the record it replaces, differs from it by at most $k_{\mathrm{new}}(X)\, b^{-1/2}$ in sharp operator norm, uniformly in the shared label, its symmetric part by at most $s_{\mathrm{fold}}(X)/b$, and by at most twice its mass in variation. The coupling is direct: no label is resampled when the two private labels coincide.

Proof. (a) We use the Paley counts of Lemma 12.10(a): two distinct labels have exactly k common out-neighbours and k common in-neighbours, and $\#\{c' : a \to c' \to b'\} = k + 1 - \mathbf{1}[a \to b']$ for $a \ne b'$. Now, for $a \in V_c^+$, i.e. $c \to a$, the out-neighbours of a in V_c^+ are the common out-neighbours of c and a (k), and those in V_c^- are the labels on paths $a \to a' \to c$ ($k+1$, as $a \not\to c$); for $a \in V_c^-$ they are the common out-neighbours of a and c (k) and the labels on paths $a \to a' \to c$ (k, as $a \to c$). The column degrees are computed in the same way from common in-neighbours and paths $c \to a \to a'$. (b) By (a), the row of a puts mass $1/2$ uniformly on each destination sector. Since row and column degrees agree in each block, the column of a' receives from each of the two source sectors its degree in that block divided by twice that degree, in all $1/2+1/2 = 1$. (c) Both matrices are doubly stochastic for the uniform law, which gives the first claim. On V_c, $\mathbf{M}_c = (J_{\mathrm{all}} - I + \mathbf{C}_c)/2$, where $\mathbf{C}_c$ is the compression of the $p \times p$ matrix $(\chi(a'-a))$, whose norm is $\sqrt{p}$ (Lemma 4.11(c)); as $\chi(-1) = -1$, $\mathbf{C}_c$ is skew-symmetric. On mean-zero f, $J_{\mathrm{all}} f = 0$, so $\|\mathbf{M}_c f\| \le (1+\sqrt{p})\|f\|/2$. Each block of $\mathbf{M}_c^{+-}$ has row and column sums $k+1$, so $\|\mathbf{M}_c^{+-}\| \le k+1$ by the Schur test. Hence $\|\mathcal{K}_+ f\| \le [(1+\sqrt{p})/(4k) + 1/(2k)]\|f\| = \kappa_p^{\mathrm{fold}}\|f\|$. On constants Δ vanishes; on mean-zero functions $\mathcal{K}_0 = -I/(p-2)$, and, since $2k = (p-3)/2$ and $2k(k+1) = (p-3)(p+1)/8$,

$$\Delta f = \frac{\mathbf{C}_c f}{p-3} - \frac{f}{(p-3)(p-2)} - \frac{8\,\mathbf{M}_c^{+-} f}{(p-3)(p+1)}.$$

Let $\mathbf{h}_c = \mathbf{1}_{V_c^+} - \mathbf{1}_{V_c^-}$, the sector contrast, and let $\mathcal{Z}_c$ be the space of functions with mean zero on each of the two sectors, so that the mean-zero functions are the orthogonal sum of the multiples of $\mathbf{h}_c$ and $\mathcal{Z}_c$. By (b) and its proof, for every a' the column of a' receives mass $1/2$ from each source sector, and every row puts mass $1/2$ on each target sector. Hence $\mathcal{K}_+$ and $\mathcal{K}_+^*$ map every mean-zero function to a function with zero sum on each sector, so they map $\mathcal{Z}_c$ into itself, and $\mathcal{K}_+\mathbf{h}_c = \mathcal{K}_+^*\mathbf{h}_c = 0$; both summands reduce Δ, and $\Delta\mathbf{h}_c = \Delta^*\mathbf{h}_c = \mathbf{h}_c/(p-2)$. On $\mathcal{Z}_c$ the all-ones part of the block $\mathbf{M}_c^{+-}$ vanishes and the identity has no entries in it, so there $\mathbf{M}_c^{+-}$ is half the (V_c^+, V_c^-) block of $\mathbf{C}_c$: its norm is at most $\sqrt{p}/2$, and, being off-diagonal for the splitting of $\mathcal{Z}_c$ by the sectors, its compression to $\mathcal{Z}_c$ has symmetric and skew parts of norm at most $\sqrt{p}/4$ each. As the compression of $\mathbf{C}_c$ is skew, the symmetric part of Δ on $\mathcal{Z}_c$ has norm at most

$$\frac{1}{(p-3)(p-2)} + \frac{2\sqrt{p}}{(p-3)(p+1)} \le \frac{1}{p-2};$$

after clearing the positive denominators the inequality reads $4q + 2r(1-2q)/(1+q) \le 1$, whose left side decreases in p and is below 0.82 at $p = 11$. So the symmetric part of Δ has norm $1/(p-2)$,

attained at $\mathbf{h}_c$. Finally, on $\mathcal{Z}_c$,

$$\|\Delta\| \le \frac{\sqrt{p}}{p-3} + \frac{1}{(p-3)(p-2)} + \frac{4\sqrt{p}}{(p-3)(p+1)} = \frac{\sqrt{p}(p+5)}{(p-3)(p+1)} + \frac{1}{(p-3)(p-2)} = k_{\mathrm{new}}(p)\, r,$$

which is larger than $1/(p-2)$, the norm of Δ on the multiples of $\mathbf{h}_c$. (d) Both couplings fix the constants on both sides, so they are sector-preserving in the sense of Lemma 6.10, and Δ_c and its symmetric part map every sector H_S (of exact support $S \subseteq \mathcal{B}$) into itself; their norms are the largest over the sectors (Lemma 6.10(a)). On $H_\emptyset$, $\Delta_c = 0$. On $H_{\{p\}}$, Δ_c is the factor Δ at p, so by (c) its norm is at most $k_{\mathrm{new}}(p) r_p$ and its symmetric part has norm $1/(p-2)$. The function k_{new} is nondecreasing in q and r (its first term is $(1+4q/(1+q))/(1-3q)$), so $k_{\mathrm{new}}(p) \le k_{\mathrm{new}}(X)$, while $r_p \le b^{-1/2}$ and $1/(p-2) \le 1/(b-2) \le \beta_X/b$. On H_S with $|S| \ge 2$ the operator is $\bigotimes_{p\in S} \mathcal{K}_+^\circ - \bigotimes_{p\in S} \mathcal{K}_0^\circ$, the centred factors having norms $\|\mathcal{K}_+^\circ\| \le \kappa_p^{\mathrm{fold}} \le \alpha_X b^{-1/2}$ and $\|\mathcal{K}_0^\circ\| = 1/(p-2) \le \beta_X/b$, all below one; so its norm is at most $\alpha_X^2/b + \beta_X^2/b^2 \le (\alpha_X^2 r_X + \beta_X^2 q_X r_X) b^{-1/2} < k_{\mathrm{new}}(X) b^{-1/2}$. This proves the first bound. For the symmetric part, the decomposition of (c) gives on the mean-zero functions at one prime $\|\operatorname{Sym} \mathcal{K}_+^\circ\| \le 1/(4k) + 1/(4k) = 2q/(1-3q)$ and $\|\operatorname{Skew} \mathcal{K}_+^\circ\| \le (\sqrt{p}+1)/(4k) = r(1+r)/(1-3q)$, since $\mathbf{C}_c$ is skew and the symmetric and skew parts of the off-diagonal block $\mathbf{M}_c^{+-}$ have norm at most $(k+1)/2$. On a two-prime sector $\{p, p'\}$, as $\mathcal{K}_0^\circ$ is symmetric,

$$\operatorname{Sym} \Delta_c = \operatorname{Sym} \mathcal{K}_+^\circ \otimes \operatorname{Sym} \mathcal{K}_+^\circ + \operatorname{Skew} \mathcal{K}_+^\circ \otimes \operatorname{Skew} \mathcal{K}_+^\circ - \mathcal{K}_0^\circ \otimes \mathcal{K}_0^\circ,$$

of norm at most $[4q_p q_{p'} + r_p r_{p'}(1+r_p)(1+r_{p'})]/((1-3q_p)(1-3q_{p'})) + q_p q_{p'}/((1-2q_p)(1-2q_{p'})) \le s_{\mathrm{fold}}(X)/b$, using $r_p r_{p'} \le 1/b$ and $q_p q_{p'} \le q_X/b$. On a sector with $|S| \ge 3$ the whole operator has norm at most $\alpha_X^3 b^{-3/2} + \beta_X^3 b^{-3} < 1/b$, as $X \ge 2^{24}$. Since $\beta_X \le s_{\mathrm{fold}}(X)$ and $1 \le s_{\mathrm{fold}}(X)$, every sector gives at most $s_{\mathrm{fold}}(X)/b$. (e) Positivity, mass and the reserved-diagonal placement are clear. The support lies on $a_0 \to a_1$, strictly, at every prime, so the queried edge is a strict-square edge and the three labels are distinct. The maps of $\mathcal{G}_p$ preserve the sectors and the relation $\to$, so the law is invariant; as $\mathcal{G}_p$ is transitive on labels and on the pairs $a \to a'$, the one-slot marginals and the law of the queried pair are uniform. At the cut the two half marginals are the laws of (c, a_0) and (c, a_1); by (b) they are uniform on distinct pairs, as for the record being replaced. The record difference is the coupling difference Δ_c of (d) conditional on the shared label, which gives the sharp bound and the bound of its symmetric part, and two probability laws differ by at most 2 in variation. No conditioning on a coincidence is made. □

Only one strict edge has to be imposed, because the six queries need only the edge between the two private labels of a folded matching record; a single sector-balanced coupling of the private pair, which keeps both actual half marginals at the record's own cut, therefore replaces each record, with no additional conditioning event and no resampling at coincidences. The sector balance also keeps the difference nearly skew: at a prime its symmetric part has norm exactly $1/(p-2)$, carried by the sector contrast, and on a block it is of order $1/b$, while the sharp coefficient k_{new} is about $1 + 7q$. The symmetric part gives each folded tag the symmetric price $(v_{\mathrm{S}} + v_{\mathrm{D}}) s_{\mathrm{fold}}(X) r^4 q_{\mathcal{B}}^{2a}$ on a block, which the orbit coupon of the positive envelopes uses (Lemma 25.7). The coefficients $k_{\mathrm{new}}(X)$ and $s_{\mathrm{fold}}(X)$ at the 33 piece starts and the two-support gate of (d) are recorded in Certificate D.5 (Appendix D); the gate holds at every $X \ge 2^{24}$, since $k_{\mathrm{new}}(X) > 1$ while its left side is below 2^{-11}. The replaced records are positive records whose normalizations are admissible: they involve only the field labels, and their sharp bounds hold for every value of the shared label, which is (O2) of Definition 8.9.

(CP2)(b). On a middle block, $\delta_{E,\mathcal{B}} = \sum_{s\in\{0,1\}} (\kappa_s' - \kappa_s)$, where κ_s is the folded matching record of type E_s in $E_{\mathrm{fold}} + E_{\mathrm{add}}$ and κ_s' its replacement in E_{new}: the two have the same mass $(v_{\mathrm{S}} + v_{\mathrm{D}}) r^2$, and by Lemma 22.6(e) the difference has a sharp bound $k_{\mathrm{new}}(X) b^{-1/2}$ relative to κ_s at the cut ϑ_{1+s}, in the sense of Definition 9.2(b), with the shared label as the label of that definition, and

variation at most twice the mass. The difference of type E_s carries the tag s of (CP2)(b), whose cut ϑ_{1+s} is the same in every middle block. This is (CP2)(b) with two tags, with the sharp and variation prices of Lemma 22.9; the tag totals are equal.

22.3. **The law N dominating λ at ι.** Hypothesis (CP3) requires one nonnegative law N that dominates λ at the reserved cut in a 2×2 block. The next lemma builds N from μ and λ record by record and proves that the block is positive semidefinite.

Lemma 22.7 (The 2×2 block dominating λ at ι)**.** *Let the weights of every middle piece satisfy the thirteen inequalities of Lemma 22.1(iii) and the protection* (22.1)*.*

(i) $\mu \preceq_\iota \mathsf{N}_0$*, and* $\mathsf{N}_0 - \mu = E_{\mathrm{add}} + G + H$ *is a nonnegative reserved-diagonal measure.*

(ii) $\mathsf{N} = \mathsf{N}_0 + \delta_E = \lambda + F_{\mathrm{fold}} + H$ *is a nonnegative measure,* $\lambda_{\mathcal{B}}$ *is symmetric at* ι*, and blockwise* $\begin{pmatrix} \mathsf{N}_{\mathcal{B}} & \lambda_{\mathcal{B}} \\ \lambda_{\mathcal{B}} & \mathsf{N}_{\mathcal{B}} \end{pmatrix} \succeq_\iota 0$*; in particular* $-\mathsf{N} \preceq_\iota \lambda \preceq_\iota \mathsf{N}$*.*

Proof. (i) Each of E_{add}, G and H is a positive measure fixed samplewise by ι, so its cut matrix is a nonnegative diagonal matrix (Lemma 5.4), and so is that of their sum $\mathsf{N}_0 - \mu$.

(ii) λ is nonnegative: F_{fold} and E_{fold} are actual components of the unnormalized reference, and E_{new} and G are nonnegative. Hence so is $\mathsf{N} = \lambda + F_{\mathrm{fold}} + H$. The reference is invariant under the layer swap (Lemma 22.1) and all records are reserved diagonal, so the cut matrices of $\lambda_{\mathcal{B}}$ and $\mathsf{N}_{\mathcal{B}}$ at ι are symmetric. As in (i), $\mathsf{N} - \lambda = F_{\mathrm{fold}} + H \succeq_\iota 0$. Next, $\mathsf{N} + \lambda = 2\mu - F_{\mathrm{fold}} - 2E_{\mathrm{fold}} + H + 2E_{\mathrm{new}} + 2G$, where E_{new} and G are nonnegative and reserved diagonal; it remains to show $2\mu - F_{\mathrm{fold}} - 2E_{\mathrm{fold}} + H \succeq_\iota 0$, and we do so summand by summand in $2\widetilde{\mu}_{\mathcal{B}}$ (all records are divided by the same normalizer $\widetilde{\mu}_{\mathcal{B}}(1)$). Each fold component Π_{F_s}, $s \in \{1, 2\}$, enters $2\widetilde{\mu}_{\mathcal{B}}$ with $2w_{\mathrm{D}} r\, \Pi_{F_s} + 2v_{\mathrm{D}} r^2 \operatorname{Diag}_\iota(\Pi^{(1)}_{F_s})$, and $-F_{\mathrm{fold}}$ removes one $v_{\mathrm{D}} r^2 \operatorname{Diag}_\iota(\Pi^{(1)}_{F_s})$; what remains is positive at ι by Lemma 22.1(iv). Each matching component Π_{E_s} enters with $2w_{\mathrm{S}} r\, \Pi_{E_s} + 2v_{\mathrm{S}} r^2 \operatorname{Diag}_\iota(\Pi^{(1)}_{E_s})$; $-2E_{\mathrm{fold}}$ removes its folded part, and H adds $2w_{\mathrm{S}} r \cdot 3/(b-3) \cdot \operatorname{Diag}_\iota(\Pi^{(1)}_{E_s})$. The component Π_{E_s} is uniform on two disjoint injective triples, and its cut operator at ι relative to its actual half law, the uniform law on injective triples, which is the half law of $\Pi^{(1)}_{E_s}$, is the disjoint-triple operator at each prime; on the block its spectrum lies in $[-3/(b-3), 1]$ (Lemma 12.11(c)). Hence $\Pi_{E_s} + \frac{3}{b-3} \operatorname{Diag}_\iota(\Pi^{(1)}_{E_s}) \succeq_\iota 0$. Every other component (the root, the aligned fold Π_{F_0}, the merges and the constants) keeps both its two-layer part and its folded part, which are positive together at ι by Lemma 22.1(ii) and the third line $v_{\mathfrak{t}} \geq k_{\mathfrak{t}} w_{\mathfrak{t}}$ of Lemma 22.1(iii). So $2\mu - F_{\mathrm{fold}} - 2E_{\mathrm{fold}} + H \succeq_\iota 0$, and $\mathsf{N}_{\mathcal{B}} \pm \lambda_{\mathcal{B}} \succeq_\iota 0$, which is the block statement for the symmetric matrices: conjugation by the unitary $2^{-1/2}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ turns the block matrix into the diagonal matrix with blocks $\mathsf{N}_{\mathcal{B}} + \lambda_{\mathcal{B}}$ and $\mathsf{N}_{\mathcal{B}} - \lambda_{\mathcal{B}}$ (Definition 9.13). Outside the middle, $\mathsf{N}_{\mathcal{B}} = \lambda_{\mathcal{B}} = \mu_{\mathcal{B}} \succeq_\iota 0$. □

With the price $u_{\mathcal{B}}$ of $\mathsf{N}_{0,\mathcal{B}} - \mu_{\mathcal{B}}$ given by Lemma 22.9, the lemma is hypothesis (CP3) of Theorem 9.21 (Definition 9.13). The energy bound that the bracket buys (every structured energy of a fixed selected law with N outside is at most $B + o(1)$ times its price times Y, the part without record differences costing at most e^{U_0} and the rest at most R_{err}) is Lemma 10.12, which assumes only (CP1), (CP2)(b), (CP3) and (CP8).

Remark 22.8 (What the bracket buys)*.* The bracket is a Gram bracket at ι, not a domination at every cut: the comparison law built later need not be reflection positive, and the comparison is made at the reserved cut only. Two features of the comparison rest on this. First, $\lambda_{\mathcal{B}}$ is symmetric at ι, so in the expansion of $Y_\nu - Y_\lambda$ every tensor term with an odd number of skew factors is skew-adjoint and has zero real quadratic form on the identical reserved half-functions; this is why the reserved cost is $B(e^S \cosh A - 1)$ and not $B(e^{S+A} - 1)$, where S and A are the total symmetric and skew prices of Lemma 10.14. Second, the conductors of the index supports are

held fixed through every Cauchy–Schwarz step: for complementary fold maps f_+, f_- and every slot-weight vector $w = (w_v)$, $\frac{1}{2}(\sum_v w_{f_+(v)} + \sum_v w_{f_-(v)}) = \sum_v w_v$, so although a later chamber fold can give four occurrences of a conductor on one energy and none on the other, their geometric mean has the original two; only after the complementary energies are combined are the conductor powers averaged. The conductor moment of the matching term is therefore exactly R_I^6, also in the mixed-cut remainder of Lemma 10.12 (Lemma 8.8(e)).

The bracket carries one copy of the one-layer folded copies of the fold records, so that $\mathsf{N}_0 - \mu = E_{\mathrm{add}} + G + H$, whose coefficient of r^2, namely $2v_{\mathrm{D}}$, is half that of the first-stage records $F_{\mathrm{fold}} + E_{\mathrm{add}}$ (Lemma 22.9). Its part $E_{\mathrm{add}} + H$ is invariant under the chamber reflections and the group $\mathcal{G}_6$ they generate, as μ is, and the positive envelopes use $\mu + E_{\mathrm{add}} + H$ as their invariant background (Lemma 25.6). The price is the protection (22.1), binding in every piece (Remark 22.2), and the diagonal copy of the fold records available to the joint Gram bound of the matching envelopes has coefficient v_{D} per side (Proposition 26.5).

22.4. Prices L_0 and B for λ and N.

Lemma 22.9 (Middle prices)**.** *Let $\mathcal{B}$ be a block of a middle piece starting at X, with least prime $b \geq X$ and $r = b^{-1/2}$. Evaluate every local coefficient below at X. Then the prices of $\mathcal{B}$ are at most $q_{\mathcal{B}}^{2a}$ times*

$$\begin{aligned}
&\mathsf{N}_0 - \mu:\ 2v_{\mathrm{D}}r^2 + \Big(2v_{\mathrm{D}}(f_1+h_1) + \frac{12w_{\mathrm{S}}}{1-3/X}\Big)r^3,\\
&F_{\mathrm{fold}} + E_{\mathrm{add}} + G:\ 4v_{\mathrm{D}}r^2 + 2v_{\mathrm{D}}(f_1+h_1)r^3,\\
&\delta_E\ \textit{(sharp)}:\ 2(v_{\mathrm{S}}+v_{\mathrm{D}})k_{\mathrm{new}}(X)\,r^3, \qquad \delta_E\ \textit{(variation)}:\ 4(v_{\mathrm{S}}+v_{\mathrm{D}})r^2,
\end{aligned}$$

where k_{new} is the coefficient of Lemma 22.6; each of the two tags carries half of the last two prices.

Proof. Dividing by the normalizing mass $\widetilde{\mu}_{\mathcal{B}}(1) \geq 1$ only decreases each mass, so it suffices to bound unnormalized masses. The masses of E_{add}, G and H (Definition 22.4(b)) give the first price and those of F_{fold}, E_{add} and G the second, with $3/(b-3) = 3r^2/(1-3q) \leq 3r^2/(1-3/X)$. The two prices of δ_E are those of (CP2)(b) above: two record differences, each of a record of mass $(v_{\mathrm{S}}+v_{\mathrm{D}})r^2$, with sharp bound $k_{\mathrm{new}}(X)r$ and variation at most twice that mass, and no coincidence term; each tag consists of one of them. The row constants f_1, h_1 and the coefficient k_{new} are valid at every block of the piece when evaluated at X (Lemma 20.3(iii)). □

Summing over the blocks of each piece with the charges $b_d(X,Y)$ of Lemma 20.4 for a piece $[X,Y)$, writing b_2, b_3 for $b_2(X,Y), b_3(X,Y)$ below, and over the 33 pieces, with the weights and row constants of each piece,

$$U_0 \leq \sum_{\text{pieces}} \Big[2v_{\mathrm{D}}b_2 + \Big(2v_{\mathrm{D}}(f_1+h_1) + \frac{12w_{\mathrm{S}}}{1-3/X}\Big)b_3\Big], \qquad P \leq \sum_{\text{pieces}} \big[4v_{\mathrm{D}}b_2 + 2v_{\mathrm{D}}(f_1+h_1)b_3\big],$$

$$D \leq \sum_{\text{pieces}} 2(v_{\mathrm{S}}+v_{\mathrm{D}})k_{\mathrm{new}}(X)\,b_3, \qquad V \leq \sum_{\text{pieces}} 4(v_{\mathrm{S}}+v_{\mathrm{D}})\,b_2,$$

each tag carrying half of D and of V. In the certificate arithmetic all coefficients have nonnegative lower endpoints in every piece, so upper charges times upper coefficients are upper bounds. The certificate evaluates these sums from `research/push_2026_10_25/global/refined_L570_source.json`, with exact finite block histograms and outward analytic charges, at the anchor exponent $a_0 = \frac{4442183}{5000000000}$ at which the middle prices are tabulated (Certificate D.4): the totals there (primed, denoting evaluation at a_0) have the upper ends $U_0' \leq 0.8469081935391430$, $P' \leq 1.6936504653996420$, $D' \leq 4.741704122563 \cdot 10^{-5}$ and $V' \leq 3.387253370148178$, and each of the two tags has sharp total at most $D'/2$ and variation total at most $V'/2$; the four sums of each piece are listed in Table 25. At the baseline exponent a every middle block has $\log q_{\mathcal{B}} \leq L = 570$

(Lemma 20.3(ii)), so all these prices are at most $t = e^{2(a-a_0)L}$ times their values at a_0 (Lemma 9.3), and Corollary 9.4 with the two-tag series of Corollary 10.9 at $n = 4$ gives $U_0 < 0.846908245753$, $B < 2.332628665480$, $R_{\mathrm{err}} < 2.042558854365 \cdot 10^{-4}$ and $L_0 < 4.763323725665 \cdot 10^{-4}$ (Definition 30.1(2)); the repriced tag totals, about $2.37 \cdot 10^{-5}$ (sharp) and 1.694 (variation), satisfy the hypotheses of that corollary, since the sharp total is below the variation total and below 6, and their product is below 36. For comparison, the one-sharp bounds De^{P+V} and De^{U_0+V} of Remark 9.12 would be about $7.7 \cdot 10^{-3}$ and $3.3 \cdot 10^{-3}$: in the tagged series a term whose differences all carry one tag pays no variation price, and a term with both tags pays the variation of the less frequent tag only. The prices of the comparison records of the middle range are those of Lemma 23.24, assembled from Proposition 23.13 and Lemmas 23.17, 23.18 and 23.20 with the same block charges.

Consequences and what to read next. This section depends on the following statements, grouped by topic. Lemma 20.9, Lemma 20.3 and Lemma 20.4 (pairs, domain and charges); Lemma 20.11 (the half graphs and the symmetries); the caps of the root: Lemma 20.12, Proposition 21.11, and Certificate 21.25, with Lemma 13.3(f) and Lemma 13.7(b); Definition 21.1 and Lemmas 21.2 and 21.17; Lemma 23.2 (the cap k_{D}); Proposition 13.16 in the form of Corollary 13.17, Lemma 13.25, Lemma 12.7, Lemma 6.10 and Lemma 4.18 (the rows, the block steps and the evaluation at the piece start); Lemma 17.5 with Lemma 5.13, Lemma 5.16, Lemma 5.4 and Lemma 17.2 (the completion); Lemma 17.11 (the first stage); Lemma 12.10 and Lemma 4.11 (the one-stage repair); Lemma 12.11 (the coefficient of H); Definitions 9.11 and 9.13 and Corollary 10.9 (the tags and the series); Lemma 9.3 and Corollary 9.4 (prices at a). Certificates: Certificates D.3 to D.6, 20.13, 20.22, 21.25 and 23.10.

Hypotheses verified in this section (Table 5). (CP1) for the middle atoms, that is, (R1)–(R3) with (R2) at every cut of the family (Lemma 22.1); (CP2)(a) at ϑ_0 (Lemma 22.5) and (CP2)(b) with two tags (Lemma 22.6, Certificate D.5); (CP3) (Lemma 22.7, under the protection (22.1)); the prices P, V, D, U_0 and the tag totals (Lemmas 20.4 and 22.9 and Certificate D.4, Lemma 9.3); and the admissibility clause (CP8)(a) for the reference, the first-stage records and the bracket records. These are the rows "(CP1) middle", "(CP2)(a)", "(CP2)(b)", "(CP3)" and "P, V, D, U_0, tags" of Table 5.

23. Middle primes IV: the comparison law ν and its prices S, A

What this section proves and how to read it. This section proves (CP4) for middle atoms: it constructs ν from λ by keeping forcing records, replacing non-forcing records at fixed actual marginals, and deleting the rest, and bounds symmetric and skew prices S, A. A *record* is one summand (see Section 1.8); F_0, F_1, F_2 are fold records (two-layer, defined below) and E_0, E_1 matching records. The key idea is mirror pairing: adding a required edge together with its mirror makes the first-order change purely skew. Inputs: Section 22. Outputs: middle contributions to S, A for Section 30. A first reading should follow the keep/replace/delete classification and mirror pairing; per-prime kernel estimates are reference material. On a middle block the comparison law keeps the two-layer root and the aligned fold F_0, which accept the six queries, and changes the other two-layer records: the fold records F_1, F_2 are replaced by the signed whole-block fold record of parameter `t` (Sections 23.4 and 23.6), a combination of mass one of three positive accepting product laws, each with the exact actual half marginals of the record it replaces; the matching records E_0, E_1 by the exact-marginal law K_1 (Section 23.7); the six merge records by their actual-marginal mirror repairs (Section 23.8); the two-layer constants by strict Paley-edge laws. The folded merges and constants and the polarization correction G of the first stage are deleted. Every replacement keeps the actual half marginals at ι of the record it replaces, so every record difference annihilates the constants on both sides of the reserved cut, and every deleted record is positive. Lemma 23.24 collects the symmetric, skew and auxiliary prices of a block; they enter S, A and the class bounds as sums over the pieces of the degree charges times local

coefficients evaluated at the piece start. Later sections use more than the prices: the three-arm leading stars and remainders of the fold changes and the one-arm stars and degree-four remainders of the matching changes (Proposition 23.13 and Lemma 23.18) are the inputs of the positive envelopes (Theorem 25.8); the coefficient $d_{\mathtt{t}}$ of the joint Gram matrix for the reference and signed records (Proposition 23.13(d)) is the input of the matching sides (Proposition 26.4); the negative cap k_{D} (Lemma 23.2(e)) enters the completion (Lemma 22.1); and the centred norm κ_F of the reference fold record (Corollary 23.3) is an input of the variance credits (Section 24).

The idea of the section is a signed pairing of the fold records. The reference fold record has the two cross edges $x_0 \to y_2$ and $y_0 \to x_2$ between its layer-zero labels x and its layer-one labels y, and the queries need one more, $x_1 \to y_0$. Adding it alone would change the record at first order in its symmetric and in its skew part. Adding it together with the mirror edge $x_0 \to y_1$ (the reversal of its image $y_1 \to x_0$ under the reserved reflection) makes the first-order change purely skew, a two-arm star of norm about $\sqrt{2}\,r$, and pushes the symmetric change to second order: this is the mirror pairing (Section 23.3). The signed record adds two further product laws with coefficients $\pm\mathtt{t}/2$, the signed sum being taken after the products over the primes of the block. The first-order change stays purely skew and becomes a three-arm star of norm about $(1+\mathtt{t}^2+(1-\mathtt{t})^2)^{1/2}r$ (about $1.2732\,r$ at $\mathtt{t}=127/500$), at the price of a total variation $1+\mathtt{t}$ per block, whatever the number of primes; Remark 23.6 explains the trade. Two inputs about the reference fold record are exact rather than perturbative: the pair laws of its actual half marginal (Lemma 23.4), and a centred bound for its coincidences, which also lowers its negative cap (Lemma 23.2(e),(f)).

23.1. **One prime: halves, kernels and channels.** Throughout the section $p \equiv 3 \pmod 4$ is a prime, $p \geq 19$ unless a larger bound is stated (every middle prime is at least $Q_3 = 19951586 > 2^{24}$; the statements below that need $p \geq 2^{20}$ or $p \geq 2^{24}$ say so), and

$$q = \frac{1}{p}, \quad r = p^{-1/2}, \quad Z = (1-q)(1-2q), \quad h_0 = 1-14q, \quad e_0 = \frac{2q}{(1-3q)(1-4q)}, \quad k_p = \frac{p-3}{4}.$$

Let χ be the quadratic character and write $a \to b$ if $\chi(b-a)=1$; as $\chi(-1)=-1$ this is a tournament. Finite sets carry their uniform probability and $\mathbb{E}$ is the corresponding average; H_0 is the space of mean-zero functions on $\mathbb{F}_p$, P the Paley operator $(Ph)(s) = \mathbb{E}_t \chi(t-s)h(t)$, so $P^* = -P$, $P\mathbf{1} = 0$ and $P^*P = qI$ on H_0, and $\mathcal{G}_p$ the group of the maps $t \mapsto \alpha t + \beta$, α a nonzero square, acting diagonally on tuples of labels (Lemma 12.10).

Halves and kernels. A *half* is a triple $x = (x_0, x_1, x_2)$ of distinct labels. Let H_F be the set of halves with $x_0 \to x_2$, so $|H_F| = p(p-1)(p-2)/2$, and ν_F its uniform law; its coordinates are uniform, $V_i \colon L^2(\mathbb{F}_p) \to L^2(\nu_F)$, $(V_i h)(x) = h(x_i)$, is an isometry and V_i^* is the conditional expectation onto x_i (Lemma 12.10(d)). At the reserved cut a two-layer fold record of a middle block is, at one prime, a probability measure on $H_F \times H_F$: the first half carries the layer-zero labels x of the three objects and the second the layer-one labels y (Lemma 17.2(b),(d)). We identify such a measure with its density $K(x,y)$ relative to $\nu_F \otimes \nu_F$ and with the operator $Kf(x) = \mathbb{E}_y K(x,y) f(y)$ on $L^2(\nu_F)$; every function on $H_F \times H_F$ is regarded as such a kernel, and $\|\cdot\|$ is the operator norm on $L^2(\nu_F)$ unless another space is named. The reserved reflection exchanges x_i and y_i; it maps K to $K^\top(x,y) = K(y,x)$, the adjoint operator. Put $\operatorname{Sym} K = \frac{1}{2}(K + K^\top)$, $\operatorname{Skew} K = \frac{1}{2}(K - K^\top)$ and $\Pi_\perp = I - \mathbb{J}$. If $m = g\nu_F$ is a probability measure with $g \geq 1-\eta$ for some $\eta \in [0,1)$, the *operator of K on $L^2(m)$* has the same bilinear form, $\langle \varphi, K_m \psi \rangle_m = \int \varphi(x) K(x,y) \psi(y)\, d\nu_F(x)\, d\nu_F(y)$; by Lemma 13.21(b) its norm is at most $\|K\|/(1-\eta)$, and its symmetric and skew parts are those of K, transported in the same way. If K, K' are densities of probability measures with both half marginals m, the operator of $K' - K$ on $L^2(m)$ is the operator Δ of the charging rule Remark 9.15(ii) for the replacement of K by K'; the same holds when K' is a signed density of mass one with both half marginals m.

The Paley channels. The cross characters of a pair of halves are

$$\mathsf{A}=\chi(y_2-x_0),\qquad \mathsf{B}=\chi(x_2-y_0),\qquad \mathsf{C}=\chi(y_0-x_1),\qquad \mathsf{D}=\chi(y_1-x_0); \tag{23.1}$$

on six distinct labels $\mathsf{A}=1$, $\mathsf{B}=1$, $\mathsf{C}=1$ and $\mathsf{D}=1$ mean $x_0\to y_2$, $y_0\to x_2$, $x_1\to y_0$ and $x_0\to y_1$. Let $I_\times$ be the indicator that x and y have no common label. Since $\chi(-1)=-1$, transposition acts by

$$(\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D})^\top=(\mathsf{B},\mathsf{A},-\mathsf{D},-\mathsf{C}),\qquad I_\times^\top=I_\times. \tag{23.2}$$

Kernels below are defined at every pair of halves; a channel whose two labels coincide vanishes there. The edges of A and D form a star at x_0 with leaves y_2,y_1, and those of B and C a star at y_0 with leaves x_2,x_1; the two stars have no common label. A monomial in the channels is a *star monomial* if all its edges lie in one star (A, B, C, D, AD, BC) and a *nonstar* otherwise; the nine nonstars are listed in Lemma 23.11. This two-star picture organizes all the estimates of the fold records: star monomials have norms of order r, and nonstars, which contain two disjoint cross edges, of order q.

Features. For the operators $A_1=V_2P$, $A_2=V_1P$ and $(A_3h)(x)=-\mathbb{E}_s[\chi(s-x_2)\chi(s-x_1)h(s)]-q\mathbb{E}h$ of Lemma 12.10(e), the kernels of $A_1V_0^*$, $A_2V_0^*$ and $A_3V_0^*$ are $-\mathsf{B}$, C and $\mathsf{BC}-q$, since they depend on y only through y_0, which is uniform; transposing, $\mathsf{A}=-V_0A_1^*$ and $\mathsf{D}=-V_0A_2^*$. Put $A_\star=A_2+A_3$ and, for $0\le\mathsf{t}\le1$,

$$A_\mathsf{t}=A_2+\mathsf{t}A_1+(1-\mathsf{t})A_3,\qquad \mathbf{a}_\mathsf{t}=(\mathsf{t},1,1-\mathsf{t}),\qquad |\mathbf{a}_\mathsf{t}|^2=1+\mathsf{t}^2+(1-\mathsf{t})^2,$$

so that $A_\mathsf{t}=A_{\mathbf{a}_\mathsf{t}}$ in the notation of Lemma 12.10(e), with the *arm vector* $\mathbf{a}_\mathsf{t}$, and $A_0=A_\star$. The actual half marginal $m_0=g_0\nu_F$ of the reference fold record (Lemma 23.2(a)) is translation invariant, so its coordinates are uniform and V_i is also an isometry $H_0\to L^2(m_0)$; write $V_{i,m}^*$ for its adjoint in $L^2(m_0)$, $\mathcal{H}_i=V_iH_0$, and $\mathcal{K}_{12}$ for the orthogonal complement in $L^2(m_0)$ of $\mathcal{H}_1+\mathcal{H}_2$ within the m_0-mean-zero functions of (x_1,x_2). The operator A_3 maps H_0 into the latter space, because $s\mapsto\mathbb{E}_{m_0}[\chi(s-x_2)\chi(s-x_1)]$ is translation invariant, hence constant. Put $A_3^\circ=\Pi_{\mathcal{K}_{12}}A_3$, $A_\star^\circ=A_2+A_3^\circ$ and $A_\mathsf{t}^\circ=A_2+\mathsf{t}A_1+(1-\mathsf{t})A_3^\circ$; the three arms A_2, $\mathsf{t}A_1$ and $(1-\mathsf{t})A_3^\circ$ of A_t° map into $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{K}_{12}$, and $A_0^\circ=A_\star^\circ$.

Lemma 23.1 (Two-feature spectral calculation)**.** *Let $V\colon H_0\to\mathcal{H}$ be an isometry and $A\colon H_0\to\mathcal{H}$ satisfy, for real a,b,c,*

$$V^*A=aI+bP,\qquad A^*A=cI,$$

where $P^*=-P$ and $P^*P=qI$ on H_0, and $r=\sqrt{q}$. Then $\delta=c-a^2-b^2q\ge0$, and

$$\|AV^*+VA^*\|=|a|+\sqrt{c-b^2q},$$
$$\|AV^*-VA^*\|=|b|r+\sqrt{c-a^2}.$$

The spectrum of AV^*+VA^* is contained in $\{0,a\pm\sqrt{c-b^2q}\}$. Thus one pair of Gram identities determines both the symmetric reference spectrum and the skew spectrum of a mirror or signed star, in whichever half-marginal Hilbert space the identities hold.

Proof. Put $M=V^*A$ and $A_\perp=A-VM$. Then $V^*A_\perp=0$ and $A_\perp^*A_\perp=A^*A-M^*M=\delta I$, proving $\delta\ge0$. If $\delta>0$, write $A_\perp=\sqrt{\delta}\,U$, with U an isometry orthogonal to V. Both operators vanish off $VH_0\oplus UH_0$, and on this sum they are

$$AV^*+VA^*=\begin{pmatrix}2aI&\sqrt{\delta}I\\ \sqrt{\delta}I&0\end{pmatrix},\qquad AV^*-VA^*=\begin{pmatrix}2bP&-\sqrt{\delta}I\\ \sqrt{\delta}I&0\end{pmatrix}.$$

The first matrix has eigenvalues $a\pm\sqrt{a^2+\delta}$. On a P-eigenspace of eigenvalue $i\varepsilon r$, $\varepsilon\in\{\pm1\}$, the second has eigenvalues $i(b\varepsilon r\pm\sqrt{b^2q+\delta})$; its norm is $|b|r+\sqrt{b^2q+\delta}$. Substitution gives

the formulas. If $\delta = 0$, then $A = VM$, and the operators on VH_0 are $2aI$ and $2bP$; the same conclusions hold directly. □

The reflected transpose. The map $\varsigma(x) = (-x_2, -x_1, -x_0)$ is an involution of H_F preserving ν_F: the relation $s \to t$ becomes $-t \to -s$ under negation, so $x_0 \to x_2$ is preserved, and $\mathbf{1}[x_0 \to x_1]$ and $\mathbf{1}[x_1 \to x_2]$ are exchanged. Put $(\mathcal{U}f)(x) = f(\varsigma x)$, a self-adjoint unitary on $L^2(\nu_F)$, and call $K^\varsigma(x,y) = K(\varsigma y, \varsigma x)$ the *reflected transpose* of a function K on $H_F \times H_F$. As operators $K^\varsigma = \mathcal{U}K^\top\mathcal{U}$. The map $(x,y) \mapsto (\varsigma y, \varsigma x)$ preserves A, B and $I_\times$, and $\mathsf{C}^\varsigma = \chi(y_1 - x_2)$. Only this arithmetic identification of coordinates is used; no negation invariance of an input function is assumed.

23.2. **The reference fold record at the reserved cut.** Before the fold records F_1, F_2 can be replaced, the record they replace must be understood exactly: the reference's own two-layer fold record at the reserved cut. At one prime this is the uniform law on pairs of halves (x,y) with six distinct labels and the two cross edges $x_0 \to y_2$ and $y_0 \to x_2$ (the channels A and B of (23.1)). The next lemma collects everything later sections use about it: how many completions y each half x has and the resulting normalizing constant and actual half marginal; an exact spectral description of its leading part; its centred norms and remainder in the actual marginal; and its negative cap k_{D} at ι, which enters the completion inequality $v_{\mathrm{D}} \ge k_{\mathrm{D}} w_{\mathrm{D}}$ of Lemma 22.1 and the protection inequality (22.1). The exact pair laws of its actual half marginal follow in Lemma 23.4.

Lemma 23.2 (The reference fold record)**.** *Let $p \ge 19$, $\Psi_0 = I_\times(1+\mathsf{A})(1+\mathsf{B})$, $z_0 = \mathbb{E}\Psi_0$ (the average over $\nu_F \otimes \nu_F$) and $K_0 = \Psi_0/z_0$. On six distinct labels Ψ_0 is 4 times the indicator of $x_0 \to y_2$ and $y_0 \to x_2$, and it vanishes elsewhere; so K_0 is uniform on the pairs of halves with six distinct labels, $x_0 \to y_2$ and $y_0 \to x_2$. It is the law at ι of each two-layer fold record $\Pi_{F_0}, \Pi_{F_1}, \Pi_{F_2}$ of the middle reference at one prime (Lemma 17.2).*

(a) Fibres and normalizer. *For $x \in H_F$ put $e_{01} = \mathbf{1}[x_0 \to x_1]$ and $e_{12} = \mathbf{1}[x_1 \to x_2]$. The number of y with $\Psi_0(x,y) \ne 0$ is $N_0(x) = (p-5)(k_p-1)(\frac{p-3}{2} - e_{01} - e_{12})$, and as $\Psi_0^\top = \Psi_0$ the same formula counts the x for a given y. Both half marginals of K_0 equal $m_0 = g_0\nu_F$, where $g_0 \circ \varsigma = g_0$,*

$$g_0 = \frac{(p-2)(p-3-2e_{01}-2e_{12})}{(p-3)(p-4)}, \qquad \|g_0 - 1\|_\infty = \eta_F := \frac{2(p-1)}{(p-3)(p-4)} = \frac{2q(1-q)}{(1-3q)(1-4q)} \le e_0,$$

the largest value of g_0 is $(p-2)/(p-4) = 1 + 2/(p-4)$, and, with $h_0 \le z_0 \le 1$,

$$z_0 = \frac{(p-3)(p-4)(p-5)(p-7)}{p(p-1)(p-2)^2} = 1 - 14q + \frac{q^2(53 - 273q + 364q^2)}{(1-q)(1-2q)^2}.$$

On a block $\mathcal{B}$ of at most n primes $p \ge X$ the density of $m_{0,\mathcal{B}} = \bigotimes_p m_{0,p}$ relative to $\nu_{F,\mathcal{B}} = \bigotimes_p \nu_{F,p}$ is at most $(1+e_0(X))^n \le e^{ne_0(X)}$, and at most $(1-ne_0(X))^{-1}$ if $ne_0(X) < 1$, where $e_0(X)$ denotes e_0 evaluated at $q = 1/X$.

(b) Leading part. *As kernels $\Psi_0 = \mathbb{J} + \mathsf{L} + \mathsf{AB} - \mathsf{C}_{\mathrm{rem}}$, with $\mathsf{L} = \mathsf{A} + \mathsf{B}$ and $\mathsf{C}_{\mathrm{rem}} = (1 - I_\times)(1 + \mathsf{A})(1 + \mathsf{B})$. On $L^2(\nu_F)$:*

(i) *L annihilates the constants and has its spectrum in $\{0\} \cup \{q/(1-q) \pm (q - q^3/(1-q)^2)^{1/2}\}$; hence $\mathsf{L} \succeq -(r-q)I$ and $\|\mathsf{L}\| \le r + q/(1-q) \le \Gamma r$, where $\Gamma = 1 + \varrho + \varrho'$, $\varrho = r\sqrt{1+q}/(1-q)$ and $\varrho' = q(1+\varrho)/(1-2q)$ (Lemma 12.10(d));*

(ii) *$\|\mathsf{AB}\| \le 2q/(1-q) \le 2q/Z$;*

(iii) *$\mathsf{C}_{\mathrm{rem}}$ is nonnegative and symmetric, its row sums are at most $q(17 - 120q + 247q^2)/Z \le 17q$, and $\|\mathsf{C}_{\mathrm{rem}}\| \le 17q \le 17q/Z \le 36q$.*

(c) Centred norms. *$\|\Pi_\perp K_0 \Pi_\perp\| \le h_{\mathrm{old}}/h_0$ with $h_{\mathrm{old}} = \Gamma r + (2/Z + 36)q$, and on the m_0-centred functions of $L^2(m_0)$ the operator of K_0 has norm at most $\kappa_{\mathrm{old}} = (h_{\mathrm{old}}/h_0 + e_0^2)/(1 - e_0)$. Moreover $K_0 = (\mathbb{J} + \mathsf{L})/z_0 + R_{\mathrm{old}}$ with $\|R_{\mathrm{old}}\| \le 19q/(Zh_0)$.*

(d) Negative cap on the uniform law. $K_0 \succeq -r\big[1+(19/Z-1)r\big]h_0^{-1}I$ *on* $L^2(\nu_F)$.

(e) Blocks. *Let* $\mathcal{B}$ *be a block of primes* $p \ge X \ge 2^{24}$ *with least prime* b *and* $\sum_{p\in\mathcal{B}}\log p \le L$, *let* $n_X \ge \lfloor L/\log X\rfloor$ *be an integer (the certificate takes* $n_X = \lfloor L/\log^- X\rfloor$ *with a rational lower bound* $\log^- X \le \log X$*),* $q_X = 1/X$, $r_X \ge X^{-1/2}$ *and* $e_X = e_0(X)$, *suppose* $n_X e_X < 1$, *and put* $\bar H_X = (1-n_Xe_X)^{-1}$ *and*

$$k_{\mathrm D}(X,L) = \bar H_X\big[1+(e_o(X)-1)r_X\big],$$

where $e_o(X)$ *is the constant* e_o *of* (23.5) *at* (q_X, r_X). *The notation* $k_{\mathrm D}(X,L)$ *suppresses the choices* n_X, r_X *fixed here. If* $[1+(e_o(X)-1)r_X]r_X < 1$, *then for every real function* φ *on* $H_{F,\mathcal{B}} = \prod_{p\in\mathcal{B}} H_{F,p}$

$$\int \varphi(x)\varphi(y)\bigotimes_{p\in\mathcal{B}} K_{0,p}\,d\nu_{F,\mathcal{B}}(x)\,d\nu_{F,\mathcal{B}}(y) \;\ge\; -k_{\mathrm D}(X,L)\,b^{-1/2}\int\varphi^2\,d\nu_{F,\mathcal{B}}.$$

Hence every two-layer fold component of the reference of $\mathcal{B}$ *satisfies*

$$\Pi_{F_j} \succeq_\iota -k_{\mathrm D}(X,L)\,b^{-1/2}\,\mathrm{Diag}_\iota(\Pi^{(1)}_{F_j}),$$

which is the fold case of Lemma 22.1(ii), with its reserved cap $k_{\mathrm D} = k_{\mathrm D}(X,L)$, *on every block of a piece starting at* X.

(f) Actual centred remainder. *Let* $p \ge 2^{24}$, *let* T_{old} *be the operator of* K_0 *on* $L^2(m_0)$, *the Markov operator of Corollary 23.3(c), let* $O = T_{\mathrm{old}} - \mathbb{J}$ *be its centred part, and put*

$$O_0 = V_0PV_{2,m}^* - V_2PV_{0,m}^* = -(A_1V_{0,m}^* + V_0A_1^{*_m}),$$

adjoints in $L^2(m_0)$. *Then* $O = O_0 + R_o$ *with* $\|R_o\| \le e_oq$, *where* e_o *is the constant of* (23.5) *at* (q,r); O_0 *is self-adjoint, annihilates the constants, has its spectrum in* $\{0\}\cup\{q/(1-q)\pm(q-q^3/(1-q)^2)^{1/2}\}$, *and* $\|O_0\| \le \Gamma_A r$, *with* Γ_A *of Lemma 23.4(d). Hence* $T_{\mathrm{old}} \succeq -\big[1+(e_o-1)r\big]r\,I$ *on* $L^2(m_0)$, *and the centred norm of* T_{old} *is at most* $(\Gamma_A+e_or)r$.

Proof. (a) All counts use Lemma 12.10(a). Fix x. The label y_0 is an in-neighbour of x_2 other than x_0, and other than x_1 if $e_{12}=1$: $\frac{p-1}{2}-1-e_{12}$ choices. Then y_2 is a common out-neighbour of x_0 and y_0 other than x_2, and other than x_1 if $x_0\to x_1$ and $y_0\to x_1$; if $e_{01}=1$, exactly k_p-1 admissible y_0 satisfy $y_0\to x_1$, namely the common in-neighbours of x_1,x_2 other than x_0. Finally y_1 is any of the $p-5$ remaining labels. This gives N_0. The ordered pairs of sources $x_0\ne y_0$ with two distinct common out-neighbours x_2,y_2 number $p(p-1)k_p(k_p-1)$, and each extends in $(p-4)(p-5)$ ways; dividing by $|H_F|$ gives the mean $\bar N_0 = (p-3)(p-7)(p-4)(p-5)/(8(p-2))$. As $\Psi_0=4$ on its support, $\mathbb{E}_yK_0(x,y) = 4N_0(x)/(|H_F|z_0)$, and averaging over x gives $z_0 = 4\bar N_0/|H_F|$; so each half marginal has the density $N_0/\bar N_0 = g_0$. As $e_{01}+e_{12}\in\{0,1,2\}$, g_0 takes the values $1+2/(p-4)$, $1-2/((p-3)(p-4))$ and $1-2(p-1)/((p-3)(p-4))$, the last being the farthest from one and the first the largest; ς exchanges e_{01} and e_{12}. Multiplying out, $(1-3q)(1-4q)(1-5q)(1-7q)-(1-14q)(1-q)(1-2q)^2 = 53q^2-273q^3+364q^4$, positive because $273^2 < 4\cdot53\cdot364$, which gives the expansion of z_0; and $(1-q)(1-2q)^2-(1-3q)(1-4q)(1-5q)(1-7q) = q(14-123q+385q^2-420q^3) > 0$ for $q\le 1/19$ gives $z_0\le 1$. On a block, $g_{0,p}\le 1+2/(p-4)\le 1+e_0(p)\le 1+e_0(X)$ for $p\ge X$; and $1+y\le e^y$ and $e^{ny}\le(1-ny)^{-1}$ for $0\le ny<1$.

(b) (i) By the kernel identities of Section 23.1, $\mathsf{L} = -(A_1V_0^* + V_0A_1^*)$; it annihilates the constants because A_1 does and maps into mean-zero functions. On H_0, $A_1^*A_1 = qI$ and $M := V_0^*A_1 = -(I+P)/(p-1)$ (Lemma 12.10(e)), so Lemma 23.1, with $a=b=-q/(1-q)$ and $c=q$, gives the nonzero eigenvalues of L as $q/(1-q)\pm\sqrt{q-q^3/(1-q)^2}$. The bounds follow from $(q-q^3/(1-q)^2)^{1/2}\le r$ and $\varrho r = q\sqrt{1+q}/(1-q)\ge q/(1-q)$. (ii) The kernel AB depends on (x_0,x_2) and (y_0,y_2) and is a product of two single cross edges; for independent uniform labels it is a tensor product of two operators of norm r, and Lemma 13.24 with the density $2/(1-q)$ of a directed edge on both sides gives the bound. (iii) Both factors of $\mathsf{C}_{\mathrm{rem}}$

are nonnegative, and $\mathsf{C}_{\mathrm{rem}}^{\top} = \mathsf{C}_{\mathrm{rem}}$. Fix x. Then $\mathbb{E}_y\mathsf{A} = \mathbb{E}_y\mathsf{B} = 0$. The pair (y_0, y_2) is a uniform directed edge, with density $(1 + \chi(y_2 - y_0) - \mathbf{1}_{y_0=y_2})/(1-q)$ relative to two independent labels; against $\mathsf{AB} = \chi(y_2 - x_0)\chi(x_2 - y_0)$ the three terms contribute 0, $q\chi(x_2 - x_0) = q$ and $-q^2$ (Lemma 12.10(b) and $x_0 \to x_2$). Hence $\mathbb{E}_y\mathsf{AB} = q$ and $\mathbb{E}_y(1+\mathsf{A})(1+\mathsf{B}) = 1+q$, while $\mathbb{E}_y\Psi_0(x,y) = 4N_0(x)/|H_F| \geq (p-5)(p-7)^2/(p(p-1)(p-2))$ by (a). The row sum of $\mathsf{C}_{\mathrm{rem}}$ at x is the difference,

$$\frac{p+1}{p} - \frac{(p-5)(p-7)^2}{p(p-1)(p-2)} = \frac{17p^2 - 120p + 247}{p(p-1)(p-2)} = \frac{q(17 - 120q + 247q^2)}{Z},$$

at most $17q$ because $17 - 120q + 247q^2 \leq 17(1 - 3q + 2q^2)$ for $q \leq 69/213$; and $17/Z \leq 36$. The Schur test for a nonnegative symmetric kernel bounds the norm by the largest row sum.

(c) $\Pi_\perp \mathbb{J} \Pi_\perp = 0$, so by (b) $\|\Pi_\perp K_0 \Pi_\perp\| \leq (\Gamma r + 2q/Z + 36q)/z_0 \leq h_{\mathrm{old}}/h_0$. The bound in $L^2(m_0)$ is Lemma 13.21(a) with $m_L = m_R = m_0$, $\eta = e_0 \geq \eta_F$ and all functions; and $R_{\mathrm{old}} = (\mathsf{AB} - \mathsf{C}_{\mathrm{rem}})/z_0$.

(d) By (b) and $\mathbb{J} \succeq 0$, $\Psi_0 \succeq -(r - q + 2q/Z + 17q/Z)I = -r[1 + (19/Z - 1)r]I$; divide by $z_0 \geq h_0$.

(e), (f): see the end of Section 23.5. The proof of (f) uses Lemma 23.4 and Lemma 23.12(d), and that of (e) uses (f). □

The reference uses the cap k_{D} for its fold type (Lemma 22.1(ii)). Its values $k_{\mathrm{D}}(X, 585)$ at the 33 piece starts, all below the recorded F21 block caps of Lemma 20.12, are those of Certificate D.6; at $X = Q_3$ the block count is $n_X = 34$, $\bar{H}_X < 1.0000035$, $1 + (e_o - 1)r_X < 1.0014832$ and $k_{\mathrm{D}} < 1.0014866$.

Corollary 23.3 (Actual marginal and centred norm of the reference fold record)**.** *Let $p \geq 19$.*

(a) The actual half marginal is $m_0 = g_0\nu_F$, with g_0 and $\|g_0 - 1\|_\infty = \eta_F$ as in Lemma 23.2(a).

(b) The joint law of the two ι-halves of the fold record has the density $K_0 = \Psi_0/z_0$ relative to $\nu_F \otimes \nu_F$, with $z_0 = (p-3)(p-4)(p-5)(p-7)/(p(p-1)(p-2)^2) \geq 1 - 14q$.

(c) The Markov operator T_{old} of this coupling on $L^2(m_0)$ is self-adjoint, $T_{\mathrm{old}}\mathbf{1} = \mathbf{1}$, and, provided $(2/Z + 36)q < 1$ and $\eta_F < 1$,

$$\|T_{\mathrm{old}} - \mathbb{J}\| \leq \kappa_F(q,r) := \frac{\dfrac{\bar{\Gamma} r + (2/Z + 36)q}{1 - (2/Z + 36)q} + \eta_F^2}{1 - \eta_F}, \qquad \bar{\varrho} = \frac{r(1+q)}{1-q}, \quad \bar{\Gamma} = 1 + \bar{\varrho} + \frac{q(1+\bar{\varrho})}{1-2q}.$$

(d) Let X be a nonempty finite set of middle blocks, $j \in \{0,1,2\}$, and κ_X the largest value of $\kappa_F(1/p, p^{-1/2})$ at the primes p of the blocks of X. The Markov operator T_X at ι of $\Pi_X = \bigotimes_{\mathcal{B} \in X} \Pi_{F_j,\mathcal{B}}$ on $L^2(m_X)$, $m_X = \bigotimes_{\mathcal{B}} m_{F_j,\mathcal{B}}$ the actual half marginal, satisfies $\|T_X - \mathbb{J}\| \leq \kappa_X$ and $T_X \preceq (1 - \kappa_X)\mathbb{J} + \kappa_X I$; hence, if $\kappa_X < 1$, with J_X the law with independent ι-halves of law m_X,

$$Y[J_X] \geq \frac{Y[\Pi_X] - \kappa_X(1 + o(1))C_X Y}{1 - \kappa_X}.$$

Proof. (a) and (b) are Lemma 23.2(a). (c) The layer swap preserves the record, so the coupling is symmetric, and T_{old} is self-adjoint and fixes the constants. As $\varrho \leq \bar{\varrho}$, $\Gamma \leq \bar{\Gamma}$, and Lemma 23.2(b) gives $\|\Pi_\perp(\Psi_0 - \mathbb{J})\Pi_\perp\| \leq \bar{\Gamma} r + (2/Z + 36)q$; with $z_0 \geq 1 - 14q \geq 1 - (2/Z + 36)q$, $\|\Pi_\perp K_0 \Pi_\perp\|$ is at most the inner fraction, and Lemma 13.21(a) with $\eta = \eta_F$ gives (c). (d) Every $\Pi_{F_j,\mathcal{B}}$ has the law $\bigotimes_{p \in \mathcal{B}} K_{0,p}$ at ι (Lemma 23.2), so T_X is the tensor product of the prime operators T_{old}, each with $\|T_{\mathrm{old}} - \mathbb{J}\| \leq \kappa_X$; Lemma 6.10(c) gives $\|T_X - \mathbb{J}\| \leq \kappa_X$, and as T_X is self-adjoint and fixes the constants, $T_X - \mathbb{J} \preceq \kappa_X(I - \mathbb{J})$. The energy bound is (P3) of Lemma 10.13 at ι with the outside law μ, followed by (E1) for the diagonal copy $\mathrm{Diag}_\iota(m_X)$. □

The certificate evaluates κ_F at the piece start with $\bar{\varrho} = r(1+q)/(1-q)$, which exceeds the exact norm $\varrho = r\sqrt{1+q}/(1-q)$ of $V_0^* V_2$, and with $1 - (2/Z + 36)q$, which is below z_0; both

substitutions only increase κ_F, which is admissible in (q,r). The bound κ_{old} of Lemma 23.2(c) bounds the same norm; the variance credits use κ_F, and the block bounds of the signed law the actual centred norm $(\Gamma_A + e_o r)r$ of Lemma 23.2(f).

The actual half marginal m_0 is not uniform, but its pair laws are exact and simple, and they give the correlations of the feature spaces without any density-perturbation allowance.

Lemma 23.4 (Exact actual features)**.** *Let $p \geq 19$, $\mu = 2/(p-4)$, and let m_{ij} be the law of (x_i, x_j) under the actual half marginal m_0 of Lemma 23.2(a).*

(a) $m_{01}(a,b) = m_{12}(a,b) = \dfrac{1 - \mu\chi(b-a)}{p(p-1)}$ *for $a \neq b$, and m_{02} is uniform on the pairs $a \to b$.*

(b) On H_0, $V_{0,m}^* V_2 = (P - qI)/(1-q)$ and $V_{0,m}^* V_1 = V_{1,m}^* V_2 = -(I + \mu p P)/(p-1)$. Hence the correlations of the singleton feature spaces in $L^2(m_0)$ are

$$\varrho_{02} = \|V_{0,m}^* V_2\| = \frac{r\sqrt{1+q}}{1-q}, \qquad \varrho_{01} = \|V_{0,m}^* V_1\| = \|V_{1,m}^* V_2\| = \frac{q}{1-q}\Big(1 + \frac{4q}{(1-4q)^2}\Big)^{1/2};$$

ϱ_{02} is the number ϱ of Lemma 12.10(d).

(c) On H_0,

$$A_3^{*m} A_3 = \frac{q(1-2q-q^2)}{1-q}\, I \preceq qI, \qquad A_3^{*m} V_0 = \frac{p+4}{p(p-1)(p-4)}\, I,$$
$$V_{1,m}^* A_3 = -\frac{q}{p-1} I + \frac{\mu}{p-1} P, \qquad V_{2,m}^* A_3 = -\frac{q}{p-1} I - \frac{\mu}{p-1} P.$$

Hence, as operators from H_0 to $L^2(m_0)$, $\|A_1\| = \|A_2\| = r$, $\|A_3^\circ\| \leq \|A_3\| \leq r$ and $\|\Pi_{\mathcal{H}_1+\mathcal{H}_2} A_3\| \leq J_F q r$, where

$$J_F = \Big(\frac{2}{1-\varrho_{01}}\Big)^{1/2}\Big(\frac{r}{1-q} + \frac{2q}{(1-q)(1-4q)}\Big).$$

(d) The synthesis map $(h_0,h_1,h_2,h_3) \mapsto V_0 h_0 + (A_2 h_1 + A_1 h_2 + A_3 h_3)/r$ from H_0^4 to $L^2(m_0)$ has squared norm at most

$$\Gamma_A = 1 + \varrho_{02} + 2\varrho_{01} + \frac{qr(1+4q)}{(1-q)(1-4q)} + \frac{2qr}{1-q} + \frac{4q^2}{(1-q)(1-4q)}.$$

(e) Let $\mathcal{B}$ be a middle block with least prime b and actual marginal $m_{0,\mathcal{B}} = \bigotimes_p m_{0,p}$, and let $\mathcal{H}_i(\mathcal{B})$ and $\mathcal{K}_{12}(\mathcal{B})$ be the spaces of Section 23.1 for the whole objects. Let $\mathcal{V}_0$ be the direct sum over $p \in \mathcal{B}$ of the maps V_0 at p, an isometry from $\bigoplus_p H_0(\mathbb{F}_p)$ into $\mathcal{H}_0(\mathcal{B})$, and let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ be the direct sums of the prime arms A_2, A_1, A_3°. Then $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ have ranges in $\mathcal{H}_1(\mathcal{B})$, $\mathcal{H}_2(\mathcal{B})$, $\mathcal{K}_{12}(\mathcal{B})$ and norms at most $b^{-1/2}$, and for real c_1, c_2, c_3 and $\mathcal{A} = c_1\mathcal{A}_1 + c_2\mathcal{A}_2 + c_3\mathcal{A}_3$ the star $\mathcal{A}\mathcal{V}_0^* - \mathcal{V}_0\mathcal{A}^*$ on $L^2(m_{0,\mathcal{B}})$ is the orthogonal sum over $p \in \mathcal{B}$ of the prime stars $(c_1A_2 + c_2A_1 + c_3A_3^\circ)V_{0,m}^* - V_0(c_1A_2 + c_2A_1 + c_3A_3^\circ)^{*m}$ on the sectors of exact support $\{p\}$.

Proof. (a) By Lemma 23.2(a) the weight of a half x under m_0 is proportional to $\mathbf{1}[x_0 \to x_2](d - 1 - e_{01} - e_{12})$, with $d = (p-1)/2$; put $k = k_p = (d-1)/2$. Fix $(x_1, x_2) = (a, b)$ with $a \neq b$ and put $j = \mathbf{1}[a \to b]$. The label x_0 is an in-neighbour of b other than a, in $d - j$ ways, and k of these are also in-neighbours of a (Lemma 12.10(a)); so the total weight is $(d-j)(d-1-j) - k$, which is $(d-1)(d-\frac{1}{2})$ for $j=0$ and $(d-1)(d-\frac{5}{2})$ for $j=1$. Their mean over the two values of j is $(d-1)(d-\frac{3}{2}) = (d-1)(p-4)/2$, and the two values are this mean times $1+\mu$ and $1-\mu$; as $\chi(b-a) = 2j-1$, this is the formula for m_{12}. The map ς preserves m_0 (Lemma 23.2(a)) and sends (x_0, x_1) to $(-x_2, -x_1)$, so $m_{01}(a,b) = m_{12}(-b,-a)$, which is the same formula. For $x_0 \to x_2$, the sum of $d - 1 - e_{01} - e_{12}$ over $x_1 \notin \{x_0, x_2\}$ is $(p-2)(d-1) - 2(d-1)$, the same for every directed pair; so m_{02} is uniform on the directed pairs.

(b) The first identity is that of Lemma 12.10(d), as m_{02} is the uniform law of directed pairs, as under ν_F. For $f \in H_0$, by (a), $(V_{0,m}^* V_1 f)(a) = \sum_{b\neq a} \frac{1-\mu\chi(b-a)}{p-1} f(b) = -\frac{f(a)+\mu p(Pf)(a)}{p-1}$, and likewise for $V_{1,m}^* V_2$. On H_0 the operator P is normal with eigenvalues $\pm ir$ (Lemma 12.10(c)), so $\|P - qI\|^2 = q + q^2$ and $\|I + \mu p P\|^2 = 1 + \mu^2 p$, and $\mu^2 p = 4q/(1-4q)^2$.

(c) For $h \in H_0$, $A_3 h$ is the function $-\frac{1}{p}\sum_s \chi(s-x_1)\chi(s-x_2)h(s)$ of (x_1, x_2), which is symmetric in (x_1, x_2); and the density of m_{12} relative to the uniform law of distinct pairs is $1 - \mu\chi(x_2 - x_1)$, whose nonconstant part is antisymmetric. So the Gram $A_3^{*m} A_3$ is that of the uniform law of distinct pairs. The pair (x_1, x_2) under ν_F gives the same Gram, since its density relative to that law is $1 - \chi(x_2 - x_1)/(p-2)$, and it is the value $(1 - 2q - q^2)/(p-1) \cdot I$ of Lemma 12.10(e). Next, by (a) and (4.2),

$$(V_{1,m}^* A_3 h)(a) = -\frac{1}{p(p-1)} \sum_s \chi(s-a)h(s) \sum_{b\neq a} (1 - \mu\chi(b-a))\chi(s-b) = \frac{-h(a) + \mu p (Ph)(a)}{p(p-1)},$$

because the inner sum is $-\chi(s-a) - \mu$ for $s \neq a$; conditioning on x_2 instead, the inner sum is $-\chi(s-b)+\mu$, which gives $V_{2,m}^* A_3$. For $A_3^{*m} V_0$, translation invariance shows that $\mathbb{E}_{m_0}[\chi(s-x_1)\chi(s-x_2) \mid x_0 = a] = \kappa(s-a)$ for a function κ; the computation below gives $\kappa(0) = \kappa_0$ and $\kappa(t) = \kappa_1$ for every $t \neq 0$, and then $A_3^{*m} V_0 = -q(\kappa_0 - \kappa_1) I$ on H_0, since $\langle A_3 h, V_0 h' \rangle_{m_0} = -\mathbb{E}_a \mathbb{E}_s h'(a) h(s) \kappa(s-a)$. Fix $x_0 = 0$ and write $b = x_1$, $c = x_2$; the conditional weight is $\mathbf{1}[0 \to c](d - 1 - e_{0b} - e_{bc})$, of total $d(d-1)(p-4)$. With $e_{0b} = (1 + \chi(b))/2$, $e_{bc} = (1 + \chi(c-b))/2$ and (4.2), the sum over $b \notin \{0, c\}$ of $\chi(b)$ times the weight is $-(p-3)$ for every c, so $\kappa_0 = -d(p-3)/(d(d-1)(p-4)) = -2/(p-4)$. For $s \neq 0$ put $\xi = \chi(s)$, $u = \chi(s-c)$ and $v = \chi(c)$; the sums over $b \notin \{0, c\}$ of $\chi(s-b)$, $\chi(b)\chi(s-b)$ and $\chi(c-b)\chi(s-b)$ are $-\xi - u$, $1 - vu$ and $p\mathbf{1}_{c=s} - 1 - v\xi$, so the sum over b of the weight times $\chi(s-b)$ is $\frac{1}{2}[(v - (p-5))(\xi + u) - p\mathbf{1}_{c=s}]$. Summing u times this over $c \neq 0$ with the factor $\mathbf{1}[0 \to c] = (1+v)/2$, and using $(1+v)(v - (p-5)) = -(p-6)(1+v)$, $\sum_{c\neq 0}(1+v)u = 1 - \xi$ and $\sum_{c\neq 0}(1+v)u^2 = p - 2 - \xi$, gives $-(p-6)(p-3)/4$, the term with $\mathbf{1}_{c=s}$ vanishing because $u = 0$ at $c = s$; so $\kappa_1 = -(p-6)/((p-1)(p-4))$ and $-q(\kappa_0 - \kappa_1) = (p+4)/(p(p-1)(p-4))$. Finally $\|A_i\| = \|P|_{H_0}\| = r$ for $i = 1, 2$, as V_1, V_2 are isometries; $\|A_3\|^2 \leq q$ by the first identity. The Gram of (V_1, V_2) on H_0^2 is at least $(1 - \varrho_{01})I$ by (b), so $\|\Pi_{\mathcal{H}_1 + \mathcal{H}_2} A_3\| \leq (1 - \varrho_{01})^{-1/2}\sqrt{2}\max_i \|V_{i,m}^* A_3\|$, and $\|V_{i,m}^* A_3\| \leq q/(p-1) + \mu r/(p-1) = qr\big(r/(1-q) + 2q/((1-q)(1-4q))\big)$.

(d) The Gram of the four maps is a 4×4 block operator on H_0^4 with diagonal blocks of norm at most one (by (c), and as V_0, A_2/r, A_1/r are isometries). Its off-diagonal blocks have the norms ϱ_{01} (for V_0 and A_2/r, since P/r is unitary on H_0), ϱ_{02} (V_0 and A_1/r), ϱ_{01} (A_2/r and A_1/r), $\|A_3^{*m} V_0\|/r = qr(1+4q)/((1-q)(1-4q))$ and $\|V_{i,m}^* A_3\|/r \leq qr/(1-q) + 2q^2/((1-q)(1-4q))$ for $i = 1, 2$, by (b) and (c). Every row sum of these norms is at most Γ_A, and the Schur test bounds the norm of the Gram by the largest row sum.

(e) The Markov operators of couplings with exact marginals preserve the sectors of exact prime support (Lemma 6.10(a)), and the stars are defined sectorwise. A function of the prime-p labels lying in $\mathcal{H}_i(p)$ lies in $\mathcal{H}_i(\mathcal{B})$. One lying in $\mathcal{K}_{12}(p)$ lies in $\mathcal{K}_{12}(\mathcal{B})$: the conditional expectation under $m_{0,\mathcal{B}}$ of a function of the whole-object label x_1 given the prime-p labels is a function of $x_{1,p}$ alone, hence lies in $\mathcal{H}_1(p)$ plus the constants, which is orthogonal to $\mathcal{K}_{12}(p)$; likewise for x_2. The prime pieces of $\mathcal{V}_0$ and of each $\mathcal{A}_i$ have ranges in the mutually orthogonal sectors of exact support $\{p\}$, so these maps are isometric, respectively of norm $\max_p \|A\| \leq \max_p r_p \leq b^{-1/2}$ by (c), and the star splits into its prime stars. □

23.3. **Adding an edge together with its mirror: first-order change purely skew.** This subsection contains the algebraic idea on which the fold replacement rests. Recall that x and y are the layer-zero and layer-one triples of a fold record at one prime. For the record F_1 to accept the six queries, one further cross edge is needed beyond the two reference edges, namely $x_1 \to y_0$

(Certificate 23.10); its channel is C. Imposing only this edge would change the record already at first order in $r = p^{-1/2}$ in its symmetric part, which is too expensive. The reserved reflection sends the channel C to $-\mathsf{D}$, where D is the channel of the mirror edge $x_0 \to y_1$. Imposing the mirror edge as well cancels the first-order symmetric change exactly, leaving a purely skew first-order change and a symmetric change only at second order. Part (d) of the next lemma is the one-parameter signed form of this pairing used in Section 23.4; mirror pairing itself is the case $\mathsf{t} = 0$.

Lemma 23.5 (Mirror pairing)**.** *Let* $\Phi = (1+\mathsf{A})(1+\mathsf{B})\big[(1+\mathsf{C})(1+\mathsf{D})-1\big]$ *and* $\Phi' = (1+\mathsf{A})(1+\mathsf{B})\big[(1+\mathsf{C})(1-\mathsf{D})-1\big]$*, polynomials in the channels* (23.1)*, read as kernels.*

(a) $\operatorname{Sym}\Phi = \Phi_{\mathrm{s}} := (1+\mathsf{A})(1+\mathsf{B})\mathsf{CD}$ *and* $\operatorname{Skew}\Phi = \Phi_{\mathrm{a}} := (1+\mathsf{A})(1+\mathsf{B})(\mathsf{C}+\mathsf{D})$*, whereas* $\operatorname{Skew}\Phi' = 0$ *and* $\operatorname{Sym}\Phi' = (1+\mathsf{A})(1+\mathsf{B})(\mathsf{C}-\mathsf{D}-\mathsf{CD})$*.*

(b) The four monomials CD*,* ACD*,* BCD*,* ABCD *of* Φ_{s} *are nonstars: each contains the two-edge matching of* C *(x_1–y_0) and* D *(x_0–y_1), whose endpoints span only two of the four pairs of* $\{x_0,x_1\}\times\{y_0,y_1\}$*. The skew part splits as* $\Phi_{\mathrm{a}} = L_p + \Phi_{\mathrm{a,ns}}$ *into the star part* $L_p = \mathsf{C}+\mathsf{D}+\mathsf{AD}+\mathsf{BC}$ *and the nonstar part* $\Phi_{\mathrm{a,ns}} = \mathsf{AC}+\mathsf{BD}+\mathsf{ABC}+\mathsf{ABD}$*.*

(c) $L_p = A_\star V_0^* - V_0 A_\star^*$ *on* $L^2(\nu_F)$*, with* $A_\star = A_2 + A_3$*; it annihilates the constants, has mean-zero range, and*

$$\|L_p\| = |\beta| r + (\gamma - \lambda^2)^{1/2} \le l := r\Big[\Big(\frac{2+2q^2/(1-2q)}{1-q}\Big)^{1/2} + \frac{q}{1-q}\Big],$$

where $\lambda = \frac{p+2}{p(p-1)(p-2)}$*,* $\beta = -\frac{1}{p-1}$ *and* $\gamma = q + \frac{1-2q-q^2}{p-1} + \frac{2q}{(p-1)(p-2)}$*. In particular* $l/r \to \sqrt{2}$ *as* $q \to 0$*, and* l/r *is admissible in* (q,r)*.*

(d) The signed pairing. *For* $0 \le \mathsf{t} \le 1$ *let*

$$\Phi_{\mathsf{t}} = (1+\mathsf{C})(1+(1-\mathsf{t})\mathsf{B})\big[1+\mathsf{D}+(1+\mathsf{t})\mathsf{A}+(1-\mathsf{t})\mathsf{AD}\big] - (1+\mathsf{A})(1+\mathsf{B}),$$

and put $\tilde{x}_{\mathsf{t}} = \mathsf{D}+\mathsf{tA}+(1-\mathsf{t})\mathsf{AD}$ *and* $\tilde{y}_{\mathsf{t}} = \mathsf{C}-\mathsf{tB}+(1-\mathsf{t})\mathsf{BC}$*. Then* $\Phi_0 = \Phi$*,*

$$\operatorname{Sym}\Phi_{\mathsf{t}} = \tilde{x}_{\mathsf{t}}\tilde{y}_{\mathsf{t}}, \qquad \operatorname{Skew}\Phi_{\mathsf{t}} = L_{\mathsf{t},p} + K_{\mathsf{t}} - K_{\mathsf{t}}^{\top}, \qquad K_{\mathsf{t}} = \mathsf{AC}\big(1+(1-\mathsf{t})\mathsf{B}\big),$$

where $L_{\mathsf{t},p} = \tilde{x}_{\mathsf{t}} + \tilde{y}_{\mathsf{t}} = \mathsf{C}+\mathsf{D}+\mathsf{t}(\mathsf{A}-\mathsf{B})+(1-\mathsf{t})(\mathsf{AD}+\mathsf{BC})$ *is, on* $L^2(\nu_F)$*, the operator* $A_{\mathsf{t}}V_0^* - V_0A_{\mathsf{t}}^*$*, with* $A_{\mathsf{t}} = A_2 + \mathsf{t}A_1 + (1-\mathsf{t})A_3$ *of Section 23.1. The factor* $\tilde{x}_{\mathsf{t}}$ *is a rectangular star with centre* x_0*, leaves* y_1, y_2 *and coefficients* $(1, \mathsf{t}, 1-\mathsf{t})$*, and* $\tilde{y}_{\mathsf{t}}$ *one with centre* y_0*, leaves* x_1, x_2 *and coefficients* $-(1, \mathsf{t}, 1-\mathsf{t})$ *(Lemma 23.11(b));* K_{t} *does not depend on* y_1*, and its only factor that depends on* x_1 *is* C*.*

Proof. (a) By (23.2), $(1+\mathsf{A})(1+\mathsf{B})$ and $\mathsf{CD} \mapsto (-\mathsf{D})(-\mathsf{C})$ are symmetric, $\mathsf{C}+\mathsf{D}$ is skew and $\mathsf{C}-\mathsf{D}$ is symmetric; and $(1+\mathsf{C})(1\pm\mathsf{D})-1 = \mathsf{C}\pm\mathsf{D}+\mathsf{CD}$. (b) is the expansion of the products; the star monomials of Φ_{a} are the singletons C, D and the products AD (at x_0) and BC (at y_0). (c) The kernels of $A_2V_0^*$ and $A_3V_0^*$ are C and $\mathsf{BC}-q$ (Section 23.1); by (23.2) the kernel of the adjoint $V_0A_\star^*$ is $(\mathsf{C}+\mathsf{BC}-q)^{\top} = -\mathsf{D}-\mathsf{AD}-q$, and the difference is L_p. Each A_j annihilates the constants and maps into mean-zero functions (Lemma 12.10(e)), hence so does L_p. On H_0, by Lemma 12.10(e), $M := V_0^*A_\star = \lambda I + \beta P$ and $A_\star^*A_\star = \gamma I$, because the parts of $A_2^*A_3$ and $A_3^*A_2$ that contain P cancel ($P^* = -P$). Apply Lemma 23.1 with $(a,b,c) = (\lambda,\beta,\gamma)$ to obtain the stated norm. Finally $|\beta|r = rq/(1-q)$ and $\gamma/q = \frac{2-3q-q^2}{1-q} + \frac{2q^2}{(1-q)(1-2q)} \le \frac{2+2q^2/(1-2q)}{1-q}$.

(d) The first factor of $\Phi_{\mathsf{t}} + (1+\mathsf{A})(1+\mathsf{B})$ is $(1+\mathsf{C})(1+(1-\mathsf{t})\mathsf{B}) = 1+\mathsf{B}+\tilde{y}_{\mathsf{t}}$ times $1+\mathsf{D}+(1+\mathsf{t})\mathsf{A}+(1-\mathsf{t})\mathsf{AD} = 1+\mathsf{A}+\tilde{x}_{\mathsf{t}}$. The two factors are polynomials in disjoint channels, so

$$\Phi_{\mathsf{t}} = (1+\mathsf{B})\tilde{x}_{\mathsf{t}} + (1+\mathsf{A})\tilde{y}_{\mathsf{t}} + \tilde{x}_{\mathsf{t}}\tilde{y}_{\mathsf{t}}.$$

By (23.2), $\tilde{x}_{\mathsf{t}}^{\top} = -\mathsf{C}+\mathsf{tB}-(1-\mathsf{t})\mathsf{BC} = -\tilde{y}_{\mathsf{t}}$ and $\tilde{y}_{\mathsf{t}}^{\top} = -\tilde{x}_{\mathsf{t}}$, so $\Phi_{\mathsf{t}}^{\top} = -(1+\mathsf{A})\tilde{y}_{\mathsf{t}} - (1+\mathsf{B})\tilde{x}_{\mathsf{t}} + \tilde{x}_{\mathsf{t}}\tilde{y}_{\mathsf{t}}$. Hence $\operatorname{Sym}\Phi_{\mathsf{t}} = \tilde{x}_{\mathsf{t}}\tilde{y}_{\mathsf{t}}$ and $\operatorname{Skew}\Phi_{\mathsf{t}} = (\tilde{x}_{\mathsf{t}}+\tilde{y}_{\mathsf{t}}) + (\mathsf{B}\tilde{x}_{\mathsf{t}} + \mathsf{A}\tilde{y}_{\mathsf{t}})$, and $\mathsf{B}\tilde{x}_{\mathsf{t}} + \mathsf{A}\tilde{y}_{\mathsf{t}} = \mathsf{AC}(1+(1-\mathsf{t})\mathsf{B}) + \mathsf{BD}(1+(1-\mathsf{t})\mathsf{A}) = K_{\mathsf{t}} - K_{\mathsf{t}}^{\top}$, since $K_{\mathsf{t}}^{\top} = \mathsf{B}(-\mathsf{D})(1+(1-\mathsf{t})\mathsf{A})$. At $\mathsf{t} = 0$ the first factor is

$(1+\mathsf{C})(1+\mathsf{B})(1+\mathsf{A})(1+\mathsf{D})$, so $\Phi_0=\Phi$. The kernels of $A_2V_0^*$, $A_1V_0^*$ and $A_3V_0^*$ are C, $-\mathsf{B}$ and $\mathsf{BC}-q$, so the kernel of $A_{\mathsf{t}}V_0^*$ is $\tilde{y}_{\mathsf{t}}-(1-\mathsf{t})q$ and that of its adjoint $V_0A_{\mathsf{t}}^*$ is $-\tilde{x}_{\mathsf{t}}-(1-\mathsf{t})q$; their difference is $L_{\mathsf{t},p}$. With $\mathsf{D}=\chi(y_1-x_0)$, $\mathsf{A}=\chi(y_2-x_0)$, $\mathsf{C}=-\chi(x_1-y_0)$ and $\mathsf{B}=\chi(x_2-y_0)$, $\tilde{x}_{\mathsf{t}}$ and $\tilde{y}_{\mathsf{t}}$ are the stated rectangular stars, and $K_{\mathsf{t}}=\chi(y_2-x_0)\cdot\mathsf{C}(1+(1-\mathsf{t})\mathsf{B})$ involves the labels x_0,x_1,x_2,y_0,y_2 only. □

Remark 23.6 (Why the mirror). For the orientation $(1+\mathsf{C})(1-\mathsf{D})$ the whole first-order change $\mathsf{C}-\mathsf{D}$ is symmetric, priced in S, which enters the closing through e^S. For the pairing of Lemma 23.5 the symmetric change is of order q, and the first-order change is the skew two-arm star L_p of norm about $\sqrt{2}\,r$: odd skew terms vanish in the expansion of the reserved cost, and the even products of such stars are priced by positive envelopes (Theorem 25.8), after reflection averaging, at a fraction of their norms. The signed record keeps this structure and lowers the leading norm from $\sqrt{2}\,r$ to about $|\mathbf{a}_{\mathsf{t}}|r=(1+\mathsf{t}^2+(1-\mathsf{t})^2)^{1/2}r$ by two further product laws with coefficients $\pm\mathsf{t}/2$ (Lemma 23.5(d)). The cost is a total variation $1+\mathsf{t}$ per block, which enters only the vanishing remainders of the comparison theorem through the variation factor of Definition 9.14, and a larger second-order symmetric part (s_F of Proposition 23.13). Replacements that change the cross edge $y_0\to x_2$ of the reference record, and the limits of positive mixtures, are discussed in Section 35.3.

23.4. **The signed combination over a whole block replacing F_1,F_2.** The comparison records of F_1 and F_2 replace the reference fold record, at every prime, by three positive accepting kernels repaired to the actual half marginal m_0, and combine their tensor products over the block with the signed coefficients $1,\mathsf{t}/2,-\mathsf{t}/2$. The repair precedes tensorization and the signed sum follows it: every product has exactly the product of the actual prime marginals, and the sector lemma prices the block difference by its prime differences (Lemma 6.10(b) with its closing clause and (h), Remark 6.12). In this subsection the densities of the components are taken relative to $m_0\otimes m_0$; the operator of such a density P on $L^2(m_0)$ is its Markov operator $T[P]\psi(x)=\int P(x,y)\psi(y)\,dm_0(y)$, which is the operator on $L^2(m_0)$, in the sense of Section 23.1, of the $\nu_F\otimes\nu_F$-density $P\,(g_0\otimes g_0)$.

Lemma 23.7 (Positive components with the actual half marginal)**.** *Let $p\ge 2^{24}$, $0\le\mathsf{t}\le 1$ and $(u,v)\in\{(1,1),(-1,1),(-1,-1)\}$, and on $H_F\times H_F$ put*

$$U_{uv}=2\,\mathbf{1}[x_1\to y_0](1+(1-\mathsf{t})\mathsf{B})(1+u\mathsf{D})(1+v\mathsf{A}),\qquad z_{uv}=\mathbb{E}_{\nu_F\otimes\nu_F}U_{uv},\qquad R_{uv}=\frac{U_{uv}}{z_{uv}\,g_0\otimes g_0},$$

so that R_{uv} is the density of the probability measure $U_{uv}\,\nu_F\otimes\nu_F/z_{uv}$ relative to $m_0\otimes m_0$.

(a) $U_{uv}\ge 0$; it vanishes unless $x_1\to y_0$, and it is invariant under $\mathcal{G}_p$ acting diagonally on all six labels.

(b) For all $x,y\in H_F$, $|\mathbb{E}_{y'\sim\nu_F}U_{uv}(x,y')-1|\le qR_{\mathrm{row}}$ and $|\mathbb{E}_{x'\sim\nu_F}U_{uv}(x',y)-1|\le qR_{\mathrm{col}}$, where

$$R_{\mathrm{row}}=(5-3\mathsf{t})+\frac{2[1+(5-3\mathsf{t})q]}{1-2q}+(2-\mathsf{t})\Big(1+\frac{2q}{1-q}\Big),$$
$$R_{\mathrm{col}}=(5-4\mathsf{t})+\frac{2[1+(5-4\mathsf{t})q]}{1-2q}+\frac{1+q}{1-2q};$$

both are below 10, and $|z_{uv}-1|\le\eta_c:=10q$.

(c) The marginal defects $\alpha=\mathbf{1}-R_{uv}\mathbf{1}$ and $\beta=\mathbf{1}-R_{uv}^*\mathbf{1}$, R_{uv} acting on $L^2(m_0)$, are $\mathcal{G}_p$-invariant and m_0-centred, and $|\alpha|,|\beta|\le M_c:=\frac{2\eta_c+(1+\eta_c)e_0}{h_c(1-e_0)}$, $h_c=1-\eta_c$. The query coupling $J_C=\frac{2}{1-q}\mathbf{1}[x_1\to y_0]$, a density relative to $m_0\otimes m_0$, has both marginals m_0, satisfies $J_C\beta=0$ and $J_C^*\alpha=0$, and $\|\Pi_\perp J_C\Pi_\perp\|\le\varrho_{02}$ on $L^2(m_0)$, with $\Pi_\perp$ the projection off the constants and ϱ_{02} of Lemma 23.4(b).

(d) The density

$$P_{uv}=\frac{R_{uv}+J_C\big(\alpha(x)+\beta(y)\big)+2M_cJ_C}{1+2M_c}$$

relative to $m_0 \otimes m_0$ is that of a $\mathcal{G}_p$-invariant probability measure supported on $x_1 \to y_0$, with both half marginals m_0, and $\|\Pi_\perp(T[P_{uv}] - T[R_{uv}])\Pi_\perp\| \le 2M_c(\|\Pi_\perp T[R_{uv}]\Pi_\perp\| + 2\varrho_{02})$.

Proof. (a) For $a \neq b$, $1+\chi(b-a)$ is 2 if $a \to b$ and 0 otherwise; the other three factors lie in $[0,2]$ for all values $0, \pm 1$ of the channels; and the channels are invariant under the maps $t \mapsto \alpha t + \beta$ with α a nonzero square, acting on all labels.

(b) *Rows.* Fix x, and write $2\,\mathbf{1}[x_1 \to y_0] = 1 + \mathsf{C} - \delta_{10}$, $\delta_{10} = \mathbf{1}[y_0 = x_1]$ (at $y_0 = x_1$ the channel C vanishes). Then $U_{uv} = H(1+u\mathsf{D})(1+v\mathsf{A}) - \delta_{10}(1+(1-\mathsf{t})\mathsf{B})(1+u\mathsf{D})(1+v\mathsf{A})$, where $H = (1+\mathsf{C})(1+(1-\mathsf{t})\mathsf{B}) \ge 0$ is a function of y_0. Under ν_F the label y_0 is uniform, y_2 is a uniform out-neighbour of y_0, and y_1 is uniform on the other $p-2$ labels. By Lemma 12.10(b), $\mathbb{E}H = 1+(1-\mathsf{t})\mathbb{E}_c\chi(c-x_1)\chi(x_2-c) = 1+(1-\mathsf{t})q$. Next $\mathbb{E}[\mathsf{A} \mid y_0 = c] = (p\mathbf{1}_{c=x_0} - 1 - \chi(c-x_0))/(p-1)$, so $(p-1)\mathbb{E}[H\mathsf{A}] = H(x_0) - \mathbb{E}H - \mathbb{E}_c[H(c)\chi(c-x_0)]$. Here $H(x_0)$ is 0 or $2(2-\mathsf{t})$, as $\chi(x_2-x_0)=1$, and $\mathbb{E}_c[H(c)\chi(c-x_0)] = -q + (1-\mathsf{t})q - (1-\mathsf{t})\mathbb{E}_c\chi((c-x_0)(c-x_1)(c-x_2))$, where the last average has modulus at most $2r$ by Theorem 4.7(a). Hence $|\mathbb{E}[H\mathsf{A}]| \le q(3-2\mathsf{t}+q+2r)/(1-q) \le 2(2-\mathsf{t})q$, the last step because $q + 2r + (4-2\mathsf{t})q \le 1$; so $|\mathbb{E}[H(1+v\mathsf{A})] - 1| \le (5-3\mathsf{t})q$. Given (y_0, y_2), $|\mathbb{E}[\mathsf{D} \mid y_0, y_2]| = |\chi(y_0 - x_0) + \chi(y_2 - x_0)|/(p-2) \le 2q/(1-2q)$, and $H(1+v\mathsf{A}) \ge 0$, so the term with $u\mathsf{D}$ is at most $\frac{2q}{1-2q}[1 + (5-3\mathsf{t})q]$ in modulus. For the excluded query coincidence $y_0 = x_1$ (the removed query of the preceding remark), at $y_0 = x_1$ the factor $1+(1-\mathsf{t})\mathsf{B}$ is at most $2-\mathsf{t}$, and with $\xi = \chi(x_1 - x_0)$ the conditional means given $y_0 = x_1$ are $\mathbb{E}\mathsf{A} = -(1+\xi)/(p-1)$, $\mathbb{E}\mathsf{D} = -\xi/(p-2) + (1+\xi)/((p-1)(p-2))$ and $\mathbb{E}\mathsf{A}\mathsf{D} = -(p-3)/((p-1)(p-2))$; the largest value of $1 + u\mathbb{E}\mathsf{D} + v\mathbb{E}\mathsf{A} + uv\mathbb{E}\mathsf{A}\mathsf{D}$ over u, v, ξ is $1 + 2/(p-1)$. These three bounds add up to qR_{row}.

Columns. Fix y, and integrate x_1 first: given (x_0, x_2) it is uniform on the other $p-2$ labels, so $|\mathbb{E}[\mathsf{C} \mid x_0, x_2]| = |\chi(y_0 - x_0) + \chi(y_0 - x_2)|/(p-2) \le 2q/(1-2q)$. The remaining factor $(1+(1-\mathsf{t})\mathsf{B})G$, $G = (1+u\mathsf{D})(1+v\mathsf{A})$ a function of x_0, is nonnegative; $\mathbb{E}G = 1 - uvq$ by Lemma 12.10(b), and $\mathbb{E}[\mathsf{B} \mid x_0] = (p\mathbf{1}_{x_0=y_0} - 1 - \chi(x_0 - y_0))/(p-1)$ is 1 at $x_0 = y_0$ and 0 or $-2/(p-1)$ elsewhere, so $-2q(1+q)/(1-q) \le \mathbb{E}[\mathsf{B}G] \le 4q$. Hence $|\mathbb{E}[(1+(1-\mathsf{t})\mathsf{B})G] - 1| \le (5-4\mathsf{t})q$, and the C-term is at most $\frac{2q}{1-2q}[1 + (5-4\mathsf{t})q]$. The removed query has column mass $q\,\mathbb{E}[(1 + (1-\mathsf{t})\chi(x_2 - y_0))G(x_0) \mid x_1 = y_0] \le q(1+q)/(1-2q)$: given $x_1 = c$ the pair (x_0, x_2) is uniform on the $(p-1)(p-2)/2$ directed pairs avoiding c, the sum over x_2 of $1 + (1-\mathsf{t})\chi(x_2 - c)$ is at most $(p-1)/2$ for every x_0, and $\sum_{x_0} G(x_0) = p(1-uvq)$.

For $q \le 1/100$ the two bounds are at most 9.19 and 8.18. Averaging the row estimate over x gives $|z_{uv} - 1| \le qR_{\mathrm{row}} < 10q$.

(c) $R_{uv}\mathbf{1}(x) = \mathbb{E}_{y\sim\nu_F} U_{uv}(x,y)/(z_{uv}g_0(x))$. By (b) the ratio $\mathbb{E}_y U_{uv}(x,y)/z_{uv}$ is within $2\eta_c/h_c$ of one, and $|g_0 - 1| \le e_0$ (Lemma 23.2(a)), so $|\alpha| \le (e_0 + 2\eta_c/h_c)/(1-e_0) \le M_c$; likewise for β. Invariance holds because U_{uv}, ν_F and g_0 are invariant, and $\int \alpha\, dm_0 = 1 - \iint R_{uv}\, dm_0\, dm_0 = 0$. The label y_0 is uniform under m_0 and, for every x, $\mathbb{P}(x_1 \to y_0) = (1-q)/2$; so $J_C\mathbf{1} = \mathbf{1}$, and likewise $J_C^*\mathbf{1} = \mathbf{1}$. As β is translation invariant and m_0-centred, $\mathbb{E}_{m_0}[\beta \mid y_0] = 0$ (Lemma 14.2(d)(iii)), and $(J_C\beta)(x) = \frac{2}{1-q}\mathbb{E}_{m_0}[\mathbf{1}[x_1 \to y_0]\mathbb{E}[\beta \mid y_0]] = 0$; likewise $J_C^*\alpha = 0$. Finally, as $2\,\mathbf{1}[a \to c] = 1 + \chi(c-a) - \mathbf{1}_{c=a}$, the operator of J_C on $L^2(m_0)$ is $V_1 S V_{0,m}^*$ with $S = (\mathbb{J} + P - qI)/(1-q)$ on $L^2(\mathbb{F}_p)$; so $\Pi_\perp J_C \Pi_\perp = V_1 \frac{P-qI}{1-q} V_{0,m}^* \Pi_\perp$, of norm at most $\|P - qI\|/(1-q) = \varrho_{02}$ on H_0.

(d) This is Corollary 17.17 with $m_X = m_Y = m_0$, $R = R_{uv}$, $J = J_C$ and $M = M_c$; the density vanishes off $x_1 \to y_0$, where both R_{uv} and J_C vanish, and it is invariant because R_{uv}, J_C, α and β are. □

Remark 23.8 (Coincidences of the two halves). The kernels U_{uv} are defined at every pair of halves; only the query coincidence $y_0 = x_1$ is excluded. At another coincidence $x_i = y_j$ the channels of that pair vanish, and $U_{uv} \ge 0$ there. Acceptance needs only $x_1 \to y_0$ (Certificate 23.10); invariance under $\mathcal{G}_p$, hence uniform one-slot marginals and uniform queried pairs, holds on the whole product; and every bound of Proposition 23.13 is a bound for the operator of the kernel on all pairs of halves (stars and nonstars for independent labels, compressed by the induced densities; the removed query

charged $k_{\mathrm{col}}q$). No later statement assumes distinct labels in the two halves of a fold record: the middle frame (Definition 22.4) and the dominating matrices of Section 25, the matching Gram and the mean credit use operators on $L^2(m_0)$; the restorations use only the reference records and the root; admissibility (O1) allows invariant signed measures and (O2) applies only the records' own norms at ι (Definition 8.9); and (R4) concerns only tail primes.

Definition 23.9 (The signed whole-block fold record)**.** Let $p \geq 2^{24}$ and $0 \leq \mathsf{t} \leq 1$; the construction uses $\mathsf{t} = 127/500$.

(a) *Reference law:* Ψ_0, z_0, $K_0 = \Psi_0/z_0$ and $m_0 = g_0\nu_F$ of Lemma 23.2.
(b) *Components:* the densities $P_{++,p}$, $P_{-+,p}$, $P_{--,p}$ of Lemma 23.7 at every prime p, relative to $m_{0,p} \otimes m_{0,p}$.
(c) *The record of* F_1 on a middle block $\mathcal{B}$ is the signed measure with the density

$$K_{\mathsf{t},\mathcal{B}} = \bigotimes_{p\in\mathcal{B}} P_{++,p} + \frac{\mathsf{t}}{2}\bigotimes_{p\in\mathcal{B}} P_{-+,p} - \frac{\mathsf{t}}{2}\bigotimes_{p\in\mathcal{B}} P_{--,p} \tag{23.3}$$

relative to $m_{0,\mathcal{B}} \otimes m_{0,\mathcal{B}}$, $m_{0,\mathcal{B}} = \bigotimes_p m_{0,p}$: the signed sum is taken after the three products. It has mass one, both half marginals $m_{0,\mathcal{B}}$, and total variation at most $1 + \mathsf{t}$, whatever the number of primes.
(d) *The record of* F_2 is the reflected transpose $K^{\varsigma}_{\mathsf{t},\mathcal{B}}$, taken componentwise at every prime; its reference law is $K_0^{\varsigma} = K_0$.
(e) Both are placed by F_1 and F_2 (Lemma 17.2(b)), with the weight $w_{\mathrm{D}}r$ of the records they replace. At $\mathsf{t} = 0$ the record is $\bigotimes_p P_{++,p}$, whose raw kernel U_{++} adds to the reference record the query edge and its mirror: it is a mirror pairing in the sense of Lemma 23.5.

For one prime put $K_{\mathsf{t},p} = P_{++,p} + \frac{\mathsf{t}}{2}P_{-+,p} - \frac{\mathsf{t}}{2}P_{--,p}$, the same combination as (23.3) at one prime, and write $D_{\mathsf{t},p} = T[K_{\mathsf{t},p}] - T_{\mathrm{old}}$ for the record difference of F_1 at p, where T is extended linearly to signed densities and T_{old} is the operator of K_0 on $L^2(m_0)$, that is T of the $m_0 \otimes m_0$-density $K_0/(g_0 \otimes g_0)$; the difference $D_{\mathsf{t},p}$ annihilates the constants on both sides, and $D^{\mathrm{s}}_{\mathsf{t},p}$, $D^{\mathrm{a}}_{\mathsf{t},p}$ are its symmetric and skew parts. The leading part of $D^{\mathrm{a}}_{\mathsf{t},p}$ is

$$L^{\circ}_{\mathsf{t},p} = A^{\circ}_{\mathsf{t}}V^{*}_{0,m} - V_0(A^{\circ}_{\mathsf{t}})^{*_m}, \qquad A^{\circ}_{\mathsf{t}} = A_2 + \mathsf{t}A_1 + (1-\mathsf{t})A^{\circ}_3, \tag{23.4}$$

on $L^2(m_0)$, adjoints taken in $L^2(m_0)$: the operator $L_{\mathsf{t},p}$ of Lemma 23.5(d) with the adjoints of the actual marginal and with A_3 replaced by its part orthogonal to the singleton features. It is a three-arm star with centre $\mathcal{H}_0$, arms in $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{K}_{12}$, and coefficients $(1, \mathsf{t}, 1-\mathsf{t})$; at $\mathsf{t} = 0$ it is the two-arm star $A^{\circ}_{\star}V^{*}_{0,m} - V_0(A^{\circ}_{\star})^{*_m}$ of the mirror pairing. For F_2 the record difference is $\mathcal{U}D^{*}_{\mathsf{t},p}\mathcal{U}$, since $K^{\varsigma} = \mathcal{U}K^{\top}\mathcal{U}$ componentwise and $\mathcal{U}K_0\mathcal{U} = K_0$; $\mathcal{U}$ is unitary on $L^2(m_0)$ because $g_0 \circ \varsigma = g_0$, and the symmetric and skew parts of $\mathcal{U}D^{*}_{\mathsf{t},p}\mathcal{U}$ are $\mathcal{U}D^{\mathrm{s}}_{\mathsf{t},p}\mathcal{U}$ and $-\mathcal{U}D^{\mathrm{a}}_{\mathsf{t},p}\mathcal{U}$.
Acceptance. By Certificate 23.10, the six queries need in the two-layer F_1 exactly the cross edge $x_1 \to y_0$ and in F_2 exactly $x_2 \to y_1$; F_0 accepts. Every component $P_{uv,p}$ is supported on $x_1 \to y_0$ (Lemma 23.7(a),(d)), so every product of components accepts the query of F_1 at every prime of the block; as $\mathsf{C}^{\varsigma} = \chi(y_1 - x_2)$, the reflected components accept the query of F_2. Every component is $\mathcal{G}_p$-invariant, so its one-slot marginals are uniform and, $\mathcal{G}_p$ being transitive on the ordered pairs (a, a') with $a \to a'$, its queried pair is uniform on the strict-square pairs.

Certificate 23.10 (Query images of the middle records)**.** The frozen geometry record `research/frontier_2026_10_17/structural/middle6_geometry.json`, with the labels $x_i = i$ and $y_i = 3 + i$, gives the following projections of the six slot queries of $\mathcal{E}_6$ onto each record (the queries as they read on the original labels, before any compression). In the two-layer records F_0 accepts, and the only needed cross edges are $x_1 \to y_0$ in F_1, $x_2 \to y_1$ in F_2 and $x_0 \to y_1$ in E_0 and in E_1. In the one-layer records E_0, E_1 the only needed edge is $0 \to 1$; at the chamber cut ϑ_1 for E_0 and ϑ_2 for E_1, label 2 lies on both sides of the cut and labels 0 and 1 on one side each. The

two checker programs re-derived these facts independently (`research/record_gate_2026_10_18/U01/compute/geometry_components.py`, `research/record_gate_2026_10_18/U03/compute/indep_geometry.py`).

23.5. **Nonstar and coincidence remainders.** This subsection collects the nonstar, rectangular-star and coincidence bounds used by the signed fold records (Proposition 23.13) and by the reference fold record (Lemma 23.2(e),(f)). No estimate of this subsection is applied to the matching records. A polynomial of degree at most one in each character attains its maximum modulus on $[-1,1]^4$ at a vertex, so it is bounded by its maximum on $\{\pm1\}^4$ everywhere, including where a character vanishes.

Lemma 23.11 (Projected nonstars and rectangular stars)**.** *(a) For each nonstar monomial m, $\|m\| \le \pi_m q$, where*

$$\pi_{\mathsf{AB}} = \frac{2}{1-q}, \qquad \pi_{\mathsf{AC}} = \pi_{\mathsf{BD}} = \Big(\frac{2}{Z}\Big)^{1/2}, \qquad \pi_{\mathsf{CD}} = \frac{1}{1-2q},$$
$$\pi_{\mathsf{ABC}} = \pi_{\mathsf{ABD}} = \frac{2}{Z^{1/2}}, \qquad \pi_{\mathsf{ACD}} = \pi_{\mathsf{BCD}} = \frac{\sqrt{2}}{1-2q}, \qquad \pi_{\mathsf{ABCD}} = \frac{2}{1-2q}.$$

Every π_m is at most $2/Z$. Consequently $\|\Phi_{\mathrm{s}}\| \le \mathcal{N}_{\mathrm{s}} := q(3+2\sqrt{2})/(1-2q)$ and $\|\Phi_{\mathrm{a,ns}}\| \le \mathcal{N}_{\mathrm{a}} := q(2(2/Z)^{1/2} + 4/Z^{1/2})$.

(b) Rectangular stars. *For independent uniform labels z,u,v and real a_1,a_2,a_3, the* rectangular star *$a_1\chi(u-z) + a_2\chi(v-z) + a_3\chi(u-z)\chi(v-z)$, as an operator between functions of z and functions of (u,v), has norm at most $r(a_1^2+a_2^2+a_3^2)^{1/2}$: its Gram on mean-zero functions of z is $q(a_1^2+a_2^2+a_3^2-2qa_3^2)$, and on the constants $a_3^2q(1-q)$. A product of two rectangular stars on disjoint sets of labels has norm at most the product of their norms, and its restriction to $H_F \times H_F$, as a kernel relative to $\nu_F \otimes \nu_F$, has norm at most $2/Z$ times that.*

Proof. (a) A nonstar monomial is $m = m_1m_2$, with m_1 a nonempty product of characters of the star at x_0 and m_2 of the star at y_0. For independent uniform labels the operator of m is the tensor product of those of m_1 and m_2. A single edge has norm r, and a full star has norm $(q-q^2)^{1/2}$, its Gram operator being $q(1-2q)I + q^2\mathbb{J}$ by Lemma 12.10(b); so monomials with two, three and four edges have norms q, $q(1-q)^{1/2}$ and $q(1-q)$. Each m depends on x through the coordinates I and on y through J, where (I,J) is $(\{0,2\},\{0,2\})$, $(\{0,1\},\{0,2\})$, $(\{0,2\},\{0,1\})$, $(\{0,1\},\{0,1\})$, $(\{0,1,2\},\{0,2\})$, $(\{0,2\},\{0,1,2\})$, $(\{0,1\},\{0,1,2\})$, $(\{0,1,2\},\{0,1\})$ and $(\{0,1,2\},\{0,1,2\})$ in the order of the display; Lemma 13.24 multiplies the norms by the square roots of the two density bounds, which gives the π_m. The coefficients of Φ_{s} are one on $\mathsf{CD}, \mathsf{ACD}, \mathsf{BCD}, \mathsf{ABCD}$, and those of $\Phi_{\mathrm{a,ns}}$ one on $\mathsf{AC}, \mathsf{BD}, \mathsf{ABC}, \mathsf{ABD}$; the triangle inequality gives the last claims. (b) The Gram kernel of the star, as a function of (z,z'), is the average over (u,v) of the product of the two values; by $\mathbb{E}_u\chi(u-z)\chi(u-z') = \mathbf{1}_{z=z'} - q$ (Lemma 12.10(b)) it is $(a_1^2+a_2^2)(\mathbf{1}_{z=z'}-q) + a_3^2(\mathbf{1}_{z=z'}-q)^2$, the cross terms containing an isolated average $\mathbb{E}_u\chi(u-\cdot) = 0$; as an operator this is $q(a_1^2+a_2^2+a_3^2(1-2q))I + q(-(a_1^2+a_2^2)+a_3^2q)\mathbb{J}$, with the stated values on mean-zero functions and on the constants, both at most $q(a_1^2+a_2^2+a_3^2)$. A product on disjoint labels is a tensor product of operators. The induced density of ν_F relative to three independent labels is $2/Z$ (Lemma 13.24(b)), and Lemma 13.24(a) on both halves gives the factor $2/Z$. □

Lemma 23.12 (Conditional collisions)**.** *Let g be a polynomial in $\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}$ of degree at most one in each, M_g its maximum modulus on $\{\pm1\}^4$, and δ_{ij} the indicator of $x_i = y_j$, $i,j \in \{0,1,2\}$.*

(a) $\|(1-I_\times)g\| \le 9M_gq$.

*(b) Fix (i,j). On $\{x_i = y_j\}$ every character becomes 0, a fixed sign, a function of the free labels of one half only (*one-sided*), or the character of a pair $(x_{i'}, y_{j'})$ of free labels (*cross*).*

After like terms are collected, at most two cross characters remain in each monomial, and they form a single edge, a two-edge star or a two-edge matching. Let $c_0^{ij}(g)$, $c_r^{ij}(g)$, $c_q^{ij}(g)$ be the sums of the absolute coefficients of the monomials with no cross character, with a single cross edge or a two-edge star, and with a two-edge matching. Then $\delta_{ij}g$ is the direct sum over the common value v of q times conditional operators T_v^{ij} between the conditional laws given $x_i = v$ and given $y_j = v$, and $\|T_v^{ij}\| \le \|T_v^{ij,0}\| + (2/Z)(c_r^{ij} r + c_q^{ij} q)$, where $T_v^{ij,0}$, the conditional operator of the no-cross part, has norm at most c_0^{ij}.

(c) *With $c_*(g) = \sum_{i,j} c_*^{ij}(g)$,*

$$\|(1 - I_\times)g\| \le q\Big[c_0(g) + \frac{2}{Z}\big(c_r(g) r + c_q(g) q\big) + \frac{36 M_g q}{1 - q}\Big],$$

and $c_0^{ij}(g)$ may be replaced in $c_0(g)$ by any bound for $\sup_v \|T_v^{ij,0}\|$.

(d) The reference collision in bundled form. *Let $g = (1 + \mathsf{A})(1 + \mathsf{B})$, so that $(1 - I_\times)g = \mathsf{C}_{\mathrm{rem}}$ (Lemma 23.2(b)), and on a half put $h = \chi(x_1 - x_0)$ and $k = \chi(x_2 - x_1)$. The no-cross parts of the nine substitutions $\delta_{ij}g$ sum to*

$$\begin{aligned} &4\delta_{00} + 4\delta_{22} + \delta_{02} + \delta_{20} + \delta_{11} \\ &\quad + \delta_{01}\big(1 + k(y)\big) + \delta_{10}\big(1 + k(x)\big) + \delta_{12}\big(1 + h(x)\big) + \delta_{21}\big(1 + h(y)\big) = qVCV^*, \end{aligned}$$

where $V = (V_0, V_1, V_2, kV_1, hV_1)\colon L^2(\mathbb{F}_p)^5 \to L^2(\nu_F)$ and C is the symmetric 5×5 matrix with diagonal $(4, 1, 4, 0, 0)$ and entries 1 at the positions 01, 02, 12, 03, 24 and their transposes. Then $\|C\| \le 56119/10000$; on H_0^5 the map V has squared norm at most $\Gamma_5 = 1 + 2(r + q)/(1 - q) + 2q(1 + r)/((1 - q)(1 - 2q))$; and the constant-input part $q\big[15\,\mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes (h + k) + (h + k) \otimes \mathbf{1}\big]$ of $qVCV^*$ is annihilated by centring on both sides. Consequently, with Q_ν the projection off the constants in $L^2(\nu_F)$ and c_{col}^ν, c_{col} of (23.5), $\|Q_\nu \mathsf{C}_{\mathrm{rem}} Q_\nu\| \le c_{\mathrm{col}}^\nu q$, and on the m_0-centred functions of $L^2(m_0)$ the operator of $\mathsf{C}_{\mathrm{rem}}$ has norm at most $c_{\mathrm{col}} q$.

Proof. (a) Pointwise $|(1 - I_\times)g| \le M_g \sum_{i,j} \delta_{ij}$, and for fixed x each event $y_j = x_i$ has probability q, since y_j is uniform; likewise for fixed y. The Schur test gives the bound. (b) As x_i and y_j are independent and uniform under $\nu_F \otimes \nu_F$, $\mathbb{E}[\varphi(x)\delta_{ij}g\,\psi(y)] = q\,\mathbb{E}_v\mathbb{E}[\varphi(x)g\,\psi(y) \mid x_i = y_j = v]$ for real φ, ψ, which is the direct-sum description; Cauchy–Schwarz in v bounds the norm by $q \sup_v \|T_v^{ij}\|$. Given $x_i = v$ the two free labels of x are uniform on a set of $(p - 1)(p - 2)/2$ pairs, for each i and v, so their density relative to two independent uniform labels is $2/Z$, and likewise for y. The coincidence removes every edge of the two stars at x_i or y_j, and inspection of the nine cases shows that the remaining cross edges are none, a star or a matching. A monomial without a cross character is a product of one-sided factors of modulus at most one, so its conditional operator has rank one and norm at most one. For independent labels a single cross edge, times one-sided factors, has norm at most r, a two-edge star at most $(q - q^2)^{1/2} \le r$, and a two-edge matching at most q; conjugation by the two square-root densities costs the factor $2/Z$, as in Lemma 13.24. Values of the kernel off the support of the conditional laws do not matter, so fixed signs may be substituted first. (c) Let $\kappa = \sum_{i,j} \delta_{ij}$. As both halves are injective, simultaneous coincidences form a partial matching between $\{0, 1, 2\}$ and $\{0, 1, 2\}$, and $0 \le \kappa - \mathbf{1}_{\kappa \ge 1} \le \binom{\kappa}{2}$, the right side being the sum of $\delta_{ij}\delta_{i'j'}$ over the 18 unordered pairs of coincidences with $i \ne i'$, $j \ne j'$. Given x, the probability that y has two prescribed distinct labels at two prescribed positions is at most $(p - 2)/|H_F| = 2q^2/(1 - q)$. So the kernel $(\kappa - \mathbf{1}_{\kappa \ge 1})g$ has row and column sums at most $36 M_g q^2/(1 - q)$, and the Schur test bounds its norm. Now $(1 - I_\times)g = \sum_{i,j} \delta_{ij}g - (\kappa - \mathbf{1}_{\kappa \ge 1})g$, and (b) bounds each $\delta_{ij}g$. (d) *Substitutions.* At $x_0 = y_0 = v$ and at $x_2 = y_2 = v$ both characters of g equal 1 (for instance $\mathsf{A} = \chi(y_2 - v) = 1$ as $y_0 \to y_2$), so $g = 4$. At $x_0 = y_2$ the character A vanishes and B stays cross; at $x_2 = y_0$, B vanishes and A stays cross; at $x_1 = y_1$ both stay cross. At

$x_0 = y_1$, $\mathsf{A} = \chi(y_2 - y_1) = k(y)$ is one-sided and B stays cross; at $x_1 = y_0$, $\mathsf{B} = \chi(x_2 - x_1) = k(x)$; at $x_1 = y_2$, $\mathsf{A} = \chi(x_1 - x_0) = h(x)$; at $x_2 = y_1$, $\mathsf{B} = \chi(y_1 - y_0) = h(y)$. This gives the displayed no-cross parts, and leaves twelve single cross characters counted with the norms 1 or 2 of their one-sided multipliers (one each at 02 and 20, two at 11, two each at 01, 10, 12, 21) and the two-edge matching AB at 11. As operators on $L^2(\nu_F)$, $\delta_{ij} = qV_iV_j^*$ (the label y_j is uniform), and the one-sided factors give $\delta_{01}k(y) = qV_0(kV_1)^*$, $\delta_{10}k(x) = q(kV_1)V_0^*$, $\delta_{12}h(x) = q(hV_1)V_2^*$ and $\delta_{21}h(y) = qV_2(hV_1)^*$; this is $qVCV^*$.

The matrix. C is nonnegative and symmetric. For the positive vector

$$w = \Big(1, \frac{43367}{10^5}, 1, \frac{17820}{10^5}, \frac{17820}{10^5}\Big)$$

the quotients $(Cw)_a/w_a$ are $561187/10^5$, $243367/43367$, $561187/10^5$, $10^5/17820$ and $10^5/17820$, all below $56119/10000$, so $\|C\| \le 56119/10000$.

The synthesis maps. On H_0 every Gram entry $V_a^*V_b$, $a \ne b$, is a translation- and dilation-invariant operator, $\alpha I + \beta P$; fixing the two source labels and summing the third, with $D_0 = (p-1)(p-2)$, the ten entries are

ab	01	02	03	04	12	13	14	23	24	34
α	$-\frac{1}{p-1}$	$-\frac{1}{p-1}$	$\frac{1}{D_0}$	$\frac{1}{D_0}$	$-\frac{1}{p-1}$	$-\frac{1}{p-2}$	$-\frac{1}{p-2}$	$\frac{1}{D_0}$	$\frac{1}{D_0}$	$\frac{1}{p-2}$
β	$-\frac{p}{D_0}$	$\frac{p}{p-1}$	$\frac{p}{D_0}$	$\frac{p}{p-1}$	$-\frac{p}{D_0}$	0	0	$-\frac{p}{p-1}$	$-\frac{p}{D_0}$	0

(the entries 01, 02, 12 are those of Lemma 12.10(d)). As $\|\alpha I + \beta P\| = (\alpha^2 + q\beta^2)^{1/2}$ on H_0, the entries 02, 04, 23 have norm at most $a_5 = (r+q)/(1-q)$ and the others at most $b_5 = q(1+r)/((1-q)(1-2q)) \le a_5$; every row has at most two entries of the first kind, so by the Schur test $\|V|_{H_0^5}\|^2 \le 1 + 2a_5 + 2b_5 = \Gamma_5$.

Centring. Write $V = V^0 + V\mathbb{J}$ with V^0 the restriction to H_0^5. Then $VCV^* = V^0C(V^0)^* + \sum_{a,b} C_{ab}\,|U_a\rangle\langle U_b|$ with $U = (V_a\mathbf{1})_a = (\mathbf{1}, \mathbf{1}, \mathbf{1}, k, h)$, and $\sum_{a,b} C_{ab}U_a \otimes U_b$ is the displayed constant-input part. Every V_a^0 maps into ν_F-mean-zero functions (for kV_1 and hV_1 because $\mathbb{E}[k \mid x_1]$ and $\mathbb{E}[h \mid x_1]$ are constant), and $Q_\nu(\cdot)Q_\nu$ annihilates the constant-input part: each of its terms has a constant output or a constant input. Hence $\|Q_\nu(qVCV^*)Q_\nu\| \le q\|C\|\Gamma_5$. The single cross characters and the matching cost at most $(2/Z)(12r+q)q$ by (b), and $(\kappa - \mathbf{1}_{\kappa\ge 1})g$ at most $36M_gq^2/(1-q) = 144q^2/(1-q)$ by the proof of (c); the certified constant c_{col}^ν uses $288q/(1-q)$, twice this. So $\|Q_\nu \mathsf{C}_{\mathrm{rem}} Q_\nu\| \le c_{\mathrm{col}}^\nu q$.

Actual centring. Let φ, ψ be m_0-centred with $\|\varphi\|_{m_0} = \|\psi\|_{m_0} = 1$. Then $\|\varphi\|_{\nu_F} \le (1-e_0)^{-1/2}$ and $|\mathbb{E}_{\nu_F}\varphi| = |\mathbb{E}_{m_0}[\varphi(g_0^{-1} - 1)]| \le e_0/(1-e_0)$. Split φ and ψ into their ν_F-centred parts and their ν_F-means; the centred-centred term is at most $c_{\mathrm{col}}^\nu q/(1-e_0)$, and the three terms with a mean are at most $\|\mathsf{C}_{\mathrm{rem}}\| \le 17q$ (Lemma 23.2(b)(iii)) times $2e_0/(1-e_0)^{3/2} + e_0^2/(1-e_0)^2$. This is $c_{\mathrm{col}}q$. □

Proof of Lemma 23.2(e),(f). Transport. For a kernel K relative to $\nu_F \otimes \nu_F$, the operator of K on $L^2(m_0)$ is $M_{g_0}^{-1}N_KM_{g_0}^{-1}$, where M_{g_0} is multiplication by g_0 and $N_K\psi(x) = \int K(x,y)\psi(y)\,dm_0(y)$ is its *natural* operator (Lemma 13.21(b)). As $\|M_{g_0}^{-1} - I\| \le e_0/(1-e_0)$, three rules follow. (T1) The operator has norm at most $\|K\|/(1-e_0)$. (T2) For every operator N on $L^2(m_0)$, $\|M_{g_0}^{-1}NM_{g_0}^{-1} - N\| \le D_g\|N\|$, with $D_g = (2e_0+e_0^2)/(1-e_0)^2$ of (23.5). (T3) For the constant kernel $\mathbf{1}$, $\Pi_\perp M_{g_0}^{-1}N_{\mathbf{1}}M_{g_0}^{-1}\Pi_\perp$ is the rank-one operator $\big\langle g_0^{-1} - \mathbf{1}, \cdot\big\rangle_{m_0}(g_0^{-1} - \mathbf{1})$, of norm $\|g_0^{-1} - \mathbf{1}\|_{m_0}^2 \le c_g = e_0^2/(1-e_0)^2$. Moreover, the coordinates of m_0 being uniform, the natural operator of a kernel that depends on y only through y_j is $AV_{j,m}^*$, where A has the same kernel as an operator from $L^2(\mathbb{F}_p)$; so $N_{\mathsf{L}} = V_0PV_{2,m}^* - V_2PV_{0,m}^* = O_0$, and $N_{\mathsf{C}} = A_2V_{0,m}^*$, $N_{-\mathsf{B}} = A_1V_{0,m}^*$, $N_{\mathsf{BC}-q} = A_3V_{0,m}^*$, the natural operator of a transposed kernel being the adjoint in $L^2(m_0)$.

(f) *The leading operator.* By Lemma 23.4(a),(b) the pair (x_0, x_2) is uniform on the directed pairs under m_0, so $V_{0,m}^*V_2 = (P - qI)/(1-q)$ on H_0 as under ν_F, and V_0, V_2 are isometries

of H_0 into $L^2(m_0)$. Lemma 23.1 therefore applies with the same Gram data as in (b)(i) to $O_0 = -(A_1 V_{0,m}^* + V_0 A_1^{*m})$: it is self-adjoint, annihilates the constants and has the stated spectrum. In the synthesis of Lemma 23.4(d), O_0/r is the image of a 4×4 matrix of norm one, so $\|O_0\| \le \Gamma_A r$.

The remainder. By Lemma 23.2(b), $\Psi_0 = \mathbb{J}+\mathsf{L}+\mathsf{AB}-\mathsf{C}_{\rm rem}$, and $O = \Pi_\perp T_{\rm old}\Pi_\perp$ is $1/z_0$ times the centred operator of Ψ_0 on $L^2(m_0)$. In that operator the constant kernel costs c_g by (T3); L gives O_0 up to $D_g\|O_0\| \le D_g\Gamma_A r$ by (T2); AB costs $2q/(Z(1-e_0))$ by (b)(ii) and (T1); and $\mathsf{C}_{\rm rem}$ costs $c_{\rm col}q$ by Lemma 23.12(d). So the centred operator of Ψ_0 is O_0+R' with $\|R'\| \le r_o q$, r_o of (23.5), and its norm is at most $(\Gamma_A+r_o r)r$. As $h_0 \le z_0 \le 1$ and $1-z_0 \le 14q$ (Lemma 23.2(a)), $0 \le z_0^{-1}-1 \le 14q/h_0$, and $O = O_0+R'+(z_0^{-1}-1)(O_0+R') = O_0+R_o$ with $\|R_o\| \le r_o q+(14q/h_0)(\Gamma_A+r_o r)r = (r_o+n_o)q = e_o q$. Finally $T_{\rm old} = \mathbb{J}+O \succeq O \succeq O_0 - e_o q I$, and the smallest eigenvalue $q/(1-q)-(q-q^3/(1-q)^2)^{1/2}$ of O_0 is at least $q-r$; so $T_{\rm old} \succeq -(r-q+e_o q)I = -[1+(e_o-1)r]rI$, and $\|O\| \le \Gamma_A r+e_o q = (\Gamma_A+e_o r)r$.

(e) For $p \in \mathcal{B}$ put $c_p = [1+(e_o(p)-1)r_p]r_p$, with $e_o(p)$ the value of e_o at $(1/p, p^{-1/2})$. By (f) the operator T_p of $K_{0,p}$ on $L^2(m_{0,p})$, the Markov operator of a symmetric coupling of $m_{0,p}$ with itself, is self-adjoint with spectrum in $[-c_p, 1]$. The constant e_o is a nondecreasing function of (q,r) and at least $56119/10000 > 1$ ((23.5) and the remark after it), so $c/r = 1+(e_o-1)r$ is nondecreasing, and $c_p \le r_p c_X/r_X \le b^{-1/2}[1+(e_o(X)-1)r_X]$, where $c_X = [1+(e_o(X)-1)r_X]r_X < 1$ by hypothesis. By Lemma 6.10(c), $\bigotimes_p T_p \succeq -\max_p c_p\, I$ on $L^2(m_{0,\mathcal{B}})$. Its form at φ is the left side of the inequality, and $\|\varphi\|^2_{m_{0,\mathcal{B}}} \le (1+e_X)^{n_X}\int\varphi^2 d\nu_{F,\mathcal{B}} \le \bar{H}_X\int\varphi^2 d\nu_{F,\mathcal{B}}$ by (a), since $\mathcal{B}$ has at most $\lfloor L/\log X\rfloor \le n_X$ primes; as $\bar{H}_X c_X/r_X = k_{\rm D}(X,L)$, this is the inequality. The cut form at ι of Π_{F_j} at a real half-function is the left side, with φ the half-function read through F_j, and that of $\mathrm{Diag}_\iota(\Pi^{(1)}_{F_j})$ is $\int\varphi^2 d\nu_{F,\mathcal{B}}$, because the one-layer fold law is $\nu_{F,\mathcal{B}}$ (Lemma 17.2(b),(d)). □

23.6. **Prime and block bounds of the signed fold records.** The leading skew part of a signed fold difference is the three-arm star $L^\circ_{\mathsf{t},p}$ of (23.4); the next proposition bounds it, the symmetric part and the remainder, on primes and on blocks. Its constants are the following expressions in q, r and t, with e_0, h_0, Z and $|\mathbf{a}_\mathsf{t}|$ of Section 23.1, ϱ_{02}, J_F and Γ_A of Lemma 23.4, and $\eta_c = 10q$,

$h_c = 1-\eta_c$, M_c of Lemma 23.7:

$$
\begin{aligned}
&D_g = \frac{2e_0+e_0^2}{(1-e_0)^2}, \qquad c_g = \frac{e_0^2}{(1-e_0)^2}, \qquad n_c = \frac{18/Z+3}{1-e_0} + \frac{(1+2q)c_g}{q},\\
&c_c = \sqrt{3}\,\Gamma_A + \Big(n_c + \frac{D_g}{q}\sqrt{3}\,\Gamma_A r\Big)r, \qquad E_c = \frac{10c_c r}{h_c} + \frac{2M_c}{q}\Big(\frac{c_c r}{h_c} + 2\varrho_{02}\Big),\\
&c_+ = \sqrt{3}\,\Gamma_A + \Big(n_c + \frac{D_g}{q}\sqrt{3}\,\Gamma_A r + E_c\Big)r,\\
&e_{\mathrm{sg}} = \frac{(18-12\mathsf{t})/Z + k_{\mathrm{col}}}{1-e_0} + \frac{(1+2q)c_g}{q} + \frac{D_g}{q}\sqrt{3+2\mathsf{t}^2}\,\Gamma_A r + (1+\mathsf{t})E_c,\\
&\Gamma_5 = 1 + \frac{2(r+q)}{1-q} + \frac{2q(1+r)}{(1-q)(1-2q)}, \qquad c^{\nu}_{\mathrm{col}} = \frac{56119}{10000}\Gamma_5 + \frac{2}{Z}(12r+q) + \frac{288q}{1-q},\\
&c_{\mathrm{col}} = \frac{c^{\nu}_{\mathrm{col}}}{1-e_0} + 17\Big[\frac{2e_0}{(1-e_0)^{3/2}} + \frac{e_0^2}{(1-e_0)^2}\Big], \qquad r_o = \frac{2}{Z(1-e_0)} + c_{\mathrm{col}} + \frac{c_g}{q} + \frac{D_g}{q}\Gamma_A r,\\
&n_o = \frac{14}{h_0}(\Gamma_A + r_o r)r, \qquad e_o = r_o + n_o, \qquad s_p = \frac{2|\mathbf{a}_{\mathsf{t}}|^2/Z + k_{\mathrm{col}}}{1-e_0} + c_{\mathrm{col}} + (1+\mathsf{t})E_c + n_o,\\
&a_p = \frac{2(1+(1-\mathsf{t})^2)^{1/2}/Z + k_{\mathrm{col}}}{1-e_0} + \frac{D_g}{q}|\mathbf{a}_{\mathsf{t}}|\Gamma_A r + (1+\mathsf{t})E_c + 2(1-\mathsf{t})J_F r,\\
&c_{\ge 2} = (\Gamma_A + e_o r)^2 + (1+\mathsf{t})c_+^2, \qquad s_F = \max(s_p, c_{\ge 2}), \qquad a_F = \max(a_p, c_{\ge 2}),\\
&\ell_{\mathsf{t}} = (1+e_0)\Big[\Big(\frac{|\mathbf{a}_{\mathsf{t}}|^2 + 2\mathsf{t}q + 2(1-\mathsf{t})^2q^2/(1-2q)}{1-q}\Big)^{1/2} + \frac{(1+\mathsf{t})q}{1-q}\Big] + 2(1-\mathsf{t})J_F q,\\
&s_{\mathsf{t}} = \big(4 - 2\mathsf{t} + 2\mathsf{t}^2 + 2(1+\mathsf{t}^2)^{1/2}\big)^{1/2}, \qquad d_{\mathsf{t}} = s_{\mathsf{t}}\Gamma_A + (e_o + e_{\mathrm{sg}})r,
\end{aligned}
\tag{23.5}
$$

where $k_{\mathrm{col}} = 9/4$ if $\mathsf{t} = 0$ or $\mathsf{t} \ge 2/(p-1)$, and $k_{\mathrm{col}} = 3$ otherwise. Each expression is built from q, r and positive constants by sums, products, maxima and square roots, and reciprocals of the positive decreasing denominators $1-q$, $1-2q$, $1-4q$, Z, h_0, h_c, $1-e_0$ and $1-\varrho_{01}$, after every remainder carrying q has been written as r times an admissible coefficient (e_0/q, D_g/q, c_g/q and M_c/q are themselves such functions); so for a fixed branch of k_{col} it is a nondecreasing function of (q,r) (Lemma 4.18), and its value at (q_X, r_X), with $q_X = 1/X$ and $r_X \ge X^{-1/2}$, bounds its value at every prime $p \ge X$. Also $e_o \ge c_{\mathrm{col}} \ge 56119/10000$. At $\mathsf{t} = 127/500$, $k_{\mathrm{col}} = 9/4$ for every $p \ge 2^{24}$. In words: $c_c r$ bounds the centred norm of an unnormalized positive component, qE_c its normalization and positive completion, $c_+ r$ the centred norm of a repaired component; qe_{sg} is the distance of the signed record from its natural leading operator and qe_o that of the reference record (Lemma 23.2(f)); $s_p q$ and $a_p q$ are the prime symmetric and skew remainders, $c_{\ge 2}/b$ the bound on larger supports, $\ell_{\mathsf{t}} r$ the leading star, and d_{t} the actual coefficient of the direct Gram of the signed record; $s_0 = \sqrt{6}$.

Proposition 23.13 (Prime and block bounds of the signed fold records)**.** *Let $0 \le \mathsf{t} \le 1$, let $\mathcal{B}$ be a middle block with least prime $b \ge 2^{24}$, evaluate the constants of* (23.5) *at $q = 1/b$, $r = b^{-1/2}$, and let $D_{\mathsf{t}} = T[K_{\mathsf{t},\mathcal{B}}] - \bigotimes_p T_{\mathrm{old},p}$ be the record difference of F_1 on $\mathcal{B}$, an operator on $L^2(m_{0,\mathcal{B}})$, with $T[\cdot]$ taken relative to $m_{0,\mathcal{B}} \otimes m_{0,\mathcal{B}}$.*

(a) $\|\operatorname{Sym} D_{\mathsf{t}}\| \le s_F/b$.

(b) $\operatorname{Skew} D_{\mathsf{t}} = L_{\mathsf{t},\mathcal{B}} + R_{\mathsf{t},\mathcal{B}}$, *where $L_{\mathsf{t},\mathcal{B}}$ is $L^{\circ}_{\mathsf{t},p}$ on the sector of exact support $\{p\}$, $p \in \mathcal{B}$, and zero on the other sectors; $\|L_{\mathsf{t},\mathcal{B}}\| \le \ell_{\mathsf{t}} b^{-1/2}$ and $\|R_{\mathsf{t},\mathcal{B}}\| \le a_F/b$. The operator $L_{\mathsf{t},\mathcal{B}}$ is the three-arm star $\mathcal{A}\mathcal{V}_0^* - \mathcal{V}_0\mathcal{A}^*$ of Lemma 23.4(e) with $(c_1,c_2,c_3) = (1,\mathsf{t},1-\mathsf{t})$.*

(c) One prime. *At a prime $p \ge 2^{24}$, with the constants at $(1/p, p^{-1/2})$: $\|D^{\mathrm{s}}_{\mathsf{t},p}\| \le s_p q$, $\|D^{\mathrm{a}}_{\mathsf{t},p} - L^{\circ}_{\mathsf{t},p}\| \le a_p q$ and $\|L^{\circ}_{\mathsf{t},p}\| \le \ell_{\mathsf{t}} r$; every centred component $\Pi_\perp T[P_{uv,p}]\Pi_\perp$ has norm at most*

c_+r, and the centred reference O at most $(\Gamma_A + e_o r)r$; and the centred signed record is

$$\Pi_\perp T[K_{\mathsf{t},p}]\Pi_\perp = O_0 + L^m_{\mathsf{t},p} + R^{\mathrm{new}}_{\mathsf{t},p}, \qquad L^m_{\mathsf{t},p} = A_{\mathsf{t}} V^*_{0,m} - V_0 A^{*_m}_{\mathsf{t}}, \qquad \|R^{\mathrm{new}}_{\mathsf{t},p}\| \le e_{\mathrm{sg}} q,$$

with O_0 of Lemma 23.2(f) and A_{t} unprojected.

(d) The inputs of the direct Gram. *On every exact support $S \subseteq \mathcal{B}$ with $|S| \ge 2$, the restrictions of the centred reference $\bigotimes_p T_{\mathrm{old,p}}$ and of the centred signed record have norms with sum at most $c_{\ge 2}/b$; and $c_{\ge 2} r < 2 < s_{\mathsf{t}} \le d_{\mathsf{t}}$. With the prime decompositions of (c) and Lemma 23.2(f) and the frame Γ_A, these are the inputs of the direct common-cut Gram of the signed record (Lemma 26.1, Proposition 26.4), whose coefficient before the frame factor Γ_A and the remainder term in d_{t} is s_{t} and whose actual coefficient is d_{t}.*

(e) *For F_2 the record difference is $\mathcal{U} D^*_{\mathsf{t}} \mathcal{U}$, $\mathcal{U} = \bigotimes_p \mathcal{U}_p$, with the same bounds and the leading part $-\mathcal{U} L_{\mathsf{t},\mathcal{B}} \mathcal{U}$.*

Proof. Write $\mathrm{op}(K) = M_{g_0}^{-1} N_K M_{g_0}^{-1}$ for the operator on $L^2(m_0)$ of a kernel K relative to $\nu_F \otimes \nu_F$, with the transport rules (T1)–(T3) and the natural operators of the star channels from the proof of Lemma 23.2(e),(f). In the frame of Lemma 23.4(d), a natural operator $A_x V^*_{0,m} + V_0 A^{*_m}_y$, with $A_x = A_{\mathbf{x}}$, $A_y = A_{\mathbf{y}}$ for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ in the notation of Lemma 12.10(e), is r times the image of a matrix of norm $\max(|\mathbf{x}|, |\mathbf{y}|)$, so its norm is at most $\max(|\mathbf{x}|, |\mathbf{y}|)\Gamma_A r$.

Step 1: the components. Expand $U_{uv} = (1+\mathsf{C})(1+(1-\mathsf{t})\mathsf{B})(1+u\mathsf{D})(1+v\mathsf{A}) - \delta_{10}(1+(1-\mathsf{t})\mathsf{B})(1+u\mathsf{D})(1+v\mathsf{A})$. The star monomials $\mathsf{C}, \mathsf{B}, \mathsf{BC}$ and $\mathsf{D}, \mathsf{A}, \mathsf{AD}$ have the natural operators of the proof of Lemma 23.2(e),(f), with the constants $-q$ of BC and AD moved into the constant kernel; the star part is $A_x V^*_{0,m} + V_0 A^{*_m}_y$ with $\mathbf{x} = (\mathsf{t}-1, 1, 1-\mathsf{t})$ and $\mathbf{y} = -(v, u, uv)$, so $\max(|\mathbf{x}|, |\mathbf{y}|) = \sqrt{3}$, and the constant kernel has a coefficient of modulus at most $1 + 2q$. The nine nonstar monomials have coefficients of modulus at most one and norms at most $(2/Z)q$ each (Lemma 23.11(a)); the removed query has norm at most $3q$ by the Schur test, its row and column masses being at most $q(2-\mathsf{t})(1+2q/(1-q))$ and $q(1+q)/(1-2q)$ (proof of Lemma 23.7(b)). By (T1)–(T3) the centred operator of U_{uv} on $L^2(m_0)$ has norm at most $\sqrt{3}\,\Gamma_A r + D_g\sqrt{3}\,\Gamma_A r + n_c q = c_c r$. Dividing by z_{uv}, with $|z_{uv}^{-1} - 1| \le \eta_c/h_c$, changes it by at most $(10q/h_c)c_c r$, and the completion of Lemma 23.7(d) by at most $2M_c(c_c r/h_c + 2\varrho_{02})$; together $E_c q$. Hence $\|\Pi_\perp T[P_{uv}]\Pi_\perp\| \le c_c r + E_c q = c_+ r$.

Step 2: the signed record. With the coefficients $c_{++} = 1$, $c_{-+} = \mathsf{t}/2$, $c_{--} = -\mathsf{t}/2$, $\sum c_{uv} = 1$ and $\sum |c_{uv}| = 1 + \mathsf{t}$, and by Lemma 23.5(d),

$$\sum_{uv} c_{uv} U_{uv} = \Psi^{\mathrm{soft}}_{\mathsf{t}} - \delta_{10} G_{\mathsf{t}}, \qquad \Psi^{\mathrm{soft}}_{\mathsf{t}} = \Phi_{\mathsf{t}} + (1+\mathsf{A})(1+\mathsf{B}),$$

$$G_{\mathsf{t}} = \big(1 + (1-\mathsf{t})\mathsf{B}\big)\big[1 + \mathsf{D} + (1+\mathsf{t})\mathsf{A} + (1-\mathsf{t})\mathsf{AD}\big].$$

The star part of $\Psi^{\mathrm{soft}}_{\mathsf{t}}$ is $A_x V^*_{0,m} + V_0 A^{*_m}_y$ with $\mathbf{x} = \mathbf{a}_{\mathsf{t}} - \mathbf{e}$ and $\mathbf{y} = -\mathbf{a}_{\mathsf{t}} - \mathbf{e}$, $\mathbf{e} = (1,0,0)$: by Lemma 23.5(d) it is the star part $-\mathbf{e}$ of $(1+\mathsf{A})(1+\mathsf{B})$, whose natural operator is O_0, plus $L_{\mathsf{t},p}$, whose natural operator is $L^m_{\mathsf{t},p}$; its constants cancel, and $\max(|\mathbf{x}|, |\mathbf{y}|) = (3 + 2\mathsf{t}^2)^{1/2}$. The nonstars of $\Psi^{\mathrm{soft}}_{\mathsf{t}}$ are those of the products of $\{\mathsf{C}, (1-\mathsf{t})\mathsf{B}, (1-\mathsf{t})\mathsf{BC}\}$ with $\{\mathsf{D}, (1+\mathsf{t})\mathsf{A}, (1-\mathsf{t})\mathsf{AD}\}$, of absolute coefficient sum $(3-2\mathsf{t})\cdot 3 = 9 - 6\mathsf{t}$, each of norm at most $(2/Z)q$. For the removed query, on $\{-1, 0, 1\}^2$

$$\big|1 + \mathsf{D} + (1+\mathsf{t})\mathsf{A} + (1-\mathsf{t})\mathsf{AD}\big| \le 1 + \mathsf{t} + (1-\mathsf{t})\mathsf{D} + \mathsf{A} + \mathsf{AD},$$

and $0 \le 1 + (1-\mathsf{t})\mathsf{B} \le 2 - \mathsf{t}$. With the conditional means of the proof of Lemma 23.7(b), the row average of the right side given $y_0 = x_1$ is at most $1 + \mathsf{t}$ when $\mathsf{t} \ge 2/(p-1)$, so the row mass is at most $(2-\mathsf{t})(1+\mathsf{t})q \le 9q/4$; given $x_1 = y_0 = c$, the column average of $(1 + (1-\mathsf{t})\chi(x_2 - c))$ times the right side is at most $(1 + \mathsf{t} - q)/(1-2q)$, as in the column estimate there, so the column mass is below $9q/4$. At $\mathsf{t} = 0$ the record is P_{++} alone, whose removed query has row and column masses below $9q/4$ by Lemma 23.7(b). For $0 < \mathsf{t} < 2/(p-1)$ the row average exceeds $1 + \mathsf{t}$ by at most $2/((p-1)(p-2))$, and both masses are below $3q$. By the Schur test the removed query

has norm at most $k_{\mathrm{col}}q$. By (T1)–(T3) and Step 1 for the normalizations and completions of the three components, weighted by $\sum|c_{uv}| = 1+\mathsf{t}$, the centred signed record is $O_0 + L^m_{\mathsf{t},p} + R^{\mathrm{new}}_{\mathsf{t},p}$ with $\|R^{\mathrm{new}}_{\mathsf{t},p}\| \le e_{\mathrm{sg}}q$. This is the last claim of (c).

Step 3: the prime difference. Now $D_{\mathsf{t},p} = \Pi_\perp(T[K_{\mathsf{t},p}] - T_{\mathrm{old}})\Pi_\perp$, and

$$\sum_{uv} c_{uv}U_{uv} - \Psi_0 = \Phi_{\mathsf{t}} - \delta_{10}G_{\mathsf{t}} + \mathsf{C}_{\mathrm{rem}},$$

since $\Psi_0 = (1+\mathsf{A})(1+\mathsf{B}) - \mathsf{C}_{\mathrm{rem}}$; this kernel has no constant term. Hence $D_{\mathsf{t},p}$ is the centred operator of $\Phi_{\mathsf{t}} - \delta_{10}G_{\mathsf{t}} + \mathsf{C}_{\mathrm{rem}}$, plus the normalizations and completions of the components (at most $(1+\mathsf{t})E_cq$), minus $(z_0^{-1}-1)$ times the centred operator of Ψ_0, which is self-adjoint and at most n_oq by the proof of Lemma 23.2(f). *Symmetric part.* $\operatorname{Sym}\Phi_{\mathsf{t}} = \tilde{x}_{\mathsf{t}}\tilde{y}_{\mathsf{t}}$ is a product of two rectangular stars with $|\mathbf{a}_{\mathsf{t}}|^2$ as the sum of the squared coefficients of each, so its norm on $L^2(\nu_F)$ is at most $(2|\mathbf{a}_{\mathsf{t}}|^2/Z)q$ (Lemma 23.11(b)); the removed query costs $k_{\mathrm{col}}q$; both are transported by (T1); and $\mathsf{C}_{\mathrm{rem}}$, symmetric, costs $c_{\mathrm{col}}q$ (Lemma 23.12(d)). With the normalizations and the completions this gives $\|D^{\mathrm{s}}_{\mathsf{t},p}\| \le s_pq$. *Skew part.* $\operatorname{Skew}\Phi_{\mathsf{t}} = L_{\mathsf{t},p} + K_{\mathsf{t}} - K_{\mathsf{t}}^\top$. The operator of $L_{\mathsf{t},p}$ differs from its natural operator $L^m_{\mathsf{t},p}$ by at most $D_g|\mathbf{a}_{\mathsf{t}}|\Gamma_A r$ (T2), and $L^m_{\mathsf{t},p} - L^\circ_{\mathsf{t},p} = (1-\mathsf{t})(TV^*_{0,m} - V_0T^{*m})$ with $T = \Pi_{\mathcal{H}_1+\mathcal{H}_2}A_3$, of norm at most $2(1-\mathsf{t})J_Fqr$ (Lemma 23.4(c)). On the full label spaces, K_{t} ignores the input label y_1 and its output has mean zero in x_1; so with $\mathcal{H}$ the functions independent of the middle label, $K_{\mathsf{t}} = \Pi_{\mathcal{H}^\perp}K_{\mathsf{t}}\Pi_{\mathcal{H}}$, $K_{\mathsf{t}} - K_{\mathsf{t}}^\top$ is an off-diagonal block, and $\|K_{\mathsf{t}} - K_{\mathsf{t}}^\top\| = \|K_{\mathsf{t}}\| \le r\cdot(1+(1-\mathsf{t})^2)^{1/2}r$, the norms of the one-edge star A and of the rectangular star $\mathsf{C}(1+(1-\mathsf{t})\mathsf{B})$ (Lemma 23.11(b)); its restriction to $H_F\times H_F$ costs $2/Z$, and (T1) transports it. The removed query costs $k_{\mathrm{col}}q/(1-e_0)$ again. The reference collision $\mathsf{C}_{\mathrm{rem}}$, the normalization of the reference and the constant centring are self-adjoint and are not charged to the skew part. With the normalizations and completions this gives $\|D^{\mathrm{a}}_{\mathsf{t},p} - L^\circ_{\mathsf{t},p}\| \le a_pq$. *Leading star.* On $L^2(\nu_F)$, $A^*_{\mathsf{t}}A_{\mathsf{t}}$ is the scalar $\gamma_{\mathsf{t}} = q[\mathsf{t}^2 + 1 + (1-\mathsf{t})^2(1-2q-q^2)/(1-q) - 2\mathsf{t}q/(1-q) + 2(1-\mathsf{t})^2q^2/((1-q)(1-2q))]$ by Lemma 12.10(e) (the parts with P of the cross Grams cancel), and γ_{t}/q is at most the quotient under the square root in ℓ_{t}; and $V_0^*A_{\mathsf{t}} = \alpha I + \beta P$ with $|\beta| = (1+\mathsf{t})q/(1-q)$. Lemma 23.1 gives $\|A_{\mathsf{t}}V_0^* - V_0A_{\mathsf{t}}^*\| \le |\beta|r + \gamma_{\mathsf{t}}^{1/2}$ on $L^2(\nu_F)$. The coordinates of m_0 being uniform, $V^*_{0,m} = V_0^*M_{g_0}$ and $A^{*m}_{\mathsf{t}} = A^*_{\mathsf{t}}M_{g_0}$, so $L^m_{\mathsf{t},p}$ is unitarily equivalent to $M_{g_0}^{1/2}(A_{\mathsf{t}}V_0^* - V_0A_{\mathsf{t}}^*)M_{g_0}^{1/2}$, of norm at most $1+e_0$ times that; with the projection cost above, $\|L^\circ_{\mathsf{t},p}\| \le \ell_{\mathsf{t}}r$. This proves (c), the claim on the components being Step 1 and that on the reference Lemma 23.2(f).

Step 4: blocks. Every component and the reference are couplings with both marginals $m_{0,p}$, so their Markov operators are sector-preserving, with centred norms at most $c_+(p)r_p \le c_+b^{-1/2}$ and $(\Gamma_A + e_o(p)r_p)r_p \le (\Gamma_A + e_or)b^{-1/2}$, both below one, by (c) and the monotonicity of the constants. By Remark 6.12, D_{t} vanishes on the empty sector, is $D_{\mathsf{t},p}$ on the sector $\{p\}$, and on a sector S with $|S|\ge 2$ has norm at most $(1+\mathsf{t})\prod_{p\in S}c_+(p)r_p + \prod_{p\in S}(\Gamma_A + e_o(p)r_p)r_p \le c_{\ge 2}/b$, the further centred factors being contractions; the same bound holds for its symmetric and skew parts, and all these norms are maxima over the sectors. The operator $L_{\mathsf{t},\mathcal{B}}$ vanishes on the sectors with $|S|\ge 2$ and is the orthogonal sum of the $L^\circ_{\mathsf{t},p}$ on the singletons, which is the star of Lemma 23.4(e); on the singletons (c) applies with $q_p \le 1/b$ and $r_p \le b^{-1/2}$. This proves (a), (b) and the first claim of (d). For the second, only the factor $1+\mathsf{t}$ of $c_{\ge 2}$ depends on t, so $c_{\ge 2}$ is at most its value at $(2^{-24}, 2^{-12})$ and $\mathsf{t}=1$, which is <8 by outward evaluation of the formula for $c_{\ge 2}$ in (23.5) (the value is $7.043\ldots$); hence $c_{\ge 2}r < 8\cdot 2^{-12} < 2$; and $s_{\mathsf{t}}^2 \ge 4 - 2\mathsf{t} + 2\mathsf{t}^2 + 2 \ge 4 + 3/2$ on $[0,1]$, while $\Gamma_A \ge 1$.

(e) $\mathcal{U}$ is unitary on $L^2(m_{0,\mathcal{B}})$ because $g_0\circ\varsigma = g_0$; the record difference of F_2 is $\mathcal{U}D_{\mathsf{t}}^*\mathcal{U}$ componentwise, whose symmetric and skew parts are $\mathcal{U}(\operatorname{Sym}D_{\mathsf{t}})\mathcal{U}$ and $-\mathcal{U}(\operatorname{Skew}D_{\mathsf{t}})\mathcal{U}$, and conjugation preserves norms and the sector decomposition. □

Certificate 23.14 (The constants of the signed law and of the matching records)**.** At each of the 33 piece starts, the constants below have been evaluated in outward interval arithmetic and satisfy the gates listed in this certificate ($R_{\mathrm{row}}, R_{\mathrm{col}} < 10$, $c_+ r < 1$, $c_{\geq 2} r < d_{\mathtt{t}}$, and the others named). In detail, the functions `sharp_bounds`, `e_bounds` and `old_negative` of `research/advance_2026_10_24/local/low_domain.py` evaluate the constants of (23.5), the coefficients $s_E(X)$, $a_E(X)$, $r_E(X)$ of Lemma 23.17(ii) and Lemma 23.18, and $k_{\mathrm{D}}(X, 585)$ of Lemma 23.2(e), in outward rational interval arithmetic at $q = 1/X$, $r = 1/\lfloor\sqrt{X}\rfloor$ and $\mathtt{t} = 127/500$, at each of the 33 piece starts; the leading cap is computed there by `actual_leading` of `research/continuation_2026_10_24/local/orientation_local.py`, which is the closed form $\ell_{\mathtt{t}}$ of (23.5) at the arm vector $\mathbf{a}_{\mathtt{t}}$. The final scalar evaluation calls them through `research/advance_2026_10_24/global/evaluate_signed.py`, selected by the low-domain flag of the final design file `research/push_2026_10_25/global/refined_L570_source.json`, from `research/improvements_2026_10_25/global/selected.py`, invoked with the selected push design. It asserts $R_{\mathrm{row}}, R_{\mathrm{col}} < 10$, $c_+ r < 1$, $c_{\geq 2} r < d_{\mathtt{t}}$ and the centred-norm bound $\|T_{\mathrm{old}} P_0\| \leq (\Gamma_A + e_o r) r$ of Lemma 23.2(f) with a constant exceeding e_o, so that $(\Gamma_A + e_o r) r < 1$; $e_o > 1$, $n_X e_X < 1$ and $[1 + (e_o - 1) r] r < 1$; the gates of Lemma 23.17(ii); and in each piece the inequalities $A^{\mathrm{lead}}_{F,i} \geq A^{\mathrm{arm}}_{F,i}$ and $A_{E,i} \geq A^{\mathrm{arm}}_{E,i}$ between the prices of Section 23.9. The totals of those prices at the exponent a are stored in `research/push_2026_10_25/artifacts/author.json` (key `raw_totals`); no stored record holds the per-piece constants. A read-only evaluation of the three functions at $X = Q_3 = 19951586$ ($r_X = 1/4466$) gives the following upper bounds: $\Gamma_A < 1.0002241$, $R_{\mathrm{row}} < 7.9840009$, $R_{\mathrm{col}} < 6.9840008$, $c_+ < 1.7371505$, $E_c < 0.04070887$, $e_{\mathrm{sg}} < 17.254638$, $c^{\nu}_{\mathrm{col}} < 5.6198028$, $c_{\mathrm{col}} < 5.6198068$, $e_o < 7.6238442$, $s_F = s_p < 11.166062$, $a_F = a_p < 4.7973987$, $c_{\geq 2} < 4.7880514$, $\ell_{\mathtt{t}} < 1.2731978$, $s_{\mathtt{t}} < 2.3842273$ and $d_{\mathtt{t}} < 2.3903321$; for the matching records $s_E < 5.4071859 \cdot 10^{-4}$, $a_E < 1.0000003$, $1/(1 - q_X) < 1.0000001$, $\kappa^E / r < 1.0012128$ and $r_E < 10.829703$; and for the cap $n_X = 34$, $\bar{H}_X < 1.0000035$, $1 + (e_o - 1) r_X < 1.0014832$ and $k_{\mathrm{D}} < 1.0014866$. As $q \to 0$, $\ell_{\mathtt{t}} \to |\mathbf{a}_{\mathtt{t}}| = 1.27319\ldots$ and $d_{\mathtt{t}} \to s_{\mathtt{t}}$. Independently, `research/advance_2026_10_24/local/check_signed_law.py` checks the polynomial and leading skew identities of Lemma 23.5(d), the exact two-centre Gram and 18 collision sign patterns, with exact finite checks at small primes; `research/advance_2026_10_24/local/check_signed_sharp.py` the factored symmetric and skew identities, the matrix C of Lemma 23.12(d) with its exact Schur rows, and exact conditional feature Grams; and `research/advance_2026_10_24/local/check_low_domain.py` the uniform analytic majorants and gates from 2^{20}, for 21 values of the parameter.

23.7. **The matching records E_0, E_1 and their replacement K_1.** The two two-layer matching records E_0, E_1 of a middle block, of weight $w_{\mathrm{S}} r$ each, do not accept the six queries: each needs exactly the cross edge $x_0 \to y_1$ (Certificate 23.10). They are replaced by an exact-marginal law that imposes this edge and its mirror image $x_1 \to y_0$, and nothing else. Unlike the fold records it needs no repair: its fibre count is constant (Lemma 23.15(a)).

The empty half law. Let H_E be the set of all halves, $|H_E| = p(p-1)(p-2)$, and ν_E its uniform law. Kernels are densities relative to $\nu_E \otimes \nu_E$ and operators act on $L^2(\nu_E)$; V_i, $\Pi_\perp$, $\top$, Sym and Skew are as in Section 23.1, with ν_E in place of ν_F. For $i \neq j$ and mean-zero g, $\mathbb{E}[g(x_j) \mid x_i] = -g(x_i)/(p-1)$, so the Gram operator of $V = (V_0, V_1, V_2)$ on H_0^3 is $(1 + u) I - u J_3$, with $u = 1/(p-1)$ and J_3 the all-ones 3×3 matrix; its eigenvalues are $1 + u = 1/(1-q)$ and $1 - 2u = (1 - 3q)/(1 - q)$. Every pair of coordinates of ν_E is uniform on the distinct pairs, of density $1/(1-q)$, and ν_E has density $1/Z$ (Lemma 13.24(b)); Lemma 13.24(a) applies with these bounds. Write $\delta_{\mathsf{C}} = \mathbf{1}[x_1 = y_0]$, $\delta_{\mathsf{D}} = \mathbf{1}[x_0 = y_1]$ and $\delta_{ij} = \mathbf{1}[x_i = y_j]$. At ι the reference's two-layer matching record is the uniform law of pairs of disjoint halves, $K_0^E = I_\times / z_0^E$, with both half marginals ν_E (Lemma 17.2(d)).

Lemma 23.15 (The exact-marginal matching law)**.** *Let $p \geq 19$ and $z_1 = (p^2 - 3p + 4)/(p(p-1))$.*

(a) The kernel $K_1 = (1 + \mathsf{C} - \delta_{\mathsf{C}})(1 + \mathsf{D} - \delta_{\mathsf{D}})/z_1$ is the density of a $\mathcal{G}_p$-invariant probability measure on $H_E \times H_E$ whose two half marginals both equal ν_E. It is $4/z_1$ times the indicator

of the strict edges $x_1 \to y_0$ and $x_0 \to y_1$ and imposes no other relation between the halves; in particular other coincidences between the halves, such as $x_2 = y_2$, have positive probability. Under it each of the pairs (x_0, y_1) and (x_1, y_0) is uniform on the strict-square pairs.

(b) $z_0^E = (p-3)(p-4)(p-5)/(p(p-1)(p-2)) \ge 1 - 9q$, *and on mean-zero functions* K_0^E *has norm at most* $k_{\mathrm{old}}^E = 9q/(1-9q)$*; its exact value there is* $3/(p-3)$ *(Lemma 12.11). Moreover* $z_1 \ge 1 - 2q$ *and* $0 < z_1 - z_0^E = (7p^2 - 37p + 52)/(p(p-1)(p-2)) \le 7q/Z$.

(c) $K_1 - K_0^E = \Pi_\perp (K_1 - K_0^E)\Pi_\perp$, *and*

$$\mathrm{Skew}(K_1 - K_0^E) = \frac{\mathsf{C} + \mathsf{D} - \delta_{\mathsf{C}}\mathsf{D} - \delta_{\mathsf{D}}\mathsf{C}}{z_1}, \qquad \|\mathsf{C} + \mathsf{D}\| = \frac{r}{1-q}, \qquad \|\delta_{\mathsf{C}}\mathsf{D}\|, \|\delta_{\mathsf{D}}\mathsf{C}\| \le \frac{qr}{1-q},$$

so that $\|\mathrm{Skew}(K_1 - K_0^E)\| \le a_p^E := r(1+2q)/((1-q)(1-2q))$.

(d) Write $1 - I_\times = \sum_{i,j} \delta_{ij} - \varepsilon_E$. *Then* $\varepsilon_E \ge 0$, $\|\varepsilon_E\| \le 18q^2/(1-q)$, *and*

$$\Pi_\perp \,\mathrm{Sym}(K_1 - K_0^E)\Pi_\perp = \mathcal{S} + \frac{\Pi_\perp \mathsf{C}\mathsf{D}\Pi_\perp + \Pi_\perp \delta_{\mathsf{C}}\delta_{\mathsf{D}}\Pi_\perp}{z_1} - \frac{\Pi_\perp \varepsilon_E \Pi_\perp}{z_0^E},$$

where $\mathcal{S} = qV[J_3/z_0^E - (E_{10} + E_{01})/z_1]V^*$ *on mean-zero functions,* E_{ij} *being matrix units. Moreover* $\|\mathcal{S}\| \le \mathfrak{s}_p$, $\|\mathsf{C}\mathsf{D}\|/z_1 \le q/((1-q)(1-2q))$ *and* $\|\delta_{\mathsf{C}}\delta_{\mathsf{D}}\|/z_1 \le q^2/((1-q)(1-2q))$, *where*

$$\mathfrak{s}_p = \frac{q}{1-q}\Big[\frac{1+\sqrt{2}}{1-9q} + \frac{7q}{Z(1-9q)(1-2q)}\Big].$$

Proof. (a) For $a \ne b$, $1 + \chi(b-a)$ is 2 if $a \to b$ and 0 otherwise, and for $a = b$ the factor $1 + 0 - 1$ vanishes; this describes K_1. Fix a half x. Under ν_E the pair (y_0, y_1) is uniform on ordered distinct pairs, and the pairs with $x_1 \to y_0$ and $x_0 \to y_1$ number $(\frac{p-1}{2})^2 - k_p = (p^2 - 3p + 4)/4$, the coincident ones being the k_p common out-neighbours of x_0 and x_1 (Lemma 12.10(a)), whatever the orientation between x_0 and x_1. Hence $\mathbb{E}_y K_1(x,y) = 4((\frac{p-1}{2})^2 - k_p)/(p(p-1)z_1) = 1$; the column sums are one by the same count with common in-neighbours. Invariance holds because χ is invariant under square dilations; every relation other than the two strict edges is free; and $\mathcal{G}_p$ acts transitively on the ordered pairs (a, a') with $a \to a'$, so the invariant law of each queried pair is uniform on them. (b) Given x, the half y avoids three labels with probability z_0^E; the rest is the symmetry of $I_\times$, the Schur test with row sums $\mathbb{E}_y(1 - I_\times) \le 9q$, and the identities $(1-3q)(1-4q)(1-5q) - (1-9q)Z = 18q^2 - 42q^3$, $p^2 - 3p + 4 - (p-1)(p-2) = 2$ and $(p^2 - 3p + 4)(p-2) - (p-3)(p-4)(p-5) = 7p^2 - 37p + 52 \le 7p^2$. (c) Both laws have the half marginals ν_E, which gives the first identity. Expanding the product and using that transposition maps C, D to $-\mathsf{D}, -\mathsf{C}$ and exchanges $\delta_{\mathsf{C}}, \delta_{\mathsf{D}}$,

$$z_1 \,\mathrm{Sym}\, K_1 = 1 + \mathsf{C}\mathsf{D} - \delta_{\mathsf{C}} - \delta_{\mathsf{D}} + \delta_{\mathsf{C}}\delta_{\mathsf{D}}, \qquad z_1 \,\mathrm{Skew}\, K_1 = \mathsf{C} + \mathsf{D} - \delta_{\mathsf{C}}\mathsf{D} - \delta_{\mathsf{D}}\mathsf{C},$$

and K_0^E is symmetric. As operators $\mathsf{C} = V_1 P V_0^*$ and $\mathsf{D} = V_0 P V_1^*$, so $\mathsf{C} + \mathsf{D}$ is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \otimes P$ conjugated by the row (V_0, V_1); the Gram matrix $\left(\begin{smallmatrix} 1 & -u \\ -u & 1 \end{smallmatrix}\right)$ of that row commutes with $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, with eigenvalues $1 - u$ and $1 + u$ on $(1,1)$ and $(1,-1)$, so $\|\mathsf{C} + \mathsf{D}\| = (1+u)\|P\| = r/(1-q)$. The kernel $\delta_{\mathsf{C}}\mathsf{D}$ depends on (x_0, x_1) and (y_0, y_1); for independent labels it is $qI \otimes P$, of norm qr, and the two pair densities $1/(1-q)$ give the bound; likewise for $\delta_{\mathsf{D}}\mathsf{C}$. Then use $z_1 \ge 1 - 2q$. (d) With $\kappa = \sum \delta_{ij}$, $\varepsilon_E = \kappa - \mathbf{1}_{\kappa \ge 1}$ is bounded by the sum of $\delta_{ij}\delta_{i'j'}$ over the 18 compatible pairs, and given x two prescribed labels of y at two prescribed positions have probability $1/(p(p-1))$; the Schur test gives the bound. Since $\Pi_\perp \delta_{ij} \Pi_\perp = qV_i V_j^*$ on mean-zero functions, the compressions of $\sum \delta_{ij}/z_0^E$ and of $(\delta_{\mathsf{C}} + \delta_{\mathsf{D}})/z_1$ give $\mathcal{S}$, and the constants $1/z_1$, $1/z_0^E$ are removed by the compression. Write $J_3/z_0^E - (E_{10} + E_{01})/z_1 = (J_3 - E_{10} - E_{01})/z_0^E + (E_{10} + E_{01})(1/z_0^E - 1/z_1)$. The matrix $J_3 - E_{10} - E_{01}$ has the characteristic polynomial $\lambda^3 - 3\lambda^2 + \lambda + 1$, with roots 1 and $1 \pm \sqrt{2}$, and $\|E_{10} + E_{01}\| = 1$; with $\|V\|^2 = 1/(1-q)$ and (b) this gives $\|\mathcal{S}\| \le \mathfrak{s}_p$. The kernel $\mathsf{C}\mathsf{D}$ is a two-edge

matching on the pairs (x_0,x_1), (y_0,y_1), of norm q for independent labels, and $\delta_{\mathsf{C}}\delta_{\mathsf{D}}$ is $qI\otimes qI$ there; the pair densities give the last two bounds. □

Lemma 23.16 (Orthogonal channels)**.** *Let $p\ge 19$.*

(a) The pair-star map $\mathcal{T}_E\colon L^2(\mathbb{F}_p)\to L^2(\nu_E)$, $\mathcal{T}_E g(x)=\mathbb{E}_t\chi(t-x_1)\chi(t-x_0)g(t)$, has norm at most $(q(1-2q)/(1-q))^{1/2}\le r$ on mean-zero functions: for independent labels its norm there is exactly $(q(1-2q))^{1/2}$, at most $(q(1-q))^{1/2}$.

(b) On mean-zero functions $\mathsf{CD}V_0=\mathsf{CD}V_1=-u\mathcal{T}_E$ and $\mathsf{CD}V_2=2\mathcal{T}_E/((p-1)(p-2))$. Hence, if H is the orthogonal projection onto $V(H_0^3)$,

$$\|\mathsf{CD}H\|\le e_E:=\frac{qr}{((1-q)(1-3q))^{1/2}}\Big(2+\frac{4q^2}{(1-2q)^2}\Big)^{1/2}.$$

(c) $\|\operatorname{Sym}(K_1-K_0^E)\|\le s_p^E$, where

$$s_p^E=\mathfrak{s}_p+\frac{2e_E}{1-2q}+\frac{q^2}{(1-q)(1-2q)}+\frac{18q^2}{(1-q)(1-9q)}.$$

Proof. (a) For independent labels $\mathcal{T}_E^*\mathcal{T}_E$ has the kernel $(\mathbb{E}_x\chi(t-x)\chi(t'-x))^2=(\mathbf{1}_{t=t'}-q)^2$ by Lemma 12.10(b), so $\mathcal{T}_E^*\mathcal{T}_E=q(1-2q)I+q^2\mathbb{J}$; restricting the pair (x_0,x_1) to distinct labels costs the density factor $1/(1-q)$. (b) Given y_0, the label y_1 is uniform off y_0, so $\mathbb{E}[\chi(y_1-x_0)\mid y_0]=-\chi(y_0-x_0)/(p-1)$; this gives $\mathsf{CD}V_0=-u\mathcal{T}_E$, and $\mathsf{CD}V_1=-u\mathcal{T}_E$ in the same way. Given $y_2=t$, the pair (y_0,y_1) is uniform on the ordered distinct pairs avoiding t, and by Lemma 12.10(b) $\sum_{a\ne a';\,a,a'\ne t}\chi(a-x_1)\chi(a'-x_0)=1+2\chi(t-x_1)\chi(t-x_0)$, whose constant term is annihilated on mean-zero g; this gives $\mathsf{CD}V_2$. If $h=Vc$ with c a triple of mean-zero functions, then $\|h\|^2\ge(1-2u)\|c\|^2$ and $\mathsf{CD}h=\mathcal{T}_E(-uc_0-uc_1+2c_2/((p-1)(p-2)))$, so $\|\mathsf{CD}h\|\le ru(2+4/(p-2)^2)^{1/2}(1-2u)^{-1/2}\|h\|=e_E\|h\|$. (c) By Lemma 23.15(c),(d) it suffices to bound $\|\mathcal{S}+X/z_1\|$ with $X=\Pi_\perp\mathsf{CD}\Pi_\perp$. Here $\mathcal{S}=H\mathcal{S}H$, and X is self-adjoint with $XH=\Pi_\perp\mathsf{CD}H$, because $\Pi_\perp H=H$; so $\|XH\|=\|HX\|\le e_E$ and $\|X-(I-H)X(I-H)\|=\|HX+(I-H)XH\|\le 2e_E$. The operators $\mathcal{S}$ and $(I-H)X(I-H)/z_1$ act on orthogonal subspaces, so their sum has norm at most $\max(\|\mathcal{S}\|,\|X\|/z_1)\le\mathfrak{s}_p$, because $(1+\sqrt{2})/(1-9q)\ge 1/(1-2q)$. Hence $\|\mathcal{S}+X/z_1\|\le\mathfrak{s}_p+2e_E/z_1$, and the other two terms of Lemma 23.15(d) contribute $q^2/((1-q)(1-2q))$ and $18q^2/((1-q)(1-9q))$. □

Lemma 23.17 (Prices of the matching records)**.** *Let $p\ge 2^{20}$. In the comparison law each two-layer matching record E_0,E_1 of a middle block is replaced, at every prime, by the law K_1 of Lemma 23.15, and the block record is the tensor product over the primes.*

(i) On $L^2(\nu_E)$, $\|\operatorname{Sym}(K_1-K_0^E)\|\le s_p^E$ and $\|\operatorname{Skew}(K_1-K_0^E)\|\le a_p^E$. The centred operator of K_0^E is self-adjoint, of norm $k_p^E=3/(p-3)=3q/(1-3q)$ (Lemma 12.11), so the centred symmetric part of K_1 has norm at most $c_E^{\mathrm{s}}q:=k_p^E+s_p^E$ and the centred operator of K_1 norm at most $\kappa_p^E:=k_p^E+s_p^E+a_p^E$. Each of s_p^E/r, a_p^E/r, c_E^{s} and κ_p^E/r is a nondecreasing function of (q,r).

(ii) Let $\mathcal{B}$ be a block of a piece starting at $X\ge 2^{20}$, with least prime b, let $q_X=1/X$ and $r_X\ge X^{-1/2}$, and put, with every quantity of (i) at (q_X,r_X),

$$s_E(X)=\frac{s^E}{r},\qquad a_E(X)=\frac{a^E}{r}=\frac{1+2q_X}{(1-q_X)(1-2q_X)},$$

$$c_E^{\mathrm{s}}(X)=c_E^{\mathrm{s}},\qquad \kappa_E(X)=\frac{\kappa^E}{r},\qquad r_E(X)=2c_E^{\mathrm{s}}(X)a_E(X).$$

Suppose that, at (q_X, r_X), $3q/(1-3q) < 1$, $\kappa_E r < 1$, $r_E r^2 < 1$, $4/((1-q)(1-2q)) < r_E$, $\kappa_E^3 < r_E$,

$$s^E_{(2)} := \frac{6r}{1-3q} s_E + s_E^2 + a_E^2 < 2 \qquad \text{and} \qquad s^E_{(3)} := \Big[\Big(\frac{3r}{1-3q}\Big)^3 + \kappa_E^3\Big] r < 2.$$

Then the block differences $\bigotimes_{p\in\mathcal{B}} K_{1,p} - \bigotimes_{p\in\mathcal{B}} K^E_{0,p}$ of E_0 and E_1 have, on $L^2(\bigotimes_p \nu_{E,p})$, symmetric part of norm at most $s_E(X) b^{-1/2}$ and skew part of norm at most $a_E(X) b^{-1/2}$. The block law is invariant under independent maps of $\mathcal{G}_p$ at the primes of $\mathcal{B}$, accepts the six queries, and its queried pair is uniform on the pairs that are strict-square at every prime of $\mathcal{B}$.

Proof. (i) The bounds are Lemma 23.15(c) and Lemma 23.16(c); the centred operator of K_1 is that of K_0^E plus the difference, and its symmetric part is that of K_0^E plus the symmetric part of the difference. The monotonicity holds because every term is a positive expression in (q, r) with nonnegative coefficients and positive decreasing denominators, after the powers of r are cancelled ($\mathfrak{s}_p/q$, e_E/q and the last two terms of s_p^E divided by q are such expressions). (ii) For $p \in \mathcal{B}$, $q_p \le q_X$ and $r_p \le b^{-1/2} \le X^{-1/2} \le r_X$, so by (i) $s_p^E \le s_E(X) r_p$, $a_p^E \le a_E(X) r_p$, the centred symmetric part of $K_{1,p}$ has norm at most $c_E^{\mathrm{s}}(X) q_p$, the centred operator of $K_{1,p}$ norm at most $\kappa_E(X) r_p < 1$, and the centred reference $T_{0,p}$, self-adjoint, norm $k_p^E \le (3r_X/(1-3q_X)) r_p < 1$. The prime operators are Markov operators of couplings with exact marginals $\nu_{E,p}$, so by Lemma 6.10(a),(b) the block difference vanishes on the empty sector, is the prime difference on the singleton sectors, and the norms of its symmetric and skew parts are the maxima over the sectors. On a singleton $\{p\}$ they are at most $s_E(X) b^{-1/2}$ and $a_E(X) b^{-1/2}$. On a sector $\{p, p'\}$, Lemma 6.10(h) gives the symmetric part $T^\circ_{0,p} \otimes S^\circ_{p'} + S^\circ_p \otimes T^\circ_{0,p'} + S^\circ_p \otimes S^\circ_{p'} + A^\circ_p \otimes A^\circ_{p'}$, of norm at most $s^E_{(2)} r_p r_{p'} \le s^E_{(2)}/b$, and the skew part $(T_{0,p} + S_p)^\circ \otimes A^\circ_{p'} + A^\circ_p \otimes (T_{0,p'} + S_{p'})^\circ$, of norm at most $2 c_E^{\mathrm{s}}(X) a_E(X) b^{-3/2} = r_E(X) b^{-3/2}$; here S_p, A_p are the symmetric and skew parts of the prime difference. On a sector S with $|S| \ge 3$ the reference tensor is self-adjoint, so the skew part is that of the new tensor (Lemma 6.10(b)), of norm at most $(\kappa_E(X) b^{-1/2})^3 < r_E(X) b^{-3/2}$; and the symmetric part has norm at most the sum of the norms of the two tensors, $[(3r_X/(1-3q_X))^3 + \kappa_E(X)^3] b^{-3/2} \le s^E_{(3)}/b$; the further centred factors are contractions. Since $s_b^E \ge \mathfrak{s}_b \ge (1+\sqrt{2}) q_b$ and $s_b^E \le s_E(X) r_b$, the bounds $s^E_{(2)}/b, s^E_{(3)}/b < 2/b$ are below $s_E(X) b^{-1/2}$; and $r_E(X) b^{-3/2} \le r_E(X) r_X^2 b^{-1/2} < b^{-1/2} \le a_E(X) b^{-1/2}$. Invariance, acceptance and the queried pairs hold primewise by Lemma 23.15(a). □

The values of $s_E(X)$, $a_E(X)$, $r_E(X)$ at the 33 piece starts, with $r_X = 1/\lfloor\sqrt{X}\rfloor$, and the gates of (ii) are those of Certificate D.7; at Q_3 they are those of Certificate 23.14.

Lemma 23.18 (Matching one-arm stars)**.** *Let $\mathcal{B}$ be a block of a piece starting at $X \ge 2^{20}$, with least prime b, under the hypotheses of Lemma 23.17(ii), and let $D^{\mathrm{a}}_{E,\mathcal{B}}$ be the skew part of the block difference of a matching record. The matching coefficients of the piece are*

$$s_E(X) = \frac{s^E}{r}, \qquad a_E(X) = \frac{1+2q}{(1-q)(1-2q)}, \qquad r_E(X) = 2\Big(\frac{3}{1-3q} + \frac{s^E}{q}\Big) a_E(X),$$

at $(q, r) = (q_X, r_X)$, with $s^E = s_p^E$ of Lemma 23.16(c) there. Then $D^{\mathrm{a}}_{E,\mathcal{B}} = L_{E,\mathcal{B}} + R_{E,\mathcal{B}}$, where $L_{E,\mathcal{B}}$ equals $L_{E,p} = \mathsf{C} + \mathsf{D} = V_1 P V_0^* - V_0 (V_1 P)^*$ on the sector of exact support $\{p\}$, for $p \in \mathcal{B}$, and zero on every other sector. The leading term is a one-arm star with centre $\mathcal{H}_0$ and arm in $\mathcal{H}_1$, of arm norm at most $b^{-1/2}$ and norm at most $b^{-1/2}/(1-b^{-1})$; and $\|R_{E,\mathcal{B}}\| \le r_E(X) b^{-3/2}$, where $r_E(X) \ge 2(4+\sqrt{2})$, and $r_E(X) < 11$ if $r_X \le 2^{-10}$, for instance for $r_X = 1/\lfloor\sqrt{X}\rfloor$.

Proof. At a prime, Lemma 23.15(c) gives $D^{\mathrm{a}}_{E,p} - L_{E,p} = (z_1^{-1} - 1)(\mathsf{C} + \mathsf{D}) - (\delta_{\mathsf{C}} \mathsf{D} + \delta_{\mathsf{D}} \mathsf{C})/z_1$, and $1 - 2q \le z_1 \le 1$, so $\|D^{\mathrm{a}}_{E,p} - L_{E,p}\| \le \frac{r}{1-q} \cdot \frac{2q}{1-2q} + \frac{2qr}{(1-q)(1-2q)} = \frac{4qr}{(1-q)(1-2q)}$, which is below $r_E(X) b^{-3/2}$

by the gate $4/((1-q)(1-2q)) < r_E$ and monotonicity. On the sectors of size at least two the skew part of the block difference is at most $r_E(X)b^{-3/2}$ by the proof of Lemma 23.17(ii), and $L_{E,\mathcal{B}}$ vanishes there. The sectors are orthogonal, which gives the bound on $R_{E,\mathcal{B}}$. The arm V_1P has norm $r_p \le b^{-1/2}$, and $\|L_{E,p}\| = r_p/(1-q_p) \le b^{-1/2}/(1-b^{-1})$ (Lemma 23.15(c)); $L_{E,\mathcal{B}}$ is the orthogonal sum of the $L_{E,p}$ over the singleton sectors. Finally $3/(1-3q) \ge 3$, $s^E/q \ge \mathfrak{s}_p/q \ge 1+\sqrt{2}$ and $a_E \ge 1$ give $r_E \ge 2(4+\sqrt{2})$; and r_E is a nondecreasing function of (q,r) whose value at $(2^{-20}, 2^{-10})$ is $10.834\ldots < 11$ by outward evaluation of its formula, so $r_E(X) < 11$ when $r_X \le 2^{-10}$. □

23.8. **Merges replaced at fixed marginals, and deleted records.** The six two-layer merge records of a middle block have weight $w_{\mathrm{R}}r^2$ each. We replace them by positive laws which accept the six queries and preserve their actual half marginals. Four records use a pair of mirror edges; the other two use one edge with the unused labels drawn conditionally from their original half laws. The mirror construction cancels the leading symmetric error. The two-layer constant records also admit an accepting replacement, so only their folded counterparts are deleted.

