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Closing the identification step in the GFF limit: Uniqueness of the zero-slope six-vertex phase for $-1 \leq \Delta < \frac{1}{2}$

Yan Ru Pei

yanrpei@gmail.com

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Abstract

For the six-vertex model with $a = b = 1$ and $c \in (1, 2]$, that is $-1 \leq \Delta < \frac{1}{2}$, we prove that the plane measure, the limit of balanced torus measures, is the only translation-ergodic gradient Gibbs measure of slope zero, and hence the unique translation-invariant minimiser of the specific free energy. Both statements had been asserted by reference to Sheffield's uniqueness theory, which is proved for nearest-neighbour potentials; we show that for $c \neq 1$ the six-vertex weights, which couple faces across the diagonals of each vertex, are not a nearest-neighbour potential in height coordinates. The proof extends Sheffield's cluster swapping to these diagonal interactions by means of residual energies on the diagonals, shows with the Lanford–Ruelle theorem for subshifts of finite type and a Burton–Keane argument that two distinct ergodic zero-slope phases are gradients of Gibbs measures on heights, and shows from the spin representation in the form given by Duminil-Copin, Kozlowski, Lammers and Manolescu that the plane measure is not.

This supplies a step that we found missing in the chain of recent results on the Gaussian free field limit of the six-vertex height function. An unrefereed manuscript of OpenAI states the full-plane limit for every $c \in (0, 2]$; for $\sqrt{2} \leq c < 2$ it identifies a state built from nearly balanced sectors with the plane measure through the characterisation above. Our result gives that identification, given two properties of the state stated in the manuscript. We also check that the manuscript and the work of Averous, Duminil-Copin, He, Lammers and Manolescu concern the same measure, so that their convergence in domains holds, for $c \in [1, 2]$ and with constants depending on Δ, wherever the manuscript's full-plane theorem holds.

Contents

1 Introduction **2**

2 The six-vertex model as a gradient model **7**

3 Swaps and the case distinction **10**

4 The fold, and the proof of Theorem 1.4 **15**

5 The plane measure: proof of Theorem 1.5 **24**

6 Proofs of Theorem 1.2 and Corollaries 1.3 and 1.6 **27**

7 The limit in domains **28**

A Two tools 32

1 Introduction

1.1 Where things stand

The six-vertex model on $\mathbb{Z}^2$ assigns an arrow to every edge so that each vertex has two incoming and two outgoing arrows. We take the weights $a = b = 1$ and $c > 0$: a vertex has weight c if its two horizontal arrows both point into it or both point out of it, and weight 1 otherwise. We write $\Delta := 1 - c^2/2$, so that $c \in [1, 2]$ corresponds to $\Delta \in [-1, \frac{1}{2}]$. An arrow configuration has a *height function* h on the faces of $\mathbb{Z}^2$, defined up to an additive constant: adjacent faces differ by 1, and the face to the left of an arrow is the higher one.

For these weights, three recent works together come close to proving that the height function converges to a Gaussian free field throughout $-1 \le \Delta \le \frac{1}{2}$:

- Duminil-Copin, Kozlowski, Lammers and Manolescu [5] construct the *plane measure* $\mathbb{P}_c$, the limit of the model on a torus conditioned to have no net horizontal flux (Definition 1.1), work with a spin representation of it that goes back to Lis [13, 14], and prove the Gaussian free field limit of its height function for $-1 \le \Delta \le -\frac{1}{2}$.
- Averous, Duminil-Copin, He, Lammers and Manolescu [1] prove that the height function in discrete approximations of simply connected domains converges to a Gaussian free field at every Δ for which the full-plane correlations of $\mathbb{P}_c$ converge to those of a Gaussian free field.
- A manuscript of OpenAI [15], which is not refereed, states that full-plane input for every $c \in (0, 2]$, computing correlations by Bethe ansatz.

Figure 1 shows how these pieces fit. Checking them against each other, we found one step that is not proved in the literature. For $\sqrt{2} \le c < 2$ the Bethe ansatz of [15] is carried out in sectors that are only nearly balanced, and the state ρ_c it produces is identified with $\mathbb{P}_c$ by citing a characterisation of $\mathbb{P}_c$ as the unique minimiser of the specific free energy. That characterisation, and uniqueness of the zero-slope phase more generally, is stated in [5] as a remark without proof, and by Ryan [16] with a sketch that applies Sheffield's uniqueness theory for simply attractive potentials [17]. Sheffield's theory is proved for potentials that couple nearest neighbours, and the six-vertex model is not of that form in height coordinates when $c \neq 1$ (Proposition 2.3 and Figure 2): its weight is an interaction along the diagonals of every vertex.

The main result of this paper proves the uniqueness for $1 < c \le 2$ (Theorem 1.2), and with it the characterisation (Corollary 1.3). This supplies the missing identification, given two properties of the near-balanced state stated in [15] (Corollary 1.6). The remaining checks, that the three works concern the same measure and that one step of [1] at positive mesh follows from their hypotheses, are in Section 7.

1.2 Main results

A probability measure on arrow configurations is a *gradient Gibbs measure* of the six-vertex model if, for every finite set Λ of faces, the conditional law of the heights on Λ given the heights off Λ is the six-vertex weight restricted to the admissible fillings (Section 2; this is Sheffield's notion [17] for the six-vertex potential). Such a measure is *smooth* if it is the gradient of a Gibbs measure on heights and *rough* otherwise [17, p. 10]. For a measure invariant under the translations of $\mathbb{Z}^2$, the

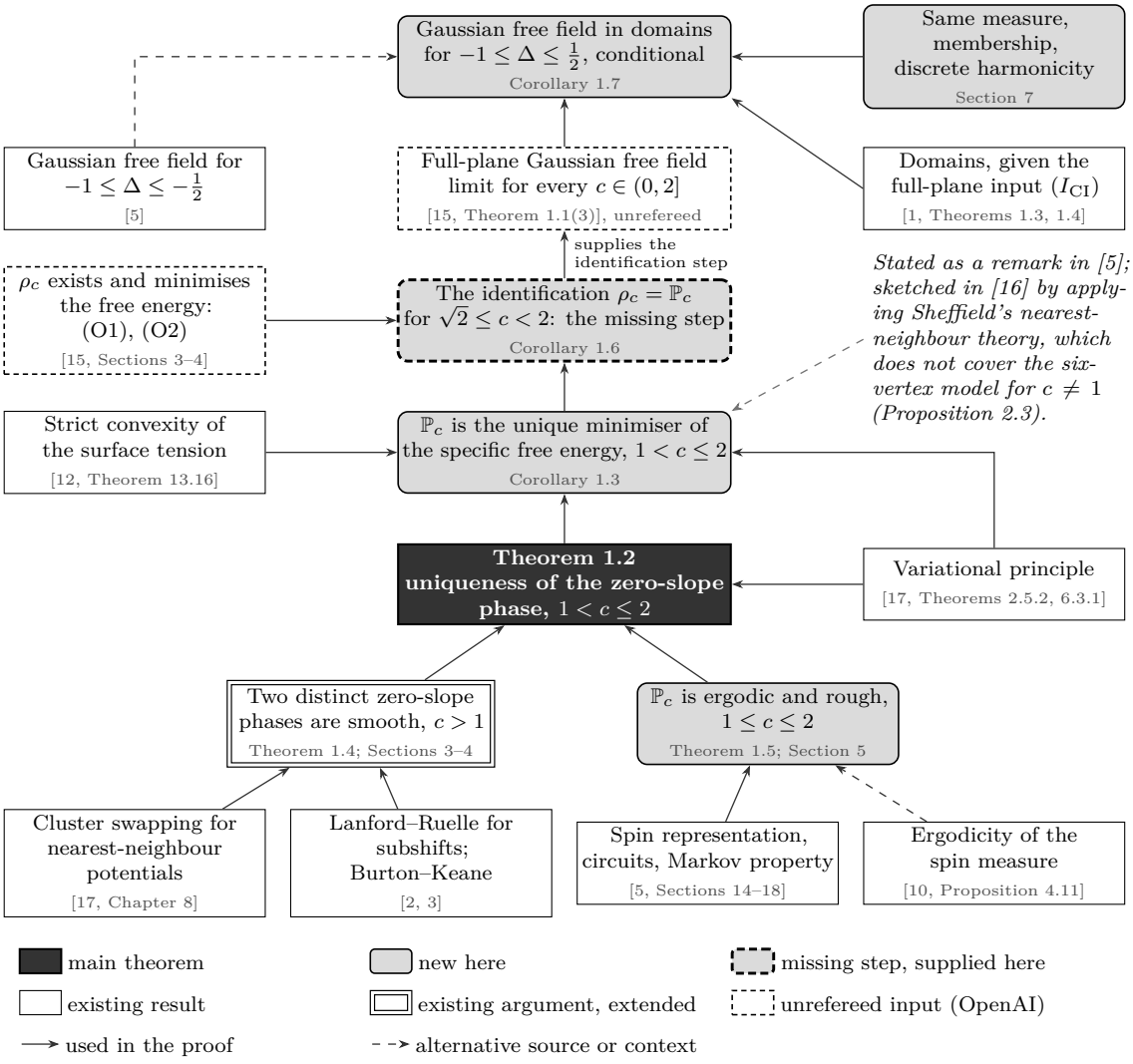


Figure 1: How the pieces fit. Read from the bottom up: an arrow means that its source is used to prove its target; only the main dependencies are drawn. Box styles give provenance, and grey labels give sources and where each item is proved. Here $c \in [1, 2]$ and $\Delta = 1 - c^2/2$, so that $c = 1, \sqrt{2}, \sqrt{3}, 2$ correspond to $\Delta = \frac{1}{2}, 0, -\frac{1}{2}, -1$; ρ_c is the state constructed in [15] from nearly balanced sectors, (O1) and (O2) are the two properties of it that we use (Section 1.3), and a *phase* is a translation-ergodic gradient Gibbs measure. The one step of the chain from the plane measure to the Gaussian free field that we found not to be proved in the literature is the identification in the dashed rounded box; it matters for $-\frac{1}{2} < \Delta \le 0$, and we supply it given (O1) and (O2). For $-1 \le \Delta \le -\frac{1}{2}$ the full-plane input is the main theorem of [5]. Everything above the identification is as strong as its inputs, some of which come from an unrefereed manuscript.

slope is the vector of mean height increments $(\mathbb{E}[h(e_1) - h(0)], \mathbb{E}[h(e_2) - h(0)])$. We call a gradient Gibbs measure that is invariant and ergodic under the translations of $\mathbb{Z}^2$ a *phase*.

For integers $N \geq 1$ and L even, let $\mathbb{P}^{\mathrm{bal}}_{N,L}$ be the six-vertex measure on the torus with N columns and L rows conditioned on every vertical column of horizontal edges having equally many left and right arrows.

Definition 1.1 (The plane measure)**.** For $c \in [1, 2]$,

$$\mathbb{P}_c := \lim_{L \to \infty,\ L \text{ even}} \lim_{N \to \infty} \mathbb{P}^{\mathrm{bal}}_{N,L}$$

in the local weak topology. The limit exists and is invariant under the automorphisms of $\mathbb{Z}^2$ by [5, Theorem 2.2], where it is denoted $\mathbb{P}_{\mathbb{Z}^2}$.

Theorem 1.2 (Uniqueness of the zero-slope phase)**.** *Let $a = b = 1$ and $1 < c \leq 2$. Every gradient Gibbs measure of the six-vertex model that is invariant and ergodic under the translations of $\mathbb{Z}^2$ and has slope 0 is equal to $\mathbb{P}_c$.*

The *specific free energy* of a translation-invariant measure μ on arrow configurations is $\mathrm{SFE}(\mu) = -\mathsf{h}(\mu) - \mathbb{E}_\mu[\log w_0] + \mathrm{const}$, with h the entropy per face and w_0 the weight of the vertex at the origin (Section 2.3).

Corollary 1.3 (Variational characterisation)**.** *Let $a = b = 1$ and $1 < c \leq 2$. The plane measure $\mathbb{P}_c$ is the only probability measure on arrow configurations that is invariant under the translations of $\mathbb{Z}^2$ and minimises the specific free energy.*

Corollary 1.3 follows from Theorem 1.2 in three steps (Section 6): almost every ergodic component of a minimiser is a minimiser; it has slope 0 because the surface tension is strictly convex [12, Theorem 13.16]; and it is a Gibbs measure by Sheffield's variational principle [17, Theorem 2.5.2]. A translation-invariant measure that is *flip-symmetric*, that is, invariant under $h \mapsto -h$, has slope 0, so Theorem 1.2 also gives, in the gradient formulation, the form of the remark in [5] that concerns flip-symmetric ergodic Gibbs measures. At $c = 1$ the conclusion of Theorem 1.2 follows from Sheffield's theorem for nearest-neighbour potentials (Remark 6.1).

Theorem 1.2 has two ingredients.