Definition 23.19 (Merge records and their designated edges)**.** Let $E_0 = (0,1,1,2,2,0)$ and $E_1 = (0,0,1,1,2,2)$ be the matching patterns and, for $s \in \{0,1\}$ and $k \in \{0,1,2\}$, let $G_{sk}(j) = \mathbf{1}[E_s(j) = k]$ be the merge patterns, with objects $0,1$ (Definition 17.1). At a prime $p \equiv 3 \pmod 4$, the two-layer merge record π_{sk} is uniform on injective four-label tuples (x_0, x_1, y_0, y_1), where x_a and y_a are the labels of object a in layers zero and one. At the reserved cut both half marginals are the uniform law m_p on ordered distinct pairs. For this pattern G_{sk}, every query of $\mathcal{E}_6$ reads the same designated pair $x_{G_{sk}(0)} \to y_{G_{sk}(2)}$, determined by G_{sk}.

For $(s,k) \in \{(0,0),(0,1),(1,0),(1,1)\}$ replace π_{sk} by the uniform injective four-label law satisfying the two mirror edges $x_1 \to y_0$ and $x_0 \to y_1$. For $(s,k) \in \{(0,2),(1,2)\}$ use instead the law with density

$$\frac{2}{1-1/p}\mathbf{1}[x_0 \to y_0] \quad \text{relative to } m_p \otimes m_p.$$

The latter law allows coincidences between the two halves, but each half remains an ordered distinct pair. Block replacements are products of these laws over the primes of the block.

record	pattern	designated $\mathcal{E}_6$ edge	replacement
G_{00}	$(1,0,0,0,0,1)$	$x_1 \to y_0$	both mirror edges
G_{01}	$(0,1,1,0,0,0)$	$x_0 \to y_1$	both mirror edges
G_{02}	$(0,0,0,1,1,0)$	$x_0 \to y_0$	first-coordinate edge
G_{10}	$(1,1,0,0,0,0)$	$x_1 \to y_0$	both mirror edges
G_{11}	$(0,0,1,1,0,0)$	$x_0 \to y_1$	both mirror edges
G_{12}	$(0,0,0,0,1,1)$	$x_0 \to y_0$	first-coordinate edge

TABLE 11. The six merge records and their accepting replacements. Each replacement has exactly the original ordered-distinct-pair half laws.

Lemma 23.20 (Exact merge repairs)**.** *The replacements of Definition 23.19 are positive probability laws with both actual half marginals equal to m_p. They are invariant under $\mathcal{G}_p$, accept every query of $\mathcal{E}_6$, and have uniform queried pairs on the ordered strict-square pairs.*

Write D_p for the old Markov operator on $L^2(m_p)$, and P_p^{c}, P_p^{s} for the mirror-edge and first-coordinate replacements. For $p \geq 2^{24}$ put

$$d_p = \frac{2}{p-2}, \qquad \eta_p^{\mathrm{G}} = \frac{1}{p-1}, \qquad \gamma_p^{\mathrm{G}} = \frac{\sqrt{p}}{p-1}, \qquad \rho_p = \frac{\sqrt{p+1}}{p-1},$$
$$s_p^{\mathrm{G}} = \frac{5p-6}{(p-2)(p-3)}, \qquad a_p^{\mathrm{G}} = \frac{p^{-1/2}[1+2p^{-1/2}(4-6/p)]}{(1-2/p)(1-3/p)}.$$

For a block $\mathcal{B}$ with least prime b, the differences between the product replacement and the product source have symmetric and skew norms at most

	symmetric	*skew*
mirror-edge replacement	s_b	a_b
first-coordinate replacement	$d_b + \eta_b$	γ_b.

All norms are relative to the product of the actual half laws m_p.

Proof. Half marginals and acceptance. Fix $x_0 \neq x_1$ and put $k = (p-1)/2$. Among labels different from both x_0, x_1, the two allowed out-neighbour sets for y_0, y_1 have sizes k and $k-1$ in some order. Their intersection has size $(p-3)/4$. Thus the number of admissible distinct pairs (y_0, y_1) is

$$k(k-1) - (p-3)/4 = (p-2)(p-3)/4,$$

independent of the fixed half. The incoming-neighbour count is identical, so the mirror law has both marginals m_p. The first-coordinate law has the same marginals because each coordinate of m_p is uniform and a strict Paley edge has uniform endpoints. Each designated edge in Table 11 holds under its replacement. Square-affine invariance and transitivity on ordered strict-square pairs give the claimed queried-pair law. The construction uses only $\mathcal{E}_6$.

The source and the first-coordinate law. The source operator averages over ordered pairs disjoint from its input pair. Its eigenvalues are 1 on constants, zero on antisymmetric pair functions, $-2/(p-2)$ on $u(x_0) + u(x_1)$ with $\mathbb{E}u = 0$, and $2/((p-2)(p-3))$ on symmetric functions with zero row sums. These mutually orthogonal spaces exhaust $L^2(m_p)$: summing a function over all pairs and subtracting the pairs meeting the input gives the displayed values. In particular $\|D_p - \mathbb{J}\| = d_p$.

Let V_i embed a singleton function as a function of coordinate i of m_p. The normalized strict Paley operator P has symmetric part $\mathbb{J} - \eta_p^{\mathrm{G}}(I - \mathbb{J})$, skew norm γ_p^{G}, and centered norm ρ_p. Since $P_p^{\mathrm{s}} = V_0 P V_0^*$, its difference from D_p has symmetric norm at most $d_p + \eta_p^{\mathrm{G}}$, skew norm γ_p^{G}, and centered replacement norm ρ_p.

The mirror law. Write $C = \chi(y_0 - x_1)$ and $H = \chi(y_1 - x_0)$. On the injective source, P_p^{c} has density $(1+C)(1+H)$ relative to D_p. Transposition exchanges C, H with $-H, -C$, respectively. Hence, as identities of kernels relative to $m_p \otimes m_p$,

$$\mathrm{Sym}(P_p^{\mathrm{c}} - D_p) = D_p CH, \qquad \mathrm{Skew}(P_p^{\mathrm{c}} - D_p) = D_p(C+H), \tag{23.6}$$

where the products on the right are pointwise products of densities. Put $z = (p-2)(p-3)/(p(p-1))$ and $\tau = 1 - z$. Then D_p is the indicator of disjoint halves divided by z, and the deleted collision kernel has constant row and column mass τ. On independent field pairs the two-edge character kernel CH has norm $1/p$; restricting both halves to distinct pairs costs at most $(1-1/p)^{-1}$. Schur's bound on the collision kernel gives

$$\|D_p CH\| \leq \frac{1/(p-1) + \tau}{z} = \frac{5p-6}{(p-2)(p-3)} = s_p^{\mathrm{G}}.$$

For the linear term, the singleton embeddings on mean-zero functions satisfy $V_i^* V_i = I$ and $V_0^* V_1 = V_1^* V_0 = -\eta_p^{\mathrm{G}} I$. If $Kf(x) = \mathbb{E}_y \chi(y-x) f(y)$, then $C + H = V_1 K V_0^* + V_0 K V_1^*$. Orthogonalizing

V_0+V_1 and V_0-V_1 gives the two blocks $(1-\eta_p^{\mathrm{G}})K$ and $-(1+\eta_p^{\mathrm{G}})K$, so the norm is $(1+\eta_p^{\mathrm{G}})/\sqrt{p}=\gamma_p^{\mathrm{G}}$. The collision part has norm at most 2τ. Consequently

$$\|D_p(C+H)\| \le \frac{\gamma_p^{\mathrm{G}}+2\tau}{z} = a_p^{\mathrm{G}}, \qquad \|P_p^{\mathrm{c}}-\mathbb{J}\| \le \alpha_p^{\mathrm{G}} := d_p + s_p^{\mathrm{G}} + a_p^{\mathrm{G}}.$$

Products over primes. Every operator is Markov in the same actual half law, so it preserves the orthogonal decomposition by exact nonconstant prime support. For $p \ge 2^{24}$ the displayed formulas imply

$$\alpha_p^{\mathrm{G}} \le 2p^{-1/2}, \quad d_p \le p^{-1/2}, \quad s_p^{\mathrm{G}} \ge 5/p, \quad a_p^{\mathrm{G}} \ge 4/p, \qquad \rho_p^2 + d_p^2 \le d_p + \eta_p^{\mathrm{G}}, \quad \rho_p^2 \le \gamma_p^{\mathrm{G}}.$$

For example, the first two follow by substituting $r \le 2^{-12}$ in their positive rational majorants; the other four follow directly after clearing denominators. All the bounds decrease with p. On singleton supports use the prime bounds. On a support of size at least two, the mirror difference has symmetric norm at most $\alpha_b^2 + d_b^2 \le 5/b \le s_b$, and skew norm at most $\alpha_b^2 \le 4/b \le a_b$, since the source is self-adjoint. Further support factors are contractions. For the first-coordinate difference the corresponding bounds are $\rho_b^2 + d_b^2$ and ρ_b^2. Taking the maximum over the orthogonal supports proves the block claims. □

Corollary 23.21 (The six merge prices)**.** *For a middle piece starting at X, set $q_X = 1/X$, $r_X = 1/\lfloor\sqrt{X}\rfloor$, and*

$$g_{\mathrm{S}}(X) = \frac{20}{(1-2q_X)(1-3q_X)} + \frac{4}{1-2q_X} + \frac{2}{1-q_X},$$
$$g_{\mathrm{A}}(X) = \frac{4(1+8r_X)}{(1-2q_X)(1-3q_X)} + \frac{2}{1-q_X}.$$

On a block of that piece, with least prime b and $r = b^{-1/2}$, the six merge changes have total symmetric and skew prices at most

$$w_{\mathrm{R}} g_{\mathrm{S}}(X) r^4 q_{\mathcal{B}}^{2a}, \qquad w_{\mathrm{R}} g_{\mathrm{A}}(X) r^3 q_{\mathcal{B}}^{2a},$$

respectively. Their norm envelopes are positive reserved diagonals of their actual ordered-distinct-pair half laws.

Proof. There are four mirror and two first-coordinate records, each of mass $w_{\mathrm{R}} r^2$ before the common normalization. In the formulas of Lemma 23.20, drop the negative numerator corrections and use $1/b \le q_X$, $b^{-1/2} \le r_X$. Summing gives the displayed coefficients. The charging rule of Remark 9.15(ii) applies relative to the unchanged actual half marginals. Omitting the division by the reference normalizer (a number ≥ 1, so its reciprocal is ≤ 1) only enlarges the prices. □

Certificate 23.22 (The merge coefficients)**.** The coefficients $g_{\mathrm{S}}(X)$ and $g_{\mathrm{A}}(X)$ are the explicit positive rational functions of Corollary 23.21; no graph-count approximation is needed to define them. At $X = Q_3 = 19951586$ they are, respectively, $26.0000055133\ldots$ and $6.0071663530\ldots$, and they decrease to 26 and 6. The exact checker `research/four_routes_2026_10_26/pair_transfer/actual_g_mirror_check.py` verifies the rational endpoint gates at $p = 2^{24}$ and the physical marginals, designated edges and mirror identities by exact enumeration at $p = 7, 11, 19$. It also checks the source spectral polynomial at $p = 7, 11$. The uniform proof and the operator identities were independently reviewed in `research/four_routes_2026_10_26/folded_states/INDEPENDENT_ACTUAL_G_MIRROR_REVIEW.md`. The original source weights and reference laws are unchanged; these coefficients replace only the merge comparison prices.

Lemma 23.23 (Accepting two-layer constants)**.** *Each of the two two-layer constant records of a block can be replaced by a positive accepting law with the same actual half marginals. If $b \ge 7$ is*

the least prime (as for every middle block) and $r = b^{-1/2}$, the symmetric and skew norms of the normalized replacement minus the source are at most

$$\frac{r^2}{(1-1/b)^2}, \qquad \frac{r}{1-1/b},$$

respectively. For a piece starting at X, the two records together have symmetric and skew prices at most

$$\frac{2w_{\mathrm{C}}}{(1-1/X)^2} r^5 q_{\mathcal{B}}^{2a}, \qquad \frac{2w_{\mathrm{C}}}{1-1/X} r^4 q_{\mathcal{B}}^{2a}.$$

Both norm envelopes are uniform one-label reserved diagonals.

Proof. Write (a_p, b_p) for the labels of the single object of either constant record in layers 0 and 1 at the prime p (the labels seen at the reserved cut). They are uniform ordered distinct labels: this is the one-object marginal of the original injective matching law. All six queries read $a_p \to b_p$. Replace this law by the uniform strict Paley edge, independently at every prime. It is square-affine invariant, accepts all six queries and has the same uniform endpoint marginals. The two records give the same physical law, but both of their weights are retained.

On uniform field L^2, the old and new operators are $D_p = \mathbb{J} - \eta_p(I - \mathbb{J})$ and $P_p = D_p + \frac{p}{p-1} K_p$, where $K_p f(a) = \mathbb{E}_b \chi(b-a) f(b)$, $K_p^* = -K_p$ and $K_p^* K_p = p^{-1}(I - \mathbb{J})$. They commute and are normal. On a nonconstant complex Fourier mode their eigenvalues are $-\eta_p$ and $-\eta_p \pm i\gamma_p$. The singleton-support difference is therefore purely skew, of norm γ_p. On a support of exactly two primes the symmetric difference is $\pm\gamma_p\gamma_{p'}$. On a support of size at least three its symmetric norm is at most $\rho_b^3 + \eta_b^3 \le \gamma_b^2$; the last inequality follows from $(b+1)^{3/2} + 1 \le b(b-1)$ for $b \ge 7$. On every support of size at least two the skew norm is at most $\rho_b^2 \le \gamma_b$. Constants cancel. These exact-support bounds prove the asserted whole-block norms without a factor for the number of primes. Each original record has mass $w_{\mathrm{C}} r^3$ before normalization, so the two norm prices follow from Remark 9.15(ii). The comparison replacement changes neither the reference nor its first stage or Gram bracket. □

Deletions. The comparison law deletes the six folded merge records ($6v_{\mathrm{R}} r^3$), the polarization correction G of the first stage ($2v_{\mathrm{D}}(f_1 + h_1) r^3$), and the two folded constant records ($2v_{\mathrm{C}} r^4$). These are positive records, symmetric at ι (the folded ones reserved diagonal), and a deleted positive record is charged its mass to the symmetric part only (Remark 9.15(i)). The two-layer constant records are replaced as in Lemma 23.23 and are not included in these deletions.

23.9. **The comparison law of a middle block and its prices.** On a middle block $\mathcal{B}$ the comparison law is

$$\nu_{\mathcal{B}} = \lambda_{\mathcal{B}} + \sum_{t \in \mathcal{T}_{\mathcal{B}}} \zeta_t,$$

where ζ_t is the replacement of the record t minus t, or minus t if t is deleted, each with its weight in $\lambda_{\mathcal{B}}$, divided by the normalizing mass $\widetilde{\mu}_{\mathcal{B}}(1)$ of the reference (Lemma 22.1). The records $t \in \mathcal{T}_{\mathcal{B}}$ and their replacements are: the two two-layer fold records F_1, F_2 (weight $w_{\mathrm{D}} r$), replaced by the signed records of Definition 23.9(c),(d); the two two-layer matching records (weight $w_{\mathrm{S}} r$), replaced by the block laws of Lemma 23.17; the six two-layer merge records (weight $w_{\mathrm{R}} r^2$), replaced by the repairs of Corollary 23.21; the two two-layer constant records (weight $w_{\mathrm{C}} r^3$), replaced by Lemma 23.23; and the deleted records listed above. The first stage changes only one-layer folded records (Definition 22.4), so the two-layer records enter $\lambda_{\mathcal{B}}$ with the weights of $\mu_{\mathcal{B}}$. The retained records of $\nu_{\mathcal{B}}$ are the two-layer root and F_0, the folded root and the folded F_0, the record E_{new} of the first stage, and the replacements; each accepts the six queries: the root and F_0 contain the edge $0 \to 2$ of the first coordinate in both layers, and so do their diagonal copies (on the pieces from 2^{30} the root is the positive root of Section 21, which has the support of the uniform root and

accepts the six queries, Lemma 21.2, Section 21.9, and Certificate 21.25); E_{new} by Lemma 22.6; the replacements by Lemmas 23.7, 23.17 and 23.23 and Corollary 23.21. The comparison law of the block is a signed measure,

$$\nu_{\mathcal{B}} = \nu^{+}_{\mathcal{B}} - \frac{\mathtt{t}}{2}\, w\, \Pi^{--}_{F_1,\mathcal{B}} - \frac{\mathtt{t}}{2}\, w\, \Pi^{--}_{F_2,\mathcal{B}}, \qquad w = \frac{w_{\mathrm{D}} r}{\widetilde{\mu}_{\mathcal{B}}(1)},$$

where $\Pi^{--}_{F_1,\mathcal{B}}$ and $\Pi^{--}_{F_2,\mathcal{B}}$ are the probability records placed by F_1 and F_2 with the laws $\bigotimes_p P_{--,p}$ and $\bigotimes_p P^{\varsigma}_{--,p}$, and $\nu^{+}_{\mathcal{B}} \geq 0$ consists of all other records of $\nu_{\mathcal{B}}$, among them the components $\bigotimes_p P_{++,p}$ and $\frac{\mathtt{t}}{2}\bigotimes_p P_{-+,p}$ of the two signed records.

Lemma 23.24 (Comparison prices of a middle block)**.** *Let $\mathcal{B}$ be a block of the middle piece i starting at X, with least prime $b \geq X$ and $r = b^{-1/2}$, and evaluate the constants of* (23.5) *and the matching coefficients of Lemma 23.17(ii) at the piece start, at (q_X, r_X) with $q_X = 1/X$ and $r_X = 1/\lfloor\sqrt{X}\rfloor$. Then $\nu_{\mathcal{B}}$ is a finite real combination of the three nonnegative measures $\nu^{+}_{\mathcal{B}}$, $\Pi^{--}_{F_1,\mathcal{B}}$ and $\Pi^{--}_{F_2,\mathcal{B}}$, each invariant under independent square-affine maps at the primes of $\mathcal{B}$ and accepting the six queries, with $\nu_{\mathcal{B}}(1) > 0$ and total variation $|\nu|_{\mathcal{B}} \leq \nu_{\mathcal{B}}(1) + 2\mathtt{t}w$; it satisfies (a) of Definition 9.14, and its changes satisfy (b) there with norm envelopes* Diag_{ι} *of their actual half marginals and prices*

$$s_{\mathcal{B}} \leq (S_2 r^2 + S_3 r^3 + S_4 r^4 + S_5 r^5) q_{\mathcal{B}}^{2a}, \qquad a_{\mathcal{B}} \leq (A_2 r^2 + A_3 r^3 + A_4 r^4) q_{\mathcal{B}}^{2a},$$

where

$$\begin{aligned} S_2 &= 2w_{\mathrm{S}} s_E, & A_2 &= 2w_{\mathrm{D}}\ell_{\mathtt{t}} + 2w_{\mathrm{S}} a_E,\\ S_3 &= 2w_{\mathrm{D}} s_F + 6v_{\mathrm{R}} + 2v_{\mathrm{D}}(f_1 + h_1), & A_3 &= 2w_{\mathrm{D}} a_F + w_{\mathrm{R}} g_{\mathrm{A}},\\ S_4 &= 2v_{\mathrm{C}} + w_{\mathrm{R}} g_{\mathrm{S}}, & A_4 &= \frac{2w_{\mathrm{C}}}{1 - q_X},\\ S_5 &= \frac{2w_{\mathrm{C}}}{(1-q_X)^2}. \end{aligned}$$

with $s_F, a_F, \ell_{\mathtt{t}}$ of (23.5) *(Proposition 23.13), $s_E = s_E(X)$, $a_E = a_E(X)$ of Lemma 23.17(ii), and $g_{\mathrm{S}} = g_{\mathrm{S}}(X)$, $g_{\mathrm{A}} = g_{\mathrm{A}}(X)$ of Corollary 23.21. The auxiliary prices (Definition 9.16) of the skew change of each of the two fold records are, times $q_{\mathcal{B}}^{2a}$, the raw-arm price $w_{\mathrm{D}} r^2$, the leading price $w_{\mathrm{D}}\ell_{\mathtt{t}} r^2$ and the remainder price $w_{\mathrm{D}} a_F r^3$; those of each of the two matching records are $w_{\mathrm{S}} r^2$, $w_{\mathrm{S}} r^2/(1 - q_X)$ and $w_{\mathrm{S}} r_E r^4$, with $r_E = r_E(X)$ of Lemma 23.18. Summed over the two records of each kind they are $2w_{\mathrm{D}} r^2$, $2w_{\mathrm{D}}\ell_{\mathtt{t}} r^2$, $2w_{\mathrm{D}} a_F r^3$ and $2w_{\mathrm{S}} r^2$, $2w_{\mathrm{S}} r^2/(1 - q_X)$, $2w_{\mathrm{S}} r_E r^4$. Every selected law of these changes is admissible (Definition 8.9).*

Proof. Dividing by $\widetilde{\mu}_{\mathcal{B}}(1) \geq 1$ only decreases the weights, so it suffices to charge the unnormalized ones. *Decomposition and (a).* The records of $\lambda_{\mathcal{B}}$ that the comparison law keeps, the replacements of the matching and merge records, the accepting two-layer constants, and the components $\bigotimes_p P_{++,p}$, $\frac{\mathtt{t}}{2}\bigotimes_p P_{-+,p}$ of the two signed records are nonnegative measures; their sum is $\nu^{+}_{\mathcal{B}}$, and $\nu_{\mathcal{B}}(1) = \nu^{+}_{\mathcal{B}}(1) - \mathtt{t}w$, which gives the total variation. The two-layer root has weight one in $\widetilde{\mu}_{\mathcal{B}}$ and is retained, so $\nu^{+}_{\mathcal{B}}(1) \geq 1/\widetilde{\mu}_{\mathcal{B}}(1)$ and $\nu_{\mathcal{B}}(1) \geq (1 - \mathtt{t}w_{\mathrm{D}} r)/\widetilde{\mu}_{\mathcal{B}}(1) > 0$, as $w_{\mathrm{D}} < 6$ (Remark 22.2, Table 26) and $r < 2^{-12}$. Every record and component is invariant under independent square-affine maps at the primes of $\mathcal{B}$ (Lemma 23.7(a),(d), Lemma 23.15(a), Corollary 23.21 and Lemma 23.23; for the unchanged records Lemma 22.1 and Lemma 22.6), so its one-slot marginals are uniform and its queried pairs are its mass times the uniform law on strict-square pairs; and each accepts the six queries, as listed above. This is (a) and the clause (R5) for each of the three components. *Prices.* Each replacement has the same actual half marginals on both halves of ι as the record it replaces, so Remark 9.15(ii) charges it its weight times the norms of the symmetric and skew parts of the difference; for the signed records the proof of (ii) uses only that the new record has mass one

and the half marginals of the old one, not that it is nonnegative. The two fold records cost at most $w_{\mathrm{D}} r \cdot s_F r^2$ and $w_{\mathrm{D}} r \cdot (\ell_{\mathrm{t}} r + a_F r^2)$ each (Proposition 23.13(a),(b),(e)), the two matching records $w_{\mathrm{S}} r \cdot s_E r$ and $w_{\mathrm{S}} r \cdot a_E r$ each (Lemma 23.17(ii)), and the six merge records together $w_{\mathrm{R}} g_{\mathrm{S}} r^4$ to the symmetric part and $w_{\mathrm{R}} g_{\mathrm{A}} r^3$ to the skew part (Corollary 23.21). The two-layer constants add $S_5 r^5$ and $A_4 r^4$, respectively (Lemma 23.23). The deleted records are charged their masses to the symmetric part (Remark 9.15(i)). The skew change of a fold record splits into its leading three-arm star, of norm at most $\ell_{\mathrm{t}} r$ with arms of norm at most r (Proposition 23.13(b), Lemma 23.4(e)), and a remainder of norm at most $a_F r^2$; that of a matching record into its one-arm star, of norm at most $r/(1-q_X)$ with arm norm at most r, and a remainder of norm at most $r_E r^3$ (Lemma 23.18); times the weights these are the auxiliary prices. The coefficients are evaluated at the piece start and bound the norms at every block of the piece (Lemma 4.18, the constants being nondecreasing in (q, r)), and the weights and the row constants f_1, h_1 are those of the piece (Lemma 20.3(iii)). *Admissibility.* The changes are invariant signed measures on the field labels only, independent of the indices, so (O1) holds; and the only operator bounds applied to them are their own norms at ι, with no shared label, so (O2) holds. □

Summing over the blocks of each piece with the charges of Lemma 20.4 and over the pieces gives the middle parts of S and A and the totals of the auxiliary prices directly. In the piece i, with start X_i, weights $w_{\mathrm{D},i} = w_{\mathrm{S},i}$, the coefficients of Lemma 23.24 at X_i, and the certified degree charges $\bar I_{d,i} \ge \sum_{\mathcal B} b_{\mathcal B}^{-d/2} q_{\mathcal B}^{2a_0}$ of Table 24, the prices at the anchor a_0 are

$$
\begin{aligned}
&A^{\mathrm{arm}}_{F,i} = \mathrm{up}(2w_{\mathrm{D},i}\bar I_{2,i}), \quad A^{\mathrm{lead}}_{F,i} = \mathrm{up}(2w_{\mathrm{D},i}\ell_{\mathrm{t}}(X_i)\bar I_{2,i}), \quad A^{\mathrm{rem}}_{F,i} = \mathrm{up}(2w_{\mathrm{D},i}a_F(X_i)\bar I_{3,i}),\\
&S_{F,i} = \mathrm{up}(2w_{\mathrm{D},i}s_F(X_i)\bar I_{3,i}), \qquad A_{F,i} = A^{\mathrm{lead}}_{F,i} + A^{\mathrm{rem}}_{F,i},\\
&A^{\mathrm{arm}}_{E,i} = \mathrm{up}(2w_{\mathrm{S},i}\bar I_{2,i}), \quad A^{\mathrm{lead}}_{E,i} = \mathrm{up}\Big(\frac{2w_{\mathrm{S},i}\bar I_{2,i}}{1-q_{X_i}}\Big), \quad A^{\mathrm{rem}}_{E,i} = \mathrm{up}(2w_{\mathrm{S},i}r_E(X_i)\bar I_{4,i}),\\
&A_{E,i} = 2\,\mathrm{up}(w_{\mathrm{S},i}a_E(X_i)\bar I_{2,i}), \qquad S_{E,i} = \mathrm{up}(2w_{\mathrm{S},i}s_E(X_i)\bar I_{2,i}),\\
&S_{\mathrm{oth},i} = \sigma_{s,i} + S^{\mathrm{oth}}_{3,i}\bar I_{3,i} + S_{4,i}\bar I_{4,i} + S_{5,i}\bar I_{3,i}/X_i,\\
&A_{\mathrm{oth},i} = \sigma_{a,i} + w_{\mathrm{R},i}g_{\mathrm{A}}(X_i)\bar I_{3,i} + A_{4,i}\bar I_{4,i},
\end{aligned}
\tag{23.7}
$$

with $\mathrm{up}(x) = 10^{-30}\lceil 10^{30}x\rceil$ and $S_3^{\mathrm{oth}} = S_3 - 2w_{\mathrm{D}}s_F = 6v_{\mathrm{R}} + 2v_{\mathrm{D}}(f_1+h_1)$. Here $\sigma_{s,i}, \sigma_{a,i} \in [0, 10^{-30})$ are the rounding slacks of the corresponding merge and two-layer-constant rows of the authenticated middle-price source named in Definition 30.1(3): each is its stored upward rounding minus its exact unrounded expression. Keeping these slacks makes the update an exact rational replacement of the two-layer constant and merge contributions as stored there. The degree-five charge needs no new prime enumeration, since $b^{-5/2} \le X_i^{-1} b^{-3/2}$. The totals at the exponent a are t times the sums over the 33 pieces, $t = e^{2(a-a_0)L}$ with $L = 570$ (Lemma 9.3, Definition 30.1): $A_F^{\mathrm{arm}} = t\sum_i A_{F,i}^{\mathrm{arm}}$ and so on for A_F^{lead}, A_F^{rem}, A_F, S_F, A_E^{arm}, A_E^{lead}, A_E^{rem}, A_E, S_E, S_{oth} and A_{oth}, with the middle totals $S_{\mathrm{mid}} = S_F + S_E + S_{\mathrm{oth}}$ and $A_{\mathrm{mid}} = A_F + A_E + A_{\mathrm{oth}}$. The merge and constant replacements have different symmetric and skew prices, as displayed in (23.7). The full skew price A_E of the matching records is bounded directly; A_F is the sum of the leading and remainder prices, and $A_F^{\mathrm{lead}} \ge A_F^{\mathrm{arm}}$, $A_E \ge A_E^{\mathrm{arm}}$ (Certificate 23.14). At the density-linear baseline exponent $a = \frac{5422587}{6103515625}$, the following conservative upper totals remain valid: the generic symmetric and skew prices $S_{\mathrm{oth},i}$ and $A_{\mathrm{oth},i}$ in (23.7) are each no larger than the corresponding generic rows of Definition 30.1(3). The totals at a, which are the baseline totals used in Section 30, are $A_F^{\mathrm{arm}} = A_E^{\mathrm{arm}} < 0.42332183839047$, $A_F^{\mathrm{lead}} < 0.53897233394172$, $A_F^{\mathrm{rem}} < 0.00005658950959$, $A_F < 0.53902892345131$, $A_E^{\mathrm{lead}} < 0.42332183982722$, $A_E^{\mathrm{rem}} < 1.436778E-8$, $A_E < 0.42332184557419$, $S_F < 0.00013191008864$, $S_E < 0.00003092662078$, $S_{\mathrm{oth}} < 0.00052581574396$, $A_{\mathrm{oth}} < 0.00023870593198$, $S_{\mathrm{mid}} < 0.00068865245337$ and $A_{\mathrm{mid}} <$

0.96258947495747 (`research/push_2026_10_25/artifacts/author.json`, key `raw_totals`; Table 29). The final construction uses the smaller prices in (23.7) and also retains the common reference normalization; its complete new totals are given in Section 33.

Consequences and what to read next. From Part II: Lemma 4.18 (evaluation at the piece start), Lemma 6.10 (sectors of block differences, Markov tensor maximum, the closing clause of (b) and the two-support step (h)) and Remark 6.12 (signed sums of products), Definition 8.9, and the Weil bound Theorem 4.7. From Part III: the signed comparison laws of Definition 9.14, the auxiliary prices of Definition 9.16, the charging rules Remark 9.15, the energy toolkit Lemma 10.13 (for Corollary 23.3(d)), Lemma 9.3. From Part IV: Lemma 17.2 (the records at ι and their half marginals), Lemma 12.10 (Paley counts, the Paley operator, directed triples and their features), Lemma 14.2 (invariant functions), Lemmas 12.2 and 12.5 (count errors), Lemma 12.11 (the disjoint-triple operator), Proposition 13.16 and Corollary 13.17 (correlations of the merge graph laws), Lemmas 13.21 and 13.24 (actual marginals, projected kernels), and the repair package Lemma 17.14 and Corollary 17.17. From Sections 20 and 22: the reference (Lemma 22.1) and its weights (Remark 22.2), the frame (Definition 22.4), the block charges, the local domain and the recorded F21 caps (Lemma 20.12); from Section 21, the support of the positive root. Data and certificates: Certificates D.6, D.7, 23.10, 23.14 and 23.22; the totals of Table 29.

Hypotheses verified in this section (Table 5). (CP4) for the middle atoms, that is, the signed decomposition of $\nu_{\mathcal{B}}$ into nonnegative accepting components with uniform one-slot marginals, its variation and (R5), and the brackets of every change with the prices and auxiliary prices of Lemma 23.24; the negative cap $k_{\mathrm{D}} = k_{\mathrm{D}}(X, L)$ used in (CP1) for the middle atoms (Lemma 23.2(e)); the centred norm κ_F used by the variance credits (Corollary 23.3); the direct-Gram coefficient d_{t} (Proposition 23.13(d)); and the admissibility clause (CP8)(a) for the changes of the middle records.

24. The variance identity: savings D_{var} from fixed marginals

What this section proves and how to read it. When a replacement keeps the same one-half marginals, its energy against N equals a diagonal energy minus a product energy (a variance). The trivial bound keeps only the diagonal ($\le BC_XY$); this section bounds the product energy below by gC_XY, giving the improved class bound $B - g$ allowed by (CP5). The total saving D_{var} is subtracted in Θ as $(63/64)D$ (a *saving*, called a *credit* in the text; the word *coupon* below means one proved term in a single coefficient: a subtracted saving g for price coefficients, or an added energy lower bound for the mean constant m). Four records per middle block are treated (*eligible* means: replaced at fixed marginals and annihilating constants). Reading-order note: the final D_{var} in Corollary 24.18 uses dominator caps from Section 25, and Section 25 in turn uses the variance savings for aligned classes; the per-block inequalities here do not depend on Section 25, only the final assembly does—read the two sections as: per-block identities first, assembly last. Inputs: Section 23. Output: $D_{\mathrm{var}} > 0.0099$ (baseline).

The lower bounds are not positive semidefinite orders at ι. They restore the product energy from the root of the reference in four steps. (A) At the chamber cut of the fold that produces the record, the actual half law of the root and the law induced by the record are mutually absolutely continuous, with a pointwise density bound and a chi-square bound (Lemma 24.7; for the positive roots of the late pieces, Lemma 24.7(iv)); the chi-square bound rests on the second moment of a neighbourhood count, which a modular evaluation of a character sum makes small (Lemma 24.3). (B) The root is a probability coupling at that cut, so its energy is at most that of its own half diagonal; step (A) compares that diagonal with the record. (C) The reference is a positive mixture of the root and of laws of controlled energy, so a stated positive fraction of Y is recovered from the root energy (Lemma 24.8). (D) At the reserved cut the record's own Markov operator, in its own actual marginal space, has centred norm $k_\phi < 1$ (Section 24.4), i.e. it is a contraction towards

the projection onto constants. The two matching records are not folds; they are reached from the two fold records by a polarization at the chamber cut ϑ_0 (Lemmas 24.10 and 24.11). Lemma 24.12 turns the four steps into a gain. For a selected matching product the exterior bracket adds positive background terms, which a joint restoration bounds below (Proposition 24.17). The energies $Y[\kappa]$, $Y^{\mathsf{N}}[\kappa]$, $Y^{\mathsf{N}_0}[\kappa]$ of selected laws, the price $C_X = \prod_{\mathcal{B}\in X} q_{\mathcal{B}}^{2a}$, the diagonal copies Diag_ϑ, the product laws J_ϑ and the relations (E1)–(E3), (P1)–(P4) and (P3′) are those of Section 10 (Lemma 10.13); $Y = Y_\mu$.

24.1. **The four records treated here.** The variance saving applies to only four records of a middle block: the two nonaligned folds F_1, F_2 and the two matchings E_0, E_1, called the *eligible* records. Each is replaced by a law with exactly the same actual half marginals at ι, so the difference annihilates constants and part of its price can be recovered as a variance, as the following subsections show. This subsection fixes the four laws, their actual half marginals, and the two comparison laws—the diagonal copy and the product of the marginals—between which the variance is measured.

Let $\mathcal{B}$ be a middle block (Definition 20.1) with least prime $b = b_{\mathcal{B}}$, and put $r = r_{\mathcal{B}} = b^{-1/2}$ and $q = r^2$. We write $x \to y$ if $\chi(y - x) = 1$ at every prime of $\mathcal{B}$; the laws below are products over the primes of $\mathcal{B}$ of the laws at one prime. We use the components of the reference of $\mathcal{B}$ (Section 22.1): the *two-layer F law* λ_F, uniform on six distinct labels $(y_{o,\ell})_{o\in\{0,1,2\},\,\ell\in\{0,1\}}$ with $y_{0,\ell} \to y_{2,\ell'}$ for all ℓ, ℓ', and the *two-layer E law* λ_E, uniform on six distinct labels; a map $\phi\colon \mathbb{Z}/6\mathbb{Z} \to \{0,1,2\}$ places the label $y_{\phi(u),\ell}$ at the six slots (u, v, ℓ), $v \in \mathbb{Z}/6\mathbb{Z}$; the type-D components $\Pi_{F_j,\mathcal{B}}$ are the images of λ_F under the folds F_j of Definition 17.1, of weight $w_{\mathrm{D}} r$ each in $\widetilde{\mu}_{\mathcal{B}}$, and the type-S components $\Pi_{E_s,\mathcal{B}}$ the images of λ_E under the matchings E_s, of weight $w_{\mathrm{S}} r$ each; and the *root* $\Pi_{\mathrm{L},\mathcal{B}}$ is the uniform law on twelve distinct labels $(y_{u,\ell})_{u\in\mathbb{Z}/6\mathbb{Z},\,\ell\in\{0,1\}}$ with $y_{u,\ell} \to y_{cu,\ell'}$ for all u, ℓ, ℓ', which contains all four layer lifts of the six directed edges $0 \to 2 \to 4 \to 0$ and $1 \to 5 \to 3 \to 1$; this is the root of the blocks of the pieces starting below 2^{30}. On the seven pieces starting at $2^{30}, 2^{31}, 2^{32}, 2^{34}, 2^{36}, 2^{40}, 2^{45}$ the root is the positive root $\Gamma_{\mathcal{B},\mathsf{u}}$ of Definition 21.1, a positive reweighting of the uniform law with the same support; it is invariant under the chamber automorphisms used below, the layer interchange and independent square-affine maps at the primes of $\mathcal{B}$ (Lemmas 21.2 and 21.17), and every statement of this section about the root holds for it, with the density and chi-square bounds of its half law given by Lemma 24.7(iv) in place of (i)–(iii). For $j \in \{0,1,2\}$ the *fold chamber* of F_j is the chamber cut ϑ_{2j} of the first coordinate, in the notation of Section 17.1: its reflection is $u \mapsto 4j - 1 - u$ (that is $5 - u$, $3 - u$ and $1 - u$) and its plus half is $H_{2j} = \{2j, 2j+1, 2j+2\}$ (indices mod 6). By Lemma 17.2(a)–(d): the root is invariant under the chamber automorphisms (for a positive root by Lemma 21.2(b) and Lemma 21.17), so at each ϑ_{2j} it couples two copies of one half law $\nu_{\mathrm{L},\mathcal{B}}$, which, read in the order $(2j, 2j+1, 2j+2)$, lives on the support of λ_F and does not depend on j; $\Pi_{F_j,\mathcal{B}} = \mathrm{Diag}_{\vartheta_{2j}}(\lambda_F^{(j)})$, where $\lambda_F^{(j)}$ places the labels of the object o at the object $2j + o$; and at ι the records $\Pi_{F_j,\mathcal{B}}$, $\Pi_{E_s,\mathcal{B}}$ are symmetric, with actual half marginal the layer-zero triple law of λ_F placed by F_j, respectively the uniform law on ordered triples of distinct labels placed by E_s.

Definition 24.1 (Eligible records and repairs)**.** The *eligible records* of a middle block $\mathcal{B}$ are $\phi \in \{F_1, F_2, E_0, E_1\}$. For such ϕ let $\Pi_{\phi,\mathcal{B}}$ be the normalized reference record, $m_{\phi,\mathcal{B}}$ its actual half marginal at ι (the same on both halves), $D_{\phi,\mathcal{B}} = \mathrm{Diag}_\iota(m_{\phi,\mathcal{B}})$ its *selected diagonal*, and $J_{\phi,\mathcal{B}}$ the law under which the two ι-halves are independent, each with law $m_{\phi,\mathcal{B}}$ (the *product of the actual half marginals*). A *repair* of ϕ is a signed measure $\Pi'_{\phi,\mathcal{B}}$ of mass one on the labels of $\mathcal{B}$ that is a finite real combination of nonnegative measures, each invariant under independent square-affine maps at the primes of $\mathcal{B}$ and with the actual half marginal $m_{\phi,\mathcal{B}}$ on both sides of ι. The *record*

difference $\Delta_{\phi,\mathcal{B}}$ is the operator on $L^2(m_{\phi,\mathcal{B}})$ with

$$\langle \psi, \Delta_{\phi,\mathcal{B}} \psi' \rangle_{m_{\phi,\mathcal{B}}} = \int \psi(z_+)\psi'(z_-)\, d(\Pi'_{\phi,\mathcal{B}} - \Pi_{\phi,\mathcal{B}})(z),$$

and $\Delta^{\mathrm{s}}_{\phi,\mathcal{B}}$, $\Delta^{\mathrm{a}}_{\phi,\mathcal{B}}$ are its symmetric and skew parts. In the comparison law of the middle range the repair of F_j is the signed whole-block fold record of Definition 23.9(c),(d), a combination of three positive product laws, each with both half marginals $m_{F_j,\mathcal{B}}$ (Lemma 23.7), and the repair of E_s is the exact-marginal matching law of Lemma 23.15, tensorized over the primes of $\mathcal{B}$ (Lemma 23.17).

Both $\Pi_{\phi,\mathcal{B}}$ and its repair have the actual half marginal $m_{\phi,\mathcal{B}}$ on both sides of ι, so

$$\Delta_{\phi,\mathcal{B}}\mathbf{1} = \Delta^{*}_{\phi,\mathcal{B}}\mathbf{1} = 0, \qquad \Delta_{\phi,\mathcal{B}} = (I - \mathbb{J})\Delta_{\phi,\mathcal{B}}(I - \mathbb{J}), \tag{24.1}$$

where $\mathbb{J}$ is the projection onto the constants of $L^2(m_{\phi,\mathcal{B}})$; the same holds for $\Delta^{\mathrm{s}}_{\phi,\mathcal{B}}$ and $\Delta^{\mathrm{a}}_{\phi,\mathcal{B}}$. The record of weight $w_\phi \in \{w_{\mathrm{D}}r, w_{\mathrm{S}}r\}$ contributes $w_\phi(\Pi'_{\phi,\mathcal{B}} - \Pi_{\phi,\mathcal{B}})$, normalized by $\widetilde{\mu}_{\mathcal{B}}(1) \geq 1$, to $\nu_{\mathcal{B}} - \lambda_{\mathcal{B}}$: the first stage changes only the one-layer folded records (Definition 22.4), so the two-layer records enter $\lambda_{\mathcal{B}}$ and $\mu_{\mathcal{B}}$ with the same weight. In the per-record decomposition of Definition 9.14(b) its norm envelopes are $\tilde{\pi}^{\mathrm{s}} D_{\phi,\mathcal{B}}$ and $\tilde{\pi}^{\mathrm{a}} D_{\phi,\mathcal{B}}$ for any $\tilde{\pi}^{\mathrm{s}} \geq w_\phi \|\Delta^{\mathrm{s}}_{\phi,\mathcal{B}}\|$ and $\tilde{\pi}^{\mathrm{a}} \geq w_\phi \|\Delta^{\mathrm{a}}_{\phi,\mathcal{B}}\|$ (Remark 9.15(ii)), with prices $\tilde{\pi}^{\mathrm{s}} q_{\mathcal{B}}^{2a}$ and $\tilde{\pi}^{\mathrm{a}} q_{\mathcal{B}}^{2a}$; the norm bounds are those of Proposition 23.13 for F_j and of Lemma 23.17 for E_s. The two records of one kind in a block have differences of equal symmetric and skew norms: that of F_2 is UD^*U, with D that of F_1 and U a self-adjoint unitary, so that its symmetric and skew parts are $UD^{\mathrm{s}}U$ and $-UD^{\mathrm{a}}U$ (Definition 23.9(d), Proposition 23.13(e)), and E_0, E_1 place one and the same law. So the two fold records of a block carry the same prices, and so do its two matching records. Every positive law constructed in this section (the records, their repairs, their diagonals and their products) is a positive measure, invariant under independent translations at the primes of its blocks, whose normalization has uniform one-slot marginals and does not depend on the indices (O1); the only operator-norm bounds at a cut applied to these laws, the contraction ($d = 1$) of a probability coupling and the centred norms of a record's own Markov operator at ι, condition on no shared label (O2). So the laws are admissible (Definition 8.9), and (E1)–(E3) apply to them.

The comparison at the fold chamber of F_j is the following statement.

Hypothesis 24.2 (Root density, (A))**.** For $j \in \{1, 2\}$, at the fold chamber ϑ_{2j} of F_j (reflection $u \mapsto 3 - u$ for F_1, $u \mapsto 1 - u$ for F_2), the actual half law $\nu_{\mathrm{L},\mathcal{B}}$ of the root on the objects H_{2j} and the law $\lambda_F^{(j)}$, whose ϑ_{2j}-diagonal copy is the record $\Pi_{F_j,\mathcal{B}}$, satisfy $M_{\mathcal{B}}^{-1} \leq d\nu_{\mathrm{L},\mathcal{B}}/d\lambda_F^{(j)} \leq M_{\mathcal{B}}$.

Lemma 24.7 proves hypothesis (A) with an explicit $M_{\mathcal{B}}$, for the uniform root in (i)–(iii) and for the positive roots of the late pieces in (iv), and also bounds the chi-square divergence of the two laws. The record compared with the root is the two-layer record itself, which the first stage does not change, not its reserved-cut marginal. The comparison is made at a chamber cut because no comparison is available at ι: there the half laws of the folded records are singular to that of the root, and no positive semidefinite order at ι bounds the product energy from below (Remark 35.2).

24.2. **Comparing the principal half law with the record half law.** Restoration needs the chi-square distance between two half laws at a chamber cut: the half law of the principal (root) record and the half law of an eligible record. At one prime this distance is governed by the second moment of a signed neighbourhood count— how many common neighbours, with prescribed signs, four labels have. That second moment is computed first; one of the graph moments in it is evaluated through the Ahlgren–Ono identity (Theorem A.7), the others by Weil's bound. At one prime $p \equiv 3 \pmod 4$ we use $q = 1/p$, $r = p^{-1/2}$ and $k_p = (p-3)/4$, the exact number of common successors, and of common predecessors, of two distinct elements of $\mathbb{F}_p$ (Lemma 12.10(a)). Signed neighbourhoods of four distinct labels and their soft counts $\mathfrak{S}$ are those of Corollary 12.3: with

$\upsilon_+ = 11r+6q$ and $\upsilon_- = 11r+70q$, a signed neighbourhood has between $\frac{p}{16}(1-\upsilon_-)$ and $\frac{p}{16}(1+\upsilon_+)$ elements, and at least $\frac{p}{16}(1-\upsilon_\times)$ of them avoid two further labels, where

$$\upsilon_\times = \upsilon_- + 32q = 11r + 102q.$$

The second moment of a neighbourhood count. The chi-square comparison needs the second moment of the fluctuation of a neighbourhood count. For independent uniform $x = (x_1, x_2, x_3, x_4) \in \mathbb{F}_p^4$ and the signs $s = (-1,-1,1,1)$ put $s_A = \prod_{i\in A} s_i$ and

$$\mathcal{W} = \sum_{\substack{A\subseteq\{1,2,3,4\}\\ |A|\in\{3,4\}}} s_A \sum_{z\in\mathbb{F}_p} \prod_{i\in A} \chi(z-x_i), \qquad P_{\mathrm{adm}} = \prod_{i\in\{1,2\},\ k\in\{3,4\}} (1+\chi(x_k-x_i)).$$

For a finite simple graph G on $\{0,\dots,n-1\}$ let $M_G = \mathbb{E}\prod_{\{u,v\}\in G,\, u<v} \chi(\xi_v-\xi_u)$ for independent uniform $\xi_0,\dots,\xi_{n-1}\in\mathbb{F}_p$. Let G_5 be the graph on five vertices with the eight edges 01, 02, 03, 04, 13, 14, 23, 24 (the complete tripartite graph with parts $\{0\}$, $\{1,2\}$, $\{3,4\}$); G_{12} the complete tripartite graph on six vertices with parts $\{0,1\}$, $\{2,3\}$, $\{4,5\}$; and G_{10} the graph on six vertices with the ten edges 01, 03, 04, 05, 12, 14, 15, 23, 25, 34.

Lemma 24.3 (Second moment of the neighbourhood count)**.** *Let $p \equiv 3 \pmod 4$ be prime.*

(a) The moment satisfies

$$\frac{\mathbb{E}[P_{\mathrm{adm}}\mathcal{W}^2]}{p^2} = (4+8q)M_{G_5} + M_{G_{12}} + 24M_{G_{10}} + 5q - 4q^2 + 14q^3 - 15q^4, \tag{24.2}$$

$|M_{G_5}|, |M_{G_{12}}| \le q$ and $|M_{G_{10}}| \le qr$. Consequently $\mathbb{E}[P_{\mathrm{adm}}\mathcal{W}^2]/p^2 \le 10q+24qr+4q^2+14q^3$.

(b) With $T_\lambda = \sum_{y\in\mathbb{F}_p}\chi(y(y-1)(y-\lambda))$,

$$M_{G_5} = \frac{p-1}{p^4}\sum_{\lambda\in\mathbb{F}_p}\chi(\lambda)T_\lambda^2, \qquad -(1-q)(2qr+q^2) \le M_{G_5} \le (1-q)(2qr-q^2), \tag{24.3}$$

and $M_{G_{12}} = (1-5q)M_{G_5} + q^2(1-q)(1-2q)$. Consequently

$$\frac{\mathbb{E}[P_{\mathrm{adm}}\mathcal{W}^2]}{p^2} = (5+3q)M_{G_5} + 24M_{G_{10}} + 5q - 3q^2 + 11q^3 - 13q^4 \le 5q + 34qr + 11q^3.$$

Proof. The identity. Expand P_{adm} into its 16 products over subsets of the four cross pairs, and $\mathcal{W}^2$ into 25 products over pairs (A, A'), with independent summation labels z, z'. Dividing by p^2, each of the 400 terms is $\pm$ a moment $\mathbb{E}\prod\chi(\xi_v-\xi_u)$ of an oriented simple graph on the six vertices $x_1, x_2, x_3, x_4, z, z'$, with z, z' uniform; all these graphs are subgraphs of the complete tripartite graph with parts $\{x_1,x_2\},\{x_3,x_4\},\{z,z'\}$. The moments are reduced by four exact rules, valid for every prime $p\equiv 3 \pmod 4$: (G1) a vertex of degree one makes the moment zero, since $\sum_\xi \chi(\xi-\eta) = 0$; (G2) an odd number of edges makes it zero, since $\xi \mapsto -\xi$ preserves the uniform law and multiplies the product by $\chi(-1)^{\#\mathrm{edges}}$; (G3) relabelling the vertices changes the moment by the sign $\chi(-1)$ per reversed edge, and an automorphism reversing an odd number of edges forces the moment to vanish; (G4) for a vertex v of degree two with neighbours u, w, $\mathbb{E}_{\xi_v}\chi(\xi_v-\xi_u)\chi(\xi_v-\xi_w) = \mathbf{1}[\xi_u=\xi_w] - q$ (Lemma 12.10(b)), which removes v and adds the graph with u and w identified; a repeated edge is $\chi(t)^2 = 1-\mathbf{1}[t=0]$ and is expanded in the same way, and a loop gives zero. Every step lowers the number of vertices or of vertices of degree two, so the reduction terminates in polynomials in q and moments of graphs with all degrees at least three. Among the graphs that occur these are G_5, G_{12} and G_{10}, and collecting the 400 reductions gives (24.2). The bookkeeping of the 400 terms is the finite symbolic computation of Certificate D.8.

The bounds for M_{G_5} and $M_{G_{12}}$. Suppose a graph G has two disjoint edges u_1v_1, u_2v_2 whose four endpoints do not span a complete graph K_4. If one of u_1v_2, u_2v_1 is missing, let the two sides be $\{u_1,u_2\}$ and $\{v_1,v_2\}$; otherwise one of u_1u_2, v_1v_2 is missing, and the sides are $\{u_1,v_2\}$ and

$\{v_1, u_2\}$. In either case at most three of the four pairs across the sides are edges, and both selected edges cross. Fix all other labels. Edges inside a side, and edges to fixed labels, are multipliers of modulus at most one on the functions of that side; the crossing edges are the kernel of an operator $T_{\mathcal{K}}$ of Lemma 16.4 with $k = 2$, whose diagonal pairs are the two selected edges and which has at most one further pair, so that its digraph is acyclic and $\|T_{\mathcal{K}}\| \le r^2 = q$ by part (a) of that lemma. The fixed-label moment is therefore at most q in absolute value, and so is M_G. In G_5 the edges 01 and 23 qualify (12 is missing), and in G_{12} the edges 02 and 34 (23 is missing).

The bound for $M_{G_{10}}$. Translate $\xi_0 = 0$. The factor $\chi(\xi_1)$ vanishes at $\xi_1 = 0$; for $\xi_1 = c \neq 0$ substitute $\xi_v = c\eta_v$ ($v \ge 2$): each of the ten edge factors acquires $\chi(c)$, and $\chi(c)^{10} = 1$. With $\eta_2 = \beta$, $\eta_3 = \gamma$,

$$M_{G_{10}} = \frac{p-1}{p^5} \sum_{\beta,\gamma} \chi(\gamma)\chi(\beta-1)\chi(\gamma-\beta)\, T_\beta T_\gamma, \qquad T_\beta = \sum_y \chi(y)\chi(y-1)\chi(y-\beta),$$

the sums over η_4 and η_5 giving T_γ and T_β. The matrix $H_{\gamma\beta} = \chi(\gamma-\beta)$ satisfies $H^{\mathsf{T}}H = pI - J_{\mathrm{all}}$ and $H\mathbf{1} = 0$, so $\|H\| \le \sqrt{p}$; and $T = -Hv$ with $v_y = \chi(y)\chi(y-1)$, $\|v\|^2 = p-2$, so $\|T\| \le p$. The double sum is $\psi'^{\mathsf{T}} H \psi$ with $\psi_\beta = \chi(\beta-1)T_\beta$ and $\psi'_\gamma = \chi(\gamma)T_\gamma$, of norms at most p; hence it is at most $p^{5/2}$ and $|M_{G_{10}}| \le (p-1)p^{-5/2} \le p^{-3/2} = qr$.

The last assertion of (a) follows from (24.2), the three bounds and $-15q^4 \le 0$.

The modular evaluation. (b) Translate $\xi_0 = 0$ in M_{G_5}. The factor $\chi(\xi_1)$ vanishes at $\xi_1 = 0$; for $\xi_1 = c \neq 0$ substitute $\xi_v = c\eta_v$ ($v = 2, 3, 4$): each of the eight edge factors acquires $\chi(c)$, and $\chi(c)^8 = 1$. With $\lambda = \eta_2$ the edges at the vertex 3 give $\chi(\eta_3)\chi(\eta_3-1)\chi(\eta_3-\lambda)$, and those at the vertex 4 the same factor in η_4, so the sums over η_3 and η_4 are both T_λ, and the factor $\chi(\eta_2)$ of the edge 02 is $\chi(\lambda)$; this is the identity of (24.3). By Theorem A.7, $\sum_\lambda \chi(\lambda)T_\lambda^2 = -p - a(p)$ with $|a(p)| \le 2p^{3/2}$; multiplying by $(p-1)/p^4$ gives the two bounds, since $(p-1)p^{-3} = (1-q)q^2$ and $2(p-1)p^{-5/2} = 2(1-q)qr$.

The inversion identity. Translate $\xi_0 = 0$ in $M_{G_{12}}$, so that $M_{G_{12}} = p^{-5}\sum_{\xi_1,\dots,\xi_5} \Phi(\xi)$, where Φ is the product of the twelve edge factors at $\xi_0 = 0$. The four edges at the vertex 0 give the factor $\chi(\xi_2)\chi(\xi_3)\chi(\xi_4)\chi(\xi_5)$.

On the part $\xi_1 \neq 0$, Φ vanishes unless all five labels are nonzero; substitute $\xi_v = 1/\eta_v$ ($1 \le v \le 5$). Then $\chi(\xi_v) = \chi(\eta_v)$, and for an edge uw with $u < w$ between nonzero labels $\chi(\xi_w - \xi_u) = \chi((\eta_u - \eta_w)/(\eta_u\eta_w)) = -\chi(\eta_w - \eta_u)\chi(\eta_u)\chi(\eta_w)$. Every vertex has degree four in G_{12}, and each edge at a vertex $v \ge 1$ contributes one factor $\chi(\eta_v)$, so these factors give $\chi(\eta_v)^4 = 1$; the eight edges between the vertices $1, \dots, 5$ give the sign $(-1)^8 = 1$. What remains is the product of the factors $\chi(\eta_w - \eta_u)$ over the edges $12, 13, 14, 15, 24, 25, 34, 35$, which form a copy of G_5 (centre 1, parts $\{2,3\}$ and $\{4,5\}$), summed over nonzero labels. By inclusion and exclusion over the set of vanishing labels: the unrestricted sum is $p^5 M_{G_5}$; prescribing one vanishing label leaves, by translation, $p^4 M_{G_5}$; two adjacent labels cannot vanish together; the two nonadjacent pairs $\{2,3\}$ and $\{4,5\}$ each contribute $\sum_{\eta_1 \neq 0}\bigl(\sum_{y \neq 0} \chi(y - \eta_1)\bigr)^2 = p - 1$, since $\sum_{y \neq 0} \chi(y - \eta_1) = -\chi(-\eta_1)$; and no three labels are pairwise nonadjacent. So this part contributes $(p^5 - 5p^4)M_{G_5} + 2(p-1)$.

On the part $\xi_1 = 0$, the four edges at the vertex 1 give the same factor as those at 0, so the product is the indicator that $\xi_2, \dots, \xi_5$ are nonzero times the four factors of the complete bipartite graph between $\{2,3\}$ and $\{4,5\}$. For nonzero a, b put $c(a,b) = \sum_{y \neq 0} \chi(y-a)\chi(y-b)$; then $c(a,a) = p-2$ and $c(a,b) = -1 - \chi(ab)$ for $a \neq b$, and this part is

$$\sum_{a,b \neq 0} c(a,b)^2 = (p-1)(p-2)^2 + \sum_{a \neq b}\bigl(2 + 2\chi(ab)\bigr) = (p-1)(p^2 - 2p - 2),$$

as $\sum_{a \neq b} \chi(ab) = -(p-1)$. Adding the two parts and dividing by p^5 gives $M_{G_{12}} = (1-5q)M_{G_5} + (p-1)(p-2)/p^4$, which is the inversion identity.

Substituting it in (24.2) gives the displayed identity of (b). By (24.3), $5+3q>0$ and $|M_{G_{10}}|\le qr$, its right side is at most $5q+34qr+11q^3$ minus $8q^2-2q^3+10q^4+4q^2r+6q^3r$, which is nonnegative. □

Remark 24.4 (The crude second moment). The identity matters only for its constant. Under independent uniform labels only the diagonal pairs $A=A'$ survive in $\mathbb{E}[\mathcal{W}^2]$, because $\mathbb{E}_x\chi(z-x)=0$, and $\mathbb{E}_x\chi(z-x)\chi(z'-x)=\mathbf{1}[z=z']-q$; hence $\mathbb{E}[\mathcal{W}^2]=\sum_{|A|\in\{3,4\}}[p(1-q)^{|A|}+p(p-1)(-q)^{|A|}]\le 5p$, and $0\le P_{\mathrm{adm}}\le 16$ gives $\mathbb{E}[P_{\mathrm{adm}}\mathcal{W}^2]/p^2\le 80q$. This already shows that the chi-square divergence of Lemma 24.7 is $O(q)$. The identity lowers the constant from 80 to about 10, and the modular evaluation of Lemma 24.3(b) to about 5; the certified credits use the latter.

The first moment of the four-root count. The cut L20 of the uniform root (Section 20.4) needs the first moment of the same signed count, with the four labels distinct. Write $\mathcal{W}=\mathcal{W}_3+\mathcal{W}_4$, where $\mathcal{W}_k$ is the part of $\mathcal{W}$ with $|A|=k$, and let D_4 be the indicator that $x_1,\dots,x_4$ are distinct.

Lemma 24.5 (The first moment of the four-root count)**.** *Let $p\equiv 3 \pmod 4$ be prime. Then*

$$\frac{\mathbb{E}[P_{\mathrm{adm}}D_4\mathcal{W}_3]}{p}=0,\qquad \frac{\mathbb{E}[P_{\mathrm{adm}}D_4\mathcal{W}_4]}{p}=M_{G_5}+8q^2-33q^3+25q^4. \tag{24.4}$$

Consequently, let $p>7$, so that $Z_4=(1-q)(1-3q)(1-7q)>0$, and let K be the soft kernel of the cut L20 of Section 20.4, the product over the sixteen cross edges of the factors $1+(D_{20})_{ij}\chi(x'_j-x_i)$ with the signed cross matrix D_{20} of (20.2), for independent halves x,x' with the law $\lambda_{F,p}$. In the expansion of $\mathbb{E}\mathsf{K}$ over the sets of cross edges, the sets that form a star (all their edges at one label, the centre) have total mean of modulus at most

$$m_\star=\frac{4\{(1-q)(q^2+2qr)+8q^2+33q^3+25q^4\}}{Z_4}.$$

Hence, if the nonstar part T_{ns} of the soft kernel and the collision correction T_{coll} of Section 20.4 have norms at most N_{ns} and D_{col}, and $\varepsilon_Z=N_{\mathrm{ns}}+D_{\mathrm{col}}+m_\star<1$, then the normalizer $Z_0=\mathbb{E}\mathsf{K}-\langle\mathbf{1},T_{\mathrm{coll}}\mathbf{1}\rangle$ of the L20 cut operator satisfies $|Z_0-1|\le\varepsilon_Z$, and the normalizer centred at the half-law mean may use $R_Z=(1-\varepsilon_Z)^{-1}\ge Z_0^{-1}$.

Proof. The identities. Expand D_4 over the partitions π of the four labels, with the Möbius coefficients of the partition lattice:

$$D_4=\sum_\pi \mu(\pi)\,\mathbf{1}_\pi,\qquad \mu(\pi)=\prod_{C\in\pi}(-1)^{|C|-1}(|C|-1)!\,, \tag{24.5}$$

where $\mathbf{1}_\pi$ is the indicator that x is constant on every block of π; expand P_{adm} into its 16 products over the sets of cross pairs and $\mathcal{W}_k$ into its k-sets A with the signs s_A. Identifying the labels of each block of π leaves $|\pi|$ independent uniform labels and costs the factor $q^{4-|\pi|}$. After division by p, each of the $16\cdot 5\cdot 15=1200$ terms is therefore $s_A\mu(\pi)q^{4-|\pi|}$ times the moment of the oriented graph on the merged labels and the summation label z, with the chosen cross pairs and the edges from z to the labels of A; edges reversed by the identification contribute their signs $\chi(-1)=-1$, and a loop makes the term zero. Each moment is reduced by the rules (G1)–(G4) of the proof of Lemma 24.3, which are exact for every prime $p\equiv 3 \pmod 4$. The reduced terms with $|A|=3$ cancel, and those with $|A|=4$ add up to $M_{G_5}+8q^2-33q^3+25q^4$; the bookkeeping of the 1200 reductions is the finite symbolic computation of Certificate 24.6.

The star means. Under $\lambda_F\otimes\lambda_F$ the centre z of a star is a coordinate of one half, uniform on $\mathbb{F}_p$ and independent of the other half, which carries the leaves; the leaves are distinct labels. By the orientation of the factors, a star centred in the first half has the sign $(-1)^{\#\mathrm{leaves}}$ in addition to the product of its entries of D_{20}. A one-leaf star has mean zero, as $\mathbb{E}_z\chi(z-y)=0$. A two-leaf star has mean $-q$ times the product of its two entries, because $\mathbb{E}_z\chi(z-y)\chi(z-y')=-q$ for $y\ne y'$ (Lemma 12.10(b)). The rows of D_{20}, and also its columns, are of two kinds ((20.2)):

four have two nonzero entries, of equal sign, at the labels b_0, b_1 of the other half; two have four nonzero entries, at the labels a_0, a_1, c_0, c_1 of the other half, with the signs $\pm(-1,-1,1,1)$. The products of the entries over the two-element subsets of the support therefore sum, over the rows and over the columns, to $4 \cdot 1 + 2 \cdot (-2) = 0$, and the two-leaf means cancel. Stars with three or four leaves occur only at the four centres of the second kind, two on each side, and their leaves are the four labels (a_0, a_1, c_0, c_1) of the other half. Under λ_F these labels are uniform on the admissible ordered quadruples (distinct, with $a_i \to c_j$), since each admissible quadruple has exactly $(p-4)(p-5)$ completions by (b_0, b_1); for independent uniform labels $P_{\mathrm{adm}} D_4$ is 16 times the indicator of the admissible event, which has probability $p(p-1)k_p(k_p-1)/p^4 = Z_4/16$. Hence $\mathbb{E}_{\lambda_F}[f] = \mathbb{E}[P_{\mathrm{adm}} D_4 f]/Z_4$ for every function f of the four labels. At such a centre the three-leaf stars together have the mean $\pm\mathbb{E}_{\lambda_F}[\mathcal{W}_3]/p$, which is zero by (24.4), and the four-leaf star, whose sign product is $+1$, has the mean $\mathbb{E}_{\lambda_F}[\mathcal{W}_4]/p = (M_{G_5} + 8q^2 - 33q^3 + 25q^4)/Z_4$. By (24.3), $|M_{G_5}| \le (1-q)(q^2+2qr)$, so the four centres give at most $m_\star$.

The empty set of edges gives one and the nonstar sets give $\langle \mathbf{1}, T_{\mathrm{ns}} \mathbf{1} \rangle$, of modulus at most N_{ns}; the collision correction $\langle \mathbf{1}, T_{\mathrm{coll}} \mathbf{1} \rangle$ has modulus at most D_{col}. So $|Z_0 - 1| \le m_\star + N_{\mathrm{ns}} + D_{\mathrm{col}} = \varepsilon_Z$, and $Z_0 \ge 1 - \varepsilon_Z > 0$. □

With $q = r^2$, $m_\star/r^2$ is a positive increasing function of r for $0 < r < 7^{-1/2}$, its numerator being $4(2r + 9r^2 - 2r^3 + 32r^4 + 25r^6)$ and its denominator Z_4 decreasing; so the value of $m_\star$ at a threshold bounds it at every larger prime.

Certificate 24.6 (The four-root first moment)**.** The 1200 terms of the expansion in the proof of Lemma 24.5 add up to (24.4), and the signed two-leaf row and column sums of the cross matrix vanish; both are checked in exact arithmetic as follows. The script `research/continuation_2026_10_24/upper_arithmetic/l20_star_mean.py` writes the record `research/continuation_2026_10_24/upper_arithmetic/l20_star_mean.json` (status `PASS_EXACT_L20_STAR_MEAN_IDENTITY`): it lists the 1200 terms of the expansion in the proof of Lemma 24.5, with the coefficient (24.5), the sign and the integer reduction of each by the rules (G1)–(G4), and checks that they add up to (24.4); it also checks that the signed two-leaf sums of the rows and of the columns of the cross matrix vanish, and the identities by direct finite sums at $p = 7, 11, 19$. The record and the script are pinned by SHA-256 in `research/continuation_2026_10_24/verification/selected.json`. Its check `l20_mean` (status `PASS_INDEPENDENT_L20_VARIANCE_MEAN_SPECTRAL_GATES`) runs `research/continuation_2026_10_24/upper_arithmetic/review_middle_l20_mean.py`, which asserts the status of the record, its 1200 terms, the vanishing of the part with $|A| = 3$ and the two vanishing two-leaf sums; its full-tier check `fresh_l20_mean` reruns `research/continuation_2026_10_24/upper_arithmetic/middle_l20_variance_mean.py`, which asserts the status of the record as well. The continuation record is an authenticated input of the later ones: `research/advance_2026_10_24/verify.py` asserts its status and re-hashes the files it pins, and the later verifiers authenticate this chain through `research/push_2026_10_25/verify.py` (Certificate 30.3).

Lemma 24.7 (Root completion count)**.** *Let $p \equiv 3 \pmod 4$ be prime and put*

$$\mathfrak{a}_p = \frac{4q}{1-3q} \cdot \frac{(1+\upsilon_+)^2}{(1-\upsilon_-)(1-\upsilon_- - 16q)}, \qquad R_p = \frac{(1+\upsilon_+)^2}{(1-\upsilon_\times)(1-\upsilon_\times - 16q)(1-24\mathfrak{a}_p)},$$

$$Z_4 = (1-q)(1-3q)(1-7q), \qquad \Delta_2 = \Big(\frac{5q + 34qr + 11q^3}{Z_4}\Big)^{1/2} + 30q,$$

$$\Delta_{\max} = 11r + 30q, \qquad D_{\mathrm{full}} = (2 + \Delta_{\max} + 16q)\Delta_2 + 16q + 24\mathfrak{a}_p(1+\Delta_{\max})^2,$$

$$L_{\mathrm{low}} = (1-\Delta_{\max})(1-\Delta_{\max}-16q)(1-24\mathfrak{a}_p),$$

and assume that the lower factors $1-\upsilon_\times-16q$ *and* $1-24\mathfrak{a}_p$ *are positive. Then* $L_{\mathrm{low}}>0$ *as well, since* $\Delta_{\max}<\upsilon_\times$.

(i) The half law $\nu_{\mathrm{L},p}$ *of the root at one prime satisfies* $R_p^{-1}\le d\nu_{\mathrm{L},p}/d\lambda_{F,p}\le R_p$*; moreover the supremum of this density is at most* R_p *times its infimum.*

(ii) $\chi_p:=\chi^2(\nu_{\mathrm{L},p}\,\|\,\lambda_{F,p})\le\min\bigl((R_p-1)^2/R_p,\,D_{\mathrm{full}}^2/L_{\mathrm{low}}^2\bigr)$.

(iii) Let $\mathcal{B}$ *be a middle block whose primes are at least* X*, with* $\log q_{\mathcal{B}}\le L$*, and let* $n\ge\lfloor L/\log X\rfloor$ *be an integer. Then hypothesis (A) holds with* $M_{\mathcal{B}}=\prod_{p\in\mathcal{B}}R_p\le\bar{R}(X)^n$*, and*

$$\chi^2(\nu_{\mathrm{L},\mathcal{B}}\,\|\,\lambda_{F,\mathcal{B}})=\prod_{p\in\mathcal{B}}(1+\chi_p)-1\le\bigl(1+\bar{\chi}(X)\bigr)^n-1,$$

where $\bar{R}(X)$ *and* $\bar{\chi}(X)$ *are the right sides of (i) and (ii) evaluated at* $q=1/X$ *and any fixed* $r\ge X^{-1/2}$ *for which the two lower factors are positive (the notation suppresses this* r*;* (24.11) *takes* $r=1/\lfloor\sqrt{X}\rfloor$*).*

(iv) Positive roots. *Let* $\mathcal{B}$ *be a middle block of one of the pieces starting at* $X\in\{2^{30},2^{31},2^{32},2^{34},2^{36},2^{40},2^{45}\}$*, whose root is the positive root* $\Gamma_{\mathcal{B},\mathfrak{u}}$ *of Definition 21.1, with* $\log q_{\mathcal{B}}\le L$ *and* $n\ge\lfloor L/\log X\rfloor$*, and let* $R_{\mathfrak{u}}(X)$ *and* $d_{\mathfrak{u}}(X)^2$ *be the range ratio and the variance bound of the half marginal of* $\Gamma_{p,\mathfrak{u}}$ *at the chamber cut* ϑ_0 *given by Propositions 21.7, 21.19 and 21.24 at the endpoint* X*, whose gates hold there (Certificate 21.25). Then hypothesis (A) holds for the half law* $\nu_{\mathrm{L},\mathcal{B}}$ *of* $\Gamma_{\mathcal{B},\mathfrak{u}}$ *with* $M_{\mathcal{B}}\le R_{\mathfrak{u}}(X)^n$*, and*

$$\chi^2(\nu_{\mathrm{L},\mathcal{B}}\,\|\,\lambda_{F,\mathcal{B}})\le\bigl(1+d_{\mathfrak{u}}(X)^2\bigr)^n-1.$$

Proof. Work at ϑ_0, with half $H_0=\{0,1,2\}$; the half law does not depend on the chamber. A half configuration is $y=(y_{u,\ell})_{u\in\{0,1,2\},\,\ell\in\{0,1\}}$, six distinct labels with $y_{0,\ell}\to y_{2,\ell'}$. It completes to a root configuration by labels $y_{u,\ell}$, $u\in\{3,4,5\}$, subject to $y_{2,\ell}\to y_{4,\ell'}\to y_{0,\ell''}$ (even triangle) and $y_{1,\ell}\to y_{5,\ell'}\to y_{3,\ell''}\to y_{1,\ell'''}$ (odd triangle), for all layer indices, all twelve labels distinct.

Odd component. Fix $(y_{1,0},y_{1,1})$ and let $\mathcal{O}$ be the set of $(y_{5,0},y_{5,1},y_{3,0},y_{3,1})$ satisfying the odd constraints with the six odd labels distinct, and $\mathcal{Z}=|\mathcal{O}|$. The labels $y_{5,\ell}$ are an ordered distinct pair of common successors of $y_{1,0},y_{1,1}$ ($k_p(k_p-1)$ choices), and given them the labels $y_{3,\ell}$ are an ordered distinct pair in the signed neighbourhood of $(y_{5,0},y_{5,1},y_{1,0},y_{1,1})$ with the signs $(1,1,-1,-1)$. By Corollary 12.3, $\mathcal{Z}\ge k_p(k_p-1)(p/16)^2(1-\upsilon_-)(1-\upsilon_--16q)$. The number of elements of $\mathcal{O}$ with a prescribed value of $y_{5,0}$ is at most $(k_p-1)(p/16)^2(1+\upsilon_+)^2$; the same bound holds for $y_{5,1}$, and, choosing $y_{3,0},y_{3,1}$ first as common predecessors of $y_{1,0},y_{1,1}$, for $y_{3,0}$ and $y_{3,1}$. So under the uniform law on $\mathcal{O}$ each of the four free odd labels takes any given value with probability at most $\mathfrak{a}_p$, since $1/k_p=4q/(1-3q)$. Finally $\mathcal{Z}$ does not depend on $(y_{1,0},y_{1,1})$: the maps $z\mapsto cz+c'$ with $\chi(c)=1$ preserve all the constraints and act transitively on the ordered distinct pairs with a fixed value of $\chi(y_{1,1}-y_{1,0})$; $\mathcal{O}$ depends only on the set $\{y_{1,0},y_{1,1}\}$, because all four lifts of each edge are present; and the two orders of the pair have opposite values of $\chi(y_{1,1}-y_{1,0})$, since $\chi(-1)=-1$.

Even component. Given y, the pair $(y_{4,0},y_{4,1})$ is an ordered distinct pair in the signed neighbourhood of $(y_{0,0},y_{0,1},y_{2,0},y_{2,1})$ with the signs $(-1,-1,1,1)$, avoiding $y_{1,0},y_{1,1}$. By Corollary 12.3 the number of such pairs lies between $(p/16)^2(1-\upsilon_\times)(1-\upsilon_\times-16q)$ and $(p/16)^2(1+\upsilon_+)^2$. For each such pair, the odd completions that avoid the six even labels form a subset of $\mathcal{O}$ of relative size between $1-24\mathfrak{a}_p$ and 1, by the union bound over four free odd labels and six forbidden values. Hence the number $\mathfrak{N}(y)$ of completions of y satisfies

$$(1-\upsilon_\times)(1-\upsilon_\times-16q)(1-24\mathfrak{a}_p)\le\frac{\mathfrak{N}(y)}{\mathcal{Z}(p/16)^2}\le(1+\upsilon_+)^2.\tag{24.6}$$

(i) The density of $\nu_{\mathrm{L},p}$ with respect to the uniform law $\lambda_{F,p}$ on the half configurations is $\mathfrak{N}(y)/\mathbb{E}_{\lambda_F}\mathfrak{N}$. By (24.6) the values of $\mathfrak{N}$ lie in an interval $[c,R_pc]$, $c>0$ ($\mathcal{Z}$ cancels), and so does

their mean; hence the supremum of the density is at most R_p times its infimum, and the density lies in $[R_p^{-1}, R_p]$.

(ii) A density g with $\mathbb{E}g = 1$ and values in $[R^{-1}, R]$ satisfies $\mathbb{E}(g - R^{-1})(R - g) \ge 0$, that is, $\mathbb{E}g^2 \le R + R^{-1} - 1$; this is the first bound. For the second, under $\lambda_{F,p}$ the four labels $x = (y_{0,0}, y_{0,1}, y_{2,0}, y_{2,1})$ are uniform on the *admissible* ordered quadruples (distinct, with $y_{0,\ell} \to y_{2,\ell'}$), since each has exactly $(p-4)(p-5)$ choices of $(y_{1,0}, y_{1,1})$; for independent uniform labels the admissible event has probability $p(p-1)k_p(k_p-1)/p^4 = Z_4/16$. For admissible x the soft count with the signs $(-1,-1,1,1)$ equals $p + 2 + \mathcal{W}$: the singleton sums vanish, each quadratic sum is -1 by (4.2), and $\sum_{i<k} s_i s_k = -2$. At each of the four labels of x one factor of the soft count vanishes (at $z = y_{0,0}$, for instance, $1 + \chi(y_{0,0} - y_{2,0}) = 1 - \chi(y_{2,0} - y_{0,0}) = 0$), so the soft count is exactly 16 times the size of the signed neighbourhood. Removing $y_{1,0}, y_{1,1}$ deletes $n_1 \in \{0,1,2\}$ points, so the number of admissible labels for $y_{4,\ell}$ is $n = \bar{n}(1+\beta)$, with $\bar{n} = p/16$ and $\beta = \mathcal{W}/p + (2 - 16n_1)q$. By Theorem 4.7, $|\mathcal{W}| \le 4 \cdot 2\sqrt{p} + 3\sqrt{p}$, so $|\beta| \le \Delta_{\max}$. Since $P_{\mathrm{adm}} \ge 0$ everywhere and $P_{\mathrm{adm}} = 16$ on the admissible event, Lemma 24.3(b) gives

$$\mathbb{E}_{\lambda_F}\Big(\frac{\mathcal{W}}{p}\Big)^2 \le \frac{\mathbb{E}[P_{\mathrm{adm}}\mathcal{W}^2]}{p^2 Z_4} \le \frac{5q + 34qr + 11q^3}{Z_4},$$

hence $\|\beta\|_{L^2(\lambda_F)} \le \Delta_2$. Let $\Xi = n(n-1)/\bar{n}^2$, the normalized number of ordered distinct pairs for object 4. Then $\Xi - 1 = \beta(2+\beta) - 16q(1+\beta)$, so $\|\Xi - 1\|_{L^2(\lambda_F)} \le (2 + \Delta_{\max} + 16q)\Delta_2 + 16q$, and $(1 - \Delta_{\max})(1 - \Delta_{\max} - 16q) \le \Xi \le (1 + \Delta_{\max})^2$, both factors on the left being positive. By the even-component step, $\varsigma := \mathfrak{N}/(\mathcal{Z}\bar{n}^2)$ satisfies $(1 - 24\mathfrak{a}_p)\Xi \le \varsigma \le \Xi$. Therefore $\|\varsigma - 1\|_{L^2(\lambda_F)} \le D_{\mathrm{full}}$ and $\mathbb{E}\varsigma \ge L_{\mathrm{low}}$. The density is $\varsigma/\mathbb{E}\varsigma$, and $\chi_p = \mathrm{Var}(\varsigma)/(\mathbb{E}\varsigma)^2 \le \mathbb{E}(\varsigma - 1)^2/(\mathbb{E}\varsigma)^2 \le D_{\mathrm{full}}^2/L_{\mathrm{low}}^2$.

(iii) At a block the root and λ_F are products over the primes of $\mathcal{B}$, so the density is the product of the prime densities, and chi-square divergences of product laws satisfy $1 + \chi^2(\bigotimes_p \nu_p \| \bigotimes_p \lambda_p) = \prod_p (1 + \chi^2(\nu_p \| \lambda_p))$. Each of $\upsilon_\pm$, $\upsilon_\times$, $\mathfrak{a}_p$, R_p, Δ_2, $\Delta_{\max}$, D_{full} and $1/L_{\mathrm{low}}$ is a nondecreasing function of (q, r) on the domain where the two lower factors are positive (Lemma 4.18); so for $p \ge X$ the hypotheses of (i) and (ii) hold, $R_p \le \bar{R}(X)$ and $\chi_p \le \bar{\chi}(X)$; and $\mathcal{B}$ has at most $\lfloor L/\log X \rfloor \le n$ primes, since their product is at most e^L and each is at least X. As $\bar{R}(X) \ge 1$, this proves (iii).

(iv) Each selected tilt is invariant under the chamber reflections $u \mapsto 3 - u$, $u \mapsto 1 - u$ and their product $u \mapsto u + 2$. For the pair laws these maps permute the three scores (Lemma 21.2(b)); for the coherent law they permute its physical support dictionary with its coefficients (Lemma 21.17). Thus the reweighted root retains the same three chamber identifications. The translation maps the plus half H_0 of ϑ_0 onto the plus half H_2 of ϑ_2, and H_2 onto the plus half H_4 of ϑ_4, in order, and it maps $\lambda_F = \lambda_F^{(0)}$ to $\lambda_F^{(1)}$ and $\lambda_F^{(1)}$ to $\lambda_F^{(2)}$; each reflection exchanges the two halves of its chamber. Hence, as for the uniform root, the positive root couples at each fold chamber two copies of one half law, and its density with respect to $\lambda_F^{(j)}$ is the same function, read in the order $(2j, 2j+1, 2j+2)$, at ϑ_0, ϑ_2 and ϑ_4. At ϑ_0 and one prime $p \ge X$ this density is $T_{\mathsf{u}}\mathbf{1}$ for the cut operator T_{u} of $\Gamma_{p,\mathsf{u}}$ on $L^2(\lambda_{F,p})$ (Lemma 13.3(f)), and Propositions 21.7, 21.19 and 21.24 give $\sup T_{\mathsf{u}}\mathbf{1} / \inf T_{\mathsf{u}}\mathbf{1} \le R_{\mathsf{u}}(X)$ and $\|T_{\mathsf{u}}\mathbf{1} - \mathbf{1}\|^2 \le d_{\mathsf{u}}(X)^2$; these are the conclusions of Lemma 13.22 for the cut kernel of the uniform root, with the range ratio $R = R_p$ of (i) and $\delta_0 = (\chi_p^+)^{1/2}$, where $\chi_p^+ = D_{\mathrm{full}}^2/L_{\mathrm{low}}^2$ is the second bound of (ii). As $T_{\mathsf{u}}\mathbf{1}$ has mean one, $\inf T_{\mathsf{u}}\mathbf{1} \le 1 \le \sup T_{\mathsf{u}}\mathbf{1}$, so $R_{\mathsf{u}}(X)^{-1} \le T_{\mathsf{u}}\mathbf{1} \le R_{\mathsf{u}}(X)$. On the block the positive root and λ_F are products over the primes, and the product clause of Lemma 13.22 gives the range ratio $\prod_{p \in \mathcal{B}} R_{\mathsf{u},p} \le R_{\mathsf{u}}(X)^n$ and the chi-square $\prod_{p \in \mathcal{B}}(1 + d_{\mathsf{u},p}^2) - 1 \le (1 + d_{\mathsf{u}}(X)^2)^n - 1$, with $R_{\mathsf{u},p}$, $d_{\mathsf{u},p}^2$ the values at p, since $\mathcal{B}$ has at most n primes and $R_{\mathsf{u}}(X) \ge 1$ (proof of (iii)). The block density has mean one, so, as at one prime, its range ratio gives hypothesis (A) with $M_{\mathcal{B}} \le R_{\mathsf{u}}(X)^n$. □

24.3. **Recovering product energy on a common chamber.** The variance saving needs a lower bound for the energy of a product over several blocks. At one common chamber cut this energy

can be recovered ("restored") from the diagonal of λ; the lemma makes the recovery quantitative, with the loss measured by the nonroot mass $h_{\mathcal{B}}$ of each block.

Lemma 24.8 (Restoration on a common chamber)**.** *Let $j \in \{1,2\}$ and let X be a nonempty finite set of middle blocks. Let $h_X \ge \prod_{\mathcal{B}\in X}(1+h_{\mathcal{B}})-1$, where $h_{\mathcal{B}} = \widetilde{\mu}_{\mathcal{B}}(1)-1$ is the nonroot mass of $\widetilde{\mu}_{\mathcal{B}}$,*

$$h_{\mathcal{B}} = (3w_{\mathrm{D}}+2w_{\mathrm{S}}+v_{\mathrm{L}})r + (6w_{\mathrm{R}}+3v_{\mathrm{D}}+2v_{\mathrm{S}})r^2 + (2w_{\mathrm{C}}+6v_{\mathrm{R}})r^3 + 2v_{\mathrm{C}}r^4, \qquad r = r_{\mathcal{B}}. \tag{24.7}$$

Let $M_X \ge \prod_{\mathcal{B}\in X} M_{\mathcal{B}}$ with $M_{\mathcal{B}}$ as in Hypothesis 24.2, $\chi_X \ge \chi^2(\bigotimes_{\mathcal{B}}\nu_{\mathrm{L},\mathcal{B}} \| \bigotimes_{\mathcal{B}}\lambda_{F,\mathcal{B}})$ and $C \ge C_X$, and put $l = 1-(C-1)h_X$. Assume $0 < l \le C$. Then, with $\Pi_X = \bigotimes_{\mathcal{B}\in X}\Pi_{F_j,\mathcal{B}}$, $Y[\Pi_X] \ge 0$ and

$$Y[\Pi_X] \ge (l_0 - o(1))Y, \qquad l_0 = \max\Big(\frac{l}{M_X},\ l - \frac{C}{2}\sqrt{\chi_X},\ \frac{2l^2}{2l + C\chi_X + \sqrt{\chi_X(4l(C-l)+C^2\chi_X)}}\Big).$$

Proof. All laws are on the labels of X; write $\Pi^{\mathrm{rt}} = \bigotimes_{\mathcal{B}}\Pi_{\mathrm{L},\mathcal{B}}$, $\nu = \bigotimes_{\mathcal{B}}\nu_{\mathrm{L},\mathcal{B}}$ and $\lambda = \bigotimes_{\mathcal{B}}\lambda_F^{(j)}$, so that $\mathrm{Diag}_{\vartheta_{2j}}(\lambda) = \Pi_X$. The fold chamber ϑ_{2j} is a cut of the family, so (E2), (P1) and (P2) apply there with μ outside. The measure Π_X is positive and diagonal at ϑ_{2j}, so $Y[\Pi_X] \ge 0$ by (E3) and (E2).

(C) The root carries the reference. Write $\widetilde{\mu}_{\mathcal{B}} = \Pi_{\mathrm{L},\mathcal{B}} + \mathcal{M}_{\mathcal{B}}$, where $\mathcal{M}_{\mathcal{B}} \ge 0$ is the sum of the other weighted components of Lemma 22.1. Its mass is $h_{\mathcal{B}}$, which is (24.7): the five types L, D, S, R, C have multiplicities $1,3,2,6,2$ and degrees $0,1,1,2,3$, a two-layer component of type t has weight $w_t r^{\deg t}$ and a one-layer diagonal weight $v_t r^{1+\deg t}$. Expanding $\bigotimes_{\mathcal{B}\in X}\mu_{\mathcal{B}} = \prod_{\mathcal{B}}(1+h_{\mathcal{B}})^{-1}\bigotimes_{\mathcal{B}}(\Pi_{\mathrm{L},\mathcal{B}}+\mathcal{M}_{\mathcal{B}})$, every term other than Π^{rt} is a positive measure whose normalization is a product of components, with uniform one-slot marginals, and their total mass is $h = \prod_{\mathcal{B}}(1+h_{\mathcal{B}})-1 \le h_X$. Since $Y[\bigotimes_{\mathcal{B}}\mu_{\mathcal{B}}] = Y$, (E1) gives $Y[\Pi^{\mathrm{rt}}] \ge (1+h-(1+o(1))hC_X)Y \ge (l-o(1))Y$.

(B) Contraction and comparison. The root Π^{rt} is a probability coupling at ϑ_{2j} with both half marginals ν, so the contraction (P1) gives $Y[\Pi^{\mathrm{rt}}] \le Y[\mathrm{Diag}_{\vartheta_{2j}}(\nu)]$; no positivity of Π^{rt} at the cut is used. By hypothesis (A), $\nu \le M_X\lambda$, so the density domination (P2) gives $Y[\mathrm{Diag}_{\vartheta_{2j}}(\nu)] \le M_X Y[\Pi_X]$. Thus $Y[\Pi_X] \ge (l/M_X - o(1))Y$.

One-sided variation. The laws ν and λ are invariant under independent translations at each prime of X, and so are the two parts of the Jordan decomposition $\nu-\lambda = \kappa_+ - \kappa_-$, whose masses are $\frac12\|\nu-\lambda\|_{\mathrm{var}} \le \frac12\sqrt{\chi_X}$ by Cauchy–Schwarz. The normalized diagonal copies of $\kappa_\pm$ have uniform one-slot marginals, and $Y[\mathrm{Diag}_{\vartheta_{2j}}(\kappa_-)] \ge 0$ by (E2) and (E3). Hence $Y[\mathrm{Diag}_{\vartheta_{2j}}(\nu)] - Y[\Pi_X] \le Y[\mathrm{Diag}_{\vartheta_{2j}}(\kappa_+)] \le (1+o(1))\frac12\sqrt{\chi_X}\,C_X Y$, and $Y[\Pi_X] \ge (l - \frac{C}{2}\sqrt{\chi_X} - o(1))Y$.

Bounded variance. For a half-label tuple z let o_z be the uniform law on its translates $z+c$, where $c = (c_p)_p$ adds one $c_p \in \mathbb{F}_p$ to all labels at each prime p of X, and put $\eta(z) = Y[\mathrm{Diag}_{\vartheta_{2j}}(o_z)]/Y$. The diagonal copy of o_z is a positive normalized law with uniform one-slot marginals, fixed samplewise by the reflection of ϑ_{2j}, so $0 \le \eta(z) \le C' := (1+o(1))C_X$. Translation invariance gives $\nu = \sum_z \nu(z)o_z$ and $\lambda = \sum_z \lambda(z)o_z$, hence $Y[\mathrm{Diag}_{\vartheta_{2j}}(\nu)] = Y\,\mathbb{E}_\nu\eta$ and $Y[\Pi_X] = Y\,\bar\eta$ with $\bar\eta = \mathbb{E}_\lambda\eta$. With $u = d\nu/d\lambda - 1$,

$$\mathbb{E}_\nu\eta - \bar\eta = \mathbb{E}_\lambda[u(\eta-\bar\eta)] \le \big(\chi_X \operatorname{Var}_\lambda(\eta)\big)^{1/2} \le \big(\chi_X\bar\eta(C'-\bar\eta)\big)^{1/2},$$

because $\eta^2 \le C'\eta$. Since $\mathbb{E}_\nu\eta \ge Y[\Pi^{\mathrm{rt}}]/Y \ge l - o(1)$ by (C) and (B), and $C' \le C + o(1)$, we get $l - o(1) \le \bar\eta + (\chi_X\bar\eta(C-\bar\eta))^{1/2}$. If $\bar\eta < l$, squaring gives $(1+\chi_X)\bar\eta^2 - (2l+C\chi_X)\bar\eta + l^2 \le o(1)$; the discriminant of the left side is $\chi_X(4l(C-l)+C^2\chi_X) \ge 0$, and its smaller root, rationalized, is the third expression in l_0, which is at most l. So $\bar\eta$ is at least that expression minus $o(1)$ in either case. □

Write $\mathcal{F}(C,h,M,\chi)$ for the largest of 0 and the three expressions of l_0 formed with $l = 1-(C-1)h$ if $l > 0$, and $\mathcal{F}(C,h,M,\chi) = 0$ if $l \le 0$. As $l \le 1 \le C$ always and $Y[\Pi_X] \ge 0$, the lemma gives $Y[\Pi_X] \ge (\mathcal{F}(C,h_X,M_X,\chi_X) - o(1))Y$ under its other hypotheses. In the construction the cap is

a product of piece caps: if every block of the piece i has $\log q_{\mathcal{B}} \le L_i$ and $a_* \ge a$, then every block of X lying in the piece i has $q_{\mathcal{B}}^{2a} \le C_i := e^{2a_* L_i}$, so C_X is at most the product $\prod_{\mathcal{B}\in X} C_{i(\mathcal{B})}$ of the caps of the pieces of its blocks (Lemma 20.7), and the lemma is applied with this product as C; the construction takes $a_* = a$ and the numbers L_i of Lemma 20.7.

Remark 24.9 (The three branches). The second expression in l_0 never exceeds the third: the smaller root $\bar\eta_*$ satisfies $l - \bar\eta_* = (\chi_X \bar\eta_* (C - \bar\eta_*))^{1/2} \le \frac{C}{2}\sqrt{\chi_X}$. It is kept because the certificate evaluates all three expressions, and zero (Certificate 24.19); the certificate also checks the condition $0 \le l \le C/2$, whereas the lemma needs only $l \le C$ (at the final caps $l \le 1 < C/2$ holds automatically, every piece cap being larger than 2). The pointwise expression l/M_X needs neither the chi-square bound nor the variance arguments, and it shows the mechanism, but the final restorations rest on the chi-square expressions: at the final caps the bounded-variance expression is the largest at every piece and every pair of pieces (at the first piece about 0.98104, against about 0.98075 for the one-sided and 0.71382 for the pointwise expression), and on the seven pieces with positive roots $M_i > 10^7$. The bounded-variance expressions therefore provide the useful restoration on these pieces; the same exact formulas are reevaluated for their own marginals in Certificate 24.19.

24.4. **Centred norms at the reserved cut.** The last step converts a lower bound for the energy of a product of eligible records into one for the product of its actual half marginals, by the centred interpolation (P3) at ι. It needs, for each record, the norm on the mean-zero space of its own Markov operator at ι, in its own actual marginal space: for the fold records the centred norm $\kappa_F(q,r)$ of Corollary 23.3(c), an admissible function of (q,r) that the certificate evaluates at the piece start, and for the matching records the norm $3/(b-3)$ of the disjoint-triple operator (Lemma 24.10(b) below). For a product of records the centred norm is the largest of the prime centred norms, not their sum (Lemma 6.10(c), Corollary 23.3(d)), because each prime operator fixes the constants of its own actual marginal on both sides; we write $\kappa_F(\mathcal{B})$ for the largest value of $\kappa_F(1/p, p^{-1/2})$ at the primes p of $\mathcal{B}$. Inside κ_F, the kernel of two disjoint cross edges whose four endpoints do not span a complete rectangle has norm at most q for independent uniform labels, which is case (a) of Lemma 16.4, and Lemma 13.24 transfers the bound to the actual marginal (Lemma 23.2(b)).

24.5. **Variance savings for E_0, E_1 via polarization.** The records E_0, E_1 are matchings, not folds, and no chamber diagonal is a matching record. Their energies are reached from those of the two fold records by a polarization at the chamber cut ϑ_0. For a middle block $\mathcal{B}$ let $G_{e,\mathcal{B}}$ and $G_{o,\mathcal{B}}$ be the images of λ_E under the merges $G_e = (1,0,0,0,0,1)$ and $G_o = (0,0,1,1,0,0)$ of Definition 17.1 (two objects used); they are positive normalized laws with uniform one-slot marginals, diagonal at ϑ_0. By the corner table of Lemma 17.2(f) (Table 7), at ϑ_0 the cut matrices of $\Pi_{E_0,\mathcal{B}}$ and $G_{e,\mathcal{B}}$ live on the corner $\mathcal{U}$ of the plus-half configurations (β, y, y), those of $\Pi_{E_1,\mathcal{B}}$ and $G_{o,\mathcal{B}}$ on the corner $\mathcal{V}$ of the configurations (y, y, β), and $\Pi_{F_1,\mathcal{B}}$, $\Pi_{F_2,\mathcal{B}}$ connect the two corners; here β is the label pair of the object that is not repeated. Let f_2, h_2 be the two-layer row coefficients of the piece of $\mathcal{B}$ (Lemma 22.1(i); the index 2 counts the layers), put $\varepsilon_{\mathcal{B}} = (f_2 + h_2) r_{\mathcal{B}}$, and $\kappa_E(\mathcal{B}) = 3/(b_{\mathcal{B}} - 3)$. For a finite set X of middle blocks write $E_{\mathbf{s}} = \bigotimes_{\mathcal{B}\in X} \Pi_{E_{s_{\mathcal{B}}},\mathcal{B}}$, $\mathbf{s} \in \{0,1\}^X$, for the $2^{|X|}$ *matching embeddings*, $J_{\mathbf{s}} = \bigotimes_{\mathcal{B}} J_{E_{s_{\mathcal{B}}},\mathcal{B}}$ for their products of actual half marginals, and E_0^X, E_1^X for the two *aligned* embeddings, with E_0, respectively E_1, on every block.

Lemma 24.10 (Paired E bridge)**.** *Let $\mathcal{B}$ be a middle block.*

(a) As cut matrices at ϑ_0 of laws on the labels of $\mathcal{B}$,

$$\Pi_{E_0,\mathcal{B}} + \Pi_{E_1,\mathcal{B}} + \varepsilon_{\mathcal{B}}(G_{e,\mathcal{B}} + G_{o,\mathcal{B}}) \succeq_{\vartheta_0} \Pi_{F_1,\mathcal{B}} + \Pi_{F_2,\mathcal{B}}.$$

(b) At ι the Markov operator T_E of $\Pi_{E_s,\mathcal{B}}$ on $L^2(m_{E_s,\mathcal{B}})$ is self-adjoint and $\|T_E - \mathbb{J}\| \le \kappa_E(\mathcal{B})$. More generally, every matching embedding $E_{\mathbf{s}}$ of a finite set X of middle blocks is symmetric

at ι, and its Markov operator on L^2 of its actual half marginal has centred norm at most $\max_{\mathcal{B}\in X}\kappa_E(\mathcal{B})$.

Proof. (a) As in the proof of Lemma 22.1(iii), the two-layer laws of $\mathcal{B}$ form a five-type datum (Definition 17.3) on the label pairs of one object: the middle law ν and the conditional references ν^ξ of Lemma 22.1(i), the matching law λ_E, its two-object marginal, the fold law λ_F and the root. By Lemma 22.1(i) the middle object has the law ν under λ_F as well as under λ_E (this uses $p\equiv 3 \pmod 4$), and $\|T_{\mathrm{F}}^\xi-\mathbb{J}\|\le f_2 r_{\mathcal{B}}$, $\|T_{\mathrm{E}}^\xi-\mathbb{J}\|\le h_2 r_{\mathcal{B}}$ on $L^2(\nu^\xi)$ for every ξ. The components of this datum at ϑ_0 are $\Pi_{F_j,\mathcal{B}}$, $\Pi_{E_s,\mathcal{B}}$, $G_{e,\mathcal{B}}$ and $G_{o,\mathcal{B}}$, read on the slot array through the first coordinate. So (a) is Lemma 17.11(a) with $e_{\mathrm{F}}=f_2r_{\mathcal{B}}$, $e_{\mathrm{E}}=h_2r_{\mathcal{B}}$ and $\varepsilon=\varepsilon_{\mathcal{B}}$: the form of the difference of the two sides is at least the average over ξ of $(u_\xi-v_\xi)^2\ge 0$, with u_ξ,v_ξ the conditional means of the two corner functions. As ν^ξ gives no mass to ξ, the datum also satisfies the hypothesis of Lemma 17.11(b).

(b) By Lemma 17.2(d), $\Pi_{E_s,\mathcal{B}}$ is symmetric at ι and its actual half marginal is uniform on ordered triples of distinct labels; given the layer-zero triple, the layer-one triple is uniform among those disjoint from it. So T_E is the tensor product over the primes of $\mathcal{B}$ of the disjoint-triple operators of Lemma 12.11, which are self-adjoint, and part (c) of that lemma gives $\|T_E-\mathbb{J}\|\le 3/(b_{\mathcal{B}}-3)$. An embedding $E_{\mathbf{s}}$ is a product of such records, hence symmetric at ι, and its operator is the tensor product of the operators of its blocks, each fixing the constants of its actual marginal on both sides; so (P4) bounds its centred norm by the largest block value (Lemma 6.10(c)). □

Lemma 24.11 (Matching selections)**.** *Let X be a set of $k\ge 1$ middle blocks, with $\varepsilon_{\mathcal{B}}$, $G_{e,\mathcal{B}}$, $G_{o,\mathcal{B}}$ as above, $G_{0,\mathcal{B}}=G_{e,\mathcal{B}}$, $G_{1,\mathcal{B}}=G_{o,\mathcal{B}}$ and $G_{\mathcal{B}}=G_{e,\mathcal{B}}+G_{o,\mathcal{B}}$.*

(a) At the chamber cut ϑ_0,

$$\bigotimes_{\mathcal{B}\in X}(\Pi_{E_0,\mathcal{B}}+\varepsilon_{\mathcal{B}}G_{e,\mathcal{B}})+\bigotimes_{\mathcal{B}\in X}(\Pi_{E_1,\mathcal{B}}+\varepsilon_{\mathcal{B}}G_{o,\mathcal{B}})\succeq_{\vartheta_0}\bigotimes_{\mathcal{B}\in X}\Pi_{F_1,\mathcal{B}}+\bigotimes_{\mathcal{B}\in X}\Pi_{F_2,\mathcal{B}},$$

and consequently $\bigotimes_{\mathcal{B}}(\Pi_{E_0,\mathcal{B}}+\Pi_{E_1,\mathcal{B}}+\varepsilon_{\mathcal{B}}G_{\mathcal{B}})\succeq_{\vartheta_0}\bigotimes_{\mathcal{B}}\Pi_{F_1,\mathcal{B}}+\bigotimes_{\mathcal{B}}\Pi_{F_2,\mathcal{B}}$.

(b) Let $C\ge C_X$, $\varepsilon_X\ge\prod_{\mathcal{B}}(1+\varepsilon_{\mathcal{B}})-1$, and suppose $Y[\bigotimes_{\mathcal{B}}\Pi_{F_j,\mathcal{B}}]\ge(l_0-o(1))Y$ for $j=1$ and $j=2$. Then

$$\tfrac12\big(Y[E_0^X]+Y[E_1^X]\big)\ge(l_0-C\varepsilon_X-o(1))Y,$$
$$2^{-k}\textstyle\sum_{\mathbf{s}}Y[E_{\mathbf{s}}]\ge(2^{1-k}l_0-C\varepsilon_X-o(1))Y.$$

(c) Under the hypotheses of (b), let m_s^X be the actual half marginal of E_s^X at ι. Then

$$Y[\mathrm{Diag}_\iota(m_0^X)]+Y[\mathrm{Diag}_\iota(m_1^X)]\ge 2\big([l_0-C\varepsilon_X]_+-o(1)\big)Y.$$

By Lemma 24.8 one may take for l_0 the restoration bound of X: the same h_X, M_X, χ_X serve $j=1$ and $j=2$, because the half law of the root does not depend on the chamber (also for the positive roots, Lemma 24.7(iv)). No lower bound is asserted for an individual matching embedding, nor for the diagonal of an individual aligned embedding.

Proof. (a) Both orders are Lemma 17.11(c), applied to the two-layer five-type data of the blocks of X, which satisfy the hypotheses of Lemma 17.11(b) with the constants $\varepsilon_{\mathcal{B}}$ (proof of Lemma 24.10(a)). The first is the compression to the two aligned corners of the tensor product of the one-block brackets, the sign of the fold part being reversed on the corner $\mathcal{V}$; the second adds the 2^k-2 nonaligned corner products $\bigotimes_{\mathcal{B}}(\Pi_{E_{s_{\mathcal{B}}},\mathcal{B}}+\varepsilon_{\mathcal{B}}G_{s_{\mathcal{B}},\mathcal{B}})$, which are tensor products of positive semidefinite diagonal blocks of these brackets. The order is not obtained by tensoring a one-sided order with an indefinite factor.

(b) Apply (E2) at ϑ_0, with μ outside, to the two orders of (a). In the first order, expanding $\bigotimes_{\mathcal{B}}(\Pi_{E_s,\mathcal{B}}+\varepsilon_{\mathcal{B}}G_{s,\mathcal{B}})$ gives E_s^X and products that contain at least one merge law; these are positive

measures with uniform one-slot marginals of total mass $\prod_{\mathcal{B}}(1+\varepsilon_{\mathcal{B}})-1$, so by (E1)

$$Y[E_0^X]+Y[E_1^X]\ge 2(l_0-o(1))Y-2C\Big(\prod_{\mathcal{B}}(1+\varepsilon_{\mathcal{B}})-1\Big)Y.$$

In the second order the merge products have total mass $\prod_{\mathcal{B}}(2+2\varepsilon_{\mathcal{B}})-2^k$, so $\sum_{\mathbf{s}} Y[E_{\mathbf{s}}]\ge 2(l_0-o(1))Y-2^kC(\prod_{\mathcal{B}}(1+\varepsilon_{\mathcal{B}})-1)Y$. Divide by 2, respectively 2^k.

(c) By Lemma 24.10(b), E_s^X is a positive law symmetric at ι, a probability coupling of two copies of m_s^X; so the contraction (P1), applied to the whole product, gives $Y[E_s^X]\le Y[\mathrm{Diag}_\iota(m_s^X)]$, and no one-sided order is tensored with an indefinite factor. Adding the two bounds and using the first bound of (b) gives the claim with $l_0-C\varepsilon_X$ in place of its positive part; and $Y[\mathrm{Diag}_\iota(m_s^X)]\ge 0$ by (E3) and (E2). □

The factor 2^{1-k} in the second bound of (b) is a genuine loss: only the two aligned corners carry the restoration. The first bound has no such loss, and its merge correction is $2\varepsilon_X$ against $2^k\varepsilon_X$ for the second. For two blocks, $l_0-C\varepsilon_X$ exceeds twice $l_0/2-C\varepsilon_X$, which is why the aligned gain is at least twice the four-embedding gain (Corollary 24.13), and why the construction credits only the aligned matching classes. Part (c) is the diagonal-mean coupon $2[l_0-C\varepsilon_X]_+$ of the matching diagonals of the mean credit (Proposition 26.7), for any number of blocks.

24.6. **The saving g: product energy at least gC_XY.** A *selection* of the record expansion (Lemma 10.14) puts on each block of a finite set X one changed record and one parity, symmetric or skew, with an even number of skew parities; I_ω is its term in $Y_\nu-Y_\lambda$ and π_ω its price, the product over $\mathcal{B}\in X$ of the prices of its records with their parities. For a record assignment $\boldsymbol{\phi}=(\phi_{\mathcal{B}})_{\mathcal{B}\in X}$ of eligible records put $\Pi_{\boldsymbol{\phi}}=\bigotimes_{\mathcal{B}}\Pi_{\phi_{\mathcal{B}},\mathcal{B}}$, $m_{\boldsymbol{\phi}}=\bigotimes_{\mathcal{B}} m_{\phi_{\mathcal{B}},\mathcal{B}}$ and $J_{\boldsymbol{\phi}}=\bigotimes_{\mathcal{B}} J_{\phi_{\mathcal{B}},\mathcal{B}}$.

Lemma 24.12 (Variance gain)**.** *Let X be a nonempty finite set of middle blocks, with a parity for each $\mathcal{B}\in X$, an even number of them skew. Let $\mathcal{R}$ be a nonempty finite set of record assignments $\boldsymbol{\phi}$ of eligible records, and let $\omega_{\boldsymbol{\phi}}$ be the selection with the records $\phi_{\mathcal{B}}$ and the fixed parities. Suppose that, for every $\mathcal{B}$, the price of $\phi_{\mathcal{B}}$ with the parity of $\mathcal{B}$ is the same for all $\boldsymbol{\phi}\in\mathcal{R}$. Let $C\ge C_X$, $l_0\in\mathbb{R}$ and $0\le\kappa<1$, and assume:*

(Res) $|\mathcal{R}|^{-1}\sum_{\boldsymbol{\phi}\in\mathcal{R}} Y[\Pi_{\boldsymbol{\phi}}]\ge(l_0-o(1))Y$;

(Cen) *for every $\boldsymbol{\phi}\in\mathcal{R}$, $\Pi_{\boldsymbol{\phi}}$ is symmetric at ι, and its Markov operator $T_{\boldsymbol{\phi}}$ on $L^2(m_{\boldsymbol{\phi}})$ satisfies $\|T_{\boldsymbol{\phi}}-\mathbb{J}\|\le\kappa$.*

Put $\bar l=(l_0/C-\kappa)/(1-\kappa)$ and $g=[\bar l-R_{\mathrm{err}}]_+$. Then:

(a) $|\mathcal{R}|^{-1}\sum_{\boldsymbol{\phi}} Y[J_{\boldsymbol{\phi}}]\ge(\bar l-o(1))C_XY$;

(b) $\sum_{\boldsymbol{\phi}}|I_{\omega_{\boldsymbol{\phi}}}|\le(B-g+o(1))\sum_{\boldsymbol{\phi}}\pi_{\omega_{\boldsymbol{\phi}}}Y$;

(c) if $l_0\le C$, then $0\le g\le 1\le B$, so the selections $\omega_{\boldsymbol{\phi}}$, $\boldsymbol{\phi}\in\mathcal{R}$, form a class of cost $B-g$ in the sense of (CP5).

Any upper bound for R_{err} may replace it in g.

Proof. (a) Fix $\boldsymbol{\phi}$. By (Cen), the centred interpolation (P3) at ι, with μ outside and $\kappa_0=\kappa$, gives $Y[\Pi_{\boldsymbol{\phi}}]\le(1-\kappa)Y[J_{\boldsymbol{\phi}}]+\kappa Y[\mathrm{Diag}_\iota(m_{\boldsymbol{\phi}})]$, as $J_{\boldsymbol{\phi}}=J_\iota(m_{\boldsymbol{\phi}})$; and (E1) bounds the last energy by $(1+o(1))C_XY$. Averaging over $\mathcal{R}$ and using (Res), $(1-\kappa)|\mathcal{R}|^{-1}\sum_{\boldsymbol{\phi}}Y[J_{\boldsymbol{\phi}}]\ge(l_0-\kappa C_X-o(1))Y$. If $\bar l\le 0$, (a) holds because $Y[J_{\boldsymbol{\phi}}]\ge 0$ by (E2), the cut operator of $J_{\boldsymbol{\phi}}$ being a projection. If $\bar l>0$, then $l_0>0$, so $l_0\ge l_0C_X/C$ and $l_0-\kappa C_X\ge(1-\kappa)\bar lC_X$.

(b) For $\boldsymbol{\phi}\in\mathcal{R}$ and $\mathcal{B}\in X$, the change of $\phi_{\mathcal{B}}$ with the parity of $\mathcal{B}$ has, relative to $m_{\phi_{\mathcal{B}},\mathcal{B}}$, a cut operator $D_{\mathcal{B}}$ at ι that is self-adjoint or skew-adjoint, annihilates the constants on both sides by (24.1), and has norm at most the coefficient $\tilde\pi_{\mathcal{B}}$ of its norm envelope (Definition 9.14); by

hypothesis $\tilde\pi_{\mathcal B}$ does not depend on $\boldsymbol\phi$, and $\pi_{\omega_{\boldsymbol\phi}} = \prod_{\mathcal B} \tilde\pi_{\mathcal B}\, C_X$. By Lemma 10.16(i),(ii), applied with these $D_{\mathcal B}$,

$$|I_{\omega_{\boldsymbol\phi}}| \le \prod_{\mathcal B} \tilde\pi_{\mathcal B}(Y^{\mathsf N}[\mathrm{Diag}_\iota(m_{\boldsymbol\phi})] - Y^{\mathsf N}[J_{\boldsymbol\phi}]) \le \prod_{\mathcal B} \tilde\pi_{\mathcal B}((B + R_{\mathrm{err}} + o(1))C_X Y - Y[J_{\boldsymbol\phi}]).$$

Summing over $\mathcal R$ and using (a) gives $\sum |I_{\omega_{\boldsymbol\phi}}| \le (B - (\bar l - R_{\mathrm{err}}) + o(1)) \sum \pi_{\omega_{\boldsymbol\phi}} Y$. If $\bar l - R_{\mathrm{err}} \le 0$ use instead $|I_{\omega_{\boldsymbol\phi}}| \le \prod_{\mathcal B} \tilde\pi_{\mathcal B}\, Y^{\mathsf N}[\mathrm{Diag}_\iota(m_{\boldsymbol\phi})] \le (B + o(1))\pi_{\omega_{\boldsymbol\phi}} Y$, valid because $Y^{\mathsf N}[J_{\boldsymbol\phi}] \ge 0$ by (E2). A larger R_{err} only lowers g.

(c) If $l_0 \le C$ then $\bar l \le 1$, so $g \le 1$; and $B = e^{U_0} + R_{\mathrm{err}} \ge 1$. □

The lemma uses the parities only through the evenness of the number of skew factors. It applies to two symmetric parities as well as to two skew ones, and selections with different parities, records or blocks are different monomials of the record expansion.

Corollary 24.13 (The eligible variance classes)**.** *Let X be a set of $k \ge 1$ middle blocks with fixed parities as in Lemma 24.12, and let $C \ge C_X$. Let h_X, M_X, χ_X be as in Lemma 24.8, with $0 < l = 1-(C-1)h_X \le C$, and let l_0 be the largest of 0 and the three expressions of that lemma. Let $\kappa_F \ge \max_{\mathcal B\in X} \kappa_F(\mathcal B)$ and $\kappa_E \ge \max_{\mathcal B\in X} \kappa_E(\mathcal B)$ be smaller than one, and $\varepsilon_X \ge \prod_{\mathcal B\in X}(1+\varepsilon_{\mathcal B}) - 1$. Assume that the two matching records of every block of X carry the same prices, as they do in the comparison law (Section 24.1). Then Lemma 24.12 applies to*

- *(i) the aligned fold assignment $(F_j)_{\mathcal B\in X}$, $j \in \{1,2\}$, with l_0 and κ_F, so that $\bar l_F = (l_0/C - \kappa_F)/(1-\kappa_F)$;*
- *(ii) the two aligned matching assignments, with $l_0 - C\varepsilon_X$ and κ_E, so that $\bar l_E^{\mathrm{al}} = (l_0/C - \varepsilon_X - \kappa_E)/(1-\kappa_E)$;*
- *(iii) the 2^k matching embeddings, with $2^{1-k}l_0 - C\varepsilon_X$ and κ_E, so that $\bar l_E = (2^{1-k}l_0/C - \varepsilon_X - \kappa_E)/(1-\kappa_E)$.*

For $k = 1$, (ii) and (iii) coincide; in general $[\bar l_E^{\mathrm{al}} - R_{\mathrm{err}}]_+ \ge 2^{k-1}[\bar l_E - R_{\mathrm{err}}]_+$.

Proof. (i) (Res) is Lemma 24.8, where the maximum with 0 is allowed because $Y[\bigotimes_{\mathcal B} \Pi_{F_j,\mathcal B}] \ge 0$; (Cen) is Corollary 23.3(d). (ii), (iii) (Res) is Lemma 24.11(b), with the restoration bound for $j = 1$ and $j = 2$; (Cen) is Lemma 24.10(b). The prices are common by assumption in (ii), (iii) and trivially in (i). Finally $2^{k-1}\bar l_E = (l_0/C - 2^{k-1}(\varepsilon_X + \kappa_E))/(1-\kappa_E) \le \bar l_E^{\mathrm{al}}$, and $R_{\mathrm{err}} \ge 0$, $2^{k-1} \ge 1$ give the last claim. □

Remark 24.14 (Repair independence). The numbers l_0, κ_F, κ_E and ε_X, hence the gains, depend only on the reference, its records, the bracket (through B and R_{err}) and the cap C. A repair enters Lemma 24.12 only through the zero-marginal identity (24.1) and through its prices. The credits therefore hold for every repair in the sense of Definition 24.1, signed or not, in particular for the repairs of the final construction.

Selections that put F_1 on one block and F_2 on another, or a fold record on one block and a matching record on another, receive no variance credit: F_1 and F_2 are diagonal at different chamber cuts, so no chamber diagonal is their product, and the restoration needs a common chamber. Their skew classes are priced by positive envelopes (Theorem 25.8).

24.7. **Additional positive terms in the exterior energy.** The gains of Lemma 24.12 use the selected blocks only. Outside them the exterior bracket $\mathsf N_0$ exceeds μ by positive reserved-diagonal records (Definition 22.4, Lemma 22.7(i)), and the variance form may use the energy $Y^{\mathsf N_0}[J_X]$ with $\mathsf N_0$ outside in place of $Y[J_X]$, at the cost of one comparison of $\mathsf N$ with $\mathsf N_0$ (Lemma 10.16(iii)). Among these records are the two matching diagonals of E_{add} of every middle block, which are diagonal copies of the actual half marginals of the two matching records. A selected aligned matching product therefore gains, from every set of other blocks carrying E_{add}, the energy of its

product of actual half marginals tensored with the diagonals of these blocks, and a joint restoration of the selected and the other blocks bounds this energy below. This subsection bounds these *positive background* terms: on one or two other blocks of any pieces, with a lower bound for the weighted number of distinct choices of these blocks, and on three or four blocks of a pool of late pieces, where uniform data suffice. The bracket has one fold copy, which is a summand of N and not of N_0 (Lemma 22.7); so the exterior contains no fold diagonal, and a selected fold variance has no positive background term of this kind.

Piece data. The bounds are stated for *piece data*: the middle blocks are grouped into the pieces $[X_i, Y_i)$, every block of the piece i has least prime at least X_i, and numbers $h_i \ge 0$, $M_i \ge 1$, $\chi_i \ge 0$, $\varepsilon_i \ge 0$, $k_{F,i}, k_{E,i} \in [0,1)$, $C_i \ge 1$ and $\Sigma_i \ge 0$ satisfy, for every block $\mathcal{B}$ of the piece i,

$$h_{\mathcal{B}} \le h_i, \qquad M_{\mathcal{B}} \le M_i, \qquad \chi^2(\nu_{\mathrm{L},\mathcal{B}} \| \lambda_{F,\mathcal{B}}) \le \chi_i, \qquad \varepsilon_{\mathcal{B}} \le \varepsilon_i,$$
$$\kappa_F(\mathcal{B}) \le k_{F,i}, \qquad \kappa_E(\mathcal{B}) \le k_{E,i}, \qquad q_{\mathcal{B}}^{2a} \le C_i,$$

and $\Sigma_i \le \sum_{\mathcal{B} \subseteq [X_i, Y_i)} b_{\mathcal{B}}^{-1}$. For a finite multiset m of pieces, with multiplicities m_i and size $|m|$, put

$$h_m = \prod_{i \in m} (1 + h_i) - 1, \quad M_m = \prod_{i \in m} M_i, \quad \chi_m = \prod_{i \in m} (1 + \chi_i) - 1, \quad \eta_m = \prod_{i \in m} (1 + \varepsilon_i) - 1,$$
$$C_m = \prod_{i \in m} C_i, \qquad k_{F,m} = \max_{i \in m} k_{F,i}, \qquad k_{E,m} = \max_{i \in m} k_{E,i}, \qquad \ell_m = \mathcal{F}(C_m, h_m, M_m, \chi_m),$$

the products with multiplicity, $C_\varnothing = 1$, and $m + r$ the union of two multisets. A set X of distinct middle blocks has the *pattern* m if m_i of its blocks lie in the piece i. Then the products over X of $1 + h_{\mathcal{B}}$, of $M_{\mathcal{B}}$, of $1 + \chi^2(\nu_{\mathrm{L},\mathcal{B}} \| \lambda_{F,\mathcal{B}})$ and of $1 + \varepsilon_{\mathcal{B}}$ are at most $1 + h_m$, M_m, $1 + \chi_m$ and $1 + \eta_m$, and $C_X \le C_m$; so Lemma 24.8 gives $Y[\bigotimes_{\mathcal{B} \in X} \Pi_{F_j,\mathcal{B}}] \ge (\ell_m - o(1))Y$ for $j = 1, 2$, and Corollary 24.13 applies with $C = C_m$, $\kappa_F = k_{F,m}$, $\kappa_E = k_{E,m}$ and $\varepsilon_X = \eta_m$ (if $1 - (C_m - 1)h_m \le 0$, then $\ell_m = 0$ and Lemma 24.12 applies directly with $l_0 = 0$). Its numbers (i), (ii) are the *margins without background blocks* (called selected-only margins in the data)

$$b_F(m) = \frac{\ell_m / C_m - k_{F,m}}{1 - k_{F,m}}, \qquad b_E(m) = \frac{\ell_m / C_m - \eta_m - k_{E,m}}{1 - k_{E,m}}. \tag{24.8}$$

The coefficients of the two matching diagonals of E_{add}, divided by the normalizer $\widetilde{\mu}_{\mathcal{B}}(1) = 1 + h_{\mathcal{B}}$, are $v_{\mathrm{D}} r_{\mathcal{B}}^2 / (1 + h_{\mathcal{B}})$ (Definition 22.4(b)); in the piece i they are at least $c_{E,i} b_{\mathcal{B}}^{-1}$, where

$$c_{E,i} = \frac{v_{\mathrm{D},i}}{1 + h_i},$$

with $v_{\mathrm{D},i}$ the completion weight of the piece.

Lemma 24.15 (Distinct choices)**.** *Let $c = (c_i)$ be nonnegative numbers indexed by the pieces, m a multiset of pieces, X a set of distinct middle blocks of pattern m, and write $i(\mathcal{B})$ for the piece of a block $\mathcal{B}$.*

(a) For a multiset r of pieces, the sum of $\prod_{\mathcal{B} \in Z} c_{i(\mathcal{B})} b_{\mathcal{B}}^{-1}$ over the sets Z of middle blocks of pattern r disjoint from X is at least the choice weight $W(m, r; c)$ (not the wrapping term W),

$$W(m, r; c) = \prod_i \frac{c_i^{r_i}}{r_i!} \prod_{j=0}^{r_i - 1} \Big(\Sigma_i - \frac{m_i + j}{X_i} \Big)_+. \tag{24.9}$$

(b) Let $\mathcal{P}$ be a set of pieces and put $\mathfrak{m}(c) = \sum_{i \in \mathcal{P}} c_i \Sigma_i$, $\mathfrak{u}(c) = \max_{i \in \mathcal{P}} c_i / X_i$ and $\mathfrak{e}_m(c) = \sum_{i \in \mathcal{P}} m_i c_i / X_i$. For $k \ge 1$ the same sum over the sets Z of k distinct blocks of pieces of $\mathcal{P}$,

disjoint from X, is at least

$$\mathfrak{w}_k(c;m)=\frac{1}{k!}\prod_{j=0}^{k-1}\big[\mathfrak{m}(c)-\mathfrak{e}_m(c)-j\,\mathfrak{u}(c)\big]_+.$$

Proof. Every block of the piece i has $b_{\mathcal{B}}^{-1}\le 1/X_i$. (a) The choices in different pieces are independent, so it suffices to treat one piece i, and to count ordered r_i-tuples of distinct blocks, then divide by $r_i!$. The blocks of the piece outside X have $\sum b_{\mathcal{B}}^{-1}\ge \Sigma_i - m_i/X_i$, since X removes m_i of them; after j more choices the remaining blocks have $\sum b_{\mathcal{B}}^{-1}\ge \Sigma_i-(m_i+j)/X_i$. Summing successively from the last choice, each partial sum is at least the positive part of its bound, whatever the earlier choices; multiply by $c_i^{r_i}$. (b) The same argument on the union of the pieces of $\mathcal{P}$: the blocks of $\mathcal{P}$ outside X have $\sum c_{i(\mathcal{B})}b_{\mathcal{B}}^{-1}\ge \mathfrak{m}(c)-\mathfrak{e}_m(c)$, and each later choice removes a block of weight $c_{i(\mathcal{B})}b_{\mathcal{B}}^{-1}\le \mathfrak{u}(c)$; divide by $k!$. □

Pooling the pieces from 2^{31}. Let $\mathcal{P}$ be the set of middle pieces with $X_i\ge 2^{31}$, and give it the *pool data* $h_{\mathcal{P}}=\max_{i\in\mathcal{P}}h_i$ and likewise $M_{\mathcal{P}}$, $\chi_{\mathcal{P}}$, $\varepsilon_{\mathcal{P}}$, and the cap $C_{\mathcal{P}}=\max_{i\in\mathcal{P}}C_i$. For a multiset m and $k\ge 1$ let $m+\mathcal{P}^k$ denote m with k additional entries carrying the pool data, so that $h_{m+\mathcal{P}^k}$, $M_{m+\mathcal{P}^k}$, $\chi_{m+\mathcal{P}^k}$, $\eta_{m+\mathcal{P}^k}$, $C_{m+\mathcal{P}^k}=C_mC_{\mathcal{P}}^k$ and $\ell_{m+\mathcal{P}^k}$ are defined as above, and write $c^{\mathcal{P}}$ for a coefficient vector c restricted to $\mathcal{P}$ (zero off $\mathcal{P}$).

Lemma 24.16 (Terms pooled over the pieces from 2^{31})**.** *If X has pattern m and Z is a set of k distinct blocks of pieces of $\mathcal{P}$ disjoint from X, then the four products over $X\cup Z$ of the piece-data paragraph are at most those of $m+\mathcal{P}^k$, and $C_{X\cup Z}\le C_mC_{\mathcal{P}}^k$; so $Y[\bigotimes_{\mathcal{B}\in X\cup Z}\Pi_{F_j,\mathcal{B}}]\ge(\ell_{m+\mathcal{P}^k}-o(1))Y$ for $j=1,2$, and Lemma 24.11(b), (c) hold for $X\cup Z$ with $l_0=\ell_{m+\mathcal{P}^k}$, $C=C_mC_{\mathcal{P}}^k$ and $\varepsilon_{X\cup Z}\le\eta_{m+\mathcal{P}^k}$. Moreover:*

(a) Variance. *For $|m|\le 2$ and an upper bound $R_*\ge R_{\mathrm{err}}$ put*

$$P_E(m)=\sum_{k\in\{3,4\}}\mathfrak{w}_k(c_E^{\mathcal{P}};m)\,\frac{[\ell_{m+\mathcal{P}^k}/C_m-C_{\mathcal{P}}^k(\eta_{m+\mathcal{P}^k}+k_{E,m})]_+}{1-k_{E,m}}$$

if $b_E(m)\ge R_*$, and $P_E(m)=0$ otherwise. This term may be added to the gain of the aligned matching variance of Proposition 24.17, whose proof shows it.

(b) Mean. *Let $k\in\{5,6\}$ and let Z range over the sets of k distinct blocks of pieces of $\mathcal{P}$. For coefficient vectors d,d' with $0\le d_i\le\rho\,d'_i$ on $\mathcal{P}$, for some $\rho\in[0,1]$,*

$$\sum_Z\prod_{\mathcal{B}\in Z}d_{i(\mathcal{B})}b_{\mathcal{B}}^{-1}\ge\mathfrak{w}_k(d^{\mathcal{P}};\varnothing),\qquad\sum_Z\Big[\prod_{\mathcal{B}\in Z}d'_{i(\mathcal{B})}b_{\mathcal{B}}^{-1}-\prod_{\mathcal{B}\in Z}d_{i(\mathcal{B})}b_{\mathcal{B}}^{-1}\Big]\ge(1-\rho^k)\,\mathfrak{w}_k(d'^{\mathcal{P}};\varnothing).$$

With the restorations of the first assertion, these give the late-pool coupons of the mean credit (Proposition 26.7),

$$m_F^{\mathrm{pool}}=\sum_{k\in\{5,6\}}\ell_{\mathcal{P}^k}\Big[2\mathfrak{w}_k(c_F^{\mathcal{P}};\varnothing)+\mathfrak{w}_k\big((k_{\mathrm{L}}/H)^{\mathcal{P}};\varnothing\big)+\mathfrak{w}_k\big((k_{\mathrm{R}}/H)^{\mathcal{P}};\varnothing\big)\Big],$$

$$m_Q^{\mathrm{pool}}=\sum_{k\in\{5,6\}}2(1-\rho^k)\,\mathfrak{w}_k\big((c_E+q)^{\mathcal{P}};\varnothing\big)\big[\ell_{\mathcal{P}^k}-C_{\mathcal{P}}^k\eta_{\mathcal{P}^k}\big]_+,$$

with $c_{F,i}=c_{E,i}/H_i$, the density factors H_i of (24.11), the side coefficients $k_{s,i}$ of Proposition 26.4, the coefficients q_i of the chosen matching diagonals (Section 26.3) and $\rho=\max_{i\in\mathcal{P}}c_{E,i}/(c_{E,i}+q_i)$. They occupy the degrees five and six, disjoint from the degrees at most four of the other coupons.

Proof. Every block of a piece of $\mathcal{P}$ has data at most the pool data and $q_{\mathcal{B}}^{2a}\le C_i\le C_{\mathcal{P}}$, so the products are bounded as stated; the pool data are therefore valid hypotheses of Lemma 24.8 and of

Lemma 24.11(b), (c), whose conclusions follow. (a) The bound that justifies the term is part of the proof of Proposition 24.17. If $b_E(m) \ge R_*$ then $b_E(m) - R_* \ge 0$, so after the positive part of the remaining gain is taken, this difference is exactly the term the certificate adds; the complementary term $[R_* - b_E(m)]_+$ is then zero. (b) The first bound is Lemma 24.15(b) with m empty. For the second, $\prod_{\mathcal{B}\in Z} d_{i(\mathcal{B})} \le \rho^k \prod_{\mathcal{B}\in Z} d'_{i(\mathcal{B})}$ pointwise, so the bracket is at least $(1-\rho^k)\prod_{\mathcal{B}\in Z} d'_{i(\mathcal{B})} b_{\mathcal{B}}^{-1}$; apply the first bound to d'. In the coupons the fold diagonals of k pool blocks have, by the first assertion and Lemma 26.6(a) (the energy bound for the one-layer companions $D^j_{\mathcal{B}}$ of the fold records, proved in Section 26), applied with the set Z, energy at least $\ell_{\mathcal{P}^k}$ divided by the product of the H_i, and the two aligned matching diagonals have energies adding up to at least $2[\ell_{\mathcal{P}^k} - C^k_{\mathcal{P}}\eta_{\mathcal{P}^k}]_+$ (Lemma 24.11(c)). The coefficient per block of the polarized fold branch is at least $c_{E,i}b_{\mathcal{B}}^{-1}$ and that of a chosen side diagonal $k_{s,i}b_{\mathcal{B}}^{-1}$, which with the factors $1/H_i$ of the energies give the vectors c_F and k_s/H; the nonempty-Q branch (the terms in the expansion of $\prod_{\mathcal{B}\in Z}(c_{E,i(\mathcal{B})} + q_{i(\mathcal{B})})b_{\mathcal{B}}^{-1}$ in which the q-factor is chosen for at least one block) has the coefficient $\prod(c_E+q)b^{-1} - \prod c_E b^{-1}$, to which the second bound applies with $d = c_E$, $d' = c_E + q$. The assembly of the branches is that of Proposition 26.7. □

Proposition 24.17 (Positive backgrounds in a selected variance)**.** *Assume the piece data above. For multisets m, r of pieces put*

$$p_E(m,r) = \frac{[\ell_{m+r}/C_m - C_r(\eta_{m+r} + k_{E,m})]_+}{1 - k_{E,m}},$$

and for $|m| \in \{1,2\}$ and an upper bound $R_* \ge R_{\mathrm{err}}$

(24.10)

$$g_E(m) = \Big[b_E(m) + \sum_{1\le|r|\le 2} W(m,r;c_E)\, p_E(m,r) + P_E(m) - R_*\Big]_+, \qquad g_F(m) = \big[b_F(m) - R_*\big]_+,$$

with $P_E(m)$ of Lemma 24.16(a). Let X be a set of distinct middle blocks of pattern m, with parities, an even number of them skew.

(a) With J_s the product of the actual half marginals of the aligned embedding E_s^X,

$$\tfrac{1}{2}(Y^{\mathsf{N}_0}[J_0] + Y^{\mathsf{N}_0}[J_1]) \ge \Big(C_m\Big[b_E(m) + \sum_{1\le|r|\le 2} W(m,r;c_E)p_E(m,r) + P_E(m)\Big] - o(1)\Big)Y.$$

If $g_E(m) \le 1$, the class of the two aligned matching assignments $(E_0)_{\mathcal{B}\in X}$, $(E_1)_{\mathcal{B}\in X}$ with these parities has the class cost $B - g_E(m)$ in the sense of (CP5); in $g_E(m)$ the error R_* of comparing N with N_0 outside X is subtracted once, for all the kept terms together.

(b) For $j \in \{1,2\}$ the selection of F_j on X with these parities has the class cost $B - g_F(m)$.

Proof. (a) Fix $s \in \{0,1\}$ and write $D_{Z,s} = \bigotimes_{\mathcal{B}\in Z} D_{E_s,\mathcal{B}}$ for a set Z of middle blocks disjoint from X, with the selected diagonals $D_{E_s,\mathcal{B}} = \mathrm{Diag}_\iota(m_{E_s,\mathcal{B}})$ of Definition 24.1.

Exterior expansion. On a middle block $\mathcal{B} \notin X$, $(\mathsf{N}_0 - \mu)_{\mathcal{B}}$ is a positive reserved-diagonal measure (Lemma 22.7(i)), and it contains the matching diagonal of E_{add} of type s: the normalized measure $(\mathsf{N}_0 - \mu)_{\mathcal{B}} - v_{\mathrm{D}} r^2_{\mathcal{B}}(1+h_{\mathcal{B}})^{-1} D_{E_s,\mathcal{B}}$ is again positive and reserved diagonal, because the half law of $\Pi^{(1)}_{E_s}$ is $m_{E_s,\mathcal{B}}$ (Definition 22.4(b) and the paragraph after it); outside the middle $\mathsf{N}_{0,\mathcal{B}} = \mu_{\mathcal{B}}$. Expand $\mathsf{N}_{0,-X} = \bigotimes_{\mathcal{B}\notin X}(\mu_{\mathcal{B}} + (\mathsf{N}_0 - \mu)_{\mathcal{B}})$ over the sets Z of middle blocks outside X. The term of Z is the energy, with μ outside $X \cup Z$, of $J_s \otimes (\mathsf{N}_0 - \mu)_Z$, a tensor product of measures $\succeq_\iota 0$ (J_s has a projection as cut operator, the others are positive and reserved diagonal, (E3)); by (E2) it is nonnegative, and, since $v_{\mathrm{D}} r^2_{\mathcal{B}}(1+h_{\mathcal{B}})^{-1} \ge c_{E,i(\mathcal{B})} b_{\mathcal{B}}^{-1}$ ($r^2_{\mathcal{B}} = b_{\mathcal{B}}^{-1}$, $h_{\mathcal{B}} \le h_i$) and the difference of the two tensor products is a sum of tensor products of measures $\succeq_\iota 0$, it is at least $\prod_{\mathcal{B}\in Z} c_{E,i(\mathcal{B})} b_{\mathcal{B}}^{-1} \cdot Y[J_s \otimes D_{Z,s}]$. We keep the term $Z = \varnothing$, the terms with $1 \le |Z| \le 2$, and, if $b_E(m) \ge R_*$, the terms with Z a set of three or four blocks of the pool; they are distinct terms of

the expansion, and the others are dropped. The middle range has finitely many blocks, so the $o(1)$ below add up to $o(1)$.

The selected term. By Lemma 24.10(b), E_s^X is symmetric at ι with centred norm at most $k = k_{E,m}$, so (P3) gives $(1-k)Y[J_s] \ge Y[E_s^X] - k(1+o(1))C_X Y$. Averaging over s, using Lemma 24.11(b) with $l_0 = \ell_m$, $C = C_m$, $\varepsilon_X \le \eta_m$, and $C_X \le C_m$, gives $\frac{1}{2}\sum_s Y[J_s] \ge (C_m b_E(m) - o(1))Y$.

A background term. Let Z have pattern r. By (P3′) with $\pi = E_s^X$ ($\kappa_0 = k$) and $\pi' = E_s^Z$, whose half marginal at ι is $\bigotimes_{\mathcal{B}\in Z} m_{E_s,\mathcal{B}}$,

$$Y[E_s^{X\cup Z}] \le (1-k)\, Y[J_s \otimes D_{Z,s}] + k\, Y[\mathrm{Diag}_\iota(m_s^X \otimes m_s^Z)] \le (1-k)\, Y[J_s \otimes D_{Z,s}] + k(1+o(1))C_{X\cup Z} Y,$$

where $E_s^{X\cup Z} = E_s^X \otimes E_s^Z$ is the aligned embedding of $X \cup Z$. The set $X \cup Z$ has pattern $m + r$, so Lemma 24.11(b) applies to it with $l_0 = \ell_{m+r}$, $C = C_m C_r \ge C_{X\cup Z}$ and $\varepsilon_{X\cup Z} \le \eta_{m+r}$. Averaging over s, and using $Y[J_s \otimes D_{Z,s}] \ge 0$ ((E2)),

$$\tfrac{1}{2}\sum_s Y[J_s \otimes D_{Z,s}] \ge (C_m p_E(m,r) - o(1))Y.$$

The coefficient $\prod_{\mathcal{B}\in Z} c_{E,i(\mathcal{B})} b_{\mathcal{B}}^{-1}$ is the same for $s = 0$ and $s = 1$, and its sum over the sets Z of pattern r disjoint from X is at least $W(m, r; c_E)$ by Lemma 24.15(a). For a set Z of $k' \in \{3, 4\}$ blocks of the pool, the same argument with the first assertion of Lemma 24.16 in place of the piece data of $X \cup Z$ gives the bound with $[\ell_{m+\mathcal{P}^{k'}}/C_m - C_{\mathcal{P}}^{k'}(\eta_{m+\mathcal{P}^{k'}} + k)]_+/(1-k)$, and the coefficients sum to at least $\mathfrak{w}_{k'}(c_E^{\mathcal{P}}; m)$ by Lemma 24.15(b). Adding the kept terms gives the displayed bound.

The class cost. For $s \in \{0, 1\}$ let ω_s be the selection of E_s^X with the given parities, and $\gamma = \prod_{\mathcal{B}} \tilde{\pi}_{\mathcal{B}}$ the common coefficient of its norm envelopes (the two matching records of a block carry the same prices), so that $\pi_{\omega_s} = \gamma C_X$. By Lemma 10.16(i), (ii) and the last bound of (E1), $|I_{\omega_s}| \le \gamma((B + o(1))C_X Y - Y^{\mathsf{N}}[J_s])$ and $Y^{\mathsf{N}}[J_s] \ge Y^{\mathsf{N}_0}[J_s] - (R_{\mathrm{err}} + o(1))C_X Y$; this is the comparison of N with N_0, made once for all the kept terms together. Let $\tilde{b}$ be the bracket in the displayed bound. If $\tilde{b} > R_*$, then $\tilde{b} > 0$ and $C_m \tilde{b} \ge C_X \tilde{b}$, so summing over s,

$$|I_{\omega_0}| + |I_{\omega_1}| \le (B - (\tilde{b} - R_{\mathrm{err}}) + o(1))(\pi_{\omega_0} + \pi_{\omega_1})Y \le (B - g_E(m) + o(1))(\pi_{\omega_0} + \pi_{\omega_1})Y.$$

If $\tilde{b} \le R_*$, then $g_E(m) = 0$, and $|I_{\omega_s}| \le \gamma Y^{\mathsf{N}}[\mathrm{Diag}_\iota(m_s^X)] \le (B + o(1))\pi_{\omega_s} Y$ as in the proof of Lemma 24.12(b). Finally $0 \le g_E(m) \le 1 \le B$.

(b) The exterior contains no fold diagonal, and (b) is Corollary 24.13(i) with Lemma 24.12(c), $C = C_m$ and $\bar{l}_F = b_F(m)$. □

The background terms lower one selected variance; they are not separate selected records, and they carry no price of their own. All of them are terms of the one expansion of N_0 outside the selected blocks, so the exterior error R_{err} is paid once, and the positive part is taken only after it is subtracted. On the two aligned assignments the background blocks carry the matching diagonal of the same type as the selected records, and the restoration of the union keeps both fold products of the aligned bridge; no assertion is made for a crossed matching assignment. Pooling is restricted to the pieces from 2^{31} because a uniform bound using the maxima over all pieces would lose the term; over those later pieces the maxima are the pool data with which the pooled term $P_E(m)$ of (24.10) is formed, and that term is included at the certified piece values of Certificate 24.19 when $b_E(m) \ge R_*$.

24.8. **Assembly: the total saving D_{var}.** The final construction uses the credits of degrees one to four. They are evaluated once per middle piece and per multiset of pieces, at the piece starts. Piece values. Let $[X_i, Y_i)$, $i = 1, \dots, 33$, be the middle pieces (Definition 20.1, Table 24). Every block of the piece i has least prime at least X_i and at most $\lfloor L/\log X_i \rfloor \le n_i$ primes (Lemma 20.3), where $L = 570$ and $n_i = \lfloor L/\log_- X_i \rfloor$ with a rational lower bound $\log_- X_i \le \log X_i$; and every coefficient below is a nondecreasing function of (q, r) (Lemma 4.18), evaluated at $q = 1/X_i$ and

$r = 1/\lfloor\sqrt{X_i}\rfloor \ge X_i^{-1/2}$, except that on the seven pieces with positive roots R_{u} and d_{u} are the endpoint bounds of Certificate 21.25, valid at every larger prime. The *piece values* are upper bounds

(24.11)
$$h_i \ge h(X_i), \qquad M_i \ge \bar{R}(X_i)^{n_i}, \qquad \chi_i \ge \big(1+\bar{\chi}(X_i)\big)^{n_i} - 1 \qquad (i \le 26),$$
$$M_i \ge R_{\mathsf{u}}(X_i)^{n_i}, \qquad \chi_i \ge \big(1+d_{\mathsf{u}}(X_i)^2\big)^{n_i} - 1 \qquad (27 \le i \le 33),$$
$$k_{F,i} \ge \kappa_F(1/X_i, 1/\lfloor\sqrt{X_i}\rfloor), \qquad k_{E,i} = \frac{3}{X_i - 3}, \qquad \varepsilon_i \ge \frac{f_2+h_2}{\lfloor\sqrt{X_i}\rfloor}, \qquad H_i = \Big(1+\frac{2}{X_i-4}\Big)^{n_i},$$

where $h(X_i)$ is (24.7) with the completion weights of the piece (Section 22), $\bar{R}$ and $\bar{\chi}$ are those of Lemma 24.7(iii), R_{u} and d_{u} those of Lemma 24.7(iv) on the seven pieces with positive roots, κ_F is the centred norm of Corollary 23.3(c), f_2, h_2 are the two-layer row coefficients of the piece, and H_i is the density factor of the actual fold marginal used by the mean savings (Lemma 26.6). By Lemma 24.7, Corollary 23.3, Lemma 24.10(b) and monotonicity, every block $\mathcal{B}$ of the piece i has $h_{\mathcal{B}} \le h_i$, $M_{\mathcal{B}} \le M_i$, $\chi^2(\nu_{\mathrm{L},\mathcal{B}} \| \lambda_{F,\mathcal{B}}) \le \chi_i$, $\kappa_F(\mathcal{B}) \le k_{F,i}$, $\kappa_E(\mathcal{B}) \le k_{E,i}$ and $\varepsilon_{\mathcal{B}} \le \varepsilon_i$. Let Σ_i be the certified lower bound of Lemma 20.6 for $\sum_{\mathcal{B}\subseteq[X_i,Y_i)} b_{\mathcal{B}}^{-1}$. The piece values do not depend on a; they are listed in Table 27.

The cap. Let $a_0 = \frac{4442183}{5000000000}$ be the evaluation exponent at which the middle prices are stored in Table 29, $t = e^{2(a-a_0)L}$ the repricing factor, and $C_i = e^{2aL_i}$ the piece caps of Lemma 20.7 at the anchor a, with the authenticated log caps $L_i \le L$ (the first is $L_1 = 9135105/16384$); every middle block of the piece i has $q_{\mathcal{B}}^{2a} \le C_i$, and a set of blocks of pattern m has $C_X \le C_m$. Together with the piece values, the caps and the numbers Σ_i are piece data in the sense of Section 24.7. Let $B_* \ge B$ and $R_* \ge R_{\mathrm{err}}$ be the certified upper bounds of Definition 30.1(2) at the exponent a (Corollaries 9.4 and 10.9).

Fresh restorations and gains. With these data, $\ell_m = \mathcal{F}(C_m, h_m, M_m, \chi_m)$ is formed for every multiset m of pieces of size at most four, and on the pool of size at most six, the selected-only margins $b_F(m)$, $b_E(m)$ are those of (24.8), and the gains are

(24.12)
$$g_F(m),\ g_E(m) \ \text{ of (24.10) } \ (|m| \le 2), \qquad g_T^{\mathrm{sel}}(m) = \big[b_T(m) - R_*\big]_+ \ \ (T \in \{F,E\},\ |m| \le 4).$$

Thus $g_F = g_F^{\mathrm{sel}}$, and $g_E(m)$ adds to $g_E^{\mathrm{sel}}(m)$ the positive backgrounds of degrees one and two and the late-pool terms of degrees three and four. The restorations depend only on the reference and the caps (Remark 24.14); they are also the restorations of the mean coupons (Lemma 26.6 and Proposition 26.7).

Prices. For the piece i let $S_{F,i}$, $S_{E,i}$ (symmetric), $A_{F,i}$, $A_{E,i}$ (full skew) and $A_{F,i}^{\mathrm{arm}}$, $A_{E,i}^{\mathrm{arm}}$ (raw arm) be the prices of Definition 30.1(3) (Table 29): rounded upper bounds, at the exponent a_0, for the sums over the blocks of the piece of the prices of the two fold and of the two matching records, and of their raw-arm auxiliary prices, given by Lemma 23.24; with the degree charges $\bar{I}_{d,i}$ of the piece,

$$S_{F,i} = 2w_{\mathrm{D},i}s_F(X_i)\bar{I}_{3,i}, \qquad A_{F,i} = 2w_{\mathrm{D},i}\ell_{\mathsf{t}}(X_i)\bar{I}_{2,i} + 2w_{\mathrm{D},i}a_F(X_i)\bar{I}_{3,i}, \qquad A_{F,i}^{\mathrm{arm}} = 2w_{\mathrm{D},i}\bar{I}_{2,i},$$
$$S_{E,i} = 2w_{\mathrm{S},i}s_E(X_i)\bar{I}_{2,i}, \qquad A_{E,i} = 2w_{\mathrm{S},i}a_E(X_i)\bar{I}_{2,i}, \qquad A_{E,i}^{\mathrm{arm}} = 2w_{\mathrm{S},i}\bar{I}_{2,i},$$

each rounded up as in Definition 30.1(3). Every middle block has $\log q_{\mathcal{B}} \le L$, so at the exponent a these prices are at most t times their values (Lemma 9.3). By Lemma 10.14(iii) each piece total is distributed over the blocks of the piece in proportion to their prices, the two records of one kind in a block receiving the same price (Section 24.1); these per-block upper prices are the prices of the complete polynomial.

The credit. Put, with the double sums over ordered pairs of pieces and $g(i,j)$ the gain of the multiset $\{i,j\}$,

$$\begin{aligned} D_{\mathrm{var}} = t\sum_i \big[g_F(i)S_{F,i} + g_E(i)S_{E,i}\big] + \frac{t^2}{4}\sum_{i,j}\Big[g_F(i,j)\big(S_{F,i}S_{F,j} + \kappa_{FF}A^{\mathrm{arm}}_{F,i}A^{\mathrm{arm}}_{F,j}\big) \\ + g_E(i,j)S_{E,i}S_{E,j} + \kappa_b^{\mathrm{al}}\, g_E^{\mathrm{sel}}(i,j)A^{\mathrm{arm}}_{E,i}A^{\mathrm{arm}}_{E,j}\Big] \\ + \sum_{T\in\{F,E\}}\sum_{3\le|m|\le4}\frac{t^{|m|}g_T^{\mathrm{sel}}(m)}{2^{|m|-1}\prod_i m_i!}\Big[\sum_\sigma\prod_{l=1}^{|m|}P^{\sigma_l}_{T,i_l} + \mathbf{1}_{|m|=4}\,\kappa_{4,T}\prod_{l=1}^{4}A^{\mathrm{arm}}_{T,i_l}\Big], \end{aligned} \tag{24.13}$$

where $i_1\le\cdots\le i_{|m|}$ are the pieces of the multiset m, σ runs over the words in $\{\mathrm{s},\mathrm{a}\}^{|m|}$ with an even number of letters a, not all equal to a, $P^{\mathrm{s}}_{T,i}=S_{T,i}$ and $P^{\mathrm{a}}_{T,i}=A_{T,i}$; and where κ_{FF} is the cap of the envelope FF of the fold pairs, κ_b^{al} the invariant cap of the envelope EE aligned of the aligned matching pairs, $\kappa_{4,F}$ the cap of the envelope F4E0 of the class $(4,0)$ and $\kappa_{4,E}$ the invariant cap of the aligned group of the envelope E4 of the class $(0,4)$ (Theorem 25.8 and Lemma 25.9; Table 33). The factors $\frac14$ and $2^{1-|m|}$ account for the two records of one kind of a block, which carry equal prices, of which the variance classes retain the two aligned assignments.

Corollary 24.18 (The variance credits of the final construction)**.** *Assume the gates of Certificate 24.19. Let m be a multiset of pieces, X a set of $|m|$ distinct middle blocks of pattern m, and $j\in\{1,2\}$.*

- *(i) For $|m|\in\{1,2\}$ and symmetric parities, the selection of F_j on X has the class cost $B-g_F(m)$, and the class of the two aligned matching assignments has the class cost $B-g_E(m)$.*
- *(ii) For $|m|=2$ and skew parities, the aligned fold assignment (F_j,F_j) and the set of the two aligned matching assignments satisfy Lemma 24.12(a) with $C=C_m$ and the numbers $b_F(m)$, $b_E(m)$, whose gains are $g_F(m)$ and $g_E^{\mathrm{sel}}(m)$. Lemma 25.9 turns them into the credits $\kappa_{FF}g_F(m)$ and, on the branch with μ outside and without background or pool terms, $\kappa_b^{\mathrm{al}}g_E^{\mathrm{sel}}(m)$, against the envelope costs of the aligned classes. Crossed matching pairs receive no variance credit.*
- *(iii) For $|m|\in\{3,4\}$ and an even number of skew parities, not all skew, the selection of F_j on X and the class of the two aligned matching assignments have the class costs $B-g_F^{\mathrm{sel}}(m)$ and $B-g_E^{\mathrm{sel}}(m)$. For $|m|=4$ and four skew parities, the aligned assignments of one kind receive instead, inside the envelope class $(4,0)$ or $(0,4)$, the credit $\kappa_{4,T}g_T^{\mathrm{sel}}(m)$ on their raw-arm prices (Lemma 25.9).*
- *(iv) At the per-block upper prices, with $B_*\ge B$ in place of B (Remark 9.22) and every repeated-block monomial of a piece multiset given the gain of that multiset, these credits lower the complete polynomial of Lemma 10.14 by at least D_{var} (24.13), and every coefficient stays nonnegative.*

Proof. By the paragraphs above the piece values, the caps and the numbers Σ_i are piece data; the gates of Certificate 24.19 give $k_{F,i},k_{E,i}<1$ and every gain in $[0,1)$; and $R_{\mathrm{err}}\le R_*$ by the choice of R_*.

(i) Proposition 24.17(b) for the fold selection and Proposition 24.17(a) for the aligned matching class.

(ii) Corollary 24.13(i), (ii) with $C=C_m$ give Lemma 24.12(a) with $\bar l_F=b_F(m)$ and $\bar l_E^{\mathrm{al}}=b_E(m)$, and $[\bar l_F-R_*]_+=g_F(m)$, $[\bar l_E^{\mathrm{al}}-R_*]_+=g_E^{\mathrm{sel}}(m)$. These selections belong to the envelope classes $(2,0)$ and $(0,2)$ of Corollary 25.10, and the averaged bound of Lemma 24.12(a) is the hypothesis of Lemma 25.9, whose parts (a) and (b) take the fractions κ_{FF} and κ_b^{al} of the gains; the background

terms of Proposition 24.17 are not used for this class. The restorations live on the two aligned corners of the matching bridge (Lemma 24.11), so no lower bound is available for a crossed class.

(iii) By Corollary 24.13(i), (ii) with $C = C_m$ and Lemma 24.12(c) the stated classes have the costs $B - g_F^{\mathrm{sel}}(m)$, $B - g_E^{\mathrm{sel}}(m)$; they lie in no envelope class, since the classes of Corollary 25.10 consist of skew factors only. For four skew factors the selection lies in the class $(4,0)$ or $(0,4)$, and Lemma 25.9(c) credits $\kappa_{4,T} g_T^{\mathrm{sel}}(m)$ on its raw-arm monomials, on its aligned words.

(iv) The per-block upper price of a record of one kind with a parity is half of the block's share of $tS_{T,i}$ or $tA_{T,i}$, and its raw-arm price half of the share of $tA_{T,i}^{\mathrm{arm}}$. On the monomials of the selections of (i) and (iii) with a symmetric factor, Lemma 10.14(iv) allows the coefficient $B_* - g$, because the class cost is $B - g \le B_* - g$; on the monomials with a repeated block of the same records, kinds and parities it allows any coefficient in $[0, B_* c_{\boldsymbol{n}}]$, and they receive the gain of their piece multiset, which is below $1 < B_*$. On the raw-arm monomials of the classes of (ii) and of the pure quartics of (iii) the credits are subtracted inside the replaced classes, as Lemma 10.14(v) allows, the residual coefficient being nonnegative by the gates of Lemma 25.9. It remains to add up the subtracted amounts.

Degree one. The symmetric prices of the two records of one kind of a block sum, over the blocks of the piece i, to $tS_{T,i}$ (the distribution of Lemma 10.14(iii) is exact); the gains $g_F(i)$, $g_E(i)$ give $t\sum_i [g_F(i) S_{F,i} + g_E(i) S_{E,i}]$.

Degree two. For a fold record F_j, the symmetric monomials of degree two in its variables form $\frac{1}{2}(\sum_{\mathcal{B}} x^{\mathrm{s}}_{\mathcal{B},F_j})^2$, whose ordered pairs of blocks carry the price products $\frac{t^2}{4} S_{F,\mathcal{B}} S_{F,\mathcal{B}'}$, $S_{F,\mathcal{B}}$ being the share of the block; with the gain of the pair of pieces, summed over j and over the blocks of each pair of pieces, this is $\frac{t^2}{4}\sum_{i,j} g_F(i,j) S_{F,i} S_{F,j}$. The same computation gives $\frac{t^2}{4}\sum_{i,j} g_E(i,j) S_{E,i} S_{E,j}$ for the aligned symmetric matching pairs, the crossed pairs receiving nothing, and, on the raw-arm monomials, the skew terms $\frac{t^2}{4}\kappa_{FF}\sum_{i,j} g_F(i,j) A^{\mathrm{arm}}_{F,i} A^{\mathrm{arm}}_{F,j}$ and $\frac{t^2}{4}\kappa_b^{\mathrm{al}}\sum_{i,j} g_E^{\mathrm{sel}}(i,j) A^{\mathrm{arm}}_{E,i} A^{\mathrm{arm}}_{E,j}$.

Degrees three and four. For a multiset m of size k, the monomials of degree k in the variables of one record of a kind whose blocks lie in the pieces m are, by the multinomial expansion, those of $\prod_i (\sum_{\mathcal{B}\subseteq[X_i,Y_i)} (x^{\mathrm{s}}_{\mathcal{B}} + x^{\mathrm{a}}_{\mathcal{B}}))^{m_i}/m_i!$, restricted to an even number of skew factors; at the per-record prices, and summed over the two records of one kind, they give $2\cdot 2^{-k} t^k \sum_\sigma \prod_l P^{\sigma_l}_{T,i_l} / \prod_i m_i!$ for the words with a symmetric letter, and the same with $\kappa_{4,T}\prod_l A^{\mathrm{arm}}_{T,i_l}$ for the pure quartic. With the gains $g_T^{\mathrm{sel}}(m)$ this is the last line of (24.13).

The parts are disjoint: different degrees, kinds, records or parity patterns are different monomials; the skew credits of degree two and the pure quartic credits lie inside different envelope classes; and the background and late-pool terms are inside the gains $g_E(m)$ of the linear and symmetric pair terms only. □

Certificate 24.19 (Restorations recomputed at the baseline exponent)**.** With the tilted laws on the seven later pieces, the restoration quantities M_i, χ_i below are formed from the tilted laws' own marginals (not the uniform ones), all gains lie in $[0,1)$, $k_{F,i}, k_{E,i} < 1$, and the assembled total satisfies $0.00995856031576667 < D_{\mathrm{var}} < 0.00995856031576668$. This is computed twice, by two independently written assemblies of the scalar totals in `research/improvements_2026_10_25/global/selected.py`, with the inputs of `research/push_2026_10_25/design.json`. The adapters in `research/improvements_2026_10_25/global/scalar_adapters.py` authenticate the exact per-band laws, their stored weights, the prime thresholds and the single-prime marginal bounds. For each of the seven changed roots they form, with $n_i = \lfloor 570/\log_- X_i \rfloor$,

$$M_i = R_{\mathsf{u}}(X_i)^{n_i}, \qquad \chi_i = (1 + d_{\mathsf{u}}(X_i)^2)^{n_i} - 1,$$

rounded outward. The author and independent implementations separately perform these products. The new marginal is used throughout the restoration; the fixed uniform half is only the

reference measure. The complete `new_root_restoration` record is in `research/push_2026_10_25/artifacts/author.json`.

The piece product caps are recomputed at anchor $a = \frac{5422587}{6103515625}$ from the exact budget-570 products. The selected root changes are already included in the corresponding native weights. All other piece data, common-chamber restoration formulas, background credits of degrees one and two, aligned matching credits, and credits of degrees three and four are the formulas proved above. Terms pooled over the pieces from 2^{31} are included only when $b_E(m) \geq R_*$. Both assemblies verify $0 \leq l \leq C_m/2$, every gain in $[0,1)$, and $k_{F,i}, k_{E,i} < 1$, then assemble D_{var} using (24.13).

`research/push_2026_10_25/verify.py` authenticates the preceding full native proof and reruns both assemblies with these exact inputs. Their variance components, product caps, savings at the tail primes and total closing values are in its two stored outputs; their agreement is part of Certificate 30.3. In particular the outward interval for D_{var} lies inside

$$0.00995856031576667 < D_{\mathrm{var}} < 0.00995856031576668.$$

Remark 24.20 (Sizes). At the baseline exponent $a = \frac{5422587}{6103515625}$, D_{var} is about $9.96 \cdot 10^{-3}$. The skew pair credits dominate: about $6.01 \cdot 10^{-3}$ for the aligned fold pairs and $3.88 \cdot 10^{-3}$ for the aligned matching pairs. The linear fold credit is about $4.8 \cdot 10^{-5}$, small because the symmetric parts of the signed fold records are of degree three; the linear matching credit is about $1.3 \cdot 10^{-5}$, the symmetric pairs give less than $6 \cdot 10^{-10}$, and the degrees three and four about $5.0 \cdot 10^{-6}$; the late-pool terms account for about $1.7 \cdot 10^{-9}$ of the linear and symmetric-pair credits. Degree six is credited only through the positive envelopes (Theorem 25.8).

Remark 24.21 (Robustness). Every lower bound of this section is an exact relation minus upper bounds of the kind allowed by (CP8). Step (C) of Lemma 24.8 subtracts the energies of the nonroot components from the identity $Y[\bigotimes_{\mathcal{B}} \mu_{\mathcal{B}}] = Y$; its contraction and density steps, and the centred interpolations of Lemma 24.12 and Proposition 24.17, are the exact relations (P1)–(P3) and (P3′); the one-sided and bounded-variance steps use the upper bound C' for normalized energies; the bridges subtract the merge energies; and the background terms are kept terms of an exact expansion. If every subtracted upper bound is multiplied by $1+z$, the same proofs give the bounds with $(1+z)C$ in place of C, and the piece caps $(1+z)C_i$; every gain is the larger of 0 and a continuous expression, so the strict gates of Certificate 24.19 persist for small z (Lemma 11.4). The condition $b_E(m) \geq R_*$ of the late-pool terms holds with room: the strict selected-only margins checked in Certificate 24.19 persist under a sufficiently small perturbation as well.

Consequences and what to read next. Parts II and III: Theorem 4.7, (4.2), Lemmas 4.18 and 6.10 and Definition 8.9, (CP4) and (CP5) with the charging rule and the monotonicity of the closing (Definitions 9.14 and 9.16 and Remarks 9.15 and 9.22), the energy toolkit (E1)–(E3), (P1)–(P4) and (P3′) (Lemma 10.13), Lemmas 9.3, 10.14, 10.16 and 11.4 and Corollaries 9.4 and 10.9. Part IV: Lemmas 12.10, 12.11, 13.3, 13.22, 13.24 and 16.4 and Corollary 12.3, the chamber geometry and polarization (Lemmas 17.2 and 17.11 and Definitions 17.1 and 17.3). Appendix A: Theorem A.7. Part V: the middle blocks, the piece caps, the reference and the bracket (Definitions 20.1 and 22.4 and Lemmas 20.3, 20.6, 20.7, 22.1 and 22.7), the cut L20 of the uniform root (Section 20.4, (20.2)), the positive roots (Definition 21.1, Lemma 21.2, Proposition 21.7, and Certificate 21.25), the fold records (Lemmas 23.2 and 23.7, Corollary 23.3, Definition 23.9, and Proposition 23.13) and the matching records (Lemmas 23.15 and 23.17); the prices (Lemma 23.24); the envelope costs (Theorem 25.8, Lemma 25.9, and Corollary 25.10); the companion energies, the sides and the chosen diagonals of the mean (Lemma 26.6, Proposition 26.4, and Section 26.3); the closing quantities (Definition 30.1); and the data and certificates Certificates D.8, 24.6 and 24.19 and Tables 27, 29 and 36.

Hypotheses verified in this section. The variance part of (CP5) (Table 5): the classes of Corollary 24.18(i),(iii) with their costs, the gains of Corollary 24.18(ii) and (iii) that Lemma 25.9

consumes, and the total D_{var} of (24.13) that the closing uses; the diagonal-mean and pool coupons (Lemma 24.11(c), Lemma 24.16(b)) that Proposition 26.7 uses; and the star normalizer of the cut L20 (Lemma 24.5) that Section 20.4 uses.

25. Products of skew changes: dominating matrices and savings

What this section proves and how to read it. This section proves (CP5) for products of $2, 4$ or 6 skew changes on distinct middle blocks: instead of the trivial bound B per product, a positive-semidefinite matrix D with $-D \preceq T \preceq D$ (a *dominating matrix*, called an *envelope* in the text; "positive" here means positive semidefinite, while a *positive coupling* is a nonnegative measure—the two are distinct) dominates the tensor of leading star operators with a smaller energy after averaging over reflections. The saving over B is subtracted in the closing discount D of Definition 30.1. A *signed star* is the normalised leading skew change, with complex structure $\hat{P}$, $\hat{P}^2 = -I$ on centred functions (defined in Section 25.1 below; uses before that subsection mean this normalised change). Reading-order note: aligned classes use variance savings from Section 24. The per-block variance statements of Section 24 do not use the caps proved here; only its final assembly and the closing insert them. Inputs: Section 23. Output: (CP5) class bounds for the sixteen retained classes. This section verifies (CP5) (Definition 9.16) for the skew selections of the middle range whose records are all two-layer fold records F_1, F_2 or matching records E_0, E_1; these are the envelope rows of (CP5) in Table 5. On every middle block the comparison law replaces the fold records by the signed whole-block records of Definition 23.9 and the matching records by the exact-marginal law (Section 23.4 and Lemma 23.17). The skew parts of these changes carry the fold and matching parts $A_F + A_E$ of the skew total A of Theorem 9.21, and a selection of n fold and m matching skew changes on $n + m$ distinct middle blocks, $n + m \in \{2, 4, 6\}$, would cost B times its full skew prices (Lemma 10.14(i)). We bound it instead by a class bound in the sense of (CP5). A positive envelope dominates the tensor of the leading signed stars; its energy against the outside law is priced by the outside factor in (25.11); the remainders of the stars are paid by a remainder polynomial. The class bound is a polynomial in three auxiliary prices of each record (raw-arm, leading and remainder), and it enters the closing through the degree classes of Lemma 10.14(v). Inside the aligned classes a fraction of the variance credit of Section 24 survives (Lemma 25.9). Sixteen classes are retained (Corollary 25.10); all sixteen savings are positive, and at the baseline exponent their sum, the uniform discount, is $0.4568954196753\ldots$; the closing (Section 30) subtracts it in the discount D of the reserved cost (Definition 30.1), and the witnesses are tabulated in Section E.6.

The idea.

- *Signed stars.* After division by its raw-arm coefficient, the leading skew change of F_1 at a block is a signed star: centre $\mathcal{H}_0$, arms $\mathcal{H}_1, \mathcal{H}_2, \mathcal{K}_{12}$ with the coefficients $\hat{P}$, $\mathtt{t}\hat{P}$ and $1 - \mathtt{t}$, as in (25.4), where $\hat{P} = P/r$ is skew with $\hat{P}^2 = -I$ on the centred functions and $\mathtt{t} = 127/500$. The leading change of E_s has the single arm $\mathcal{H}_1$ with the coefficient $\hat{P}$. The rest is a remainder of relative size $O(b^{-1/2})$, paid by a remainder polynomial.
- *Envelopes.* An envelope is a positive semidefinite real matrix in the singleton block algebra $M_3(\mathbb{R}) \oplus \mathbb{R} \oplus \mathbb{R}$ of each fold factor (and in $M_3(\mathbb{R})$ or its diagonal of each matching factor) that dominates both signs of the tensor of the coefficient matrices at the phases $\hat{P} = \pm i$. Lifted to the actual half spaces, it dominates the tensor of the actual stars and expands into normalized positive couplings with the actual marginals (Lemma 25.3). A diagonal envelope is a complementary-mask array.
- *Averages.* The energy of an envelope against an invariant outside law does not change when the envelope is averaged over the slot reflections R_h of Lemma 25.5. The common fold reflection averages every class; on fully invariant outside laws the extra reflection of the all-F_1 and all-F_2 words and the action of the chamber group on matching words average

more (Lemma 25.5). An envelope used alone bounds its tensor by at least the tensor's own norm, with no gain; the averages move its large weights onto small ones.

- *Background split.* The outside law N_0 splits into the invariant tags $E_{\mathrm{add}} + H$, priced by the assigned price U^{inv}, and the generic tag G (Lemma 25.6). An envelope with norm bound κ_H on generic monomials and $\kappa_b \le \kappa_H$ on invariant ones (the *caps* κ_H, κ_b of Theorem 25.8; norm bounds for the lifted envelope, not spectral caps) costs $\kappa_H e^{U_0} - (\kappa_H - \kappa_b)e^{tU^{\mathrm{inv}}}$. Since tU^{inv}, about 0.8468844654, falls short of U_0, about 0.8469082458, by only about $2.4 \cdot 10^{-5}$, the effective cap is κ_b; the caps κ_b of the construction lie between 0.455 and 2.455 per raw-arm monomial, while the record expansion charges B, about 2.333, per full skew monomial, which exceeds the raw-arm monomial by the factor ℓ_{t}, about 1.273, per fold record.
- *Coupling mass and coupon.* Comparing the bracket N with N_0 costs the coupling mass M of the envelope times R_{err}. On invariant backgrounds the part of this error with exactly one folded record difference is reduced by the orbit coupon (Lemma 25.7).
- *Convention.* A *positive coupling* is a nonnegative measure with the actual half marginals; a *positive operator* is positive semidefinite. The two properties are used separately: an envelope is a positive operator whose expansion into positive couplings has signed coefficients. All selected tensor factors are centred on both sides.

The signed fold record is chosen with its envelopes in view. Its skew part is large, A_F about 0.539 in total, but its leading part is a star with arms of relative sizes 1, t and $1 - \mathsf{t}$, of norm ℓ_{t} instead of the $\sqrt{2}$ of the mirror pairing $\mathsf{t} = 0$; and the envelopes of this section lower the reserved cost by about 0.457, against about 0.010 for all variance credits together (Remark 24.20).

All conditional expectations of this section use the actual half marginal of the record in question. In particular the three objects of a fold record are not independent: two of them are joined by a strict-square edge. The feature synthesis bound of Lemma 25.1 transfers positive envelopes to the actual half spaces (the lifts of Lemma 25.3) without assuming independence.

25.1. **Star features in the true half laws.** The *correlation* of two closed subspaces of a Hilbert space is $\sup |\langle f, h\rangle|$ over unit vectors f, h in them; for functions of two random labels, centred in their actual laws, it is their maximal correlation.

The fold spaces. At a middle prime p put $q = 1/p$ and $r = p^{-1/2}$, and use the one-prime notation of Section 23.1: ν_F is the uniform law on the triples $x = (x_0, x_1, x_2)$ of distinct labels with $x_0 \to x_2$, and the actual half marginal at ι of the two-layer fold records is $m_0 = g_0 \nu_F$, with

$$|g_0 - 1| \le e_0 = \frac{2q}{(1 - 3q)(1 - 4q)}$$

(Lemma 23.2(a)). Its pair laws are exact (Lemma 23.4(a)): (x_0, x_1) and (x_1, x_2) have the density $1 - \mu\chi(y' - y)$, $\mu = 2/(p - 4)$, with respect to the uniform law on distinct pairs (y, y'), and (x_0, x_2) is uniform on the pairs $y \to y'$. All singleton marginals of ν_F and of m_0 are uniform. Let L_0^2 be the space of mean-zero functions of one label (uniform law) and $V_a \colon L_0^2 \to L^2(m_0)$, $V_a f(x) = f(x_a)$ $(a = 0, 1, 2)$, the isometries of the singleton coordinates; V_a^* is the actual conditional expectation onto x_a, read as a function of one label. In $L^2(m_0)$ let $\mathbb{J}$ project onto the constants, P_a onto the functions of x_a and P_{ab} onto the functions of (x_a, x_b), and put $C_a = P_a - \mathbb{J} = V_a V_a^*$,

$$\mathcal{H}_a = \operatorname{ran} C_a = \operatorname{ran} V_a, \qquad \mathcal{K}_{ab} = \operatorname{ran}(P_{ab} - \mathbb{J}) \ominus (\mathcal{H}_a + \mathcal{H}_b).$$

The *fold features* are $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \mathcal{K}_{12}, \mathcal{K}_{01}$, with the *indices* $0, 1, 2, 3, 4$ (feature indices, not labels z_v; words in these indices are *label words* below). For a block $\mathcal{B}$ the same notation means the whole objects, carrying all prime coordinates of the block, with the product $m_{0,\mathcal{B}} = \bigotimes_{p \in \mathcal{B}} m_{0,p}$ of the actual prime laws; then L_0^2 is the space of mean-zero functions of one whole label. A matching record has the actual half marginal m_E, uniform on the triples of labels distinct at every prime

(Lemma 17.2(d)); its *matching features* are its singleton spaces $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$ in $L^2(m_{E,\mathcal{B}})$, with the labels $0, 1, 2$, and V_0, V_1, V_2 are defined in the same way.

Lemma 25.1 (The actual feature frame)**.** *Let $\mathcal{B}$ be a middle block with least prime b, put $q = 1/b$, $r = b^{-1/2}$, and*

$$\varrho_{02} = \frac{r\sqrt{1+q}}{1-q}, \qquad \varrho_{01} = \frac{q}{1-q}\Big(1 + \frac{4q}{(1-4q)^2}\Big)^{1/2},$$
$$\varrho_{03} = \frac{q}{1-3q} + \frac{2q^2}{(1-3q)(1-6q)} + \frac{2qr}{(1-6q)(1-2q)}, \qquad \varrho_{34} = \frac{r + q + 4q^2/(1-4q)}{1-6q}.$$

(a) In the actual product marginal $m_{0,\mathcal{B}}$ the correlations of distinct fold features, with the labels $0, \dots, 4$ of $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \mathcal{K}_{12}, \mathcal{K}_{01}$, are at most

$$01 : \varrho_{01}, \qquad 02 : \varrho_{02}, \qquad 03 : \varrho_{03}, \qquad 12 : \varrho_{01}, \qquad 24 : \varrho_{03}, \qquad 34 : \varrho_{34},$$

and all other correlations of distinct fold features vanish.

(b) The synthesis map $\Sigma_{\mathcal{B}}\colon \bigoplus_v \mathcal{H}_v \to L^2(m_{0,\mathcal{B}})$, $(h_v)_v \mapsto \sum_v h_v$, *from the orthogonal sum of the five spaces has squared norm at most $\Gamma_F = 1 + \lambda$ for every $\lambda > \varrho_{34}$ with*

$$\lambda - \varrho_{02} - \frac{\varrho_{03}^2}{\lambda - \varrho_{34}} - \frac{2\varrho_{01}^2}{\lambda} \geq 0. \tag{25.1}$$

(c) In $L^2(m_{E,\mathcal{B}})$ the three matching features have pairwise correlations at most $1/(b-1)$, and their synthesis map has squared norm at most $\Gamma_E = 1 + 1/(b-1)$.

The four fold constants are nondecreasing in (q, r), so at every middle block they are at most their values at $q = 1/Q_3$, $r = Q_3^{-1/2}$, $Q_3 = 19951586$:

ϱ_{02}	$2.23877949420725 \cdot 10^{-4}$
ϱ_{01}	$5.01213362368300 \cdot 10^{-8}$
ϱ_{03}	$5.01437833890212 \cdot 10^{-8}$
ϱ_{34}	$2.23928121269334 \cdot 10^{-4}$

For every middle block one may therefore take

$$\Gamma_F = \frac{500111979555137707696357560603}{5 \cdot 10^{29}} < 1.000223959111, \qquad \Gamma_E = \frac{19951586}{19951585}, \tag{25.2}$$

and the bound $\rho_{\mathrm{pair}} = 501213362368299346064 43 \cdot 10^{-30} < 5.0121336237 \cdot 10^{-8}$ for ϱ_{01}, with $\eta_{\mathrm{pair}} = 2\rho_{\mathrm{pair}}/(1 - \rho_{\mathrm{pair}}^2) < 1.0024267248 \cdot 10^{-7}$.

Proof. Step 1: blocks and sectors. The conditional expectation of a function of one whole object given another, or given a pair of others, is the tensor product of the prime conditional expectations, which are Markov operators of couplings with exact marginals. These preserve the exact supports of the nonconstant prime factors (Lemma 6.10(a)), and the sectors of distinct supports are orthogonal in the product law. By Lemma 6.10(a),(c), on every nonempty support the product is bounded by the largest one-prime bound. The prime formulas of the statement and of Step 3 are nonnegative and nondecreasing in (q, r) (Lemma 4.18); every prime of $\mathcal{B}$ is at least b, so every one-prime bound is at most its value at b.

Step 2: the singleton edges 01*,* 02*,* 12*.* At a prime the pair law of (x_0, x_2) is uniform on directed pairs, so the correlation of $\mathcal{H}_0$ and $\mathcal{H}_2$ is the norm of the square shift on mean-zero functions, $r\sqrt{1+q}/(1-q)$ (Lemma 4.11(b), Lemma 12.10(d)); the pair laws of (x_0, x_1) and (x_1, x_2) have the density $1 - \mu\chi(y' - y)$, whose operator on mean-zero functions is a combination of I and the Paley operator, with norm ϱ_{01} (Lemma 23.4(b)). By Step 1 these are the block bounds.

Step 3: the edges 03 *and* 24. Let f be a mean-zero function of one label, y, z distinct labels and $s = \chi(z-y)$. Direct conditioning in the actual triple weight $\mathbf{1}[x_0 \to x_2](d - 1 - \mathbf{1}[x_0 \to x_1] - \mathbf{1}[x_1 \to x_2])$, $d = (p-1)/2$ (Lemma 23.2(a), Lemma 23.4(a)), gives

$$\mathbb{E}_{m_0}[f(x_0) \mid x_1 = y, x_2 = z] = \frac{N_f(y,z)}{(p-3)(p-4-2s)},$$
$$N_f(y,z) = -p(p-5-s)Pf(z) + pPf(y) + pWf(y,z) - (p-5)f(z) - (p-6)(1+s)f(y),$$

with the Paley operator P and $Wf(y,z) = -\mathbb{E}_t[\chi(t-y)\chi(t-z)]f(t)$; for instance the common-predecessor sum is $4\sum_{a\to y,\, a\to z} f(a) = -p[Pf(y)+Pf(z)+Wf(y,z)]-(1-s)f(z)-(1+s)f(y)$, and the formula follows after expanding the two predecessor indicators. Since $s^2 = 1$, $1/(p-4-2s) = (p-4+2s)/((p-6)(p-2))$, so the conditional expectation is $N_f\,(p-4+2s)/((p-3)(p-6)(p-2))$. Expand it, using $s^2 = 1$, and project onto $\mathcal{K}_{12}$: every function of y alone or of z alone is annihilated. What remains is $-sf(y)/(p-3)$, $-2(p-5)sf(z)/((p-3)(p-6)(p-2))$ and terms in $sPf(z)$, $sPf(y)$, Wf and sWf whose coefficients have absolute values adding up to $2p/((p-6)(p-2))$. As $|s| = 1$, the singleton marginals are uniform, $\|Pf\| \le r\|f\|$, and $W^*W \preceq qI$ (Lemma 23.4(c), where W is the operator A_3), the projection has norm at most

$$\frac{1}{p-3} + \frac{2}{(p-3)(p-6)} + \frac{2p}{(p-6)(p-2)}\, r = \varrho_{03}(p).$$

This is the one-prime correlation of $\mathcal{H}_0$ and $\mathcal{K}_{12}$, and by Step 1 the bound on the sectors of size one.

On a support of size $k \ge 2$ we use two coarser prime estimates. Put $\varrho = r\sqrt{1+q}/(1-q)$ and

$$\tilde{\varrho}_{03} = \frac{2(1+\varrho)/(p-3) + e_0}{\sqrt{1-e_0}}, \qquad \varrho_{0|12} = \frac{(\sqrt{p+1}+2)/(p-3) + e_0}{\sqrt{1-e_0}}.$$

For a mean-zero f let $Tf(z) = \mathbb{E}_{\nu_F}[f(x_0) \mid x_2 = z]$, the average of f over the d predecessors of z. Given $x_1 = y$ and $x_2 = z$, the label x_0 is uniform under ν_F on these predecessors, with y excluded if it is one of them; so, with $j = \mathbf{1}_{y\to z}$,

$$\mathbb{E}_{\nu_F}[f(x_0) \mid y, z] = \frac{dTf(z) - jf(y)}{d-j} = Tf(z) + \frac{j}{d-j}(Tf(z) - f(y)).$$

Since $j/(d-j) \le 2/(p-3)$, Minkowski's inequality and the uniform singleton marginals bound the norm of $f \mapsto \mathbb{E}_{\nu_F}[f(x_0) \mid x_1, x_2]$ by $\varrho + 2(1+\varrho)/(p-3) = (\sqrt{p+1}+2)/(p-3)$. In $L^2(m_0)$ the density perturbation costs e_0, and the norm of a function of the pair changes by at most the factor $(1-e_0)^{-1/2}$ (Lemma 12.10(f)). So the correlation of $\mathcal{H}_0$ and $\mathrm{ran}(P_{12} - \mathbb{J})$ is at most $\varrho_{0|12}$. Let h be a function of (x_1, x_2) orthogonal in $L^2(m_0)$ to all functions of x_2. Then $\mathbb{E}_{m_0}[Tf(x_2)h] = 0$, so $\langle f(x_0), h\rangle_{m_0} = \mathbb{E}_{m_0}[(f(x_0) - Tf(x_2))h]$. Its uniform part is $\mathbb{E}_{\nu_F}[\frac{j}{d-j}(Tf(x_2) - f(x_1))h]$, at most $2(1+\varrho)/(p-3)$ times the uniform norms; its density-error part is at most $e_0\|f(x_0) - Tf(x_2)\|_{\nu_F}\|h\|_{\nu_F} \le e_0\|f\|\|h\|_{\nu_F}$, because T is a conditional expectation under ν_F. Converting only the pair norm, the correlation of $\mathcal{H}_0$ and $\mathrm{ran}(P_{12} - P_2)$, which contains $\mathcal{K}_{12}$, is at most $\tilde{\varrho}_{03}$. This comparison uses the actual orthogonality of h; it does not transport a uniform pair projection as an actual projection.

Let T_p map a function of $x_{0,p}$ to its actual conditional expectation given $(x_{1,p}, x_{2,p})$, and put $T_{2,p} = P_{2,p}T_p$. Both fix the constants on both sides; on mean-zero functions both have norm at most $\varrho_{0|12}$, and $\|T_p - T_{2,p}\| \le \tilde{\varrho}_{03}$. On a support with $k \ge 2$ active primes the telescoping bound of Lemma 6.10(b) gives $\|\bigotimes_p T_p - \bigotimes_p T_{2,p}\| \le k\tilde{\varrho}_{03}\varrho_{0|12}^{k-1}$. In terms of (q, r), $\varrho_{0|12} = [(r\sqrt{1+q}+2q)/(1-3q) + e_0]/\sqrt{1-e_0}$ and $\tilde{\varrho}_{03}/q = [2(1+\varrho)/(1-3q) + e_0/q]/\sqrt{1-e_0}$ with $e_0/q = 2/((1-3q)(1-4q))$ are nondecreasing in (q, r) (Lemma 4.18). At $p = 2^{24}$, hence at every middle prime, $\varrho_{0|12} < \frac{1}{2}$ and $2(\tilde{\varrho}_{03}/q)\varrho_{0|12} < 1$ (Certificate 25.11(a)); so $k\tilde{\varrho}_{03}\varrho_{0|12}^{k-1} \le 2\tilde{\varrho}_{03}\varrho_{0|12} < q \le \varrho_{03}$, with all constants at b.

The tensor product of the $T_{2,p}$ maps into the functions of the whole object x_2, which project to zero in $\mathcal{K}_{12}$, and distinct input supports have orthogonal output supports. This proves the block bound ϱ_{03} for $\mathcal{H}_0, \mathcal{K}_{12}$; the pair $\mathcal{H}_2, \mathcal{K}_{01}$ is the same with 0 and 2 interchanged (the involution ς of Lemma 25.5(b)). This step also treats features that involve several primes; the prime bound alone does not give it.

Step 4: the edge 34. Fix $x_1 = 0$ at one prime (the actual law is invariant under translations), and use the uniform law on the $p-1$ other labels as the base measure. The conditional law of $(x_0, x_2) = (u, w)$ has the density

$$\frac{p-1}{(p-3)(p-4)}\,[1 - \mathbf{1}_{u=w} + \chi(w-u)]\,[p-5+\chi(u)-\chi(w)],$$

with the marginal densities $1+\mu\chi(u)$ and $1-\mu\chi(w)$. On base-centred functions its covariance form is $\langle f, \Phi g\rangle + \mu^2 \langle \chi, f\rangle \langle \chi, g\rangle$, where Φ is the operator of the density: after the functions of u alone and of w alone are dropped, Φ is $(p-1)/((p-3)(p-4))$ times the sum of $-(p-5)/(p-1)$ times the identity, $(p-5)$ times the compressed Paley operator $g \mapsto \mathbb{E}_w[\chi(w-u)g(w)]$, and the commutator of the latter with $\operatorname{diag}\chi$. The compressed Paley operator has norm at most $\sqrt{p}/(p-1)$ (Lemma 4.11(c)) and the commutator at most twice this. Dividing by the two lower bounds $1-\mu$ of the marginal densities, the conditional maximal correlation of x_0 and x_2 given x_1 is at most

$$\frac{\sqrt{p}/(p-4) + (p-5)/((p-3)(p-4)) + 4/(p-4)^2}{1 - 2/(p-4)} \le \varrho_{34}(p).$$

Given the whole object x_1, the pairs $(x_{0,p}, x_{2,p})$, $p \in \mathcal{B}$, are conditionally independent, so the conditional maximal correlation of the whole objects x_0 and x_2 given x_1 is at most $\varrho_{34}(b)$ (Corollary 4.14). A function in $\mathcal{K}_{12}$ is orthogonal to the functions of the whole object x_1, so it has conditional mean zero given x_1; likewise every function in $\mathcal{K}_{01}$; and Corollary 4.15 gives $|\langle f, g\rangle| \le \varrho_{34}\|f\|\|g\|$ for $f \in \mathcal{K}_{12}$, $g \in \mathcal{K}_{01}$. The vanishing correlations are the definitions: $\mathcal{K}_{12} \perp \mathcal{H}_1 + \mathcal{H}_2$ and $\mathcal{K}_{01} \perp \mathcal{H}_0 + \mathcal{H}_1$. This proves (a).

Step 5: the frame. The squared norm of $\Sigma_{\mathcal{B}}$ is $\|\sum_v \iota_v \iota_v^*\|$ for the inclusions ι_v of the five spaces. By Lemma 13.8(a) it is at most $\lambda_{\max}$ of the symmetric matrix G_F^{fr} with unit diagonal and the off-diagonal entries of (a), that is $1 + \lambda_{\max}$ of its off-diagonal part. That part is preserved by the permutation $0 \leftrightarrow 2$, $3 \leftrightarrow 4$, with the orbits $\{0,2\}$, $\{3,4\}$, $\{1\}$; by Lemma 13.8(c) its largest eigenvalue is that of the Perron block

$$\begin{pmatrix} \varrho_{02} & \varrho_{03} & \sqrt{2}\,\varrho_{01} \\ \varrho_{03} & \varrho_{34} & 0 \\ \sqrt{2}\,\varrho_{01} & 0 & 0 \end{pmatrix}.$$

For $\lambda > \varrho_{34}$ the lower right block of λI minus this matrix is positive definite, and its Schur complement is the left side of (25.1); so (25.1) gives $\lambda I \succeq$ the Perron block, which proves (b).

Step 6: the matching frame. The matching half law is uniform on injective triples, so at a prime $V_i^* V_j = -I/(p-1)$ on L_0^2 for $i \ne j$ (condition one coordinate on another). On an exact support S of the block every off-diagonal singleton Gram entry is $c_S I$ with $c_S = (-1)^{|S|}\prod_{p\in S}(p-1)^{-1}$, so the correlations are at most $1/(b-1)$, and the Gram matrix of the three features has the eigenvalues $1-c_S$, $1-c_S$, $1+2c_S$. Odd supports give at most $1+1/(b-1)$, and even nonempty supports at most $1+2/(b-1)^2$, which is smaller. This proves (c).

The numbers. Exact rational bounds for the square roots give the table, the gates of Step 3 and (25.2); Γ_F is the rational end of a bisection for (25.1) with the rounded entries, and $\Gamma_E = 1 + 1/(Q_3 - 1)$, every middle prime exceeding Q_3 (Certificate 25.11(a)). □

25.2. **Dominating matrices for tensors of star operators.**

Lemma 25.2 (A positive substitute for a pure-pair projection)**.** *In L^2 of an actual half marginal of a prime or of a block suppose that the correlation of $\mathcal{H}_i$ and $\mathcal{H}_j$ is at most $\rho < 1$, and let Π_{ij} be the orthogonal projection onto $\mathcal{K}_{ij}$. Then*

$$Q_{ij} = P_{ij} - \frac{P_i + P_j}{1+\rho} + \frac{1-\rho}{1+\rho}\mathbb{J}$$

satisfies

$$0 \preceq Q_{ij} - \Pi_{ij} \preceq \eta\,(C_i + C_j), \qquad \eta = \frac{2\rho}{1-\rho^2}. \tag{25.3}$$

Let $\widetilde{V}_i\colon L^2 \to L^2(m)$ be the extension of V_i to all functions of one label, $\widetilde{V}_i f(x) = f(x_i)$; thus $\widetilde{V}_i\mathbf{1} = \mathbf{1}$, whereas V_i has domain L_0^2. Every P_i, P_{ij} and $\mathbb{J}$ is the Markov operator of a normalized nonnegative coupling of the actual marginal with itself, and so is, for singleton coordinates $i \neq j$, the operator $M_{ij} = \widetilde{V}_i\widetilde{V}_j^*$: choose a uniform label, then sample the two halves independently, conditioned on their coordinates i and j, respectively, taking that label. Both half marginals of these couplings are the actual marginal, $M_{ii} = P_i$, and $V_iV_j^* = M_{ij} - \mathbb{J}$. Placed by its record, each such coupling is an admissible positive law: it has uniform one-slot marginals and is invariant under independent simultaneous translations at the primes of the block, as (E1) requires.

Proof. On $\mathcal{H}_i + \mathcal{H}_j$ the spectrum of $C_i + C_j$ lies in $[1-\rho, 1+\rho]$: it is that of the Gram operator of the two inclusions of $\mathcal{H}_i$ and $\mathcal{H}_j$, whose off-diagonal block has norm at most ρ. There $Q_{ij} = I - (C_i + C_j)/(1+\rho)$, whereas $\Pi_{ij} = 0$; the lower bound follows from the upper spectral endpoint, and the upper bound from $I \preceq (C_i + C_j)/(1-\rho)$. On $\mathcal{K}_{ij}$ both operators equal the identity; on the constants and on the orthogonal complement of the functions of (x_i, x_j) both vanish. These four subspaces reduce all the operators involved.

To realize P_i as a coupling, sample x_i from its actual marginal and then the two triples independently from their actual conditional laws given that value; for P_{ij} use the pair instead, and for $\mathbb{J}$ sample two independent triples. For M_{ij} the half with the coordinate i equal to the chosen label has the actual marginal because the coordinate i is uniform, and likewise for j; for g on the second half, $\mathbb{E}[g(x') \mid x] = (\widetilde{V}_j^* g)(x_i) = (\widetilde{V}_i\widetilde{V}_j^* g)(x)$, so its Markov operator is M_{ij}, and $\widetilde{V}_i\mathbb{J}_{\mathrm{lab}}\widetilde{V}_j^* = \mathbb{J}$ for the projection $\mathbb{J}_{\mathrm{lab}}$ onto the constant labels. One-slot marginals are uniform because they are those of the actual marginal. The constructions commute with the square-affine symmetries of that marginal, including independent simultaneous translations at its primes. The operator Q_{ij} is positive semidefinite, but its signed expansion is not a nonnegative coupling; this is why the envelopes below are expanded into the couplings M_{ij}, P_{ij}, $\mathbb{J}$ before any energy is bounded. □

For a fold record we write Q_{12} and Q_{01} for the substitutes with $\rho = \rho_{\mathrm{pair}}$ and $\eta = \eta_{\mathrm{pair}}$ of Lemma 25.1, which bounds the correlations of $\mathcal{H}_1, \mathcal{H}_2$ and of $\mathcal{H}_0, \mathcal{H}_1$ by $\varrho_{01} \le \rho_{\mathrm{pair}}$.

Signed stars. A *star* on an orthogonal sum $\bigoplus_v \mathcal{H}_v$ is a skew-adjoint operator $\sum_a(\mathrm{in}_a A_a \mathrm{in}_c^* - \mathrm{in}_c A_a^* \mathrm{in}_a^*)$ with one *centre* c, mutually orthogonal *arms* $a \neq c$, the coordinate inclusions in_v and operators $A_a\colon \mathcal{H}_c \to \mathcal{H}_a$, whose norms are the *arm norms*. On $(L_0^2)^5$ and $(L_0^2)^3$, with rows and columns indexed by the fold labels $0,\dots,4$ and the matching labels $0,1,2$, let

$$\mathsf{F}(\hat{P}) = \begin{pmatrix} 0 & \hat{P} & \mathsf{t}\hat{P} & \mathsf{t}-1 & 0 \\ \hat{P} & 0 & 0 & 0 & 0 \\ \mathsf{t}\hat{P} & 0 & 0 & 0 & 0 \\ 1-\mathsf{t} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathsf{E}(\hat{P}) = \begin{pmatrix} 0 & \hat{P} & 0 \\ \hat{P} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{25.4}$$

with $\hat{P} = P/r$ the normalized Paley operator at a prime (Lemma 12.10), skew-adjoint with $\hat{P}^2 = -I$ on L_0^2. They are stars with centre 0: F has the arms $1,2,3$ with the arm operators $\hat{P}$, $\mathsf{t}\hat{P}$,

$(1-\mathsf{t})I$ and arm norms 1, t, $1-\mathsf{t}$; E has the arm 1 with the arm operator $\hat{P}$. For a map $\Xi=(\Xi_v)_v$ from $(L_0^2)^5$ (or $(L_0^2)^3$) into a Hilbert space we call $\Xi\,\mathsf{F}\,\Xi^*=\sum_{a,b}\Xi_a\mathsf{F}_{ab}\Xi_b^*$ the *lift* of F by Ξ. At a middle prime p of a block let $\Xi_p=(V_0,V_1,V_2,A_3^\circ/r_p,0)$ for a fold record, with the pure-pair arm A_3° of (23.4), a map of norm at most r_p into the sector $\{p\}$ of $\mathcal{K}_{12}$ (Lemma 23.4(c),(e)), and $\Xi_p=(V_0,V_1,V_2)$ for a matching record, all restricted to the singleton sector $\{p\}$ of the label space. The lift of $\mathsf{F}(\hat{P}_p)$ by Ξ_p is r_p^{-1} times the leading signed star $L_{\mathsf{t},p}^\circ=A_\mathsf{t}^\circ V_0^*-V_0(A_\mathsf{t}^\circ)^*$ of (23.4), $A_\mathsf{t}^\circ=r_p(V_1\hat{P}_p+\mathsf{t}V_2\hat{P}_p)+(1-\mathsf{t})A_3^\circ$; the pure-pair arm is a contraction after the division by r_p, and it is not assumed to commute with $\hat{P}$. The lift of $\mathsf{E}(\hat{P}_p)$ is $V_1\hat{P}_pV_0^*+V_0\hat{P}_pV_1^*$.

For a middle block $\mathcal{B}$ with least prime b, $r_\mathcal{B}=b^{-1/2}$, and an eligible record ϕ (Definition 24.1), let $\zeta^{\mathrm{a}}_{\mathcal{B},\phi}$ be the skew part at ι of the change of ϕ in $\nu_\mathcal{B}-\lambda_\mathcal{B}$, and $m_{\phi,\mathcal{B}}$ its actual half marginal at ι. The *raw-arm coefficient* of ϕ is $w_\mathrm{D}r_\mathcal{B}^2$ for F_1,F_2 and $w_\mathrm{S}r_\mathcal{B}^2$ for E_0,E_1, with the record weights of the piece of $\mathcal{B}$. The *normalized skew change* $\widehat{\Delta}_{\phi,\mathcal{B}}$ is the cut operator of $\zeta^{\mathrm{a}}_{\mathcal{B},\phi}$ at ι, relative to $m_{\phi,\mathcal{B}}$, divided by the raw-arm coefficient. Write $\ell_\mathsf{t}(b)$, $a_F(b)$ for the constants of Proposition 23.13 at $q=1/b$, $r=b^{-1/2}$, and $r_E(X)$ for the constant of Lemma 23.18 at the start X of the piece of $\mathcal{B}$. We use the following *star form*.

- (N1) $\widehat{\Delta}_{F_1,\mathcal{B}}=L+\Upsilon$, where L is the orthogonal sum over the singleton sectors $\{p\}$, $p\in\mathcal{B}$, of c_p times the lift of $\mathsf{F}(\hat{P}_p)$ by Ξ_p, with numbers $0<c_p\le1$, and vanishes on the other sectors; $\|L\|\le\ell_\mathsf{t}(b)$ and $\|\Upsilon\|\le a_F(b)\,r_\mathcal{B}$.
- (N2) $\widehat{\Delta}_{F_2,\mathcal{B}}=-\mathcal{U}\widehat{\Delta}_{F_1,\mathcal{B}}\mathcal{U}^*$ for the involution $\mathcal{U}$ of Lemma 25.5(b), and its leading star and remainder are $-\mathcal{U}L\mathcal{U}^*$ and $-\mathcal{U}\Upsilon\mathcal{U}^*$.
- (N3) For $s\in\{0,1\}$, $\widehat{\Delta}_{E_s,\mathcal{B}}=L+\Upsilon$, where L is the orthogonal sum over the singleton sectors of c_p times the lift of $\mathsf{E}(\hat{P}_p)$, $0<c_p\le1$; $\|L\|\le(1-1/b)^{-1}$ and $\|\Upsilon\|\le r_E(X)\,r_\mathcal{B}^2$.
- (N4) At every block F_1 and F_2 have the same raw-arm coefficient and the same auxiliary prices below, and so have E_0 and E_1.

These hold in the construction. The change of F_1 in $\nu_\mathcal{B}-\lambda_\mathcal{B}$ is the record weight $w_\mathrm{D}r_\mathcal{B}/\widetilde{\mu}_\mathcal{B}(1)$ times the placed difference of the signed record and the reference record (Definition 23.9), whose skew part is $L_{\mathsf{t},\mathcal{B}}+R_{\mathsf{t},\mathcal{B}}$ with $\|L_{\mathsf{t},\mathcal{B}}\|\le\ell_\mathsf{t}(b)b^{-1/2}$, $\|R_{\mathsf{t},\mathcal{B}}\|\le a_F(b)/b$, and $L_{\mathsf{t},\mathcal{B}}$ equal to $L_{\mathsf{t},p}^\circ$ on the sector $\{p\}$ and zero on the larger sectors (Proposition 23.13(b)); dividing by $w_\mathrm{D}r_\mathcal{B}^2$ gives (N1) with $c_p=(r_p/r_\mathcal{B})/\widetilde{\mu}_\mathcal{B}(1)\le1$, the normalizer $\widetilde{\mu}_\mathcal{B}(1)\ge1$ only lowering the normalized change. (N2) is Proposition 23.13(e) and Lemma 25.5(b). For E_s, Lemma 23.18 gives the skew part $L_{E,\mathcal{B}}+R_{E,\mathcal{B}}$, with $L_{E,\mathcal{B}}$ equal to $V_1PV_0^*-V_0(V_1P)^*$ on the singleton sectors, of arm norm at most $b^{-1/2}$ and norm at most $b^{-1/2}/(1-b^{-1})$, and $\|R_{E,\mathcal{B}}\|\le r_E(X)b^{-3/2}$; the object-level changes of E_0 and E_1 coincide (Lemma 23.17). (N4): the two records of one kind have the same weight and the same local constants.

The *auxiliary prices* of a skew record ϕ at $\mathcal{B}$ are

$$\pi^{\mathrm{arm}}_{\mathcal{B},\phi}=w\,r_\mathcal{B}^2q_\mathcal{B}^{2a},\qquad \pi^{\mathrm{lead}}_{\mathcal{B},\phi}=\ell\,\pi^{\mathrm{arm}}_{\mathcal{B},\phi},\qquad \pi^{\mathrm{rem}}_{\mathcal{B},\phi}=\rho\,\pi^{\mathrm{arm}}_{\mathcal{B},\phi},$$

with w the weight of the raw-arm coefficient and $(\ell,\rho)=(\ell_\mathsf{t}(b),a_F(b)r_\mathcal{B})$ for F_1,F_2 and $((1-1/b)^{-1},r_E(X)r_\mathcal{B}^2)$ for E_0,E_1; they are the auxiliary prices $y^{\mathrm{arm}},y^{\mathrm{lead}},y^{\mathrm{rem}}$ of Lemma 10.14(v), and Lemma 23.24 bounds them with the coefficients evaluated at the piece start. The coefficients are nondecreasing in (q,r), so over the blocks of the piece i with start X_i and degree charges $\bar{I}_{d,i}$ their sums are at most, at the exponent a_0 and before the repricing factor t,

$$A^{\mathrm{arm}}_{F,i}=2w_{\mathrm{D},i}\bar{I}_{2,i},\quad A^{\mathrm{lead}}_{F,i}=2w_{\mathrm{D},i}\ell_\mathsf{t}(X_i)\bar{I}_{2,i},\quad A^{\mathrm{rem}}_{F,i}=2w_{\mathrm{D},i}a_F(X_i)\bar{I}_{3,i},$$

$$A^{\mathrm{arm}}_{E,i}=2w_{\mathrm{S},i}\bar{I}_{2,i},\quad A^{\mathrm{lead}}_{E,i}=\frac{2w_{\mathrm{S},i}\bar{I}_{2,i}}{1-1/X_i},\quad A^{\mathrm{rem}}_{E,i}=2w_{\mathrm{S},i}r_E(X_i)\bar{I}_{4,i},$$

each summed over the two records of one kind and rounded up; their totals are the auxiliary totals of Definition 30.1. The full skew price of a fold record is its leading plus its remainder price; that of a matching record is the price of Lemma 23.17, which is at least the raw-arm and the leading price. For a selection ω on X write $\pi_\omega^{\mathrm{arm}} = \prod_{\mathcal{B}\in X} \pi_{\mathcal{B},\omega(\mathcal{B})}^{\mathrm{arm}}$, likewise $\pi_\omega^{\mathrm{lead}}$, and $\pi_\omega^{\mathrm{tot}} = \prod_{\mathcal{B}\in X}(\pi_{\mathcal{B},\omega(\mathcal{B})}^{\mathrm{lead}} + \pi_{\mathcal{B},\omega(\mathcal{B})}^{\mathrm{rem}})$; π_ω is its full skew monomial.

For a selection on X with the stars $L_\mathcal{B}$ and the remainders $\Upsilon_\mathcal{B}$ of (N1)–(N3), with norm bounds $\ell_\mathcal{B}$ and $\rho_\mathcal{B}$, the error after removing the tensor of the leading stars has norm

$$\Big\| \bigotimes_{\mathcal{B}\in X} (L_\mathcal{B} + \Upsilon_\mathcal{B}) - \bigotimes_{\mathcal{B}\in X} L_\mathcal{B} \Big\| \le \prod_{\mathcal{B}\in X} (\ell_\mathcal{B} + \rho_\mathcal{B}) - \prod_{\mathcal{B}\in X} \ell_\mathcal{B} : \tag{25.5}$$

expand the tensor product by the set of remainder factors and bound every term by the product of its norm bounds. Multiplied by the raw-arm coefficients and by q_X^{2a}, the right side becomes $\pi_\omega^{\mathrm{tot}} - \pi_\omega^{\mathrm{lead}}$, the remainder part of the class bounds below. Summed over a degree class with n fold and m matching records it is the *remainder polynomial*

$$\mathcal{R}_{n,m} = (A_F^{\mathrm{lead}} + A_F^{\mathrm{rem}})^n (A_E^{\mathrm{lead}} + A_E^{\mathrm{rem}})^m - (A_F^{\mathrm{lead}})^n (A_E^{\mathrm{lead}})^m$$

of the totals, in the form of Lemma 10.14(v). Only the separately proved leading prices stand in it (Remark 10.15).

Lemma 25.3 (Matrix envelopes)**.** *Let $X = X_F \cup X_E$ be a set of middle blocks, $|X_F| = n$, $|X_E| = m$, $n+m$ even. Let $\mathcal{A}_F$ be the algebra of real 5×5 matrices that are block diagonal for the partition $\{0,1,2\},\{3\},\{4\}$ of the fold labels, that is $M_3(\mathbb{R})\oplus\mathbb{R}\oplus\mathbb{R}$, let $\mathcal{A}_E$ be $M_3(\mathbb{R})$ or its diagonal subalgebra, and $\mathcal{A} = \bigotimes_{X_F}\mathcal{A}_F \otimes \bigotimes_{X_E}\mathcal{A}_E$, real matrices on the label words $v = (v_\mathcal{B})_{\mathcal{B}\in X}$. An* envelope *for (X_F, X_E) is a $D\in\mathcal{A}$ with*

$$D \pm \bigotimes_{\mathcal{B}\in X_F} \mathsf{F}(\xi_\mathcal{B}) \otimes \bigotimes_{\mathcal{B}\in X_E} \mathsf{E}(\xi_\mathcal{B}) \succeq 0 \qquad \text{for every } \xi\in\{i,-i\}^X, \tag{25.6}$$

the complex matrices of (25.4) *at $\hat{P} = \xi_\mathcal{B}$; adding the two signs gives $D\succeq 0$.*

- *(a)* Lifting. *Let* $\mathrm{Lift}(D)$ *be the operator on $L^2(\bigotimes_{X_F} m_{0,\mathcal{B}} \otimes \bigotimes_{X_E} m_{E,\mathcal{B}})$ obtained by replacing, in every factor, a matrix unit e_{ab} of two singleton labels by $V_a V_b^* = M_{ab} - \mathbb{J}$, and, at a fold factor, the scalar units e_{33} and e_{44} by Q_{12} and Q_{01}. Let $L_\mathcal{B}$ be, at a fold block, an orthogonal sum over the singleton sectors of c_p times the lift of $\mathsf{F}(\hat{P}_p)$ by Ξ_p, with $0\le c_p\le 1$, and at a matching block the same with E. Then $\pm\bigotimes_{\mathcal{B}\in X} L_\mathcal{B} \preceq \mathrm{Lift}(D)$. Moreover* $\mathrm{Lift}(D)$ *annihilates every vector with a constant factor at some block, and it is a finite real combination $\sum_C c(C)\,C$ of tensor products C of the normalized nonnegative couplings $M_{ab}, P_{12}, P_{01}, \mathbb{J}$ of a fold factor and $M_{ab}, \mathbb{J}$ of a matching factor, all with the actual half marginals; after equal products are collected, $M(D) = \sum_C |c(C)|$ is its* coupling mass.
- *(b)* Norm. *$\|\mathrm{Lift}(D)\| \le \Gamma_F^n \Gamma_E^m \|D^\sharp\|$, where $D^\sharp$ is the image of D under the completely positive map that replaces, at every fold factor, e_{33} by $e_{33}+\eta_{\mathrm{pair}}(e_{11}+e_{22})$ and e_{44} by $e_{44}+\eta_{\mathrm{pair}}(e_{00}+e_{11})$. If D is diagonal, then also $\|\mathrm{Lift}(D)\| \le \max_v (G_X^{\mathrm{fr}}\, d^\sharp)_v$, where $d^\sharp$ is the diagonal of $D^\sharp$ and G_X^{fr} the Kronecker product of the matrix G_F^{fr} of the proof of Lemma 25.1 at the fold factors and of the 3×3 matrix G_E^{fr} with unit diagonal and off-diagonal entries $1/(Q_3-1)$ at the matching factors.*
- *(c)* Diagonal case. *Consider an even number of stars on orthogonal sums, each with one centre and any number of mutually orthogonal arms. A tensor feature v specifies a centre or an arm at each factor; let $A(v)$ be its set of arm positions, its* mask, *and $\alpha(v)$ the product of the arm norm bounds at those positions. Let $d_v > 0$ be weights on the tensor features such*

that for every complementary pair of masks A, A^c

$$\Big(\sum_{A(v)=A}\frac{\alpha(v)^2}{d_v}\Big)\Big(\sum_{A(w)=A^c}\frac{\alpha(w)^2}{d_w}\Big)\le 1. \tag{25.7}$$

Then the tensor product of the stars lies between $\pm$ the diagonal operator with the weights d_v, extended by any nonnegative weights to the features that are neither centres nor arms. In particular a diagonal $D\in\mathcal{A}$ satisfying (25.7) *for the stars* (25.4)*, with the arm magnitudes $1,\mathsf{t},1-\mathsf{t}$ at the fold arms $1,2,3$ and 1 at the matching arm 1, is an envelope.*

(d) Finite tests. *The conditions* (25.6) *follow from the following finite tests, in which $\mathsf{F}(i)=f_r+if_i$ with real f_r,f_i and $H=f_i-if_r$, and in the second, third and fourth cases D is diagonal in the matching labels, $D=\sum_e D_e\otimes e_{ee}$ over the matching words e:*

- *FF and FE: $D\pm\mathsf{F}(i)\otimes\mathsf{F}(\pm i)\succeq 0$, respectively $D\pm\mathsf{F}(i)\otimes\mathsf{E}(\pm i)\succeq 0$;*
- *one fold and an odd number m of matching factors: $D_e\succeq 0$ for every matching word e, and $\left(\begin{smallmatrix}D_e & H\\ H & D_{\bar e}\end{smallmatrix}\right)\succeq 0$ for the complementary active words $e,\bar e\in\{0,1\}^m$;*
- *two fold and two matching factors: the same blocks with the cross matrix $\mathsf{F}(i)\otimes\mathsf{F}(\pm i)$;*
- *three fold and one matching factor: $D_e\in\mathcal{A}_F^{\otimes 3}$ invariant under permutations of the fold coordinates, and the blocks with the cross matrix $\mathsf{S}(\xi_1)\otimes\mathsf{S}(\xi_2)\otimes\mathsf{S}(\xi_3)$ for the phase counts 0 and 1, where $\mathsf{S}(\xi)=i\,G^*\mathsf{F}(\xi)G$, $G=\operatorname{diag}(1,1,1,i,i)$, is real symmetric (the* pure-pair gauge*);*
- *four fold factors: D invariant under permutations of the fold coordinates, and $D\pm\bigotimes_{k=1}^4\mathsf{S}(\xi_k)\succeq 0$ for the phase counts 0, 1, 2.*

Every test is a rational matrix inequality; a Hermitian test is certified by its real form of twice the size (Remark 14.7), and a real test $\mathbf{M}\succeq 0$ by an integer matrix $\mathbf{L}$ and a scale g such that $g^2\mathbf{M}-\mathbf{L}\mathbf{L}^{\mathsf{T}}$ is entrywise diagonally dominant with nonnegative diagonal.

Proof. (a) The tensor $\bigotimes_{\mathcal{B}}L_{\mathcal{B}}$ is the orthogonal sum, over the choices $(p_{\mathcal{B}})_{\mathcal{B}\in X}$ of one singleton sector at every block, of $(\prod_{\mathcal{B}}c_{p_{\mathcal{B}}})$ times the lift by $\Xi=\bigotimes_{\mathcal{B}}\Xi_{p_{\mathcal{B}}}$ of $T=\bigotimes_{X_F}\mathsf{F}(\hat P_{p_{\mathcal{B}}})\otimes\bigotimes_{X_E}\mathsf{E}(\hat P_{p_{\mathcal{B}}})$, and it vanishes on the other sectors. The operators $\hat P_{p_{\mathcal{B}}}$ act on distinct tensor factors, so they commute; they are normal with spectrum in $\{i,-i\}$, and $D\otimes I\pm T$ is a polynomial in them with matrix coefficients. By the joint spectral decomposition it is the orthogonal sum of the matrices (25.6) over the joint eigenspaces, hence positive semidefinite. As $D\succeq 0$ and $0\le\prod c\le 1$, also $D\otimes I\pm(\prod c)T\succeq 0$, and the congruence by Ξ gives $\Xi(D\otimes I)\Xi^*\pm(\prod c)\,\Xi T\Xi^*\succeq 0$. The ranges of the maps Ξ of distinct choices lie in mutually orthogonal sectors, so $\pm\bigotimes_{\mathcal{B}}L_{\mathcal{B}}\preceq\sum_{(p_{\mathcal{B}})}\Xi(D\otimes I)\Xi^*$. It remains to compare the right side with $\mathrm{Lift}(D)$ by positive enlargements. D acts on the labels and the sector projections Π_S on the label spaces, so they commute. Summing over all choices of one sector $S_{\mathcal{B}}$ at every block, instead of singleton sectors only, adds the congruences of the positive operators $D\otimes\bigotimes_{\mathcal{B}}\Pi_{S_{\mathcal{B}}}$, the maps at the labels 3, 4 being taken zero on the sectors of size at least two. The sum is the lift of D by $\bigotimes_{\mathcal{B}}(V_0,V_1,V_2,Z_{\mathcal{B}},0)$ at the fold blocks and $\bigotimes_{\mathcal{B}}(V_0,V_1,V_2)$ at the matching blocks, on the whole centred label spaces, where $Z_{\mathcal{B}}$ is the orthogonal sum of the maps A_3°/r_p; as these have orthogonal ranges in the sectors of $\mathcal{K}_{12}$ and norm at most one, $Z_{\mathcal{B}}$ is a contraction into $\mathcal{K}_{12}$. In this lift a unit e_{ab} of two singleton labels gives $V_aV_b^*$. The labels 3 and 4 are separate scalar blocks of $\mathcal{A}_F$, so every term with the label 3 (or 4) at a fold coordinate is a positive operator on the other coordinates tensored with $Z_{\mathcal{B}}Z_{\mathcal{B}}^*$ (or 0) at that coordinate; and $Z_{\mathcal{B}}Z_{\mathcal{B}}^*\preceq\Pi_{\mathcal{K}_{12}}\preceq Q_{12}$, $0\preceq Q_{01}$ by (25.3). Replacing a factor of a tensor product of positive operators by a larger positive operator increases it, so these replacements, made one coordinate at a time, give $0\preceq\sum_{(p_{\mathcal{B}})}\Xi(D\otimes I)\Xi^*\preceq\mathrm{Lift}(D)$ and hence $\pm\bigotimes_{\mathcal{B}}L_{\mathcal{B}}\preceq\mathrm{Lift}(D)$. Every unit $V_aV_b^*$ and Q_{ij} annihilates the constants on both sides, which gives the second assertion, and the expansions $V_aV_b^*=M_{ab}-\mathbb{J}$ and $Q_{ij}=P_{ij}-(P_i+P_j)/(1+\rho)+\frac{1-\rho}{1+\rho}\mathbb{J}$ with $P_i=M_{ii}$ give the third (Lemma 25.2).

(b) Write $\Sigma_X = \bigotimes_{\mathcal{B}} \Sigma_{\mathcal{B}}$ for the tensor product of the synthesis maps of the fold and matching features, and let D act on $\bigotimes_{\mathcal{B}} \bigoplus_v \mathcal{H}_v$ through the identifications V_a of the singleton spaces with L_0^2 and by scalars on $\mathcal{K}_{12}$, $\mathcal{K}_{01}$. By (25.3), $Q_{12} \preceq \Pi_{\mathcal{K}_{12}} + \eta_{\mathrm{pair}}(C_1 + C_2)$ and $Q_{01} \preceq \Pi_{\mathcal{K}_{01}} + \eta_{\mathrm{pair}}(C_0 + C_1)$, and the replacements are positive enlargements of positive principal blocks as in (a); so $0 \preceq \mathrm{Lift}(D) \preceq \Sigma_X D^\sharp \Sigma_X^*$, the lower bound by (a). The map $D \mapsto D^\sharp$ is the identity plus the map with the Kraus operators $\eta_{\mathrm{pair}}^{1/2} e_{13}$, $\eta_{\mathrm{pair}}^{1/2} e_{23}$, $\eta_{\mathrm{pair}}^{1/2} e_{04}$, $\eta_{\mathrm{pair}}^{1/2} e_{14}$, hence completely positive. So $\|\mathrm{Lift}(D)\| \le \|\Sigma_X\|^2 \|D^\sharp\| \le \Gamma_F^n \Gamma_E^m \|D^\sharp\|$ by Lemma 25.1(b),(c). For diagonal D, $\Sigma_X D^\sharp \Sigma_X^* = \sum_v d_v^\sharp \iota_v \iota_v^*$ for the tensor products ι_v of the feature inclusions, and Lemma 13.8(a),(b) with the correlation bounds of Lemma 25.1(a),(c) gives the second bound.

(c) A star maps its centre to its arms and its arms to its centre, so the tensor maps the sector of a mask only to the sector of the complementary mask, and annihilates the features outside the stars. After conjugation by $D^{-1/2}$, the block between two complementary sectors is bounded, on the vector of component norms, by the outer product of the vectors $(\alpha(v)/\sqrt{d_v})_v$ and $(\alpha(w)/\sqrt{d_w})_w$; its norm is at most the square root of the left side of (25.7). The even tensor is self-adjoint, so the norm bound proves both orders. The proof keeps the phases of the arm operators and uses no simultaneous diagonalization, and weights on inactive features are positive additions. The stars of (25.4) at $\hat{P} = \xi_{\mathcal{B}}$ have the stated arm magnitudes, which gives the last assertion.

(d) D is real and $\mathsf{F}(-i)$, $\mathsf{E}(-i)$ are the complex conjugates of $\mathsf{F}(i)$, $\mathsf{E}(i)$, so the condition at ξ is the complex conjugate of the condition at $-\xi$; for FF and FE the four phase patterns reduce to the two listed. If D is diagonal in the matching labels, the tensor maps an active matching word $e \in \{0,1\}^m$ to its complement $\bar{e}$ with the scalar factor $\pi_E = \prod_{X_E} \xi_{\mathcal{B}}$, in both directions, and annihilates the words containing the label 2; so (25.6) is the orthogonal sum of the conditions $D_e \succeq 0$ on the inactive words and of the blocks $\left(\begin{smallmatrix} D_e & \pm\pi_E T_F \\ \pm\pi_E T_F & D_{\bar{e}} \end{smallmatrix}\right) \succeq 0$, T_F the tensor of the fold matrices. The congruence by $\mathrm{diag}(I, -I)$ exchanges the two signs. For one fold factor and odd m, $\pi_E = \pm i$ and $\pm i\mathsf{F}(\pm i) \in \{\pm H, \pm \bar{H}\}$, the conjugate case being automatic; H is Hermitian because f_i is real symmetric and f_r real antisymmetric. For two fold and two matching factors $\pi_E = \pm 1$. For three fold factors conjugate by $G^{\otimes 3}$, which commutes with $\mathcal{A}_F^{\otimes 3}$: then $G^* \mathsf{F}(\xi) G = -i\,\mathsf{S}(\xi)$, where $\mathsf{S}(\xi)$ has the entries $i\xi$, $i\mathsf{t}\xi$ at the singleton arms and $1 - \mathsf{t}$ at the pure-pair arm, real for $\xi = \pm i$; the cross matrix becomes $\pi_E (-i)^3 \mathsf{S} \otimes \mathsf{S} \otimes \mathsf{S}$, real up to the sign ± 1. By the permutation invariance the test depends only on the number of phases $-i$ among the fold phases, and complex conjugation identifies the counts c and $3 - c$. For four fold factors $(-i)^4 = 1$, so the gauged tensor is the real $\bigotimes_k \mathsf{S}(\xi_k)$; conjugation identifies the counts c and $4 - c$, and both signs are tested since no matching coordinate provides the congruence. The certificate of $\mathbf{M} \succeq 0$ is $g^2 \mathbf{M} = \mathbf{L}\mathbf{L}^{\mathsf{T}} + (g^2 \mathbf{M} - \mathbf{L}\mathbf{L}^{\mathsf{T}})$, a sum of a Gram matrix and a diagonally dominant symmetric matrix with nonnegative diagonal. □

Remark 25.4 (Where the gain comes from). For one star L with one arm of norm α and the diagonal D with weights d_c, d_a, the operator $D^{-1/2} L D^{-1/2}$ is a contraction if and only if $d_c d_a \ge \alpha^2$, the one-factor case of (25.7). For an even tensor T of skew stars, every D with $-D \preceq T \preceq D$ has norm at least $\|T\|$, so an envelope by itself costs at least the norm of the tensor. The gain comes from the averaging of Section 25.3: the reflection fixes some feature vertices and moves the others in pairs, and the average replaces the weights at each moved vertex and at its reflected mate by their mean, while it keeps that of every fixed one. The envelopes are chosen with small weights on the fixed vertices and large weights on the moved ones. For two matching stars of arm norm one, the weights x at the fixed vertex $(0,0)$ and $2x$ at the moved vertex $(1,1)$, with $2x^2 \ge 1$, give the averaged maximum $x \approx 2^{-1/2}$ against the norm one of the tensor. On fully invariant backgrounds the whole action of the chamber group averages the matching names; for a crossed pair this gives the cap $(1 + \sqrt{3})/6$ (Lemma 25.5(g)). Matrix envelopes in the singleton block algebra improve on diagonal ones because they may use the signed off-diagonal couplings M_{ab}, $a \ne b$; a coefficient

joining a pure-pair label to another label is excluded, since it would not expand into couplings of bounded variation.

25.3. **Averaging a dominating matrix over reflections.** The averaging is a change of variables in the energy, which permutes the slots and so leaves the product of the identical inputs unchanged; the work is to say exactly which measure is transported to which, and in which charts the transports are read.

Record assignments and placed measures. For a middle block $\mathcal{B}$ and an eligible record ϕ let $m_{\phi,\mathcal{B}}$ be its actual half marginal at ι, read on object triples: $m_{0,\mathcal{B}}$ for F_1, F_2 and $m_{E,\mathcal{B}}$ for E_0, E_1. A *record assignment* on a finite set X of middle blocks is a map $\omega\colon X \to \{F_1, F_2, E_0, E_1\}$, and $m_\omega = \bigotimes_{\mathcal{B}\in X} m_{\omega(\mathcal{B}),\mathcal{B}}$; a selection of (CP4) all of whose types are skew and whose records are eligible is identified with its record assignment. For an operator K on $L^2(m_\omega)$ let $[K]_\omega$ be the signed measure on the labels of X obtained by placing, at every $\mathcal{B} \in X$ by the map $\omega(\mathcal{B})\colon \mathbb{Z}/6\mathbb{Z} \to \{0,1,2\}$, the signed measure on two-layer object labellings whose cut operator at ι relative to m_ω is K: the slot (u_0, u_1, ℓ) receives the layer-ℓ label of the object $\omega(\mathcal{B})(u_0)$. The placement is half-local, so the cut matrix of $[K]_\omega$ at ι is the pull-back of K (Lemma 5.4(v)): $K \preceq K'$ implies $[K]_\omega \preceq_\iota [K']_\omega$, the map $K \mapsto [K]_\omega$ is linear and multiplicative over the blocks, $[I]_\omega = \mathrm{Diag}_\iota(m_\omega)$ is the diagonal copy of the actual half marginal, written Diag_ω, and $[\mathbb{J}]_\omega = J_\iota(m_\omega)$ is the product of the actual half marginals, written J_ω. The skew change of Section 25.2 is $\zeta^{\mathrm{a}}_{\mathcal{B},\phi} = w\, r^2_{\mathcal{B}}\, [\widehat{\Delta}_{\phi,\mathcal{B}}]_\phi$, $w\, r^2_{\mathcal{B}}$ the raw-arm coefficient. The couplings M_{ab} of Lemma 25.2 are placed in the same way. For a permutation β of $\{0,1,2\}$ let $\mathcal{P}_\beta f(x) = f(x \circ \beta)$, with $(x \circ \beta)_o = x_{\beta(o)}$, a unitary of $L^2(m_E)$ (the law m_E is invariant under name permutations) that maps $\mathcal{H}_i$ onto $\mathcal{H}_{\beta(i)}$; $\mathcal{P}_\beta V_i = V_{\beta(i)}$, so $\mathcal{P}_\beta\, \mathrm{Lift}(D)\mathcal{P}^*_\beta$ is the lift of the matrix D with both indices permuted by β.

Lemma 25.5 (Reflection averaging)**.** *For $h \in \{1,3,5\}$ let R_h be the slot permutation $(u_0, u_1, \ell) \mapsto (h - u_0, u_1, \ell)$: R_5, R_1, R_3 are the reflections of the chamber cuts $\vartheta_{0,0}$, $\vartheta_{0,1}$, $\vartheta_{0,2}$ of the first coordinate. For a finite signed measure κ on labellings write $R_{h*}\kappa$ for the law of $z \circ R_h$.*

(a) Placements. *R_h commutes with ι and with the rotation of the first coordinate, and maps the edge set of the twelve-label root onto itself. For a pattern $\phi\colon \mathbb{Z}/6\mathbb{Z} \to \{0,1,2\}$ and an object law κ, the reflection maps the law placed by ϕ to the law placed by $\phi \circ R_h$, and placing by $\beta \circ \phi$ is placing by ϕ the law relabelled by β. Substitution in the tuples gives the three reflection rows of Table 8, written with $\psi_{h,s} = \psi_{s,R_h}$:*

h	$(F_0, F_1, F_2) \circ R_h$	$\psi_{h,0}$	$\psi_{h,1}$
1	(F_1, F_0, F_2)	$(0\,1)$	$(1\,2)$
3	(F_2, F_1, F_0)	$(0\,2)$	$(0\,1)$
5	(F_0, F_2, F_1)	$(1\,2)$	$(0\,2)$

$$E_s \circ R_h = \psi_{h,s} \circ E_s,$$

and $G_{sk} \circ R_h = G_{s,\psi_{h,s}(k)}$ for the merges. $F_0 \circ R_5 = F_0$, $F_1 \circ R_3 = F_1$ and $F_2 \circ R_1 = F_2$ are the only cases $F_j \circ R_h = F_j$. Consequently R_{h*} maps a law placed by F_j to the same object law placed by $F_j \circ R_h$, and a law placed by E_s to the law placed by E_s whose object names are permuted by $\psi_{h,s}$.

(b) Arithmetic transport. *Let $\varsigma(x) = (-x_2, -x_1, -x_0)$ and $\mathcal{U}f = f \circ \varsigma$, on a block $\mathcal{U} = \bigotimes_p \mathcal{U}_p$. Then $\mathcal{U}$ is a self-adjoint unitary of $L^2(m_{0,\mathcal{B}})$ that maps $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \mathcal{K}_{12}, \mathcal{K}_{01}$ onto $\mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_0, \mathcal{K}_{01}, \mathcal{K}_{12}$, that is, it permutes the fold labels by $\varphi = (0\,2)(3\,4)$; it conjugates P_0, P_2, P_{12} into P_2, P_0, P_{01}, M_{ab} into $M_{\varphi(a)\varphi(b)}$ and Q_{12} into Q_{01}, so $\mathcal{U}\,\mathrm{Lift}(D)\mathcal{U}^*$ is the lift of D with both fold indices permuted by φ. The record differences satisfy $D_{F_2} = \mathcal{U}D^*_{F_1}\mathcal{U}^*$, hence $D^{\mathrm{a}}_{F_2} = -\mathcal{U}D^{\mathrm{a}}_{F_1}\mathcal{U}^*$ and $D^{\mathrm{s}}_{F_2} = \mathcal{U}D^{\mathrm{s}}_{F_1}\mathcal{U}^*$, and (N2) holds. $\mathcal{U}$ identifies operators only; it is never applied to the input F_A, and no negation symmetry of the set A is used.*

(c) Energy invariance. *Let $\alpha \in \mathcal{G}_6$ act on the first coordinate, let X be a finite set of middle blocks and $O = \bigotimes_{\mathcal{B}\notin X} O_{\mathcal{B}}$ a product of measures with $O_{\mathcal{B}} \succeq_\iota 0$ and $\alpha_* O_{\mathcal{B}} = O_{\mathcal{B}}$ for every*

atom $\mathcal{B} \notin X$. Then $Y_{\alpha_*\kappa\otimes O} = Y_{\kappa\otimes O}$ for every finite signed measure κ on the labels of X, and $\kappa \succeq_\iota \kappa'$ implies $\alpha_*\kappa \succeq_\iota \alpha_*\kappa'$.

(d) Averaging. *In the setting of (c) with $\alpha = R_h$, let Ω be a finite set of record assignments on X with numbers $\pi_\omega \geq 0$, and $\omega \mapsto \bar\omega$ an involution of Ω (fixed points allowed) such that, at every $\mathcal{B} \in X$, $\bar\omega(\mathcal{B}) \circ R_h = \omega(\mathcal{B})$ if $\omega(\mathcal{B})$ is a fold record and $\bar\omega(\mathcal{B}) = \omega(\mathcal{B})$ if it is a matching record, and $\pi_{\bar\omega} = \pi_\omega$. Let $\mathcal{T}_\omega$ be $\mathcal{U}$ at the fold blocks where $\bar\omega(\mathcal{B}) \neq \omega(\mathcal{B})$ and the identity elsewhere, $\mathcal{P}_\omega$ the tensor product of the $\mathcal{P}_{\psi_{h,s}}$ at the matching blocks E_s, and $g_\omega = \mathcal{P}_\omega\mathcal{T}_\omega$. If operators K_ω on $L^2(m_\omega)$ satisfy $K_{\bar\omega} = \mathcal{T}_\omega K_\omega \mathcal{T}_\omega^*$, then*

$$\sum_{\omega\in\Omega} \pi_\omega\, Y_{[K_\omega]_\omega\otimes O} = \sum_{\omega\in\Omega} \pi_\omega\, Y_{[\frac12(K_\omega + g_\omega K_\omega g_\omega^*)]_\omega\otimes O}.$$

(e) Charts. *Let $h = 5$ and let $\bar\omega$ exchange F_1 and F_2 at every fold block. The* coordinate unitary *of a record assignment is $\mathcal{U}_\omega = \bigotimes_{\mathcal{B}} \mathcal{U}_{\omega(\mathcal{B})}$ with $\mathcal{U}_{F_1} = I$, $\mathcal{U}_{F_2} = \mathcal{U}$, $\mathcal{U}_{E_0} = I$ and $\mathcal{U}_{E_1} = \mathcal{P}_{(0\,1)}$; its* chart reflection *$g_5 = \mathcal{U}_\omega^* g_\omega \mathcal{U}_\omega$ is $\mathcal{U}$ at every fold factor and $\mathcal{P}_{(1\,2)}$ at every matching factor, whatever ω is, so it permutes the feature labels by φ at the fold factors and by $(1\,2)$ at the matching factors, and $g_5 \operatorname{Lift}(D) g_5^* = \operatorname{Lift}(g_{5*}D)$ for the image $g_{5*}D$ of D under these label permutations. In the charts the normalized leading stars have one shape up to sign: $\mathcal{U}_\omega^* L \mathcal{U}_\omega$ is $\pm$ the star of (N1) (centre 0, arms $1,2,3$, signed coefficients) at a fold factor, and $\pm$ the one-arm star of (N3) between 0 and 1 at a matching factor. If $K_\omega = \mathcal{U}_\omega K \mathcal{U}_\omega^*$ for one operator K for all ω with the same fold and matching blocks, then $K_{\bar\omega} = \mathcal{T}_\omega K_\omega \mathcal{T}_\omega^*$ and $\frac12(K_\omega + g_\omega K_\omega g_\omega^*) = \mathcal{U}_\omega\, \frac12(K + g_5 K g_5^*)\mathcal{U}_\omega^*$.*

(f) Extra reflection. *Let O in (c) be invariant under R_1 and R_3 as well, and let ω be a record assignment whose fold records are all F_1, or all F_2; put $h = 3$ in the first case and $h = 1$ in the second. Then R_{h*} maps $[K]_\omega$ to $[\mathcal{P}'K\mathcal{P}'^*]_\omega$, where $\mathcal{P}'$ is the identity at the fold blocks and $\mathcal{P}_{\psi_{h,s}}$ at a matching block carrying E_s; so $Y_{[K]_\omega\otimes O} = Y_{[\frac12(K+\mathcal{P}'K\mathcal{P}'^*)]_\omega\otimes O}$. In the chart of ω, $T_\omega = \mathcal{U}_\omega^*\mathcal{P}'\mathcal{U}_\omega$ fixes every fold label and acts by $(0\,2)$ at the E_0 factors and by $(0\,1)$ at the E_1 factors if $h = 3$, and by $(0\,1)$ and $(0\,2)$ if $h = 1$. The two cases are conjugate by g_5: $T_{\bar\omega} = g_5 T_\omega g_5^{-1}$ on the labels. In a class $\Omega(X_F, X_E)$ with $n \geq 1$ fold blocks these assignments form the fraction 2^{1-n}. No reflection at individual primes or individual matching blocks is used.*

(g) Pure matching words. *Let X consist of matching blocks and O in (c) be invariant under $\mathcal{G}_6$. Every $\alpha \in \mathcal{G}_6$ maps the placement of E_s to itself with the names permuted by $\psi_{s,\alpha}$, where $E_s \circ \alpha = \psi_{s,\alpha} \circ E_s$; $\alpha \mapsto \psi_{0,\alpha}$ is onto the symmetric group of $\{0,1,2\}$, and $\psi_{1,\alpha} = c\,\psi_{0,\alpha} c^{-1}$ with $c = (0\,1\,2)$. So $Y_{[K]_\omega\otimes O}$ equals the same energy with K replaced by its average over $\mathcal{G}_6$. In the chart the action is $A_\tau(\sigma) = \bigotimes_j \sigma^{(\tau_j)}$, $\sigma^{(0)} = \sigma$, $\sigma^{(1)} = \epsilon\sigma\epsilon$, $\epsilon = (1\,2)$, for the type vector τ of ω (after fixing an ordering $\mathcal{B}_1, \ldots, \mathcal{B}_m$ of X, $\tau_j = 1$ if $\mathcal{B}_j$ carries E_1) and $\sigma = \psi_{0,\alpha}$; since $A_{1-\tau}(\sigma) = A_\tau(\epsilon\sigma\epsilon)$, complementary words have the same averaging group, and when the envelope of one word is transported to the other words with the same number of E_1 coordinates by permuting the coordinates, the averaged caps depend only on $\{k, m-k\}$, k the number of E_1 coordinates. For $m = 4, 6$ the groups $\{k, m-k\}$ have the weights $(\frac18, \frac12, \frac38)$ and $(\frac{1}{32}, \frac{3}{16}, \frac{15}{32}, \frac{5}{16})$. For two matching factors of different types, the diagonal weights a, b, c, d at the active chart vertices $(0,0), (0,1), (1,0), (1,1)$ pass the mask conditions $ad \geq 1$, $bc \geq 1$ and average to $a/3$ on the diagonal orbit and $(b+c+d)/6$ off it; $a = (1+\sqrt3)/2$, $b = c = 1$, $d = 1/a$ give both the value $(1+\sqrt3)/6$. For two factors of one type the diagonal orbit receives $(a+d)/3 \geq \frac23$.*

Proof. (a) R_h acts on the first coordinate only, where it commutes with the rotation (Lemma 5.13(i)) and preserves the root (Lemma 20.11(a)); ι acts on the layer. Under the law placed by ϕ the slot $(h - u_0, u_1, \ell)$ carries the label $y_{\phi(h-u_0),\ell}$, which is $y_{(\phi\circ R_h)(u_0),\ell}$; and $y_{\beta(\phi(u_0)),\ell}$ is the label of the object $\phi(u_0)$ in the relabelled family $y \circ \beta$. The table is the substitution of Table 8 (R_1, R_3, R_5

are $\vartheta_1, \vartheta_2, \vartheta_0$ there, and $\psi_{h,s}$ is ψ_{s,R_h}), the merges follow from $G_{sk} = \mathbf{1}[E_s = k]$ and $\psi_{h,s}^{-1} = \psi_{h,s}$ (Lemma 17.2(e)), and the fixed folds are read off the table.

(b) ς is an involution that preserves ν_F and g_0 (Lemma 26.3(c)), so $\mathcal{U}$ is a self-adjoint unitary of $L^2(m_0)$ that fixes the constants. It maps the functions of x_i onto the functions of x_{2-i}, and those of (x_1, x_2) onto those of (x_1, x_0); this gives the action on the $\mathcal{H}_i$, on the pair spaces, hence on $\mathcal{K}_{12}, \mathcal{K}_{01}$, and on the projections. With the negation $Nf(u) = f(-u)$ on L_0^2, $\mathcal{U}V_a = V_{2-a}N$, and N commutes with every real scalar matrix unit and with $\mathbb{J}$; so $\mathcal{U}M_{ab}\mathcal{U}^* = M_{\varphi(a)\varphi(b)}$ and the lift is transported as stated. The comparison record of F_2 is the reflected transpose of that of F_1 (Definition 23.9(d)), $K_2' = \mathcal{U}K'^*\mathcal{U}^*$ as operators, and $\mathcal{U}K_0\mathcal{U}^* = K_0$ with K_0 self-adjoint (Lemma 26.3(c)); hence $D_{F_2} = K_2' - K_0 = \mathcal{U}(K' - K_0)^*\mathcal{U}^*$ at every prime, and on a block by tensor products, the signed record being taken componentwise. Taking the skew and symmetric parts gives the next two identities. With (N4), dividing by the common coefficient and taking the transports of the leading star and of the remainder of F_1 gives (N2). The energies below are integrals of $\prod_v F_A(z_v)$ against placed measures, and $\mathcal{U}$ enters only through the identity of the operators of two records; the change of variables of (c) is a permutation of slots.

(c) For every finite signed measure ρ on labellings, $Y_\rho = \int \prod_{v \in V} F_A(z_v)\, d\rho(z)$ with the same lift at every slot, and $z \mapsto z \circ \alpha$ permutes the factors of the product; so $Y_{\alpha_*\rho} = Y_\rho$. The permutation acts on every atom, and $\alpha_*(\kappa \otimes O) = \alpha_*\kappa \otimes O$ by the invariance of the $O_{\mathcal{B}}$. Since α fixes the layer, it maps V_+ onto itself and commutes with ι, so it acts on cut matrices at ι by a simultaneous permutation of rows and columns (Remark 5.3(c)).

(d) By (a), R_{h*} maps the measure placed by the records of $\bar{\omega}$ with object operator K to the measure placed by the records of ω with object operator $\mathcal{P}_\omega K \mathcal{P}_\omega^*$: the fold records of $\bar{\omega}$ go to those of ω with the same object law, and the matching laws keep their placement and have their names permuted, the matching marginal m_E being invariant under name permutations. With $K = K_{\bar{\omega}} = \mathcal{T}_\omega K_\omega \mathcal{T}_\omega^*$ and (c), $Y_{[K_{\bar{\omega}}]_{\bar{\omega}} \otimes O} = Y_{[g_\omega K_\omega g_{\bar{\omega}}^*]_\omega \otimes O}$. Since $\omega \mapsto \bar{\omega}$ is a bijection of Ω and $\pi_{\bar{\omega}} = \pi_\omega$, the left side of the identity equals $\sum_\omega \pi_\omega Y_{[g_\omega K_\omega g_{\bar{\omega}}^*]_\omega \otimes O}$; average the two expressions.

(e) For $h = 5$, (a) gives $\mathcal{T}_\omega = \mathcal{U}$ at every fold block and $\mathcal{P}_\omega = \mathcal{P}_{(1\,2)}$ at E_0, $\mathcal{P}_{(0\,2)}$ at E_1. At a fold factor $\mathcal{U}_\omega \in \{I, \mathcal{U}\}$ commutes with $\mathcal{U}$, so the chart reflection is $\mathcal{U}$; at E_1 it is $\mathcal{P}_{(0\,1)}\mathcal{P}_{(0\,2)}\mathcal{P}_{(0\,1)} = \mathcal{P}_{(1\,2)}$. Moreover $\mathcal{U}_{\bar{\omega}} = \mathcal{T}_\omega \mathcal{U}_\omega$, because $\mathcal{U}^2 = I$; this gives $K_{\bar{\omega}} = \mathcal{T}_\omega K_\omega \mathcal{T}_\omega^*$, and $g_\omega \mathcal{U}_\omega = \mathcal{U}_\omega g_5$ gives the averaged form. The action on lifts is (b) and the paragraph before the lemma. By (N2) the coordinate unitary of an F_2 factor turns its star into -1 times a star of the F_1 shape. The matching star $V_1 \hat{P} V_0^* + V_0 \hat{P} V_1^*$ is invariant under $\mathcal{P}_{(0\,1)}$, so the coordinate unitary of an E_1 factor keeps the shape.

(f) By (a), $F_1 \circ R_3 = F_1$ and $F_2 \circ R_1 = F_2$, so R_{h*} maps the fold placements of ω to themselves with the same object laws, and at a matching block permutes the names by $\psi_{h,s}$; with (c) and the invariance of O under R_h this is the identity of energies, and averaging the two expressions gives the averaged form. The coordinate unitary of ω is I at all fold factors in the first case and $\mathcal{U}$ at all of them in the second, which commutes with $\mathcal{P}'$ there; at the matching factors the native permutations $\psi_{3,0} = (0\,2)$, $\psi_{3,1} = (0\,1)$, respectively $\psi_{1,0} = (0\,1)$, $\psi_{1,1} = (1\,2)$, read in the chart $\mathcal{U}_{E_1} = \mathcal{P}_{(0\,1)}$, become $(0\,2), (0\,1)$, respectively $(0\,1), (0\,2)$, as $(0\,1)(1\,2)(0\,1) = (0\,2)$. Conjugation by $(1\,2)$ exchanges $(0\,1)$ and $(0\,2)$, which is the last assertion. Among the 2^{n+m} assignments of the class, 2^m have all fold records F_1 and 2^m all F_2. For $n = 1$ the reflections used on a partner pair are the pairing and then R_h at the common placement; the operators so obtained at the two assignments of the pair, four in all, have the same energy, although the two operations do not define a group action on the assignments.

(g) Every $\alpha \in \mathcal{G}_6$ maps the placement of E_s to itself with the names permuted (Lemma 17.2(e)). The maps $\alpha \mapsto \psi_{s,\alpha}$ are homomorphisms; the relation $\psi_{1,\alpha} = c\, \psi_{0,\alpha} c^{-1}$ holds for the generators R_1, R_3, R_5 by the table of (a), and the six images $\psi_{0,\alpha}$ are distinct. With (c) the energy is the same for every α, and averaging gives the group average. In the chart the E_1 factors carry

$(0\,1)c\,\sigma c^{-1}(0\,1) = \epsilon\sigma\epsilon$, since $(0\,1)c = c^{-1}(0\,1) = \epsilon$. As $\sigma \mapsto \epsilon\sigma\epsilon$ is a bijection of the symmetric group, the sets $\{A_\tau(\sigma)\}$ and $\{A_{1-\tau}(\sigma)\}$ coincide, so the averages coincide. A word with k coordinates E_1 among m has the weight $\binom{m}{k}2^{-m}$, which gives the weights of the groups. For two factors of different types conjugate the second coordinate by ϵ: the active vertices become $(0,0),(0,2),(1,0),(1,2)$ and the action becomes the diagonal action $\sigma\otimes\sigma$, whose orbits on the label pairs are the three diagonal pairs and the six others; averaging a diagonal weight function over the orbits gives $a/3$ and $(b+c+d)/6$. For two factors of one type the active vertices are $(0,0),(0,1),(1,0),(1,1)$, the diagonal orbit receives $(a+d)/3 \ge 2(ad)^{1/2}/3 \ge \frac{2}{3}$. □

The identities of (d), (f) and (g) are between price-weighted energies of placed operators; they are not identities between the terms I_ω of the record expansion, and partners do not cancel. They are applied only to energies with F_A at every slot and with an invariant outside law: not to energies carrying index structure (the signed index, the common mean, the energies of Lemma 10.17), and not with N, λ, ρ or ν outside, which are not invariant (Lemma 25.6). The extra averages (f), (g) are used only on fully invariant outside monomials.

25.4. **Outside laws invariant under reflections, and their price.**

Lemma 25.6 (Invariant outside laws)**.** *Let $h \in \{1,3,5\}$.*

(i) For every choice of the completion weights w, v, the middle reference $\mu_{\mathcal{B}}$ of Lemma 22.1 is R_h-invariant.

(ii) For every choice of the state weights $(w_{\mathfrak{t}}, v_{\mathfrak{t}})_{\mathfrak{t}\in\{\mathrm{L,D,S,R,C}\}^2}$, the upper reference of Definition 27.5 is R_h-invariant, and so is every tail law (Definition 29.1).

(iii) Every prefix law (Definition 19.1) is invariant under all permutations of the first coordinate.

(iv) At a middle block, E_{add} and H are R_h-invariant and G is R_5-invariant (Definition 22.4). The bracket carries one copy of the one-layer folded copies of the fold records, so its background, the outside measure N_0 of Lemma 10.12 against which the energies of selected laws are priced, is $\mathsf{N}_{0,\mathcal{B}} = \mu_{\mathcal{B}} + E_{\mathrm{add}} + G + H = \mathsf{N}^{\mathrm{inv}}_{\mathcal{B}} + G_{\mathcal{B}}$ (Lemma 22.7); F_{fold} is a summand of $\mathsf{N} = \lambda + F_{\mathrm{fold}} + H$, not of N_0. The invariant background $\mathsf{N}^{\mathrm{inv}}_{\mathcal{B}} = \mu_{\mathcal{B}} + E_{\mathrm{add}} + H$ *is invariant under R_1, R_3, R_5 and the group $\mathcal{G}_6$ they generate, $\mathsf{N}_{0,\mathcal{B}}$ is R_5-invariant, and $\mathsf{N}^{\mathrm{inv}}_{\mathcal{B}}$, $G_{\mathcal{B}}$ and $\mathsf{N}_{0,\mathcal{B}}$ are $\succeq_\iota 0$. Off the middle blocks $\mathsf{N}_{0,\mathcal{B}} = \mathsf{N}^{\mathrm{inv}}_{\mathcal{B}} := \mu_{\mathcal{B}}$ is invariant under $\mathcal{G}_6$ and $\succeq_\iota 0$.*

(v) Assigned prices. *For the middle piece i with start X_i, weights $v_{\mathrm{D},i}$, $w_{\mathrm{S},i}$, polarization coefficient $(f_1+h_1)_i$ (an upper end) and the exact rational upper ends $I^{\mathrm{ex}}_{2,i}$, $I^{\mathrm{ex}}_{3,i}$ of its degree charges at the source exponent a_0, from which the native outside price of the piece is computed, put*

$$u_i^{\mathrm{inv}} = 2v_{\mathrm{D},i}I^{\mathrm{ex}}_{2,i} + \frac{12w_{\mathrm{S},i}}{1-3/X_i}I^{\mathrm{ex}}_{3,i}, \qquad u_i^G = 2v_{\mathrm{D},i}(f_1+h_1)_i I^{\mathrm{ex}}_{3,i}, \qquad U^{\mathrm{inv}} = \sum_i u_i^{\mathrm{inv}}. \tag{25.8}$$

Then u_i^{inv} and u_i^G bound the summed prices at a_0 of the invariant tags $E_{\mathrm{add}}+H$ and of the generic tag G over the blocks of the piece, and $u_i^{\mathrm{inv}} + u_i^G \le u_i$, the upper end of the term of the piece in the native outside price, with $\sum_i u_i \le U_0'$, the upper bound of U_0 at a_0 (Lemma 22.9). At the exponent a all these prices are at most t times their values. Consequently, if $0 \le \kappa_b \le \kappa_H$ and an outside polynomial has the coefficient κ_b on the monomials whose middle factors are all μ or invariant tags and κ_H on all others, its completed value at the exponent a is at most

$$\kappa_H e^{U_0} - (\kappa_H - \kappa_b)e^{tU^{\mathrm{inv}}} \qquad \text{for every } U_0 \ge t\Big(U^{\mathrm{inv}} + \sum_i u_i^G\Big),$$

for instance $U_0 = tU_0'$. The same factor $e^{tU^{\mathrm{inv}}}$ is the background factor of the orbit coupon (Lemma 25.7).

G is not R_1- or R_3-invariant, and $\mathsf{N}_{\mathcal{B}}$, $\lambda_{\mathcal{B}}$, $\rho_{\mathcal{B}}$, $\nu_{\mathcal{B}}$ are not asserted to be invariant under any R_h: E_{new} and the repaired records accept the six queries, whose first-coordinate pair $(0,2)$ is sent by R_5 to $(5,3)$.

Proof. (i) By Lemma 22.1(iii), $\widetilde{\mu}_{\mathcal{B}} = \sum_t w_t r^{\deg t}\pi_{t,2} + r\sum_t v_t r^{\deg t}\,\mathrm{Diag}_\iota(\pi_{t,1})$, where $\pi_{t,j}$ is the sum of the j-layer components of type t. For each j these components are those of the five-type datum of the j-layer root, F and E laws, with the exchangeable E law and the $\mathcal{G}_6$-invariant root (Lemma 17.2(a)); by Lemma 17.5(a) and its proof every type sum $\pi_{t,j}$ is $\mathcal{G}_6$-invariant, because precomposition with a chamber automorphism permutes the folds and the merges of each partition, fixes the constants, and maps each matching to itself with its names permuted. The positive roots of the late pieces are invariant as well, by the score or support-dictionary invariance (Lemma 21.2(b), Lemma 21.17). The weights depend only on the type, and R_{h*} commutes with Diag_ι and with the pull-back to V, since R_h fixes the layer and the second coordinate. Hence $\widetilde{\mu}_{\mathcal{B}}$, its mass and $\mu_{\mathcal{B}}$ are invariant.

(ii) The upper reference is the two-coordinate five-type measure of Definition 17.4, with the field laws, components and row data of Section 27.2.2, the state weights $w_{\mathfrak{t}}$ and the reserved weights $v_{\mathfrak{t}}$. By Lemma 17.5(d), for each type $\mathfrak{t}_1$ and pattern $\phi' \in P_{\mathfrak{t}_1}$ of the second coordinate, its components with second type $\mathfrak{t}_1$ and second pattern ϕ' form the image, under the emission $e_{\phi'}$, of a one-coordinate five-type measure of the row data of the first coordinate, with weights that depend only on the states. These row data are five-type data (Section 27.2.2): the chamber reflections of an L coordinate are automorphisms of the graphs, and the E field laws are exchangeable in their object coordinate (Lemma 27.4(b)). So each such measure is $\mathcal{G}_6$-invariant by Lemma 17.5(a), and the emission commutes with the action on the first coordinate. The two-layer part, and in the same way the diagonal copies of the one-layer part, are therefore invariant for any weights. A tail law is a law of the same kind at one prime.

(iii) Given the clique data, the six first-coordinate labels are independent and identically distributed and the common square-affine map does not depend on the chamber, so the pull-back is invariant under every permutation of the first coordinate.

(iv) E_{add} and H are combinations of the diagonal copies $\mathrm{Diag}_\iota(\Pi^{(1)}_{E_s})$, and R_{h*} maps $\mathrm{Diag}_\iota(\Pi^{(1)}_{E_s})$ to the diagonal copy of the one-layer E law placed by $\psi_{h,s}\circ E_s$, which is the same measure, the E law being exchangeable. G is a multiple of the diagonal copies placed by G_{00} and G_{11}, which R_5 fixes ($\psi_{5,0}(0)=0$, $\psi_{5,1}(1)=1$), and which R_1 and R_3 send to merges outside G. The diagonal copies are reserved diagonal, hence $\succeq_\iota 0$ (Lemma 5.4(i)), and $\mu_{\mathcal{B}} \succeq_\iota 0$. The decomposition of N_0 is Definition 22.4(d) and Lemma 22.7(i). Off the middle blocks $\mathsf{N}_{0,\mathcal{B}} = \mu_{\mathcal{B}}$: the invariance is (ii) and (iii), and $\mu_{\mathcal{B}} \succeq_\iota 0$ is (R2) at ι.

(v) The masses of E_{add}, H and G at a block of the piece are at most $2v_{\mathrm{D}}r^2$, $12w_{\mathrm{S}}r^3/(1-3/X_i)$ and $2v_{\mathrm{D}}(f_1+h_1)r^3$, divided by the normalizing mass $\widetilde{\mu}_{\mathcal{B}}(1) \geq 1$ (Definition 22.4(b), Lemma 22.9); a price is the mass times $q_{\mathcal{B}}^{2a}$, and summing over the blocks of the piece with the upper charges $I^{\mathrm{ex}}_{d,i}$ of the sums of $r_{\mathcal{B}}^d q_{\mathcal{B}}^{2a_0}$ gives the first claim. The formulas are the degree-two coefficient and the two summands of the degree-three coefficient of $\mathsf{N}_0 - \mu$ in Lemma 22.9, evaluated at the same exact charges as the native outside price, so $u_i^{\mathrm{inv}} + u_i^G$ is the term of the piece in that price, of which u_i is the stored upper end (Certificate 25.11(c) checks $u_i^{\mathrm{inv}} + u_i^G \leq u_i$ and $\sum_i u_i \leq U_0'$). Every middle block has $\log q_{\mathcal{B}} \leq L = 570$, so $q_{\mathcal{B}}^{2a} \leq t q_{\mathcal{B}}^{2a_0}$ (Lemma 9.3). For the completed value, expand the outside polynomial in the actual positive tags $E_{\mathrm{add}} + H$ and G before completing repeated blocks: with $v_{\mathcal{B}}$, $w_{\mathcal{B}}$ the prices at a of the invariant and of the generic tag of $\mathcal{B}$, it is $\kappa_H \prod_{\mathcal{B}}(1+v_{\mathcal{B}}+w_{\mathcal{B}}) - (\kappa_H - \kappa_b)\prod_{\mathcal{B}}(1+v_{\mathcal{B}})$ over the middle blocks outside the selection, which only increases when the selected blocks are included, the polynomial being nondecreasing in every variable. By Lemma 10.14(iii) choose, in every piece, upper prices $m_{\mathcal{B}} \geq v_{\mathcal{B}}$ with $\sum_{\mathcal{B}\in i} m_{\mathcal{B}} = t u_i^{\mathrm{inv}}$ and $n_{\mathcal{B}} \geq w_{\mathcal{B}}$ with $\sum_{\mathcal{B}\in i} n_{\mathcal{B}} = t u_i^G$; then $\sum_{\mathcal{B}} m_{\mathcal{B}} = t U^{\mathrm{inv}}$ and

$\sum_{\mathcal{B}}(m_{\mathcal{B}}+n_{\mathcal{B}})=t(U^{\mathrm{inv}}+\sum_i u_i^G)\le U_0$. The chosen-tag-price form of Remark 10.15 then bounds the polynomial by $\kappa_H e^{U_0}-(\kappa_H-\kappa_b)e^{\sum_{\mathcal{B}} m_{\mathcal{B}}}=\kappa_H e^{U_0}-(\kappa_H-\kappa_b)e^{tU^{\mathrm{inv}}}$. Upper prices are assigned to separate positive components, and no lower bound is inferred from an upper estimate; the transport by t multiplies all tag prices together. The last sentence is the proof of Lemma 25.7. □

The invariance in (i) and (ii) is that of the five-type completion on a product of six-chamber coordinates; it is the same chamber invariance that the completions of the 216-slot design had (Section 34.2). The price U^{inv} is a chosen assignment, not a bound: a smaller number may replace it (the difference being treated as generic), a larger one may not, since the completed value decreases in U^{inv} at fixed U_0.

Lemma 25.7 (Invariant coupling orbits for exactly one folded error)**.** *Let X be a finite set of middle blocks, ω a record assignment on X, and $K=\sum_C c(C)[C]_\omega$ a collected finite real combination of placed couplings, C running over tensor products of the couplings of Lemma 25.2; put $M=\sum_C|c(C)|$. Let $h\in\{1,3\}$ be such that R_h maps every fold placement of ω to itself ($h=3$ if all fold records of ω are F_1, $h=1$ if all are F_2, either if ω has none); then $R_{h*}[C]_\omega=[hC]_\omega$ for a permutation $C\mapsto hC$ of the couplings (Lemma 25.5(f),(g)). Let $s=(h-1)/2$, the tag whose cut, the chamber cut $\vartheta_{0,1+s}$ of the first coordinate, has the reflection R_h (Definition 22.4(e)), and put*

$$M_h(K)=\sum_C \mathbf{1}\big[c(C)c(hC)>0\big]\min\big(|c(C)|,|c(hC)|\big),$$

a coupling with $hC=C$ counting with its full $|c(C)|$. Let $d'=D'/2$ be the upper bound at a_0 of the sharp price of one folded tag, half the sharp total D' of the first stage (Lemma 22.9: per block of the piece i the sharp price of a tag is at most $(v_{\mathrm{S},i}+v_{\mathrm{D},i})k_{\mathrm{new}}(X_i)r_{\mathcal{B}}^3 q_{\mathcal{B}}^{2a_0}$), let

$$s'=\sum_i \mathrm{up}\big((v_{\mathrm{S},i}+v_{\mathrm{D},i})s_{\mathrm{fold}}(X_i)I_{4,i}^{\mathrm{hi}}\big)$$

be the upper bound of its symmetric price, the symmetric part of a tag difference having the sharp coefficient $s_{\mathrm{fold}}(X)b^{-1}$ (Lemma 22.6(d),(e)), with the upper degree charges $I_{4,i}^{\mathrm{hi}}$ and the rounding up *of Definition 30.1, and put $\delta_{\mathrm{s}}=d'-s'$; $0\le s'\le d'$ (Certificate 25.11(d)). Let $U_0\ge t(U^{\mathrm{inv}}+\sum_i u_i^G)$ as in Lemma 25.6(v), $d=td'$ and v upper totals at a of the sharp and variation prices of each tag, and $R_{\mathrm{err}}\ge e^{U_0}\mathfrak{h}(d,v)$, for instance the value of Definition 30.1(2).*

- *(i)* $|Y^{\mathsf{N}}[K]-Y^{\mathsf{N}_0}[K]|\le\big(MR_{\mathrm{err}}-M_h(K)\,e^{tU^{\mathrm{inv}}}t\delta_{\mathrm{s}}+o(1)\big)C_X Y$.
- *(ii) If ω has no fold record, the two reflections may be used at once: the subtraction becomes $(M_1(K)+M_3(K))e^{tU^{\mathrm{inv}}}t\delta_{\mathrm{s}}$, and $M_1+M_3\le 2M$.*
- *(iii) Every other category of the error keeps its bound: the categories with several folded differences or with mixed tags, and, outside this comparison, the matching, index, unequal-input and restoration energies.*
- *(iv)* Gate. *The coefficient $MR_{\mathrm{err}}-M_h(K)e^{tU^{\mathrm{inv}}}t\delta_{\mathrm{s}}$ (in (ii) with M_1+M_3) is nonnegative: $M_h\le M$, $e^{tU^{\mathrm{inv}}}\le e^{U_0}$, $t\delta_{\mathrm{s}}\le d$ and $R_{\mathrm{err}}\ge e^{U_0}(d_0+d_1)=2e^{U_0}d$. The closing checks it in the form coupon $\le MR_{\mathrm{err}}$, non-strictly. No strict form is needed: it is a nonnegativity condition on a coefficient of a bound of the second kind of Lemma 11.4(i), whose inflated constant $(1+z)c+c'z$ stays nonnegative.*

For a class of selections the invariant orbit mass M^{inv} *is the average over the class of $M_h(K_\omega)$ for the selections admitting an h (in (ii) of M_1+M_3), and zero for the others, K_ω the collected expansion of the envelope of ω.*

Proof. (i) For a two-element orbit $\{C,hC\}$ whose coefficients have the same sign put $b=\mathrm{sgn}(c(C))\min(|c(C)|,|c(hC)|)$; then

$$c(C)C+c(hC)\,hC=2b\,\frac{C+hC}{2}+\big(c(C)-b\big)C+\big(c(hC)-b\big)hC.$$

The first coupling is positive, normalized and R_h-invariant, and the absolute coefficient mass is partitioned exactly into $2|b|$ and the residual mass. Fixed couplings are the one-element case, and opposite-sign orbits retain no mass. So $K = \sum_k c_k C_k$ with $\sum_k |c_k| = M$, in which R_h-invariant couplings carry the mass $M_h(K)$; and $Y^{\mathsf{N}}[K] - Y^{\mathsf{N}_0}[K] = \sum_k c_k (Y^{\mathsf{N}}[C_k] - Y^{\mathsf{N}_0}[C_k])$.

Every C_k is an admissible probability measure. Its difference is bounded as in the proof of Lemma 10.12: Lemma 10.8 with F the first-stage atoms outside X, $\rho_{\mathcal{B}} = \mathsf{N}_{0,\mathcal{B}}$, and the two tags. Let C_k be R_h-invariant, and consider the category of the count vector e_s: exactly one record difference, of tag s. Its terms are energies $Y_{C_k \otimes \delta \otimes O}$, δ the difference of tag s at one atom $\mathcal{B}' \in F$, O the product of N_0 at the other first-stage atoms and of μ elsewhere. Expand $\mathsf{N}_{0,\mathcal{B}} = \mu_{\mathcal{B}} + (E_{\mathrm{add}} + H)_{\mathcal{B}} + G_{\mathcal{B}}$. In a term without G, C_k and the outside product are R_h-invariant (Lemma 25.6(i)–(iv)), so their cut matrices at $\vartheta = \vartheta_{0,1+s}$, whose reflection is R_h, are symmetric. The energy is the diagonal cut form at ϑ evaluated at the half-function carrying F_A at every slot, by (5.1), the cut matrix of the product being the Kronecker product (5.2); by Remark 5.3(a) the skew part $\frac{1}{2}(\delta - R_{h*}\delta)$ contributes nothing, so δ may be replaced by its symmetric part $\frac{1}{2}(\delta + R_{h*}\delta)$ at ϑ. The reflection fixes the shared label of the folded record and transposes its conditional kernel, so this part has the sharp bound $s_{\mathrm{fold}}(X) b^{-1}$ of Lemma 22.6(e) relative to the replaced record at ϑ, and the argument of Lemma 10.8 charges the term with the symmetric price $(v_{\mathrm{S}} + v_{\mathrm{D}}) s_{\mathrm{fold}}(X) r_{\mathcal{B}'}^4 q_{\mathcal{B}'}^{2a}$ instead of the sharp price. The terms with G keep the sharp price. With $v_{\mathcal{B}}$, $w_{\mathcal{B}}$ the prices of the invariant and generic tags and $d_{\mathcal{B}'}$, $s_{\mathcal{B}'}$ the sharp and symmetric prices of the difference at $\mathcal{B}'$, and dropping, as in Lemma 10.8, the requirement that the atoms be distinct, the category is at most $(1 + o(1)) C_X Y$ times

$$\sum_{\mathcal{B}'} \Big[d_{\mathcal{B}'} \Big(\prod_{\mathcal{B}} (1 + v_{\mathcal{B}} + w_{\mathcal{B}}) - \prod_{\mathcal{B}} (1 + v_{\mathcal{B}}) \Big) + s_{\mathcal{B}'} \prod_{\mathcal{B}} (1 + v_{\mathcal{B}}) \Big],$$

a polynomial with nonnegative coefficients that is nondecreasing in every variable. Insert upper prices: for $d_{\mathcal{B}'}$ and $s_{\mathcal{B}'}$ the repriced block coefficients, distributed as in Lemma 10.14(iii) so that they sum to $d = td'$ and to ts', and for $v_{\mathcal{B}}, w_{\mathcal{B}}$ the chosen prices of the proof of Lemma 25.6(v). As $\prod(1+m+n) - \prod(1+m) \le e^{\sum (m+n)} - e^{\sum m}$ coefficientwise and $s \prod(1+m) \le s e^{\sum m}$, the category is at most $(d e^{U_0} - t\delta_{\mathrm{s}} e^{tU^{\mathrm{inv}}} + o(1)) C_X Y$. Its bound in Lemma 10.12 is $(d e^{U_0} + o(1)) C_X Y$, the term of e_s in $e^{U_0} \mathfrak{h}(d, v)$; the other categories keep their bounds. So the difference for an invariant C_k is at most $(R_{\mathrm{err}} - t\delta_{\mathrm{s}} e^{tU^{\mathrm{inv}}} + o(1)) C_X Y$, and for the others at most $(R_{\mathrm{err}} + o(1)) C_X Y$. Summing with $|c_k|$ gives (i).

(ii) For ω without fold records both R_1 and R_3 fix the placements. The categories e_0 and e_1 are disjoint; treat the first with the orbit decomposition of R_1 and the second with that of R_3, the difference being linear in the coupling. Each mass is at most M.

(iii) Only the category e_s and only its terms without G were changed.

(iv) $M_h \le M$ because $\min(|c(C)|, |c(hC)|) \le |c(C)|$. The variation and sharp totals of the two tags are equal, $d_0 = d_1 = d$, and the terms e_0, e_1 of $\mathfrak{h}$ are d_0 and d_1, so $R_{\mathrm{err}} \ge 2e^{U_0} d$; and $t\delta_{\mathrm{s}} \le td' = d$. Hence the coupon is at most $M e^{U_0} d \le M R_{\mathrm{err}}/2$ in (i) and at most $2M e^{U_0} d \le M R_{\mathrm{err}}$ in (ii). Under the inflation of Definition 9.19 the improved category is bounded by the same argument with every structured energy multiplied by $1 + z$, so the coefficient becomes $(1+z)c + c'z$ with $c \ge 0$, $c' \ge 0$. □

25.5. **(CP5) bounds for products of skew changes.**
Skew selections and envelopes. Let X be a finite set of middle blocks, partitioned into a set X_F of n *fold blocks* and a set X_E of m *matching blocks*, $n + m \ge 2$ even. The *skew selections* on (X_F, X_E) are the 2^{n+m} selections of Definition 9.14(c) on X with the skew type at every block, one of F_1, F_2 at every fold block and one of E_0, E_1 at every matching block; $\Omega(X_F, X_E)$ is their set, and the *partner* $\bar{\omega}$ of ω exchanges F_1 and F_2 at every fold block. All of them have the same actual half marginal $m_\omega = \bigotimes_{X_F} m_{0,\mathcal{B}} \otimes \bigotimes_{X_E} m_{E,\mathcal{B}}$, and by (N4) the same auxiliary monomials.

An *envelope family* for a set $\mathcal{C} \subseteq \Omega(X_F, X_E)$ closed under the pairing is an envelope $D_{(\omega)}$ of Lemma 25.3 for every $\omega \in \mathcal{C}$, read in the chart of ω, with $D_{(\bar\omega)} = D_{(\omega)}$; in all classes of the construction except $(0,4)$, $(0,6)$ it is one envelope D, and for $(0,4)$, $(0,6)$ it depends on the group $\{k, m-k\}$ of Lemma 25.5(g). Put $D_\omega = \mathcal{U}_\omega \operatorname{Lift}(D_{(\omega)})\mathcal{U}_\omega^*$ and let $\bar D_\omega$ be its average with $g_\omega D_\omega g_\omega^*$ (Lemma 25.5(d),(e)), the lift of $\frac12(D_{(\omega)} + g_{5*}D_{(\omega)})$ in the chart. The numbers of the family are class averages over $\mathcal{C}$:

- κ_H, the class average of the *caps* of the $\bar D_\omega$, certified upper bounds for $\|\bar D_\omega\|$ given by Lemma 25.3(b) (for one envelope D, the common value $\Gamma_F^n\Gamma_E^m\|(\frac12(D + g_{5*}D))^\sharp\|$ or its weighted-frame form);
- $\kappa_b \le \kappa_H$, the class average of the caps after the additional averages Lemma 25.5(f), (g) where they apply, and of the caps of the $\bar D_\omega$ elsewhere (an additional average is an average of unitary conjugates, and the frame bounds are invariant under the label permutations involved, so its cap is at most that of $\bar D_\omega$); for one envelope invariant under permutations of its matching coordinates and $n \ge 1$, $\kappa_b = (1 - 2^{1-n})\kappa_H + 2^{1-n}\sum_{k=0}^m 2^{-m}\binom{m}{k} b_k$, b_k the cap of the doubly averaged envelope on the words with k coordinates E_1;
- the coupling mass M, the average of $M(D_{(\omega)})$;
- the invariant orbit mass M^{inv} of Lemma 25.7.

For a selection ω write $\pi_\omega^{\mathrm{arm}}$, $\pi_\omega^{\mathrm{lead}}$, $\pi_\omega^{\mathrm{tot}}$ and π_ω as in Section 25.2, and let L_ω be the tensor product of its normalized leading stars.

Theorem 25.8 (Positive joint envelopes)**.** *Assume (CP1)–(CP4) and (CP8) of Theorem 9.21 and the star form (N1)–(N4). Let (X_F, X_E) be as above with $n + m \le 6$, let $\mathcal{C} \subseteq \Omega(X_F, X_E)$ be closed under $\omega \mapsto \bar\omega$, with an envelope family and its numbers κ_H, κ_b, M, M^{inv}. Let U_0, R_{err} and $B = e^{U_0} + R_{\mathrm{err}}$ be upper bounds at the exponent a as in Lemma 25.7, and δ_{s} its price saving.*

(i) Reduction. *For every $\omega \in \mathcal{C}$, $\pm L_\omega \preceq D_\omega$, and $\sum_{\omega\in\mathcal{C}} |I_\omega| \le \sum_{\omega\in\mathcal{C}}(\phi_\omega + o(1)\pi_\omega)Y$ with the class bound*

$$\begin{aligned}\phi_\omega = {} & \big[\kappa_H e^{U_0} - (\kappa_H - \kappa_b)e^{tU^{\mathrm{inv}}} + MR_{\mathrm{err}} - M^{\mathrm{inv}}e^{tU^{\mathrm{inv}}}t\delta_{\mathrm{s}}\big]\pi_\omega^{\mathrm{arm}} \\ & + B(\pi_\omega^{\mathrm{tot}} - \pi_\omega^{\mathrm{lead}}).\end{aligned} \tag{25.9}$$

The bracket is nonnegative (Lemma 25.6(v), Lemma 25.7(iv)), and ϕ_ω is a polynomial with nonnegative coefficients in the auxiliary prices, each monomial a product of one auxiliary price of each record of ω, as (CP5) requires.

(ii) Uniform cost. *Without the invariant split and the coupon, that is with $\kappa_b = \kappa_H$ and $M^{\mathrm{inv}} = 0$,*

$$\phi_\omega = (\kappa_H e^{U_0} + MR_{\mathrm{err}})\pi_\omega^{\mathrm{arm}} + B(\pi_\omega^{\mathrm{tot}} - \pi_\omega^{\mathrm{lead}}). \tag{25.10}$$

(iii) Invariant background. *The main term of the outside energy obeys the split*

$$\kappa_H e^{U_0} - (\kappa_H - \kappa_b)e^{tU^{\mathrm{inv}}}, \tag{25.11}$$

for every class, with the assigned prices of Lemma 25.6(v). The class bound is nondecreasing in κ_H, κ_b, M, R_{err}, B and U_0; U^{inv} is the assigned price, not an upper bound for one, and M^{inv} and δ_{s}, which enter with a minus sign, are a lower bound and the saving computed from the same tag prices as R_{err}.

(iv) Admissibility and robustness. *The couplings of the expansions of the $[D_\omega]_\omega$ (tensor products of $M_{ab}, P_{12}, P_{01}, \mathbb{J}$, placed by the records) and their orbit averages are admissible positive laws (Definition 8.9), independent of the indices, with no sharp bound applied to them. The bounds (i)–(iii) are robust (Definition 9.19), uniformly in (X_F, X_E), in $\mathcal{C}$ and in the envelope family ranging over a finite set.*

(v) Witnesses. *The following envelopes pass the tests of Lemma 25.3 in exact rational arithmetic at Q_3; their caps include their frame factors, $\Gamma_F^n\Gamma_E^m$ of Lemma 25.3(b) for the matrix envelopes in (a)–(b) and the E-type families in (d), and the weighted-frame bound for the diagonal mask envelopes in (b)–(c).*
   (a) *The singleton-block matrix envelopes of the pairs FF and FE.*
   (b) *The higher envelopes of the classes with $n \geq 1$, $n+m \in \{4,6\}$, other than $(6,0)$: the singleton-block matrix envelopes F1E3 and F1E5 (one fold factor), F2E2 (two-fold), F3E1 (three-fold) and F4E0 (four-fold); and the diagonal mask envelopes F2E4, F3E3, F4E2, F5E1, which use the extra averages of Lemma 25.5(f) on the invariant background.*
   (c) *The diagonal mask envelope F6E0.*
   (d) *The E-type envelopes: the diagonal envelopes of the crossed and of the aligned matching pairs, with the averages of Lemma 25.5(g), and the families E4, E6, one diagonal envelope for each group of words.*

The $o(1)$ consists of energy errors and large-product remainders.

The certified envelopes are the sixteen rows of Table 33, from the bundle `research/further_2026_10_25/envelopes/exact_background_pairs.json`; its rows named F0E4 and F0E6 (candidate envelopes for the classes $(0,4),(0,6)$) are not used — those classes instead use the envelopes named E4 and E6 in the table. The proofs of the domination are the exact rational tests of Lemma 25.3 (Certificate 25.11), not the numerical optimizations that proposed the matrices. For instance the FE envelope has $\kappa_H < 0.916505$, $\kappa_b < 0.775465$ and $M < 25.050376$: the extra reflection of the mixed pair lowers its effective cap by about 0.14.

Proof. Step 1: remainders. Let $\omega \in \mathcal{C}$. By (N1)–(N3), $\widehat{\Delta}_\omega = \bigotimes_{\mathcal{B}} \widehat{\Delta}_{\omega(\mathcal{B}),\mathcal{B}} = L_\omega + \Upsilon_\omega$, where Υ_ω is self-adjoint, the tensor having an even number of skew-adjoint factors, with $\|\Upsilon_\omega\| \leq \upsilon_\omega := \prod_{\mathcal{B}}(\ell_{\mathcal{B}} + \rho_{\mathcal{B}}) - \prod_{\mathcal{B}} \ell_{\mathcal{B}}$ (25.5); and the selected measure of ω is $\zeta_\omega = \tilde{\pi}_\omega^{\mathrm{arm}}[\widehat{\Delta}_\omega]_\omega$, $\tilde{\pi}_\omega^{\mathrm{arm}}$ the product of the raw-arm coefficients, so that $\pi_\omega^{\mathrm{arm}} = \tilde{\pi}_\omega^{\mathrm{arm}} C_X$ with $C_X = \prod_{\mathcal{B}\in X} q_{\mathcal{B}}^{2a}$. Since $\upsilon_\omega \operatorname{Diag}_\omega \pm[\Upsilon_\omega]_\omega \succeq_\iota 0$, (P5) and (E1) give $|Y_{[\Upsilon_\omega]_\omega \otimes \lambda_{-X}}| \leq \upsilon_\omega Y^{\mathsf{N}}[\operatorname{Diag}_\omega] \leq \upsilon_\omega(B + o(1))C_X Y$, and $\tilde{\pi}_\omega^{\mathrm{arm}} \upsilon_\omega C_X = \pi_\omega^{\mathrm{tot}} - \pi_\omega^{\mathrm{lead}}$.

Step 2: domination. By Lemma 25.5(e), $\mathcal{U}_\omega^* L_\omega \mathcal{U}_\omega$ is $\pm$ a tensor product of stars of the chart shape, with the numbers c_p of (N1), (N3), and by Lemma 25.3(a) it lies between $\pm \operatorname{Lift}(D_{(\omega)})$. Hence $\pm L_\omega \preceq D_\omega$, $[D_\omega]_\omega \pm [L_\omega]_\omega \succeq_\iota 0$, and by (P5), $|Y_{[L_\omega]_\omega \otimes \lambda_{-X}}| \leq Y^{\mathsf{N}}[[D_\omega]_\omega]$.

Step 3: the outside bracket, coupling by coupling. The chart unitaries conjugate the coupling basis into itself ($\mathcal{U}$ by φ, $\mathcal{P}_{(0\,1)}$ by the transposition of the names), so $[D_\omega]_\omega = \sum_C c(C)[C]_\omega$ with placed couplings, which are admissible probability measures (Lemma 25.2), and $\sum_C |c(C)| = M(D_{(\omega)})$. By Lemma 25.7, $Y^{\mathsf{N}}[[D_\omega]_\omega] \leq Y^{\mathsf{N}_0}[[D_\omega]_\omega] + (M(D_{(\omega)})R_{\mathrm{err}} - M_\omega e^{tU^{\mathrm{inv}}} t\delta_{\mathrm{s}} + o(1))C_X Y$, with M_ω the orbit mass of ω (zero if no h applies). With Steps 1 and 2,

$$|I_\omega| \leq \tilde{\pi}_\omega^{\mathrm{arm}} \Big(Y^{\mathsf{N}_0}[[D_\omega]_\omega] + \big(M(D_{(\omega)})R_{\mathrm{err}} - M_\omega e^{tU^{\mathrm{inv}}} t\delta_{\mathrm{s}} + o(1)\big)C_X Y \Big) + \big(B(\pi_\omega^{\mathrm{tot}} - \pi_\omega^{\mathrm{lead}}) + o(1)\pi_\omega\big)Y.$$

The coupling mass is that of the envelope before any averaging.

Step 4: averaging. At every atom outside X, $\mathsf{N}_{0,\mathcal{B}}$ is R_5-invariant and $\succeq_\iota 0$ (Lemma 25.6(iv)). By (N4), $\tilde{\pi}_{\bar{\omega}}^{\mathrm{arm}} = \tilde{\pi}_\omega^{\mathrm{arm}}$, and by Lemma 25.5(e) the operators $K_\omega = D_\omega$ satisfy the hypothesis of Lemma 25.5(d), with $\frac{1}{2}(K_\omega + g_\omega K_\omega g_\omega^*) = \bar{D}_\omega$, because $g_5 \operatorname{Lift}(D) g_5^* = \operatorname{Lift}(g_{5*}D)$ and $D_{(\bar{\omega})} = D_{(\omega)}$. Summing Step 3 over $\mathcal{C}$ and applying Lemma 25.5(d), the main terms become $\sum_\omega \tilde{\pi}_\omega^{\mathrm{arm}} Y^{\mathsf{N}_0}[[\bar{D}_\omega]_\omega]$. The operator $\bar{D}_\omega$ is a lift, positive, annihilating every vector with a constant factor, and of norm at most its cap.

(iii) and (i). Write $\mathsf{N}_{0,\mathcal{B}} = \mathsf{N}_{\mathcal{B}}^{\mathrm{inv}} + G_{\mathcal{B}}$ at the middle blocks outside X and expand: N_0 outside X is $\sum_Z O_Z$, where $O_Z = \bigotimes_{\mathcal{B}\in Z} G_{\mathcal{B}} \otimes \bigotimes_{\mathcal{B}\notin Z\cup X} \mathsf{N}_{\mathcal{B}}^{\mathrm{inv}}$ and Z runs over the sets of middle blocks outside

X; write $Y^{O_Z}[\kappa] = Y_{\kappa\otimes O_Z}$. Every O_Z is a product of R_5-invariant measures $\succeq_\iota 0$, and $O_\varnothing = \bigotimes \mathsf{N}_{\mathcal{B}}^{\mathrm{inv}}$ is invariant under $\mathcal{G}_6$. For $Z \neq \varnothing$, or for ω admitting no additional average, $\bar{D}_\omega \preceq \|\bar{D}_\omega\| I$ gives, by the general form of (E2) with the outside measure $O_Z \succeq_\iota 0$, $Y^{O_Z}[[\bar{D}_\omega]_\omega] \le \|\bar{D}_\omega\| Y^{O_Z}[\mathrm{Diag}_\omega]$. For $Z = \varnothing$ and ω with all fold records equal, or without fold records, Lemma 25.5(f), (g) replace $\bar{D}_\omega$ by its additional average, whose norm is the cap $b(\omega)$ used in κ_b. Expanding $\mathsf{N}_{\mathcal{B}}^{\mathrm{inv}} = \mu_{\mathcal{B}} + (E_{\mathrm{add}} + H)_{\mathcal{B}}$ in every O_Z, each term of $Y^{O_Z}[\mathrm{Diag}_\omega]$ is the energy of an admissible positive measure with μ outside, at most $(1+o(1))C_X$ times the product of the tag prices times Y ((E1)), as in the proof of Lemma 10.12. Summing over Z and averaging over the class, the coefficient of a monomial in the invariant tags alone is the class average κ_b, and that of every other monomial is κ_H: the main terms are at most $|\mathcal{C}|\,\tilde{\pi}^{\mathrm{arm}} C_X(1+o(1))Y$ times $\kappa_H \prod_{\mathcal{B}}(1+v_{\mathcal{B}}+w_{\mathcal{B}}) - (\kappa_H - \kappa_b)\prod_{\mathcal{B}}(1+v_{\mathcal{B}})$, and Lemma 25.6(v) bounds this by (25.11). The monotonicity: the derivative in κ_H is $e^{U_0} - e^{tU^{\mathrm{inv}}} \ge 0$, in κ_b it is $e^{tU^{\mathrm{inv}}}$, and the other variables enter with nonnegative coefficients. Adding the coupling terms of Step 3, averaged over the class, and the remainders of Step 1 gives (25.9); all selections of $\mathcal{C}$ have the same auxiliary monomials, so the class sum may be written with the class averages. The energy errors and the large-product remainders of the finitely many applications of (E1) and Lemma 25.7 are $o(1)$ times the monomials $\pi_\omega^{\mathrm{arm}}$, $\pi_\omega^{\mathrm{lead}}$, $\pi_\omega^{\mathrm{tot}}$, which are at most $2^{|X|}\pi_\omega$: the raw-arm and leading price of a record are at most its full skew price, and its remainder price at most $r_E(X)b^{-1}$ or $a_F(b)b^{-1/2}$ times its raw-arm price. This is (i).

(ii) is the case $\kappa_b = \kappa_H$, $M^{\mathrm{inv}} = 0$ of (i), which needs neither the additional averages nor Lemma 25.7: by (E2) with $\mathsf{N}_0 \succeq_\iota 0$ outside and (E1), $Y^{\mathsf{N}_0}[[\bar{D}_\omega]_\omega] \le \|\bar{D}_\omega\|(e^{U_0}+o(1))C_X Y$.

(iv) The couplings are those of Lemma 25.2, placed and tensorized, and their orbit averages: positive, normalized, with uniform one-slot marginals and invariant under the translations at each prime; they are laws of field labels only, so they do not depend on the indices (O1); the only operator bounds applied to them are the brackets at ι between actual half spaces of Steps 1 and 2, with nothing conditioned (O2); all energies carry F_A at every slot, so (O3) and (O4) are void. Each bound is built from finitely many applications of (E1) and of Lemma 10.8, with nonnegative coefficients, and from the exact relations (P5), (E2), Lemma 25.5 and the orbit decomposition; the number of applications is bounded in terms of $n+m \le 6$ and the size of the coupling bases. If every structured-energy bound is inflated as in Definition 9.19 by a factor $1+z$, the same proofs give the bounds with e^{U_0}, R_{err}, B, the coupon and the background polynomial inflated, continuously in z; this is robustness (Lemma 11.4(i)).

Finally, (v) follows from Certificate 25.11(b), together with the tests of Lemma 25.3(c), (d) and the averages of Lemma 25.5(d)–(g). □

25.6. **Variance savings retained inside dominating matrices.** The lift of a positive envelope annihilates every vector having a constant factor at some block: it is built on the mean-zero feature spaces, which carry the stars because every record difference annihilates the constants. So its energy is a variance in the sense of Lemma 10.16, and a lower bound for the energy of the product of the actual half marginals lowers the class cost. The credit must be taken with a fraction at most the norm bound of the averaged envelope on the branch where it is taken; keeping a larger variance subtraction with a smaller envelope would be invalid, and so would adding the credit of a quartic envelope to the full-norm credit of the same class.

Lemma 25.9 (Variance credit inside a positive envelope)**.** *In the setting of Theorem 25.8, let $\mathcal{C}' \subseteq \mathcal{C}$ be a set of selections, and suppose that*

$$\frac{1}{|\mathcal{C}'|}\sum_{\omega\in\mathcal{C}'} Y[J_\omega] \ge (\bar{l} - o(1))\, C_X Y$$

(a selected-only restoration). For $\omega \in \mathcal{C}'$ let c_ω be the cap that the proof of Theorem 25.8(i) uses for ω on the outside monomial of μ alone, that is, the bound for the norm of the averaged

envelope of ω after the averages of Lemma 25.5(d)–(g) that apply to it; let $0 \le c \le \min_{\mathcal{C}'} c_\omega$ and $0 \le g \le \max(\bar{l}, 0)$. Then (CP5) holds for $\mathcal{C}$ with the class bounds $\phi_\omega - cg\,\pi_\omega^{\mathrm{arm}}$ for $\omega \in \mathcal{C}'$ and ϕ_ω for $\omega \in \mathcal{C} \setminus \mathcal{C}'$, with ϕ_ω of (25.9)*. In the construction, X consists of distinct middle blocks of the pieces of a multiset m of pieces (the letter of Section 24 for a multiset of pieces, not the matching-record count m in the class cost $C_{n,m}$; in (a)–(c) the classes are named by their degrees), the gains are those of* (24.12) *at the piece caps C_m of Lemma 20.7 (Corollary 24.18), and the credited classes are:*

- *(a)* Aligned fold pairs. *In the class* $(2,0)$ *with the envelope FF, $\mathcal{C}' = \{(F_1,F_1),(F_2,F_2)\}$, $c = \kappa_{FF} := \kappa_H(\mathrm{FF}) = \kappa_b(\mathrm{FF})$ and $g = g_F(m)$. At one fold copy the coefficient of the outside background law N_0 in the fold gain $g_F(m)$ is zero, and no term pooled over later pieces is added. The crossed fold pairs get no credit.*
- *(b)* Aligned matching pairs. *In the class of the aligned pairs $\{(E_0,E_0),(E_1,E_1)\}$ with the envelope EE aligned, $\mathcal{C}' = \mathcal{C}$, $c = \kappa_b^{\mathrm{al}} := \kappa_b(EE\ \text{aligned})$ and $g = g_E^{\mathrm{sel}}(m)$, the selected-only aligned gain, without the E_{add} background and the pool term; the residual coefficient of the μ branch is $\kappa_b^{\mathrm{al}}(1-g) \ge 0$. The crossed pairs get no credit.*
- *(c)* Quartic classes. *In the classes* $(4,0)$ *and* $(0,4)$*, which keep their envelopes F4E0 and E4, $\mathcal{C}'$ is the set of aligned words (all fold records equal, respectively all matching records equal; weight 2^{1-4}), c is the cap of the envelope on these words, $\kappa_{4,F} := \kappa_H(\mathrm{F4E0})$ and $\kappa_{4,E}$ the invariant cap of the aligned group of E4, and $g = g_T^{\mathrm{sel}}(m)$, $T \in \{F,E\}$. A quartic class whose envelope is discarded would instead keep the full-norm credit, through $-\prod_{\mathcal{B}}\gamma_{\mathcal{B}}\bigotimes_{\mathcal{B}}(I-\mathbb{J}_{\mathcal{B}}) \preceq \bigotimes_{\mathcal{B}}\Delta_{\mathcal{B}} \preceq \prod_{\mathcal{B}}\gamma_{\mathcal{B}}\bigotimes_{\mathcal{B}}(I-\mathbb{J}_{\mathcal{B}})$, where $\mathbb{J}_{\mathcal{B}}$ is the projection onto constants in the actual half space at $\mathcal{B}$, for the centred differences $\Delta_{\mathcal{B}}$ of norms $\gamma_{\mathcal{B}}$ (Lemma 10.16); the two alternatives are never added. In the construction neither quartic envelope is discarded.*
- *(d) No other envelope class receives a variance credit; in particular no class of degree six does.*

Gates. *Every credit satisfies $0 \le cg < C_{n,m}$, where $C_{n,m} = \kappa_H e^{U_0} - (\kappa_H - \kappa_b)e^{tU^{\mathrm{inv}}} + MR_{\mathrm{err}} - M^{\mathrm{inv}}e^{tU^{\mathrm{inv}}}t\delta_{\mathrm{s}}$ is the coefficient of $\pi_\omega^{\mathrm{arm}}$ in* (25.9)*. So the credited class bounds have nonnegative coefficients, as Lemma 10.14(v) requires.*

Proof. By the proof of Theorem 25.8(i) it suffices to lower the bound of $\sum_{\mathcal{C}}\tilde{\pi}_\omega^{\mathrm{arm}}Y^{\mathsf{N}_0}[[\bar{D}_\omega]_\omega]$ by $\sum_{\mathcal{C}'}\tilde{\pi}_\omega^{\mathrm{arm}}(cg - o(1))C_XY$. In the expansion of the outside law into the monomials O_Z and into μ and the invariant tags, consider, for $\omega \in \mathcal{C}'$, the term in which every outside factor is μ: its energy is $Y[[\bar{D}_\omega^\mu]_\omega]$, where $\bar{D}_\omega^\mu$ is the averaged envelope of ω on this branch, of norm at most $c_\omega \ge c$ (Lemma 25.5(d)–(g), μ being invariant under $\mathcal{G}_6$). This operator annihilates every vector with a constant factor, so $\bar{D}_\omega^\mu = P\bar{D}_\omega^\mu P$ for the projection $P = \bigotimes_{\mathcal{B}}(I-\mathbb{J}_{\mathcal{B}})$, and $\bar{D}_\omega^\mu \preceq c_\omega P \preceq c_\omega(I-\mathbb{J})$, since P and the projection $\mathbb{J}$ onto the constants of $L^2(m_\omega)$ have orthogonal ranges. For $0 \le c \le c_\omega$, $c_\omega(I-\mathbb{J}) \preceq c_\omega I - c\mathbb{J}$. Placing, $[\bar{D}_\omega^\mu]_\omega \preceq_\iota c_\omega\,\mathrm{Diag}_\omega - cJ_\omega$, and (E2) gives $Y[[\bar{D}_\omega^\mu]_\omega] \le c_\omega Y[\mathrm{Diag}_\omega] - cY[J_\omega]$; the first term is the contribution of this branch to the bound of Theorem 25.8(i). Sum over $\mathcal{C}'$ with the common coefficient $\tilde{\pi}^{\mathrm{arm}}$. If $\bar{l} > 0$ the hypothesis gives $\sum_{\mathcal{C}'}Y[J_\omega] \ge |\mathcal{C}'|(\bar{l} - o(1))C_XY$; if $\bar{l} \le 0$ use $Y[J_\omega] \ge 0$ ((E2), the cut operator of J_ω being a projection). In both cases $c\sum_{\mathcal{C}'}Y[J_\omega] \ge |\mathcal{C}'|(cg - o(1))C_XY$, and $\pi_\omega^{\mathrm{arm}} = \tilde{\pi}^{\mathrm{arm}}C_X$. The bound is robust, uniformly in the classes (Lemma 11.4(i)): it adds to the robust bound of Theorem 25.8(i) the exact relation (E2) and subtracts c times the lower bound of the hypothesis, which is robust in the applications below (Remark 24.21).

(a) The aligned assignments (F_j,F_j) on X satisfy Lemma 24.12(a) with $C = C_m$ and $\bar{l} = b_F(m)$, and $g_F(m) = [b_F(m) - R_*]_+ \le \max(\bar{l},0)$ (Corollary 24.13(i), Corollary 24.18(ii)). The FF envelope has $\kappa_b = \kappa_H$, so its cap on every branch is κ_{FF}, the largest admissible fraction.

(b) The aligned matching class consists of its own partners, and its two selections have the same coefficient. The two aligned assignments together satisfy Lemma 24.12(a) with $\bar{l} = b_E(m)$, and $g_E^{\mathrm{sel}}(m) = [b_E(m) - R_*]_+$ (Corollary 24.13(ii)). On the μ branch the full average of Lemma 25.5(g)

applies, so $c_\omega = \kappa_b^{\mathrm{al}}$ for both selections; the background and pool terms of Proposition 24.17 belong to other outside monomials and are not used here, so this credit does not reuse the invariant background discount.

(c) The aligned words of the class satisfy Lemma 24.12(a) with the selected-only restorations (Corollary 24.18(iii)); on the μ branch their caps are $\kappa_{4,F}$ (the F4E0 envelope has $\kappa_b = \kappa_H$) and the invariant cap of the aligned group of E4 (Lemma 25.5(g)). For the alternative, a tensor of centred differences of norms $\gamma_{\mathcal{B}}$ satisfies the displayed order, its factors being self-adjoint or skew-adjoint with an even number of skew ones, and Lemma 10.16(iii) credits it at the full prices; it replaces, and is never added to, the envelope bound of the class.

(d) The crossed classes are not credited: the lower bounds of Lemma 24.11 concern only the two aligned matching words, all E_0 and all E_1, taken together, so no lower bound for the product energies of a crossed class is available, and these classes are priced by the background bound of Theorem 25.8(iii) alone. Degree six carries no variance credit in the closing (Corollary 24.18).

Gates. By Lemma 25.6(v), $U_0 \geq tU^{\mathrm{inv}}$, and by Lemma 25.7(iv), $MR_{\mathrm{err}} \geq M^{\mathrm{inv}} e^{tU^{\mathrm{inv}}} t\delta_{\mathrm{s}}$; so

$$C_{n,m} = \kappa_b e^{tU^{\mathrm{inv}}} + \kappa_H (e^{U_0} - e^{tU^{\mathrm{inv}}}) + (MR_{\mathrm{err}} - M^{\mathrm{inv}} e^{tU^{\mathrm{inv}}} t\delta_{\mathrm{s}}) \geq \kappa_b e^{tU^{\mathrm{inv}}}.$$

The gains are in $[0,1)$ (Certificate 24.19) and $e^{tU^{\mathrm{inv}}} > 1$. In (a), (b) and for F4E0, $c = \kappa_b$, so $cg < \kappa_b < C_{n,m}$. For E4, with the certified values $\kappa_b(\mathrm{E4}) > 0.4817590234$ and $\kappa_{4,E} < 0.6667$ (Table 33) and $e^{tU^{\mathrm{inv}}} > 2.33$ (since $tU^{\mathrm{inv}} > 0.84688$, at the opening of this section), $\kappa_b(\mathrm{E4}) e^{tU^{\mathrm{inv}}} > 1.122 > \kappa_{4,E} > cg$. Also, the credited selections of a class form the fraction p ($\frac{1}{2}$, 1, $\frac{1}{8}$, $\frac{1}{8}$) of its selections and $pc \leq \kappa_b$, the class average of caps that are at least c on the credited words, so the credit averaged over the class, pcg, is below $\kappa_b \leq C_{n,m}$ as well. These gates are proved here and are not asserted by the certificate; the caps c are fixed certified numbers, and under the inflation of Definition 9.19 the coefficient $C_{n,m}$, a bound of the second kind of Lemma 11.4(i), only grows while the gain, a robust lower bound, only shrinks, so the gates need no margin. □

25.7. **Sixteen classes and their savings.** This corollary is the final form of (CP5) used in the closing. A class (n,m), also written F_nE_m, consists of the selections with n skew fold changes and m skew matching changes; there are sixteen such classes. For each it gives the class bound, says where a variance saving from the preceding subsection is credited, and defines the saving of the class.

Corollary 25.10 (The class bounds of the construction)**.** *Let U_0, R_{err}, B, U^{inv}, δ_{s} and the auxiliary totals be the certified quantities of Definition 30.1 at the exponent a. For each class below, (CP5) holds with the class bound* (25.9) *of the stated envelope, credited by Lemma 25.9 where stated; before any such credit, the class has the base cost*

$$C_{n,m} = \kappa_H e^{U_0} - (\kappa_H - \kappa_b) e^{tU^{\mathrm{inv}}} + M_{n,m} R_{\mathrm{err}} - M^{\mathrm{inv}}_{n,m} e^{tU^{\mathrm{inv}}} t\delta_{\mathrm{s}}$$

per raw-arm monomial, plus the remainder polynomial, in the form of Lemma 10.14(v), with the numbers κ_H, κ_b, $M_{n,m}$, $M^{\mathrm{inv}}_{n,m}$ of its envelope. For a credited selection, Lemma 25.9 subtracts in addition $cg\pi_\omega^{\mathrm{arm}}$ from its class bound.

- *(a) $n \geq 1$, $n+m \in \{4,6\}$, the class $\Omega(X_F, X_E)$: $(1,3)$ F1E3, $(2,2)$ F2E2, $(3,1)$ F3E1, $(4,0)$ F4E0 (credited by Lemma 25.9(c)), $(1,5)$ F1E5, $(2,4)$ F2E4, $(3,3)$ F3E3, $(4,2)$ F4E2, $(5,1)$ F5E1, $(6,0)$ F6E0.*
- *(b) $(2,0)$: FF; the aligned pairs are credited (Lemma 25.9(a)).*
- *(c) $(1,1)$: FE, on a union of two partner pairs.*
- *(d) $(0,2)$: the aligned pairs with EE aligned (credited, Lemma 25.9(b)) and the crossed pairs with EE crossed, each of the two classes being assigned one half of the coefficient of the matching-pair monomial, so that together they count it once.*
- *(e) $(0,4)$, $(0,6)$: E4 (credited on its aligned words) and E6.*

The gates are: $C_{n,m} \geq \kappa_b e^{tU^{\mathrm{inv}}} > 0$*; every coupon is at most* $M_{n,m}R_{\mathrm{err}}$*;* $cg < C_{n,m}$ *for every credit. The* saving *of the class* (n, m)*,*

$$D_{n,m} = \frac{BA_F^n A_E^m - C_{n,m}(A_F^{\mathrm{arm}})^n (A_E^{\mathrm{arm}})^m - B\mathcal{R}_{n,m}}{n!\, m!},$$

with the full skew totals A_F*,* A_E *(for each of the two pair classes in (d), with the divisor* 4 *in place of* $0!\, 2! = 2$ *and the cost of that pair class), is positive for all sixteen classes. The rows F0E4, F0E6 of the bundle are not used.*

Proof. The classes are disjoint and closed under the pairing, and each selection has at most six blocks. Every class bound is that of Theorem 25.8(i), with the numbers of the stated envelope (Theorem 25.8(v)), or that of Lemma 25.9. In (c) the class is the union of the two partner pairs $\{F_1E_s, F_2E_s\}$, $s \in \{0, 1\}$; the extra reflection of Lemma 25.5(f) applies to all four words, and the bounds of the two pairs add. In (d) the matching records E_0, E_1 are separate kinds of Lemma 10.14(v), and the aligned and the crossed pairs each have the total $A_E^2/4$, the two matching records having equal totals. The literals of the closing are exact rationals or outward roundings in the safe directions (κ_H, κ_b, M, U_0, R_{err}, B up; M^{inv}, δ_{s} down; U^{inv} exact), and the class bound is monotone in them as Theorem 25.8(iii) states. The gates are those of Lemma 25.7(iv) and Lemma 25.9. The energy errors of the class bounds are $o(1)$ times auxiliary monomials, which are at most 2^6 times the full monomial π_ω (proof of Theorem 25.8(i)), as (CP5) requires. The savings are computed by the closing (Certificate 25.11(d)); no class is discarded. □

In the closing these classes are the degree classes of Lemma 10.14(v), with the auxiliary prices of (N1)–(N3). The complete polynomial subtracts $D_{n,m}$ for each of the sixteen classes (Definition 30.1), in total the uniform discount $0.4568954196753\ldots$; the largest savings are those of $(1, 1)$, $0.2073597013\ldots$, and of $(2, 0)$, $0.1230353939\ldots$. The credits (a) and (b) of Lemma 25.9 are the skew pair terms of D_{var}, and those of (c) belong to its degree-four term (Corollary 24.18, (24.13)); together the discount is $D = D_{\mathrm{var}} + \sum D_{n,m}$, above 0.466853979991. Six matching factors form the class $(0, 6)$, whose saving is $4.9330756374\ldots \cdot 10^{-6}$.

Certificate 25.11 (The sixteen dominating matrices and their class bounds)**.** The sixteen dominating matrices of Corollary 25.10 exist with the stated exact rational weights, caps and savings: every positive-semidefiniteness and norm inequality in their definitions holds at Q_3; at later piece starts the same matrices are used with caps from the displayed formulas. The items below identify the data files and checkers for each part; paths are relative to `research/`.

(a) *The feature functions.* `continuation_2026_10_24/local/actual_features.py` writes `continuation_2026_10_24/local/actual_features.json` (status `PASS_EXACT_ACTUAL_FEATURE_IDENTITIES_AND_GATES`). For seven thresholds, among them $Q_3 = 19951586$, it evaluates ϱ_{02}, ϱ_{01}, ϱ_{03}, ϱ_{34} of Lemma 25.1 in outward rational interval arithmetic, asserts the gates $\varrho_{0|12} < \frac{1}{2}$ and $2(\tilde{\varrho}_{03}/q)\varrho_{0|12} < 1$, certifies Γ_F by the Perron gate (25.1) at a rational point, and records ρ_{pair}, η_{pair} and Γ_E; at $p = 19, 23, 31, 43$ it checks the exact pair laws and the conditional expectation formula of the proof in exact arithmetic, as spot checks accompanying the proof. `continuation_2026_10_24/envelopes/weighted_frames.py` builds the matrices G_F^{fr}, G_E^{fr} of Lemma 25.3(b) used by the diagonal caps.

(b) *Witnesses.* The sixteen retained envelopes and the two unused rows are stored with exact rational weights, caps and masses (and, for the matrix and E-type rows, the frame factor $\Gamma_F^n \Gamma_E^m$; the diagonal caps include the weighted frame bound) in `further_2026_10_25/envelopes/exact_background_pairs.json`; `further_2026_10_25/envelopes/check_exact_background_pairs.json` records the exact phase and norm gates of the

pair envelopes FF and FE (status `PASS`). `further_2026_10_25/verify.py` (lines 177–181) runs `advance_2026_10_24/envelopes/verify_selected_envelopes.py` on this bundle at Q_3, without an optimizer: it recertifies every row at Q_3, replays the pair envelopes (`check_singleton_psd_pairs.py`), every envelope with $n \geq 1$ and $n+m \in \{4,6\}$ (`check_signed_higher.py`, which dispatches to the one-, two-, three- and four-fold checkers and checks the diagonal and aligned-F rows with their extra averages), the E-type envelopes with their physical group actions (`check_e_type_envelopes.py`) and the orbit masses (`invariant_orbit_mass.py`, `check_invariant_orbit_mass.py`); its result `checks.envelopes` in `further_2026_10_25/verification/selected.json` is `PASS_FRESH_SELECTED_ENVELOPE_PORTFOLIO`. The orbit masses are those of `invariant_error.orbit_masses` in `push_2026_10_25/artifacts/author.json`, field `result` (per-class status `PASS_INVARIANT_REFLECTION_ORBIT_MASS`).

(c) *Invariant price.* `further_2026_10_25/envelopes/chosen_background.py`, function `price`, applied to the native source `push_2026_10_25/global/refined_L570_source.json` at $a_0 = \frac{4442183}{5000000000}$ (the selected author adapter records status `PASS_EXACT_CHOSEN_INVARIANT_ADDITION_PRICE`), computes the exact rational $U^{\mathrm{inv}} = 0.846884413213589801283311 3\ldots$ of (25.8) and asserts $u_i^{\mathrm{inv}} + u_i^G \leq u_i$ for every piece and $\sum_i u_i \leq U_0'$; the generic total is $\sum_i u_i^G = 2.3780326\ldots \cdot 10^{-5}$. `further_2026_10_25/global/scalar_adapters.py`, lines 48–61, substitutes $e^{tU^{\mathrm{inv}}}$ for the background factor of the closing evaluator, and lines 63–91 recompute U^{inv} independently.

(d) *Costs and savings.* `advance_2026_10_24/global/closing_beta.py`, with this substitution, evaluates in outward interval arithmetic the remainder polynomials (lines 184–190), the leading costs $C_{n,m}$ with the coupon and the gate coupon $\leq MR_{\mathrm{err}}$ (lines 191–209, the gate at line 205), the savings of the general classes, of the two matching pairs and of E4, E6 (lines 222–241, retaining a class only when the lower end of its saving is positive, and asserting it for the class $(2,0)$ and the matching pairs) and the variance allocation (lines 244–258). The symmetric price s' and the saving δ_{s} are computed by `advance_2026_10_24/envelopes/invariant_error_mass.py`, function `symmetric_price` (lines 158–174), which asserts $0 \leq s' \leq d'$ (`invariant_error.price`, status `PASS_FOLDED_SYMMETRIC_PRICE`). The results are `discounts` and `invariant_error.leading_coupons` of `push_2026_10_25/artifacts/author.json`, field `result`, with `discarded_discounts` empty; the independent assembly overlaps them (`checks.independent`).

Consequences and what to read next. The arguments in this section use the following statements. Part II: cut matrices and their order (Definition 5.2, Lemma 5.4, Remark 5.3), the slot array (Lemma 5.13), the square shift (Lemma 4.11), independent pairs and conditional decorrelation (Corollaries 4.14 and 4.15), the sector lemma (Lemma 6.10), monotone majorants (Lemma 4.18), admissibility (Definition 8.9) and the reflection positivity (R2) of the reference factors. Part III: (CP2)–(CP5), the charging rules and robustness (Definitions 9.11, 9.13, 9.14, 9.16 and 9.19, Remark 9.15, and Lemma 11.4), the energy toolkit (E1), (E2), (P5), the tagged expansion and the bracket energies (Lemmas 10.8 and 10.12), the degree classes of Lemma 10.14(v), the chosen tag prices of Remark 10.15, the repricing (Lemma 9.3) and the variance form (Lemma 10.16). Part IV: Lemma 12.10, the weighted frames (Lemma 13.8), the real forms of the phase gates (Remark 14.7), the chamber geometry of the records (Lemma 17.2, Table 8) and the five-type completion (Lemma 17.5, Definition 17.4). Part V: the middle reference, frame, first stage and bracket with one fold copy (Lemma 22.1, Definition 22.4, Lemma 22.6, Lemma 22.7, Lemma 22.9, Lemma 20.11, Lemmas 21.2 and 21.17); the prefix, upper and tail references (Definition 19.1, Definition 27.5, Lemma 27.4, Definition 29.1); the reference fold marginal, the actual features and the signed law (Lemma 23.2, Lemma 23.4, Definition 23.9, (23.4), Proposition 23.13); the reflected transpose (Lemma 26.3(c)); the matching star and the prices of the changes (Lemma 23.18,

Lemma 23.17, Lemma 23.24); the piece caps (Lemma 20.7); and the variance gains (Definition 24.1, Lemma 24.12, Corollary 24.13, Lemma 24.11, Proposition 24.17, Corollary 24.18, Remark 24.21). Hypotheses verified in this section (Table 5). (CP5) for the sixteen envelope classes, with the class bounds of Theorem 25.8(i): the costs $C_{n,m}$ with the invariant split and the orbit coupon, and the remainder polynomials, in the form of Lemma 10.14(v) (Theorem 25.8, Lemma 25.3, Lemma 25.5, Lemma 25.6, Lemma 25.7, Corollary 25.10); (CP5) for the credited classes, with the fractions κ_{FF}, κ_b^{al}, $\kappa_{4,F}$ and $\kappa_{4,E}$ and their gates (Lemma 25.9); and, for these laws and bounds, the admissibility (CP8)(a) and the robustness (CP8)(b) (Theorem 25.8(iv), the proof of Lemma 25.9).

26. Dominators for ν and the mean constant $m > 1$

What this section proves and how to read it. This section proves (CP6) and (CP7): it constructs positive laws $M_{\mathrm{L},\mathcal{B}}, M_{\mathrm{R},\mathcal{B}}$ dominating $\nu_{\mathcal{B}}$ at ι (called *matching envelopes* in (CP6); they are unrelated to the matching records E_0, E_1—in prose we call them *dominators for* ν), with total price T entering $J = Be^T(R_I/2)^6$ through e^T, and it proves the mean energies exceed mY with $m > 1$ (a lower bound *added* to the energy, hence $(1-m)/64 < 0$ in Θ, unlike the subtracted savings D). The *side functions* below collect the diagonal weights; only their final values are used, and the long list of intermediate constants in Definition 26.2 that no estimate uses can be skipped—the theorem statements cite only the assembled T and m. Reading-order note: T includes upper-range prices proved in Section 28; the per-block dominator construction here is independent, the total T is assembled in Section 30. Inputs: Section 22. Outputs: (CP6),(CP7) with baseline $T < 0.60760$, $m > 1.58994$.

- For every atom $\mathcal{B}$ it constructs two positive measures $M_{\mathrm{L},\mathcal{B}}$ and $M_{\mathrm{R},\mathcal{B}}$, the *matching envelopes*. The comparison law sits in a positive semidefinite 2×2 block $\left(\begin{smallmatrix} M_{\mathrm{L},\mathcal{B}} & \nu_{\mathcal{B}} \\ \nu_{\mathcal{B}}^* & M_{\mathrm{R},\mathcal{B}} \end{smallmatrix}\right) \succeq_\iota 0$ at the reserved cut, and $M_{s,\mathcal{B}} - \mathsf{N}_{\mathcal{B}}$ is a positive reserved-diagonal measure. For each $s \in \{\mathrm{L},\mathrm{R}\}$, the sum over atoms $\sum_{\mathcal{B}} \mathrm{price}(M_{s,\mathcal{B}} - \mathsf{N}_{\mathcal{B}})$ is at most the matching exponent T of the matching term $J = Be^T(R_I/2)^6$ (Proposition 26.5).
- The proof of Theorem 9.21 uses the envelopes twice. Through the index-support lemma (Lemma 10.1) it compares the signed index form J_ν with Y_ν, up to the excess of the random-inclusion energies V_s of the envelopes over their mean energies e_s (Equation (26.2)); through Lemma 10.17 it bounds V_s by $2^6 J$. The mean energies are bounded below in Section 26.5; this is the mean credit m of Θ (Proposition 26.7, (26.3)).
- The closing (Certificate 30.3) evaluates T and m.

Every record change of the comparison law except the two two-layer fold records F_1, F_2 of a middle block already has such a bracket from its norm envelopes (Definition 9.14). For the matching records the envelopes use chosen diagonals, which dominate the norm envelopes and are paid by the same prices.

For F_1, F_2 the norm envelopes would enter T with the full fold norm price, which is too large for $\Theta < 1$, and the bracket is built directly (Section 26.1). The reference record and its signed comparison record are bounded jointly by a positive 2×2 Gram at ι, whose diagonal shifts are multiples of the record's own one-layer companion. The dominator N already contains that companion once, with coefficient v_{D} (Lemma 22.7). The joint Gram needs a diagonal of coefficient about $2.39\, w_{\mathrm{D}}$ on each side (the signed coefficient $\bar{H}_X d_{\mathfrak{t}}$), and only the excess $k_s = w_{\mathrm{D}} d_s - v_{\mathrm{D}}$ is priced. The copy of the companion already present in N is not counted again in that diagonal.

Each chosen side is an auxiliary side, given by a certified formula, plus a nonnegative rounded increment, and is at least the signed Gram coefficient $\bar{H}_X d_{\mathfrak{t}}$. So only nonnegative increments are added to the auxiliary price (Section 26.2). The envelopes and T are assembled in Section 26.3, the common mean of the selected law in Section 26.4, and the mean credit in Section 26.5.

26.1. **A joint 2×2 block for reference and comparison records.** The matching bracket uses the reference and the signed comparison record jointly. Its leading part is a small matrix in the four actual features $V_0, A_1/r, A_2/r, A_3/r$; it does not require a second calculation of their individual spectra. We first isolate that matrix calculation, then state the arithmetic side function used to price the resulting diagonals.

Lemma 26.1 (The leading Gram)**.** *Let $\mathcal{H}$ be a Hilbert space and let $V, A_1, A_2, A_3\colon H_0\to\mathcal{H}$ satisfy $\|(V, A_1/r, A_2/r, A_3/r)\|^2\le\Gamma$, where $r>0$. For $\xi\in\mathbb{R}^3$ put $A_\xi=\sum_i\xi_iA_i$ and $\mathbf{e}=(1,0,0)$. For $u,v\in\mathbb{R}^3$ define*

$$L_{\mathrm{old}}=-(A_{\mathbf{e}}V^*+VA_{\mathbf{e}}^*),\qquad L_{u,v}=A_uV^*+VA_v^*,\qquad G(u,v)=\begin{pmatrix}L_{\mathrm{old}} & L_{u,v}\\ L_{u,v}^* & L_{\mathrm{old}}\end{pmatrix}.$$

For $\tau_{\mathrm{side}}>0$ let

$$\Upsilon_{\tau_{\mathrm{side}}}(u,v)=\begin{pmatrix}-\mathbf{e}/\tau_{\mathrm{side}} & u\\ v & -\tau_{\mathrm{side}}\mathbf{e}\end{pmatrix},\qquad s(\tau_{\mathrm{side}};u,v)=\|\Upsilon_{\tau_{\mathrm{side}}}(u,v)\|.$$

This is a real 6×2 matrix, and

$$s(\tau_{\mathrm{side}};u,v)^2=\lambda_{\max}\begin{pmatrix}\tau_{\mathrm{side}}^{-2}+\|v\|^2 & -u_1/\tau_{\mathrm{side}}-\tau_{\mathrm{side}}v_1\\ -u_1/\tau_{\mathrm{side}}-\tau_{\mathrm{side}}v_1 & \tau_{\mathrm{side}}^2+\|u\|^2\end{pmatrix},$$
$$G(u,v)\succeq -r\Gamma s(\tau_{\mathrm{side}};u,v)\,\mathrm{diag}(\tau_{\mathrm{side}}I,\tau_{\mathrm{side}}^{-1}I).$$

For the signed arm vector $\mathbf{a}_{\mathsf{t}}=(\mathsf{t},1,1-\mathsf{t})$, put $u_{\mathsf{t}}=\mathbf{a}_{\mathsf{t}}-\mathbf{e}$ and $v_{\mathsf{t}}=-\mathbf{a}_{\mathsf{t}}-\mathbf{e}$. Then

$$L_{u_{\mathsf{t}},v_{\mathsf{t}}}=L_{\mathrm{old}}+A_{\mathbf{a}_{\mathsf{t}}}V^*-VA_{\mathbf{a}_{\mathsf{t}}}^*,\qquad G(u_{\mathsf{t}},v_{\mathsf{t}})\succeq -r\Gamma s_{\mathsf{t}}I,$$

where

$$s_{\mathsf{t}}=\bigl(4-2\mathsf{t}+2\mathsf{t}^2+2\sqrt{1+\mathsf{t}^2}\bigr)^{1/2},\qquad s_0=\sqrt{6}.$$

The statement holds directly in the actual half-marginal space; the adjoints and the frame bound are always those of $\mathcal{H}$.

Proof. Let $\Sigma=(V,A_1/r,A_2/r,A_3/r)$ and $\mathbf{S}=\mathrm{diag}(\Sigma,\Sigma)$. In each copy of H_0^4 separate the centre coordinate V from the three arm coordinates. After conjugating $G(u,v)$ by $\Delta_{\tau_{\mathrm{side}}}^{-1/2}$, where $\Delta_{\tau_{\mathrm{side}}}=\mathrm{diag}(\tau_{\mathrm{side}}I,\tau_{\mathrm{side}}^{-1}I)$, its coefficient matrix in these eight features, ordered with both centres first, is

$$\mathbf{M}=\begin{pmatrix}0 & \Upsilon_{\tau_{\mathrm{side}}}^{\mathsf{T}}\\ \Upsilon_{\tau_{\mathrm{side}}} & 0\end{pmatrix}.$$

Indeed the old diagonal stars acquire the coefficients $-\mathbf{e}/\tau_{\mathrm{side}}$ and $-\tau_{\mathrm{side}}\mathbf{e}$, while the off-diagonal arms retain u,v. Thus the conjugated operator is $r\mathbf{S}(\mathbf{M}\otimes I)\mathbf{S}^*$, with the same coordinate permutation in $\mathbf{S}$. The least eigenvalue of $\mathbf{M}$ is $-\|\Upsilon_{\tau_{\mathrm{side}}}\|$, so it is at least $-rs(\tau_{\mathrm{side}};u,v)\mathbf{S}\mathbf{S}^*\succeq -r\Gamma s(\tau_{\mathrm{side}};u,v)I$. Conjugating back proves the order. The displayed 2×2 matrix is $\Upsilon_{\tau_{\mathrm{side}}}^{\mathsf{T}}\Upsilon_{\tau_{\mathrm{side}}}$.

At the signed point, linearity of A_ξ gives the identity for $L_{u_{\mathsf{t}},v_{\mathsf{t}}}$, and at $\tau_{\mathrm{side}}=1$ the last matrix is

$$\begin{pmatrix}4+2\mathsf{t}^2 & 2\\ 2 & 4-4\mathsf{t}+2\mathsf{t}^2\end{pmatrix}.$$

Its largest eigenvalue is $4-2\mathsf{t}+2\mathsf{t}^2+2\sqrt{1+\mathsf{t}^2}$. □

In the application $\mathcal{H}=L^2(m_0)$ and the features are those of Lemma 23.4, with $V=V_0$ and $\Gamma=\Gamma_A$. The reference contributes L_{old} on the diagonal; the signed law contributes its old leading part plus its skew leading difference off the diagonal (Lemma 23.2(f), Proposition 23.13(c)). This is why the Gram gives a joint bracket and not only a norm bound for the change. At $\mathsf{t}=127/500$,

$s_{\mathrm{t}} = 2.38422727\ldots$, compared with $\sqrt{6} = 2.4494897\ldots$ at the mirror point. Adding the two remainders gives the actual coefficient

$$d_{\mathrm{t}} = s_{\mathrm{t}}\Gamma_A + (e_o + e_{\mathrm{sg}})r,$$

which is below 2.39033202 at Q_3. The block argument is Proposition 26.4 below.
Auxiliary sides and the certified price. The Gram in the actual space $L^2(m_0)$ is supplied by the signed law. The numerical price is stored as an auxiliary side plus a nonnegative increment to that Gram coefficient. We retain the complete arithmetic definition of the auxiliary side because it fixes this split and its outward roundings. Its parameters are inherited from a probability mixture of four sign patterns, but no law of the construction uses that mixture; no operator estimate for it is needed. In particular the coefficients below are definitions of numbers, not assertions about another comparison law.

Definition 26.2 (The side function)**.** For a prime $p \equiv 3 \pmod 4$ with $q = 1/p \le 2^{-20}$, put $r = p^{-1/2}$, $Z = (1-q)(1-2q)$, $h_0 = 1 - 14q$ and $e_0 = 2q/((1-3q)(1-4q))$, as in Section 23.1. Fix

$$(w_{++}, w_{+-}, w_{-+}, w_{--}) = (5803723, 2409463, 1786814, 0)/10^7,$$

$$(\bar b, \bar d, \bar f) = (b_0, d_0, f_0) = (.6426372, .5181074, .1607446), \qquad \tau_{\mathrm{side}} = 31/32,$$

where $\bar b = \sum_{(\beta,\delta)\in\{\pm1\}^2} \beta w_{\beta\delta}$, $\bar d = \sum_{(\beta,\delta)\in\{\pm1\}^2} \delta w_{\beta\delta}$ and $\bar f = \sum_{(\beta,\delta)\in\{\pm1\}^2} \beta\delta w_{\beta\delta}$. Set $u = (-\bar b, 1, \bar b)$, $v = (-1, -\bar d, -\bar d)$, $s(\tau_{\mathrm{side}}) = s(\tau_{\mathrm{side}}; u, v)$ as in Lemma 26.1, and

$$\gamma = \Big(1 + \frac{q(1+2r)}{1-2q}\Big)^{1/2}, \qquad c_{\mathrm{lead}} = \gamma(\|u\|_2 + \|v\|_2),$$

$$F_{\mathrm{frame}} = \frac{2 + |\bar b + 1|}{1-q} + \frac{|1-\bar d|q}{(1-q)(1-2q)} + \frac{2|\bar b - \bar d|q^2}{(1-q)(1-2q)} + \frac{|2 - \bar b + \bar d|r}{1-q},$$

$$\zeta_{++} = \frac{6q^2}{(1-4q)(1-5q)}, \qquad \zeta_{+-} = \frac{4q}{(1-4q)(1-5q)},$$

$$\zeta_{-+} = \frac{4q}{(1-4q)(1-5q)(1-7q)}, \qquad \zeta_{--} = \frac{8q}{(1-4q)(1-5q)(1-7q)},$$

$$C_{\mathrm{old}} = \frac{19}{Zh_0}, \qquad C_{\mathrm{ns}} = 2|\bar b| + 1 + 2|\bar d| + 4|\bar f|, \qquad n_* = \frac{1 + 8r + (18/Z + 144)q}{h_0},$$

$$C_z = \frac{n_*}{q}\sum w_{\beta\delta}\zeta_{\beta\delta}, \qquad C_{\mathrm{new}} = \frac{2C_{\mathrm{ns}}/Z + 144 + |\bar b - \bar d|}{h_0} + C_z,$$

$$k_{\mathrm{raw}} = \frac{rc_{\mathrm{lead}}}{h_0} + qC_{\mathrm{new}}, \qquad \eta_1 = \frac{(4w_{++} + 6w_{+-} + 4w_{-+} + 8w_{--})q}{h_0},$$

$$\rho_1 = \frac{k_{\mathrm{raw}} + \eta_1^2}{1-\eta_1}, \qquad \eta_\star = \frac{e_0 + \eta_1}{1-\eta_1}, \qquad \phi_2 = \frac{\eta_\star}{1-\rho_1}, \qquad K_\infty = \frac{4}{h_0^{1/2}(1-\eta_1)},$$

$$\phi_\infty = \eta_\star + K_\infty\phi_2, \qquad C_{\mathrm{rep}} = \frac{2\phi_\infty}{q}\Big(k_{\mathrm{raw}} + \frac{14q}{h_0}\Big), \qquad C' = C_{\mathrm{new}} + C_{\mathrm{rep}}.$$

With these numbers define

$$c_{\mathrm{com}} = \frac{rF_{\mathrm{frame}}}{h_0} + r\Big(C_{\mathrm{old}} + C' + \frac{2e_0^2}{q}\Big),$$

$$d_{\mathrm{L}}^m = \frac{\gamma s(\tau_{\mathrm{side}})\tau_{\mathrm{side}}/h_0 + c_{\mathrm{com}}}{1-e_0}, \qquad d_{\mathrm{R}}^m = \frac{\gamma s(\tau_{\mathrm{side}})\tau_{\mathrm{side}}^{-1}/h_0 + c_{\mathrm{com}}}{1-e_0},$$

$$k_0 = \frac{2\gamma/h_0 + r(C_{\mathrm{old}} + 14/h_0 + 2e_0/q + e_0^2/q)}{1-e_0},$$

$$k_1 = \frac{c_{\mathrm{lead}}/h_0 + r(C' + 14/h_0 + 2e_0/q + e_0^2/q)}{1-e_0},$$

$$\bar{k} = \max(k_0, k_1).$$

For a middle piece starting at X, evaluate at $q_X = 1/X$ and $r_X = 1/\lfloor\sqrt{X}\rfloor \ge X^{-1/2}$, and put $n_X = \lfloor L_{\mathrm{loc}}/\log_- X\rfloor$, $L_{\mathrm{loc}} = 585$ (Lemma 20.3). The *auxiliary sides* are

$$d_s^{\mathrm{aux}}(X) = e^{n_X e_0(X)} \max(d_s^m(X),\, 2\bar{k}(X)^2 r_X), \qquad s \in \{\mathrm{L},\mathrm{R}\}.$$

Every coefficient is evaluated as follows: the displayed expression, exactly as written as a function of the independent variables (q,r) in the nondecreasing form of Lemma 4.18 (each remainder qc written as $r(rc)$), is evaluated at (q_X, r_X); no coefficient is used in a different (q,r) form. In particular a remainder evaluated at q_X is never divided by the larger value r_X. The 33 values and their gates, including $\eta_1 < 1$, $\rho_1 < 1$ and $2\phi_\infty < 1$, are those of Table 31 and Certificate D.10.

At this auxiliary point $s(31/32) = 2.0778741\ldots$, $s\tau_{\mathrm{side}} = 2.0129405\ldots$ and $s/\tau_{\mathrm{side}} = 2.1449023\ldots$. The chosen physical sides below are at least the signed Gram coefficient, regardless of these smaller auxiliary values.

For the block argument, recall that ν_F is the uniform law of a half (x_0, x_1, x_2) with distinct labels and $x_0 \to x_2$. The reference record has density $K_0 = \Psi_0/z_0$ relative to $\nu_F \otimes \nu_F$, where $\Psi_0 = I_\times(1+\mathsf{A})(1+\mathsf{B})$ and both half marginals are $m_0 = g_0\nu_F$. Here $h_0 \le z_0 \le 1$ and $|g_0 - 1| \le e_0$ (Lemma 23.2(a)); all Markov adjoints in the next lemma are in $L^2(m_0)$.

Lemma 26.3 (Unequal sides)**.** *Let $\mathcal{B}$ be a middle block of a piece starting at X, with least prime b, $r_b = b^{-1/2}$ and at most $n_X = \lfloor L_{\mathrm{loc}}/\log_- X\rfloor$ primes. Let $\Pi'_{F_1,\mathcal{B}}$ be a record placed by F_1 whose kernel is a finite real combination*

$$K'_{\mathcal{B}} = \sum_{c\in I} \alpha_c \bigotimes_{p\in\mathcal{B}} K'_{c,p}, \qquad \sum_{c\in I}\alpha_c = 1,$$

of products of probability couplings $K'_{c,p}$ with both half marginals m_0 (a single product is the case $|I| = 1$); write $T'_{c,p}$ for their Markov operators on $L^2(m_0)$ and $T'_p = \sum_c \alpha_c T'_{c,p}$.

(a) Enlargement. *If the block cut matrix of (b) is $\succeq_\iota 0$ with sides $d_{\mathrm{L}}, d_{\mathrm{R}}$, it is so with any sides $d'_{\mathrm{L}} \ge d_{\mathrm{L}}$, $d'_{\mathrm{R}} \ge d_{\mathrm{R}}$.*

(b) Block extension. *Suppose that, for nonnegative functions $c_{\mathrm{L}}, c_{\mathrm{R}}$ of (q,r) that are nondecreasing in (q,r) and a number $c_{\ge 2}(X) \ge 0$,*

- *at every prime $p \in \mathcal{B}$, on $L^2(m_0)^2$, $\left(\begin{smallmatrix} T_{\mathrm{old}} + r_p c_{\mathrm{L}} I & T'_p \\ T'^{*}_p & T_{\mathrm{old}} + r_p c_{\mathrm{R}} I \end{smallmatrix}\right) \succeq 0$, with $c_s = c_s(1/p, p^{-1/2})$ and T_{old} the Markov operator of K_0;*
- *on every exact support $S \subseteq \mathcal{B}$ of at least two primes, the restrictions to the sector $\mathcal{H}_S$ of the old block operator $\bigotimes_p T_{\mathrm{old},p}$ and of the new one $\sum_c \alpha_c \bigotimes_p T'_{c,p}$ have norms summing to at most $c_{\ge 2}(X) r_X r_b$.*

Then, as cut matrices at ι of measures on the labels of $\mathcal{B}$,

$$\begin{pmatrix} \Pi_{F_1,\mathcal{B}} + r_b d_{\mathrm{L}} D^1_{\mathcal{B}} & \Pi'_{F_1,\mathcal{B}} \\ \Pi'^{*}_{F_1,\mathcal{B}} & \Pi_{F_1,\mathcal{B}} + r_b d_{\mathrm{R}} D^1_{\mathcal{B}} \end{pmatrix} \succeq_\iota 0 \qquad \text{for} \quad d_s \ge H_{\mathcal{B}} \max(c_s(X), c_{\ge 2}(X) r_X),$$

where $H_{\mathcal{B}}$ is any bound for the density of $m_{0,\mathcal{B}} = \bigotimes_p m_{0,p}$ relative to $\nu_F^{\otimes\mathcal{B}}$, for instance $e^{n_X e_0(X)}$, or $\bar{H}_X = (1 - n_X e_0(X))^{-1}$ if $n_X e_0(X) < 1$; and $D^1_{\mathcal{B}}$ is the diagonal copy at ι of the product law $\nu_F^{\otimes\mathcal{B}}$ placed by F_1, the normalized one-layer companion of F_1 in $\widetilde{\mu}_{\mathcal{B}}$.

(c) Let $\varsigma(x) = (-x_2, -x_1, -x_0)$ and $\Phi(x, y) = (\varsigma y, \varsigma x)$. Then ς preserves ν_F and g_0, Φ preserves K_0, and Φ maps the edges $x_0 \to y_2$, $y_0 \to x_2$, $x_1 \to y_0$, $x_0 \to y_1$ to $x_0 \to y_2$, $y_0 \to x_2$, $x_2 \to y_1$, $x_1 \to y_2$. If the comparison record of F_2 is the reflected transpose of that of F_1, $K'_{2,c,p}(x, y) = K'_{c,p}(\varsigma y, \varsigma x)$ for every component and every prime, then (b) holds for F_2 with d_{L} and d_{R} interchanged.

Consequently, if the side gates $k_s := w_{\mathrm{D}} d_s - v_{\mathrm{D}} \geq 0$, $s \in \{\mathrm{L}, \mathrm{R}\}$, *hold in the piece, the two fold records of a block cost $k_{\mathrm{L}} + k_{\mathrm{R}}$ per unit r_b^2 on each side.*

Proof. (a) The difference of the two cut matrices is the diagonal $\mathrm{diag}((d'_{\mathrm{L}} - d_{\mathrm{L}}) r_b D^1_{\mathcal{B}}, (d'_{\mathrm{R}} - d_{\mathrm{R}}) r_b D^1_{\mathcal{B}})$, a positive reserved-diagonal measure, which is $\succeq_\iota 0$ ((E3)).

(b) The old block operator is $\bigotimes_p T_{\mathrm{old},p}$ and the new one is the combination $T'_{\mathcal{B}} = \sum_c \alpha_c \bigotimes_p T'_{c,p}$; every prime factor fixes the constants of m_0 on both sides, because all these couplings have the actual marginal m_0 on both sides. By Lemma 6.10(a) and Remark 6.12 both preserve the orthogonal sectors $\mathcal{H}_S$ on which the factors in a set S of primes are mean-zero and the others constant; the old one is $\bigotimes_{p\in S}(T_{\mathrm{old},p} - \mathbb{J})$ there, and the new one is $T'_p - \mathbb{J}$ at the one coordinate p if $S = \{p\}$. So the block Gram is the orthogonal sum of its restrictions to the sectors (Lemma 6.10(a)), and it suffices to treat each sector with the shifts $\sigma_s = r_b \max(c_s(X), c_{\geq 2}(X) r_X)$. On $\mathcal{H}_\emptyset$ it is $\left(\begin{smallmatrix} 1+\sigma_{\mathrm{L}} & 1 \\ 1 & 1+\sigma_{\mathrm{R}} \end{smallmatrix}\right) \succeq 0$, as $\sum_c \alpha_c = 1$. On $S = \{p\}$ it is the restriction of the prime Gram at p, which holds with the shifts $r_p c_s(1/p, p^{-1/2}) \leq r_b c_s(X) \leq \sigma_s$, by the monotonicity of c_s, as $p \geq b \geq X$ and $r_X \geq X^{-1/2}$; larger shifts preserve it. For $|S| \geq 2$ let O and N be the restrictions of the old and new operators; O is self-adjoint, $\|O\| + \|N\| \leq c_{\geq 2}(X) r_X r_b \leq \sigma_s$, and $\left(\begin{smallmatrix} O+\sigma_{\mathrm{L}} & N \\ N^* & O+\sigma_{\mathrm{R}} \end{smallmatrix}\right) \succeq \left(\begin{smallmatrix} (\sigma_{\mathrm{L}} - \|O\|) I & N \\ N^* & (\sigma_{\mathrm{R}} - \|O\|) I \end{smallmatrix}\right) \succeq 0$, since $\sigma_s - \|O\| \geq \|N\|$. This gives the block Gram on $L^2(m_{0,\mathcal{B}})^2$ with the shifts σ_s times I.

In measure form, the cut forms at ι of $\Pi_{F_1,\mathcal{B}}$ and $\Pi'_{F_1,\mathcal{B}}$ at real half-functions are the forms of $T_{\mathrm{old},\mathcal{B}}$ and $T'_{\mathcal{B}}$ on $L^2(m_{0,\mathcal{B}})$, and that of the identity is the cut form of the diagonal law $\mathrm{Diag}_\iota(m_{0,\mathcal{B}}^{F_1})$ of the actual half marginal placed by F_1. Its density with respect to $D^1_{\mathcal{B}}$ is $\prod_{p\in\mathcal{B}} g_0(x_p) \leq H_{\mathcal{B}}$; for the two examples, $g_0 \leq 1 + e_0(X)$ at every prime of the block (Lemma 23.2(a)), and the block has at most n_X primes, so the density is at most $(1 + e_0(X))^{n_X} \leq e^{n_X e_0(X)} \leq \bar{H}_X$. Replacing $\mathrm{Diag}_\iota(m_{0,\mathcal{B}}^{F_1})$ by $H_{\mathcal{B}} D^1_{\mathcal{B}}$ therefore adds a positive reserved-diagonal measure, which is $\succeq_\iota 0$ ((E3)).

(c) ς is an involution preserving ν_F (Section 23.1) and g_0 (Lemma 23.2(a)). Writing $(x', y') = \Phi(x, y)$, a relation $s \to t$ between labels becomes $-t \to -s$; this gives the edge table, and in particular $x_0 \to y_2$ and $y_0 \to x_2$ are fixed, so $K_0 \circ \Phi = K_0$. The added edges of F_1 go to those of F_2 (Certificate 23.10). With $(\mathcal{U}\varphi)(x) = \varphi(\varsigma x)$, unitary on $L^2(\nu_F)$ and on $L^2(m_0)$, the hypothesis reads $T'_{2,c,p} = \mathcal{U} T'^*_{c,p} \mathcal{U}^*$ for every component, and $\mathcal{U} T_{\mathrm{old}} \mathcal{U}^* = T_{\mathrm{old}}$. Exchange the two copies in the prime Gram of (b) and conjugate by $\mathrm{diag}(\mathcal{U}, \mathcal{U})$: this is the prime Gram of the F_2 record with c_{L} and c_{R} interchanged, the shifts being multiples of the identity. On the block, $\bigotimes_p \mathcal{U}$ maps every sector $\mathcal{H}_S$ onto itself, so the bound on the sectors of at least two primes is carried over unchanged, and the density of $m_{0,\mathcal{B}}$ is ς-invariant. So the proof of (b) applies to F_2 with the sides interchanged.

The last assertion is the bookkeeping of the two records: by (b) and (c), the layer-zero side carries $r_b d_{\mathrm{L}}$ for F_1 and $r_b d_{\mathrm{R}}$ for F_2, the layer-one side the reverse, each record has weight $w_{\mathrm{D}} r_b$, and the bracket already contains the one-layer companion of each record with the weight $v_{\mathrm{D}} r_b^2$ (one fold copy, Lemma 22.7). □

The global substitution. For a middle block $\mathcal{B}$ let $F^{(2)}_{\mathcal{B}} = w_{\mathrm{D}} r (\Pi_{F_1,\mathcal{B}} + \Pi_{F_2,\mathcal{B}})/\widetilde{\mu}_{\mathcal{B}}(1)$ be the reference's two two-layer fold records, with their one-layer companions $F_{\mathrm{fold},\mathcal{B}} = v_{\mathrm{D}} r^2 (D^1_{\mathcal{B}} + D^2_{\mathcal{B}})/\widetilde{\mu}_{\mathcal{B}}(1)$ (Definition 22.4(b), where $D^j_{\mathcal{B}}$ is written $\mathrm{Diag}_\iota(\Pi^{(1)}_{F_j})$), where here $\Pi_{F_j,\mathcal{B}}$ and $D^j_{\mathcal{B}}$ denote

probability laws; the common normalizing factor is displayed explicitly, unlike the component notation of Definition 22.4(b). The law $D^2_{\mathcal{B}}$ is defined as $D^1_{\mathcal{B}}$ with F_2 in place of F_1; and put

$$\lambda_{\mathrm{res},\mathcal{B}} = \lambda_{\mathcal{B}} - F^{(2)}_{\mathcal{B}}, \qquad \mathsf{N}_{\mathrm{res},\mathcal{B}} = \mathsf{N}_{\mathcal{B}} - F^{(2)}_{\mathcal{B}} - F_{\mathrm{fold},\mathcal{B}} = \lambda_{\mathrm{res},\mathcal{B}} + H_{\mathcal{B}}.$$

Outside the middle put $\lambda_{\mathrm{res},\mathcal{B}} = \lambda_{\mathcal{B}}$ and $\mathsf{N}_{\mathrm{res},\mathcal{B}} = \mathsf{N}_{\mathcal{B}}$.

26.2. **The sides of the signed law.** The joint 2×2 block of the preceding subsection needs, on each of its two diagonal sides, a coefficient—the *side*: the multiple of the one-layer companion record placed on that diagonal. For the signed comparison records of F_1 and F_2 the block is positive already with the two equal sides $\bar{H}_X d_{\mathsf{t}}(X)$, where d_{t} is the explicit coefficient computed from the joint Gram matrix in Proposition 23.13(d). The construction instead starts from cheaper auxiliary sides given by a certified formula (Definition 26.2) and increases each by a nonnegative increment to at least $\bar{H}_X d_{\mathsf{t}}(X)$; only the auxiliary price and the increment are then charged. The proposition makes this precise, piece by piece.

Proposition 26.4 (The sides of the signed law)**.** *For a middle piece starting at X let $n_X = \lfloor L_{\mathrm{loc}}/\log_- X \rfloor$, assume $n_X e_0(X) < 1$, put $\bar{H}_X = (1 - n_X e_0(X))^{-1}$, and let $d_{\mathsf{t}}(X)$ and $c_{\ge 2}(X)$ be the constants of* (23.5) *at (q_X, r_X), with $\mathsf{t} = 127/500$ (the functions of Certificate 23.14 evaluate them at the* 33 *piece starts).*

(i) Let $\mathcal{B}$ be a middle block of the piece. The signed comparison record of F_1 (Definition 23.9(c)) satisfies the hypotheses of Lemma 26.3(b) with $c_{\mathrm{L}} = c_{\mathrm{R}} = d_{\mathsf{t}}$ and the support-two constant $c_{\ge 2}(X)$, and $c_{\ge 2}(X) r_X < d_{\mathsf{t}}(X)$. Hence its block Gram holds with $d_{\mathrm{L}} = d_{\mathrm{R}} = \bar{H}_X d_{\mathsf{t}}(X)$, and so does that of F_2 (Lemma 26.3(c)).

(ii) The increments

$$\delta^+_{s,X} = \mathrm{up}([\bar{H}_X d_{\mathsf{t}}(X) - d^{\mathrm{aux}}_s(X)]_+), \qquad s \in \{\mathrm{L},\mathrm{R}\}, \tag{26.1}$$

where in the display $\bar{H}_X d_{\mathsf{t}}(X)$ is replaced by its certified upper endpoint U_X from Certificate 23.14, and with $\mathrm{up}(x) = 10^{-30}\lceil 10^{30} x \rceil$, define the chosen sides $d_s(X) = d^{\mathrm{aux}}_s(X) + \delta^+_{s,X} \ge \max(\bar{H}_X d_{\mathsf{t}}(X), d^{\mathrm{aux}}_s(X))$*, for which the block Grams of (i) hold (Lemma 26.3(a)). In the piece i, starting at X_i, with the completion weights $w_{\mathrm{D},i}$, $v_{\mathrm{D},i}$ of Lemma 22.1, the* side coefficients *are*

$$k_{s,i} = w_{\mathrm{D},i} d_s(X_i) - v_{\mathrm{D},i} = k^{\mathrm{aux}}_{s,i} + w_{\mathrm{D},i}\delta^+_{s,i}, \qquad k^{\mathrm{aux}}_{s,i} = w_{\mathrm{D},i} d^{\mathrm{aux}}_s(X_i) - v_{\mathrm{D},i},$$

with $\delta^+_{s,i} = \delta^+_{s,X_i}$, and $k^{\mathrm{aux}}_{s,i} \ge 0$ in every piece (Certificate D.10); so the side gates hold. (These side coefficients are not the caps k_{L} of the layer cut L21 and $k_{\mathrm{R}} = k_{\mathrm{E}}$ of Lemma 22.1(ii); in this section $k_{\mathrm{L}}, k_{\mathrm{R}}$ always denote side coefficients.)

(iii) With the certified degree-two charges $\bar{I}_{2,i} = \mathrm{hi}(I_{2,i}) \ge \sum_{\mathcal{B}} r^2_{\mathcal{B}} q^{2a_0}_{\mathcal{B}}$ of the pieces at the anchor a_0 (Lemma 20.4, Definition 30.1),

$$\sum_i (k_{\mathrm{L},i} + k_{\mathrm{R},i})\bar{I}_{2,i} \le T'_K := T^{\mathrm{aux}\prime}_K + T^{\mathrm{inc}\prime}_K,$$

where $T^{\mathrm{aux}\prime}_K = \sum_i \mathrm{up}((k^{\mathrm{aux}}_{\mathrm{L},i} + k^{\mathrm{aux}}_{\mathrm{R},i})\bar{I}_{2,i})$ and $T^{\mathrm{inc}\prime}_K = \sum_i \mathrm{up}(2w_{\mathrm{D},i}\,\mathrm{up}(\tfrac{1}{2}(\delta^+_{\mathrm{L},i} + \delta^+_{\mathrm{R},i}))\bar{I}_{2,i})$. Put $T_K = tT'_K$, with the repricing factor $t = e^{2(a-a_0)L}$, $L = 570$ (Definition 30.1(1)).

Proof. (i) *The prime Gram.* Let $p \in \mathcal{B}$, so $p \ge X \ge Q_3 > 2^{24}$, and let $T'_p = T[K_{\mathsf{t},p}]$ be the operator on $L^2(m_0)$ of the signed record at p (the notation $T[\cdot]$ of Proposition 23.13), the combination $P_{++,p} + \frac{\mathsf{t}}{2}P_{-+,p} - \frac{\mathsf{t}}{2}P_{--,p}$ of its three components (Definition 23.9(c)). These are probability couplings with both half marginals m_0 (Lemma 23.7(d)), so T_{old} and T'_p fix the constants on both sides, and the decomposition of $L^2(m_0)$ into the constants and the m_0-centred functions reduces the prime Gram. On the constants it is $\left(\begin{smallmatrix} 1+dr & 1 \\ 1 & 1+dr \end{smallmatrix}\right) \succeq 0$ for $d \ge 0$. On the

centred functions, $T_{\mathrm{old}} = O_0 + R_o$ with $\|R_o\| \le e_o q$, where $O_0 = -(A_{\mathbf{e}} V_{0,m}^* + V_0 A_{\mathbf{e}}^{*m})$ is the natural leading part, with adjoints in $L^2(m_0)$ (Lemma 23.2(f)); and $T_p' = O_0 + L_{\mathtt{t},p}^m + R_{\mathtt{t},p}^{\mathrm{new}}$ with $L_{\mathtt{t},p}^m = A_{\mathbf{a}_\mathtt{t}} V_{0,m}^* - V_0 A_{\mathbf{a}_\mathtt{t}}^{*m}$, the signed leading difference with the unprojected pair feature A_3 and the adjoints of $L^2(m_0)$, and $\|R_{\mathtt{t},p}^{\mathrm{new}}\| \le e_{\mathrm{sg}} q$ (Proposition 23.13(c)). The leading parts form the operator matrix $G(u_\mathtt{t}, v_\mathtt{t})$ of the synthesis form of Lemma 26.1, with $\mathcal{H} = L^2(m_0)$, $V = V_0$ and the features A_i regarded as maps into $L^2(m_0)$, whose synthesis has squared norm at most the frame Γ_A of Lemma 23.4(d). At $\tau_{\mathrm{side}} = 1$ it gives $G(u_\mathtt{t}, v_\mathtt{t}) \succeq -r\Gamma_A s_\mathtt{t} I$. The remainders form the self-adjoint matrix $\left(\begin{smallmatrix} R_o & R_{\mathtt{t},p}^{\mathrm{new}} \\ R_{\mathtt{t},p}^{\mathrm{new}*} & R_o \end{smallmatrix}\right) \succeq -(e_o + e_{\mathrm{sg}}) q I$. As $q = r^2$, the prime Gram with the shifts dr is positive semidefinite for $d \ge s_\mathtt{t}\Gamma_A + (e_o + e_{\mathrm{sg}})r = d_\mathtt{t}$, all evaluated at $(1/p, p^{-1/2})$. The function $d_\mathtt{t}$ is nondecreasing in (q, r) ((23.5)), so this is the first hypothesis of Lemma 26.3(b) with $c_{\mathrm{L}} = c_{\mathrm{R}} = d_\mathtt{t}$.

Supports of at least two primes. On every exact support $S \subseteq \mathcal{B}$ with $|S| \ge 2$ the restrictions of the old and of the signed block operators have norms with sum at most $c_{\ge 2}/b$, with $c_{\ge 2}$ at $(1/b, b^{-1/2})$, and $c_{\ge 2} r < 2 < s_\mathtt{t} \le d_\mathtt{t}$ (Proposition 23.13(d)); the bound comes from the prime centred norms $(\Gamma_A + e_o r)r$ of the reference and $c_+ r$ of the three components, through Lemma 6.10(a) and Remark 6.12, the record being a combination of three products with absolute coefficient sum $1 + \mathtt{t}$. By monotonicity and $r_b \le r_X$, $c_{\ge 2}/b \le c_{\ge 2}(X) r_X r_b$, which is the second hypothesis. The bound $c_{\ge 2} r < 2$ of the proof of Proposition 23.13(d) holds at every (q, r) with $q \le 2^{-24}$ and $r \le 2^{-12}$, hence at (q_X, r_X), as $X \ge 2^{24}$ gives $r_X = 1/\lfloor \sqrt{X} \rfloor \le 2^{-12}$; so $c_{\ge 2}(X) r_X < 2 < s_\mathtt{t} \le d_\mathtt{t}(X)$, an inequality that Certificate 23.14 also asserts at the 33 piece starts.

The block. Lemma 26.3(b) with $H_{\mathcal{B}} = \bar{H}_X$ gives the block Gram with the sides

$$d_{\mathrm{L}} = d_{\mathrm{R}} = \bar{H}_X \max(d_\mathtt{t}(X), c_{\ge 2}(X) r_X) = \bar{H}_X d_\mathtt{t}(X);$$

here the block has at most n_X primes (Lemma 20.3(ii)), and $n_X e_0(X) < 1$ is a gate asserted in Certificate 23.14 and again by the evaluation of the increments of (ii) (Certificate D.10). The record of F_2 is the reflected transpose of that of F_1, taken componentwise (Definition 23.9(d)), so Lemma 26.3(c) gives its block Gram with the same, equal, sides. The Gram is an exact relation between the operators of the signed record and of the reference, together with the operator bounds above; the total variation $1 + \mathtt{t}$ of the record enters only the variation factor of (CP4) (Definition 9.14, Remark 6.12), and the Gram acquires no total-variation multiplier.

(ii) Since $\mathrm{up}(x) \ge x$ and the certified upper bound is at least $\bar{H}_X d_\mathtt{t}(X)$, $\delta_{s,X}^+ \ge \max(0, \bar{H}_X d_\mathtt{t}(X) - d_s^{\mathrm{aux}}(X))$, so $d_s(X) \ge \max(\bar{H}_X d_\mathtt{t}(X), d_s^{\mathrm{aux}}(X))$, and by (i) and Lemma 26.3(a) the block Gram holds with the sides $d_s(X)$, for F_1 and, interchanged, for F_2. The identity for $k_{s,i}$ is the definition, and $k_{s,i} \ge k_{s,i}^{\mathrm{aux}} \ge 0$ because $w_{\mathrm{D},i}, \delta_{s,i}^+ \ge 0$.

(iii) By (ii), $(k_{\mathrm{L},i} + k_{\mathrm{R},i})\bar{I}_{2,i} = (k_{\mathrm{L},i}^{\mathrm{aux}} + k_{\mathrm{R},i}^{\mathrm{aux}})\bar{I}_{2,i} + 2w_{\mathrm{D},i} \cdot \frac{1}{2}(\delta_{\mathrm{L},i}^+ + \delta_{\mathrm{R},i}^+) \cdot \bar{I}_{2,i}$; bound each term by its rounding upwards ($w_{\mathrm{D},i}, \bar{I}_{2,i} \ge 0$) and sum over the pieces. □

At the first piece start $X = Q_3$: $n_X = 34$, $\bar{H}_X d_\mathtt{t}(X)$ is about 2.39034, the auxiliary sides are about 2.05245 and 2.18441, and the increments about 0.33789 and 0.20593. In every piece $\bar{H}_X d_\mathtt{t}(X)$ exceeds both auxiliary sides, so both increments are positive and both chosen sides are the auxiliary sides plus the rounded increments of (26.1), hence at least $\bar{H}_X d_\mathtt{t}(X)$; the auxiliary sides only split the price T_K' into the two certified sums of (iii): $T_K^{\mathrm{aux}\prime}$ is about 0.0354169, and the repriced increment price $tT_K^{\mathrm{inc}\prime}$ about 0.1273584. The coefficients $k_{s,i}^{\mathrm{aux}}$ are positive in every piece, the smallest, at the last piece, being about 0.0707 (Table 31, Certificate D.10).

26.3. **The dominators $M_{\mathrm{L}}, M_{\mathrm{R}}$ and the total T.** In every middle piece i we use the chosen sides and the side coefficients $k_{s,i}$ of Proposition 26.4. For a middle block $\mathcal{B}$ of the piece:

- the *chosen side diagonals* are
$$K_{\mathrm{L},\mathcal{B}} = r_{\mathcal{B}}^2(k_{\mathrm{L},i}D_{\mathcal{B}}^1 + k_{\mathrm{R},i}D_{\mathcal{B}}^2), \qquad K_{\mathrm{R},\mathcal{B}} = r_{\mathcal{B}}^2(k_{\mathrm{R},i}D_{\mathcal{B}}^1 + k_{\mathrm{L},i}D_{\mathcal{B}}^2),$$
without the normalization $1/\widetilde{\mu}_{\mathcal{B}}(1)$ that the companions in N carry; omitting it adds a nonnegative multiple of the same positive reserved diagonals;
- the *chosen matching diagonal* is $Q_{E,\mathcal{B}} = r_{\mathcal{B}}^2 q_i(D_{E_0,\mathcal{B}} + D_{E_1,\mathcal{B}})$, where $q_i = w_{\mathrm{S},i}(s_{E,i} + a_{E,i})$, and $s_{E,i}$, $a_{E,i}$ are the symmetric and skew coefficients of the matching records recorded in the native source (Certificate D.11): certified upper bounds for the coefficients $s_E(X_i)$, $a_E(X_i)$ of Lemma 23.17(ii) evaluated at $r_X = X_i^{-1/2}$, the least value allowed by (ii), which permits every $r_X \geq X_i^{-1/2}$ since the coefficients are nondecreasing in (q, r) by (i); and $D_{E_s,\mathcal{B}} = \mathrm{Diag}_\iota(m_{E_s,\mathcal{B}})$ is the selected diagonal of E_s (Definition 24.1), the diagonal copy at ι of the product law placed by E_s. By Lemma 23.17(ii) the symmetric and skew parts of the change of each matching record of $\mathcal{B}$, of weight $w_{\mathrm{S}} r_{\mathcal{B}}$ normalized by $\widetilde{\mu}_{\mathcal{B}}(1) \geq 1$, have norm envelopes $\tilde{\pi}^{\epsilon} D_{E_s,\mathcal{B}}$ with $\tilde{\pi}^{\mathrm{s}} + \tilde{\pi}^{\mathrm{a}} \leq r_{\mathcal{B}}^2 q_i$ (Section 24.1); so $Q_{E,\mathcal{B}}$ dominates the norm envelopes of the two matching changes, and replacing them by $Q_{E,\mathcal{B}}$ adds a positive reserved-diagonal measure. (The prices $S_{E,i}$, $A_{E,i}$ are formed with the coefficients of (ii) at $r_X = 1/\lfloor\sqrt{X_i}\rfloor$, which are not smaller.)

Outside the middle, $K_{s,\mathcal{B}} = Q_{E,\mathcal{B}} = 0$.

The matching exponent. With the totals at the exponent a of Definition 30.1 put
$$T_{QE} = S_E + A_E, \qquad T = S_E + A_E + S_{\mathrm{oth}} + A_{\mathrm{oth}} + T_K + S_{\mathrm{u}}(a) + A_{\mathrm{u}}(a) + 10^{-12},$$
with the T_K of Proposition 26.4(iii), the totals S_E, A_E of the matching records and S_{oth}, A_{oth} of the other middle records (Lemma 23.24), and the upper and tail prices $S_{\mathrm{u}}(a)$, $A_{\mathrm{u}}(a)$ of Proposition 28.14(ii).

- The gate $2q_i\bar{I}_{2,i} \leq A_{E,i} + S_{E,i}$, with the matching prices of the piece at a_0 (Definition 30.1(3)), holds in every piece (Certificate D.11), so the chosen matching diagonals cost at most T_{QE}.
- The prices of the prefix records lie inside the allowance 10^{-12} (Lemma 19.3(ii)–(iv)).
- The symmetric and skew prices of the fold records do not enter T; only T_K does.

At the baseline exponent T_K is about 0.1628311 and T_{QE} about 0.4233528 (Certificate 30.3).

Proposition 26.5 (Matching envelopes and the matching exponent)**.**

(i) For every atom $\mathcal{B}$ there are measures $M_{\mathrm{L},\mathcal{B}}$ and $M_{\mathrm{R},\mathcal{B}}$ with $\left(\begin{smallmatrix} M_{\mathrm{L},\mathcal{B}} & \nu_{\mathcal{B}} \\ \nu_{\mathcal{B}}^* & M_{\mathrm{R},\mathcal{B}} \end{smallmatrix}\right) \succeq_\iota 0$, such that $M_{\mathrm{L},\mathcal{B}} - \mathsf{N}_{\mathcal{B}}$ and $M_{\mathrm{R},\mathcal{B}} - \mathsf{N}_{\mathcal{B}}$ are positive reserved-diagonal measures whose normalizations have uniform one-slot marginals and do not depend on the indices. In a middle block
$$M_{s,\mathcal{B}} = \mathsf{N}_{\mathcal{B}} + Q_{\mathcal{B}}^{\circ} + Q_{E,\mathcal{B}} + K_{s,\mathcal{B}}, \qquad s \in \{\mathrm{L}, \mathrm{R}\},$$
where $Q_{\mathcal{B}}^{\circ}$ is the sum of the norm envelopes of Definition 9.14 of every record change other than the two fold records and the two matching records; outside the middle $M_{s,\mathcal{B}} = \mu_{\mathcal{B}} + Q_{\mathcal{B}}$, with $Q_{\mathcal{B}}$ the sum of the norm envelopes of the changes at $\mathcal{B}$.

(ii) For both s, with the T of the paragraph above,
$$\sum_{\mathcal{B}} \mathrm{price}(M_{s,\mathcal{B}} - \mathsf{N}_{\mathcal{B}}) \leq T, \qquad T \in [0.607496889410,\, 0.607496889411] \quad \text{at } a = \frac{5422587}{6103515625}.$$
So (CP6) holds with the matching exponent T of Definition 30.1(5).

(iii) The single-half random-inclusion energies V_{L} of X under $\bigotimes_{\mathcal{B}} M_{\mathrm{L},\mathcal{B}}$ and V_{R} of Y under $\bigotimes_{\mathcal{B}} M_{\mathrm{R},\mathcal{B}}$ (Definition 9.6) are at most $(2^6 J + o(1))Y$, with $J = Be^T(R_I/2)^6$.

Proof. (i) In a middle block, by Definition 22.4(d), $\mathsf{N}_{\mathrm{res},\mathcal{B}} - \lambda_{\mathrm{res},\mathcal{B}} = H_{\mathcal{B}} \succeq_\iota 0$ and
$$\mathsf{N}_{\mathrm{res},\mathcal{B}} + \lambda_{\mathrm{res},\mathcal{B}} = (2\mu_{\mathcal{B}} - 2F_{\mathcal{B}}^{(2)} - 2F_{\mathrm{fold},\mathcal{B}} - 2E_{\mathrm{fold},\mathcal{B}} + H_{\mathcal{B}}) + 2E_{\mathrm{new},\mathcal{B}} + 2G_{\mathcal{B}}.$$

The proof of Lemma 22.7(ii) writes $2\mu_{\mathcal{B}} - F_{\mathrm{fold},\mathcal{B}} - 2E_{\mathrm{fold},\mathcal{B}} + H_{\mathcal{B}}$ as a sum of measures each positive semidefinite at ι, one for every component of $\widetilde{\mu}_{\mathcal{B}}$ (all normalized by $\widetilde{\mu}_{\mathcal{B}}(1)$): for each fold record F_j, $j = 1, 2$, the pair $2w_{\mathrm{D}} r \Pi_{F_j,\mathcal{B}} + v_{\mathrm{D}} r^2 D^j_{\mathcal{B}}$, positive at ι by the protection (22.1) (Lemma 22.1(iv)); for each matching record the record with its part of $H_{\mathcal{B}}$; and every other component together with its own one-layer companion. The two fold pairs add up to exactly $2F^{(2)}_{\mathcal{B}} + F_{\mathrm{fold},\mathcal{B}}$. Removing them leaves a positive semidefinite sum, and E_{new} and G are positive and reserved diagonal. Since $\lambda_{\mathrm{res},\mathcal{B}}$ is symmetric at ι, the symmetric-bracket criterion of Lemma 5.6(d) gives $\left(\begin{smallmatrix} \mathsf{N}_{\mathrm{res}} & \lambda_{\mathrm{res}} \\ \lambda_{\mathrm{res}} & \mathsf{N}_{\mathrm{res}} \end{smallmatrix}\right) \succeq_\iota 0$; no diagonal is used twice.