Theorem 1.4 (Two distinct zero-slope phases are smooth)**.** *Let $a = b = 1$ and $c > 1$. Let $\mu_1 \neq \mu_2$ be gradient Gibbs measures of the six-vertex model that are invariant and ergodic under the translations of $\mathbb{Z}^2$ and have slope 0. Then μ_1 and μ_2 are both smooth.*

Theorem 1.5 (The plane measure is ergodic and rough)**.** *Let $a = b = 1$ and $1 \leq c \leq 2$. The plane measure $\mathbb{P}_c$ is a gradient Gibbs measure of the six-vertex model, it is trivial on the tail σ-field and hence ergodic under the translations of $\mathbb{Z}^2$, it has slope 0, and it is rough: no Gibbs measure on heights, translation-invariant or not, has gradient law $\mathbb{P}_c$.*

Theorem 1.2 follows at once: if $\mu \neq \mathbb{P}_c$, both would be smooth by Theorem 1.4, but $\mathbb{P}_c$ is rough. Theorem 1.4 is the six-vertex form of the part of Sheffield's Chapter 8 that makes two distinct ergodic phases of the same slope smooth [17, Sections 8.4–8.6]. It holds for every $c > 1$; the hypothesis $c \leq 2$ enters only through Theorem 1.5, which rests on the spin representation of [5, Section 14].

1.3 The identification step and the limit in domains

The manuscript [15] computes correlations by Bethe ansatz, which works in one sector of the transfer matrix at a time. For $1 \le c < \sqrt{2}$ its construction uses the balanced sector, so it concerns $\mathbb{P}_c$ directly (see Remark 7.4 for the endpoint $c = 1$). For $\sqrt{2} \le c < 2$ it uses the sectors with $n_L := L/2 - \lfloor L^{\varepsilon} \rfloor$ right arrows per column, for a fixed small $\varepsilon > 0$ [15, Section 3], and studies

$$\rho_c := \lim_{L \to \infty,\ L \text{ even}} \ \lim_{N \to \infty} \ \mathbb{P}_{N,L}^{(n_L)}, \tag{1.1}$$

where $\mathbb{P}_{N,L}^{(n)}$ is the torus measure conditioned on n right arrows per column. The theorem about $\mathbb{P}_c$ then requires the identification

$$\rho_c = \mathbb{P}_c. \tag{ID}$$

In [15, Section 4, "Identification when $c \ge 1$"] this is derived from the characterisation of $\mathbb{P}_c$ as the unique minimiser of the free energy, with a reference to the remark of [5]. Corollary 1.3 proves that characterisation. We use two statements of [15], which we do not reprove:

(O1) the limit (1.1) exists [15, Sections 3 and 4];

(O2) ρ_c minimises the specific free energy among translation-invariant measures [15, Section 4, the rectangle bound and the tiling argument].

Corollary 1.6 (Identification)**.** *Let $a = b = 1$ and $\sqrt{2} \le c < 2$. If* (O1) *and* (O2) *hold, then $\rho_c = \mathbb{P}_c$.*

The argument is the one of [15, Section 4], with Corollary 1.3 in place of the remark it cites. For $\sqrt{3} \le c \le 2$, that is $-1 \le \Delta \le -\frac{1}{2}$, the full-plane statement is the main theorem of [5] and does not depend on [15]. The range in which the manuscript's full-plane theorem depended on the identification is therefore $\sqrt{2} \le c < \sqrt{3}$, that is $-\frac{1}{2} < \Delta \le 0$. There it no longer rests on an unproved characterisation; it remains only as strong as the rest of [15], which we do not assess.

Averous, Duminil-Copin, He, Lammers and Manolescu [1] denote by I_{CI} the set of Δ at which the full-plane correlations of the height function under $\mathbb{P}_c$ converge to those of a Gaussian free field [1, (11)]. Section 7 checks that [15] and [1] concern the same full-plane measure and observables, which gives the following.

Corollary 1.7 (Domains)**.** *Let $c \in [1, 2]$ and $\Delta = 1 - c^2/2$. If the full-plane statement of [15, Theorem 1.1(3)] holds at c (Definition 7.2), then $\Delta \in I_{\mathrm{CI}}$, and [1, Theorems 1.3 and 1.4] hold at Δ with the constant δ_0 of [1, Theorem 1.3] allowed to depend on Δ.*

Corollary 1.7 is an implication between the results of three preprints and is only as strong as they are. For $0 < \Delta \le \frac{1}{2}$ the construction of [15] uses the balanced sector; for $-\frac{1}{2} < \Delta \le 0$ it uses the identification of Corollary 1.6; for $-1 \le \Delta \le -\frac{1}{2}$ the full-plane statement is the main theorem of [5] (which [1] record for $-1 < \Delta \le -\frac{1}{2}$).

1.4 Strategy of the proofs

Theorem 1.4. We follow Sheffield's cluster-swapping argument [17, Chapter 8], which compares two Gibbs measures through independent samples ψ_1, ψ_2 of each and their difference $D = \psi_2 - \psi_1$. For a level ℓ, exchanging the two samples on a region where $D \ne \ell$, bounded by faces where $D = \ell$, preserves the nearest-neighbour constraint. It preserves the joint weight only up to the diagonal interactions that straddle the boundary of the region. Call a diagonal across which $D - \ell$ has the

same sign, while $D = \ell$ on the other diagonal, a *saddle*: separating its two faces costs a factor $q = c^{-2}$. We pay for it with an independent field of nonnegative integer *residual energies* on the diagonals, in the spirit of the coupling of Fortuin–Kasteleyn and Edwards–Sokal [9, 7]: separating a saddle lowers its residual energy by one, so a saddle whose residual energy is 0 cannot be separated and is an *open bridge*. The clusters of $\{D \neq \ell\}$ joined by open bridges are exactly the sets that can be exchanged without changing the joint weight (Figure 3). This is where the argument departs from Sheffield's: his residual energies live on the nearest-neighbour edges, ours on the diagonals, and clusters must be joined across saddles.

The swap identities sort the possible patterns of infinite clusters (Section 3):

- no infinite cluster at any level: then the two copies are exchangeable and $\mu_1 = \mu_2$;
- infinite clusters of one sign only, at every level: impossible;
- a highest level with an infinite plus cluster, or a lowest level with an infinite minus cluster: that level defines a canonical height, and both measures are smooth;
- infinite clusters of both signs at every level: the remaining case.

In the remaining case we fold the pair (Section 4): on alternate bands of levels the copies are exchanged, so that the difference is folded into a strip (Figure 4). The folded pair has the same free energy as the original, so it is again an equilibrium state; since the six-vertex model has hard constraints, we deduce that it is a Gibbs measure from the Lanford–Ruelle theorem for subshifts of finite type [2]. Each infinite cluster of the folded pair lies in a *double band*, a pair of adjacent bands, and a Burton–Keane argument [3] shows that there are finitely many. If some double band carries an infinite cluster, the lowest such one defines a canonical height and both measures are smooth; if none does, a symmetry of the folded pair gives a contradiction.

Theorem 1.5. In the spin representation in the form of [5, Section 14], the plane measure has infinitely many nested open circuits around every face, with a Markov property across each circuit and a reflection symmetry inside it. If $\mathbb{P}_c$ were the gradient of a Gibbs measure ν on heights, the height at a face would be the height of an outer circuit plus a sum of increments whose signs are independent fair coin flips given everything else, and anti-concentration [8] would make its law spread out without bound (Figure 5).

1.5 Relation to earlier work

Uniqueness. At $c = 1$ the six-vertex model is the uniform measure on graph homomorphisms, a nearest-neighbour simply attractive model, and uniqueness of the zero-slope phase follows from [17, Theorem 9.1.1]; Chandgotia, Peled, Sheffield and Tassy prove delocalisation and the absence of translation-invariant Gibbs measures on heights there [4]. At $c = 2$ Glazman and Peled prove uniqueness of the flat Gibbs state [11, Corollary 2.1], and for $c > 2$ they prove that the height function is localised: its extremal Gibbs states are Gibbs measures on heights, invariant under parity-preserving translations [11, Theorem 2]. A proof of uniqueness through roughness must therefore use $c \leq 2$. For $1 < c < 2$, uniqueness of the zero-slope phase is asserted in [5] (remark after (6)) and in [16, Section 2.1], in both cases by reference to Sheffield's theory; Ryan's sketch regards the six-vertex potential as simply attractive and applies [17, Lemma 8.7.1]. A remark of Lammers and Tassy after [12, Theorem 13.16] states that the six-vertex potential can be written as a simply attractive potential when $c_+c_- \geq \max\{a_+a_-, b_+b_-\}$ in their notation, which reads $c \geq 1$ when $a = b = 1$; we do not know which representation is intended, and Proposition 2.3 shows that for $a = b = 1$ it cannot use nearest-neighbour interactions between face heights when $c \neq 1$. We use

the strict convexity proved in [12, Theorem 13.16] itself, not the remark. The planar arguments of [17, Chapter 9] do not extend beyond nearest-neighbour interactions [17, Section 10.4.3]; our proof extends the non-planar ones of Chapter 8. Cluster swapping has recently been extended to spin models by van Engelenburg, Heeney and Lis [18].

The plane measure. Ergodicity of the full-plane spin measure is known [10], and for $c \in [\sqrt{3}, 2]$ Lis proves that the torus measures have an ergodic limit [13] (Section 5.3). Delocalisation, in the form of variance growth, is proved by Duminil-Copin, Karrila, Manolescu and Oulamara for $1 \le c \le 2$ [6] and by Glazman and Lammers [10], whose random walk of circuit heights [10, Section 4] is also the mechanism of our proof of roughness. Roughness in Sheffield's sense, that no Gibbs measure on heights has gradient law $\mathbb{P}_c$, is what the proof of Theorem 1.2 needs. Ryan states, citing Lammers and Ott, that for $a, b \le c$ roughness in this sense is equivalent to delocalisation [16, Section 2.1]; we give a direct proof.

Variational principles. Lammers and Tassy prove that minimisers of the specific free energy are Gibbs measures for a class of Lipschitz height models that includes the six-vertex model [12, Theorem 4.4]. We need the same conclusion for the folded pair of Section 4, which carries residual energies and a hard constraint between the two copies, and obtain it from the Lanford–Ruelle theorem of [2].

1.6 Organisation

Sections 2–4 prove Theorem 1.4, Section 5 proves Theorem 1.5, Section 6 proves Theorem 1.2 and Corollaries 1.3 and 1.6, and Section 7 treats the limit in domains.

2 The six-vertex model as a gradient model

This section places the six-vertex model in Sheffield's framework [17]. In a shifted height variable the ice rule becomes a nearest-neighbour constraint and the weights become an interaction along the diagonals of each vertex (Lemma 2.1, Figure 2). This makes the model a perturbed Lipschitz simply attractive potential, so Sheffield's variational principle applies (Proposition 2.2, Lemma 2.4); but for $c \neq 1$ no nearest-neighbour potential has the same specification (Proposition 2.3), so his uniqueness theorems do not apply. The last subsection introduces the pairs with residual energies on which the swaps of Section 3 act.

2.1 Heights, blocks and weights

The faces of $\mathbb{Z}^2$ form the set $F = \mathbb{Z}^2$, a face being indexed by its lower left corner. For $x \in F$ the *block* $B(x) = \{x, x+e_1, x+e_2, x+e_1+e_2\}$ is the set of four faces around one vertex of $\mathbb{Z}^2$. Its two *diagonals* are $d_x^+ = \{x, x+e_1+e_2\}$ and $d_x^- = \{x+e_1, x+e_2\}$; $\mathfrak{D}$ is the set of all diagonals and $d^\perp$ denotes the other diagonal of the block of d. Faces are *NN-adjacent* if $|x-y|_1 = 1$, and *$*$-adjacent* if they are NN-adjacent or form a diagonal.

Fix the parity convention $h(x) \equiv x_1 + x_2 \pmod 2$ and put $\psi(x) := (h(x) + x_1 + x_2)/2$, the representation of [12, Section 4.4]. Let Ω be the set of $\psi : F \to \mathbb{Z}$ with $\psi(x+e_i) - \psi(x) \in \{0, 1\}$ for all x and $i = 1, 2$. Height functions of arrow configurations and elements of Ω correspond bijectively up to additive constants, and arrow configurations correspond to gradients of elements of Ω, equivariantly under translations. For $\psi \in \Omega$ and a diagonal d put

$$\mathsf{a}(\psi; d_x^+) := \mathbf{1}\{\psi(x+e_1+e_2) - \psi(x) = 1\}, \qquad \mathsf{a}(\psi; d_x^-) := \mathbf{1}\{\psi(x+e_2) = \psi(x+e_1)\};$$

in both cases $\mathsf{a}(\psi; d) = 1$ exactly when h takes the same value on the two faces of d.

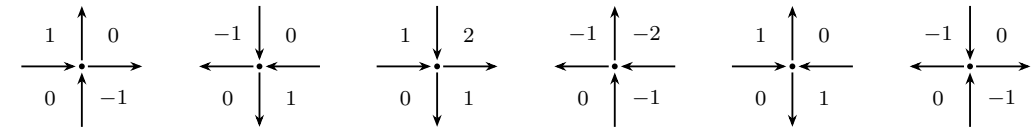


Figure 2: The six vertex configurations with $a = b = 1$, and the heights of the four surrounding faces, normalised to 0 on the lower left face (the face to the left of an arrow is one unit higher). A vertex has weight c exactly when the heights are equal across both diagonals of its block, and weight 1 when they are equal across one diagonal only (Lemma 2.1). In height coordinates the weight is therefore an interaction along the diagonals, not between neighbouring faces.