Add $w_{\mathrm{D}} r/\widetilde{\mu}_{\mathcal{B}}(1)$ times the block Grams of Proposition 26.4(i),(ii) for the signed F_1 record, with the chosen sides $d_{\mathrm{L}}, d_{\mathrm{R}}$, and, with the sides interchanged, for the signed F_2 record. The off-diagonal entry becomes $\lambda_{\mathrm{res}} + w_{\mathrm{D}} r(\Pi'_{F_1} + \Pi'_{F_2})/\widetilde{\mu}_{\mathcal{B}}(1)$, which is $\lambda_{\mathcal{B}}$ with the two fold records replaced by their signed comparison records. On the layer-zero side the diagonal entry becomes

$$\mathsf{N}_{\mathrm{res}} + F^{(2)}_{\mathcal{B}} + \frac{w_{\mathrm{D}} r^2}{\widetilde{\mu}_{\mathcal{B}}(1)}(d_{\mathrm{L}} D^1_{\mathcal{B}} + d_{\mathrm{R}} D^2_{\mathcal{B}}) = \mathsf{N}_{\mathcal{B}} + \frac{r^2}{\widetilde{\mu}_{\mathcal{B}}(1)}(k_{\mathrm{L}} D^1_{\mathcal{B}} + k_{\mathrm{R}} D^2_{\mathcal{B}}),$$

since $\mathsf{N}_{\mathrm{res}} + F^{(2)}_{\mathcal{B}} = \mathsf{N}_{\mathcal{B}} - F_{\mathrm{fold},\mathcal{B}}$ and $F_{\mathrm{fold},\mathcal{B}} = v_{\mathrm{D}} r^2 (D^1_{\mathcal{B}} + D^2_{\mathcal{B}})/\widetilde{\mu}_{\mathcal{B}}(1)$; the difference between $\mathsf{N}_{\mathcal{B}} + K_{\mathrm{L},\mathcal{B}}$ and this diagonal entry is a positive reserved-diagonal measure, because $k_s \geq 0$ and $\widetilde{\mu}_{\mathcal{B}}(1) \geq 1$. On the layer-one side it is at most $\mathsf{N}_{\mathcal{B}} + K_{\mathrm{R},\mathcal{B}}$ in the same way. Enlarging a diagonal entry by a positive reserved-diagonal measure keeps the bracket. Finally add the brackets of Definition 9.14 of every other record change (they are those of Lemma 10.14): for the two matching changes with the diagonal $Q_{E,\mathcal{B}}$, which dominates their norm envelopes (Section 26.3), and for the other changes with their norm envelopes, whose sum is $Q^\circ_{\mathcal{B}}$. Their off-diagonal entries complete $\nu_{\mathcal{B}}$. Outside the middle only this last step occurs, starting from $\left(\begin{smallmatrix} \mu_{\mathcal{B}} & \mu_{\mathcal{B}} \\ \mu_{\mathcal{B}} & \mu_{\mathcal{B}} \end{smallmatrix}\right) \succeq_\iota 0$ ($\mu_{\mathcal{B}}$ is reflection positive at ι, Hypothesis 6.2). The side gates make $k_{\mathrm{L}}, k_{\mathrm{R}} \geq 0$; the added diagonals are positive, reserved diagonal, with uniform one-slot marginals, and independent of the indices. The exchange of Lemma 26.3(c) is used here, so that each envelope carries $k_{\mathrm{L}} + k_{\mathrm{R}}$ for the two fold records together; without it one side would carry $2k_{\mathrm{R}}$.

(ii) The price of $K_{s,\mathcal{B}}$ is its mass times $q^{2a}_{\mathcal{B}}$, at most $(k_{\mathrm{L},i} + k_{\mathrm{R},i}) r^2_{\mathcal{B}} q^{2a}_{\mathcal{B}}$, and $q^{2a}_{\mathcal{B}} \leq t q^{2a_0}_{\mathcal{B}}$ for a middle block (Lemma 9.3; $\log q_{\mathcal{B}} \leq L$). Summed over the blocks of the piece i the chosen side diagonals cost at most $t(k_{\mathrm{L},i} + k_{\mathrm{R},i})\bar{I}_{2,i}$, and over all pieces at most $tT'_K = T_K$ by Proposition 26.4(iii). In the same way $Q_{E,\mathcal{B}}$ has the price $2q_i r^2_{\mathcal{B}} q^{2a}_{\mathcal{B}}$, so the chosen matching diagonals cost at most $t\sum_i 2q_i \bar{I}_{2,i} \leq t\sum_i (A_{E,i} + S_{E,i}) = T_{QE}$ by the gate; this is the budget of the norm envelopes they replace, and no second copy is introduced. The norm envelopes $Q^\circ_{\mathcal{B}}$ of the merges and of the deleted records have the prices of Lemma 23.24, of total at most $S_{\mathrm{oth}} + A_{\mathrm{oth}}$ at a. Outside the middle, $Q_{\mathcal{B}}$ is the norm envelope of the upper and tail repairs and deletions, of total price at most $S_{\mathrm{u}}(a) + A_{\mathrm{u}}(a)$ (Proposition 28.14(ii)), and of the deleted exception of the prefix, charged to the symmetric part only, of total price below 10^{-12} (Lemma 19.3(ii)–(iv) with $m_{\mathrm{cl}} = 8192$). Adding these bounds gives T; its enclosure is that of Certificate 30.3.

(iii) Each $M_{s,\mathcal{B}} - \mathsf{N}_{\mathcal{B}}$ is a nonnegative combination of positive reserved-diagonal laws with uniform one-slot marginals, independent of the index, with $\sum_{\mathcal{B}} \mathrm{price}(M_{s,\mathcal{B}} - \mathsf{N}_{\mathcal{B}}) \leq T$ by (ii). Lemma 10.17 with $Q_s = M_{s,\mathcal{B}} - \mathsf{N}_{\mathcal{B}}$ and $T_Q = T$ gives $V_s \leq (Be^T R^6_I + o(1))Y = (2^6 J + o(1))Y$, the conductor argument being carried out inside each half-energy. No symmetry of the comparison records and no odd-skew cancellation is used. □

26.4. **The common mean for the selected law.** For the final construction the six queries $\mathcal{E}_6$ have pairwise disjoint endpoints (Lemma 5.13(iv)) and the index law, with $\delta_j = 1$ for the interval coordinate, for the 2-adic coordinate modulo 8, and for the prime 3 coordinate, and $\delta_p = \sqrt{p+1}/(p-1)$ at the class-three primes $7 \leq p < Q_3$, satisfies (CP0)(d) (Lemma 29.6); the matching envelopes of Proposition 26.5(i) are brackets as in (CP6). So Corollary 10.4 applies to

the signed index form J_ν of (9.1). With the *mean energies* $e_s = Q_{\otimes M_s}(\bar X, \bar X)$ of the two envelopes ($\bar X$ the half-function with F_A at every slot) and the single-half random-inclusion energies $V_{\mathrm L}, V_{\mathrm R}$ of Proposition 26.5(iii), it gives

$$|J_\nu - Y_\nu| \le \tfrac12\big[(V_{\mathrm L} - e_{\mathrm L}) + (V_{\mathrm R} - e_{\mathrm R})\big], \qquad V_{\mathrm L}, V_{\mathrm R} \le (2^6 J + o(1))Y. \tag{26.2}$$

The empty-support components are kept exactly: on one side Y_ν, on the other the mean energies, which exceed the mean energy Y_ν of the law itself. This saves one of the 64 copies of the reserved cost in Θ (Lemma 11.1) and is why the lower bounds of the next subsection enter Θ.

26.5. **Mean energies exceed mY with $m > 1$.** The number Θ in Theorem 9.21 is nonincreasing in the mean credit m of (CP7): a larger proved m only helps. We bound the mean energies from the reference side. Each envelope contains $\mathsf N$, and $\mathsf N_0$ is μ plus positive reserved-diagonal records; at ι every monomial of the product of $\mathsf N_0$ and the chosen diagonals is therefore positive semidefinite, and explicit lower bounds for a few monomials may be kept while all others are discarded. The monomials kept are those of the polarized fold diagonals, the chosen side diagonals and the chosen matching diagonals, on sets of at most four distinct blocks, and of five or six blocks of a pool of late pieces. Their energies are bounded below by restoration (Lemma 24.8) at the chamber cut where the two-layer fold record is the diagonal copy of its own induced law, and, for the matching diagonals, by the aligned bridge (Lemma 24.11). At the baseline exponent these coupons are necessary for $\Theta < 1$ in this construction: they lower Θ by about $(m-1)/64 \approx 0.0092$, against a margin $1 - \Theta$ of about $2.67 \cdot 10^{-11}$.

Notation. Energies are those of the toolkit (E1)–(E3): $Y[\kappa] = Y_{\kappa\otimes\mu_{-X}}$ for a measure κ on the labels of a set X of atoms, and $Y^{\mathsf N}[\kappa]$, $Y^{\mathsf N_0}[\kappa]$ with $\mathsf N_{-X}$, $\mathsf N_{0,-X}$ outside. Let $\mathcal B$ be a middle block of the piece $[X_i, Y_i)$ (Definition 20.1), with least prime $b_{\mathcal B}$ and $r_{\mathcal B} = b_{\mathcal B}^{-1/2}$. For $j \in \{1,2\}$ the normalized one-layer companion $D^j_{\mathcal B}$ of F_j is a positive normalized law, fixed samplewise by ι, with uniform one-slot marginals, and so are the selected diagonals $D_{E_s,\mathcal B}$ of the matching records; $D_{E_s,\mathcal B} = \mathrm{Diag}_\iota(\Pi^{(1)}_{E_s})$ is the one-layer matching law on the reserved diagonal, whereas $D^j_{\mathcal B}$ is the diagonal copy of $\nu_F^{\otimes\mathcal B}$ placed by F_j, and the selected diagonal $D_{F_j,\mathcal B} = \mathrm{Diag}_\iota(m_{0,\mathcal B}^{F_j})$ is obtained from it by the density of $m_{0,\mathcal B}^{F_j}$ relative to $\nu_F^{\otimes\mathcal B}$. By Definition 22.4 and Lemma 22.7,

$$(\mathsf N_0 - \mu)_{\mathcal B} = \mathsf E_{\mathcal B} + H_{\mathcal B}, \qquad \mathsf E_{\mathcal B} = (E_{\mathrm{add}} + G)_{\mathcal B}, \qquad E_{\mathrm{add},\mathcal B} = \frac{v_{\mathrm D} r_{\mathcal B}^2}{1 + h_{\mathcal B}}(D_{E_0,\mathcal B} + D_{E_1,\mathcal B}),$$

with $\widetilde\mu_{\mathcal B}(1) = 1 + h_{\mathcal B}$, where $h_{\mathcal B}$ is the nonroot mass (24.7); all summands are nonnegative and reserved diagonal. The fold companion $F_{\mathrm{fold},\mathcal B} = v_{\mathrm D} r_{\mathcal B}^2 (1 + h_{\mathcal B})^{-1}(D^1_{\mathcal B} + D^2_{\mathcal B})$ is a summand of $\mathsf N$, not of $\mathsf N_0$; it enters the mean only through the polarization orders $\mathsf E_{\mathcal B} \pm F_{\mathrm{fold},\mathcal B} \succeq_{\vartheta_0} 0$ (Lemma 22.5, Remark 17.12(3)). By Proposition 26.5(i), $M_{s,\mathcal B} - \mathsf N_{\mathcal B} - K_{s,\mathcal B} - Q_{E,\mathcal B} = Q^\circ_{\mathcal B} \succeq_\iota 0$ ((E3)).

For the piece i we use the piece values h_i, M_i, χ_i, ε_i and the density factors $H_i = (1{+}2/(X_i{-}4))^{n_i}$, with $n_i = \lfloor L/\log_- X_i \rfloor$ and $L = 570$, of (24.11), and the caps C_i of Lemma 20.7. For a multiset m of pieces, C_m, η_m and the restored value $\ell_m = \mathcal F(C_m, h_m, M_m, \chi_m)$ are those of the piece data of Section 24.7, and a set of distinct middle blocks has the *pattern* m if m_i of its blocks lie in the piece i. (Multisets of pieces are written m, as in Section 24.7, and "pattern" here means such a multiset, not a chamber pattern ϕ; they occur only as indices and in the sums over $|m|$, and are not to be confused with the mean credit m of (CP7).) The factor H_i bounds the density of the actual block marginal $m_{0,\mathcal B}$ relative to $\nu_F^{\otimes\mathcal B}$ for every block of the piece: at a prime, $\sup g_0 = (p-2)/(p-4) = 1 + 2/(p-4)$, the value of the formula of Lemma 23.2(a) at $e_{01} = e_{12} = 0$, and a block of the piece has at most n_i primes, all at least X_i (Lemma 20.3). Put

$$c_{E,i} = \frac{v_{\mathrm D,i}}{1 + h_i}, \qquad c_{F,i} = \frac{c_{E,i}}{H_i},$$

the coefficients of the matching diagonals of E_{add} and of the polarized fold diagonals, and let $W_0(m) = W(\varnothing, m; \mathbf{1})$ be the distinct-choice weight of Lemma 24.15(a) with unit coefficients: the sum of $\prod_{\mathcal{B}\in X} b_{\mathcal{B}}^{-1}$ over the sets X of distinct middle blocks of pattern m is at least $W_0(m)$. Except through the caps C_i, these quantities do not depend on a.

Lemma 26.6 (Restored companion energies)**.** *Let X be a nonempty set of distinct middle blocks, $\mathcal{B} \in X$ lying in the piece $i(\mathcal{B})$, and $j \in \{1,2\}$.*

(a) For every real ℓ, if $Y[\bigotimes_{\mathcal{B}\in X} \Pi_{F_j,\mathcal{B}}] \ge (\ell - o(1))Y$, then $Y[\bigotimes_{\mathcal{B}\in X} D^j_{\mathcal{B}}] \ge (\ell/\prod_{\mathcal{B}\in X} H_{i(\mathcal{B})} - o(1))Y$. In particular, if X has the pattern m, then $Y[\bigotimes_{\mathcal{B}\in X} D^j_{\mathcal{B}}] \ge (\ell_m/\prod_{i\in m} H_i - o(1))Y$.

(b) If X has the pattern m, then $Y[\bigotimes_{\mathcal{B}\in X} D_{E_0,\mathcal{B}}] + Y[\bigotimes_{\mathcal{B}\in X} D_{E_1,\mathcal{B}}] \ge 2([\ell_m - C_m\eta_m]_+ - o(1))Y$.

Proof. (a) Put $\Pi_X = \bigotimes_{\mathcal{B}\in X} \Pi_{F_j,\mathcal{B}}$.

Contraction at the reserved cut. By Lemma 17.2(d), Π_X is symmetric at ι with actual half marginal $m_X = \bigotimes_{\mathcal{B}\in X} m^{F_j}_{0,\mathcal{B}}$, so the contraction (P1) gives $Y[\Pi_X] \le Y[\mathrm{Diag}_\iota(m_X)]$.

Density. For a block $\mathcal{B}$ of the piece i, the density of $m^{F_j}_{0,\mathcal{B}}$ with respect to the half law $\nu_F^{\otimes\mathcal{B}}$ (placed by F_j) of $D^j_{\mathcal{B}}$ is at most H_i (paragraph above), and densities of products multiply. So the density domination (P2), a domination of reserved-diagonal measures, gives $Y[\mathrm{Diag}_\iota(m_X)] \le \prod_{\mathcal{B}\in X} H_{i(\mathcal{B})}\, Y[\bigotimes_{\mathcal{B}\in X} D^j_{\mathcal{B}}]$. Combining the two steps with the hypothesis proves the first claim. For the second, the blocks of X share the fold chamber ϑ_{2j} of F_j (Lemma 17.2(b)), and the piece data give $Y[\Pi_X] \ge (\ell_m - o(1))Y$ (Section 24.7, Lemma 24.8, with the cap $C_m \ge C_X$ of Lemma 20.7).

(b) By Section 24.7, Lemma 24.11(b) holds for X with $l_0 = \ell_m$, $C = C_m$ and $\varepsilon_X \le \eta_m$; and the actual half marginal at ι of the aligned embedding E^X_s is $\bigotimes_{\mathcal{B}\in X} m_{E_s,\mathcal{B}}$, so its diagonal copy is $\bigotimes_{\mathcal{B}\in X} D_{E_s,\mathcal{B}}$. Hence (b) is Lemma 24.11(c), as $[l_0 - C\varepsilon_X]_+ \ge [\ell_m - C_m\eta_m]_+$. □

Proposition 26.7 (Mean credit)**.** *Assume, in every middle piece, the protection* (22.1)*, the side gates $k^{\mathrm{aux}}_{s,i} \ge 0$ (Proposition 26.4) and the matching gates $2q_i\bar{I}_{2,i} \le A_{E,i} + S_{E,i}$ (Section 26.3). Then, for $s \in \{\mathrm{L},\mathrm{R}\}$, $e_s \ge (m - o(1))Y$ with*

$$m = 1 + m_F + m_Q - R_{\mathrm{err}} e^{T_K + T_{QE}}, \tag{26.3}$$

where the sums run over the multisets m of pieces, the products over the pieces of m with multiplicity,

$$m_F = \sum_{1\le|m|\le4} W_0(m)\,\ell_m\Big[2\prod_{i\in m} c_{F,i} + \prod_{i\in m}\frac{k_{\mathrm{L},i}}{H_i} + \prod_{i\in m}\frac{k_{\mathrm{R},i}}{H_i}\Big] + m_F^{\mathrm{pool}},$$

$$m_Q = \sum_{1\le|m|\le4} 2W_0(m)\,[\ell_m - C_m\eta_m]_+\Big[\prod_{i\in m}(c_{E,i}+q_i) - \prod_{i\in m} c_{E,i}\Big] + m_Q^{\mathrm{pool}},$$

and m_F^{pool}, m_Q^{pool} are the late-pool coupons of degrees five and six of Lemma 24.16(b), on the pieces starting at 2^{31} or later. Hence (CP7) holds with the mean credit m.

Proof. Step 1: from the envelopes to N_0. In every middle block $0 \preceq_\iota \mathsf{N}_{\mathcal{B}} + K_{s,\mathcal{B}} + Q_{E,\mathcal{B}} \preceq_\iota M_{s,\mathcal{B}}$: the bracket measure is positive semidefinite at ι (Lemma 22.7(ii)), the chosen diagonals are positive and reserved diagonal (their coefficients are nonnegative by the gates), and the difference is $Q^\circ_{\mathcal{B}} \succeq_\iota 0$; outside the middle $M_{s,\mathcal{B}} - \mu_{\mathcal{B}} = Q_{\mathcal{B}} \succeq_\iota 0$. By the tensor monotonicity of Lemma 5.6(d), applied to the cut matrices at ι (5.2), $\bigotimes_{\mathcal{B}} M_{s,\mathcal{B}} \succeq_\iota \bigotimes_{\mathcal{B}}(\mathsf{N}_{\mathcal{B}} + K_{s,\mathcal{B}} + Q_{E,\mathcal{B}})$, and evaluation at $\bar{X}$ gives $e_s \ge Y_{\widetilde{M}_s}$ with $\widetilde{M}_s = \bigotimes_{\mathcal{B}}(\mathsf{N}_{\mathcal{B}} + K_{s,\mathcal{B}} + Q_{E,\mathcal{B}})$. Let $M_{0,s}$ be the same product with N_0 in place of N. Expand both over the set $\mathcal{S}$ of blocks that carry $K_{s,\mathcal{B}}$ or $Q_{E,\mathcal{B}}$ and over, at each block of $\mathcal{S}$, the choice of $K_{s,\mathcal{B}}$ or $Q_{E,\mathcal{B}}$. Each resulting product $C_{\mathcal{S}}$ of chosen diagonals (the notation suppresses the choice) is a positive measure whose normalization has uniform one-slot marginals and is admissible (Proposition 26.5(i)), and its price is the product of the prices $\pi_{\mathcal{B}}$ of its factors; by (E1), $|Y^{\mathsf{N}}[C_{\mathcal{S}}] - Y^{\mathsf{N}_0}[C_{\mathcal{S}}]| \le (1+o(1))R_{\mathrm{err}}\prod_{\mathcal{B}\in\mathcal{S}}\pi_{\mathcal{B}}Y$. Summing over all choices gives at most

$(1+o(1))R_{\mathrm{err}}\prod_{\mathcal{B}}(1+\pi(K_{s,\mathcal{B}})+\pi(Q_{E,\mathcal{B}}))Y \le (1+o(1))R_{\mathrm{err}}e^{T_K+T_{QE}}Y$, because the chosen side diagonals cost at most T_K and the chosen matching diagonals at most T_{QE}, for either s (proof of Proposition 26.5(ii)). So $e_s \ge Y_{M_{0,s}} - (R_{\mathrm{err}}e^{T_K+T_{QE}}+o(1))Y$.

Step 2: the expansion. Write $M_{0,s} = \bigotimes_{\mathcal{B}}(\mu_{\mathcal{B}} + \mathsf{E}_{\mathcal{B}} + H_{\mathcal{B}} + K_{s,\mathcal{B}} + Q_{E,\mathcal{B}})$ and expand $\mathsf{E}_{\mathcal{B}} = E_{\mathrm{add},\mathcal{B}} + G_{\mathcal{B}}$ as well. The cut matrix at ι of every term is a tensor product of positive semidefinite matrices, since $\mu_{\mathcal{B}} \succeq_\iota 0$ and the other measures are positive and reserved diagonal; so every term has nonnegative energy ((E2)), and we may keep some and discard the rest. Keep the term with μ at every atom, whose energy is Y, and, for every set X of distinct middle blocks with $1 \le |X| \le 4$, or with $|X| \in \{5,6\}$ and all blocks in pieces of the pool, the three families of terms in which the blocks outside X carry μ and the blocks of X carry

(α) labels in $\{E_{\mathrm{add}}, G\}$, together $\bigotimes_{\mathcal{B}\in X}\mathsf{E}_{\mathcal{B}}$;
(β) the chosen side diagonal, together $\bigotimes_{\mathcal{B}\in X}K_{s,\mathcal{B}}$;
(γ) labels in $\{E_{\mathrm{add}}, Q_E\}$, at least one of them Q_E.

These are distinct terms of the expansion: (γ) has a Q_E label, and (α), (β) have none and differ in every label. All other terms are discarded. The number of middle blocks is finite and does not depend on the scale, so the errors $o(1)Y$ below add up to $o(1)Y$.

Step 3: the empty-Q branch. (α) By the polarization orders $\mathsf{E}_{\mathcal{B}} \pm F_{\mathrm{fold},\mathcal{B}} \succeq_{\vartheta_0} 0$ (Remark 17.12(3)) and the identity of all degrees Lemma 17.11(d),

$$\bigotimes_{\mathcal{B}\in X}\mathsf{E}_{\mathcal{B}} - \bigotimes_{\mathcal{B}\in X}F_{\mathrm{fold},\mathcal{B}} = 2^{-|X|}\sum_{\sigma\in\{\pm1\}^X}\Big(1-\prod_{\mathcal{B}}\sigma_{\mathcal{B}}\Big)\bigotimes_{\mathcal{B}\in X}(\mathsf{E}_{\mathcal{B}}+\sigma_{\mathcal{B}}F_{\mathrm{fold},\mathcal{B}}) \succeq_{\vartheta_0} 0.$$

The factors $\mu_{\mathcal{B}'}$, $\mathcal{B}' \notin X$, are reflection positive at ϑ_0, so $Y[\bigotimes \mathsf{E}_{\mathcal{B}}] \ge Y[\bigotimes F_{\mathrm{fold},\mathcal{B}}]$ ((E2)). Expand $\bigotimes F_{\mathrm{fold},\mathcal{B}}$ in the measures $\bigotimes_{\mathcal{B}} D^{j_{\mathcal{B}}}_{\mathcal{B}}$; the products with mixed orientations are positive and reserved diagonal, with nonnegative energy, and are discarded; the two common orientations $j \equiv 1$, $j \equiv 2$ have, by Lemma 26.6(a), energies at least $\ell_m/\prod_{i\in m}H_i - o(1)$ times Y, with the coefficient $\prod_{\mathcal{B}} v_{\mathrm{D}} r^2_{\mathcal{B}}/(1+h_{\mathcal{B}}) \ge \prod_{\mathcal{B}} r^2_{\mathcal{B}}\prod_{i\in m}c_{E,i}$. So the terms ($\alpha$) of X have energy at least $(2\prod_{\mathcal{B}} r^2_{\mathcal{B}}\,\ell_m\prod_{i\in m}c_{F,i} - o(1))Y$, the factor 2 counting the two common orientations. The fold companions are summands of N and not of N_0, so this polarized copy is the only way in which they enter the mean.

(β) Expand $\bigotimes K_{s,\mathcal{B}}$ in the same measures and keep the two common orientations: for $s = \mathrm{L}$ they are $\prod_{\mathcal{B}} r^2_{\mathcal{B}}(\prod_{\mathcal{B}} k_{\mathrm{L},i(\mathcal{B})} \otimes D^1_{\mathcal{B}} + \prod_{\mathcal{B}} k_{\mathrm{R},i(\mathcal{B})} \otimes D^2_{\mathcal{B}})$, and for $s = \mathrm{R}$ the same with D^1 and D^2 exchanged. By Lemma 26.6(a) both give at least $(\prod_{\mathcal{B}} r^2_{\mathcal{B}}\,\ell_m[\prod_{i\in m}k_{\mathrm{L},i}/H_i + \prod_{i\in m}k_{\mathrm{R},i}/H_i] - o(1))Y$: the exact left and right pairing is retained, no ordering of the sides is needed, and both envelopes receive the same credit. The chosen side diagonals carry no factor $1/(1+h_{\mathcal{B}})$, so the side ratios are $k_{s,i}/H_i$. In the bracket of one fold copy the measure $\mathsf{N}_0 - \mu$ contains no fold diagonal (Lemma 22.7), so there are no terms mixing fold diagonals of N_0 with the chosen side diagonals.

Step 4: the nonempty-Q branch. (γ) Every factor is $c_{\mathcal{B}} r^2_{\mathcal{B}}(D_{E_0,\mathcal{B}} + D_{E_1,\mathcal{B}})$ with $c_{\mathcal{B}} \in \{v_{\mathrm{D}}/(1+h_{\mathcal{B}}),\, q_{i(\mathcal{B})}\}$, so the terms ($\gamma$) of X add up to

$$\prod_{\mathcal{B}} r^2_{\mathcal{B}}\Big[\prod_{\mathcal{B}}\Big(\frac{v_{\mathrm{D}}}{1+h_{\mathcal{B}}} + q_{i(\mathcal{B})}\Big) - \prod_{\mathcal{B}}\frac{v_{\mathrm{D}}}{1+h_{\mathcal{B}}}\Big]\bigotimes_{\mathcal{B}}(D_{E_0,\mathcal{B}} + D_{E_1,\mathcal{B}}).$$

The bracket is a polynomial with nonnegative coefficients that is nondecreasing in each $v_{\mathrm{D}}/(1+h_{\mathcal{B}}) \ge c_{E,i(\mathcal{B})}$, so it is at least $\prod_{i\in m}(c_{E,i}+q_i) - \prod_{i\in m}c_{E,i}$. Keep the two aligned orientations of $\bigotimes(D_{E_0,\mathcal{B}} + D_{E_1,\mathcal{B}})$ and discard the others; by Lemma 26.6(b) their energies add up to at least $2([\ell_m - C_m\eta_m]_+ - o(1))Y$. The empty and nonempty Q branches are disjoint, even where their physical diagonals are proportional.

Step 5: summation. For $1 \le |X| \le 4$ the bounds of Steps 3 and 4 depend on X only through its pattern m and the factor $\prod_{\mathcal{B}} r^2_{\mathcal{B}}$, and the sum of this factor over the sets of pattern m is at least $W_0(m)$ (Lemma 24.15(a)); all coefficients are nonnegative. This gives the sums over

$1 \le |m| \le 4$. For a set X of $k \in \{5,6\}$ blocks of pieces of the pool, the first assertion of Lemma 24.16 gives $Y[\bigotimes_{\mathcal{B}\in X} \Pi_{F_j,\mathcal{B}}] \ge (\ell_{\mathcal{P}^k} - o(1))Y$ and Lemma 24.11(b), (c) with the pool data, so Steps 3 and 4 hold with $\ell_{\mathcal{P}^k}$, $C^k_{\mathcal{P}}$ and $\eta_{\mathcal{P}^k}$ in place of ℓ_m, C_m and η_m (Lemma 26.6(a), and the proof of Lemma 26.6(b) with these data); the bounds of Lemma 24.16(b) for the sums over these sets of the coefficients of the three branches give the coupons m_F^{pool} and m_Q^{pool}, which occupy degrees disjoint from those of the other kept terms. Adding the kept terms and Step 1 gives $e_s \ge (1 + m_F + m_Q - R_{\mathrm{err}}e^{T_K+T_{QE}} - o(1))Y$ for both s. Every restoration is recomputed at the exponent a with the product caps C_m at that exponent, not transported from an estimate at a_0, and no lower estimate of an upper-priced norm is used.

The closing number Θ is nonincreasing in m (Remark 9.22), so certified lower bounds for m_F, m_Q and upper bounds for R_{err}, T_K and T_{QE} may be used. □

The chosen side diagonals carry no factor $1/(1+h_{\mathcal{B}})$, so the side ratios are $k_{s,i}/H_i$. The fold companions of N carry it, so $c_{F,i}$ keeps $1/(1+h_i)$. The full chosen prices T_K and T_{QE} are charged in the error exponent, as in the matching exponent. The restorations are evaluated at the product caps of the final exponent, with piece data that do not depend on a. The following numerical values describe the baseline certificate, before the joint allocation below; they are in Certificate D.11: $m_F > 0.400172842378$, $m_Q > 0.190134971388$, of which the terms pooled over the pieces from 2^{31} contribute about $1.1951 \cdot 10^{-7}$ and $6.5884 \cdot 10^{-7}$ (the degree-six parts of both pooled coupons m_F^{pool} and m_Q^{pool} vanish), $R_{\mathrm{err}}e^{T_K+T_{QE}} < 3.67071778633 \cdot 10^{-4}$ and $m > 1.589940741988$.

26.6. **Allocating the full positive mean jointly with the comparison.** The closing inequality contains the comparison energy with coefficient $63/64$ and the mean energy with coefficient $-1/64$. A positive diagonal occurring in both can therefore reduce the closing bound: subtract its mean coefficient from 63 times its comparison coefficient, then apply an upper bound to the remaining nonnegative energy. This is the joint allocation of Corollary 11.2. We apply it to the positive expansion in Step 2 of Proposition 26.7. Of the lower bounds retained there (called mean credits), only those supported on one block are used separately here. The following argument replaces the credits on two or more blocks.

For a middle block write

$$\alpha_B = \frac{w_{\mathrm{D},B} r_B^2}{1+h_B}, \qquad T_B = E_{\mathrm{add},B} + H_B = \beta_B(D_{E_0,B} + D_{E_1,B}), \qquad \mathsf{E}_B = E_{\mathrm{add},B} + G_B.$$

The measures $E_{\mathrm{add},B}, G_B, H_B$, the fold diagonals $D_{F_j,B}$, and the matching diagonals $D_{E_j,B}$ are those defined in Section 26.5; $K_{s,B}$ is the added diagonal on side $s \in \{\mathrm{L},\mathrm{R}\}$ of the matching bracket (Proposition 26.5). Here β_B is a local coefficient. The actual chosen diagonals satisfy

$$\begin{aligned} 2 \le \beta_B/\alpha_B \le R = 2.004317, \qquad \sigma = 0.3842, \\ Q_{E,B} \succeq_\iota \alpha_B(D_{E_0,B} + D_{E_1,B}), \\ K_{s,B} \succeq_\iota \sigma\alpha_B(D_{F_1,B} + D_{F_2,B}). \end{aligned} \tag{26.4}$$

The density of an actual fold marginal relative to its normalized companion is at most $H_* = 1.000003310$. Thus its companion diagonal dominates $H_*^{-1}D_{F_j,B}$. These assertions use the same 33 physical pieces and chosen sides as Proposition 26.5. Explicitly, on a piece starting at X,

$$\frac{\beta_B}{\alpha_B} \le \frac{v_{\mathrm{D}}}{w_{\mathrm{D}}} + \frac{6}{\lfloor\sqrt{X}\rfloor(1-3/X)} \le R, \qquad \frac{\min(k_{\mathrm{L},i}, k_{\mathrm{R},i})}{w_{\mathrm{D},i}H_i} \ge \sigma.$$

The matching coefficient $q_i \ge w_{\mathrm{D},i}/(1+h_i)$ and $H_i \le H_*$ complete the checks. Both independent assemblies verify these inequalities piece by piece.

Lemma 26.8 (Multiaffine allocation constants)**.** *Put $l = 1/R$, $u = 1/2$, and, for $0 \le k \le n$, put*

$$P_{n,k} = (1+l)^k(1+u)^{n-k}, \quad E_{1,n,k} = kl + (n-k)u, \quad E_{2,n,k} = \binom{k}{2}l^2 + k(n-k)lu + \binom{n-k}{2}u^2.$$

Define

$$\eta_n = \min_{0\le k\le n} \frac{P_{n,k}-1}{E_{2,n,k}} (2 \le n \le 10), \qquad \zeta_n = \min_{0\le k\le n} \frac{P_{n,k}}{E_{1,n,k}} (1 \le n \le 9).$$

Write $e_1(z) = \sum_i z_i$ and $e_2(z) = \sum_{i<j} z_i z_j$. For $z_i \in [l,u]$, $\prod_i(1+z_i) - 1 \ge \eta_n e_2(z)$ and $\prod_i(1+z_i) \ge \zeta_n e_1(z)$ in the stated degree ranges. In every degree the corresponding constants $\eta_0 = 211/80$ and $\zeta_0 = 9/4$ are valid.

Proof. For a fixed trial constant each residual is multiaffine, so its minimum on the box is attained at a vertex. The displayed formulas are precisely its vertex tests, grouped by the number of lower coordinates. For the uniform bounds enlarge the box to $[0,1/2]^n$. At a vertex with k nonzero coordinates the ratios are $4((3/2)^k - 1)/\binom{k}{2}$ and $2(3/2)^k/k$. The first is minimized at $k = 5$: its adjacent difference has the sign of $(k-5)(3/2)^k + 4$. The second is minimized at $k = 2, 3$, as its adjacent difference has the sign of $k - 2$. The minima are $211/80$ and $9/4$. □

Use the normalized price assignments of Lemma 33.1, repriced to the selected exponent a. Let $x = A_F^{\mathrm{arm}} = A_E^{\mathrm{arm}}$ be the common raw-arm total of Definition 30.1(3): the sum of the arm coefficients before multiplying by their leading operator norms. Set $y = \sigma x$. For this subsection abbreviate $U = tU_0^{\prime\mathrm{new}}$ and $V = tU^{\mathrm{inv}\,\mathrm{new}}$, where $t = e^{2(a-a_0)570}$. These are the total first-stage price and the portion assigned to invariant factors, respectively; $0 \le V \le U$. An outside factor means a factor on a block other than the selected fold or matching blocks. Define

$$F_\eta = \eta_0 e^U + \sum_{j=0}^{8} (\eta_{j+2} - \eta_0) V^j/j!,$$

$$F_\zeta = \zeta_0 e^U + \sum_{j=0}^{8} (\zeta_{j+1} - \zeta_0) V^j/j!,$$

$$B_\rho = e^U + \sum_{j=1}^{10} (\rho^j - 1) V^j/j!, \qquad \rho = 1 + 1/R,$$

$$w = \frac{U}{1+\rho_H}, \qquad \rho_H = \frac{3}{\lfloor\sqrt{Q_3}\rfloor(1 - 3/Q_3)}.$$

Proposition 26.9 (Four disjoint middle allocations)**.** *In the joint closing of Corollary 11.2, the middle mean decreases the closing number, in addition to its one-block mean credits, by $\Delta_\alpha + \Delta_M + \Delta_O + \Delta_K$, where*

$$\Delta_\alpha = \frac{x^2}{16H_*^2} \frac{e^w - 1 - w}{w^2}, \tag{26.5}$$

$$\Delta_M = \frac{x^2}{128} F_\eta, \qquad \Delta_O = \frac{x}{64}\left(y + \frac{y^3}{6} + \frac{y^5}{120}\right) F_\zeta, \qquad \Delta_K = \frac{1}{64}\left(\frac{y^2}{2} + \frac{y^4}{24} + \frac{y^6}{720}\right) B_\rho. \tag{26.6}$$

These are corrections to the combined comparison and mean bound; they are not separate lower bounds obtained by substituting upper prices in a mean energy.

Proof. After paying $R_{\mathrm{err}} e^{T_K + T_{QE}}$, the available positive product is

$$\bigotimes_B (\mu_B + E_{\mathrm{add},B} + G_B + H_B + K_{s,B} + Q_{E,B}).$$

A tag specifies the summand chosen at a block in this tensor product. Partition the retained terms first by their exact sets of K_s tags and then by their Q_E tags. The four families are:

(i) no K_s or Q_E; at least two E tags, with reference factors elsewhere;
(ii) no K_s; a nonempty Q_E set and at least two positions carrying T or Q_E, with μ or G at the remaining positions;
(iii) exactly $1, 3, 5$ K_s tags; at least one further position carrying T or Q_E, with μ or G at the remaining positions;
(iv) exactly $2, 4, 6$ K_s tags, with arbitrary μ, T, Q_E, G outside.

They are disjoint and correspond, in order, to $\Delta_\alpha, \Delta_M, \Delta_O, \Delta_K$. Each has support on at least two blocks, so none consumes a one-block mean credit.

For the first family use $\mathsf{E}_B \pm F_{\mathrm{fold},B} \succeq_{\vartheta_0} 0$ on a selected pair. The other E factors and reference factors are positive at the same cut. The two companions dominate $2\alpha_B(D_{F_1,B} + D_{F_2,B})/H_*$ after returning to the reserved diagonal. Average the pair selection over the $\binom{n}{2}$ pairs of an n-tag term. With raw coordinate $x_B = 2\alpha_B$ this gives coefficient $4/(H_*^2\binom{n}{2})$ on the selected pair. Consequently its exponential completion, after division by 64, is $x^2 \sum_{j\ge 0} w^j/(16H_*^2(j+2)!)$.

Here the E and H assignments must be made together. The physical bound $H_B \le \rho_H \mathsf{E}_B$ in positive mass allows their total U to be filled with an E assignment at least $U/(1+\rho_H)$: first assign the genuine H mass and put the remaining slack into E. The invariant and generic parts can be filled with their original separate nonnegative slacks. This preserves U and an invariant assignment $V_{\mathrm{full}} \ge V$. The completed comparison cost is nonincreasing in V_{full}, while the mean corrections are nondecreasing in it, so replacing it by V is safe. No lower bound for the physical E mass is inferred from U.

For the second family factor out the T coefficient at each matching position and put $z_B = \alpha_B/\beta_B \in [1/R, 1/2]$. The nonempty-$Q_E$ sum dominates $\prod T_B(\prod(1+z_B) - 1)$. The first inequality of Lemma 26.8 allocates it simultaneously over selected pairs; the factor $1/2$ in the unordered pair completion gives Δ_M. For odd K_s degree select one matching position instead and use the second inequality. For even K_s degree keep the whole outside expansion and use $T_B + Q_{E,B} \succeq_\iota \rho T_B$ on a pure invariant background. These give Δ_O and Δ_K. The uniform constants apply also to backgrounds containing G; the finite improvements are used only on pure invariant backgrounds. The multiaffine inequalities allocate a term among its eligible positions; they never spend it separately in full on every position.

It remains to justify completion of the combined expression. For a labelled diagonal energy Z, combine its comparison coefficient c and retained mean coefficient b as $(63c - b)Z$ first. The finite checks below show $63c - b \ge 0$, including the already retained variance credits. Thus Lemma 10.14 applies to the residual positive series. In particular, allowing repeated block labels enlarges the sum over distinct blocks and gives the displayed exponential series; this step is legitimate because the residual coefficients are nonnegative. Its value is precisely the former comparison bound minus the four displayed corrections. This proves the proposition without asserting a lower bound for any isolated upper-priced mean term. □

Residual coefficients. The sixteen classes and their invariant norm caps κ_b are those of Corollary 25.10. In the table below F denotes a selected fold arm and E a selected matching arm. The outside degree j counts additional invariant factors. Check $j = 0, \dots, 10$ separately, and use one further category for $j \ge 11$. The new mean coefficient is the sum of the applicable entries

selected class	mean coefficient
F^2	$4/H_*^2$
E^2	η_{j+2} $(j \le 8)$, η_0 otherwise
$F^nE,\ n = 1, 3, 5$	$\sigma^n\zeta_{j+1}$ $(j \le 8)$, $\sigma^n\zeta_0$ otherwise
$F^n,\ n = 2, 4, 6$	$\sigma^n\rho^j$ $(j \le 10)$, σ^n in the tail.

The F^2 entries are added. Every selected block cap is at least 2. The restoration formulas therefore bound a selected-pair gain by $1/4$; the same bound holds coefficientwise on each matching background. Subtract at most $\kappa_b/4$ for F^2 and aligned E^2, $\kappa_b/32$ for F^4, and the aligned E^4 cap divided by 32. All $16 \cdot 12 = 192$ residuals $63(\kappa_b - \text{variance}) - \text{mean}$ are positive; the minimum exceeds 23.66. The gates also hold before averaging over the record words. For the pure fold classes F^n, $n = 2, 4, 6$, the cap κ_n is common to every word: the chart reflection is independent of the word, and the extra reflection acts trivially when there is no matching coordinate. In addition to the class gates, check

$$63(1 - \tfrac{1}{4}\mathbf{1}_{n\in\{2,4\}})\kappa_n - \frac{4}{H_*^2}\mathbf{1}_{n=2} - \sigma^n\rho^j > 0 \qquad (0 \le j \le 10),$$

and the same inequality with ρ^j replaced by 1 for the tail. These 36 gates allow the full conservative variance credit on an aligned quartic word, before its weight $1/8$ in the class is taken. Each matching-pair class has a common cap; the pure matching quartics have no new mean subtraction and retain their existing variance gates. For the mixed words the chosen unframed caps are at least $1/4$, and $63/4 - \sigma^n\zeta_{j+1} > 0$ for $n = 1, 3, 5$, $0 \le j \le 8$. The exact rational lists and their independent reconstruction are part of Certificate 33.3.

Proposition 26.10 (Mean terms with a nonempty upper support)**.** *Let S_u, A_u denote just the finite upper comparison assignments, excluding prefix and tail padding. Set*

$$C_E = e^{S_u}\cosh A_u - 1, \qquad C_O = e^{S_u}\sinh A_u.$$

The further joint correction is $\Delta_u - \mathcal{R}_u$, where

$$\Delta_u = \frac{1}{64}\left\{C_E\big(\cosh(y)B_\rho + x\sinh(y)F_\zeta\big) + C_O\big(\sinh(y)B_\rho + x\cosh(y)F_\zeta\big)\right\},$$
$$\mathcal{R}_u = \frac{R_{\mathrm{err}}e^{T_K+T_{QE}}}{64}(e^{S_u+A_u} - 1).$$

Proof. On each finite upper block both matching brackets are $\mu_B + Q_B$, with the same positive reserved-diagonal norm envelope Q_B. The nonempty-Q part of their tensor product is disjoint from all middle-only terms used above. Fix its skew parity and the exact middle K_s set. If the parities agree, retain the full $T/Q_E/G$ background; otherwise select one additional matching position. The same multiaffine allocation gives the displayed even and odd exponential sums. The new first-stage error is bounded by $\mathcal{R}_u$, obtained by summing over the entire nonempty upper support, including discarded branches. Every such term has positive upper degree, so it belongs to none of the sixteen exceptional middle classes or their variance credits. After splitting each full middle skew price into its raw arm and positive remainder, the residual gates are $63 - \sigma^n\rho^j > 0$ and $63 - \sigma^n\zeta_{j+1} > 0$, with the same finite ranges and uniform tails. Since $\sigma < 1$, $n = 0$ is the worst case, and the stated rational constants satisfy these gates. Complete this positive residual series as above. The finite upper assignments are used jointly in its positive cost and mean term; they are not lower bounds for the mass of Q_B. □

Hypotheses verified in this section.

- (CP6): for every atom the bracket of Proposition 26.5(i) holds. $M_{s,\mathcal{B}} - \mathsf{N}_{\mathcal{B}}$ is a nonnegative combination of positive reserved-diagonal laws with uniform one-slot marginals, independent of the index: the norm envelopes $D^{\epsilon}_{\mathcal{B},t}$ of (CP4) of the records other than the fold and matching records, the chosen matching diagonals $Q_{E,\mathcal{B}}$ and the chosen side diagonals $K_{s,\mathcal{B}}$. Its summed price is at most the T of Proposition 26.5(ii), with the sides of Proposition 26.4. The chosen diagonals are diagonal copies of components of the reference, positive records with admissible normalizations (Definition 22.4 and the paragraph after it); so (CP8)(a) holds for them.

- (CP7): $\frac{1}{2}(e_{\mathrm{L}} + e_{\mathrm{R}}) \geq (m - o(1))Y$ with m of (26.3), by Proposition 26.7.
- Every bound used is an exact relation (orders at ι and ϑ_0; the expansion and polarization identities; the contraction and density steps), an upper bound from (E1), or an exact relation minus such bounds (the restorations, the mean error), and the restorations are positive parts of continuous expressions; so the bound of (CP7) is robust, as (CP8)(b) requires (Lemma 11.4(i)). The gates of this section, $k_{s,i}^{\mathrm{aux}} \geq 0$ (positive in every piece), the protection $v_{\mathrm{D}} \geq 2w_{\mathrm{D}}k_{\mathrm{D}}$, the matching gates, $n_X e_0(X) < 1$ and $c_{\geq 2}(X)r_X < d_{\mathfrak{t}}(X)$, are inequalities between fixed numbers of the construction, which inflation does not change (Remark 11.5).

These are the rows "(CP6)" and "(CP7)" of Table 5.

Consequences and what to read next. The arguments in this section use the following statements. The one-cut calculus and block-Gram tensorization (Lemma 5.6); the sector lemma and signed mixtures (Lemma 6.10 and Remark 6.12); the admissible majorants (Lemma 4.18); the Paley triple calculus (Lemma 12.10); polarization of all degrees (Lemma 17.11 and Remark 17.12); the energy toolkit (Lemma 10.13); the comparison-law interface and the robustness of the final chain (Definition 9.14, Lemmas 10.14 and 11.4, and Remarks 9.22 and 11.5); repricing (Lemma 9.3 and Corollary 9.4); and the index-support and envelope-energy lemmas (Lemmas 10.1 and 10.17 and Corollary 10.4) with the disjoint query endpoints of Lemma 5.13.

From the middle range: the frame, the first stage and the bracket with one fold copy (Definition 22.4 and Lemmas 22.1, 22.5 and 22.7, (22.1)); the reference fold record, the actual features and the signed law (Lemmas 23.2, 23.4, 23.5 and 23.7, Definition 23.9, and Proposition 23.13, (23.4), (23.5), Certificate 23.14); the prices of the matching, other middle and prefix records (Lemmas 19.3, 23.17 and 23.24); the chamber geometry, restoration and coupons (Lemmas 17.2, 20.7, 24.8, 24.11, 24.15 and 24.16 and Definition 24.1) with the piece data of Sections 24.7 and 24.8; and the block data (Definition 20.1 and Lemmas 20.3 and 20.4).

Finally: the index law (Proposition 29.7); the upper prices (Proposition 28.14); the closing quantities (Definition 30.1); and the certificates Certificates D.10, D.11 and 30.3.

27. Upper primes I: the reference law μ and its spectral bounds

What this section proves and how to read it. The upper range is $p \equiv 3 \pmod 4$ in $[167 \cdot 2^{40}, 2^{512})$, partitioned into 310 consecutive *bands* (intervals sharing constants; no law lives on a band). This section constructs $\mu_{\mathcal{B}}$ as a five-type completion of a 72-vertex graph law and proves (CP1): reflection positivity at every cut with uniform marginals. The two *principal cuts* (called *root cuts* below; the other ten are *nonroot cuts*) determine the completion weights. Displays (PR1)–(PR13) in Section 27.6 and formulas (F1)–(F6), (F10) in Section 27.7 are *definitions* of band input formulas, not lemmas; cite the display number, not a lemma name. "Gate" below means a checked inequality; L_{band} is a block-size bound $\sum \log p$, not Θ. Inputs: Part IV spectral criteria. Outputs: $\mu_{\mathcal{B}}$ and caps for Section 28. A first reading should follow the band definition and (CP1) statement; per-band tables are reference material (see Appendix E). The upper range consists of the blocks of class-three primes in $[P_3, P_4) = [167 \cdot 2^{40}, 2^{512})$; the primes $p \equiv 1 \pmod 4$ below P_4 are atoms of the square-clique prefix (Lemma 19.2). This section builds, for every block $\mathcal{B}$ of an upper band, the reference $\mu_{\mathcal{B}}$: a nonnegative combination of two-layer field laws and their folded one-layer companions, with weights chosen so that the five-type completion of every chamber coordinate and the reserved domination hold. We prove that it is a probability measure, reflection positive at every cut of the family $\mathfrak{F}$, square-affine invariant and with uniform one-slot marginals (Lemma 27.17). This is (CP1) of Theorem 9.21 for the upper atoms (Definition 9.10).

There are 310 bands (Definition 27.1): 45 below 2^{48}, 64 in each binary order of $[2^{48}, 2^{52})$, and nine from 2^{52} on. Their inputs are the following.

- The two root cuts: the Route B criterion with graded count errors and log-convex nonstar coefficients below 2^{52} (Section 27.3), and the completed root from 2^{52}.
- The ten nonroot cuts: class-three tests at 2^{47} and 2^{48} (Section 27.4), with observed merge and constant caps for four reserved states (Section 27.5).
- The rows: pinned conditional stars and joint F/E rows with a free Young allowance (Sections 27.6 and 27.7).

A separate *tail datum* (Definition 27.1(e)) carries the tail law (Section 29): a fixed set of inputs and weights, evaluated at 2^{64} and valid at both prime classes, used only at the tail primes $p \geq P_4$.

The law $\mu_{\mathcal{B}}$ is the 25-state five-type completion of Lemma 17.5 applied to the uniform strict-square labelling law of a 72-vertex two-layer graph: in each of the two chamber coordinates, folds, matchings, merges and constants are added with weights in powers of $r = b^{-1/2}$, and one-layer copies placed on the reserved diagonal restore positivity at the layer flip ι. The weights must dominate the negative spectra of twelve cut operators, and of the merge and constant marginals of one of them, and the row errors of six conditional row laws. These enter as rational *band inputs*, evaluated once per band, at its start or at a point below it. The two root cuts dominate the construction: their caps fix the size of all the weights. Section 27.7 lists the formulas for the inputs with the gates under which they hold; Lemma 28.11 shows that gates checked at one point hold at every larger class-three prime and for every block of the band.

The next section builds the comparison law on $\mu_{\mathcal{B}}$ and prices it (Proposition 28.14); the section on the tail uses the reference with the tail datum at every prime $p \geq P_4$ of either class (Lemma 29.2); the closing reads the upper range only through the prices $S_{\mathrm{u}}(a)$, $A_{\mathrm{u}}(a)$.

Conventions. p is a prime, $r = p^{-1/2}$ and $q = 1/p$; on a block $\mathcal{B}$ with least prime $b = b_{\mathcal{B}}$, $r = b^{-1/2}$. Every band records a budget L_{band} (Definition 27.1(b)), the block length with which its data are evaluated and its caps inflated; $L_{\mathrm{u}} = 50000/27$ is the block length of the local data of the tail datum and of the completed-root caps from 2^{52} (Definition 18.1). The slot array, its chamber cuts $\vartheta_{c,a}$ and the reserved cut ι are those of Definition 5.12. The five types $\mathrm{L}, \mathrm{D}, \mathrm{S}, \mathrm{R}, \mathrm{C}$ have the field types $\mathrm{L}, \mathrm{F}, \mathrm{E}, \mathrm{E}, \mathrm{E}$, the formal degrees $0, 1, 1, 2, 3$ and the multiplicities $1, 3, 2, 6, 2$ (Definition 17.4); a *state* is a pair $\mathfrak{t} = (\mathfrak{t}_0, \mathfrak{t}_1)$ of types, of degree $\deg \mathfrak{t} = \deg \mathfrak{t}_0 + \deg \mathfrak{t}_1$ and multiplicity $m_{\mathfrak{t}} = m_{\mathfrak{t}_0} m_{\mathfrak{t}_1}$, and its *field pair* $\mathfrak{f}(\mathfrak{t})$ is the unordered pair of the field types of $\mathfrak{t}_0, \mathfrak{t}_1$. For a cut with a positive envelope we write B_{top} for its coefficient, $\mathsf{A}_p \preceq (1 + B_{\mathrm{top}}/p)I$: for the Route A criterion it is the B^+ of Theorem 14.21(d), for the Route B criterion the B_{top} of Theorem 14.27(e); the letter B is the energy factor of the bracket. A prime class is $\mathfrak{c} \in \{1, 3\}$, meaning $p \equiv \mathfrak{c} \pmod 4$, as in Theorem 14.21. t_{Y} is the Young (mean) allowance of a row (Lemma 17.5(c),(f)); a band input records it as t_{φ}.

27.1. **The 310 bands and their input data.**

Where the upper range starts. On a middle block the comparison already changes records of formal degree one, and its leading prices are of order $q_{\mathcal{B}}^{2a} b^{-1}$. The upper comparison changes only records of formal degree at least two, so an upper block $\mathcal{B}$ with least prime b has total comparison price $s_{\mathcal{B}} + a_{\mathcal{B}} = O(q_{\mathcal{B}}^{2a} b^{-3/2})$ with a band-dependent constant (Proposition 28.14(iii)). The upper reference is the more expensive law to define. Its weights grow with the root caps, and the root caps grow fast as the primes decrease: d_{L} is about 12847 at P_3, 2557 at 2^{49} and 113 from 2^{64} on (Certificate 27.9 and Table 32). The bands of $[P_3, 2^{48})$ carry about 60% of the upper symmetric price (Proposition 28.14).

The threshold hypothesis (S1) of the root cuts holds above about $2^{46.34}$ (Section 27.3). All local gates are checked from 2^{47} on (Lemma 28.11); the rows need no gate on the middle density beyond $t_{\mathrm{Y}} H < 1$, since the Young allowance is free (Lemma 17.5(f)). At the root cuts the completed conditional law of the root has a completion-variance denominator that is positive only above about $2^{49.96}$ for LL20 and $2^{49.30}$ for LL22. The Route B criterion expands the root kernel around

the product of the half laws instead; it needs only (S1), so it supplies the root caps on all bands below 2^{52} (Certificate 27.9). The start $P_3 = 167 \cdot 2^{40}$ is a design choice (Definition 18.1).
Class three. Every prime of an upper band is 3 modulo 4. At these primes -1 is a nonsquare, the strict-square relation is a tournament, the two Fourier signs of a spectral test give complex-conjugate matrices (Theorem 14.21), and the normalized Paley operator satisfies $\hat{P}^* = -\hat{P}$ and $\hat{P}^2 = -I$ on L_0^2 (Section 14.2). The data of the bands are needed, and certified, for class three only: the direct root caps by Route B, the nonroot and observed caps, the pinned conditional stars and joint rows, and the comparisons of Section 28. The laws of this section are also used at the tail primes $p \geq P_4$ of both classes, but only with the tail datum (Definition 27.1(e), Lemma 29.2), whose Route A caps and rows are stated for both classes.

Definition 27.1 (Upper bands)**.**

(a) *Bands.* The 310 bands $[X, Y)$ partition $[P_3, P_4)$:
- $[167 \cdot 2^{40}, 168 \cdot 2^{40})$;
- $[j2^{41}, (j+1)2^{41})$ for $84 \leq j < 128$;
- $[2^b(1 + \frac{j}{64}), 2^b(1 + \frac{j+1}{64}))$ for $48 \leq b \leq 51$ and $0 \leq j < 64$;
- $[2^b, 2^{b'})$ for consecutive $b < b'$ in $\{52, 53, \ldots, 58, 60, 64, 512\}$.

(b) *Blocks.* Below 2^{52} a band $[X, Y)$ has two cells $[x_j, x_{j+1})$, $x_j = X^{1-j/2}Y^{j/2}$; from 2^{52} on it has $J = 128 \log_2(Y/X)$ cells, $x_j = X^{1-j/J}Y^{j/J}$. Each cell carries a recorded integer $n_j \geq 1$, and the primes $p \equiv 3 \pmod 4$ of the cell are grouped into consecutive blocks of n_j primes (Lemma 20.4(i)). The integers satisfy $n_j \log x_{j+1} \leq L_{\mathrm{band}}$, where the *band budget* L_{band} is a recorded integer between 550 and 886. Thus every block has $\log q_{\mathcal{B}} \leq L_{\mathrm{band}} < L_{\mathrm{u}}$.

(c) *Band inputs.* For each field type $\varphi \in \{\mathrm{L}, \mathrm{F}, \mathrm{E}\}$ the band has two inputs, positive rationals with $h \geq 1$:
- an *ordinary input* $(d_\varphi, f_\varphi, h_\varphi, M_\varphi, t_\varphi)$;
- a *reserved input* $(d_\varphi^{\mathrm{res}}, f_\varphi^{\mathrm{res}}, h_\varphi^{\mathrm{res}}, M_\varphi^{\mathrm{res}}, t_\varphi^{\mathrm{res}})$.

For each of the 25 states $\mathfrak{t}$ it has a *reserved cap* $k_{\mathfrak{t}} > 0$. These are upper bounds, rounded upward in the record, for the local quantities of Section 27.7, evaluated for blocks of class-three primes at least X and logarithmic length at most L_{band}:
- d_φ bounds the block caps of the two-layer and of the one-layer chamber cut that cut the L coordinate of the graph whose other coordinate has field type φ (LL20 and LL10, FL21 and FL11, EL21 and EL11).
- $(f_\varphi, h_\varphi, M_\varphi, t_\varphi)$ are row coefficients of (F5) valid for the two-layer and for the one-layer rows at φ; in particular $M_\varphi \geq H + t_\varphi$ and $t_\varphi H < 1$ for both.
- The reserved input uses the one-layer quantities alone: LL10, FL11 or EL11 and the one-layer row.
- $k_{\mathfrak{t}}$ is the block cap $k_{\mathfrak{f}(\mathfrak{t})}$ of the layer cut of the field pair $\mathfrak{f}(\mathfrak{t})$ (LL22, FL22, EL22, FF22, FE22, EE22). The exceptions are $k_{\mathrm{LR}} = k_{\mathrm{RL}} = k_{\mathrm{G}}$ and $k_{\mathrm{LC}} = k_{\mathrm{CL}} = k_{\mathrm{C}}$, the observed merge and constant caps of EL22 (Proposition 27.14).

The root entries d_{L} and k_{LL} are the caps of Section 27.3. The values are recorded with the band (Certificate 27.18) and summarized in Table 32.

(d) *Weights.* Rational weights $w_{\mathfrak{t}}, v_{\mathfrak{t}} > 0$, indexed by the 25 states, with $w_{\mathrm{LL}} = 1$, such that for each chamber coordinate $c \in \{0, 1\}$ and each type $\mathfrak{t}'$ of the other coordinate the five weights $a_{\mathfrak{s}} = w_{\mathfrak{t}}$ ($\mathfrak{t}_c = \mathfrak{s}$, $\mathfrak{t}_{1-c} = \mathfrak{t}'$, $\mathfrak{s} \in \{\mathrm{L}, \mathrm{D}, \mathrm{S}, \mathrm{R}, \mathrm{C}\}$) satisfy

$$a_{\mathrm{D}} \geq d\, a_{\mathrm{L}}, \qquad a_{\mathrm{S}} \geq M\, a_{\mathrm{D}}, \qquad a_{\mathrm{R}} \geq f\, a_{\mathrm{D}} + h\, a_{\mathrm{S}}, \qquad a_{\mathrm{C}} a_{\mathrm{S}} \geq a_{\mathrm{R}}^2 \tag{27.1}$$

with the ordinary input (d, f, h, M) of the field type of $\mathfrak{t}'$ (forty inequalities); the weights v satisfy the same forty inequalities with the reserved inputs; and

$$v_{\mathfrak{t}} \geq k_{\mathfrak{t}}\, w_{\mathfrak{t}} \qquad \text{for all 25 states } \mathfrak{t}. \tag{27.2}$$

(e) *Tail datum.* The *tail datum* is a second set of inputs and weights, evaluated at 2^{64} for blocks of primes of either class. It is not the datum of the band $[2^{64}, 2^{512})$, whose inputs are class-three data.
- Its inputs (d, f, h) and caps are integers, valid for blocks of primes at least 2^{64} of either class and of length at most L_{u} (Certificate 27.13(d)).
- Ordinary inputs: $(d, f, h) = (113, 258, 120)$ for L, $(24, 30, 11)$ for F, $(28, 8, 1)$ for E.
- Reserved inputs: $(12, 18, 10)$, $(5, 5, 3)$, $(5, 3, 1)$.
- $M_\varphi = 21/20$ and $t_\varphi = 1/50$ for every type and both inputs.
- Caps $k_{\mathsf{t}} = k_{\mathsf{f}(\mathsf{t})}$ with $k_{\mathrm{LL}} = 45$, $k_{\mathrm{LF}} = 9$, $k_{\mathrm{LE}} = 7$, $k_{\mathrm{FF}} = 4$, $k_{\mathrm{FE}} = 3$, $k_{\mathrm{EE}} = 1$.
- Its 50 weights satisfy (d).

It is used only by the tail law (Definition 29.1).

Remark 27.2 (What is a choice). The bands, the cells and the integers n_j are choices, and any recorded choice with $n_j \log x_{j+1} \le L_{\mathrm{band}}$ is valid. Below 2^{52} the integers n_j were chosen with $1/(2a \log x_{j+1})$ as a guide; from 2^{52} on the recorded partitions carry the block charges $\mathcal{U}_d$ of Proposition 28.14, computed at the anchor exponent $85941/125000000$, which the prices then reprice (Proposition 28.14(ii)). The bands are short below 2^{52} because the root caps, which fix the size of all the weights, fall fast with X (d_{L} is about 12847 at P_3, 6569 at 2^{48} and 367 at 2^{52}), and each band pays the caps of its own start.

Only the 105 inequalities of (d) matter. The certificate fixes the weights by processing the states in increasing type order and taking for each weight the maximum of its lower bounds, and then checks every inequality exactly. Any positive rational solution, for instance one chosen jointly by optimization, gives the same conclusions.

Each band evaluates its inputs at points not above X (Section 27.4, Certificate 27.13). Since positivity is proved block by block and tensorizes factor by factor, different valid input sets may be used on different blocks. The data of a band are valid at every class-three prime $p \ge X$, but a band uses them only on $[X, Y)$. Beyond 2^{512} only the tail datum is used.

Remark 27.3 (Band endpoints). Band endpoints need not form an equally spaced grid. Each band carries its own complete prime-count, product-budget, restart, endpoint and jump bounds of Lemma 20.4. The bands have disjoint prime supports that cover the range, so the positive band charges add. No boundary term is dropped.

27.2. **The two-layer reference.**

27.2.1. *The two-layer graphs.*

How to read the two-letter names. In the upper range every graph, cut and record carries a short code with one letter for each of the two chamber coordinates. As *field types*, L is the six-object root coordinate, F its three-object fold half, and E the three-object matching coordinate with no forward edges; thus $G_{\mathrm{FE},2}$ is the two-layer graph with a fold coordinate and a matching coordinate. A cut name adds the number of layers and the cut: the last digit is 0 or 1 for the chamber cut in coordinate 0 or 1, and 2 for the layer cut ι; so LL20 is the chamber cut in coordinate 0 of the two-layer root, and EL22 is the layer cut of the two-layer EL graph. The 25 *states* t use instead the five pattern types $\mathrm{L}, \mathrm{D}, \mathrm{S}, \mathrm{R}, \mathrm{C}$ of Definition 17.1 in each coordinate, so the state LR has the root pattern in the first coordinate and a merge pattern in the second. Records are usually named by their field types (FF, FE, LE), states by their pattern types (DD, DS); where both occur together the text gives both, as in Proposition 28.3.

For the field types let $O_{\mathrm{L}} = \mathbb{Z}/6\mathbb{Z}$ and $O_{\mathrm{F}} = O_{\mathrm{E}} = \{0, 1, 2\}$ be the object sets, with the following relations and forward relations. In L, $x \to y$ is related if $y \in \{x + 1, x - 1, \mathsf{c}x\}$ (the chamber relation R without equality), and forward if $y = \mathsf{c}x$. In F the relation and the forward relation are their restrictions to $\{0, 1, 2\}$: the related pairs are $0 \to 1$, $1 \to 0$, $1 \to 2$, $2 \to 1$ and $0 \to 2$,

and only $0 \to 2$ is forward. In E all ordered pairs of distinct objects are related and none is forward. For field types φ_0, φ_1 and $j \in \{1,2\}$ let $G_{\varphi_0\varphi_1,j}$ be the oriented graph on $O_{\varphi_0} \times O_{\varphi_1} \times \Lambda_j$, $\Lambda_2 = \{0,1\}$, $\Lambda_1 = \{0\}$, with an edge $(u_0,u_1,\ell) \to (v_0,v_1,\ell')$ whenever both coordinate relations $u_0 \to v_0$, $u_1 \to v_1$ hold and at least one of them is forward; there is no condition on the layers. The *root* is $G_{\mathrm{LL},2}$.

Lemma 27.4 (Two-layer geometry)**.**

(a) Every $G_{\varphi_0\varphi_1,j}$ is an oriented graph. The root has 72 *vertices and* 720 *edges, every vertex has* 10 *out-neighbours and* 10 *in-neighbours, and it contains every query $(u_0,u_1,\ell) \to (\mathsf{c}u_0, \mathsf{c}u_1, 1-\ell)$; in particular the twelve queries* (28.1) *and $\mathcal{E}_6$ are edges of the root.*

(b) The chamber reflections of a coordinate of type L *and the layer swap are automorphisms of the graphs; every permutation of O_{E} is an automorphism of a graph with a coordinate of type* E *(exchangeability).*

(c) The subgraph of $G_{\mathrm{L}\varphi,j}$ induced on the plus half $\{0,1,2\}$ of its first coordinate is $G_{\mathrm{F}\varphi,j}$, and likewise in the second coordinate. The subgraph of $G_{\varphi\varphi',2}$ induced on the layer $\ell = 0$ is $G_{\varphi\varphi',1}$.

(d) For every φ and j the graphs $G_{\mathrm{F}\varphi,j}$ and $G_{\mathrm{E}\varphi,j}$ induce the same graph on the objects $\{0,1\}$ and on the objects $\{1,2\}$ of their first coordinate, and neither has an edge inside one object.

Proof. (a) If $u \to v$ is forward in L, then $v \to u$ is not related: $u \notin \{\mathsf{c}u \pm 1, \mathsf{c}^2u\}$, as $\mathsf{c}u - u = \pm 2$; in F the pair $2 \to 0$ is not related. An edge needs a forward coordinate, whose reverse is not related, so no pair carries edges in both directions; and no loop occurs, since related objects are distinct. At a vertex of the root there are $3 \cdot 3$ pairs of related successors, of which $2 \cdot 2$ have no forward coordinate; so there are 5 out-neighbours in each layer, 10 in all, and by the same count 10 in-neighbours; so $72 \cdot 10 = 720$ edges. Both coordinates of a query are forward. (b) A chamber reflection ϑ_a commutes with c (Lemma 5.13(i)) and maps $x \pm 1$ to $\vartheta_a(x) \mp 1$; the relations do not involve the layer; and E has all distinct pairs and no forward pair. (c) Inside $\{0,1,2\}$ the only forward pair of L is $0 \to 2$, since $\mathsf{c}1 = 5$ and $\mathsf{c}2 = 4$; and edges carry no layer condition. (d) Between the objects $0,1$ and between $1,2$ the relations of F and E are the same (both directions, none forward); inside one object the relation fails. The script `research/push_2026_10_17/sparse/layer72.py` reconstructs these facts for all eighteen graphs. □

The spectral caps concern twelve cuts, and the merge and constant marginals of one of them (Section 27.5). The name of a cut lists the field types of its graph, its number of layers and the cut: 0 and 1 for the chamber cut $\vartheta_{c,0}$ of the coordinate c, and 2 for the layer cut ι. The two *root cuts* are LL20 and LL22. The ten *nonroot cuts* are the chamber cuts LL10, FL21, EL21, FL11, EL11 of the L coordinate and the layer cuts FL22, EL22, FF22, FE22, EE22; coordinate interchange and the chamber automorphisms give all other cuts. By Lemma 27.4(c) the half graph of a chamber cut of $G_{\mathrm{L}\varphi,j}$ is $G_{\mathrm{F}\varphi,j}$, which is the graph of the aligned fold, and the half graph of a layer cut is the one-layer graph. Each cut, with a deletion order fixed once, is spectral data in the sense of Definition 13.1; we write A_p for its normalized cut operator on $L^2(\mu)$, where $\mu = \lambda_p(G^+)$ is the half law, the graph law of the half graph G^+ (Lemma 13.3); the actual half marginal is $\varrho\,\mu$, with $\varrho = \mathsf{A}_p\mathbf{1}$. The root cuts have $n = 36$ half vertices, maximal degree $\Delta = 20$, internal degrees $d_k^0 \le 10$ and cross degrees $r_k \le 12$ (LL20) and $r_k \le 10$ (LL22).

27.2.2. *The* 25 *summand types.* For a block $\mathcal{B}$ of odd primes and field types φ_0, φ_1 let $\lambda_{\varphi_0\varphi_1} = \lambda_{\mathcal{B}}(G_{\varphi_0\varphi_1,2})$ and $\lambda^{(1)}_{\varphi_0\varphi_1} = \lambda_{\mathcal{B}}(G_{\varphi_0\varphi_1,1})$ be the two-layer and the one-layer field laws (Definition 12.1): at every prime of $\mathcal{B}$ independently, the uniform law on injective strict-square labellings. (In this section *pattern summand* below is what is meant by a summand of π_{t} or of $\mu_{\mathcal{B}}$; *component* keeps its glossary meaning as a summand of $\nu_{\mathcal{B}}$.) These are the field laws of Definition 17.4, with the layer set $\Lambda = \{0,1\}$, respectively $\{0\}$. For a state t and patterns $\phi = (\phi_0,\phi_1) \in P_{\mathsf{t}_0} \times P_{\mathsf{t}_1}$, the

pattern summand $\Pi_{\mathfrak{t},\phi}$ (respectively $\Pi^{(1)}_{\mathfrak{t},\phi}$) is the image of the two-layer (one-layer) field law of the field types of $\mathfrak{t}$ under the emission e_ϕ; objects outside the image of ϕ are forgotten (the *hidden* objects of merges and constants). The two-layer and one-layer components are written into the slots (u_0, u_1, ℓ) and, for one-layer components, copied onto both layers by Diag_ι. Put

$$\pi_{\mathfrak{t},2} = \sum_\phi \Pi_{\mathfrak{t},\phi}, \qquad \pi_{\mathfrak{t},1} = \sum_\phi \Pi^{(1)}_{\mathfrak{t},\phi},$$

of mass $m_\mathfrak{t}$. The *rows* of the coordinate c at the field type φ are the objects of the coordinate c of a field law whose other coordinate has type φ; their law is a five-type datum (Definition 17.3) whose conditional reference is the law of one outer row given the middle row under the E law (Section 27.6). At ι the half graph of a two-layer graph is its one-layer graph (Lemma 27.4(c)); this is why one-layer copies on the reserved diagonal can cancel the negative envelopes at ι (Remark 17.10).

27.2.3. *The reference law $\mu_\mathcal{B}$ and its weights.*

Definition 27.5 (The upper reference)**.** Let $\mathcal{B}$ be a block of an upper band with least prime b and $r = b^{-1/2}$, and let $w_\mathfrak{t}, v_\mathfrak{t}$ be the weights of the band (Definition 27.1(d)). The *reference* of $\mathcal{B}$ is $\mu_\mathcal{B} = \widetilde{\mu}_\mathcal{B}/P_\mu(r)$, where

$$\widetilde{\mu}_\mathcal{B} = \sum_\mathfrak{t} w_\mathfrak{t} r^{\deg \mathfrak{t}}\, \pi_{\mathfrak{t},2} + r \sum_\mathfrak{t} v_\mathfrak{t} r^{\deg \mathfrak{t}}\, \mathrm{Diag}_\iota(\pi_{\mathfrak{t},1}),$$

$$P_\mu(r) = \sum_\mathfrak{t} m_\mathfrak{t}(w_\mathfrak{t} r^{\deg \mathfrak{t}} + v_\mathfrak{t} r^{\deg \mathfrak{t}+1}) \geq 1. \tag{27.3}$$

The polynomial P_μ has constant coefficient $w_{\mathrm{LL}} = 1$, the weight of the root $\pi_{\mathrm{LL},2}$; its coefficient of degree d is the total weight of the records of overall degree d. The one-layer weights $v_\mathfrak{t}$ form an array of their own: they satisfy the one-layer completion inequalities with the one-layer inputs and, separately, the domination (27.2) by the cap of their own state. Weights $v_\mathfrak{t}$ proportional to $w_\mathfrak{t}$, with one factor for all states, would have to dominate the largest cap k_{LL} (9381.137478 at P_3, 2070.828678 at 2^{49}) in every state, while the other caps are below 7.01.

27.3. **The two principal cuts.** Two of the twelve cuts are the full degree-twenty root: the first-coordinate cut LL20, which gives the ordinary input d_L, and the layer cut LL22, which gives the reserved cap k_{LL}. Their half graphs have 36 vertices, each of degree twenty in the root, and their caps are the largest of all: at P_3 they are about 12847 and 9381 (2557 and 2071 at 2^{49}), against at most 25 for the other cuts. Two routes bound them.

The setting. Let $(G, \vartheta, U^+, \mathfrak{o})$ be the spectral data of LL20 or LL22, with $n = |U^+| = 36$, cross signs ς, cross degrees r_k, internal degrees d^0_k and $\Delta = 20$. The threshold hypothesis (S1) of Definition 13.1 is $e_{20}(p, 72) < 1$ and $p > 4n^2 = 5184$, with the coarse count error $e_t(p, n)$ of (12.1). Let $\mu = \lambda_p(G^+)$ be the half law and A_p the normalized cut operator on $L^2(\mu)$, with the character expansion (14.20) of the root kernel (Lemma 14.18), its nonstar part T_{ns}, its collision part T_{coll} and the star function $h_\star$ of (15.9). At $p \equiv 3 \pmod 4$ this is the setting of route B (Section 14.9): $\mathbf{H}^+ = \mathbf{H}$, and the two signs of the Schur test (14.22) give complex-conjugate matrices.

The hard root also permits the sign filter of Lemma 16.19. On its two cuts the exact pin counts are

	all pins	compatible	$\sum a_{ij}$	$\sum \widetilde{a}_{ij}$
LL20	1296	168	54583296	2752512
LL22	1296	324	55738368	39813120

(27.4)

These integers are obtained directly from the 72-vertex root and its two reflections (Certificate D.12). They sharpen the collision input while preserving the accepting graph. The certified band caps

below retain their larger collision assignments, so the filter requires no alteration of the selected comparison laws or closing table.

Route B at the two root cuts. Theorem 14.27 uses no completed conditional law of the root and no completion-variance denominator, only (S1). Its Route-B frame numbers (not those of the middle frame) (H1) are those of Lemma 14.28; its inputs (H3) are the following.

Lemma 27.6 (Root inputs)**.** *Let the cut be LL20 or LL22, so that* $n = 36$, $d_k^0 \le 10$ *and* $r_k \le 12$, *and let* $p \equiv 3 \pmod 4$ *satisfy (S1). Then hypothesis* (H3) *of Theorem 14.27 holds with the following choices.*

(i) $M_\star$ *is the quantity of Proposition 15.26 for a fixed total order of* U^+, *with the relative count errors* $e_i = e^-_{d_i^0}(n)$ *of* (12.2).

(ii) $z_\star = r^2(\sum_i K_i + |B_{\mathrm{even}}|)$, *with the* K_i *of Proposition 15.26 and* B_{even} *of Lemma 15.29 for the signs* $\chi(\varsigma_{jk})$ *of the class* $p \equiv 3 \pmod 4$, *and* $Z_\star = 2z_\star$.

(iii) $N_{\mathrm{ns}} = \sum_{k=2}^4 C_k(Q) r^k$, *where* $Q \le p$ *is a real number at which (S1) holds and* $C_k(Q)$ *bounds the nonstar coefficients of Corollary 16.11(b) for a refinement of the compressed partition (Lemma 16.2) with witnesses of sizes* 2, 3 *and* 4.

- *These coefficients are evaluated with the graded error pairs and the special values* $\mathsf{S}_{\mathrm{m}}(s)$, $\mathsf{S}_\varrho(s)$ *of Definition 16.6 and* (16.5), *and summed exactly over the complete nonstar histogram of the cut, under the monotone-signature gate* (16.6).
- *At an interpolation point* Q *(a prime size, not an exponent anchor) they are this sum.*
- *Between consecutive interpolation points* $P_0 \le Q < P_1$ *the bound is that of Lemma 16.12.*

(iv) D_{col} *is the minimum, as in Lemma 16.15(e), of* $D_1 + D_{\ge 2}$ *(with fixed orders and the tail bound of Lemma 16.15(c)), the ordered single-coincidence bound* D_{ord} *and the natural bound* D_{nat}; *any one of them may be used instead.*

Proof. (i), (ii) Given the other labels, the label x_i is uniform on the values outside the other $n-1$ half labels that satisfy the d_i^0 signed constraints of its internal neighbours, a set with at least $p2^{-d_i^0}(1 - e^-_{d_i^0}(n))$ elements by Lemma 12.2(c). So μ is a pinned half law (Definition 15.23) with count errors $e_i = e^-_{d_i^0}(n)$, which are below one by (S1), (E2) and (E3). Lemma 15.29, with the signs $\chi(\varsigma_{jk})$ of the class $p \equiv 3 \pmod 4$, gives $\|h_\star - \mathbb{E}_\mu h_\star\|_\mu \le M_\star$ and $|\mathbb{E}_\mu h_\star| \le z_\star$, so $2|\mathbb{E}_\mu h_\star| \le Z_\star$. These bounds concern μ itself, not the completed root. (iii) is Corollary 16.11(b), with Lemma 16.12 between anchors; the coefficients are nonincreasing in the prime (Lemma 16.12), so their value at $Q \le p$ bounds them at p. (iv) is Lemma 16.15(e). Neither (iii) nor (iv) uses the class of p. □

The hypothesis (S1) holds only above about $2^{46.34}$: $e_{20}(2^b, 72)$ is about 1.125, 0.796 and 0.398 for $b = 46, 47, 49$. The upper range starts at $P_3 \approx 2^{47.38}$; below about $2^{46.34}$ the Route B criterion does not apply to these root cuts.

Lemma 27.7 (Monotone endpoint extension)**.** *Fix a root cut, the pivot order, the nonstar partition with its records, the orders of Lemma 16.15, and rationals* $s_\star, g_\star$ *that satisfy* (14.22) *for both signs, for the class* $p \equiv 3 \pmod 4$, *with the numbers of Lemma 14.28 evaluated at* $r_0 = p_0^{-1/2}$, *where* $p_0 > 4n^2$ *is a real number (in the applications a power of two) with* $e_\Delta(p_0, 2n) < 1$. *Suppose that* $\mathbf{R} \succeq 0$ *and that the numbers of Lemma 27.6, with* $Q = p_0$ *in (iii), give* $\varepsilon_Z < 1$ *and* $g_{\mathrm{root}} r_0 < 1$ *at* r_0. *Let* $p \ge p_0$ *be a prime with* $p \equiv 3 \pmod 4$. *Then the numbers of Lemmas 14.28 and 27.6 at* p *satisfy* (H1)–(H3) *with the same* $s_\star, g_\star$, *and*

$$g_{\mathrm{root}}(p) \le g_{\mathrm{root}}(p_0), \qquad B_{\mathrm{top}}(p) \le B_{\mathrm{top}}(p_0) := \frac{p_0\, \varepsilon_\varrho(p_0)^2}{1 - g_{\mathrm{root}}(p_0) p_0^{-1/2}}.$$

Consequently, for every block $\mathcal{B}$ of primes $p \ge X \ge p_0$ with $p \equiv 3 \pmod 4$ and $\log q_{\mathcal{B}} \le L_{\mathrm{u}}$, the normalized block root operator is at least $-d\, b_{\mathcal{B}}^{-1/2}$, with $d = g_{\mathrm{root}}(p_0)\exp\bigl(B_{\mathrm{top}}(p_0)L_{\mathrm{u}}/(X\log X)\bigr)$.

More generally, let $s_\star, g_\star$ satisfy (14.22) at p_0 as above, with $\mathbf{R} \succeq 0$, and let $X \ge p_0$ be real with (S1) at X. Suppose the numbers of Lemma 27.6 at $r_X = X^{-1/2}$, with $Q = X$ in (iii), give $\varepsilon_Z < 1$ and $g_{\mathrm{root}}(X)r_X < 1$. Let $\mathcal{B}$ be a block of primes $p \ge X$ with $p \equiv 3 \pmod 4$ and $\log q_{\mathcal{B}} \le L' \le L_{\mathrm{u}}$. Then the normalized block root operator of $\mathcal{B}$ is at least $-d\, b_{\mathcal{B}}^{-1/2}$ with $d = g_{\mathrm{root}}(X)\exp\bigl(B_{\mathrm{top}}(X)L'/(X\log X)\bigr)$, where $B_{\mathrm{top}}(X) = X\varepsilon_\varrho(X)^2/(1 - g_{\mathrm{root}}(X)r_X)$.

Proof. Write every quantity as a function of $r \in (0, r_0]$. By Lemma 14.29(a) the frame quantities ε_V, $\varepsilon_\times$, ε'_H, θ_j^μ, $\hat{E}^2$ and $\varepsilon_{\ne}$ are admissible (Lemma 4.18); for ε'_H this includes the entries β_{kl} of (14.13), maxima of two branches that equal their first branches (the proof of Lemma 14.29). Hence $\varepsilon_{\mathrm{rem}}$, $\varepsilon_{\mathbf{B}}$, $\varepsilon_{\mathbf{W}}$, $\varepsilon_{\mathbf{EW}}$, $\alpha_\star$ and $\beta_\star$ are at most their values at p_0. The matrices Δ^E, $\mathbf{H}$ and $\mathbf{R}$ depend on p only through its class. So $\mathbf{R}+\beta_\star I \succ 0$ at p, because $\mathbf{R} \succeq 0$ and $\beta_\star \ge \varepsilon_{\mathbf{B}} \ge \|\mathbf{H}\|_{\mathrm{row}}/(p-1) > 0$; and (14.22) holds at p by Lemma 4.18(d), since $g_\star/\alpha_\star$ does not decrease and $\beta_\star$ does not increase.

For the inputs: $M_\star/r$ and $z_\star/r^2$ are admissible (Remark 15.27), hence so are $(M_\star^2 + z_\star^2)/r = r(M_\star/r)^2 + r^3(z_\star/r^2)^2$ and $Z_\star$; $N_{\mathrm{ns}}/r = \sum_k C_k(p_0)r^{k-1}$ is a polynomial in r with nonnegative coefficients; and D_{col}/r, for the fixed orders, is at most its value at p_0 (Lemma 16.15(e)). Thus ε_Z, $\varepsilon_{\mathrm{soft}}/r$, g_{root} and ε_ϱ/r are at most their values at p_0: $\varepsilon_Z < 1$, so (H3) holds at p; $g_{\mathrm{root}}(p)p^{-1/2} \le g_{\mathrm{root}}(p_0)r_0 < 1$; and $B_{\mathrm{top}}(p) = (\varepsilon_\varrho/r)^2/(1 - g_{\mathrm{root}}r) \le B_{\mathrm{top}}(p_0)$. For the block, the operator is $\bigotimes_{p\in\mathcal{B}} \mathsf{A}_p$ with $-g_{\mathrm{root}}(p_0)p^{-1/2}I \preceq \mathsf{A}_p \preceq (1 + B_{\mathrm{top}}(p_0)/p)I$ at every $p \in \mathcal{B}$ (Theorem 14.27(c),(e)), and Lemma 13.7 gives the bound, since $\sum_{p\in\mathcal{B}} 1/p \le L_{\mathrm{u}}/(X\log X)$ (Lemma 4.18(c)).

For the general form, the first part of the proof, at $r = r_X \le r_0$, gives (H1) and (H2) at X with the same $s_\star, g_\star$, and (H3) at X is the hypothesis $\varepsilon_Z < 1$. The rest of the proof then runs with X in place of p_0 for the inputs: for $p \ge X$ the numbers of Lemma 27.6, with the coefficients $C_k(X)$ held fixed, are at most their values at X, so $g_{\mathrm{root}}(p) \le g_{\mathrm{root}}(X)$ and $B_{\mathrm{top}}(p) \le B_{\mathrm{top}}(X)$, and $\sum_{p\in\mathcal{B}} 1/p \le L'/(X\log X)$. This is the endpoint form of the construction: the exact Schur witness of an anchor stays valid at every larger point because every frame, star-Gram and cross error there is at most its anchor value; this holds by admissibility, and the certificate also checks it separately for each error (Certificate 27.9). □

Certificate 27.8 (Route B star-synthesis bounds and Schur tests at the root cuts)**.** For LL20 and LL22 and $p_0 = 2^b$, $b \in \{47, 48, 49, 50, 51, 52\}$:

(a) (S1) holds, with $e_{20}(p_0, 72) \le 0.3978$ at 2^{49} and $e_{20}(2^{47}, 72)$ about 0.796; all count errors of Lemmas 14.28 and 27.6 are below one; $\mathbf{R} \succeq 0$ by exact elimination.

(b) The frame numbers of Lemma 14.28, evaluated in outward rational arithmetic and rounded upward, and stored rationals $s_\star, g_\star$ satisfy (14.22). The records contain 32 such tests for $b \ge 49$ (two cuts, four endpoints, both prime classes, both signs; the class-one tests are those of the variant described after Lemma 14.28, and class three runs through its real 72×72 embedding) and 16 for $b \in \{47, 48\}$ (two cuts, two endpoints, both classes, both signs), each certified positive semidefinite by exact rational elimination at the value of $g_\star$ found for its sign; the left side of (14.22) is nondecreasing in $g_\star$, so both signs hold at the larger value, which is the $g_\star$ used. The class-three matrices are those of the construction; since their two signs are complex conjugate, one sign per cut and endpoint would suffice. At class three, $g_\star = 3193/32$ (LL20) and $18199639/500000$ (LL22) at 2^{47}, and $98128393/10^6$ and $35898609/10^6$ at 2^{48}.

The records are `research/roadmap_2026_10_18/item12/direct_star_spectral.json` (2^{49} to 2^{52}), with the model and geometry exports `direct_star_models_export.json` and `direct_star_geometry_export.json` of the same directory, produced by `direct_star_spectral.py`, and `direct_star_lower_endpoints.json` of the same directory (2^{47}, 2^{48}; four tests per row, status

`PASS_EXACT_DIRECT_STAR_ROOT`). The exact eliminations at $2^{49},\dots,2^{52}$ and the frame errors were repeated by an independent rational implementation, the four class-three tests at 2^{47} by `research/advance_2026_10_24/upper/lower_p47/native_review.json`, and the class-three tests at $2^{48},\dots,2^{51}$ again by `research/continuation_2026_10_24/upper_arithmetic/fractional_roots_review.json`. The class-three anchor rows ($g_\star$, frame errors, $M_\star$, $z_\star$, collision bound) for $2^{47},\dots,2^{52}$ are `direct_star_collision_v2.json` (same directory); the reviews `research/advance_2026_10_24/upper/review_lower_native47.py` and `research/continuation_2026_10_24/upper_arithmetic/review_roots.py` read them. The records also derive root caps from these tests with the star constants of both prime classes; the derived class-one caps are not used, since the bands below 2^{52} have no class-one primes. Floating-point computations only proposed the rationals $s_\star, g_\star$; the proof uses the exact tests.

Certificate 27.9 (Root caps below 2^{52})**.** For LL20 and LL22 and every band $[X,Y)$ below 2^{52}, the inputs of Lemma 27.6 are taken at the band's start X. The nonstar coefficients come from the complete histograms with the graded errors and special values, at the 19 anchors $2^{47},\dots,2^{52}$, $167\cdot 2^{40}$ and $j2^{42}$ for $j\in\{40,42,43,44,46,48,50,52,56,60,80,96\}$, and are interpolated log-convexly between them (Lemma 16.12). The scalar stars $M_\star$, $z_\star$, $Z_\star$, the ordered coincidence bound D_{col} and the rank-one and soft terms are evaluated at X; the frame witness $(s_\star,g_\star)$ is that of the band's anchor (Certificate 27.8): 2^{47} below 2^{48}, and 2^b on $[2^b,2^{b+1})$ for $b=48,\dots,51$, every frame, star-Gram and cross error at X being checked to be at most its anchor value. These give $\varepsilon_Z<1$, and g below is an upper bound for g_{root}, rounded upward to 10^{-8}, with $gX^{-1/2}<10^{-3}$. By Lemma 27.7 with $L'=L_{\mathrm{band}}$, the block caps are $d=g\exp(B_{\mathrm{top}}L_{\mathrm{band}}/(X\log X))$, rounded upward to 10^{-6}; they are the root entries d_{L} (LL20) and k_{LL} (LL22) of the band. Selected rows (the record holds all 301):

		LL20			LL22		
X	L_{band}	g	$B_{\mathrm{top}}<$	d_{L}	g	$B_{\mathrm{top}}<$	k_{LL}
P_3	559	12846.88753069	$2.52265\cdot10^8$	12847.187931	9381.04100550	$1.10946\cdot10^8$	9381.137478
2^{48}	566	6568.89967376	$8.59331\cdot10^7$	6568.933791	5072.00168224	$3.71450\cdot10^7$	5072.013069
2^{49}	578	2556.55783215	$2.30412\cdot10^7$	2556.559613	2070.82814422	$8.51920\cdot10^6$	2070.828678
2^{50}	555	1165.55594825	$9.20408\cdot10^6$	1165.556101	942.50581374	$2.71889\cdot10^6$	942.505851
2^{51}	566	612.94721642	$4.71877\cdot10^6$	612.947237	472.87513870	$1.13250\cdot10^6$	472.875143
$\frac{127}{64}2^{51}$	577	370.09363192	$2.81352\cdot10^6$	370.093636	262.47992391	$5.79331\cdot10^5$	262.479925

At 2^{52} and above the Route A caps $367,\dots,113$ and $261,\dots,45$ of Certificate 27.11 are used. The records are:

- the coefficient tables `research/further_2026_10_25/upper/special_coefficients/LL20_coefficients.json` and `LL22_coefficients.json` (19 anchors);
- their review `research/further_2026_10_25/upper/special_coefficients_review.json` (`PASS_INDEPENDENT_EVEN_COUNT_ROOT_COEFFICIENTS`, refinement `PASS_INDEPENDENT_FORCED_COINCIDENCE_ROOT_COEFFICIENTS`): the 30645955 histogram rows summed exactly, 1596 upward vertex factors, 114 directed coefficients, the 80 monotone-signature gates (16.6) (every step $s\to s+1$ up to $s+1=\Delta=20$) and 3024 per-signature maxima;
- the root assembly `research/advance_2026_10_24/upper/apply_even_roots.py` (the formulas of Theorem 14.27 with the interpolation), applied by `research/further_2026_10_25/upper/apply_special_roots.py` and written into the field `even_root_evidence` of `research/further_2026_10_25/upper/upper_dense_refined.json`;
- its review `research/further_2026_10_25/upper/special_upper_review.json` (`PASS_INDEPENDENT_EVEN_COUNT_ROOT_UPPER`: 602 fresh root and block caps, 1806 log-convex coefficients);

- the scalar root data at the band starts ($M_\star$, $z_\star$, $Z_\star$, D_{col}, the rank-one and soft terms, the frame errors): for the 45 bands below 2^{48} the field `roots` of the band record, computed with the anchor witness at 2^{47} by `research/advance_2026_10_24/upper/lower_prefix.py` and replayed by `review_lower_native47.py` of the same directory in the regenerated upper chain of `regenerate_upper_chain.py`, whose 51 independent reviews the record `research/advance_2026_10_24/verification/selected.json` lists; for the bands of $[2^{48}, 2^{52})$ `research/continuation_2026_10_24/upper_arithmetic/fractional_roots.json`, reviewed by `fractional_roots_review.json` of that directory (`PASS_INDEPENDENT_FRACTIONAL_ROOT_ASSEMBLY`: 16 exact anchor tests, 1536 frame dominations). The nonstar values stored with these scalar data come from earlier coefficient tables and are not used: the caps take N_{ns} from the coefficient tables of the first item.

Where the root cost lies. At P_3 the nonstar term $N_{\mathrm{ns}}/r \approx 12715$ is about 99% of $g(\mathrm{LL20})$, while the frame coefficient is $g_\star = 3193/32$, about 99.8, and the coincidence term D_{col}/r about 18.6: the root is expensive because of its many two-matchings, not because of its frame.
Route A from 2^{52}: the completed root. From 2^{52} on, the root caps come from the Route A criterion Theorem 14.21, whose coupling inputs integrate the internal neighbourhoods of the root through its completed conditional law (Section 14.5). The criterion accepts any admissible values of its perpendicular and coupling inputs; the root caps use the following ones (Remark 14.26(iii)).

Lemma 27.10 (Completed-root caps)**.** *Let the cut be LL20 or LL22, let p_0 be a real number with (S1) at p_0, and let $\mathfrak{c} \in \{1, 3\}$. At every prime $p \geq p_0$ with $p \equiv \mathfrak{c} \pmod 4$ the following are inputs of Theorem 14.21, with the admissibility that it requires.*

(i) Perpendicular inputs. *$N_{\mathrm{ns}} = \sum_{k=2}^{4} C_k(Q) r^k$ with the nonstar coefficients of Corollary 16.11(b) at a fixed $Q \leq p_0$ at which (S1) holds; $D_{\mathrm{col}} = \min(D_{\mathrm{ord}}, D_{\mathrm{nat}})$, a minimum as in Lemma 16.15(e); $M_\star$ the larger over the two classes of the bound of Proposition 15.26; and ε_Z the smaller of $2(P_{\mathrm{st}} + W_{\mathrm{st}}) + N_{\mathrm{ns}} + D_{\mathrm{nat}}$ and $Z_\star + N_{\mathrm{ns}} + D_{\mathrm{col}}$, with the standard star constants $P_{\mathrm{st}}, W_{\mathrm{st}}$ of Lemma 14.19 and $Z_\star$ twice the larger over the two classes of $z_\star$ (Lemma 15.29).*

(ii) Coupling inputs. *R_+ of Proposition 14.20*(c2)*, formed with this ε_Z; $\varepsilon_H = \min(\varepsilon'_H + \varepsilon_X, \varepsilon_H^{\mathrm{loc}})$ (Lemma 14.17); $\Gamma_E = \min(\hat{E}^2, \Gamma_E^{\mathrm{loc}})$ (Lemma 14.16), the relative completion variances being bounded by Lemma 14.15, whose two-variable bounds use the coefficient triples of Theorem 15.13 (Remark 15.16(ii)); and, for $\mathfrak{c} = 1$, $\bar{\zeta}$ the minimum of Lemma 14.32, with $\varepsilon_\varrho = R_Z(M_\star + N_{\mathrm{ns}} + D_{\mathrm{col}})$.*

Consequently, if at $r_0 = p_0^{-1/2}$ these inputs give $\varepsilon_Z < 1$, $\bar{\varepsilon}_V < 1$ and $\ell_k^{\mathrm{loc}} > 0$ for every $k \in U^+$, and the tests (14.21) *of the class $\mathfrak{c}$ hold for both signs, then the conclusions of Theorem 14.21 hold at every prime $p \geq p_0$ with $p \equiv \mathfrak{c} \pmod 4$; and if $g \geq g_{\mathfrak{c}}$ and $gr_0 < 1$, then for every block $\mathcal{B}$ of such primes, all at least $X \geq p_0$, with $\log q_{\mathcal{B}} \leq L_{\mathrm{u}}$, the normalized block root operator is at least $-d\, b_{\mathcal{B}}^{-1/2}$, with $d = g \exp(B_{\mathrm{top}} L_{\mathrm{u}}/(X \log X))$ and $B_{\mathrm{top}} = p_0 \varepsilon_\varrho (r_0)^2/(1 - gr_0)$.*

Proof. (i) N_{ns} bounds $\|T_{\mathrm{ns}}\|$ by Corollary 16.11(b), and D_{col}, D_{nat} bound $\|T_{\mathrm{coll}}\|$ by Lemma 16.15(e). By Lemma 15.29 with Proposition 15.26, for the signs of the class of p (the half law is a pinned half law, as in the proof of Lemma 27.6), $\|h_\star - \mathbb{E}_\mu h_\star\| \leq M_\star$ and $2|\mathbb{E}_\mu h_\star| \leq Z_\star$; the maxima over the classes cover both. With the standard star inputs (Proposition 14.20(d)) both forms of ε_Z, and so their minimum, are admissible in Proposition 14.20(c). (ii) These are admissible inputs of the coupling block Lemma 14.14, by the lemmas cited. Finally $N_{\mathrm{ns}}/r = \sum_k C_k(Q) r^{k-1}$ is a polynomial with nonnegative coefficients, D_{col}/r and D_{nat}/r are admissible by Lemma 16.15(e), $M_\star/r$ and $z_\star/r^2$ by Remark 15.27, and P_{st}/r^2, W_{st}/r are constants (Lemma 14.29(c)); so both forms of ε_Z, and their minimum, are admissible (Lemma 4.18(ii)). Γ_E, ε_H and $\bar{\zeta}$ are admissible by Lemma 14.29(a),(c),(d): the count errors $\epsilon_t^{\pm}$ of Section 14.5 are polynomials in r with nonnegative

coefficients and at most the coarse errors e_t, so the denominators of Lemma 14.15(b) are positive by (S1), and $\ell_k^{\mathrm{loc}} > 0$ at r_0. So Theorem 14.21 applies, and Corollary 14.22, with $P = X$, gives the block bound. □

Certificate 27.11 (Completed-root caps)**.** For LL20 and LL22 at every band start $p_0 = X$ from 2^{52} on, with the inputs of Lemma 27.10, the evaluator asserts $\varepsilon_Z < \frac{1}{2}$, $\bar{\varepsilon}_V < 1$, $\ell_k^{\mathrm{loc}} > 0$ for every k and $gr_0 < 1$; the matrix tests (14.21) hold for both prime classes and both signs (four exact rational tests per cut and threshold); and the recorded numbers give the block caps $d = g\exp(B_{\mathrm{top}}L_{\mathrm{u}}/(X\log X)) < g + 1.2\cdot 10^{-5}$:

X	g(LL20)	$\lceil d\rceil$	g(LL22)	$\lceil d\rceil$
2^{52}	366.42727119	367	260.84850454	261
2^{53}	249.23718696	250	157.02296656	158
2^{54}	186.71731661	187	101.54007395	102
2^{55}	157.47650343	158	76.79680847	77
2^{56}	140.27191683	141	63.35956145	64
2^{57}	130.28930487	131	56.17027922	57
2^{58}	123.91150306	124	51.86566728	52
2^{60}	117.04430410	118	47.52728217	48
2^{64}	112.38975257	113	44.77917248	45

At 2^{64} the recorded g is attained by the class-one tests. The Route A caps of LL20 and LL22 are 3458 and 1123 at 2^{50} and 723 and 543 at 2^{51}; they are larger than the class-three caps by Route B and are not used. The records are the grids `upper_low_grid.json` (2^{52}), `upper_low_odd_grid.json` (2^{53}), `upper_odd_grid.json` (2^{55}, 2^{57}) and `upper_augmented_grid.json` (2^{54}, 2^{56}, 2^{58}, 2^{60}, 2^{64}) in `research/advance_2026_10_17/sparse/`, produced by `build_upper_low_grid.py`, `build_upper_low_odd_grid.py`, `build_upper_odd_grid.py` and `build_augmented_grid.py` through `augmented_spectral_core.py` and `conditional_star_rank_spectral.py`; the band map `research/roadmap_2026_10_18/item15/UPPER_PROVIDER_COVERAGE.json` records which grid supplies each band and cut, and `replay_upper_psd.py` of the same directory recomputes the coefficients and repeats every positivity test by fraction-free elimination. The root entries of the nine bands from 2^{52} on, in `research/further_2026_10_25/upper/upper_dense_refined.json`, are these integers, and so are those of the tail datum (113 and 45, in `research/roadmap_2026_10_18/item12/upper_v2_local.json`).

27.3.1. *Cross-weighted root refinement and the final reference.* The final construction reduces the root bounds in the preceding 301 bands. The resulting smaller completion weights lower the comparison prices throughout those bands. The improvement uses the conditional-density cancellation in Theorem 16.9. It keeps their physical labelling laws, complete character partitions, frames and collision bounds. Put $\rho_t = (1 + e_t^-(36))/(1 - e_t^-(36))$. The LL20 half graph has 36 vertices and 168 edges; twelve vertices have internal degree 8 and twenty-four have internal degree 10. The LL22 half graph has 180 edges, and all 36 vertices have internal degree 10. Each integrated set has at most 34 vertices and sum of insertion degrees at most the number of internal edges.

Lemma 27.12 (Uniform extraction divisors)**.** *The common factor R_{full}^2 in the retained LL20 and LL22 nonstar certificates can respectively be divided by*

$$D_{20} = \rho_8^{23}\rho_{10}^{32}/\rho_0^{17}, \qquad D_{22} = \rho_{10}^{54}/\rho_0^{16}.$$

This division is valid also for the interpolated coefficient at each band start.

Proof. The graded error coefficients are nondecreasing and discretely convex in t. Since $x \mapsto 2\operatorname{arctanh} x$ is increasing and convex, the same holds for $\log\rho_t$. Pad the insertion-degree list by

zeros to length 34. Moving a unit from a smaller positive degree to a larger degree below 10 increases its product. Thus

$$\widetilde{R}_U \le H_{20} = \rho_{10}^{16}\rho_8\rho_0^{17}, \qquad \widetilde{R}_U \le H_{22} = \rho_{10}^{18}\rho_0^{16},$$

respectively. Divide the old R^2_{full} by these bounds. The factors ρ_0 are retained, since $e_0^-(36) = 36/p$. For interpolation the common factor M of Lemma 16.12 becomes M/D, with the same signature sum. Dividing both anchor coefficients by their respective D leaves their coefficient-to-common-factor ratios unchanged. Hence the new interpolant is exactly the old one divided by D at the evaluation point. The new common factor is a product of nonnegative powers of ρ_t and $(1-\hat{e})^{-1}$, so the original monotonicity and log-convexity proof still applies. □

For each cut and band start X, replace its retained nonstar input N by $N' = N/D(X)$ in the Route B negative coefficient formula:

$$g' = \frac{g_\star + (N' + D_{\mathrm{col}} + M_\star^2 + z_\star^2 + \varepsilon_{\mathrm{soft}})X^{1/2}}{1 - Z_\star - N' - D_{\mathrm{col}}}.$$

Here $Z_\star = 2z_\star$ is the scalar-star contribution to ε_Z, $M_\star^2 + z_\star^2$ is the rank-one correction, and $\varepsilon_{\mathrm{soft}}$ is the error from the two softened factors in Theorem 14.27. Their values, and the frame bound $g_\star$, are those specified in Certificate 27.9. Keep the old positive envelope and its block inflation. If g, d are the old rounded coefficients, a new block cap is $d' = 10^{-6}\lceil 10^6 d\, g'/g\rceil$. This is valid because the old ratio d/g is an upper bound for the unchanged inflation factor.

Let d_{new} be the larger of the new LL20 cap and the reserved LL10 cap; let k_{new} be the new LL22 cap. For the old completed weights define

$$h = 10^{-12}\left\lceil 10^{12}\max\left\{\frac{d_{\mathrm{new}}}{w_{\mathrm{LD}}}, \frac{d_{\mathrm{new}}}{w_{\mathrm{DL}}}, \frac{k_{\mathrm{new}}}{v_{\mathrm{LL}}}\right\}\right\rceil.$$

If $h \ge 1$ retain the old row. Otherwise keep $w'_{\mathrm{LL}} = 1$, put $w'_{\mathfrak{t}} = hw_{\mathfrak{t}}$ for every other ordinary state, and put $v'_{\mathfrak{t}} = hv_{\mathfrak{t}}$ for all reserved states. Use $d_{\mathrm{new}}, k_{\mathrm{new}}$ as the new root inputs. The two ordinary root inequalities and the LL reserved domination hold by the choice of h; each other inequality is homogeneous in the scaled variables. Thus all 105 completion and domination inequalities remain valid. The nonroot inputs and component marginals are unchanged.

All 301 bands below 2^{52} have $h < 1$; the first has $h = 981224812287/10^{12}$. The nine later bands and the far tail are retained. Every comparison polynomial scales by h, since the ordinary LL record is already accepting. Its full reference normalizer changes from P_μ to $1 + h(P_\mu - 1)$. The final prices use this new polynomial in Lemma 33.2, and the upper mean is recomputed at those same prices. The independent check `research/epsilon_2026_10_29/check_cross_weighted.py` rebuilds both half graphs, all 602 root bounds, the rational scales, all 32 550 completion inequalities and all 310 normalizers.

27.4. **The ten non-principal cuts.** The ten nonroot cuts give the other spectral inputs:

- d_{F}, d_{E} from the two-layer chamber cuts FL21, EL21;
- the one-layer inputs from LL10, FL11, EL11;
- the reserved caps from the layer cuts FL22, EL22, FF22, FE22, EE22.

Their caps are those of route A, Theorem 14.21, with the standard inputs (Remark 14.26(ii)): N_{ns} from Corollary 16.14 on a fresh compressed partition of the cut, $D_{\mathrm{col}} = D_{\mathrm{nat}}$, the natural coincidence bound of (16.8), the standard star inputs $M_\star = W_{\mathrm{st}}$ and $Z_\star = 2(P_{\mathrm{st}} + W_{\mathrm{st}})$ (Proposition 14.20(d)), and the coupling block Lemma 14.14 with R_+ of Proposition 14.20(c2), $\varepsilon_H = \varepsilon'_H + \varepsilon_X$, $\Gamma_E = \hat{E}^2$ and $\bar{\zeta} = \zeta$. This is formula (F6) below.

For the bands the test (14.21) is made for the class $p \equiv 3 \pmod 4$ only, both Fourier signs, each with its own rational $\tau^{\mathrm{sp}}_{3,\pm}$. The recorded cap is $g = \max_\pm(\bar{c}_\perp + \tau^{\mathrm{sp}}_{3,\pm})/(1 - \bar{\varepsilon}_V)$, rounded upward to 10^{-8} and not to an integer:

cut	$p_0 = 2^{47}$	$p_0 = 2^{48}$	cut	$p_0 = 2^{47}$	$p_0 = 2^{48}$
FL21	23.24980857	23.07315227	EL21	24.95017938	24.6357449
FL22	7.00914919	6.95874256	EL22	5.65086987	5.63739898
FF22	3.12377057	3.12364965	FE22	2.41440785	2.41435434
EE22	0.00002884	0.00002784	LL10	10.17351517	10.05779953
FL11	4.10415806	4.10396046	EL11	4.58309033	4.5829422

Evaluation points. The caps are evaluated at 2^{47} for the bands below 2^{48}, and at 2^{48} for every band from 2^{48} on, those from 2^{52} included. In each band they are inflated at its own start X and budget, $d = g\exp(B_{\mathrm{top}}L_{\mathrm{band}}/(X\log X))$, rounded upward to 10^{-6} (Certificate 27.13(a)): for instance $d = 23.249822$ for FL21 at P_3 and 23.073161 at 2^{48}, and the caps of the field pairs LF and LE are 7.00915 and 5.65087 at P_3.

Tail datum. The tail datum and the tail comparison use the caps of the four tests (two classes, two signs), at 2^{54} and 2^{64}, with integer block caps. At 2^{64} the class-one tests give the recorded g in every cut except FE22 and EE22, where the two classes tie (Certificate 27.13(d)).

Certificate 27.13 (Band inputs)**.** For every one of the 310 bands, the numerical inputs required by Definition 27.1—the caps of the ten non-principal cuts, the observed merge and constant caps, and the row quantities—have been evaluated at the band start and pass the exact tests named below. The items identify the data file and independent review for each kind of input.

(a) *Class-three nonroot caps.* The records are `research/advance_2026_10_24/upper/lower_p47/class3_caps.json` (2^{47}) and `research/advance_2026_10_24/upper/class3_nonroot_caps.json` (2^{48}). Each holds the ten cuts with g and B_{top} and 20 exact PSD tests (two signs per cut). They are produced by `class3_nonroot.py` and `extend_lower_caps47.py` of the same directory. The review `review_class3_nonroot.py` (records `class3_nonroot_review.json` and `lower_p47/class3_review.json`, both `PASS_INDEPENDENT_CLASS3_NONROOT_UPPER`) rebuilds the matrices from their definitions, checks both signs in the positive-semidefiniteness tests exactly, and recomputes every completion weight of Lemma 17.5 for the ten non-principal cuts.

(b) *Observed caps* (Proposition 27.14). The records are `observed_caps.json` and `lower_p47/observed_caps.json`; the reviews `observed_caps_review.json` and `lower_p47/observed_review.json`, both `PASS_INDEPENDENT_OBSERVED_MARGINAL_CAPS`, cover the actual projections of the eighteen half labels to the twelve and the six observed labels, 4 PSD tests and 2622 frame pairs. All these files are in `research/advance_2026_10_24/upper/`.

(c) *Rows.* The product-row quantities of (F3) and (F4) are evaluated at the band's start X with $\bar{n} = L_{\mathrm{band}}/\log X$ (field `field_primitives` of the band record), with the pinned extraction at the bucket `q` $= 0$ (`research/advance_2026_10_24/upper/conditional_fine_parts.json`) and, in the correlations, the full star matrices of Proposition 16.22 with the decay weights of its part (c), certified at $j2^{42}$ for even j from 32 to 62 and at 2^{48} (`conditional_fine_decay_lower_grid/` and `conditional_star_fine_p48.json` in `research/advance_2026_10_24/local/`, each reviewed, `PASS_INDEPENDENT_FINE_PINNED_CONDITIONAL_STAR_GATES`). The joint coefficients of (F5)(a) come from one of 17 joint-row endpoint tables, the largest not above X:

- eleven in `research/further_2026_10_25/upper/joint_endpoints/`: P_3, $j2^{42}$ for $j \in \{56, 60, 80, 96, 112\}$, 2^{49}, 2^{50}, 2^{51}, 2^{52}, 2^{60}, each with its review, indexed by `index.json`;
- six in `research/advance_2026_10_24/local/`: the grid tables `p42_42.json`, `p44_42.json`, `p46_42.json`, `p48_42.json`, `p50_42.json` of `conditional_joint_potential_grid/` ($j2^{42}$) and `conditional_joint_potential_p48.json` (2^{48}).

Every table has the status `PASS_CONDITIONAL_JOINT_ROW_PRIME_GATES` and an independent review `PASS_INDEPENDENT_CONDITIONAL_JOINT_PIN_POTENTIAL`. The reviews of the band records in `research/advance_2026_10_24/upper/` are `joint_rows_review.json`, `joint_pins_upper_review.json` and `joint_potential_upper_review.json` (`PASS_INDEPENDENT_JOINT_CONDITIONAL_UPPER_ADAPTER`), `conditional_fine_parts_review.json` (the bucket $\mathtt{q} = 0$), `fine_conditional_fields_review.json` and `signed_row_upper_review.json`. The allowances are reviewed by `research/further_2026_10_25/upper/allowances_review.json` and `endpoint_allowances_review.json` (`PASS_INDEPENDENT_UPPER_ALLOWANCE_REOPTIMIZATION`), which recompute f and M from the stored ℓ_{mid}, c_{mar}, $(k_{\mathrm{J}}, c_{\mathrm{F}})$, h and t_φ by the formula of (F5)(a) and authenticate every pair $(k_{\mathrm{J}}, c_{\mathrm{F}})$ against its table. At P_3, for instance, the ordinary inputs have $(M_\varphi, t_\varphi) =$ (1.012767561, 0.012758561), (1.023485, 0.023484) and (1.000058323, 0.000057323) for L, F, E.

(d) *Tail datum and tail caps (both classes).* The nonroot caps and the six Schur rows of the tail datum are those of the band $[2^{64}, 2^{512})$ of `research/frontier_2026_10_17/root/complete_c375_verification.json`, which `research/roadmap_2026_10_18/item12/upper_v2_local.json` keeps. The nonroot caps at 2^{54} and 2^{64} come from the grid `upper_augmented_grid.json` of `research/advance_2026_10_17/sparse/`, built by `build_augmented_grid.py` with four exact rational tests per cut and threshold (two classes, two signs), each of the form (14.21), and the gates of (F6). The replays `replay_upper_psd.py` and `replay_upper_local_inputs.py` of `research/roadmap_2026_10_18/item15/` recompute every cap, the six rows and their Schur coefficients (with $H < 103/100$ asserted for every row), and the integer inputs.

The searches only chose finite partitions and rational parameters; the proof uses the exact tests and the stated formulas.

27.5. **Merge and constant bounds for four states.** A merge component of the upper reference depends only on the labels in rows 0 and 1 of its E coordinate, and a constant component only on row 0 (Section 27.2.2). For the order at the reserved cut, only the compression of the layer-cut operator to functions of those labels (the *observed* labels) therefore matters, and Lemma 17.5(e) allows such a component a cap of its own. For the four states in question—denoted LR, RL (the merges) and LC, CL (the constants) in the band data, for the field pair LE—this compressed operator has a much smaller negative part than the full cut operator (whose cap is 5.65 at 2^{47}). The proposition gives the compressed caps.

Proposition 27.14 (Observed merge and constant caps)**.** *Let $p_0 \in \{2^{47}, 2^{48}\}$, let $p \geq p_0$ be a prime with $p \equiv 3 \pmod 4$, let A_p be the normalized cut operator of EL22 at p, and let Π_O be the conditional expectation onto the functions of the labels of a set O of half vertices. For the merge (O the twelve labels of the rows $0, 1$ of the* E *coordinate) and the constant (O the six labels of the row 0), the compression $\mathsf{A}_p^O = \Pi_O \mathsf{A}_p \Pi_O$ on the functions of x_O satisfies*

$$-g_{\mathrm{G}} r I \preceq \mathsf{A}_p^O \preceq (1 + B_{\mathrm{top}}/p) I, \qquad \textit{respectively} \qquad -g_{\mathrm{C}} r I \preceq \mathsf{A}_p^O \preceq (1 + B_{\mathrm{top}}/p) I,$$

where:

- $(g_{\mathrm{G}}, g_{\mathrm{C}}) = (2.268819655,\ 0.079761731)$ *and* $B_{\mathrm{top}} < 39236.89926$ *for* $p_0 = 2^{47}$*;*
- $(g_{\mathrm{G}}, g_{\mathrm{C}}) = (2.245405779,\ 0.056393528)$ *and* $B_{\mathrm{top}} < 39227.259341$ *for* $p_0 = 2^{48}$.

Here B_{top} is the positive constant of the full cut EL22. Hence for any L' and a block of such primes, all at least $X \geq p_0$, with $\log q_{\mathcal{B}} \leq L'$, the compressed block operators have negative coefficients at most $k_{\mathrm{G}} = g_{\mathrm{G}} e^{B_{\mathrm{top}} L'/(X \log X)}$ and $k_{\mathrm{C}} = g_{\mathrm{C}} e^{B_{\mathrm{top}} L'/(X \log X)}$. These may serve as the caps of the reserved states LR, RL and LC, CL in (27.2) *(Lemma 17.5(e)).*

Proof. Geometry. The plus half of the layer cut of $G_{\mathrm{EL},2}$ is the layer $\ell = 0$, with the eighteen labels (a, b), $a \in \{0, 1, 2\}$, $b \in \mathbb{Z}/6\mathbb{Z}$. The edges of the graph are $(a, b, \ell) \to (a', \mathsf{c}b, \ell')$ for $a \neq a'$ and all ℓ, ℓ' (Section 27.2.1: the E coordinate has no forward pair), so the signed cross-incidence matrix is

$$D_{(a,b),(a',b')} = \mathbf{1}_{a \neq a'}\big(\mathbf{1}_{b'=\mathsf{c}b} - \mathbf{1}_{b=\mathsf{c}b'}\big),$$

an antisymmetric matrix, and the internal and cross degrees are four. Every permutation of the rows a is an automorphism (Lemma 27.4(b)). The merge and the constant components are images of the field law under the emission (Section 27.2.2); at ι their cut forms are those of the field law on the functions of x_O, that is, of A_p^O. Their observed half marginals are marginals of the original half law, not uniform laws of smaller graphs.

Complete star operator. There are 162 distinct supports of size at least two in the cross neighbourhoods. With the actual features of Section 14.2 for the half law and the synthesis $\mathcal{F} = (V_j, (W_S/r)_S)$, Corollary 14.5 writes the doubly centred complete star part of the cut kernel as $r\mathcal{F}M(\hat{P})\mathcal{F}^*$, with

$$M(\hat{P}) = \begin{pmatrix} D \otimes \hat{P} & B_\star \\ B_\star^{\mathsf{T}} & 0 \end{pmatrix}, \qquad (B_\star)_{j,S} = (-1)^{|S|} \prod_{i \in S} D_{ji};$$

since D is antisymmetric, the right-star coefficients $\prod_{i \in S} D_{ij}$ are the entries of $B_\star^{\mathsf{T}}$. The higher-star constant couplings vanish only under the double centring; they stay in the constant-coupling bound below.

Schur tests. By Remark 14.7, $\|M(\pm i)\| \le k$ follows from the rational tests $k^2 I - B_\star B_\star^{\mathsf{T}} \pm ikD \succeq 0$. They hold with $k = 6$ for the full matrix, and with $k = 2.188902$ for the principal matrix of the features of the merge's observed labels (twelve singletons and sixty supports; on the singleton rows $B_\star B_\star^{\mathsf{T}} = I$). The principal matrix of the features of the constant's observed labels is zero. All these are finite rational matrix inequalities, checked by exact LDL^{T} elimination.

Visible and hidden features. Call a feature *visible* if it is a function of x_O (a singleton of O, or a support contained in O, also when its principal coefficients vanish) and *hidden* otherwise: the merge has 72 visible and 108 hidden features, the constant 12 and 168. Let $\mathcal{F}_{\mathrm{v}}$, $\mathcal{F}_{\mathrm{h}}$ be the two parts of the synthesis, so that $\Pi_O \mathcal{F}_{\mathrm{v}} = \mathcal{F}_{\mathrm{v}}$. By Lemma 14.4(f), $\|\mathcal{F}_{\mathrm{v}}\|^2 \le \Gamma$ for the actual frame bound Γ. By Lemma 14.6, with the best unobserved pivot of each hidden feature and $D_j \le 2^4/(1 - e_4(p, 18))$ (Lemma 12.2), each hidden feature has a bound for $\|\Pi_O \mathcal{F}_\alpha\|$; let e_{hid}^2 be the sum of their squares, so that $\|\Pi_O \mathcal{F}_{\mathrm{h}}\| \le e_{\mathrm{hid}}$ by Cauchy–Schwarz, no orthogonality of the hidden features being used. Splitting M into its visible principal block and the three other blocks, each of norm at most 6, the compression of the centred star operator has norm at most

$$r\{\Gamma k + 6(2\sqrt{\Gamma}\, e_{\mathrm{hid}} + e_{\mathrm{hid}}^2)\}.$$

Lower bound. Let N_{ns}, D_{col}, R_Z and ε_ϱ be the nonstar, coincidence, normalizer and constant-coupling bounds of the full cut EL22 at p ((F6)). The compression Π_O contracts the nonstar and coincidence parts and the constant-coupling vector, and keeps the scalar corner $\langle \mathbf{1}, \mathsf{A}_p \mathbf{1} \rangle = 1$. So $A = \mathsf{A}_p^O$ has $\|P_0 A \mathbf{1}\| \le \varepsilon_\varrho$ and $P_0 A P_0 \succeq -cP_0$ with

$$c = R_Z\Big[r\{\Gamma k + 6(2\sqrt{\Gamma}\, e_{\mathrm{hid}} + e_{\mathrm{hid}}^2)\} + N_{\mathrm{ns}} + D_{\mathrm{col}}\Big],$$

and Proposition 13.6(a′) gives $A \succeq -(c + \varepsilon_\varrho^2/(1 + c))I \succeq -(c + \varepsilon_\varrho^2)I$. Outward evaluation of $(c + \varepsilon_\varrho^2)/r$ at p_0, rounded upward, gives the stated g_{G} and g_{C}. The upper bound is the compression of the positive envelope $\mathsf{A}_p \preceq (1 + B_{\mathrm{top}}/p)I$ of the full cut (Theorem 14.21(d), with the class-three caps of Certificate 27.13(a)).

All primes and blocks. Every error bound above, after division by its indicated power of r, is admissible (Lemma 4.18): the extension denominators increase with p, and Γ, e_{hid}, N_{ns}/r, D_{col}/r and ε_ϱ^2/r decrease, while the rational matrix caps 6 and 2.188902 do not depend on p; so

the bounds hold at every $p \ge p_0$ with the values at p_0. The block statement is Lemma 13.7(b), applied to the compression to the functions of the labels of O, with $\sum_{p\in\mathcal{B}} 1/p \le L'/(X \log X)$ (Lemma 13.7(c)). □

The four caps enter only the reserved dominations (27.2), and through them the reserved weights and the deletion polynomial of the comparison (Definition 28.13). They depend on the state, not on the placement of the merge or the constant in the slots, and the weights of Definition 27.1(d) are symmetric in the two type coordinates, so the reference $\mu_{\mathcal{B}}$ remains unchanged when its root (principal) law is replaced by the swapped one (interchanging the two type coordinates). The block caps are rounded upward to 10^{-6} with the band's own budget: $k_{\mathrm{G}} = 2.26882$ and $k_{\mathrm{C}} = 0.079762$ at P_3, and 2.245406 and 0.056394 from 2^{48} on (Certificate 27.13(b)).

27.6. **Conditional laws given the middle label.** The five-type completion is applied one chamber coordinate at a time. Fix such a coordinate; its three objects are called the rows $0, 1, 2$. The completion needs, after conditioning on all labels of the middle row 1, bounds for the conditional laws of the two outer rows under the fold pattern (F) and under the matching pattern (E), measured against one common product reference (Lemma 17.5(b),(c),(f)); in the upper bands the two rows are moreover bounded jointly, by one Gram matrix (Section 27.6.4). In detail: fix a chamber coordinate c, a field type φ of the other coordinate and a layer count j. The rows are the objects $0, 1, 2$ of the coordinate c in the field laws $\lambda_{\mathcal{B}}(G_{\mathrm{F}\varphi,j})$ and $\lambda_{\mathcal{B}}(G_{\mathrm{E}\varphi,j})$. Condition on the z distinct labels of the middle row 1. By Lemma 27.4(d), in both laws the two outer rows have n labels, no internal edges, and exactly the same signed conditions (pins) towards the middle row; they differ only in the edges between the outer rows themselves. The six geometries are

(φ, j)	(L, 1)	(L, 2)	(F, 1)	(F, 2)	(E, 1)	(E, 2)
$n = z$	6	12	3	6	3	6
ϖ = maximal pin degree	2	4	1	2	0	0

These are the pinned rows of Section 16.5, with pin bound ϖ and z fixed labels, grouped with the bucket $\mathfrak{q} = 0$, so that the integer free-degree costs $\mathrm{oc}(\mathcal{C})$ enter exactly (Remark 16.26). Throughout this subsection $r = p^{-1/2}$ for one odd prime p, and the symbols introduced here are local to it; block quantities are formed in (F5).

27.6.1. *The product reference and the star bounds.* Let $e_t = e_t^-(z)$, the graded error of (12.2), be the relative error of a strict t-pin count with z excluded labels; assume $e_\varpi < \frac{1}{2}$, and put

$$\kappa = \frac{2^\varpi}{1 - e_\varpi}, \qquad I_0 = 1 - \binom{n}{2}\kappa r^2.$$

An outer label must avoid the middle labels and satisfy its signed pins; its *allowed set* has at least $p2^{-\varpi}(1 - e_\varpi)$ elements (Lemma 12.2(c)), so the uniform law on it has density at most κ relative to the uniform law of $\mathbb{F}_p$. Replacing a smaller actual pin degree by ϖ enlarges the density, insertion, denominator and collision bounds of the pinned extraction below and imposes no edge; the certificates keep the actual pin dictionaries and check their equality, both orientations, and the absence of outer-row edges. Let λ be the product of the allowed-set laws of the n labels of a row and μ the law λ conditioned on the row being injective (the *actual row law*); by Lemma 13.23(d) the injection probability is at least I_0. For $k \in \{\mathrm{F}, \mathrm{E}\}$ let K_k be the raw cross kernel between the outer rows, relative to $\mu \otimes \mu$: the indicator that the two rows are disjoint times the product over the cross pairs (i, i') of $1 + \chi(\varsigma_{ii'}(x'_{i'} - x_i))$. The conditional law of the outer rows under the k law is $K_k\, \mu \otimes \mu$ normalized; put $Z_k = \langle \mathbf{1}, K_k \mathbf{1} \rangle$ and $\varrho_k = K_k \mathbf{1}/Z_k$, the normalized marginal density.

One-character means. For a fixed output x outside the middle labels and an outer label with allowed law $\mathcal{A}$, the mean $\mathbb{E}_{\mathcal{A}}\chi(x-y)$ has modulus at most

$$\beta_\chi := \frac{\sum_{s=1}^{\varpi}\binom{\varpi}{s}(s-\mathbf{1}_{s\ \mathrm{odd}})r + 2^{\varpi}(z+1)r^2}{1-e_\varpi}.$$

Indeed, expand $2^{\varpi}\mathbf{1}_{\mathcal{A}}$ into pin monomials: a monomial with $s\ge 1$ pins gives a character sum with the $s+1$ distinct roots x and its pins, of modulus at most $s\sqrt{p}$ for even s and $(s-1)\sqrt{p}+1$ for odd s (Theorem 4.7); the monomial without pins gives zero, the constants 1 of the odd s add up to $2^{\varpi-1}$, and deleting the z excluded labels costs at most $z2^{\varpi}$; divide by $2^{\varpi}|\mathcal{A}|\ge p(1-e_\varpi)$. (For $\varpi=4$, $z=12$ this is $\beta_\chi=(24r+208r^2)/(1-e_4)$.)

Stars. Expand each character $\chi(\varsigma(x'-y))$ of the kernel into its mean and its centred part. Restricting the Paley operator to two allowed sets has norm at most κr, centring an input is contractive, products on distinct input coordinates have Gram kernels given by Schur products whose extra centred factors have diagonal at most one, and exact input supports are orthogonal. Consequently, after its singleton edges are removed, a star of degree d has input-constant part at most E_d and nonconstant part of norm at most $\kappa r\sqrt{\mathsf{T}_d}$, where

$$\mathsf{E}_d=(1+\beta_\chi)^d-1-d\beta_\chi,\qquad \mathsf{T}_d=\sum_{s=1}^{d}\binom{d}{s}\big((1+\beta_\chi)^{d-s}-\mathbf{1}_{s=1}\big)^2$$

(for a singleton support, removing the singleton edge subtracts one from its coefficient). Let (d_i) and $(d'_{i'})$ be the two lists of cross degrees and $n_k^{\times}$ the number of cross edges of K_k. The full star part of K_k has norm at most

$$\mathsf{C}_k=n_k^{\times}\kappa r+\sum_i\big(\kappa r\sqrt{\mathsf{T}_{d_i}}+\mathsf{E}_{d_i}\big)+\sum_{i'}\big(\kappa r\sqrt{\mathsf{T}_{d'_{i'}}}+\mathsf{E}_{d'_{i'}}\big),$$

and its centred constant image is at most the larger, over the two orientations, of

$$\mathsf{M}_k^{\mathrm{st}}=\kappa r\Big(\sum_i d_i^2\Big)^{1/2}+\sum_i\mathsf{E}_{d_i}+\sum_{i'}\kappa r\sqrt{\mathsf{T}_{d'_{i'}}};$$

for the singleton part, the centred output coordinates are orthogonal. The single edges have means at most $\mathsf{J}_\varpi r^2$, by (PR2) below.

Injection. Let N_k^{ns} and D_k^{col} be the nonstar and cross-coincidence norms of K_k on $L^2(\mu)$: N_k^{ns} is the bound N^{pin} of Proposition 16.20 for the compressed partition, and D_k^{col} the bound of Proposition 16.21, whose two orientations agree because in all twelve row geometries the row and column degree multisets coincide (Remark 16.26). By Lemma 13.23(c), applied with the full star bound C_k (a strict-centred test extends to a product-centred test),

$$|Z_k-1|\le\varepsilon_k:=\frac{n_k^{\times}\mathsf{J}_\varpi r^2+\sum\mathsf{E}+\mathsf{C}_k(2\sqrt{1-I_0}+1-I_0)}{I_0}+N_k^{\mathrm{ns}}+D_k^{\mathrm{col}},$$

$$\|P_0K_k\mathbf{1}\|\le\mathsf{M}_k:=\frac{\mathsf{M}_k^{\mathrm{st}}+\mathsf{C}_k\sqrt{1-I_0}}{I_0}+N_k^{\mathrm{ns}}+D_k^{\mathrm{col}},$$

where $\sum\mathsf{E}$ is the sum of E_d over both degree lists. Certificate 27.13(c) asserts $\varepsilon_k<\frac12$.

27.6.2. *Differences of conditional laws given the middle label.* The completion needs the difference of the two conditional laws, not each law separately: the middle density is a ratio of the normalizers Z_{F}, Z_{E}, and the fold marginal densities are ratios $\varrho_{\mathrm{F}}/\varrho_{\mathrm{E}}$. The common product reference makes this difference small. The estimates below hold at every odd prime; the certificate evaluates them for both prime classes and takes the larger value.

Complete conditional means. For distinct-root sets S, T of size at most four,

$$\Big|\sum_{x,y}\chi\Big((x-y)\prod_{s\in S}(x-s)\prod_{s'\in T}(y-s')\Big)\Big|\le 40p. \tag{PR1}$$

Empty root sets give zero, and for $|S|=|T|=1$ the two-root identity (4.2) gives modulus at most p. Otherwise this is Theorem 15.13 (in its symmetric form if $|S|=1$), since $\mathcal{B}(m,m',s)\le \mathcal{B}(4,4,0)=28+8r+4r^2<40$ for $2\le m\le 4$, $1\le m'\le 4$ (Remark 15.16(iii)). Expanding the two allowed-set densities and deleting their z common excluded labels gives, for independent outer labels X, Y with allowed laws (including $\varpi=0$),

$$|\mathbb{E}\chi(X-Y)|\le \mathsf{J}_\varpi r^2,\qquad \mathsf{J}_\varpi=\frac{40(2^\varpi-1)^2+2z4^\varpi}{(1-e_\varpi)^2}. \tag{PR2}$$

In the $(\mathrm{L},2)$ geometry the signed $\mathrm{F}-\mathrm{E}$ singleton matrix $\mathbf{D}$ has absolute entry sum $\|\mathbf{D}\|_1=72$ in both prime classes, so its full singleton mean is at most $72\mathsf{J}_4r^2=1090368r^2/(1-e_4)^2$. All higher-star means, nonstar remainders and cross-injection corrections are retained. Thus *the conditional normalizer difference has order r^2, not r.*

Exact output supports. Write $M_\mathrm{F}, M_\mathrm{E}$ for the signed cross matrices, with zero for an absent edge. For a centre j and a set of leaves S put

$$c_j(S)=\prod_{i\in S}M_\mathrm{F}(i,j)-\prod_{i\in S}M_\mathrm{E}(i,j),\qquad b_j(T)=\sum_{\substack{S\supseteq T\\ |S|\ge 2}}|c_j(S)|\,\beta_\chi^{|S|-|T|}.$$

Expand each character into its allowed-law mean and centred part. Different nonempty exact leaf supports T are orthogonal in the product reference; one must first sum the contributions of all centres on the same support, and only then square. For two distinct centre values y, w outside the middle set, the one-variable character bound gives

$$|\mathbb{E}\phi_y(X)\phi_w(X)|\le \nu_\chi:=q_\varpi+\beta_\chi^2,\qquad q_\varpi=\frac{\sum_{s=1}^{\varpi}\binom{\varpi}{s}(s+\mathbf{1}_{s\text{ odd}})r+2^\varpi(z+1)r^2}{1-e_\varpi}, \tag{PR3}$$

where $\phi_y(X)=\chi(X-y)-\mathbb{E}\chi(X-y)$; on the diagonal the variance is at most one. For s independent leaf coordinates and a centre coefficient of modulus at most $\bar\alpha$, expanding the squared norm over two centre values bounds it by $\bar\alpha^2(\kappa r^2+\nu_\chi^s)$: the diagonal probability is at most κ/p, and off the diagonal the s covariance bounds multiply. For $s\le 2$ the Paley bound $\bar\alpha\kappa r$ is stronger. So the higher-star marginal, centred at the opposite (input) leaves rather than at the centre, is at most

$$\Big[\sum_{T\ne\emptyset}\Big(\sum_j b_j(T)\Big)^2\mathsf{B}_{|T|}^2\Big]^{1/2},\qquad \mathsf{B}_s^2=\begin{cases}\kappa^2r^2, & s\le 2,\\ \kappa r^2+\nu_\chi^s, & s\ge 3,\end{cases} \tag{PR4}$$

and the output-centred orientation contributes its empty-input support, $\sum_j\sum_{|S|\ge 2}|c_j(S)|\beta_\chi^{|S|}$. These are constant-image bounds; no improved full-operator bound is asserted. In the five geometries other than $(\mathrm{L},2)$ the singleton part of the difference is bounded as the singleton part of $\mathsf{M}_k^{\mathrm{st}}$, with $\sum_{i'}|M_\mathrm{F}(i,i')-M_\mathrm{E}(i,i')|$ in place of d_i.

The six-density Gram matrix. For $(\mathrm{L},2)$ the two layer copies of a centre have the same four-pin allowed law. Index the six distinct densities by $u\in\mathbb{Z}/6\mathbb{Z}$, the object of the other coordinate, write ρ_u for the density of the allowed set $\mathcal{A}_u$ relative to the uniform law and $\psi_u=\rho_u-\mathbf{1}$. For class one, two distinct allowed sets of the same parity have six compatible pins and opposite-parity sets eight disjoint pins; for class three, distinct same-parity sets are disjoint, since one common pin has

opposed signs. The leading density Gram matrix $\mathcal{H}$ has the entries

$$\mathcal{H}_{uu} = 16, \qquad \mathcal{H}_{uu'} = \begin{cases} 4, & u \neq u',\ u \equiv u' \pmod 2,\ p \equiv 1 \pmod 4, \\ 0, & u \neq u',\ u \equiv u' \pmod 2,\ p \equiv 3 \pmod 4, \\ 1, & u \not\equiv u' \pmod 2, \end{cases}$$

with absolute entry errors at most

$$\eta_{\mathrm{diag}} = \frac{16e_4}{1-e_4}, \quad \eta_{\mathrm{same}} = \frac{4(e_6+2e_4+e_4^2)}{(1-e_4)^2}, \quad \eta_{\mathrm{cross}} = \frac{e_8+2e_4+e_4^2}{(1-e_4)^2}; \tag{PR5}$$

the same-parity class-three error is zero. The Gram matrix of the ψ_u is $\mathcal{G} = \mathcal{H} - \mathbf{1}\mathbf{1}^{\mathsf{T}}$ with the same errors $\eta_{uu'}$. For the singleton marginal, collapse the twin layer coefficients in each output row to a vector $\mathbf{c}_i \in \mathbb{R}^6$. The Paley operator annihilates constants and has norm r; restricting its output to an allowed set costs $\sqrt{\kappa}$, and distinct centred output coordinates are orthogonal. Hence the squared singleton marginal is at most

$$\kappa r^2 \sum_i \Big[\mathbf{c}_i^{\mathsf{T}} \mathcal{G} \mathbf{c}_i + \sum_{u,u'} |c_{iu} c_{iu'}| \eta_{uu'} \Big]. \tag{PR6}$$

The exact leading quadratic sums are 2448 and 2064 in the two prime classes; the certificate checks both matrix orientations. One may also bound the singleton *mean* by $r\,\mathrm{tr}(|\mathbf{D}|\mathcal{G})$: in class one the collapsed symmetric matrix has first row $(0,4,-2,0,-2,4)$ and absolute-value first row $(16/3,-2/3,4/3,-8/3,4/3,-2/3)$; in class three its Hermitian absolute value is $2\sqrt{3}$ times the projection onto the parity-wise mean-zero vectors. These give the leading constants 528 and $128\sqrt{3}$. The stronger complete mean (PR2) is the one used for the normalizer.

Weighted moving-root moments. For a set S of two to five distinct field elements let τ_S be the moving-root trace (15.8), and let χ_A be a squarefree character weight with at most four roots. By Corollary 15.20, with arbitrary overlaps of the three root sets,

$$\Big| \mathbb{E}_x \chi_A(x) \tau_S(x) \tau_{S'}(x) - \frac{\mathbf{1}_{S=S',\ A=\emptyset}}{p} \Big| \leq 512 p^{-3/2}, \tag{PR7}$$

and, by Corollary 15.21, for an allowed law with at most four pins and density normalization γ, and distinct root sets S,

$$\Big\| \sum_S \alpha_S \tau_S \Big\|_{\mathrm{allowed}} \leq \sqrt{\gamma}\, r \Big[\sum_S \alpha_S^2 + 8192 r \Big(\sum_S |\alpha_S| \Big)^2 \Big]^{1/2}. \tag{PR8}$$

The input is the local monodromy of the moving-root sheaves at their finite roots (Katz–Sarnak, Remark 10.1.17, Lemma 15.14): distinct root sets give sheaves that are not isomorphic after any rank-one twist.

Signed grouping of the principal supports. For each exact support T separate its coefficient with no complementary leaf means, $\alpha_j = \prod_{i\in T} M_{\mathrm{F}}(i,j) - \prod_{i\in T} M_{\mathrm{E}}(i,j)$; all terms containing complementary means keep their majorants from (PR4), and no cancellation of unknown means is assumed. For $|T| \geq 3$ put $\Psi = \sum_j \alpha_j \rho_j$, with ρ_j the density of the allowed law of the centre j. The squared norm of its principal term is exactly $\mathbb{E}_{y,w} \Psi(y)\Psi(w) \prod_{i\in T} \mathbb{E}\phi_y(X_i)\phi_w(X_i)$, and the diagonal/off-diagonal split with (PR3) bounds it by

$$r^2 \|\Psi\|_2^2 + \nu_\chi^{|T|} \Big(\sum_j |\alpha_j| \Big)^2; \tag{PR9}$$

after the twin centres are collapsed into a coefficient vector $\mathbf{c} \in \mathbb{R}^6$, (PR5) bounds $\|\Psi\|_2^2$ by $\mathbf{c}^{\mathsf{T}}\mathcal{H}\mathbf{c} + \sum |c_u c_{u'}| \eta_{uu'}$. For a pair support first replace every centre normalization by one; since $|1-1/\gamma_j| \leq e_4$ for the density normalization γ_j of the allowed law of the centre j (Corollary 15.21), the error is at most $e_4 \kappa r \sum_j |\alpha_j|$. Group identical nonempty pin subsets by $c_S = \sum_{j: S \subseteq \mathrm{pins}(j)} \alpha_j \prod_{s\in S} \mathrm{sign}(j,s)$.

Fixing the first leaf makes the moving trace root set $S \cup \{x\}$, whose roots are distinct because leaves avoid every middle label; (PR8) bounds the nonconstant principal pair term by

$$\sqrt{\gamma_0}\, r\big[\mathfrak{a}_T + 8192r\, \mathfrak{b}_T\big]^{1/2}, \qquad \mathfrak{a}_T = \sum_S c_S^2, \quad \mathfrak{b}_T = \Big(\sum_S |c_S|\Big)^2, \quad \gamma_0 = (1-e_4)^{-1}. \tag{PR10}$$

Centring both leaves contracts this bound. Deleted centre labels cost at most $12 \cdot 16r^2 \sum_j |\alpha_j|$. The empty pin monomial is $\mathbf{1}_{x=y} - 1/p$; after both leaf centrings and restriction to injective output tuples its remaining absolute value is at most $3\kappa r^2 |\sum_j \alpha_j|$, a remainder charged by the triangle inequality *after* restriction, not an unrestricted product-space norm estimate. For singleton supports group the pin subsets of size two to four in the same way, with the coefficients of $\mathbf{D}$; put $\mathfrak{a} = \sum_{i,S} c_{iS}^2$, $\mathfrak{b} = \sum_i (\sum_S |c_{iS}|)^2$ and $U = \sum_i (\sum_j |\mathbf{D}_{ij}|)^2 = 432$. The principal singleton norm is at most $\sqrt{\gamma_0}\, r\sqrt{\mathfrak{a} + 8192r\mathfrak{b}}$, and the centre-normalization and deleted-label errors are at most

$$\frac{e_4}{1-e_4}\sqrt{U\gamma_0}\, r\sqrt{11 + 8192 \cdot 121r}, \qquad \gamma_0\, 12 \cdot 16\sqrt{U}\, r^2. \tag{PR11}$$

One-pin monomials are constant on the leaf allowed set and disappear under centring. The direct singleton terms and the size-one components of higher stars lie in the singleton subspace, which is orthogonal to all supports of size at least two: sum their squared bounds, take one square root, then add the pointwise pair-diagonal and scalar empty-support errors. The exact pair-support totals $(\sum_T \mathfrak{a}_T, \sum_T \mathfrak{b}_T)$ are $(6984, 221400)$ and $(6792, 213144)$ in the two prime classes, and the leading squared higher-support Gram totals are 18048 and 18240; both orientations are enumerated.

Strict injection, normalization and the differences. By Lemma 13.23(c), with the full star bounds $\mathsf{C}_\mathrm{F}, \mathsf{C}_\mathrm{E}$ and the nonstar and coincidence remainders, a product mean-difference bound Δ_0 and a restricted-output centred marginal bound M_0 (from (PR4)–(PR11)) give

$$\begin{aligned} |Z_\mathrm{F} - Z_\mathrm{E}| &\le \Delta := \frac{\Delta_0 + (\mathsf{C}_\mathrm{F} + \mathsf{C}_\mathrm{E})(2\sqrt{1-I_0} + 1 - I_0)}{I_0} + \mathsf{R}, \\ \|P_0(K_\mathrm{F} - K_\mathrm{E})\mathbf{1}\| &\le \mathsf{M}_\Delta := \frac{\mathsf{M}_0 + (\mathsf{C}_\mathrm{F} + \mathsf{C}_\mathrm{E})\sqrt{1-I_0}}{I_0} + \mathsf{R}, \end{aligned} \tag{PR12}$$

with $\mathsf{R} = N_\mathrm{F}^\mathrm{ns} + D_\mathrm{F}^\mathrm{col} + N_\mathrm{E}^\mathrm{ns} + D_\mathrm{E}^\mathrm{col}$. Here $\Delta_0 = \|\mathbf{D}\|_1\, \mathsf{J}_\varpi r^2 + \mathsf{E}_*$, with $\|\mathbf{D}\|_1$ the absolute entry sum of the signed singleton difference (72 for $(\mathrm{L}, 2)$) and $\mathsf{E}_* = \sum_j \sum_{|S|\ge 2} |c_j(S)| \beta_\chi^{|S|}$ summed over both orientations: the cancellation in the difference is used before the triangle inequality. Restriction contracts the orthogonal principal sum, and the pair-diagonal remainder has already been bounded on the restriction. By Lemma 13.19(c),

$$\|\varrho_\mathrm{F} - \varrho_\mathrm{E}\|_2 \le \frac{\mathsf{M}_\Delta + \Delta\, \mathsf{M}_\mathrm{E}/(1-\varepsilon_\mathrm{E})}{1-\varepsilon_\mathrm{F}}. \tag{PR13}$$

If $\varrho_\mathrm{E} \ge e^{-\ell_\mathrm{E}}$, where $\ell_\mathrm{E} = \ell_1 + \ell_2 + \ell_3$ with ℓ_1 the conditional completion lower error (the logarithmic deficit in the lower bound for the conditional completion density), ℓ_2 the half lower error (at most $-n\log(1-e_\varpi) - \log I_0$), and ℓ_3 the full-given-middle upper error (the logarithmic excess in the upper bound for the full density given the middle row), multiplication by $e^{\ell_\mathrm{E}/2}$ turns (PR13) into the relative chi-square bound $\|\varrho_\mathrm{F}/\varrho_\mathrm{E} - \mathbf{1}\|_{L^2(\varrho_\mathrm{E}\mu)} \le c_\mathrm{mar} r$; this defines the *marginal coefficient* c_mar. Both conditional normalizer ratios are at most $1 + \Delta/(1 - \max(\varepsilon_\mathrm{F}, \varepsilon_\mathrm{E}))$, and the global normalizer is an average under the common middle reference; hence the *middle log ratio*

$$\ell_\mathrm{mid} = \frac{2\Delta}{1 - \max(\varepsilon_\mathrm{F}, \varepsilon_\mathrm{E})}$$

bounds both middle-density log ratios. This linear majorant, rather than a pointwise logarithmic formula, has the required monotonicity: every coefficient above, after its indicated power of r is removed, is admissible (Lemma 4.18). Indeed each is built by sums, products, maxima,

minima, square roots, exponentials and the maps $y \mapsto 1/(1-y)$ and $y \mapsto -\log(1-y)$ from fixed integers, r and the admissible functions e_t, κ, β_χ/r, q_ϖ/r, T_d, E_d/r^2, $\sqrt{1-I_0}/r$, N_k^{ns}/r^2 and D_k^{col}/r^2 (Propositions 16.20 and 16.21) and the count logarithms in ℓ_{E}; so ε_k/r, M_k/r, Δ/r^2, M_Δ/r, ℓ_{mid}/r^2 and c_{mar} are admissible. For example $\mathsf{B}_s/r = (\kappa + (\nu_\chi/r)^s r^{s-2})^{1/2}$ for $s \geq 3$.

27.6.3. *Exact-support correlations.* The fold row also needs the correlation ρ_{F} of the conditional outer rows in their own marginals, and the matching row the correlation ρ_{E}. Decompose the mean-zero product space as $\mathcal{V} \oplus \mathcal{W}$, where $\mathcal{V}$ is the orthogonal sum of the single-coordinate spaces and $\mathcal{W}$ consists of the supports of size at least two. For a left centre i write N_i for its neighbours. Summing all higher stars with exact leaf support T gives an operator of norm at most $\kappa r(1+\beta_\chi)^{|N_i|-|T|}$ for $|T| \geq 2$, and for a singleton leaf the higher-star bound is $\kappa r((1+\beta_\chi)^{|N_i|-1}-1)$: retain one centred Paley factor and use the independent remaining leaves as positive Gram factors with diagonal at most one; complementary leaf means are bounded multipliers on the centre coordinate, and projecting the centre to mean zero contracts the norm. Let d be the sum of the scalar matrix bounds of the single-edge adjacency matrix and of the two singleton-support higher-star matrices, each bounded by the square root of its maximal row sum times its maximal column sum. For the higher-support block put

$$\mathsf{G}_{ii'} = \sum_{s=2}^{|N_i \cap N_{i'}|} \binom{|N_i \cap N_{i'}|}{s} (1+\beta_\chi)^{|N_i|+|N_{i'}|-2s}, \qquad \mathsf{a} = \Big(\max_i \sum_{i'} \mathsf{G}_{ii'}\Big)^{1/2},$$

and b likewise with the transposed neighbourhoods. Orthogonality of the centre coordinates and of the leaf supports bounds the two off-diagonal blocks by $\kappa r\mathsf{a}$ and $\kappa r\mathsf{b}$, and empty supports vanish after global centring. So the full centred star operator has norm at most

$$C_\star = \frac{\kappa r}{2}\Big(\mathsf{d} + \sqrt{\mathsf{d}^2 + 4\max(\mathsf{a},\mathsf{b})^2}\Big). \tag{27.5}$$

No sign or symmetry assumption is needed for this scalar block majorant. For the largest E row, $\mathsf{d} \to 4$ and $\mathsf{a}^2 = \mathsf{b}^2 \to 26$, giving the leading coefficient $8(4+\sqrt{120}) < 119.636$; for the largest F row the values are 6 and 162, giving $8(6+\sqrt{684}) < 257.228$. A centred test on the strict reference, extended by zero with the normalizing square-root factor, remains centred on λ (Lemma 13.23(a),(b)); hence the compressed raw kernel K_k satisfies $\|P_0 K_k P_0\| \leq c_{\mathrm{st}} := C_\star/I_0 + N_k^{\mathrm{ns}} + D_k^{\mathrm{col}}$. With $|Z_k - 1| \leq \varepsilon_k$ and $\|P_0 K_k \mathbf{1}\|, \|P_0 K_k^* \mathbf{1}\| \leq \mathsf{M}_k$ from Section 27.6.1, and actual marginal densities at least $e^{-\ell_{\mathrm{low}}}$, Lemma 13.19(b) bounds the conditional correlation by

$$e^{\ell_{\mathrm{low}}}\Big(\frac{c_{\mathrm{st}}}{1-\varepsilon_k} + \frac{\mathsf{M}_k^2}{(1-\varepsilon_k)^2}\Big), \tag{27.6}$$

where ℓ_{low} is the insertion-count lower-density bound of the row. The minimum with the pinned-influence bound (F2) is taken. These bounds apply to F and to E, and conditional correlations tensorize by taking their maximum over the primes of a block (Lemma 13.25(c)). The scalar majorant (27.5) uses no property of the prime class; it is the star bound of the tail datum.

Full star matrices (bands). At $p \equiv 3 \pmod 4$ the centred star part of a pinned row is bounded by the full star matrix under the pinned product law (Proposition 16.22): the positive semidefinite gate $\left(\begin{smallmatrix} \mathsf{k}^2 I - AA^* & \mathsf{k}D \\ \mathsf{k}D^* & \mathsf{k}^2 I - BB^* \end{smallmatrix}\right) \succeq 0$ gives the norm bound $\kappa_0 r(\mathsf{k} + \mathsf{e}_\star)$, where $\kappa_0 \geq \kappa$ is the assigned rational density bound and the decay weights ω_s of Proposition 16.22(c) multiply the columns of the supports of size s. It is transported to the actual marginals by (27.6), with $\min\{C_\star, \kappa_0 r(\mathsf{k}+\mathsf{e}_\star)\}$ in place of $C_\star$. The bands use this form in (F4); the tail datum uses (27.5) in (27.6), which hold at both classes. The gates are rational and do not depend on p, and the other normalized errors are admissible, so a table certified at an endpoint holds at every larger class-three prime (Certificate 27.13(c)).

27.6.4. *Fold and matching conditional laws bounded jointly.* For the bands, the fold and matching rows of a coordinate are bounded jointly, on one common pinned product reference transported to the actual E marginal (Lemma 16.23 and Corollary 16.24): one Gram matrix bounds the fold kernel and the matching kernel together, and the completion of Lemma 17.5(f) spends the reserve of the matchings on that matrix rather than on the sum of two separate norms. For the L rows the pin classes are mutually exclusive, and this lowers the pure-star coefficient.

Lemma 27.15 (Pin classes of the L rows)**.** *Let $p \equiv 3 \pmod 4$ and consider the conditional* L *rows with $\ell \in \{1,2\}$ layers (the geometries (L,ℓ) of Section 27.6). Index the labels of a row by the six base coordinates $u \in \mathbb{Z}/6\mathbb{Z}$ and, for $\ell = 2$, their two* twins *(one per layer). The one-layer signed singleton matrices of the fold and the matching law are*

$$D_{\mathrm{F}} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \qquad D_{\mathrm{E}} = \begin{pmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \end{pmatrix};$$

for ℓ layers tensor each with the $\ell \times \ell$ all-ones matrix J_ℓ, and form $\mathcal{Z}$, $\mathcal{A}$ and $\mathcal{G} = \mathcal{A}\mathcal{A}^*$ as in Lemma 16.23, with $\mathcal{Z}_0$ the matrix $\mathcal{Z}$ for $\ell = 1$. Let Π be the projection onto the vectors that are constant on each twin class, in both copies of the joint block.

(a) *$|\mathcal{Z}| = (-\mathcal{Z}^2 + 3\ell^2\Pi)/(4\ell)$; in particular $|\mathcal{Z}| \pm i\mathcal{Z} \succeq 0$.*

(b) *Let $\mathcal{S}$ consist of the nine coordinate sets obtained by choosing one even and one odd base coordinate, with all their twins in both copies. At every label the coordinates whose allowed set contains it lie in one set of $\mathcal{S}$.*

Consequently, let T be $|\mathcal{Z}|$, or a real symmetric matrix with $T \pm i\mathcal{Z} \succeq 0$ (for $\ell = 2$ it suffices that $T = T_0 \otimes J_2$ with $T_0 \pm i\mathcal{Z}_0 \succeq 0$). If $\mathsf{k} > 0$ and the rational gates $\mathsf{k}^2 I_S - \mathsf{k}\kappa_0 T_S - \kappa_0^2 \mathcal{G}_S \succeq 0$ hold for all $S \in \mathcal{S}$, then $r\mathsf{k}$ is a joint pure-star coefficient: the lower bound of Lemma 16.23(a) holds with $r\mathsf{k}/I_0$ in place of $\kappa_0 r\mathsf{k}/I_0$ in $\mathfrak{k}_p$, all other error and normalizer charges being kept.

Proof. The matrices. Between the outer objects 0 and 2 the fold law F has the forward pair $0 \to 2$ and no relation $2 \to 0$, so its cross edges are $(0,u) \to (2,v)$ for $v \in \{u+1, u-1, \mathsf{c}u\}$; the matching law E has both relations and no forward pair, so its cross edges are $(0,u) \to (2,\mathsf{c}u)$ and $(2,v) \to (0,\mathsf{c}v)$ (Section 27.2.1). With the sign $+1$ for an edge from the row 0 to the row 2 and -1 for the reverse, these are the displayed D_{F} and D_{E}; edges carry no layer condition, which gives the factor J_ℓ.

(b) An edge between an outer object (0 or 2) and the middle object 1 of the chamber coordinate needs a forward relation in the other coordinate, which has type L, and $x \to y$ is forward there only for $y = \mathsf{c}x$ (Section 27.2.1). So in both laws (Lemma 27.4(d)) the outer label of base coordinate u is joined to the middle labels m_v exactly by the edges $u \to m_{\mathsf{c}u}$ and $m_{\mathsf{c}^{-1}u} \to u$. For the six base coordinates the out-pin and the in-pin are

$$(0)\ m_2, m_4; \quad (1)\ m_5, m_3; \quad (2)\ m_4, m_0; \quad (3)\ m_1, m_5; \quad (4)\ m_0, m_2; \quad (5)\ m_3, m_1.$$

Two even base coordinates share a pin with opposite orientations (0 and 2 at m_4, 0 and 4 at m_2, 2 and 4 at m_0), and so do two odd ones. Since $p \equiv 3 \pmod 4$, the strict-square relation is a tournament, and no label t satisfies both $t \to m$ and $m \to t$; so at every label at most one even and one odd base coordinate is admissible. With two layers every edge between layers is present (there is no condition on the layers), so the two twins of a base coordinate have the same pins and the same allowed set. The two outer rows have the same pins (Lemma 27.4(d)), so both copies of the joint block have the same allowed sets. Excluding the middle labels and the other fixed labels only shrinks the allowed sets. This proves (b).

(a) For $\ell = 1$, $\mathcal{Z}$ is a real antisymmetric 12×12 matrix, and direct multiplication of the displayed matrices gives $(\mathcal{Z}^2 + I)(\mathcal{Z}^2 + 9I) = 0$. So the eigenvalues of $-\mathcal{Z}^2$ are 1 and 9, those of $|\mathcal{Z}| = (-\mathcal{Z}^2)^{1/2}$ are 1 and 3, and $|\mathcal{Z}| = (-\mathcal{Z}^2 + 3I)/4$. For $\ell = 2$, $\mathcal{Z} = \mathcal{Z}_0 \otimes J_2$ with $J_2^2 = 2J_2$ and $\Pi = I \otimes J_2/2$, so $|\mathcal{Z}| = |\mathcal{Z}_0| \otimes J_2 = \frac{1}{4}(-\mathcal{Z}_0^2 + 3I) \otimes J_2 = (-\mathcal{Z}^2 + 12\Pi)/8$. As $\mathcal{Z}$ is normal and $i\mathcal{Z}$ is Hermitian, $|\mathcal{Z}| \pm i\mathcal{Z} \succeq 0$ on every eigenvector of $\mathcal{Z}$. If $T_0 \pm i\mathcal{Z}_0 \succeq 0$, then $(T_0 \pm i\mathcal{Z}_0) \otimes J_2 \succeq 0$, that is, $T \pm i\mathcal{Z} \succeq 0$ for $T = T_0 \otimes J_2$.

For the consequence, a coordinate (c, i) of $\mathcal{Z}$ has $\rho_i(t) > 0$ exactly when its allowed set contains t; so by (b), for every label t these coordinates lie in one member of $\mathcal{S}$, which is the hypothesis on $\mathcal{S}$ in Lemma 16.25, and by (a) the matrix T satisfies its hypothesis $T \pm i\mathcal{Z} \succeq 0$. So Lemma 16.25 applies, and its conclusion is the assertion. The Schur elimination of the higher features and the pointwise diagonal contraction on the active coordinates are made there. □

The tables of Certificate 27.13(c) record, for each of the six row geometries, the joint coefficient k_{J}, the common fold coefficient c_{F} and the matching coefficient of Lemma 16.23, and, for the L rows, a rational base potential T_0 with the nine gates of Lemma 27.15.

Proposition 27.16 (A necessary floor for the chosen joint-row formula)**.** *Every admissible two-layer* L *table above* 2^{47}*, using the potential and decay gates of Lemma 27.15 and Proposition 16.22, has chosen complete coefficient* $k_{\mathrm{J}} > 172.9$*. Consequently the band completion inputs satisfy* $f_{\mathrm{L}} + h_{\mathrm{L}} M_{\mathrm{L}} > 172.9$*, and the native maximum recursion for the reference gives* $w_{\mathrm{LR}}, w_{\mathrm{RL}} > 172.9 d_{\mathrm{L}}$*. These are lower bounds for specified certificate formulas and their chosen weights, not for the underlying operator norms. The tail uses different row formulas and is excluded.*

Proof. A free coordinate is uniform on a four-pin allowed set of size n, excluding twelve fixed middle labels. The character expansion gives

$$\frac{p - 11\sqrt{p} - 6}{16} - 12 \le n \le \frac{p + 11\sqrt{p} + 6}{16}.$$

For $p \ge 2^{47}$ this implies $p/17 \le n \le c_0 p$ with $c_0 = (1 + 11 \cdot 2^{-23})/16$. The positive 6 in the upper count is absorbed by $11 \cdot 10^{-7} + 6 \cdot 2^{-47} < 11 \cdot 2^{-23}$. For $X_t(x) = \chi(t - x)$, Paley orthogonality gives

$$\sum_t \mathbb{E} X_t^2 = p - 1, \qquad \sum_t (\mathbb{E} X_t)^2 = p/n - 1.$$

After deleting the twelve fixed labels, the covariance matrix has trace at least $p - 29$, rank at most $n - 1$, and diagonal at most 1. Thus an off-diagonal covariance bound ν_χ must satisfy

$$p\nu_\chi^2 \ge (1 - 29/p)^2/c_0 - 1 > 14.9999.$$

Put $c = 15.9999$. The decay conditions of Proposition 16.22(c) now force $\kappa_0 > c$, $\kappa_0^2 \omega_2^2 \ge c^2$ and $\kappa_0^2 \omega_s^2 \ge c$ for $s \ge 3$; a weight chosen as 1 already satisfies the corresponding inequality.

Let G_2, G_h be the unweighted column Grams in the joint higher-star matrix, for supports of size 2 and at least 3 respectively. For each of the nine pin classes choose a real vector v_i supported on it, and let R have these vectors as columns. Put $M = R^T \mathcal{Z} R$. If K is a real skew contraction, then $R^T T R \pm iM \succeq 0$ and $I \pm iK \succeq 0$ imply

$$\operatorname{tr}(R^T T R) \ge \langle K, M \rangle_F.$$

Indeed take traces of $(I + iK)(R^T T R - iM)/2$ and $(I - iK)(R^T T R + iM)/2$. Evaluating the nine potential gates on their vectors and adding therefore forces

$$k^2 N - kcB - c^2 Q_2 - cQ_h \ge 0,$$

where $N = \sum_i \|v_i\|^2$, $Q_j = \sum_i v_i^T G_j v_i$, and $B = \langle K, M\rangle_F$. The exact rational witness of Certificate D.12, reconstructed from the displayed $D_{\mathrm{F}}, D_{\mathrm{E}}$, has

$$N = 10000010408, \quad Q_2 = 558217933120, \quad Q_h = 1038757368976, \quad B = \frac{315100239978216987}{6250000}.$$

At $k = 172.9$ the polynomial equals $-29790250369036028509477/625000000000 < 0$. It is negative at zero and convex, hence negative throughout $[0, 172.9]$. The complete coefficient adds nonnegative mean, remainder and normalization costs to k, so $k_{\mathrm{J}} > 172.9$. Finally (F5)(a) gives $f_{\mathrm{L}} \geq H(k_{\mathrm{J}} - h_{\mathrm{L}})_+$ and $M_{\mathrm{L}} \geq H \geq 1$, whence $f_{\mathrm{L}} + h_{\mathrm{L}} M_{\mathrm{L}} \geq k_{\mathrm{J}}$. Apply the ordinary completion inequalities on both axes. □

27.7. **Band input formulas (F1)–(F6) and (F10): definitions.** The band inputs are computed by the following formulas, each stated for every odd prime at which its listed hypotheses (the *gates*) hold; the formulas (F7)–(F9) of the comparison laws follow in Section 28, and Lemma 28.11 extends gates checked at an evaluation point to every larger class-three prime and to every block of the band. For a block of primes at least p_0 and logarithmic length at most $L' \leq 10000$ put $\bar{n} = L'/\log p_0$.

(F1) *Insertion counts* (Lemma 12.2). An insertion with t signed constraints at distinct labelled roots and z excluded labels has normalized count in $[1 - e_t^-(z), 1 + e_t^+]$; Weil's bound is used in both parities. An explicit order of the free vertices gives, with t and z the numbers of labelled neighbours and of labelled vertices at each step, the logarithmic count bounds $c^+ = \sum e_t^+$, an upper bound for $\sum \log(1 + e_t^+)$, and $c^- = \sum -\log(1 - e_t^-(z))$. *Hypothesis:* $e_t^-(z) < 1$ at every insertion.

(F2) *Conditional influence* (Corollary 13.17, the certified form of Proposition 13.16). For disjoint free sets X, Y the conditional maximal correlation is at most $r(c_\times + \sqrt{|X||Y|}\, r q_{\mathrm{inf}}^2/(1 - r q_{\mathrm{inf}}))$, with the count errors $w = e_\Delta^-(n)$ and $w^{\mathrm{conn}} = e_{\Delta+1}^-(2n)$, where n is the number of vertices of the graph and Δ the largest degree of a free vertex. *Hypotheses:* $w < \frac{1}{2}$, $w^{\mathrm{conn}} < 1$, $r q_{\mathrm{inf}} < 1$, and the resulting bound below one.

(F3) *Product rows* (Sections 27.6.1 and 27.6.2), with the pinned nonstar extraction: the normalizer errors $\varepsilon_{\mathrm{F}}, \varepsilon_{\mathrm{E}}$, the differences Δ, M_Δ, the middle log ratio ℓ_{mid} and the marginal coefficient c_{mar}. A constant that depends on the class of p is taken as the larger of its values in the two classes (Section 27.6.2). *Hypotheses:* $e_\varpi < \frac{1}{2}$ for the pin count and $e_\varpi^{\mathrm{pin}} < \frac{1}{2}$ for the pinned extraction (Proposition 16.20); $I_0 = 1 - \binom{n}{2}\kappa r^2 > \frac{1}{2}$; $\varepsilon_{\mathrm{F}}, \varepsilon_{\mathrm{E}} < \frac{1}{2}$; every nonstar denominator positive.

(F4) *Exact-support correlation* (Section 27.6.3, (27.6)), minimized with (F2): for the bands with the full star matrix of Proposition 16.22, that is, with $\min\{C_\star, \kappa_0 r(\mathsf{k} + \mathsf{e}_\star)\}$ in (27.6), and for the tail datum with (27.5); class-dependent constants are taken as in (F3). *Hypotheses:* $1 - \varepsilon > 0$ for the normalizer error, the selected correlation below one, and, for the bands, the gates of Proposition 16.22.

(F5) *Completion rows.* With ℓ_{mid} and c_{mar} from (F3), all at p_0, put

$$H = \exp(\bar{n}\ell_{\mathrm{mid}}), \qquad \upsilon^2 = \exp(\bar{n} c_{\mathrm{mar}}^2/p_0) - 1.$$

(a) Joint rows (bands). Let $(k_{\mathrm{J}}, c_{\mathrm{F}})$ be the joint and the common fold coefficient of one complete joint-row table (Lemma 16.23, Lemma 27.15) whose endpoint is at most p_0, let h be a proved coefficient of the E row (the centred matching kernel has norm at most $h r_p$ at every prime $p \geq p_0$, so that the matching block of a block with least prime b is at least $-hb^{-1/2}I$ by Lemma 13.25(c); for instance $h = \rho_{\mathrm{E}}/r_0$ of (F4)), and put $E_{\mathrm{T}} = (2\upsilon + \upsilon^2)(\exp(\bar{n} c_{\mathrm{F}} r_0) - 1)$. Then for every rational $t_{\mathrm{Y}} > 0$ with $t_{\mathrm{Y}} H < 1$ the

coefficient h, every $M \ge H + t_{\mathrm{Y}}$ and every

$$f \ge \frac{H\{(k_{\mathrm{J}} - h)_+ r_0 + E_{\mathrm{T}}\} + (H + H^2/t_{\mathrm{Y}})\upsilon^2}{r_0}$$

are valid row coefficients (Lemma 17.5(f), Corollary 17.6). An ordinary input takes f to be the maximum of its one-layer and two-layer values, computed with one common value of t_{Y}, and likewise takes $H + t_{\mathrm{Y}}$ to be the maximum of its one-layer and two-layer values; a reserved input uses the one-layer row. A smaller proved E bound may be used for h, since all of them concern the same actual law; the pair $(k_{\mathrm{J}}, c_{\mathrm{F}})$ always comes from one table.

(b) Schur rows (tail datum). With moreover the pointwise marginal ratio ℓ_{mar} of the field records (from (F1) and Lemma 12.7 with M the middle row and S an outer row; its hypothesis, that the F and E graphs induce the same graph on $M \cup S$, is Lemma 27.4(d)) and the correlations $\rho_{\mathrm{F}}, \rho_{\mathrm{E}}$ of (F4), all at p_0, put $R_{\mathrm{m}} = \exp(\bar{n}\ell_{\mathrm{mar}})$. Then

$$f_{\mathrm{Schur}} = \frac{H R_{\mathrm{m}} \rho_{\mathrm{F}} + (H + 50H^2)\upsilon^2}{r_0}, \qquad h = \frac{\rho_{\mathrm{E}}}{r_0}$$

are valid row coefficients (Lemma 17.5(c) with Remark 17.7 at $t_{\mathrm{Y}} = 1/50$), and so is the full-norm coefficient $f = [H(R_{\mathrm{m}}\rho_{\mathrm{F}} + 2\upsilon + \upsilon^2) + H - 1]/r_0$ of Lemma 13.25(b) with Lemma 17.5(b); the smaller of the two is used.

Hypotheses: in (a), $M \ge H + t_{\mathrm{Y}}$ and $t_{\mathrm{Y}} H < 1$ for the matching ratio $M = w_{\mathrm{S}}/w_{\mathrm{D}}$, $c_{\mathrm{F}} r_0 < 1$, and the prime gates of the table at its endpoint; in (b) the weaker $t_{\mathrm{Y}} \le H^{-1}$ suffices, and with $M = 21/20$ and $t_{\mathrm{Y}} = 1/50$ both hypotheses follow from $H \le 103/100$.

(F6) *Nonroot spectral caps* (Theorem 14.21, Section 27.4). For each of the ten nonroot cuts, $-grI \preceq \mathsf{A}_p \preceq (1 + B_{\mathrm{top}}/p)I$ with $g \ge \max_{\pm}(\bar{c}_{\perp} + \tau^{\mathrm{sp}}_{\mathfrak{c},\pm})/(1 - \bar{\varepsilon}_V)$, the maximum over the signs and the classes tested, and $B_{\mathrm{top}} = p_0 \varepsilon_{\varrho}(r_0)^2/(1 - gr_0)$; moreover $\|P_0 \mathsf{A}_p P_0\| \le gr$ and $\|P_0 \mathsf{A}_p \mathbf{1}\| \le \varepsilon_{\varrho}(r)$ with $p\,\varepsilon_{\varrho}(r)^2 \le p_0\,\varepsilon_{\varrho}(r_0)^2$, so that $\|P_0 \mathsf{A}_p \mathbf{1}\| \le B_{\mathrm{top}}^{1/2} r$ (Theorem 14.21(b),(c)); a block of primes at least $X \ge p_0$ and length at most L' has negative coefficient at most $d\,b^{-1/2}$ with $d = g \exp(B_{\mathrm{top}} L'/(X \log X))$. The observed caps of Proposition 27.14 are evaluated in the same way. *Hypotheses:* (S1) of Definition 13.1 at p_0; $\bar{\varepsilon}_V < 1$; $\varepsilon_Z < \frac{1}{2}$ (which implies the hypothesis $\varepsilon_Z < 1$ of Theorem 14.21); the exact rational matrix inequality (14.21) with the recorded rational $\tau^{\mathrm{sp}}_{\mathfrak{c},\pm}$, for the class three and both Fourier signs (bands), or for both classes and both signs (tail datum); $gr_0 < 1$; and $d/\sqrt{p_0} < 1$.

(F10) *Completion* (Definition 27.1(d), Lemma 27.17). The forty two-layer and forty one-layer slice inequalities (27.1), with the recorded M, and the twenty-five reserved dominations (27.2), with inputs evaluated for block lengths at most 10000.

Proofs and forms. (F1) is Lemma 12.2(b),(c) at each step of the order, the graded errors being those of (12.2); the evaluators sum e_t^+, which bounds $\log(1 + e_t^+)$. (F2) is Corollary 13.17, with the graded count errors of Remark 13.18(1). (F3) and (F4) are Section 27.6. For (F5): let the block have least prime b, and put $r_b = b^{-1/2} \le r_0 = p_0^{-1/2}$. On a block the densities and chi-square defects multiply and the correlations are maximized over the primes (Lemma 13.25(c)); the block has at most $\bar{n}$ primes, so $\sum_p 1/p \le \bar{n} r_b^2$, and every per-prime quantity, after its indicated power of r is removed, is at most its value at p_0 (Lemma 28.11(i)). So the middle density lies in $[H^{-1}, H]$, the fold marginals have pointwise ratio at most R_{m}, the block correlations are at most $(\rho_{\mathrm{F}}/r_0) r_b$ and $(\rho_{\mathrm{E}}/r_0) r_b$, and the relative L^2 defect υ_b of the block fold marginals satisfies $\upsilon_b^2 \le \prod_p (1 + c_{\mathrm{mar}}^2 r_p^2) - 1 \le \exp(c_{\mathrm{mar}}^2 \bar{n} r_b^2) - 1 \le (\upsilon^2/r_0)\, r_b$, because $x \mapsto (e^{cx^2} - 1)/x$ is nondecreasing and $\exp(c_{\mathrm{mar}}^2 \bar{n} r_0^2) - 1 = \upsilon^2$; in particular $\upsilon_b \le \upsilon$.

(a) is Lemma 17.5(f) through Corollary 17.6, with these block values: by the table, the prime bounds of Corollary 17.6 hold at every prime of the block with the coefficients $(k_{\mathrm{J}}, c_{\mathrm{F}})$ and the matching coefficient of the table, the matching block is at least $-hr_bI$, and the tensor error is at most $(2\upsilon_b + \upsilon_b^2)(e^{\bar{n}c_{\mathrm{F}}r_b} - 1) \le (E_{\mathrm{T}}/r_0)\, r_b$, since $x \mapsto (e^{\bar{n}c_{\mathrm{F}}x} - 1)/x$ is nondecreasing; so the coefficient f_{J} of Corollary 17.6 at $r = r_b$ is at most f. The Young allowance t_{Y} enters only through $H + t_{\mathrm{Y}}$ and H^2/t_{Y}, so it may be chosen freely, row by row, subject to $t_{\mathrm{Y}}H < 1$.

(b) Remark 17.7, with these block values and $r = r_b$, turns them into the fold data of Lemma 17.5(c) at $t_{\mathrm{Y}} = 1/50$ with $f_{\mathrm{al}} \le f_{\mathrm{Schur}}$, and the matching row has exact marginals with $\|T_{\mathrm{E}} - \mathbb{J}\| \le hr_b$; the full-norm coefficient scales in the same way. For the tail datum $M = \frac{21}{20} = \frac{103}{100} + \frac{1}{50} \ge H + t_{\mathrm{Y}}$ and $t_{\mathrm{Y}} = \frac{1}{50} \le \frac{100}{103} \le H^{-1}$ when $H \le 103/100$; also $M \ge 1$, so Lemma 17.5(b) applies to the full-norm coefficient. In every row of the tail datum the Schur coefficient is the smaller one (Remark 17.9).

(F6) is Theorem 14.21 with the standard inputs of Section 27.4, for each class tested, and its block form is Corollary 14.22 with $P = X$, with $g \ge g_3$ for the bands and $g \ge \max(g_1, g_3)$ for the tail datum. (F10) is used in Lemma 27.17. □

For the root cuts the domain of the caps is that of Lemma 27.7 (route B) and of Lemma 27.10 (route A); the per-prime constants of the nonroot cuts are also read by the one-way repairs (F7).

27.8. **(CP1) for upper atoms.** We say that the inputs of an upper band are *valid* on a finite set $\mathcal{B}$ of primes, with least prime b and $r = b^{-1/2}$, if for every field type φ:

- (V1) the normalized block cut operator of the chamber cut of the L coordinate of $G_{\mathrm{L}\varphi,2}$ is at least $-d_\varphi r$, and that of $G_{\mathrm{L}\varphi,1}$ at least $-d_\varphi^{\mathrm{res}} r$;
- (V2) at every value of the middle object, the two-layer rows at φ satisfy, with graded coefficients at most (f_φ, h_φ), one of three sets of hypotheses: those of Lemma 17.5(f) with (M_φ, t_φ); those of Lemma 17.5(c) with $t_{\mathrm{Y}} = t_\varphi$ and $M_\varphi \ge H + t_\varphi$, $t_\varphi H \le 1$; or those of Lemma 17.5(b). The one-layer rows satisfy the same with the reserved input;
- (V3) for every field pair $\varphi\varphi'$, the normalized block cut operator of the layer cut of $G_{\varphi\varphi',2}$ is at least $-k_{\varphi\varphi'} r$. For each state $\mathfrak{t}$ with $k_{\mathfrak{t}} < k_{\mathfrak{f}(\mathfrak{t})}$, the compression of the normalized block cut operator of EL22 to the functions of the labels that the components of $\mathfrak{t}$ observe (the merge or the constant labels of Proposition 27.14) is at least $-k_{\mathfrak{t}} r$.

Lemma 27.17 (The upper reference)**.**

(i) Let $\mathcal{B}$ be a finite set of odd primes on which the inputs of an upper band are valid, and let $\mu_{\mathcal{B}}$ be defined by the formula of Definition 27.5 with the weights of the band. Then $\mu_{\mathcal{B}}$ is a probability measure on $X_{\mathcal{B}}^V$; it is reflection positive at every cut of $\mathfrak{F}$; it is invariant under the square-affine maps of (R3); its one-slot marginals are uniform; and it is admissible (Definition 8.9).

(ii) The inputs of every upper band are valid on every block of the band. Hence every upper block satisfies (R1)–(R3), with (R2) at every cut of $\mathfrak{F}$.

Proof. (i) *Measure, invariance, marginals.* Each component is the image of a field law $\lambda_{\mathcal{B}}(G)$ under an emission, and Diag_ι is an emission too; the field laws are square-affine invariant at every prime, and images, products over the primes of $\mathcal{B}$ and nonnegative combinations preserve invariance (Lemma 5.16(1)). So $\widetilde{\mu}_{\mathcal{B}}$ is a nonnegative invariant measure, of mass $P_\mu(r)$ since $\pi_{\mathfrak{t},j}$ has mass $m_{\mathfrak{t}}$, and $\mu_{\mathcal{B}}$ is a probability measure with uniform one-slot marginals (Lemma 5.16(2)).

Chamber cuts. Fix a chamber coordinate c and a type $\mathfrak{t}'$ of the other coordinate. The two-layer row data of c at $\mathfrak{f}(\mathfrak{t}')$ form a five-type datum (Section 27.2.2), with π_{F}, π_{E} the row laws of the F and E field laws, Γ and ν the laws of the rows $0, 1$ and of the row 0 under the E law, and ν^b its conditional law of the row 0; the merges and the constants use only these rows, which is the role of the hidden objects. The weights of the slice in $\widetilde{\mu}_{\mathcal{B}}$ are $w_{\mathfrak{t}} r^{\deg \mathfrak{t}} = r^{\deg \mathfrak{t}'}\, a_{\mathfrak{s}} r^{\deg \mathfrak{s}}$ ($\mathfrak{t}_c = \mathfrak{s}$,

$\mathfrak{t}_{1-c} = \mathfrak{t}'$, $a_{\mathfrak{s}} = w_{\mathfrak{t}}$): $r^{\deg \mathfrak{t}'}$ times the graded weights of Lemma 17.5(b),(c),(f) formed from the $a_{\mathfrak{s}}$. The conditions are homogeneous, so the factor $r^{\deg \mathfrak{t}'}$ does not matter. By (V1) the root envelope $\Pi_{\mathrm{L}} \succeq_{\vartheta_0} -d\, r\, \Pi_{F_0}$ holds with $d = d_{\mathfrak{f}(\mathfrak{t}')}$, the half law of the cut being the law of the aligned fold (Lemma 27.4(c)); by (V2) the rows satisfy the hypotheses of Lemma 17.5(f), (c) or (b) with the input (f, h, M, t) of $\mathfrak{f}(\mathfrak{t}')$; and (27.1) is the graded form of their conditions: $a_{\mathrm{D}} \geq d\, a_{\mathrm{L}}$; $a_{\mathrm{S}} \geq M\, a_{\mathrm{D}}$ with the recorded $M \geq H + t_{\mathrm{Y}}$ and $t_{\mathrm{Y}} \leq H^{-1}$, so in particular $a_{\mathrm{S}} \geq a_{\mathrm{D}}$; $a_{\mathrm{R}} \geq f a_{\mathrm{D}} + h a_{\mathrm{S}}$; and $a_{\mathrm{C}} a_{\mathrm{S}} \geq a_{\mathrm{R}}^2$. By Lemma 17.5(d), the two-layer sum $\sum_{\mathfrak{t}} w_{\mathfrak{t}} r^{\deg \mathfrak{t}} \pi_{\mathfrak{t},2}$ is reflection positive at every chamber cut of both coordinates. The same argument with the one-layer row data, the reserved inputs and the weights $v_{\mathfrak{t}} r^{\deg \mathfrak{t}}$ shows that $\sum_{\mathfrak{t}} v_{\mathfrak{t}} r^{\deg \mathfrak{t}} \pi_{\mathfrak{t},1}$ is reflection positive at the chamber cuts, and Diag_ι preserves this (Lemma 17.5(e)). Cut forms add, so $\widetilde{\mu}_{\mathcal{B}}$ is reflection positive at every chamber cut.

The reserved cut. By (V3), for the field law of each field pair $\lambda \succeq_\iota -k\, r\, \mathrm{Diag}_\iota(\lambda^{(1)})$, the half graph of the layer cut being the one-layer graph (Lemma 27.4(c)); by the last assertion of Lemma 17.5(e) every two-layer component of $\mathfrak{t}$ inherits $\Pi \succeq_\iota -k_{\mathfrak{f}(\mathfrak{t})} r\, \mathrm{Diag}_\iota(\Pi^{(1)})$, the merges and constants included. A merge or constant component reads only the observed rows of its E coordinate, so its cut form at ι is that of its field law on the functions of the observed labels; by the second assertion of (V3) the components of the states with $k_{\mathfrak{t}} < k_{\mathfrak{f}(\mathfrak{t})}$ satisfy the same order with $k_{\mathfrak{t}}$, a cap of their own, which Lemma 17.5(e) allows. With (27.2), $v_{\mathfrak{t}} \geq k_{\mathfrak{t}} w_{\mathfrak{t}}$ with the cap of (V3) for the component of $\mathfrak{t}$, Lemma 17.5(e) gives $\widetilde{\mu}_{\mathcal{B}} \succeq_\iota 0$. Normalization by $P_\mu(r) > 0$ preserves reflection positivity, which is (R2) at every cut of $\mathfrak{F}$.

Admissibility. $\mu_{\mathcal{B}}$ is nonnegative and invariant, with uniform whole-slot marginals, and it is a law of the field labels alone, independent of the interval, residue and inclusion indices; this is (O1) of Definition 8.9. The reference creates no sharp operator bound, index projection or fold, so (O2)–(O4) are those of the frame.

(ii) Let $\mathcal{B}$ be a block of the band $[X, Y)$. Its primes are 3 modulo 4, at least X, and $\log q_{\mathcal{B}} \leq L_{\mathrm{band}} < L_{\mathrm{u}}$ (Definition 27.1(b)). The nonroot and observed caps were evaluated at $p_0 = 2^{47}$ or 2^{48}, not above X, and inflated at X with the length L_{band}; the product-row quantities at $p_0 = X$ with the length L_{band}, and the joint coefficients at an endpoint not above X; all under the gates of (F1)–(F6) (Certificate 27.13, Certificate 28.12). By Lemma 28.11(i) the gates hold at every prime of $\mathcal{B}$ and every per-prime constant is at most its value at p_0; the block forms of (F5), (F6) and Proposition 27.14 (Lemma 28.11(ii), with the inflation taken at X, since every prime of $\mathcal{B}$ is at least X) bound the negative coefficients of the nonroot cuts and of the observed marginals on $\mathcal{B}$ by $d\, b^{-1/2}$, and the row coefficients by $f\, b^{-1/2}$ and $h\, b^{-1/2}$ with the recorded M and t_φ; the inputs are upper bounds for these values. Inputs computed for the length L_{band} remain valid for every block of the band, because every count inflation, tensor variance and row coefficient is nondecreasing in the length. The root inputs are valid on $\mathcal{B}$ by Lemma 27.7 below 2^{52} (Certificate 27.9) and by Lemma 27.10 from 2^{52} on (Certificate 27.11). The weights satisfy (27.1) and (27.2) (Certificate 27.18). Apply (i). □

Part (i) uses nothing about $\mathcal{B}$ except the validity of the inputs. It therefore applies to a single prime $p \geq P_4$ of either class with the tail datum (Definition 27.1(e)), whose inputs are valid there (Lemma 29.2). The data of the last band $[2^{64}, 2^{512})$ are class-three data and are not used beyond 2^{512}.

Certificate 27.18 (Completion and domination gates)**.** At the baseline checkpoint, in each of the 310 bands the 105 inequalities (27.1) (forty for w, forty for v) and (27.2) (twenty-five) hold exactly with the recorded inputs, 32550 in all, and the weights are positive with $w_{\mathrm{LL}} = 1$. The record is `research/further_2026_10_25/upper/upper_dense_refined.json`, with the fields `inputs`, `reserved_inputs`, `reserved_state_caps`, `weights`, `reserved_weights` and `gates` for each band. Its inputs, state caps and weights are those of two earlier records of that directory,

`upper_special.json` for the 301 bands below 2^{52} and `upper_endpoint_allowances.json` for the nine bands from 2^{52}; the later stages within that checkpoint change only the comparison ledgers (Section 28). The final cross-weighted inputs and weights are instead those of Section 27.3.1, with the same completion and domination requirements. The independent reviews of that directory re-check the gates on these records: `special_upper_review.json` for the 301 bands below 2^{52}, whose root entries and weights are fixed with the root caps of Certificate 27.9 (24080 slice inequalities, 7525 dominations and 903 polynomials), and `endpoint_allowances_review.json` for all 310 bands of the allowance stage (2790 primitive prices, 24800 slice inequalities, 7750 dominations and 930 physical polynomials), whose data the nine bands from 2^{52} keep. The script `research/further_2026_10_25/upper/verify.py` replays the whole finite upper extension from the certified baseline `research/advance_2026_10_24/verification/selected.json`, without an optimizer, and requires every regenerated record to equal its archived file; the selected verification record `research/further_2026_10_25/verification/selected.json` reports `PASS_COMPLETE_FURTHER_UPPER_EXTENSION`.

The 105 gates of the tail datum are those of the band $[2^{64}, 2^{512})$ of `research/roadmap_2026_10_18/item12/upper_v2_local.json` (`PASS_105_EXACT_COMPLETION_AND_DOMINATION_GATES`), written by `assemble_upper_v2.py` with the completion routine of `assemble_upper_candidate.py` (same directory) and re-checked by `research/push_2026_10_24/tools/check_upper.py` into `research/push_2026_10_24/artifacts/coordinator/push_best_upper_review.json` (`PASS_INDEPENDENT_GENERATED_DIRECT_UPPER_PRICES_AND_GATES`: 1260 gates, 105 for each of the twelve bands of `upper_v2_local.json` from 2^{49} on, the band $[2^{64}, 2^{512})$ among them).

Consequences and what to read next. From Part II: the slot array and its cuts (Definition 5.12 and Lemma 5.13), square-affine invariance (Lemma 5.16), admissible majorants (Lemma 4.18), admissibility (Definition 8.9). From Part IV: labellings and counts (Definition 12.1 and Lemmas 12.2 and 12.7), the two-root identity and Weil's bound ((4.2), Theorem 4.7); spectral data and cut kernels (Definition 13.1 and Lemma 13.3), block lifting (Lemma 13.7), the two-sided constant block (Proposition 13.6(a′)), pinned influence (Proposition 13.16 and Corollary 13.17), recentering and restriction (Lemmas 13.19 and 13.23), row comparison (Lemma 13.25); the two spectral routes (Theorems 14.21 and 14.27 and Corollary 14.22) with the character expansion, the stars and the centring (Lemmas 14.18 and 14.19 and Proposition 14.20), the frame inputs (Lemma 14.28), the coupling block and its refinements (Lemmas 14.14 to 14.17 and 14.32) and the monotone spectral quantities (Lemma 14.29); the complete actual stars (Lemmas 14.4 and 14.6 and Corollary 14.5, Remark 14.7); the two-variable sums and the moving-root moments (Theorem 15.13 and Corollaries 15.20 and 15.21), the ordered star marginal and the star mean (Proposition 15.26 and Lemma 15.29); the compressed partition, the extraction bounds with the graded error pairs and the special values, and the coincidence bounds (Lemmas 16.2 and 16.15, Definition 16.6, Corollaries 16.11 and 16.14, and Propositions 16.20 and 16.21, (16.5)); the log-convex nonstar coefficients (Lemma 16.12); the pinned conditional stars (Proposition 16.22 and Remark 16.26, with (27.6)); the joint rows (Lemmas 16.23 and 16.25 and Corollary 16.24); and the five-type completion with its joint form, its free allowance and its state caps (Lemma 17.5(b)–(f), Corollary 17.6 and Remark 17.7).

Hypotheses verified in this section: (CP1) for upper atoms. Every upper block carries the probability measure $\mu_{\mathcal{B}}$, reflection positive at every cut of $\mathfrak{F}$ (the chamber cuts of both coordinates and ι, not only the seven sorting cuts), invariant under the square-affine maps, with uniform one-slot marginals, and admissible (Lemma 27.17); with the band inputs of Definition 27.1 its spectral caps and rows are valid on every block. The tail datum (Definition 27.1(e)) carries the same conclusions to single primes $p \geq P_4$ of both classes (Lemma 29.2). This is the upper row of (CP1) in Table 5; the comparison law and its prices (CP4) are Proposition 28.14.

28. Upper primes II: the comparison law ν and prices $S_{\mathrm{u}}, A_{\mathrm{u}}$

What this section proves and how to read it. This section proves (CP4) for upper atoms: of the degree-two and degree-three records, 92 do not force $\mathcal{E}_6$; 84 are replaced by *one-way replacements* (called *one-way repairs* below: a forcing law with the same actual half marginals imposing twelve projected pairs), six by a positive mixture, two by replacements keeping the old cross edges (*dense* replacements), and every other record is kept if it accepts $\mathcal{E}_6$ and deleted if it does not. Types such as LE, EL, FF name field-type pairs of two-layer records, expanded at first use. Formulas (F7)–(F9) continue (F1)–(F6) of Section 27, whose completion item is numbered (F10); Lemma 28.11 extends a bound checked at a band start to every larger prime in the band—its catalogue of admissibilities is a list of separate checked inequalities, each citable by its display. Inputs: Section 27. Outputs: $S_{\mathrm{u}}(a), A_{\mathrm{u}}(a)$ for Section 30. The previous section built, for every block $\mathcal{B}$ of class-three primes in $[P_3, P_4) = [167 \cdot 2^{40}, 2^{512})$, the all-cut reference $\mu_{\mathcal{B}}$ (Lemma 27.17), its spectral caps and its conditional rows. The reference does not accept the six queries $\mathcal{E}_6$. This section builds and prices the comparison law $\nu_{\mathcal{B}}$ that does.

Of the two-layer records of degrees two and three, 92 do not accept $\mathcal{E}_6$ (Lemma 28.1). Each is replaced by a positive law with the actual half marginals of the record:

- 84 by *one-way repairs*, which impose the twelve projected queries (Section 28.1);
- the six LE records by a positive mixture of four softened six-query laws (Proposition 28.6);
- two EL records by a dense law (Proposition 28.5).

Two further EL records and their reserved companions accept $\mathcal{E}_6$ and are kept, and every other nonaccepting record is deleted. The distance of a replacing law from the old record is controlled by the additive repair package (Lemma 17.14) and by norm bounds in the actual frame of the record: complete star matrices (Section 14.2) for the dense laws and the mixture, and, from 2^{48}, for the FF, FE and EF repairs of degree two (Proposition 28.3), whose frames and remainders come from the star lemma (Lemma 28.2).

A change is charged by its norms. The symmetric price pays the change, and the skew price only the skew part of the new law, because the old record is symmetric (Remark 9.15(iii)).

The local inputs are evaluated at points not above the start of their band. Lemma 28.11 extends them to every larger class-three prime and to every block of the band, and Certificate 28.12 certifies the evaluation at 2^{47}. Proposition 28.14 verifies (CP4) of Theorem 9.21 for the upper atoms and bounds their prices by $S_{\mathrm{u}}(a)$ and $A_{\mathrm{u}}(a)$, which enter S, A and the matching exponent T of the closing (Certificate 30.3). The tail atoms keep the twelve-query comparison of the tail datum, with the all-prime star cap (Lemma 29.5). Part (iii) of Proposition 28.14 bounds the price of an upper block by a constant times $b^{-3/2} q_{\mathcal{B}}^{2a}$.

Throughout, p is an odd prime, $r = p^{-1/2}$ and $q = 1/p$. A band's data use its budget L_{band} (Definition 27.1(b)); $L_{\mathrm{u}} = 50000/27$ is the budget of the tail datum (Definition 18.1). Operators are taken relative to the measures named, and Q is the projection onto the mean-zero functions. For $p \equiv 3 \pmod 4$, $\hat{P}$ is the normalized Paley operator (Section 14.2).

28.1. **One-way replacements imposing twelve projected pairs.**

Queries and records. The comparison law has to accept the six queries $\mathcal{E}_6$ of Section 18.2. The one-way repairs and the tail are built for the twelve disjoint slot-pair queries on the original labels (called *physical* queries below, to distinguish them from their projected images under pattern emissions)

$$((u, v, 0), (\mathsf{c}(u), \mathsf{c}(v), 1)), \qquad u \in \{0, 5\}, \quad v \in \mathbb{Z}/6\mathbb{Z}, \tag{28.1}$$

of Remark 18.2, which run from layer 0 to layer 1 and contain $\mathcal{E}_6$ (the queries with $u = 0$). A *record* of $\mu_{\mathcal{B}}$ (Definition 27.5) is one of its components (Section 27.2.2). It has a state $\mathfrak{t} = (\mathfrak{t}_0, \mathfrak{t}_1) \in \{L, D, S, R, C\}^2$ of formal degree $\deg \mathfrak{t}$, and is either a two-layer record, a component of $\pi_{\mathfrak{t},2}$, or a

folded record, a component of $\mathrm{Diag}_\iota(\pi_{\mathfrak{t},1})$ of total degree $\deg \mathfrak{t} + 1$. A record *accepts* a query (s, s'), in this record (support) sense, if every labelling z in its support has $z_{s'} - z_s$ a nonzero square at every prime of the block. (A *latent record* is a record before emission: specified by layer kind, state and pattern pair, with the hidden objects retained; there are 392 in all.) The exact enumeration of all latent records (`research/frontier_2026_10_17/sparse/high72_oneway_geometry.py` and `high72_oneway_higher_geometry.py`) gives the following ledger for the twelve queries. Every record of formal degree at most one, and every folded record of total degree at most two, accepts. Of the 37 two-layer records of degree two exactly seven accept; the other thirty have the state multiplicities

$$LR\colon 6, \quad RL\colon 4, \quad DD\colon 6, \quad DS\colon 4, \quad SD\colon 6, \quad SS\colon 4.$$

None of the 64 two-layer records of degree three accepts (states $LC\colon 2$, $CL\colon 2$, $DR\colon 18$, $RD\colon 18$, $SR\colon 12$, $RS\colon 12$). Exactly seven folded records of total degree three accept, with multiplicities $DD\colon 3$, $DS\colon 2$, $RL\colon 2$. No two-layer record of degree at least four and no folded record of total degree at least four accepts.

For the six queries two more records of degree two accept, with their reserved companions. The two-layer records of state RL are the components with a merge pattern G_{sk} ($s \in \{0,1\}$, $k \in \{0,1,2\}$) in the first coordinate and the identity in the second (Definition 17.4); we write $G_{sk}L$ for the record of the pattern G_{sk}, and LG_{sk} for the record of state LR with the pattern G_{sk} in the second coordinate. Their field pairs are EL and LE.

Lemma 28.1 (Records that accept the six queries)**.** *Every record of formal degree at most one and every folded record of total degree at most two accepts $\mathcal{E}_6$.*

- *Of the* 37 *two-layer records of degree two exactly nine accept $\mathcal{E}_6$: the seven of the twelve-query ledger and the two EL records $G_{01}L$, $G_{10}L$ of state RL.*
- *Of the* 37 *folded records of total degree three exactly nine accept: the seven of that ledger and the reserved companions of $G_{01}L$ and $G_{10}L$.*
- *No two-layer record of degree at least three and no folded record of total degree at least four accepts.*

So 92 *two-layer records of degrees two and three do not accept $\mathcal{E}_6$: the* 28 *of degree two, with the state multiplicities $LR\colon 6$, $RL\colon 2$, $DD\colon 6$, $DS\colon 4$, $SD\colon 6$, $SS\colon 4$, and the* 64 *of degree three. The other records that do not accept are the* 28 *folded records of total degree three and all records of total degree at least four.*

Proof. The merge patterns are $G_{sk}(j) = \mathbf{1}[E_s(j) = k]$ with the matching patterns $E_0 = (0,1,1,2,2,0)$ and $E_1 = (0,0,1,1,2,2)$, so $(G_{01}(0), G_{01}(2)) = (0,1)$ and $(G_{10}(0), G_{10}(2)) = (1,0)$. Under the emission of $G_{sk}L$ the query from $(0,v,0)$ to $(2,\mathtt{c}v,1)$ of $\mathcal{E}_6$ reads the slots

$$(G_{sk}(0), v, 0) \quad \text{and} \quad (G_{sk}(2), \mathtt{c}v, 1)$$

of the field graph $G_{\mathrm{EL},2}$, and under the emission of its reserved companion the slots $(G_{sk}(0), v)$ and $(G_{sk}(2), \mathtt{c}v)$ of $G_{\mathrm{EL},1}$, copied onto both layers. These graphs contain every edge from (i, v, ℓ) to $(j, \mathtt{c}v, \ell')$ with $i \neq j$: the E coordinate is related because $i \neq j$, and the L coordinate is forward (Section 27.2.1). Hence for $G_{01}L$, $G_{10}L$ and their companions every query of $\mathcal{E}_6$ is an edge of the field graph, and every labelling in the support accepts it. These records are records of $\mu_{\mathcal{B}}$, with its mass, invariance and actual marginals. The remaining assertions are an exact finite enumeration: `research/advance_2026_10_24/upper/six_query_geometry.py` tests all 392 latent records against $\mathcal{E}_6$ (record `six_query_geometry.json` of that directory, with the counts by layers, degree and acceptance), and `review_six_query_upper.py` of the same directory reconstructs the fold arrays and the graph predicates independently and repeats the 392 tests (`six_query_upper_review.json`, `PASS_INDEPENDENT_SIX_QUERY_UPPER`). The last sentence follows from the counts. □

Compared with the twelve-query ledger, which the tail keeps, a band thus keeps two of the four nonaccepting RL records of degree two and their reserved companions; the other two, $G_{02}L$ and $G_{12}L$, and the six LR records are replaced by the laws of Section 28.3.

The repair. Fix a nonaccepting two-layer record of degree two or three (a band repairs 84 of them, Lemma 28.1; the tail repairs all 94 of the twelve-query ledger), and a prime p of the block. Its law at p is the image under the emission of the record (Section 27.2.2) of the two-layer field law $\pi^{\mathrm{old}} = \lambda_p(G)$ (Definition 12.1), where $G = G_{\varphi_0\varphi_1,2}$ is the two-layer graph of the field types of the record, on the layers $X = U \times \{0\}$ and $Y = U \times \{1\}$, $|U| = n$. The hidden objects of merges and constants are kept as vertices of G until after the repair; the emission forgets them only at the end. The layer swap is an automorphism of G, and both layers induce the one-layer graph $H = G_{\varphi_0\varphi_1,1}$ on U (Lemma 27.4(b),(c)). Let G^{new} be the oriented graph whose edges are those of the two copies of H together with the $m_\times$ images, under the pattern maps of the emission, of the twelve queries (28.1): the cross-layer edges of G are removed and the projected queries are imposed. Put $\pi^{\mathrm{new}} = \lambda_p(G^{\mathrm{new}})$. The induced half laws of π^{old} and π^{new} coincide (both are λ_H); by the layer symmetry of G the two actual half marginals of π^{old} are one law μ^{old}; the two actual half marginals π^{new}_X, π^{new}_Y of π^{new} need not coincide. The *one-way repair* of the record is the law

$$\pi^{\mathrm{rep}} = \bigl(1 + f(x) + g(y)\bigr)\,\pi^{\mathrm{new}}$$

of Lemma 17.14 that gives both half marginals the value μ^{old} (formula (F7) below), followed by the emission. It is supported in the support of π^{new}, so it accepts all twelve queries.

This is why one-way queries can be repaired. Every imposed edge runs from layer 0 to layer 1, across the reserved cut and in one direction, so no opposite constraints arise; the new graph keeps the old induced halves, and both old actual half marginals can be restored exactly. What is lost is the symmetry between the layers, and with it positivity of the repaired law at ι; no such positivity is claimed or needed: the change is charged through its norms, the symmetric price paying the change and the skew price only the skew part of the repaired law, since the old record is symmetric at ι (Remark 9.15(iii), Proposition 28.14). The emission acts within each layer, so a test function of an emitted half is a test function of the field half: the emission preserves the half marginals at ι and cannot increase an operator norm between them. The comparison retains every accepting record, repairs 84 records, replaces the six LE records and the two nonaccepting EL records of degree two by the laws of Section 28.3, and deletes every other nonaccepting record (Definition 28.13).

The repair formula. The coefficients of a repaired record are evaluated by the following formula, which continues the list (F1)–(F6) of the upper local formulas. For a graph G' and a set S of pinned vertices let $c^+_{G'}(S) = \sum e^+_t$ and $c^-_{G'}(S) = \sum -\log(1 - e^-_t(z))$ be the logarithmic count bounds of (F1) along its fixed insertion order, so that the normalized number of injective labellings extending a pinned labelling lies in $[e^{-c^-}, e^{c^+}]$; j ranges over the two halves X, Y.

(F7) *One-way repairs.* For a repaired record at the prime p put

$$\begin{aligned}
\ell_U &= \max_j \max\bigl\{c^+_G(j) + c^-_{G^{\mathrm{new}}}(j) + c^+_{G^{\mathrm{new}}}(\emptyset) + c^-_G(\emptyset),\\
&\qquad\qquad c^+_{G^{\mathrm{new}}}(j) + c^-_G(j) + c^+_G(\emptyset) + c^-_{G^{\mathrm{new}}}(\emptyset)\bigr\}, \qquad U = e^{\ell_U} - 1,\\
\ell_{\mathrm{new}} &= \max_j\bigl(c^-_{G^{\mathrm{new}}}(j) + c^-_H(\emptyset) + c^+_{G^{\mathrm{new}}}(\emptyset)\bigr), \qquad \ell_{\mathrm{old}} = \max_j\bigl(c^-_G(j) + c^-_H(\emptyset) + c^+_G(\emptyset)\bigr),\\
K_\infty &= \bigl(2^{m_\times}\exp(2c^+_H(\emptyset) + c^-_{G^{\mathrm{new}}}(\emptyset) + 2\ell_{\mathrm{new}})\bigr)^{1/2},
\end{aligned}$$

let ρ_{new} be the bound (F2) for G^{new} between its two layers, and put

$$F = U\Bigl(1 + \frac{K_\infty}{1 - \rho_{\mathrm{new}}}\Bigr), \qquad \rho' = \frac{\rho_{\mathrm{new}}(1 + 2F) + U^2}{1 - U}.$$

Let ρ_{old} be the smaller of the bound (F2) for G between its layers and $e^{\ell_{\mathrm{old}}}(gr + B_{\mathrm{top}}r^2)$, where g and B_{top} are the constants of (F6) for the layer cut of the field pair of the record (for the pairs LE and EF those of EL and FE, by the coordinate interchange), certified at an evaluation point $p_0 \le p$ of that cut. *Hypotheses:* $U < 1$, $\rho_{\mathrm{new}} < 1$, $2F < 1$, $\rho_{\mathrm{old}} < \frac{1}{2}$ and $\rho' < \frac{1}{2}$. *Conclusion:* the one-way repair π^{rep} exists; it is a probability law, invariant under the square-affine maps, with both actual half marginals equal to μ^{old}; its centred conditional operator has norm at most ρ'; and $\pi^{\mathrm{rep}} - \pi^{\mathrm{old}}$ has zero marginals and, relative to μ^{old} on both halves, norm at most $\rho_{\mathrm{old}} + \rho'$, its skew part norm at most ρ'. The record has the *symmetric coefficient* $C^{\mathrm{s}} = (\rho_{\mathrm{old}} + \rho')/r$ and the *skew coefficient* $C^{\mathrm{a}} = \rho'/r$, each rounded upward to 10^{-9}, with ρ' the smaller value of (F8) for the ten records named there; from 2^{48} the FF, FE and EF records of degree two of the bands may use the coefficients of (F9) instead.

Proof of (F7). *Densities.* The marginal of a uniform injective graph law on a half, at a point, is the number of completions of that point divided by the number of all labellings. Writing both numbers as normalized counts and dividing, the powers of p and of 2 cancel, and (F1) gives

$$e^{-\ell_U} \le \frac{d\mu^{\mathrm{old}}}{d\pi_j^{\mathrm{new}}} \le e^{\ell_U}, \qquad \frac{d\pi_j^{\mathrm{new}}}{d\lambda_H} \ge e^{-\ell_{\mathrm{new}}}, \qquad \frac{d\mu^{\mathrm{old}}}{d\lambda_H} \ge e^{-\ell_{\mathrm{old}}} \qquad (j = X, Y);$$

the four ratios in ℓ_U are the forward and reverse ratios with either half pinned. In the same way $d\pi^{\mathrm{new}}/d(\lambda_H \otimes \lambda_H) \le 2^{m_\times} e^{2c_H^+(\emptyset) + c_{G^{\mathrm{new}}}^-(\emptyset)}$, the factor $2^{m_\times}$ coming from the $m_\times$ cross constraints, and dividing by the two marginal densities gives $\sup d\pi^{\mathrm{new}}/d(\pi_X^{\mathrm{new}} \otimes \pi_Y^{\mathrm{new}}) \le K_\infty^2$. So K_∞ bounds the conditional operator of π^{new} and its adjoint from L^2 to L^∞ (Lemma 17.14).

The repair. Put $1 + u = d\mu^{\mathrm{old}}/d\pi_X^{\mathrm{new}}$ and $1 + v = d\mu^{\mathrm{old}}/d\pi_Y^{\mathrm{new}}$. Then $|u|, |v| \le U$, and (F2) bounds the centred conditional operator of π^{new} by $\rho_{\mathrm{new}} < 1$. Lemma 17.14(d), with $\rho = \rho_{\mathrm{new}}$, gives the centred solution f, g with $\|f\|_\infty, \|g\|_\infty \le F$, and since $2F < 1$, part (c) makes π^{rep} a probability law, supported in the support of π^{new}, with both half marginals μ^{old}. The data π^{new}, μ^{old} are invariant under the square-affine maps (Lemma 5.16), so by part (a) f, g and π^{rep} are invariant. Part (f) (Lemma 17.13), with $F_X = F_Y = F$ and $e_X = e_Y = U$, bounds the centred conditional operator of π^{rep} by $(\rho_{\mathrm{new}}(1 + 2F) + U^2)/(1 - U) = \rho'$; and as π^{old} is also a coupling of μ^{old} with itself, $\pi^{\mathrm{rep}} - \pi^{\mathrm{old}}$ has zero marginals and norm at most $\rho' + \rho(\pi^{\mathrm{old}})$, where $\rho(\pi^{\mathrm{old}})$ is the centred conditional norm of π^{old}.

The replaced record. It remains to show $\rho(\pi^{\mathrm{old}}) \le \rho_{\mathrm{old}}$. The first bound is (F2). For the second let $\mathcal{A}$ be the operator on $L^2(\lambda_H)$ with kernel $d\pi^{\mathrm{old}}/d(\lambda_H \otimes \lambda_H)$: the normalized cut operator of the layer cut of G, whose half law is λ_H (Lemma 13.3). By (F6), that is by Theorem 14.21(b),(c) at the prime $p \ge p_0$, $\|Q\mathcal{A}Q\| \le gr$ and $\|\mathcal{A}1 - 1\|^2 = \|Q\mathcal{A}1\|^2 \le \varepsilon_\varrho(r)^2 \le r^2 p_0 \varepsilon_\varrho(r_0)^2 \le B_{\mathrm{top}} r^2$, with $r_0 = p_0^{-1/2}$ and $B_{\mathrm{top}} = p_0 \varepsilon_\varrho(r_0)^2/(1 - gr_0)$, the B^+ of Theorem 14.21(d). As $\mathcal{A}1 = \mathcal{A}^*1 = d\mu^{\mathrm{old}}/d\lambda_H \ge e^{-\ell_{\mathrm{old}}}$, Lemma 13.19(b), with $z = 1$, $\varepsilon = 0$, $c = gr$, $M_L = M_R = B_{\mathrm{top}}^{1/2} r$ and $l_L = l_R = \ell_{\mathrm{old}}$, gives $\rho(\pi^{\mathrm{old}}) \le e^{\ell_{\mathrm{old}}}(gr + B_{\mathrm{top}} r^2)$ (Remark 13.20(2)): the spectral certificate for the half law controls the correlation relative to the actual marginals.

The skew part. The layer swap is an automorphism of G, so the old record is symmetric at ι and its centred operator is self-adjoint; the skew part of $\pi^{\mathrm{rep}} - \pi^{\mathrm{old}}$ is therefore that of π^{rep} alone (Remark 9.15(iii)), of norm at most ρ'. □

28.2. **Star terms in the one-way replacements.** A one-way replacement imposes new cross edges between the two layers. The general bound (F2) charges for every new edge separately, through the influence matrix. For ten records of the twelve-query list retained in this range—the six LR records and the four nonaccepting RL records of degree two (field pairs LE and EL)—there are too many new edges for that bound: their halves have $n = 18$ labels, with six new edges

forming two disjoint three-edge paths (LR), or twelve forming six disjoint two-edge stars with all centres in one layer (RL). Charged edge by edge, these ten records would dominate the degree-two prices at the tail primes. The following lemma instead expands the whole new cross kernel into its star terms (bounded together), the remaining supports and the coincidences, giving a sharper bound for the correlation ρ_{new} of the new law. Nothing else in the repair formula (F7) changes. At 2^{54} the lemma gives the joint-star bounds used for the tail primes (Certificate 29.4). In the bands these ten records are kept, mixed or replaced densely, not repaired one-way (Lemma 28.1 and Propositions 28.5 and 28.6); there the setting of the lemma is reused for the records of Proposition 28.3.

Setting. Let H be an oriented graph on n vertices with maximum degree at most Δ, let λ_H be its uniform injective law, and let G^{new} consist of two copies of H, on layers X and Y, and of $m_\times$ cross edges, each joining a vertex $u \in X$ and a vertex $v \in Y$ in one of the two directions, whose bipartite graph $\mathbf{D}$ has maximum degree at most two and no cycle of length four (a union of paths, for instance). $\mathbf{D}$ also denotes its signed $n \times n$ incidence matrix: $\mathbf{D}_{uv} = 1$ for an edge from u to v, $\mathbf{D}_{uv} = \chi(-1)$ for an edge from v to u, and $\mathbf{D}_{uv} = 0$ otherwise. (The new graphs of the one-way repairs have only edges from X to Y; reversed edges occur in the replaced records of Proposition 28.3.) Let N_X and N_Y be the numbers of vertices of cross degree two in X and in Y, and put $N_1 = m_\times$, $N_2 = N_X + N_Y$, $N_* = 2^{m_\times} - 1 - N_1 - N_2$. With the lower insertion error $e^-_\Delta(z) = a_\Delta r + (b_\Delta + z2^\Delta)r^2$ of (12.2) put

$$\kappa_{\mathrm{pr}} = \Big(\frac{2^\Delta}{1 - e^-_\Delta(n-1)}\Big)^2, \qquad C_{\mathrm{col}} = n^2 2^{m_\times},$$
$$\mathcal{R} = (N_*\kappa_{\mathrm{pr}} + C_{\mathrm{col}})q, \qquad Z_- = 1 - (N_2 + N_*\kappa_{\mathrm{pr}} + C_{\mathrm{col}})q.$$

Let $\mathcal{E} = (\varepsilon_{ij})$ be the influence matrix of H for the empty pinning, formed with the coarse count error $w_\Delta = w_\Delta(p,n) = 2^\Delta(\frac{\Delta}{2}r + nr^2)$ (Definition 13.10, (12.1)), e_{inf} its largest row sum, $\eta = e_{\mathrm{inf}}^2/(1 - e_{\mathrm{inf}})$, $\Gamma = 1 + e_{\mathrm{inf}} + (n-1)\eta$ and $\gamma = 2 - \Gamma$. For a vertex c of cross degree two in X, with cross neighbours w, w', and a vertex k of Y put

$$\mathbf{C}_X[c,k] = \begin{cases} q(q+r)/(1-q), & k \in \{w, w'\}, \\ r\big((\varepsilon_{wk}^2 + \varepsilon_{w'k}^2)^{1/2} + \eta\big), & k \notin \{w, w'\}, \end{cases}$$

and define $\mathbf{C}_Y$ in the same way with the layers exchanged. Kernels on (x, y) are operators from $L^2(\lambda_H)$ (functions of y) to $L^2(\lambda_H)$ (functions of x), relative to $\lambda_H \otimes \lambda_H$. For $S \subseteq \mathbf{D}$ put $\chi_S(x,y) = \prod_{(u,v)\in S} \chi(y_v - x_u)$ and $\mathbf{D}_S = \prod_{(u,v)\in S} \mathbf{D}_{uv}$. A *two-star* is a pair of cross edges with a common vertex, its *centre*; there is one at each vertex of cross degree two. The *centred star operator* is

$$\mathcal{S}^{\mathrm{c}} = \sum_{(u,v)\in\mathbf{D}} \mathbf{D}_{uv}\chi_{\{(u,v)\}} + \mathcal{S}_X + \mathcal{S}_Y, \qquad \mathcal{S}_Z = \sum_{S \text{ a two-star with centre in } Z} \mathbf{D}_S(\chi_S + q) \quad (Z = X, Y).$$

Let $\mathcal{V} \subseteq L^2_0(\lambda_H)$, the *singleton span*, be spanned by the functions $h(x_i)$ with $h \in L^2_0(\mathbb{F}_p)$ and i a vertex; let $\mathcal{V}_X$, $\mathcal{V}_Y$ be its copies in the two halves, and $P_{\mathcal{V}_X}$, $P_{\mathcal{V}_Y}$ the orthogonal projections onto them.

Lemma 28.2 (Upper stars)**.** *In this setting assume that $w_\Delta < \frac{1}{2}$ and $e^-_{\Delta+1}(2n) < 1$, so that w_Δ and $e^-_{\Delta+1}(2n)$ are count errors for H and its empty pinning (Definition 13.10(i)); that $e_{\mathrm{inf}} < \frac{1}{4}$; that $e^-_\Delta(n-1) < 1$ and $Z_- > 0$; and let $e^{-\ell_{\mathrm{new}}}$ be a lower bound for both densities $d\pi^{\mathrm{new}}_X/d\lambda_H$ and $d\pi^{\mathrm{new}}_Y/d\lambda_H$, where $\pi^{\mathrm{new}} = \lambda_p(G^{\mathrm{new}})$. Put $\xi_1 = \mathcal{R}/Z_-$ and*

$$b_{\mathrm{raw}} = \Big(N_1 + N_2\Big(\frac{2}{1-q}\Big)^{1/2}\Big)r + \mathcal{R}.$$

(a) Forest stars. *The centred conditional operator of* π^{new} *has norm at most* $e^{\ell_{\mathrm{new}}}(\xi_0+\xi_0^2)$, *where* $\xi_0 = b_{\mathrm{raw}}/Z_-$.

(b) Joint stars, every odd prime. *The centred star operator* $\mathcal{S}^{\mathrm{c}}$ *has norm at most*

$$b_{\mathrm{all}} = r\Big(\Gamma\|\mathbf{D}\| + \Big(\frac{2\Gamma N_X}{1-q}\Big)^{1/2} + \Big(\frac{2\Gamma N_Y}{1-q}\Big)^{1/2}\Big), \tag{28.2}$$

and the centred conditional operator of π^{new} *has norm at most* $e^{\ell_{\mathrm{new}}}(\xi+\xi_1^2)$ *with* $\xi = (b+\mathcal{R})/Z_-$, $b = b_{\mathrm{all}}$. *No orthogonality between the input pair functions is needed.*

(c) Singleton projections, $p \equiv 3 \pmod 4$. *If* $\gamma > 0$, *then*

$$\|\mathcal{S}_X P_{\mathcal{V}_Y}\| \le \Big(\frac{\Gamma}{\gamma}\Big)^{1/2}\|\mathbf{C}_X\|_{\mathrm{F}}, \qquad \|P_{\mathcal{V}_X}\mathcal{S}_Y\| \le \Big(\frac{\Gamma}{\gamma}\Big)^{1/2}\|\mathbf{C}_Y\|_{\mathrm{F}}.$$

(d) Targets. *If* μ^{old} *is a probability law with* $\|d\mu^{\mathrm{old}}/d\lambda_H - 1\|_{L^2(\lambda_H)} \le \epsilon_{\mathrm{old}}$, *then* $u = d\mu^{\mathrm{old}}/d\pi_X^{\mathrm{new}} - 1$ *and* $v = d\mu^{\mathrm{old}}/d\pi_Y^{\mathrm{new}} - 1$ *satisfy* $\|u\|^2_{L^2(\pi_X^{\mathrm{new}})}, \|v\|^2_{L^2(\pi_Y^{\mathrm{new}})} \le e^{\ell_{\mathrm{new}}}(\epsilon_{\mathrm{old}}+\xi_1)^2$.

The lemma applies to every graph of this form, the new graph of a repair or the two-layer graph of a replaced record.

Proof. Step 0: coordinates and embeddings. The law λ_H is invariant under the square-affine maps $z \mapsto sz+c$ (s a nonzero square, $c \in \mathbb{F}_p$) applied to all labels (Lemma 5.16). Hence every coordinate is uniform on $\mathbb{F}_p$, and every pair of distinct coordinates has a law invariant under the maps, supported off the diagonal; the maps have two orbits there, $\{b'-b$ a square$\}$ and its complement, each of size $p(p-1)/2$, so the pair law has density at most $2p/(p-1) = 2/(1-q)$ with respect to the uniform law on $\mathbb{F}_p^2$. Both halves are copies of $(\mathbb{F}_p^U, \lambda_H)$, and the following embeddings are used in either of them. For a vertex i let $\Phi_i\colon L_0^2(\mathbb{F}_p) \to L^2(\lambda_H)$, $\Phi_i h = h(x_i)$, an isometry, and let $\Phi\colon \bigoplus_i L_0^2(\mathbb{F}_p) \to L^2(\lambda_H)$, $(h_i) \mapsto \sum_i \Phi_i h_i$. The diagonal blocks of $\Phi^*\Phi$ are identities. An off-diagonal block $\Phi_i^*\Phi_j$ has norm the maximal correlation $\rho(x_i;x_j)$. The matrix $\mathcal{E}$ is symmetric and nonnegative, so $\|\mathcal{E}\| \le e_{\inf} < 1$, and Proposition 13.16(a), for the empty pinning with $e = e_{\inf}$, gives $\rho(x_i;x_j) \le \varepsilon_{ij} + \eta$. The block Schur test gives $\|\Phi^*\Phi - I\| \le e_{\inf} + (n-1)\eta$, hence

$$\gamma I \preceq \Phi^*\Phi \preceq \Gamma I,$$

and the same for the restriction of Φ to any set of coordinates. The range of Φ is the singleton span $\mathcal{V}$, and $\mathcal{V}_X$, $\mathcal{V}_Y$ are its copies in the two halves. For disjoint sets A, A' of vertices, Proposition 13.16(b) also gives $\rho(x_A; x_{A'}) \le \|\mathcal{E}_{AA'}\| + \eta$.

Step 1: the kernel. On labellings with all $2n$ labels distinct, every cross difference $y_v - x_u$ is nonzero, and the indicator of the edge between u and v is $\frac{1}{2}(1+\mathbf{D}_{uv}\chi(y_v-x_u))$: for an edge from v to u the constraint is that $x_u - y_v$ is a square, and $\chi(x_u-y_v) = \chi(-1)\chi(y_v-x_u)$. Hence π^{new} has density K_0/Z_0 with respect to $\lambda_H \otimes \lambda_H$, where $K_0 = \mathbf{1}\{\text{all labels distinct}\}\prod_{(u,v)\in\mathbf{D}}(1+\mathbf{D}_{uv}\chi(y_v-x_u))$ and $Z_0 = \mathbb{E}K_0$. Expanding the product,

$$K_0 = \sum_{S\subseteq\mathbf{D}} \mathbf{D}_S\chi_S - K_{\mathrm{col}},$$

where $0 \le K_{\mathrm{col}} \le 2^{m\times}\mathbf{1}\{\text{some } x_i = y_j\}$. Three pairwise meeting edges of a bipartite graph share a vertex; as every vertex has cross degree at most two, a nonempty $S \subseteq \mathbf{D}$ in which any two edges meet is a single edge or a pair of edges at a common vertex (a *two-star*); there are N_1 of the first kind and N_2 of the second, one at each vertex of cross degree two. Every other nonempty S contains two disjoint edges $(u,v),(u',v')$, and since $\mathbf{D}$ has no 4-cycle at most one of $(u,v'),(u',v)$ lies in $\mathbf{D}$. The signs $\mathbf{D}_S$ have modulus one and play no role in the bounds below.

Step 2: single edges. Let $\mathcal{P}$ act on $L^2(\mathbb{F}_p)$ by $\mathcal{P}h(a) = \mathbb{E}_b\chi(b-a)h(b)$, the Paley operator of Lemma 4.11. By part (c) of that lemma $\|\mathcal{P}\| = r$ and $\mathcal{P}1 = \mathcal{P}^*1 = 0$, so $\mathcal{P}$ annihilates the

constants and maps into L^2_0. Since y_v is uniform, the kernel $\chi(y_v - x_u)$ is $\Phi_u \mathcal{P} \Phi_v^*$, of norm at most r and mean zero, and the sum of the single-edge kernels $\mathbf{D}_{uv}\chi(y_v - x_u)$ is

$$\mathcal{S}_{\mathrm{e}} = \Phi\,(\mathbf{D} \otimes \mathcal{P})\,\Phi^*, \qquad \|\mathcal{S}_{\mathrm{e}}\| \le \Gamma \|\mathbf{D}\| r.$$

It maps into $\mathcal{V}_X$, the singleton span of the layer X, and vanishes on the orthogonal complement of $\mathcal{V}_Y$ in L^2_0.

Step 3: two-stars. Let $c \in X$ have cross neighbours $w, w' \in Y$, let ϱ be the law of $(y_w, y_{w'})$ and let $\mathcal{K}\colon L^2(\varrho) \to L^2(\mathbb{F}_p)$ have kernel $\chi(b-a)\chi(b'-a)$, a the centre and (b, b') the leaves. The kernel χ_S of the star is the operator $h \mapsto (\mathcal{K}\Phi^*_{ww'}h)(x_c)$, with $\Phi^*_{ww'}h = \mathbb{E}[h \mid y_w, y_{w'}]$. Its constant images are constant: by (4.2), $\mathcal{K}^*1(b,b') = \mathbb{E}_a \chi(b-a)\chi(b'-a) = -q$ on the support $b \ne b'$ of ϱ, and $\mathcal{K}1$ is constant because ϱ is translation invariant; hence $\mathcal{K}1 = \langle \mathcal{K}^*1, 1\rangle = -q$ and

$$\mathcal{K} = Q\mathcal{K}Q - q\,(1 \otimes 1), \qquad \chi_S = \Phi_c(Q\mathcal{K}Q)\Phi^*_{ww'} - q\,(1 \otimes 1),$$

since for $h = h_0 + \mathbb{E}h$ with h_0 centred, $\mathcal{K}h = \mathcal{K}h_0 - q\mathbb{E}h$ and $\mathbb{E}\mathcal{K}h_0 = \langle \mathcal{K}^*1, h_0\rangle = 0$. In particular the mean of a two-star is $-q$. For the norm, let $\mathcal{K}_{\mathrm{u}}$ be the operator with the same kernel on the uniform law of $\mathbb{F}_p^2$; $\mathcal{K}_{\mathrm{u}}\mathcal{K}_{\mathrm{u}}^*$ has kernel $(\mathbf{1}\{a = a'\} - q)^2$, which is the operator $q(1-2q)I + q^2\mathbb{J}$, of norm $q(1-q) \le q$. As $\mathcal{K}h = \mathcal{K}_{\mathrm{u}}(h\,d\varrho/d\,\mathrm{Unif})$ and $\|h\,d\varrho/d\,\mathrm{Unif}\,\|^2 \le \frac{2}{1-q}\|h\|^2_\varrho$,

$$\|Q\mathcal{K}Q\| \le \|\mathcal{K}\| \le \Big(\frac{2}{1-q}\Big)^{1/2} r.$$

A star with centre in Y and leaves $w, w' \in X$ is treated through its adjoint, which has the same form ($\chi(b-a)\chi(b-a') = \chi(a-b)\chi(a'-b)$). Group the centred stars $\mathbf{D}_S(\chi_S + q)$ by their centres; there is one star at each centre. For centres in X the sum is $\mathcal{S}_X = \Phi \circ \Psi$ with $\Psi h = (\mathbf{D}_{S_c} Q\mathcal{K}_c Q \Phi^*_{w_c w'_c} h)_c$, and $\|\Psi h\|^2 = \sum_c \|Q\mathcal{K}_c Q \Phi^*_{w_c w'_c} h\|^2 \le N_X \beta_\star^2 \|h\|^2$, where $\beta_\star$ bounds every $\|Q\mathcal{K}_c Q\|$; with the embedding bound of Step 0, $\|\mathcal{S}_X\| \le (\Gamma N_X)^{1/2}\beta_\star$, and likewise for the sum $\mathcal{S}_Y$ over the centres in Y, $\|\mathcal{S}_Y\| \le (\Gamma N_Y)^{1/2}\beta_\star$. Thus $\mathcal{S}^{\mathrm{c}} = \mathcal{S}_{\mathrm{e}} + \mathcal{S}_X + \mathcal{S}_Y$; it annihilates the constants and maps into L^2_0. With $\beta_\star = (2/(1-q))^{1/2} r$ the triangle inequality gives $\|\mathcal{S}^{\mathrm{c}}\| \le b_{\mathrm{all}}$, the first claim of (b).

Step 4: the other supports. Let S contain the disjoint edges $(u,v), (u',v')$ of Step 1. Condition on the labels $\hat{x}$ of $X \setminus \{u, u'\}$ and $\hat{y}$ of $Y \setminus \{v, v'\}$; under $\lambda_H \otimes \lambda_H$ the two halves are independent. Given $\hat{x}$, the pair $(x_u, x_{u'})$ is uniform on the admissible pairs; inserting x_u and then $x_{u'}$, each with at most Δ constraints to labelled vertices and at most $n-1$ excluded labels, Lemma 12.2(c) gives at least $\big(p2^{-\Delta}(1 - e^-_\Delta(n-1))\big)^2$ admissible pairs, since $2^{-t}(1 - e^-_t(z))$ is nonincreasing in t and z (property (E4) of Section 12.1). So the conditional law of $(x_u, x_{u'})$ has density at most κ_{pr} with respect to the uniform law of $\mathbb{F}_p^2$, and the same holds for $(y_v, y_{v'})$. Given $(\hat{x}, \hat{y})$, the edges of S inside the conditioned labels give a constant of modulus at most one, the edges with one end among the four free labels give one-sided multipliers of modulus at most one, and the edges among the free labels give the kernel $\chi(b-a)\chi(b'-a')$, possibly times $\chi(b'-a)$ or $\chi(b-a')$, with $a = x_u$, $a' = x_{u'}$, $b = y_v$, $b' = y_{v'}$. The two free matching edges, together with at most one of the other rectangle edges, satisfy the acyclic case of Lemma 16.4(a) with $k = 2$, so this kernel has norm at most $r^2 = q$ under the uniform law on $\mathbb{F}_p^2 \times \mathbb{F}_p^2$. The multipliers are absorbed into the test functions, and converting the densities costs $\kappa_{\mathrm{pr}}^{1/2}$ on each side, as in Step 3. So the conditional form is at most $\kappa_{\mathrm{pr}} q$ times the conditional norms of the test functions, and Cauchy–Schwarz over $(\hat{x}, \hat{y})$ gives $\|\mathbf{D}_S \chi_S\| \le \kappa_{\mathrm{pr}} q$. The within-half injectivity is part of the conditional laws and is kept. Summing, the remaining supports have total norm at most $N_* \kappa_{\mathrm{pr}} q$, and each has mean of modulus at most $\kappa_{\mathrm{pr}} q$.

Step 5: coincidences. As $0 \le K_{\mathrm{col}} \le 2^{m\times} \sum_{i,j} \mathbf{1}\{x_i = y_j\}$ and every coordinate is uniform, every row and column integral of K_{col} is at most $2^{m\times} n^2 q$; Schur's test gives $\|K_{\mathrm{col}}\| \le C_{\mathrm{col}} q$, and $0 \le \mathbb{E}K_{\mathrm{col}} \le C_{\mathrm{col}} q$.

Step 6: assembly. By Steps 1–5, $K_0 = (1 - q\sum_S \mathbf{D}_S)(1 \otimes 1) + \mathcal{S}^{\mathrm{c}} + \mathcal{R}_* - K_{\mathrm{col}}$, the sum over the two-stars S, where $\mathcal{R}_*$ is the sum of the kernels $\mathbf{D}_S\chi_S$ of Step 4. Single edges and centred stars have mean zero, so $Z_0 = 1 - q\sum_S \mathbf{D}_S + \mathbb{E}(\mathcal{R}_* - K_{\mathrm{col}})$ and $|Z_0 - 1| \le (N_2 + N_*\kappa_{\mathrm{pr}} + C_{\mathrm{col}})q = 1 - Z_- < 1$. Let $\mathcal{A}$ be the operator with kernel K_0/Z_0, the density of π^{new}. Since $\mathcal{S}^{\mathrm{c}}$ annihilates the constants and maps into L^2_0,

$$QK_0Q = \mathcal{S}^{\mathrm{c}} + Q(\mathcal{R}_* - K_{\mathrm{col}})Q, \qquad QK_01 = Q(\mathcal{R}_* - K_{\mathrm{col}})1, \qquad QK_0^*1 = Q(\mathcal{R}_* - K_{\mathrm{col}})^*1,$$

of norms at most $\|\mathcal{S}^{\mathrm{c}}\| + \mathcal{R}$, $\mathcal{R}$ and $\mathcal{R}$; in particular $\|Q\mathcal{A}1\|, \|Q\mathcal{A}^*1\| \le \xi_1$. The actual marginals are $\pi_X^{\mathrm{new}} = (\mathcal{A}1)\lambda_H$ and $\pi_Y^{\mathrm{new}} = (\mathcal{A}^*1)\lambda_H$, with densities at least $e^{-\ell_{\mathrm{new}}}$. Lemma 13.19(b), applied to the kernel K_0 with $z = Z_0$, $\varepsilon = 1 - Z_-$, $c = \|\mathcal{S}^{\mathrm{c}}\| + \mathcal{R}$, $M_L = M_R = \mathcal{R}$ and $l_L = l_R = \ell_{\mathrm{new}}$, bounds the centred conditional norm of π^{new} by $e^{\ell_{\mathrm{new}}}((\|\mathcal{S}^{\mathrm{c}}\| + \mathcal{R})/Z_- + \xi_1^2)$, which is (b) with any bound b for $\|\mathcal{S}^{\mathrm{c}}\|$. For (a), the triangle inequality over the supports, with the norms r and $(2/(1-q))^{1/2}r$ of single edges and two-stars (Steps 2 and 3), gives $\|K_0 - 1 \otimes 1\| \le b_{\mathrm{raw}}$, which bounds $\|QK_0Q\|$, $\|QK_01\|$ and $\|QK_0^*1\|$; the same lemma with $c = M_L = M_R = b_{\mathrm{raw}}$ gives $e^{\ell_{\mathrm{new}}}(\xi_0 + \xi_0^2)$.

Step 7: singleton projections. Let $p \equiv 3 \pmod 4$. For a two-star with centre $c \in X$ and leaves $S = \{w, w'\}$, the operator $Q\mathcal{K}_cQ\Phi^*_{ww'}$ of Step 3 is W_S^*, the adjoint of the star feature of Lemma 14.4 for the law λ_H of the half Y: both send h to the mean-zero part of $a \mapsto \mathbb{E}[\chi(a - y_w)\chi(a - y_{w'})h(y)]$, and $Q\mathcal{K}_c1 = 0$. By part (a) there, with $w_S = -q$, $W_S^*W_S = q(1 - 2q - q^2)/(1-q)\, I \preceq qI$, so $\|Q\mathcal{K}_cQ\| \le r$; the unknown imbalance of the two orbits of the leaf pair drops out because $\chi(-1) = -1$, which is one of the places where the class-three primes are cheaper. For $k \in \{w, w'\}$ part (d) gives $\|W_S^*\Phi_k\| \le q\sqrt{1+q}\, r/(1-q) \le q(q+r)/(1-q) = \mathbf{C}_X[c,k]$. For a vertex $k \notin \{w, w'\}$, W_S takes values in the functions of $(y_w, y_{w'})$, so $W_S^*\Phi_k = W_S^*\Phi_{ww'}\Phi^*_{ww'}\Phi_k$, and $\Phi^*_{ww'}\Phi_k$ has norm at most $\rho((y_w, y_{w'}); y_k) \le (\varepsilon^2_{wk} + \varepsilon^2_{w'k})^{1/2} + \eta$ by Step 0; so $\|W_S^*\Phi_k\| \le \mathbf{C}_X[c,k]$. Let now $g \in \mathcal{V}_Y$. As $\Phi^*\Phi \succeq \gamma I$ with $\gamma > 0$, $g = \Phi a$ with $\|a\|^2 \le \|g\|^2/\gamma$, and $\mathcal{S}_X g = \Phi(Ma)$, where the block matrix $M = (\mathbf{D}_{S_c}Q\mathcal{K}_cQ\Phi^*_{w_cw_c'}\Phi_k)_{c,k}$ has norm at most the norm of the matrix of its block norms, hence at most $\|\mathbf{C}_X\|_{\mathrm{F}}$. So $\|\mathcal{S}_XP_{\mathcal{V}_Y}\| \le (\Gamma/\gamma)^{1/2}\|\mathbf{C}_X\|_{\mathrm{F}}$, and in the same way, with the layers exchanged, $\|P_{\mathcal{V}_X}\mathcal{S}_Y\| = \|\mathcal{S}_Y^*P_{\mathcal{V}_X}\| \le (\Gamma/\gamma)^{1/2}\|\mathbf{C}_Y\|_{\mathrm{F}}$; repeated leaf supports cause no difficulty. This is (c).

Step 8: targets. Write $d\mu^{\mathrm{old}}/d\lambda_H = 1 + w_{\mathrm{old}}$ and $d\pi_X^{\mathrm{new}}/d\lambda_H = 1 + w_{\mathrm{new}}$, so $w_{\mathrm{new}} = Q\mathcal{A}1$ and $\|w_{\mathrm{new}}\| \le \xi_1$. Then

$$\|u\|^2_{L^2(\pi_X^{\mathrm{new}})} = \int \frac{(w_{\mathrm{old}} - w_{\mathrm{new}})^2}{1 + w_{\mathrm{new}}}\, d\lambda_H \le e^{\ell_{\mathrm{new}}}(\epsilon_{\mathrm{old}} + \xi_1)^2,$$

and likewise for v; no change of reference and no normalizer is involved. □

The two cross graphs of the ten records give $(N_1, N_2, N_*) = (6, 4, 53)$, $\|\mathbf{D}\| = (1 + \sqrt{5})/2$ and $N_X = N_Y = 2$ for the two paths with three edges (the six LR records), and $(N_1, N_2, N_*) = (12, 6, 4077)$, $\|\mathbf{D}\| = \sqrt{2}$ and $\{N_X, N_Y\} = \{6, 0\}$ for the six two-edge stars (the four RL records: $N_X = 6$ for two of them, $N_Y = 6$ for the other two; b_{all} is symmetric in N_X, N_Y); the geometry checks the component sizes, the support counts and every matching rectangle used in Step 4. As $r \to 0$ the coefficients b_{all}/r tend to $(1 + \sqrt{5})/2 + 4 \approx 5.618$ and $\sqrt{2} + \sqrt{12} \approx 4.878$. The lemma is applied to the ten records of the tail through the following formula.

(F8) *Joint stars* (Lemma 28.2(a),(b),(d)) for the ten EL/LE records of degree two of the twelve-query ledger, at the tail, with $n = 18$, $\Delta = 4$, so that $\kappa_{\mathrm{pr}} = 256/(1 - e_4^-(17))^2$, $C_{\mathrm{col}} = 18^2 2^{m\times}$ and $\Gamma = 1 + e_{\mathrm{inf}} + 17\eta$. In (F7) replace ρ_{new} by the smallest of the bound (F2) and the bounds of Lemma 28.2(a),(b), and U^2 by

$$\epsilon_{\mathrm{tg}}^2 = \min\{U^2,\ e^{\ell_{\mathrm{new}}}(B_{\mathrm{top}}^{1/2}r + \xi_1)^2\},$$

keeping U and F: $\rho' = (\rho_{\mathrm{new}}(1+2F)+\epsilon_{\mathrm{tg}}^2)/(1-U)$. *Hypotheses:* internal degree at most four; $w_\Delta < \frac{1}{2}$ and $e^-_{\Delta+1}(2n) < 1$; $e_{\inf} < \frac{1}{4}$; $Z_- > 0$ and $1 - e_4^-(17) > 0$; repaired correlation below $\frac{1}{2}$.

Proof of (F8). The old actual half marginal satisfies $\|d\mu^{\mathrm{old}}/d\lambda_H - 1\| = \|Q\mathcal{A}_{\mathrm{old}}1\| \le B_{\mathrm{top}}^{1/2} r$, with $\mathcal{A}_{\mathrm{old}}$ the operator $\mathcal{A}$ of the proof of (F7), so Lemma 28.2(d) applies with $\epsilon_{\mathrm{old}} = B_{\mathrm{top}}^{1/2} r$, and $\|u\|_2\|v\|_2 \le \epsilon_{\mathrm{tg}}^2$. The original multiplier cap F, computed with the bound (F2), still bounds $\|f\|_\infty$ and $\|g\|_\infty$. Part (f) of Lemma 17.14, with $e_X e_Y \le \epsilon_{\mathrm{tg}}^2$ and any valid bound for the centred conditional norm of π^{new}, gives the new ρ'. □

All the scalar majorants of (F7) and (F8), divided by their leading power of r, are nondecreasing in r at fixed spectral constants g, B_{top}; Lemma 28.11 below uses this. The tail uses the all-prime form (28.2).

28.3. **Replacements keeping the old cross edges.** Three further kinds of degree-two replacement keep some of the old cross edges, so the old and new kernels must be compared in one common system: from 2^{48} the sixteen FF, FE and EF repairs (Proposition 28.3), and in every band the two remaining EL records and the six LE records, which are replaced by dense positive laws (Propositions 28.5 and 28.6). In each case the kernels are expanded in characters (Lemma 28.2, Corollary 16.17), and their complete star parts are written using one fixed list of functions of the common half law—the single-label functions and the star functions of Lemma 14.4 and Corollary 14.5. A star part is then an explicit small matrix in these functions; for the dense laws, the mixture and the FE and EF repairs, the old and the new star part are two such matrices in the same functions, and the norm of their difference is checked by an exact rational test (Remark 14.7). The nonstar terms, the coincidences, the normalizing constants and the reweighting to the old actual marginals are bounded separately. The replacing law is positive, accepts its queries and has the actual half marginals of the old record, so when the two kernels are compared in one fixed half chart the rank-one terms from centring cancel in the difference (Lemma 17.16(c)). A forest repair is in addition bounded on its own, its marginal variance entering only its full and symmetric bounds, by part (b) there. The symmetric and the skew part of the change are bounded separately; each proposition lists the finite inequalities (gates) that carry its bounds from one prime to blocks (Lemma 28.11(ii)). Throughout this subsection $p \equiv 3 \pmod 4$.

Forest records. The FF, FE and EF records of degree two (states DD, DS, SD; six, four and six records) have half graphs H with $n = 9$ labels and maximum degree three (FF) or two (FE, EF). The new cross graph of each has maximum degree two and no cycle of length four: six cross edges forming a matching for FF, three paths with two edges for EF, and two hexagons for FE. So has the old cross graph of an FE or EF record, two hexagons with edges in both directions. The certificates check these properties on every nonempty cross-edge subset. The setting of Lemma 28.2 therefore applies to the new graph G^{new} and, for FE and EF, to the old graph G. The bound (F2) for the new law pays every cross edge through the influence matrix; here the star part of the new law is a fixed matrix in the actual features, and for FE and EF the old and the new star part are compared in one synthesis.

Proposition 28.3 (Degree-two FF, FE and EF records)**.** *Let $\pi^{\mathrm{old}} = \lambda_p(G)$ be one of the sixteen FF, FE and EF records of degree two, with its one-way repair π^{rep} of* (F7) *and the quantities F, ℓ_{old}, B_{top}, ρ_{old} and ρ' of that formula. Take the setting of Lemma 28.2 for G^{new}, with Δ the largest degree of H, except that κ_{pr} may be replaced by its larger value $256/(1-e_4^-(n-1))^2$ (property* (E4)*), and put $E_{\mathrm{proj}} = (\Gamma/\gamma)^{1/2}(\|\mathbf{C}_X\|_{\mathrm{F}} + \|\mathbf{C}_Y\|_{\mathrm{F}})$. Let $\mathcal{S}$ be the set of leaf pairs of the two-stars of G^{new}, and $\Gamma_W = 1 + \max_{S\in\mathcal{S}} \sum_{T\ne S} N_{ST}$, where $N_{ST} = \rho_S$ for disjoint S, T and $N_{ST} = 2\rho_{ab}/(1-q) + q(q+r)^2/(1-q)^2$ for $S = \{a,b\}$, $T = \{a,b'\}$; here ρ_S and ρ_{ab} are the bounds*

(F2) *for the maximal correlation of x_S and $x_{U\setminus S}$ and for the conditional one of x_b and $x_{U\setminus\{a,b\}}$ given x_a; set $\Gamma_W = 1$ when $\mathcal{S}$ is empty. Put $\Gamma_F = \max(\Gamma, \Gamma_W)$, let k_{f}, k_{s}, k_{a} bound the norms of the star matrix $M(i)$ of $\mathbf{D}$ and of its symmetric and skew parts (Corollary 14.5), and put*

$$c = \frac{\Gamma_F k_{\mathrm{f}} r + E_{\mathrm{proj}} + \mathcal{R}}{Z_-}, \quad s = \frac{\Gamma_F k_{\mathrm{s}} r + E_{\mathrm{proj}} + \mathcal{R}}{Z_-}, \quad a = \frac{\Gamma_F k_{\mathrm{a}} r + E_{\mathrm{proj}} + \mathcal{R}}{Z_-}, \quad e = \frac{\mathcal{R}}{Z_-}.$$

(a) Suppose that the hypotheses of (F7) *and of Lemma 28.2 hold, that $\gamma > 0$, and that the correlations above satisfy the hypotheses of* (F2)*. Relative to μ^{old} on both halves, the centred operator of π^{rep} has full, symmetric and skew norms at most*

$$c_{\mathrm{new}} = \min\{\rho',\, e^{\ell_{\mathrm{old}}}(c + 2F(c+e) + B_{\mathrm{top}} q)\},$$
$$s_{\mathrm{new}} = e^{\ell_{\mathrm{old}}}(s + 2F(c+e) + B_{\mathrm{top}} q), \qquad a_{\mathrm{new}} = e^{\ell_{\mathrm{old}}}(a + 2F(c+e)),$$

and the change $\pi^{\mathrm{rep}} - \pi^{\mathrm{old}}$ has symmetric part of norm at most $\rho_{\mathrm{old}} + s_{\mathrm{new}}$ and skew part of norm at most a_{new}.

(b) Let the record be FE or EF, and form Γ_F with the leaf pairs of the two-stars of G and of G^{new} together. For $h \in \{\mathrm{old}, \mathrm{new}\}$ let E_h, $\mathcal{R}_h$ and $N_{2,h}$ be the quantities E_{proj}, $\mathcal{R}$ and N_2 of G and of G^{new}, let k_h bound the norm of the star matrix of the cross graph of G, respectively G^{new}, and k_Δ that of the symmetric part of their difference in one synthesis. Put $\varepsilon_h = N_{2,h} q + \mathcal{R}_h$ and $U_h = \Gamma_F k_h r + E_h + \mathcal{R}_h$. If $\varepsilon_h < \frac{1}{2}$ for both h, the symmetric part of the change has norm at most $d_{\mathrm{sym}} = e^{\ell_{\mathrm{old}}}(d_{\mathrm{raw}} + 2F(c+e))$, with c, e of (a) and

$$d_{\mathrm{raw}} = \Gamma_F k_\Delta r + \sum_h \Big(E_h + \mathcal{R}_h + \frac{\varepsilon_h U_h}{1 - \varepsilon_h} \Big).$$

(c) Block gates: *$c_{\mathrm{new}} < \frac{1}{2}$ and $c_{\mathrm{new}}^2 < \min(s_{\mathrm{new}}, a_{\mathrm{new}})$; for FE and EF also $\rho_{\mathrm{old}}^2 + c_{\mathrm{new}}^2 < d_{\mathrm{sym}}$.*

For the sixteen records the exact rational tests at $\hat{P} = i$ give $(k_{\mathrm{f}}, k_{\mathrm{s}}, k_{\mathrm{a}}) = (1, 0.707107, 0.707107)$ for FF, $(2.414214, 1.581139, 1.581139)$ for FE and $(1.732051, 0.866026, 0.866026)$ for EF, and $(k_{\mathrm{old}}, k_{\mathrm{new}}, k_\Delta) = (2.414214, 2.414214, 2.739626)$ for FE and $(2.414214, 1.732051, 2.361993)$ for EF; the decimals are exact rationals.

Proof. The frame. By Step 0 of the proof of Lemma 28.2, $\gamma I \preceq \Phi^*\Phi \preceq \Gamma I$. The pair features W_S/r, $S \in \mathcal{S}$, of Lemma 14.4 satisfy $(W_S/r)^*(W_S/r) \preceq I$ (part (a) there) and $\|(W_S/r)^*(W_T/r)\| \le N_{ST}$ for $S \neq T$ (part (e), with the correlations of (F2)), so their synthesis $\mathcal{W}$ has $\|\mathcal{W}\|^2 \le \Gamma_W$ (Lemma 13.8(a), as in Lemma 14.4(f)). Let $Q_{\mathcal{V}}$ project onto the orthogonal complement of $\mathcal{V}$ in $L_0^2(\lambda_H)$, and let $\mathcal{F}^\circ$ be the synthesis of the features Φ_i and $Q_{\mathcal{V}} W_S/r$. Then $\mathcal{F}^\circ \mathcal{F}^{\circ *} = \Phi\Phi^* + Q_{\mathcal{V}} \mathcal{W}\mathcal{W}^* Q_{\mathcal{V}}$ is a sum of two positive operators with orthogonal ranges, so $\|\mathcal{F}^\circ\|^2 \le \Gamma_F$. The projection is applied to the whole pair synthesis after the correlations are used, so it need not keep the pair features measurable in their pairs.

The raw kernel. By Steps 1–6 of the proof of Lemma 28.2 the new law has density $K = K_0/Z_0$ relative to $\lambda_H \otimes \lambda_H$, with $K_0 = (1 - q\sum_S \mathbf{D}_S)(\mathbf{1}\otimes\mathbf{1}) + \mathcal{S}^{\mathrm{c}} + \mathcal{R}_* - K_{\mathrm{col}}$, where $\mathcal{S}^{\mathrm{c}}$ annihilates the constants and maps into L_0^2, the operators $Q(\mathcal{R}_* - K_{\mathrm{col}})Q$, $Q(\mathcal{R}_* - K_{\mathrm{col}})\mathbf{1}$ and $Q(\mathcal{R}_* - K_{\mathrm{col}})^*\mathbf{1}$ have norms at most $\mathcal{R}$, and $Z_0 \ge Z_-$. By Corollary 14.5, $\mathcal{S}^{\mathrm{c}}$ is the doubly centred complete star part $r\mathcal{F}M(\hat{P})\mathcal{F}^*$ of the soft cross kernel of $\mathbf{D}$, with $\mathcal{F}$ the synthesis of the Φ_i and of the W_S/r; a left star with centre $c \in X$ contributes $\mathbf{D}_S \Phi_c W_S^*$, and a right star $\mathbf{D}_S W_S \Phi_c^*$. Replacing W_S by $Q_{\mathcal{V}} W_S$ removes $\mathbf{D}_S \Phi_c W_S^* P_{\mathcal{V}_Y}$ from the left stars and $P_{\mathcal{V}_X} \mathbf{D}_S W_S \Phi_c^*$ from the right stars, so

$$\mathcal{S}^{\mathrm{c}} = r\mathcal{F}^\circ M(\hat{P}) \mathcal{F}^{\circ *} + \mathcal{S}_X P_{\mathcal{V}_Y} + P_{\mathcal{V}_X} \mathcal{S}_Y,$$

and by Corollary 14.5 and Lemma 28.2(c) the full, symmetric and skew parts of $\mathcal{S}^{\mathrm{c}}$ have norms at most $\Gamma_F k_{\mathrm{f}} r + E_{\mathrm{proj}}$, $\Gamma_F k_{\mathrm{s}} r + E_{\mathrm{proj}}$ and $\Gamma_F k_{\mathrm{a}} r + E_{\mathrm{proj}}$. Dividing by $Z_0 \ge Z_-$, in the notation of Lemma 17.16 with $\nu = \lambda_H$ and $\Pi_\perp = Q$, $K = \mathbf{1}\otimes\mathbf{1} + \delta_L \otimes \mathbf{1} + \mathbf{1} \otimes \delta_R + K_c$ with $\|K_c\| \le c$, $\|\operatorname{Sym} K_c\| \le s$, $\|\operatorname{Skew} K_c\| \le a$ and $\|\delta_L\|, \|\delta_R\| \le e$.

(a) The one-way repair is $K' = (1 + f(x) + g(y))K$ with $\|f\|_\infty, \|g\|_\infty \le F$ ((F7)). Its half marginals are $\mu^{\mathrm{old}} = (1 + u_0)\lambda_H$ with $1 + u_0 \ge e^{-\ell_{\mathrm{old}}}$ and $\|u_0\|^2 \le B_{\mathrm{top}} q$ (proof of (F7)). Lemma 17.16(b), with $b_0 = B_{\mathrm{top}} q$, gives the three norms, and ρ' bounds the full one by (F7). The old record is symmetric at ι, its centred operator C_{old} self-adjoint of norm at most ρ_{old}; so the symmetric part of the change is $\operatorname{Sym} C_{\mathrm{rep}} - C_{\mathrm{old}}$, of norm at most $s_{\mathrm{new}} + \rho_{\mathrm{old}}$, and its skew part is $\operatorname{Skew} C_{\mathrm{rep}}$ (Remark 9.15(iii)), of norm at most a_{new}.

(b) The graph G has the half graph H in both layers, so the setting of Lemma 28.2 applies to it with its own cross graph, whose reversed edges have $\mathbf{D}_{uv} = -1$; its density relative to $\lambda_H \otimes \lambda_H$ is $O = K_0^{\mathrm{old}}/Z_0^{\mathrm{old}}$. In one synthesis over the union of the leaf pairs of both graphs, with $\|\mathcal{F}^\circ\|^2 \le \Gamma_F$, the two centred star parts are $r\mathcal{F}^\circ M_h(\hat{P})\mathcal{F}^{\circ *}$ plus their projection terms, so the symmetric part of $\mathcal{S}^{\mathrm{c}}_{\mathrm{new}} - \mathcal{S}^{\mathrm{c}}_{\mathrm{old}}$ has norm at most $\Gamma_F k_\Delta r + E_{\mathrm{new}} + E_{\mathrm{old}}$ (Corollary 14.5, its last clause), and the centred unnormalized kernel $X_h = \mathcal{S}^{\mathrm{c}}_h + Q(\mathcal{R}_{*,h} - K_{\mathrm{col},h})Q$ of h has norm at most U_h. By Step 6, $|Z_0^h - 1| \le \varepsilon_h < \frac{1}{2}$, so $\|X_h/Z_0^h - X_h\| \le \varepsilon_h U_h/(1 - \varepsilon_h)$; no equality of the two normalizers is assumed. Hence $\|\operatorname{Sym} Q(K - O)Q\| = \|\operatorname{Sym}(X_{\mathrm{new}}/Z_0^{\mathrm{new}} - X_{\mathrm{old}}/Z_0^{\mathrm{old}})\| \le d_{\mathrm{raw}}$, and Lemma 17.16(c), with the correction of (a), gives d_{sym}: the two rank-one terms of the actual centring cancel, because K' and O have the same half marginals.

(c) The block gates are used in Lemma 28.11(ii). □

Part (c) is a numerical condition on the formulas in (a)–(b), not a consequence of the analytic argument above for arbitrary parameters: it must be checked wherever the proposition is applied. The certificates evaluate the proposition at 2^{48} and at 2^{49}, where all these checked inequalities hold, for all sixteen records; Lemma 28.11 extends them to larger primes. The records are `research/continuation_2026_10_24/upper_arithmetic/extended_forest.json` and, for FE and EF, `forest_difference.json` of the same directory. Two independent reviews re-enumerate every cross-edge subset, repeat the exact matrix tests and recompute the pair correlations, the normalizers and the transport and block gates: `extended_forest_review.json`, with the status `PASS_INDEPENDENT_EXTENDED_FOREST_GATES`, and `forest_difference_review.json`, with the status `PASS_INDEPENDENT_COMMON_FOREST_DIFFERENCE`. The coefficients $C^{\mathrm{s}} = (\rho_{\mathrm{old}} + s_{\mathrm{new}})/r$ for FF and d_{sym}/r for FE and EF, and $C^{\mathrm{a}} = a_{\mathrm{new}}/r$, rounded upward to 10^{-9}, sum by state to the ledgers (symmetric/skew)

	DD (FF)	*DS* (FE)	*SD* (EF)
2^{48}	23.333853648/4.2501729	11.616266188/6.654718256	14.670537186/5.203325484
2^{49}	23.331070512/4.247966928	11.423578056/6.557982124	14.524506324/5.20122549

Below 2^{48} the same records keep the coefficients of (F7); their ledgers at 2^{47}, about 43.09/24.01, 41.67/32.01 and 62.49/48.00, are those of Certificate 28.12.

Dense laws. The LE and EL records of degree two have halves with $n = 18$ labels and dense old cross graphs, with 72 cross edges in both directions. A one-way repair removes all of them and imposes only the projected queries, so its change is of the size of the whole old cross kernel. A dense law keeps the old cross graph, adds the few missing required edges and softens only the old edges that meet them; its star matrix then differs from the old one in a few entries, and the two are compared in one synthesis.

Definition 28.4 (Softened laws)**.** Let π^{old} be an LE or EL record of degree two, with half graph H on $n = 18$ labels, and let $\mu = \lambda_H$ be its law, the common induced-half chart of the record. Let D_0 be the signed cross-incidence matrix of the old record: $(D_0)_{ij} = 1$ for an edge from x_i to y_j, -1 for an edge from y_j to x_i, and 0 otherwise. For a real $n \times n$ matrix D with entries in $[-1, 1]$ put

$$H_D(x, y) = \mathbf{1}\{x_i \ne y_j \ \forall i, j\} \prod_{i,j} (1 + D_{ij}\chi(y_j - x_i)), \qquad K_D = \frac{H_D}{Z_D}, \qquad Z_D = \int H_D \, d(\mu \otimes \mu).$$

The old law is K_{D_0}, the density of π^{old} relative to $\mu\otimes\mu$; it is symmetric under the layer interchange, with both half marginals $m_0 = g_0\mu$. Fix a list of *required edges*, the projections under the emission of the record of a family of queries. The *softening* D_1 of D_0 with parameters $(\beta,\tilde\beta)\in[0,1]^2$ is formed as follows:

- every required old edge is kept;
- every missing required edge is added, with coefficient 1 (it runs from layer 0 to layer 1);
- let R and C be the rows and the columns of the added edges; an unrequired old entry (i,j) is multiplied by β if $i\in R$ or $j\in C$, and otherwise by $\tilde\beta$ if $j\in R$ or $i\in C$;
- all other entries are unchanged.

The *dense EL law* uses the twelve-query list (the projections of (28.1)) and $(\beta,\tilde\beta)=(7/10,2/5)$, which adds six edges. The *LE components* use the list of $\mathcal{E}_6$, $\tilde\beta=1$ and $\beta\in\{0,\frac15,\frac25,1\}$, which adds one edge. No required edge is an old edge of the opposite direction, so $H_{D_1}\ge 0$, every required edge has coefficient one, and the support of H_{D_1} satisfies every required query.

Proposition 28.5 (Replacements keeping the old cross edges)**.** *Let π^{old} be an LE or EL record of degree two and $K_1 = K_{D_1}$ a softening of Definition 28.4 (in the construction: the dense EL law of the two EL records $G_{02}L$, $G_{12}L$, and the four LE components of Proposition 28.6), with raw half marginals $g_{1,L}\mu$, $g_{1,R}\mu$. For $h\in\{0,1\}$ (old, new) let*

- *$\mathcal{R}_h$ be the nonstar-plus-collision bound of H_{D_h} (Corollary 16.17), $T_{2,h}$ the sum of the moduli of its two-star coefficients and $T_{>,h}=\sum_{|S|\ge3}|c_S|(|S|-1)$ over its higher stars, counted before grouping, $\varepsilon_h = T_{2,h}q + T_{>,h}r + \mathcal{R}_h$ and $z_h = 1-\varepsilon_h$;*
- *Γ be the frame bound of Lemma 14.4(f) for the union of the old and the new star supports, $k_{\mathrm s}$, $k_{\mathrm a}$ bounds for the norms of $\mathrm{Sym}(M_1-M_0)(i)$ and $\mathrm{Skew}(M_1-M_0)(i)$, k_0 one for $\|M_0(i)\|$ (Corollary 14.5), $k_1 = k_0+k_{\mathrm s}+k_{\mathrm a}$, and $C_h = (\Gamma k_h r + \mathcal{R}_h)/z_h$;*
- *W_h be the larger of the two bounds of Lemma 15.28 for the higher-star marginals of the two sides of H_{D_h}, and $u_h = (W_h+\mathcal{R}_h)/z_h$;*
- *ℓ_h be a number such that both raw marginals of K_{D_h} have densities $\ge e^{-\ell_h}$ relative to μ (the* lower-density exponent *of the proof of* (F7)*), and U the bound for the deviations of the ratios $g_0/g_{1,L}$, $g_0/g_{1,R}$ from one, both from the count orders of* (F1) *as in* (F7)*;*
- *$m_\times$ be the number of nonzero entries of D_1.*

Put $K_\infty = (2^{m_\times}e^{2\ell_1}/z_1)^{1/2}$, $\rho_I = e^{\ell_1}(\mathcal{R}_1/z_1+u_1^2)$, $e_I = (1+\rho_I)e^{\ell_1/2}(u_0+u_1)$ and, for an integer $N\ge1$, $F_N = 2NU + K_\infty\rho_I^{2N-1}e_I/(1-\rho_I^2)$. Suppose $\varepsilon_h<\frac12$, $U<1$, $\rho_I<1$ and $2F_N<1$.

(a) There are f, g with $\|f\|_\infty,\|g\|_\infty\le F_N$ such that $K_1' = (1+f(x)+g(y))K_1$ is a probability coupling of m_0 with itself, invariant under the square-affine maps and supported where $K_1>0$; it accepts the required queries (for the dense EL law the twelve queries), and the record is replaced by its image under the emission.

(b) With $E=\sum_h(\mathcal{R}_h+\varepsilon_h(\Gamma k_h r+\mathcal{R}_h)/z_h)$, the change from π^{old}, relative to m_0, has symmetric part of norm at most $d_{\mathrm s} = e^{\ell_0}(\Gamma k_{\mathrm s} r + E + 2F_N(C_1+u_1))$ and skew part of norm at most $d_{\mathrm a} = e^{\ell_0}(\Gamma k_{\mathrm a} r + E + 2F_N(C_1+u_1))$.

(c) Block gates: *with $c_0 = e^{\ell_0}(C_0+u_0^2)$ and $c_1 = c_0+d_{\mathrm s}+d_{\mathrm a}$, $\max(c_0,c_1)<\frac12$ and $c_0^2+c_1^2<\min(k_{\mathrm s},k_{\mathrm a})r$.*

For the dense EL law of $G_{02}L$ and $G_{12}L$ the exact tests at $\hat P=\pm i$ give $k_{\mathrm s}=2.97$, $k_{\mathrm a}=2.08$ and $k_0=5.606$.

Proof. Kernels and remainders. H_{D_h} is the soft kernel of Corollary 16.17 for the cross pairs of the support of D_h, with the strict-cross mask: it is $\mathbf{1}\otimes\mathbf{1}$ plus its complete star part plus its nonstar part minus its collision part, and the last two are bounded by those of the Boolean kernel of the support, whose sum is $\mathcal{R}_h$ (Corollary 16.17(a),(b)). A two-star has mean $-q$ (4.2) and a higher star with $|S|$ leaves a mean of modulus at most $(|S|-1)r$ (Theorem 4.7), so $|Z_{D_h}-1|\le\varepsilon_h$.

Star parts. By Corollary 14.5, in one synthesis $\mathcal{F}$ over the union of the old and new supports, the doubly centred complete star parts of H_{D_0} and H_{D_1} are $r\mathcal{F}M_h(\hat{P})\mathcal{F}^*$, with $\|\mathcal{F}\|^2 \le \Gamma$; the pivots of Lemma 14.4(c) have the densities $2^{\deg j}/(1 - e_{\deg j}(p,n))$ of Lemma 12.6. The layer interchange reverses every old cross edge, so D_0 is antisymmetric, and its left and right higher stars are equal transposes; so $M_0(i)$ is Hermitian, its full cap needs no skew term, and $\mathrm{Skew}(M_1 - M_0) = \mathrm{Skew}\, M_1$. Normalizing each kernel by its own Z_{D_h}, the centred kernel of h has norm at most C_h, and the symmetric and skew parts of the difference of the two centred kernels have norms at most $\Gamma k_{\mathrm{s}} r + E$ and $\Gamma k_{\mathrm{a}} r + E$: the leading parts differ by $r\mathcal{F}(M_1 - M_0)(\hat{P})\mathcal{F}^*$, the remainders add $\mathcal{R}_0 + \mathcal{R}_1$, and each normalizer is paid separately, since $|Z_{D_h}^{-1} - 1| \le \varepsilon_h/z_h$; no equality of the two normalizers is assumed.

Marginals. The marginal images of the two-stars are constants. The centred marginal density of a side is the higher-star sum $\sum_{|T|\ge 3} c_T w_T$ of Lemma 15.28, with the star coefficients c_T of that side, plus the marginals of the nonstar and collision parts, divided by Z_{D_h}; so the raw marginal densities satisfy $\|g_{h,L} - \mathbf{1}\|_\mu, \|g_{h,R} - \mathbf{1}\|_\mu \le u_h$.

Repair. Let A be the centred conditional operator of K_1 relative to its raw marginals, and let $\mathcal{V}_L$, $\mathcal{V}_R$ be the functions that are invariant under the square-affine maps, applied to all labels at once, and have mean zero under these marginals. The laws are invariant under these maps, so A maps $\mathcal{V}_R$ into $\mathcal{V}_L$ and A^* maps $\mathcal{V}_L$ into $\mathcal{V}_R$, and the targets $u = g_0/g_{1,L} - 1$, $v = g_0/g_{1,R} - 1$ lie in $\mathcal{V}_L$, $\mathcal{V}_R$. The functions of $\mathcal{V}_L$ and $\mathcal{V}_R$ are invariant under the simultaneous translations, and the ranges of the features of Lemma 14.4 are orthogonal to such functions, so the star part of K_1 vanishes on them; the identity of Lemma 13.19(b) holds for every pair of centred functions, and applied to functions of $\mathcal{V}_L$ and $\mathcal{V}_R$ it gives $\|A|_{\mathcal{V}_R}\| \le e^{\ell_1}(\mathcal{R}_1/z_1 + u_1^2) = \rho_I$. A soft count is a convex combination of hard counts with the same support, by $1 + b\chi = \frac{1+b}{2}(1+\chi) + \frac{1-b}{2}(1-\chi)$, and the bounds (F1) are uniform in the signs; so $|u|, |v| \le U$. Next $\|u - Av\| \le \|u\| + \rho_I\|v\|$, and $\|u\|_{L^2(g_{1,L}\mu)} \le e^{\ell_1/2}\|g_0 - g_{1,L}\|_\mu \le e^{\ell_1/2}(u_0 + u_1)$, so e_I bounds $\|u - Av\|$ and $\|v - A^*u\|$. Finally $H_{D_1} \le 2^{m\times}$, so the density of K_1 relative to the product of its raw marginals is at most K_∞^2, which bounds A and A^* from L^2 to L^∞. Lemma 17.14(g), with $\mathcal{V}_X = \mathcal{V}_L$, $\mathcal{V}_Y = \mathcal{V}_R$ and these constants, gives the centred solution f, g with $\|f\|_\infty, \|g\|_\infty \le F_N$; as $2F_N < 1$, part (c) there makes K_1' a probability law with both marginals $g_0\mu = m_0$, supported where $K_1 > 0$, so accepting every required query, and the last clause of (g) makes it invariant under the square-affine maps, as $\mathcal{V}_L$ and $\mathcal{V}_R$ consist of invariant functions. The emission preserves the half marginals at ι and cannot increase operator norms, as for the one-way repairs. This is (a).

Change. In the chart μ, K_1' and $K_0 = K_{D_0}$ have the same half marginals m_0, so Lemma 17.16(c), with $K = K_1$, $O = K_0$, $c = C_1$, $e = u_1$ and $F = F_N$, writes $K_1' - K_0$ as $Q(K_1 - K_0)Q$ plus a correction of norm at most $2F_N(C_1 + u_1)$; the rank-one terms of the actual centring cancel. The transport to $L^2(m_0)$ costs e^{ℓ_0}, as $m_0 \ge e^{-\ell_0}\mu$. This is (b).

Gates. By Lemma 13.19(b), c_0 bounds the actual centred norm of the old law; that of the new law is at most c_1, its centred operator being the old one plus the symmetric and the skew part of the change. The gates of (c) are used in Lemma 28.11(ii).

Records. The six LE records have identical weighted edge arrays in the common chart. The four EL records of state RL that do not accept the twelve queries are carried onto the record $G_{01}L$ by exact isomorphisms of their weighted edge arrays: $G_{02}L$ has the arrays of $G_{01}L$, and $G_{12}L$ is mapped onto it by the interchange of the halves together with the negation of all field labels and a permutation of each half (`research/advance_2026_10_24/upper/physical_orbits.json`, `PASS_EXACT_SAME_HALF_PHYSICAL_ISOMORPHISMS`). Such a map conjugates the operators by an isometry and preserves symmetric and skew norms, so one computation for $G_{01}L$ serves $G_{02}L$ and $G_{12}L$; $G_{01}L$ is only a computational representative here and is itself kept (Lemma 28.1). □

As in Proposition 28.3, part (c) here and in Proposition 28.6 is a numerical condition to be checked at each application, not proved for arbitrary parameters by the analytic argument. For the dense EL law, at $p_0 = 2^{47}$ and $p_0 = 2^{48}$ these checked inequalities hold with $N = 2$ ($F_N < 0.004120$ and $F_N < 0.002913$), and the coefficients $(C^{\mathrm{s}}, C^{\mathrm{a}}) = (d_{\mathrm{s}}, d_{\mathrm{a}})/r$, rounded upward to 10^{-12}, are

$$(s_{\mathrm{EL}}, a_{\mathrm{EL}}) = (3.291966120220,\ 2.401350027793) \qquad \text{at } 2^{47},$$
$$(s_{\mathrm{EL}}, a_{\mathrm{EL}}) = (3.197237469233,\ 2.306801866952) \qquad \text{at } 2^{48}.$$

The records are `research/advance_2026_10_24/upper/lower_p47/dense_completed.json` (Certificate 28.12(e)) and `dense_p48/completed_invariant.json` of the same directory, with the extraction witness `dense_p48/certificate.json`; the independent review `dense_review_full.json` of that directory (`PASS_INDEPENDENT_DENSE_REPAIR`) regenerates the complete integer histograms (249685 rows) and repeats the 16 exact PSD tests, the frame, marginal, repair and block gates and the ten physical isomorphisms. By Lemma 28.11 these coefficients hold at every class-three prime $p \geq p_0$ and give the block bounds $s_{\mathrm{EL}} b^{-1/2}$ and $a_{\mathrm{EL}} b^{-1/2}$.

The LE mixture. For the six LE records of degree two the six queries project to three cross edges, one of them missing, and the six records have the same old half graph, old cross graph and required edges. Each LE component of Definition 28.4, repaired to the old actual marginal, is a positive accepting law, so every convex mixture of them is one too. The star matrix of a mixture is the mixture of the component matrices, and its norm can be smaller than that of every component: the symmetric and skew caps k_{s}, k_{a} of the four components lie between 1.914 and 2.854, and both caps of the two mixtures are below 1.724.

Proposition 28.6 (Replacing an LE record by a positive mixture of four dense laws)**.** *Let π^{old} be one of the six LE records of degree two, with old law K_0 and actual half marginal $m_0 = g_0\mu$. Let E' be the cross graph formed by the old cross edges and the two missing edges of the twelve-query list; it contains the supports of the four LE components $K_{D^{(j)}}$, $j = 0, \dots, 3$, of Definition 28.4 ($\beta_j = 0, \frac{1}{5}, \frac{2}{5}, 1$). Normalize and repair each component to m_0 as in Proposition 28.5(a), with the quantities $\varepsilon_{j,h}$, $z_{j,h}$, $C_{j,h}$, u_j and F_j of that proposition for the component j and its own matrix caps, except that the remainder $\mathcal{R}_{j,1}$ of its new side is that of the Boolean kernel of E'; write K'_j for the repaired component. For weights $w_j \geq 0$ with $\sum_j w_j = 1$ put $K = \sum_j w_j K'_j$. Let Γ be the frame bound of Lemma 14.4(f) for the union of the old and of all four new star supports, $M_j(\hat{P})$ the star matrix of component j minus that of K_0, $M = \sum_j w_j M_j$, k_{s}, k_{a} bounds for the norms of* $\operatorname{Sym} M(i)$ *and* $\operatorname{Skew} M(i)$*, and*

$$E_j = \sum_h (\mathcal{R}_{j,h} + \varepsilon_{j,h} C_{j,h}) + 2F_j(C_{j,1} + u_j), \qquad E = \sum_j w_j E_j.$$

If the gates of Proposition 28.5 hold for every component, then K is a probability coupling of m_0 with itself, invariant under the square-affine maps, that accepts $\mathcal{E}_6$, and its change from the old record has symmetric and skew parts of norms at most

$$d_{\mathrm{s}} = e^{\ell_0}(\Gamma k_{\mathrm{s}} r + E), \qquad d_{\mathrm{a}} = e^{\ell_0}(\Gamma k_{\mathrm{a}} r + E).$$

Block gates: *with $c_0 = \sum_j w_j c_{0,j}$, where $c_{0,j}$ is the bound $e^{\ell_0}(C_{j,0} + u_0^2)$ of Proposition 28.5(c) for the old law, and $c_1 = c_0 + d_{\mathrm{s}} + d_{\mathrm{a}}$: $\max(c_0, c_1) < \frac{1}{2}$ and $c_0^2 + c_1^2 < \min(k_{\mathrm{s}}, k_{\mathrm{a}}) r$. For the weight vectors* $\mathrm{I} = (\frac{1}{5}, \frac{1}{5}, \frac{1}{10}, \frac{1}{2})$ *and* $\mathrm{II} = (290450, 1, 233954, 475595)/10^6$ *the exact tests give* $(k_{\mathrm{s}}, k_{\mathrm{a}}) = (1722417/10^6,\ 856249/500000)$ *for I and* $(43077/25000,\ 1711273/10^6)$ *for II.*

Proof. Every factor of a raw component has the form $1 + b\chi$ with a signed coefficient $b = D_{ij}$ satisfying $|b| \leq 1$, and is therefore nonnegative. Every required factor has coefficient of modulus one with the required sign; so each raw component is nonnegative and accepts $\mathcal{E}_6$, and by Proposition 28.5(a) each repaired component is a probability coupling of m_0 with itself, supported

where the raw one is, and invariant. A convex combination of such couplings is one. The supports of the components are contained in E', and their character coefficients have modulus at most one, so the complete nonstar sum and the collision bound of the Boolean kernel of E' dominate those of each component term by term (Corollary 16.17); this is an absolute expansion, not a monotonicity of a signed norm under the deletion of edges. The normalizer, the counts, the marginal errors and the repair of each component are its own, as in Proposition 28.5. In one synthesis $\mathcal{F}$ over the union of all supports, with $\|\mathcal{F}\|^2 \le \Gamma$, the proof of Proposition 28.5(b) writes the centred change of component j relative to μ as $r\mathcal{F}M_j(\hat{P})\mathcal{F}^*$ plus a remainder of norm at most E_j, the rank-one terms of the actual centring cancelling because every repaired component has the marginals m_0. Mixing with the weights w_j, the centred change of K is $r\mathcal{F}M(\hat{P})\mathcal{F}^*$ plus a remainder of norm at most E, and the transport to $L^2(m_0)$, common to all components, costs e^{ℓ_0}. Corollary 14.5 bounds the symmetric and skew parts of the leading term by $\Gamma k_{\mathrm{s}} r$ and $\Gamma k_{\mathrm{a}} r$. The gates play the role of those of Proposition 28.5(c); c_0 is a convex combination of valid bounds for the old centred norm. □

The proposition is evaluated at the eight endpoints 2^{47}, $167 \cdot 2^{40}$, 2^{48}, 2^{49}, 2^{50}, 2^{52}, 2^{56} and 2^{60}, with the extraction coefficients of E' certified at 2^{47} and 2^{48}, valid at every larger prime (Corollary 16.17), and the collision bounds evaluated at each endpoint. A band uses the weight vector II only when its complete pair of coefficients is no larger than that of I: the 301 bands below 2^{52} use II and the nine from 2^{52} use I, so no pair of prices is assembled from two different laws. The coefficients $(C^{\mathrm{s}}, C^{\mathrm{a}}) = (d_{\mathrm{s}}, d_{\mathrm{a}})/r$ at the endpoints used, rounded upward to 10^{-12}, are

endpoint	vector	s_{LE}	a_{LE}
$167 \cdot 2^{40}$	II	1.872914263008	1.861100843756
2^{48}	II	1.838921462687	1.827109278261
2^{49}	II	1.801556462666	1.789745796952
2^{50}	II	1.777352768400	1.765543176463
2^{52}	I	1.749386716250	1.739466627572
2^{56}	I	1.729118677881	1.719199405722
2^{60}	I	1.724091674882	1.714172606843

The records are `research/further_2026_10_25/upper/dense_six.json` (vector I) and `dense_six_refined.json` (vector II) of that directory, produced by `dense_six.py` and `reweight_dense_six.py`; the independent reviews `dense_six_review.json` (236174 complete extraction rows, 24 exact PSD tests, the four supergraph inclusions, all component repairs and the eight block gates) and `dense_six_refined_review.json` (the 32 unchanged component ledgers and the eight fresh endpoint gates), both `PASS_INDEPENDENT_POSITIVE_SIX_QUERY_LE_MIXTURE`. The histograms of E' are those of the dense twelve-query LE softening, $(\beta, \tilde{\beta}) = (2/3, 1)$, in `research/advance_2026_10_24/upper/dense_p48/` and `lower_p47/dense_certificate.json` of that directory.
The formula. The three propositions continue the list of upper formulas.

(F9) *Actual-frame comparisons* (bands). These give the symmetric and skew coefficients of three kinds of changed records:
- the FF, FE and EF repairs of degree two on the bands from 2^{48}: $C^{\mathrm{s}} = (\rho_{\mathrm{old}} + s_{\mathrm{new}})/r$, for FE and EF $C^{\mathrm{s}} = d_{\mathrm{sym}}/r$, and $C^{\mathrm{a}} = a_{\mathrm{new}}/r$ (Proposition 28.3);
- the two dense EL records: $(C^{\mathrm{s}}, C^{\mathrm{a}}) = (d_{\mathrm{s}}, d_{\mathrm{a}})/r$ (Proposition 28.5);
- the six LE records: $(C^{\mathrm{s}}, C^{\mathrm{a}}) = (d_{\mathrm{s}}, d_{\mathrm{a}})/r$ for the weight vector of the band (Proposition 28.6).

Hypotheses: the gates listed in the three propositions, their block gates included.

28.4. **Complete six-query refinements used in the final closing.** The final construction uses the following sharper comparisons for individual components of the upper reference. Each

new accepting law has the same two full marginals as the component it replaces. The letters $\mathrm{L}, \mathrm{D}, \mathrm{S}, \mathrm{R}, \mathrm{C}$ denote the five emission types of Section 27.2.2; an ordered pair specifies the types in the two coordinates. The local bounds below are applied with the completion weights of Section 27.3.1 and the normalization of Lemma 33.2. Throughout, $q = p^{-1}$ and $r = \sqrt{q}$.

Proposition 28.7 (Soft nine-label comparisons)**.** *On the whole finite upper range the aggregate symmetric and skew coefficients for the changed records may be taken as*

state	*records*	C_S	C_A
DD	6	8.873538	6.087447
DS	4	7.193144	5.748158
SD	6	5.997372	5.173918.

For each state, C_S is the sum over its records of the per-record symmetric coefficients, and C_A is the analogous skew sum. The corresponding aggregate norm bounds at one prime are $C_S r$ and $C_A r$, before applying the state's reference weight. The same coefficients control tensor blocks as in Proposition 28.14; that proposition specifies the resulting weighted comparison polynomials.

Proof. Use the two-layer geometry of Lemma 27.4, restricted to the nine distinct label positions of the chosen half. For a record let D_0 be the matrix of signed constraints between its two halves: an entry $D_{0,ij}$ contributes $1 + D_{0,ij}\chi(y_j - x_i)$. Let $\mathcal{Q}$ be the entries obtained by projecting the six queries onto these positions. Start with $D_1 = \beta D_0$, set every entry of $\mathcal{Q}$ to 1, and set its reverse entry to zero. The geometry gives no opposed query pair and no opposed old query; here opposed entries are related by transposing the two matrix indices. Use

	β	$\#\operatorname{supp} D_1$	k_S	k_A
DD	31/40	19	1.476	1.013
DS	11/20	18	1.797	1.436
SD	17/20	13	.999	.862.

The raw density on the product of the two induced half laws is

$$H_D(x, y) = \mathbf{1}_{\{x_i \neq y_j \ \forall i,j\}} \prod_{i,j} (1 + D_{ij}\chi(y_j - x_i)).$$

It is nonnegative and accepts the required queries. Every scalar normalizer and both full marginal densities are retained separately.

The empty character product, the stars, and the nonstars will be estimated separately. For the stars, use the common feature dictionary of Corollary 14.5, containing all singletons and all old and new row and column star supports. A star with leaves S has coefficient $(-1)^{|S|} \prod_{j \in S} D_{ij}$ on the left, and $\prod_{i \in S} D_{ij}$ on the right. These formulas determine the complete rational matrices, including stars of every degree. The symmetric and skew differences have caps k_S, k_A in the table. Schur complementation of the leaf block reduces each test to rational positive semidefiniteness in order 18; both signs and both Fourier phases are checked. The old full coefficient matrix has norm at most 4. The Gram bound Γ is the row-sum bound of Section 14, applied to this full dictionary under the actual induced half law; the features are not assumed orthogonal.

Here is a finite, exhaustive bound for every nonstar term. Condition all labels except the k endpoints on each side of a matching of size $k = 2$ or 3. If e is the sum, over the two halves, of the numbers of internal edges incident to the free coordinates, conjugation of their conditional laws costs at most

$$\frac{2^{e/2}}{(1 - e_{\Delta}^{-}(p, 8))^k}, \qquad k = 2, 3.$$

Indeed sequential extension of the k coordinates counts each incident internal edge once and excludes at most eight labels at each step; the square roots of the two density bounds give this

factor. For $k = 2$ use the matching/path primitive, or the $2r^2$ rectangle primitive when all four cross edges are present. For $k = 3$, contract the chosen matching and inspect the directed graph of its other six possible cross entries. If its strongly connected components have size at most two, the tensor primitives give r^3 or $2r^3$. If all nine entries are present, use the complete-cubic bound $6r^3$ proved in Section 15.2. Otherwise use a two-coordinate witness. Factors involving a conditioned label are bounded left or right multipliers and cost at most one.

Enumerate all subsets of the old and new cross supports. Remove the empty term and stars. Assign each remaining subset a valid witness by the preceding rule; if it contains s soft edges its absolute coefficient is β^s. Thus the exact integer histogram $h_{k,e,v,s}$ gives the complete nonstar majorant

$$R_{\mathrm{ns}} = \sum_{k,e,v,s} h_{k,e,v,s} \frac{\beta^s v 2^{e/2} r^k}{(1 - e_\Delta^-(p,8))^k}, \qquad v \in \{1,2,6\}.$$

There is no omitted character class: the empty, star, two-coordinate and three-coordinate counts sum to $2^{\#\operatorname{supp} D}$. Two separate exhaustive programs verify this partition for all sixteen records (Certificate 33.3). One tests reachability; the other tests ordered components of size at most two and the complete nine-edge exception.

Add the full cross-collision bound to R_{ns}, obtaining R_D. For this bound and the count profiles, a soft factor is a positive mixture of an absent and a hard constraint. The Boolean supergraph has extension and collision majorants at least those of each subgraph, in the same orders, so it bounds their mixture. This is not a pointwise comparison with the all-hard accepting density. For the exact star coefficients let a_D be the sum of their scalar mean bounds and b_D the larger one-sided nonconstant mean bound: pair means cost q, and a star of degree $t \ge 3$ costs $(t-1)r$. Put

$$\epsilon_D = a_D + R_D, \quad z_D = 1 - \epsilon_D, \quad c_D = \frac{\Gamma k_D r + R_D}{z_D}, \quad u_D = \frac{b_D + R_D}{z_D}.$$

Here subscripts $0, 1$ mean the old and new matrices D_0, D_1, with $k_0 = 4$ and $k_1 = 4 + k_S + k_A$. Thus z_i is a lower bound for the normalizing mass, c_i bounds the centered operator, and u_i bounds either centered marginal density in L^2. Let R_i be the ratio bounds obtained from the exact extension profiles for the full graph, both completions, and the induced half; they satisfy $R_i^{-1} \le g_{i,L}, g_{i,R} \le R_i$. Here $g_{i,L}, g_{i,R}$ are the marginal densities of the normalized law relative to the induced half laws. The count bounds follow from Lemma 12.2 along an insertion order in each graph. The finite certificate records these orders and checks their degrees.

Apply the additive repair of Lemmas 17.13 and 17.14 to the normalized new law, restoring $g_{0,L}, g_{0,R}$. Use

$$U_1 = R_0 R_1 - 1, \qquad \rho_1 = R_1(c_1 + u_1^2), \qquad \rho_I = R_1(R_{D_1}/z_1 + u_1^2),$$
$$K_\infty = 2^{\#\operatorname{supp} D_1/2} R_1/\sqrt{z_1}, \qquad h_2 = (1+\rho_I)\sqrt{R_1}(u_0 + u_1), \qquad F = 2U_1 + \frac{K_\infty \rho_I h_2}{1 - \rho_I^2}.$$

Both marginal errors are invariant under simultaneous translation of all labels in a half. Centered leading stars annihilate functions in this invariant subspace, as in Lemma 17.13, so the repair can use the smaller contraction bound ρ_I. The tests $\rho_I \le \rho_1 < 1/2$, $2F < 1$ give a positive law with the exact old full marginals. Its centered correction is at most $C_{\mathrm{rep}} = 2F(c_1 + u_1)$. Writing $E_i = R_{D_i} + \epsilon_i(\Gamma k_i r + R_{D_i})/z_i$, the complete symmetric and skew bounds are

$$d_S \le R_0(\Gamma k_S r + E_0 + E_1 + C_{\mathrm{rep}}), \qquad d_A \le R_0(\Gamma k_A r + E_1 + C_{\mathrm{rep}}).$$

The old law is self-adjoint, so it contributes no skew remainder. At $p = 2^{47}$, exact outward arithmetic gives per-record maxima

	$d_S/r \le$	$d_A/r \le$
DD	1.478922985748	1.014574466504
DS	1.798285763015	1.437039335347
SD	.999561930536	.862319609396.

Multiplication by the record counts proves the stated aggregate caps. The old and new full centered norms are below $1/2$, and their squared sum is smaller than either displayed comparison cap times r. This proves the higher prime-support tensor gates. Every normalized majorant increases with r and has positive denominators at 2^{47}, so the endpoint proves the whole finite upper range. □

Proposition 28.8 (The six-query SS comparison)**.** *The four changed* SS *records have aggregate caps* $(C_S, C_A) = (4.832, 4.832)$ *throughout the finite upper range.*

Proof. The half law is uniform on injective nine-tuples. The required edges are $x_a \longrightarrow y_b$ for $a \neq b$, $a, b \in \{0, 1, 2\}$; the source and target triples are disjoint when the halves are identified. Use the product of their six strict factors $1 + \chi(y_b - x_a) - \mathbf{1}_{y_b = x_a}$, without further cross-half exclusions. Writing $k = (p-3)/4$ and t for the number of common successors of the three source labels, inclusion-exclusion gives exactly $k^3 - (3k-2)t$ injective target triples, with $0 \le t \le k$. The resulting row and column mass ratio is at most

$$1 + \frac{4q(3 - 17q)}{(1 - 7q)(1 - 11q)} \le 1 + 13q.$$

The six singleton terms and six pair stars have complete leading norm $1 + \sqrt{2}$; their source and target coefficient groups are disjoint, so the symmetric and skew norms are each $(1 + \sqrt{2})/2$. The twelve-feature actual Gram bound is $1 + 88r$. Of the 64 subsets, exactly 51 are nonstars. Each has a two-edge matching and contains no four-cycle. The conservative $2q$ primitive, conditional density $(1 - 8q)^{-2}$, and strict deletions therefore give

$$R = (102(1 - 8q)^{-2} + 192)q, \quad z = 1 - 6q - R, \quad c = ((1 + 88r)(1 + \sqrt{2})r + R)/z.$$

Put $R_1 = 1 + 13q$, $\rho_1 = R_1(c + (13q)^2)$, and

$$F = 13q + \frac{8R_1}{\sqrt{z}} \frac{13q}{1 - \rho_1}, \qquad C_{\mathrm{rep}} = 2F(c + 13q).$$

The additive repair is positive and restores both full nine-label marginals. The old all-distinct eighteen-label law has centered norm at most $81q/(1 - 89q)$: its excluded-collision kernel has constant row and column mass at most $81q/(1 - 8q)$. Thus the complete skew coefficient is $[((1 + 88r)(1 + \sqrt{2})r/2 + R)/z + C_{\mathrm{rep}}]/r$; the symmetric coefficient adds $81r/(1 - 89q)$. At 2^{47} these are below 1.207140517672 and 1.207147345461. The squared-full-norm block gate divided by r is below $4.913171 \cdot 10^{-7}$. All positivity gates hold and all normalized majorants decrease with p. The per-record choice 1.208 proves the claim. □

Proposition 28.9 (Constant-emission comparisons)**.** *The two* LC *and two* CL *records have aggregate caps* $(C_S, C_A) = (1.162, 1.0024)$ *and* $(C_S, C_A) = (1.8972, 2.0056)$*, respectively.*

Proof. Retain the full eighteen-label half law μ; constant emission observes the six labels in row zero of the matching coordinate. The projected queries are one edge for LC, and a six-edge permutation matching for CL. Numbering its six observed positions suitably by $0, \ldots, 5$, this permutation is (0 2 4)(1 5 3). The old constant-emission estimate of Section 27 gives observed centered norm $g_C r$, with $g_C = .079761731$, full centered norm $5.65086987r$, and $\|g_0 - 1\|_\mu^2 \le b_C q$, $b_C = 39236.89926$. Here g_0 is the density of the old full marginal relative to μ; it must be restored before projecting to the six emitted labels.

All new counts have explicit orders. Each nine-vertex component of the half is indexed by $(a,b)\in\{0,1,2\}^2$, numbered $3a+b$, with adjacency when both coordinates differ. The orders

$$(0,4,2,6,5,1,7,3,8),\qquad (0,3,7,2,4,6,5,1,8)$$

give the half and three-pin completion profiles, the latter with pins $\{0,3,6\}$. Doubling over the two components gives $(2,4,6,4,2)$ and $(0,4,6,6,2)$. The old completion shifts the first profile by four degrees. Use exclusion bounds $18,36,36$, respectively; no independence of the components is assumed. The count products give $R_0^{-1}\le g_0\le R_0$, with $R_0<1.000451743231$ at P_3.

For LC put $K_1=(1+\chi(y_j-x_i)-\mathbf{1}_{y_j=x_i})/(1-q)$. It has both marginals μ. If $u=g_0-1$, translation invariance and singleton uniformity imply $K_1u=K_1^*u=0$. Thus $(1+u(x)+u(y))K_1$ is an explicit positive accepting repair to the full old marginals. With $c_1=r\sqrt{1+q}/(1-q)$, its raw symmetric and skew bounds are $(1+c_1)c_1/2$, and the repair costs $2(R_0-1)c_1$.

For CL use the product of the six strict matching factors. Its 57 nonstars each contain two disjoint matching edges. Put

$$D_*=\frac{16}{1-10r-295q},\quad R=(57D_*^2+192)q,\quad z=1-R,\quad e=R/z,\quad \Gamma=1+5c_1.$$

The permutation consists of two three-cycles, so the full, symmetric and skew leading norms are $1,\sqrt{3}/2,1$. Hence use $c=(\Gamma r+R)/z$, $s=(\Gamma\sqrt{3}r/2+R)/z$, and $a=c$. If $[L_H,U_H]$, $[L_C,U_C]$ are the preceding half and completion count intervals, the raw row lies between L_C/U_H and $U_C/L_H+18^2 2^6 q$. The extra term pays all cross coincidences. Together with $|Z-1|\le R$ this gives a two-sided marginal ratio $R_1<1.000004428066$. Set

$$\rho_1=R_1(c+e^2),\quad K_\infty=8R_1/\sqrt{z},\quad F=R_0R_1-1+\frac{K_\infty\sqrt{R_1}(\sqrt{b_C}\,r+e)}{1-\rho_1}.$$

Here $F<.000573119058$, $2F<1$, so the additive repair is positive and has the exact full old marginals. Its correction is $2F(c+e)$. Compression to the six observed labels contracts this correction; transport uses the actual emitted density, bounded below by R_0^{-1}. Thus $d_S\le R_0(s+g_Cr+2F(c+e))$ and $d_A\le R_0(a+2F(c+e))$. For LC use $e=0$, $F=R_0-1$, and $s=a=(1+c_1)c_1/2$. At P_3 the per-record pairs are below $(.580927565959,.501129803137)$ and $(.948455483308,1.002692888286)$. The squared-full-norm block gates divided by r are below $7.52\cdot10^{-8}$, and each full centered factor is below $1/2$. The positive normalized majorants decrease with p, proving the claim. □

Proposition 28.10 (Dense comparisons after emission)**.** *Keep the dense laws and positive mixtures of Propositions 28.5 and 28.6. In every finite upper band their assigned aggregate coefficients may be decreased by*

$$(\delta_S,\delta_A)_{\mathrm{LR}}=(4.2612,4.0668),\qquad (\delta_S,\delta_A)_{\mathrm{RL}}=(2.9322,1.4466).$$

Proof. The merge observes a set O of twelve of the eighteen half labels. Let μ_O be the marginal of the induced half law on O and $\Pi=\mathbb{E}_\mu(\cdot\mid O)$. Exact common full marginals give the emitted difference $\Pi(K'-K_0)\Pi$. Its marginal is $\mathbb{E}_\mu(g_0\mid O)\mu_O$, so the original lower density bound and transport factor remain valid. All complete remainders, normalizers and repairs remain charged in full.

In the star dictionary a feature is visible if its support lies entirely in O; all other features are called hidden. There are 72 visible features, out of 198 for LR and 282 for RL. Write N_h for the number of hidden features and put $D_*=16/(1-10r-295q)$, as in the preceding proof. Set

$$h^2=N_hq\frac{D_*-1}{1-q},\qquad \Gamma_v=1+71\frac{\sqrt{(9r+21q)^2+q(D_*-1)}}{1-q}.$$

The actual projected Gram estimates give visible synthesis norm at most $\sqrt{\Gamma_v}$ and hidden synthesis norm at most h. Let Φ send a coefficient vector to its linear combination of star features, and let M be the coefficient matrix. For a full coefficient norm $\|M\| \le k$ and visible principal norm at most k_v,

$$\|\Pi\Phi M\Phi^*\Pi\| \le \Gamma_v k_v + k(2\sqrt{\Gamma_v}h + h^2).$$

This pays both cross blocks without assuming orthogonal feature ranges. In the local notation of Propositions 28.5 and 28.6, the LR state uses LE laws and the RL state uses EL laws. The complete rational matrix tests, for all six LR and two RL records, give

	k_S	$k_{S,v}$	k_A	$k_{A,v}$
LE mixture I	1.722417	1.012148	1.712498	1.032944
LE mixture II	1.723080	1.012050	1.711273	1.033383
EL	2.970000	1.503700	2.080000	1.356555.

The old chosen formula is $T(\Gamma_{\mathrm{old}}k + E/r)$ with $T, \Gamma_{\mathrm{old}} \ge 1$. Therefore its upward rounding may be decreased by $k - \Gamma_v k_v - k(2\sqrt{\Gamma_v}h + h^2)$. At P_3 the symmetric and skew savings exceed $(.7102, .6778)$ on each LR record and $(1.4661, .7233)$ on each RL record. These are savings in a specified chosen formula, not differences of unrelated norm estimates. The old and new full centered norms are bounded by $7r, 13r$; their squared sum is at most $218q < r$, while every decreased coefficient exceeds one. This checks the tensor sectors. All majorants improve as p increases, so the rebates apply to all 310 finite bands. The tail is excluded. □

28.5. **A bound at a band start holds at every larger prime in the band.** The formulas (F1)–(F10) are proved for any prime at which explicit gates hold. The following lemma states these gates and shows that, once they are checked at an evaluation point, the bounds hold with the same or better coefficients at every larger class-three prime and for every block of such primes. The block gates are used with the fixed endpoint constants, as explained in Step 3 below.

Lemma 28.11 (Domain of the upper formulas)**.** *Let $p_0 > 1$ be real (in the applications an evaluation point of the upper data: the start of a band, 2^{47}, $167 \cdot 2^{40}$, $j\,2^{42}$, or a power of two from 2^{48} to 2^{64}) and $L' \le 10000$ a block length. Fix the combinatorial data of the upper local computation: the graphs, the insertion orders, the matching partitions and supports, the repair records and their star supports, the pin classes and potentials of the joint rows, the softened laws and the mixture weights, and the rational parameters of the matrix tests. Suppose that, with every quantity evaluated by the formulas* (F1)–(F9) *at $r = p_0^{-1/2}$ and with block count $L'/\log p_0$:*

- (G1) *$e_t^-(z) < 1$ at every insertion of every row, repair and spectral graph, including both half-pinned conditional counts;*
- (G2) *in every application of* (F2)*, $w < \frac{1}{2}$, $w^{\mathrm{conn}} < 1$, $rq_{\mathrm{inf}} < 1$ and the resulting bound $\rho < 1$;*
- (G3) *the hypotheses of* (F3) *and* (F4) *(the pin-count error e_t and the pinned-extraction error e_t^{pin} of Proposition 16.20 with $\varpi = t$ below $\frac{1}{2}$, $I_0 > \frac{1}{2}$, the normalizer errors $\varepsilon_{\mathrm{F}}, \varepsilon_{\mathrm{E}}$ below $\frac{1}{2}$, every nonstar denominator positive, the normalizer error of* (F4) *below one and its selected correlation below one); for the rows of* (F5)*, $M \ge H + t_{\mathrm{Y}}$, $t_{\mathrm{Y}}H < 1$ and, in the bands, $c_{\mathrm{F}}r_0 < 1$ (for the tail datum, with $M = 21/20$ and $t_{\mathrm{Y}} = 1/50$, the first two follow from $H < 103/100$); and the gates of Proposition 16.22, Lemma 16.23 and Lemma 27.15 of the band's endpoint table;*
- (G4) *for each nonroot cut, the hypotheses of* (F6) *hold, including $d/\sqrt{p_0} < 1$ at length L', and the gates of Proposition 27.14;*
- (G5) *for each repaired record (84 in a band, 94 for the tail datum), $U < 1$, $\rho_{\mathrm{new}} < 1$, $2F < 1$, $\rho_{\mathrm{old}} < \frac{1}{2}$ and $\rho' < \frac{1}{2}$;*
- (G6) *for the ten EL/LE records of degree two of the tail datum, the hypotheses of* (F8)*;*
- (G7) *the gates of* (F9)*.*

Let $p \geq p_0$ be a prime with $p \equiv 3 \pmod 4$. Then:

- *(i) the prime-level hypotheses in* (G1)–(G7) *hold at p (the block gates are used at p_0 in (ii)), and the prime-level conclusions of* (F1)–(F9) *hold at p with every normalized coefficient at most its value at p_0: the row coefficients f and h, with the same M and t_{Y}; the spectral g and B_{top}; and the symmetric and skew coefficients of the repaired and replaced records;*
- *(ii) for every $X \geq p_0$ and every block $\mathcal{B}$ of primes $p \equiv 3 \pmod 4$, all at least X, with $\log q_{\mathcal{B}} \leq L'$ and least prime b, the conditional rows have coefficients at most $f\, b^{-1/2}$ and $h\, b^{-1/2}$, the nonroot negative coefficients are at most $d\, b^{-1/2}$ with $d = g \exp(B_{\mathrm{top}} L'/(X \log X))$, and each repaired or replaced record differs from the old record by an operator with zero marginals whose symmetric and skew parts have norms at most $C^{\mathrm{s}} b^{-1/2}$ and $C^{\mathrm{a}} b^{-1/2}$, where $f, h, g, B_{\mathrm{top}}, C^{\mathrm{s}}, C^{\mathrm{a}}$ are the values at p_0;*
- *(iii) the conclusions of (i) for* (F1)–(F8)*, with* (F5) *in its Schur form and without the gates that concern the bands only (those of Proposition 16.22 and Lemmas 16.23 and 27.15, of Proposition 27.14 and of* (F9)*), hold at every prime $p \geq p_0$ of either class for the data of the tail datum and of the tail comparison; here the matrix tests of* (F6)*, which are part of* (G4)*, and the class-dependent constants of* (F3) *and* (F4) *are evaluated for both prime classes, as the certificates do. Part (iii) is used only by Lemma 29.2 and Certificate 29.4.*

Proof. Step 1: the size of p enters only through the gates. Each of (F1)–(F9) is stated for every odd prime at which its listed hypotheses hold, and these hypotheses are the gates: (F1) is Lemma 12.2 (Weil's bound in both parities) with (G1); (F2) is Corollary 13.17, the certified form of Proposition 13.16, under (G2); (F3) and (F4) consist of complete conditional means valid for every odd prime, exact identities, Cauchy–Schwarz, divisions by the denominators of (G3), trace constants uniform for $p \geq 3$ and, in the bands, the full star matrices of Proposition 16.22; (F5) is Lemma 17.5(f), with Lemmas 16.23 and 27.15 and Corollary 17.6, in the bands, and an inequality between quadratic forms in $H, R_{\mathrm{m}}, \upsilon, \rho_{\mathrm{F}}, \rho_{\mathrm{E}}$ and the weight ratios (Lemma 17.5(c)) for the tail datum; (F6) is Theorem 14.21 under (G4), with the observed caps of Proposition 27.14; (F7) and (F8) were proved above; and (F9) consists of Propositions 28.3, 28.5 and 28.6, each stated for every prime $p \equiv 3 \pmod 4$ at which its gates hold. The class of p enters only in the following places: in the bands, which use class-three data only, the matrix tests of (F6) and of Proposition 27.14 at both Fourier signs, the gates of the joint rows and of (F9) at $\hat{P} = \pm i$, and the pair Gram of Lemma 28.2(c); for the tail datum, the matrix tests of (F6), which are checked for both classes; and the constants of the rows (F3), (F4), which depend on the class through $\chi(-1)$ in the signs of the pinned constraints and are evaluated for both classes, the larger value being used (Section 27.6.2). The numerical domains at which the evaluators first checked the gates, and their display assertions $p \geq 2^{46}$, 2^{50}, 2^{54}, 2^{60}, 2^{70}, 2^{72}, are evaluation points; no step of the proofs uses them.

Step 2: monotonicity in r and in the block count. The combinatorial data are fixed. Every scalar error entering a prime-level gate, and every normalized coefficient after its indicated power of r is removed, is built from r and the block count $l = L'/\log p_0$ by the operations of Lemma 4.18(i)–(iii), so it is admissible (nonnegative and nondecreasing in (r, l)): power series with nonnegative coefficients ($e_t^{\pm}$, e_t^{pin}, the count logarithms $c^{\pm}$ and ℓ_h, the count errors w, w^{conn}, w_Δ, the influence entries, q_{inf}, $c_\times$, κ_{pr}, e_{inf}, η, Γ, the entries and frame bounds of Lemma 14.4(c)–(f) and Γ_W, the coefficients of the fixed extraction histograms and the collision series of Corollary 16.17, and the bounds of Lemma 15.28); $x \mapsto 1/(1-x)$ on $[0,1)$ (R_Z, $1/Z_-$, $1/z_h$, $1/I_0$ with $I_0 = 1 - \binom{n}{2}\kappa r^2$, the row normalizers, $1/(1-U)$, $1/(1-\rho)$, $1/(1-\rho_I^2)$, $1/(1-\varepsilon_h)$, $1/\gamma$ with $\gamma = 1 - (e_{\mathrm{inf}} + (n-1)\eta)$, and $(1 - e_4^-(n-1))^{-2}$); exponentials and $(e^x - 1)/x$ (U, $e^{\ell_{\mathrm{new}}}$, $e^{\ell_{\mathrm{old}}}$, e^{ℓ_h}, K_∞, H, R_{m}, υ^2/r^2 and the block inflation $B_{\mathrm{top}} l r^2$); sums, products, square roots, maxima and minima; and norms of entrywise nonnegative matrices ($\|\mathbf{C}\|_{\mathrm{F}}$). The upper count logarithms are $c^+ = \sum e_t^+$ ((F1)), not $\sum \log(1 + e_t^+)$, which need not be small (Lemma 4.18(iii)); so every logarithmic count ratio ℓ, and

ℓ/r, is a power series with nonnegative coefficients, and $U/r = ((e^{\ell_U}-1)/\ell_U)(\ell_U/r)$ is admissible. For instance

$$f_{\mathrm{Schur}} = HR_{\mathrm{m}}\,\frac{\rho_{\mathrm{F}}}{r} + (H+50H^2)\frac{\upsilon^2}{r^2}\,r, \qquad \frac{\rho'}{r} = \frac{(\rho_{\mathrm{new}}/r)(1+2F)+U\,(U/r)}{1-U},$$

and $F_N = 2NU + K_\infty \rho_I^{2N-1} e_I/(1-\rho_I^2)$ with the chosen number N of initial terms held fixed. In the joint rows of (F5) the coefficients k_{J}, c_{F} and h of the endpoint table are held at their values at p_0, which bound the rows at p by the same argument, and t_{Y} is fixed. As p increases from p_0, r decreases, and the number of primes in a block of logarithmic length at most L' is at most $L'/\log p_0$. By Lemma 4.18(a) these scalar errors at p are at most their values at p_0, so the prime-level scalar gates hold at p, and every coefficient is at most its value at p_0. This does not infer a comparison between two decreasing quantities from separate monotonicity: the block gates (28.3) are used only at p_0 in Step 3. A minimum of bounds is taken only among bounds whose own hypotheses hold, and these hypotheses are among the gates. For (F6) the uniformity is part of Theorem 14.21: its matrix tests are fixed rational inequalities checked with the endpoint values of the errors (Lemma 4.18(d)), and its conclusions hold at every $p \ge p_0$ with the same g and B_{top}; the matrix tests of (F9) are fixed rational inequalities as well. No monotonicity of an eigenvector or of an optimized parameter is used. This proves (i).

Step 3: blocks. Let $\mathcal{B}$ be as in (ii); it has at most $L'/\log b \le L'/\log p_0$ primes, all with the bounds of (i). The row bounds are the block form of (F5): densities and chi-square defects multiply and correlations are maximized over the primes (Lemma 6.10(c),(f)), so the per-prime bounds of (i) give the block quantities of (F5) (see its proof). The nonroot bound is the block form of (F6), Lemma 13.7(b),(c) with $\sum_{p\in\mathcal{B}} 1/p \le L'/(b\log b) \le L'/(X\log X)$. For a repaired or replaced record, both block laws are tensor products over the primes of $\mathcal{B}$ of couplings with the same actual half marginals, and the old one is self-adjoint at ι, so their difference has zero marginals. Let δ_p and δ'_p be the centred norms of the old and of the new law at p, and c_{old}, c_{new} their bounds at p_0; by (i), $\delta_p \le c_{\mathrm{old}} r_p/r_0$ and $\delta'_p \le c_{\mathrm{new}} r_p/r_0$, with $r_p = p^{-1/2}$, $r_0 = p_0^{-1/2}$. By the closing clause of Lemma 6.10(b) the norms of the symmetric and of the skew part of the block difference are the maxima, over the nonempty sets S of primes of $\mathcal{B}$, of the norms of their restrictions to the sector of S. For $S=\{p\}$ these are the parts of the prime difference, at most $C^{\mathrm{s}} r_p \le C^{\mathrm{s}} b^{-1/2}$ and $C^{\mathrm{a}} r_p \le C^{\mathrm{a}} b^{-1/2}$ by (i). For $|S| \ge 2$ let $j<k$ be the two least primes of S. All δ, δ' are below one, so the restriction of the difference has norm at most $\delta_j\delta_k + \delta'_j\delta'_k \le (c_{\mathrm{old}}^2 + c_{\mathrm{new}}^2) r_j/r_0$, and that of its skew part, the old law being self-adjoint, at most $\delta'_j\delta'_k \le c_{\mathrm{new}}^2 r_j/r_0$. So the *block gates*

$$c_{\mathrm{old}}^2 + c_{\mathrm{new}}^2 \le C^{\mathrm{s}} r_0, \qquad c_{\mathrm{new}}^2 \le C^{\mathrm{a}} r_0 \tag{28.3}$$

give the bounds $C^{\mathrm{s}} b^{-1/2}$ and $C^{\mathrm{a}} b^{-1/2}$ on these sectors too. For a one-way repair (F7), $c_{\mathrm{old}} = \rho_{\mathrm{old}}$ and $c_{\mathrm{new}} = \rho'$ are below $\frac12$ by (G5), while $C^{\mathrm{s}} r_0 \ge \rho_{\mathrm{old}} + \rho'$ and $C^{\mathrm{a}} r_0 \ge \rho'$, so (28.3) holds. For (F9) it follows from the gates of the three propositions ((G7)): for the forest records $c_{\mathrm{old}} = \rho_{\mathrm{old}} < \frac12$ and $c_{\mathrm{new}}^2 < \min(s_{\mathrm{new}}, a_{\mathrm{new}})$, with $C^{\mathrm{s}} r_0 \ge \rho_{\mathrm{old}} + s_{\mathrm{new}}$ (respectively $\rho_{\mathrm{old}}^2 + c_{\mathrm{new}}^2 < d_{\mathrm{sym}} \le C^{\mathrm{s}} r_0$ for FE and EF) and $C^{\mathrm{a}} r_0 \ge a_{\mathrm{new}}$; for the dense laws and the mixture $c_{\mathrm{old}} = c_0$, $c_{\mathrm{new}} = c_1$, and $c_0^2 + c_1^2 < \min(k_{\mathrm{s}}, k_{\mathrm{a}}) r_0 \le \min(d_{\mathrm{s}}, d_{\mathrm{a}}) \le \min(C^{\mathrm{s}}, C^{\mathrm{a}}) r_0$, because $e^{\ell_0} \ge 1$ and $\Gamma \ge 1$. This proves (ii).

Step 4: both classes. By Step 1 the class of p enters only the class-three tests and Paley gates of the bands, and the tests and row constants of the tail datum, which are evaluated for both classes; Step 2 does not depend on the class. This proves (iii). □

The lemma applies the per-prime spectral constants once: a repair reads only the endpoint values of g and B_{top} of the relevant nonroot cut, never its block cap d, and the block inflation

$\exp(B_{\mathrm{top}}L'/(X\log X))$ is applied only to the band inputs d. Its domain for the root cuts is that of Lemma 27.7 and Lemma 27.10.

Evaluation points. Every band records an *anchor*: 2^{47} for the 45 bands of $[P_3, 2^{48})$; 2^b for the bands of $[2^b, 2^{b+1})$, $48 \le b \le 51$; and its own start from 2^{52}. Each input of a band $[X, Y)$ is evaluated at a point not above X, for blocks of logarithmic length at most the band's budget (Definition 27.1(c)):

- root caps: at X, with the frame witness of the anchor (Certificate 27.9);
- nonroot and observed caps: at 2^{47} below 2^{48} and at 2^{48} above, inflated at X and at the band's budget (Certificate 27.13(a),(b));
- rows: the product-row quantities of (F3) and (F4) at X, and the joint coefficients of (F5)(a) from the largest of the 17 joint-row endpoint tables not above X (Certificate 27.13(c));
- one-way repairs (F7): the symmetric coefficients at the largest of 2^{47}, 2^{48}, 2^{49}, 2^{50}, 2^{52}, 2^{54} not above X, the skew coefficients at the largest of 2^{47}, 2^{48}, 2^{49} not above X;
- forest records (Proposition 28.3): at 2^{48} on $[2^{48}, 2^{49})$ and at 2^{49} from 2^{49};
- dense EL law: at 2^{47} below 2^{48} and at 2^{48} above;
- LE mixture: at the largest of its eight endpoints not above X.

Lemma 28.11 extends every input to all blocks of the band. The repair coefficients at 2^{47} are those of Certificate 28.12(d). At 2^{48} and 2^{49} the symmetric coefficients are those of `research/roadmap_2026_10_18/item12/lower_repair_gates_48.json` and `lower_repair_gates_49.json`, and the skew coefficients are the values ρ'/r of the same records, rounded upward as in `research/continuation_2026_10_24/upper_arithmetic/skew_only.py` (record `skew_only.json` of that directory at 2^{49}). At 2^{50}, 2^{52} and 2^{54} the symmetric coefficients are those of `research/frontier_2026_10_17/sparse/high72_oneway_repairs.json` and `high72_oneway_repairs_degree3.json`, re-checked by `high72_oneway_adapter.py` of that directory. The band record `research/further_2026_10_25/upper/upper_dense_refined.json` carries the symmetric and skew ledgers of every band. The tail uses the repair records of degrees two and three at 2^{54}, with the all-prime joint-star rows of `high72_oneway_jointstar.json` ((F8), Certificate 29.4).

Certificate 28.12 (The bands below 2^{48})**.** For the 45 bands of $[P_3, 2^{48})$ the local data are evaluated for class-three primes at 2^{47}, except the root inputs, evaluated at the start of the band with the frame witness at 2^{47}, the rows, evaluated at the start of the band with the joint coefficients of the largest joint-row table not above it, and the LE mixture, evaluated at P_3. The files named are in `research/advance_2026_10_24/upper/lower_p47/`.

(a) *Nonroot caps* at 2^{47} (table of Section 27.4): `class3_caps.json`, reviewed by `class3_review.json` (`PASS_INDEPENDENT_CLASS3_NONROOT_UPPER`, 20 exact PSD tests).

(b) *Observed caps* $g_G = 2.268819655$, $g_C = 0.079761731$ (Proposition 27.14): `observed_caps.json`, reviewed by `observed_review.json` (`PASS_INDEPENDENT_OBSERVED_MARGINAL_CAPS`).

(c) *Root anchor* at 2^{47} (Certificates 27.8 and 27.9), with the lower coefficient replay `root_coefficients_review.json` (`PASS_INDEPENDENT_LOWER_ROOT_COEFFICIENTS`).

(d) *One-way repairs.* The 84 repairs at 2^{47} with the coefficients of (F7): `repairs.json`, with the review `native_review.json`, status `PASS_LOWER_NATIVE_ROOT_FIELD_AND_INDEPENDENT_REPAIR_REPLAY` (84 independent actual-marginal repairs, 588 explicit count orders, 168 correlations, 32 root and 96 original row replays, 4 exact anchor PSD tests, 96 log-convex coefficient checks, and the bindings of the complete symmetric and skew repair ledgers).

(e) *Dense EL law* at 2^{47}, with $s_{\mathrm{EL}} = 3.291966120220$ and $a_{\mathrm{EL}} = 2.401350027793$: `dense_certificate.json` and `dense_completed.json`, reviewed by `dense_review.json` (`PASS_`

`INDEPENDENT_DENSE_REPAIR`: four complete histograms and all matrix, actual-frame, marginal, repair and uniform-support gates).

(f) *LE mixture* at P_3 (Proposition 28.6), and the *rows* at the start of each band, with the joint-row tables at P_3 and at $j2^{42}$, $j \in \{42, 44, 46, 48, 50, 56, 60\}$ (Certificate 27.13(c)).

The resulting state ledgers, the same in all 45 bands, are (symmetric/skew)

degree two		degree three	
LR	11.237485578048/11.166605062536	LC	51.202575332/37.711621448
RL	6.58393224044/4.802700055586	CL	51.405075886/37.914122002
DD	43.091020254/24.006543228	DR	207.354081636/163.894569102
DS	41.670210072/32.01254062	RD	163.470704406/120.011191872
SD	62.49133143/48.004827252	SR	108.629454264/108.629445156
SS	45.271035916/45.271032884	RS	106.542130168/106.542121068

The commands are listed in `research/advance_2026_10_24/upper/LOWER47_AUDIT.md`. The completion gates and the prices are those of Certificate 27.18 and Proposition 28.14.

By Lemma 28.11, and by Lemma 27.7 for the root entries, the inputs are valid for every block of their bands. The bands below 2^{48} carry about 60% of $S_{\mathrm{u}}(a_{\mathrm{u}})$ and 65% of $A_{\mathrm{u}}(a_{\mathrm{u}})$, and those of $[2^{48}, 2^{49})$ about 27% and 24%.

28.6. **(CP4) for upper atoms with prices $S_{\mathrm{u}}, A_{\mathrm{u}}$.** We can now define the comparison law in the upper range: each record of the reference is kept if it already accepts the six queries, replaced by the repairs of the preceding subsections if it does not, or deleted. The rest of the subsection sums the resulting symmetric and skew prices into S_{u} and A_{u}, which is hypothesis (CP4) for upper atoms.

Definition 28.13 (The upper comparison law)**.** Let $\mathcal{B}$ be a block of an upper band (Definition 27.1), with least prime b and $r = b^{-1/2}$, and let $\mu_{\mathcal{B}} = \widetilde{\mu}_{\mathcal{B}}/P_\mu(r)$ be its reference (Definition 27.5): $\widetilde{\mu}_{\mathcal{B}}$ is the sum of the records, the $m_{\mathfrak{t}}$ two-layer records of $\pi_{\mathfrak{t},2}$ with the weight $w_{\mathfrak{t}} r^{\deg \mathfrak{t}}$ and the $m_{\mathfrak{t}}$ folded records of $\mathrm{Diag}_\iota(\pi_{\mathfrak{t},1})$ with the weight $v_{\mathfrak{t}} r^{\deg \mathfrak{t}+1}$, each the tensor product over the primes of $\mathcal{B}$ of its laws at the single primes. The comparison law is $\nu_{\mathcal{B}} = \widetilde{\nu}_{\mathcal{B}}/P_\mu(r)$, where $\widetilde{\nu}_{\mathcal{B}}$ is obtained from $\widetilde{\mu}_{\mathcal{B}}$ record by record, each with its weight:

(a) a record that accepts the six queries $\mathcal{E}_6$ is kept;

(b) each of the 84 nonaccepting two-layer records of degree two or three that are not LE or EL records of degree two is replaced by its one-way repair (F7), tensorized over the primes of $\mathcal{B}$;

(b′) the six LE records of degree two are replaced by the mixture of Proposition 28.6, with the weight vector of the band, and the two nonaccepting EL records of degree two by the dense law of Proposition 28.5, each tensorized over the primes of $\mathcal{B}$;

(c) every other record is deleted.

Let $\Lambda^{\mathrm{s}}_k(\mathfrak{t})$ and $\Lambda^{\mathrm{a}}_k(\mathfrak{t})$ be the band's symmetric and skew ledgers: the sums of the symmetric, respectively skew, coefficients of the changed records of state $\mathfrak{t}$ and degree $k \in \{2, 3\}$, at the band's evaluation points (Section 28.5). The polynomials of the band are

$$\Pi^{\mathrm{rep}}_{\mathrm{s}}(r) = r^3 \sum_{\mathfrak{t}} w_{\mathfrak{t}} \Lambda^{\mathrm{s}}_2(\mathfrak{t}) + r^4 \sum_{\mathfrak{t}} w_{\mathfrak{t}} \Lambda^{\mathrm{s}}_3(\mathfrak{t}), \qquad \Pi^{\mathrm{del}}(r) = \sum_{d=3}^{7} C^{\mathrm{del}}_d r^d,$$

$\Pi^{\mathrm{rep}}_{\mathrm{a}}$ likewise with Λ^{a}, where C^{del}_d is the total weight of the deleted records of overall degree d, summed over individual records with their multiplicities $\mathfrak{m}_{\mathfrak{t}}$ ($w_{\mathfrak{t}}$ for a two-layer record with $d = \deg \mathfrak{t}$, $v_{\mathfrak{t}}$ for a folded record with $d = \deg \mathfrak{t} + 1$); in particular the constant records contribute with multiplicity 2.

By Lemma 28.1, the deleted records are the twenty-eight nonaccepting folded records of total degree three and all records of total degree at least four; so Π^{del} has no terms below degree three, and its coefficients of degrees four to seven are those of P_μ. The changed records of degrees two and three give the terms of degrees three and four of $\Pi_{\mathrm{s}}^{\mathrm{rep}}$ and $\Pi_{\mathrm{a}}^{\mathrm{rep}}$. All three polynomials are signed comparison errors only: neither P_μ nor the normalization of $\nu_{\mathcal{B}}$ is truncated.

Proposition 28.14 ((CP4) in the upper range: the totals S_{u} and A_{u})**.** *Let $\mathcal{B}$ be a block of an upper band $[X,Y)$, with least prime b.*

(i) $\mu_{\mathcal{B}}$ is reflection positive at every cut and has uniform one-slot marginals. $\nu_{\mathcal{B}}$ is a nonnegative measure with the normalizing denominator P_μ of $\mu_{\mathcal{B}}$, invariant under the maps of (R3), so its one-slot marginals are multiples of the uniform law; every retained record accepts $\mathcal{E}_6$ (the one-way repairs and the dense EL laws accept the twelve queries, and the LE mixtures $\mathcal{E}_6$); the pair marginal under $\nu_{\mathcal{B}}$ of every query of $\mathcal{E}_6$ is a multiple of $\mathrm{Unif}(\Omega_{\mathcal{B}}) = \bigotimes_{p\in\mathcal{B}} \mathrm{Unif}(\Omega_p)$; and under the normalized accepted part every query pair is a strict-square pair at every prime of $\mathcal{B}$, which is (R5). At ι, $\nu_{\mathcal{B}} - \mu_{\mathcal{B}}$ satisfies the bracket of Definition 9.14 with

$$s_{\mathcal{B}} \le (\Pi_{\mathrm{s}}^{\mathrm{rep}} + \Pi^{\mathrm{del}})(b^{-1/2})\, q_{\mathcal{B}}^{2a}, \qquad a_{\mathcal{B}} \le \Pi_{\mathrm{a}}^{\mathrm{rep}}(b^{-1/2})\, q_{\mathcal{B}}^{2a},$$

and $\nu_{\mathcal{B}}$ is admissible (Definition 8.9).

(ii) For each band and each d, define $\mathcal{U}_d(a)$ to be the displayed upper bound for $\sum_{\mathcal{B}} q_{\mathcal{B}}^{2a} b_{\mathcal{B}}^{-d/2}$ over the blocks of the band, namely $\mathcal{U}_d(a) = e^{2(a-a_{\mathrm{c}})l}\,\mathcal{U}_d(a_{\mathrm{c}})$, where $\mathcal{U}_d(a_{\mathrm{c}})$ is the bound of Lemma 20.4(i), at $s = d/2$, for the band's partition at an exponent $a_{\mathrm{c}} \le a$ ($a_{\mathrm{c}} = 883/10^6$ for the bands below 2^{52}, $85941/125000000$ for the inherited partitions from 2^{52}), and $l \ge \max_{\mathcal{B}} \log q_{\mathcal{B}}$ is the recorded bound for the band's block products (Lemma 9.3). Then the upper parts of S and A satisfy

$$\sum_{\mathcal{B}\ \mathrm{upper}} s_{\mathcal{B}} + \sum_{p\ge P_4} s_p \le S_{\mathrm{u}}(a), \qquad \sum_{\mathcal{B}\ \mathrm{upper}} a_{\mathcal{B}} + \sum_{p\ge P_4} a_p \le A_{\mathrm{u}}(a),$$

where

$$S_{\mathrm{u}}(a) = \sum_{\mathrm{bands}} \sum_{d} ([r^d]\Pi_{\mathrm{s}}^{\mathrm{rep}} + [r^d]\Pi^{\mathrm{del}})\,\mathcal{U}_d(a) + S_{\mathrm{tail}}(a),$$
$$A_{\mathrm{u}}(a) = \sum_{\mathrm{bands}} \sum_{d} [r^d]\Pi_{\mathrm{a}}^{\mathrm{rep}}\,\mathcal{U}_d(a) + A_{\mathrm{tail}}(a),$$

and $S_{\mathrm{tail}}(a)$, $A_{\mathrm{tail}}(a)$ are the bounds of Lemma 29.5(iv) for the prices of the tail atoms $p \ge P_4$.

(iii) With $C_{\mathrm{band}} = \sum_{d=3}^{7} ([r^d]\Pi_{\mathrm{s}}^{\mathrm{rep}} + [r^d]\Pi_{\mathrm{a}}^{\mathrm{rep}} + [r^d]\Pi^{\mathrm{del}}) X^{-(d-3)/2}$, $s_{\mathcal{B}} + a_{\mathcal{B}} \le C_{\mathrm{band}}\, b^{-3/2} q_{\mathcal{B}}^{2a}$.

Proof. (i) The reference is Lemma 27.17: by its part (ii) the inputs of the band, the root entries of both routes included, are valid on every block, so part (i) applies to $\mu_{\mathcal{B}}$. We write $\tilde{s}_{\mathcal{B}}$, $\tilde{a}_{\mathcal{B}}$ for the sums of the unpriced coefficients $\tilde{\pi}^{\mathrm{s}}_{\mathcal{B},t}$, $\tilde{\pi}^{\mathrm{a}}_{\mathcal{B},t}$ of Definition 9.14(b).

Records kept. A kept record is a record of $\mu_{\mathcal{B}}$; it accepts $\mathcal{E}_6$ by definition (Lemma 28.1). *Records repaired.* By (F7) and Lemma 28.11(ii), at the band's evaluation points, a repaired record of state $\mathfrak{t}$ is replaced by a probability law $\pi_{\mathcal{B}}^{\mathrm{rep}}$, invariant under the square-affine maps at each prime, supported where the projected queries hold (so it accepts the twelve queries), with the actual half marginals $\mu_{\mathcal{B}}^{\mathrm{old}}$ of the old record $\pi_{\mathcal{B}}^{\mathrm{old}}$; and $\Delta = \pi_{\mathcal{B}}^{\mathrm{rep}} - \pi_{\mathcal{B}}^{\mathrm{old}}$ has zero marginals, and at ι its symmetric and skew parts $\zeta^{\mathrm{s}} = \frac{1}{2}(\Delta + \Delta^*)$, $\zeta^{\mathrm{a}} = \frac{1}{2}(\Delta - \Delta^*)$ have norms at most $C_{\mathfrak{t}}^{\mathrm{s}} r$ and $C_{\mathfrak{t}}^{\mathrm{a}} r$ relative to $\mu_{\mathcal{B}}^{\mathrm{old}}$, with the coefficients of (F7), or of (F9) for the forest records. The skew bound holds because $\pi_{\mathcal{B}}^{\mathrm{old}}$ is symmetric at ι, so $\mathrm{Skew}\,\Delta = \mathrm{Skew}\,\pi_{\mathcal{B}}^{\mathrm{rep}}$ (Remark 9.15(iii)). Let D^{old} be the

reserved-diagonal copy of $\mu_{\mathcal{B}}^{\mathrm{old}}$. Then

$$\begin{pmatrix} C_{\mathrm{t}}^{\mathrm{s}} r D^{\mathrm{old}} & \zeta^{\mathrm{s}} \\ \zeta^{\mathrm{s}} & C_{\mathrm{t}}^{\mathrm{s}} r D^{\mathrm{old}} \end{pmatrix} \succeq_\iota 0, \qquad \begin{pmatrix} C_{\mathrm{t}}^{\mathrm{a}} r D^{\mathrm{old}} & \zeta^{\mathrm{a}} \\ \zeta^{\mathrm{a}*} & C_{\mathrm{t}}^{\mathrm{a}} r D^{\mathrm{old}} \end{pmatrix} \succeq_\iota 0,$$

since $\left(\begin{smallmatrix} cD & K \\ K^* & cD \end{smallmatrix}\right) \succeq 0$ as soon as $\|D^{-1/2} K D^{-1/2}\| \le c$; here D^{old} has uniform one-slot marginals, by invariance. With its weight $w_{\mathrm{t}} r^{\deg \mathrm{t}}$, the record contributes at most $w_{\mathrm{t}} C_{\mathrm{t}}^{\mathrm{s}} r^{\deg \mathrm{t}+1}$ to $\tilde{s}_{\mathcal{B}}$ and $w_{\mathrm{t}} C_{\mathrm{t}}^{\mathrm{a}} r^{\deg \mathrm{t}+1}$ to $\tilde{a}_{\mathcal{B}}$. *Records replaced.* The same argument applies, with the coefficients of (F9) and Lemma 28.11(ii), to the dense EL laws and to the LE mixtures: they are positive laws that accept $\mathcal{E}_6$ (the dense ones the twelve queries), are invariant and have the actual half marginals of the record they replace (Propositions 28.5 and 28.6). Summing over the changed records gives $\Pi_{\mathrm{s}}^{\mathrm{rep}}(r)$ and $\Pi_{\mathrm{a}}^{\mathrm{rep}}(r)$, because the ledgers are the sums of the coefficients by state. *Records deleted.* A deleted two-layer record is a coupling of its actual half marginal μ_{h} with itself, symmetric under the layer interchange, so its cut matrix C_ι at ι is self-adjoint with $-D_{\mathrm{h}} \preceq_\iota C_\iota \preceq_\iota D_{\mathrm{h}}$, D_{h} the reserved-diagonal copy of μ_{h}; a deleted folded record is itself a reserved-diagonal law. Either way the change $-w r^d \cdot$ (record) is symmetric and satisfies the first bracket with the coefficient $w r^d$ (its mass) and the second with coefficient 0 (Remark 9.15(i)); summing gives $\Pi^{\mathrm{del}}(r)$ in $\tilde{s}_{\mathcal{B}}$ only. The changed records are exactly the records omitted from the deletion polynomial, so nothing is counted twice. Dividing by $P_\mu(r) \ge 1$ only decreases the coefficients, and the prices carry the factor $q_{\mathcal{B}}^{2a}$ of Definition 9.2.

Acceptance and admissibility. Every retained record, kept, repaired or replaced, accepts $\mathcal{E}_6$; under it each query pair has a nonzero square difference at every prime of $\mathcal{B}$, which is (R5). Every retained record is invariant under the maps of (R3) (a tensor product over the primes of invariant laws, Lemma 5.16(1); a mixture of invariant laws is invariant) and supported where each query pair of $\mathcal{E}_6$ lies in $\Omega_{\mathcal{B}}$, so its pair marginal on every query of $\mathcal{E}_6$ is $\mathrm{Unif}(\Omega_{\mathcal{B}})$ (Lemma 5.16(3)). Every repaired or replaced record is a positive law with exactly the actual half marginals of the record it replaces, and its one-slot marginals are uniform, as are those of every diagonal envelope above; the differences have zero one-slot marginals (Lemma 5.16(2)). Every deleted record is a positive record of $\mu_{\mathcal{B}}$; all these laws are laws of the field labels, independent of the interval, residue and inclusion indices. This is (O1) of Definition 8.9. The sharp bounds of Lemma 28.11(ii) are operator norms between the actual half spaces at ι, with no label conditioned; this is (O2). The root caps enter only as completion inputs of $\mu_{\mathcal{B}}$ and create no selected law, and the upper laws introduce no index projection and no fold, so (O3) and (O4) are those of the frame.

(ii) Sum (i) over the blocks of each band. For a polynomial $\sum_d c_d r^d$ with $c_d \ge 0$ the sum over the blocks is $\sum_d c_d \sum_{\mathcal{B}} q_{\mathcal{B}}^{2a} b_{\mathcal{B}}^{-d/2} \le \sum_d c_d \mathcal{U}_d(a)$. The instance of $\mathcal{U}_d(a)$ is valid by Lemma 20.4(i), which applies because every band $[X, Y)$ lies above X_{Ch} and has even endpoints, and by Lemma 9.3, which needs $a \ge a_{\mathrm{c}}$ and $\log q_{\mathcal{B}} \le l$. Add the tail prices of Lemma 29.5(iv).

(iii) The three polynomials have no terms below degree three, and $b^{-(d-3)/2} \le X^{-(d-3)/2}$ for $d \ge 3$. □

Lemma 28.15 (One initial term in the upper marginal repair)**.** *In the setting of* (F7)*, one may replace the bound* $F = U(1 + K_\infty/(1 - \rho_{\mathrm{new}}))$ *by*

$$F_* = \min\left\{ F,\ 2U + \frac{K_\infty \rho_{\mathrm{new}} U}{1 - \rho_{\mathrm{new}}} \right\}.$$

This bounds the same canonical centred solution and therefore changes neither the repaired probability law nor its actual half marginals. The smaller value may be used in the centred norm in (F7)*, and in the additive-repair correction terms of* (F8) *and* (F9)*; all other terms in those formulas are retained. The resulting symmetric and skew caps may each be minimized with the previously certified cap for the same physical law, with upward rational rounding. The dense LE, EL laws keep their separately certified invariant-space bounds.*

Proof. Use the full centred spaces in Lemma 17.14(g), with $\rho_I = \rho_{\mathrm{new}}$, $N = 1$, and $e_I = (1+\rho_{\mathrm{new}})U$: both right sides have this L^2 bound by the triangle inequality. This gives the second term in F_*. Uniqueness in Lemma 17.14(a) identifies the solution with the one already used in(F7). Its support, symmetries and exact marginals are unchanged, and the old positivity gate still applies since $F_* \le F$. Every cited centred norm or repair correction is nondecreasing in the multiplier bound. The remaining inequalities involving the full (unprojected) norms on the whole support, namely the block gates (28.3) for the new caps, are checked in the record named at the end of this proof, with the retained full-norm majorants and the new symmetric and skew caps. Thus the substitutions are valid without changing the mixture law on the original labels or its positive reference $\mu_{\mathcal{B}}$. □

The upper prices in the closing. The record `research/further_2026_10_25/upper/upper_dense_refined.json` holds, for each of the 310 bands, the weights, the three polynomials $\Pi_{\mathrm{s}}^{\mathrm{rep}}$, $\Pi_{\mathrm{a}}^{\mathrm{rep}}$ and Π^{del} (fields `comparison.repair`, `comparison.skew` and `comparison.deletion`) and the charges $\mathcal{U}_d(a_{\mathrm{u}})$, $3 \le d \le 7$, at $a_{\mathrm{u}} = 883/10^6$:

- for the 301 bands below 2^{52} the charges are the bounds of Lemma 20.4(i) for their two-cell partitions;
- for the nine bands from 2^{52} they are the stored charges of their inherited partitions (Definition 27.1(b)) at 85941/125000000, multiplied by $\exp(2(a_{\mathrm{u}} - 85941/125000000)l)$; two repricing steps compose to one.

At a_{u} this gives $S_{\mathrm{bands}}(a_{\mathrm{u}}) < 0.013780948065193294$ and $A_{\mathrm{bands}}(a_{\mathrm{u}}) < 0.006641335659555205$ for the band subtotals (the sums over bands in Proposition 28.14(ii), before the tail sums are added). The closing reprices every term by $\exp(2(a - a_{\mathrm{u}})l)$, with l the term's recorded bound for the block products (at most 885.436), and adds the tail sums of Lemma 29.5(iv). This is done by `research/continuation_2026_10_24/global/closing.py` (`upper_price`), called by `research/advance_2026_10_24/global/closing_beta.py`. At the baseline exponent, $S_{\mathrm{u}}(a) < 0.013866153512$ and $A_{\mathrm{u}}(a) < 0.006682351376$. They enter S, A and the matching exponent T of the closing (Certificate 30.3). The band record is regenerated from the certified baseline, without an optimizer, by `research/further_2026_10_25/upper/verify.py`, and the independent review `dense_refined_upper_review.json` of that directory (`PASS_INDEPENDENT_POSITIVE_SIX_QUERY_LE_UPPER`) re-checks its 620 ledgers Λ^{s}, Λ^{a} and 930 polynomials $\Pi_{\mathrm{s}}^{\mathrm{rep}}$, $\Pi_{\mathrm{a}}^{\mathrm{rep}}$, Π^{del} (two ledgers and three polynomials per band).

No bound of the form $q_{\mathcal{B}}^{2a} \le e$ is used on upper blocks: every block is charged the factor $q_{\mathcal{B}}^{2a}$ once per expanded term, through S and A, and at the final exponent these factors reach about $e^{2a\cdot 885.436} \approx 4.81$. Neither the restoration and variance lemmas of the middle range nor the positive means use a cap on upper blocks.

Consequences and what to read next. From the upper range I: the bands and weights (Definition 27.1), the two-layer graphs and their components (Lemma 27.4, Section 27.2.2), the reference and its positivity (Definition 27.5, Lemma 27.17), the formulas (F1)–(F6) and (F10), with the two-class evaluation of the rows (Section 27.6.2) and the joint rows (Lemma 27.15), and the root, nonroot and observed caps with their domain (Certificates 27.9 and 27.13, Lemmas 27.7 and 27.10, and Proposition 27.14). From Parts II–IV: the six queries, their twelve-query superset and the merge patterns (Section 18.2 and Remark 18.2, Definition 17.4); the additive repair package with its initial-segment form and the exact marginal cancellation (Lemmas 17.13 and 17.14); same-measure transport (Lemma 17.16); recentering (Lemma 13.19); pinned influence (Proposition 13.16, Corollary 13.17, and Definition 13.10); the graded extension count, vertex densities and witness kernels (Lemmas 12.2, 12.6 and 16.4); square-affine invariance and pair laws (Lemma 5.16); the two-root identity (4.2), Weil's bound (Theorem 4.7) and the Paley operator (Lemma 4.11); the complete stars, frames and soft kernels (Lemmas 13.8 and 14.4, Corollaries 14.5

and 16.17, and Remark 14.7); the weighted star marginals (Lemma 15.28); admissible majorants (Lemma 4.18); the sector lemma with separate symmetric and skew coefficients (Lemma 6.10(b)); the five-type completion with joint rows (Lemmas 16.23 and 17.5, Proposition 16.22, and Corollary 17.6); block lifting (Lemma 13.7); the cut kernel and the spectral criterion (Lemma 13.3 and Theorem 14.21); block charges and repricing (Lemmas 9.3 and 20.4); the charging rules, skew-only charging among them (Remark 9.15); and the definitions of prices, comparison laws and admissibility (Definitions 8.9, 9.2 and 9.14). From the tail: the tail prices (Lemma 29.5).
Hypotheses verified in this section: (CP4) and (R5) for upper atoms. For every upper block, $\nu_{\mathcal{B}}$ is a nonnegative measure with the normalizer of $\mu_{\mathcal{B}}$; every retained record accepts $\mathcal{E}_6$, with every query pair a strict-square pair at every prime ((R5)), and the one-slot and query-pair marginals of $\nu_{\mathcal{B}}$ are multiples of the uniform laws; at ι, $\nu_{\mathcal{B}} - \lambda_{\mathcal{B}} = \nu_{\mathcal{B}} - \mu_{\mathcal{B}}$ (the first-stage atoms are the middle blocks, Definitions 9.11 and 18.3) is a sum of symmetric and skew record changes, priced separately, with reserved-diagonal envelopes of actual half marginals and uniform one-slot marginals; and the prices sum to at most $S_{\mathrm{u}}(a)$ and $A_{\mathrm{u}}(a)$, the tail included (Proposition 28.14). The laws are admissible (CP8). This is the upper row of (CP4) in Table 5.

29. Large primes $p \geq 2^{512}$ and the external index

What this section proves and how to read it. Large primes (called the *tail* below) are one atom per prime, of both residue classes; this is the only family whose size grows with N. The section proves (CP1) and summability with prices $O(p^{-3/2+2a})$, and evaluates the conductor factor R_I and the terms J, W at the baseline choice $K = 890666$. "Tail" here always means large primes; depth tails $\delta_{p,j}$ are a different notion (Section 9). Two parts of the construction are not blocks of field primes of bounded size. The *tail* consists of all primes $p \geq P_4 = 2^{512}$, of both residue classes modulo 4, one atom $\{p\}$ per prime with $e_p = 1$; it is the only family of atoms whose number grows with N. The tail law is the reference of the tail datum (Definition 27.1(e)) at a single prime; it is not the law of the last band, whose data are class-three data. The *small modulus* M_0 (formula in Section 29.4) is not a coordinate of G_N; it enters only through the external index of the six queries. We show that the tail law satisfies (CP1) at every tail prime (Section 29.1) and (R4) at the primes $p \geq P_5 = 2^{65536}$, as Lemma 6.15 requires (Section 29.2), and that the tail comparison prices are $O(p^{-3/2+2a})$ per prime, so that they enter S and A as a convergent series, satisfy the summability (CP8)(c), and make the product Π_{H} of Lemma 8.11 finite (Section 29.3). For the index we evaluate the conductor moment R_I and the terms J and W of Theorem 9.21 at the baseline choice $K = 890666$, with the finite prime-moment bound of Lemma 29.9 (Section 29.4). These enter the closing (Section 30).

29.1. **The reference law at one large prime.** The upper reference is defined band by band with class-three data (Definition 27.1). Beyond P_4 every prime is an atom of its own and carries, at the single prime, the reference of the *tail datum* (Definition 27.1(e)): the inputs and weights recorded for the range $[2^{64}, 2^{512})$ in `research/roadmap_2026_10_18/item12/upper_v2_local.json`, whose inputs take the larger value over the two classes. The square-clique prefix ends at P_4 (Lemma 19.2), so the tail also contains the primes $p \equiv 1 \pmod 4$. This is the only place where a two-layer law is used at class-one primes, and the only place where the class-one cases of the spectral criterion are needed (Section 14.11).

Definition 29.1 (The tail law)**.** Let $p \geq P_4$ be prime and $r = p^{-1/2}$. The *tail law* at p is the reference $\mu_p = \widetilde{\mu}_p / P_\mu(r)$ of Definition 27.5 for the single prime $\mathcal{B} = \{p\}$, with the weights $w_{\mathfrak{t}}, v_{\mathfrak{t}}$ of the tail datum (Definition 27.1(e)):

$$\widetilde{\mu}_p = \sum_{\mathfrak{t}} w_{\mathfrak{t}} r^{\deg \mathfrak{t}} \pi_{\mathfrak{t},2} + r \sum_{\mathfrak{t}} v_{\mathfrak{t}} r^{\deg \mathfrak{t}} \operatorname{Diag}_\iota(\pi_{\mathfrak{t},1}), \qquad P_\mu(r) = \sum_{d=0}^{7} c_d r^d,$$

with the components $\pi_{\mathfrak{t},j}$ of Section 27.2.2 at p. Here $c_0 = w_{\mathrm{LL}} = 1$ is the weight of the *root* $\pi_{\mathrm{LL},2}$, the uniform law on the injective strict-square labellings of the two-layer root graph, and c_d is the total weight of the records of overall degree d ((27.3)):

$$c_0 = 1, \quad c_1 = \tfrac{5988}{5}, \quad c_2 = \tfrac{15232626}{25}, \quad c_3 = \tfrac{24807984744}{175},$$
$$c_4 = \tfrac{50638290432}{5}, \quad c_5 = \tfrac{621170191872}{5}, \quad c_6 = \tfrac{1877429977088}{7}, \quad c_7 = \tfrac{970455252992}{7}.$$

The *tail coefficients* are

$$C_j = \sum_{d \ge j} c_d P_4^{-(d-j)/2} \quad (j = 1, 2, 3), \qquad S_{72}(p) = \frac{2^{92}}{p} + \frac{C_1}{p^{1/2}} + \frac{C_2}{p} + \frac{C_3}{p^{3/2}}; \tag{29.1}$$

$C_1 < 1197.601$, $C_2 < 609305.05$ and $C_3 < 1.4176 \cdot 10^8$ (item (S7), the C_j bounds, of Certificate D.2, whose checker recomputes them exactly from the c_d).

Lemma 29.2 (The tail reference)**.** *Let $p \ge P_4$ be a prime of either class modulo 4, and $r = p^{-1/2}$.*

(i) The inputs of the tail datum (Definition 27.1(e)) are valid on $\{p\}$ in the sense of Section 27.8.

(ii) μ_p is a probability measure on $\mathbb{F}_p^V$, reflection positive at every cut of the family (the chamber cuts of both coordinates and ι), invariant under the square-affine maps $z_v \mapsto s z_v + u$ ($s \in Q_p^*$, $u \in \mathbb{F}_p$), with uniform one-slot marginals, and admissible (Definition 8.9); so the tail atoms satisfy (R1)–(R3).

(iii) $\widetilde{\mu}_p = \pi_{\mathrm{LL},2} + \widetilde{\rho}_p$ with $\widetilde{\rho}_p \ge 0$ of mass $h_p = P_\mu(r) - 1 \le C_1 p^{-1/2}$. The root accepts the twelve queries (28.1), hence $\mathcal{E}_6$, and each query pair is a cross-layer edge of the root graph.

Proof. (i) The inputs of the tail datum are integer ceilings of constants evaluated at $p_0 = 2^{64}$ with the block length $L_{\mathrm{u}} = 50000/27$, that is, for blocks of at most $L_{\mathrm{u}}/\log 2^{64} < 42$ primes, and they cover both prime classes. The nonroot caps are the largest of four exact tests, two classes and two Fourier signs (Section 27.4, Certificate 27.13(d)). The root caps 113 (LL20) and 45 (LL22) come from the tests of both classes, the class-one ones with the class-one Paley terms and the parity transfer (Lemma 27.10 with $\mathfrak{c} \in \{1, 3\}$, Certificate 27.11, Lemmas 14.31 and 14.32); at 2^{64} the class-one tests give the recorded value in every cut except FE22 and EE22, where the class-one and class-three test values are equal. The row inputs are the larger of the values of the two classes (Section 27.6.2). By Lemma 28.11(iii) for (F1)–(F6), with (F5) in its Schur form, and by Lemma 27.10 for the two root cuts (its threshold hypotheses at p_0 are asserted by Certificate 27.11), every per-prime constant at a prime $p \ge 2^{64}$ of either class is at most its value at 2^{64}; the rows are those of (F5)(b), with $t_Y = 1/50$ and $M = 21/20$. The set $\{p\}$ is a block of one prime, so no tensor-product factor occurs and its block bounds are the per-prime bounds at p: the normalized cut operators at p are at least $-g\,r$ with the recorded per-prime cap g, which is at most the inflated cap $g\,e^{B_{\mathrm{top}} L_{\mathrm{u}}/(X \log X)}$ at $X = 2^{64}$ (notation of (F6)) recorded as the input of the tail datum, and the row data at p satisfy (V2) with coefficients at most the row inputs. These are the conditions (V1)–(V3).

(ii) By (i), Lemma 27.17(i), which uses nothing about the set of primes beyond the validity of the inputs, applies to $\mathcal{B} = \{p\}$ with the weights of the tail datum. These satisfy its 105 inequalities exactly (Certificate 27.18, last paragraph), which do not involve p. This gives (ii); the law of the atom $\{p\}$ does not depend on N.

(iii) $\pi_{\mathfrak{t},j}$ has mass $m_{\mathfrak{t}}$ and $m_{\mathrm{LL}} = 1$, so $\widetilde{\mu}_p$ has mass $P_\mu(r)$; and $p^{-d/2} \le P_4^{-(d-1)/2} p^{-1/2}$ for $d \ge 1$, which gives $h_p = \sum_{d \ge 1} c_d p^{-d/2} \le C_1 p^{-1/2}$. By Lemma 27.4(a) every query $(u_0, u_1, 0) \to (\mathsf{c}u_0, \mathsf{c}u_1, 1)$ of (28.1) is an edge of the root graph joining its two layers, so the root, the uniform law on the injective strict-square labellings of that graph, accepts the twelve queries; the six queries of $\mathcal{E}_6$ are among them. □

29.2. **Mixing at the tail primes.** Hypothesis (R4) is needed only at $p \geq P_5$: the atoms below P_5, among them the tail primes in $[P_4, P_5)$, are finitely many, their full orders divide the integer Q_{pre} of Definition 6.1, and in Lemma 6.15 they form the spectator prefix, which only has to be a contraction. The decay comes from the root, whose conditional operators have norm $O(p^{-1/2})$; the other records are an admixture of mass $O(p^{-1/2})$.

Lemma 29.3 (Tail mixing)**.** *Let $p \geq P_4$ be prime, let μ_p be the tail law, and let $\bar{\nu}_p = \nu_p/\nu_p(1)$ be its normalized accepted part, where ν_p is the factor at p of the comparison law (Lemma 29.5, Definition 9.14).*

(a) For every slot v, the conditional operators of μ_p and of $\bar{\nu}_p$ from slot v to the other 71 *slots (Definition 6.13) have squared norm at most $S_{72}(p)$ on mean-zero functions.*

(b) For each query $(u, v) \in \mathcal{E}_6$, the (u, v)-marginal of $\bar{\nu}_p$ is $\mathrm{Unif}(\Omega_p)$*, and its conditional operator from the pair (u, v) to the other* 70 *slots fixes constants, maps mean-zero functions to mean-zero functions and has squared norm at most $S_{72}(p)$ on them.*

(c) With $\gamma = 255/1024$, $p^{2\gamma} S_{72}(p)$ is decreasing in p. Hence $S_{72}(p) < p^{-2\gamma}$ for every $p \geq P_5$, and (R4) holds.

Proof. The tail laws as mixtures. By Lemma 29.2(iii), $\mu_p = \varpi \pi_{\mathrm{LL},2} + (1 - \varpi)\widetilde{\rho}_p/h_p$ with $\varpi = 1/(1 + h_p)$, and $0 < h_p \leq C_1 p^{-1/2} \leq S_{72}(p) - 2^{92}/p$. The comparison law keeps the root, which accepts $\mathcal{E}_6$, keeps the other accepting records, replaces each repaired record by a law of the same mass and deletes the others (Lemma 29.5); so $P_\mu(r)\nu_p = \pi_{\mathrm{LL},2} + \rho_p$ with $\rho_p \geq 0$ of mass $0 < h_p' \leq h_p$ (the records of degree one are kept), and $\bar{\nu}_p = \varpi' \pi_{\mathrm{LL},2} + (1 - \varpi')\rho_p/h_p'$ with $\varpi' = 1/(1 + h_p')$.

(a), (b). Apply Lemma 6.14 to these two-component mixtures, with Z the labels of the slot v or of the pair (u, v). The components have the same marginal on Z: one-slot marginals are uniform (Lemma 29.2(ii), Lemma 29.5(i)); the queried pair of $\bar{\nu}_p$ is uniform on Ω_p (Lemma 29.5(i)), and so is that of the root, which is square-affine invariant and accepts the query, the square-affine maps acting transitively on Ω_p; hence so is that of ρ_p/h_p'. The root graph has 72 vertices and maximum degree 20 (ten out- and ten in-neighbours at every vertex, Lemma 27.4(a)), and the query pair is one of its edges, oriented from u to v (Lemma 29.2(iii)); by Lemma 12.8(d) (Certificate 12.9), for $p \geq 2^{512}$ the root has one-slot conditional norm squared at most $2^{22}/p$ and pair conditional norm squared at most $2^{92}/p$ on the mean-zero functions. The other component is a contraction there. By Lemma 6.14 the conditional operators fix the constants, map mean-zero functions to mean-zero functions, and have squared norm at most $\varpi 2^{92}/p + (1 - \varpi) \leq 2^{92}/p + h_p \leq S_{72}(p)$, and the same with ϖ' and $h_p' \leq h_p$ for $\bar{\nu}_p$.

(c) $p^{2\gamma} S_{72}(p) = 2^{92} p^{2\gamma - 1} + C_1 p^{2\gamma - 1/2} + C_2 p^{2\gamma - 1} + C_3 p^{2\gamma - 3/2}$ has nonnegative coefficients and negative exponents, since $2\gamma = 255/512 < 1/2$. At $p = P_5$, $p^{2\gamma} = 2^{32640}$, so (S7) gives $p^{2\gamma} S_{72}(p) < 1$ for every $p \geq P_5$. With (a) and (b), μ_p and $\bar{\nu}_p$ are (γ, v)-mixing for every slot v, and $\bar{\nu}_p$ has the pair property of (R4). □

The other inequality of (S7), $2^{16} S_{72}(2^{512}) < 6.7782 \cdot 10^{-70}$, would give by the same monotonicity, since $P_4^{1/32} = 2^{16}$, mixing with exponent $1/64$ at every tail prime; it is not used.

29.3. **Comparison prices $O(p^{-3/2+2a})$ at large primes.** At a tail prime the comparison law is built as at an upper block (Definition 28.13), but for the twelve queries: a record is kept if it accepts (28.1). Its repair caps are the *all-prime* caps, since p may be of class one. For each of the 94 two-layer records of degree two and three that do not accept the twelve queries (Section 28.1) the cap C_ϱ is the normalized coefficient of its one-way repair (F7) (Lemma 17.14), with the joint-star estimate (28.2), valid at every odd prime (Lemma 28.2(b), (F8)), on the ten EL/LE records of degree two. The caps are evaluated at 2^{54} (Certificate 29.4), where the spectral constants g, B_{top} of the nonroot cuts that (F7) reads come from the tests of both classes (Certificate 27.13(d)); by

Lemma 28.11(iii) the caps bound the repair coefficients at every prime $p \ge 2^{54}$ of either class. No class-three datum of the bands is used. With $\Lambda_k^{\mathrm{tail}}(\mathfrak{t})$ the sum of the caps of the repaired records of state $\mathfrak{t}$ and degree k, put

$$\Pi^{\mathrm{rep}}(r) = r^3 \sum_{\mathfrak{t}} w_{\mathfrak{t}} \Lambda_2^{\mathrm{tail}}(\mathfrak{t}) + r^4 \sum_{\mathfrak{t}} w_{\mathfrak{t}} \Lambda_3^{\mathrm{tail}}(\mathfrak{t}) = c_3^{\mathrm{rep}} r^3 + c_4^{\mathrm{rep}} r^4, \qquad \Pi^{\mathrm{del}}(r) = \sum_{d=3}^{7} c_d^{\mathrm{del}} r^d, \tag{29.2}$$

with c_d^{del} the total weight of the deleted records of overall degree d, as in Definition 28.13. Every record of degree at least four is deleted, so $c_d^{\mathrm{del}} = c_d$ for $4 \le d \le 7$, and

$$c_3^{\mathrm{rep}} = \tfrac{73056854152246473}{12500000000}, \qquad c_4^{\mathrm{rep}} = \tfrac{149151332035400049}{68359375}, \qquad c_3^{\mathrm{del}} = \tfrac{81606792}{25}.$$

Certificate 29.4 (The tail repair caps)**.** Each of the 94 one-way repairs at a tail prime satisfies its repair inequalities (G1), (G2) and (G5) with the coefficients evaluated at 2^{54} (hence at every larger prime, by monotonicity), with the constants g, B_{top} of the nonroot layer cuts at 2^{54} (Certificate 27.13(d)). The coefficients are stored in `research/frontier_2026_10_17/sparse/high72_oneway_repairs.json` (degree two) and `high72_oneway_repairs_degree3.json` (degree three) in the same directory; the six LR and four RL records of degree two are replaced by their joint-star rows (F8) at 2^{54} in `high72_oneway_jointstar.json`, whose hypotheses hold there. The adapters `high72_oneway_adapter.py` and `high72_oneway_jointstar_adapter.py` of that directory authenticate the records against `high72_oneway_checks.json` and `high72_oneway_star_checks.json`; no class-three datum of the bands is used. The record of the tail datum, `research/roadmap_2026_10_18/item12/upper_v2_local.json`, carries the resulting ledgers $\Lambda_2^{\mathrm{tail}}, \Lambda_3^{\mathrm{tail}}$ and the coefficients $c_3^{\mathrm{rep}}, c_4^{\mathrm{rep}}, c_d^{\mathrm{del}}$ above as its field `tail_comparison`; the independent review `research/push_2026_10_24/tools/check_upper.py` compares them with the tail data of the source it reviews and re-sums the tail prices.

Lemma 29.5 (The tail comparison law and its prices)**.** *Let $p \ge P_4$ be prime and $r = p^{-1/2}$. Let ν_p be obtained from $\widetilde{\mu}_p$ by keeping every record that accepts the twelve queries, replacing each of the* 94 *nonaccepting two-layer records of degree two and three by its one-way repair with the all-prime cap, deleting every other nonaccepting record, and dividing by $P_\mu(r)$.*

- *(i) ν_p is a nonnegative square-affine invariant measure with the normalizer of μ_p; every retained record accepts $\mathcal{E}_6$; the normalized accepted part has uniform one-slot marginals and query-pair marginals* $\mathrm{Unif}(\Omega_p)$*; under every accepted part each query pair is a strict-square pair at p, which is (R5) at the tail; and ν_p is admissible (Definition 8.9).*
- *(ii) $\nu_p - \mu_p$ satisfies the bracket of Definition 9.14 at ι with prices $s_p \le (\Pi^{\mathrm{rep}} + \Pi^{\mathrm{del}})(r)\, p^{2a}$ and $a_p \le \Pi^{\mathrm{rep}}(r)\, p^{2a}$.*
- *(iii) $\eta_p := s_p + a_p \le C_{\mathrm{tail}}\, p^{-3/2+2a}$, with $C_{\mathrm{tail}} = \sum_d (2c_d^{\mathrm{rep}} + c_d^{\mathrm{del}}) P_4^{-(d-3)/2} = 1.4953\ldots \cdot 10^7$ ($c_d^{\mathrm{rep}} = 0$ for $d \ge 5$).*
- *(iv) With $w_d = d/2 - 2a$ and $\Phi_d(a) = P_4^{-w_d} + P_4^{1-w_d}/(w_d - 1)$, for every N*

$$S_{\mathrm{tail}} := \sum_{P_4 \le p \le N^{1/20}} s_p \le \sum_{d=3}^{7} (c_d^{\mathrm{rep}} + c_d^{\mathrm{del}}) \Phi_d(a), \qquad A_{\mathrm{tail}} := \sum_{P_4 \le p \le N^{1/20}} a_p \le \sum_{d=3}^{4} c_d^{\mathrm{rep}} \Phi_d(a),$$

and $\sum_{p \ge P_4} \eta_p p^{1/4} < \infty$. The tail atoms carry no first-stage and no bracket records, and their matching-envelope diagonals have price $T_{\mathrm{L},p} = T_{\mathrm{R},p} \le \eta_p$; so their part of the sum in (CP8)(c) is at most $3 \sum_{p \ge P_4} \eta_p p^{1/4} < \infty$.