Lemma 2.1 (Vertex weights)**.** *In every block* $\mathsf{a}(\psi; d_x^+) + \mathsf{a}(\psi; d_x^-) \in \{1, 2\}$*, and the weight of the vertex surrounded by* $B(x)$ *is*

$$w_x(\psi) = c^{\,\mathsf{a}(\psi; d_x^+) + \mathsf{a}(\psi; d_x^-) - 1}.$$

Proof. Going around the vertex, the height makes four steps ± 1 with sum zero. If the steps alternate in sign, the heights are $m, m+1, m, m+1$, both diagonal pairs are equal, and these are the two configurations in which the horizontal arrows both point in or both point out. Otherwise the heights are $m, m+1, m+2, m+1$ up to rotation, one diagonal pair is equal and the other differs by 2, and the weight is 1. □

This is the form of the weight used in [5, (248)], where the weight of a configuration is c to the number of diagonals across which the height is constant.

For a finite set $\Lambda \subset F$ and $\psi \in \Omega$, let $\gamma_\Lambda(\cdot \mid \psi)$ be the probability measure on the elements of Ω that agree with ψ off Λ, with weights proportional to $\prod_{x:\, B(x) \cap \Lambda \neq \emptyset} w_x$. The kernels γ_Λ form the *six-vertex specification*; they commute with adding a constant to ψ. A *Gibbs measure on heights* is a probability measure ν on Ω with $\nu\gamma_\Lambda = \nu$ for every finite Λ. A *gradient Gibbs measure* is a probability measure μ on gradient configurations $\nabla\psi$ (equivalently, on arrow configurations) with $\mu\gamma_\Lambda = \mu$ for every finite Λ, where γ_Λ acts on ψ modulo constants. A gradient Gibbs measure is *smooth* if it is the gradient law of a Gibbs measure on heights, and *rough* otherwise [17, p. 10].

2.2 Sheffield's framework

Proposition 2.2 (A perturbed Lipschitz simply attractive potential)**.** *For every* $c > 0$ *the six-vertex specification is, in the variable* ψ *with values in* $\mathbb{Z}$*, the gradient specification of a perturbed Lipschitz simply attractive potential in the sense of [17, Section 1.1.4].*

Proof. Fix $K > 2|\log c|$. Let Φ be the nearest-neighbour potential whose pair function on each pair $(x, x + e_i)$ is $V(t) = K$ for $t \in [0, 1]$ and $V(t) = +\infty$ otherwise. It is positive, convex and invariant under $\mathbb{Z}^2$, and it confines $\psi(x + e_i) - \psi(x)$ to $[0, 1]$, hence to $\{0, 1\}$: a Lipschitz simply attractive potential. Let Ψ assign to every diagonal d the energy $-\log c \cdot \mathsf{a}(\psi; d)$. By Lemma 2.1, the exponential of minus the sum of Ψ over the two diagonals of a block is c times the vertex weight, so $\Phi + \Psi$ has the six-vertex specification. The potential Ψ has finite range and is invariant under $\mathbb{Z}^2$, and Φ strictly dominates it: for a finite nonempty Λ and a configuration of finite Φ-energy, the Φ-energy of the pairs meeting Λ is at least $2K|\Lambda|$, since every face of Λ is the lower endpoint of two nearest-neighbour pairs, while the Ψ-energy of the pairs meeting Λ is at most $4|\log c|\,|\Lambda|$ in absolute value, since every face lies in four diagonals. The ratio is at most $2|\log c|/K < 1$. □

Proposition 2.3 (Not a nearest-neighbour model in heights)**.** *For every $c \neq 1$, no nearest-neighbour gradient potential on the heights of faces has the six-vertex specification. For $c = 1$ the six-vertex model is a Lipschitz simply attractive potential.*

Proof. Let x be a face whose four neighbours have height m; then $h(x) \in \{m-1, m+1\}$. Each of the four vertices of x is surrounded by x, two of these neighbours, which are diagonal to each other and have equal heights, and one face y diagonal to x. By Lemma 2.1 its weight is c if $h(x) = h(y)$ and 1 otherwise. If k_+ and k_- of the four diagonal faces have heights $m+1$ and $m-1$, then, given the configuration outside x, the ratio of the probabilities of $h(x) = m+1$ and $h(x) = m-1$ is $c^{k_+ - k_-}$. Both $k_+ = 4$ and $k_- = 4$ occur: take all faces of the parity class of x other than x at height $m+1$, respectively $m-1$, and all faces of the other class at height m. For a nearest-neighbour potential the conditional law of $h(x)$ depends only on the heights of its four neighbours, so the ratio would be the same in the two cases, forcing $c^4 = c^{-4}$. For $c = 1$ we have $\Psi = 0$ in the proof of Proposition 2.2. □

2.3 Free energy

Let $\mathbb{X}$ be the subshift of finite type on the alphabet $\{0,1\}^2$ (the two increments of ψ at a face) whose configurations are those in which the signed cyclic sum of the four increments around every block vanishes, that is, the gradients of elements of Ω; its constraint involves the faces x, $x + e_1$ and $x + e_2$. For a translation-invariant probability measure μ on $\mathbb{X}$ put

$$P(\mu) := \mathsf{h}(\mu) + \mu(\log w_0), \qquad P^* := \sup_{\mu} P(\mu),$$

with $\mathsf{h}(\mu)$ the Kolmogorov–Sinai entropy of the $\mathbb{Z}^2$-action. We call μ an *equilibrium state* if $P(\mu) = P^*$. Sheffield's *specific free energy* SFE and *surface tension* $\sigma_c(u)$, the infimum of SFE over translation-invariant gradient measures of slope u, are defined in [17, Section 2.3 and Chapter 4].

Lemma 2.4 (Free energy and equilibrium states)**.**

(i) For translation-invariant measures supported on admissible configurations, $\mathrm{SFE}(\mu) = -P(\mu) + \mathrm{const}$ *with a constant that does not depend on μ.*

(ii) σ_c is convex and even, and $\sigma_c(0) = \min_u \sigma_c(u) = -P^* + \mathrm{const}$. Hence $\mathrm{SFE}(\mu) = \sigma_c(0)$ if and only if μ is an equilibrium state.

(iii) Every gradient Gibbs measure of the six-vertex model that is invariant and ergodic under the translations of $\mathbb{Z}^2$ and has slope 0 is an equilibrium state.

Proof. (i) Sheffield defines $\mathrm{SFE}(\mu)$ as the limit of $|\Lambda_n|^{-1}$ times the relative entropy of the restriction of μ to Λ_n with respect to $e^{-H^\circ_{\Lambda_n}}$ times counting measure [17, Section 2.3]. For a measure on admissible configurations this is the Shannon entropy of the gradient pattern on Λ_n with a minus sign, plus $\mu(H^\circ_{\Lambda_n})$, and $H^\circ_{\Lambda_n} = K \cdot \#\{\text{NN pairs in } \Lambda_n\} - \log c \cdot \#\{d \subset \Lambda_n : \mathsf{a}(\psi; d) = 1\}$ in the notation of Proposition 2.2. Dividing by $|\Lambda_n|$ gives $\mathrm{SFE}(\mu) = -P(\mu) + 2K - \log c$. (ii) SFE is affine, so σ_c is convex [17, Chapter 4]. The map $\psi \mapsto -\psi + x_1 + x_2$, that is $h \mapsto -h$, preserves Ω and every w_x and reverses the slope, so σ_c is even and $\sigma_c(0) \le \sigma_c(u)$ for all u. The infimum of SFE over all translation-invariant measures is $\inf_u \sigma_c(u) = \sigma_c(0)$, which is $-P^*$ plus the constant of (i). (iii) By Proposition 2.2, [17, Theorem 6.3.1] applies: an ergodic gradient Gibbs measure of finite slope u has $\mathrm{SFE} = \sigma_c(u)$. (Slope 0 is $(\frac{1}{2}, \frac{1}{2})$ in the coordinates ψ, an interior point of the set $U_\Phi = (0,1)^2$ of slopes of finite surface tension.) With $u = 0$ and (ii) the measure is an equilibrium state. □

2.4 Triples: pairs with residual energies

From now on until the end of Section 4, $c > 1$ and $q := c^{-2} \in (0,1)$.

Sheffield compares two Gibbs measures through triplets made of two independent samples and an independent field of residual energies on the edges [17, Sections 8.1.3 and 8.3]. For the six-vertex model the interaction lives on the diagonals, so the residual energies do too, and since the weight on a diagonal takes two values, an integer-valued residual energy suffices. Its role is the one of the edge variable in the Edwards–Sokal coupling: given the two samples, it decides which diagonals may be cut by a swap without changing the joint weight (Lemma 3.4).

A *triple* is $\bar\omega = (\psi_1, \psi_2, n)$ with $\psi_1, \psi_2 \in \Omega$ and $n : \mathfrak{D} \to \mathbb{Z}_{\geq 0}$, taken modulo the common shifts $(\psi_1, \psi_2) \mapsto (\psi_1 + t, \psi_2 + t)$, so that the *difference* $D := \psi_2 - \psi_1$ is an *honest* integer-valued function, not only a gradient. A *lift* of a gradient triple $(\nabla\psi_1, \nabla\psi_2, n)$ is a triple with these gradients; its lifts differ by the shifts $(\psi_1, \psi_2) \mapsto (\psi_1, \psi_2 + t)$, $t \in \mathbb{Z}$. Let $\bar\Omega$ be the space of triples. The number n_d is the *residual energy* of the diagonal d, and we put

$$\tau_d(\bar\omega) := n_d - \tfrac{1}{2}\big(\mathsf{a}(\psi_1; d) + \mathsf{a}(\psi_2; d)\big).$$

Let π be the product over $\mathfrak{D}$ of the geometric law $\pi_0(j) = (1-q)q^j$, $j \geq 0$. For finite $\Lambda \subset F$ let $E(\Lambda)$ be the set of diagonals with a face in Λ, and let $\bar\gamma_\Lambda(\cdot \mid \bar\omega)$ be the kernel that resamples ψ_1 and ψ_2 on Λ from $\gamma_\Lambda(\cdot \mid \psi_1) \otimes \gamma_\Lambda(\cdot \mid \psi_2)$ and n on $E(\Lambda)$ from π. Since $c^{\mathsf{a}} q^n = q^{n - \mathsf{a}/2}$, the density of $\bar\gamma_\Lambda(\cdot \mid \bar\omega)$ on the countable set of triples that agree with $\bar\omega$ off $(\Lambda, E(\Lambda))$ is proportional to

$$\prod_{d \in E(\Lambda)} q^{\tau_d}. \tag{2.1}$$

If μ_1, μ_2 are gradient Gibbs measures, the law $\bar\mu := \mu_1 \otimes \mu_2 \otimes \pi$ of the *gradient triple* $(\nabla\psi_1, \nabla\psi_2, n)$ is invariant under every $\bar\gamma_\Lambda$.

3 Swaps and the case distinction

This section is the six-vertex form of Sheffield's cluster swapping [17, Sections 8.1–8.5]. Subsection 3.1 records what the difference of two height functions looks like around a vertex. Subsection 3.2 defines the clusters that can be swapped and shows that swapping them preserves the joint law (Figure 3); the new ingredient is the role of the saddles. Subsections 3.3 and 3.4 turn this into identities for pairs of Gibbs measures and use them to sort the possible patterns of infinite clusters: all but one pattern either cannot occur or makes both measures smooth. The remaining pattern is treated in Section 4.

3.1 Local combinatorics

Lemma 3.1. *If $\psi, \psi' \in \Omega$, then $\psi \vee \psi'$ and $\psi \wedge \psi'$ are in Ω.*

Proof. For $\alpha, \beta \in \{0,1\}$, $\max(s+\alpha, t+\beta) - \max(s,t) \in \{0,1\}$, and likewise for the minimum. □

Lemma 3.2 (Difference of two admissible functions)**.** *Let $\psi_1, \psi_2 \in \Omega$ and $D = \psi_2 - \psi_1$.*

(i) $|D(x) - D(y)| \leq 1$ for NN-adjacent faces, and ≤ 2 for the two faces of a diagonal.

(ii) Let $d = \{x, y\}$ be a diagonal, $\ell \in \mathbb{Z}$, and suppose $D = \ell$ on $d^\perp$. If $D(x) = D(y) = \ell + s$ with $s \in \{+1, -1\}$ (d is an ℓ-saddle of sign s), then $\mathsf{a}(\psi_1; d) = \mathsf{a}(\psi_2; d) = 1$. If $\{D(x), D(y)\} = \{\ell + 1, \ell - 1\}$ (d is ℓ-monotone), then $\mathsf{a}(\psi_1; d) = \mathsf{a}(\psi_2; d) = 0$.

Proof. (i) is immediate. (ii) Go around the block in the cyclic order x, z, y, z' with $d^\perp = \{z, z'\}$. Let $\varepsilon_1, \dots, \varepsilon_4 \in \{\pm 1\}$ be the steps of $h_1 = 2\psi_1 - x_1 - x_2$; the steps of h_2 are $\varepsilon_i + 2\delta_i \in \{\pm 1\}$, where the δ_i are the steps of D. For a saddle, $\delta = (-s, s, -s, s)$, which forces $\varepsilon = (s, -s, s, -s)$: both h_1 and h_2 alternate, so both are equal across d. For the monotone pattern $D = (\ell+1, \ell, \ell-1, \ell)$ along x, z, y, z', $\delta = (-1, -1, 1, 1)$, which forces $\varepsilon = (1, 1, -1, -1)$, so $h_1(y) = h_1(x) + 2$ and $h_2(y) = h_2(x) - 2$. □

For $\ell \in \mathbb{Z}$ and $A \subset F$ let $s_A^\ell(\psi_1, \psi_2) = (\psi_1', \psi_2')$ be given by $(\psi_1', \psi_2')(x) = (\psi_2(x) - \ell, \psi_1(x) + \ell)$ for $x \in A$ and $(\psi_1', \psi_2')(x) = (\psi_1, \psi_2)(x)$ otherwise. Then $D' - \ell = -(D - \ell)$ on A, and s_A^ℓ acts trivially on faces where $D = \ell$. Call A *ℓ-separated* if there is no NN-adjacent pair $x \in A$, $y \notin A$ with $D(x) \neq \ell \neq D(y)$.

Lemma 3.3 (Admissibility of a swap)**.**

(i) $s_A^\ell(\psi_1, \psi_2) \in \Omega \times \Omega$ *if and only if* A *is* ℓ*-separated.*

(ii) If A *is* ℓ*-separated, then for every diagonal* d *the number* $\chi_d := [\mathsf{a}(\psi_1'; d) + \mathsf{a}(\psi_2'; d)] - [\mathsf{a}(\psi_1; d) + \mathsf{a}(\psi_2; d)]$ *equals* -2 *if* d *is an* ℓ*-saddle with exactly one face in* A*,* $+2$ *if* d *is* ℓ*-monotone with exactly one face in* A*, and* 0 *otherwise.*

Proof. Write $D_\ell = D - \ell$. (i) Let $u \in A$, $v \notin A$ be NN-adjacent with $v = u + e_i$ (the case $v = u - e_i$ is symmetric), and $\eta = \psi_1(v) - \psi_1(u) \in \{0, 1\}$. Then $\psi_1'(v) - \psi_1'(u) = \eta - D_\ell(u)$ and $\psi_2'(v) - \psi_2'(u) = \eta + D_\ell(v)$, while $\psi_2(v) - \psi_2(u) = \eta + D_\ell(v) - D_\ell(u) \in \{0, 1\}$. If $D_\ell(u) = 0$ or $D_\ell(v) = 0$, the new increments are the old ones, in the same or the exchanged order. If both are nonzero, they have the same sign by Lemma 3.2(i), so the two new increments differ by at least 2 and cannot both lie in $\{0, 1\}$. Pairs inside A or inside its complement keep their increments up to exchange. (ii) If both faces of d are in A, the copies are exchanged on d up to the common shift ℓ; if neither is, nothing changes; in both cases $\chi_d = 0$. Let $x \in A$, $y \notin A$. If $D_\ell(x) = 0$ nothing changes at x, and if $D_\ell(y) = 0$ then $\psi_1' = \psi_2 - \ell$ and $\psi_2' = \psi_1 + \ell$ on both faces of d; in both cases $\chi_d = 0$. If both are nonzero, let $z \in d^\perp$. If $D_\ell(z) \neq 0$, separation applied to the NN pairs $\{x, z\}$ and $\{z, y\}$ puts z and then y in A, a contradiction. So $D_\ell = 0$ on $d^\perp$ and $|D_\ell(x)| = |D_\ell(y)| = 1$: d is an ℓ-saddle or ℓ-monotone, and the swap at x turns one into the other. Lemma 3.2(ii) gives a-sums 2 and 0 respectively. □

3.2 FK clusters and swaps

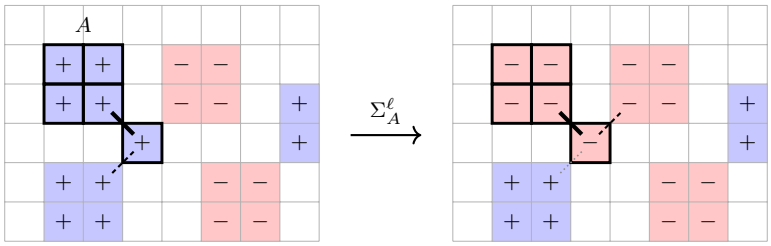
For a triple $\bar\omega$ and $\ell \in \mathbb{Z}$ let $G_\ell(\bar\omega)$ be the graph whose vertices are the faces with $D \neq \ell$, with an edge between NN-adjacent vertices (they then have the same sign of $D - \ell$), and with an edge, an *open bridge*, for every ℓ-saddle d with $n_d = 0$. Its components are the *FK clusters at level* ℓ, plus or minus according to the sign of $D - \ell$ on them. An ℓ-saddle with $n_d = 0$ is *open* and one with $n_d \geq 1$ is *closed*: by Lemma 3.4 below, a swap can separate the two faces of a closed saddle, at the price of lowering n_d by one, but not those of an open one. For a nearest-neighbour hard-constraint potential, such as the six-vertex model at $c = 1$, Sheffield's open clusters [17, Section 8.1.3] are the components of $\{D \neq \ell\}$; the open bridges are what the diagonal interaction adds.

Lemma 3.4 (FK swap)**.** *Let* A *be a union of FK clusters at level* ℓ *of* $\bar\omega$*, and define* $\Sigma_A^\ell(\bar\omega) := (s_A^\ell(\psi_1, \psi_2), n')$ *with* $n_d' := n_d + \chi_d/2$*. Then:*

(i) $\Sigma_A^\ell(\bar\omega)$ *is a triple, that is* $n' \geq 0$*, and* τ_d *is unchanged for every* d*;*

(ii) $G_\ell(\Sigma_A^\ell \bar\omega) = G_\ell(\bar\omega)$ *as graphs, with the sign reversed exactly on* A*;*

(iii) Σ_A^ℓ *is an involution on the set of triples for which* A *is a union of FK clusters at level* ℓ*.*



(a) level ℓ of $\bar{\omega}$ (b) after the swap

Figure 3: FK clusters at level ℓ for a pair of height functions with difference D. Faces with $D > \ell$ are marked $+$, faces with $D < \ell$ are marked $-$, and blank faces have $D = \ell$. Two faces of a diagonal across which $D - \ell$ has the same sign, while $D = \ell$ on the other diagonal, form a *saddle*. Separating the two faces of a saddle costs a factor $q = c^{-2}$ in the joint weight, paid by lowering the residual energy of the diagonal by one; so a saddle with residual energy 0 cannot be separated and is an *open bridge* (thick line), and otherwise it is *closed* (dashed). In (a) the outlined set A is an FK cluster: four faces at the top left and the face reached through the open bridge; the closed saddle below does not join it to the cluster there. In (b) the two copies are exchanged on A, which reverses the sign of $D - \ell$ there. The open bridge inside A is unaffected. The closed saddle of (a) has become a diagonal across which $D - \ell$ changes sign (dotted), and its residual energy was lowered by one. Conversely, the diagonal between A and the minus block, across which $D - \ell$ changed sign in (a), has become a saddle whose residual energy was raised by one, so it is closed (dashed, right). The joint weight is unchanged (Lemma 3.4).

Proof. A is ℓ-separated, so Lemma 3.3 applies. (i) $\chi_d = -2$ only at ℓ-saddles with exactly one face in A; such a saddle is not an open bridge, so $n_d \geq 1$ and $n'_d = n_d - 1 \geq 0$. The definition of n' makes τ invariant. (ii) The vertex set and the NN edges depend only on $\{D = \ell\}$. An open bridge of $\bar{\omega}$ has both faces in A or both outside, stays an ℓ-saddle and keeps $n = 0$. A saddle with exactly one face in A becomes monotone. A monotone diagonal with exactly one face in A becomes a saddle with $n' = n + 1 \geq 1$, a closed one. All other saddles keep their n. (iii) follows from (ii) and the formula for n'. □

The next lemma is the six-vertex form of [17, Lemmas 8.1.5 and 8.1.6].

Lemma 3.5 (Finite swaps preserve the kernel)**.** *Let Λ be finite, $\ell \in \mathbb{Z}$, and let $\bar{\omega} \mapsto A(\bar{\omega})$ be a rule that selects a union of FK clusters at level ℓ contained in Λ and depends only on $G_\ell(\bar{\omega})$ and on the restriction of $\bar{\omega}$ to the complement of $(\Lambda, E(\Lambda))$. Then $\Theta(\bar{\omega}) := \Sigma^{\ell}_{A(\bar{\omega})}(\bar{\omega})$ is an involution that changes $\bar{\omega}$ only on $(\Lambda, E(\Lambda))$, and $\bar{\gamma}_\Lambda(\Theta^{-1}(\cdot) \mid \bar{\omega}) = \bar{\gamma}_\Lambda(\cdot \mid \bar{\omega})$ for every $\bar{\omega}$.*

Proof. By Lemma 3.4, $A(\Theta\bar{\omega}) = A(\bar{\omega})$, Θ is an involution of the set of triples that agree with $\bar{\omega}$ off $(\Lambda, E(\Lambda))$, and it preserves (2.1). □

We use two rules: "the FK cluster at level ℓ of a given face, if it is contained in Λ", and "all FK clusters at level ℓ contained in Λ".

Lemma 3.6 (Monotonicity in the level)**.** *If C is a plus FK cluster at level ℓ and $\ell' < \ell$, then C is contained in one plus FK cluster at level ℓ'. If C is a minus FK cluster at level ℓ and $\ell' > \ell$, then C is contained in one minus FK cluster at level ℓ'.*

Proof. Consider the minus case. Faces of C have $D < \ell < \ell'$, and NN edges of C are NN edges at level ℓ'. An open bridge $\{u, v\}$ of C has $D = \ell$ on $d^\perp = \{z, z'\}$, and $D(z) = \ell < \ell'$, so u, z, v is an NN path in $G_{\ell'}$. □

3.3 Pairs of Gibbs measures

Lemmas 3.7–3.9 and Proposition 3.11 below are the six-vertex forms of the corresponding steps of [17, Sections 8.4–8.5]. Sheffield fixes the height of one copy; we instead average over all relative heights of the two copies with the σ-finite measure $\mathbb{M}$, which makes the levels interchangeable.

In this subsection μ_1, μ_2 are translation-invariant gradient Gibbs measures and $\bar\mu = \mu_1 \otimes \mu_2 \otimes \pi$. For a gradient triple and $j \in \mathbb{Z}$ let $\bar\omega^{(j)} \in \bar\Omega$ be the triple with $D(0) = j$, and let

$$\mathbb{M} := \sum_{j \in \mathbb{Z}} \mathrm{law}(\bar\omega^{(j)}),$$

a σ-finite measure on $\bar\Omega$ whose restriction to $\{D(0) = j\}$ is a copy of $\bar\mu$ for every j. Let V_t be the vertical shift $(\psi_1, \psi_2, n) \mapsto (\psi_1, \psi_2 + t, n)$. An event $E \subset \bar\Omega$ is *robust* if it is invariant under every V_t, under translations, and under every change of the triple on finitely many faces and diagonals.

Lemma 3.7 (The measure $\mathbb{M}$)**.**

(i) $\mathbb{M}$ *is invariant under translations and under every* V_t*; for a* V*-invariant event* E*,* $\mathbb{M}(E \cap \{D(0) = j\}) = \bar\mu(E)$ *for every* j*.*

(ii) $\mathbb{M}\bar\gamma_\Lambda = \mathbb{M}$ *for every finite* Λ*.*

(iii) If $\Theta : \bar\Omega \to \bar\Omega$ *is measurable and* $\bar\gamma_\Lambda(\Theta^{-1}(\cdot) \mid \bar\omega) = \bar\gamma_\Lambda(\cdot \mid \bar\omega)$ *for every* $\bar\omega$ *and some finite* Λ*, then* $\mathbb{M} \circ \Theta^{-1} = \mathbb{M}$*.*

Proof. (i) $\mathbb{M} = \sum_j \mathrm{law}(\text{lift with } D(x) = j)$ for any face x. (ii) Fix a face $b \notin \Lambda$ and write $\mathbb{M} = \sum_j \mathbb{M}_j$ with $\mathbb{M}_j$ the law of the lift with $D(b) = j$. Resampling in Λ does not change $D(b)$, and $\bar\mu$ is invariant under $\bar\gamma_\Lambda$. (iii) $\mathbb{M}(f \circ \Theta) = \int \mathbb{M}(\mathrm{d}\bar\omega)\, \bar\gamma_\Lambda(f \circ \Theta \mid \bar\omega) = \int \mathbb{M}(\mathrm{d}\bar\omega)\, \bar\gamma_\Lambda(f \mid \bar\omega) = \mathbb{M}(f)$ for $f \geq 0$. □

Lemma 3.8 (Swap identity)**.** *Let* $\ell \in \mathbb{Z}$*,* $k \geq 1$*,* E *robust, and let* $C_\ell(0)$ *be the FK cluster at level* ℓ *of the face* 0*. Then*

$$\mathbb{M}(D(0) = \ell + k,\ |C_\ell(0)| < \infty,\ E) = \mathbb{M}(D(0) = \ell - k,\ |C_\ell(0)| < \infty,\ E).$$