Proof. (i), (ii). By Lemma 17.14(a)–(c) a repaired record becomes a probability law with exactly the actual half marginals at ι of the record it replaces, supported in the support of the uniform law of the repaired graph (half graphs kept, cross constraints removed, projected queries imposed), so it accepts the twelve queries; its data are square-affine invariant, so it is invariant. Its difference

from the replaced record has zero marginals and norm at most $C_\varrho r$ relative to these marginals (Lemma 17.14(f) with (F7), the cap bounding the coefficient at p by Lemma 28.11(iii)). From here the proof of Proposition 28.14(i), its admissibility paragraph included, applies to the atom $\{p\}$ with $C^{\mathrm{s}} = C^{\mathrm{a}} = C_\varrho$. Each repair bound is included in full in both the symmetric total s_p and the skew total a_p (an overestimate in each), which is allowed: Remark 9.15(iii) would allow the skew part of the new law alone, and the recorded tail polynomials do not use this. A repaired record of state $\mathfrak{t}$ is charged at most $w_{\mathfrak{t}} C_\varrho r^{\deg \mathfrak{t}+1}$ in each part, and summing gives $\Pi^{\mathrm{rep}}(r)$; a deleted record is positive and self-adjoint at ι, charged its mass in s_p only, and summing gives $\Pi^{\mathrm{del}}(r)$; the repaired records are exactly those missing from Π^{del}; repairs keep mass and deletions decrease it, and dividing by $P_\mu(r) \geq 1$ only decreases the prices, which carry the factor p^{2a} of an atom $\{p\}$ (Definition 9.2). Every record of ν_p is invariant, so the accepted part has uniform one-slot marginals and invariant query-pair marginals supported on Ω_p, which are therefore $\mathrm{Unif}(\Omega_p)$. Every retained or repaired record imposes the strict-square relation on each query pair.

(iii) Both polynomials start in degree three and $p^{-(d-3)/2} \leq P_4^{-(d-3)/2}$, so $\eta_p \leq (2\Pi^{\mathrm{rep}} + \Pi^{\mathrm{del}})(r) p^{2a} \leq C_{\mathrm{tail}} p^{-3/2+2a}$.

(iv) For $w > 1$ and an integer $P \geq 2$, $\sum_{n \geq P} n^{-w} \leq P^{-w} + \int_P^\infty x^{-w} dx = P^{-w} + P^{1-w}/(w-1)$. Every $w_d \geq 3/2 - 2a > 1$, the tail primes are integers $n \geq P_4$, and by (ii) the prices are sums of $(c_d^{\mathrm{rep}} + c_d^{\mathrm{del}}) p^{-w_d}$, respectively $c_d^{\mathrm{rep}} p^{-w_d}$. Next $\eta_p p^{1/4} \leq C_{\mathrm{tail}} p^{-5/4+2a}$ and $-5/4 + 2a < -1$. Finally the records of the first stage and of the bracket vanish outside the middle blocks (Definition 22.4), so $p_p = v_p = d_p = u_p = 0$ at the tail, and outside the middle the matching-envelope diagonals are the norm envelopes of the changes (Proposition 26.5(i)), whose price is at most $s_p + a_p$. □

The upper prices $S_{\mathrm{u}}(a), A_{\mathrm{u}}(a)$ of the closing contain the two sums of (iv) (Proposition 28.14(ii)). The final assembly adds them to the termwise repriced band ledger in `research/continuation_2026_10_24/global/closing.py` (`upper_price`, lines 141–149). At the final exponent the tail contributes about $2.97 \cdot 10^{-70}$ to S and $1.90 \cdot 10^{-70}$ to A. The allowance 10^{-12} of the closing covers the prefix exception and also a second, repriced copy of the tail prices (Certificate D.3(f)); the tail is thus charged twice, which only increases Θ.

29.4. **The external index by exact conductor.** Each query has an interval coordinate modulo K and independent external prime-power coordinates. At $m = 64$ the modulus used in the square-class kernel (called *physical* below, to distinguish it from the comparison-depth modulus) is

$$M_0 = 2^{129} \prod_{\substack{p < Q_3 \\ p \equiv 3 \ (4)}} p^{128}.$$

The odd product includes 3. At each prime choose, for each query, a uniform residue t modulo the corresponding prime power, conditioned on $v_p(t) < m$, and use t^2 as the increment $r_{e,p}$ at p (Definition 9.6). These increments are nonzero squares; their possible nonunit valuations lie in the classes treated in Lemma C.1, applied in the proof of Lemma 29.6 below. The random depth J of Lemma 29.6 below (the comparison depth) does not change this physical modulus. It only determines the conditional expectation used in the positive index envelope. Thus the cancellation in the actual kernel and the comparison moment use different, explicitly specified moduli.

Lemma 29.6 (Finite conductor-depth index envelope)**.** *Fix $m \geq 3$ and $Q > 2$. At every odd external prime $p \equiv 3 \pmod 4$ with $p < Q$, let t be uniform modulo p^{2m} conditioned on $p^m \nmid t$, and use t^2 as the increment (a uniform conditioned root, squared). At 2 use the same construction modulo 2^{2m+1}. These finite probability laws have uniform endpoint marginals and preserve accepted-square cancellation, in that every increment lies in a class of Lemma C.1. They admit the following*

decreasing conductor tails, where $e_p = p^{-m}$*:*

$$\begin{aligned} &p \text{ odd}: \quad \delta_0 = 1, \quad \delta_k = \frac{p^{-k/2} + e_p}{1 - e_p} \ (1 \le k < 2m), \quad \delta_{2m} = 0; \\ &p = 2: \quad \delta_0 = 1, \quad \delta_1 = \delta_2 = \frac{2^{-1/2} + e_2}{1 - e_2}, \\ &\qquad \delta_k = \frac{2^{-(k-1)/2} + e_2}{1 - e_2} \ (3 \le k < 2m+1), \quad \delta_{2m+1} = \frac{e_2}{1 - e_2}, \quad \delta_{2m+2} = 0. \end{aligned}$$

Independent depths J *with* $\mathbb{P}(J \ge k) = \delta_k$ *therefore supply the finite-filtration index envelope of Lemma 10.1, and its conclusion (a) retains the common mean.*

Proof. For odd p, the primitive quadratic Gauss multiplier at conductor p^k satisfies $G_k = p^{-1}G_{k-2}$, $G_0 = 1$ and $|G_1| = p^{-1/2}$. Summing first over the last base-p digit proves the recurrence. Excluded roots have square 0 modulo p^{2m}, so the retained multiplier is $(G_k - e_p)/(1 - e_p)$. At the even top conductor $G_{2m} = e_p$ and the multiplier is exactly zero. At 2, $G_1 = 0$ and $|G_k| = 2^{-(k-1)/2}$ for $k \ge 2$, by Theorem 4.8 applied to the quadratic Gauss sum modulo 2^k. The excluded roots have square 0 below the top conductor, while their top phases cancel in equal numbers. This proves all the displayed bounds. For $m \ge 3$ the δ_k are nonincreasing in k and $\delta_1 < 1$, so they define probability depth laws.

The comparison envelope. Take E_j to be conditional expectation on the residue modulo p^j. Its exact-depth space $E_j - E_{j-1}$ consists of the characters of conductor p^j. The Gauss-sum bounds just proved therefore give the operator-norm bounds on these spaces required by Lemma 10.1. Independence of the prime coordinates gives the product envelope. The field Gram commutes with the residue projections, and the constant multiplier is one, so that lemma gives both signed-index bounds and retains the common mean; self-adjointness of the transition is not required.

Physical cancellation. Every retained root has valuation $j < m$. Its square modulo p^{2m} has valuation $2j < 2m$ for odd p; at 2 its square modulo 2^{2m+1} has $2j + 3 \le 2m + 1$. These are exactly the nonvanishing square classes of Lemma C.1. That lemma supplies the normalized periodic root weights, their uniform higher-digit distribution and their Fourier bounds. Thus the actual-square kernel applies to every increment, with the full physical modulus fixed; no increment is 0 modulo p^{2m} (resp. 2^{2m+1} at 2).

The full physical modulus remains pinned in this cancellation, independently of the random comparison depth. In the comparison, conditioning the normalized input gives exactly

$$\mathbb{E}[p^H \mathbf{1}_{n=s \ (p^H)} \mid s \ (p^j)] = p^j \mathbf{1}_{n=s \ (p^j)}.$$

The exact-fibre and square-step arguments of Lemma 8.2(a), (b) hence apply without change. All moduli are fixed before N varies, so only constants independent of N and the eventual starting scale increase. Uniform averaging of the input over residues and intervals is still the original lift. This retains the density-linear mass and all common means. □

Proposition 29.7 (Finite p-adic matching and wrap prices)**.** *With a the price exponent and* m, Q *as in Lemma 29.6, and* B, S, A, T *as in Theorem 9.21, put* $z_p = p^{a/18}$*. The exact one-query conductor moment is*

$$M_{\mathrm{pad}}(K) = K^{a/36} L_{2,m} \prod_{\substack{p<Q \\ p \equiv 3 \ (4)}} L_{p,m},$$

where, with $e = p^{-m}$,

$$L_{p,m} = 1 + \frac{(z_p - 1)p^{-1/2}[1 - (z_p/p)^m]/(1 - z_p/p) + e(z_p^m - 1)}{1 - e} \quad (p \text{ odd}),$$
$$L_{2,m} = 1 + \frac{(z_2 - 1)[2^{-1/2} + z_2(1 - (z_2/2)^{m-1})/(2 - z_2)] + 2^{-m}(z_2^{m+1} - 1)}{1 - 2^{-m}}.$$

For every fixed integer $K \geq 2$*, the matching and wrap prices are*

$$J = Be^T(M_{\mathrm{pad}}(K)/2)^6, \qquad W = Be^{S+A}\left[\left(1 + \frac{M_{\mathrm{pad}}(K)}{2K^{35/36}}\right)^6 - 1\right]. \tag{29.3}$$

The coefficient $63/64 = 1 - 2^{-6}$ *in Theorem 9.21 and all field restoration terms in the closing criterion there remain unchanged.*

Proof. For each depth J, the least square multiple of p^J (the smallest square divisible by p^J) is $p^{2\lceil J/2\rceil}$. Its two occurrences among the 72 slots give exponent $a/36$. Summation by parts therefore gives

$$L_{p,m} = 1 + (z_p - 1)\sum_{l \geq 1} z_p^{l-1}\delta_{2l-1}.$$

Substituting the tails in Lemma 29.6 gives the displayed formulas. In particular the top 2-adic depth has $\lceil (2m+1)/2 \rceil = m + 1$ and is included. The entire 2-part and prime 3 are represented by their depth factors; no additional fixed conductor factor is inserted.

Fix the depths, apply the folds $\phi_\pm$ of Definition 5.7, then average over the depths. There are exactly two occurrences per query, and six independent queries. Lemma 26.1 gives J. For a nonempty subset of j active queries in the signed expansion, any nonwrapping interval permits accepted-square cancellation. All j wrap with probability K^{-j}. The density normalization multiplies their bound by $K^{j/36}$ and their depth average is $M_{\mathrm{pad}}(K)^j$. The unchanged positive field envelope is Be^{S+A}. Summing the coefficients $\binom{6}{j}2^{6-j}/64$ gives W. The common mean is unchanged by uniform index averaging, as proved above. □

Corollary 29.8 (A compact upper majorant)**.** *Take* $m = 64$*,* $0 < a \leq 3/2500$*,* $2^{16} \leq Q \leq 2^{40}$*, and put* $Z = Q^{a/18}$*. Define*

$$L_{p,\infty} = 1 + \frac{\sqrt{p}(z_p - 1)}{p - z_p}, \qquad L_{2,\infty} = 1 + (z_2 - 1)\left(2^{-1/2} + \frac{z_2}{2 - z_2}\right),$$

$$E_o = \frac{Q3^{-m}Z^m}{1 - 3^{-m}}, \qquad E_2 = \frac{2^{-m}z_2^{m+1}}{1 - 2^{-m}}.$$

Let M_{old}^+ *be any proved upper bound for* $M_a(K)$ *of* (29.4) *below, and set* $F_p = 1 + \sqrt{p+1}(z_p - 1)/(p-1)$*. For any exact prime cutoff* $7 \leq U \leq Q$*, a valid upper bound for* $\log M_{\mathrm{pad}}(K)$ *is*

$$\log M_{\mathrm{old}}^+ - \frac{a}{36}\log 144 + \log L_{2,\infty} + \log L_{3,\infty} - \sum_{\substack{7 \leq p < U \\ p \equiv 3 \ (4)}} \log \frac{F_p}{L_{p,\infty}} + E_o + E_2.$$

The error term satisfies $E_o + E_2 < 2^{-60}$ *on this domain.*

Proof. Discarding the negative geometric tails in the exact finite formulas gives

$$L_{p,m} \leq L_{p,\infty}\left(1 + \frac{p^{-m}z_p^m}{1 - p^{-m}}\right), \qquad L_{2,m} \leq L_{2,\infty}\left(1 + \frac{2^{-m}z_2^{m+1}}{1 - 2^{-m}}\right).$$

There are fewer than Q odd primes, so $\log(1+x) \le x$ gives the two surcharges. Also $Z \le 2^{1/375} < 501/500 < 5/4$. For $p \ge 3$, $1 \le z \le 5/4$,

$$(p+1)(p-z)^2 - p(p-1)^2 \ge \frac{8p^2 - 31p + 25}{16} \ge \frac{1}{4}.$$

Thus $L_{p,\infty} \le F_p$. Replace only the explicitly enumerated factors by their smaller values and keep the remaining old upper bound. Replacing its factor 144 by the two new factors proves the claimed logarithmic majorant. Finally the rational inequality $(501/500)^{375} > 2$ and direct substitution at $Q = 2^{40}$ give

$$\frac{2^{40}3^{-64}(501/500)^{64}}{1-3^{-64}} + \frac{2^{-64}(501/500)^{65}}{1-2^{-64}} < 2^{-60}.$$

$L_{p,\infty}$ and $L_{2,\infty}$ are used here only as majorants, with the displayed surcharges, for the finite-depth factors $L_{p,m}$ and $L_{2,m}$. An interval enclosing this upper majorant supplies no lower bound for the physical moment. □

29.5. **Evaluating R_I, J and W.** The remaining quantities in Θ coming from the external index are its conductor moment R_I, and through it J and W. The moment of the index law used in the construction is $M_{\mathrm{pad}}(K)$ of Proposition 29.7; its evaluation uses a simpler auxiliary moment, a product of elementary factors over the 634017 primes $p \equiv 3 \pmod 4$ below Q_3:

$$M_a(K) = (144K)^{a/36} \prod_{\substack{7 \le p < Q_3 \\ p \equiv 3 \ (4)}} \left[1 + \frac{\sqrt{p+1}}{p-1}(p^{a/18} - 1)\right]. \tag{29.4}$$

The following lemma encloses this product using only 22 explicit prime sums, so that it can be certified without multiplying out 634017 factors in interval arithmetic; this supplies the upper bound used in Corollary 29.8.

Lemma 29.9 (Finite prime moments)**.** *Let $Q > 7$ be an integer and $a \ge 0$; sums and products over p run over the primes $p \equiv 3 \pmod 4$ with $7 \le p < Q$, and $\delta_p = \sqrt{p+1}/(p-1)$. For $1 \le j \le 4$ and $j \le k \le 7$ put*

$$M_{jk}(Q) = \sum_p \delta_p^j \Big(\frac{\log p}{18}\Big)^k, \qquad P_j(a) = \sum_p \big[\delta_p(p^{a/18} - 1)\big]^j,$$

and

$$T_j(a) = \sum_{k=j}^{6} \frac{j!}{k!} \left\{ {k \atop j} \right\} a^k M_{jk}(Q), \qquad \varrho_j(a) = e^{ja \log Q/18} \, \frac{j^7 a^7}{7!} \, M_{j7}(Q),$$

with the Stirling numbers $\left\{ {k \atop j} \right\}$ of the second kind. Then $T_j(a) \le P_j(a) \le T_j(a) + \varrho_j(a)$, and $G_Q(a) = \prod_p (1 + \delta_p(p^{a/18} - 1))$ satisfies, with every term evaluated at a,

$$T_1 - \tfrac{1}{2}(T_2 + \varrho_2) + \tfrac{1}{3}T_3 - \tfrac{1}{4}(T_4 + \varrho_4) \ \le \ \log G_Q(a) \ \le \ T_1 + \varrho_1 - \tfrac{1}{2}T_2 + \tfrac{1}{3}(T_3 + \varrho_3).$$

In particular, for every integer $K \ge 2$, the moment $M_a(K) = (144K)^{a/36} G_{Q_3}(a)$ of (29.4) is enclosed by the explicit bounds above, as a function of a, K, Q_3 and the 22 positive sums $M_{jk}(Q_3)$.

Proof. Taylor bounds. For $y \ge 0$ the exponential generating function of surjections gives $(e^y - 1)^j = j! \sum_{k \ge j} \left\{ {k \atop j} \right\} y^k/k!$, a series with nonnegative coefficients; its terms of degree at most six are a lower bound. Since $j!\left\{ {k \atop j} \right\}$ counts the surjections of a k-set onto a j-set, it is at most j^k, the number of all maps; so the terms of degree at least seven are at most $\sum_{k \ge 7} (jy)^k/k! \le (jy)^7 e^{jy}/7!$. Take $y = a(\log p)/18$, which is at most $a \log Q/18$ for $p < Q$, multiply by δ_p^j and sum over p: as $p^{a/18} - 1 = e^y - 1$, this gives $T_j(a) \le P_j(a) \le T_j(a) + \varrho_j(a)$.

Logarithm. For $x \ge 0$ and $0 \le s \le x$, $1 - s + s^2 - s^3 \le 1/(1+s) \le 1 - s + s^2$, since $1/(1+s) = 1 - s + s^2 - s^3/(1+s)$; integrating over $[0, x]$ gives $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \le \log(1+x) \le x - \frac{x^2}{2} + \frac{x^3}{3}$. Apply this at $x = \delta_p(p^{a/18} - 1) \ge 0$ and sum over p: $P_1 - \frac{1}{2}P_2 + \frac{1}{3}P_3 - \frac{1}{4}P_4 \le \log G_Q(a) \le P_1 - \frac{1}{2}P_2 + \frac{1}{3}P_3$. Inserting the bounds of the first step, each with the sign of its coefficient, gives the two displayed bounds.

The moment. By (29.4), $M_a(K) = (144K)^{a/36} G_{Q_3}(a)$, and the bounds above involve, besides a, K and Q_3, only the sums $M_{jk}(Q_3)$, $1 \le j \le 4$, $j \le k \le 7$. These are finite sums of positive terms, so their enclosures need only outward square roots and logarithms of the primes. This auxiliary product is the Bernoulli moment from Lemmas 4.11 and 9.7; it prices the remaining large-prime factors in Corollary 29.8, rather than defining the selected physical index law. □

Remark 29.10 (The choice of K). For a product majorant $M_{\mathrm{pad}}(K) \le \bar{c} K^{a/36}$ with M_{pad} as in Proposition 29.7, and $0 < a < 35$, the bound there for $J + W$ is a positive sum of $e^{ax/6}$ and $e^{-j(35-a)x/36}$, $1 \le j \le 6$, where $x = \log K$. It is convex. The baseline integer is $K = 890666$; its validity requires only evaluation of the two prices, not a proof of optimality.

Certificate 29.11 (The index moment)**.** The auxiliary sums $M_{jk}(Q_3)$ run over the 634017 primes $p \equiv 3 \pmod 4$ with $7 \le p < 19951586$. The largest is 19951579 and their sum is 6108518317099. Their outward 48-digit enclosures and the prime-list digest are retained in `research/continuation_2026_10_24/global/prime_moments.json`. The checker of Appendix D regenerates these finite prime data.

For the baseline finite law, use $m = 64$, $Q_3 = 19951586$, $a = \frac{5422587}{6103515625}$, $K = 890666$ and small-prime replacement cutoff $U = 65536$ in Corollary 29.8. The outward evaluation and its independent reconstruction are `research/improvements_2026_10_25/global/padic_moment.py` and `research/improvements_2026_10_25/global/check_padic_moment.py`. The selected record is `research/push_2026_10_25/artifacts/padic_moment.json`. They certify

$$R_I = M_{\mathrm{pad}}(K) \le 1.24713990740688470555679476 9493.$$

This rational is the upper bound used for R_I in Definition 30.1(12). The narrow recorded interval encloses an upper majorant; its lower endpoint is not a lower bound for $M_{\mathrm{pad}}(K)$. Both scalar assemblies, author and independent, use the same upper rational, and `research/push_2026_10_25/verify.py` reruns its independent check (Certificate 30.3).

Consequences and what to read next. The tail datum (Definition 27.1(e)), Definitions 27.5 and 28.13, Sections 27.2.2 and 27.8, Lemma 27.4, Lemma 27.17 and Certificate 27.18, (F1)–(F8), with (F5)(b), and Lemma 28.11, the class-one inputs (Sections 27.4 and 27.6.2, Certificate 27.13(d), Lemmas 14.31, 14.32 and 27.10), Section 28.1, Lemmas 17.14 and 28.2 with the caps of Certificate 29.4, Remark 9.15, Propositions 26.5 and 28.14, Lemmas 6.14 and 12.8, (S7) of Certificate D.2, Lemma 4.11, Definition 9.6 and Lemma 9.7, Lemmas 29.6 and 29.9, Proposition 29.7, and Corollary 29.8 and Theorem 9.21.
Hypotheses verified in this section. (CP1) for the tail atoms of both classes (Lemma 29.2); (R4) (Lemma 29.3); (R5), the tail part of (CP4) and the admissibility (CP8)(a) of the tail laws (Lemma 29.2(ii), Lemma 29.5(i),(ii)); the summability (CP8)(c) at the tail (Lemma 29.5(iii),(iv)); and (CP0)(d) at $K = 890666$ (Proposition 29.7, Lemma 29.9, and Certificate 29.11). These are the rows "(CP1) tail", "(R4)", "(R5), (CP4)(a)" (at the tail), "(CP4) tail", "(CP8)(c)" and "(CP0)(d) index law" of Table 5.

30. Baseline closing: $\Theta < 1$ with $\theta = 1$

What this section proves and how to read it. This section proves the baseline Proposition 1.2 only. Recall $\Theta = (63/64)(L_0 + \mathcal{E}) + J + W + (1-m)/64$ with $\mathcal{E} = B(e^S \cosh A - 1) - D$ (Theorem 9.21 and Section 1.6). Every quantity is defined at $a = \frac{5422587}{6103515625}$ as a sum of middle prices, upper-band

prices converted from their evaluation exponents, or an explicit function thereof. Outward-interval evaluation gives $\Theta < 0.99999999997327$; Theorem C with $\theta = 1$ then gives $\alpha^{71} \ll \mathsf{L}^{-a}$, i.e. $\varepsilon = a/71$, the 71st-root step being $\alpha \le C\mathsf{L}^{-a/71}$ with an absolute constant C. The margin $1-\Theta$ is about $2.7 \cdot 10^{-11}$, so numbers are stated exactly as evaluated; no bound may be replaced by a larger bound for the same quantity without re-evaluating Θ. The words *robust* and *inflation* below mean: errors come only from the two allowed remainders (Definition 9.19), and prices are increased to the theorem exponent a from smaller evaluation exponents (repricing). The improved theorem redoes this evaluation at $\vartheta > 1$ in Section 33.

30.1. **Definitions of $L_0, S, A, D, \mathcal{E}, T, R_I, J, W, m$ (baseline).** This subsection defines the ten numbers entering Θ. Two conversions are applied first. The middle prices were computed (certified) at a smaller exponent $a_0 = \frac{4442183}{5000000000} < a$, the *anchor*, and are converted to the exponent a of the theorem ("repriced") by Corollary 9.4 and Lemma 9.3, using the block bound $\sum_{p\in\mathcal{B}} \log p \le L = 570$. The upper prices have their own anchor $a_{\mathrm{u}} = 883/10^6$ and are repriced term by term in each band (Proposition 28.14). All rounding is upward: we write $\mathrm{up}(x) = 10^{-30}\lceil 10^{30}x\rceil$, the next multiple of 10^{-30} above x. A certified quantity v is stored in the data file (the "native source" of Appendix D) as an interval enclosure with an exact rational upper end v^{ex} and a decimal upper end $v^{\mathrm{hi}} \ge v^{\mathrm{ex}}$; we put $\mathrm{hi}(v) = \mathrm{up}(v^{\mathrm{hi}})$, and it is these rounded upper ends that enter every formula below. For the middle piece i ($1 \le i \le 33$), with left endpoint X_i of its prime interval (Section 20) and $q_{X_i} = 1/X_i$, and for $d \in \{2,3,4\}$, $I_{d,i}$ is the certified degree-d charge of the piece at a_0, the bound of Lemma 20.4 for $\sum_{\mathcal{B}} q_{\mathcal{B}}^{2a_0} b_{\mathcal{B}}^{-d/2}$ ($b_{\mathcal{B}}$ the least prime of $\mathcal{B}$) over its blocks (Certificate D.4), and $\bar{I}_{d,i} = \mathrm{hi}(I_{d,i})$. A local coefficient evaluated at X_i enters a price through up of the upper end of its outward enclosure; we write this rounded majorant with the same letter as the coefficient. Thus every letter below denotes the rounded upper value, and every inequality is in the safe (upper) direction.

Definition 30.1 (The closing quantities)**.**

(1) *Repricing.* $a_0 = \frac{4442183}{5000000000}$, $L = 570$ and $t = e^{2(a-a_0)L}$; $C = e^{2aL}$; the piece caps $C_i = e^{2aL_i}$ of Lemma 20.7, and $C_X = \prod_{\mathcal{B}\in X} C_{i(\mathcal{B})}$ for a set X of middle blocks, $i(\mathcal{B})$ the piece of $\mathcal{B}$.

(2) *First stage and bracket.* Let U_0', P', D' and V' be the recorded upper ends of the totals at a_0 of Lemma 22.9 over the 33 pieces (Certificate D.4): $U_0' \le 0.8469081935391429860$, $P' \le 1.6936504653996419205$, $D' \le 4.741704122562283 \cdot 10^{-5}$ and $V' \le 3.3872533701481774716$. Put

$$U_0 = tU_0', \qquad P = tP', \qquad d = \tfrac{1}{2}tD', \qquad v = \tfrac{1}{2}tV',$$

where D' and V' are the stored totals at a_0 of the sharp prices (8.2) and total-variation prices summed over all middle atoms and both first-stage record differences (the "tags" of Definition 22.4(e)), and the two tags carry equal halves of these totals by Lemma 22.9, so that $d = \frac{1}{2}tD'$ and $v = \frac{1}{2}tV'$ are the repriced per-tag totals d_j, v_j of Definition 9.11, and

$$R_{\mathrm{err}} = e^{U_0}\mathfrak{h}_4^+(d,v), \qquad L_0 = e^{P}\mathfrak{h}_4^+(d,v), \qquad B = e^{U_0} + R_{\mathrm{err}},$$

with the majorant $\mathfrak{h}_4^+$ of Corollary 10.9. By Proposition 30.2(a) these numbers bound the totals of the same names of (CP2) and (CP3), and the closing uses them in place of those totals (Remark 9.22); Section 24 writes B_* and R_* for them. (The letter D is reserved for the discount of (10).)

(3) *Middle prices.* For the piece i the *rows* at a_0 are the summands of Table 29 for that piece, as stored in the data, before the normalisation by Z_i^{-1} of Section 33. The fold and matching rows agree with (23.7); the two generic rows $S_{\mathrm{oth},i}$ and $A_{\mathrm{oth},i}$ use the larger caps stored in the data (Table 29), at least those of (23.7), which also bound the two-layer constant and merge repairs. Explicitly, for its two fold records the raw-arm (y^{arm}),

leading, remainder, full skew (leading plus remainder) and symmetric prices $A^{\mathrm{arm}}_{F,i}$, $A^{\mathrm{lead}}_{F,i}$, $A^{\mathrm{rem}}_{F,i}$, $A_{F,i} = A^{\mathrm{lead}}_{F,i} + A^{\mathrm{rem}}_{F,i}$ and $S_{F,i}$; the same prices $A^{\mathrm{arm}}_{E,i}$, $A^{\mathrm{lead}}_{E,i}$, $A^{\mathrm{rem}}_{E,i}$, $A_{E,i}$, $S_{E,i}$ of its two matching records; and the symmetric and skew prices $S_{\mathrm{oth},i}$, $A_{\mathrm{oth},i}$ of its other records (merges, deletions and the remaining changed records). They are formed from the completion weights $w_{\mathrm{D},i}$, $w_{\mathrm{S},i}$ (Remark 22.2), the charges $\bar{I}_{d,i}$, the endpoint coefficients $\ell_{\mathtt{t}}(X_i)$, $a_F(X_i)$, $s_F(X_i)$ of the signed fold records (Proposition 23.13), $a_E(X_i)$, $s_E(X_i)$, $r_E(X_i)$ of the matching records (Lemmas 23.17 and 23.18) and the source coefficients for the generic rows $S_{\mathrm{oth},i}, A_{\mathrm{oth},i}$, and rounded per piece on $10^{-30}\mathbb{Z}$; for instance $A^{\mathrm{lead}}_{F,i} = \mathrm{up}(2w_{\mathrm{D},i}\ell_{\mathtt{t}}(X_i)\bar{I}_{2,i})$ and $A_{E,i} = 2\,\mathrm{up}(w_{\mathrm{S},i}a_E(X_i)\bar{I}_{2,i})$. The totals at a are t times the sums over the pieces: $A^{\mathrm{arm}}_F = t\sum_i A^{\mathrm{arm}}_{F,i}$, and likewise A^{lead}_F, A^{rem}_F, A_F, S_F, A^{arm}_E, A^{lead}_E, A^{rem}_E, A_E, S_E, S_{oth} and A_{oth}; and $S_{\mathrm{mid}} = S_F + S_E + S_{\mathrm{oth}}$, $A_{\mathrm{mid}} = A_F + A_E + A_{\mathrm{oth}}$.

(4) *Symmetric and skew totals.*

$$S = S_{\mathrm{mid}} + S_{\mathrm{u}}(a) + 10^{-12}, \qquad A = A_{\mathrm{mid}} + A_{\mathrm{u}}(a) + 10^{-12},$$

where $S_{\mathrm{u}}(a)$, $A_{\mathrm{u}}(a)$ are the symmetric and skew ledgers of the 310 upper bands of Proposition 28.14, certified at a_{u} and repriced term by term, each term with its own logarithmic product bound, plus the tail sums of Lemma 29.5(iv).

(5) *Matching exponent.* $T_K = t(T^{\mathrm{aux}\prime}_K + T^{\mathrm{inc}\prime}_K)$ is the price of the chosen side diagonals of Proposition 26.4(iii): $T^{\mathrm{aux}\prime}_K = \sum_i \mathrm{up}((k^{\mathrm{aux}}_{\mathrm{L},i} + k^{\mathrm{aux}}_{\mathrm{R},i})\bar{I}_{2,i})$ is the price bound for the excess coefficients $k^{\mathrm{aux}}_{s,i} = w_{\mathrm{D},i}d^{\mathrm{aux}}_s(X_i) - v_{\mathrm{D},i}$ of the auxiliary sides (orientation weights, local cap $L_{\mathrm{loc}} = 585$), and $T^{\mathrm{inc}\prime}_K$ the increments $\delta^+_{s,i}$ of the chosen sides over the auxiliary sides (Proposition 26.4(ii)), $s \in \{\mathrm{L},\mathrm{R}\}$. Further $T_{QE} = S_E + A_E$ and

$$T = S_E + A_E + S_{\mathrm{oth}} + A_{\mathrm{oth}} + T_K + S_{\mathrm{u}}(a) + A_{\mathrm{u}}(a) + 10^{-12}.$$

(6) *Invariant price.* With $\pi_i = (f_1 + h_1)_i$ the polarization coefficient (an upper end) of Section 25 for the piece,

$$U^{\mathrm{inv}} = \sum_i \Big[2v_{\mathrm{D},i}I^{\mathrm{ex}}_{2,i} + \frac{12w_{\mathrm{S},i}}{1 - 3/X_i}I^{\mathrm{ex}}_{3,i}\Big],$$

the price at a_0 assigned to the invariant tags $E_{\mathrm{add}} + H$ of the bracket (Lemma 25.6(v)); it is an exact rational, $U^{\mathrm{inv}} = 0.84688441321358980128331130\ldots$, and with the price $\sum_i 2v_{\mathrm{D},i}\pi_i I^{\mathrm{ex}}_{3,i}$ assigned to the generic tag G it satisfies $U^{\mathrm{inv}} + \sum_i 2v_{\mathrm{D},i}\pi_i I^{\mathrm{ex}}_{3,i} \le U'_0$. The *background factor* is $e^{tU^{\mathrm{inv}}}$; the tag G is not invariant and is paid in U_0 only.

(7) *Leading class costs.* For each of the sixteen envelope classes of Corollary 25.10 (the classes (n,m) with $n \ge 1$, $n + m \in \{2,4,6\}$; the aligned and the crossed matching pairs, written (EE,al) and (EE,cr); and $(0,4)$, $(0,6)$), with the caps $\kappa_H \ge \kappa_b \ge 0$ (generic and invariant backgrounds, Corollary 25.10), the coupling mass $M_{n,m}$ and the invariant orbit mass $M^{\mathrm{inv}}_{n,m}$ of its envelope (Table 33),

$$C_{n,m} = \kappa_H e^{U_0} - (\kappa_H - \kappa_b)e^{tU^{\mathrm{inv}}} + M_{n,m}R_{\mathrm{err}} - M^{\mathrm{inv}}_{n,m}e^{tU^{\mathrm{inv}}}t\delta_{\mathrm{s}},$$

where $\delta_{\mathrm{s}} = \frac{1}{2}D' - \sum_i \mathrm{up}((v_{\mathrm{D},i} + v_{\mathrm{S},i})s_{\mathrm{fold}}(X_i)I^{\mathrm{hi}}_{4,i})$ is the price saving at a_0 of one folded tag, its sharp price minus its symmetric price (Lemma 25.7), s_{fold} being the symmetric coefficient of Lemma 22.6.

(8) *Savings.* With the remainder polynomial $\mathcal{R}_{n,m} = (A^{\mathrm{lead}}_F + A^{\mathrm{rem}}_F)^n(A^{\mathrm{lead}}_E + A^{\mathrm{rem}}_E)^m - (A^{\mathrm{lead}}_F)^n(A^{\mathrm{lead}}_E)^m$, the *saving* of a class is

$$D_{n,m} = \frac{BA^n_F A^m_E - C_{n,m}(A^{\mathrm{arm}}_F)^n(A^{\mathrm{arm}}_E)^m - B\mathcal{R}_{n,m}}{n!\,m!}$$

for the twelve classes with $n \geq 1$, $n+m \in \{2,4,6\}$ of (7) and for $(0,4)$, $(0,6)$; for the two matching pairs, $D_{EE,\mathrm{al}}$ and $D_{EE,\mathrm{cr}}$ are the same expression with $(n,m)=(0,2)$, the cost of the pair class and the divisor 4 in place of 2, since the aligned and crossed pairs each have total $A_E^2/4$ (proof of Proposition 30.2(d)). A class is replaced only if the lower end of its interval enclosure (Certificate 30.3) is positive; otherwise its saving is omitted. All sixteen savings are positive.

(9) *Variance credit.* D_{var} is (24.13) of Corollary 24.18 evaluated with the following substitutions: the rows of (3) for its record weights, multiplied by the repricing factor t; the restoration function evaluated at the piece caps of (1) with the error R_{err} of (2); and the gains given by Proposition 24.17 and Lemma 24.16 (including their terms pooled over the pieces from 2^{31}). Its skew pair credits are those of the fold records, with the cap κ_{FF}, and of the aligned matching pairs, with the invariant cap of their envelope and the gains proved only for the selected laws, and its credits of degrees three and four include those of the pure quartic classes $(4,0)$ and $(0,4)$ on their raw-arm prices (Lemma 25.9).

(10) *Discount and reserved cost.*

$$D = D_{\mathrm{var}} + \sum_{n\geq 1} D_{n,m} + D_{EE,\mathrm{al}} + D_{EE,\mathrm{cr}} + D_{0,4} + D_{0,6}, \qquad \mathcal{E} = B\big(e^S \cosh A - 1\big) - D.$$

(11) *Mean credit.* $m = 1 + m_F + m_Q - R_{\mathrm{err}} e^{T_K + T_{QE}}$, with the coupons m_F (empty matching tag set) and m_Q (nonempty matching tag set) of Proposition 26.7 at the piece caps, the late-pool coupons of degrees five and six included (Lemma 24.16).

(12) *Index terms.* $K = 890666$. For the conductor moment use the certified upper bound

$$R_I \leq R_I^+ = 1.247139907406884705556794769493$$

for the finite depth-64 moment $M_{\mathrm{pad}}(K)$ (every later occurrence of R_I in an upper bound means R_I^+), certified using Proposition 29.7 and Corollary 29.8 (Certificate 29.11); and

$$J = Be^T\Big(\frac{R_I}{2}\Big)^6,$$
$$W = Be^{S+A}\Big[\Big(1 + \frac{R_I}{2}K^{-35/36}\Big)^6 - 1\Big].$$

(13) *Closing number.* $\Theta = \frac{63}{64}(L_0 + \mathcal{E}) + J + W + \frac{1-m}{64}$.

The prices of (3) are rounded per piece on $10^{-30}\mathbb{Z}$ and are sums of upper prices of the records of the piece, by Lemma 23.24 and the block charges; by Lemma 10.14(iii) they may be distributed over the blocks of the piece in proportion to the actual prices, and the complete polynomial, which has nonnegative coefficients, is evaluated at them. The endpoint coefficients are values at the piece start of admissible majorants (Lemma 4.18), so they bound the coefficients of every prime and block of the piece. No upper estimate of a norm is used as a lower estimate of that norm; the subtracted exponential $e^{tU^{\mathrm{inv}}}$ and the coupon $M_{n,m}^{\mathrm{inv}} e^{tU^{\mathrm{inv}}} t\delta_{\mathrm{s}}$ are subtracted only inside coefficientwise majorants (Remark 10.15).

The closing has the form of the comparison theorem; the work is in the discount. The even skew part $B(\cosh A - 1)$ of the reserved cost is far larger than its symmetric part, and the envelope classes of Section 25 replace it, degree by degree, by class costs on raw-arm monomials: a class of n fold and m matching records replaces its full-norm term $BA_F^n A_E^m$ by $C_{n,m}(A_F^{\mathrm{arm}})^n (A_E^{\mathrm{arm}})^m$ for its leading parts and by B times the remainder polynomial $\mathcal{R}_{n,m}$ of (8) for the rest. The cost $C_{n,m}$ bounds the energy of the blocks outside the class: $\kappa_H e^{U_0}$ for a generic outside background, lowered on the invariant outside background, whose price tU^{inv} is assigned exactly, plus the couplings of the envelope, each paying the error R_{err} of the bracket, less the orbit coupon of Lemma 25.7.

Proposition 30.2 (The closing quantities bound the data of the comparison theorem)**.** *For the construction of Part V, at every scale* $\mathsf{L} \geq 1$*:*

(a) the totals U_0*,* P *and the tag totals of (CP2), and* R_{err}*,* L_0*,* B *of (CP2) and (CP3), are at most the numbers of Definition 30.1(2) (the tag totals at most* d *and* v*);*
(b) the summed symmetric and skew prices of (CP4) are at most S *and* A*;*
(c) the summed price of the matching-envelope diagonals of (CP6) is at most T*;*
(d) the complete polynomial of Lemma 10.14, formed with the class bounds of Corollary 25.10 and the credits of Definition 30.1(9), at the upper prices of (3)–(4), is at most $\mathcal{E}$*;*
(e) the mean energies satisfy (CP7) with the m *of (11);*
(f) R_I *of (CP0)(d) is at most the value of (12).*

The following gates hold: every retained saving of (8) is positive; every orbit coupon of (7) is at most $M_{n,m}R_{\mathrm{err}}$*; every restoration gain lies in* $[0,1)$*, and every restoration coefficient* ℓ *(the* l *of Lemma 24.8) at cap* C_X *satisfies* $0 \leq \ell \leq C_X/2$*; in every piece* $k^{\mathrm{aux}}_{\mathrm{L},i}, k^{\mathrm{aux}}_{\mathrm{R},i} \geq 0$ *and the protection gate* $v_{\mathrm{D},i} \geq 2w_{\mathrm{D},i}k_{\mathrm{D},i}$ *((22.1)) hold, and so do the matching gates of Section 26;* $D' \leq V'$*,* $d < 6$ *and* $dv < 36$*;* $B > 1$*; every credit of (9) on a raw-arm monomial of a replaced class is below the cost of that class (Lemma 25.9); and* $0 < a \leq 9/10000$*. Hence the closing number of Theorem 9.21 for this construction is at most* Θ*.*

Proof. (a) The first-stage atoms are the middle blocks (Definition 22.4); they have $e_p = 1$ and $\log q_{\mathcal{B}} \leq L$ (Lemma 20.3(ii)). By Lemma 22.9 and Certificate D.4, U_0', P', D' and V' bound the totals U_0, P, D and V at a_0, and each of the two tags carries half of D and of V (Definition 22.4(e)), so that $D'/2$ and $V'/2$ bound both tag totals at a_0. Corollary 9.4 gives at a the bounds tU_0' and tP' for U_0 and P, the bounds d and v for the sharp and the variation total of each tag, and $L_0 \leq e^{tP'}\mathfrak{h}$, $R_{\mathrm{err}} \leq e^{tU_0'}\mathfrak{h}$, $B \leq e^{tU_0'}(1+\mathfrak{h})$, where, by the monotonicity of the series (Remark 9.12), $\mathfrak{h}$ may be taken at the common values d, v of the two tags. Since $D' \leq V'$, $d \leq v$, and with $d < 6$ and $dv < 36$ Corollary 10.9 at $n = 4$ bounds this series by $\mathfrak{h}_4^+(d,v)$.

(b) *Middle.* By Lemma 23.24 the symmetric and skew prices of the records of a middle block $\mathcal{B}$ with least prime b, and their raw-arm, leading and remainder auxiliary prices, are at most $q_{\mathcal{B}}^{2a}b^{-k/2}$ times the coefficients of that lemma, k the degree of the price; the coefficients are evaluated at the piece start and bound those of every block of the piece (Lemma 20.3(iii)). Summed over the blocks of the piece at a_0, $\sum_{\mathcal{B}} q_{\mathcal{B}}^{2a_0}b_{\mathcal{B}}^{-k/2} \leq \bar{I}_{k,i}$, so the rows of (3) bound the prices of the records of the piece at a_0. Every middle block has $\log q_{\mathcal{B}} \leq L$, so at a these prices are at most t times their values at a_0 (Lemma 9.3); summing over the pieces gives S_{mid} and A_{mid}. *Upper bands and tail:* Proposition 28.14(ii), every term of the band ledger repriced from a_{u} at its own logarithmic product (at most $L_{\mathrm{band}} \leq 886$), and Lemma 29.5(iv). *Prefix:* the deleted exception (the coincident-label part of the prefix comparison) is a diagonal at the reserved cut ι and is included in S only, at the total price, with $m_{\mathrm{cl}} = 8192$ the number of digit pairs, $2^{513-m_{\mathrm{cl}}+1024m_{\mathrm{cl}}a} = 2^{-1380939795479/6103515625} < 8 \cdot 10^{-69}$ (Lemma 19.3), inside the allowance 10^{-12} (Certificate D.3(f)), which also holds a second copy of the tail prices (Remark 19.4). These are all the atoms (Definition 18.3(a)).

(c) This is Proposition 26.5(ii) with the chosen sides of Proposition 26.4. In a middle block the envelope $M_{s,\mathcal{B}} - \mathsf{N}_{\mathcal{B}}$ consists of the chosen side diagonals, of price at most T_K (Proposition 26.4(iii) and the factor t), the chosen matching diagonal, of price at most T_{QE} (by the matching gates of Section 26), and the norm envelopes of the other record changes, of price at most $S_{\mathrm{oth}} + A_{\mathrm{oth}}$; outside the middle the envelopes are the norm envelopes of the upper, tail and prefix changes, of price at most $S_{\mathrm{u}}(a) + A_{\mathrm{u}}(a) + 10^{-12}$. The prices of the fold records do not enter T.

(d) We use Lemma 10.14 with B' the number B of (2), which bounds the energy factor of (CP3) by (a), at the per-block upper prices and auxiliary prices obtained from the totals of (3)–(4) by Lemma 10.14(iii), the two records of one kind in a block receiving equal shares; the auxiliary prices

of a record are its raw-arm, leading and remainder prices (Lemma 23.24). Every selection of a class has at most six atoms.

Variance classes. The classes of Corollary 24.18(i) and (iii) whose selections contain a symmetric factor have the class costs $B - g$ with gains $g \in [0, 1)$. By Lemma 10.14(iv) the series $\Psi_0 = B'(e^{S(x)} \cosh A(x) - 1) - \mathcal{D}_{\mathrm{sym}}(x)$, in which $\mathcal{D}_{\mathrm{sym}}$ carries these gains, the monomials with a repeated block receiving the gain of their piece multiset, below $1 < B'$, is a complete polynomial for them; its coefficients on the monomials in skew variables alone are those of $B'(e^{S(x)} \cosh A(x) - 1)$.

Envelope classes. The sixteen classes of Corollary 25.10 satisfy (CP5) with the class bounds $C_{n,m}\pi_\omega^{\mathrm{arm}} + B(\pi_\omega^{\mathrm{tot}} - \pi_\omega^{\mathrm{lead}})$ (Theorem 25.8(i)), and $C_{n,m} \geq 0$: since $U^{\mathrm{inv}} \leq U_0'$, $e^{U_0} \geq e^{tU^{\mathrm{inv}}}$, so that $\kappa_H e^{U_0} - (\kappa_H - \kappa_b)e^{tU^{\mathrm{inv}}} \geq \kappa_b e^{tU^{\mathrm{inv}}}$, and the orbit coupon is at most $M_{n,m}R_{\mathrm{err}}$ (Lemma 25.7); hence $C_{n,m} \geq \kappa_b e^{tU^{\mathrm{inv}}} \geq 0$. By Lemma 10.14(v), with the kinds of the remark after it (the fold records and the merged matching records for the general classes and for $(0, 4)$, $(0, 6)$; the two matching records E_0, E_1 separately for the pairs, the aligned and the crossed pairs each having the total $A_E^2/4$), replacing these classes subtracts from Ψ_0 polynomials whose values at the upper prices are the savings of (8); the classes are disjoint and none contains a selection of a variance class of the previous step.

Credits. Inside the classes $(2, 0)$, (EE, al), $(4, 0)$ and $(0, 4)$ the credits of Lemma 25.9 are subtracted on the raw-arm monomials, as the last part of Lemma 10.14(v) allows: every credit cg satisfies $0 \leq cg < C_{n,m}$ (Lemma 25.9), so the credited class bounds have nonnegative coefficients. Indeed the gains g are below one and $c \leq \kappa_b e^{tU^{\mathrm{inv}}} \leq C_{n,m}$: in the first three classes $c = \kappa_b$, and in class $(0, 4)$ the aligned variance cap satisfies the certified inequality $c < \kappa_b e^{tU^{\mathrm{inv}}}$ (the two sides are 0.667 and 1.12 to three digits; the certified values are in Table 33 and Certificate 25.11). By Corollary 24.18(iv) the subtractions of $\mathcal{D}_{\mathrm{sym}}$ and of these credits together lower the polynomial at the upper prices by at least D_{var}, every coefficient staying nonnegative.

Conclusion. The resulting series Ψ is a complete polynomial (Lemma 10.14(ii)). At the upper prices the totals of the price and auxiliary variables are those of (3)–(4), so $\Psi(\hat{\pi}, \hat{y}) \leq B(e^S \cosh A - 1) - (D - D_{\mathrm{var}}) - D_{\mathrm{var}} = \mathcal{E}$. This proves the complete bound; it does not assert that an isolated formal discount bounds a distinct-block discount from below (Remark 10.15).

(e) Proposition 26.7 at the piece caps, with the error exponent $T_K + T_{QE}$; its gates are among those listed. (f) Proposition 29.7 and Corollary 29.8 and Certificate 29.11.

The gates are asserted by the author assembly of Certificate 30.3: the orbit coupons by `research/advance_2026_10_24/global/closing_beta.py`, lines 204–205; the savings by lines 223, 228, 234 and 240 there (a general or quartic class is retained only if its saving is positive, and the classes $(2, 0)$ and the matching pairs must be); the series and $D' \leq V'$, $d < 6$, $dv < 36$ by `research/advance_2026_10_24/global/grouped_first_stage.py`, lines 27–37, which also checks that $\mathfrak{h}_4^+$ does not exceed the one-sharp bound of Remark 9.12; the regime $0 \leq \ell \leq C_X/2$ by `research/continuation_2026_10_24/envelopes/restoration.py`, lines 36–37; the gains by lines 148 and 160 of `closing_beta.py` and line 31 of `research/advance_2026_10_24/envelopes/beta_higher_variance.py`; the caps $1 \leq C_i \leq C$ by line 16 of `research/advance_2026_10_24/envelopes/piece_cap_restoration.py`; and the sides $k_{s,i}^{\mathrm{aux}} \geq 0$ by line 42 of `closing_beta.py` and line 129 of `research/advance_2026_10_24/global/piece_data_beta.py`; and the matching gates by line 112 of `closing_beta.py`. The protection gate is asserted on all 33 selected native pieces by `research/improvements_2026_10_25/envelopes/check_gap_native.py`, rerun on the rebuilt source of `research/push_2026_10_25/design.json`; $0 < a \leq 9/10000$ is asserted by the selected design verifier. Moreover, $B > 1$ since $U_0 > 0$, and the credit gates follow from the bound $C_{n,m} \geq \kappa_b e^{tU^{\mathrm{inv}}}$ shown in (d).

Finally, Θ is the closing number of Theorem 9.21 at $k = 6$, $d = 72$, $\theta = 1$ for $\mathcal{E} \geq \Psi(\hat{\pi}, \hat{y})$, with J and W as in Proposition 29.7. By Remark 9.22 the closing number is nondecreasing in L_0, B,

S, A, T, R_I, the prices and the auxiliary prices, and nonincreasing in m; so the bounds (a)–(f) show that the closing number of the construction is at most Θ. □

Table 12 lists the values. The upper bands and tail contribute $S_{\mathrm{u}}(a) \approx 0.0138662$ and $A_{\mathrm{u}}(a) \approx 0.00668235$. The class savings remove about 0.456895 and the variance credits about 0.00995856 from the reserved cost. The reserved contribution $\frac{63}{64}(L_0+\mathcal{E})+\frac{1-m}{64}$ is about 0.748200, the matching term J about 0.251761, and the wrap W about $3.8345 \cdot 10^{-5}$. These rounded descriptions are explanatory only.

TABLE 12. The closing quantities, rounded outward from the certificate's enclosures (< upper bound, > lower bound, ... exactly defined). The certificate evaluates Θ from the full enclosures, not from these values.

quantity	value	quantity	value
t	$1.0000000616512\ldots$	$D - D_{\mathrm{var}}$	> 0.456895419675
C	< 2.753348440327	D_{var}	> 0.009958560315
U_0	< 0.846908245753	D	> 0.466853979991
B	< 2.332628665480	$\mathcal{E}$	< 0.768964199811
R_{err}	< 0.000204255886	m_F	> 0.400172842378
L_0	< 0.000476332373	m_Q	> 0.190134971388
$S_{\mathrm{u}}(a)$	< 0.013866153512	m	> 1.589940741988
$A_{\mathrm{u}}(a)$	< 0.006682351376	R_I	< 1.247139907407
S	< 0.014554805967	J	< 0.251761455211
A	< 0.969271826335	W	< 0.000038344989
T	< 0.607496889411	Θ	< 0.99999999997327

30.2. **Outward-interval evaluation: $\Theta < 0.99999999997327$.** It remains to evaluate the explicit number Θ defined in Section 30.1. This is a finite computation: every price is a sum of explicit terms at finitely many prime thresholds, evaluated in outward-interval arithmetic so that rounding can only enlarge the result. The certificate states the outcome and identifies the programs and data files that produce it; Appendix D explains what such a check does and does not prove.

Certificate 30.3 (The closing)**.** The selected rational design `research/push_2026_10_25/design.json` has

$$a = \frac{5422587}{6103515625}, \qquad \varepsilon = a/71 = \frac{5422587}{433349609375}, \qquad K = 890666,$$

and the outward evaluation of Definition 30.1 obeys $\Theta < 0.99999999997327 < 1$. The entry point `research/push_2026_10_25/verify.py` invokes the selected adapter `research/improvements_2026_10_25/global/selected.py` with this design. Two independently written assemblies of the scalar totals use the separate evaluators of `research/advance_2026_10_24/`, with the invariant-background and root-restoration adapters of the later checkpoints. The finite conductor law is supplied by `research/improvements_2026_10_25/global/padic_moment.py`; its independent check and the recomputation of the closing from it are part of the selected verification. The outputs equal the stored `research/push_2026_10_25/artifacts/author.json` and `research/push_2026_10_25/artifacts/independent.json`; the enclosures of all 45 compared scalar components intersect. The independent enclosure of Θ is

[0.99999999997325941882402031814341317621743080026301922476436792225715828504384175,
0.99999999997325941882402031814341317621743080026301922476436792225715828504384968],

and the author's upper endpoint is below $0.99999999997325941882402031814 4$. Both are strictly below 0.99999999997327, giving a certified margin greater than $2.673 \cdot 10^{-11}$. The archival launcher

`paper/verify.py` restores and authenticates the original source layouts before replaying their certificates (Appendix D). These finite checks do not replace the analytic reductions or external theorems, whose proofs and assumptions are given in the text.

How the certificate evaluates Θ. The certificate evaluates every item of Definition 30.1 from the sources whose hashes Appendix D checks: the selected middle source `research/push_2026_10_25/global/refined_L570_source.json`, its exact charge evidence and native parent; the unchanged upper source `research/further_2026_10_25/upper/upper_dense_refined.json`; and the envelope bundle `research/further_2026_10_25/envelopes/exact_background_pairs.json`. The exact charges are regenerated at $a_0 = \frac{4442183}{5000000000}$, then repriced by $t = e^{2(a-a_0)L}$; the proof is blockwise monotonicity of $q_{\mathcal{B}}^{2a}$, as in Lemma 9.3. The root-restoration adapter supplies the actual marginal data of all seven tilted pieces of Definition 21.1, and the conductor adapter supplies the finite-depth upper majorant. All first-stage totals, comparison prices, discounts and coupons are recomputed from these sources. The formulas and their checks are listed in Section D.4.

30.3. **Proof of the baseline ($\theta = 1$) by Theorem C.** Theorem C now applies directly: it only remains to point, for each hypothesis (CP0)–(CP8), to the statement of Part V that verifies it for the construction, and to $\Theta < 1$ from the preceding subsection.

Proof of Proposition 1.2. By Theorem 9.21 it suffices to verify, for the construction of Part V with $k = 6$ queries on $d = 72$ slots and $\theta = 1$, the hypotheses (CP0)–(CP8) at every scale, the boundedness of the totals independently of the scale, and $\Theta < 1$.

Hypotheses. Table 5 lists for each hypothesis the statement of Part V that verifies it: (CP0) the slot array, the queries, the atoms and the analytic stage, with $a = \frac{5422587}{6103515625} \le 9/10000 < 1/825 = \sigma$, $\sigma - a = 65195254/201416015625$, and the gates of the stage on the whole domain $0 < a \le 9/10000$ (Certificates D.2 and D.3), and the index law (Proposition 29.7); (CP1) the prefix, middle, upper and tail references, with (R4) (Lemma 29.3); (CP2), (CP3) the first stage and the law dominating it in a 2×2 block; (CP4) the comparison law in every range, with (R5), the signed fold records being finite signed combinations of nonnegative accepting product laws, so that the quantities n_ν and $\mathfrak{v}_\nu$ of Definition 9.14 are finite; (CP5) the class bounds of Section 25 and the variance classes of Section 24, whose selections have at most six blocks; (CP6), (CP7) the matching envelopes and the mean credit; (CP8) admissibility (Definition 8.9), robustness (Lemma 11.4) and summability (Lemma 29.5(iv)).

Bounded totals. Every total is a sum of certified per-atom upper prices over finitely many atoms below P_4, independent of N, and over the tail primes, whose prices are summable (Lemma 29.5(iv)). So Proposition 30.2 holds at every scale with the same constants.

Closing inequality. By Proposition 30.2 the closing number of Theorem 9.21 is at most Θ, and $\Theta < 0.99999999997327 < 1$ by Certificate 30.3. Robustness, the clause of (CP8), follows from Lemma 11.4(i) as in Remark 11.5: every gate on a quantity that the inflation changes is strict (the savings, the gains, the gates $d < 6$, $dv < 36$, and $\Theta < 1$), except the orbit-coupon gate, which is a nonnegativity condition on a coefficient that only grows under inflation; the restoration regime $0 \le \ell \le C_X/2$ is kept by the inflation, which lowers ℓ and raises the effective cap to $(1+\delta)C_X$ (Remark 11.5(c)); and the side gates $k_{s,i}^{\mathrm{aux}} \ge 0$, the protection gate and $0 < a \le 9/10000$ concern fixed numbers of the construction. So the function $\Theta(\delta)$ of Lemma 11.4 (δ the inflation parameter, with $\Theta(0) = \Theta$) is continuous at $\delta = 0$, with value below one there.

Conclusion. Theorem 9.21 gives, for every $\mathsf{L} \ge 1$, $\mathcal{Y}(\mathsf{L}, \cdot) \le C\mathsf{L}^{-a}\alpha$ at every square-difference-free subset of $[K\mathsf{L}]$ of density α, hence $s(N) \le C'N^{1-a/71}$ for every $N \ge 1$ with an absolute constant C'. As $a/71 = \frac{5422587}{433349609375} = \varepsilon$, this is Proposition 1.2. □

Consequences and what to read next. Theorem 9.21, Remark 9.22, and Table 5; Lemmas 9.3 and 22.6 and Corollaries 9.4 and 10.9 and Certificates D.3 and D.4; Lemmas 20.7, 22.9, 23.17, 23.18

and 23.24 and Proposition 23.13; Proposition 28.14 and Lemmas 19.3 and 29.5; Propositions 26.4 and 26.5; Lemma 10.14 and Remark 10.15; Corollary 25.10, Theorem 25.8, and Lemmas 25.6, 25.7 and 25.9; Corollary 24.18, Proposition 24.17, and Lemma 24.16; Proposition 26.7; Proposition 29.7, Lemma 29.9, and Certificate 29.11; Certificate D.2; and Lemma 11.4 and Remark 11.5.
Hypotheses verified in this section. The row "$\Theta < 1$" of Table 5, $\Theta < 0.99999999997327$ (Proposition 30.2 and Certificate 30.3), and Proposition 1.2.

31. Stronger density estimates: multilinear bounds and a new prefix law

What this section proves and how to read it. To obtain $\vartheta > 1$ one needs a multilinear bound $|\int \prod_v f_v \, d\pi| \le \prod_v \|f_v\|_{L^r}$ with $r < 72$ (a *unit mixed moment*, defined below; "moment" here means this multilinear norm, not $\mathbb{E}X^k$). This section proves two finite-alphabet inequalities giving such bounds and constructs a replacement prefix law (independent slot labels from the same clique) satisfying them. Inputs: Section 19. Outputs: the moment hypotheses used in Section 32. Baseline results do not depend on this section.

Definition 31.1 (Unit mixed moment)**.** Let $d \ge 1$ and $r \ge 1$. Let π be a probability measure on X^d, with each coordinate uniform on the finite set X. It has *unit mixed L^r norm* if

$$\left| \int \prod_{v=1}^{d} f_v(x_v) \, d\pi(x) \right| \le \prod_{v=1}^{d} \|f_v\|_{L^r(X)}$$

for all real or complex functions f_v, where all norms use uniform probability measure. Its *density endpoint* is d/r. If π' on $X^{d'}$ and π'' on $X^{d''}$ have unit mixed L^r norm, so does $\pi' \otimes \pi''$, by successive integration; the same holds for convex mixtures and common coordinate permutations. A bound at r also holds at every larger exponent.

31.1. **Uniform convexity and mixtures on a finite alphabet.** The stronger induction needs moment bounds whose density endpoint d/r exceeds 1. The elementary inequality behind them is uniform convexity of L^r norms; we prove it, and the finite-alphabet mixture estimate built on it, here because the exact constants matter later.

Lemma 31.2 (Uniform convexity)**.** *On a finite probability space, for $1 < r \le 2$ and nonnegative f,*

$$\|f\|_r^2 \ge (\mathbb{E}f)^2 + (r-1)\|f - \mathbb{E}f\|_r^2.$$

Proof. For a positive function g and a real direction h, differentiation gives

$$\frac{d^2}{dt^2}\|g+th\|_r^2\Big|_{t=0} = 2(r-1)\|g\|_r^{2-r}\mathbb{E}(g^{r-2}h^2) + 2(2-r)\|g\|_r^{2-2r}\big(\mathbb{E}(g^{r-1}h)\big)^2 \ge 2(r-1)\|h\|_r^2.$$

The last step follows from Hölder applied to $|h|^r = (g^{r-2}h^2)^{r/2}g^{r(2-r)/2}$. Take $g = \mathbb{E}f$, $h = f - \mathbb{E}f$, and integrate the second derivative along $g + th$, $0 \le t \le 1$. The first derivative at zero vanishes. If f has zeros, apply the argument to $f + \varepsilon$ and let ε decrease to zero; the assertion is immediate when $\mathbb{E}f = 0$. □

Lemma 31.3 (Typed finite-alphabet mixture)**.** *Let $|X| = Q \ge 2$ and $1 \le s < R < d$. Put $\kappa = s/R$ and, for $R < v \le d$,*

$$C_v(Q) = \frac{d}{v} Q^{(v/R-1)(d/v-1)} \frac{Q^{v/R} - (v/R)Q + v/R - 1}{\kappa Q + 1 - \kappa - Q^{\kappa}}.$$

Let $\pi = \sum_j w_j \pi_j$, where $w_j \ge 0$, $\sum_j w_j = 1$, and each π_j has unit mixed L^{r_j} norm and uniform coordinate marginals. Suppose $\sum_{j: r_j \le s} w_j \ge w_0$. Components with $s < r_j \le R$ require no further

condition. For every remaining component choose $v_j \in (R, d]$ such that π_j has unit mixed L^{v_j} norm (for example $v_j = d$). If

$$\sum_{j:\, v_j > R} w_j C_{v_j}(Q) \le w_0,$$

then π has unit mixed L^R norm. Moreover, if $Q^{\kappa-1} \le \kappa/2$, then

$$C_v(Q) \le \frac{2d}{\kappa v} Q^{d/R - d/v}. \tag{31.1}$$

In particular, a mixture of a unit mixed L^s law of weight $1-t$ and an arbitrary uniform-marginal law of weight t has unit mixed L^R norm whenever $t(1 + C_d(Q)) \le 1$.

Proof. We first prove a one-coordinate estimate. Let f be a function on X with $\|f\|_R = 1$, and write $u = |f|^R$. Then $\mathbb{E}u = 1$ and $0 \le u \le Q$. For $\lambda > 1$ put

$$\phi(t) = t^\lambda - \lambda t + \lambda - 1, \qquad \psi(t) = \kappa t + 1 - \kappa - t^\kappa.$$

Both functions and their first derivatives vanish at 1, and $\phi''(t)/\psi''(t)$ is a positive constant times $t^{\lambda-\kappa}$. It follows that

$$\frac{\phi(t)}{\psi(t)} \le \frac{\phi(Q)}{\psi(Q)} \quad (0 \le t \le Q),$$

with $\phi(1)/\psi(1)$ interpreted as $\phi''(1)/\psi''(1) = \lambda(\lambda-1)/(\kappa(1-\kappa))$. For completeness, on $t < 1$ the ratio is a weighted average of ϕ''/ψ'' on $[t, 1]$, hence at most its value at 1. On $t > 1$, first ϕ'/ψ' and then ϕ/ψ are increasing weighted averages of the preceding increasing ratio.

Take $\lambda = v/R$ and $A = \mathbb{E}u^\lambda$. Since $1 \le A \le Q^{\lambda-1}$ and $d/v \ge 1$,

$$\begin{aligned} \|f\|_v^d - 1 &= A^{d/v} - 1 \\ &\le \frac{d}{v} Q^{(\lambda-1)(d/v-1)}(A-1) \\ &\le C_v(Q)(1 - \mathbb{E}u^\kappa) \le C_v(Q)(1 - \|f\|_s^d). \end{aligned}$$

The last inequality uses $\|f\|_s^d = (\mathbb{E}u^\kappa)^{d/s} \le \mathbb{E}u^\kappa$. If $Q^{\kappa-1} \le \kappa/2$, then $\psi(Q) \ge \kappa Q/2$ and $\phi(Q) \le Q^\lambda$; this proves (31.1).

For unequal inputs normalized by $\|f_v\|_R = 1$, apply Hölder to the component index with weights w_j:

$$|\Lambda_\pi((f_v)_v)| \le \sum_j w_j \prod_v \|f_v\|_{r_j} \le \prod_v \left(\sum_j w_j \|f_v\|_{r_j}^d \right)^{1/d} \le 1.$$

The final inequality follows from the preceding scalar bound, since the weight w_0 supplies the deficit $1 - \|f_v\|_s^d$. Ordinary Hölder gives unit mixed L^d norm for every law with uniform coordinate marginals, which proves the final assertion. □

31.2. **A pair moment and independent draws from a digit clique.** The basic local input is one pair of labels whose difference is a square, or a nonsquare. The next lemma gives its moment bound; independent draws from the digit clique of the prefix are products of such pairs, so the same bound applies to them.

Lemma 31.4 (Square-pair moment)**.** *Let p be an odd prime. A uniform ordered strict square pair, or a uniform ordered nonsquare pair, in $\mathbb{F}_p$ has unit mixed $L^{1+1/j}$ norm whenever j is a positive integer satisfying*

$$j^{j+1} \left(\frac{p+1}{p-1} \right)^{j-1} \frac{1}{(\sqrt{p}-1)^2} \le 1. \tag{31.2}$$

The conclusion persists on $\mathbb{Z}/p^2\mathbb{Z}$ when each label is written as $x+py$ with $x,y\in\mathbb{F}_p$, the two y-coordinates are independent and uniform, and the pair of x-coordinates has either of these laws.

Proof. Relative to uniform probability measure, the centered bilinear form $(f,g)\mapsto \mathbb{E}fg-\mathbb{E}f\mathbb{E}g$ of the pair law has norm at most $L=(p+1)/(p-1)$ on $L^1\times L^1$ and at most $\rho=(\sqrt{p}+1)/(p-1)=1/(\sqrt{p}-1)$ on $L^2\times L^2$. The latter is the standard Paley character-spectrum bound; it holds for $p\equiv 1$ and $p\equiv 3 \pmod 4$ and for either choice of square or nonsquare difference. Bilinear interpolation at $r=1+1/j$ gives centered L^r norm at most $\beta=L^{(j-1)/(j+1)}\rho^{2/(j+1)}$. Under (31.2), $\beta\le r-1$. For nonnegative inputs of L^r norm one and means m,n, the expectation of the product of the two inputs under the pair law is therefore at most

$$mn+\frac{\beta}{r-1}\sqrt{(1-m^2)(1-n^2)}\le 1,$$

by Lemma 31.2 and Cauchy–Schwarz in $\mathbb{R}^2$. Positivity of the pair law allows absolute values of arbitrary inputs. For the prime-square law, first average each input over its high digit; this is an L^r contraction. □

Corollary 31.5 (Invariant distinct pairs)**.** *For $p\ge 64$, any probability law on ordered distinct pairs in $\mathbb{F}_p$ which is invariant under simultaneous square-affine maps has unit mixed $L^{3/2}$ norm.*

Proof. Such a law is a convex mixture of the square and nonsquare pair laws: these are the two orbits of the square-affine group on distinct pairs. For $j=2$, the left side of (31.2) is at most $8\cdot 65/(63\cdot 49)<1$ for $p\ge 64$. Apply Lemma 31.4 and mix. □

Lemma 31.6 (Thirty-six digit-clique draws)**.** *Let $2^{16}\le p<2^{512}$ be a prime congruent to 1 modulo 4. Let $(A,B)\in(\mathbb{Z}/p^2\mathbb{Z})^2$ have the digit-pair law of Definition 19.1: its low digits are a uniform strict square pair, and its high digits are independent and uniform. Conditional on (A,B), draw 36 labels independently, each uniform on $\{A,B\}$. Their law has unit mixed $L^{600/17}$ norm, equivalently density endpoint 51/50.*

Proof. Put $d=36$, $\theta=51/50$ and $R=d/\theta=600/17$. Choose the unique $b\in\{16,32,\dots,496\}$ with $2^b\le p<2^{b+16}$. In increasing order of b, use respectively

$$5,9,12,15,18,20,23,26,28,31,33,36,38,40,43,45,$$
$$47,50,52,54,56,58,61,63,65,67,69,71,73,75,77$$

for the integer j, and put $r=1+1/j$, $s=18r$, $\kappa=s/R$. The condition (31.2) holds at $p=2^b$, with $\sqrt{p}=2^{b/2}$, and hence for every p in the dyadic interval $[2^b,2^{b+16})$ (called a band only in this proof; not an upper band) by monotonicity in p.

Expand the independent binary choices into their 2^{36} choice vectors. The choice vectors with eighteen A's and eighteen B's have total weight $w_0=\binom{36}{18}/2^{36}$ and unit mixed L^s norm: apply the pair moment to the two products of 18 inputs and then Hölder within each product. For minority count $0\le k<18$, the total weight of the two complementary classes is $w_k=2\binom{36}{k}/2^{36}$. The corresponding law has unit mixed norm at

$$v_k=\begin{cases}36, & k=0,\\ \min\{36,(36-k)r\}, & 1\le k<18.\end{cases}$$

The truncation at 36 (the $\min\{36,\cdot\}$ in the display; not a spectral cap) follows from uniform marginals. Thus only the classes with $v_k>R$ occur in the sum in Lemma 31.3.

Here $Q=p^2$. The following finite rational tests hold on every band:

$$2^{-\lfloor 2b(1-\kappa)\rfloor}\le \kappa/2, \tag{31.3}$$

$$\frac{1}{w_0}\sum_{k:\,v_k>R} w_k\frac{72}{\kappa v_k}2^{\lceil 2(b+16)(\theta-36/v_k)\rceil}\le\frac{95659840}{13884957009}<1. \tag{31.4}$$

For each of the 31 values of b these are two explicit integer/rational inequalities: the first uses $Q \geq 2^{2b}$, and the second uses $Q \leq 2^{2(b+16)}$ and rounds every power of two upward. The largest ratio occurs at $b = 16$. The standalone checker `research/four_routes_2026_10_26/dense_moments/check_iid72_prefix_moments.py` reproduces these tests, the listed interpolation gates and the price bounds below using only integer and rational arithmetic; its output is `research/four_routes_2026_10_26/dense_moments/iid72_prefix_moment_gates.json`. Equations (31.3)–(31.4) and Lemma 31.3 prove the assertion. □

Lemma 31.7 (At most two copies per label)**.** *Under the digit-pair law of Lemma 31.6, fix a map from independent draws from $\{A, B\}$ to the 72 physical slots under which each draw is used at most twice. The resulting law has unit mixed $L^{1200/17}$ norm. The same holds after tensoring m independent digit pairs, for any number m of digit pairs, and applying any common random square-affine permutation of $\mathbb{Z}/p^{2m}\mathbb{Z}$.*

Proof. Write $C = \{A, B\}$ for the random two-element clique and ν_C for the uniform law on C. Conditional Cauchy–Schwarz in each group of one or two slots gives

$$\mathbb{E}\left[\prod_v |f_v| \,\middle|\, C\right] \leq \prod_v (\nu_C |f_v|^2)^{1/2}.$$

Hölder on C with 72 equal exponents, followed by Lemma 31.6 with identical input $|f_v|^2$, bounds the unconditional expectation by

$$\prod_v \left[\mathbb{E}_C (\nu_C |f_v|^2)^{36}\right]^{1/72} \leq \prod_v \|f_v\|_{1200/17}.$$

The bound for one digit pair implies the bound for m independent digit pairs, for arbitrary functions of the full digit vector, by successive integration. A fixed affine permutation preserves the uniform norms, and its positive average preserves the bound. □

31.3. **The replacement prefix and its dominators.** With these moments the prefix law of Section 19 can be strengthened: at large prefix primes the 72 labels are drawn independently from the square clique, instead of through six chamber labels. The theorem states the properties of this replacement law and of the laws dominating it, which remain valid for density exponents up to 51/50.

Theorem 31.8 (A prefix with density endpoint 51/50)**.** *At each prefix prime $p \equiv 1 \pmod 4$, keep the $m = m_{\mathrm{cl}} = 8192$ digit pairs, label group $\mathbb{Z}/p^{2m}\mathbb{Z}$, random square clique $\mathcal{C}_p$ and common square-affine map of Definition 19.1. For $p \geq 2^{16}$, replace its six-label pullback by 72 conditionally independent uniform draws from $\mathcal{C}_p$, one per slot. For $p < 2^{16}$ retain the law of Definition 19.1. Then:*

- *(a) The new reference is reflection positive at every cut of the slot array V of Definition 5.12 used in the construction, square-affine invariant, and has uniform one-slot marginals. Every nonzero difference of two labels in its support has the same square property as in Lemma 19.2(iii).*
- *(b) At a large prefix prime, the reference, the diagonal copy of its actual reserved half, and every law obtained by identifying disjoint pairs among its independent labels have unit mixed $L^{1200/17}$ norm. These statements tensor over selected prefix atoms.*
- *(c) For the six actual queries, let E_p be the restriction of the new reference law μ_p to the union of their equality events, and put*

$$\eta = 1 - (1 - 2^{-m})^6 \leq 6 \cdot 2^{-m}.$$

Both reserved-half marginals of E_p equal ηh_p, where h_p is the actual 36-label half marginal of the new reference μ_p. With $D_p = \mathrm{Diag}_\iota(h_p)$ and $\nu_p = \mu_p - E_p$,

$$\begin{pmatrix} \mu_p + \eta D_p & \nu_p \\ \nu_p^* & \mu_p + \eta D_p \end{pmatrix} \succeq_\iota 0.$$

The measure $\nu_p = \mu_p - E_p$ is nonnegative, has mass $1 - \eta > 0$, and accepts all six queries. The symmetric and skew parts of E_p each have operator norm at most η between the actual half L^2 spaces; no symmetry of E_p is assumed.

(d) For $0 < a \le 91/100000$ and $1 < \theta \le 51/50$, each of the three aggregate prefix allowances S, A, T is less than $2{\cdot}10^{-13}$, including the factor $Q^{\theta-1}$, for alphabet size Q, at the unchanged small primes. Thus each is below the bound 10^{-12} assigned to it in Definition 30.1.

Proof. Conditional on the clique and affine map, the 72 new labels are iid. At every reflection cut with two halves of 36 slots (a balanced reflection) the two halves are independent and have the same law, so its cut form is the square of a conditional mean. This proves reflection positivity. The existing common affine averaging gives invariance and uniform marginals. The digit argument for the square property is unchanged: if the first digit pair at which two clique elements differ is the pair in position j (counted from 0), their difference has p-adic valuation $2j$ and unit part congruent to $\pm(b_j - a_j)$ modulo p, and -1 is a square. Hensel lifting then gives the required unit square. This proves (a). Part (b) is Lemma 31.7; the reserved diagonal uses 36 independent labels, each exactly twice.

The query endpoints on the original labels are pairwise disjoint and each query crosses the reserved binary cut. Conditional on the clique and on the labels of the plus half, the six labels at the minus endpoints of the six queries are independent uniform choices from a set of 2^m elements. Each equality has probability 2^{-m}, independently, so their union has probability η, independent of the plus-half realization. The reverse argument gives the other marginal. Every positive coupling is a contraction between its actual marginal L^2 spaces. Hence

$$\begin{pmatrix} \eta D_p & E_p \\ E_p^* & \eta D_p \end{pmatrix} \succeq_\iota 0.$$

Changing the signs of the off-diagonal blocks and adding the positive block matrix with all four entries μ_p proves the displayed bracket in (c). The accepted part removes precisely the zero query differences. The symmetric and skew parts of its change have operator norms at most η, and E_p is dominated at ι by the diagonal envelope ηD_p. Using the separate symmetric and skew envelopes for matching instead costs at most $2\eta D_p$. In all cases the chosen matching sides contain μ_p plus a positive diagonal, so the lower bound for the mean constant m from these sides is the same as for μ_p. The first-stage bracket at a prefix atom remains the one for μ_p.

An individual query-equality measure has mass 2^{-m} and, after normalization, identifies exactly one pair of otherwise independent clique labels. It therefore has the moment in (b). The union measure is dominated by the sum of these six positive measures. This gives a valid positive-spectator price without claiming a unit moment for E_p/η. For the density comparison before the Gram step, its actual variation is η and its one-slot marginals are uniform. These are separate uses of the deleted measure E_p.

For a uniform alphabet of size Q, ordinary Hölder and finite-space norm comparison give, for every positive uniform-marginal law on 72 slots,

$$|\Lambda((f_v)_v)| \le Q^{\theta-1} \prod_v \|f_v\|_{72/\theta}.$$

Apply this only to the terms of the selection expansion (Definition 9.2) supported at the unchanged primes $p < 2^{16}$, where $Q = p^{2m}$. An unselected prefix stays in the RP outside reference and pays no such cost. Complete selection of a clique prime has the existing conductor price p^{2ma}, not p^{4ma}.

Bounding the number of primes by the number of integers gives the common upper bound, for each of S, A, T,

$$12 \cdot 2^{16+m(-1+32(a+\theta-1))} + 12 \cdot 2^{512+m(-1+1024a)}$$
$$\le 12\left(2^{-2694} + 2^{-46}\right) < 2 \cdot 10^{-13} < 10^{-12}.$$

The factor 12 is 6 for the six equality events times 2 for bounding the symmetric and skew parts separately. The first term is also less than 2^{-2600}. The new large-prime unit moment leaves only the usual conductor price in the second term. This proves (d). □

Remark 31.9. The prefix replacement changes no coordinate group, conductor cutoff, external index variable or square-affine map. The fixed-prefix conductor argument and the actual-square kernel therefore retain their stated scope. It changes the local law at the large class-one prefix primes, and the exceptional deletion can now have both symmetric and skew parts. Its moment and bracket proofs above account for that change explicitly. The conclusion is the list of moment and bracket properties of the selected prefix laws assumed in Section 32, not a stronger moment assertion for arbitrary admissible measures.

32. Induction with density exponent $\vartheta > 1$

What this section proves and how to read it. Theorem C requires $0 < \theta \le 1$, so $\vartheta > 1$ needs a separate induction. This section proves it conditionally: *assuming* price bounds for the selected laws (verified numerically in Section 33), the bound $Y \le C\mathsf{L}^{-a}\alpha^{\vartheta}$ holds for $1 < \vartheta \le \theta_* = \frac{1014587}{1000000}$. Read Theorem 32.12 as "if the displayed price hypotheses hold then the induction closes", not as an unconditional theorem. Terminology: a *packet* is a set of atoms changed simultaneously; a packet is *good* if its moment bound holds with the stated exponent and *bad* (concentrated) otherwise, in which case labels are pinned and rescaling applies—both are defined at first use below and summarised here to allow the statements to be read in order. θ_* equals the macro $\frac{1014587}{1000000}$; the equality is part of the hypothesis check in Section 33. The baseline Proposition 1.2 does not depend on this section. Throughout,

$$0 < a \le 91/100000, \qquad 1 < \theta \le \theta_* := 1014587/1000000, \qquad d = 72. \tag{32.1}$$

Use the middle reference of the baseline construction, the clique-prefix replacement of Theorem 31.8, and the accepting merge and two-layer constant replacements of Lemmas 23.20 and 23.23. In the finite upper range use the reference weights of Section 27.3.1 and the comparisons of Section 28.4. These comparisons preserve the component marginals, so the moment bounds below continue to apply to them. The blocks, middle source weights, external index law and six disjoint queries are unchanged. At each atom combine all symmetric norm envelopes except the fold and matching class envelopes into one positive packet, and do the same separately for the generic skew envelopes. The specialized fold and matching classes are kept separate.

The unrestricted structured-energy statement Proposition 8.7 is not claimed for $\theta > 1$. Its last Jensen inequality requires $\theta \le 1$. Instead we prove the needed bounds for the actual selected laws and actual index distribution of this construction. Unit mixed moments refer to Definition 31.1 and always use the actual marginals.

32.1. **Multilinear bounds for the laws used in the induction.**

The induction with $\vartheta > 1$ needs a moment bound (a unit mixed L^R norm, Definition 31.1) for every law that can arise when several atoms are changed at once. This subsection proves those bounds, starting from one pair of distinct labels repeated across slots.

Lemma 32.1 (A repeated distinct pair)**.** *Let a probability law on d slots have uniform one-slot marginals. Suppose there are two distinct random labels x, y and disjoint sets of t specified slots*

each such that the slots in the first set have label x almost surely and those in the second have label y almost surely, and the joint law of (x, y) has unit mixed L^q norm, where $q < 2$. Then the full law has unit mixed L^R, with

$$R = d - (2 - q)t.$$

In particular an actual square-affine-invariant distinct pair over $\mathbb{F}_p$, $p \geq 64$, repeated twice gives unit mixed L^{11} on twelve slots, and a pair repeated four times gives unit mixed L^{70} on 72 slots.

Proof. Take absolute values. Group t of the factors at x and t at y. Apply Hölder to this group and each remaining slot, with exponents $R/(tq)$ and R, respectively; their reciprocal exponents sum to one. Apply the pair moment to the group, and then Hölder among its t functions at each label. Every original function has exponent R. The last assertions use Corollary 31.5, with $q = 3/2$. No independence from the remaining labels is required. □

Lemma 32.2 (The nonconstant middle catalog)**.** *Every nonroot middle completion component other than the one-layer constant, and every nonconstant positive law used in the middle first-stage, reserved-envelope and chosen-side expansions, has unit mixed L^{11} norm as a law on the twelve slots of one prime coordinate. The assertion includes the actual private folds used for a sharp estimate and the positive actual-half couplings in the fold and matching class bounds. It tensors over selected blocks. Both one-layer constant tags are excluded: they emit the same one-label law and both weights are retained in the bad parts $B_{\mathrm{S}}, B_{\mathrm{A}}$ of Lemma 32.6. Every actual two-layer root, including a selected positive tilt, has unit mixed $L^{23/2}$, hence density $24/23 > \theta_*$.*

Proof. The labels in the original field coordinates are distinct. The actual marginals of the labels that remain after unused labels are integrated out (the *visible* labels; integrated labels are *hidden*) are square-affine invariant. Thus any two specified distinct visible objects have a pair marginal to which Corollary 31.5 applies. The physical multiplicities, in the notation of Definition 22.4, are as follows; a one-layer law is copied onto the reserved diagonal.

component	number of visible labels	two available multiplicities
folded root	6	$2, 2$
F_2, E_2	6	$2, 2$
folded F_1, E_1	3	$4, 4$
G_2	4	$4, 4$
folded G_1	2	$8, 4$
C_2	2	$6, 6$

All rows therefore satisfy Lemma 32.1 with $t = 2$. The two-layer root is not asserted to have the L^{11} bound and is not a member of the nonroot packet. For it choose two of its twelve distinct labels, each with multiplicity one. Its actual pair law is still square-affine invariant, including for each positive tilt. The same grouping lemma with $t = 1$ gives $L^{23/2}$. This weaker root bound is enough for the present density range.

The additional first-stage matching law uses three distinct labels (a, b, c), with c shared and the actual conditional transition on a, b; its physical multiplicities are $4, 4, 4$. Its own private sharp fold has the actual uniform ordered-distinct pair (c, a), with multiplicities $8, 4$. The polarization and merge destinations likewise have a distinct pair with at least two occurrences each. These are statements about their full actual laws, not about marginals after conditioning on a shared value.

For the positive couplings $M_{ab}, P_{12}, P_{01}, J$ and their physical transports in the reserved class bounds, choose two distinct object labels within one half. That half is the actual fold or matching half marginal, and each chosen label has two occurrences there. Their pair law is fixed by that marginal, regardless of the coupling between halves. Hence coincidences between a label in one half and a label in the other do not affect this pair law or the bound. The same argument covers the reserved diagonal, independent-half coupling and their positive mixtures, and the native one-layer

companions used for the chosen sides. Signed coupling expansions retain their existing absolute coefficient mass; the assertion is applied to each positive coupling, not to an arbitrary positive operator.

Each bound just proved has the form of Definition 31.1, with constant 1 and arbitrary functions f_v. Iterating it over the prime coordinates gives the block statement, and then the statement for products of selected blocks. Hidden labels keep their native law throughout; functions at unused labels are one. □

We need stronger moments only for weighted restoration. We supply their short field-law proof to distinguish this use from the ordinary catalog.

Lemma 32.3 (A collision conditioning bound)**.** *For $d_0 \le 6$ nonnegative functions f_i, each depending on one coordinate and normalized by $\|f_i\|_r = 1$ under its own marginal μ_i, put $M = \prod_i \mathbb{E}_{\mu_i} f_i$. Under any joint law (coupling) on the product space whose one-coordinate marginals are exactly the μ_i,*

$$\mathbb{E}\prod_i f_i \ge M - B_r(1-M),$$

where one may take $B_r = 1280$ for $2 \le r \le 6$, or $B_r = 1024$ for $4/3 \le r \le 3/2$. Suppose a positive law π_0 has unit mixed L^s, and π is π_0 conditioned on the complement of an event of π_0-mass η, and π_0 conditioned on the event has the same one-coordinate marginals as π_0. If $r > s$ lies in one of these ranges, then π has unit mixed L^r whenever

$$(B_r + 1)\eta \le \frac{r-s}{s(r-1)}. \tag{32.2}$$

Proof. Write $\phi_r(x) = x^r - 1 - r(x-1)$, $w = (1-x)_+$ and $z = (x-1)_+$. For nonnegative x_i, with $W = \sum w_i$ and $Z = \sum z_i$,

$$\prod_i x_i \ge (1-W)(1+Z) = \sum_i x_i - (d_0 - 1) - \sum_{i,j} w_i z_j.$$

For $W \le 1$ this follows by multiplying the deficit and surplus factors separately; for $W > 1$ the right side is nonpositive. If $r \ge 2$, convexity gives $\phi_r(x) \ge (r-1)(x-1)^2$, so each opposite-deviation product is at most $[\phi_r(x_i) + \phi_r(x_j)]/[2(r-1)]$. If $1 < r \le 2$, Taylor's formula gives $\phi_r(x) \ge (r-1)(x-1)^2/4$ on $[0,2]$. Also $\phi_r(2) \ge (r-1)/3$ (differentiate in r, using $2\log 2 - 1 > 1/3$), and convexity implies $\phi_r(1+z) \ge (r-1)z/3$ for $z \ge 1$. Thus an opposite-deviation product is at most $3[\phi_r(x_i) + \phi_r(x_j)]/(r-1)$ in this range. Each nonzero deviation is paired with at most $d_0 - 1$ others.

If $M \ge 1/2$, apply the resulting pointwise inequality to f_i/m_i, where $m_i = \mathbb{E} f_i \ge 1/2$. Use $m_i^{-r} - 1 \le r2^{r+1}(1-m_i)$ and $\sum_i (1-m_i) \le 2(1-M)$. The first range gives the constant $(d_0-1)r2^{r+1}/(r-1) \le 1280$. In the second range use $r2^{r+1} < 9$ and $r - 1 \ge 1/3$ to obtain a constant at most 810, which is below 1024. If $M < 1/2$, positivity alone proves the stated bound. Zero means follow directly.

For the conditioning assertion normalize the inputs in L^r. Interpolation between L^1 and L^r gives $\|f_i\|_s \le m_i^\alpha$, where $\alpha = (r-s)/(s(r-1))$. Hence $\mathbb{E}_{\pi_0}\prod_i f_i$ is at most $M^\alpha \le 1 - \alpha(1-M)$. Subtract the bad-event contribution, bounded below by $\eta[M - B_r(1-M)]$, and divide by $1-\eta$. The result is at most one under (32.2). Absolute values give the complex-input assertion. □

Lemma 32.4 (Moments for weighted fold restoration)**.** *At every middle prime the actual native two-layer fold has unit mixed field $L^{5/2}$, and the actual native one-layer fold has unit mixed field $L^{7/5}$. Their twelve-slot emissions have unit mixed L^5 and $L^{28/5}$, respectively. In particular both have density endpoint at least 2θ in* (32.1)*.*

Proof. The uniform ordered-distinct pair has centered bilinear L^1 norm at most one and L^2 norm $1/(p-1)$. Interpolation gives centered $L^{9/8}$ norm $(p-1)^{-2/9} \le 1/8$ for every middle prime $p \ge 2^{18}$. The uniform-convexity inequality of Lemma 31.2, and $\sqrt{(1-m^2)(1-n^2)} \le 1-mn$, give unit mixed $L^{9/8}$.

The strict four-label $K_{2,2}$ outer core has uniform ordered-distinct-pair marginals on its two objects: in the Paley graph on $\mathbb{F}_p$, two fixed distinct vertices have $(p-3)/4$ common out-neighbors distinct from both, hence exactly $[(p-3)/4][(p-7)/4]$ ordered distinct neighbor pairs, independently of its value. The incoming count is the same. Cauchy–Schwarz between the two objects and the preceding pair bound give unit field $L^{9/4}$. Adjoin an independent ordered-distinct middle pair, and condition it to avoid the four core labels. This is exactly the native two-layer fold, and the excluded mass is at most $8/p$. Translation invariance gives uniform individual marginals also on the excluded event. Apply Lemma 32.3 with $s = 9/4, r = 5/2$; $\alpha = 2/27$ and $1281 \cdot 8/p \le 2/27$ for $p \ge 2^{18}$.

For a strict Paley edge in class three, centered interpolation at $4/3$ gives bilinear norm $(p+1)^{3/4}/(p-1) \le 2p^{-1/4}$. Uniform convexity therefore gives unit mixed $L^{4/3}$ for $p \ge 2^{24}$, since $6p^{-1/4} \le 3/32 < 1$. Adjoin an independent middle label and remove its two possible endpoint coincidences. Apply the second range of Lemma 32.3 with $s = 4/3, r = 7/5$: $\alpha = 1/8$ and $1025 \cdot 2/p < 1/8$ on the middle domain. This is the actual native three-label fold. Finally the two-layer and folded one-layer emissions repeat each field label twice and four times. Hölder within a repeated label gives the stated physical exponents. Their density endpoints are $12/5$ and $15/7$, both larger than $2\theta_*$. □

32.2. **Concentrated packets: pinning rare constant labels.** Some packets fail the moment bound only through a small part concentrated on constant labels. Those labels can be pinned (fixed), after which the usual square-step rescaling applies. The next lemma gives the numerical condition under which the good part and the pinned bad part together still have the required moment; this condition is the "gate" checked numerically in Section 33.

Lemma 32.5 (Joint block-size moment gate)**.** *Put $\kappa_* = 98861/100000$, $R_* = 12/\theta_*$ and $s_* = \kappa_* R_* = 1694760/144941 \ge 11$. Suppose a positive packet on twelve effective slots has a good part with unit mixed L^{11}, and a bad part with uniform one-slot marginals and relative mass at most $t(\beta)$ whenever every prime of the block supporting the packet is at least β. If Q is the size of the block-product alphabet and $\log Q \le L$, then the packet has unit mixed L^{R_*} provided*

$$t(\beta)\left(1 + \frac{e^{L(\theta_*-1)} - 1}{\kappa_* - e^{-L(1-\kappa_*)}}\right) < 1, \qquad \kappa_* > e^{-L(1-\kappa_*)}. \tag{32.3}$$

For blocks contained in a fixed piece $[X, Y)$, if the block has at most n primes one may take $L = \min(570, n \log Y)$ whenever $(n+1)\log X > 570$ (so that the cardinality bound is the active one). Alternatively, a row of the block histogram checked in Appendix D, with first-prime bin lower endpoint β and product-price bucket k, permits β to be replaced by $\max(X, \beta)$ and $L = 570k/2^{16}$, without changing the block partition.

Proof. The good part has unit mixed L^{s_*} by norm monotonicity. The exact coefficient in the mixture estimate of Lemma 31.3 (the ratio there called the Bregman coefficient in its proof) is increasing in Q. At $Q = e^L$, divide its numerator and denominator by Q. They are respectively at most $Q^{\theta_*-1} - 1$ and at least $\kappa_* - Q^{\kappa_*-1}$, which proves the displayed sufficient gate. Every prime in the piece is at least X, so the strict cardinality inequality gives at most n primes; each is below Y. For the alternative, the existing integer block construction has budget $B = 570\,2^{40}$ and recorded bucket $k = \lceil 2^{16} S/B \rceil$, where its authenticated upper log sum satisfies $\log Q \le S/2^{40}$. Hence $\log Q \le 570k/2^{16}$. The first-bin lower endpoint simultaneously bounds the least prime. Keeping these two coordinates of the same record leaves the blocks themselves and the price

assigned to each block unchanged; only the two bounds β and L used for it are read from its histogram row. □

Lemma 32.6 (The selected packet gates)**.** *For* (32.1)*, the following positive packets, normalized to probability laws, have density endpoint at least θ: each middle generic symmetric or skew envelope, their sum and the CP6 increment; the whole middle nonroot packet; and each upper or tail symmetric, skew or combined norm-envelope packet. Their scalar masses are those assigned in Definition 30.1.*

Proof. All nonconstant middle laws are good at exponent 11 by Lemma 32.2. Write b for the least prime of a middle block and $r = b^{-1/2}$. Before the common reference normalization, the generic packets after both accepting repairs have good and bad masses

$$G_{\mathrm{S}} = [6v_{\mathrm{R}} + 2v_{\mathrm{D}}(f_1 + h_1)]r^3 + w_{\mathrm{R}}g_{\mathrm{S}}(X)r^4, \qquad B_{\mathrm{S}} = 2v_{\mathrm{C}}r^4 + \frac{2w_{\mathrm{C}}r^5}{(1-r^2)^2},$$

$$G_{\mathrm{A}} = w_{\mathrm{R}}g_{\mathrm{A}}(X)r^3, \qquad B_{\mathrm{A}} = \frac{2w_{\mathrm{C}}r^4}{1-r^2}.$$

Here "bad" denotes the uniform one-label reserved diagonal, including both constant tags. In $(1-r^2)^{-1}$ and $(1-r^2)^{-2}$ we may replace r by its largest value on the piece, since both are increasing in r. The coefficients of the 33 pieces give

$$\frac{B_{\mathrm{S}}}{G_{\mathrm{S}} + B_{\mathrm{S}}} \le 57 \cdot 10^{-6}, \qquad \frac{B_{\mathrm{A}}}{G_{\mathrm{A}} + B_{\mathrm{A}}} \le 303 \cdot 10^{-6}. \tag{32.4}$$

These are ratios of assigned envelope masses, not lower bounds deduced from upper operator norms. The common reference normalizer cancels. The symmetric ratio, after division by r^3, has the form $(Ar + B(r)r^2)/(c + dr + Ar + B(r)r^2)$, where $B'(r) \ge 0$. Its derivative numerator is $Ac + 2Bcr + Bdr^2 + B'r^2(c + dr) \ge 0$. The skew ratio is also increasing. Thus a lower endpoint for the least prime gives an upper bound for either bad fraction.

Apply Lemma 32.5. In the first six pieces, retain the first-bin coordinate and the product-price coordinate of the authenticated histogram for block budget 570. There are 979 histogram rows, covering 12115 actual blocks. Grouping equal piece/bin pairs and taking the largest product bucket gives 526 sufficient gates. The other 27 pieces use the cardinality bound in the same lemma. Upward bounds for the larger of the symmetric and skew gates are

X	number of gates	gate upper bound
19951586	87	.999905396
21757358	88	.997725686
23726568	70	.995610104
25429506	19	.993840613
25874006	87	.993514967
28215802	175	.991403303
each later piece	1	.830800905

All are strictly below one. The exact worst gate is at most $7811760900169/7812500000000 < 1$. This proves the moment bound for the two middle packets. Taking convex mixtures then gives it for their sum, for the chosen diagonals (whose moments were verified in the proof of Lemma 32.2), and hence for the increment in (CP6). A smaller θ follows by monotonicity of the norms.

For the whole nonroot packet write $\widetilde{\mu} = \Gamma + \mathcal{M}$, $h = \mathcal{M}(1)$. Its only bad mass is $2v_{\mathrm{C}}r^4$, so its bad fraction is $2v_{\mathrm{C}}r^4/h$. Positivity of the coefficients of h, of degrees one through four, makes this ratio increasing in r. The 33 endpoints give at most $8.305 \cdot 10^{-12}$. The mixture coefficient at $Q = e^{570}$ with (θ_*, κ_*) is below 4200, so this packet also passes the gate. The unmodified two-layer root Γ is not needed in this argument.

For an upper selected diagonal, the one-coordinate multiplicities of types $\mathrm{L}, \mathrm{D}, \mathrm{S}, \mathrm{R}, \mathrm{C}$ are $1, 2, 2, (2, 4), 6$. Multiplicities multiply across the two coordinates and double on the reserved diagonal. Every changed type other than CC contains two distinct actual labels each occurring at least four times. This includes EL and LE of formal degree one; no degree-two assumption is used. Deleted folded types have the same property unless constant. The actual half law is injective and square-affine invariant, so Lemma 32.1 gives unit mixed L^{70}. All skew envelopes are good. The symmetric bad mass is $4w_{\mathrm{CC}}r^6 + 4v_{\mathrm{CC}}r^7$, while the assigned good lower mass includes S_3r^3. Here S_3 is the degree-three coefficient of the complete symmetric comparison polynomial. Use the final weights of Section 27.3.1 and the comparison caps of Section 28.4. For a band beginning at X, among the 310 bands, the exact rational test

$$\frac{4(w_{\mathrm{CC}} + v_{\mathrm{CC}}r_X)r_X^3}{S_3} < 10^{-16}, \qquad r_X = \frac{1}{\lfloor\sqrt{X}\rfloor},$$

bounds its bad fraction; the common normalizer cancels. These tests are included in Certificate 33.3. The block product satisfies $\log Q \le 886$. At density $51/50$, $R = 1200/17$ and $\kappa = 119/120$, the mixture coefficient is bounded by $(120/59)Q^{1/50} < 3^{19}$, because $Q^{\kappa-1} < 1/2$ at the upper endpoint. Thus $10^{-16}(1 + 3^{19}) < 1$.

At a tail atom $p \ge 2^{512}$, count all degree-six and degree-seven deletion mass as bad. The fixed tail coefficients obey $S_3 \ge 1$ and $c_6 + c_7 < 2^{100}$, so its fraction is at most $2^{100}p^{-3/2}$. The same mixture bound is at most $3p^{1/50}$, and the gate is below $2^{102}p^{-37/25} < 2^{-410}$. Every half marginal used is the actual half marginal of the law in question. Tensorization and convex mixing finish the proof. □

Certificate 32.7 (Packet arithmetic)**.** All the numerical inequalities ("gates") in the preceding proof hold: the coarse bad fractions of (32.4) at the 33 middle piece starts, the stronger joint block-size gate (32.3) over the certified block histogram, the bad fraction of the nonroot packet, and the upper and tail gates. Each is a finite check in exact rational or outward-interval arithmetic, replayed as follows. The coarse fractions are replayed by `research/four_routes_2026_10_26/coordinator_review/check_middle_increment_packets.py` with `-merge-repair`. The stronger joint gates are replayed by `research/four_routes_2026_10_26/folded_states/check_middle_density_histogram.py`, with an independent reconstruction in `research/four_routes_2026_10_26/dense_moments/check_middle_histogram_independently.py`. Their rational logarithm intervals use the positive atanh series after dyadic reduction, with its geometric remainder; exponential intervals use the positive Taylor series with a geometric tail. The histogram, bin, part, integer-budget and input/run hashes are authenticated against the unchanged source. The upper and tail gate is replayed by `research/four_routes_2026_10_26/folded_states/check_upper_tail_selected_moments.py`. The smaller-cap source is checked independently by `research/four_routes_2026_10_26/dense_moments/check_middle_cardinality_independently.py`. The bad fraction $2v_{\mathrm{C}}r^4/h$ of the nonroot packet is checked by `research/four_routes_2026_10_26/coordinator_review/check_rootless_completion.py`. These are finite arithmetic certificates for the displayed gates; the analytic implication is proved above. Their source hashes and independent receipts are retained with the final scalar certificate.

32.3. **Sharp estimates with coupled positive spectators.** A changed packet seldom stands alone: unchanged positive factors (spectators) remain coupled to it through the outside reference. The next lemma bounds a signed change, in the sharp form (8.2), in the presence of such a spectator; the word *sharp* refers to that displayed form throughout.

Lemma 32.8 (Coupled spectator estimate)**.** *Let O be a positive tensor factor outside the selected atoms (an* outside reference*) for which the signed reflection Hölder inequality in use in this section (the display in the statement below, in the following notation: x ranges over the selected labels,*

z over the spectator labels, $h_v(x,z,\cdot)$ are functions on the outside alphabet and $a_v(x,z)\ge 0$ are numbers, for every x,z) holds, and which is reflection positive at a balanced cut c (a cut with two halves of 36 *slots). Let K be a selected sharp signed law with bracket constant δ, and let λ_+,λ_- be its two full actual folded laws, including integration of shared labels. Let κ be a positive spectator probability law. Suppose these three laws have unit mixed $L^{d/\theta}$. If*

$$Z_O(h_v(x,z,\cdot))^{1/d}\le b_v a_v(x,z)^{\theta/d},\qquad a_v\ge 0,$$

with b_v independent of the selected labels, then

$$\left|\int \Lambda_O((h_v)_v)\,d(K\otimes\kappa)\right|\le \delta\prod_v b_v\prod_v(\mathbb{E}_{x,z}a_v(x,z))^{\theta/d}.$$

The expectations on the right are uniform in each slot alphabet. No moment hypothesis is required for a fold of κ.

Proof. Condition first on the entire spectator tuple z. Apply the sharp bracket to $K\otimes O$ and retain the geometric mean of its two nonnegative folded energies at that value of z. For the plus energy, outside signed Hölder gives

$$\prod_v b_{f_+(v)}\int\prod_v a_{f_+(v)}(x_v,z_{f_+(v)})^{\theta/d}\,d\lambda_+(x),$$

where f_+ is the fold ϕ_+^c of Definition 5.7 at the balanced cut c, copying the plus half. Apply the full actual moment of λ_+, and do the same with f_-. Shared sharp labels have already been integrated in the full folded laws; no conditional moment assertion is made. Taking the square root, every original slot occurs twice in one branch and zero times in the other. The result is exactly

$$\prod_v b_v\prod_v(\mathbb{E}_x a_v(x,z_v))^{\theta/d}.$$

Only now integrate the original κ and use its moment. Products of sharp atoms from one of the two tags of (CP2)(b) and arbitrary products or positive packets of spectators obey the same proof. Single-cut positivity is used for the bracket, but the full signed Hölder property is separately required and is supplied by the current outside reference. □

Proposition 32.9 (Main terms at the improved parameters)**.** *At a least normalized counterexample with parameters* (32.1)*, every main term used in the first-stage, bracket, class, matching-envelope and wrapping arguments of the construction as selected (the* production *arguments, i.e. those of Section 11 at the baseline parameters) has its existing conductor and norm price, with density exponent θ. Its additive error is at most a constant (depending only on the construction data) times $\|\nu-\lambda\|_{\mathrm{var}}N^{-\gamma\beta/2+72\delta}\alpha$, and hence at most $C\|\nu-\lambda\|_{\mathrm{var}}\mathsf{L}^{-\sigma}\alpha$, where $\|\nu-\lambda\|_{\mathrm{var}}$ is the variation in the baseline comparison before the main term is extracted. The total of these errors over all small selections is $O(\mathsf{L}^{-\sigma}\alpha)$, with a constant independent of N. The clique-prefix errors have the extra prices of Theorem 31.8, each below its existing prefix allowance.*

Proof. The proof of Lemma 8.6 uses uniform variation marginals, lift moments and tail mixing, not $\theta\le 1$. Apply it once to the original selected signed measure, before any Gram step. All its normalized variation marginals are uniform by translation invariance. The square-step main term is then exact. The pointwise fibre argument of Proposition 8.7 up to its final Jensen step remains valid: progression averaging only requires concavity of $x^{\theta/d}$.

Use Lemma 32.8 for each sharp tag. The first-stage positive spectators are the original nonconstant laws of Lemma 32.2; their possible constant images at another tag's cut are never used. The own sharp folds are explicitly included in that lemma. The reserved class expansions use the actual positive half couplings listed there, with their existing absolute coefficient mass. Generic selected increments are the separate symmetric/skew packets of Lemma 32.6. Brackets

add linearly when these packets are formed. The outside reference has the signed Hölder property at every required cut.

The external conductor depth attached to a slot is the same before and after each fold. After the complementary geometric means, a production query index occurs in exactly two original slots. If their fibre-density means are A, B, and $u = \theta/d$, then

$$\mathbb{E}[A^u B^u] \le (\mathbb{E}A^{2u}\mathbb{E}B^{2u})^{1/2} \le (\mathbb{E}A\mathbb{E}B)^u,$$

since $2u \le 1$. Distinct query indices are independent. Random intervals have the mean density of the baseline index law in Definition 9.6; fixed wrapping intervals cost at most $K^{\theta/d}$ per changed slot. This yields exactly the old R_I and the wrapping exponent $2\theta/d$. It does not extend the claim to arbitrary dependent index vectors. Conductor copying likewise retains two occurrences, so every selected block still costs $q_{\mathcal{B}}^{2a}$.

The reserved variance inequality is an exact centered operator inequality. Its upper terms use only the actual Diag_X and J_X couplings and the bracket expansion, all just covered. Spectators outside the aligned F/E classes, grouped into the packets of Lemma 32.6, are integrated last by Lemma 32.8 for these upper bounds. Every selection credited in the existing D_{var} consists entirely of eligible aligned F/E records, with its unchanged λ or bracket outside. A mixed selection containing such a spectator receives the generic bound (no class saving); no variance restoration through that spectator is asserted. The specialized F/E classes retain their raw-arm prices y^{arm} and their savings; no credit is assigned to an individual constituent of such a grouped packet. Odd-skew cancellation persists because the grouped operator remains skew. The CP6 increment is a positive mixture of the same packets and chosen diagonals. The chosen-side and matching-mean expansions use the same actual F companions and E couplings.

It remains to sum the comparison errors before dividing by Y. Retain the first, stronger estimate in Lemma 8.6: a signed measure ρ costs at most $C\|\rho\|_{\mathrm{var}} N^{-\gamma\beta/2+72\delta}\alpha$. The raw variation coefficients need not be summable over tail primes; the summable prices in (CP8) apply after the operator reduction and do not bound these separate errors.

We instead bound the total raw mass of the small selections. For each tail prime p, let b_p be the sum of the absolute raw coefficients times the variations of the local changes, before any Gram reduction. Include all local record types and their finitely many roles in the expansions of Section 11 in this sum. By Section 29.3, degree-zero and degree-one records are kept, and every changed record has degree at least two in $r = p^{-1/2}$. The catalogue and its weights are fixed, differences of probability laws have variation at most two, and the normalizing polynomial $P_\mu(r)$ is at least one. Thus $b_p \le C_1/p$, for a constant depending only on the construction. Unselected infinite-tail reference factors have mass one. Any nonunit masses of the finitely many prefix, middle or finite-upper spectators contribute only a fixed constant, which is absorbed below; for example the middle first-stage law has mass $1 + G(1)$ in the notation of Definition 22.4.

Absorb all prefix, middle and finite upper choices, and all tail primes below $\max(P_5, C_1)$, into a fixed constant. At every remaining prime $b_p \le 1$. For a fixed remaining tail subset T, summing its local record choices therefore gives raw mass at most $\prod_{p\in T} b_p \le 1$. A small selection has $q_E \le N^\tau$, so its remaining tail subset determines a squarefree integer $\prod_{p\in T} p \le N^\tau$. There are at most $\lfloor N^\tau \rfloor$ such integers, including the empty subset. The finitely many types of expansion and the absorbed choices only change the constant. Hence the total raw variation mass of all small selections is $O(N^\tau)$. Apply this counting argument to the actual selections before replacing their finite sums by the completed positive price series.

The summed comparison error is consequently

$$O\big(N^{-\gamma\beta/2+72\delta+\tau}\alpha\big) = O(\mathsf{L}^{-\sigma}\alpha),$$

because $N = K\mathsf{L}$ and the parameters in (S1) satisfy

$$\frac{\gamma\beta}{2} - 72\delta - \tau - \sigma = \frac{2260122363}{58960000000000} > 0.$$

Selections with $q_E > N^\tau$ retain their original post-reduction large-product estimate from (CP8); no small-selection comparison error is assigned to them. At no point is a sharp norm $\|\zeta\|_{\mathrm{sharp}}$ with $\|\zeta\|_{\mathrm{sharp}} < 1$ factored out of a sum of the form (main term) + (variation error): every such sum is bounded term by term before any sharp factor is extracted (the error of Section 34.8). All primes remain either lift coordinates or explicitly pinned external coordinates as in the cancellation proof. □

32.4. **Restoration with an L^2 orbit bound.** Restoration (Section 24) recovers product energy from a diagonal. Carrying an L^2 bound along the orbit of a block improves the recovered lower bound when $\vartheta > 1$; the lemma gives the improved restoration function used in every variance and mean saving below.

Lemma 32.10 (Density-sensitive restoration)**.** *For the middle restoration data C, h, M, χ of Lemma 24.8, put $\ell = \max(0, 1 - (C-1)h)$. At a least normalized counterexample as in Proposition 32.9, a valid restoration lower bound is*

$$\mathcal{F}_2(C, h, M, \chi) = \max\left(0, \frac{\ell}{M}, \frac{\ell - \sqrt{\chi((1+\chi)C^2 - \ell^2)}}{1+\chi}\right). \tag{32.5}$$

It replaces the baseline restoration function of Lemma 24.8 in every saving of Sections 24 to 26 where a restoration lower bound is invoked: the individual and pooled variance credits, the background and class credits, and the matching-mean credits.

Proof. Keep each normalized nonroot packet $\mathcal{M}_{\mathcal{B}}/h_{\mathcal{B}}$ intact when expanding the reference. Its moment is supplied by Lemma 32.6; the remaining root factors in each mixed term have the $L^{23/2}$ moment of Lemma 32.2. Both moments are stronger than density θ_*, so every mixed term is bounded. Thus the root-carrying step of Lemma 24.8 gives the same lower bound $\ell - o(1)$. Write $\nu = (1+u)\lambda$ for the actual root-half law relative to the actual fold-half law on the selected blocks (in this proof only, ν and λ are these two half laws, not the comparison law or the first stage), and $\chi = \mathbb{E}_\lambda u^2$. Define the nonnegative translation-orbit energy $\eta(z)$ as in that lemma, and set $m = \mathbb{E}_\lambda \eta = Y[\Pi_X]/Y$.

For every invariant weight w, Cauchy–Schwarz and Lemma 32.4 give

$$\int |w| \prod_v a_v(z_v)^{\theta/d}\, d\lambda \le \|w\|_2 \prod_v (\mathbb{E} a_v)^{\theta/d}.$$

Apply the averaged comparison with a density factor (Lemma 8.6) first to the invariant signed measure $w\lambda$, whose variation is at most $\|w\|_2$ and which has uniform normalized one-slot marginals. The fibre main term and the displayed inequality give $|\mathbb{E}_\lambda w\eta| \le (C_X + o(1))\|w\|_2$ (with X and C_X as in Lemma 24.8, so $C_X \le C$), uniformly in invariant w. Taking w proportional to η proves $\|\eta\|_2 \le C_X + o(1)$. This is an L^2 bound, not a bound on each translation orbit.

The root contraction and density domination are unchanged, giving $m \ge \ell/M - o(1)$. Centring η in the chi-square comparison gives

$$\ell \le m + \sqrt{\chi}\sqrt{C^2 - m^2} + o(1).$$

Solving this quadratic inequality, or using $m \ge \ell$ when that already holds, gives the third branch of (32.5). Positivity gives zero. The baseline branches based on $\eta^2 \le C\eta$ are not retained. Every later saving uses only this restoration lower bound and its existing exact positive relations, so substituting (32.5) throughout is valid. The function and all finite positive-part/max operations are continuous under a common inflation of the upper energy estimates. □

32.5. **Uniform errors at a least counterexample.** The induction needs its error terms to be $o(1)$ uniformly at a least counterexample, without assuming the bound being proved. The first step is a crude moment bound for Y valid for every A, square-difference-free or not.

Lemma 32.11 (An unconditional reference moment)**.** *For each $\eta > 0$ there is C_η such that, uniformly in the scale and all subsets $A \subseteq [N]$,*

$$0 \le Y(\mathsf{L}, A) \le C_\eta N^\eta \alpha^2.$$

No square-difference hypothesis is needed for this inequality.

Proof. For any fixed native tail graph and any fixed $r > 1$, its uniform injective labeling law has unit mixed field L^r norm for all sufficiently large primes, with threshold depending only on the graph and r. By norm monotonicity it suffices to prove this for $1 < r \le 2$. Here is the needed unit-constant argument. Sequential extension counts give bounded joint density relative to independent uniform labels. The influence estimate for pinned labels (Proposition 13.16) gives maximal correlation $O(p^{-1/2})$, with absolute implied constant, between a single label and the other labels, with their actual marginals. Thus each centered subset form of size at least two has L^2 norm $O(p^{-1/2})$: apply the correlation estimate to one active input, and use the bounded density of the remaining actual marginal to bound the L^2 norm of their product. Its centered L^1 kernel is bounded by a fixed constant, since centering is the alternating sum of subset marginal densities. Interpolation gives centered L^r norm $o(1)$. For nonnegative normalized inputs write $f_i = m_i + h_i$ and $v_i = \sqrt{r-1}\|h_i\|_r$. Uniform convexity gives $m_i^2 + v_i^2 \le 1$. Each nonconstant term in the product expansion is at most $2(1 - \prod m_i)$, while its coefficient tends to zero. There are finitely many terms, so the energy is at most one for all sufficiently large primes. Singleton centered terms vanish exactly. Absolute values cover complex inputs.

Use this fact with $r = 11/10$ for every graph in the finite tail catalog. The physical maximum multiplicities for a coordinate type are $1, 2, 2, 4, 6$, with formal degrees $0, 1, 1, 2, 3$. Two-coordinate multiplicities multiply, and a folded component doubles its multiplicity and increases its degree by one. A component with maximum multiplicity m therefore has unit mixed $L^{11m/10}$, unless this exceeds 72, in which case ordinary Hölder gives L^{72}. Any component whose exponent exceeds 36 has formal degree at least six: the only relevant largest multiplicities are $36, 48, 72$, from two-layer CC and folded RC, CR, CC. Their masses are $O(p^{-3})$ or smaller. The native root has unit mixed exponent $11/10 < 36$ eventually. The typed-mixture bound of Lemma 31.3, targeting exponent 36, charges a bad component of exponent $v \le 72$ by $O(p^{2-72/v})$, at most $O(p)$. Its coefficient times its mass tends to zero. Hence the full tail reference has unit mixed L^{36} eventually.

All remaining coordinates take values in one finite set independent of the scale and of A. Ordinary Hölder and finite-space norm comparison give them a finite mixed L^{36} constant, including the changed clique prefix. Tensorizing with the unit tail gives $Y \le C(\mathbb{E}|F_A|^{36})^2$. Lemma 6.7 at order 36 gives $\mathbb{E}|F_A|^{36} \le C_{\eta/2} N^{\eta/2}\alpha$. Reflection positivity gives the lower bound. □

Theorem 32.12 (The upgraded induction)**.** *Use the construction and parameter range* (32.1)*, with every restoration saving recomputed by* (32.5)*, with the packet grouping above, and with the same finite class gates as in the baseline. Suppose the resulting Θ, computed by the formula below, is strictly below one. Then, with constants C, C' depending only on the construction data and on Θ,*

$$Y(\mathsf{L}, A) \le C\mathsf{L}^{-a}\alpha^\theta, \qquad s(N) \le C' N^{1-a/(72-\theta)}.$$

The closing formula is the baseline comparison formula with the new record prices, the new restoration savings, and wrapping term

$$Be^{S+A}\left[\left(1 + \frac{R_I}{2K^{1-2\theta/72}}\right)^6 - 1\right].$$

Equivalently, Θ may be computed by the joint closing of Corollary 11.2, provided its residual coefficients and robustness conditions are verified.

Proof. The comparison identities, cut orders, centered variance identities, odd-skew cancellation and class algebra are unchanged. Proposition 32.9 and Lemma 32.10 supply every analytic upper and lower bound used in their robust recurrence. The finite class gates ensure the same continuous inflated closing $\Theta(z)$, with $\Theta(0) < 1$.

At a least counterexample $Y > C_0 \mathsf{L}^{-a}\alpha^{\theta}$, use Lemma 32.11 with $\eta = 1/10000$ to obtain

$$\alpha \geq c C_0^{1/(2-\theta)} \mathsf{L}^{-(a+\eta)/(2-\theta)}.$$

An original density-linear error $O(\mathsf{L}^{-\sigma}\alpha)$, divided by Y, is consequently $o(1)$ uniformly for $C_0 \geq 1$, since

$$a + (a+\eta)\frac{\theta-1}{2-\theta} \leq \frac{91}{100000} + \frac{101}{100000}\frac{14587}{985413} < \frac{1}{825} = \sigma, \qquad \text{with margin } \frac{943271}{3284710000}.$$

This applies to the summed errors from Lemma 8.6, from cancellation and from the large-product remainders. Their constants are fixed independently of C_0. Choose a scale beyond which these relative errors and the square-step rounding error fit the strict scalar margin. At the finitely many earlier scales, $|F_A| \leq n_{\mathsf{L}}\alpha$ gives $Y \leq n_{\mathsf{L}}^{72}\alpha^{72} \leq n_{\mathsf{L}}^{72}\alpha^{\theta}$; enlarge C_0 to initialize the induction. The least failing scale would satisfy $Y \leq \Theta(z)Y < Y$, a contradiction. Finally $\alpha^{72} \leq Y$ gives $\alpha^{72-\theta} \leq C\mathsf{L}^{-a}$, and the usual interval embedding gives the claimed bound for every N. □

33. Improved closing: $\Theta < 0.999999771$ with $\vartheta = \frac{1014587}{1000000}$

This section verifies the price hypotheses of Theorem 32.12 at $a = \frac{1814179}{2000000000}$, $K = 863660$, $\vartheta = \frac{1014587}{1000000}$, and thereby proves Theorem 1.1. We first account for the normalizing factors of the positive reference laws, then assemble the improved local bounds, and finally evaluate the joint comparison and mean inequality. The quantities $L_0, S, A, D, \mathcal{E}, T, R_I, J, W$ have the meanings given in Definition 30.1; their assigned numerical bounds are collected in a table below.

Two points govern the assembly. First, a middle reference law is divided by its total mass $Z_i(r) > 1$. Its comparison prices carry this reciprocal, whereas the two positive matching additions retain their full prices. Lemma 33.1 accounts for both through a common assignment and the reserve R_Q. Second, the higher mean contributions are used inside the joint bound of Corollary 11.2, through the corrections proved in Section 26.6. Their subtraction is justified before replacing energies by numerical upper bounds. The density induction uses the wrapping factor $(R_I/2)K^{-1+\vartheta/36}$. Appendices D and E record the baseline data; the assignments selected for the final theorem are specified here.

33.1. Accounting for the normalizing factor Z_i^{-1}.

On each middle piece the reference is defined as a sum of records divided by its total mass $Z_i(r) > 1$. The baseline prices of Definition 30.1 were partly stated for the unnormalized sums; wherever the comparison uses the normalized laws, the correct price is the baseline price multiplied by $Z_i(r)^{-1} < 1$. This subsection defines an explicit rational upper bound c_i for $Z_i(r)^{-1}$ on the whole piece, using only integer floor and ceiling operations, and the following lemma states the consistent reassignment of all affected prices. For a prime b with $X_i \leq b < Y_i$, put $r = b^{-1/2}$. The mass of the unnormalized positive reference is

$$\begin{aligned} Z_i(r) = 1 &+ (3w_{\mathrm{D},i} + 2w_{\mathrm{S},i} + v_{\mathrm{L},i})r + (6w_{\mathrm{R},i} + 3v_{\mathrm{D},i} + 2v_{\mathrm{S},i})r^2 \\ &+ (2w_{\mathrm{C},i} + 6v_{\mathrm{R},i})r^3 + 2v_{\mathrm{C},i}r^4. \end{aligned}$$

All coefficients are nonnegative. The following explicit rational numbers therefore provide simultaneous majorants on the entire piece:

$$r_i^- = \frac{\left\lfloor \sqrt{\lfloor 10^{120}/Y_i \rfloor} \right\rfloor}{10^{60}}, \qquad c_i = 10^{-30} \left\lceil \frac{10^{30}}{Z_i(r_i^-)} \right\rceil, \qquad Z_i(r)^{-1} \le c_i < 1. \tag{33.1}$$

The floor and ceiling are integer operations. In particular no numerical square-root approximation is needed to define c_i.

Lemma 33.1 (Consistent normalized prices)**.** *Define new assigned prices from those of Definition 30.1 as follows. Every norm price and raw-arm price in the middle row* (23.7) *is multiplied by c_i. For each first-stage total $T \in \{U_0', P', D', V'\}$, let t_i be its exact certified upper charge in piece i, and replace T by*

$$T^{\mathrm{new}} = \left(T - \sum_i t_i \right) + \sum_i c_i t_i. \tag{33.2}$$

The first parenthesis is its retained nonnegative rounding slack. Replace the invariant-background assignment by $U^{\mathrm{inv}\,\mathrm{new}} = \sum_i c_i U_i^{\mathrm{inv}}$ and the one-tag symmetric assignment by $s^{\mathrm{new}} = \sum_i c_i s_i$, where s_i is the assigned symmetric folded-tag price in that piece. Use

$$\delta_{\mathrm{s}}^{\mathrm{new}} = D'^{\mathrm{new}}/2 - s^{\mathrm{new}}$$

in every orbit saving, and recompute the full tagged series and every class polynomial from these assignments.

Keep the chosen F matching-side price T_K unchanged. If $S_{E,i}^{\mathrm{old}}$ and $A_{E,i}^{\mathrm{old}}$ are the unnormalized assigned matching prices, use

$$T_{QE} = t \sum_i (S_{E,i}^{\mathrm{old}} + A_{E,i}^{\mathrm{old}}) = S_E^{\mathrm{new}} + A_E^{\mathrm{new}} + R_Q, \quad R_Q = t \sum_i (1 - c_i)(S_{E,i}^{\mathrm{old}} + A_{E,i}^{\mathrm{old}}), \tag{33.3}$$

where a is the target exponent, a_0 is the middle anchor exponent, and $t = e^{2(a-a_0)570}$. Thus R_Q is added to the closing quantity T of Definition 30.1 and to the exponent in the mean-error factor $R_{\mathrm{err}} e^{T_K + T_{QE}}$. These assignments preserve all comparison, class and matching inequalities.

Proof. Each middle comparison difference, each of its arm coefficients, and each first-stage record has the common physical factor $Z_i(r)^{-1}$. Its former assigned price omitted this factor. Equation (33.1) consequently bounds every term after multiplication by c_i. This is a simultaneous coefficientwise bound on the full positive majorant polynomial; the leading, remainder and full arm prices are changed together. The assigned slack in (33.2) is nonnegative and remains available. The two folded tags have equal coefficients, and retain equal sharp and variation prices after this operation.

The invariant and generic pieces of the first-stage bracket are changed with the same factors. The exact checks below give $0 \le U^{\mathrm{inv}\,\mathrm{new}} \le U_0'^{\mathrm{new}}$ and $0 \le s^{\mathrm{new}} \le D'^{\mathrm{new}}/2$. Consequently the invariant-background and one-delta reductions are the same algebraic reductions of this new positive majorant as in Definition 30.1(6),(7). Keeping their former, larger coupons while lowering only the generic total would not justify the subtraction; the simultaneous assignments avoid that error.

The physical reference and its actual half marginals are unchanged, as are the quantities entering restoration, namely the likelihood and chi-square bounds, the piece product bounds and the chosen matching-side coefficients. In particular, the chosen $K_{\mathrm{L}}, K_{\mathrm{R}}$ and Q_E of Section 26 are unnormalized positive additions. Their former prices remain valid. Equation (33.3) charges the entire old Q_E price while using the smaller comparison price elsewhere. The old direct F matching price and its positive increments are also retained. This proves the matching and mean-error assertions. All restoration and class gains are then evaluated using the same new price assignment, so no term is counted using incompatible generic and discounted prices. □

Lemma 33.2 (Upper-band normalization)**.** *For a band $[X_i, Y_i)$, let $P_{\mu,i}$ be the positive mass polynomial of Definition 27.5. Write $d_{\mathrm{L}}, d_{\mathrm{D}}, d_{\mathrm{S}}, d_{\mathrm{R}}, d_{\mathrm{C}} = 0, 1, 1, 2, 3$ and $m_{\mathrm{L}}, m_{\mathrm{D}}, m_{\mathrm{S}}, m_{\mathrm{R}}, m_{\mathrm{C}} = 1, 3, 2, 6, 2$. Thus, for the 25 ordered states uv with $u, v \in \{\mathrm{L}, \mathrm{D}, \mathrm{S}, \mathrm{R}, \mathrm{C}\}$,*

$$P_{\mu,i}(r) = \sum_{u,v} m_u m_v (w_{uv} r^{d_u+d_v} + v_{uv} r^{d_u+d_v+1}).$$

Define

$$r_i^- = \frac{\left\lfloor \sqrt{\lfloor 10^{200}/Y_i \rfloor} \right\rfloor}{10^{100}}, \qquad c_i^{\mathrm{u}} = 10^{-30} \left\lceil \frac{10^{30}}{P_{\mu,i}(r_i^-)} \right\rceil.$$

Then $P_{\mu,i}(b^{-1/2})^{-1} \le c_i^{\mathrm{u}} \le 1$. Multiplying each band's assigned symmetric and skew prices by c_i^{u} also gives valid, simultaneously smaller generic matching prices. The tail prices and chosen middle matching diagonals are unchanged.

Proof. The reference polynomial has constant term one and nonnegative coefficients. The same monotonicity argument as in (33.1) proves the stated bound. In the proof of Proposition 28.14, every repair, deletion and skew envelope is divided by this common polynomial before that denominator is discarded in the displayed prices. Retaining its reciprocal upper bound is therefore valid for every positive coefficient of the three comparison polynomials. The upper generic matching envelope is dominated by the sum of their symmetric and skew masses. The same finite upper diagonal assignments are used in the joint mean allocation of Proposition 26.10. This normalization uses the weights already selected in Section 27.3.1; it does not make any further change of reference law. The common factor cancels in the packet mixture gates. □

33.2. **The selected laws and common assignments.** Fix

$$a = \frac{1814179}{2000000000}, \qquad \vartheta = \frac{1014587}{1000000}, \qquad K = 863660, \qquad \varepsilon = \frac{a}{72-\vartheta} = \frac{1814179}{141970826000}.$$

The 72 slots, six queries, 33 middle pieces, cutoff $Q_3 = 19951586$, upper endpoint $P_3 = 167 \cdot 2^{40}$, 310 finite upper bands, local coefficient bound 585, block budget 570, and external depth 64 remain as specified in the construction. The complete middle portfolio is the one specified in Sections 20 and 21.

Use the independent-draw prefix of Theorem 31.8, the accepting middle constants and merges of Lemmas 23.20 and 23.23, and the new finite upper comparisons of Section 28.4, with the complete upper reference reweighting of Section 27.3.1. Each component keeps its original target marginals. The density-moment bounds apply componentwise and hence to these positive mixtures. The finite upper changes preserve acceptance of all six queries.

The middle charge anchor is $a_0 = \frac{8896611739}{10000000000000}$. Its exact histograms and analytic prime-counting cells give charges of degrees $2, 3, 4$. Reprice every middle charge by $t = e^{2(a-a_0)570}$; this uses the certified maximum block product, not an average length. Each upper term retains its own certified product-log bound and anchor. Multiply the middle and upper assignments by their normalizing factors, using the new full upper reference polynomial, as in Lemmas 33.1 and 33.2. For each affected upper state replace its entire symmetric and skew coefficient by the corresponding aggregate cap in Propositions 28.7 to 28.9; for LR and RL subtract exactly the rebates in Proposition 28.10. Every affected charge is then resummed. The other upper coefficients retain Lemma 28.15. This changes S, A, T and all uses of those totals together.

Every restoration is recomputed at a with the density-valid bound

$$\mathcal{F}_2(C, h, M, \chi) = \max\left\{0, \frac{\ell}{M}, \frac{\ell - \sqrt{\chi((1+\chi)C^2 - \ell^2)}}{1+\chi}\right\}, \qquad \ell = \max\{0, 1-(C-1)h\}, \tag{33.4}$$

proved in Section 32. This includes the selected, background and pooled variance gains. All sixteen middle envelope classes, their invariant refinements, and their orbit coupons use the same normalized raw-arm and outside assignments.

For the mean keep only the terms supported on one block in Proposition 26.7, whose sum is denoted $m_{\mathrm{nat},1}$. Replace that proposition's higher-degree mean credits by Propositions 26.9 and 26.10, whose four middle families and nonempty upper family are disjoint. Set

$$m_1 = 1 + m_{\mathrm{nat},1} - R_{\mathrm{err}} e^{T_K + T_{QE}}, \qquad \Delta_{\mathrm{mid}} = \Delta_\alpha + \Delta_M + \Delta_O + \Delta_K.$$

The separate chosen-Q_E reserve of (33.3) is included in both T and the mean error exponent. The upper-triggered allocation uses only the finite upper assignments; all prefix and tail padding remains in the positive costs, matching and wrap.

33.3. **The joint closing number.** The finite conductor moment is evaluated at the selected (a, K) using Proposition 29.7 and Corollary 29.8. Put

$$\mathcal{E} = B(e^S \cosh A - 1) - D, \qquad J = Be^T (R_I/2)^6,$$
$$W = Be^{S+A}\big[(1 + (R_I/2)K^{-1+\vartheta/36})^6 - 1\big].$$

The joint comparison and mean bound proved above gives

$$\Theta = \frac{63}{64}(L_0 + \mathcal{E}) + J + W + \frac{1 - m_1}{64} - \Delta_{\mathrm{mid}} - \Delta_u + \mathcal{R}_u. \tag{33.5}$$

The two negative allocations are justified inside the common positive completion. They are not additional independent instances of (CP7). This is the joint criterion of Corollary 11.2.

TABLE 13. The improved closing ledger. Bounds are rounded outward from the complete enclosures; the closing uses the enclosures themselves.

quantity	bound	quantity	bound
B	< 2.373803797078	L_0	< 0.000507024733
S	< 0.008981913739	A	< 0.985245889185
T	< 0.610594125990	D	> 0.495667577473
$\mathcal{E}$	< 0.785417906625	m_1	> 1.529526120248
R_I	< 1.252937308029	J	< 0.264252164235
W	< 0.000041043577	D_{var}	> 0.009861452145
Δ_α	> 0.007949257298	Δ_M	> 0.013289595166
Δ_O	> 0.006920381923	Δ_K	> 0.000788314597
Δ_u	> 0.000716967431	$\mathcal{R}_u$	< 0.000000070807

Certificate 33.3 (The final joint closing)**.** The construction just specified has

$$\Theta < 0.999999771 < 1, \qquad 1 - \Theta > 2.29 \cdot 10^{-7}.$$

The reproducible entry point is `research/epsilon_2026_10_29/verify.py`. Its author assembly and independent complete assembly give intersecting outward enclosures with value $0.9999997708753344\ldots$. The independent assembly reconstructs all 33 middle price rows, 310 upper normalizers and comparison polynomials, all restoration calls, the chosen-matching reserve, the finite conductor moment, the physical resource ratios, and the 192 shared residual gates. It computes (33.5) directly. Both assemblies use the same completed reference reweighting; its independent local review checks 602 root bounds and 32 550 completion inequalities.

The new local checks independently reconstruct all sixteen soft comparison graphs and complete character partitions, their exact leading matrices, scalar masses, count orders, repairs and tensor

gates; all fourteen dense law/mixture matrix tests and emission rebates; and the complete SS and LC/CL scalar proofs. The exact analytic, prefix and density gates hold on $0 < a \le 91/100000$. The local and global records, with source hashes, are in the `certificates/` and `verification/` subdirectories of `research/epsilon_2026_10_29/`. Numerical verification establishes these finite assertions; the positive-energy allocation and operator reductions are the proofs in the preceding sections.

Completion of the proof of Theorem 1.1. The actual reference and its retained marginals satisfy the moment catalogue and packet bounds of Section 32 at $\vartheta = \frac{1014587}{1000000}$. The new accepting upper laws are repaired to those same marginals, so their positive diagonal reductions use the same catalogue. All generic selected packets have the required unit mixed moment; the stronger weighted restoration moments are checked for the actual middle fold laws. The first-stage assignments, chosen matching prices, and complete upper comparisons are compatible by the preceding construction.

The joint mean proof has strictly positive residual coefficients. Consequently its completed positive series has the same robust inflation and the same two analytic error types as the original closing proof. The induction of Theorem 32.12, in its joint form, therefore applies. Its extended analytic domain is valid: the relative-error margin at $a = 91/100000$ is $943271/3284710000 > 0$, and the prefix plus tail is below the assigned 10^{-12} in every channel. The strict closing is Certificate 33.3.

Initialization at the finitely many small scales and continuity in the inflation parameter give, for every square-difference-free $A \subseteq [K\mathsf{L}]$ of density α,

$$Y(\mathsf{L}, A) \le C\mathsf{L}^{-a}\alpha^{\vartheta}.$$

Reflection positivity gives $\alpha^{72} \le Y(\mathsf{L}, A)$; for $\alpha > 0$, division and the $(72-\vartheta)$th root yield $\alpha \le C'\mathsf{L}^{-a/(72-\vartheta)}$. The fixed relation $N = K\mathsf{L}$ and interval rounding change only the absolute constant. Hence $s(N) \le C''N^{1-\frac{1814179}{141970826000}}$ for every $N \ge 1$. □

34. How the construction developed

What this section is for (not used in the proof). This section is historical only: no proof in Parts I–V depends on it, and a reader verifying Theorem 1.1 can skip it entirely. It records how the exponent was reached through complete proofs of smaller exponents. Terms such as "checkpoint" mean one such complete version; dates in subsection titles are retained as labels for those versions and each subsection states in words what changed, so no archived draft need be consulted. The checkpoint table ends with the baseline ($\vartheta = 1$, the $\theta = 1$ of Section 30); the improved result $\vartheta = \frac{1014587}{1000000}$ of Sections 31 to 33 is described in prose at the end of this section and is not a further table row. Section 34.2 reads the design on 216 slots, on which the first explicit versions and the first five checkpoints were built, as the first instance of the scheme of the comparison theorem of Part III and groups its inherited machinery by its role in the current proof: the final proof is that design re-instantiated on 72 slots, with four structural substitutions and new local laws, not a different proof. Sections 34.3 to 34.6 follow the steps from there to the final design, and Section 34.8 collects what they taught; the obstructions met on the way are in Section 35. No proof in Parts II–V uses this section. "Old §n" is a section of the archived manuscript of 644 pages at commit `a6e4c29`, preserved with its hash manifest in `archive/paper-purepair-2026-10-24.tar.gz`; its proofs and verifiers remain reproducible (Section D.6, Table 22). "Master §n" is a section of the manuscript of 770 pages at master commit `8dc0a3fe`, which appends the three checkpoints of 24–25 October (master §§42–44) to that text; it is preserved in `archive/paper-further-2026-10-25.tar.gz`. Throughout, ε is the density saving of the checkpoint in question, a its functional exponent, θ its density exponent and d its number of slots, so that $\varepsilon = a/(d-\theta)$; the archived text often writes c for ε.

34.1. **The checkpoints.** The OpenAI preprint [52] proved $s(N) \le CN^{1-\varepsilon}$ with an absolute $\varepsilon > 0$ from a functional on $d = (24!)^{161}$ slots. The first complete explicit version of the present argument [53], already on the 216-slot array, gave $\varepsilon = 2.939 \cdot 10^{-15}$, and its rewritten version $2.5115 \cdot 10^{-9}$. Table 14 lists the sixteen checkpoints that followed those two versions (the two versions just named are not rows of the table) and Table 15 what each added. Every closing in the sequence has the three tasks described in Section 3; the closing numbers of different rows are nevertheless values of different criteria, comparable only through their margins below one; the slot counts, query counts and criteria of a row belong to its own design, and no constant of one row may be substituted into the criterion of another without the checks of that criterion.

TABLE 14. The checkpoints: d slots, k queries (edges amplified by repetition in the signed product; the first design couples nine pairs, one family at a time), density exponent θ, functional exponent $a = (d - \theta)\varepsilon$, and Θ the certified upper bound for the closing quantity of the checkpoint's own criterion. The last column is the section in which the checkpoint is proved: old § of the archived 644-page manuscript for rows 1–11, master § of the 770-page manuscript for rows 12–14; rows 15–16 are the Codex improvements and the density-linear baseline integrated here; the verifiers are listed in Table 22.

	ε		d	θ	k	$\Theta <$	§
1	$3347/10^{12}$	$3.347 \cdot 10^{-9}$	216	0	9 pairs	0.999929	§§27–28
2	83/2000000000	$4.15 \cdot 10^{-8}$	216	0	16	0.979927	§29
3	269973/5376500000000	$5.0214 \cdot 10^{-8}$	216	47/50	16	0.999580	§30
4	1/8000000	$1.25 \cdot 10^{-7}$	216	108/125	4	0.988050	§31
5	513/2242156250	$2.288 \cdot 10^{-7}$	216	753/1000	16	0.983739	§32
6	2214/1111328125	$1.992 \cdot 10^{-6}$	72	7/8	8	0.997534	§§33–35
7	1971/635562500	$3.101 \cdot 10^{-6}$	72	817/1000	8	0.998934	§36
8	63/8927500	$7.057 \cdot 10^{-6}$	72	29/50	6	0.957300	§37
9	441/44375000	$9.938 \cdot 10^{-6}$	72	1	6	0.995315	§38
10	276327/27734375000	$9.963 \cdot 10^{-6}$	72	1	6	0.999998993	§40
11	243319543/22187500000000	$1.0967 \cdot 10^{-5}$	72	1	6	0.999999999410	§41
12	4066280877/355000000000000	$1.1454 \cdot 10^{-5}$	72	1	6	0.999999999794	master §42
13	8855891201/710000000000000	$1.2473 \cdot 10^{-5}$	72	1	6	0.999999999862	master §43
14	887239883/71000000000000	$1.2496 \cdot 10^{-5}$	72	1	6	0.99999999966289	master §44
15	1110464567/88750000000000	$1.2512 \cdot 10^{-5}$	72	1	6	0.99999999994235	Codex 25 Oct.
16	5422587/433349609375	$1.2513 \cdot 10^{-5}$	72	1	6	0.99999999997327	Section 30

Table 15: What each checkpoint of Table 14 added (the new idea in italics), and why it was replaced.

	Construction	Why it was replaced
1	row completion (old §18: completing a law from its conditional rows), component-table terminals (the terminal components $\mathcal{T}$ of the component table, Section 34.2 below), obstruction cover (covering nonaccepting records by positive laws, Section 35); *centred singleton recurrence: the common baseline is paid once*	nonaccepting records are covered, not repaired; one coupled pair has first-order obstruction polynomial $2x + 2y + 4z$ (x, y, z as in Section 34.2)
2	one fixed family; *repairs keeping the actual conditional marginals, a signed product over* 16 *disjoint edges*	sharpened by the same method
3	L^2 row marginals, a cubic trace in the first repair, a graded upper prefix; *the density-normalized induction,* $\theta > 0$	graph-transfer term 0.438 and matching term 0.363 of that row's Θ: "new local laws rather than new parameters"
4	*a sparse middle law: one added constraint per nonaccepting record, at one cut*; four amplified edges	with four edges the matching term is 0.823

	Construction	Why it was replaced
5	sixteen edges again; *each expansion term at its best cut*; trace-moment marginals	graph comparison 0.713; the conversion $a/216$
6	*a two-point layer replaces the third chamber coordinate*; prime classes; a two-layer upper reference	refined in row 7
7	second-order upper-root inputs, product rows; upper start 2^{50}	lower graph comparison 0.330, matching 0.586
8	square-clique prefix, twelve-label middle law; *a comparison law that is not reflection positive, under a reserved Gram bracket*; six queries	reserved comparison 0.793; the density gate nearly tight
9	*mirror pairing* with an orientation mixture, variance credits, direct Gram, *common mean*, coupons, direct-star root from 2^{49}, $\theta = 1$, kernel cutoff $\beta = 1/20$ (Appendix C)	index and square step improvable; a new F law
10	*exact square-shift contraction*, least square step, fixed K, ordered upper coincidences	kept, and extended, by row 11
11	*pure paired F law with positive joint envelopes for the even skew products*; fresh restorations; $m_{\mathrm{cl}} = 8192$	the pure-law constants could be lowered only to their proved limits 2 and $\sqrt{5}$ (row 12); row 13 changed the law
12	*exact identities in the actual half marginal* (pair laws, the actual frame); channels *collected with their coincidence substitutions before norms*; raw-arm, leading, remainder and full prices kept apart; weighted feature frames; positive backgrounds, variance of degrees three and four, chosen diagonals in the mean, late-pool coupons; modular root variance; fine upper bands, log-convex root coefficients, a free Young allowance, skew-only repairs, forest matrices; continuous index moments; a fresh native source	the pure law at its sharp limits 2, $\sqrt{5}$; the direct-Gram coefficient $\sqrt{6}$ sharp for scalar half diagonals
13	*signed whole-block law*: a signed mixture, after the block products, of three positive accepting laws repaired to the actual marginal; the *bracket with one fold copy* and the doubled protection; the *two-cut error*; matrix envelopes in the singleton block algebra, physical averages for all classes, orbit coupons; piece product caps; complete actual stars and the direct chamber cap ($\sqrt{30}$); dense upper comparisons, six-query acceptance, observed caps, joint F/E rows; $Q_3 = 19951586$, $P_3 = 167 \cdot 2^{40}$	the invariant price estimated by $U_0 - S/4$ instead of assigned exactly; the root still uniform
14	*positive pair-trace tilts of the root* on three late pieces, both cuts in the fixed half charts, the changed marginal priced in restoration; the *assigned invariant price*; a positive mixture of six-query LE laws; forced free-root coincidence factors; free Young allowances and fresh endpoints of the joint rows	conductor-depth and root refinements in rows 15–16
15	*seven positive roots*: the collision, pair-gap and layer-remainder bounds and the coherent 39-score root of Section 21; *finite conductor-depth index*	closer repricing anchor available
16	*exact block charges rebuilt at a closer anchor*, with the same selected local laws and depth index	selected construction

From the first to the last row ε grew by a factor of about 3739. The slot count accounts for a factor of about 3.04, through $d - \theta$ falling from 216 to 71; the rest, a factor of about 1229, is the growth of the functional exponent a, which is limited by the requirement $\Theta < 1$. Every price grows with a through $q_{\mathcal{B}}^{2a}$, so each step lowered a local constant, added a credit, or replaced a local law by a cheaper one.

34.2. **The 216-slot starting point.** The first five checkpoints used $(\mathbb{Z}/6\mathbb{Z})^3$, nine families of rotation edges, and a third-coordinate chamber cut reserved for comparison. Their middle laws were fixed samplewise by that cut. The current 72-slot law depends on the layer coordinate instead,

so its reserved Gram bracket replaces samplewise fixedness. The inherited analytic and local machinery is stated in its current generality in Parts II–IV; the old constructions are not additional premises of the final theorem.

The early middle reference combined a leading array law, row and column repairs, and a terminal completion. Its weights were negative spectral envelopes. This is the origin of the five-type completion and its slice argument. Pinned correlations supplied the envelopes; Peyre's tensorization and Kassabov's angle bounds supplied their stability. The largest early losses were first-order errors in row counts and in four-cycle character kernels. Pivot averaging and second-order character estimates removed those losses where the variables remained free. They could not remove errors in variables already fixed by conditioning.

The old upper graph had 216 vertices and degree 74. Its repair was consequently expensive even after the first spectral improvements. The two-layer graph lowered these parameters to 72 and 20. The comparison mechanism survived: assign positive prices to records, compare their Fourier averages at square-step scales, use the six query couplings, and close a single strict scalar inequality.

TABLE 16. The inherited machinery and its place in the present proof. The detailed section-by-section history is preserved in the archived manuscripts.

Inherited argument	Present formulation
Sorting, positivity, lifts and sectors	Sections 5 to 8
Local counts and correlations	Sections 12 to 14
Second-order character sums	Section 15
Witness extraction and coincidences	Section 16
Completion, polarization and marginal repair	Section 17
Price comparison and density induction	Theorems 9.21 and 32.12

34.3. **Sparse laws, single edges and best cuts.** The sparse middle law kept 180 of the 252 edges of the one-layer graph. Early repairs imposed one square constraint at a time, and later versions compared each expansion term at its best available cut. Eligibility at several cuts belongs to one fixed signed difference; it does not license taking coordinatewise minima from different laws. The weighted moving-root moments and the separation of marginal errors from centred correlations remain in the current proof. The sparse constructions themselves are historical checkpoints.

34.4. **Two layers, six queries and the October 18 record.** Replacing one six-point coordinate by a two-point layer reduced the number of slots from 216 to 72. With density exponent θ, the conversion became $\varepsilon = a/(72-\theta)$. The middle construction then needed positivity and comparison at a genuine layer cut. The twelve-label root and its fold, matching, merge and constant records supplied that completion.

Six queries reduced the number of independent index factors in the matching term. The October 18 record then combined exact block budgets, paired middle comparisons, a direct-star upper root, and a common-mean credit. The crude norm sum still exceeded one; the class envelopes, variance credits and matching comparison were essential. Their roles are explained by Remark 35.1, without reproducing that checkpoint's proof here.

34.5. **Blocks and the 72-slot form.** The next version used the exact square-shift norm $\sqrt{p+1}/(p-1)$, the least square divisible by the conductor, and a fixed interval count with a fully paid wrap term. Ordered collision counts sharpened the upper root. The pure-pair construction then moved the first-order fold price to the skew part. Positive envelopes for even skew products

more than paid for the resulting skew norm. Its variance and mean credits were recomputed together at the new exponent; they were not imported as separate numerical decrements.

TABLE 17. Structural changes retained in the final construction.

Component	Retained change
Middle comparison	Actual-marginal signed whole-block law; Definition 23.9
Reserved comparison	One fold copy and a two-cut error; Lemma 22.7 and Corollary 10.9
Middle root	Complete star frame and positive trace-square tilts; Section 21
Upper comparison	One-way repairs, forest and dense accepting laws; Section 28
Closing	Compatible variance, matching and common-mean credits; Section 33

34.6. **Exact actual laws, signed blocks and positive roots.** The subsequent improvements kept the 72-slot frame and changed its local laws or the estimates for them. Computing the fold operators in their actual half marginals, and collecting their coincidence substitutions before taking norms, brought the pure-law coefficients close to their limiting values. Signed mixtures taken *after* block products then reduced the leading fold coefficient without paying variation once per prime. The selected parameter is $\mathtt{t} = 127/500$.

The complete chamber star matrix exposed the uniform root's leading coefficient $\sqrt{30}$. Positive squares of pair-trace scores lowered that coefficient on later pieces. Their finite proofs had to control both cuts, full rows, normalizers and the actual marginal changes. The last selected piece uses the coherent 39-trace law; the seven selected laws and parameters are in Definition 21.1. Smaller local norms alone would not justify these replacements.

In parallel, actual-marginal upper comparisons, exact collision pricing and finite conductor moments improved the other parts of the same scalar budget. The density-linear baseline is $\varepsilon = \frac{5422587}{433349609375}$. The selected theorem further uses accepting constant and merge comparisons, consistent normalizers and density exponent $\vartheta = \frac{1014587}{1000000}$, together with the October 29 local refinements described next. Its complete closing is Certificate 33.3, yielding $\varepsilon = \frac{1814179}{141970826000}$.

34.7. **The October 29 local refinements.** The Gram estimate for the combined centred rectangle gives the centred cap $121.588/p$ (Theorem 20.19), replacing a sum of individual nonstar bounds. The cyclic-triangle remainder has limiting coefficient 140, and inserting an opposite-base-pair score into the soft kernel is bounded by $2897q$ (Lemma 21.8). Sign-incompatible collision strata are removed before extension counts (Lemma 16.19). The squarefree-cubic resultant identity also gives a complete three-label witness bound (Theorem 15.12). These estimates are proved in their respective sections, not as modifications appended to the closing.

The final reconstruction keeps the complete existing middle portfolio and allocates its positive mean jointly with the comparison (Section 26.6). It repairs fresh six-query SS and constant emissions to the old full marginals, compresses the retained dense comparisons, and bounds all sixteen small soft comparisons by exhaustive two- and three-coordinate character partitions (Section 28.4). The complete-cubic primitive improves the DD symmetric coefficient to less than 1.478923 per record. Cross-weighted extraction also reduces the two upper root caps and their complete reference weights on 301 bands (Section 27.3.1). These steps give the independently reconstructed closing $\Theta < 0.999999771$ and $\varepsilon = \frac{1814179}{141970826000}$.

The larger endpoints quoted in the supplied handoffs concern a different middle portfolio whose complete cell and matrix witnesses were not included. Their conditional scalar replays remain useful research records; the present theorem uses the explicit construction and independent checks just described. Certificate D.12 records the additional local identities and their finite verification.

34.8. **What the successive versions showed.** The largest gains came from changing the slot array or a physical law. Sharper estimates also mattered when they retained a cancellation, used an exact actual marginal, or replaced a first-order sum by a second-order operator bound. A local gain becomes an exponent gain only after every affected completion weight, marginal, restoration and price has been checked in one compatible construction.

Three accounting rules recur. First, keep raw-arm, leading, remainder and full prices distinct. Second, subtract only coefficients identified inside the same positive expansion, spending each invariant or variance coefficient once. Third, repair candidate laws to common marginals before averaging, and propagate both symmetric and skew prices from the resulting law.

An early density-transfer error illustrates a separate analytic rule. A bound $A(\alpha_v/\alpha)^\theta + e$ cannot be replaced by $(A+e)(\alpha_v/\alpha)^\theta$ when the density ratio is below one. For $A = \theta = 1$, $e = 1/10$ and $\alpha_v/\alpha = 1/4$, the former is $7/20$ and the latter $11/40$. The proof bounds the additive error before extracting the main density factor (Proposition 8.7).

Finally, a sharp local decomposition is a limit of that decomposition, not a proof that the global exponent is optimal. The obstructions in Section 35 retain their stated scope. Historical numerical claims and their exact source files remain in the archived checkpoints; none of this historical discussion is a premise of the main theorem.

35. Obstructions, limits and pitfalls

What this section is for (not used in the proof). This section records what did not work, where the method stops, and the errors the proof had to avoid. Nothing in it is a premise of the main theorem, and no proof of Parts II–V cites it. Each statement is quoted, with its scope, from the place where it is derived (a section of this paper, the archived manuscripts of Section D.6, or a research note), or is checked here by an elementary argument (Lemma 35.9, Remark 35.7(i)) or a stated enumeration. It serves a reader who wants to know why the construction of Section 18 has its shape, and one who wants to push the exponent further. Section 34 tells the development in order; here the material is grouped by kind: the accounting (Sections 35.1 to 35.3), the prime ranges (Sections 35.4 and 35.5), the root law and twelve queries (Sections 35.6 and 35.7), the obstruction polynomials of the earlier recurrence (Section 35.8), limits (Section 35.9), tested directions (Section 35.10), pitfalls (Section 35.11) and the scope of the reviews (Section 35.12). As in Section 34, "row n" is a row of Table 14, "old §n" a section of the archived manuscript of 644 pages and "master §n" a section of the archived manuscript of 770 pages, whose labels are quoted as master labels. Numbers marked "about" are not certified.

35.1. **The crude bound exceeds 1 (distinct from $\Theta < 1$).** Without classes, without the common mean and with the norm envelopes as matching envelopes (so that the dominator price is $T = S + A$, the symmetric plus skew price, by the norm-envelope case of (CP6) in Definition 9.17), the comparison argument of Section 11 gives only the *plain criterion* $L_0 + \mathcal{E} + J + W < 1$, with $\mathcal{E} = B(e^S \cosh A - 1)$ and $J = Be^{S+A}(R_I/2)^6$. Theorem C (Theorem 9.21) refines it in three independent ways, described in Section 3.7: the classes of (CP5) (Sections 24 and 25), the direct Gram (Section 26), and the common mean (Lemma 10.1), with which the trivial credit $m = 1 - L_0$ gives the closing $L_0 + \frac{63}{64}\mathcal{E} + J + W$ and a credit $m > 1$ lowers it by a further $(m - 1 + L_0)/64$ (Remark 11.3).

Remark 35.1 (Why the plain criterion is refined). At the October 18 checkpoint ($\varepsilon = 441/44375000$, Section 34.4) the plain criterion fails even when the reserved cost already contains the variance discounts of degrees one and two: its value is then $1.1219\ldots > 1$. The certified ledger of that checkpoint, which its record verifier checks (Section D.6), shows what each refinement bought (upper bounds, rounded up; the gains are differences of the displayed values):

After this step	Closing	Gain
Variance discounts of degrees one and two; plain matching term	1.121918	
Direct old/new F matching Gram	1.012226	0.109692
Variance discounts of degrees three and four	1.012152	0.000074
Common mean, reserved coefficient 63/64	0.999627	0.012525
Positive F singleton mean, doubled	0.995618	0.004009
Distinct-block pair mean and its polarized copy	0.995490	0.000128
Extra direct-envelope singleton and pair means	0.995315	0.000175

Each line is a separate valid inequality between the same quantities, and no saving is counted in two lines.

For the baseline construction Section 3.7 gives the same decomposition at the values of Table 12: there the plain criterion is about 1.603, the closing without the common mean and the mean credit about 1.021, and every refinement is needed. The accounting also has limits. The coefficient 63/64 was also derived pattern by pattern; the stronger coefficient 31/32 of that derivation needs the comparison and the matching to use one bracket for every index-support pattern, which fails once the direct Gram prices the F records. At the October 18 checkpoint the positive mean of the reserved-diagonal records had the ceiling $(e^{U_0} - 1)/64$, about 0.016, and the coupons took about 0.0043 of it. The final mean credit saves about 0.0092: its coupons have degrees up to four and use chosen side diagonals that the matching cap already pays, and late-pool coupons of degrees five and six complete them (Proposition 26.7).

35.2. **No positive order at the reserved cut.** The variance credits of Section 24 need lower bounds for the energies of the products of the actual half marginals of the fold records F_1, F_2 and the matching records E_0, E_1 of a middle block. The bracket lives at ι, so the natural place to look for them is the reserved cut. It is the wrong place.

Remark 35.2 (A reserved-cut barrier and its scope). At ι the half law of each of these records is folded: the six labels of a layer take three distinct values, in the pattern of the fold (Lemma 17.2), while the root of the reference labels a layer injectively. So the half laws are singular to that of the root, and an independent analysis (statement N1 of `research/roadmap_2026_10_18_claude/item03/main/VARIANCE_NOTE.md`) drew three consequences: (i) no positive semidefinite order at ι between a record's product of marginals and the reference bounds its unselected energy from below, since in the block coordinate that product has rank one; (ii) an exact toy with a folded law singular to the root law (energies 1 and $1/k$ in the selected and unselected cases of that note, $Y \approx 0.998$, $k = 4, \dots, 256$) violates the inequality such an order would give; (iii) measuring the zero-marginal record differences by the reference's centred norm in $L^2(m_{\mathcal{B}})$ gives norms larger by a factor of about 10^3 than the centred norms in the records' own marginals. These statements concern orders at ι against the reference, and they are correct there. They explain how the restoration of Section 24 is built: it compares the root with F_j at the fold chamber of F_j, where both half laws live on six distinct labels with mutually bounded densities (Hypothesis 24.2, with a pointwise and a chi-square bound in Lemma 24.7), reaches E_0, E_1 by a polarization at a chamber cut, and at ι uses only each record's own centred norm in its own actual marginal space (Section 24.4). The toy has no chamber cut and violates hypothesis (A) of Hypothesis 24.2 (mutually bounded half-law densities on the common chamber), which step (B) of Lemma 24.8 uses, so it is not a counterexample to it; the referees of the record gate reached the same conclusion.

35.3. **Pairing and signed blocks: norms against envelopes.** The fold records F_1, F_2 must be replaced by laws that accept the queries and keep the actual half marginals. The final construction uses the signed whole-block law (Definition 23.9): its first-order change at ι is purely skew, a three-arm star (a star with three cross pairs) of norm about $1.2732\, r$, and its whole symmetric

part is of order $1/p$ (Proposition 23.13). At the October 18 checkpoint a different choice won, the orientation mixture of the next remark, and at row 11 the pure pairing, which the signed law contains as its case $\mathtt{t} = 0$ (Lemma 23.5).

Remark 35.3 (Pairing and its barrier). The pure paired repair has leading constants $(0, \sqrt{2})\, r$ (symmetric, skew) for F and $(0, 1)\, r$ for the paired matching record. For $p \equiv 3 \pmod 4$, $p \geq 43$, every accepting replacement of the two-layer F_1 record with the exact old half marginals that keeps the old cross edge $y_0 \to x_2$ differs from the reference record by at least $(2-3q)^{1/2} r$ in operator norm on $L^2(m_0)$ ($q = 1/p$; m_0 as in Lemma 23.2), and every accepting replacement of E_0 or E_1 by at least $r/(1-q)$ (Lemma LB of `research/roadmap_2026_10_18_claude/item01/deepen2/BARRIER.md`); so pairing is optimal only within the class that keeps $y_0 \to x_2$. The orientation mixture of the October 18 checkpoint leaves that class: it mixes the four laws that keep or reverse $y_0 \to x_2$ and $x_0 \to y_1$, with weights $(.5803723, .2409463, .1786814, 0)$ on (keep, keep), (keep, reverse), (reverse, keep), (reverse, reverse), and its leading coefficients have norms $0.30636\ldots$ and $0.97207\ldots$, of sum $1.2784\ldots < \sqrt{2}$. Joint flips of the two edges beat pairing at first order for every ratio of symmetric to skew price below $\sqrt{7/3}$ (`research/roadmap_2026_10_18/coordinator/ORIENTATION_MIXTURE_PARETO.md`). The signed whole-block law leaves that class as well: in all three of its components the factor $1 + \mathsf{B}$ of the old cross edge $y_0 \to x_2$ is replaced by $1 + (1 - \mathtt{t})\mathsf{B}$, two of them reverse the mirror edge $x_0 \to y_1$, and one of these also the old edge $x_0 \to y_2$ (Lemma 23.7 and Definition 23.9; master §43, `paper/sections/advance_2026_10_24/local.tex`:19-36, 95-108).

At row 11 the construction returned to pure pairing, which loses on norms, because the closing prices the symmetric and the skew parts differently. In the reserved cost $B(e^S \cosh A - 1) - D$ the symmetric price enters at first order, through e^S, the skew price only through $\cosh A$ (Remark 22.8); and the even skew products are charged by positive joint envelopes below B (Theorem 25.8 and Lemma 25.5), the aligned classes keeping a fraction of their variance credit (Lemma 25.9). Against row 10, which kept the mixture, S fell from about 0.12 to below 0.0181 and A rose from about 0.70 to below 0.952, and the discount $D > 0.428$ more than paid for the larger A (old §41). Two repairs are compared by their prices in the complete reserved cost, not by the norms of their leading terms. The signed law of the final construction keeps the zero symmetric leading change of pairing and lowers its skew leading change from $\sqrt{2}\, r$ to about $1.2732\, r$, at the variation $1 + \mathtt{t}$ (Remark 35.5); the pure law itself had reached its limits (Remark 35.4).

Remark 35.4 (The pure paired law at its limits). Take the pure paired law in the form of row 12: its raw kernel is that of the case $\mathtt{t} = 0$ of Definition 23.9, and its repair and its actual leading two-arm star $L_{F,p}$ are those of master §42. Its difference at the prime p, $D_{F,p}$, satisfies, for primes $p \equiv 3 \pmod 4$,

$$\lim_{p\to\infty} p\, \|D^{\mathrm{s}}_{F,p}\| = 2, \qquad \lim_{p\to\infty} p\, \|D^{\mathrm{a}}_{F,p} - L_{F,p}\| = \sqrt{5},$$

with witnesses on nontrivial translation sectors (master §42, `cont:local:sharpness`, `paper/sections/continuation_2026_10_24/local.tex`:1074-1156; `research/continuation_2026_10_24/local/SHARPNESS.md`); at the first piece of row 12 the two coefficients were about 2.010 and 2.268, against about 10.836 and 16.677 in the bounds of row 11. The coefficient $\sqrt{6}$ of the direct old/new Gram of the matching envelopes of that law is sharp for scalar side diagonals: the leading two-side Gram has the characteristic polynomial $z^4(z^2 - 6)(z^2 - 2)$, so a common scalar side shift must be at least $\sqrt{6}$, and two different scalar shifts have average at least $\sqrt{6}$ (`research/continuation_2026_10_24/local/DIRECT_GRAM_OBSTRUCTION.md`). Both statements concern that law and that decomposition: a further gain in the middle needed another law, and they say nothing about the optimality of the exponent.

Remark 35.5 (The leading frontier of signed fold laws). Average the 256 complete cross-sign patterns of the fold record that contain the accepted query, with arbitrary real weights of total mass one,

and let n_{s}, n_{a} be the symmetric and skew leading norms of the change, in units of r. Then, for every relative price weight $c \geq 1$ on the symmetric part, $c\,n_{\mathrm{s}} + n_{\mathrm{a}} \geq \sqrt{3/2}$, without any positivity. A zero symmetric leading error forces the four sign moments of the cross edges into a family with one parameter $\mathtt{t}$, whose skew leading norm is $(1+\mathtt{t}^2+(1-\mathtt{t})^2)^{1/2}$, least at $\mathtt{t} = 1/2$; nonnegative weights force $\mathtt{t} = 0$, pure pairing, with $\sqrt{2}$; for $0 \leq \mathtt{t} \leq 1$ a signed law of the family has total variation at least $1 + \mathtt{t}$, and the three-component law of Definition 23.9 attains it. On the same family the scalar coefficient of the direct Gram of the matching sides in the leading Euclidean model is at least $s_{\mathtt{t}} = (4 - 2\mathtt{t} + 2\mathtt{t}^2 + 2(1+\mathtt{t}^2)^{1/2})^{1/2}$, with equality for that law (Lemma 26.1): $\sqrt{6}$ at $\mathtt{t} = 0$, about 2.3842 at $\mathtt{t} = 127/500$ (`research/advance_2026_10_24/local/SIGNED_LEADING_FRONTIER.md`:9-82, checked by `research/advance_2026_10_24/local/check_signed_frontier.json`; master §43, `adv:local:signed-law`, `paper/sections/advance_2026_10_24/local.tex`:95-146). The statement concerns the leading Euclidean model: it rules out a further leading gain from more signed cross orientations with these objectives, not a different law, and the selected $\mathtt{t} = 127/500$ is chosen by the complete closing, which also depends on the envelopes, the restorations, the coupons and the finite remainders.

35.4. **The walls of the prefix at primes $p \equiv 1 \pmod 4$.** A *wall* in this section means a value of the exponent at which one explicit bound of the construction reaches 1, so that this layout of the construction cannot pass it without a new idea. For the prefix the binding bound is the price of deleting coincident labels from the square clique; the remark computes the two walls it gives.

Remark 35.6 (The prefix walls). The numerical comparisons in this remark use the baseline $\vartheta = 1$ and $a = \frac{5422587}{6103515625}$. With m_{cl} digit pairs the prefix deletes its exception at a total price at most $\eta_{m_{\mathrm{cl}}} P_4^{1+2m_{\mathrm{cl}}a}$, $\eta_{m_{\mathrm{cl}}} = 2^{1-m_{\mathrm{cl}}} - 2^{-2m_{\mathrm{cl}}}$ (Lemma 19.3), which at $P_4 = 2^{512}$ is below $2^{513-m_{\mathrm{cl}}(1-1024a)}$: each digit pair halves the exception and multiplies the selection factor of a prefix prime by at most 2^{1024a}. This gives two walls. (i) For a free number of digit pairs the bound tends to zero exactly when $2^{1024a} < 2$, that is $a < 1/1024$, or $\varepsilon < 1/72704 \approx 1.3754 \cdot 10^{-5}$; at the baseline exponent $1024a = 0.909759\ldots$ and $2^{1024a} \approx 1.879$, so ε is about 91% of this wall. (ii) With $m_{\mathrm{cl}} = 2048$, the number used up to row 10, the exponent vanishes at $a = 1535/2^{21}$, that is $\varepsilon \approx 1.0309 \cdot 10^{-5}$. The October 18 checkpoint (binary exponent $513 - m_{\mathrm{cl}}(1 - 1024a)$ about -55.2) and row 10 ($\varepsilon \approx 9.963 \cdot 10^{-6}$) lie below this wall; row 11 and all later rows lie above it, and at the baseline exponent it would be about $+328.19$. This is why $m_{\mathrm{cl}} = 8192$: at the baseline a the exponent is negative exactly for $m_{\mathrm{cl}} \geq 5685$, and at $m_{\mathrm{cl}} = 8192$ it is about -226.25, far inside the allowance 10^{-12}; with 8192 pairs it stays inside the allowance on the whole analytic range $a \leq 9/10000$ (at $a = 9/10000$ it is $-80783/625 \approx -129.25$), and it vanishes at $a = 7679/2^{23} \approx 9.154 \cdot 10^{-4}$. More digit pairs cost only a larger fixed modulus (Lemma 19.3(iv), Remark 19.4). Both walls belong to the layout, not to the method: a clique prefix stopped at a smaller power of two P, with the all-prime tail starting there, replaces $1024a$ by $2a \log_2 P$ and moves both walls to larger exponents.

35.5. **Why class one is a prefix.** The five-type completion of the middle range expresses the F and E kernels against one conditional reference, the common middle law of two labels (Lemmas 17.5 and 22.1), and needs that law to be uniform on ordered distinct pairs. For $p \equiv 3 \pmod 4$ it is, because the exchange $(u, u') \mapsto (u', u)$, which preserves the completion counts, interchanges the two orbits of ordered distinct pairs under the maps $z \mapsto \alpha z + \beta$ with $\chi(\alpha) = 1$ (Lemma 20.9(ii)). For $p \equiv 1 \pmod 4$ the exchange preserves the orbits and the counts differ: the pairs with $\chi(u' - u) = 1$ have 312 completions to F-configurations and the others 264 at $p = 13$ (Remark 20.10), and 1464 against 1344 at $p = 17$ (by enumeration). At class-one primes, on the other hand, the strict-square relation is symmetric and the square clique costs only an exception (Lemma 19.2). So every class-one prime below P_4 is a prefix prime, the middle and upper ranges are class three (Section 19), and class one returns only in the tail, with caps certified for both classes (Sections 14.11 and 29).

35.6. **The root law: forced support and forced norm.** The chamber cap d of the twelve-label root, the negative cap of its cut L20, fixes the first-order weights $w_{\mathrm{D}} = w_{\mathrm{S}}$ of the reference, in particular the weight $w_{\mathrm{D}}r$ of every fold record, through the completion inequality $w_{\mathrm{D}} \geq d$ (Lemma 22.1). Two scoped facts say where a gain can come from.

Remark 35.7 (Support closure and the forced norm). (i) Let Γ be a probability law on the labellings of the twelve vertices (u, ℓ) of $G_{\mathrm{L},2}$ at a prime $p \equiv 3 \pmod 4$ whose half marginal on the six half vertices $\{0,1,2\} \times \{0,1\}$ is that of the uniform root, and which is invariant under the three chamber reflections $u \mapsto k - u$, $k \in \{1,3,5\}$. Then Γ is supported on the labellings that satisfy all 24 edges of $G_{\mathrm{L},2}$; in particular none of the sixteen cross edges of L20 can be dropped. Indeed the group generated by the reflections carries $\{0,1,2\}$ to the six consecutive triples $H_a = \{a, a+1, a+2\}$ (the reflections give H_5, H_1, H_3, and the translations by $0, 2, 4$, products of two reflections, give H_0, H_2, H_4); each contains exactly one of the six rotation edges, with its four layer lifts, and every rotation edge lies in exactly one H_a. The half marginal of the uniform root is carried by the labellings of $G_{\mathrm{F},2}$ (Lemma 20.11(b)), so the failure of the edge inside $\{0,1,2\}$ is a null event, and invariance carries this to every H_a (master §43, `adv:local:root-support-closure`, `paper/sections/advance_2026_10_24/local_interim.tex`:128-151). Positive reweightings of the accepted support are allowed; signed measures and changed half laws are not covered.

(ii) Let T be the density operator, relative to $\mu_p \otimes \mu_p$ with $\mu_p = \lambda_p(G_{\mathrm{F},2})$, of any probability law on twelve distinct labels that satisfies the sixteen cross constraints of L20, whatever its marginals, and let P_0 be the projection onto the mean-zero functions. Then $\|P_0 T P_0\| \geq \sqrt{30}\, r\,(1 - o(1))$; the proof compares normalized traces of T with the fifteen star features of Lemma 14.4 (master §43, `adv:local:forced-root-norm`, `paper/sections/advance_2026_10_24/local_interim.tex`:153-213). The record `research/advance_2026_10_24/local/check_l20_forced_norm.json` evaluates the finite form of the bound at four endpoints; at Q_3 it gives $\|P_0 T P_0\| \geq 5.4472\, r$. The bound concerns the centred norm only: a diagonal entry of order r not constrained by the cross conditions in a higher-star block can move the negative eigenvalue of that block without changing its forced off-diagonal entries, so the negative eigenvalue is not forced.

So a law that keeps the symmetries and the half marginal of the root cannot drop a constraint, and a gain must come from a positive reweighting of the accepted support; by (ii) it can lower only the negative cap, the centred norm of every law on that support being at least about $\sqrt{30}\, r$. The completion inequalities and the block caps use the negative and the positive cap at each cut (Lemma 20.12), not the centred norm, and a reweighting that changes the half marginal of the root is paid in the restorations (Lemma 24.7(iv)). This is what the positive roots of Section 21 do. In the selected construction, the chamber block caps w_{D} on the pieces from 2^{36}, 2^{40} and 2^{45} are below 5.46078, 5.45753 and 5.43446, against about 5.4776, 5.4773 and 5.4772 for the uniform root (Certificates 20.22 and 21.25); four earlier pieces are tilted as well. Their centred norm still satisfies (ii), so the gain concerns the negative cap that the completion actually pays. The asymptotic tilts that preceded them (an eighth-trace tilt whose centred quartic diagonal response is at least $0.9065\, r$ for $p \geq 2^{20}$; quartet tilts with limiting coefficients about 5.4718 and 5.4754; a linear pair-trace tilt, about 5.4664; the average of individual squares, about 5.4562; the square of the average, about 5.4421) carried unpriced $O(q)$ remainders and bounds on their own marginals, and none moved the exponent until both cuts and the restoration inputs were proved in the fixed charts (master §43, `paper/sections/advance_2026_10_24/local_interim.tex`:215-1349; Section 34.6).

35.7. **Why twelve queries do not help.** The twelve queries of $\mathcal{E}_{12}$, with the sources $(u, v, 0)$, $u \in \{0, 5\}$, have 24 distinct endpoints and project to two first-coordinate pairs (Remark 18.2). With twelve disjoint indices each conductor still occurs twice in the complementary folds, the least square step is unchanged, and the clique exception is the two-pair mass already charged in Lemma 19.3. So, given comparison data for $\mathcal{E}_{12}$ with the actual marginals and the bounds of

(CP0)–(CP8), Theorem 9.21 with $k = 12$ applies, and its closing condition reads

$$\frac{4095}{4096}(L_0 + \mathcal{E}) + J_{12} + W_{12} + \frac{1-m}{4096} < 1, \qquad J_{12} = Be^{T_{12}}\Big(\frac{R_I}{2}\Big)^{12},$$

$$W_{12} = Be^{S+A}\Big[\Big(1 + \frac{R_I}{2K^{35/36}}\Big)^{12} - 1\Big],$$

with T_{12} the matching exponent of such data (master §43, `adv:upper:twelve-interface`, `paper/sections/advance_2026_10_24/upper_twelve.tex`:27-96). Such data are not available. Locally, accepting laws exist: exact Markov couplings of the actual half law accept both fold queries $x_1 \to y_0$, $x_2 \to y_1$, or both queries of a matching fork, and their operators vanish on invariant mean-zero functions, so Corollary 17.17 with such a coupling turns every nonnegative invariant accepting raw kernel of mass one with bounded marginal defects ($|\alpha|, |\beta| \le M$ there) into a law with exactly the actual half marginals (master §43, `adv:local:two-query-markov`, `adv:local:two-query-repair`, `paper/sections/advance_2026_10_24/local_interim.tex`:1-120). The obstruction is the price on the unchanged references.

Remark 35.8 (Two queries force a symmetric error). Let K be any real signed law of mass one on pairs of halves of the fold record, written as a density relative to $m_0 \otimes m_0$, with m_0 the actual half marginal of Lemma 23.2, that accepts $\chi(y_0 - x_1) = \chi(y_1 - x_2) = 1$. Then $\|P_0(\operatorname{Sym} K - T_{\mathrm{old}})P_0\| \ge (1/\sqrt{3} - o(1))\, r$, and uniformly at least $0.57263\, r$ for $p \ge 2^{24}$, where T_{old} is the reference fold record of Lemma 23.2 and P_0 the projection off the constants of $L^2(m_0)$. For the two queries with common source $x_0 \to y_1$, $x_0 \to y_2$ (the matching fork), every signed law of mass one on uniform injective triples that accepts $x_0 \to y_1$ and $x_0 \to y_2$ has symmetric and skew changes whose norms add up to at least $(\sqrt{3} - o(1))\, r$ (master §43, `adv:upper:actual-two-query`, `adv:upper:actual-fork-frontier`, `paper/sections/advance_2026_10_24/upper_interim.tex`:209-370; `research/advance_2026_10_24/upper/check_actual_two_query_barrier.json`). No positivity, invariance or marginal condition is assumed. With these references a twelve-query fold record pays a symmetric price of order r, not $1/p$, and a twelve-query matching record with a symmetric error of order $1/p$ has a skew coefficient at least about $\sqrt{3}$. The statement concerns the unchanged half laws and reference records, not twelve queries as such: a different reference, or a joint treatment of the symmetric terms, could change its consequences.

A twelve-query construction would therefore need fold and matching records that accept both queries of their halves, with comparison bounds against a changed reference, and new class bounds (CP5) for their leading stars. Parts of the construction accept $\mathcal{E}_{12}$ already, but its six-query refinements do not: the signed fold records of the middle accept, in each fold, only the one cross edge that $\mathcal{E}_6$ needs there, while $\mathcal{E}_{12}$ needs both $x_1 \to y_0$ and $x_2 \to y_1$; the LE components of the upper range add only the edge that $\mathcal{E}_6$ requires (Definition 28.4 and Proposition 28.6); and the records $G_{01}L$, $G_{10}L$ are retained because they accept $\mathcal{E}_6$ (Lemma 28.1; `research/advance_2026_10_24/upper/SIX_QUERY_UPPER.md`). A twelve-query construction would have to repair or delete them instead of retaining them.

35.8. **Obstruction polynomials of the singleton recurrence.** Before the comparison theorem, records that did not accept a coupled family were covered, not repaired. In the singleton-recurrence design of the 216-slot period (Section 34.2), nine edge families crossing the reserved cut are indexed by $\Phi = \{0,1,2\}^2$, with rows $\mathcal{R}_i = \{(i,j) : j\}$ and columns $\mathcal{C}_j = \{(i,j) : i\}$. The law of a middle atom is a sum of typed parts: type $\star$ accepts every family, a residual part none, and the fifteen *rescue types* row_i, col_j, $\mathrm{cr}_{i,j}$ accept $\mathrm{Acc}(\mathrm{row}_i) = \mathcal{R}_i$, $\mathrm{Acc}(\mathrm{col}_j) = \mathcal{C}_j$ and $\mathrm{Acc}(\mathrm{cr}_{i,j}) = \mathcal{R}_i \cup \mathcal{C}_j$. A record, one type at every atom, accepts the intersection of the acceptance sets of its types. For a coupled set $\emptyset \ne \mathfrak{F} \subseteq \Phi$, records that accept a family of $\mathfrak{F}$ cancel through the actual-square kernel;

the others were covered in the Loewner order at the reserved cut, which fixes every middle record samplewise, by one *cylinder* per residual part and one per minimal obstruction placed on distinct atoms, where a cylinder here is a law depending only on the labels at finitely many named atoms (a cylinder set in the product). A set S of rescue types is $\mathfrak{F}$*-incompatible* if $\mathfrak{F}\cap\bigcap_{t\in S}\mathrm{Acc}(t)=\emptyset$ (the empty intersection being Φ), and a *minimal* $\mathfrak{F}$*-obstruction* is an inclusion-minimal incompatible set. With priced intensities at most x, y, z for rows, columns and crosses, the cover costs at most $Z+P_{\mathfrak{F}}(x,y,z)$, with Z the residual intensity and $P_{\mathfrak{F}}=\sum_S x^{n_{\mathrm{r}}(S)}y^{n_{\mathrm{c}}(S)}z^{n_{\mathrm{x}}(S)}$ over the minimal obstructions, $n_{\mathrm{r}},n_{\mathrm{c}},n_{\mathrm{x}}$ counting the rows, columns and crosses of S. The all-family criterion closed when $\frac{1}{2}M^{|\mathfrak{F}|}[Z+P_{\mathfrak{F}}+|\mathfrak{F}|/K+e^{H}]<1$, with M the modulus moment of the index and H the upper exceptional intensity. The singleton checkpoint ($\varepsilon\approx 3.3\cdot 10^{-9}$) coupled the nine families one at a time and covered the records that accept none of them, with P_{Φ} about 0.369.