Proof. Let Θ_N swap $C_\ell(0)$ at level ℓ if it is contained in $\Lambda_N := [-N, N]^2$ and do nothing otherwise. By Lemmas 3.5 and 3.7(iii), $\mathbb{M} \circ \Theta_N^{-1} = \mathbb{M}$. The map Θ_N is an involution, preserves E and the event $\{C_\ell(0) \subset \Lambda_N\}$ (Lemma 3.4(ii)), and maps $D(0) - \ell$ to its negative on that event. Hence the identity holds with $\{C_\ell(0) \subset \Lambda_N\}$ in place of $\{|C_\ell(0)| < \infty\}$; both sides are at most 1, and we let $N \to \infty$. □

Lemma 3.9 (No infinite cluster implies exchangeability)**.** *Let* E *be robust and suppose that* $\mathbb{M}$*-almost everywhere on* E *all FK clusters at level* 0 *are finite. Then* $\bar\mu(g \circ \mathrm{Ex}; E) = \bar\mu(g; E)$ *for every bounded local function* g *of the gradient pair, where* Ex *exchanges the two copies. If moreover* μ_1, μ_2 *are ergodic and* $\bar\mu(E) > 0$*, then* $\mu_1 = \mu_2$*.*

Proof. Let Θ_N swap all FK clusters at level 0 contained in Λ_N; then $\mathbb{M} \circ \Theta_N^{-1} = \mathbb{M}$ and Θ_N preserves E. Fix j_0 and put $f := g \cdot \mathbf{1}\{D(0) = j_0\}$, so that $\mathbb{M}(f \circ \Theta_N; E) = \mathbb{M}(f; E)$. For $\mathbb{M}$-almost every $\bar\omega \in E$, the finitely many clusters that meet the window of g or contain 0 are finite, so for large N the triple $\Theta_N(\bar\omega)$ coincides on that window with the full exchange (ψ_2, ψ_1, n); when every cluster is swapped, no diagonal has exactly one swapped face among two faces with $D \neq 0$, so n is unchanged. Since $|f \circ \Theta_N| \leq \|g\|\, \mathbf{1}\{|D(0)| = |j_0|\}$, which is $\mathbb{M}$-integrable, dominated convergence gives $\mathbb{M}(g \circ \mathrm{Ex} \cdot \mathbf{1}\{D(0) = -j_0\}; E) = \mathbb{M}(g \cdot \mathbf{1}\{D(0) = j_0\}; E)$, which by Lemma 3.7(i) is the first claim. For the second take $g = g_1(\nabla\psi_1)$. Then $\bar\mu(g_1(\nabla\psi_1); E) = \int \mu_1(\mathrm{d}\psi_1)\, g_1(\psi_1)\, u(\psi_1)$ with $u(\psi_1)$ the $(\mu_2 \otimes \pi)$-measure of the section of E, a translation-invariant function, hence μ_1-almost surely equal to $\bar\mu(E)$; likewise $\bar\mu(g_1(\nabla\psi_2); E) = \bar\mu(E)\, \mu_2(g_1)$. So $\mu_1(g_1) = \mu_2(g_1)$. □

3.4 The case distinction

The classification below follows [17, Section 8.5]; the measure $\mathbb{M}$ of the previous subsection, which makes the levels exchangeable, lets us state it without fixing a height.

For $\bar\omega \in \bar\Omega$ let $I^+(\bar\omega)$ (resp. $I^-(\bar\omega)$) be the set of levels ℓ at which $G_\ell(\bar\omega)$ has an infinite plus (resp. minus) cluster. By Lemma 3.6, I^+ is a down-set and I^- an up-set of $\mathbb{Z}$. Put

$$E_\emptyset := \{I^+ = \emptyset = I^-\}, \qquad E_{\mathrm{top}} := \{I^+ \neq \emptyset,\ \sup I^+ < \infty\}, \quad E_{\mathrm{bot}} := \{I^- \neq \emptyset,\ \inf I^- > -\infty\},$$
$$E_+ := \{I^+ = \mathbb{Z},\ I^- = \emptyset\}, \quad E_- := \{I^+ = \emptyset,\ I^- = \mathbb{Z}\}, \qquad E_{\mathbb{ZZ}} := \{I^+ = \mathbb{Z} = I^-\}.$$

These six events cover $\bar\Omega$.

Lemma 3.10. *If two triples agree off finitely many faces and diagonals, they have the same I^+ and I^-. In particular the six events above are robust.*

Proof. Let S be the union of the blocks that contain a changed face or a changed diagonal, a finite set. Let C be an infinite plus cluster of $G_\ell(\bar\omega)$. The set $C \setminus S$ has finitely many components in $G_\ell(\bar\omega)$, each containing a face adjacent to S, so one of them, K, is infinite. An edge of $G_\ell(\bar\omega)$ between two faces outside S is an edge of $G_\ell(\bar\omega')$: for an NN edge the two faces are unchanged; for a bridge d, the block of d contains a face outside S, hence no changed face, and d is not a changed diagonal. So K lies in an infinite plus cluster of $G_\ell(\bar\omega')$. Under V_t and translations, $I^\pm$ are shifted by t, respectively unchanged. □

Proposition 3.11 (One-sided cases)**.** $\bar\mu(E_+) = \bar\mu(E_-) = 0$.

Proof. On E_+ every minus cluster at every level is finite. By Lemmas 3.8 and 3.7(i), for all ℓ and $k \geq 1$,

$$\begin{aligned}\mathbb{M}(D(0) = \ell + k,\ |C_\ell(0)| = \infty,\ E_+) &= \bar\mu(E_+) - \mathbb{M}(D(0) = \ell + k,\ |C_\ell(0)| < \infty,\ E_+)\\ &= \bar\mu(E_+) - \mathbb{M}(D(0) = \ell - k,\ |C_\ell(0)| < \infty,\ E_+)\\ &= \bar\mu(E_+) - \bar\mu(E_+) = 0.\end{aligned}$$

So $\mathbb{M}$-almost everywhere on E_+ the face 0, and by translation invariance every face, lies in no infinite plus cluster at any level. This contradicts $I^+ = \mathbb{Z}$ unless $\bar\mu(E_+) = 0$. The case of E_- is symmetric. □

A height that is determined by the pair up to finite modifications, in the way made precise below, makes a measure smooth. This is [17, Lemma 8.4.4] in our setting.

Lemma 3.12 (A canonical lift makes a marginal smooth)**.** *Let ϖ be a probability measure on a space $\mathcal{U}$ carrying a measurable action of $\mathbb{Z}^2$ that preserves ϖ. Let μ_2 be an ergodic gradient Gibbs measure, μ_1 a translation-invariant gradient measure, and $Q_0 := \mu_1 \otimes \mu_2 \otimes \varpi$. Let $\mathcal{E}$ be an event with $Q_0(\mathcal{E}) > 0$ and let $\mathfrak{d}$ be a measurable map that assigns to each $(\nabla\psi_1, \nabla\psi_2, u) \in \mathcal{E}$ a function $\mathfrak{d} : F \to \mathbb{Z}$ with $\nabla\mathfrak{d} = \nabla\psi_2 - \nabla\psi_1$, such that*

(a) $\mathcal{E}$ is invariant under the diagonal action of $\mathbb{Z}^2$, and

(b) if $(\nabla\psi_1, \nabla\psi_2, u) \in \mathcal{E}$ and $\tilde\psi_2 \in \Omega$ differs from ψ_2 on finitely many faces, then $(\nabla\psi_1, \nabla\tilde\psi_2, u) \in \mathcal{E}$ and $\mathfrak{d}(\nabla\psi_1, \nabla\tilde\psi_2, u) = \mathfrak{d}(\nabla\psi_1, \nabla\psi_2, u) + (\tilde\psi_2 - \psi_2)$.

Then μ_2 is smooth. The same holds with the roles of the two copies exchanged.

Proof. Let $Q := Q_0(\cdot \mid \mathcal{E})$. Its second marginal has density $(\mu_1 \otimes \varpi)(\text{section of } \mathcal{E})/Q_0(\mathcal{E})$ with respect to μ_2; by (a) this density is translation-invariant, hence constant, so the second marginal is μ_2. Conditionally on $(\nabla\psi_1, u)$, the law of $\nabla\psi_2$ under Q is μ_2 conditioned on the section of $\mathcal{E}$, which by (b) is invariant under finite modifications; conditioning on such an event preserves the DLR equations. Let ψ_1^0 be the representative of $\nabla\psi_1$ with $\psi_1^0(0) = 0$ and $\hat{\psi}_2 := \psi_1^0 + \mathfrak{d}$, an admissible function with gradient $\nabla\psi_2$. By (b), replacing ψ_2 by $\tilde{\psi}_2 \sim \gamma_\Lambda(\cdot \mid \psi_2)$ replaces $\hat{\psi}_2$ by $\hat{\psi}_2 + (\tilde{\psi}_2 - \psi_2)$, whose conditional law is $\gamma_\Lambda(\cdot \mid \hat{\psi}_2)$ because the specification commutes with adding constants. Hence $\mathbb{E}_Q[f(\hat{\psi}_2)] = \mathbb{E}_Q[\gamma_\Lambda(f \mid \hat{\psi}_2)]$ for bounded measurable f: the law of $\hat{\psi}_2$ is a Gibbs measure on heights with gradient law μ_2. □

Proposition 3.13 (A top or bottom level)**.** *Let μ_1, μ_2 be ergodic. If $\bar{\mu}(E_{\text{top}} \cup E_{\text{bot}}) > 0$, then μ_1 and μ_2 are both smooth.*

Proof. On E_{top} let $\mathfrak{d} := D - \max I^+$, the lift of $\nabla\psi_2 - \nabla\psi_1$ for which $\max I^+ = 0$; on E_{bot} use $\min I^-$ instead. Condition (a) of Lemma 3.12 holds by robustness. For (b), changing ψ_2 on a finite set changes the honest difference D by $\tilde{\psi}_2 - \psi_2$ and does not change I^+ (Lemma 3.10). Apply Lemma 3.12 with $\mathcal{U}$ the space of n and $\varpi = \pi$, once as stated and once with the copies exchanged (using $-\mathfrak{d}$). □

Corollary 3.14. *Let $\mu_1 \neq \mu_2$ be ergodic translation-invariant gradient Gibbs measures. Then either both are smooth, or $\bar{\mu}(E_{\mathbb{ZZ}}) = 1$.*

Proof. $\bar{\mu}(E_+ \cup E_-) = 0$ by Proposition 3.11; $\bar{\mu}(E_\emptyset) = 0$ by Lemma 3.9, since on $E_\emptyset$ all clusters at all levels are finite; and if $\bar{\mu}(E_{\text{top}} \cup E_{\text{bot}}) > 0$, both measures are smooth by Proposition 3.13. □

4 The fold, and the proof of Theorem 1.4

Throughout this section $c > 1$, and μ_1, μ_2 are as in Theorem 1.4: ergodic translation-invariant gradient Gibbs measures of slope 0; by Lemma 2.4(iii) they are equilibrium states. By Corollary 3.14 it remains to treat the case

$$\bar{\mu}(E_{\mathbb{ZZ}}) = 1,$$

which we assume until the end of Subsection 4.6. Fix the band width $M := 2$ (any integer $M \geq 2$ would do; $M \geq 2$ is used in Lemma 4.1(iv)). Only $I^- = \mathbb{Z}$ is used in the construction.

The idea is Sheffield's repeated swap [17, Section 8.6.3]. When every level carries infinite clusters of both signs, no single swap is available, but one can exchange the two copies on infinitely many bands of levels at once, so that the difference is folded into a strip (Figure 4). The steps are:

(1) a *band index* $k(x)$, defined from the infinite minus clusters at the levels jM, which changes by at most one between neighbouring faces (Lemma 4.1);

(2) the *fold* R, which applies to each face the composition of swaps prescribed by its band index; it is admissible and preserves the joint weight (Lemma 4.2), and in the image every infinite cluster at the levels 0 and M is confined to one *double band*, a pair of adjacent bands (Lemma 4.3);

(3) a *local reconstruction*: on a box, the image and boundary data of size $O(\text{perimeter})$ determine the original, so the fold does not change the free energy per face (Lemmas 4.4–4.6);

(4) hence the image is again an equilibrium state with independent residual energies, and the Lanford–Ruelle theorem makes it a Gibbs measure (Proposition 4.7, Lemmas 4.8–4.9). Sheffield obtains this step from his variational principle [17, Lemma 8.3.2 and Theorem 2.5.2]; the

image here carries residual energies and a hard constraint between the copies, and the subshift formulation handles both;

(5) a Burton–Keane argument limits the number of infinite clusters of the image (Lemma 4.12); then either some double band carries an infinite cluster, and the lowest such double band gives a canonical height, so that both measures are smooth, or none does, and a symmetry of the image gives a contradiction.

4.1 The band index

For a triple $\bar{\omega}$ with $I^-(\bar{\omega}) = \mathbb{Z}$ and $j \in \mathbb{Z}$, let $T_j(\bar{\omega})$ be the union of the infinite minus FK clusters at level jM.