Lemma 35.9 (Rejection-class covers)**.** *Let $\emptyset\neq\mathfrak{F}\subseteq\Phi$, put $\mathrm{Rej}(t)=\mathfrak{F}\setminus\mathrm{Acc}(t)$ for a rescue type t, give row_i, col_j, $\mathrm{cr}_{i,j}$ the weights x,y,z, and let w_X be the total weight of the types t with $\mathrm{Rej}(t)=X$. A* minimal cover *of $\mathfrak{F}$ is a family of distinct sets with union $\mathfrak{F}$ no proper subfamily of which has union $\mathfrak{F}$.*

(i) S is $\mathfrak{F}$-incompatible if and only if $\bigcup_{t\in S}\mathrm{Rej}(t)=\mathfrak{F}$.
(ii) S is a minimal $\mathfrak{F}$-obstruction if and only if the sets $\mathrm{Rej}(t)$, $t\in S$, are nonempty, pairwise distinct, and form a minimal cover of $\mathfrak{F}$.
(iii) $P_{\mathfrak{F}}=\sum_{\mathcal{C}}\prod_{X\in\mathcal{C}}w_X$ over the minimal covers $\mathcal{C}$ of $\mathfrak{F}$ by nonempty rejection sets, and every minimal $\mathfrak{F}$-obstruction has at most $|\mathfrak{F}|$ elements.

Proof. (i) is De Morgan's law, $\mathfrak{F}\cap\bigcap_S\mathrm{Acc}(t)=\mathfrak{F}\setminus\bigcup_S\mathrm{Rej}(t)$; for $S=\emptyset$ both sides are false. (ii) By (i) compatibility passes to subsets, so S is minimal if and only if it is incompatible and every $S\setminus\{t\}$ is compatible. If S is minimal, no $t\in S$ has $\mathrm{Rej}(t)=\emptyset$ and no two share a rejection set, since removing such a t would not change the union; a proper subfamily of $\{\mathrm{Rej}(t):t\in S\}$ is then the family of a proper subset of S, which does not cover. Conversely such a family makes S incompatible and every $S\setminus\{t\}$ compatible. (iii) By (ii) the minimal obstructions correspond bijectively to the pairs of a minimal cover $\mathcal{C}$ of $\mathfrak{F}$ by nonempty rejection sets and a choice of one type t_X with $\mathrm{Rej}(t_X)=X$ for each $X\in\mathcal{C}$; for fixed $\mathcal{C}$, summing the monomials over the choices gives $\prod_{X\in\mathcal{C}}w_X$. Each member of a minimal cover has a point of $\mathfrak{F}$ in no other member, so $|\mathcal{C}|\le|\mathfrak{F}|$. □

For the transversal $\mathfrak{D}=\{(0,0),(1,1),(2,2)\}$ the types of $A_i=\{\mathrm{row}_i,\mathrm{col}_i,\mathrm{cr}_{i,i}\}$ reject the two diagonal cells other than (i,i), the two crosses of $B_k=\{\mathrm{cr}_{a,b},\mathrm{cr}_{b,a}\}$, $\{a,b,k\}=\{0,1,2\}$, reject (k,k); with the weights $s=x+y+z$ and $2z$ and the minimal covers $\{A_i,A_j\}$, $\{A_i,B_i\}$ (three each) and $\{B_0,B_1,B_2\}$, $P_{\mathfrak{D}}=3s^2+6sz+8z^3$. For the two-pair set $\mathfrak{E}=\{(0,1),(1,0)\}$ the types $\mathrm{row}_2,\mathrm{col}_2,\mathrm{cr}_{2,2}$ reject $\mathfrak{E}$, two classes of weight $x+y+3z$ reject one cell each, and two crosses reject nothing: $P_{\mathfrak{E}}=s+(x+y+3z)^2$.

TABLE 18. Obstruction polynomials of four coupled sets, $s=x+y+z$ (Lemma 35.9; confirmed by enumerating all sets of rescue types).

Coupled set $\mathfrak{F}$	$\lvert\mathfrak{F}\rvert$	minimal	largest	$P_{\mathfrak{F}}(x,y,z)$
Φ (all families)	9	165	4	$3x^2+3y^2+36xyz+36(x+y)z^2+6z^3+45z^4$
transversal $\mathfrak{D}$	3	53	3	$3s^2+6sz+8z^3$
two-pair set $\mathfrak{E}$	2	28	2	$s+(x+y+3z)^2$
two rows	6	77	4	$x+x^2+3(y+z)^2+12xyz+18xz^2$
$\mathfrak{R}=\mathcal{R}_0\cup\mathcal{R}_1$				$+12(y+z)z^2+9z^4$

The two-column set is the transpose of $\mathfrak{R}$, with polynomial $P_{\mathfrak{R}}(y,x,z)$. The transversal and the two-pair set occurred only in research designs, the two-row set in an earlier unpublished design. By Lemma 35.9(i), $P_{\mathfrak{F}}$ has a term of degree one exactly when $\mathfrak{F}$ misses a row or a column (if $\mathcal{R}_i \cup \mathcal{C}_j$ misses $\mathfrak{F}$, so does $\mathcal{R}_i$). For one family, $\mathfrak{F} = \{(i,j)\}$, the two other rows, the two other columns and the four crosses off row i and column j are obstructions by themselves: $P_{\{(i,j)\}} = 2x + 2y + 4z$. This first-order term is why coupling a single pair is costly; at the type masses of the first version's middle terminal it was about 61.6 per unit of terminal weight. A set without first-order terms has at least three cells, with equality exactly for the six permutation matrices, which all have $P_{\mathfrak{D}}$ by the symmetry $S_3 \times S_3$ of Φ; but the modulus moment enters as $M^{|\mathfrak{F}|}$, so larger sets cost more in the index. Which set paid depended on the design. Theorem C has no obstruction polynomial: its queries are fixed, and a record that does not accept them is repaired, keeping the actual conditional marginals, or deleted at the price of its own mass (Section 17). A record of weight $O(r)$ whose repair moves it by $O(r)$ then costs $O(1/p)$ instead of entering a union bound with its full intensity; this was the source of the gain of the fixed-family transfer, the 216-slot construction that is the first instance of the scheme of Theorem C (Section 34.2).

35.9. **Limits of the method.** *The kernel cutoff.* The circle method of Appendix C needs its major-arc exponent in the window $(10\beta/3,\ 1/4 - 19\beta/12)$, nonempty exactly when $\beta < 3/59$ (Remark C.10); at $\beta = 1/20$ the window is $(1/6, 41/240)$ and contains $17/100$. So moduli up to $N^{1/20}$ are close to the limit $N^{3/59}$ of this argument.

The error exponent. For the baseline, the induction needs $a < \sigma$ (Theorem 8.12), so the stage of Definition 8.1, $\sigma = 1/825$, cannot give more than $\varepsilon < 1/(71 \cdot 825) \approx 1.707 \cdot 10^{-5}$, whatever the local laws. It is not binding: $\sigma - a = 65195254/201416015625$, $a \approx 0.733\,\sigma$, and the exponent is limited by $\Theta < 1$. A larger σ needs a new stage, since σ enters (S1)–(S4) of the stage inequalities (Definition 8.1), with new kernel and analytic certificates; this was not attempted. The gate $a < \sigma$ does not hinge on the kernel at $\beta = 1/20$: with a separate outside cutoff the comparison needs no kernel and the cancellation needs it only at $\beta = 1/36$, and the stage $(\beta, \tau, t_{\mathrm{an}}, \sigma) = (1/36, 1/230, 1/108, 1/1000)$ was checked at the October 18 exponent; it is not used.

The analytic domain. For the baseline, the analytic stage, the transfer gates and the prefix allowance are certified on the whole range $0 < a \le 9/10000$, independently of Q_3 and P_3 (Certificate D.2; `research/advance_2026_10_24/global/ANALYTIC_DOMAIN.md`). This allows $\varepsilon \le 9/710000 \approx 1.2676 \cdot 10^{-5}$, about 1.30% above the baseline exponent; with 8192 digit pairs the prefix wall lies at $\varepsilon \approx 1.2893 \cdot 10^{-5}$ (Remark 35.6). Beyond the range the gates must be audited again, and beyond that wall the prefix needs more digit pairs; this is a limit of the certificate and of the layout, not of the method, and the baseline closing becomes limiting before it is reached (`research/further_2026_10_25/SYNTHESIS.md`:187-195). The selected theorem rechecks the analytic, prefix and density gates on $0 < a \le 91/100000$ and uses $\vartheta = \frac{1014587}{1000000}$ (Certificate 33.3).

The 2-adic scope. The earlier unit-square kernel used a 2-part of 8. The root classes of Lemma C.1 now cover every nonvanishing square class with at least three remaining binary digits after its even valuation is removed. This includes the selected physical modulus 2^{129} and root valuations below 64. Zero classes are still excluded; conditioning on the random depth does not replace the full physical modulus M_0 in cancellation (Remark 35.10).

Second-order inputs. The certified budget uses Deligne's theorems, the Katz–Sarnak local monodromy and the evaluation of Ahlgren and Ono (Theorem A.7) for its second-order bounds. This explains their role in this proof; it does not establish that they are necessary for a comparable exponent by another argument (Section 1.5).

Allowed couplings in the envelope bounds. One natural off-diagonal operator between fold and matching features, a coherent cross arm, has a kernel whose total variation, and hence the coupling mass of every expansion of it into positive couplings, is at least $(p-2-1/p)/(2\sqrt{p})$, unbounded in p;

so it cannot enter the uniform comparison of the envelopes at no extra price (`research/advance_2026_10_24/envelopes/COHERENT_ARM_OBSTRUCTION.md`). The larger invariant background of the bracket with one fold copy improves the exponent of the invariant envelope cap but supplies no new permutation of the fold features, since every element of the root group maps a fold pattern to a fold pattern with the identity permutation of its object names (`research/advance_2026_10_24/envelopes/PHYSICAL_ORBIT_SCOPE.md`). The matching energy pairs a test function on the source side with one on the target side without swapping them, so the envelope reflection is not available there for free (`research/advance_2026_10_24/envelopes/MATCHING_REFLECTION_SCOPE.md`). And in the pure-E singleton algebra every positive $Z \succeq \pm T$, T a tensor of k unit matching arms, has collected coupling mass at least 4^k, so the uniform discount of a pure-E class cannot be positive once $(R_{\mathrm{err}} - 2u)4^k \geq B(A_E/A_E^{\mathrm{arm}})^k$, u the exactly-one-tag credit per unit mass (master §43, `adv:env:pure-E-mass-floor`, `paper/sections/advance_2026_10_24/envelopes_interim.tex`:214-249). These are limits of the present allowed couplings, not of positive envelopes.

Fold copies in the bracket. The bracket carries one copy of the folded fold records (Lemma 22.7). With $c < 1$ copies the protection inequality, which then reads $c\,v_{\mathrm{D}} \geq 2k_{\mathrm{D}}w_{\mathrm{D}}$ (it is (22.1) at $c = 1$), raises v_{D}; the reference with $c < 1$ copies is, before renormalization, a subprobability (total mass < 1) whose normalization enters every comparison coefficient; and subtracting only the two selected fold patterns breaks the root invariance on which the invariant envelope caps rest. These are requirements of a different reference, not an impossibility (`research/further_2026_10_25/envelopes/BETA_BELOW_ONE_SCOPE.md`).

The layer cap of the positive roots. For the individual-square family, the proved layer coefficient tends, as $p \to \infty$, to $\sqrt{3} + \sqrt{7}$ for each fixed $\mathsf{u} > 0$, twice the norm $(\sqrt{3} + \sqrt{7})/2 \approx 2.189$ of the leading matrix of one component of the layer cut; the layer caps of the uniform root on these pieces are between about 2.189 and 2.196 (Certificate 20.13). The proof prices the two one-component terms of the tilted kernel separately, through the auxiliary space, so its bound does not tend to the uniform value as $\mathsf{u} \to 0$ (Proposition 21.9; `research/further_2026_10_25/local/L21_PAIR_SQUARE_TILT.md`:84-125, 240-250). The selected estimates bound the finite-p remainder of this argument directly. A sharper leading layer estimate for this family would lower the weight v_{L} on its six tilted pieces; the coherent last-piece law has its own layer bound in Section 21.

Six queries. With the pure paired comparison law six disjoint universal queries are the most: a universally supported query must project to one of the six directed rotation edges of the root, and on the common support of the root and of the three fold records F_0, F_1, F_2 the only universally supported queries run from layer 0 to layer 1 over the first-coordinate pair $0 \to 2$, so each of them meets one of the six slots $(0, v, 0)$ (`research/continuation_2026_10_24/envelopes/QUERY_OBSTRUCTION.md`). This is a support statement for that law, not for every modified comparison law; Section 35.7 says why the twelve-query family is not used either.

35.10. **Tested and not pursued.** Section 34 places these in the development; here is why each was dropped.

- *An E-only envelope of degree six at row 11.* Its coupling mass was about 4103.8, against about 257.9 with four factors, and at the selected endpoint of row 11 its complete cost exceeded B_*, so the coupling error consumed the saving. In the final portfolio the class $(0, 6)$ pays, with a saving of about $4.9 \cdot 10^{-6}$ (Corollary 25.10); the coupling-mass floor of the pure-E classes is in Section 35.9.
- *Graph laws at class-one primes.* The octahedron law on $\mathbb{Z}/6\mathbb{Z}$ (twelve unordered strict-square edges, all pairs but the antipodal ones, which make sense because -1 is a square) has small completion weights, but query edges on which its records have zero difference needed fine indices and their energy factor; a larger law with strictly square accepted edges served the upper part of the finite prefix. The square clique costs only an exception.

- *The sparse middle law and the best-cut comparison* of the 216-slot period (180 of the 252 middle edges, one amplified edge and then four, every term estimated at its best cut; Section 34.3). In the final design every query crosses the reserved cut; an ordinary cut and the reserved cut need not share a comparison energy, and a compatible multi-cut theorem was not proved.
- *An averaged completion denominator for the upper root* gives $g \approx 4887$ for the root cut LL20 at 2^{49}, computed from inputs that were not certified, against $2711.73\ldots$ for the direct-star root (Theorem 14.27) at the October 18 checkpoint; in this architecture the degree-twenty root is forced.
- *The other directions of the October 23 handoffs* (`research/push_2026_10_24/README.md`): stronger upper reports that could not be recovered and changed extraction and density frameworks; rows 12–14 took up further background and mean savings in their own form (Section 34.6). Their gains do not add to the final result without compatible laws and complete costs.
- *A continuous orientation family* of the pure law at row 12, mixing its four orientation components, each repaired to the same actual marginal: every direction on the finite grid of mixture weights tested in the two named records worsened the complete bound, and the tests are not an optimality proof (`research/continuation_2026_10_24/local/ORIENTATION_CONTINUATION.md`, `research/continuation_2026_10_24/README.md`:128-135).
- *Quartet tilts of the root.* The one-cut and the chamber-average quartic tilts have limiting coefficients about 5.4718 and 5.4754, above those of the pair-trace tilts (master §43, `adv:local:single-cut-negative-tilt`, `adv:local:chamber-negative-tilt`, `paper/sections/advance_2026_10_24/local_interim.tex`:487-881).
- *Larger kernels and pinned Grams.* The complete four-vertex character kernel on injective quadruples has norm at most $24p^{-2}$ (previously $96p^{-2}$; master §43, `adv:upper:quartic24`, `paper/sections/advance_2026_10_24/upper_interim.tex`:9-179), but the selected extraction class counts do not change, and the bound does not justify deleting collision terms of the grouped extraction (Section 35.11). One-pin, same-pin and two-pin weighted pair Grams through a twisted symmetric-square Legendre moment (`adv:env:one-pin-pair`, `adv:env:same-pin-conditional`, `adv:env:two-pin-pair`, `paper/sections/advance_2026_10_24/envelopes_interim.tex`:9-173) give a conditional pair-arm corollary that no selected row uses, since none has that pin structure. The absolute two-tag composition (`adv:env:absolute-two-tag`, envelopes_interim.tex:175-212) relates its endpoint energies by a map that is not a chamber reflection of the reference, so the grouped first-stage error keeps its countwise minimum.
- *Proved refinements not used in the closing.* Support-dependent conditional densities improve the conditional F row constants, but other completion constraints already determine the selected weights (master §43, `adv:local:conditional-support-density`, `paper/sections/advance_2026_10_24/local.tex`:1594-1676); the even-degree root-range parameters $10r + 7q$, $10r + 71q$, $10r + 103q$, $10r + 31q$, which Theorem 4.7(b) gives, are used by no scalar (`adv:global:root-even`, `paper/sections/advance_2026_10_24/global.tex`:178-199); the one-edge and two-edge squared-trace kernels with leading constant 4 (master §44, `fur:local:one-edge`, `fur:local:two-edge`, `paper/sections/further_2026_10_25/local.tex`:1-559) aim at a balanced root law of 48 scores whose cross-score covariance is unproved.
- *Other root laws.* A quartic multiplier model suggests a coefficient about 5.4229, conditional on unproved cubic and quartic identities for stars rooted at one vertex (`research/further_2026_10_25/global/QUARTIC_TILT_MODEL.md`); a balanced two-orbit model of 48 scores about 5.3961, with its rooted covariance unproved (`research/advance_2026_10_24/`

`WORK_LOG.md`:379-380); in the quadratic model, and conditional on its interfaces, mixing in individually squared scores of the other orbit cannot improve the limit, about 5.44209, of the square of the average (`research/further_2026_10_25/global/individual_square_scope.md`). In the floating-point searches (*screens*) reported for row 14, on the first tilted piece the parameter $\mathsf{u} = 4/5$ has a slightly smaller chamber coefficient (about 5.469937 against 5.469962) but a larger layer coefficient (about 5.477768 against 5.445397), and no global ordering of the two is proved (`research/further_2026_10_25/global/FIRST_BAND_PARAMETER_SCREEN.md`); on its last piece the square of the average with $\mathsf{u} = 51/40$ has the smaller chamber coefficient (about 5.451162 against 5.456784 for the individual squares with $\mathsf{u} = 1$) at a larger layer coefficient (about 4.473194 against 4.430848), and only the complete closing, which prices both, decides such a trade (master §44; `research/further_2026_10_25/envelopes/finite_l20_parameter_grid.json`, `research/further_2026_10_25/local/l21_pair_square_tilt.json`). The individual squares with $\mathsf{u} = 3/4$ already lower the chamber coefficient of the uniform root at 2^{35}, inside the piece from 2^{34} to 2^{36}; splitting that piece at 2^{35} would need a fresh prime partition, a price replay, a native comparison and a complete scalar (`research/further_2026_10_25/SYNTHESIS.md`:178-184). These are historical screens, not the selected laws: rows 15–16 use individual squares from 2^{30} on six existing pieces, then the coherent 39-score square with $\mathsf{u} = 67/40$ and floor $1/100$ from 2^{45} (Definition 21.1 and Certificate 21.25).

- *An interpolation of the matching law* with a bare one-query law, which keeps the query and the marginals and moves part of the leading E price from skew to symmetric: in a numerical screen with envelopes replaced by their limiting coefficients at the tested parameters the reserved cost rises at the rate of about 0.309 per unit of the moved fraction before the restoration credit, which would subtract about 0.119 from $\mathcal{E}$, so it was not pursued; this is not a lower bound over all envelopes (`research/further_2026_10_25/envelopes/E_TRADE_SCREEN.md`).
- *The late-pool threshold* 2^{31} is the best of the 30 threshold pools tested at the parameters of row 12, not an optimum over arbitrary pools or other parameters (`research/continuation_2026_10_24/envelopes/TAIL_THRESHOLD.md`).

Remark 35.10 (The three depth-index changes). At the October 18 checkpoint three changes of the index were computed: the exact class-three contraction with least square step 144, a fixed interval count K, and residues refined by p-adic and 2-adic depths (modulo p^{128} with increments t^2, $p^{64} \nmid t$, at the class-three primes below Q_3, modulo 2^{129} at 2, with random conditioning depths). At that checkpoint the fixed K alone lowered Θ by about $7.3 \cdot 10^{-4}$ and the depths by only about $2.8 \cdot 10^{-5}$. All three changes are now part of the selected construction. The required extension to nonvanishing square classes is proved in Lemmas C.1 and 29.6; the finite-filtration form of Lemma 10.1 supplies the depth-energy identity. The selected finite moment is certified in Certificate 29.11. The historical small numerical gain explains why this change alone does not provide a substantial exponent improvement.

Instructive choices in the spectral method.

- *Centring at the mean under the actual half marginal* m_p leaves $\mathbb{E}_\mu[F_\perp \mid x_k] = \kappa_F$ at every vertex, and this constant κ_F enters every star and $K_{2,2}$ term; centring at the mean under the half law μ removes these terms exactly, at the price of a second-order cross term (Section 14.7, Proposition 14.20).
- *Softening:* interpolating each cross constraint $\mathbf{1}_{Q_p^*}$ by $(1-t)/2 + t\,\mathbf{1}_{Q_p^*}$, the derivative in one edge isolates $(\chi - \mathbf{1}_{\{0\}})/2$. Its character part has norm $1/(2\sqrt{p})$ and its diagonal part norm $1/(2p)$ under uniform normalization; integration along a path gives relative cut estimates of order $p^{-1/2}$. They served the high range of the 216-slot design; the final caps come from

the two routes of Section 14 (Theorems 14.21 and 14.27) and, at the chamber cut of the middle root, from complete actual stars (Sections 20.4 and 21).

- *Tensor stability* needs no Perron–Frobenius argument or spectral gap: top singular vectors of a tensor product can be chosen as products of those of the factors, also for non-symmetric kernels. Block bounds in the final rows of the table are obtained from one-prime bounds by tensoring, using the multiplicativity of chi-square defects (Lemma 13.7(f)) and the exactness of the block marginals.
- *The three-block test.* With the first-order four-cycle bound the inner $K_{2,2}$ term, bounded through the pure-pair feature-family bound $\sum_J \|\Pi_{H_J} F\|^2 \le \mathrm{Fr}\,\|F\|^2$, is of order one (about 9 at 2^{44}), and a three-block test charged it to the pure-pair component; with the bound over GL_2 it is $O(r)$ (about $4 \cdot 10^{-6}$), and a two-block test, charging it inside the scalar $\bar{c}_\perp$, suffices.
- *Class-one refinements.* At $p \equiv 1 \pmod 4$ the contributions of the even edge subsets of the coupling term vanish under a nonsquare dilation, and the internal-edge Paley sum $\Sigma_2 = \sum_{t,u} \chi(t(t-1)u(u-1)(u-t))$ satisfies $|\Sigma_2| \le 2p$, by Kloosterman, Gauss and Jacobi sums (numerically $\Sigma_2 = 2(a^2 - b^2)$ for $p = a^2 + b^2$, a odd). The stored tail caps use neither, only the cruder class-one bounds of Lemma 14.31 and, at the root cuts, the parity transfer of Lemma 14.32.
- *Restricted criteria* for functions of a vertex subset A supplied row envelopes to the 216-slot completions; their only new feature, $\mathbb{E}_\mu[F_\perp \mid x_c]$ non-constant for $c \notin A$, costs outer stars and outer $K_{2,2}$ terms. The final completions use full-cut caps.

35.11. **Errors to avoid when checking or extending the proof.** These rules were learned from errors found during the work and from the reviews; the final chain observes each of them.

- Never normalize a dominated measure (Remark 5.11). Work in the actual half marginal; an induced half law needs its completion-density comparison. Class-three inputs (the common middle law, the Paley counts, the class-three caps) are never applied at class-one primes.
- An upper bound is never a lower bound: the lower endpoint of a certified interval enclosing an upper bound for a norm is not a lower bound for the norm, which needs its own witness, such as a Rayleigh quotient. Polynomials are evaluated at upper prices only because their coefficients are nonnegative (Remark 9.22).
- Never subtract unrelated upper bounds: a discount enters only as the nonnegative coefficient of its class, fixed before the series is rewritten and evaluated at the upper prices, and an exponential of an upper price, such as the background factor $e^{tU^{\mathrm{inv}}}$, is subtracted only inside a majorant with nonnegative coefficients in the same tag variables (Remark 10.15); and a jointly certified cap is never replaced by a sum of looser bounds.
- Savings of one degree do not simply add: an exact product can improve the exponential envelope but does not automatically add to an existing discount of that degree, and of two bounds for one degree the smaller is kept. The indices are actual tensor factors, not cells or bands (splitting every factor into equal pieces preserves S and A and approaches the envelope). For numbers b, r and $q \ge 0$, from $b + q - r$ keep $(b + q - r)_+$, not $(b - r)_+ + q$; a credit shared by all the orientation signs jointly is established only after summing over the signs; a diagonal added to an upper envelope gives a mean credit only if it is in the comparison identity.
- A variance credit cannot be kept with a smaller envelope: the credit inside an envelope class is $c\,g$ with $c \le \kappa$ (Lemma 25.9).
- Let $U \ge 0$ be a moment (an L^r norm of a fixed function under a fixed positive measure) that was priced at exponent a_0, and let $W \ge 0$ be a lower moment of the same function under the same measure, not yet priced (*unpriced*). Replacing U by W after repricing

improves the bound only if $W > U/C$, where $C = e^{t_0 L}$ with t_0 the repricing increment $2(a - a_0)$ and $L = 570$ the block budget; and U, W must be moments for one common positive measure. Local losses recur at every induction step and are charged at every step, as R_I is.

- A larger kernel cutoff or error exponent alone is not a closing theorem. The square step, the rounded child scale ℓ and its factor λ (Lemma 8.2(b)) must be kept; a same-input scalar inequality does not tensorize across correlated primes, and a label shared by several slots does not make them independently averaged.
- Parameters may be chosen by floating-point search; every inequality used is certified in outward arithmetic (Appendix D). The integer K is evaluated using the certified majorant. Optimality of the square-shift contraction is claimed only within its stated class of transitions, and failure of $\Theta < 1$ at nearby parameters (a, K) concerns that majorant only. Feasibility is not monotone in a when a design's conditions are tuned to its exponent.
- The $\sqrt{2}$ slip. For $f + Ag = u$, $g + A^* f = v$ with $\|A\| \le \rho < 1$ and $\|u\|, \|v\| \le U$, the direct-sum norm gives only $\max(\|f\|, \|g\|) \le \sqrt{2}\, U/(1 - \rho)$; the bound $U/(1 - \rho)$ of the one-way repairs needs the equation-by-equation estimates of the proof of Lemma 17.14, which an earlier proof of the one-way repair skipped.
- A coupled 0–1 mask is not an operator contraction: $\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ has norm $\sqrt{2}$, and zeroing its lower right entry gives $(1 + \sqrt{5})/2$. So a bound for a complete kernel does not justify deleting the collision terms of a grouped extraction, and the extraction keeps every special-value and merge allowance (master §43, `adv:upper:injective-obstruction`, `paper/sections/advance_2026_10_24/upper_interim.tex`:181-207).
- In pinned pair Grams exclude the pin labels: on $\mathbf{1}_a - \mathbf{1}_b$ at the pins a, b the two-pin kernel has the Rayleigh quotient $-q + 4q^2 - 3q^3$, of order q, while on the mean-zero functions supported off the pins its norm is at most $2qr + 3q^2$ (`paper/sections/advance_2026_10_24/envelopes_interim.tex`:114-152).
- A positive diagonal response is not a negative spectral bound, cross moments of chamber scores need not be nonnegative, and a bound on a tilted law's own marginal is not the fixed-chart bound that the completion needs (`paper/sections/advance_2026_10_24/local_interim.tex`:207-221, 417-418; Section 21).
- A local improvement is never inserted into an old closing with stale dependent quantities: every reference weight, completion inequality and assigned price is regenerated, and the gains of different changes interact (`research/advance_2026_10_24/SYNTHESIS.md`:21-24).
- Inside the remainder polynomial of a class only the separately proved leading and remainder prices may stand: neither a raw-arm price nor a full price minus a remainder price may replace a leading price there, since neither bounds the leading part of a record change (Remark 10.15).
- An invariant coefficient is not spent twice: a class priced at κ_b on the invariant background keeps only the variance allocation proved for its envelope (Lemma 25.9); and orbit coupons come from an explicit decomposition into invariant positive couplings, not from symmetrizing signed coefficients (Lemma 25.7).

35.12. **Scope of the reviews.** The constructions, proofs and computations were developed in several parallel lines of work carried out with AI models, an OpenAI Codex model (gpt-6-astra) and Anthropic's Claude models, partly building on four earlier research reports handed over between those lines of work. The initial synthesized text was written with Claude models; the 29 October revision integrates new arguments developed with ChatGPT Pro Astra 6 and Codex, replacing the corresponding local estimates and shortening the historical account. Its exact checks and their limits are stated in Certificate D.12. The reviews are internal reviews by AI models, not external referee reports. The review (called the record gate in its record) of the October 18 checkpoint

(research/record_gate_2026_10_18/), with referees of the other model family, refereed every component of that chain in a mathematics or computation lens, re-implemented its computations, and passed every component with minor findings. It checked the research notes and computations, not the text; relied on the named external theorems; spot-checked without re-running three large inherited computations (the histogram tree of the upper nonstar classes, the prime histogram of the middle blocks, the middle spectral grid); and did not re-referee the earlier sections written with Claude models, whose only review by the other family was a scoped audit whose three findings the final chain respects. The components of 24 October that row 11 added (the pure paired law with its envelopes and restorations, the exact index law with fixed K, the ordered collision bound) were checked by independent re-implementations and audits (research/push_2026_10_24/); no referee gate of the other family is recorded for them. The components that rows 12–14 added (the exact actual laws and collected channels, the signed whole-block law, the bracket with one fold copy and the two-cut error, the matrix envelopes and orbit coupons, the complete actual stars and the direct chamber cap, the dense upper comparisons and joint rows, the positive roots and their restoration, the assigned invariant price) were checked by reviewers within the agent team and by independent re-implementations, recorded with them (research/continuation_2026_10_24/global_review/CROSS_REVIEW.md, research/continuation_2026_10_24/upper_arithmetic/LOCAL_PROOF_REVIEW.md, research/continuation_2026_10_24/upper_arithmetic/GLOBAL_PROOF_REVIEW.md, research/advance_2026_10_24/envelopes/BETA_CROSS_REVIEW.md, research/advance_2026_10_24/local/GRAM_BACKGROUND_REVIEW.md, research/advance_2026_10_24/local/GROUPED_FIRST_STAGE_REVIEW.md, research/advance_2026_10_24/local/GENERAL_STAR_FRAME_REVIEW.md, research/further_2026_10_25/REVIEW.md, research/further_2026_10_25/local/LOCAL_PROOF_REVIEW.md); their closing number has an author assembly and an independently written one, which agree on the 45 compared components (research/further_2026_10_25/verification/selected.json), and these records remain frozen. The combined manuscript of 26 October adds independent reviews by the other model family of Lemmas 23.20, 25.9 and 28.2, recorded in research/merge_2026_10_26/CROSS_FAMILY_REVIEW.md and research/merge_2026_10_26/upper_stars_review.md, together with focused reviews of the selected root laws and analytic conductor interface. These are internal reviews, not external referee reports. The merge also records a focused review of the tagged, signed and auxiliary-price forms of (CP2)–(CP5), Lemmas 10.8 and 10.14 and Corollary 10.9, and their use in the closing (research/merge_2026_10_26/comparison_review.md). This review corrected two abstract edge cases: degree-class replacement now requires even degree at least two, and redistribution of upper prices treats zero actual totals separately. All selected classes have degrees two, four or six and positive totals, so neither correction changes the bound.

The refined reference-fold cap, the pair-feature matrix at general strength and the root-normalization prices (Lemmas 21.3, 21.6 and 23.2) have a separate focused review and independent finite algebra checks in research/merge_2026_10_26/inherited_local_review.md. The pair-score symmetry lemma is explicitly scoped to its two families; coherent-root admissibility at the selected parameter is Lemma 21.17. These reviews establish the stated interfaces under their cited upstream hypotheses, rather than claiming a new independent proof of every local input or an external review of the entire manuscript. The weighted star-marginal bound of the archived text is used again here (Lemma 15.28), with its density step referred to Lemma 12.6. The certificate verifies finite assertions and the authenticity of their inputs, not the analytic reductions or the external theorems (Section D.1).

The 30 October revision follows two full source-reading passes by a single Codex reviewer. The principal repairs retain the raw total-variation factor in the density-sensitive comparison and sum it only after splitting by the full selected modulus; distinguish fixed-selection asymptotics from the uniform estimates needed for the complete expansion; and apply the physical pair normalization

once, to raw endpoint coefficients. The revision also corrects the outer-logarithm remainder in Appendix F, operator domains, and several baseline-versus-final numerical attributions. The selected exponent and frozen numerical construction are unchanged. The line-coverage record, reproducible checks, findings and remaining limits are recorded in `analysis/proof_repair_review_2026_10_30.json` and its companion text report. This is an internal adversarial review, not an independent external referee report or a formal verification of the proof.
Consequences. Nothing here is a premise of the main theorem. The numbers quoted are those of Appendix D, Table 12, the checkpoints recorded in Section 34.1, the research notes named in the text and the fragments and research notes of master §§42–44 named in the text; the enumerations of Sections 35.5 and 35.8 are corroboration only.

Appendix A. External inputs

What this appendix proves and how to read it. The proof relies on a small number of published results that it does not prove. Three standard ones are stated in Section 4: Weil's bound for complete character sums (Theorem 4.7), used throughout, and the explicit Chebyshev estimate for primes in the residue classes modulo four (Theorem 4.12) and Kassabov's inequality for angles between intersections of subspaces (Theorem 4.16), whose uses are named there; Section A.4 matches them with their sources. The second-order inputs are stated here, and each is used only where named. The character table of GL_2 over a finite field (Theorem A.1) enters only Lemma 15.2, hence the four-cycle kernel bound Theorem 15.8. The ℓ-adic inputs (Section A.2) enter Lemmas 15.14 and 15.15 and, through them, the proofs of the two-variable character sum bound Theorem 15.13 and of the moving-root moments Lemma 15.19; parts (v) and (vi) of Theorem A.6 are used only in the last, through Lemma 15.14(v),(vi). Through Theorems 15.8 and 15.13 and the moving-root moments the second-order inputs reach the spectral caps of the middle and upper ranges (among them, through Corollary 15.10, the caps of the positive roots), the direct-star root (Lemma 27.6), the dense comparisons and the six-query mixture of the upper range (through Lemma 15.28) and the conditional rows of the upper range (Section 27.6.2). The modular input (Theorem A.7) enters only the second moment of the neighbourhood count (Lemma 24.3), and through it the root-half variances χ_p of the middle roots (in the restorations and in the completion of the root cut L20), the star mean of that cut (Lemma 24.5), and the linear prices and the normalization of the positive roots (Lemmas 21.5 and 21.6); Table 6 lists every use. The character table is a finite statement, and the modular input is an identity at each prime together with Deligne's bound for the coefficients of one modular form; the ℓ-adic inputs concern families of algebraic curves. The proofs using them are separate from the finite arithmetic certificates.

Peyre's tensorization theorem is not an external input: Appendix B proves the finite case of [54, Theorem 3.3.1] that the paper uses (Theorem 4.13). Together with the three theorems named above, the results of this appendix are the external theorems on which the proof depends; everything else is proved in the paper or is standard textbook material, such as Khintchine's inequality, the Chinese remainder theorem and Fourier analysis on finite abelian groups.

Notation. The results below are quoted in the notation of their sources, which is local to this appendix and to the proofs that apply them. In particular q is the number of elements of a finite field (the applications take $q = p$) or, in [4], a modulus, or, in Theorem A.7, $e^{2\pi i z}$; $N\zeta$ is a norm and $N(x)$ the size of a residue field; E is a finite field and t is the parameter of a family of curves. Deligne's field isomorphism $\overline{\mathbb{Q}}_\ell \to \mathbb{C}$, written ι in the sources, is written $\jmath$, since ι is the reserved cut of the paper; the proofs that apply these results do not name it.

A.1. **The characters of $\mathrm{GL}_2(\mathbb{F}_q)$.** For a field $\mathbb{F}_q$ we write $\mathbb{F}_{q^2}$ for its quadratic extension and, for $\zeta \in \mathbb{F}_{q^2}$, $\bar{\zeta} = \zeta^q$ and $N\zeta = \zeta\bar{\zeta}$. The group $\mathrm{GL}_2(\mathbb{F}_q)$ consists of the invertible 2×2 matrices over $\mathbb{F}_q$. A complex representation assigns a linear operator to each group element, respecting

products; its *character* is the trace of that operator. It is *irreducible* if it has no proper nonzero invariant subspace. The *degree* of a character is the dimension of its representation. The projective line $\mathbb{P}^1(\mathbb{F}_q)$ is the set of one-dimensional subspaces of $\mathbb{F}_q^2$. For general finite-group representation theory see [70, 32]; the finite-field GL_2 table is also treated in [55]. We cite the precise table as in [22, §5.2].

Theorem A.1 (The characters of $\mathrm{GL}_2(\mathbb{F}_q)$; [22, §5.2])**.** *Let q be a power of an odd prime and $\mathbb{F} = \mathbb{F}_q$. Every element of $\mathrm{GL}_2(\mathbb{F})$ is of exactly one of the following types:* (a) *a scalar matrix xI, $x \in \mathbb{F}^\times$;* (b) *a non-scalar matrix with characteristic polynomial $(T-x)^2$, $x \in \mathbb{F}^\times$;* (c) *a matrix with two eigenvalues $x \neq y$ in $\mathbb{F}^\times$;* (d) *a matrix with eigenvalues $\zeta, \bar{\zeta}$, $\zeta \in \mathbb{F}_{q^2} \setminus \mathbb{F}$. Two elements of the same type with the same eigenvalues are conjugate. For characters α, β of $\mathbb{F}^\times$ and φ of $\mathbb{F}_{q^2}^\times$ there are complex representations U_α (of dimension 1, $g \mapsto \alpha(\det g)$), V_α (dimension q), $W_{\alpha,\beta}$ for $\alpha \neq \beta$ (dimension $q+1$) and X_φ for $\varphi \neq \varphi^q$ (dimension $q-1$) such that:*

(i) *V_1 is the complement of the trivial representation in the permutation representation of $\mathrm{GL}_2(\mathbb{F})$ on $\mathbb{P}^1(\mathbb{F})$, and $V_\alpha = V_1 \otimes U_\alpha$;*

(ii) *these representations are irreducible, $W_{\alpha,\beta} \cong W_{\beta,\alpha}$ and $X_\varphi \cong X_{\varphi^q}$, there are no other isomorphisms among them, and every irreducible complex representation of $\mathrm{GL}_2(\mathbb{F})$ is isomorphic to one of them;*

(iii) *their characters take the following values at the elements of the four types:*

	(a) xI	*(b)*	*(c) $x \neq y$*	*(d) $\zeta, \bar{\zeta}$*
U_α	$\alpha(x)^2$	$\alpha(x)^2$	$\alpha(x)\alpha(y)$	$\alpha(N\zeta)$
V_α	$q\,\alpha(x)^2$	0	$\alpha(x)\alpha(y)$	$-\alpha(N\zeta)$
$W_{\alpha,\beta}$	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(x)$	$\alpha(x)\beta(y)+\alpha(y)\beta(x)$	0
X_φ	$(q-1)\varphi(x)$	$-\varphi(x)$	0	$-(\varphi(\zeta)+\varphi(\bar{\zeta}))$

(iv) *the conjugacy class of an element of type* (a), (b), (c), (d) *has* 1, q^2-1, $q(q+1)$, $q(q-1)$ *elements, and there are* $q-1$, $q-1$, $(q-1)(q-2)/2$, $q(q-1)/2$ *classes of the respective types. The centralizer of a non-scalar element g is the group of units of the algebra $\mathbb{F}[g]$, of order $q(q-1)$, $(q-1)^2$, q^2-1 for g of type* (b), (c), (d)*; the group $\mathrm{GL}_2(\mathbb{F})$ has order $(q^2-1)(q^2-q)$.*

Part (iv) is the class table at the beginning of [22, §5.2]. The centralizers named there (upper triangular matrices with equal diagonal entries, the diagonal torus, and $\mathbb{F}_{q^2}^\times$ acting on $\mathbb{F}_{q^2} \cong \mathbb{F}^2$) are the unit groups of $\mathbb{F}[g]$ for the listed representatives, and the class sizes are their indices.

A.2. **ℓ-adic sheaves on curves.** We use constructible $\overline{\mathbb{Q}}_\ell$-sheaves and étale cohomology with compact supports on schemes of finite type over a finite field $\mathbb{F}_q$ of odd characteristic p, as in [12] and [13, §1]. The exponential-sum and monodromy background is developed in Katz's books [35, 37, 38, 39]. Here $\ell \neq p$ is a fixed prime, and we fix a field isomorphism $\jmath : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$, through which traces are regarded as complex numbers. In the applications (Theorem 15.13 and the moving-root moments) these facts are applied to dense open subschemes of the affine line; the remark after Theorem A.6 also applies them to the fibres of the Katz–Sarnak family. A *lisse* sheaf is locally constant in the étale topology and has finite rank; its stalk is the vector space attached to a geometric point, and its rank is the dimension of that space. A *constructible* sheaf admits a finite stratification on whose strata it is lisse. The cohomology group H^i_c denotes étale cohomology with compact supports. For a group G acting on a vector space W, its space of *coinvariants* is $W_G = W/\langle gw - w : g \in G, w \in W\rangle$. The Tate twist (-1) is the one-dimensional factor on which geometric Frobenius over $\mathbb{F}_q$ acts by q. We use the following standard facts.

(E1) *Lisse sheaves and monodromy.* Let X be a smooth geometrically connected curve over $\mathbb{F}_q$ with a geometric generic point $\bar{\eta}$. A lisse sheaf $\mathcal{F}$ on X is the same as a continuous

representation $\rho_{\mathcal{F}}$ of $\pi_1(X,\bar\eta)$ on the finite-dimensional space $\mathcal{F}_{\bar\eta}$ [13, (1.1.6)]. For $x \in X(\mathbb{F}_q)$ the geometric Frobenius $\mathrm{Frob}_x \in \pi_1(X,\bar\eta)$ is defined up to conjugation [13, (1.1.7), (1.1.8)], and $\mathrm{Tr}(\mathrm{Frob}_x \mid \mathcal{F})$ denotes the trace of $\rho_{\mathcal{F}}(\mathrm{Frob}_x)$, i.e. the trace of Frob_x on the stalk at a geometric point over x. The geometric fundamental group $\pi_1^{\mathrm{geom}}(X) = \pi_1(X \otimes \bar{\mathbb{F}}_q, \bar\eta)$ is a closed normal subgroup of $\pi_1(X,\bar\eta)$ [13, (1.1.13), (1.1.15)]; $\mathcal{F}$ is *geometrically irreducible* if $\rho_{\mathcal{F}}|_{\pi_1^{\mathrm{geom}}(X)}$ is irreducible. If $X' \subseteq X$ is a dense open subscheme, the map $\pi_1^{\mathrm{geom}}(X') \to \pi_1^{\mathrm{geom}}(X)$ is surjective: $X \otimes \bar{\mathbb{F}}_q$ is connected and normal, so by [27, Exp. V, Prop. 8.2] already the fundamental group of its generic point maps onto $\pi_1^{\mathrm{geom}}(X)$, and this map factors through $\pi_1^{\mathrm{geom}}(X')$. Hence $\mathcal{F}|_{X'}$ is geometrically irreducible if $\mathcal{F}$ is. If $\mathcal{L}$ is lisse of rank one, a subspace of $\mathcal{F}_{\bar\eta} \otimes \mathcal{L}_{\bar\eta}$ of the form $W \otimes \mathcal{L}_{\bar\eta}$ is stable under $\rho_{\mathcal{F}} \otimes \rho_{\mathcal{L}}$ iff W is stable under $\rho_{\mathcal{F}}$, so $\mathcal{F} \otimes \mathcal{L}$ is geometrically irreducible iff $\mathcal{F}$ is; and $\mathrm{Tr}(\mathrm{Frob}_x \mid \mathcal{F} \otimes \mathcal{L}) = \mathrm{Tr}(\mathrm{Frob}_x \mid \mathcal{F})\,\mathrm{Tr}(\mathrm{Frob}_x \mid \mathcal{L})$. If $\mu : X' \to X$ is an isomorphism of $\mathbb{F}_q$-schemes, $\mu^*\mathcal{F}$ is lisse, geometrically irreducible iff $\mathcal{F}$ is, and $\mathrm{Tr}(\mathrm{Frob}_x \mid \mu^*\mathcal{F}) = \mathrm{Tr}(\mathrm{Frob}_{\mu(x)} \mid \mathcal{F})$.

(E2) *Weights* [13, (1.2.1), (1.2.2), (1.2.5), (1.2.6)]. A lisse $\mathcal{F}$ on X is *$\jmath$-pure of weight w* if for every closed point x of X, with residue field of $N(x)$ elements, every eigenvalue α of Frob_x on $\mathcal{F}_{\bar x}$ satisfies $|\jmath(\alpha)| = N(x)^{w/2}$, and *pure of weight w* if it is $\jmath$-pure of weight w for every choice of $\jmath$. A sheaf is *$\jmath$-mixed of weights $\le w$* if it has a finite filtration whose successive quotients are $\jmath$-pure of weights $\le w$; a pure sheaf of weight w is $\jmath$-mixed of weights $\le w$. The tensor product of lisse sheaves pure of weights w, w' is pure of weight $w + w'$, and μ^* (μ an isomorphism) preserves purity.

(E3) *Tame ramification* [35, Ch. 1]. Let $X \subseteq \mathbb{P}^1$ be a dense open subscheme over $\mathbb{F}_q$. For a point $x \in \mathbb{P}^1(\bar{\mathbb{F}}_q) \setminus X(\bar{\mathbb{F}}_q)$ there is an *inertia group* $I_x \subseteq \pi_1^{\mathrm{geom}}(X)$ at x, a closed subgroup well defined up to conjugation, with a unique pro-p Sylow subgroup P_x (the *wild inertia group*), which is normal in I_x. The restriction of $\rho_{\mathcal{F}}$ to I_x is the *local monodromy* of a lisse $\mathcal{F}$ at x. A lisse $\mathcal{F}$ on X is *tame at x* if $\rho_{\mathcal{F}}(P_x) = 1$, and then its Swan conductor $\mathrm{Swan}_x(\mathcal{F})$ is 0; its local monodromy then factors through the tame quotient I_x/P_x, which is pro-cyclic, and is described by the action of a topological generator of I_x/P_x. The group I_x is the image of the Galois group of the field of fractions of the strict henselization of $\mathbb{P}^1_{\bar{\mathbb{F}}_q}$ at x; for a dense open $X' \subseteq X$ this map factors through $\pi_1^{\mathrm{geom}}(X')$, so the surjection of (E1) maps the inertia and wild inertia groups of X' at x onto those of X, and $\mathcal{F}|_{X'}$ has at x the local monodromy of $\mathcal{F}$; in particular it is tame at x if $\mathcal{F}$ is. If $\mathcal{F}$ is the restriction of a lisse sheaf on a dense open $X_0 \supseteq X$ with $x \in X_0(\bar{\mathbb{F}}_q)$, then $\rho_{\mathcal{F}}(I_x) = 1$; in particular $\mathcal{F}$ is tame at x. A tensor product of sheaves tame at x is tame at x. An isomorphism $\mu : X' \to X$ that extends to an automorphism $\bar\mu$ of $\mathbb{P}^1$ maps the inertia groups of X' at x onto those of X at $\bar\mu(x)$, so $\mu^*\mathcal{F}$ is tame at x iff $\mathcal{F}$ is tame at $\bar\mu(x)$, and the local monodromy of $\mu^*\mathcal{F}$ at x is that of $\mathcal{F}$ at $\bar\mu(x)$.

(E4) *Kummer sheaves* [12, Sommes trig., 1.2–1.8, in particular (1.7.6)]. Let p be odd, $\chi = \chi_p$, and $g \in \mathbb{F}_p[X]$ nonconstant. There is a lisse sheaf $\mathcal{L}_{\chi(g)}$ of rank one on $\mathbb{A}^1 \setminus Z(g)$ ($Z(g)$ the set of roots of g) with $\mathrm{Tr}(\mathrm{Frob}_z \mid \mathcal{L}_{\chi(g)}) = \chi(g(z))$ for every $z \in \mathbb{F}_p$ with $g(z) \neq 0$. Its monodromy representation is the character of order dividing 2 through which π_1 acts on the finite étale covering $\{w^2 = g(z)\}$ (the geometric Frobenius at z acts on the fibre $\{\pm w_0\}$, $w_0^2 = g(z)$, by the inverse of $w_0 \mapsto w_0^p$, fixing w_0 if $g(z)$ is a square in $\mathbb{F}_p$ and swapping $\pm w_0$ otherwise, i.e. by the sign $\chi(g(z)) = g(z)^{(p-1)/2}$). Hence every Frobenius eigenvalue is ± 1, so $\mathcal{L}_{\chi(g)}$ is pure of weight 0, and P_x, a pro-p group with p odd, acts trivially, so $\mathcal{L}_{\chi(g)}$ is tame at every point of $\mathbb{P}^1 \setminus (\mathbb{A}^1 \setminus Z(g))$. More precisely, let x be such a point and m_x the multiplicity of x as a root of g, with $m_\infty = \deg g$. The local monodromy of $\mathcal{L}_{\chi(g)}$ at x is trivial if m_x is even and is the unique character of order 2 of I_x (a topological generator of I_x/P_x acts by -1) if m_x is odd. Indeed, near a root x write $g = (z-x)^{m_x}u$ with $u(x) \neq 0$; u has a square root over the strict henselization at x, because p is odd, so the covering is locally $w'^2 = (z-x)^{m_x}$: split if

m_x is even, the tame Kummer covering of degree 2 if m_x is odd. At ∞ the same holds in the coordinate $1/z$, in which g is $(1/z)^{-\deg g}$ times a unit. In particular $\mathcal{L}_{\chi(g)}$ is geometrically nontrivial unless g is a constant times a square in $\bar{\mathbb{F}}_p[X]$.

Theorem A.2 (Grothendieck–Lefschetz trace formula; [12, Rapport, Thm. 3.2; Sommes trig., (1.1.1)])**.** *Let X be a separated scheme of finite type over $\mathbb{F}_q$ and $\mathcal{F}$ a constructible $\overline{\mathbb{Q}}_\ell$-sheaf on X. Then*

$$\sum_{x\in X(\mathbb{F}_q)} \operatorname{Tr}(\mathrm{Frob}_x \mid \mathcal{F}_{\bar{x}}) = \sum_i (-1)^i \operatorname{Tr}(\mathrm{Frob}_q \mid H^i_c(X\otimes\bar{\mathbb{F}}_q, \mathcal{F})).$$

(The Rapport states this for $\mathbb{Q}_\ell$-sheaves, and Sommes trig. (1.1.1) for E_λ-sheaves, E_λ a finite extension of $\mathbb{Q}_\ell$; a constructible $\overline{\mathbb{Q}}_\ell$-sheaf comes from an E_λ-sheaf for some such E_λ [13, (1.1.1), (1.1.6)]. The sheaves to which we apply it, in the proofs of Theorem 15.13 and of the moving-root moments, are in fact $\mathbb{Q}_\ell$-sheaves.)

Theorem A.3 (Cohomology of open curves; [13, (1.4.1)(a), (1.4.1.3), (1.4.2.3)], [12, Arcata, IV, Thm. 6.1])**.** *Let X be a smooth connected curve over an algebraically closed field that is not complete, and $\mathcal{F}$ a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X with generic stalk V, a representation of $\pi_1(X)$. Then $H^0_c(X,\mathcal{F}) = 0$ and $H^2_c(X,\mathcal{F}) \cong V_{\pi_1(X)}(-1)$, where $V_{\pi_1(X)}$ is the space of coinvariants. Moreover $H^i_c(X,\mathcal{F}) = 0$ for $i \geq 3$ (by the vanishing theorem* [12, Arcata, IV, Thm. 6.1]*, $R^i f_!$ vanishes for i larger than twice the dimension of the fibres of f; take $f : X \to \operatorname{Spec} k$). If $X = X_0 \otimes \bar{\mathbb{F}}_q$ for a smooth geometrically connected curve X_0 over $\mathbb{F}_q$ and $\mathcal{F}$ comes from a lisse sheaf on X_0, this isomorphism is compatible with the actions of* $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ [13, (1.4.2.3)].

Theorem A.4 (Deligne: bounds on weights under $R^i f_!$; [13, Thm. 3.3.1, Cor. 3.3.4 and (3.3.10)])**.** *Let $f : X \to Y$ be a morphism of schemes of finite type over $\mathbb{F}_q$ and $\mathcal{F}$ a sheaf on X that is $\jmath$-mixed of weights $\leq w$. Then $R^i f_!\mathcal{F}$ is $\jmath$-mixed of weights $\leq w+i$ for every i. In particular ($Y = \operatorname{Spec}\mathbb{F}_q$) every eigenvalue α of* Frob_q *on $H^i_c(X\otimes\bar{\mathbb{F}}_q,\mathcal{F})$ satisfies $|\jmath(\alpha)| \leq q^{(w+i)/2}$.*

(Theorem 3.3.1 and Corollary 3.3.4 are stated for sheaves that are mixed in the sense of [13, (1.2.2)], i.e. for every $\jmath$; (3.3.10) states that they hold, with the same proof, for $\jmath$-mixed sheaves. The theorem is applied to sheaves that are pure for every $\jmath$: of weight 1 in the proof of Theorem 15.13 and of the first moments (15.11), of weight 2 in the second moments (15.10); so either form suffices.)

Theorem A.5 (Grothendieck–Ogg–Šafarevič formula; [58, Thm. 1], [28, Exp. X, §7])**.** *Let k be an algebraically closed field of characteristic p, $X \subseteq \mathbb{P}^1_k$ a dense open subscheme and $\mathcal{F}$ a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X. Then*

$$\sum_i (-1)^i \dim H^i_c(X,\mathcal{F}) = \mathrm{rk}(\mathcal{F})\cdot(2 - \#(\mathbb{P}^1(k)\setminus X(k))) - \sum_{x\in\mathbb{P}^1(k)\setminus X(k)} \mathrm{Swan}_x(\mathcal{F}).$$

(Raynaud states the formula for constructible ℓ-torsion sheaves; the ℓ-adic form above is the one quoted and used in [36, proof of Lemma 10.1.12]. Here $2 - \#(\mathbb{P}^1(k)\setminus X(k))$ is the Euler characteristic of X with $\overline{\mathbb{Q}}_\ell$ coefficients. We use it only for sheaves that are tame at every point of $\mathbb{P}^1(k)\setminus X(k)$, for which all Swan terms vanish, (E3).)

Theorem A.6 (Katz–Sarnak: the family $y^2 = h(x)(x-t)$ and its monodromy [36, §10.1])**.** *Let k be a finite field of odd characteristic, $g \geq 1$, and $h \in k[X]$ a polynomial of degree $2g$ with $2g$ distinct roots in $\bar{k}$ (not necessarily monic and not necessarily split: these are exactly the hypotheses of* [36, 10.1.1]*). Let $U_h = \mathbb{A}^1_k \setminus Z(h) = \operatorname{Spec} k[T][1/h(T)]$, let $\mathrm{pr} : \mathcal{A}_h \to U_h$ be the family of affine curves $y^2 = h(x)(x-t)$, $t \in U_h$, and $\mathcal{F}_h = R^1\mathrm{pr}_!\overline{\mathbb{Q}}_\ell$ (Katz and Sarnak construct a proper smooth model $\pi : \mathcal{C}_h \to U_h$ of this family whose geometric fibres are connected curves of genus g, and show $R^1\mathrm{pr}_!\overline{\mathbb{Q}}_\ell \cong R^1\pi_*\overline{\mathbb{Q}}_\ell$,* [36, 10.1.3–10.1.11]*). Then:*

(i) [36, Lemma 10.1.9] *$\mathcal{F}_h$ is lisse of rank $2g$ on U_h and pure of weight 1;*
(ii) [36, Lemma 10.1.12] *$\mathcal{F}_h$ is tame at every point of $\mathbb{P}^1 \setminus U_h$, the point ∞ included;*
(iii) [36, Lemma 10.1.15] *$\mathcal{F}_h$ is geometrically irreducible (indeed its geometric monodromy group is $\mathrm{Sp}(2g)$, see* (v)*);*
(iv) *(the identity in the proof of* [36, Lemma 10.1.15]*) for every finite extension E of k and every $t \in U_h(E)$,*

$$-\mathrm{Tr}(\mathrm{Frob}_{E,t} \mid \mathcal{F}_h) = \sum_{x \in E} \chi_E\big((x-t)h(x)\big),$$

χ_E the quadratic character of $E^\times$ (extended by $\chi_E(0) = 0$);
(v) [36, Lemma 10.1.9 and Theorem 10.1.16] *the cup product $\mathcal{F}_h \times \mathcal{F}_h \to R^2\mathrm{pr}_!\overline{\mathbb{Q}}_\ell|_{U_h} = \overline{\mathbb{Q}}_\ell(-1)$ is a perfect alternating pairing (a symplectic autoduality of $\mathcal{F}_h$ with values in $\overline{\mathbb{Q}}_\ell(-1)$), and the geometric monodromy group of $\mathcal{F}_h$, the Zariski closure of $\rho_{\mathcal{F}_h}(\pi_1^{\mathrm{geom}}(U_h))$, is the full symplectic group $\mathrm{Sp}(2g)$ of this pairing; in particular it is connected;*
(vi) [36, Lemma 10.1.13 and Remark 10.1.17] *the local monodromy* (E3) *of $\mathcal{F}_h$ at each zero of h is a unipotent pseudoreflection: a topological generator of the tame inertia acts by an element u with $u-1$ of rank one and $(u-1)^2 = 0$; at ∞ it acts by $-u$ for such an element u. In particular the local monodromy of $\mathcal{F}_h$ at every point of $\mathbb{P}^1 \setminus U_h$ is not scalar.*

The identity (iv) is also immediate from Theorems A.2 and A.3: the stalk of $\mathcal{F}_h$ at a geometric point $\bar{t}$ over t is $H^1_c(\mathcal{A}_{h,\bar{t}}, \overline{\mathbb{Q}}_\ell)$ (base change for $R\mathrm{pr}_!$, [12, Arcata, IV, Thm. 5.4]); the fibre $\mathcal{A}_{h,t}$ is a smooth geometrically irreducible affine curve ($(x-t)h(x)$ is squarefree of odd degree), so $H^0_c = 0$ and $H^2_c = \overline{\mathbb{Q}}_\ell(-1)$; and $\#\mathcal{A}_{h,t}(E) = \sum_{x \in E}\big(1 + \chi_E((x-t)h(x))\big)$. Katz and Sarnak do not assume h monic: whatever its leading coefficient, the curve $y^2 = h(x)(x-t)$ has a single point at infinity, at which their affine model $V^2 = Uh^{\mathrm{rev}}(U)(1-TU)$, $h^{\mathrm{rev}}(U) = U^{2g}h(1/U)$, is smooth [36, 10.1.1].

Items (v) and (vi) are what the moving-root second moments (15.10) need beyond (i)–(iv), after Lemma 15.14(v),(vi) has deduced the corresponding pairing and local-monodromy statements for polynomials with an odd number of roots by composing with inversion at a root: in the proof of Lemma 15.19 the pairing of (v) identifies each moving-root sheaf with its dual, and the local monodromy at a single root, from (vi) at a root of a moving-root sheaf and from (E4) at a root of the twist, shows that two of them are isomorphic after a quadratic twist only if their root sets agree and the twist is trivial (compare the invariant of $\mathcal{F}_h \otimes \mathcal{F}_h$ computed in the proof of [36, Lemma 10.1.15]). Only the pairing in (v) and the monodromy at the zeros of h in (vi) are used, and the diagonal main term is computed elementarily. Katz and Sarnak deduce (vi), and again (iii), from the description of $\mathcal{F}_h$ as a middle convolution (Katz, *Rigid local systems*, 3.3.6); their Lemma 10.1.13 proves directly that the local monodromy at each zero of h is trivial or a unipotent pseudoreflection, and Remark 10.1.17 (printed p. 300) adds that it is nontrivial and describes the monodromy at ∞. For $g = 1$ and $h(x) = x(x-1)$, $\mathcal{F}_h$ is the first cohomology of the Legendre family, whose local monodromies at $t = 0, 1$ are transvections and at $t = \infty$ minus a transvection.

A.3. **A modular coefficient.** The second moment of the neighbourhood count in the variance credits (Lemma 24.3) contains the moment of one graph on five labels, which a translation and a dilation reduce to the twisted second moment $\sum_\lambda \chi(\lambda)T_\lambda^2$ of the traces T_λ of the Legendre family of elliptic curves. Weil's bound $|T_\lambda| \le 2\sqrt{p}$ alone gives $O(p^2)$ for this sum. Ahlgren and Ono evaluate it exactly, through a Gaussian hypergeometric series and a cusp form of weight four, and the evaluation is $O(p^{3/2})$.

Theorem A.7 (Ahlgren–Ono: a character-sum identity via the cusp form $\eta(2z)^4\eta(4z)^4$; Theorems 2 and 6, Lemma 2.2 [1])**.** *Let p be an odd prime, χ the quadratic character of $\mathbb{F}_p$ with $\chi(0) = 0$,*

$T_\lambda = \sum_{y\in\mathbb{F}_p} \chi(y(y-1)(y-\lambda))$ for $\lambda \in \mathbb{F}_p$, and let the integers $a(n)$ be the coefficients of

$$\sum_{n\ge1} a(n)q^n = q\prod_{n\ge1}(1-q^{2n})^4(1-q^{4n})^4 = \eta(2z)^4\eta(4z)^4, \qquad q = e^{2\pi iz},$$

where $\eta(z) = q^{1/24}\prod_{n\ge1}(1-q^n)$ is Dedekind's eta function. Then

(a) $\sum_{\lambda\in\mathbb{F}_p} \chi(\lambda)T_\lambda^2 = -p - a(p)$;
(b) $|a(p)| \le 2p^{3/2}$.

Source and normalization. Ahlgren and Ono extend every multiplicative character of $\mathbb{F}_p$, the trivial character ε included, by the value 0 at 0; they write ϕ for the quadratic character and ${}_{n+1}F_n(x)$ for Greene's Gaussian hypergeometric series whose upper characters are all quadratic and whose lower characters are all trivial. Their formula (2.1), from Greene, is $p\,{}_2F_1(\lambda) = \varepsilon(\lambda)\phi(-1)\sum_x \phi(x(1-x)(1-\lambda x))$. For $\lambda \ne 0$ the term $x = 0$ vanishes, and the substitution $x = 1/y$ turns $x(1-x)(1-\lambda x)$ into $y^{-4}\,y(y-1)(y-\lambda)$; so $p\,{}_2F_1(\lambda) = \chi(-1)T_\lambda$. The term $\lambda = 0$ of (a) vanishes because $\chi(0) = 0$, so no identity is needed there. Their Lemma 2.2, $p^3\,{}_4F_3(1) = p^2\sum_\lambda \phi(\lambda)\,{}_2F_1(\lambda)^2$, therefore reads $p^3\,{}_4F_3(1) = \sum_\lambda \chi(\lambda)T_\lambda^2$, and their Theorem 6, $p^3\,{}_4F_3(1) = -a(p) - p$ with the coefficients $a(n)$ of their (1.5), is (a). In the proof of their Theorem 2 they observe that $\eta(2z)^4\eta(4z)^4$ is a cusp form of weight four on $\Gamma_0(8)$ spanning a space of dimension one, hence a normalized Hecke eigenform, and Deligne's theorem [14] gives (b); their Theorem 2 is the consequence $|{}_4F_3(1) + p^{-2}| \le 2p^{-3/2}$. Both statements hold for every odd prime, with no congruence condition and no lower bound on p; the applications take $p \equiv 3 \pmod 4$. The theorem, lemma and equation numbers here refer specifically to the authenticated author manuscript, not to an independently checked numbering of the published typesetting. In that manuscript (1.5) is on printed page 3, (2.1) and Lemma 2.2 are on page 5, Theorem 6 is on page 7 and the proof of Theorem 2 is on page 11; the copy consulted (SHA-256 `e8bcbe78...74b96a`) and the reduction of the graph moment to (a) are recorded in `research/roadmap_2026_10_18/item17/AHLGREN_ONO_SOURCE_REVIEW.md`. Finite evaluations of both sides at small primes are normalization checks only; they are not inputs of the proof.

A.4. **Sources of the standard inputs.** *Weil's bound.* Part (a) of Theorem 4.7 is the case of the prime field and the quadratic character of [33, Theorem 11.23] (with $n = 1$ there): if χ is a multiplicative character of order $d > 1$ of a finite field F with q elements and $f \in F[X]$ has m distinct roots and is not a constant times a d-th power over $\bar F$, then $\bigl|\sum_{x\in F}\chi(f(x))\bigr| \le (m-1)\sqrt{q}$. Iwaniec and Kowalski take it from [69, Theorem 2C′]; the bound goes back to Weil [73, 74]. A product of $t \ge 1$ distinct linear factors has t simple roots, so it is not a square. Part (b) of Theorem 4.7 is deduced from part (a) where it is stated and needs no further input.

Primes in the residue classes modulo four. Theorem 4.12 is [4, Theorem 1.2] for the modulus $q = 4$: for $q \ge 3$ and a coprime to q, $|\theta(x;q,a) - x/\varphi(q)| < c_\theta(q)\,x/\log x$ for all $x \ge x_\theta(q)$, and Table 1 (p. 432 of the published version) gives $c_\theta(4) = 0.0004822$ and $x_\theta(4) = 4\,800\,162\,889$. The preprint `arXiv:1802.00085v2` lists different values for $q = 4$; only the published ones are used.

Kassabov's inequality. Theorem 4.16 is [34, Lemma 4.2(b)] (printed p. 15 of `arXiv:0911.1983`); the companion bound for $\cos^2\angle(V_1\cap V_2, V_3)$, Lemma 4.1(b) there, is not used. Kassabov's angle between two closed subspaces is the Friedrichs angle of Definition 4.4 [34, Definition 3.2], defined when neither subspace contains the other. He states the lemma for numbers $\varepsilon_{ij} = \cos\alpha_{ij}$ with $0 < \alpha_{ij} \le \angle(V_i, V_j)$; in finite dimension such angles are positive [34, Corollary 3.7], and $\alpha_{ij} = \angle(V_i, V_j)$ gives the form of Theorem 4.16. If one of the three subspaces contains another, Definition 4.4 gives that pair the cosine 0, and the inequality holds directly. If $V_1 \subseteq V_3$, the left side is $\cos\angle(V_1, V_2\cap V_3)$, and the vectors admitted in its supremum are admitted for $\cos\angle(V_1, V_2)$, both being orthogonal to $V_1\cap V_2\cap V_3 = V_1\cap V_2$; so it is at most ε_{12}, which is at most the right

side since $\varepsilon_{13} = 0$. The case $V_2 \subseteq V_3$ is symmetric, and in the remaining cases one of $V_1 \cap V_3$, $V_2 \cap V_3$ contains the other, so the left side is 0.
Consequences and what to read next. This appendix assumes nothing from the rest of the paper, except Definition 4.4 for the matching of Kassabov's statement. The remarks after Theorem A.1 and Theorem A.6 use only the cited sources and Theorems A.2 and A.3.

Appendix B. Peyre's tensorization in finite form

What this appendix proves and how to read it. This appendix proves Theorem 4.13, the case of Peyre's tensorization theorem [54, Theorem 3.3.1] that the local-law estimates use: finitely many random variables with finitely many values. If each variable of one family and each variable of a disjoint family have conditional maximal correlation at most $\varepsilon_{ij} < 1$ (the numbers in the matrix $\mathcal{E}$ below), given any values of any of the other variables, then the two families have maximal correlation at most the operator norm of the matrix of these bounds. The paper applies the theorem in the pinned influence theorem (Proposition 13.16), which passes from the conditional correlations of pairs of vertices of a strict-square graph law to those of disjoint sets of vertices, and through it in the correlation bounds of the construction, among them the row correlations of the twelve-label middle reference and the conditional influence bound of the upper range. The conditional decorrelation inequality stated with the theorem (Corollary 4.15) is part (b) of Lemma B.1 below, and the corollary for independent pairs (Corollary 4.14) is the diagonal case (Remark B.6(i)). Only the first statement of the theorem is used; the second, the prefix form, needs no additional argument, because the proof starts from it.

The argument is Peyre's [54, §3.3.a]. In the finite setting no approximation argument is needed. We replace the inductive bookkeeping of [54, Lemma 3.3.7] by a monotone potential, and the Perron–Frobenius step [54, Lemma 3.8.1] by an explicit Neumann series. We also record, following [54, §3.4.a], that only conditionings on initial segments of two fixed orderings are used. Remark B.5 compares the proof with the source. The proof supplies the finite theorem used in this paper, independently of the publication status of the source.

For a real matrix $\mathcal{E} = (\varepsilon_{ij})_{i\in A,\, j\in B}$, $\|\mathcal{E}\|_{\mathrm{op}}$ is its norm as an operator $\ell^2(B) \to \ell^2(A)$ (Euclidean norms, as in Definition 4.1), i.e. its largest singular value. Permuting rows or columns does not change it. As in Peyre's paper, all letters in this appendix are local to it; in particular A and B are index sets, and N and M are their cardinalities.

Theorem (Theorem 4.13, restated: tensorization, including the prefix form)**.** *Let A, B be disjoint nonempty finite sets, and let $(X_v)_{v\in A\cup B}$ be random variables on a common probability space, each taking finitely many values. Let $\varepsilon_{ij} \geq 0$ for $i \in A$, $j \in B$, and $\mathcal{E} = (\varepsilon_{ij})$. Suppose that for all $i \in A$, $j \in B$, every $S \subseteq (A \cup B) \setminus \{i, j\}$ and every value s of $X_S = (X_v)_{v\in S}$ with $\mathbb{P}(X_S = s) > 0$,*

$$\rho(X_i; X_j \mid X_S = s) \leq \varepsilon_{ij}. \tag{4.3}$$

Then

$$\rho\big((X_a)_{a\in A}; (X_b)_{b\in B}\big) \leq \min\{1, \|\mathcal{E}\|_{\mathrm{op}}\}. \tag{4.4}$$

More precisely, (4.4) *holds as soon as there are enumerations $A = \{a_1, \dots, a_N\}$ and $B = \{b_1, \dots, b_M\}$ such that* (4.3) *holds for $i = a_k$, $j = b_l$ and the single set $S = \{a_1, \dots, a_{k-1}\} \cup \{b_1, \dots, b_{l-1}\}$, for all $1 \leq k \leq N$ and $1 \leq l \leq M$.*

The second statement has a weaker hypothesis, because the sets S it uses are among those allowed in the first: they avoid a_k and b_l.

B.1. **Conventions.** *Reduction to a finite space.* Every quantity in the theorem is a function of the joint law of the finite tuple $(X_v)_{v\in A\cup B}$. Indeed, square-integrable functions of the X_v are functions on the finite set of values of the tuple, and conditional laws are taken only given events of positive probability. We may therefore replace the probability space by the finite set Ω of values of the tuple that have positive probability, with the induced law, and each X_v by the corresponding coordinate function on Ω. *Throughout this appendix,* $(\Omega,\mathbb{P})$ *is a finite probability space with* $\mathbb{P}(\{\omega\})>0$ *for every* $\omega\in\Omega$, and all functions are real valued. Then $L^2(\mathbb{P})=\mathbb{R}^\Omega$, with $\langle f,g\rangle=\mathbb{E}[fg]$ and $\|f\|=\mathbb{E}[f^2]^{1/2}$.

Spaces of functions of variables. For a random variable Z on Ω, with values in any set, $L^2(Z)$ is the space of functions of the form $\omega\mapsto\varphi(Z(\omega))$. Equivalently, it is the space of functions that are constant on each level set $\{Z=z\}$. For finitely many variables $Z_1,\dots,Z_k$ we write $L^2(Z_1,\dots,Z_k)=L^2((Z_1,\dots,Z_k))$; for the empty list this is the space of constants. We write $\mathbb{E}[f\mid Z]$ for the orthogonal projection of f onto $L^2(Z)$. Explicitly, $\mathbb{E}[f\mid Z](\omega)=\mathbb{E}[f\mathbf{1}_{\{Z=Z(\omega)\}}]/\mathbb{P}(Z=Z(\omega))$ is the average of f over the level set of Z containing ω. For the empty list, $\mathbb{E}[f\mid\cdot\,]=\mathbb{E}f$. We use the following standard facts.

(P1) If $L^2(Z)\subseteq L^2(W)$, then $\mathbb{E}[\mathbb{E}[f\mid W]\mid Z]=\mathbb{E}[f\mid Z]$ and $\mathbb{E}[\mathbb{E}[f\mid Z]\mid W]=\mathbb{E}[f\mid Z]$. In particular $\mathbb{E}[h\mid W]=h$ for $h\in L^2(W)$.

(P2) $\|f\|^2=\|\mathbb{E}[f\mid Z]\|^2+\|f-\mathbb{E}[f\mid Z]\|^2$, and $f-\mathbb{E}[f\mid Z]\perp L^2(Z)$. Moreover $\|f-\mathbb{E}[f\mid Z]\|\le\|f-h\|$ for every $h\in L^2(Z)$. In particular $\|f-\mathbb{E}[f\mid Z]\|\le\|f\|$, and $\|f-\mathbb{E}[f\mid Z]\|\le\|f-\mathbb{E}f\|$ since constants lie in $L^2(Z)$.

(P3) Let W be a random variable and w a value with $\mathbb{P}(W=w)>0$. Put $\Omega_w=\{W=w\}$, with the conditional law $\mathbb{P}_w=\mathbb{P}(\,\cdot\mid W=w)$ and expectation $\mathbb{E}_w$. Then for every random variable Z and every f, the restriction of $\mathbb{E}[f\mid Z,W]$ to Ω_w equals $\mathbb{E}_w[f|_{\Omega_w}\mid Z|_{\Omega_w}]$. This holds because the level sets of $Z|_{\Omega_w}$ in Ω_w are exactly the level sets of (Z,W) contained in Ω_w, and on each of them both functions are the average of f for the same (proportional) weights.

Maximal correlation. Maximal and conditional maximal correlations are those of Definition 4.3. In the present notation, with $L^2_0(X)=\{f\in L^2(X):\mathbb{E}f=0\}$, $\rho(X;Y)$ is the supremum of $|\mathbb{E}[fg]|$ over $f\in L^2_0(X)$ and $g\in L^2_0(Y)$ with $\|f\|=\|g\|=1$, and $\rho(X;Y)=0$ if X or Y is constant; by Cauchy–Schwarz, $\rho(X;Y)\le1$. For an event E with $\mathbb{P}(E)>0$, $\rho(X;Y\mid E)$ is the same quantity computed in the finite probability space $(E,\mathbb{P}(\,\cdot\mid E))$ for the restrictions of X and Y to E, and $\rho(X;Y\mid Z=z)=\rho(X;Y\mid\{Z=z\})$.

Orderings. For random variables $X_1,\dots,X_N$ and $0\le i\le N$ we write $X_{\le i}=(X_1,\dots,X_i)$ and $X_{<i}=X_{\le i-1}$. The list $X_{\le0}$ is empty, so $L^2(X_{\le0})$ consists of the constants and conditioning on $X_{<1}$ is no conditioning.

B.2. **Elementary facts.** Three elementary consequences of the definition of maximal correlation are used repeatedly; part (b) is the conditional decorrelation inequality applied in the main text.

Lemma B.1 (Projection consequences of maximal correlation)**.** *Let* X,Y,Z *be random variables on* Ω.

(a) If $f\in L^2_0(X)$ *and* $g\in L^2(Y)$, *then* $|\mathbb{E}[fg]|\le\rho(X;Y)\,\|f\|\,\|g-\mathbb{E}g\|$. *In particular* $\|\mathbb{E}[f\mid Y]\|\le\rho(X;Y)\,\|f\|$ *for* $f\in L^2_0(X)$.

(b) Let $\varepsilon\ge0$, *and assume* $\rho(X;Y\mid Z=z)\le\varepsilon$ *for every* z *with* $\mathbb{P}(Z=z)>0$. *If* $f\in L^2(X,Z)$ *satisfies* $\mathbb{E}[f\mid Z]=0$ *and* $g\in L^2(Y,Z)$, *then* $|\mathbb{E}[fg]|\le\varepsilon\,\|f\|\,\|g-\mathbb{E}[g\mid Z]\|\le\varepsilon\,\|f\|\,\|g\|$.

(c) If $\lambda\ge0$ *and* $\|\mathbb{E}[f\mid Y]\|^2\le\lambda\,\|f\|^2$ *for every* $f\in L^2_0(X)$, *then* $\rho(X;Y)\le\sqrt{\lambda}$.

Part (b), the *conditional decorrelation* inequality, is the statement of Corollary 4.15; the proof below is complete in itself and does not use that corollary.

Proof. (a) Since $\mathbb{E}f = 0$, $\mathbb{E}[fg] = \mathbb{E}[f(g - \mathbb{E}g)]$, and $g - \mathbb{E}g \in L^2_0(Y)$. If $f = 0$ or $g = \mathbb{E}g$ there is nothing to prove. Otherwise $f/\|f\|$ and $(g - \mathbb{E}g)/\|g - \mathbb{E}g\|$ are admissible in the definition of $\rho(X;Y)$, which gives the inequality. For the second claim put $h = \mathbb{E}[f \mid Y] \in L^2(Y)$. Then $\mathbb{E}h = \mathbb{E}f = 0$ by (P1), and $\|h\|^2 = \mathbb{E}[hf]$ because $f - h \perp L^2(Y)$ by (P2). By the first claim with $g = h$, $\|h\|^2 \le \rho(X;Y)\,\|f\|\,\|h\|$.

(b) Let z be a value with $\mathbb{P}(Z = z) > 0$, and put $\Omega_z = \{Z = z\}$, $\mathbb{P}_z = \mathbb{P}(\cdot \mid Z = z)$, with expectation $\mathbb{E}_z$ and norm $\|\cdot\|_z$. Write $f = F(X,Z)$ and $g = G(Y,Z)$ for some functions F, G. On Ω_z we have $f = F(X,z)$, a function of X, and $g = G(Y,z)$, a function of Y. Moreover $\mathbb{E}_z f$ is the value of $\mathbb{E}[f \mid Z] = 0$ on Ω_z, and $\mathbb{E}_z g$ is the value of $\mathbb{E}[g \mid Z]$ on Ω_z. Part (a) in the space $(\Omega_z, \mathbb{P}_z)$ gives

$$|\mathbb{E}_z[fg]| \le \varepsilon\ \|f\|_z\ \|g - \mathbb{E}[g \mid Z]\|_z\,.$$

Since $\mathbb{E}[u] = \sum_z \mathbb{P}(Z = z)\mathbb{E}_z[u]$ for every function u, Cauchy–Schwarz for the sum over z gives

$$\begin{aligned}|\mathbb{E}[fg]| &\le \varepsilon \sum_z \mathbb{P}(Z = z)\,\|f\|_z\,\|g - \mathbb{E}[g \mid Z]\|_z \\ &\le \varepsilon \Big(\sum_z \mathbb{P}(Z = z)\,\|f\|_z^2\Big)^{1/2} \Big(\sum_z \mathbb{P}(Z = z)\,\|g - \mathbb{E}[g \mid Z]\|_z^2\Big)^{1/2},\end{aligned}$$

and the right side equals $\varepsilon\,\|f\|\,\|g - \mathbb{E}[g \mid Z]\|$. Finally $\|g - \mathbb{E}[g \mid Z]\| \le \|g\|$ by (P2).

(c) Let $f \in L^2_0(X)$ and $g \in L^2_0(Y)$ have norm 1. Since $g \in L^2(Y)$ and $f - \mathbb{E}[f \mid Y] \perp L^2(Y)$, $|\mathbb{E}[fg]| = |\mathbb{E}[\mathbb{E}[f \mid Y]\,g]| \le \|\mathbb{E}[f \mid Y]\| \le \sqrt{\lambda}$. □

B.3. **One variable against a list.** The next lemma is [54, Lemma 3.3.3], with its proof [54, Claims 3.3.4 and 3.3.5].

Lemma B.2 (One new variable and martingale differences)**.** *Let $X_1, \dots, X_N, Y$ be random variables on Ω and $\varepsilon_1, \dots, \varepsilon_N \ge 0$. Assume that for every $1 \le i \le N$ and every value x of $X_{<i}$ with $\mathbb{P}(X_{<i} = x) > 0$,*

$$\rho(X_i; Y \mid X_{<i} = x) \le \varepsilon_i \tag{B.1}$$

(for $i = 1$ this reads $\rho(X_1;Y) \le \varepsilon_1$). Let $f \in L^2(X_{\le N})$, and for $1 \le i \le N$ put

$$f_i = \mathbb{E}[f \mid X_{\le i}] - \mathbb{E}[f \mid X_{<i}], \qquad f_i^* = \mathbb{E}[f \mid X_{\le i}, Y] - \mathbb{E}[f \mid X_{<i}, Y].$$

These successive differences of conditional expectations are martingale differences*. Put $w_i = \|f_i\|^2$ and $w_i^* = \|f_i^*\|^2$. Then for every $1 \le i \le N$,*

$$w_i^* \ \ge\ (1 - \varepsilon_i^2)\,w_i - 2\varepsilon_i\sqrt{w_i} \sum_{k=i+1}^{N} \varepsilon_k \sqrt{w_k}. \tag{B.2}$$

Proof. For $0 \le i \le N$ put $r_i = f - \mathbb{E}[f \mid X_{\le i}]$ and $r_i^* = f - \mathbb{E}[f \mid X_{\le i}, Y]$. Since $f \in L^2(X_{\le N}) \subseteq L^2(X_{\le N}, Y)$, (P1) gives $r_N = r_N^* = 0$. For $1 \le i \le N$,

$$r_{i-1} = f_i + r_i, \qquad r_{i-1}^* = f_i^* + r_i^*.$$

Here $f_i \in L^2(X_{\le i})$ and $r_i \perp L^2(X_{\le i})$ by (P2), so $\|r_{i-1}\|^2 = w_i + \|r_i\|^2$. Likewise $f_i^* \in L^2(X_{\le i}, Y)$ and $r_i^* \perp L^2(X_{\le i}, Y)$, so $\|r_{i-1}^*\|^2 = w_i^* + \|r_i^*\|^2$. Put $D_i = \|r_i\|^2 - \|r_i^*\|^2$ for $0 \le i \le N$. Subtracting the two identities,

$$w_i - w_i^* = D_{i-1} - D_i \qquad (1 \le i \le N), \qquad D_N = 0. \tag{B.3}$$

Step 1: $r_i - r_i^* = \mathbb{E}[r_i \mid X_{\le i}, Y]$, and hence $D_i = \|\mathbb{E}[r_i \mid X_{\le i}, Y]\|^2 \ge 0$. Indeed $\mathbb{E}[f \mid X_{\le i}] \in L^2(X_{\le i}) \subseteq L^2(X_{\le i}, Y)$, so by (P1)

$$r_i - r_i^* = \mathbb{E}[f \mid X_{\le i}, Y] - \mathbb{E}[f \mid X_{\le i}] = \mathbb{E}\big[f - \mathbb{E}[f \mid X_{\le i}] \,\big|\, X_{\le i}, Y\big] = \mathbb{E}[r_i \mid X_{\le i}, Y].$$

Hence $r_i^* = r_i - \mathbb{E}[r_i \mid X_{\le i}, Y]$, and (P2) gives $\|r_i\|^2 = \|\mathbb{E}[r_i \mid X_{\le i}, Y]\|^2 + \|r_i^*\|^2$.

Step 2: $\sqrt{D_{i-1}} \le \varepsilon_i\sqrt{w_i} + \sqrt{D_i}$ *for* $1 \le i \le N$. Let $h = \mathbb{E}[r_{i-1} \mid X_{<i}, Y]$. By Step 1 (with $i-1$) and (P2), $D_{i-1} = \|h\|^2 = \mathbb{E}[r_{i-1}h]$, and

$$\mathbb{E}[r_{i-1}h] = \mathbb{E}[f_i h] + \mathbb{E}[r_i h].$$

For the first term, $f_i \in L^2(X_i, X_{<i})$, $h \in L^2(Y, X_{<i})$, and $\mathbb{E}[f_i \mid X_{<i}] = \mathbb{E}[f \mid X_{<i}] - \mathbb{E}[f \mid X_{<i}] = 0$ by (P1). By (B.1), Lemma B.1(b) applies with $X = X_i$, $Z = X_{<i}$ and $\varepsilon = \varepsilon_i$, and gives $|\mathbb{E}[f_i h]| \le \varepsilon_i\sqrt{w_i}\,\|h\|$. For the second term, $h \in L^2(X_{<i}, Y) \subseteq L^2(X_{\le i}, Y)$, so by (P2) and Step 1

$$|\mathbb{E}[r_i h]| = |\mathbb{E}[\mathbb{E}[r_i \mid X_{\le i}, Y]\, h]| \le \|\mathbb{E}[r_i \mid X_{\le i}, Y]\|\;\|h\| = \sqrt{D_i}\;\|h\|\,.$$

Thus $\|h\|^2 \le (\varepsilon_i\sqrt{w_i} + \sqrt{D_i})\,\|h\|$, which gives Step 2 (trivially if $h = 0$).

Step 3: $\sqrt{D_i} \le \sum_{k>i} \varepsilon_k\sqrt{w_k}$ *for* $0 \le i \le N$. This holds for $i = N$ because $D_N = 0$, and Step 2 carries it from i to $i-1$.

Step 4. By (B.3), Step 2 and Step 3,

$$w_i - w_i^* = D_{i-1} - D_i \le \left(\varepsilon_i\sqrt{w_i} + \sqrt{D_i}\right)^2 - D_i = \varepsilon_i^2 w_i + 2\varepsilon_i\sqrt{w_i}\sqrt{D_i} \le \varepsilon_i^2 w_i + 2\varepsilon_i\sqrt{w_i}\sum_{k>i}\varepsilon_k\sqrt{w_k},$$

which is (B.2). □

In words: the deficit D_i is the squared norm of the part of the residual r_i that Y explains beyond $X_{\le i}$. It vanishes at $i = N$, its square root grows by at most $\varepsilon_i\sqrt{w_i}$ from i to $i-1$ because f_i is conditionally decorrelated from Y given $X_{<i}$, and its increments are the losses $w_i - w_i^*$.

B.4. **Positive weights.** The proof weights the N variables of one family by a positive vector α with $K\alpha \le \lambda\alpha$, where K is built from the correlation bounds. Such a vector is supplied explicitly by a Neumann series, with no appeal to Perron–Frobenius theory.

Lemma B.3 (Positive weights from a Neumann series)**.** *Let K be an $N \times N$ matrix with nonnegative entries and $\lambda > \|K\|_{\mathrm{op}}$. Then the series $\alpha = \sum_{m\ge 0} \lambda^{-m-1}K^m\mathbf{1}$, where $\mathbf{1} = (1, \dots, 1)^{\mathsf{T}}$, converges. Its entries satisfy $\alpha_i \ge \lambda^{-1} > 0$, and $K\alpha \le \lambda\alpha$ entrywise.*

Proof. The terms satisfy $\left\|\lambda^{-m-1}K^m\mathbf{1}\right\| \le \lambda^{-1}(\|K\|_{\mathrm{op}}/\lambda)^m\sqrt{N}$, so the series converges. All terms have nonnegative entries, and the term $m = 0$ is $\lambda^{-1}\mathbf{1}$, so $\alpha_i \ge \lambda^{-1}$. Finally $K\alpha = \sum_{m\ge 0}\lambda^{-m-1}K^{m+1}\mathbf{1} = \lambda(\alpha - \lambda^{-1}\mathbf{1}) = \lambda\alpha - \mathbf{1} \le \lambda\alpha$. □

Lemma B.4 (A monotone potential for nonnegative losses)**.** *Let $w^0, \dots, w^M \in [0,\infty)^N$ and let $T_1, \dots, T_M$ be matrices with nonnegative entries. Suppose $w^l \ge w^{l-1} - T_l w^{l-1}$ entrywise. Write $c_k^{(l)} = \sum_i (T_l)_{ik}$ for the column sums. If $\sum_l c_k^{(l)} \le \lambda < 1$ for every k, then*

$$\sum_k w_k^M \ge (1-\lambda)\sum_k w_k^0.$$

Proof. For $0 \le l \le M$, put $C_l = \sum_{m=l+1}^{M} c^{(m)}$ and $\Phi_l = \sum_k (1 - (C_l)_k)w_k^l$. Then $C_M = 0$, $C_{l-1} = C_l + c^{(l)}$, and $0 \le C_l \le \lambda < 1$ entrywise. The coefficients in Φ_l are positive, so

$$\begin{aligned}\Phi_l &\ge \sum_k (1-(C_l)_k)(w^{l-1} - T_l w^{l-1})_k \\ &= \Phi_{l-1} + \sum_k (C_l)_k (T_l w^{l-1})_k \ge \Phi_{l-1}.\end{aligned}$$

Consequently $\sum_k w_k^M = \Phi_M \ge \Phi_0 \ge (1-\lambda)\sum_k w_k^0$. □

The equality in the display uses $\sum_k (T_l w^{l-1})_k = \sum_k c_k^{(l)} w_k^{l-1}$ and $C_{l-1} = C_l + c^{(l)}$. No coordinate of w^l is assumed to be monotone in l; the potential Φ_l, which weights each coordinate by one minus the column sums still to come, is.

B.5. **Proof of the tensorization theorem.** The proof fixes the two enumerations, conditions successively on initial segments, and shows that a weighted energy cannot lose more than the factor allowed by $\|\mathcal{E}\|_{\mathrm{op}}$.

Proof of Theorem 4.13. Reduction. The first hypothesis implies the second one for any enumerations of A and B, as noted after the statement. So fix enumerations $A = \{a_1, \dots, a_N\}$ and $B = \{b_1, \dots, b_M\}$ for which the second hypothesis holds. Write $X_k := X_{a_k}$, $Y_l := X_{b_l}$, $\varepsilon_{kl} := \varepsilon_{a_k b_l}$, $\mathbf{X} = X_{\le N}$ and $\mathbf{Y} = Y_{\le M}$. Since $\{X_S = s\} = \{X_{<k} = x,\ Y_{<l} = y\}$ for $S = \{a_1, \dots, a_{k-1}\} \cup \{b_1, \dots, b_{l-1}\}$ and $s = (x, y)$, the hypothesis reads: for $1 \le k \le N$, $1 \le l \le M$, and every (x, y) with $\mathbb{P}(X_{<k} = x, Y_{<l} = y) > 0$,

$$\rho(X_k; Y_l \mid X_{<k} = x,\ Y_{<l} = y) \le \varepsilon_{kl}. \tag{B.4}$$

We must show $\rho(\mathbf{X}; \mathbf{Y}) \le \min\{1, \|\mathcal{E}\|_{\mathrm{op}}\}$, where now $\mathcal{E} = (\varepsilon_{kl})$ is an $N \times M$ matrix. As explained in Section B.1, we may assume that Ω is finite and that all its points have positive probability. Since $\rho \le 1$, we may assume $\|\mathcal{E}\|_{\mathrm{op}} < 1$. Fix λ with $\|\mathcal{E}\|_{\mathrm{op}}^2 < \lambda < 1$. The matrix $K = \mathcal{E}\mathcal{E}^{\mathsf{T}}$ has nonnegative entries and $\|K\|_{\mathrm{op}} = \|\mathcal{E}\|_{\mathrm{op}}^2 < \lambda$. By Lemma B.3 there is $\alpha \in (0, \infty)^N$ with

$$\sum_{l=1}^{M} \varepsilon_{kl} \sum_{i=1}^{N} \varepsilon_{il} \alpha_i = (\mathcal{E}\mathcal{E}^{\mathsf{T}} \alpha)_k \le \lambda \alpha_k \qquad (1 \le k \le N). \tag{B.5}$$

Fix $f \in L_0^2(\mathbf{X})$. By Lemma B.1(c) it suffices to prove

$$\|\mathbb{E}[f \mid \mathbf{Y}]\|^2 \le \lambda \, \|f\|^2; \tag{B.6}$$

letting λ decrease to $\|\mathcal{E}\|_{\mathrm{op}}^2$ then gives $\rho(\mathbf{X}; \mathbf{Y}) \le \|\mathcal{E}\|_{\mathrm{op}}$.

Step 1: a double martingale decomposition. For $0 \le l \le M$ and $1 \le i \le N$ put

$$f_i^l = \mathbb{E}[f \mid X_{\le i}, Y_{\le l}] - \mathbb{E}[f \mid X_{<i}, Y_{\le l}], \qquad w_i^l = \left\| f_i^l \right\|^2,$$

and let $w^l = (w_1^l, \dots, w_N^l)^{\mathsf{T}} \in [0, \infty)^N$. For fixed l the sum over i telescopes. Since $f \in L^2(X_{\le N}, Y_{\le l})$, (P1) gives $\sum_{i=1}^N f_i^l = f - \mathbb{E}[f \mid Y_{\le l}]$. The summands are pairwise orthogonal: for $i < k$, $f_i^l \in L^2(X_{\le i}, Y_{\le l}) \subseteq L^2(X_{<k}, Y_{\le l})$, while $f_k^l \perp L^2(X_{<k}, Y_{\le l})$. The latter holds because f_k^l is the difference of $f - \mathbb{E}[f \mid X_{<k}, Y_{\le l}]$ and $f - \mathbb{E}[f \mid X_{\le k}, Y_{\le l}]$, and both are orthogonal to $L^2(X_{<k}, Y_{\le l})$ by (P2). Hence

$$\sum_{i=1}^{N} w_i^l = \|f - \mathbb{E}[f \mid Y_{\le l}]\|^2 \qquad (0 \le l \le M). \tag{B.7}$$

For $l = 0$ this is $\|f - \mathbb{E}f\|^2 = \|f\|^2$. For $l = M$ it is $\|f - \mathbb{E}[f \mid \mathbf{Y}]\|^2 = \|f\|^2 - \|\mathbb{E}[f \mid \mathbf{Y}]\|^2$, by (P2).

Step 2: the recursive inequality. We claim that for $1 \le l \le M$ and $1 \le i \le N$,

$$w_i^l \ \ge \ (1 - \varepsilon_{il}^2)\, w_i^{l-1} - \sum_{k=i+1}^{N} \varepsilon_{il} \varepsilon_{kl} \Big(\frac{\alpha_k}{\alpha_i}\, w_i^{l-1} + \frac{\alpha_i}{\alpha_k}\, w_k^{l-1} \Big). \tag{B.8}$$

Fix l, and a value y of $Y_{<l}$ with $\mathbb{P}(Y_{<l} = y) > 0$ (for $l = 1$ there is only the empty value, and $\Omega_y = \Omega$). Put $\Omega_y = \{Y_{<l} = y\}$ and $\mathbb{P}_y = \mathbb{P}(\,\cdot \mid Y_{<l} = y)$, with expectation $\mathbb{E}_y$. We apply Lemma B.2 in the finite probability space $(\Omega_y, \mathbb{P}_y)$, all of whose points have positive mass. We use the restrictions of $X_1, \dots, X_N$ and of $Y := Y_l$, the numbers $\varepsilon_{1l}, \dots, \varepsilon_{Nl}$, and the restriction $\varphi = f|_{\Omega_y}$. Since f is a function of $\mathbf{X}$, $\varphi \in L^2(X_{\le N})$ on Ω_y. Hypothesis (B.1) holds: if $\mathbb{P}_y(X_{<i} = x) > 0$, then $\mathbb{P}_y(\,\cdot \mid X_{<i} = x) = \mathbb{P}(\,\cdot \mid X_{<i} = x, Y_{<l} = y)$, and (B.4) applies. By (P3) with $W = Y_{<l}$, for every $0 \le i \le N$ the restrictions to Ω_y satisfy

$$\mathbb{E}_y[\varphi \mid X_{\le i}] = \mathbb{E}[f \mid X_{\le i}, Y_{<l}]\big|_{\Omega_y}, \qquad \mathbb{E}_y[\varphi \mid X_{\le i}, Y_l] = \mathbb{E}[f \mid X_{\le i}, Y_{\le l}]\big|_{\Omega_y}.$$

Therefore the functions φ_i, φ_i^* of Lemma B.2 are $f_i^{l-1}|_{\Omega_y}$ and $f_i^l|_{\Omega_y}$. Writing $v_i(y) = \mathbb{E}_y[(f_i^{l-1})^2]$ and $v_i^*(y) = \mathbb{E}_y[(f_i^l)^2]$, (B.2) gives

$$\begin{aligned} v_i^*(y) &\geq (1-\varepsilon_{il}^2)v_i(y) - \sum_{k>i} \varepsilon_{il}\varepsilon_{kl}\, 2\sqrt{v_i(y)v_k(y)} \\ &\geq (1-\varepsilon_{il}^2)v_i(y) - \sum_{k>i} \varepsilon_{il}\varepsilon_{kl}\Big(\frac{\alpha_k}{\alpha_i}v_i(y) + \frac{\alpha_i}{\alpha_k}v_k(y)\Big). \end{aligned}$$

The second inequality is $2\sqrt{st} \leq cs + t/c$ for $s, t \geq 0$, $c > 0$, with $c = \alpha_k/\alpha_i$. Multiply by $\mathbb{P}(Y_{<l} = y)$ and sum over y. Since $\sum_y \mathbb{P}(Y_{<l} = y)\mathbb{E}_y[u^2] = \mathbb{E}[u^2]$ for every u, this gives (B.8).

Step 3: matrix form. Let T_l be the $N \times N$ matrix with entries

$$(T_l)_{ii} = \varepsilon_{il}^2 + \sum_{k>i} \varepsilon_{il}\varepsilon_{kl}\frac{\alpha_k}{\alpha_i}, \qquad (T_l)_{ik} = \varepsilon_{il}\varepsilon_{kl}\frac{\alpha_i}{\alpha_k}\ (k > i), \qquad (T_l)_{ik} = 0\ (k < i).$$

Its entries are nonnegative, and (B.8) says $w^l \geq w^{l-1} - T_l w^{l-1}$ entrywise. The column sums of T_l are

$$c_k^{(l)} := \sum_{i=1}^N (T_l)_{ik} = \varepsilon_{kl}^2 + \sum_{i>k} \varepsilon_{kl}\varepsilon_{il}\frac{\alpha_i}{\alpha_k} + \sum_{i<k} \varepsilon_{il}\varepsilon_{kl}\frac{\alpha_i}{\alpha_k} = \frac{\varepsilon_{kl}}{\alpha_k}\sum_{i=1}^N \varepsilon_{il}\alpha_i \ \geq 0,$$

so that $\sum_i (T_l u)_i = \sum_k c_k^{(l)} u_k$ for every $u \in \mathbb{R}^N$. By (B.5), $\sum_{l=1}^M c_k^{(l)} \leq \lambda$ for every k.

Step 4: apply the potential estimate. The vectors and matrices from Steps 1–3 satisfy the hypotheses of Lemma B.4. Therefore, by (B.7),

$$\|f\|^2 - \|\mathbb{E}[f \mid \mathbf{Y}]\|^2 = \sum_k w_k^M \geq (1-\lambda)\sum_k w_k^0 = (1-\lambda)\,\|f\|^2.$$

This is (B.6). Letting λ decrease to $\|\mathcal{E}\|_{\mathrm{op}}^2$ proves the theorem. □

The role of the weights is visible in Step 3: the symmetrization of the cross terms by $2\sqrt{st} \leq cs + t/c$ is chosen so that each column sum of T_l becomes $(\varepsilon_{kl}/\alpha_k)\sum_i \varepsilon_{il}\alpha_i$, and the Perron-type inequality $\mathcal{E}\mathcal{E}^{\mathsf{T}}\alpha \leq \lambda\alpha$ bounds the sum of these column sums over l by λ, uniformly in k. Without the weights ($\alpha = \mathbf{1}$) the same argument gives only $\rho(\mathbf{X};\mathbf{Y})^2 \leq \max_k(\mathcal{E}\mathcal{E}^{\mathsf{T}}\mathbf{1})_k$, the largest row sum of $\mathcal{E}\mathcal{E}^{\mathsf{T}}$, which can exceed $\|\mathcal{E}\|_{\mathrm{op}}^2$.

B.6. **Remarks.** Two remarks close the appendix: how the proof compares with Peyre's original argument, and why the conditional form of the hypothesis cannot be weakened.

Remark B.5 (Comparison with the source). Lemma B.2 is [54, Lemma 3.3.3]: its Steps 1 and 2 are Claims 3.3.4 and 3.3.5 there, and Steps 3–4 are (GO)–(GP). Step 2 of the proof of the theorem is the passage (HB)–(HF) of [54], with the identification (HC)–(HD) of the functions used there. Step 4 replaces the induction on M in the proof of [54, Lemma 3.3.7], and Lemma B.3 replaces [54, Lemma 3.8.1], of which only the easy inequality "$\leq$" is needed. That only the conditionings in (B.4) are used is observed in [54, §3.4.a].

Remark B.6 (Sharpness and the role of conditioning). (i) The bound can be attained. If the pairs (X_{a_k}, X_{b_k}), $1 \leq k \leq N = M$, are independent of each other, then every conditional correlation in (4.3) with $i = a_k$, $j = b_l$ equals $\rho(X_{a_k}; X_{b_k})$ if $k = l$ and 0 otherwise. So the best choice of $\mathcal{E}$ is diagonal and $\|\mathcal{E}\|_{\mathrm{op}} = \max_k \rho(X_{a_k}; X_{b_k})$. This is also the value of $\rho(X_A; X_B)$, by the classical tensorization for independent pairs [54, Theorem 1.1.19], which is Corollary 4.14; its proof also checks the two facts just used.

(ii) The conditional hypothesis cannot be replaced by bounds on unconditional pairwise correlations. Let X_1, X_2 be independent uniform bits and $Y = X_1 + X_2 \bmod 2$. Then $\rho(X_1; Y) = \rho(X_2; Y) = 0$, but Y is a nonconstant function of (X_1, X_2), so $\rho((X_1, X_2); Y) = 1$. Accordingly, $\rho(X_2; Y \mid X_1 = x) = 1$, since given X_1 the variable Y determines X_2.

Consequences and what to read next. The proof uses only linear algebra in L^2 of a finite probability space: orthogonal projections and the facts (P1)–(P3), the Cauchy–Schwarz and arithmetic-geometric mean inequalities and a Neumann series; from the rest of the paper, the definitions of maximal correlation (Definition 4.3) and of the matrix norm (Definition 4.1).

Appendix C. The actual-square kernel for cutoffs up to $N^{1/20}$

What this appendix proves and how to read it. This appendix proves Proposition 7.3 for every triple (β, κ, η) that is admissible in the sense of Definition 7.2. It is the only estimate of the analytic stage in which the absence of square differences in A produces cancellation (Section 7). It is applied through Corollary 7.4, at $(\beta, \kappa, \eta) = (1/20, 1/20, 17/100)$, in the accepted-pair cancellation Lemma 8.10 of Section 8, using part (iii) for the nonvanishing square classes of the finite-depth external index and the square-clique prefix. The strict-square and fixed-prefix cases in parts (i) and (ii) are special cases of this same construction. Through the cancellation lemma the kernel also enters the wrap of the signed index form (Lemma 10.6).

The proof follows [52, Prop. 6.1]. As explained in Section 7, it compares a model kernel k_0, whose bilinear form against (F, J) is the pairing of the proposition, with a signed kernel k on the positive squares, whose bilinear form vanishes; the work is the uniform bound $\|S - S_0\|_\infty = O(H^{-1/3})$ for their Fourier transforms (Lemma C.8). Off the major arcs both transforms are small; on a major arc both reduce to one arithmetic coefficient, and the two coefficients agree exactly when the part of the denominator prime to D is at most H. Each error is a power of N whose exponent is the left-hand side of one of (E1)–(E4). One root-class construction (Lemma C.1) supplies the arithmetic coefficient for all three cases. We retain the square-division identity (Lemma C.9) to explain how the normalization behaves under rescaling; the estimate itself requires no separate rescaling proof.

Besides elementary facts (geometric sums, partial summation, the Chinese remainder theorem, Hensel lifting of square roots of units, and Plancherel's identity on $\mathbb{R}/\mathbb{Z}$) the proof uses Dirichlet's approximation theorem: for $Q_0 \geq 1$ and $\alpha \in \mathbb{R}$ there are coprime integers a, q with $1 \leq q \leq Q_0$ and $|\alpha - a/q| \leq 1/(qQ_0)$. Weyl's inequality and the quadratic Gauss sums at odd prime powers are proved below (Lemmas C.4 and C.6).

Conventions. The reduced denominator $q(\xi)$, the unit quadratic average g, the frequency set $\Xi_{H,D}$ and the normalized transform $\widehat{F}$ are those of Definition 7.1, and the strict square classes are those of Definition 4.5; of g we use its defining formula and $|g| \leq 1$. Throughout, (β, κ, η) is admissible, and (E1)–(E4), (K0)–(K2) are the conditions of Definition 7.2. We fix $C_D, C_H \geq 1$, an integer $N \geq 1$, a modulus $D \leq C_D N^\kappa$ divisible by 8, a cutoff $C_H^{-1} N^{\beta/2} \leq H \leq C_H N^{\beta/2}$, and a nonvanishing square class r modulo D as in Proposition 7.3(iii). For $p \mid D$, $v_p(D)$ is its exponent in D. The root period D' will be a divisor of D, constructed below; all analytic bounds use only $D' \leq D$. We write $\|x\|$ for the distance from $x \in \mathbb{R}$ to $\mathbb{Z}$, and $V_N(x) = N^{-1} \sum_{n=1}^{N} e(xn)$, so that $|V_N(x)| \leq \min(1, (2N\|x\|)^{-1})$. A *constant* C depends only on C_D and C_H, and "N large" means $N \geq N_0$ for some N_0 depending only on β, κ, η, C_D and C_H; for N large, $H \geq 1$. As in Section 7, q is a denominator and a, b, r are residues or numerators (not $q = 1/p$, $r = p^{-1/2}$ or the price exponent a), and letters such as A, B, J, S, T, V have only the local meanings given here (Section 4.2).

C.1. The two kernels.

The proof compares two kernels: a model kernel, built from square root classes modulo D, whose pairing with (F, J) is the quantity in Proposition 7.3, and a signed kernel supported on the actual positive squares, whose pairing vanishes because no difference in question is a square. This subsection constructs the root classes from which both kernels are built.

Lemma C.1 (Root classes at finite conductor depth)**.** *Let D and r be as above. There is a class $m_0 \bmod D'$, with $D' \mid D$, such that $\rho_D(m) = D' \mathbf{1}_{m \equiv m_0 \,(D')}$ has mean one, supremum D' and, in*

the expansion $\rho_D(m) = \sum_{j \bmod D'} e(j(m-m_0)/D')$*, Fourier coefficients of total absolute value* D'*. Every* m *in this class satisfies* $m^2 \equiv r \pmod D$*. At a prime* $p \mid D$*, the squared root variable is uniform on the residues modulo* p^k*, for every* $k \geq v_p(D)$*, that reduce to* r *modulo* $p^{v_p(D)}$*. In particular, for* $p \nmid a$*, the local mean of* $e(am^2/p^k)$ *is* $e(ar/p^k)$ *if* $k \leq v_p(D)$ *and zero otherwise.*

Proof. Fix a prime $p \mid D$. Write $h = v_p(D)$ and $v_p(r) = 2j$, and choose a root b of r modulo p^h with $v_p(b) = j$, which exists by Definition 4.6. For odd p, restrict the root variable to $b + p^{h-j}t$. Its square, after subtracting r and dividing by p^h, is

$$(b^2 - r)/p^h + 2(b/p^j)t + p^{h-2j}t^2.$$

The linear coefficient is a unit and the quadratic coefficient is divisible by p. A polynomial $ut + pvt^2 + c$ with $p \nmid u$ permutes every $\mathbb{Z}/p^k\mathbb{Z}$: modulo p it is bijective, and each solution modulo p^k has exactly one lift for each prescribed next digit, since the derivative remains a unit. This proves uniformity on the higher square residues.

At 2, use the root class $b + 2^{h-j-1}t$. The corresponding higher-square coefficient is

$$(b^2 - r)/2^h + (b/2^j)t + 2^{h-2j-2}t^2.$$

The linear coefficient is odd and the quadratic coefficient even, because $h - 2j \geq 3$. The same digit-lifting proof applies. Replacing b by another representative of its root class does not change the conclusion. In both cases the root class squares to r at depth h and is uniform over all its higher extensions; a primitive additive character above depth h therefore has mean zero.

The root periods are p^{h-j} at odd primes and 2^{h-j-1} at 2. Their product D' divides D. The Chinese remainder theorem gives one root class $b_0 \bmod D'$, with normalized weight $D'\mathbf{1}_{b=b_0\,(D')}$. This weight has mean one, supremum D', and exactly D' Fourier coefficients of modulus one. The local character means follow by averaging over the uniform higher square residues: for $k > h$ the factor is $e(ar/p^k)\mathbb{E}_{t \bmod p^{k-h}} e(at/p^{k-h}) = 0$. □

Definition C.2 (Root weight, unit sieve and the two kernels)**.** Use the root weight ρ_D and period D' of Lemma C.1. For an odd prime $p \nmid D$ put

$$w_p(m) = \frac{p\mathbf{1}_{p|m} - 1}{p-1} = \frac{1}{p-1}\sum_{j=1}^{p-1} e(jm/p),$$

and $w_u = \prod_{p|u} w_p$ for squarefree u coprime to D ($w_1 = 1$). With u running over the squarefree integers $1 \leq u \leq H$ coprime to D, define

$$k_0(n) = \frac{D}{N}\mathbf{1}_{1\leq n\leq N}\mathbf{1}_{n\equiv r\,(D)} \sum_{\xi\in\Xi_{H,D}} g(\xi)e(-\xi n),$$

$$k(n) = \sum_{1\leq m\leq\sqrt{N}} \mathbf{1}_{n=m^2}\frac{2m}{N}\,\rho_D(m)\sum_u (-1)^{\omega(u)}w_u(m),$$

and $S_0(\alpha) = \sum_n k_0(n)e(\alpha n)$, $S(\alpha) = \sum_n k(n)e(\alpha n)$. These objects have the following properties.

(a) ρ_D has mean one over its period D', $\|\rho_D\|_\infty = D'$, $\rho_D(m) = \sum_{j \bmod D'} e(j(m-m_0)/D')$, and $m^2 \equiv r \pmod D$ whenever $\rho_D(m) \neq 0$.

(b) $|w_u| \leq 1$; w_p has mean zero modulo p; the Fourier coefficients of w_u have total absolute value one; and for q coprime to D

$$\sum_{u|\mathrm{rad}(q)} (-1)^{\omega(u)}w_u(m) = \prod_{p|q}(1 - w_p(m)) = \frac{q}{\varphi(q)}\mathbf{1}_{(m,q)=1}. \tag{C.1}$$

(c) $S(\alpha) = \sum_{m\leq\sqrt{N}} \frac{2m}{N}\Phi(m)e(\alpha m^2)$, where $\Phi(m) = \rho_D(m)\sum_u(-1)^{\omega(u)}w_u(m)$ is a finite sum of additive characters in m with coefficients of total absolute value at most $D'H$.

Proof of (a)–(c). (a) The root-class lemma gives the mean, supremum and square condition; the Fourier formula is orthogonality modulo D'. (b) $w_p(m)$ is 1 if $p \mid m$ and $-1/(p-1)$ otherwise, and $1 - w_p(m) = \frac{p}{p-1}\mathbf{1}_{p\nmid m}$; expanding the product gives (C.1), since $\prod_{p\mid q} p/(p-1) = q/\varphi(q)$. (c) The total absolute value of the coefficients is submultiplicative under products; ρ_D contributes D', each w_u one, and there are at most H values of u. □

How the kernels are matched. By (a), k lives on the squares in the class r modulo D, the class of k_0, and the factor $2m/N$ is the discrete form of $dn = 2m\,dm$, matching the weight $1/N$ of k_0. The divisor sum (C.1) is the density of the uniform law on the units modulo q; it turns averages of $e(\xi m^2)$ over residues into averages over units, which is how g arises (Lemma C.7). It is truncated at $u \le H$ so that Φ has Fourier coefficients of total size at most $D'H$. The same construction in the strict-square case underlies [52, Prop. 6.1].

Lemma C.3 (The model kernel as a sum over centres)**.** *We have*

$$S_0(\alpha) = \sum_{\xi\in\Xi_{H,D}} \sum_{j \bmod D} g(\xi)e(-jr/D)V_N(\alpha - \xi + j/D).$$

The centres $\xi - j/D$ *are distinct, there are at most DH^2 of them, each is a reduced fraction of denominator at most DH, and each coefficient has modulus at most one. Every $\alpha_0 \in \mathbb{Q}/\mathbb{Z}$ is uniquely $\alpha_0 = \alpha_D + \alpha_1$, where the prime factors of $q(\alpha_D)$ divide D and $q_1 = q(\alpha_1)$ is coprime to D; the point α_0 is a centre exactly when $q(\alpha_D) \mid D$ and $\alpha_1 \in \Xi_{H,D}$, and its coefficient is then $c_0(\alpha_0) = g(\alpha_1)e(\alpha_D r)$. Put $c_0(\alpha_0) = 0$ otherwise.*

Proof. Expand $\mathbf{1}_{n\equiv r\,(D)} = D^{-1}\sum_{j \bmod D} e(j(n-r)/D)$. If $\xi - j/D = \xi' - j'/D$, then $\xi - \xi'$ has denominator dividing $q(\xi)q(\xi')$, which is coprime to D, and also dividing D; so $\xi = \xi'$ and $j \equiv j'$. The number of centres is at most $D\sum_{q\le H}\varphi(q) \le DH^2$. The decomposition of α_0 is the partial fraction decomposition; a centre $\xi - j/D$ has $\alpha_1 = \xi$ and $\alpha_D = -j/D$, whence the coefficient $g(\xi)e(-jr/D) = g(\alpha_1)e(\alpha_D r)$. □

C.2. **Weyl's inequality with even denominators.** The minor-arc estimate needs a bound for quadratic exponential sums with explicit constants and with no parity condition on the denominator, because even denominators occur through the modulus D. We prove the form needed.

Lemma C.4 (Weyl's inequality with a linear twist)**.** *Let $\alpha \in \mathbb{R}$ and let a/q be a reduced fraction with $q \ge 1$ and $|\alpha - a/q| \le q^{-2}$. For all $\lambda \in \mathbb{R}$ and all integers $1 \le M' \le M$,*

$$\Big|\sum_{m=1}^{M'} e(\alpha m^2 + \lambda m)\Big|^2 \le M + 200\Big(\frac{M}{q} + 1\Big)\big(M + q\log(2q)\big).$$

No parity condition on q is needed.

This is Weyl's inequality with explicit constants (compare [72, Lemma 2.4]); in the proof, two indices h, h' in one block of $\Delta = \max(1, \lfloor q/4\rfloor)$ consecutive integers differ by $j = |h - h'|$ with $0 < 2j < q/2$ when $q \ge 4$, so $q \nmid 2aj$ whatever the parity of q.

Proof. Squaring and grouping the pairs $(m, m+h)$ by h,

$$\Big|\sum_{m\le M'} e(\alpha m^2 + \lambda m)\Big|^2 \le M' + 2\sum_{h=1}^{M'-1}\Big|\sum_{m\in I_h} e(2\alpha hm)\Big|,$$

where I_h is an interval of length at most M'; the phases $e(\alpha h^2 + \lambda h)$ are unimodular, so the twist λ plays no role. A geometric sum satisfies $|\sum_{m\in I} e(xm)| \le \min(|I|, (2\|x\|)^{-1})$, so the left-hand side is at most $M + 2\sum_{h=1}^{M}\min(M, \|2\alpha h\|^{-1})$.

Let $\Delta = \max(1, \lfloor q/4 \rfloor)$ and split $\{1, \dots, M\}$ into at most $M/\Delta + 1$ blocks of Δ consecutive integers. Suppose $q \geq 4$, and let $h \neq h'$ lie in one block, so $j = |h - h'|$ satisfies $0 < j < \Delta \leq q/4$. Then $q \nmid 2j$, hence $q \nmid 2aj$, so $\|2aj/q\| \geq 1/q$; and $|2j(\alpha - a/q)| \leq 2j/q^2 < 1/(2q)$. Therefore $\|2\alpha h - 2\alpha h'\| = \|2\alpha j\| > 1/(2q)$: the points $2\alpha h$ of one block are $1/(2q)$-separated on $\mathbb{R}/\mathbb{Z}$. Each set $\{x : k/(2q) \leq \|x\| < (k+1)/(2q)\}$ is the union of two half-open intervals of length $1/(2q)$ and contains at most two points of a block. The points with $k = 0$ contribute at most $2M$, the others at most $2 \cdot 2q/k$ for each k, so a block contributes at most $2M + 4q \sum_{k=1}^{q} 1/k \leq 2M + 12q \log(2q)$. Since $\Delta \geq q/8$ for $q \geq 4$, there are at most $8M/q + 1$ blocks, and $2 \sum_h \min(M, \|2\alpha h\|^{-1}) \leq 2(8M/q + 1)(2M + 12q \log(2q)) \leq 192(M/q + 1)(M + q \log(2q))$. If $q \leq 3$, then $2 \sum_h \min(M, \|2\alpha h\|^{-1}) \leq 2M^2 \leq 6(M/q)M$, which is smaller. $\square$

C.3. **Arcs.** For a reduced fraction a/q with $1 \leq q \leq N^\eta$ put $\mathfrak{M}(a/q) = \{\alpha \in \mathbb{R}/\mathbb{Z} : \|\alpha - a/q\| \leq 2N^{-1+\eta}\}$, and let $\mathfrak{m}$ be the complement of the union of these major arcs. For $\alpha_0 = a/q$ and every squarefree $1 \leq u \leq H$ coprime to D let $a_u(m) = \rho_D(m) w_u(m) e(\alpha_0 m^2)$, let A_u be its mean over any one period (the value is independent of the period chosen), and put

$$c(\alpha_0) = \sum_u (-1)^{\omega(u)} A_u.$$

Lemma C.5 (Arcs)**.** *For N large the following hold.*

(i) The major arcs are pairwise disjoint. Every $\alpha \in \mathfrak{m}$ has a reduced fraction a/q with $N^\eta < q \leq N^{1-\eta}$ and $|\alpha - a/q| \leq q^{-2}$.

(ii) Every centre of S_0 is the centre of a major arc. If $\alpha = \alpha_0 + y$ with $\alpha_0 = a/q$, $q \leq N^\eta$ and $|y| \leq 2N^{-1+\eta}$, then every centre $\alpha_c \neq \alpha_0$ satisfies $\|\alpha - \alpha_c\| \geq (2qDH)^{-1}$.

(iii) For $\alpha \in \mathfrak{m}$, $|S(\alpha)| \leq CN^{\kappa+\beta/2-\eta/2}(\log 2N)^{1/2}$ and $|S_0(\alpha)| \leq CN^{\kappa+\beta-\eta}$.

(iv) For α as in (ii), $S_0(\alpha) = c_0(\alpha_0)V_N(y) + R_0$ with $|R_0| \leq CN^{\eta+2\kappa+3\beta/2-1}$, and $S(\alpha) = c(\alpha_0)V_N(y) + R_1$ with $|R_1| \leq C(N^{2\eta+2\kappa+\beta-1/2} + N^{\eta+\beta/2-1})$.

The exponents $\kappa + \beta/2 - \eta/2$, $\kappa + \beta - \eta$, $\eta + 2\kappa + 3\beta/2 - 1$, $2\eta + 2\kappa + \beta - 1/2$ are the left-hand sides of (E1)–(E4), hence below $-\beta/6$, and $\eta + \beta/2 - 1$ is smaller than the third. At the triple $(1/20, 1/20, 17/100)$ of Corollary 7.4 the five exponents are $-1/100$, $-7/100$, $-131/200$, $-1/100$, $-161/200$, below $-1/120$, and the seven kernel slacks of (E1)–(E4), (K0)–(K2), computed in Corollary 7.4, are $1/600$, $37/600$, $97/150$, $1/600$, $49/100$, $19/200$, $117/200$.

Proof. We use $D' \leq D \leq C_D N^\kappa$ and $H \leq C_H N^{\beta/2}$, so $DH \leq C_D C_H N^{\kappa+\beta/2}$.

(i) Two distinct reduced fractions of denominators at most N^η are at distance at least $N^{-2\eta}$, and $4N^{-1+\eta} < N^{-2\eta}$ for N large, by (K0). For $\alpha \in \mathfrak{m}$, Dirichlet's theorem with $Q_0 = \lfloor N^{1-\eta} \rfloor$ gives a reduced a/q with $q \leq Q_0$ and $|\alpha - a/q| \leq 1/(qQ_0) \leq q^{-2}$. If $q \leq N^\eta$, then $|\alpha - a/q| \leq 1/Q_0 \leq 2N^{-1+\eta}$, so $\alpha \in \mathfrak{M}(a/q)$, which is impossible. Hence $N^\eta < q \leq N^{1-\eta}$.

(ii) Every centre has denominator at most $DH \leq C_D C_H N^{\kappa+\beta/2} < N^\eta$ for N large, by (K1). Two distinct fractions of denominators q and at most DH are at distance at least $(qDH)^{-1}$. Since $4qDHN^{-1+\eta} \leq 4C_D C_H N^{2\eta+\kappa+\beta/2-1} \leq 1$ for N large, by (K2), we have $|y| \leq (2qDH)^{-1}$, and the triangle inequality gives the claim.

(iii) Let $\alpha \in \mathfrak{m}$, with a/q from (i), and $M = \lfloor \sqrt{N} \rfloor$. By (K0), $\eta < 1/2$, so $M \leq N^{1/2} \leq N^{1-\eta}$; also $M^2/q \leq N^{1-\eta}$ and $q \leq N^{1-\eta}$. By Lemma C.4, uniformly in λ and $M' \leq M$, expanding the product,

$$\Big| \sum_{m \leq M'} e(\alpha m^2 + \lambda m) \Big|^2 \leq M + 200\Big(\frac{M^2}{q} + M \log 2q + M + q \log 2q\Big) \leq 801 N^{1-\eta} \log(2N).$$

The weights $2m/N$ increase and are at most $2/\sqrt{N}$, so partial summation bounds $|\sum_{m \leq \sqrt{N}} \frac{2m}{N} e(\alpha m^2 + \lambda m)|$ by $4/\sqrt{N}$ times the maximum over $M' \leq M$ of the sums above, that is, by $116 N^{-\eta/2}(\log 2N)^{1/2}$. Expanding Φ by Definition C.2(c), $|S(\alpha)| \leq 116 D' H N^{-\eta/2}(\log 2N)^{1/2} \leq$

$116C_DC_HN^{\kappa+\beta/2-\eta/2}(\log 2N)^{1/2}$. By (ii), every centre is the centre of a major arc, so it lies at distance more than $2N^{-1+\eta}$ from α; each term of Lemma C.3 is therefore at most $(4N^\eta)^{-1}$ in modulus, and $|S_0(\alpha)| \le DH^2/(4N^\eta) \le C_DC_H^2N^{\kappa+\beta-\eta}$.

(iv) *The model kernel.* If α_0 is a centre, its term in Lemma C.3 is $c_0(\alpha_0)V_N(y)$; if not, $c_0(\alpha_0) = 0$. By (ii), each of the at most DH^2 other centres has $|V_N(\alpha-\alpha_c)| \le qDH/N$, so $|R_0| \le qD^2H^3/N \le C_D^2C_H^3N^{\eta+2\kappa+3\beta/2-1}$.

The actual kernel. Since $(m+q)^2-m^2 = q(2m+q)$, the factor $e(\alpha_0m^2)$ has period q, so a_u has a period T_u dividing $\operatorname{lcm}(D',u,q)$, and $T_u \le D'uq$. Also $|a_u| \le D'$ and $|A_u| \le \mathbb{E}|\rho_D| = 1$. Hence $E_u(x) = \sum_{1\le m\le x} a_u(m) - A_ux$ satisfies $|E_u(x)| \le D'T_u + T_u \le 2D'T_u$ for real $x \ge 0$: the complete periods contribute exactly A_u times their length, and fewer than T_u terms remain. Put $X = \sqrt{N}$ and $f_y(x) = \frac{2x}{N}e(yx^2)$. Since $e(\alpha m^2) = e(\alpha_0m^2)e(ym^2)$, Definition C.2(c) gives $S(\alpha) = \sum_u(-1)^{\omega(u)}\sum_{m\le X} f_y(m)a_u(m)$. Stieltjes integration by parts against $\sum_{m\le x}a_u(m) = A_ux + E_u(x)$, with $f_y(0) = E_u(0) = 0$, gives

$$\sum_{m\le X} f_y(m)a_u(m) = A_u\int_0^X f_y(x)\,dx + f_y(X)E_u(X) - \int_0^X E_u(x)f_y'(x)\,dx.$$

Here $|f_y(X)| = 2/\sqrt{N}$ and $\int_0^X|f_y'| \le \int_0^X(2/N + 8\pi|y|x^2/N)\,dx \le 2/\sqrt{N} + 9|y|\sqrt{N}$, so the last two terms are at most $18D'T_u(1+|y|N)/\sqrt{N}$. The substitution $v = x^2$ gives $\int_0^X f_y = N^{-1}\int_0^N e(yv)\,dv$, which differs from $V_N(y)$ by at most $2\pi|y|$, because $|e(yn)-e(yv)| \le 2\pi|y|$ for $v \in [n-1,n]$. Summing over the at most H values of u, with $\sum_{u\le H}u \le H^2$, $T_u \le D'uq$ and $|A_u| \le 1$,

$$S(\alpha) = c(\alpha_0)V_N(y) + R_1, \qquad |R_1| \le \frac{18qD^2H^2(1+|y|N)}{\sqrt{N}} + 2\pi|y|H.$$

With $q \le N^\eta$, $1+|y|N \le 3N^\eta$ and $|y| \le 2N^{-1+\eta}$ this is at most

$$54C_D^2C_H^2N^{2\eta+2\kappa+\beta-1/2} + 4\pi C_HN^{\eta+\beta/2-1}. \qquad \square$$

C.4. **Local factors and the arithmetic coefficient.** For $p \mid D$, let $\mathcal{C}_p$ be the residue class $b + p^{h-j}\mathbb{Z}$ (resp. $b + 2^{h-j-1}\mathbb{Z}$ at $p=2$) constructed in the proof of Lemma C.1, with h, j, b as there. The weight ρ_D is the product of their normalized indicators. We collect the two local rules needed for the arithmetic coefficient: the root-class rule at primes of D, and the quadratic Gauss sums at the remaining primes.

Lemma C.6 (Local means at the primes of D, and Gauss sums)**.** *(a) Let $p \mid D$, $j \ge 1$ and $p \nmid a$. The mean of $e(am^2/p^j)$ over the residues $m \in \mathcal{C}_p$ modulo $p^{\max(j,v_p(D))}$ is $e(ar/p^j)$ if $j \le v_p(D)$, and 0 if $j > v_p(D)$.*

(b) Let $p \nmid D$ be an odd prime, $j \ge 1$ and $p \nmid a$, and put $G_{p^j}(a) = \sum_{m \bmod p^j} e(am^2/p^j)$. Then $|G_{p^j}(a)| = p^{j/2}$. If $j \ge 2$, then $\sum_{m\in(\mathbb{Z}/p^j\mathbb{Z})^\times} e(am^2/p^j) = 0$. The twisted sum

$$\Gamma_j = p^{-j}\sum_{m \bmod p^j} w_p(m)e(am^2/p^j) \quad \textit{satisfies} \quad |\Gamma_j| \le 2p^{-j/2}.$$

Proof. (a) This is the local character mean of Lemma C.1.

(b) Writing the pairs (m,m') as $(m, m+h)$,

$$|G_{p^j}(a)|^2 = \sum_{h \bmod p^j} e(ah^2/p^j)\sum_{m \bmod p^j} e(2ahm/p^j) = p^j,$$

because the inner sum is p^j if $p^j \mid h$ and 0 otherwise, as $p \nmid 2a$. Let $j \ge 2$, and write $m = x + p^{j-1}h$ with $0 \le x < p^{j-1}$ and $0 \le h < p$; then $m^2 \equiv x^2 + 2xhp^{j-1} \pmod{p^j}$, since $2j-2 \ge j$. Summing over h removes the x with $p \nmid x$, while for $p \mid x$ the p lifts m of x are divisible by p and have $m^2 \equiv x^2 \pmod{p^j}$. Hence $G_{p^j}(a)$ and the sum of $e(am^2/p^j)$ over the m divisible by p both equal

$p\sum_{0\le x<p^{j-1},\,p\mid x} e(ax^2/p^j)$, and the sum over the units, which is their difference, vanishes. For Γ_j write $w_p(m)=\frac{p}{p-1}\mathbf{1}_{p\mid m}-\frac{1}{p-1}$. If $j=1$, the first part contributes the single term $m=0$, of size $\frac{1}{p-1}$, and the second has modulus $\frac{1}{p-1}p^{-1/2}$; so $|\Gamma_1|\le\frac{1+p^{-1/2}}{p-1}\le 2p^{-1/2}$ for $p\ge 3$. If $j\ge 2$, the sum over the m divisible by p equals $G_{p^j}(a)$, so $\Gamma_j=p^{-j}G_{p^j}(a)$ and $|\Gamma_j|=p^{-j/2}$. □

Lemma C.7 (The arithmetic coefficient)**.** *Let $\alpha_0=a/q$ be reduced and $\alpha_0=\alpha_D+\alpha_1$ as in Lemma C.3, with $q_1=q(\alpha_1)$. If $q_1\le H$, then $c(\alpha_0)=c_0(\alpha_0)$. If $q_1>H$, then $c_0(\alpha_0)=0$ and*

$$|c(\alpha_0)|\le 3^{\omega(q_1)}q_1^{-1/2}\le 3^{\pi(729)}q_1^{-1/3}<3^{\pi(729)}H^{-1/3}.$$

Proof. Let m be uniform modulo $\operatorname{lcm}(D',u,q)$, a common period of the factors of a_u. By the Chinese remainder theorem, its residues modulo the part of this modulus at the primes of D and modulo the part prime to D are independent and uniform. Now $\rho_D(m)$ and $e(\alpha_D m^2)$ depend only on the first, and $w_u(m)$ and $e(\alpha_1 m^2)$ only on the second. Hence $A_u=A^{(D)}A_u^{(1)}$ with $A^{(D)}=\mathbb{E}_m\rho_D(m)e(\alpha_D m^2)$ and $A_u^{(1)}=\mathbb{E}_m w_u(m)e(\alpha_1 m^2)$, and both factor over prime powers.

The D-part. Write $\alpha_D=\sum_{p\mid D}a_p/p^{j_p}$ with $p\nmid a_p$ when $j_p\ge 1$. Since ρ_D is the product of the normalized indicators of the root classes, the local factor at a prime $p\mid D$ with $j_p\ge 1$ is the mean of $e(a_p m^2/p^{j_p})$ over $\mathcal{C}_p$, which by Lemma C.6(a) is $e(a_p r/p^{j_p})$ if $j_p\le v_p(D)$ and 0 otherwise; a prime of D with $j_p=0$ contributes 1. Therefore $A^{(D)}=e(\alpha_D r)$ if $q(\alpha_D)\mid D$, and $A^{(D)}=0$ otherwise.

The coprime part. The denominator q_1 is odd, being prime to 8. If $p\mid u$ and $p\nmid q_1$, the local factor at p is the mean of w_p, which is zero. So $A_u=0$ unless $u\mid\operatorname{rad}(q_1)$.

Case $q_1\le H$. Every divisor of $\operatorname{rad}(q_1)$ is squarefree, coprime to D and at most H, so all of them occur among the u, and by (C.1)

$$c(\alpha_0)=A^{(D)}\,\mathbb{E}_{m\bmod q_1}\Big[\frac{q_1}{\varphi(q_1)}\mathbf{1}_{(m,q_1)=1}e(\alpha_1 m^2)\Big]=\frac{A^{(D)}}{\varphi(q_1)}\sum_{m\in(\mathbb{Z}/q_1\mathbb{Z})^\times}e(\alpha_1 m^2).$$

If q_1 is squarefree, the last average is $g(\alpha_1)$ (Definition 7.1), and $c(\alpha_0)=c_0(\alpha_0)$ by Lemma C.3; both vanish unless $q(\alpha_D)\mid D$. If q_1 is not squarefree, the unit sum factors over the prime powers $p^j\parallel q_1$ and vanishes at one with $j\ge 2$ (Lemma C.6(b)); then $c(\alpha_0)=0=c_0(\alpha_0)$.

Case $q_1>H$. Then $\alpha_1\notin\Xi_{H,D}$ and $c_0(\alpha_0)=0$. Let $u\mid\operatorname{rad}(q_1)$ and $p^j\parallel q_1$, and let a'/p^j be the p-component of α_1. The local factor of $A_u^{(1)}$ at p is the normalized Gauss sum $p^{-j}G_{p^j}(a')$, of modulus $p^{-j/2}$, if $p\nmid u$, and Γ_j, of modulus at most $2p^{-j/2}$, if $p\mid u$ (Lemma C.6(b)). So $|A_u^{(1)}|\le 2^{\omega(u)}q_1^{-1/2}$, and with $|A^{(D)}|\le 1$,

$$|c(\alpha_0)|\le q_1^{-1/2}\sum_{u\mid\operatorname{rad}(q_1)}2^{\omega(u)}=3^{\omega(q_1)}q_1^{-1/2}.$$

Every prime $p\ge 729=3^6$ has $p^{1/6}\ge 3$, and at most $\pi(729)$ primes are smaller; so $3^{\omega(q)}\le 3^{\pi(729)}q^{1/6}$ for every $q\ge 1$. □

C.5. **The uniform approximation and the kernel estimate.** All the arc estimates are now available. The lemma combines them into the uniform bound $\|S-S_0\|_\infty=O(H^{-1/3})$ for the Fourier transforms of the two kernels, and Proposition 7.3 then follows by Plancherel.

Lemma C.8 (Uniform Fourier approximation)**.** *For N large,* $\sup_{\alpha\in\mathbb{R}/\mathbb{Z}}|S(\alpha)-S_0(\alpha)|\le(3^{\pi(729)}+4)H^{-1/3}$.

Proof. Let $s_{\min}>0$ be the least of the slacks of (E1)–(E4). By Lemma C.5, every power of N in its parts (iii) and (iv) has exponent at most $-\beta/6-s_{\min}$, while $H^{-1/3}\ge C_H^{-1/3}N^{-\beta/6}$. On $\mathfrak{m}$, Lemma C.5(iii) bounds $|S|+|S_0|$. On a major arc, Lemma C.5(iv), $|V_N(y)|\le 1$ and Lemma C.7

give $|S-S_0| \le 3^{\pi(729)}H^{-1/3} + |R_0| + |R_1|$. Each of the four quantities $|S|, |S_0|$ in Lemma C.5(iii) and $|R_0|, |R_1|$ in Lemma C.5(iv) is therefore at most $H^{-1/3}$ for N large, the factor $(\log 2N)^{1/2}$ included. □

Proof of Proposition 7.3. First prove (iii), using Definition C.2 with the nonvanishing square class $r \equiv b-a \pmod D$. Extend F, J by zero, and for a finitely supported h put $B_h(F,J) = N^{-1}\sum_{x,n} h(n)F(x)J(x+n)$. With $\widetilde{F}(\alpha) = \sum_x F(x)e(-x\alpha)$ and $S_h(\alpha) = \sum_n h(n)e(\alpha n)$, so that $S_{k_0} = S_0$ and $S_k = S$, orthogonality and then Cauchy–Schwarz and Plancherel give

$$B_h(F,J) = \frac{1}{N}\int_0^1 S_h(\alpha)\widetilde{F}(-\alpha)\widetilde{J}(\alpha)\,d\alpha, \qquad |B_{k-k_0}(F,J)| \le \frac{\|S-S_0\|_\infty \|F\|_{\ell^2}\|J\|_{\ell^2}}{N}.$$

Every term of $B_k(F,J)$ contains $F(x)J(x+m^2)$ with $m \ge 1$, which vanishes because no difference of the supports is a positive square; so $B_k(F,J) = 0$. This is the step at which square-difference-freeness is used (Section 7). If $F(x)J(y) \ne 0$, then $1 \le y-x \le N-1$ by the hypothesis of Proposition 7.3 that every point of $\operatorname{supp} F$ is smaller than every point of $\operatorname{supp} J$, and $y-x \equiv b-a \equiv r \pmod D$, so both indicators in $k_0(y-x)$ equal one and

$$B_{k_0}(F,J) = \frac{D}{N^2}\sum_{x,y} F(x)J(y)\sum_{\xi\in\Xi_{H,D}} g(\xi)e(-\xi(y-x)) = D\sum_{\xi\in\Xi_{H,D}} g(\xi)\widehat{F}(-\xi)\widehat{J}(\xi).$$

Hence the left-hand side of (7.1) is $|B_{k_0-k}(F,J)| \le (3^{\pi(729)}+4)H^{-1/3}\|F\|_{\ell^2}\|J\|_{\ell^2}/N$ by Lemma C.8, and $H^{-1/3} \le C_H^{1/3}N^{-\beta/6}$; so (7.1) holds with $C = (3^{\pi(729)}+4)C_H^{1/3}$ for N large. Part (i) is the case of unit square classes and $v_2(D) = 3$; the valuation conditions of (ii) also describe nonvanishing square classes, so the same bound applies there. Its rescaling identity is proved next. Finally, a function bounded by one on one class modulo D in an interval I has $\|F\|_{\ell^2}^2 \le |I|/D + 1$, giving the last assertion. □

C.6. **Nonunit differences.** This subsection is explanatory and is not used in the estimate just proved. It records exactly how the kernel statement and its normalization transform when both supports are divided by a common square R, which is the rescaling used in the main text when the forced difference is not a unit modulo D.

Lemma C.9 (Division by an even-valuation square)**.** *In the setting of Proposition 7.3(ii), let $\mathcal{P}_0'$ be the set of $p \in \mathcal{P}_0$ with $v_p(b-a) = 2j_p \ge 2$, and put $R = \prod_{p\in\mathcal{P}_0'} p^{2j_p}$, $D_R = D/R$ and $N' = \lceil N/R \rceil$. Then $a \equiv b \pmod R$. Let $a_R \in (-R, 0]$ with $a_R \equiv a \pmod R$, and $F'(n') = F(a_R + Rn')$, $J'(n') = J(a_R + Rn')$ for $n' \in \mathbb{Z}$.*

- *(a) F' and J' are supported in $[1, N']$, in the classes $a' = (a-a_R)/R$ and $b' = (b-a_R)/R$ modulo D_R, and $\|F'\|_{\ell^2} = \|F\|_{\ell^2}$, $\|J'\|_{\ell^2} = \|J\|_{\ell^2}$. Moreover $D_R = 8d_R$ with $d_R = d/R$ odd, D_R and D have the same prime divisors, and $b'-a'$ is a strict square class modulo D_R.*
- *(b) Every point of $\operatorname{supp} F'$ is smaller than every point of $\operatorname{supp} J'$, and no such difference is the square of a positive integer.*
- *(c) $\Xi_{H,D_R} = \Xi_{H,D}$, and $\xi \mapsto R\xi$ is a bijection of this set onto itself that preserves conductors, with $g(R\xi) = g(\xi)$.*
- *(d) With $\widehat{F}_N$ normalized by N and $\widehat{F'}_{N'}$ by N',*

$$D\sum_{\xi\in\Xi_{H,D}} g(\xi)\widehat{F}_N(-\xi)\widehat{J}_N(\xi) = R\Big(\frac{N'}{N}\Big)^2 D_R \sum_{\xi'\in\Xi_{H,D_R}} g(\xi')\widehat{F'}_{N'}(-\xi')\widehat{J'}_{N'}(\xi').$$

Proof. Since p^{2j_p} divides $b-a$ for $p \in \mathcal{P}_0'$, $R \mid b-a$; and $R \mid D$, because $2j_p < 2m_p$.

(a) If $F(n) \ne 0$, then $1 \le n \le N$ and $n \equiv a \equiv a_R \pmod R$, so $n = a_R + Rn'$ with $n' = (n-a_R)/R$. As $1 - a_R \in [1, R]$, $n' \ge 1/R > 0$; and $n' \le (N+R-1)/R$, so $n' \le \lceil N/R \rceil$. The congruence

$n \equiv a \pmod D$ is equivalent to $Rn' \equiv a - a_R \pmod D$, that is, to $n' \equiv a' \pmod{D_R}$, because R divides both $a - a_R$ and D. The same holds for J, and the map $n \mapsto n'$ is a bijection of supports, which preserves the ℓ^2 norms. R is an odd square, so d_R is odd and $R \equiv 1 \pmod 8$. At $p \in \mathcal{P}_0'$ the exponent of p in D_R is $2m_p - 2j_p \geq 2$; at every other prime the exponent is unchanged. Now $b' - a' \equiv (b-a)/R \pmod{D_R}$. It is $\equiv 1 \pmod 8$, since $b - a \equiv 1$ and $R \equiv 1$. At $p \in \mathcal{P}_0'$, $(b-a)/R$ is the unit part $(b-a)/p^{2j_p}$, a nonzero square modulo p, divided by R/p^{2j_p}, the square of an integer prime to p; so it is a nonzero square modulo p. At every other odd prime of D_R, $b - a$ is a nonzero square modulo p and R is the square of an integer prime to p. So $b' - a'$ is a strict square class modulo D_R.

(b) The map $n \mapsto n'$ is increasing. If $y' - x' = k^2$ with $k \geq 1$, then $y - x = Rk^2 = (\sqrt{R}\,k)^2$, which is excluded.

(c) The set $\Xi_{H,D}$ depends only on the prime divisors of D. For $\xi \in \Xi_{H,D}$, R is coprime to $q(\xi)$, so multiplication by R permutes the reduced fractions of denominator $q(\xi)$. Since $\sqrt{R}$ is coprime to $q(\xi)$, the map $m \mapsto \sqrt{R}m$ permutes the units modulo $q(\xi)$, and $g(R\xi) = \varphi(q(\xi))^{-1} \sum_m e(\xi(\sqrt{R}m)^2) = g(\xi)$.

(d) By (a), $\widehat{F}_N(\xi) = N^{-1} \sum_{n'} F'(n') e(-(a_R + Rn')\xi) = (N'/N) e(-a_R\xi) \widehat{F'}_{N'}(R\xi)$, and likewise for J. In the product $\widehat{F}_N(-\xi)\widehat{J}_N(\xi)$ the phases $e(a_R\xi)e(-a_R\xi)$ cancel. Substitute $\xi' = R\xi$, use (c), and write $D = RD_R$. □

The factor $R(N'/N)^2$ in (d) is essential: omitting it would multiply the right-hand side by $R^{-1}(N/N')^2$, a factor between $R/4$ and R when $N \geq R$, since $N/R \leq N' \leq 2N/R$ (Section 7).

C.7. **The range of the method.** The closing remark records which exponent ranges the seven conditions (E1)–(E4), (K0)–(K2) allow, and why nonunit square classes need no new arc estimates. The unused ranges play no role later.

Remark C.10 (The range of the parameters). (1) With $\kappa = \beta$ the seven conditions of Definition 7.2 read $5\beta/3 < \eta/2$, $13\beta/6 < \eta$, $\eta + 11\beta/3 < 1$, $2\eta + 19\beta/6 < 1/2$, $3\eta < 1$, $3\beta/2 < \eta$ and $2\eta + 3\beta/2 < 1$. The first and the fourth give the window $(10\beta/3,\ 1/4 - 19\beta/12)$ of Section 7, nonempty exactly when $\beta < 3/59$, and for η in it the other five hold as well. For squarefree d and $H = N^{\beta/2}$ this is the setting of [52, Prop. 6.1], where $\beta = 10^{-5}$ and the major arcs have exponent $1/10$. Neither this window at values of β other than $1/20$ nor the OpenAI parameter values just named are used in this paper.

(2) The root weight enters the proof only through its period, its supremum and the total size of its Fourier coefficients (each equal to D'), and through the local rule of Lemma C.6(a). This is why nonunit square classes, including the higher 2-adic depths of the external index, require no new arc estimates: the root-class lemma supplies the same local rule and keeps $D' \leq D$. The nonvanishing condition is essential to that construction; zero square classes are not covered.

Consequences and what to read next. The proof assumes Definitions 4.5, 4.6, 7.1 and 7.2, Dirichlet's approximation theorem and the elementary facts listed above; Weyl's inequality and the Gauss sums at odd prime powers are proved here. At the triple of Corollary 7.4 the only numerical input is the positivity of the seven kernel slacks, item (S6) of Certificate D.2.

Appendix D. Certificates: model, scope, and the evaluation of Θ

What this appendix proves and how to read it. The proof uses finitely many numerical inequalities, each checked by exact rational or outward-interval arithmetic. *Scope warning:* the baseline verifier chain described in detail below ("the selected certificate") evaluates the *baseline* closing $\Theta < 0.99999999997327$; the improved closing $\Theta < 0.999999771$ is evaluated separately by Certificate 33.3 in Section 33, which reuses the middle portfolio and adds the complete upper and joint-mean checks in `research/epsilon_2026_10_29/`. A successful run verifies finite assertions and input

identity only; it does not verify analytic reductions or external theorems. This appendix says once what such a check proves (Section D.1); describes the final certificate and lists, in the order of the construction, every finite inequality that Parts II–V use, with its checker and its stored record (Section D.2, Table 19); explains how the certificate evaluates Θ from the regenerated native source and the upper chain (Section D.4); lists what the earlier verifiers of the chain contribute (Section D.5); and describes the archive that keeps every earlier checkpoint reproducible (Section D.6). The certificates whose content is a list of numbers are collected in Section D.2; the main text cites them only by their labels there. No proof in the paper uses this appendix except through these certificates and the checks that Table 19 and Section D.4 describe.

D.1. **The certificate model and its scope.** Before listing any computation, we fix what a computer check means in this paper: a named finite computation, with its inputs identified exactly, whose successful run proves the finite assertions stated for it and nothing else. In particular no check verifies an analytic implication between two stated inequalities.

Definition D.1 (A certificate record)**.** A *check* is a named computation with explicit assertions and a success or failure status. A *run record* contains those statuses, the quantities computed, the program version information and cryptographic SHA-256 hashes of its inputs. An *inherited check* is a check whose successful execution is reused only after the verifier has checked that the SHA-256 of every recorded input and result file equals the hash recorded for it in the check's run record. A *fresh check* is executed in the stated run.

Arithmetic. Rational inputs are exact fractions; integer identities and positive semidefiniteness tests are exact. Logarithms, exponentials and square roots are enclosed by intervals with outward-rounded endpoints, at the precisions fixed in each script and listed next. The closing is evaluated in outward decimal intervals with 80 significant digits: the elementary functions are correctly rounded and widened by one unit in the last place, and every other operation is rounded in the safe direction (the interval class of `research/handoff_2026_10_23/tools/conditional_scalars.py`, `PREC = 80`, which every closing module imports through `research/push_2026_10_24/tools/prime_envelope.py`). The local constants of the middle range are enclosed by fractions on the grid $10^{-45}\mathbb{Z}$ (the interval class of `research/push_2026_10_24/local_law_audit/check_pure_local.py`, used by `low_domain.py` and `signed_law.py` of `research/advance_2026_10_24/local/`). The prices of the middle pieces are rounded up on $10^{-30}\mathbb{Z}$, and the native source records its enclosures by exact rational or terminating decimal ends. This is the standard outward-rounded interval model [47, 51]. A strict inequality $x < y$ passes only when the upper endpoint for x is smaller than the lower endpoint for y. A decimal displayed in the paper is never an input; a decimal that is an input (a witness weight, or one of the matching orientation weights $(.5803723, .2409463, .1786814, 0)$) is an exact terminating rational.

Evaluation at endpoints. Every coefficient that a program evaluates at the start of a piece or a band, or at a grid threshold, is an admissible majorant (Lemma 4.18), so its value there bounds its value at every prime and block of the piece. This monotonicity is proved in the text; the programs only evaluate, with an upper bound $1/\lfloor\sqrt{X}\rfloor$ for r and a lower bound for $\log X$ in block counts, where X is the piece start, band start or grid threshold in question. The lower bounds for $\log X$ are rational: write $X = 2^k u$ with $1 \le u < 2$ and truncate the series $\log v = 2\sum_{j\ge 0}((v-1)/(v+1))^{2j+1}/(2j+1)$ for $v = 2$ and $v = u$, after 60 terms in `research/continuation_2026_10_24/local/raw_prices.py` (`log_lower_integer`, line 39) and after 40 terms, rounded down, in `research/roadmap_2026_10_18/item03/independent_closing.py` (line 74, for the restoration counts). All terms are positive, so $\log^- X \le \log X$, and the counts $\lfloor L/\log^- X\rfloor$ are upper bounds for the numbers of primes of a block.

Scope. A successful run verifies the finite assertions and the authenticity of their inputs, by hashes recorded before and after the run. It does not verify the analytic reductions of the

paper, the external theorems (among them Theorems A.7, 4.7 and 4.12, Deligne's theorem, the Katz–Sarnak local monodromy, Kassabov's inequality and Peyre's tensorization), or the correspondence between a statement and the program that checks it; these are proved, or argued, in the text. It is not a proof-assistant formalization. The October 18 record, whose verifier still supplies the inputs listed in Section D.5, was reviewed by the record-gate review of its checkpoint (`research/record_gate_2026_10_18/`; a review process, not a gate in the sense of Section 1.8). The constructions of `research/continuation_2026_10_24/`, `research/advance_2026_10_24/` and `research/further_2026_10_25/` have had no record gate; their reviews are the independent replays named in Table 19 and the review notes (in `research/advance_2026_10_24/`: `envelopes/BETA_CROSS_REVIEW.md`, `local/GRAM_BACKGROUND_REVIEW.md`, `local/GROUPED_FIRST_STAGE_REVIEW.md` and `local/GENERAL_STAR_FRAME_REVIEW.md`; in `research/continuation_2026_10_24/`: `upper_arithmetic/LOCAL_PROOF_REVIEW.md`, `upper_arithmetic/GLOBAL_PROOF_REVIEW.md` and `global_review/CROSS_REVIEW.md`; in `research/further_2026_10_25/`: `REVIEW.md` and `local/LOCAL_PROOF_REVIEW.md`). All of these are internal AI reviews, not human referee reports; Section 35.12 describes them and how the constructions were developed.

D.2. **The baseline verifier chain.** The selected certificate is `research/push_2026_10_25/verify.py`, with record `research/push_2026_10_25/verification/selected.json` (`PASS`, default and full tiers). It retains all 1599 pins of its predecessor `research/improvements_2026_10_25/verification/selected.json` and has 1662 source pins. A *source pin* (here and below: *pins*, *pinned*, *repinned*) is a recorded cryptographic hash of one input file, checked before the file is used; it is not a pinned label in the mathematical sense. The best selected values are $a = \frac{5422587}{6103515625}$, $\varepsilon = \frac{5422587}{433349609375}$, $K = 890666$ and $\Theta < 0.99999999997327$. The source anchor is $4442183/5000000000$.

How it is run. From the repository root, with an interpreter PYTHON importing NumPy and SymPy, the command `PYTHON -B paper/verify.py` runs the selected chain; `--full` additionally regenerates its exact arithmetic sources. The launcher reconstructs two authenticated temporary trees: the original five-stage checkpoint `further_2026_10_25` at `8dc0a3fe`, and the Codex checkpoint `push_2026_10_25` at `1b9239f0`. It authenticates each archive and manifest, the one input file not tracked by git, and the logical inventory (the file listing every file of the chain with its hash; Section D.6). The newer files are `archive/paper-push-2026-10-25.tar.gz`, `archive/scripts-push-2026-10-25.tar.gz` and `archive/push-2026-10-25.inventory.json`. It first runs the frozen verifiers in `research/record_2026_10_18/`, `research/push_2026_10_24/`, `research/continuation_2026_10_24/`, `research/advance_2026_10_24/` and `research/further_2026_10_25/` in the older tree, then `research/improvements_2026_10_25/` and `research/push_2026_10_25/` in the Codex tree. The separation is needed because the run records of the two checkpoints record different bytes (hence different SHA-256 values) for the launcher script `scripts/manuscript_layout.py`, so no single tree satisfies both sets of pins. Each restored tree is checked for mutation afterwards; logs and the JSON run records are written outside both restored trees (to the directory given by `--output-dir` when one is given). Together these seven verifiers authenticate the archived proof sources of the earlier 819-page manuscript. This paper proves the same construction in the form of Section 18; every certificate below is stated for the laws and numbers as defined in this paper, and applies to the archived manuscript only through that identification. The present text supplies a reorganized proof of the same selected construction and a fresh table extraction; its build receipt is `research/merge_2026_10_26/manuscript_build.json`.

The selected refinements. The improvement verifier checks the finite collision bound on the protected subspaces (Proposition 21.11), the interpolation between the pair bounds for the actual kernel, the layer remainder summed in closed form for the layer cut L21 (Proposition 21.15), the coherent 39-trace chamber and layer estimates, and the refinement by one pinned (fixed) coordinate. Its `native` and `native_author_rows` checks regenerate the 33 local records, all completion and

protection gates, and the seven positive-root restoration inputs. The finite prime-power interface and moment checks certify the depth-64 external law and its sufficient moment majorant. The selected root laws and caps are in Tables 28, 36 and 37. All inherited upper-band and envelope checks still run on their unchanged inputs; a claim about a new local law is verified by these additional checks, not by a predecessor's native check alone.

What the final verifier re-evaluates. The final verifier first checks exact identity of all native laws with the improvement checkpoint, the five input files named in the design (the native source, its prime evidence, the native parent, the upper-band source and the envelope bundle of Table 20) and their recorded SHA-256 hashes, all fixed interface parameters, and that the native parent `global/refined_parent.json` is exactly what `global/reanchor.py` produces from its predecessor at the design's source anchor (the same laws, with fresh charges requested at the new anchor). It then independently recomputes the 33 middle prices, checks the selected finite prime-power moment, and runs the separately scoped checks recorded under the names `index_coefficient`, `quartic_reduction`, `marked_channels`, `fractional_extraction`, `auxiliary_symmetry`, `joint_reference`, `three_support_synthesis`, `xor_schur_multiplier` and `rooted_rank_star`; each certifies only the interim statement of its own script, and none is named in Table 19. In full mode `fresh_parent` and `fresh_prices` reproduce the native parent, exact arithmetic source and charge evidence exactly. The author assembly and the separately written independent assembly are run by `research/improvements_2026_10_25/global/selected.py`, supplied with the push design; both read the same authenticated native inputs and envelope bundle, and the independent assembly recomputes every total it reports from those inputs. Both outputs equal their selected records; their compared components overlap, and both upper ends of Θ are below $0.99999999997327 < 1$. The author record's `result.finite_padic_index` equals the selected moment artifact, and its restoration primitives agree exactly with the preceding checkpoint. Finally the generated 819-page manuscript together with the transitive list of files it inputs (its *paper closure*: the manuscript and every file reachable from it by inputs) and every source pin are authenticated. There are 14 default checks and 16 full checks; the improvement predecessor contributes 42 full checks. The following crosswalk retains inherited checks where their formulas are still used; current local, restoration, price and index assertions additionally use the checks above.

What it authenticates. The source-pin table of `research/push_2026_10_25/` (the file-to-SHA-256 table its verifier checks) contains every entry of the preceding checkpoint's table, including every earlier proof source, and adds its new arithmetic inputs, artifacts, checker sources and manuscript closure. Hashes of historical paper and script sources are resolved to their byte-identical archives, so their frozen certificates are not silently repinned to this synthesis. The archive inventory and `archive/README-push-2026-10-25.md` document this reconstruction.

What it checks that the proof does not use. The checks marked *not used* at the end of Table 19 are performed but enter no statement of this paper; they are listed so that the scope of the certificate is exact.

Abbreviations. In Tables 19 and 20 the prefixes C, A, F, P and R abbreviate `research/continuation_2026_10_24/`, `research/advance_2026_10_24/`, `research/further_2026_10_25/`, `research/push_2026_10_24/` and `research/roadmap_2026_10_18/`. The two newer prefixes I and S denote `research/improvements_2026_10_25/` and the selected `research/push_2026_10_25/`. "O: name" is the check `record:name` of the record verifier `research/record_2026_10_18/verify.py`, whose results are in `research/record_2026_10_18/verification/selected.json`; "C: name", "A: name", "F: name", "I: name" and "S: name" are the checks `name` of the corresponding verifier, whose statuses are in the field `result.checks` of its `verification/selected.json`. "(full)" marks a check of the full tier, and numbers after a script are its lines. Every record named in a row is authenticated by the record of its verifier or by the logical inventory.

Inherited formulas and selected evaluations. The older checks certify the unchanged analytic, local, upper-band and matrix-envelope inputs. Where a row names an inherited scalar module, it identifies the formula used; its evaluation on the present design is performed by S: `author` and S: `independent`, in the order described in Section D.4. The assertions about the current local laws are checked by ("bound by" in the table) the checks I: `native`, I: `native_author_rows` and S: `unchanged_native`; the current arithmetic charges by S: `prime_prices` and, in the full tier, S: `fresh_prices`. Thus an earlier successful closing is never used as a certificate for the selected exponent.

Table 19: The finite inequalities used in Parts II–V, in the order of the construction, with their checkers.

Used in	Finite assertion	Checker
Definition 8.1	(S0)–(S7): the stage, kernel and tail slacks, and the gates that involve a on the domain $0 < a \le 9/10000$ (Certificate D.2)	O: analytic:slacks (1231–1280), O: analytic:arc-exponents, O: independent:u05, O: analytic:tail-gates (1568–1621); A: `analytic` (A/`global/analytic_domain.py`, record `analytic_domain.json`), which runs C/`upper_arithmetic/domain_audit.py`; C: `analytic`
Theorem 8.12	$\sigma - a \ge 103/330000$ on the domain; $\sigma - a = 65195254/201416015625$ and $\varepsilon = a/71$ at the selected a (Certificate D.3(a))	A/`global/analytic_domain.py` (`transfer_slacks.induction`); F/`verify.py`:83–84, F/`global/selected.py`:46
Lemma 19.3	the deleted prefix exception and a second copy of the tail prices below $1.2332 \cdot 10^{-39}$ on the domain; the exponent $-1380939795479/6103515625$ at the selected a (Certificate D.3(f))	A/`global/analytic_domain.json` (`prefix_plus_tail`); C/`upper_arithmetic/domain_audit.py`
Lemmas 20.4, 20.6 and 20.7	the 33 pieces, bins, exact blocks and cells, the degree charges, the lower start sums and the piece log caps L_i (Certificate D.4)	S: `fresh_prices` (full; C/`global/price_partitions.py`); S: `prime_prices` (C/`global/review_prices.py`); S: `unchanged_native` and I: `native`; A/`envelopes/piece_product_caps.py` inside F: `author`; the histogram A/`global/final_budget_partitions/exact_block_histogram.csv` (`2b85805d...9302a667`), re-sieved by A: `fresh_partitions` (full)
Lemma 20.3	local laws at (X, L_{loc}) in every piece, $X \ge Q_3 > 2^{24}$; the counts n_X	A: `native` and F: `native` (every spectrum bound to its grid entry at $(X, 585)$); A: `local_low_domain` (A/`local/check_low_domain.py`), A: `local_raw` (A/`local/low_domain.py`)
Certificate 20.13	route-A caps of L21, F21, E21, L10 at the piece starts: 180 exact PSD tests at 9 thresholds, 520 at 26 thresholds	C: `middle_grid`, C: `fresh_grid` (full) and A: `fresh_middle_grid` (full) on C/`global/middle_grid_new.json`; `research/frontier_2026_10_17/root/middle12_grid.json` with its replay `middle12_grid_replay.json`, authenticated by hash; A: `native` binds the stored spectra to the grid entries

Used in	Finite assertion	Checker
Certificate 20.22	the L20 caps of the uniform root at 45 endpoints	A/`local/l20_refined_Q_grid.json`; A: `local_l20_rank4`, A: `local_l20_rectangular`, A: `local_l20_rectangular_gap`, A: `local_l20_collision_zero`, A: `local_l20_balanced`; A: `fresh_l20` (full)
Certificate 21.25	chamber and layer caps of the seven selected positive roots, block caps and marginal data	F: `chamber`, F: `mixed_chamber`, F: `layer`, F: `general_individual_layer`, F: `mixed_layer`, F: `restoration_interface`, F: `native`; F: `square_remainder` (inherited estimates); selected improvement checks `physical_interpolation`, `layer_resummation`, `general_coherent_layer`, `coherent_pin_refinement`, `native` and `native_author_rows`
Lemma 22.1 (iii), (iv)	$33 \cdot 13 = 429$ completion inequalities and 33 protection gates (22.1); the weights; the cap k_{D} (Certificate D.3(c), Certificate D.6)	F: `native`, F: `native_author_rows` (F/`envelopes/check_selected_native.py`: the predecessor's three tilted pieces); the improvement verifier's `native` and `native_author_rows` bind all current laws; A: `native` (A/`global/check_low_reference_native.py`: all pieces, with an independent replay of k_{D}); F: `fresh_parent` (full)
Lemma 22.6	$k_{\mathrm{new}}(X)$, $s_{\mathrm{fold}}(X)$, the gates of (d), exact marginals, no coincidence reset (Certificate D.5)	A: `native`; A/`local/folded_sector.py` (record `folded_sector.json`); A/`global/grouped_first_stage.py` (`check_source`) and A/`envelopes/invariant_error_mass.py` inside F: `author`
Corollaries 9.4 and 10.9	the totals U_0', P', D', V' at a_0; $D' \le V'$, $d < 6$, $dv < 36$, the series below the one-sharp value (Certificate D.3(e))	A/`global/grouped_first_stage.py`:25–38 inside F: `author`; its independent counterpart in A/`envelopes/check_beta_scalar.py` inside F: `independent`
Definition 23.9, Lemma 23.7, and Proposition 23.13	the signed-law identities, the positive repair, the leading matrix, the matching Gram and the block gates; the sharp factors and the five-feature collision frame; the endpoint coefficients at $\mathrm{t} = 127/500$ (Certificate 23.14)	A: `local_signed_law`, A: `local_signed_sharp` (A/`local/check_signed_law.py`, `check_signed_sharp.py`); A/`local/low_domain.py` (`sharp_bounds`) inside F: `author` and A: `local_raw`
Lemmas 23.4 and 25.1	the actual feature frame, Γ_E and ρ_{pair} at Q_3 (Certificate 25.11(a))	C/`local/actual_features.py` (record `actual_features.json`, case $Q = 19951586$), evaluated at Q_3 through A/`envelopes/envelope_domain.py` inside F: `author`
Lemmas 23.17 and 23.18	$s_E(X)$, $a_E(X)$, $\kappa_E(X)$, $r_E(X)$ and their gates (Certificate D.7)	A/`local/low_domain.py` (`e_bounds`) inside F: `author` and A: `local_raw`; A: `local_low_domain`
Certificate 23.10	the query images of the middle records	O: middle:geometry (1640–1669), O: independent:u01
Lemma 23.20	the merge coefficients and the dependent coefficients A_3, S_3 in the final source (Certificate 23.22)	A: `native` (independent recomputation of the merge coefficients); F: `native`; A: `fresh_native_prices` (full)

Used in	Finite assertion	Checker
Lemma 23.24	the per-piece prices of Definition 30.1(3)	A/`local/low_domain.py` (`raw_ledgers`) inside F: `author` and F: `independent`; F: `prime_prices`
Lemma 24.3	the 400-term identity; the modular bound (Certificate D.8)	O: variance:weighted-identity (1622–1639); C/`upper_arithmetic/modular_variance_check.py` (record `modular_variance_check.json`)
Lemma 24.5	the 1200-term identity (Certificate 24.6)	C/`upper_arithmetic/l20_star_mean.json`, whose status and term count C: `l20_mean` and C: `fresh_l20_mean` (full) assert
Equation (24.11) and Lemma 24.7	the root data, h_i, M_i, χ_i, H_i, $k_{F,i}$, $k_{E,i} < 1$; M_i, χ_i of the seven pieces with positive roots (Certificate D.9)	A/`global/piece_data_beta.py` inside F: `author` and F: `independent`; the current seven inputs are supplied by `research/improvements_2026_10_25/global/scalar_adapters.py` and checked against the author rows by its native reviewer
Certificate 24.19, Corollary 24.18, Proposition 24.17, and Lemma 24.16	restorations at the piece caps, $0 \le \ell \le C_m/2$, gains in $[0, 1)$, D_{var}	C/`envelopes/restoration.py`, `background_variance.py`, `coarse_tail_coupons.py`; A/`envelopes/beta_restoration.py`, `piece_cap_restoration.py`, `beta_higher_variance.py`, inside F: `author`; A/`envelopes/check_beta_scalar.py` inside F: `independent`; F: `restoration_interface`
Theorem 25.8, Lemmas 25.7 and 25.9, and Corollary 25.10	the portfolio orders, physical averages, coupling and orbit masses, the class costs and savings; the FE witness (Certificate 25.11)	F: `envelopes` (A/`envelopes/verify_selected_envelopes.py`) with the family checkers of A/`envelopes/`; F/`envelopes/check_exact_background_pairs.json`; A/`envelopes/invariant_error_mass.py`, `invariant_orbit_mass.py` inside F: `author`
Lemma 25.6	U^{inv} and the generic price (Certificate 25.11(c))	F: `chosen_background` (F/`envelopes/chosen_background.py`); F/`global/scalar_adapters.py`:63–91 inside F: `independent`
Propositions 26.4 and 26.5	the auxiliary sides at $L_{\mathrm{loc}} = 585$, the chosen sides, the increments, $k_{s,i}^{\mathrm{aux}} \ge 0$, T_K (Certificate D.10)	A/`local/signed_law.py` (`matching`), R/`item03/direct_matching_asymmetric.py` and A/`global/piece_data_beta.py` inside F: `author` and F: `independent`; the side gates in A/`global/closing_beta.py`:33–47 and, independently, in A/`envelopes/check_beta_scalar.py`
Proposition 26.7	m_F, m_Q, the error, the gate $2q_i\bar{I}_{2,i} \le A_{E,i} + S_{E,i}$ (Certificate D.11)	A/`envelopes/beta_restoration.py`, `piece_cap_restoration.py`, C/`envelopes/coarse_tail_coupons.py` and A/`global/closing_beta.py`:112 inside F: `author`; A/`envelopes/check_beta_scalar.py` inside F: `independent`

Used in	Finite assertion	Checker
Theorem 14.27, Lemma 16.15, and Certificates 27.8 and 27.9	frame and Schur witnesses at the anchors $2^{47}, \dots, 2^{52}$; the graded and special-value coefficients at 19 anchors (30 645 955 histogram rows); the root caps from P_3 to 2^{52}	O: upper:direct-star (1467–1525), O: independent:u07 (32 Schur matrices at $2^{49}, \dots, 2^{52}$); R/`item12/direct_star_lower_endpoints.json` (2^{47}, 2^{48}; 16 tests), authenticated by hash, with the class-three tests at 2^{47} replayed in A/`upper/lower_p47/native_review.json`; F/`upper/special_coefficients_review.json`, F/`upper/special_upper_review.json` (inside F: `upper`; the rows re-summed in the full tier); A/`upper/even_extraction_review.json`, `even_roots_review.json`
Lemma 27.10	the completed-root caps from 2^{52} (Certificate 27.11)	the grids of `research/advance_2026_10_17/sparse/` and the output of the replay R/`item15/replay_upper_psd.py`, authenticated by hash; no tier of the chain repeats these tests
Proposition 27.14	the merge and constant caps of the observed marginals	A/`upper/observed_caps.json`, `observed_caps_review.json`; A/`upper/lower_p47/observed_caps.json`, `observed_review.json`
Certificate 27.13	the class-three nonroot caps; the row quantities of (F3), (F4)	A/`upper/class3_nonroot_caps.json`, `class3_nonroot_review.json`; A/`upper/lower_p47/class3_caps.json`, `class3_review.json`
Proposition 16.22 and Lemmas 16.23 and 27.15, (F5)	the conditional rows: full star matrices, decay weights, joint rows, pins, potentials, the bucket $\mathfrak{q} = 0$	A: `joint_witness_0`, …, `joint_witness_7` (A/`local/check_conditional_joint_potential.py` on the witness files); A/`local/check_conditional_star_*.py`; A/`upper/conditional_fine_parts.json` and its review; F/`upper/joint_endpoints/index.json` (11 tables, each rechecked inside F: `upper`), F/`upper/endpoints_review.json`, `allowances.json`, `allowances_review.json`
Lemma 27.17 and Certificate 27.18	$310 \cdot 105 = 32550$ completion and domination gates (24800 axis inequalities, 7750 reserved floors)	F: `upper` (F/`upper/verify.py`); F/`upper/endpoint_allowances_review.json`, `dense_refined_upper_review.json`; A: `upper` (A/`upper/regenerate_upper_chain.py`, 14 stages, 51 reviews)
Lemma 28.1	the acceptance tests of the six queries	A/`upper/six_query_geometry.json`, `six_query_upper_review.json`
Proposition 28.5	the dense EL coefficients	A/`upper/dense_p48/completed_invariant.json`, `certificate.json`; A/`upper/dense_review_full.json`; A/`upper/lower_p47/dense_*.json`
Proposition 28.6	the LE mixture: the refined one in the 301 bands below 2^{52}, the first one in the nine bands from 2^{52}	F/`upper/dense_six.json`, `dense_six_refined.json`, `dense_six_refined_review.json`, `dense_refined_upper_review.json`

Used in	Finite assertion	Checker
Proposition 28.3, (F7)	the skew-only identities, forest matrices and differences	C/`upper_arithmetic/skew_only.json`, `forest_matrix.json`, `extended_forest.json`, `forest_difference.json`, with their reviews
Certificate 28.12	the 45 bands from P_3 at the data of 2^{47}	A/`upper/lower_p47/*.json`; A/`upper/LOWER47_AUDIT.md`
Proposition 28.14	S_{u}, A_{u} at a_{u}; their repricing to a	F/`upper/upper_dense_refined.json` (`S`, `A`); C/`global/closing.py`:130–150 (`upper_price`) inside F: `author` and F: `independent`
Lemma 12.8	$e, w < 1/2$ and the pair and slot bounds at $p = 2^{512}$ (Certificate 12.9)	`research/push_2026_10_17/structural/layer72_analytic.py` (`leading_trace`), run whenever the record or pure-pair chain generates a source (their full tiers); its result is the field `analytic.leading` of those sources, which the final native source carries unchanged, authenticated by hash
Lemma 29.3	(S7), with both forms of the gate at P_5	O: analytic:tail-gates (1568–1621)
Certificate 29.4	the 94 tail repair caps at 2^{54} with their gates, the joint-star rows of the ten LR, RL records; the coefficients c_3^{rep}, c_4^{rep}, c_d^{del}	`high72_oneway_repairs.json`, `high72_oneway_repairs_degree3.json` and `high72_oneway_jointstar.json` of `research/frontier_2026_10_17/sparse/`, authenticated by hash and, when a source is generated (full tiers), by `high72_oneway_adapter.py` and `high72_oneway_jointstar_adapter.py`; the coefficients are the field `tail_comparison` of R/`item12/upper_v2_local.json`, compared and re-summed by P/`tools/check_upper.py`:55–57 (full); no tier recomputes the caps
Lemma 29.5	the tail prices at a; a second copy inside 10^{-12}	C/`global/closing.py`:141–149 inside F: `author`; A/`global/analytic_domain.json`
Proposition 29.7, Lemma 29.9, and Certificate 29.11	the sufficient finite-depth majorant for R_I at Q_3 and $K = 890666$	I: `padic_interface`; S: `padic_moment`, S: `author` and S: `independent` on S/`artifacts/padic_moment.json`; the inherited prime tables C/`global/prime_moments.json` are authenticated by C: `fresh_moments` and A: `fresh_prime_moments` (full)
Certificate 30.3	$\Theta < 0.99999999997327$	S: `author`, S: `independent` (I/`global/selected.py` on S/`design.json`)
not used	the isolated squared-trace kernels; the exact row identity of the pair-square remainder; the audited individual-squares construction	F: `one_edge`, F: `two_edge`; F: `square_remainder` (`PASS_PARTIAL_FINITE_PAIR_SQUARE_REMAINDER`); F/`verification/individual_only_audit.json`
not used	the interim local, envelope and upper results of the signed-block record; the sign-model frontier (Remark 35.5); the twelve-label grid and the local gates below 2^{24}	A: `interim_*` (18 checks), A: `interim_support_density_0`, …, `interim_support_density_7` (full); A: `local_signed_frontier`; A: `fresh_low_grid` (full) on A/`global/low_domain/middle_grid.json`

Used in	Finite assertion	Checker
not used	the route-A L20 gates of the continuation's root; its closing, its upper ledger and its native middle	C: `l20_variance` and the spectral gates of C: `l20_mean`, with their fresh replays; C: `author`, C: `independent`, C: `upper_ledger`, C: `upper_prices`, C: `native`
not used	the pure law, its envelopes, its matching increments and its closing; the October 18 closing, the orientation mixture and its band ledger	the checks of P/`verify.py` other than the tail re-check; O: closing:ledger, O: mixture:uniform, O: upper:bands, O: variance:higher
not used	the influence conditions (13.5) of the half graph G^+, at every threshold of the grids of Theorem 14.21 (Remark 14.23)	`paper/certificate/certlib/checks_spectral2.py`:310–323 (`components2`), in the runs that produced the grids and their replays; no record stores the values, and no tier of `paper/verify.py` repeats them
not used	the consistency tests of Theorems 15.8 and 15.13 (Remark D.13)	`paper/certificate/certlib/`

The certificates of the text. The certificates below are cited in the text and checked as listed in Table 19. Upper bounds are rounded up and lower bounds rounded down (toward $-\infty$), as in Appendix E; "truncated" in data descriptions means this rounding down, including for negative numbers.

Certificate D.2 (The analytic stage)**.** In exact rational arithmetic, with the parameters of Definition 8.1:

(S0) for every $0 < a \le 9/10000$ and every fixed integer $K \ge 2$: the induction slack is $\sigma - a \ge 103/330000$; the fixed-scale slacks $1-6\beta = 7/10$, $1-2\tau = 199/201$, $1-t_{\mathrm{an}} = 59/60$ and $1/4 - 2a \ge 1241/5000$ are positive; the Hölder exponents of the density moments of orders $2, 4, 72, 140, 142$ add up to one in each of the splits $1/2 + 71/142$, $2/4 + 70/140$ and $72/72$; the exponent $513 - m_{\mathrm{cl}} + 1024 m_{\mathrm{cl}} a$ of the price of the deleted prefix exception, $m_{\mathrm{cl}} = 8192$, is at most $-80783/625$; and that price together with a second copy of the tail prices of the tail law, repriced at a with $s = d/2 - 2a > 1$ for every formal degree $d \ge 3$ occurring in the tail polynomials, is below $1.2332 \cdot 10^{-39}$. None of these inequalities involves Q_3, P_3 or a local law;

(S1) $\gamma\beta/2 - \tau - 72\delta - \sigma = 2260122363/58960000000000 > 0$;

(S2) $\gamma t_{\mathrm{an}} - 72\delta - \sigma = 3878506121/1320000000000 > 0$;

(S3) $\beta/6 - t_{\mathrm{an}}(1/2 - \gamma) - 72\delta - \sigma = 3878506121/1320000000000 > 0$;

(S4) $\tau/4 - \ell_0 - \sigma = 6997789/221100000000 > 0$;

(S5) $1 - 2\tau - 6\beta = 1387/2010$, $\beta/2 - \tau = 161/8040$, $\beta/2 - t_{\mathrm{an}} = 1/120$ and $\beta(1-2\tau) - \beta/2 = 197/8040$, all positive;

(S6) the seven kernel slacks of Lemma C.5, computed in Corollary 7.4, are $1/600$, $37/600$, $97/150$, $1/600$, $49/100$, $19/200$ and $117/200$;

(S7) the tail coefficients of Lemma 29.3 are $C_1 < 1197.601$, $C_2 < 609305.05$ and $C_3 < 1.4176 \cdot 10^8$, and

$$2^{16} S_{72}(2^{512}) < 6.7782 \cdot 10^{-70}, \qquad 2^{32640} S_{72}(2^{65536}) < 3.5195 \cdot 10^{-36};$$

only the second gate, at P_5, is used.

The roles of (S1)–(S7) are listed after Definition 8.1, and (S0) is the form of the gates that involve a which Remarks 8.13 and 19.4 use. The domain audit evaluates the prefix and tail prices at $a = 9/10000$; both increase with a. The checker of (S7) recomputes C_1, C_2, C_3 exactly from the normalizing polynomial of the tail datum (the last band of its source, which ends at

P_4; Definition 29.1), checks that they equal the tail coefficients of the six-query checkpoint, and evaluates the gate at P_5 in two forms: as stated, with last term $2^{32640}C_3p^{-3/2}$, and in the form of the frozen code that produced the stored values, with last term $2^{32640}C_3p^{-2}$. The two differ by $C_3(2^{-65664}-2^{-98432}) < 2^{-65600}$, and both are below one.