Lemma 4.1 (The band index)**.**

(i) *$T_j \subset T_{j+1}$; $x \in T_j$ implies $D(x) < jM$; and every face lies in T_j for all large j. Hence $k(x) := \min\{j : x \in T_{j+1}\}$ is a well-defined integer, $x \in T_{k(x)+1} \setminus T_{k(x)}$, and $D(x) < (k(x)+1)M$.*

(ii) *If $x \in T_j$, y is NN-adjacent to x and $D(y) < jM$, then $y \in T_j$.*

(iii) *For NN-adjacent faces $|k(x) - k(y)| \le 1$, and if $k(y) = k(x) + 1 =: j$ then $D(x) = jM - 1$ and $D(y) = jM$.*

(iv) *For the two faces x, y of a diagonal d, $|k(x) - k(y)| \le 1$, and if $k(y) = k(x) + 1 =: j$ then exactly one of the following holds: (α) $D(y) = jM$; (β) $D(y) = jM + 1$, $D(x) = jM - 1$ and $D = jM$ on $d^\perp$, so that d is jM-monotone; (γ) $D(x) = D(y) = jM - 1$, $D = jM$ on $d^\perp$ and $n_d \ge 1$, so that d is a closed jM-saddle of sign $-$.*

(v) *$k(V_{2M}\bar{\omega}) = k(\bar{\omega}) + 2$, and k is covariant under translations.*

Proof. (i) $T_j \subset T_{j+1}$ by Lemma 3.6. Let C be an infinite minus cluster at some level $j_0 M$, $y \in C$, and $y = y_0, \dots, y_m = x$ an NN path. For $jM > \max(j_0 M, \max_i D(y_i))$ all y_i are minus faces at level jM, C lies in an infinite minus cluster at level jM, and so does x. (ii) An NN edge between two minus faces is an edge of G_{jM}. (iii) Let $k(x) < k(y)$ and $j := k(x) + 1$. Then $x \in T_j$ and $y \notin T_j$, so $D(y) \ge jM$ by (ii), $D(x) \le jM - 1$, and Lemma 3.2(i) gives equality in both. Since $x \in T_{j+1}$ and $D(y) = jM < (j+1)M$, (ii) gives $y \in T_{j+1}$, that is $k(y) = j$. (iv) Let $z \in d^\perp$. By (iii), $|k(x) - k(y)| \le 2$, and if $k(y) = k(x) + 2$ then $k(z) = k(x) + 1 =: j$, and (iii) for the pairs $\{x, z\}$ and $\{z, y\}$ gives $D(z) = jM$ and $D(z) = (j+1)M - 1$, which is impossible for $M \ge 2$. Let $k(y) = k(x) + 1 = j$; then $x \in T_j$, $y \notin T_j$ and $D(x) \le jM - 1$. If $D(y) \ge jM$, then $D(y) \le D(x) + 2$ gives (α), or $D(y) = jM + 1$ and $D(x) = jM - 1$, and then $D = jM$ on $d^\perp$ by Lemma 3.2(i), which is (β). If $D(y) < jM$: if $D(z) < jM$ for some $z \in d^\perp$, then $z \in T_j$ by (ii) applied to x, and $y \in T_j$ by (ii) applied to z, a contradiction; so $D = jM$ on $d^\perp$, $D(x) = D(y) = jM - 1$, and the bridge d is closed because $y \notin T_j$, which is (γ). (v) is clear. □

4.2 The folding map

For $k \in \mathbb{Z}$ let

$$g_k(p_1, p_2) := \begin{cases} (p_1 + kM/2,\ p_2 - kM/2) & k \text{ even}, \\ (p_2 - (k+1)M/2,\ p_1 + (k+1)M/2) & k \text{ odd}, \end{cases} \qquad s_\ell(p_1, p_2) := (p_2 - \ell,\ p_1 + \ell).$$

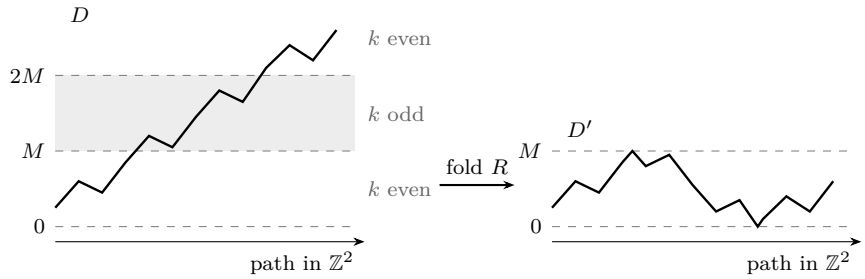


Figure 4: The fold, schematically. On the left, the difference D along a path, with the bands of levels $[kM, (k+1)M)$ shaded alternately. The fold shifts the two copies on every band and also exchanges them on the odd bands, so that $D' = D - kM$ on even bands and $D' = (k+1)M - D$ on odd bands: the difference is folded like an accordion into a strip of width about M (right). In the proof the band index $k(x)$ of a face is defined through the infinite minus clusters at the levels jM, not through the value of D at x alone, which is what makes the fold admissible and keeps the joint weight unchanged (Lemmas 4.1 and 4.2).

Then $g_j = g_{j-1} \circ s_{jM}$ for every j: crossing from band $j-1$ into band j adds one swap at level jM. Define $R(\bar\omega) := (\psi_1', \psi_2', n')$ by

$$(\psi_1', \psi_2')(x) := g_{k(x)}(\psi_1(x), \psi_2(x)), \qquad n_d' := n_d + \tfrac{1}{2}\big([\mathsf{a}(\psi_1'; d) + \mathsf{a}(\psi_2'; d)] - [\mathsf{a}(\psi_1; d) + \mathsf{a}(\psi_2; d)]\big),$$

and write $D' := \psi_2' - \psi_1'$; thus $D' = D - kM$ where k is even and $D' = (k+1)M - D$ where k is odd.

Lemma 4.2 (The fold is admissible and weight-preserving)**.**

(i) $R(\bar\omega)$ *is a triple:* $\psi_1', \psi_2' \in \Omega$ *and* $n' \geq 0$*. In the notation of Lemma 4.1(iv),* $n_d' = n_d$ *if* k *is constant on* d *or in case* (α)*,* $n_d' = n_d + 1$ *in case* (β)*, and* $n_d' = n_d - 1$ *in case* (γ)*. Moreover* $\tau_d(R\bar\omega) = \tau_d(\bar\omega)$ *for every* d*.*

(ii) $R(V_{2M}\bar\omega) = R(\bar\omega)$ *in* $\bar\Omega$*, and* R *commutes with translations, so* R *is defined on* $\bar\Omega/\langle V_{2M}\rangle$*.*

(iii) $\psi_1' + \psi_2' = \psi_1 + \psi_2$*.*

(iv) $D' \leq M - 1$ *where* k *is even and* $D' \geq 1$ *where* k *is odd. For NN-adjacent faces with* $k(y) = k(x) + 1 = j$*,* $(D'(x), D'(y)) = (1, 0)$ *if* j *is even and* $(M-1, M)$ *if* j *is odd.*

Proof. (i) If $k(x) = k(y)$, both faces are transformed by the same g_k, which shifts each copy by its own constant and, for odd k, exchanges them; NN increments and the a-sum on a diagonal are preserved. If $k(y) = k(x) + 1 = j$, then $(\psi'(x), \psi'(y)) = (g_{j-1}\psi(x), g_{j-1}s_{jM}\psi(y))$. For an NN pair, $D(y) = jM$ by Lemma 4.1(iii), so s_{jM} acts trivially at y. For a diagonal, the a-sum of the image equals that of the pair in which only y is swapped at level jM: in case (α) nothing changes, in case (β) a jM-monotone diagonal becomes a saddle $(+2)$, and in case (γ) a saddle becomes monotone (-2), by Lemma 3.2(ii); in case (γ), $n_d \geq 1$. (ii) $g_{k+2}(p_1, p_2 + 2M) = g_k(p_1, p_2) + (M, M)$. (iii) and (iv) follow from the formulas and Lemma 4.1(i),(iii). □

4.3 Structure of the image

Put $\Pi_m := \{x : k(x) \in \{2m, 2m+1\}\}$ and $\Pi_m' := \{x : k(x) \in \{2m-1, 2m\}\}$, the *double bands*. FK clusters of the image are those of the triple $R(\bar\omega)$, defined with n'.

Lemma 4.3 (Clusters of the image)**.**

(a) Every minus FK cluster at level 0 *and every plus FK cluster at level* M *of* $R(\bar\omega)$ *is finite.*

(b) Every plus FK cluster at level 0 *of* $R(\bar\omega)$ *is contained in one* Π_m*, and every minus FK cluster at level* M *of* $R(\bar\omega)$ *is contained in one* Π'_m*.*

Proof. (a) Let C' be a minus cluster at level 0 of the image. Its faces have $D' < 0$, hence even k and $D < kM$. NN-adjacent faces of C' have the same k (both even, differing by at most 1), and they are NN-adjacent minus faces at level kM of $\bar\omega$. Let $d = \{x, y\}$ be an open bridge of C': $D'(x) = D'(y) = -1$, $D' = 0$ on $d^\perp$, $n'_d = 0$. Then $k(x) = k(y) =: k$ (both even, Lemma 4.1(iv)) and $D(x) = D(y) = kM - 1$. For $z \in d^\perp$, $k(z) \in \{k-1, k, k+1\}$: $k(z) = k+1$ would give $D(z) = (k+2)M$, too large; $k(z) = k-1$ would give $D(z) = kM$ with $z \in T_k$, impossible; so $k(z) = k$, $D(z) = kM$, d is a kM-saddle of sign $-$ of $\bar\omega$, and $n_d = n'_d = 0$. So C' is contained in one minus FK cluster at level kM of $\bar\omega$, which is finite because its faces are not in T_k. Plus clusters at level M are treated in the same way with k odd: $D' > M$ if and only if $D < kM$, and a bridge has $D' = M + 1$ on d and M on $d^\perp$, while $k(z) = k \pm 1$ would give $D(z) = kM$ with $z \in T_k$, or $D(z) = (k+2)M$. (b) By Lemma 4.2(iv), an NN pair in different Π's ($k = 2m-1$ and $2m$) has $D' = (1, 0)$, so NN edges of the plus graph at level 0 stay in one Π_m. Let $d = \{x, y\}$ be a 0-saddle of sign $+$ of the image with $k(x) = 2m-1$ and $k(y) = 2m =: j$. Then $D(x) = jM - 1$ and $D(y) = jM + 1$, which is case (β) of Lemma 4.1(iv), and $n'_d = n_d + 1 \geq 1$: the bridge is closed. At level M, an NN pair with $k = 2m$ and $2m+1$ has $D' = (M-1, M)$, and a saddle of sign $-$ at level M joining $k(x) = j - 1$ (even) and $k(y) = j$ (odd) has $D(x) = jM - 1$, $D(y) = jM + 1$, case (β) again. □

4.4 Local reconstruction

Let Λ be a box, $\partial^*\Lambda$ the set of faces of Λ that are $*$-adjacent to its complement, b_0 a fixed corner of Λ, diam the NN-diameter of Λ, and $L_\Lambda := M + 2 + 2\,\mathrm{diam}$. In the following, "gradients on Λ" are the increments between pairs of faces that both lie in Λ. For $\bar\omega$ with $I^- = \mathbb{Z}$ put

- $X_\Lambda(\bar\omega) := (\nabla\psi_1, \nabla\psi_2 \text{ on } \Lambda;\ D \bmod 2M \text{ on } \Lambda;\ n_d \text{ for the diagonals } d \subset \Lambda)$,
- $Y_\Lambda(\bar\omega) := (\nabla\psi'_1, \nabla\psi'_2 \text{ on } \Lambda;\ D' \bmod 2M \text{ on } \Lambda;\ D'^{\mathrm{tr}} := \max(-1, \min(M+1, D')) \text{ on } \Lambda;\ n'_d \text{ for } d \subset \Lambda)$,
- $B_\Lambda(\bar\omega) := \big((k(b) - k(b_0))_{b \in \partial^*\Lambda};\ k(b_0) \bmod 2;\ \max(-L_\Lambda, \min(L_\Lambda, D'(b_0)))\big)$.

All three are functions on $\bar\Omega/\langle V_{2M}\rangle$. Since k is 1-Lipschitz along the NN-connected ring $\partial^*\Lambda$, B_Λ takes at most $3^{|\partial^*\Lambda|} \cdot 2 \cdot (2L_\Lambda + 1)$ values.

Lemma 4.4 (Reconstruction)**.** (X_Λ, B_Λ) *and* (Y_Λ, B_Λ) *determine each other, and* $\prod_{d\subset\Lambda} q^{\tau_d}$ *takes the same value at* X_Λ *and at* Y_Λ*.*

Proof. The statement about τ is Lemma 4.2(i).

Fact 1. If k is not constant on Λ, then $|D'(b_0)| < L_\Lambda$: some NN pair of Λ has different values of k, so $D' \in \{0, 1, M-1, M\}$ on it (Lemma 4.2(iv)), and D' is 1-Lipschitz along NN steps.

Fact 2. $T_j \cap \Lambda$ is the set of faces of Λ joined, by a path of minus faces and edges of $G_{jM}(\bar\omega)$ inside Λ, to a face $b \in \partial^*\Lambda$ with $k(b) < j$. Indeed a face joined to such a b lies in its infinite cluster; conversely, a path in an infinite cluster from $x \in \Lambda$ to the complement of Λ has a last face in Λ before its first exit, and since edges of G join $*$-adjacent faces, that face is in $\partial^*\Lambda$ and in T_j.

From (X_Λ, B_Λ) *to* Y_Λ. If the third entry of B_Λ is $\pm L_\Lambda$, then $k \equiv k(b_0)$ on Λ by Fact 1; the image on Λ is the original ($k(b_0)$ even) or the original with the copies exchanged ($k(b_0)$ odd), $D' \equiv \pm D \pmod{2M}$, D'^{tr} is the constant -1 or $M+1$ according to the sign, and $n' = n$. Otherwise $D'(b_0)$ is known; choose the representative modulo V_{2M} with $k(b_0) \in \{0,1\}$. Then $D(b_0)$, which equals $D'(b_0) + k(b_0)M$ or $(k(b_0)+1)M - D'(b_0)$, is known, hence the honest D on Λ, hence all $T_j \cap \Lambda$ by Fact 2, hence k on Λ and Y_Λ.

From (Y_Λ, B_Λ) *to* X_Λ. In the first case invert as above. Otherwise the honest D' on Λ is known, and it suffices to recover k on Λ. For $\kappa \in \mathbb{Z}$ let $\mathcal{T}_\kappa := \{x \in \Lambda : k(x) < \kappa\}$; by Fact 2, $\mathcal{T}_\kappa = \emptyset$ for $\kappa \le \min_{\partial^*\Lambda} k$. We claim that $\{x \in \Lambda : k(x) = \kappa\}$ is the set of faces of $\Lambda \setminus \mathcal{T}_\kappa$ joined to a *seed* by a path in the graph $\mathfrak{G}_\kappa$, where $\mathfrak{G}_\kappa$ is the plus graph at level 0 of the image if κ is odd, and the minus graph at level M of the image if κ is even, restricted to the faces of $\Lambda \setminus \mathcal{T}_\kappa$ and to bridges $d \subset \Lambda \setminus \mathcal{T}_\kappa$; the seeds are the faces $b \in \partial^*\Lambda$ with $k(b) = \kappa$ and the faces of $\Lambda \setminus \mathcal{T}_\kappa$ that are NN-adjacent to $\mathcal{T}_\kappa$. This determines $\mathcal{T}_{\kappa+1}$ from $\mathcal{T}_\kappa$, the image and B_Λ, hence k on Λ by induction on κ.

We prove the claim for odd κ; the even case is the same with level M. Put $j := \kappa + 1$. (1) Faces with $k = \kappa$ are plus faces at level 0 of the image ($D' \ge 1$). For two such faces, an NN edge is an edge of both $G_{jM}(\bar\omega)$ (both have $D < jM$) and $\mathfrak{G}_\kappa$. A bridge of $G_{jM}(\bar\omega)$ between them ($D = jM - 1$ on d, jM on $d^\perp$, $n_d = 0$) has $k = j$ on $d^\perp$, because faces with $D = jM$ are not in T_j and are NN-adjacent to faces with $k = \kappa$; so $D' = 1$ on d, $D' = 0$ on $d^\perp$ and $n'_d = n_d = 0$, a bridge of $\mathfrak{G}_\kappa$. Conversely, a bridge of $\mathfrak{G}_\kappa$ between two faces with $k = \kappa$ has $D = jM - 1$ on d, and for $z \in d^\perp$, where $D'(z) = 0$: $k(z) = \kappa$ is impossible (odd band, $D' \ge 1$), $k(z) = \kappa - 1$ would give $D(z) = (\kappa-1)M$, too small for $M \ge 2$, so $k(z) = j$ and $D(z) = jM$, a bridge of $G_{jM}(\bar\omega)$. (2) Seeds have $k = \kappa$: a face of $\Lambda \setminus \mathcal{T}_\kappa$ has $k \ge \kappa$, and if it is NN-adjacent to a face with $k < \kappa$, then $k = \kappa$. (3) No edge of $\mathfrak{G}_\kappa$ joins a face with $k = \kappa$ to one with $k = \kappa + 1$ (Lemma 4.3(b); its proof is local). Hence every face joined to a seed in $\mathfrak{G}_\kappa$ has $k = \kappa$. (4) Conversely, let $k(x) = \kappa$ with $x \in \Lambda$. Then x lies in an infinite minus cluster of $G_{jM}(\bar\omega)$. Follow a path in it from x until it is about to leave Λ or to enter $\mathcal{T}_\kappa$. Up to that point its faces are in $T_j \setminus \mathcal{T}_\kappa$, that is, have $k = \kappa$, and its edges are edges of $\mathfrak{G}_\kappa$ by (1). If the next step leaves Λ, the current face is a seed of the first kind. If the next step enters $\mathcal{T}_\kappa$, it is an NN step (a bridge at level jM has $D = jM - 1$ on both faces, while faces with $k < \kappa$ have $D < \kappa M \le jM - 2$), so the current face is a seed of the second kind. □

4.5 Free energy of the image

We now compare free energies per face. The original pair, seen through the pattern X_Λ, has the free energy of two independent equilibrium states (Lemma 4.6(i)); by Lemma 4.4 the image, seen through Y_Λ, has at most that much plus the entropy of the boundary data B_Λ, which is of the order of the perimeter (Lemma 4.5). Since no pair process can do better than two equilibrium states (Lemma 4.6(ii)), the image is itself optimal, and this forces its residual energies to be independent and geometric (Lemma 4.8) and its pair marginal to be a Gibbs measure (Lemma 4.9).

Let $\mathbb{P}^\sharp$ be the law on $\bar\Omega/\langle V_{2M}\rangle$ of the class of $\bar\omega^{(\vartheta)}$, where the gradient triple has law $\bar\mu$ and ϑ is uniform on $\{0, \dots, 2M-1\}$ and independent. Then $\mathbb{P}^\sharp$ is translation-invariant and $\mathbb{P}^\sharp(I^- = \mathbb{Z}) = 1$. Let

$$\bar\rho := \mathbb{P}^\sharp \circ R^{-1},$$

a translation-invariant probability measure on $\bar\Omega$ (Lemma 4.2(ii)).

For a box Λ put $W_\Lambda := \prod_{d\subset\Lambda} q^{\tau_d} = \prod_{d\subset\Lambda} c^{\mathsf{a}(\psi_1;d)+\mathsf{a}(\psi_2;d)} q^{n_d}$, $N_\Lambda := \#\{d \subset \Lambda\}$, and for the law p of a pattern Z_Λ ($Z = X$ or Y) put $\mathrm{FE}(Z_\Lambda) := \sum_z p(z)\log(p(z)/W_\Lambda(z))$. For a translation-invariant

law Q of a pair process (a gradient pair, possibly with decorations) put

$$P^{(2)}(Q) := \mathsf{h}(Q) + \log c \cdot Q\Big(\sum_{d \in \{d_0^+, d_0^-\}} \big[\mathsf{a}(\psi_1; d) + \mathsf{a}(\psi_2; d)\big] \Big).$$

Lemma 4.5 (The fold costs no free energy)**.** $\mathrm{FE}(Y_\Lambda \text{ under } \mathbb{P}^\sharp) \le \mathrm{FE}(X_\Lambda \text{ under } \mathbb{P}^\sharp) + \log\big(3^{|\partial^* \Lambda|} \cdot 2 \cdot (2L_\Lambda + 1)\big)$.

Proof. By Lemma 4.4, $(X_\Lambda, B_\Lambda) \mapsto (Y_\Lambda, B_\Lambda)$ is a bijection between the supports that preserves W_Λ. So $\sum p(x,b) \log(p(x,b)/W_\Lambda(x)) = \sum p'(y,b) \log(p'(y,b)/W_\Lambda(y)) =: J$. Now $\mathrm{FE}(X_\Lambda) = J + H(B_\Lambda \mid X_\Lambda)$ and $\mathrm{FE}(Y_\Lambda) = J + H(B_\Lambda \mid Y_\Lambda)$, with H the conditional Shannon entropy, and $0 \le H \le \log \#\{\text{values of } B_\Lambda\}$. □

Lemma 4.6 (Free energy of the original pair)**.**

(i) $\lim_\Lambda |\Lambda|^{-1} \mathrm{FE}(X_\Lambda \text{ under } \mathbb{P}^\sharp) = -(2P^* + 2\log c) + 2\log(1-q)$.

(ii) For every translation-invariant law Q of a pair process on admissible pairs, with decorations each of which takes, on a box Λ, at most $e^{o(|\Lambda|)}$ values given the gradient pair on Λ: $P^{(2)}(Q) \le 2P^* + 2\log c$.

Proof. (i) Under $\mathbb{P}^\sharp$ the pattern X_Λ consists of independent pieces with laws $\mu_1|_\Lambda$, $\mu_2|_\Lambda$, the uniform law on $\mathbb{Z}/2M$ (the value of $D \bmod 2M$ at one face), and π on the diagonals. So $\mathrm{FE}(X_\Lambda) = -H(\mu_1|_\Lambda) - H(\mu_2|_\Lambda) - \log 2M - \log c \sum_{d \subset \Lambda} (\mu_1(\mathsf{a}_d) + \mu_2(\mathsf{a}_d)) + N_\Lambda \log(1-q)$. Divide by $|\Lambda|$: $N_\Lambda/|\Lambda| \to 2$, and $\mathsf{h}(\mu_i) + \log c \cdot \mu_i(\mathsf{a}(d_0^+) + \mathsf{a}(d_0^-)) = P(\mu_i) + \log c = P^* + \log c$ because μ_i is an equilibrium state. (ii) The Shannon entropy of the decorated pattern on Λ is at most that of the two gradient patterns plus $o(|\Lambda|)$, so $\mathsf{h}(Q) \le \mathsf{h}(Q^1) + \mathsf{h}(Q^2)$ for the two gradient marginals, and $P(Q^i) \le P^*$. □

The decorations used below are $D' \bmod 2M$ and D'^{tr}. On a box Λ, given the gradient pair on Λ, the first is determined by its value at one face, and the second by $\max(-1 - \mathrm{diam}, \min(M + 1 + \mathrm{diam}, D'(x_0)))$ for a face $x_0 \in \Lambda$; so they take at most $2M(2M + 4 + 2\,\mathrm{diam})$ values, which is $e^{o(|\Lambda|)}$.

Proposition 4.7 (The image is an equilibrium state)**.** *Let Y^{pr} be the pair part of Y (the gradients of ψ_1', ψ_2', $D' \bmod 2M$ and D'^{tr}) and ρ^{pr} its law under $\bar\rho$. Then*

(i) $P^{(2)}(\rho^{\mathrm{pr}}) = 2P^* + 2\log c$*;*

(ii) $\lim_\Lambda |\Lambda|^{-1} \mathcal{H}\big(\mathrm{law}\, Y_\Lambda \,\big\|\, \mathrm{law}\, Y_\Lambda^{\mathrm{pr}} \otimes \pi_\Lambda\big) = 0$*, where $\mathcal{H}(\cdot\,\|\,\cdot)$ is relative entropy and π_Λ the product of π_0 over the diagonals in Λ.*

Proof. Since $W_\Lambda(y,n) = c^{\sum \mathsf{a}} \prod q^{n_d}$ and $\pi_\Lambda(n) = (1-q)^{N_\Lambda} \prod q^{n_d}$, the chain rule gives

$$\mathrm{FE}(Y_\Lambda) = \Big[-H(Y_\Lambda^{\mathrm{pr}}) - \log c \sum_{d \subset \Lambda} \bar\rho\big(\mathsf{a}(\psi_1'; d) + \mathsf{a}(\psi_2'; d)\big)\Big] + \mathcal{H}\big(\mathrm{law}\, Y_\Lambda \,\big\|\, \mathrm{law}\, Y_\Lambda^{\mathrm{pr}} \otimes \pi_\Lambda\big) + N_\Lambda \log(1-q).$$

The bracket divided by $|\Lambda|$ tends to $-P^{(2)}(\rho^{\mathrm{pr}})$, and the relative entropy is nonnegative. By Lemmas 4.5 and 4.6(i), $-P^{(2)}(\rho^{\mathrm{pr}}) + \limsup |\Lambda|^{-1} \mathcal{H}(\cdots) \le -(2P^* + 2\log c)$, and by Lemma 4.6(ii), $P^{(2)}(\rho^{\mathrm{pr}}) \le 2P^* + 2\log c$. Both claims follow. □

Lemma 4.8 (Residual energies of the image)**.** *Under $\bar\rho$, n' is independent and identically distributed with law π_0 and independent of the pair $(\psi_1', \psi_2') \in \bar\Omega$.*