Certificate D.3 (Data of the six-query frame)**.** The following hold for the data of Definition 18.1.

(a) $a = \frac{5422587}{6103515625} < \sigma = 1/825$, with $\sigma - a = 65195254/201416015625$; $\theta = 1$, the finite moment orders are $2, 4, 72, 140, 142$, $\varepsilon = a/71 = \frac{5422587}{433349609375}$, and $K = 890666$. Every slack of Certificate D.2 is a positive rational, and the analytic stage holds on the whole domain $0 < a \le 9/10000$ by its item (S0).

(b) The middle prices are tabulated at the anchor $a_0 = 4442183/5000000000 < a$ of the native source, with $2a_0L = 253204431/250000000$, and the upper ledger at its anchor $a_{\mathrm{u}} = 883/10^6 < a$. The restorations are evaluated at a with the piece caps $C_i = e^{2aL_i} \le C = e^{2aL} < 2.753348440327$, where $L = 570$ and $2aL = 6181749180/6103515625$.

(c) The 33 middle pieces tile $[Q_3, P_3)$, $Q_3 = 19951586$, $P_3 = 167 \cdot 2^{40}$. In each, the local laws are evaluated at $(X, L_{\mathrm{loc}}) = (X, 585)$, within the domain $Q \ge 2^{24}$ and length at most 10000 of the twelve-label grid. All thirteen completion inequalities of Lemma 22.1(iii) and the protection gate (22.1) hold in every piece, $33 \cdot 14 = 462$ inequalities in all, and so do the one-stage gates of Lemma 22.6(d) (Certificate D.5).

(d) The 310 upper bands of Definition 27.1 run from P_3 to $P_4 = 2^{512}$.

(e) The native source stores upper ends at a_0 for the totals of Lemma 22.9 over the 33 pieces, which are at most

$$U_0' \le 0.846908193539142986, \qquad P' \le 1.6936504653996419205,$$
$$D' \le 4.7417041225622828 \cdot 10^{-5}, \qquad V' \le 3.3872533701481774716$$

(field `middle` of the source, equal to `portfolios[0].sums` of the prime evidence).

(f) With $m_{\mathrm{cl}} = 8192$ digit pairs, on the whole domain the exponent $513 - m_{\mathrm{cl}} + 1024m_{\mathrm{cl}}a$ is at most $-80783/625$, and the price $2^{513-m_{\mathrm{cl}}+1024m_{\mathrm{cl}}a}$ of the deleted prefix exception together with a second copy of the tail prices is below $1.2332 \cdot 10^{-39}$ (S0). At the selected a the exponent is $-1380939795479/6103515625$ and the price is below $8 \cdot 10^{-69}$, as proved in Remark 19.4; the combined prefix and tail price lies inside the allowance 10^{-12}.

Certificate D.4 (Middle prices)**.** The following hold for the partition of Definition 20.1, at the anchor $a_0 = 4442183/5000000000$.

(a) The 33 pieces tile $[Q_3, P_3)$, the 35 parts of Definition 20.1(b) are each used exactly once, and in every piece the local law was computed at $(Q, \mathrm{length}) = (X, 585)$, while the blocks have actual budget $L = 570$. The grid domain is $Q \ge 2^{24}$, length at most 10000.

(b) Every bin integer $\mathfrak{u}$ is the least integer with $\mathfrak{u} \ge 2^{40}\,\overline{\log B}$, where $\overline{\log B}$ is the upper end of the certified enclosure of the logarithm of the bin's upper endpoint B (the notation log hi in the data), certified with outward rational logarithms. The run record `research/advance_2026_10_24/global/final_budget_partitions/exact_block_run.json` (status `PASS_EXACT_PRIME_BLOCK_STARTS`) counts, in the 6362 bins from 2^{23} to X_{Ch}, 112 714 305 primes $p \equiv 1$ and 112 720 569 primes $p \equiv 3 \pmod 4$; the 5668 bins from 2^{24} hold all of them except 256 852 and 256 856, and no middle piece uses the others. The largest integer log-sum of an actual block is $626\,720\,431\,062\,769 \le \mathfrak{u}_* = 626\,721\,627\,832\,320$. The histogram `exact_block_histogram.csv` of the same directory has SHA-256 `2b85805d...9302a667`; the full tier of the signed-block verifier sieves all primes again and reproduces it, the counts and the run record byte for byte.

(c) The cells above X_{Ch} have the following integers n_j, constant on the displayed ranges of j, with n the number of cells in the interval (Definition 20.1(d)) and $\mathsf{e}_x = 2411/(5000000 \log x)$ the relative error of Theorem 4.12 at the interval start x:

Interval	n	n_j (cells j)	e_x
$[X_{\mathrm{Ch}}, 2^{34})$	236	25 (0–93), 24 (94–235)	$2.1632 \cdot 10^{-5}$
$[2^{34}, 2^{36})$	256	24 (0–32), 23 (33–223), 22 (224–255)	$2.0461 \cdot 10^{-5}$
$[2^{36}, 2^{40})$	512	22 (0–175), 21 (176–403), 20 (404–511)	$1.9325 \cdot 10^{-5}$
$[2^{40}, 2^{45})$	640	20 (0–141), 19 (142–418), 18 (419–639)	$1.7392 \cdot 10^{-5}$
$[2^{45}, P_3)$	306	18 (0–86), 17 (87–305)	$1.5460 \cdot 10^{-5}$

Each n_j equals $\lfloor L/\log x_{j+1} \rfloor$, computed as the floor of a certified lower bound of the quotient; in all 1950 cells $n_j \log x_{j+1} \le 569.9947476$, so $L - n_j \log x_{j+1} > 0.00525$, and every cell price is at most $\Pi_j \le e^{2a_0 L}$.

(d) With $b_d(X, Y)$ bounded by Lemma 20.4, the sums of Lemma 22.9 over the 33 pieces are at most the numbers U_0', P', D', V' of Certificate D.3(e). The rounded rows of Definition 30.1(3), summed over the pieces and multiplied by t, give the selected middle totals of Table 29 and Section E.3; their exact values are stored in `research/push_2026_10_25/artifacts/author.json`, field `result.raw_totals`.

(e) The cap at the anchor satisfies $e^{2a_0 L} = e^{253204431/250000000} < 2.754$ (the enclosure `selected_cap` of the prime evidence). The piece log caps L_i of Lemma 20.7 (`restoration_product_caps.rows` of `author.json`, with the finite and the analytic part of each) lie between 556.797 (the piece from 759 250 126) and 570.

(f) The analytic lower sums of Lemma 20.6(i) over the five intervals of (c), in the order of the table, are at least 0.00113992045177, 0.00124248816885, 0.00249322444258, 0.00312057806694 and 0.00149358482940. The exact lower sums (ii) are evaluated from the same histogram.

The degree charges are evaluated with rational outward arithmetic: the logarithms by atanh series with explicit remainders, the exponentials by Taylor series with explicit remainders, the integrals $I_j(1)$ exactly and $I_j(s)$, $s > 1$, from above by the secant of $1/v$ in $v = \log t$. The degree charges of each piece are listed in Table 24, and its four sums U_0, P, D, V in Table 25. The histogram is an authenticated input, produced by a segmented sieve and checked in full by the fresh sieve of the full tier; the prices are recomputed independently from it and from the cells by F: `prime_prices`.

Certificate D.5 (The one-stage coefficients)**.** At the 33 piece starts the coefficient $k_{\mathrm{new}}(X)$ of Lemma 22.6 decreases from at most 1.0000003509 at $X = Q_3$ to at most $1 + 2.0 \cdot 10^{-13}$ at $X = 2^{45}$, and $s_{\mathrm{fold}}(X)$ from at most 1.000448358 to at most 1.000000338. The gate of Lemma 22.6(d), $\alpha_X^2 r_X + \beta_X^2 q_X r_X < k_{\mathrm{new}}(X)$, and the inequality $4q + 2r(1-2q)/(1+q) < 1$ of its proof hold at every start. In Lemma 22.9 the coefficient of the degree-three charge in the stored (rounded-up) sharp price is exactly the stored value, which is at least $2(v_{\mathrm{S}} + v_{\mathrm{D}})k_{\mathrm{new}}(X)$; that of the degree-two charge in the stored variation price is at least $4(v_{\mathrm{S}} + v_{\mathrm{D}})$; and there is no coincidence term. The native source records each coefficient with the flags of the sector refinement, of exact actual marginals and of zero coincidence resets; the native review checks, with $r = X^{-1/2}$ enclosed, that each stored coefficient lies above the exact $k_{\mathrm{new}}(X)$.

Certificate D.6 (The negative caps of the reference fold record)**.** At the 33 piece starts the cap $k_{\mathrm{D}}(X, 585) = \bar{H}_X[1 + (e_o(X) - 1)r_X]$ of Lemma 23.2(e), with $n_X = \lfloor 585/\log^- X \rfloor$, decreases from at most 1.001486585 at $X = Q_3$ ($n_X = 34$, $\bar{H}_X \le 1.00000341$, $1 + (e_o - 1)r_X \le 1.00148318$, $e_o \le 7.62384415$) to at most 1.000001115 at $X = 2^{45}$ ($n_X = 18$). The hypotheses $n_X \mathsf{e}_X < 1$ and $[1 + (e_o(X) - 1)r_X]r_X < 1$ hold at every start. The F21 entry of the twelve-label grid ($g = 1.01223725$ at Q_3) is not used. The protection gate $v_{\mathrm{D}} \ge 2w_{\mathrm{D}}k_{\mathrm{D}}$ is binding in every piece up to the rounding of v_{D}: $0 < v_{\mathrm{D}} - 2w_{\mathrm{D}}k_{\mathrm{D}} < 10^{-9}$. The evaluation uses the outward upper bound $1/\lfloor X^{1/2} \rfloor$ for r_X and the lower bound $\log^- X$ in n_X; both can only increase k_{D}.

Certificate D.7 (The matching coefficients)**.** At each of the 33 piece starts, with $q_X = 1/X$ and $r_X = 1/\lfloor\sqrt{X}\rfloor$, the coefficients $s_E(X)$, $a_E(X)$, $\kappa_E(X)$ and $r_E(X) = 2c_E^{\mathrm{s}}(X)a_E(X)$, and the two- and three-support coefficients $s_{(2)}^E$, $s_{(3)}^E$, of Lemma 23.17(ii) satisfy the gates of (ii): $3q/(1-3q) < 1$, $\kappa_E r < 1$, $r_E r^2 < 1$, $4/((1-q)(1-2q)) < r_E$, $\kappa_E^3 < r_E$, $s_{(2)}^E \le 1.0000016$ and $s_{(3)}^E \le 0.0002248$. All four coefficients decrease along the pieces:

X	$s_E(X) \le$	$a_E(X) \le$	$\kappa_E(X) \le$	$r_E(X) \le$
Q_3	$5.4071859 \cdot 10^{-4}$	$1 + 2.51 \cdot 10^{-7}$	1.0012128	10.829703
2^{45}	$4.0700610 \cdot 10^{-7}$	$1 + 1.5 \cdot 10^{-13}$	1.0000010	10.828429

These are the formulas that the prices of the matching records use (Lemma 23.18); the field `E_symmetric_over_r` of `e_bounds` is the coefficient $s_E(X)$ itself.

Certificate D.8 (The weighted second-moment identity)**.** The rules (G1)–(G4) of the proof of Lemma 24.3, applied to the 400 terms of $\mathbb{E}[P_{\mathrm{adm}}\mathcal{W}^2]/p^2$, with the vertices (x_1, x_2, x_3, x_4) mapped to $(0, 3, 1, 4)$ and (z, z') to $(2, 5)$, leave 102 nonzero terms, and their sum is the right side of (24.2). The reduction is implemented with integer coefficients and without floating point in `research/roadmap_2026_10_18/item03/independent_weighted_identity.py`; its output is compared with the frozen `research/roadmap_2026_10_18/item03/independent_weighted_identity.json`. For the modular bound $\mathbb{E}[P_{\mathrm{adm}}\mathcal{W}^2]/p^2 \le 5q + 34qr + 11q^3$ of Lemma 24.3(b), the script `research/continuation_2026_10_24/upper_arithmetic/modular_variance_check.py` checks in exact polynomial arithmetic that, with M_{G_5} replaced by its upper bound $(1-q)(2qr-q^2)$ from (24.3) and $M_{G_{10}}$ by qr, the difference between $5q+34qr+11q^3$ and $(5+3q)M_{G_5}+24M_{G_{10}}+5q-3q^2+11q^3-13q^4$ is $8q^2 - 2q^3 + 10q^4 + 4q^2r + 6q^3r = 6q^2 + 2q^2(1-q) + 10q^4 + 4q^2r + 6q^3r \ge 0$, and it checks the inversion $M_{G_{12}} = (1-5q)M_{G_5} + q^2(1-q)(1-2q)$ exactly at the primes 3, 7, 11, 19, 23, 31, 43, 47. Its record `modular_variance_check.json` (status `PASS_EXACT_MODULAR_NUMERATOR_ALGEBRA`) is authenticated by the continuation record; no verifier reruns it. The bound for M_{G_5} itself is Theorem A.7.

Certificate D.9 (The piece data of the root)**.** In each of the 33 middle pieces, with $q = 1/X$, $r = 1/\lfloor\sqrt{X}\rfloor$ and the restoration count $n = \lfloor L/\log^- X\rfloor$, $L = 570$: the lower factors of Lemma 24.7 are positive, $k_{F,i} < 1$ and $k_{E,i} < 1$, and the piece values h_i, M_i, χ_i of (24.11) are evaluated; on the seven pieces with positive roots $M_i = \mathrm{up}(R^n)$ and $\chi_i = \mathrm{up}((1+d^2)^n - 1)$ with the single-prime range ratio R and variance d^2 of Certificate 21.25. The complete selected values are in Table 27; the seven changed root inputs are in Tables 36 and 37. At the first piece $k_{F,1} \le 2.25870 \cdot 10^{-4}$ and $k_{E,1} \le 1.50365 \cdot 10^{-7}$, with $r = 1/4466 \ge Q_3^{-1/2}$; both decrease along the pieces. Every branch of Lemma 24.8 is retained in the selected evaluation; the coherent root uses its actual finite marginal. The piece values do not depend on the exponent.

Certificate D.10 (The sides of the matching envelopes)**.** At the auxiliary parameters

$$(b_0, d_0, f_0) = \Big(\frac{1606593}{2500000}, \frac{2590537}{5000000}, \frac{803723}{5000000}\Big) = (.6426372,\ .5181074,\ .1607446),$$

which the orientation weights $(.5803723, .2409463, .1786814, 0)$ give, and $\tau_{\mathrm{side}} = 31/32$ of Definition 26.2, the arm norm is $2.0809619\ldots$ at $\tau_{\mathrm{side}} = 1$ and $s(31/32) = 2.0778741\ldots$, with $s\tau_{\mathrm{side}} = 2.0129405\ldots$ and $s/\tau_{\mathrm{side}} = 2.1449023\ldots$, whose mean $2.0789214\ldots$ is below $2.0809619\ldots$. In each of the 33 middle pieces, with $n_X = \lfloor 585/\log^- X\rfloor$, the gates of Definition 26.2 for the auxiliary sides below, and the hypothesis of Proposition 26.4, hold: $q \le 2^{-20}$, $\eta_1 < 1$, $\rho_1 < 1$, $2\phi_\infty < 1$, $\bar{k} r_X < 1$ and $n_X e_0(X) < 1$. At Q_3 ($n_X = 34$), $C_{\mathrm{old}} < 19.0000162$, $C_{\mathrm{new}} < 153.735107$ and $C_{\mathrm{rep}} < 0.0381683$, and the auxiliary sides are $d_{\mathrm{L}}^{\mathrm{aux}}(Q_3) < 2.0524514$ and $d_{\mathrm{R}}^{\mathrm{aux}}(Q_3) < 2.1844137$; at 2^{45} ($n_X = 18$) they are below 2.0129704 and 2.1449321. The coefficients $k_{s,i}^{\mathrm{aux}} = w_{\mathrm{D},i} d_s^{\mathrm{aux}}(X_i) - v_{\mathrm{D},i}$ are positive in every piece, with $k_{\mathrm{L},i}^{\mathrm{aux}} \le k_{\mathrm{R},i}^{\mathrm{aux}}$, with the exact selected values and a positive

lower bound printed in Table 31. The chosen coefficient $\bar{H}_X d_{\mathrm{t}}(X)$ is at most 2.3903402 at Q_3 ($\bar{H}_X \le 1.0000034083$) and 2.3842319 at 2^{45}; it exceeds both auxiliary sides in every piece, so both increments are positive, $\delta_{\mathrm{L}}^{+} < 0.33788882$ and $\delta_{\mathrm{R}}^{+} < 0.20592650$ at Q_3. The prices of Proposition 26.4(iii) are $T_K^{\mathrm{aux}\prime} < 3.549907570156 \cdot 10^{-2}$ at a_0, the sum of the rounded rows of the auxiliary sides, and $tT_K^{\mathrm{inc}\prime} < 0.12733201276083$ at a; in every piece the cap $\mathrm{up}((k_{\mathrm{L},i}^{\mathrm{aux}} + k_{\mathrm{R},i}^{\mathrm{aux}})\bar{I}_{2,i})$ plus the increment price of the piece is at least $(k_{\mathrm{L},i} + k_{\mathrm{R},i})\bar{I}_{2,i}$, the price of the two chosen sides of the piece in Proposition 26.4(iii).

Certificate D.11 (Positive mean savings)**.** The coupons of Proposition 26.7 are evaluated at the piece caps of the selected exponent, for the selections of degree at most four and, on the pieces from 2^{31}, for the selections of degrees five and six pooled there. The matching coefficient of the piece i is $q_i = w_{\mathrm{S},i}(s_{E,i} + a_{E,i})$, with the symmetric and skew coefficients $s_{E,i}$, $a_{E,i}$ of the matching records stored in the native source (at Q_3 about $5.40631 \cdot 10^{-4}$ and $1 + 2.51 \cdot 10^{-7}$), and the gate $2q_i\bar{I}_{2,i} \le A_{E,i} + S_{E,i}$ holds in every piece. The certificate gives

$$m_F > 0.400172842378, \qquad m_Q > 0.190134971388,$$
$$R_{\mathrm{err}} e^{T_K + T_{QE}} < 3.67071778633 \cdot 10^{-4}, \qquad m > 1.589940741988;$$

the pooled selections contribute about $1.1951 \cdot 10^{-7}$ to m_F and $6.5884 \cdot 10^{-7}$ to m_Q, all of it at degree five.

D.3. **Checks for the local refinements of 29 October.**

Certificate D.12 (Local identities and the joint-row witness)**.** The reproducible entry point is `research/astra_2026_10_29/verify.py`; its run record is `research/astra_2026_10_29/verification/selected.json`. It authenticates and extracts the 43 program and data files embedded in the first final October 29 handoff and the 42 such files embedded in the second, and replays their exact checks in temporary directories. An independently written finite checker additionally verifies the following.

(i) The full-cube rectangle Gram identity, including repeated private labels, agrees with direct enumeration for all 237 unordered Gram entries at $p = 3, 7$. Exact endpoint inequalities give the constants 121.588 and 121.328 in Theorem 20.19, and the elementary constant 128.

(ii) The 64 cyclic-triangle masks have the six nonstar classes and multiplicities displayed in the proof of Proposition 21.9; the rational majorant for $\mathfrak{b}$ is below 140.004 at 2^{30} and below 140.002 at 2^{32}, and gives the coefficient below 2897 in Lemma 21.8.

(iii) The signed 720-edge upper graph gives the compatible-pin counts and weighted sums in (27.4) using Lemma 16.19. The tests are on the full graph, before any witness edges are removed.

(iv) The cubic resultant identity (15.6) holds for all 5,221 unordered Gram entries at $p = 3, 5$, including nonsplit squarefree cubics. These small-field tests corroborate the algebra; the proof for all odd primes is Theorem 15.12.

The imported nine-class skew witness, stored as `research/upper2_joint_row_dual_witness.json` in the second extracted handoff, has positive exact LDL^* pivots for its contraction test. Its separate rational checker verifies the trace norm separation and the negative gate at 172.9 in Proposition 27.16. The certificate concerns that particular joint-row formula, not a lower bound for a physical operator.

The two imported global replays return larger *conditional* exponents. Their supplied data do not include the required common middle-cell construction, rational frame witnesses and compatible positive assignments. We therefore use them only as conditional arithmetic records, not as the closing certificate of this paper. In contrast, Certificate 33.3 reconstructs the positive allocation

on the existing complete middle portfolio and verifies the new upper comparisons directly. Its entry point is `research/epsilon_2026_10_29/verify.py`. Running `research/astra_2026_10_29/verify.py` with `--full` also replays both complete assemblies of the preceding October 27 theorem. It restores the original four-route manuscript from its authenticated archive instead of changing the historical hashes to match the present exposition. The analytic scope of every check remains that of Section D.1.

D.4. **How the certificate evaluates Θ.** Every quantity of Definition 30.1 is defined directly, as a sum over the atoms of upper prices of records of the final laws, or as an explicit function of these sums with the lemma that justifies it. The certificate evaluates each of them, in the order of the author assembly `research/advance_2026_10_24/global/closing_beta.py` (`basic`, lines 120–127; `restoration`, lines 129–165; `evaluate`, lines 167–300), from three stored sources (Table 20; regenerated from first principles only in the `--full` tier, Section D.2): the native source of the middle range, with its prime evidence and its native parent; the source of the 310 upper bands; and the envelope bundle. The chain of evaluators is `research/improvements_2026_10_25/global/selected.py`, with `research/push_2026_10_25/design.json`; its adapters call the inherited signed-block assemblies. For the seven pieces with positive roots the adapters substitute $M_i = \mathrm{up}(R^n)$ and $\chi_i = \mathrm{up}((1+d^2)^n - 1)$, with R, d^2 and n of Certificate 21.25, for the uniform-root values; they use the invariant background factor $e^{tU^{\mathrm{inv}}}$ and replace the external index moment by the depth-64 finite prime-power upper majorant. All previous local and class formulas remain available; the adapters assert the exact source patterns they replace.

TABLE 20. The inputs of the selected evaluation (S denotes `research/push_2026_10_25/`; the other prefixes are as in Table 19). SHA-256 hashes are abbreviated to their first and last eight hexadecimal digits; the full hashes are pinned by the design and final record.

File	Content	SHA-256
S/`design.json`	the selected design	`8ea866f5...73ffcdff`
S/`global/refined_L570_source.json`	native source: pieces, local data, spectra, weights, charges	`0ad70bfc...c5492ba9`
S/`global/refined_L570_charges.json`	prime evidence: parts, cells, degree charges, sums	`1e4c7e76...9123b984`
S/`global/refined_parent.json`	native parent: root changes, new weights	`f89868ba...b9e7fed1`
F/`upper/upper_dense_refined.json`	the 310 upper bands	`c301f76e...8091f337`
F/`envelopes/exact_background_pairs.json`	envelope bundle	`6eb2ce2a...c0591caf`
S/`artifacts/author.json`	the author scalar	`ff9c5825...c7fb4748`
R/`item12/upper_v2_local.json`	the tail datum (`tail_comparison`)	`7ce4d475...efbb2589`
C/`global/prime_moments.json`	the prime-moment tables	`134b49af...7755e0d3`
A/`verification/selected.json`	the signed-block record (baseline)	`d90b1e06...3dbd0384`
S/`verification/selected.json`	the final record	`b9bde039...0c5f50d6`

(V1) *Repricing and caps* (lines 120–127). $t = e^{2(a-a_0)L}$ and $C = e^{2aL}$ with the actual budget $L = 570$; the piece caps $C_i = e^{2aL_i}$ of Lemma 20.7 (`research/advance_2026_10_24/envelopes/piece_product_caps.py`, `caps_at`), from the authenticated histogram and the cells.

(V2) *First stage* (lines 124–126). $U_0 = tU_0'$ and the tag totals $d = tD'/2$, $v = tV'/2$ from the totals of the source at a_0; R_{err} and L_0 by the majorant $\mathfrak{h}_4^+$ of Corollary 10.9 (`research/advance_2026_10_24/global/grouped_first_stage.py`, lines 25–38, which asserts $D' \le V'$, $d < 6$, $dv < 36$ and that the series does not exceed the one-sharp bound), and $B = e^{U_0} + R_{\mathrm{err}}$; the

dependent coefficients of the source are checked by `check_source` (lines 12–23) when the assembly is built.

(V3) *Middle prices.* `raw_ledgers` of `research/advance_2026_10_24/local/low_domain.py`, selected by the low-domain flag of the source (`research/advance_2026_10_24/global/evaluate_signed.py`, lines 19–21), forms for every piece the rows of Definition 30.1(3), rounded up on $10^{-30}\mathbb{Z}$; the totals are multiplied by t (line 171).

(V4) *Upper ledger* (`upper_price` of `research/continuation_2026_10_24/global/closing.py`, lines 130–150). In each of the 310 bands every term is its unit charge at a_{u} times $e^{2(a-a_{\mathrm{u}})\ell}$ with its recorded logarithmic product bound ℓ (between 549.40 and 885.44); $S_{\mathrm{u}}(a)$ sums the repair and deletion polynomials of the bands and $A_{\mathrm{u}}(a)$ their skew polynomials. The tail polynomials of the tail datum enter with $s = \deg/2 - 2a > 1$ at P_4: the repair and the deletion prices in $S_{\mathrm{u}}(a)$, the repair prices also in $A_{\mathrm{u}}(a)$ (Lemma 29.5).

(V5) *Totals and the matching exponent* (lines 173–176). S, A and T of Definition 30.1(4), (5), with $T_K = t(T_K^{\mathrm{aux}\prime} + T_K^{\mathrm{inc}\prime})$: $T_K^{\mathrm{aux}\prime}$ from `research/advance_2026_10_24/global/piece_data_beta.py` (lines 126–131), $T_K^{\mathrm{inc}\prime}$ from the increments of `raw_ledgers`; the chosen sides are checked by `chosen_sides` (lines 33–47) and the matching gate at line 112.

(V6) *Invariant price.* U^{inv} is the exact rational of `research/further_2026_10_25/envelopes/chosen_background.py` (`price`). The adapter `research/further_2026_10_25/global/scalar_adapters.py` (lines 48–60) makes the one substitution in `evaluate`: where the inherited assembly used the background factor $e^{U_0 - S/4}$ (the factor $\exp((U_0 - S/4)/\beta)$ at the source's bracket parameter $\beta = 1$), the selected evaluation uses $e^{tU^{\mathrm{inv}}}$, as justified by Lemma 25.6; the independent assembly uses the independent price of lines 63–91.

(V7) *Class costs and savings* (lines 184–241): the remainder polynomials, the leading costs $C_{n,m}$ with the orbit coupons and the gate at line 205, and the sixteen savings of Definition 30.1(8). The general classes are the rows with $n \geq 1$ of the bundle and its pair rows FF, FE; the matching pairs and the classes $(0,4)$, $(0,6)$ come from its E-type rows, and its two general rows without a fold record are not used. No saving is discarded.

(V8) *Restorations, gains and coupons* (lines 129–165 and 242–258): the restorations with the piece caps (`research/advance_2026_10_24/envelopes/piece_cap_restoration.py`), the gain tables with backgrounds of degrees one and two (`research/continuation_2026_10_24/envelopes/background_variance.py`), the coupons m_F, m_Q (`research/advance_2026_10_24/envelopes/beta_restoration.py`), the late-pool terms on the pieces from 2^{31}, the selected-only gains of the aligned matching pairs and the credits of degrees three and four (`research/advance_2026_10_24/envelopes/beta_higher_variance.py`); then the price tables and D_{var} of Corollary 24.18.

(V9) *Index.* The adapter `research/improvements_2026_10_25/global/reclose.py` replaces the old field moment by the finite prime-power majorant of `global/padic_moment.py` in that package, using the selected a, K. It adds the finite-depth correction to the logarithm of the old field-prime moment's upper endpoint, verifies that the resulting majorant is below that endpoint, and only the upper endpoint of the new enclosure is used.

(V10) *Closing.* The final author and independent programs evaluate D, $\mathcal{E}$, m, J, W and Θ from the selected inputs. They share primitive local prices and envelope proofs, but assemble separately each of these totals and every ledger and discount entry; the comparison passes only if every pair of corresponding enclosures, and the two enclosures of Θ, overlap. Both Θ upper endpoints are below $0.99999999997327 < 1$; the complete current numerical ledger is in Certificate 30.3.

No total of an earlier checkpoint's closing enters the evaluation: its inputs are the three authenticated sources, the tail datum (Definition 27.1(e)) and the prime-moment tables. The source also carries historical closing fields of its producer (in its field `middle`: `S`, `A`, `R`, `EY`, `J`, `first_stage_`

`loss` and `closing_without_upper`); the design records `historical_native_global_fields_used=false`, and the evaluation reads none of them. The modules of the earlier checkpoints that the evaluation calls supply formulas only: the interval arithmetic, the restoration formula (`research/continuation_2026_10_24/envelopes/restoration.py`), the gain and price tables (`research/continuation_2026_10_24/envelopes/background_variance.py`), and the class skeleton with the upper repricing (`research/continuation_2026_10_24/global/closing.py`, reached through `research/advance_2026_10_24/global/closing.py`).

D.5. **What the earlier verifiers contribute.** The first four verifiers supply the foundational checks listed in Table 21. The fifth, F, adds the positive-root estimates, invariant background price, envelope bundle and refined upper ledger named in Table 19. The selected middle laws and their restoration data are bound by I, and the reanchored charges and closing by S. The older modules remain useful because their formulas are evaluated anew on the selected inputs, as explained above.

The record verifier `research/record_2026_10_18/verify.py`, its record re-frozen at `8dc0a3fe` for the 770-page manuscript, has 62 checks (53 in the default tier, 9 more with `--full`), which all pass. The final proof uses its analytic checks, its weighted identity, its query images, its direct-star tests and its authentication of the inherited upper and tail inputs; its closing chains, middle data and band ledger belong to the October 18 construction and are not used.

The pure-pair verifier `research/push_2026_10_24/verify.py` (no named check registry) re-evaluates both closing chains of its checkpoint, reruns four checks of the record verifier on its stored source, replays its local audits, envelopes and restorations, and with `--full` regenerates its source and both closing chains, with the independent upper review `tools/check_upper.py`. The final proof uses only that re-check of the tail datum; none of its finite assertions about the pure law is used.

The continuation verifier `research/continuation_2026_10_24/verify.py` has 21 checks (11, and 10 more with `--full`). The final proof uses its analytic domain audit, its review of the nine-threshold twelve-label grid, its prime-moment tables, the status of the four-root identity record, and, as formulas, its restoration, gain and price-table code; the skew-only and forest records of its folder enter the upper chain. Its closing, its 245-band upper ledger, its native middle and the route-A L20 gates of its root are not used.

The signed-block verifier `research/advance_2026_10_24/verify.py` has 58 checks (43, and 15 more with `--full`). It supplies, among others, the analytic domain, the local laws and the L20 caps of the uniform root, the native review of all pieces, the signed law, the envelope replay and the regenerated upper chain of 310 bands, which the final upper replay takes as its baseline. Its 18 interim checks and the eight interim density checks of its full tier are not used.

Two points need a word. First, in the baseline upper source the root entries of LL20 and LL22 in the 301 bands below 2^{52} are those of the field `even_root_evidence` of each band, the caps of Certificate 27.9 (at P_3, $d = 12847.187931$ for LL20); the field `roots` of the 45 bands below 2^{48} records an earlier stage of the chain, with larger caps, and is not used; from 2^{52} on the root entries are the completed-root caps of Certificate 27.11. Second, the earlier suite `paper/certificate/` contains finite consistency tests of the second-order bounds of Section 15; they are not inputs of any proof.

Remark D.13 (Consistency tests of the second-order bounds). The numerical sharpness of Theorems 15.8 and 15.13 is recorded in Remarks 15.11 and 15.16. Two checks of the suite `paper/certificate/` test the theorems on finite cases. Neither is an input of any proof, and the launcher `paper/verify.py` does not run them. The check `check_deligne_theorem_K` (`certlib/checks_deligne.py`) verifies $2pI \mp M \succeq 0$ by exact LDL^{T} decompositions of the full matrix for $p \in \{3, 5, 7, 11\}$; the same on the T-invariant subspace for every prime $3 \le p \le 61$,

TABLE 21. The earlier verifiers of the chain and what the final proof takes from them (prefixes of the check names omitted).

Verifier	Checks	Used by the final proof
record	analytic:slacks, analytic:arc-exponents, independent:u05, analytic:tail-gates	(S1)–(S7)
	variance:weighted-identity	the 400-term identity
	middle:geometry, independent:u01	Certificate 23.10
	upper:direct-star, independent:u07	the direct-star tests at $2^{49}, 2^{50}, 2^{51}$
	authenticate, bindings, source:manifests, audit:closure	authentication of the inherited grids, direct-star records and tail files
	the other checks (closing chains, middle prices and local laws, orientation mixture, variance and means, band ledger, regenerations)	not used
pure pair	`tools/check_upper.py` (full)	the 105 gates and the coefficients of the tail datum, the tail prices
	the other checks	not used
continuation	`analytic`	(S0), through the signed-block analytic check
	`middle_grid`, `fresh_grid`	the grid of nine thresholds
	`l20_mean`, `fresh_l20_mean`	the status of the identity record of Certificate 24.6
	`fresh_moments`	the prime-moment tables
	`local`, `author`, `independent`, `native`, `prices`, `upper_ledger`, `upper_prices`, `l20_variance`, the other `fresh_*`	not used (the final verifier reruns the script of `prices` on the final source)
signed block	`analytic`, `local_*` except `local_signed_frontier`, `native`, `envelopes`, `joint_witness_*`, `upper`, `fresh_partitions`, `fresh_l20`, `fresh_middle_grid`, `fresh_native_prices`, `fresh_prime_moments`	the rows of Table 19 that name them; `upper` gives the baseline of the final upper replay
	`author`, `independent`, `prices`, `interim_*`, `local_signed_frontier`, `fresh_low_grid`, `fresh_native`	not used (the final verifier reruns the scalar and price scripts on the final inputs)

which is a complete verification for these p, since every constituent of Lemma 15.3 has a T-fixed vector and M acts on each isotypic component as the identity on π tensored with a matrix of size m_π; and, for $p \le 31$ and every row x, the row sum $\sum_z M(x,z) = -(p-1)$ and the identity $\sum_z M(x,z)f(z_1) = f(x_1) + f(x_2)$ for $f = \delta_0 - \frac{1}{p+1}$ on $\mathbb{P}^1(\mathbb{F}_p)$. The check `check_deligne_h2_bruteforce` (`certlib/checks_deligne_h2.py`) compares Theorem 15.13 with exact double sums for 3360 instances drawn by that check's sampler with fixed seed 20261011 (squarefree f, g, split or not, $2 \le \deg f \le 7$, $0 \le \deg g \le 7$, random leading coefficients and shared roots; the family also contains cases outside the hypotheses of the theorem: non-split f, g, and $\deg g = 0$, where $\Sigma = 0$) over the 42 primes $11 \le p \le 199$, and requires the bound of each fibration separately; the largest ratio $|\Sigma|/(p\gamma(m)(m+n-s-1))$ found is $0.98\ldots$. Both run, in `paper/certificate/`, with `python3 -I -B verify_all.py --only` followed by their names.

D.6. **The archive and the earlier checkpoints.** The baseline checkpoint is `push_2026_10_25` at commit `1b9239f0`. Its paper archive has 162 files and SHA-256 `6bacf686...b01d73b`; its script archive has SHA-256 `8f642802...98f4e08`. The complete inventory and the archive README give every constituent hash. The following commands run the full selected chain, rebuild its archived paper, or build this synthesis:

```
PYTHON -B paper/verify.py --checkpoint push_2026_10_25 --full \
```

```
  --output-dir DIR
python3 -B paper/build.py --checkpoint push_2026_10_25 \
  --output-dir DIR
python3 -B paper/build.py
```

The rest of this subsection records the earlier archive checkpoints and their reproducible commands. Their old exponents and source pins are preserved.

Two earlier manuscripts are archived. The manuscript as it stood before the synthesis rewriting (commit `a6e4c29`, 644 pages) is preserved in `archive/paper-purepair-2026-10-24.tar.gz`, with the launchers of that commit in `archive/scripts-purepair-2026-10-24.tar.gz`; their manifests `archive/paper-purepair-2026-10-24.manifest.json` (SHA-256 `87789fa9...ef5b0770`) and `archive/scripts-purepair-2026-10-24.manifest.json` (`b68f164b...d716a8b6`) record every member's size and hash, and the logical inventory `archive/purepair-2026-10-24.inventory.json` (`102cc539...ab754fa3`) lists the 5113 files on which the record and pure-pair verifiers depend. The 770-page manuscript on the master branch (commit `8dc0a3fe`), which the inherited five-stage chain authenticates, is preserved in `archive/paper-further-2026-10-25.tar.gz` (144 files, every committed file under `paper/` except `paper/certificate/`; manifest `896c67af...376a82b2`); the scripts archive above serves it too, as `scripts/` is byte-identical at both commits. Two further archives complete that tree: `archive/untracked-further-2026-10-25.tar.gz`, the one git-ignored input that the chain reads, whose bytes five committed records pin (manifest `39f0a1ab...b966f7b7`), and `archive/research-purepair-2026-10-24.tar.gz`, the `a6e4c29` bytes of the five research files that `8dc0a3fe` re-froze, which the older restored checkpoints need (manifest `efd1a7c9...0255cffa`). The logical inventory `archive/further-2026-10-25.inventory.json` (`e66068b1...5f5d553b`) lists the 6761 files of the chain, the hashes of the two generated archives and those of the 1640 files of the prepared frontier layout. `scripts/manuscript_layout.py` pins every manifest by its SHA-256; `archive/README-purepair-2026-10-24.md` and `archive/README-further-2026-10-25.md` document the archives, the reconstruction and its test runs. The commit of the October 18 record is `8709495`; `paper/certificate/` stays in place, untouched. From the repository root, with PYTHON as in Section D.2 and DIR an empty directory outside the checkout:

```
PYTHON -B paper/verify.py --checkpoint further_2026_10_25 \
  [--full] [--output-dir DIR]
python3 -B paper/build.py --checkpoint further_2026_10_25 \
  [--output-dir DIR]
python3 -B scripts/manuscript_layout.py \
  --checkpoint further_2026_10_25 archive --output-dir DIR \
  [--untracked-from WORKING_COPY]
python3 -B paper/verify.py --checkpoint purepair_2026_10_24 \
  [--output-dir DIR]
/usr/bin/python3 -B paper/verify.py --checkpoint purepair_2026_10_24 \
  --full [--output-dir DIR]
python3 -B paper/verify.py --checkpoint record_2026_10_18 [--full]
python3 -B scripts/manuscript_layout.py \
  --checkpoint purepair_2026_10_24 archive --output-dir DIR
python3 -B paper/verify.py --checkpoint frontier_2026_10_17
```

The first runs the previous five-stage certificate; the second rebuilds the 770-page manuscript; the third rebuilds the archives of that manuscript, the manifests of those archives and its logical inventory from commit `8dc0a3fe` (the untracked input from its own archive, or from a working copy) and compares all of them byte for byte with the committed files; the fourth and fifth run

the pure-pair checkpoint on the 644-page manuscript (its full tier compares an interpreter path and needs `/usr/bin/python3`); the sixth runs the October 18 record alone; the seventh rebuilds the archives of commit `a6e4c29`; and the last runs the six-query checkpoint from its own archive. Table 22 lists the checkpoints, which Section 34.1 describes; their verifiers read their own frozen inputs, and the first row needs separate checkouts of the two commits named there.

TABLE 22. The checkpoints of Table 14 and their entry points (from the repository root; the exponents are ε of the corresponding theorem). The last eight rows are the checkpoints whose verifiers (seven in all, the `research/push_2026_10_24/` verifier serving two of the rows) are run by `paper/verify.py` (Section D.2); the first five of the seven formed the chain at commit `8dc0a3fe`.

Design	ε	Entry point
216 slots, singleton recurrence	$3347/10^{12}$	`paper2/certificate/verify_current.py` at `ef1a0e7`, then two snapshot verifiers at `90d5b46`
216 slots, synthesized transfer	83/2000000000	`research/synthesis_2026_10_16/verify.py`
216 slots, fixed family	269973/5376500000000	`research/further_2026_10_16/verify.py`
one amplified edge	1/8000000	`research/single_edge_2026_10_17/verify_single.py`
simultaneous cuts	513/2242156250	`research/improvements_2026_10_17/verify_sixteen.py`
72 slots, two layers	2214/1111328125	`research/push_2026_10_17/verify.py`
72 slots, local refinements	1971/635562500	`research/advance_2026_10_17/verify.py`
six queries	63/8927500	`paper/verify.py --checkpoint frontier_2026_10_17` (commit `be0cc8d`)
paired repairs	441/44375000	`paper/verify.py --checkpoint record_2026_10_18` (commit `8709495`)
intermediate (exact index)	276327/27734375000	`research/push_2026_10_24/verify.py` (its first record)
pure paired law	$\frac{243319543}{22187500000000}$	`paper/verify.py --checkpoint purepair_2026_10_24`
exact actual laws	$\frac{4066280877}{355000000000000}$	`research/continuation_2026_10_24/verify.py`
signed blocks	$\frac{8855891201}{710000000000000}$	`research/advance_2026_10_24/verify.py`
positive roots (previous checkpoint)	$\frac{887239883}{71000000000000}$	`paper/verify.py --checkpoint further_2026_10_25`
collision and coherent roots	$\frac{1110464567}{88750000000000}$	`research/improvements_2026_10_25/verify.py`
exact reanchoring (baseline)	$\frac{5422587}{433349609375}$	`paper/verify.py`

Remark D.14 (Provenance). The arguments of Parts III–V were first written as research notes, which remain in the repository, and as the layered text of the 770-page manuscript on the master branch (its §§42–44); those of the October 18 record come with the referee reports of its record gate (`research/record_gate_2026_10_18/`, units `U01`–`U15` and `I1`–`I3`). They are provenance only, and every step used is proved in the text. The kernel, the prime-power interface and the moments are in `research/roadmap_2026_10_18/item05/` and `item06/`, and Appendix C follows the independently written proof `research/roadmap_2026_10_18_claude/item05/beta-global/K_general_kernel.tex`; the repairs and collisions in `item01/` and `item03/`, with the first negative cap of the reference fold record (`item03/OLD_F_NEGATIVE_CAP.md`); the weighted identity and the auxiliary sides in `item03/`; the charge theorem of the blocks in `item06/`; the tail datum and the direct-star tests in `item12/`, with the one-way repairs of the tail in `research/frontier_2026_10_17/`. The exact actual laws, the restorations, the backgrounds, the prime moments and the forest records are in `research/continuation_2026_10_24/`; the signed whole-block law, the refined negative cap (`local/OLD_F_NEGATIVE_REFINEMENT.md`), the L20 cap of the uniform root, the tagged first stage, the matrix envelopes, the piece caps and the upper

chain from P_3 in `research/advance_2026_10_24/`; and the positive roots, the assigned invariant price and the 310-band refinement in `research/further_2026_10_25/`. One charging rule of those notes is worth recording: a one-way repair of a symmetric record is charged its symmetric norm in full but only the skew part of the new law in the skew ledger (Remark 9.15(iii)).

Several generated inputs carry status labels written by their producers when they were generated, before the reviews and the verifiers ran. The signed-block record lists its check `upper` with the stage label `EVEN_COUNT_ROOT_UPPER_PENDING_INDEPENDENT_REVIEW` of the regenerated upper file (written by `upper/apply_even_roots.py`, line 111; the check also records 310 bands, 14 stages and 51 independent reviews), its check `author` with `PASS_EXECUTION`, and its check `fresh_native` with `INFEASIBLE`, the status field of the historical closing of the regenerated native source (the operative check is `fresh_native_prices`, `PASS_COMPLETE_NATIVE_REGENERATION`). The final inputs carry `NEW_EXACT_LOG_BUDGET_CONSTRUCTION_NOT_YET_RECLOSED` (the native source), `POSITIVE_SIX_QUERY_LE_UPPER_PENDING_INDEPENDENT_REVIEW` (the upper source), `EXACT_WITNESSES_REQUIRING_FRESH_SELECTED_REPLAY` (the envelope bundle), while the selected author's inner result carries `COMPLETE_PADIC_SCALAR_PENDING_CONSTRUCTION_REVIEW`. Its enclosing record has status `PASS_SELECTED_AUTHOR_SCALAR`; the independent record has `PASS_SELECTED_INDEPENDENT_SCALAR`. The twelve-label grid records carry `EXACT_PSD_WITNESSES_REPLAY_PENDING`. The older stage labels describe when those fields were produced; the operative results are the named checks of the authenticated verifier chain and the reviews in Table 19. The historical closing fields of the native source and the price fields `S`, `A` of its evidence (Section D.4), which include prices of a fold law that the construction does not use, are not inputs of any statement.

Consequences and what to read next. Read Lemma 4.18 for evaluation at endpoints; Lemma 9.3 and Corollaries 9.4 and 10.9 and Lemma 20.7 for the repricing used in Section D.4; Definition 30.1 and Proposition 30.2 for the meaning of the evaluated quantities; Propositions 26.4, 26.5 and 28.14 and Lemma 23.24 for the prices and the matching exponent; and the certificates of the other sections for the scope of the table.

Appendix E. Design data, prices and witness tables

What this appendix is for (baseline tables only). Every table in this appendix belongs to the baseline ($\theta = 1$) unless its caption states the improved closing; do not use these numbers for Theorem 1.1 without the conversion in Section 33. "Charge" in captions means a degree sum $\sum q_{\mathcal{B}}^{2a} b_{\mathcal{B}}^{-d/2}$; "witness" means a matrix D with $-D \preceq T \preceq D$; "slack" means the margin in a checked strict inequality. These are the source data before the normalizer factors of (33.1), and the prices assigned to them in the baseline closing of Section 30. The final construction retains the same middle pieces, weights and spectral witnesses, replaces the upper weights by those of Section 27.3.1, replaces the two generic rows for the constant and merge records (Definition 30.1(3)) by (23.7) and applies the explicit normalizer factors of (33.1). Its final parameters and full ledger appear in Section 33; the baseline prices below are not asserted to equal those new totals.

It lists the finite data that the construction of Part V uses as numbers: the parameters (Section E.1); the 33 middle pieces with their blocks, degree charges, first-stage sums, completion weights, piece values and the block caps of their reference (Section E.2), used by Lemmas 22.1 and 22.9, Corollary 24.18, and Proposition 26.7; the prices of the middle records, piece by piece, with the endpoint coefficients from which they are formed (Section E.3), used by Lemma 23.24 and Definition 30.1(3); the sides of the matching envelopes of Proposition 26.4 (Section E.4); the upper bands of Definition 27.1 (Section E.5); the envelopes of Theorem 25.8 (Section E.6); and the spectral grids of Lemma 20.12 and Certificates 20.22 and 21.25 and the analytic slacks of Definition 8.1 (Section E.7). Every number is an exact rational fixed by the design, an end of a certified enclosure, or the value of a formula given in the text; the statement that uses it says

which property it needs. Table 19 says which script checks each inequality, and Section D.4 how the certificate combines the numbers into Θ.

Conventions. A decimal entry is rounded in the safe direction in its last digit (upward for an upper bound, downward for a lower bound; the captions name the lower bounds); integers, fractions and entries marked exact are exact. The tables show enough digits to identify a number: the certificate uses the exact rationals of the stored records listed at the end of this appendix, and every inequality of the text is checked on them. As in Definition 30.1, $\mathrm{up}(x) = 10^{-30}\lceil 10^{30}x\rceil$, and $\mathrm{hi}(v)$ is up of the upper end of a certified enclosure v. The middle tables are evaluated at the anchor $a_0 = 4442183/5000000000 < a$: every middle block has $\log q_{\mathcal{B}} \le L = 570$ (Lemma 20.3), so by Lemma 9.3 a middle price, charge or sum at the exponent a of Proposition 1.2 is at most $t = e^{2(a-a_0)L} \in (1, 1.00001)$ times its tabulated value (for the first-stage constants, Corollary 9.4). The upper ledger is evaluated at its anchor $a_{\mathrm{u}} = 883/10^6 < a$ and repriced term by term (Section E.5). Both anchors are evaluation points, not exponents of a theorem.

E.1. **The parameters.** Table 23 restates the parameters of Definition 18.1 for reference.

Table 23: The parameters of the construction.

exponents	$a = \frac{5422587}{6103515625}$; $\varepsilon = a/71 = \frac{5422587}{433349609375} = 1.251319231098\ldots\cdot 10^{-5}$; $\theta = 1$
slots, queries	$V = (\mathbb{Z}/6\mathbb{Z})^2 \times \{0,1\}$ with the family of cuts of Definition 5.12: the six chamber cuts of each coordinate and the reserved cut $\iota(u_0,u_1,\ell) = (u_0,u_1,1-\ell)$; $\mathcal{E}_6 = \{((0,v,0),(\mathsf{c}(0),\mathsf{c}(v),1)) : v \in \mathbb{Z}/6\mathbb{Z}\}$
index	$K = 890666$; finite nonzero-square laws at depth 64, on 2^{129} and p^{128} for $p < Q_3$, $p \equiv 3 \pmod 4$; with the coarse interval factor of Remark 10.7 included; $R_I \le 1.247139907407$ (Lemma 29.9); the enclosure behind the displayed number encloses the sufficient majorant of Certificate 29.11, and its lower endpoint is not a lower bound for the moment under the depth-64 law itself
prefix	$p \equiv 1 \pmod 4$, $p < P_4 = 2^{512}$; one atom, $e_p = 2m_{\mathrm{cl}}$, $m_{\mathrm{cl}} = 8192$
middle	$p \equiv 3 \pmod 4$ in $[Q_3, P_3)$, $Q_3 = 19\,951\,586$, $P_3 = 167\cdot 2^{40} = 183\,618\,441\,838\,592$; 33 pieces; block budget $L = 570$, local cap $L_{\mathrm{loc}} = 585$; $C = e^{2aL} < 2.753348440327$; piece caps $C_i = e^{2aL_i}$ (Lemma 20.7)
fold records	the signed whole-block law with $\mathsf{t} = \frac{127}{500}$ (Definition 23.9)
roots	uniform on pieces 1–26; pair-square tilts on pieces 27–32, coherent 39-term square on piece 33, with the seven exact parameters in Tables 36 and 37
bracket	one copy of the one-layer folded copies of the fold records; protection $v_{\mathrm{D}} \ge 2w_{\mathrm{D}}k_{\mathrm{D}}$ ((22.1))
upper	$p \equiv 3 \pmod 4$ in $[P_3, P_4)$; 310 bands; band budgets 550–886; $L_{\mathrm{u}} = 50000/27$; anchor $a_{\mathrm{u}} = 883/10^6$
tail	every prime $p \ge P_4$, one atom each, with the tail datum of Definition 27.1(e); mixing for $p \ge P_5 = 2^{65536}$
analytic	$\beta = 1/20$; major-arc exponent $17/100$; $\gamma = 255/1024$; $\tau = 1/201$; $t_{\mathrm{an}} = 1/60$; $\sigma = 1/825$; $\delta = 10^{-10}$; $\ell_0 = 10^{-8}$; gates on the domain $0 < a \le 9/10000$
comparison	coefficient $63/64$; auxiliary point $(b_0,d_0,f_0) = (.6426372, .5181074, .1607446)$, that is, orientation weights $\mathbf{w}^{\mathrm{aux}} = (.5803723, .2409463, .1786814, 0)$, and $\tau_{\mathrm{side}} = 31/32$ (Definition 26.2)
closing	$\Theta < 0.99999999997327$ (Certificate 30.3)

E.2. **The middle pieces.** The 33 pieces $[X_i, Y_i) = [X_i, X_{i+1})$ of Definition 20.1 tile $[Q_3, P_3)$. Table 24 gives their numbers of exact blocks (below $X_{\mathrm{Ch}} = 4\,800\,162\,890$), of cells (above it) with the range of the cell integers n_j, and the charges $\bar{I}_{d,i} \ge \sum_{\mathcal{B}} b_{\mathcal{B}}^{-d/2} q_{\mathcal{B}}^{2a_0}$, $d = 2,3,4$, over the blocks of the piece (Lemma 20.4): $\bar{I}_{d,i} = \mathrm{hi}(I_{d,i})$, where $I_{d,i}$ is the certified enclosure of the bound of that lemma for the piece, so that $\bar{I}_{2,i}$ is the charge of Definition 30.1(3). Table 25 gives, at a_0, the four sums U_0, P, D, V of Lemma 22.9 for each piece; their certified totals (Certificate D.4) are $U_0' \le 0.846908193539142986$, $P' \le 1.6936504653996419205$, $D' \le 4.7417041225622828\cdot 10^{-5}$ and $V' \le 3.3872533701481774716$, the totals of Definition 30.1(2). The prices of the comparison records are those of Section E.3.

Table 24: Middle pieces: exact blocks, cells, the range of the cell integers n_j, and the degree-d charges $\bar{I}_{d,i}$ at a_0 (upper endpoints, rounded up). Source: `refined_L570_source.json` (`middle.pieces[i].degree_charges`) and `refined_L570_charges.json` (`bands[i]`: `finite_parts`, `analytic`).

i	X_i	Y_i	blocks	cells	n_j	degree 2	degree 3	degree 4
1	19951586	21757358	1623	–	–	$2.093 \cdot 10^{-4}$	$4.584 \cdot 10^{-8}$	$1.005 \cdot 10^{-11}$
2	21757358	23726568	1763	–	–	$2.094 \cdot 10^{-4}$	$4.394 \cdot 10^{-8}$	$9.219 \cdot 10^{-12}$
3	23726568	25429506	1516	–	–	$1.674 \cdot 10^{-4}$	$3.377 \cdot 10^{-8}$	$6.814 \cdot 10^{-12}$
4	25429506	25874006	397	–	–	$4.204 \cdot 10^{-5}$	$8.300 \cdot 10^{-9}$	$1.639 \cdot 10^{-12}$
5	25874006	28215802	2073	–	–	$2.092 \cdot 10^{-4}$	$4.025 \cdot 10^{-8}$	$7.744 \cdot 10^{-12}$
6	28215802	2^{25}	4743	–	–	$4.188 \cdot 10^{-4}$	$7.551 \cdot 10^{-8}$	$1.363 \cdot 10^{-11}$
7	2^{25}	39903170	5693	–	–	$4.184 \cdot 10^{-4}$	$6.918 \cdot 10^{-8}$	$1.145 \cdot 10^{-11}$
8	39903170	47453134	6708	–	–	$4.187 \cdot 10^{-4}$	$6.348 \cdot 10^{-8}$	$9.632 \cdot 10^{-12}$
9	47453134	56431604	7952	–	–	$4.185 \cdot 10^{-4}$	$5.819 \cdot 10^{-8}$	$8.096 \cdot 10^{-12}$
10	56431604	2^{26}	9608	–	–	$4.189 \cdot 10^{-4}$	$5.342 \cdot 10^{-8}$	$6.816 \cdot 10^{-12}$
11	2^{26}	79806340	11303	–	–	$4.184 \cdot 10^{-4}$	$4.892 \cdot 10^{-8}$	$5.724 \cdot 10^{-12}$
12	79806340	94906266	13317	–	–	$4.185 \cdot 10^{-4}$	$4.487 \cdot 10^{-8}$	$4.814 \cdot 10^{-12}$
13	94906266	112863208	16166	–	–	$4.187 \cdot 10^{-4}$	$4.117 \cdot 10^{-8}$	$4.051 \cdot 10^{-12}$
14	112863208	2^{27}	19097	–	–	$4.184 \cdot 10^{-4}$	$3.773 \cdot 10^{-8}$	$3.404 \cdot 10^{-12}$
15	2^{27}	159612678	22517	–	–	$4.187 \cdot 10^{-4}$	$3.462 \cdot 10^{-8}$	$2.864 \cdot 10^{-12}$
16	159612678	189812532	26870	–	–	$4.187 \cdot 10^{-4}$	$3.175 \cdot 10^{-8}$	$2.409 \cdot 10^{-12}$
17	189812532	225726414	32323	–	–	$4.185 \cdot 10^{-4}$	$2.910 \cdot 10^{-8}$	$2.024 \cdot 10^{-12}$
18	225726414	2^{28}	38106	–	–	$4.186 \cdot 10^{-4}$	$2.669 \cdot 10^{-8}$	$1.703 \cdot 10^{-12}$
19	2^{28}	319225356	44916	–	–	$4.186 \cdot 10^{-4}$	$2.447 \cdot 10^{-8}$	$1.432 \cdot 10^{-12}$
20	319225356	379625064	54085	–	–	$4.187 \cdot 10^{-4}$	$2.245 \cdot 10^{-8}$	$1.205 \cdot 10^{-12}$
21	379625064	451452826	64634	–	–	$4.186 \cdot 10^{-4}$	$2.058 \cdot 10^{-8}$	$1.013 \cdot 10^{-12}$
22	451452826	2^{29}	76200	–	–	$4.186 \cdot 10^{-4}$	$1.887 \cdot 10^{-8}$	$8.512 \cdot 10^{-13}$
23	2^{29}	638450710	89850	–	–	$4.186 \cdot 10^{-4}$	$1.731 \cdot 10^{-8}$	$7.158 \cdot 10^{-13}$
24	638450710	759250126	108058	–	–	$4.186 \cdot 10^{-4}$	$1.587 \cdot 10^{-8}$	$6.019 \cdot 10^{-13}$
25	759250126	902905652	129512	–	–	$4.185 \cdot 10^{-4}$	$1.455 \cdot 10^{-8}$	$5.061 \cdot 10^{-13}$
26	902905652	2^{30}	152792	–	–	$4.187 \cdot 10^{-4}$	$1.335 \cdot 10^{-8}$	$4.257 \cdot 10^{-13}$
27	2^{30}	2^{31}	961376	–	–	$1.675 \cdot 10^{-3}$	$4.318 \cdot 10^{-8}$	$1.125 \cdot 10^{-12}$
28	2^{31}	2^{32}	1922275	–	–	$1.675 \cdot 10^{-3}$	$3.054 \cdot 10^{-8}$	$5.624 \cdot 10^{-13}$
29	2^{32}	2^{34}	454440	236	24–25	$3.350 \cdot 10^{-3}$	$3.687 \cdot 10^{-8}$	$4.219 \cdot 10^{-13}$
30	2^{34}	2^{36}	–	256	22–24	$3.350 \cdot 10^{-3}$	$1.844 \cdot 10^{-8}$	$1.055 \cdot 10^{-13}$
31	2^{36}	2^{40}	–	512	20–22	$6.699 \cdot 10^{-3}$	$1.383 \cdot 10^{-8}$	$3.296 \cdot 10^{-14}$
32	2^{40}	2^{45}	–	640	18–20	$8.373 \cdot 10^{-3}$	$3.794 \cdot 10^{-9}$	$2.129 \cdot 10^{-15}$
33	2^{45}	$167 \cdot 2^{40}$	–	306	17–18	$3.992 \cdot 10^{-3}$	$4.581 \cdot 10^{-10}$	$5.552 \cdot 10^{-17}$

Table 25: The four first-stage sums of each piece at a_0 (upper endpoints, rounded up); the totals are the enclosures of the sums, not the sums of the displayed values. Source: `refined_L570_charges.json` (`bands[i].charges`, `sums`).

i	U_0	P	D	V
1	$4.599 \cdot 10^{-3}$	$9.190 \cdot 10^{-3}$	$2.014 \cdot 10^{-6}$	$1.838 \cdot 10^{-2}$
2	$4.602 \cdot 10^{-3}$	$9.196 \cdot 10^{-3}$	$1.930 \cdot 10^{-6}$	$1.839 \cdot 10^{-2}$
3	$3.677 \cdot 10^{-3}$	$7.348 \cdot 10^{-3}$	$1.483 \cdot 10^{-6}$	$1.470 \cdot 10^{-2}$
4	$9.236 \cdot 10^{-4}$	$1.846 \cdot 10^{-3}$	$3.645 \cdot 10^{-7}$	$3.692 \cdot 10^{-3}$
5	$4.595 \cdot 10^{-3}$	$9.184 \cdot 10^{-3}$	$1.767 \cdot 10^{-6}$	$1.837 \cdot 10^{-2}$
6	$9.198 \cdot 10^{-3}$	$1.839 \cdot 10^{-2}$	$3.315 \cdot 10^{-6}$	$3.677 \cdot 10^{-2}$
7	$9.187 \cdot 10^{-3}$	$1.837 \cdot 10^{-2}$	$3.037 \cdot 10^{-6}$	$3.672 \cdot 10^{-2}$
8	$9.191 \cdot 10^{-3}$	$1.838 \cdot 10^{-2}$	$2.786 \cdot 10^{-6}$	$3.674 \cdot 10^{-2}$
9	$9.185 \cdot 10^{-3}$	$1.836 \cdot 10^{-2}$	$2.553 \cdot 10^{-6}$	$3.672 \cdot 10^{-2}$
10	$9.193 \cdot 10^{-3}$	$1.838 \cdot 10^{-2}$	$2.344 \cdot 10^{-6}$	$3.676 \cdot 10^{-2}$
11	$9.179 \cdot 10^{-3}$	$1.836 \cdot 10^{-2}$	$2.146 \cdot 10^{-6}$	$3.670 \cdot 10^{-2}$
12	$9.180 \cdot 10^{-3}$	$1.836 \cdot 10^{-2}$	$1.968 \cdot 10^{-6}$	$3.671 \cdot 10^{-2}$
13	$9.184 \cdot 10^{-3}$	$1.837 \cdot 10^{-2}$	$1.806 \cdot 10^{-6}$	$3.673 \cdot 10^{-2}$
14	$9.178 \cdot 10^{-3}$	$1.835 \cdot 10^{-2}$	$1.655 \cdot 10^{-6}$	$3.670 \cdot 10^{-2}$
15	$9.183 \cdot 10^{-3}$	$1.836 \cdot 10^{-2}$	$1.518 \cdot 10^{-6}$	$3.672 \cdot 10^{-2}$
16	$9.181 \cdot 10^{-3}$	$1.836 \cdot 10^{-2}$	$1.392 \cdot 10^{-6}$	$3.671 \cdot 10^{-2}$
17	$9.176 \cdot 10^{-3}$	$1.835 \cdot 10^{-2}$	$1.276 \cdot 10^{-6}$	$3.669 \cdot 10^{-2}$
18	$9.177 \cdot 10^{-3}$	$1.835 \cdot 10^{-2}$	$1.170 \cdot 10^{-6}$	$3.670 \cdot 10^{-2}$
19	$9.177 \cdot 10^{-3}$	$1.835 \cdot 10^{-2}$	$1.073 \cdot 10^{-6}$	$3.670 \cdot 10^{-2}$

i	U_0	P	D	V
20	$9.179 \cdot 10^{-3}$	$1.836 \cdot 10^{-2}$	$9.840 \cdot 10^{-7}$	$3.671 \cdot 10^{-2}$
21	$9.175 \cdot 10^{-3}$	$1.835 \cdot 10^{-2}$	$9.020 \cdot 10^{-7}$	$3.670 \cdot 10^{-2}$
22	$9.175 \cdot 10^{-3}$	$1.835 \cdot 10^{-2}$	$8.271 \cdot 10^{-7}$	$3.669 \cdot 10^{-2}$
23	$9.175 \cdot 10^{-3}$	$1.835 \cdot 10^{-2}$	$7.585 \cdot 10^{-7}$	$3.670 \cdot 10^{-2}$
24	$9.174 \cdot 10^{-3}$	$1.835 \cdot 10^{-2}$	$6.955 \cdot 10^{-7}$	$3.670 \cdot 10^{-2}$
25	$9.173 \cdot 10^{-3}$	$1.835 \cdot 10^{-2}$	$6.377 \cdot 10^{-7}$	$3.669 \cdot 10^{-2}$
26	$9.175 \cdot 10^{-3}$	$1.835 \cdot 10^{-2}$	$5.849 \cdot 10^{-7}$	$3.670 \cdot 10^{-2}$
27	$3.669 \cdot 10^{-2}$	$7.337 \cdot 10^{-2}$	$1.893 \cdot 10^{-6}$	$1.468 \cdot 10^{-1}$
28	$3.667 \cdot 10^{-2}$	$7.334 \cdot 10^{-2}$	$1.338 \cdot 10^{-6}$	$1.467 \cdot 10^{-1}$
29	$7.331 \cdot 10^{-2}$	$1.467 \cdot 10^{-1}$	$1.614 \cdot 10^{-6}$	$2.933 \cdot 10^{-1}$
30	$7.321 \cdot 10^{-2}$	$1.465 \cdot 10^{-1}$	$8.058 \cdot 10^{-7}$	$2.929 \cdot 10^{-1}$
31	$1.464 \cdot 10^{-1}$	$2.927 \cdot 10^{-1}$	$6.040 \cdot 10^{-7}$	$5.853 \cdot 10^{-1}$
32	$1.828 \cdot 10^{-1}$	$3.656 \cdot 10^{-1}$	$1.657 \cdot 10^{-7}$	$7.312 \cdot 10^{-1}$
33	$8.678 \cdot 10^{-2}$	$1.736 \cdot 10^{-1}$	$1.992 \cdot 10^{-8}$	$3.472 \cdot 10^{-1}$
$\sum$	0.8469081935392	1.6936504654	$4.741704122563 \cdot 10^{-5}$	3.387253370149

The histogram of the exact blocks. The charges of Lemma 20.4(ii) and the start sums of Lemma 20.6(ii) need, for every part of Definition 20.1(c), first-prime bin and price bucket $k = k_{\mathcal{B}}$, only the number of blocks and the sums of the start weights $w_d(\mathcal{B})$, $d = 2, 3, 4$. These are the rows of the histogram `research/advance_2026_10_24/global/final_budget_partitions/exact_block_histogram.csv`: 35 145 rows for the three budgets 568, 570 and 572, of which the 11 708 rows of the budget 570 (that is, $\mathfrak{u}_* = 570 \cdot 2^{40}$) are those of the construction, the 10 099 rows of its parts in $[Q_3, X_{\mathrm{Ch}})$ counting the 4 279 913 exact blocks of the pieces; SHA-256 `2b85805d1fd33962e018766248ec0bfbd9c209524689ccf7b04c3a6b9302a667`. It was produced by the integer sieve `research/roadmap_2026_10_18/item06/exact_block_starts.cpp` from the inputs in the same folder as the histogram: the 37 parts of `exact_block_parts.csv`, from 2^{23} to X_{Ch} (the 35 parts of Definition 20.1(b) in $[Q_3, X_{\mathrm{Ch}})$, plus two parts below Q_3 used only for the local-domain audit); the 6362 count bins of `exact_block_bins.csv`, from 2^{23}, of which the 5668 bins of Definition 20.1(c) cover $[2^{24}, X_{\mathrm{Ch}})$; and the budgets of `exact_block_budgets.txt` (inputs and run records `exact_block_inputs.json`, `exact_block_run.json`). The proof note is `research/roadmap_2026_10_18/item06/EXACT_FINITE_BLOCK_STARTS.md`, and the full tier of the final verification re-sieves the histogram in exact integer arithmetic. Its prime counts, its largest block log-sum and the regeneration are stated in Certificate D.4, as are the cells of the pieces above X_{Ch}. The rows, parts and bins below Q_3 belong to the audit of the local domain and are not used.

Completion weights and piece values. Table 26 lists the weights of the twelve-label reference of each piece (Lemma 22.1(iii), Remark 22.2), fixed rationals with $w_{\mathrm{L}} = 1$, $w_{\mathrm{S}} = w_{\mathrm{D}}$ and $v_{\mathrm{S}} = v_{\mathrm{D}}$. Three of them are fixed by caps of the piece: w_{D} is the block cap d of L20 and v_{L} the block cap k_{L} of L21, both rounded up (Table 28; on the pieces 27–33 the caps of the positive root), and v_{D} is $2w_{\mathrm{D}}k_{\mathrm{D}}$ rounded up, so that the protection (22.1) is binding up to the rounding of v_{D}: $0 \le v_{\mathrm{D}} - 2w_{\mathrm{D}}k_{\mathrm{D}} < 10^{-9}$ in every piece. Table 27 lists the piece values n_i, h_i, M_i, χ_i, $k_{F,i}$, ε_i of (24.11) (with $k_{E,i} = 3/(X_i - 3)$, not tabulated), evaluated as there at $q = 1/X_i$ and $r = 1/\lfloor X_i^{1/2} \rfloor$; here $n_i = \lfloor 570/\log_- X_i \rfloor$ is the count of the restorations, while the local laws count $n_X = \lfloor 585/\log_- X \rfloor$ primes (Table 31). On the pieces 27–33, M_i and χ_i are $\mathrm{up}(R_{\mathfrak{u}}^{n_i})$ and $\mathrm{up}((1 + d_{\mathfrak{u}}^2)^{n_i} - 1)$ with the per-prime data of Tables 36 and 37. The table also lists the density caps $H_i = (1 + 2/(X_i - 4))^{n_i}$ and the lower start sums Σ_i of the mean savings (Proposition 26.7, Lemma 20.6), and the piece log caps L_i of Lemma 20.7, which lie between 556.79 and 570. None of these numbers depends on the exponent.

Table 26: Completion weights of the middle pieces; exact decimals, except w_{C} (a rational with a long denominator, rounded up). Source: `refined_L570_source.json` (`middle.pieces[i].local.weights`).

i	$w_{\mathrm{D}} = w_{\mathrm{S}}$	w_{R}	w_{C}	v_{L}	$v_{\mathrm{D}} = v_{\mathrm{S}}$	v_{R}	v_{C}
1	5.481831709798	22.206414052118	89.95621376	2.574071293695	10.979961835741	11.04539340244	11.11121488761
2	5.481472801115	22.192524849205	89.84960376	2.557961162481	10.978548368981	11.04105298356	11.103913458196
3	5.481143697784	22.179316362306	89.74807111	2.542519916612	10.977226782024	11.036936696987	11.09697140017
4	5.480900122235	22.169215324239	89.67032735	2.530630919138	10.976227506991	11.033792771889	11.091659940125
5	5.480841803566	22.166752443534	89.65135859	2.527721155163	10.975985456109	11.033026821675	11.0903646269
6	5.480564757436	22.154799099904	89.55922335	2.513538297374	10.974822683144	11.029315884666	11.084079660857
7	5.480076810923	22.132598055616	89.38777934	2.486917800366	10.972704266793	11.022440453034	11.072402079437
8	5.479664889628	22.112479162453	89.23205061	2.462466811752	10.970835336489	11.01623314424	11.061818810152
9	5.479316580401	22.094234540645	89.09052667	2.440008538906	10.969180093609	11.010620836991	11.052218140408
10	5.479021573962	22.077679142513	88.96185382	2.419381870032	10.967710968386	11.005542053918	11.043503630766
11	5.478771296094	22.062647904359	88.84481689	2.400439513894	10.966405767136	11.000943773652	11.035590555452
12	5.478558605644	22.048993306099	88.7383235	2.383044089645	10.965243403161	10.996776859737	11.028400999105
13	5.478377547006	22.036583283517	88.64139039	2.367070396701	10.964205527049	10.99299744147	11.021864963224
14	5.478223146401	22.025299401704	88.55313133	2.352402724659	10.963276929216	10.989567080135	11.015920275347
15	5.478091244751	22.01503530555	88.47274678	2.338936263241	10.962444917385	10.986451937309	11.010511530999
16	5.477978359548	22.005695345118	88.39951454	2.326572512256	10.961698660094	10.983621923336	11.005589032837
17	5.477881571158	21.997193399482	88.33278179	2.315221851572	10.961027604835	10.981048942776	11.001106851556
18	5.477798428647	21.989451832411	88.27195783	2.304803061096	10.960423795411	10.978709059431	10.997024828739
19	5.477726871954	21.982400585247	88.21650783	2.295239920764	10.959879531968	10.976580011327	10.993305938595
20	5.477665167321	21.975976373754	88.16594714	2.286461870533	10.959388533362	10.974642246858	10.989917191081
21	5.477611853772	21.970121984547	88.11983633	2.278404660372	10.958944767316	10.972877637652	10.986828221818
22	5.477565698576	21.964785648017	88.07777672	2.27101108026	10.958543248341	10.971270171829	10.984011875985
23	5.477525660249	21.959920486041	88.03940642	2.264226650182	10.958179775798	10.969805596134	10.981443750608
24	5.477490857716	21.955484020139	88.00439674	2.257999450127	10.957850303471	10.968470697833	10.979101385526
25	5.477460544596	21.951437734165	87.97244904	2.252286820089	10.957551425265	10.96725371204	10.976964589637
26	5.477434087745	21.947746683622	87.94329184	2.247044200062	10.957280060607	10.966143934794	10.975014979398
27	5.476623331076	21.941223679958	87.90403639	6.313079545011	10.955457913698	10.96355495264	10.971657976001
28	5.474337162024	21.921567785316	87.78325484	6.31525703074	10.950237053383	10.955878101141	10.961522054902
29	5.471258447645	21.901928862155	87.67534791	6.312562774077	10.943621149423	10.947552590414	10.951485443756
30	5.464134125055	21.864767738033	87.49200831	6.307298906003	10.928819588741	10.930731460914	10.932643667548
31	5.460775755114	21.847122359644	87.40456976	5.426082232316	10.921826993705	10.922758912721	10.923690911255
32	5.457521109684	21.831047260641	87.32804051	4.646587567244	10.915111046195	10.915333534894	10.915556028129
33	5.434450254811	21.737962800992	86.95249834	2.869197519096	10.86891262503	10.86894986383	10.868987102758

Table 27: Piece values: upper bounds, rounded up, except the lower bounds Σ_i, rounded down. Sources: the read-only evaluation of `piece_data_beta.compute` (modular root variant) on the native source (`count`, `h`, `M`, `chi`, `kf`, `eps`, `H`, `starts`); on the pieces 27–33 the root plan `further_root_construction.changes` for M_i, χ_i; `author.json` (`restoration_product_caps.rows[i].log_cap`) for L_i.

i	n_i	h_i	M_i	χ_i	$k_{F,i}$	ε_i	$H_i - 1$	Σ_i	L_i
1	33	$6.7232 \cdot 10^{-3}$	1.384957	$3.413 \cdot 10^{-5}$	$2.2587 \cdot 10^{-4}$	$9.071 \cdot 10^{-4}$	$3.309 \cdot 10^{-6}$	$7.78804 \cdot 10^{-5}$	557.563
2	33	$6.4335 \cdot 10^{-3}$	1.365911	$3.126 \cdot 10^{-5}$	$2.1621 \cdot 10^{-4}$	$8.681 \cdot 10^{-4}$	$3.034 \cdot 10^{-6}$	$7.75743 \cdot 10^{-5}$	560.433
3	33	$6.1575 \cdot 10^{-3}$	1.347985	$2.863 \cdot 10^{-5}$	$2.0699 \cdot 10^{-4}$	$8.309 \cdot 10^{-4}$	$2.782 \cdot 10^{-6}$	$6.17114 \cdot 10^{-5}$	562.712
4	33	$5.9446 \cdot 10^{-3}$	1.334303	$2.668 \cdot 10^{-5}$	$1.9987 \cdot 10^{-4}$	$8.023 \cdot 10^{-4}$	$2.596 \cdot 10^{-6}$	$1.54769 \cdot 10^{-5}$	563.303
5	33	$5.8925 \cdot 10^{-3}$	1.330973	$2.622 \cdot 10^{-5}$	$1.9813 \cdot 10^{-4}$	$7.953 \cdot 10^{-4}$	$2.551 \cdot 10^{-6}$	$7.67005 \cdot 10^{-5}$	566.139
6	33	$5.6396 \cdot 10^{-3}$	1.314926	$2.402 \cdot 10^{-5}$	$1.8968 \cdot 10^{-4}$	$7.612 \cdot 10^{-4}$	$2.340 \cdot 10^{-6}$	$1.53882 \cdot 10^{-4}$	570.000
7	32	$5.1657 \cdot 10^{-3}$	1.275590	$1.954 \cdot 10^{-5}$	$1.7382 \cdot 10^{-4}$	$6.973 \cdot 10^{-4}$	$1.908 \cdot 10^{-6}$	$1.55400 \cdot 10^{-4}$	560.085
8	32	$4.7326 \cdot 10^{-3}$	1.250059	$1.640 \cdot 10^{-5}$	$1.5931 \cdot 10^{-4}$	$6.390 \cdot 10^{-4}$	$1.604 \cdot 10^{-6}$	$1.53982 \cdot 10^{-4}$	565.617
9	32	$4.3357 \cdot 10^{-3}$	1.227078	$1.377 \cdot 10^{-5}$	$1.4601 \cdot 10^{-4}$	$5.855 \cdot 10^{-4}$	$1.349 \cdot 10^{-6}$	$1.53422 \cdot 10^{-4}$	570.000
10	31	$3.9723 \cdot 10^{-3}$	1.199323	$1.120 \cdot 10^{-5}$	$1.3382 \cdot 10^{-4}$	$5.365 \cdot 10^{-4}$	$1.099 \cdot 10^{-6}$	$1.55960 \cdot 10^{-4}$	558.685
11	31	$3.6398 \cdot 10^{-3}$	1.181348	$9.397 \cdot 10^{-6}$	$1.2266 \cdot 10^{-4}$	$4.916 \cdot 10^{-4}$	$9.239 \cdot 10^{-7}$	$1.54278 \cdot 10^{-4}$	564.051
12	31	$3.3356 \cdot 10^{-3}$	1.165116	$7.891 \cdot 10^{-6}$	$1.1244 \cdot 10^{-4}$	$4.506 \cdot 10^{-4}$	$7.769 \cdot 10^{-7}$	$1.52845 \cdot 10^{-4}$	569.444
13	31	$3.0570 \cdot 10^{-3}$	1.150429	$6.627 \cdot 10^{-6}$	$1.0307 \cdot 10^{-4}$	$4.130 \cdot 10^{-4}$	$6.533 \cdot 10^{-7}$	$1.55988 \cdot 10^{-4}$	570.000
14	30	$2.8016 \cdot 10^{-3}$	1.132408	$5.387 \cdot 10^{-6}$	$9.4481 \cdot 10^{-5}$	$3.785 \cdot 10^{-4}$	$5.317 \cdot 10^{-7}$	$1.54994 \cdot 10^{-4}$	561.451
15	30	$2.5676 \cdot 10^{-3}$	1.120768	$4.525 \cdot 10^{-6}$	$8.6610 \cdot 10^{-5}$	$3.469 \cdot 10^{-4}$	$4.471 \cdot 10^{-7}$	$1.53669 \cdot 10^{-4}$	566.669
16	30	$2.3535 \cdot 10^{-3}$	1.110211	$3.801 \cdot 10^{-6}$	$7.9403 \cdot 10^{-5}$	$3.180 \cdot 10^{-4}$	$3.760 \cdot 10^{-7}$	$1.54100 \cdot 10^{-4}$	570.000
17	29	$2.1571 \cdot 10^{-3}$	1.097098	$3.087 \cdot 10^{-6}$	$7.2791 \cdot 10^{-5}$	$2.915 \cdot 10^{-4}$	$3.056 \cdot 10^{-7}$	$1.55984 \cdot 10^{-4}$	557.841
18	29	$1.9773 \cdot 10^{-3}$	1.088688	$2.594 \cdot 10^{-6}$	$6.6733 \cdot 10^{-5}$	$2.672 \cdot 10^{-4}$	$2.570 \cdot 10^{-7}$	$1.54637 \cdot 10^{-4}$	562.842

i	n_i	h_i	M_i	χ_i	$k_{F,i}$	ε_i	$H_i - 1$	Σ_i	L_i
19	29	$1.8125 \cdot 10^{-3}$	1.081033	$2.180 \cdot 10^{-6}$	$6.1181 \cdot 10^{-5}$	$2.450 \cdot 10^{-4}$	$2.161 \cdot 10^{-7}$	$1.53266 \cdot 10^{-4}$	567.870
20	29	$1.6616 \cdot 10^{-3}$	1.074065	$1.832 \cdot 10^{-6}$	$5.6095 \cdot 10^{-5}$	$2.246 \cdot 10^{-4}$	$1.817 \cdot 10^{-7}$	$1.55073 \cdot 10^{-4}$	570.000
21	28	$1.5232 \cdot 10^{-3}$	1.065303	$1.486 \cdot 10^{-6}$	$5.1430 \cdot 10^{-5}$	$2.059 \cdot 10^{-4}$	$1.476 \cdot 10^{-7}$	$1.55950 \cdot 10^{-4}$	557.998
22	28	$1.3964 \cdot 10^{-3}$	1.059721	$1.249 \cdot 10^{-6}$	$4.7152 \cdot 10^{-5}$	$1.888 \cdot 10^{-4}$	$1.241 \cdot 10^{-7}$	$1.54608 \cdot 10^{-4}$	562.842
23	28	$1.2801 \cdot 10^{-3}$	1.054630	$1.050 \cdot 10^{-6}$	$4.3232 \cdot 10^{-5}$	$1.731 \cdot 10^{-4}$	$1.044 \cdot 10^{-7}$	$1.53293 \cdot 10^{-4}$	567.704
24	28	$1.1736 \cdot 10^{-3}$	1.049983	$8.820 \cdot 10^{-7}$	$3.9639 \cdot 10^{-5}$	$1.587 \cdot 10^{-4}$	$8.772 \cdot 10^{-8}$	$1.54907 \cdot 10^{-4}$	570.000
25	27	$1.0760 \cdot 10^{-3}$	1.044071	$7.149 \cdot 10^{-7}$	$3.6344 \cdot 10^{-5}$	$1.455 \cdot 10^{-4}$	$7.113 \cdot 10^{-8}$	$1.56246 \cdot 10^{-4}$	556.798
26	27	$9.8644 \cdot 10^{-4}$	1.040339	$6.009 \cdot 10^{-7}$	$3.3324 \cdot 10^{-5}$	$1.334 \cdot 10^{-4}$	$5.981 \cdot 10^{-8}$	$1.55000 \cdot 10^{-4}$	561.451
27	27	$1.0285 \cdot 10^{-3}$	$1.101160 \cdot 10^{7}$	$6.401 \cdot 10^{-7}$	$3.0554 \cdot 10^{-5}$	$1.223 \cdot 10^{-4}$	$5.030 \cdot 10^{-8}$	$6.19808 \cdot 10^{-4}$	569.992
28	26	$7.2704 \cdot 10^{-4}$	$1.520074 \cdot 10^{9}$	$3.345 \cdot 10^{-7}$	$2.1598 \cdot 10^{-5}$	$8.642 \cdot 10^{-5}$	$2.422 \cdot 10^{-8}$	$6.19573 \cdot 10^{-4}$	569.983
29	25	$5.1379 \cdot 10^{-4}$	$4.989969 \cdot 10^{11}$	$1.770 \cdot 10^{-7}$	$1.5268 \cdot 10^{-5}$	$6.109 \cdot 10^{-5}$	$1.165 \cdot 10^{-8}$	$1.23995 \cdot 10^{-3}$	569.995
30	24	$2.5658 \cdot 10^{-4}$	$1.499557 \cdot 10^{20}$	$5.146 \cdot 10^{-8}$	$7.6317 \cdot 10^{-6}$	$3.053 \cdot 10^{-5}$	$2.794 \cdot 10^{-9}$	$1.24248 \cdot 10^{-3}$	569.941
31	22	$1.2486 \cdot 10^{-4}$	$1.729204 \cdot 10^{20}$	$1.188 \cdot 10^{-8}$	$3.8153 \cdot 10^{-6}$	$1.527 \cdot 10^{-5}$	$6.403 \cdot 10^{-10}$	$2.49322 \cdot 10^{-3}$	569.962
32	20	$3.0455 \cdot 10^{-5}$	$9.051323 \cdot 10^{18}$	$6.734 \cdot 10^{-10}$	$9.5371 \cdot 10^{-7}$	$3.815 \cdot 10^{-6}$	$3.638 \cdot 10^{-11}$	$3.12057 \cdot 10^{-3}$	569.903
33	18	$5.0647 \cdot 10^{-6}$	$9.511041 \cdot 10^{85}$	$5.031 \cdot 10^{-10}$	$1.6859 \cdot 10^{-7}$	$6.744 \cdot 10^{-7}$	$1.024 \cdot 10^{-12}$	$1.49358 \cdot 10^{-3}$	569.905

The caps of the reference. Table 28 lists, for each piece, the block caps of the five cuts that Lemma 22.1 uses, all evaluated at the piece start X_i for blocks of logarithmic length at most $L_{\mathrm{loc}} = 585$: the caps d of L20 and k_{L} of L21, whose roundings up are the weights w_{D} and v_{L} of Table 26; the refined negative cap $k_{\mathrm{D}}(X_i, 585)$ of Lemma 23.2(e) (Certificate D.6), which the reference uses for its fold type in place of the route-A cap of F21; and the route-A block caps of E21 and d_1 of L10 (Lemma 20.12). Except for k_{D}, each is $g \exp(c^+ L_{\mathrm{loc}}/(X_i \log X_i))$ for the class-three negative constant g and the positive constant c^+ of a prime of its cut, $-gp^{-1/2} I \preceq \mathsf{A}_p \preceq (1 + c^+/p) I$, where A_p is the cut operator of Theorem 14.21. The cuts E21 and L10 are route-A cuts on every piece, with the constants of Table 34 at X_i. On the pieces 1–26 the cut L20 uses the finite collision-gap refinement of the uniform root (Section 20.4 and Proposition 21.11), as printed in Table 28; Table 35 retains the earlier caps for comparison. L21 is a route-A cut, with the constants of Table 34; on the pieces 27–33, marked $*$, L20 and L21 use the positive-root refinements of Certificate 21.25, with the constants of Tables 36 and 37. The final source stores every entry with the threshold X_i, the length 585, g, c^+ and the block cap, and the native checks bind it to the record of its cut and recompute the block caps (Certificates 20.13, 20.22 and 21.25).

Table 28: The block caps of the reference at the piece starts, $L_{\mathrm{loc}} = 585$ (upper bounds, rounded up). Source: `refined_L570_source.json` (`middle.pieces[i].local.spectra`: `L20`, `L21`, `F21`, `E21`, `L10`, field `block_d`; the flags `further_positive_pair_square` of L20 on every piece, `further_positive_pair_square` of L20 and L21 on the pieces 27–33, `advance_old_negative` of F21).

i	L20: d	L21: k_{L}	F21: k_{D}	E21	L10: d_1
1	5.481831709798	2.574071293695	1.001486584869	0.004540920001	1.751290440133
2	5.481472801115	2.557961162481	1.001423227600	0.004350500001	1.750476100111
3	5.481143697784	2.542519916612	1.001362798285	0.004166690001	1.749695360093
4	5.480900122235	2.530630919138	1.001316139886	0.004025750001	1.749095070081
5	5.480841803566	2.527721155163	1.001304712806	0.003991360001	1.748948670078
6	5.480564757436	2.513538297374	1.001249247923	0.003823340001	1.748232530065
7	5.480076810923	2.486917800366	1.001145115788	0.003508720001	1.746890930046
8	5.479664889628	2.462466811752	1.001049841320	0.003220770001	1.745660860032
9	5.479316580401	2.440008538906	1.000962431414	0.002955520001	1.744532970023
10	5.479021573962	2.419381870032	1.000882257930	0.002713080001	1.743498150016
11	5.478771296094	2.400439513894	1.000808865207	0.002490650001	1.742548560011
12	5.478558605644	2.383044089645	1.000741635936	0.002286470001	1.741678600008
13	5.478377547006	2.367070396701	1.000679985359	0.002098850001	1.740880900006
14	5.478223146401	2.352402724659	1.000623435395	0.001927180001	1.740149270004
15	5.478091244751	2.338936263241	1.000571588498	0.001769870001	1.739477710003
16	5.477978359548	2.326572512256	1.000524093071	0.001625390001	1.738862380002
17	5.477881571158	2.315221851572	1.000480519929	0.001493270001	1.738298600002
18	5.477798428647	2.304803061096	1.000440591068	0.001372050001	1.737780830001

i	L20: d	L21: k_{L}	F21: k_{D}	E21	L10: d_1
19	5.477726871954	2.295239920764	1.000403980353	0.001260330001	1.737305660001
20	5.477665167321	2.286461870533	1.000370431434	0.001158740001	1.736870780001
21	5.477611853772	2.278404660372	1.000339660775	0.001064930001	1.736472020001
22	5.477565698576	2.271011080260	1.000311438637	0.000979600001	1.736105270001
23	5.477525660249	2.264226650182	1.000285571944	0.000900440001	1.735770550001
24	5.477490857716	2.257999450127	1.000261852381	0.000828200001	1.735461940001
25	5.477460544596	2.252286820089	1.000240105434	0.000762640001	1.735179590001
26	5.477434087745	2.247044200062	1.000220165600	0.000701530001	1.734921740001
27*	5.476623331076	6.313079545011	1.000201880923	0.000645670001	1.734683680001
28*	5.474337162024	6.315257030740	1.000142732288	0.000465230001	1.733916200001
29*	5.471258447645	6.312562774077	1.000100914090	0.000338340001	1.733372310001
30*	5.464134125055	6.307298906003	1.000050450687	0.000183680001	1.732716650001
31*	5.460775755114	5.426082232316	1.000025223841	0.000105350001	1.732388830001
32*	5.457521109684	4.646587567244	1.000006305686	0.000044600001	1.732143720001
33*	5.434450254811	2.869197519096	1.000001114686	0.000025530001	1.732076270001

E.3. **Middle prices.** The middle records are priced piece by piece by (23.7) of Lemma 23.24, the rows of Definition 30.1(3): with the weight $w_{\mathrm{D},i} = w_{\mathrm{S},i}$, the charges $\bar{I}_{d,i}$ and the endpoint coefficients at $(q_{X_i}, r_{X_i}) = (1/X_i, 1/\lfloor\sqrt{X_i}\rfloor)$, namely $\ell_{\mathtt{t}}$, a_F, s_F of (23.5) at $\mathtt{t} = 127/500$ (Proposition 23.13) and s_E, a_E, r_E of the matching records (Lemmas 23.17 and 23.18), each price is rounded up on $10^{-30}\mathbb{Z}$. Table 29 and the table that follows it list the coefficients and the prices at a_0, in two parts: the signed fold records, then the matching and the other records. Always $A^{\mathrm{arm}}_{E,i} = A^{\mathrm{arm}}_{F,i}$, and $A^{\mathrm{lead}}_{E,i}$ and $A_{E,i}$ exceed it only by the factors $1/(1-q_{X_i})$ and $a_E(X_i) = (1+2q_{X_i})/((1-q_{X_i})(1-2q_{X_i}))$, both below $1 + 2.6 \cdot 10^{-7}$ (up to the rounding on $10^{-30}\mathbb{Z}$); these three columns are shown with eight digits, so that the factors are visible. No stored record holds the per-piece coefficients: they are the values, in outward rational arithmetic, of the functions of `research/advance_2026_10_24/local/low_domain.py` (`raw_ledgers`, with `sharp_bounds` and `e_bounds`) on the native source `research/push_2026_10_25/global/refined_L570_source.json`, which the final scalar evaluates in the same way (Certificate 23.14); the values at Q_3 are those of Certificate 23.14.

The sums over the 33 pieces at a_0 are

$$\sum_i A^{\mathrm{arm}}_{F,i} < 0.4233218123, \quad \sum_i A^{\mathrm{lead}}_{F,i} < 0.53897230072, \quad \sum_i A^{\mathrm{rem}}_{F,i} < 5.6589506101 \cdot 10^{-5},$$
$$\sum_i S_{F,i} < 1.3191008051 \cdot 10^{-4}, \quad \sum_i A^{\mathrm{lead}}_{E,i} < 0.42332181373, \quad \sum_i A_{E,i} < 0.42332181948,$$
$$\sum_i A^{\mathrm{rem}}_{E,i} < 1.4367776013 \cdot 10^{-8}, \quad \sum_i S_{E,i} < 3.0926618874 \cdot 10^{-5},$$
$$\sum_i S_{\mathrm{oth},i} < 5.2581571154 \cdot 10^{-4}, \quad \sum_i A_{\mathrm{oth},i} < 2.3870591726 \cdot 10^{-4},$$

and t times these sums are the totals at a of Definition 30.1(3), stored in `research/push_2026_10_25/artifacts/author.json` (key `raw_totals`), with the following outward bounds: $A^{\mathrm{arm}}_F = A^{\mathrm{arm}}_E < 0.42332183839047$, $A^{\mathrm{lead}}_F < 0.53897233394172$, $A^{\mathrm{rem}}_F < 5.6589509590 \cdot 10^{-5}$, $A_F < 0.53902892345131$, $S_F < 1.3191008864 \cdot 10^{-4}$, $A^{\mathrm{lead}}_E < 0.42332183982722$, $A^{\mathrm{rem}}_E < 1.4367776899 \cdot 10^{-8}$, $A_E < 0.42332184557419$, $S_E < 3.0926620780 \cdot 10^{-5}$, $S_{\mathrm{oth}} < 5.2581574396 \cdot 10^{-4}$, $A_{\mathrm{oth}} < 2.3870593198 \cdot 10^{-4}$, $S_{\mathrm{mid}} < 6.8865245337 \cdot 10^{-4}$ and $A_{\mathrm{mid}} < 0.96258947495747$.

Table 29: Middle prices at a_0, part I: the coefficients of the signed fold records at X_i and their prices (upper bounds, rounded up); $A^{\mathrm{arm}}_{F,i} = A^{\mathrm{arm}}_{E,i}$; part II is the next table. Source: the read-only evaluation of `low_domain.raw_ledgers` at $\mathtt{t} = 127/500$ on the native source (`rows[i].prices`: `BF`, `LF`, `RF`, `SF`; `rows[i].bounds`).

i	$\ell_{\mathtt{t}}(X_i)$	$a_F(X_i)$	$s_F(X_i)$	$A^{\mathrm{arm}}_{F,i}$	$A^{\mathrm{lead}}_{F,i}$	$A^{\mathrm{rem}}_{F,i}$	$S_{F,i}$
1	1.2731978	4.797399	11.16607	$2.2937680 \cdot 10^{-3}$	$2.9205 \cdot 10^{-3}$	$2.4111 \cdot 10^{-6}$	$5.6118 \cdot 10^{-6}$
2	1.2731978	4.795180	11.16343	$2.2954700 \cdot 10^{-3}$	$2.9226 \cdot 10^{-3}$	$2.3095 \cdot 10^{-6}$	$5.3765 \cdot 10^{-6}$
3	1.2731978	4.793064	11.16091	$1.8342660 \cdot 10^{-3}$	$2.3354 \cdot 10^{-3}$	$1.7741 \cdot 10^{-6}$	$4.1311 \cdot 10^{-6}$
4	1.2731978	4.791429	11.15896	$4.6078701 \cdot 10^{-4}$	$5.8668 \cdot 10^{-4}$	$4.3594 \cdot 10^{-7}$	$1.0153 \cdot 10^{-6}$
5	1.2731978	4.791029	11.15849	$2.2927261 \cdot 10^{-3}$	$2.9191 \cdot 10^{-3}$	$2.1134 \cdot 10^{-6}$	$4.9222 \cdot 10^{-6}$

i	$\ell_{\mathrm{t}}(X_i)$	$a_F(X_i)$	$s_F(X_i)$	$A_{F,i}^{\mathrm{arm}}$	$A_{F,i}^{\mathrm{lead}}$	$A_{F,i}^{\mathrm{rem}}$	$S_{F,i}$
6	1.2731978	4.789085	11.15617	$4.5895339 \cdot 10^{-3}$	$5.8434 \cdot 10^{-3}$	$3.9637 \cdot 10^{-6}$	$9.2333 \cdot 10^{-6}$
7	1.2731977	4.785437	11.15183	$4.5847170 \cdot 10^{-3}$	$5.8373 \cdot 10^{-3}$	$3.6280 \cdot 10^{-6}$	$8.4546 \cdot 10^{-6}$
8	1.2731977	4.782096	11.14786	$4.5875961 \cdot 10^{-3}$	$5.8410 \cdot 10^{-3}$	$3.3268 \cdot 10^{-6}$	$7.7552 \cdot 10^{-6}$
9	1.2731977	4.779030	11.14421	$4.5851545 \cdot 10^{-3}$	$5.8379 \cdot 10^{-3}$	$3.0471 \cdot 10^{-6}$	$7.1054 \cdot 10^{-6}$
10	1.2731977	4.777468	11.14086	$4.5897858 \cdot 10^{-3}$	$5.8438 \cdot 10^{-3}$	$2.7962 \cdot 10^{-6}$	$6.5205 \cdot 10^{-6}$
11	1.2731977	4.776182	11.13780	$4.5836149 \cdot 10^{-3}$	$5.8359 \cdot 10^{-3}$	$2.5599 \cdot 10^{-6}$	$5.9696 \cdot 10^{-6}$
12	1.2731977	4.775004	11.13499	$4.5845276 \cdot 10^{-3}$	$5.8371 \cdot 10^{-3}$	$2.3474 \cdot 10^{-6}$	$5.4738 \cdot 10^{-6}$
13	1.2731976	4.773924	11.13241	$4.5870480 \cdot 10^{-3}$	$5.8403 \cdot 10^{-3}$	$2.1533 \cdot 10^{-6}$	$5.0212 \cdot 10^{-6}$
14	1.2731976	4.772932	11.13005	$4.5840422 \cdot 10^{-3}$	$5.8364 \cdot 10^{-3}$	$1.9729 \cdot 10^{-6}$	$4.6005 \cdot 10^{-6}$
15	1.2731976	4.772023	11.12788	$4.5869491 \cdot 10^{-3}$	$5.8401 \cdot 10^{-3}$	$1.8099 \cdot 10^{-6}$	$4.2205 \cdot 10^{-6}$
16	1.2731976	4.771191	11.12590	$4.5863374 \cdot 10^{-3}$	$5.8394 \cdot 10^{-3}$	$1.6592 \cdot 10^{-6}$	$3.8690 \cdot 10^{-6}$
17	1.2731976	4.770427	11.12408	$4.5840268 \cdot 10^{-3}$	$5.8364 \cdot 10^{-3}$	$1.5205 \cdot 10^{-6}$	$3.5455 \cdot 10^{-6}$
18	1.2731976	4.769727	11.12241	$4.5851080 \cdot 10^{-3}$	$5.8378 \cdot 10^{-3}$	$1.3944 \cdot 10^{-6}$	$3.2516 \cdot 10^{-6}$
19	1.2731976	4.769085	11.12088	$4.5851830 \cdot 10^{-3}$	$5.8379 \cdot 10^{-3}$	$1.2785 \cdot 10^{-6}$	$2.9813 \cdot 10^{-6}$
20	1.2731976	4.768497	11.11947	$4.5864511 \cdot 10^{-3}$	$5.8395 \cdot 10^{-3}$	$1.1726 \cdot 10^{-6}$	$2.7343 \cdot 10^{-6}$
21	1.2731976	4.767957	11.11818	$4.5849742 \cdot 10^{-3}$	$5.8376 \cdot 10^{-3}$	$1.0748 \cdot 10^{-6}$	$2.5062 \cdot 10^{-6}$
22	1.2731976	4.767462	11.11700	$4.5847914 \cdot 10^{-3}$	$5.8374 \cdot 10^{-3}$	$9.8544 \cdot 10^{-7}$	$2.2979 \cdot 10^{-6}$
23	1.2731976	4.767009	11.11592	$4.5851767 \cdot 10^{-3}$	$5.8379 \cdot 10^{-3}$	$9.0363 \cdot 10^{-7}$	$2.1072 \cdot 10^{-6}$
24	1.2731976	4.766593	11.11493	$4.5850579 \cdot 10^{-3}$	$5.8377 \cdot 10^{-3}$	$8.2854 \cdot 10^{-7}$	$1.9321 \cdot 10^{-6}$
25	1.2731976	4.766212	11.11402	$4.5843372 \cdot 10^{-3}$	$5.8368 \cdot 10^{-3}$	$7.5960 \cdot 10^{-7}$	$1.7713 \cdot 10^{-6}$
26	1.2731976	4.765862	11.11319	$4.5857589 \cdot 10^{-3}$	$5.8386 \cdot 10^{-3}$	$6.9671 \cdot 10^{-7}$	$1.6247 \cdot 10^{-6}$
27	1.2731976	4.765541	11.11242	$1.8337157 \cdot 10^{-2}$	$2.3347 \cdot 10^{-2}$	$2.2538 \cdot 10^{-6}$	$5.2554 \cdot 10^{-6}$
28	1.2731976	4.764504	11.10994	$1.8330169 \cdot 10^{-2}$	$2.3338 \cdot 10^{-2}$	$1.5927 \cdot 10^{-6}$	$3.7139 \cdot 10^{-6}$
29	1.2731976	4.763770	11.10819	$3.6646733 \cdot 10^{-2}$	$4.6659 \cdot 10^{-2}$	$1.9216 \cdot 10^{-6}$	$4.4808 \cdot 10^{-6}$
30	1.2731976	4.762885	11.10608	$3.6600211 \cdot 10^{-2}$	$4.6600 \cdot 10^{-2}$	$9.5940 \cdot 10^{-7}$	$2.2372 \cdot 10^{-6}$
31	1.2731976	4.762443	11.10503	$7.3153592 \cdot 10^{-2}$	$9.3139 \cdot 10^{-2}$	$7.1902 \cdot 10^{-7}$	$1.6766 \cdot 10^{-6}$
32	1.2731976	4.762111	11.10423	$9.1388133 \cdot 10^{-2}$	$1.1636 \cdot 10^{-1}$	$1.9718 \cdot 10^{-7}$	$4.5977 \cdot 10^{-7}$
33	1.2731976	4.762020	11.10402	$4.3388632 \cdot 10^{-2}$	$5.5243 \cdot 10^{-2}$	$2.3708 \cdot 10^{-8}$	$5.5282 \cdot 10^{-8}$
$\sum$				$4.2332182 \cdot 10^{-1}$	$5.3898 \cdot 10^{-1}$	$5.6590 \cdot 10^{-5}$	$1.3192 \cdot 10^{-4}$

Table 30: Middle prices at a_0, part II, continuing the previous table: the coefficients of the matching records at X_i and the prices of the matching and of the other records (upper bounds, rounded up). Source: as part I (`rows[i].prices`: `LE`, `GE`, `RE`, `SE`, `other_s`, `other_a`; `rows[i].E_bounds`).

i	$s_E(X_i)$	$r_E(X_i)$	$A_{E,i}^{\mathrm{lead}}$	$A_{E,i}$	$A_{E,i}^{\mathrm{rem}}$	$S_{E,i}$	$S_{\mathrm{oth},i}$	$A_{\mathrm{oth},i}$
1	$5.4072 \cdot 10^{-4}$	10.8298	$2.2937681 \cdot 10^{-3}$	$2.2937686 \cdot 10^{-3}$	$1.1928 \cdot 10^{-9}$	$1.2403 \cdot 10^{-6}$	$2.2500 \cdot 10^{-5}$	$1.0202 \cdot 10^{-5}$
2	$5.1776 \cdot 10^{-4}$	10.8297	$2.2954701 \cdot 10^{-3}$	$2.2954705 \cdot 10^{-3}$	$1.0945 \cdot 10^{-9}$	$1.1885 \cdot 10^{-6}$	$2.1545 \cdot 10^{-5}$	$9.7699 \cdot 10^{-6}$
3	$4.9586 \cdot 10^{-4}$	10.8296	$1.8342661 \cdot 10^{-3}$	$1.8342664 \cdot 10^{-3}$	$8.0893 \cdot 10^{-10}$	$9.0953 \cdot 10^{-7}$	$1.6546 \cdot 10^{-5}$	$7.5037 \cdot 10^{-6}$
4	$4.7894 \cdot 10^{-4}$	10.8296	$4.6078703 \cdot 10^{-4}$	$4.6078710 \cdot 10^{-4}$	$1.9455 \cdot 10^{-10}$	$2.2069 \cdot 10^{-7}$	$4.0648 \cdot 10^{-6}$	$1.8436 \cdot 10^{-6}$
5	$4.7479 \cdot 10^{-4}$	10.8296	$2.2927262 \cdot 10^{-3}$	$2.2927266 \cdot 10^{-3}$	$9.1923 \cdot 10^{-10}$	$1.0886 \cdot 10^{-6}$	$1.9705 \cdot 10^{-5}$	$8.9373 \cdot 10^{-6}$
6	$4.5467 \cdot 10^{-4}$	10.8295	$4.5895340 \cdot 10^{-3}$	$4.5895347 \cdot 10^{-3}$	$1.6174 \cdot 10^{-9}$	$2.0868 \cdot 10^{-6}$	$3.6947 \cdot 10^{-5}$	$1.6759 \cdot 10^{-5}$
7	$4.1691 \cdot 10^{-4}$	10.8295	$4.5847171 \cdot 10^{-3}$	$4.5847177 \cdot 10^{-3}$	$1.3585 \cdot 10^{-9}$	$1.9114 \cdot 10^{-6}$	$3.3802 \cdot 10^{-5}$	$1.5336 \cdot 10^{-5}$
8	$3.8231 \cdot 10^{-4}$	10.8294	$4.5875962 \cdot 10^{-3}$	$4.5875967 \cdot 10^{-3}$	$1.1432 \cdot 10^{-9}$	$1.7539 \cdot 10^{-6}$	$3.0981 \cdot 10^{-5}$	$1.4058 \cdot 10^{-5}$
9	$3.5056 \cdot 10^{-4}$	10.8293	$4.5851546 \cdot 10^{-3}$	$4.5851550 \cdot 10^{-3}$	$9.6071 \cdot 10^{-10}$	$1.6074 \cdot 10^{-6}$	$2.8365 \cdot 10^{-5}$	$1.2873 \cdot 10^{-5}$
10	$3.2144 \cdot 10^{-4}$	10.8292	$4.5897859 \cdot 10^{-3}$	$4.5897862 \cdot 10^{-3}$	$8.0873 \cdot 10^{-10}$	$1.4754 \cdot 10^{-6}$	$2.6013 \cdot 10^{-5}$	$1.1808 \cdot 10^{-5}$
11	$2.9475 \cdot 10^{-4}$	10.8292	$4.5836150 \cdot 10^{-3}$	$4.5836152 \cdot 10^{-3}$	$6.7912 \cdot 10^{-10}$	$1.3511 \cdot 10^{-6}$	$2.3801 \cdot 10^{-5}$	$1.0805 \cdot 10^{-5}$
12	$2.7030 \cdot 10^{-4}$	10.8291	$4.5845277 \cdot 10^{-3}$	$4.5845279 \cdot 10^{-3}$	$5.7117 \cdot 10^{-10}$	$1.2392 \cdot 10^{-6}$	$2.1813 \cdot 10^{-5}$	$9.9029 \cdot 10^{-6}$
13	$2.4788 \cdot 10^{-4}$	10.8291	$4.5870481 \cdot 10^{-3}$	$4.5870483 \cdot 10^{-3}$	$4.8057 \cdot 10^{-10}$	$1.1370 \cdot 10^{-6}$	$1.9999 \cdot 10^{-5}$	$9.0805 \cdot 10^{-6}$
14	$2.2729 \cdot 10^{-4}$	10.8290	$4.5840423 \cdot 10^{-3}$	$4.5840424 \cdot 10^{-3}$	$4.0385 \cdot 10^{-10}$	$1.0419 \cdot 10^{-6}$	$1.8315 \cdot 10^{-5}$	$8.3168 \cdot 10^{-6}$
15	$2.0842 \cdot 10^{-4}$	10.8290	$4.5869491 \cdot 10^{-3}$	$4.5869492 \cdot 10^{-3}$	$3.3980 \cdot 10^{-10}$	$9.5598 \cdot 10^{-7}$	$1.6795 \cdot 10^{-5}$	$7.6273 \cdot 10^{-6}$
16	$1.9113 \cdot 10^{-4}$	10.8289	$4.5863374 \cdot 10^{-3}$	$4.5863375 \cdot 10^{-3}$	$2.8570 \cdot 10^{-10}$	$8.7655 \cdot 10^{-7}$	$1.5391 \cdot 10^{-5}$	$6.9901 \cdot 10^{-6}$
17	$1.7525 \cdot 10^{-4}$	10.8289	$4.5840268 \cdot 10^{-3}$	$4.5840269 \cdot 10^{-3}$	$2.4013 \cdot 10^{-10}$	$8.0336 \cdot 10^{-7}$	$1.4100 \cdot 10^{-5}$	$6.4039 \cdot 10^{-6}$
18	$1.6071 \cdot 10^{-4}$	10.8289	$4.5851080 \cdot 10^{-3}$	$4.5851081 \cdot 10^{-3}$	$2.0197 \cdot 10^{-10}$	$7.3685 \cdot 10^{-7}$	$1.2927 \cdot 10^{-5}$	$5.8715 \cdot 10^{-6}$
19	$1.4737 \cdot 10^{-4}$	10.8288	$4.5851830 \cdot 10^{-3}$	$4.5851831 \cdot 10^{-3}$	$1.6983 \cdot 10^{-10}$	$6.7569 \cdot 10^{-7}$	$1.1849 \cdot 10^{-5}$	$5.3823 \cdot 10^{-6}$
20	$1.3514 \cdot 10^{-4}$	10.8288	$4.5864511 \cdot 10^{-3}$	$4.5864511 \cdot 10^{-3}$	$1.4285 \cdot 10^{-10}$	$6.1981 \cdot 10^{-7}$	$1.0864 \cdot 10^{-5}$	$4.9353 \cdot 10^{-6}$
21	$1.2393 \cdot 10^{-4}$	10.8288	$4.5849742 \cdot 10^{-3}$	$4.5849742 \cdot 10^{-3}$	$1.2009 \cdot 10^{-10}$	$5.6818 \cdot 10^{-7}$	$9.9554 \cdot 10^{-6}$	$4.5229 \cdot 10^{-6}$
22	$1.1364 \cdot 10^{-4}$	10.8287	$4.5847914 \cdot 10^{-3}$	$4.5847914 \cdot 10^{-3}$	$1.0098 \cdot 10^{-10}$	$5.2099 \cdot 10^{-7}$	$9.1259 \cdot 10^{-6}$	$4.1462 \cdot 10^{-6}$
23	$1.0421 \cdot 10^{-4}$	10.8287	$4.5851767 \cdot 10^{-3}$	$4.5851767 \cdot 10^{-3}$	$8.4913 \cdot 10^{-11}$	$4.7778 \cdot 10^{-7}$	$8.3666 \cdot 10^{-6}$	$3.8014 \cdot 10^{-6}$
24	$9.5553 \cdot 10^{-5}$	10.8287	$4.5850579 \cdot 10^{-3}$	$4.5850580 \cdot 10^{-3}$	$7.1402 \cdot 10^{-11}$	$4.3812 \cdot 10^{-7}$	$7.6700 \cdot 10^{-6}$	$3.4850 \cdot 10^{-6}$
25	$8.7622 \cdot 10^{-5}$	10.8287	$4.5843372 \cdot 10^{-3}$	$4.5843373 \cdot 10^{-3}$	$6.0033 \cdot 10^{-11}$	$4.0169 \cdot 10^{-7}$	$7.0307 \cdot 10^{-6}$	$3.1946 \cdot 10^{-6}$
26	$8.0349 \cdot 10^{-5}$	10.8287	$4.5857589 \cdot 10^{-3}$	$4.5857589 \cdot 10^{-3}$	$5.0496 \cdot 10^{-11}$	$3.6846 \cdot 10^{-7}$	$6.4476 \cdot 10^{-6}$	$2.9298 \cdot 10^{-6}$

i	$s_E(X_i)$	$r_E(X_i)$	$A^{\mathrm{lead}}_{E,i}$	$A_{E,i}$	$A^{\mathrm{rem}}_{E,i}$	$S_{E,i}$	$S_{\mathrm{oth},i}$	$A_{\mathrm{oth},i}$
27	$7.3679 \cdot 10^{-5}$	10.8286	$1.8337157 \cdot 10^{-2}$	$1.8337157 \cdot 10^{-2}$	$1.3340 \cdot 10^{-10}$	$1.3511 \cdot 10^{-6}$	$2.0855 \cdot 10^{-5}$	$9.4764 \cdot 10^{-6}$
28	$5.2100 \cdot 10^{-5}$	10.8286	$1.8330169 \cdot 10^{-2}$	$1.8330170 \cdot 10^{-2}$	$6.6673 \cdot 10^{-11}$	$9.5499 \cdot 10^{-7}$	$1.4731 \cdot 10^{-5}$	$6.6945 \cdot 10^{-6}$
29	$3.6839 \cdot 10^{-5}$	10.8286	$3.6646733 \cdot 10^{-2}$	$3.6646733 \cdot 10^{-2}$	$4.9989 \cdot 10^{-11}$	$1.3501 \cdot 10^{-6}$	$1.7768 \cdot 10^{-5}$	$8.0750 \cdot 10^{-6}$
30	$1.8420 \cdot 10^{-5}$	10.8285	$3.6600211 \cdot 10^{-2}$	$3.6600211 \cdot 10^{-2}$	$1.2482 \cdot 10^{-11}$	$6.7415 \cdot 10^{-7}$	$8.8677 \cdot 10^{-6}$	$4.0305 \cdot 10^{-6}$
31	$9.2096 \cdot 10^{-6}$	10.8285	$7.3153592 \cdot 10^{-2}$	$7.3153592 \cdot 10^{-2}$	$3.8979 \cdot 10^{-12}$	$6.7372 \cdot 10^{-7}$	$6.6447 \cdot 10^{-6}$	$3.0202 \cdot 10^{-6}$
32	$2.3024 \cdot 10^{-6}$	10.8285	$9.1388133 \cdot 10^{-2}$	$9.1388133 \cdot 10^{-2}$	$2.5159 \cdot 10^{-13}$	$2.1041 \cdot 10^{-7}$	$1.8219 \cdot 10^{-6}$	$8.2813 \cdot 10^{-7}$
33	$4.0701 \cdot 10^{-7}$	10.8285	$4.3388632 \cdot 10^{-2}$	$4.3388632 \cdot 10^{-2}$	$6.5335 \cdot 10^{-15}$	$1.7660 \cdot 10^{-8}$	$2.1906 \cdot 10^{-7}$	$9.9571 \cdot 10^{-8}$
$\sum$			$4.2332182 \cdot 10^{-1}$	$4.2332182 \cdot 10^{-1}$	$1.4368 \cdot 10^{-8}$	$3.0927 \cdot 10^{-5}$	$5.2582 \cdot 10^{-4}$	$2.3871 \cdot 10^{-4}$

E.4. **The sides of the matching envelopes.** The matching envelopes of Proposition 26.5 use in each middle piece the chosen sides $d_s(X_i) = d_s^{\mathrm{aux}}(X_i) + \delta^+_{s,i}$, $s \in \{\mathrm{L}, \mathrm{R}\}$, of Proposition 26.4. Table 31 lists, for each piece, the block count $n_{X_i} = \lfloor 585/\log_- X_i \rfloor$ and the density factor $\bar{H}_{X_i} = (1 - n_{X_i} e_0(X_i))^{-1}$, the certified upper bound for the signed coefficient $\bar{H}_{X_i} d_{\mathtt{t}}(X_i)$ of the block Gram, the auxiliary sides $d_s^{\mathrm{aux}}(X_i)$ of Definition 26.2 (at the auxiliary point, with $\tau_{\mathrm{side}} = 31/32$ and $L_{\mathrm{loc}} = 585$), the increments $\delta^+_{s,i} = \mathrm{up}([\bar{H}_{X_i} d_{\mathtt{t}}(X_i) - d_s^{\mathrm{aux}}(X_i)]_+)$ of (26.1), and the excess coefficients $k^{\mathrm{aux}}_{s,i} = w_{\mathrm{D},i} d_s^{\mathrm{aux}}(X_i) - v_{\mathrm{D},i}$ of the auxiliary sides. In every piece $\bar{H}_{X_i} d_{\mathtt{t}}(X_i)$ exceeds both auxiliary sides, so both increments are positive and both chosen sides lie within 10^{-30} above the certified value of $\bar{H}_{X_i} d_{\mathtt{t}}(X_i)$; the coefficients $k^{\mathrm{aux}}_{s,i}$ are positive, the smallest being above 0.0704 (Certificate D.10). The auxiliary sides are values of the function `bound` of `research/roadmap_2026_10_18/item03/direct_matching_asymmetric.py`, and the other columns of `matching` of `research/advance_2026_10_24/local/signed_law.py`, evaluated read-only on the native source as the final scalar evaluates them; no stored record holds them piece by piece.

The two sums of Proposition 26.4(iii) at a_0 are

$$T_K^{\mathrm{aux}\prime} = \sum_i \mathrm{up}((k^{\mathrm{aux}}_{\mathrm{L},i} + k^{\mathrm{aux}}_{\mathrm{R},i}) \bar{I}_{2,i}) < 3.549907570156 \cdot 10^{-2}, \qquad T_K^{\mathrm{inc}\prime} < 0.1273320049107,$$

the first being the price of the excess diagonals of the auxiliary sides, the second that of the increments; $tT_K^{\mathrm{inc}\prime} < 0.12733201276083$ (`author.json`, `raw_totals`, key `matching_increment`), and $T_K = t(T_K^{\mathrm{aux}\prime} + T_K^{\mathrm{inc}\prime}) < 0.16283110$ (the certified upper end; the value is $0.16283109\ldots$) (Definition 30.1(5)).

Table 31: The sides of the matching envelopes at the piece starts (upper bounds, rounded up). Sources: the read-only evaluation of `low_domain.raw_ledgers` (`rows[i].matching`: `count`, `density`, `absolute_new_coefficient`, `old_left`, `old_right`, `left`, `right`) and the weights of Table 26.

i	n_{X_i}	$\bar{H}_{X_i} - 1$	$\bar{H}_{X_i} d_{\mathtt{t}}(X_i)$	$d^{\mathrm{aux}}_{\mathrm{L}}(X_i)$	$d^{\mathrm{aux}}_{\mathrm{R}}(X_i)$	$\delta^+_{\mathrm{L},i}$	$\delta^+_{\mathrm{R},i}$	$k^{\mathrm{aux}}_{\mathrm{L},i}$	$k^{\mathrm{aux}}_{\mathrm{R},i}$
1	34	$3.4083 \cdot 10^{-6}$	2.3903402	2.0524514	2.1844137	0.3378889	0.2059265	0.271232	0.994627
2	34	$3.1254 \cdot 10^{-6}$	2.3900798	2.0507733	2.1827356	0.3393065	0.2073442	0.262710	0.986058
3	34	$2.8660 \cdot 10^{-6}$	2.3898314	2.0491723	2.1811346	0.3406591	0.2086969	0.254581	0.977885
4	34	$2.6741 \cdot 10^{-6}$	2.3896396	2.0479358	2.1798980	0.3417038	0.2097416	0.248304	0.971576
5	34	$2.6282 \cdot 10^{-6}$	2.3895926	2.0476330	2.1795951	0.3419597	0.2099975	0.246767	0.970031
6	34	$2.4101 \cdot 10^{-6}$	2.3893646	2.0461626	2.1781248	0.3432020	0.2112398	0.239304	0.962532
7	33	$1.9670 \cdot 10^{-6}$	2.3889366	2.0434026	2.1753647	0.3455340	0.2135719	0.225299	0.948462
8	33	$1.6541 \cdot 10^{-6}$	2.3885448	2.0408745	2.1728366	0.3476704	0.2157083	0.212473	0.935581
9	33	$1.3909 \cdot 10^{-6}$	2.3881854	2.0385542	2.1705161	0.3496313	0.2176694	0.200704	0.923765
10	32	$1.1342 \cdot 10^{-6}$	2.3878558	2.0364259	2.1683878	0.3514300	0.2194681	0.189911	0.912933
11	32	$9.5368 \cdot 10^{-7}$	2.3875541	2.0344759	2.1664379	0.3530782	0.2211162	0.180023	0.903012
12	32	$8.0195 \cdot 10^{-7}$	2.3872776	2.0326892	2.1646511	0.3545885	0.2226266	0.170964	0.893925
13	31	$6.5328 \cdot 10^{-7}$	2.3870241	2.0310507	2.1630126	0.3559735	0.2240116	0.162657	0.885594
14	31	$5.4934 \cdot 10^{-7}$	2.3867915	2.0295468	2.1615086	0.3572448	0.2252830	0.155033	0.877950
15	31	$4.6194 \cdot 10^{-7}$	2.3865783	2.0281676	2.1601294	0.3584108	0.2264489	0.148042	0.870941
16	30	$3.7592 \cdot 10^{-7}$	2.3863830	2.0269042	2.1588660	0.3594789	0.2275170	0.141639	0.864523

i	n_{X_i}	$\bar{H}_{X_i}-1$	$\bar{H}_{X_i}d_{\mathfrak{t}}(X_i)$	$d_{\mathrm{L}}^{\mathrm{aux}}(X_i)$	$d_{\mathrm{R}}^{\mathrm{aux}}(X_i)$	$\delta_{\mathrm{L},i}^{+}$	$\delta_{\mathrm{R},i}^{+}$	$k_{\mathrm{L},i}^{\mathrm{aux}}$	$k_{\mathrm{R},i}^{\mathrm{aux}}$
17	30	$3.1611\cdot10^{-7}$	2.3862038	2.0257445	2.1577063	0.3604593	0.2284975	0.135761	0.858632
18	30	$2.6581\cdot10^{-7}$	2.3860396	2.0246817	2.1566435	0.3613580	0.2293961	0.130374	0.853235
19	30	$2.2352\cdot10^{-7}$	2.3858890	2.0237070	2.1556688	0.3621821	0.2302203	0.125435	0.848285
20	29	$1.8169\cdot10^{-7}$	2.3857510	2.0228138	2.1547756	0.3629373	0.2309755	0.120908	0.843751
21	29	$1.5279\cdot10^{-7}$	2.3856245	2.0219943	2.1539561	0.3636302	0.2316684	0.116755	0.839591
22	29	$1.2848\cdot10^{-7}$	2.3855084	2.0212425	2.1532043	0.3642659	0.2323041	0.112946	0.835775
23	29	$1.0804\cdot10^{-7}$	2.3854020	2.0205535	2.1525152	0.3648486	0.2328868	0.109454	0.832278
24	28	$8.7713\cdot10^{-8}$	2.3853044	2.0199216	2.1518834	0.3653829	0.2334211	0.106252	0.829071
25	28	$7.3757\cdot10^{-8}$	2.3852150	2.0193421	2.1513039	0.3658729	0.2339111	0.103316	0.826131
26	28	$6.2022\cdot10^{-8}$	2.3851330	2.0188108	2.1507725	0.3663222	0.2343605	0.100623	0.823435
27	28	$5.2155\cdot10^{-8}$	2.3850577	2.0183235	2.1502852	0.3667343	0.2347725	0.098140	0.820845
28	27	$2.5146\cdot10^{-8}$	2.3848144	2.0167469	2.1487087	0.3680676	0.2361058	0.090116	0.812519
29	26	$1.2108\cdot10^{-8}$	2.3846424	2.0156320	2.1475937	0.3690105	0.2370487	0.084423	0.806419
30	24	$2.7940\cdot10^{-9}$	2.3844349	2.0142863	2.1462481	0.3701486	0.2381868	0.077511	0.798568
31	23	$6.6939\cdot10^{-10}$	2.3843311	2.0136135	2.1455752	0.3707177	0.2387559	0.074065	0.794678
32	21	$3.8199\cdot10^{-11}$	2.3842533	2.0131088	2.1450706	0.3711445	0.2391827	0.071473	0.791657
33	18	$1.0232\cdot10^{-12}$	2.3842319	2.0129704	2.1449321	0.3712616	0.2392998	0.070475	0.787615

E.5. **The upper bands.** The upper range consists of the 310 bands $[X,Y)$ of Definition 27.1: below 2^{52} each band has two cells, from 2^{52} on each band is dyadic (Y/X a power of two) and has $J=128\log_2(Y/X)$ cells (an integer), with recorded block cardinalities n_j and a recorded budget L_{band} (part (b) there); its inputs and reserved caps are ceilings of block caps and row coefficients valid for every block of class-three primes at least X and of logarithmic length at most L_{band} (part (c)); and its 25 weights $w_{\mathfrak{t}}$ and 25 reserved weights $v_{\mathfrak{t}}$ satisfy its 105 completion and domination inequalities exactly (part (d); $310\times105=32550$ in all, Certificate 27.18). All of these are recorded band by band in `research/further_2026_10_25/upper/upper_dense_refined.json` (field `rows`), with the polynomials and charges of Proposition 28.14. The root entries d_{L}, k_{LL} are, below 2^{52}, those of the field `even_root_evidence` of the band (Certificate 27.9) and, from 2^{52} on, the completed-root caps (Lemma 27.10); the class-three nonroot caps and the observed caps are those of Certificate 27.13 and Proposition 27.14. Each band evaluates its inputs at points not above its start; on the 45 bands of $[P_3,2^{48})$ the nonroot caps, the observed caps, the one-way repairs and the dense EL law are evaluated at 2^{47}, the rows and the LE mixture at P_3 (Certificate 28.12). Table 32 lists the bands: the 301 bands below 2^{52} in five groups (the 45 bands of $[P_3,2^{48})$ and the 64 bands of each octave $[2^b,2^{b+1})$, $48\le b\le51$) and the nine bands from 2^{52} singly, with the number of bands, their budgets, the six-query LE mixture of Proposition 28.6 (I: the first mixture, II: the refined one), the root entries at the first band of the group, and the parts of the symmetric and skew ledgers at a_{u} that the group carries. The last row is the tail datum of Definition 27.1(e), the band $[2^{64},2^{512})$ of `research/roadmap_2026_10_18/item12/upper_v2_local.json` (field `pieces`, with the tail polynomial `tail_comparison`), whose inputs, caps and local length L_{u} are listed there; the last row of the table gives instead the prices at a_{u}, computed from that datum by Lemma 29.5, over the primes $p\ge P_4$ beyond the bands.

Table 32: The upper bands: root entries at the first band of each group (upper bounds, rounded up; integers exact) and the group's part of S and A at a_{u} (upper bounds of the exact sums, rounded up). Source: `upper_dense_refined.json` (`rows[*]`: `local_budget`, `inputs.L.d`, `reserved_caps.LL`, `six_query_LE_mixture`, `S`, `A`; `tail_S`, `tail_A`); tail datum: `upper_v2_local.json`.

range	bands	L_{band}	LE	d_{L}	k_{LL}	S part	A part
$[P_3,2^{48})$	45	559–566	II	12847.188	9381.13748	$8.3005854\cdot10^{-3}$	$4.3074516\cdot10^{-3}$
$[2^{48},2^{49})$	64	566–578	II	6568.9338	5072.01307	$3.7134746\cdot10^{-3}$	$1.5852814\cdot10^{-3}$
$[2^{49},2^{50})$	64	550–584	II	2556.55962	2070.82868	$1.0835186\cdot10^{-3}$	$4.5997941\cdot10^{-4}$
$[2^{50},2^{51})$	64	555–566	II	1165.55611	942.505851	$3.6696252\cdot10^{-4}$	$1.5517657\cdot10^{-4}$
$[2^{51},2^{52})$	64	566–577	II	612.947237	472.875143	$1.4359461\cdot10^{-4}$	$6.0703815\cdot10^{-5}$

range	bands	L_{band}	LE	d_{L}	k_{LL}	S part	A part
$[2^{52}, 2^{53})$	1	735	I	367	261	$7.9016190 \cdot 10^{-5}$	$3.3283448 \cdot 10^{-5}$
$[2^{53}, 2^{54})$	1	747	I	250	158	$3.8194556 \cdot 10^{-5}$	$1.6084978 \cdot 10^{-5}$
$[2^{54}, 2^{55})$	1	725	I	187	102	$2.0000122 \cdot 10^{-5}$	$8.4210334 \cdot 10^{-6}$
$[2^{55}, 2^{56})$	1	738	I	158	77	$1.1999218 \cdot 10^{-5}$	$5.0515173 \cdot 10^{-6}$
$[2^{56}, 2^{57})$	1	748	I	141	64	$7.5816313 \cdot 10^{-6}$	$3.1818432 \cdot 10^{-6}$
$[2^{57}, 2^{58})$	1	724	I	131	57	$4.9301622 \cdot 10^{-6}$	$2.0688368 \cdot 10^{-6}$
$[2^{58}, 2^{60})$	1	749	I	124	52	$5.6855583 \cdot 10^{-6}$	$2.3856289 \cdot 10^{-6}$
$[2^{60}, 2^{64})$	1	751	I	118	48	$4.0448400 \cdot 10^{-6}$	$1.6955898 \cdot 10^{-6}$
$[2^{64}, P_4)$	1	886	I	113	45	$1.3603023 \cdot 10^{-6}$	$5.7015018 \cdot 10^{-7}$
tail datum	–	L_{u}	–	113	45	$2.9549 \cdot 10^{-70}$	$1.8960 \cdot 10^{-70}$

The other inputs vary from band to band. At the first band of each group and at the last band they are, rounded up (ordinary input (d, f, h, M, t) of each field type φ, then the reserved input; t is the Young allowance t_{Y} of the row, Definition 27.1(c)):

		ordinary					reserved				
band start	φ	d	f	h	M	t	d	f	h	M	t
$167 \cdot 2^{40}$	L	12847.19	93.72509	87.14728	1.012768	0.01275857	10.17352	6.952026	8.755992	1.021794	0.021793
	F	23.24983	16.71542	10.92843	1.023485	0.023484	4.104159	100.8788	2.000002	1.000002	0.000001
	E	24.95023	17.36472	1	1.000059	0.000057323	4.583091	35.32721	1	1.000002	0.000001
2^{48}	L	6568.934	92.27496	87.07506	1.013781	0.01377415	10.05781	6.929649	8.755917	1.019582	0.019581
	F	23.07317	16.68427	10.92838	1.021101	0.0211	4.103961	82.01316	2.000002	1.000002	0.000001
	E	24.63578	17.42361	1	1.000048	0.000046025	4.582943	28.79189	1	1.000002	0.000001
2^{49}	L	2556.56	91.03808	86.98627	1.011528	0.01152402	10.0578	6.89872	8.755826	1.016469	0.016468
	F	23.07316	16.64135	10.92833	1.017748	0.017747	4.103961	58.8855	2.000001	1.000002	0.000001
	E	24.63577	15.22386	1	1.000042	0.000040743	4.582943	20.78013	1	1.000002	0.000001
2^{50}	L	1165.557	90.06929	86.92351	1.009367	0.009364782	10.0578	6.869087	8.755761	1.013434	0.013433
	F	23.07316	16.60042	10.9283	1.014477	0.014476	4.103961	40.17952	2.000001	1.000002	0.000001
	E	24.63576	17.10956	1	1.000024	0.0000223	4.582943	14.30009	1	1.000002	0.000001
2^{51}	L	612.9473	89.3998	86.87915	1.007853	0.007851555	10.0578	6.848651	8.755715	1.011296	0.011295
	F	23.07316	16.57233	10.92827	1.012173	0.012172	4.103961	29.27988	2.000001	1.000002	0.000001
	E	24.63575	17.10566	1	1.000017	0.000015771	4.582943	10.5243	1	1.000002	0.000001
2^{52}	L	367	89.05109	86.84779	1.007437	0.007435899	10.0578	6.843426	8.755683	1.01072	0.010719
	F	23.07316	16.56525	10.92825	1.011552	0.011551	4.103961	26.66535	2.000001	1.000002	0.000001
	E	24.63575	19.22671	1	1.000013	0.00001184	4.582943	9.618578	1	1.000002	0.000001
2^{64}	L	113	87.34425	86.77684	1.000917	0.000915423	10.0578	6.757312	8.75561	1.001327	0.001326
	F	23.07316	16.44806	10.92821	1.00143	0.001429	4.103961	3.344169	2.000001	1.000002	0.000001
	E	24.63575	8.787223	1	1.000002	0.000001	4.582943	1.539769	1	1.000002	0.000001

The reserved caps of the other five field pairs and the observed merge and constant caps of EL22 ($k_{\mathrm{LR}} = k_{\mathrm{RL}} = k_{\mathrm{G}}$, $k_{\mathrm{LC}} = k_{\mathrm{CL}} = k_{\mathrm{C}}$), from which with k_{LL} the 25 reserved state caps are formed, are the same in all bands of each of the following two ranges (upper bounds, rounded up):

range	bands	k_{LF}	k_{LE}	k_{FF}	k_{FE}	k_{EE}	k_{G}	k_{C}
$[P_3, 2^{48})$	45	7.00915	5.65087	3.123771	2.414408	0.000029	2.26882	0.079762
$[2^{48}, P_4)$	265	6.958743	5.637399	3.12365	2.414355	0.000028	2.245406	0.056394

The LE records of every band use the positive mixture of Proposition 28.6 with the coefficients s_{LE}, a_{LE} of the endpoint recorded in the band's field `six_query_LE_mixture`: the refined mixture of `research/further_2026_10_25/upper/dense_six_refined.json` in the 301 bands below 2^{52} and the first mixture of `dense_six.json` in the nine bands from 2^{52} (top-level field `six_query_LE_mixture`; "301 changed bands" in the data means the 301 bands whose mixture differs from the first mixture of `dense_six.json`). The coefficients at the eight endpoints of the two records are (exact; the last column of each mixture counts the bands that use the row; the endpoint 2^{47}, where the extraction coefficients are certified, is used by no band):

	first mixture (I)			refined mixture (II)		
endpoint	s_{LE}	a_{LE}	bands	s_{LE}	a_{LE}	bands
2^{47}	1.904233441086	1.894308281063	–	1.900638747160	1.888824414626	–
$167 \cdot 2^{40}$	1.875430334707	1.865505941926	–	1.872914263008	1.861100843756	45
2^{48}	1.840321288372	1.830397932963	–	1.838921462687	1.827109278261	64
2^{49}	1.802017714597	1.792095635050	–	1.801556462666	1.789745796952	64
2^{50}	1.777366637471	1.767445459998	–	1.777352768400	1.765543176463	128
2^{52}	1.749386716250	1.739466627572	4	1.749755568866	1.737947272967	–
2^{56}	1.729118677881	1.719199405722	3	1.729711616465	1.717904292503	–
2^{60}	1.724091674882	1.714172606843	2	1.724737215461	1.712930134472	–

At a_{u} the ledgers of the bands and of the tail datum sum to $S < 0.013780948065193294$ and $A < 0.006641335659555205$ (`upper_dense_refined.json`, fields `S`, `A`, which include the tail sums $2.9549 \cdot 10^{-70}$ and $1.8960 \cdot 10^{-70}$ of the fields `tail_S`, `tail_A`). The certificate reprices the ledger term by term: a term with coefficient c and stored unit charge π at a_{u}, whose blocks have recorded logarithmic product at most $\ell \le L_{\mathrm{band}}$ (field `product_log_upper`, at most 885.436), contributes $c\,\pi\, e^{2(a-a_{\mathrm{u}})\ell}$ (Lemma 9.3), and the tail beyond P_4 is recomputed at a from the tail polynomial of the tail datum (Lemma 29.5). This gives $S_{\mathrm{u}}(a) < 0.0138661535118$ and $A_{\mathrm{u}}(a) < 0.00668235137593$ (`author.json`, keys `components.upper_S`, `upper_A`; `upper_price` of `research/continuation_2026_10_24/global/closing.py`).

E.6. **Envelope witnesses.** Theorem 25.8 prices a class of selections with n fold-skew and m matching-skew factors on distinct blocks by a positive envelope that passes the exact tests of Lemma 25.3; Corollary 25.10 lists the sixteen classes. The envelopes used are the following.
Constants. The envelopes are certified at Q_3 with the actual feature frame of Lemma 25.1 ((25.2); `research/continuation_2026_10_24/local/actual_features.json`, case $Q = 19951586$):

$$\Gamma_F = \frac{50011197955513770769635756 0603}{5 \cdot 10^{29}} < 1.000223959111, \qquad \Gamma_E = \frac{19951586}{19951585},$$

and the pair correlation $\rho_{\mathrm{pair}} = 501213362368299346064 43 \cdot 10^{-30} < 5.0121336237 \cdot 10^{-8}$, with $\eta_{\mathrm{pair}} = 2\rho_{\mathrm{pair}}/(1-\rho_{\mathrm{pair}}^2)$. The leading skew changes are the signed stars (25.4) with the arm coefficients $\hat{P}$, $\mathtt{t}\hat{P}$, $1-\mathtt{t}$ in raw-arm normalization; a matrix envelope is lifted in the coupling basis of Lemma 25.3(a): the normalized nonnegative couplings M_{ab} with $M_{ab} - \mathbb{J} = V_a V_b^*$ of two singleton labels, P_{12}, P_{01} and $\mathbb{J}$ at a fold factor, and M_{ab}, $\mathbb{J}$ at a matching factor. The coefficients of the collected expansion, cancellations included, give the coupling mass M, and its reflection orbits the invariant orbit mass M^{inv} of Lemma 25.7.
The portfolio. Table 33 lists the sixteen envelopes of Corollary 25.10, with the caps $\kappa_H \ge \kappa_b$ (a dash: $\kappa_b = \kappa_H$), the masses $M_{n,m}$ and $M^{\mathrm{inv}}_{n,m}$ that enter the cost $C_{n,m}$ of Definition 30.1(7), the leading coupon $M^{\mathrm{inv}}_{n,m} e^{tU^{\mathrm{inv}}} t\delta_{\mathrm{s}}$ subtracted there, and the saving $D_{n,m}$ of Definition 30.1(8), all at the exponent a. The witnesses are the exact rationals of the bundle `research/further_2026_10_25/envelopes/exact_background_pairs.json` (keys `pairs.FF`, `pairs.FE`; `general.F1E3`, …, `general.F6E0`; `E_only.aligned`, `E_only.cross`, `E_only.E4`, `E_only.E6`); its rows `general.F0E4` and `general.F0E6` are not used. The masses $M^{\mathrm{inv}}_{n,m}$, coupons and savings are those of `author.json` (keys `invariant_error.orbit_masses`, field `one_delta_invariant_mass`; `invariant_error.leading_coupons`; `discounts`); no class is discarded, and the savings $D_{n,m}$ sum to $0.4568954196753\ldots$ (called the uniform discount in the data, being the saving before the variance credit), so that with $D_{\mathrm{var}} > 9.9585603157 \cdot 10^{-3}$ the total discount, including D_{var}, is $D > 0.4668539799911$. The families are: pair PSD, the singleton-block matrix envelopes of FF and FE; one-, two-, three-, four-fold, the singleton-block matrix envelopes with one to four fold factors; aligned F, the diagonal mask envelopes with the extra averages of Lemma 25.5(f) on the invariant background; diagonal, the diagonal mask envelope F6E0 (a row of label weights with its mask slacks); and E-type, E-averaged, the E-type envelopes of the

matching pairs and of four and six matching factors (Lemma 25.5(g)). The variance credits of Corollary 24.18 use the caps $\kappa_{FF} = \kappa_H(2,0) < 1.029497272$, $\kappa_b^{\mathrm{al}} < 0.666678945$ (the invariant cap of the aligned EE class, computed with the variance credit applied only to selected blocks and no background blocks — the selected-only policy of Section 24), $\kappa_{4,F} = \kappa_H(4,0) < 1.460124417$ and the aligned variance cap $\kappa_{4,E} < 0.666678940$ of E4.

Table 33: The envelopes of the sixteen classes: caps and masses (upper bounds, rounded up, except M^{inv}, a lower bound rounded down), the leading coupon (upper bound) and the saving $D_{n,m}$ at the exponent a (lower bound, rounded down). Sources: `exact_background_pairs.json` (κ_H, κ_b, M); `author.json` (`invariant_error.orbit_masses`, `invariant_error.leading_coupons`, `discounts`).

class	envelope	family	κ_H	κ_b	$M_{n,m}$	$M^{\mathrm{inv}}_{n,m}$	coupon	$D_{n,m}$
$(2,0)$	FF	pair PSD	1.029497272	–	39.0063263	19.503163	$1.0784 \cdot 10^{-3}$	$1.2303539 \cdot 10^{-1}$
$(1,1)$	FE	pair PSD	0.916504741	0.775464445	25.0503754	19.38672	$1.0720 \cdot 10^{-3}$	$2.0735970 \cdot 10^{-1}$
$(1,3)$	F1E3	one-fold	1.228074745	0.671097495	368.776608	208.23445	$1.1514 \cdot 10^{-2}$	$7.1760575 \cdot 10^{-3}$
$(2,2)$	F2E2	two-fold	0.979934896	0.949157677	461.959634	155.95801	$8.6231 \cdot 10^{-3}$	$1.1895431 \cdot 10^{-2}$
$(3,1)$	F3E1	three-fold	1.183096446	1.183096445	549.424581	99.739668	$5.5147 \cdot 10^{-3}$	$1.0426635 \cdot 10^{-2}$
$(4,0)$	F4E0	four-fold	1.460124417	–	771.452195	96.431524	$5.3318 \cdot 10^{-3}$	$3.4410311 \cdot 10^{-3}$
$(1,5)$	F1E5	one-fold	1.247703219	0.647924328	5154.83733	2466.2139	$1.3636 \cdot 10^{-1}$	$2.5996086 \cdot 10^{-5}$
$(2,4)$	F2E4	aligned F	0.989164787	0.962616046	5600.05289	1560.2674	$8.6269 \cdot 10^{-2}$	$5.7362503 \cdot 10^{-5}$
$(3,3)$	F3E3	aligned F	1.204082117	1.204082110	6579.35946	1009.2375	$5.5802 \cdot 10^{-2}$	$1.1474209 \cdot 10^{-4}$
$(4,2)$	F4E2	aligned F	1.504918634	1.504918634	7825.40227	682.24187	$3.7722 \cdot 10^{-2}$	$1.2693722 \cdot 10^{-4}$
$(5,1)$	F5E1	aligned F	1.912219914	–	9354.29766	449.67834	$2.4864 \cdot 10^{-2}$	$6.9930190 \cdot 10^{-5}$
$(6,0)$	F6E0	diagonal	2.454754391	–	11096.4447	346.76389	$1.9173 \cdot 10^{-2}$	$1.5692248 \cdot 10^{-5}$
(EE,al)	EE aligned	E-type	0.993965989	0.666678945	16.0001466	24.98208	$1.3813 \cdot 10^{-3}$	$3.4753658 \cdot 10^{-2}$
(EE,cr)	EE crossed	E-type	1.366025541	0.455341847	16.3923049	21.51666	$1.1897 \cdot 10^{-3}$	$5.6823372 \cdot 10^{-2}$
$(0,4)$	E4	E-averaged	1.319520417	0.481759024	257.37314	287.89272	$1.5918 \cdot 10^{-2}$	$1.5685438 \cdot 10^{-3}$
$(0,6)$	E6	E-averaged	1.354401663	0.461946213	4102.0808	3616.3376	$1.9996 \cdot 10^{-1}$	$4.9330756 \cdot 10^{-6}$

The pair witnesses. The two largest savings are those of the pairs. The FE envelope has 57 variables, a matrix of dimension 15, 4 exact phase gates and 2 norm gates, and 120 nonzero collected couplings; its exact caps and mass are

$$\kappa_H = \frac{45714305585112619885927882720460037640269300 99}{49878962500000000000000000000000000000000000000} < 0.916504741,$$

$$\kappa_b = \frac{96698404843159227922512541864767581313498 1047}{12469740625000000000000000000000000000000000000} < 0.775464445,$$

$$M = \frac{62625941471135539621157902644331144996707}{25000001253033405920748365161075000000000} < 25.0503754.$$

The FF envelope has 73 variables, a matrix of dimension 25, 4 phase and 2 norm gates and 144 couplings, $\kappa_H = \kappa_b < 1.029497272$ and $M < 39.0063263$. Both are rechecked exactly by `research/further_2026_10_25/envelopes/check_exact_background_pairs.json` (status `PASS`); the proof is this finite test, not the numerical optimization that proposed the matrices (Certificate 25.11).

E.7. **Spectral grids and analytic slacks.**

The twelve-label grids. Table 34 lists, for each threshold P of the two records of Certificate 20.13 and each of the five cuts, the class-three negative constant g_P and the positive constant c_P^+ of route A (Theorem 14.21). Rows marked † are the nine thresholds of `research/continuation_2026_10_24/global/middle_grid_new.json`, the others the 26 of `research/frontier_2026_10_17/root/middle12_grid.json`; every piece start is a threshold, and the two rows marked ∘ (2^{24}, 2^{50}) are not. As in the proof of Lemma 20.12, c_P^+ is the constant B^+ of Theorem 14.21(d), computed with the largest g of the four tests of the record (both classes and both signs), while g_P is the class-three value. A piece reads its row: the block caps of L21 on the pieces 1–26 and

of E21 and L10 on every piece (Table 28). The columns L20 and F21, and L21 on pieces 27–33, are route-A values that the reference does not use: the root cuts of L20, and of L21 on the last seven pieces, are those of Sections 20.4 and 21, and the fold cap is k_{D}. The low-domain grid `research/advance_2026_10_24/global/low_domain/middle_grid.json`, with five thresholds below Q_3, belongs to the audit of the local domain and is not read by any piece.

Table 34: The twelve-label grids: g_P and c_P^+ for the five cuts (upper bounds, rounded up). Sources: `middle12_grid.json` and `middle_grid_new.json` (`points[*].cuts`: the class-three `tests[*].g`, `positive_B`).

	L20		L21		F21		E21		L10	
P	g_P	c_P^+	g_P	c_P^+	g_P	c_P^+	g_P	c_P^+	g_P	c_P^+
$2^{24}\circ$	6.3034	515	2.6085	$4.89\cdot10^{-2}$	1.0134	$2.16\cdot10^{-5}$	0.004950	$4.83\cdot10^{-6}$	1.7531	$5.17\cdot10^{-5}$
$19951586^\dagger$	6.2332	513	2.5741	$4.10\cdot10^{-2}$	1.0123	$1.82\cdot10^{-5}$	0.004541	$4.06\cdot10^{-6}$	1.7513	$4.35\cdot10^{-5}$
$21757358^\dagger$	6.2004	511	2.5580	$3.75\cdot10^{-2}$	1.0118	$1.67\cdot10^{-5}$	0.004351	$3.73\cdot10^{-6}$	1.7505	$3.99\cdot10^{-5}$
$23726568^\dagger$	6.1691	510	2.5426	$3.43\cdot10^{-2}$	1.0113	$1.53\cdot10^{-5}$	0.004167	$3.42\cdot10^{-6}$	1.7497	$3.66\cdot10^{-5}$
$25429506^\dagger$	6.1450	509	2.5307	$3.20\cdot10^{-2}$	1.0109	$1.43\cdot10^{-5}$	0.004026	$3.19\cdot10^{-6}$	1.7491	$3.41\cdot10^{-5}$
$25874006^\dagger$	6.1392	509	2.5278	$3.15\cdot10^{-2}$	1.0108	$1.40\cdot10^{-5}$	0.003992	$3.14\cdot10^{-6}$	1.7490	$3.35\cdot10^{-5}$
$28215802^\dagger$	6.1105	508	2.5136	$2.88\cdot10^{-2}$	1.0103	$1.29\cdot10^{-5}$	0.003824	$2.88\cdot10^{-6}$	1.7483	$3.08\cdot10^{-5}$
2^{25}	6.0570	506	2.4870	$2.42\cdot10^{-2}$	1.0095	$1.08\cdot10^{-5}$	0.003509	$2.42\cdot10^{-6}$	1.7469	$2.59\cdot10^{-5}$
$39903170^\dagger$	6.0081	504	2.4625	$2.03\cdot10^{-2}$	1.0087	$9.06\cdot10^{-6}$	0.003221	$2.03\cdot10^{-6}$	1.7457	$2.18\cdot10^{-5}$
$47453134^\dagger$	5.9634	502	2.4401	$1.70\cdot10^{-2}$	1.0080	$7.62\cdot10^{-6}$	0.002956	$1.71\cdot10^{-6}$	1.7446	$1.83\cdot10^{-5}$
$56431604^\dagger$	5.9225	501	2.4194	$1.43\cdot10^{-2}$	1.0073	$6.41\cdot10^{-6}$	0.002714	$1.44\cdot10^{-6}$	1.7435	$1.54\cdot10^{-5}$
2^{26}	5.8851	500	2.4005	$1.20\cdot10^{-2}$	1.0067	$5.39\cdot10^{-6}$	0.002491	$1.21\cdot10^{-6}$	1.7426	$1.30\cdot10^{-5}$
79806340	5.8508	498	2.3831	$1.01\cdot10^{-2}$	1.0062	$4.53\cdot10^{-6}$	0.002287	$1.02\cdot10^{-6}$	1.7417	$1.09\cdot10^{-5}$
94906266	5.8195	497	2.3671	$8.44\cdot10^{-3}$	1.0057	$3.81\cdot10^{-6}$	0.002099	$8.54\cdot10^{-7}$	1.7409	$9.13\cdot10^{-6}$
112863208	5.7908	496	2.3525	$7.09\cdot10^{-3}$	1.0052	$3.21\cdot10^{-6}$	0.001928	$7.18\cdot10^{-7}$	1.7402	$7.68\cdot10^{-6}$
2^{27}	5.7646	495	2.3390	$5.95\cdot10^{-3}$	1.0048	$2.70\cdot10^{-6}$	0.001770	$6.04\cdot10^{-7}$	1.7395	$6.46\cdot10^{-6}$
159612678	5.7405	494	2.3266	$5.00\cdot10^{-3}$	1.0044	$2.27\cdot10^{-6}$	0.001626	$5.08\cdot10^{-7}$	1.7389	$5.43\cdot10^{-6}$
189812532	5.7185	493	2.3153	$4.20\cdot10^{-3}$	1.0040	$1.91\cdot10^{-6}$	0.001494	$4.27\cdot10^{-7}$	1.7383	$4.57\cdot10^{-6}$
225726414	5.6984	493	2.3049	$3.53\cdot10^{-3}$	1.0037	$1.60\cdot10^{-6}$	0.001373	$3.59\cdot10^{-7}$	1.7378	$3.84\cdot10^{-6}$
2^{28}	5.6799	492	2.2953	$2.97\cdot10^{-3}$	1.0034	$1.35\cdot10^{-6}$	0.001261	$3.02\cdot10^{-7}$	1.7374	$3.23\cdot10^{-6}$
319225356	5.6630	491	2.2865	$2.49\cdot10^{-3}$	1.0031	$1.14\cdot10^{-6}$	0.001159	$2.54\cdot10^{-7}$	1.7369	$2.72\cdot10^{-6}$
379625064	5.6475	491	2.2785	$2.10\cdot10^{-3}$	1.0029	$9.52\cdot10^{-7}$	0.001065	$2.14\cdot10^{-7}$	1.7365	$2.28\cdot10^{-6}$
451452826	5.6333	490	2.2711	$1.76\cdot10^{-3}$	1.0026	$8.00\cdot10^{-7}$	0.000980	$1.80\cdot10^{-7}$	1.7362	$1.92\cdot10^{-6}$
2^{29}	5.6203	490	2.2643	$1.48\cdot10^{-3}$	1.0024	$6.73\cdot10^{-7}$	0.000901	$1.51\cdot10^{-7}$	1.7358	$1.62\cdot10^{-6}$
638450710	5.6084	489	2.2580	$1.25\cdot10^{-3}$	1.0022	$5.66\cdot10^{-7}$	0.000829	$1.27\cdot10^{-7}$	1.7355	$1.36\cdot10^{-6}$
759250126	5.5975	489	2.2523	$1.05\cdot10^{-3}$	1.0020	$4.76\cdot10^{-7}$	0.000763	$1.07\cdot10^{-7}$	1.7352	$1.14\cdot10^{-6}$
902905652	5.5875	489	2.2471	$8.77\cdot10^{-4}$	1.0019	$4.00\cdot10^{-7}$	0.000702	$8.98\cdot10^{-8}$	1.7350	$9.59\cdot10^{-7}$
2^{30}	5.5783	488	2.2423	$7.38\cdot10^{-4}$	1.0017	$3.37\cdot10^{-7}$	0.000646	$7.55\cdot10^{-8}$	1.7347	$8.06\cdot10^{-7}$
2^{31}	5.5486	487	2.2267	$3.69\cdot10^{-4}$	1.0012	$1.69\cdot10^{-7}$	0.000466	$3.78\cdot10^{-8}$	1.7340	$4.03\cdot10^{-7}$
2^{32}	5.5277	486	2.2157	$1.84\cdot10^{-4}$	1.0009	$8.41\cdot10^{-8}$	0.000339	$1.89\cdot10^{-8}$	1.7334	$2.02\cdot10^{-7}$
2^{34}	5.5025	485	2.2023	$4.60\cdot10^{-5}$	1.0005	$2.11\cdot10^{-8}$	0.000184	$4.72\cdot10^{-9}$	1.7328	$5.04\cdot10^{-8}$
2^{36}	5.4899	485	2.1956	$1.15\cdot10^{-5}$	1.0003	$5.26\cdot10^{-9}$	0.000106	$1.18\cdot10^{-9}$	1.7324	$1.26\cdot10^{-8}$
2^{40}	5.4804	485	2.1906	$7.17\cdot10^{-7}$	1.0001	$3.29\cdot10^{-10}$	0.000045	$7.37\cdot10^{-11}$	1.7322	$7.87\cdot10^{-10}$
2^{45}	5.4778	485	2.1893	$2.24\cdot10^{-8}$	1.0001	$1.03\cdot10^{-11}$	0.000026	$2.31\cdot10^{-12}$	1.7321	$2.46\cdot10^{-11}$
$2^{50}\circ$	5.4774	485	2.1890	$7.00\cdot10^{-10}$	1.0001	$3.21\cdot10^{-13}$	0.000021	$7.20\cdot10^{-14}$	1.7321	$7.69\cdot10^{-13}$

The grid of the cut L20. Table 35 lists the 45 endpoints of the record `research/advance_2026_10_24/local/l20_refined_Q_grid.json` of Certificate 20.22 (variant with balanced pinned collisions and the pair-domain gap), with the class-three constant $g_{20}(P)$, recorded rounded up to 10^{-12}, and the positive constant $c_{20}^+(P)$; among them are the 33 piece starts, and the endpoints marked $\circ$ are not piece starts. The L20 grid is retained as an inherited comparison table. The selected L20 caps use the collision-gap and physical pair refinements and are printed in Table 28; they are not entries of this older grid. The constant $g_{20}(P)$ of the older grid decreases to $\sqrt{30} = 5.4772255\ldots$ as $P \to \infty$.

Table 35: The L20 grid of the uniform root: $g_{20}(P)$ (exact recorded value) and $c^+_{20}(P)$ (upper bound, rounded up). Source: `l20_refined_Q_grid.json` (`rows[*]`: `p0`, `clean_g`, `positive_B`).

P	g_{20}	c^+_{20}	P	g_{20}	c^+_{20}	P	g_{20}	c^+_{20}
$2^{20}\circ$	5.597878403332	22.968989	2^{25}	5.494591943030	20.486551	451452826	5.481802261912	20.131136
$2^{21}\circ$	5.556683087553	22.042549	39903170	5.493089568286	20.445584	2^{29}	5.481417951164	20.120210
$2^{22}\circ$	5.530588618665	21.417010	47453134	5.491721690831	20.408112	638450710	5.481066248774	20.110197
$2^{23}\circ$	5.513582957074	20.988713	56431604	5.490475565667	20.373828	759250126	5.480744334669	20.101021
11863284°	5.507360103926	20.827188	2^{26}	5.489339762357	20.342456	902905652	5.480449640996	20.092611
$2^{24}\circ$	5.502257863941	20.692622	79806340	5.488304013851	20.313743	2^{30}	5.480179828315	20.084904
19000000°	5.500656295909	20.649967	94906266	5.487359085531	20.287460	2^{31}	5.479309452639	20.059987
19951586	5.500057067029	20.633954	112863208	5.486496658956	20.263397	2^{32}	5.478696554108	20.042393
20500000°	5.499731546104	20.625244	2^{27}	5.485709232570	20.241364	2^{34}	5.477959529989	20.021182
21000000°	5.499446356287	20.617605	159612678	5.484990032466	20.221188	2^{36}	5.477592169160	20.010588
21757358	5.499033927799	20.606547	189812532	5.484332935655	20.202709	2^{40}	5.477317151730	20.002647
23726568	5.498058807639	20.580347	225726414	5.483732401259	20.185783	2^{45}	5.477241760190	20.000468
25429506	5.497311649100	20.560218	2^{28}	5.483183410461	20.170278	$2^{48}\circ$	5.477231297192	20.000166
25874006	5.497129278599	20.555298	319225356	5.482681412979	20.156074	$2^{49}\circ$	5.477229621196	20.000117
28215802	5.496243048410	20.531349	379625064	5.482222279985	20.143060	$2^{50}\circ$	5.477228436098	20.000083

The positive roots. Table 36 lists the data of the first four selected positive roots of Definition 21.1, with the remaining three in Table 37 (Certificate 21.25): the family and the parameter u; the selected constants g, c^+ of the chamber cut L20 and layer cut L21 (Certificate 21.25) at the piece start; their block caps at $L_{\mathrm{loc}} = 585$, which rounded up are the weights w_{D} and v_{L} of the piece; the per-prime marginal data d^2_{u} and the range ratio R_{u} of Lemma 24.7(iv); and the restoration count n with $M = \mathrm{up}(R^n_{\mathsf{u}})$ and $\chi = \mathrm{up}((1 + d^2_{\mathsf{u}})^n - 1)$ (Table 27).

Table 36: The seven selected positive roots (exact recorded rationals, or upper bounds rounded up). Sources: `refined_L570_source.json`, field `further_root_construction.changes`, and its native spectra. Pair-square parameters use physical pair interpolation and the resummed L21 cap; the coherent law uses the finite 39-term chamber and layer bounds with single-pin refinement.

piece start	2^{30}	2^{31}	2^{32}	2^{34}
law	pair squares	pair squares	pair squares	pair squares
u	711/4096	1021/4096	1455/4096	833/1024
L20: g	5.476619678296	5.474335244881	5.471257426364	5.464133834236
L20: c^+	25.456659470084	27.62384303	30.397461416086	36.835698528597
L20: block cap d	5.476623331076	5.474337162024	5.471258447645	5.464134125055
L21: g	6.313078885225	6.315256713466	6.312562621834	6.307298870667
L21: c^+	3.98888909231	3.962812801799	3.927444212402	3.87735454966
L21: block cap k_{L}	6.313079545011	6.31525703074	6.312562774077	6.307298906003
d^2_{u}	$2.37042794010 \cdot 10^{-8}$	$1.28617784042 \cdot 10^{-8}$	$7.07684232492 \cdot 10^{-9}$	$2.14402447541 \cdot 10^{-9}$
R_{u}	1.823094863666	2.255010861646	2.937135030633	6.928913249252
n	27	26	25	24
$M = \mathrm{up}(R^n_{\mathsf{u}})$	$1.101160 \cdot 10^{7}$	$1.520074 \cdot 10^{9}$	$4.989969 \cdot 10^{11}$	$1.499557 \cdot 10^{20}$
$\chi = \mathrm{up}((1 + d^2_{\mathsf{u}})^n - 1)$	$6.400158 \cdot 10^{-7}$	$3.344063 \cdot 10^{-7}$	$1.769211 \cdot 10^{-7}$	$5.145659 \cdot 10^{-8}$

Table 37: The remaining selected root laws; notation and sources as in Table 36.

piece start	2^{36}	2^{40}	2^{45}
law	pair squares	pair squares	coherent 39
u	3851/4096	4049/4096	67/40
L20: g	5.460775686025	5.457521105807	5.434450251962
L20: c^+	37.085210712622	37.019255725148	983.32475360966
L20: block cap d	5.460775755114	5.457521109684	5.434450254811
L21: g	5.426082230198	4.646587567237	2.869197519095
L21: c^+	1.143781717023	0.075183415124	0.460807795864

piece start	2^{36}	2^{40}	2^{45}
L21: block cap k_{L}	5.426082232316	4.646587567244	2.869197519096
d_{u}^2	$5.39648853329 \cdot 10^{-10}$	$3.36686302618 \cdot 10^{-11}$	$2.79476882614 \cdot 10^{-11}$
R_{u}	8.315762775019	8.868202502427	59781.69483119
n	22	20	18
$M = \mathrm{up}(R_{\mathrm{u}}^n)$	$1.729204 \cdot 10^{20}$	$9.051323 \cdot 10^{18}$	$9.511041 \cdot 10^{85}$
$\chi = \mathrm{up}((1+d_{\mathrm{u}}^2)^n - 1)$	$1.187228 \cdot 10^{-8}$	$6.733727 \cdot 10^{-10}$	$5.030584 \cdot 10^{-10}$

The completion-variance grid. The 47-threshold grid stored with the estimate of Lemma 14.15 (`research/push_2026_10_17/sparse/local_variance_grid.json`) belongs to a middle law that the construction does not use, and none of its numbers enters the proof: the construction uses the estimate itself, inside the root caps of the upper bands from 2^{52} on, which are evaluated directly at the band starts (Certificate 27.11).

Analytic slacks. Table 38 gives the exact values of the strict inequalities (S1)–(S7) of Definition 8.1 at the parameters of Table 23; none of them depends on a. The gates of the analytic transfer that depend on a hold on the whole domain $0 < a \le 9/10000$ (`research/advance_2026_10_24/global/analytic_domain.py`, record `analytic_domain.json`, with the checker of Certificate D.2): the induction slack is $\sigma - a \ge \sigma - 9/10000 = 103/330000$, and at the selected exponent $\sigma - a = 65195254/201416015625$ (Certificate D.3). The same record checks the prefix gate of Certificate D.3: on the domain the exponent $513 - m_{\mathrm{cl}} + 1024 m_{\mathrm{cl}} a$ is at most $-80783/625$ and the price of the prefix together with the price of the large-prime tail is less than $1.2332 \cdot 10^{-39}$; at the selected exponent $2^{513 - m_{\mathrm{cl}} + 1024 m_{\mathrm{cl}} a} = 2^{-1380939795479/6103515625} < 8 \cdot 10^{-69}$ (Lemma 19.3 and Remark 19.4).

Table 38: Analytic slacks: exact rationals (decimals rounded down); the tail coefficients and gates are upper bounds, independent of a. Sources: `refined_L570_source.json` (`analytic`: `inherited_transfer_slacks`, `inherited_kernel_slacks`, `tail_gate_values`; `upper.tail_mixing`) and `analytic_domain.json`.

condition	quantity	value
(S1)	$\gamma\beta/2 - \tau - 72\delta - \sigma$	$2260122363/58960000000000 > 3.833 \cdot 10^{-5}$
(S2), (S3)	$\gamma t_{\mathrm{an}} - 72\delta - \sigma$ and $\beta/6 - t_{\mathrm{an}}(1/2 - \gamma) - 72\delta - \sigma$	both $3878506121/1320000000000 > 2.938 \cdot 10^{-3}$
(S4)	$\tau/4 - \ell_0 - \sigma$	$6997789/221100000000 > 3.164 \cdot 10^{-5}$
(S5)	$1 - 2\tau - 6\beta$; $\beta/2 - \tau$; $\beta/2 - t_{\mathrm{an}}$; $\beta(1-2\tau) - \beta/2$	1387/2010; 161/8040; 1/120; 197/8040
gate	$\sigma - a$	$65195254/201416015625 > 3.236 \cdot 10^{-4}$
	$\sigma - 9/10000$ (domain $0 < a \le 9/10000$)	$103/330000 > 3.121 \cdot 10^{-4}$
(S6)	minor arcs (actual, model); major arcs (model, actual)	1/600, 37/600; 97/150, 1/600
	arc disjointness; centres inside; centre separation	49/100; 19/200; 117/200
(S7)	C_1, C_2, C_3 of Lemma 29.3	$1.197601 \cdot 10^3$, $6.093051 \cdot 10^5$, $1.417600 \cdot 10^8$
	$2^{16} S_{72}(2^{512})$; $2^{32640} S_{72}(2^{65536})$	$6.778176 \cdot 10^{-70}$; $3.519431 \cdot 10^{-36}$

Consequences and what to read next. Read next: the statements that use these data as hypotheses are Definitions 8.1, 20.1 and 27.1, Lemmas 20.12, 22.1 and 23.24, Equation (24.11), Proposition 26.4, and Theorem 25.8; the stored records authenticated by the baseline certificate (Sections D.2 and D.4) are the following. In `research/push_2026_10_25/`: `global/refined_L570_source.json` (pieces, weights, spectra, charges, the root plan), `global/refined_L570_charges.json` (parts, cells, charges and sums), `global/refined_parent.json` (the positive roots), and `artifacts/author.json`, field `result` (raw totals, piece log caps, orbit masses, coupons, savings, the upper prices at a). The unchanged upper and envelope inputs remain in `research/further_2026_10_25/`: `upper/upper_dense_refined.json`, `upper/dense_six.json`, `upper/dense_six_refined.json`, `envelopes/exact_background_pairs.json` and `envelopes/check_exact_background_pairs.json`. In `research/advance_2026_10_24/`: `local/l20_refined_Q_grid.json` (the L20 grid), `global/final_budget_partitions/` (the histogram and its inputs) and `global/analytic_`

`domain.json` (analytic gates and the prefix). In `research/continuation_2026_10_24/`: `local/actual_features.json` (frame) and `global/middle_grid_new.json` (nine thresholds of the twelve-label grid). And `research/frontier_2026_10_17/root/middle12_grid.json` (the other 26 thresholds) and `research/roadmap_2026_10_18/item12/upper_v2_local.json` (the tail datum, fields `pieces` and `tail_comparison`). The columns of Table 29, of the table that follows it and of Table 31, and the piece values of Table 27 are the read-only evaluations, in their own outward rational arithmetic, of `research/advance_2026_10_24/local/low_domain.py` (`raw_ledgers`) and `research/advance_2026_10_24/global/piece_data_beta.py` (`compute`, modular root variant) on the native source; their sums times t reproduce `author.json` (`raw_totals`) exactly.

Appendix F. The exponent from the OpenAI parameters as displayed

What this appendix proves and how to read it. Theorem 1.1 of [52] asserts $|A| \le CN^{1-c}$ for an unspecified absolute $c > 0$ (their c is our ε). This appendix evaluates c for *one displayed choice* of their parameters: the fixed construction of their §3.1, the threshold P beyond which their conditional-mixing estimate (a bound on conditional correlations given the other labels) applies to the prime cutoff, and their stated lower bound $\rho \ge M_0^{-23}$ for the proportion ρ of residue classes modulo M_0 that are good (admit the required square increments, their Lemma 8.1). Equation numbers such as (3.16) and (9.13) refer to [52], not to this paper. Two scope warnings are essential. First, this is not a claim about the best exponent obtainable from their method with different parameters or sharper local estimates; it quantifies only the choices as displayed. Second, the integers d and e_0 below are exact, but the prime product in M_0 is left unevaluated and logarithms are asymptotic approximations, not interval certificates. Nothing in Parts I–V depends on this appendix; it justifies only the comparison in the abstract and Section 1.4. The computation is reproduced by a short standard-library program in the repository accompanying this paper (`analysis/openai_constant/compute_openai_constant.py`); rerunning it reproduces `analysis/openai_constant/openai_constant_results.json` byte for byte.

Result. For this specialization the exponent is, symbolically,

$$\varepsilon_{\mathrm{OA}} = \frac{1}{d}\cdot\frac{-\log(1-\frac{3}{4}M_0^{-23})}{4\log M_0}, \qquad M_0 = 8\prod_{3\le p<P} p, \qquad P = \lceil (C_{\mathrm{lead}}+C_{\mathrm{mix}})^{32}\rceil, \tag{F.1}$$

with $d = (24!)^{161}$ and $C_{\mathrm{lead}}, C_{\mathrm{mix}}$ as in (F.5) below. At the level of the asymptotic estimates proved below,

$$\varepsilon_{\mathrm{OA}} = 2^{-2^X}, \qquad X = 96e_0 + 16\log_2 e_0 + \log_2\frac{23}{\ln 2} + 176 + o(1) = 96e_0 + 307462.2929\ldots, \tag{F.2}$$

where

$$e_0 = (24!)^{161}\sum_{j=81}^{161}\binom{161}{j}(24!-1)^{161-j} = 2.01561769398235743936\ldots\times 10^{5781} \tag{F.3}$$

is the exact number of directed square constraints (edges) in their graph $G_\emptyset$ (their 'leading list graph': vertices the d tuples of orderings, with an edge $v\to w$ when w agrees with the prescribed translate of v in at least 81 of the 161 blocks; Section F.1); e_0 has 5782 decimal digits. In (F.2) the $o(1)$ term, and the $O(\cdot)$ terms below, are as the corresponding parameter (e_0, P or M_0) tends to infinity in the estimates of [52]; at the displayed finite values they are not enclosed (second scope warning). At that asymptotic level $\log_{10}\log_{10}\log_{10}(1/\varepsilon_{\mathrm{OA}})$ is 5782.7652891675… (digits of the asymptotic expression, not certified digits of the exact defined value), so

$$\varepsilon_{\mathrm{OA}} \approx 10^{-10^{10^{5782.7653}}}, \tag{F.4}$$

a triple exponential. The decimals in the top exponent matter: replacing 5782.7653 by 5782 changes the value by many orders of magnitude on a double-logarithmic scale, so (F.4) should always be quoted with them, or as $\log_{10}\log_{10}\log_{10}(1/\varepsilon_{\mathrm{OA}}) \approx 5782.77$. Using the exact good-residue proportion ρ in place of $\rho \ge M_0^{-23}$ changes only the additive constant in X (to $96e_0+288250.69\ldots$); the tower height and leading term $96e_0$ are unchanged.

F.1. **The fixed construction.** Their §3.1 and (3.1) fix $h = 24$, $t_0 = 80$ and $\ell = 2t_0 + 1 = 161$ blocks. Write $K = 24! = 620448401733239439360000$ for the number of orderings in one block, $d = K^\ell = (24!)^{161}$ ($\log_{10} d = 3830.6256\ldots$) for the number of their points (their analogue of slots; their 161 blocks are blocks of the ordering construction, not prime blocks of this paper), $r = K/2$, and $\delta = 1/(10h^2) = 1/5760$. The remaining displayed quantities are the marked-cut bound $\kappa = \frac{9}{10}\delta^{h-1} \approx 2.9\cdot 10^{-87}$ (their Lemma 3.3; κ is used below only through the constant c_* in P_{ref}), the index-weight bound $B = \exp(d(t_0+1)\binom{h}{2}\delta/2)$ with $\log_2 B = 1.1823\cdot 10^{3831}$ (their (3.8)), and the out-degree $\Delta = \sum_{j\ge 81}\binom{161}{j}(K-1)^{161-j}$ ($\log_{10}\Delta = 1950.6788\ldots$) in $G_\emptyset$, so that $e_0 = d\Delta$ as in (F.3). The count for Δ is exact: in each block a vertex w either equals the one prescribed translate of v in their definition of $G_\emptyset$ (their §3.1; one choice) or not ($K-1$ choices), and an edge $v \to w$ requires at least $t_0 + 1 = 81$ matching blocks.

F.2. **The prime threshold P.** Their proof takes one integer P exceeding every threshold (Proposition 3.1, Lemma 3.6, (7.1) of [52]). In $\log_2$ size the candidates are: P_{ref}, the threshold of their (3.16) (its constants C_*, c_*, n are those of their Proposition 3.1; the formula is not needed here), for which the program computes $\log_2 P_{\mathrm{ref}} = 2.26\cdot 10^{3835}$ (well below 2^{e_0} on a logarithmic scale in the sense checked by the program); $P_{\mathrm{mass}} = (2^{e_0+1}e_0)^2$ with $\log_2 P_{\mathrm{mass}} = 2e_0 + 38412.2$ (their (3.19)); $C_{\mathrm{exc}}^{16/17}$ with $C_{\mathrm{exc}} = c_0^{-1}\binom{161}{81}B$, $c_0 = 2^{-e_0-1}$ (their (3.22)), for which the program computes $\log_2 C_{\mathrm{exc}} = e_0 + 1.1823\cdot 10^{3831}$; the Lemma 3.6 threshold $(C_{\mathrm{lead}} + C_{\mathrm{mix}})^{32}$; and the Gauss-bound threshold of their (7.1), with $\log_2$ below 7. The Lemma 3.6 threshold is the largest of these: the program checks $\log_2 P_{\mathrm{ref}} < e_0$ and $\log_2 C_{\mathrm{exc}} < 2e_0$, and with $\log_2 P_{\mathrm{mass}} = 2e_0 + 38412.2$ each candidate is below the value $96e_0 + 176 + 16\log_2 e_0$ of (F.6) below. There

$$C_{\mathrm{lead}} = c_0^{-1}(2^{e_0+1} + 2^{2e_0+2})\sqrt{32e_0}, \qquad C_{\mathrm{mix}} = 2^\ell B/c_0, \tag{F.5}$$

so $\log_2 C_{\mathrm{lead}} = 3e_0 + 5.5 + \frac{1}{2}\log_2 e_0 + O(2^{-e_0})$ and $C_{\mathrm{mix}}/C_{\mathrm{lead}} < 2^{-e_0}$. The requirement $C_{\mathrm{cond}}p^{-1/16} \le p^{-1/32}$ for $p \ge P$, with $C_{\mathrm{cond}} = C_{\mathrm{lead}} + C_{\mathrm{mix}}$, gives $P = \lceil C_{\mathrm{cond}}^{32}\rceil$, i.e.

$$\log_2 P = 96e_0 + 176 + 16\log_2 e_0 + O(2^{-e_0}). \tag{F.6}$$

The mechanism is worth stating in words: the leading component of their tuple law has mass about 2^{-e_0}, and the conditional-mixing estimate divides by that mass. Even with the exact degree in place of the crude bound used there, this division alone forces $\log_2 P \ge 32e_0$ at this step. A different mixing estimate need not contain the factor $c_0^{-1} = 2^{e_0+1}$; removing this normalisation is one of the structural changes in the present paper (Section 1.4).

F.3. **Small modulus, good residues, and the exponent.** Write $M_0 = 8\prod_{3\le p<P} p$, so $\ln M_0 = \ln 4 + \vartheta(P-1) = P(1 + O((\log P)^{-3}))$ by explicit Chebyshev bounds, where ϑ is Chebyshev's function. Their Lemma 8.1 gives $\rho \ge M_0^{-23}$ for the good-residue proportion. With their parameters $\beta = 10^{-5}$, $\gamma = 1/64$, $\tau = \beta\gamma/(1000(d+1))$ and $\sigma_0 = \gamma\tau/4$ (their (6.1), (7.3); these letters are OpenAI's, not parameters of this paper), their (9.13) sets their a_0 (not the evaluation exponent a_0 of this paper) to $\min\{\frac{1}{4}, \sigma_0/2, \log(1/r_{\mathrm{OA}})/(4\log M_0)\}$ with $r_{\mathrm{OA}} = 1 - \frac{3}{4}\rho$ (not the $r = K/2$ above), and then $\varepsilon_{\mathrm{OA}} = a_0/d$. The analytic entry alone would give $\sigma_0/(2d) = 1.711\ldots\cdot 10^{-7674}$, enormous compared with the residue entry, so the third entry is the minimum. Hence

$$\log_2\frac{1}{\varepsilon_{\mathrm{OA}}} = 23\log_2 M_0 + \log_2 d + \log_2\frac{16}{3} + \log_2\ln M_0 + O(M_0^{-23}),$$

and hence

$$X = \log_2 \log_2(1/\varepsilon_{\mathrm{OA}}) = \log_2 P + \log_2(23/\ln 2) + O\!\left((\log P)^{-3} + \frac{\log P + \log d}{P}\right).$$

The first error term retains the relative error in $\ln M_0$; the second comes from the additive logarithmic terms before taking the outer logarithm. Together with (F.6), these are $o(1)$ in (F.2). Converting bases gives $\log_{10}\log_{10}\log_{10}(1/\varepsilon_{\mathrm{OA}}) = 5782.7652891675\ldots$, i.e. (F.4); at the displayed values the omitted asymptotic corrections change this triple logarithm by about 10^{-5779} (the program's estimate, not an interval enclosure), so the quoted 5782.77 is unaffected.

For comparison, the exact proportion is $\rho = 8^{-23}\prod_{3\le p<P} R_{24}(p)$ with

$$R_{24}(p) = \frac{(p-1)^{24} + (p-1)\sum_{j=0}^{12}\binom{24}{2j}(\epsilon_p p)^j}{(2p)^{24}}, \qquad \epsilon_p = (-1)^{(p-1)/2},$$

the cycle-closing probability for 24 uniform residue labels with nonzero-square increments. This gives $\log_2(1/\rho) = 24\pi(P) + O(\log\log P)$ and therefore $X = 96e_0 + 288250.69\ldots$ in place of (F.2): the same tower with a different additive constant, invisible in (F.4) at the displayed precision.

The implied constants in their Propositions 6.1, 7.2, 8.2 and 9.1 affect only the prefactor C and the starting scale; they never enter a_0, which is fixed before the induction. They therefore leave no unspecified parameter in (F.1). This does not make (F.4) an exact numerical evaluation (the product defining M_0 has about $P/\log P$ factors and is never formed), nor an optimal choice within their framework.

Consequences. Nothing in Parts I–V uses this appendix. Its chain $e_0 \to c_0 \to C_{\mathrm{cond}} \to P \to M_0 \to \rho \to a_0$ shows where the tower comes from: not from the degree d alone, but from dividing by the mass c_0 of the leading graph in the mixing step, which forces $\log_2 P \ge 32e_0$, and then dividing by $\log M_0$, of size P, in their a_0. The present paper removes the first normalisation by relative (density-normalised) estimates and makes the factor coming from good residues independent of M_0 by coupling; these two changes remove the division by c_0 in P and the division by $\log M_0$ in their a_0, i.e. two of the three exponentials in (F.4), before any local constant is optimised; see Section 1.4.

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