Proof. Fix a box Λ_0 and let $H_0 := \mathcal{H}(\operatorname{law} Y_{\Lambda_0} \,\|\, \operatorname{law} Y^{\mathrm{pr}}_{\Lambda_0} \otimes \pi_{\Lambda_0})$. For a large box Λ containing m disjoint translates $\Lambda^{(1)}, \dots, \Lambda^{(m)}$ of Λ_0, with $n^{(i)}$ the restriction of n' to the diagonals in $\Lambda^{(i)}$,

$$\begin{aligned}\mathcal{H}(\operatorname{law} Y_\Lambda \,\|\, \operatorname{law} Y^{\mathrm{pr}}_\Lambda \otimes \pi_\Lambda) &\geq \sum_i \mathbb{E}\Big[\mathcal{H}\big(\operatorname{law}(n^{(i)} \mid Y^{\mathrm{pr}}_\Lambda, n^{(1)}, \dots, n^{(i-1)}) \,\|\, \pi\big)\Big] \\ &\geq \sum_i \mathbb{E}\Big[\mathcal{H}\big(\operatorname{law}(n^{(i)} \mid Y^{\mathrm{pr}}_{\Lambda^{(i)}}) \,\|\, \pi\big)\Big] = mH_0,\end{aligned}$$

by monotonicity of relative entropy under marginalisation, the chain rule, convexity (conditioning on less decreases the expected relative entropy) and translation invariance. Taking $m|\Lambda_0|/|\Lambda| \to 1$ and using Proposition 4.7(ii) gives $H_0 = 0$. So for every box Λ_0, n' on the diagonals of Λ_0 has law π and is independent of $Y^{\mathrm{pr}}_{\Lambda_0}$; applying this to boxes $\Lambda_1 \supset \Lambda_0$ and letting $\Lambda_1 \uparrow F$, n' on Λ_0 is independent of the whole process Y^{pr}. Finally the pair $(\psi'_1, \psi'_2) \in \bar{\Omega}$ is a measurable function of Y^{pr}: the honest D' is determined by the gradients and by D'^{tr} at any face where $0 \leq D' \leq M$, and such faces exist because k is not constant (Lemma 4.2(iv)). □

Lemma 4.9 (Equilibrium implies Gibbs)**.** *The pair marginal of $\bar{\rho}$ satisfies the DLR equations with honest difference: for every finite $A \subset F$ and every bounded measurable function f of pairs (ψ'_1, ψ'_2) modulo common shifts, $\bar{\rho}(f) = \bar{\rho}((\gamma_A \otimes \gamma_A)f)$.*

Proof. Fix an integer $N \geq 3$. Let $\mathbb{Y}_N \subset ((\mathbb{Z}/N)^2)^F$ be the set of pairs (y_1, y_2) with $y_i(x+e_j) - y_i(x) \in \{0, 1\}$ mod N for all x, i, j; it is a nearest-neighbour subshift of finite type, hence has the topological Markov property [2, p. 11]. Every element of $\mathbb{Y}_N$ is the reduction mod N of a pair of admissible functions, unique up to adding multiples of N to each: the four increments around a block lie in $\{0, 1\}$ and sum to 0 mod N, hence to 0, so the increments integrate on $\mathbb{Z}^2$. Let Φ be the interaction with $\Phi_{B(x)}(y) := -\log w_x(y_1) - \log w_x(y_2)$ and $\Phi_A := 0$ for every other A; it is bounded, of finite range and translation-invariant, hence absolutely summable.

Let ρ_N be the law of $((\psi'_1 + U) \bmod N, (\psi'_2 + U) \bmod N)$, with (ψ'_1, ψ'_2) a representative of the pair under $\bar{\rho}$ and U uniform on $\mathbb{Z}/N$ and independent. It is translation-invariant and its entropy is $\mathsf{h}(\rho^{\mathrm{pr}})$, since both processes are the gradient pair with decorations of subexponential complexity. By Proposition 4.7(i), $\mathsf{h}(\rho_N) - \rho_N(f_\Phi) = P^{(2)}(\rho^{\mathrm{pr}}) - 2\log c = 2P^*$, and by Lemma 4.6(ii) no invariant measure on $\mathbb{Y}_N$ does better, so ρ_N is an equilibrium measure for f_Φ. By the Lanford–Ruelle theorem for subshifts of finite type (Theorem A.1), ρ_N is a Gibbs measure for Φ: for every finite A, the conditional law of the pattern on A given the pattern off A is the Boltzmann distribution on the patterns u with $u \vee y_{A^c} \in \mathbb{Y}_N$.

Let A be a box. Its complement is connected, so the patterns u with $u \vee y_{A^c} \in \mathbb{Y}_N$ correspond, by the lifting property applied to the whole configuration, to the admissible fillings of A for each copy, and the Boltzmann weights are those of $\gamma_A \otimes \gamma_A$. Hence $\bar{\rho}(f) = \bar{\rho}((\gamma_A \otimes \gamma_A)f)$ for every f that is a bounded function of the gradient pair on a finite window and of $D'(x_0)$ mod N. For a bounded f of the gradient pair on a window and of the honest $D'(x_0)$, let f_N be the same function evaluated at the representative of $D'(x_0)$ mod N in $(-N/2, N/2]$; then $f_N \to f$ and $(\gamma_A \otimes \gamma_A)f_N \to (\gamma_A \otimes \gamma_A)f$ pointwise and boundedly as $N \to \infty$, since the kernel changes $D'(x_0)$ by at most $|A|$. Such f generate the σ-field, which proves the claim for boxes. For a general finite A choose a box $B \supset A$; by consistency of the specification, $(\gamma_B \otimes \gamma_B)(\gamma_A \otimes \gamma_A) = \gamma_B \otimes \gamma_B$, so $\bar{\rho}(\gamma_A \otimes \gamma_A) = \bar{\rho}(\gamma_B \otimes \gamma_B)(\gamma_A \otimes \gamma_A) = \bar{\rho}(\gamma_B \otimes \gamma_B) = \bar{\rho}$. □

For a finite A whose complement is not connected the correspondence used for boxes fails: the mod-N kernel allows the heights inside a hole to shift by a multiple of N. This is why boxes are treated first.

Proposition 4.10 (The image is a Gibbs measure)**.** *$\bar\rho$ is a translation-invariant probability measure on $\bar\Omega$ with $\bar\rho\bar\gamma_\Lambda = \bar\rho$ for every finite Λ.*

Proof. Lemmas 4.8 and 4.9. □

4.6 Burton–Keane, and the proof of Theorem 1.4

Statements about "the image" concern $R(\bar\omega)$ with $\bar\omega$ distributed according to $\mathbb{P}^\sharp$. Its law $\bar\rho$ satisfies Proposition 4.10, so Lemmas 3.5 and 3.7–3.9 apply to $\bar\rho$ directly; no lift is needed, since $\bar\rho$ is a probability measure on triples with honest difference.

Lemma 4.11 (Sublinear growth along the axes)**.** *$\mathbb{P}^\sharp$-almost surely, for $i = 1, 2$ and both signs, $h_1(\pm me_i)/m \to 0$, $h_2(\pm me_i)/m \to 0$ and $D'(\pm me_i)/m \to 0$ as $m \to \infty$, where $h_i := 2\psi_i - x_1 - x_2$ is normalised by $h_i(0) = 0$. Consequently $h_1' := 2\psi_1' - x_1 - x_2 = (h_1 + h_2)/2 - D'$ and $h_2' := (h_1 + h_2)/2 + D'$ satisfy the same.*

Proof. For μ_i: $\xi := h(e_1) - h(0)$ is bounded, and by Birkhoff's theorem $(h(me_1) - h(0))/m$ converges almost surely to $f := \mathbb{E}[\xi \mid \text{invariant } \sigma\text{-field of the shift by } e_1]$. Since $|h(me_1 + e_2) - h(me_1)| \le 1$, f is also invariant under the shift by e_2, hence constant by ergodicity, and equal to the slope 0. For D': it is an integer field with translation-invariant law and NN increments bounded by 1; Birkhoff's theorem gives almost sure convergence of $D'(me_1)/m$, and the limit is 0 because $D'(me_1)$ has the law of $D'(0)$. The formulas for h_i' follow from Lemma 4.2(iii). □

Lemma 4.12 (Finitely many infinite clusters)**.** *$\bar\rho$-almost surely the plus graph at level 0 has finitely many infinite clusters, and so has the minus graph at level M.*

Proof. We treat the plus graph $\mathfrak{G}$ at level 0; the other case is symmetric (exchange the copies and shift by M). Suppose that with positive probability there are infinitely many infinite clusters, and choose r such that with positive probability $\Lambda_r := [-r, r]^2$ meets at least three of them.

Modification. Let A_0 be the smallest even integer at least $\max_{\Lambda_r} h_1' + 2r + 2$, let $G(x) := A_0 - |x_1| - |x_2|$, an admissible height in h-coordinates, and $S_G := \{x : G(x) \ge h_1'(x) + 2\}$; then $\Lambda_r \subset S_G$. By Lemma 4.11 the set $\{x : h_2'(x) + |x|_1 < A_0\}$ is finite almost surely: if $h_2'(x) - h_2'(0) \le -|x|_1 + t$, then comparing along $0 \to (x_1, 0) \to x$ and $0 \to (0, x_2) \to x$ and using $|h_2'(y) - h_2'(y')| \le |y - y'|_1$ gives $h_2'(x_1, 0) - h_2'(0) \le -|x_1| + t$ and $h_2'(0, x_2) - h_2'(0) \le -|x_2| + t$, which by Lemma 4.11 bounds $|x_1|$ and $|x_2|$. So we can fix n such that with positive probability Λ_r meets at least three infinite clusters and $G \le h_2'$ off Λ_n; call this event $\mathcal{A}$. On $\mathcal{A}$ define the modified triple $m(\bar\omega')$ by replacing h_2' with $h_2'' := h_2' \vee (G \wedge (h_1' + 2))$ and keeping ψ_1' and n'. By Lemma 3.1 it is admissible. It differs from $\bar\omega'$ only on faces of Λ_n; its difference $D'' = \max(D', \min((G - h_1')/2, 1)) \ge D'$; $D'' \ge 1$ on $S_G \cap \Lambda_n$; and faces with $D'' \ge 1 > D'$ lie in $S_G \cap \Lambda_n$. Since $\bar\gamma_{\Lambda_n}(\{m(\bar\omega')\} \mid \bar\omega') > 0$ and $\bar\rho\bar\gamma_{\Lambda_n} = \bar\rho$, we have $\bar\rho(\mathcal{T}) \ge \int_{\mathcal{A}} \bar\gamma_{\Lambda_n}(\{m(\bar\omega')\} \mid \bar\omega')\, \bar\rho(\mathrm{d}\bar\omega') > 0$ for every event $\mathcal{T}$ that contains all $m(\bar\omega')$ with $\bar\omega' \in \mathcal{A}$.

Trifurcation. Let $\mathcal{T}$ be the event that there is a finite connected set $\hat Q$ of vertices of $\mathfrak{G}$, contained in Λ_n, such that $\mathfrak{G} \setminus \hat Q$ has at least three infinite components adjacent to $\hat Q$. We claim that $m(\bar\omega') \in \mathcal{T}$ for $\bar\omega' \in \mathcal{A}$. Write $\mathfrak{G}'$ and $\mathfrak{G}''$ for the graphs of $\bar\omega'$ and $m(\bar\omega')$. The set $S_G \cap \Lambda_n$ is NN-connected, since $G - h_1'$ does not decrease along an NN step that decreases $|x|_1$, so every monotone path to the origin stays in S_G; and it consists of plus faces of $\mathfrak{G}''$. Let $\hat Q$ be the component of $\mathfrak{G}''$ restricted to Λ_n that contains it. (1) Every vertex of $\mathfrak{G}''$ outside $\hat Q$ is a plus face of $\bar\omega'$ with $D'' = D'$, and every edge of $\mathfrak{G}''$ between two such vertices is an edge of $\mathfrak{G}'$: for an NN edge this is clear; for a bridge d, $D'' = 1$ on d and $D'' = 0$ on $d^\perp$ force $D' = 1$ on d and then $D' = 0$ on $d^\perp$ (since $D' \le D''$ and by Lemma 3.2(i)), and n' is unchanged. Hence every component of $\mathfrak{G}'' \setminus \hat Q$ is contained in one cluster of

$\mathfrak{G}'$. (2) If C is an infinite cluster of $\mathfrak{G}'$ meeting Λ_r, then $C \cup \hat{Q}$ is connected in $\mathfrak{G}''$: faces of C stay plus faces, NN edges stay, and for a bridge d of C either $D'' = 0$ on $d^\perp$ and it stays an open bridge, or some $z \in d^\perp$ has $D''(z) \geq 1$ and the two faces of d are joined through z. Removing the finite set $\hat{Q}$ from the connected, infinite, locally finite graph on $C \cup \hat{Q}$ leaves finitely many components, one of which is infinite, adjacent to $\hat{Q}$, and contained in C. Three clusters C give three distinct such components by (1).

Counting. By translation invariance each translate $\Lambda_n + (2n+1)z$ carries the translated event with probability $\delta := \bar{\rho}(\mathcal{T}) > 0$, so the expected number of such tiles inside Λ_N grows like $\delta N^2/(2n+1)^2$. The sets $\hat{Q}$ of distinct tiles are disjoint and connected, and each other $\hat{Q}$ lies in one component of $\mathfrak{G} \setminus \hat{Q}$; so after contracting all of them to points, each point still separates at least three infinite components. The counting argument of Burton and Keane [3] (count the leaves of the resulting forest on the boundary) bounds their number by the number $O(N)$ of faces $*$-adjacent to Λ_N, since bridges join diagonal faces. This is a contradiction for large N. □

Proof of Theorem 1.4. Let $\mu_1 \neq \mu_2$ be as in the theorem. By Corollary 3.14 either both are smooth or $\bar{\mu}(E_{\mathbb{ZZ}}) = 1$; assume the latter. Let $\mathcal{M}$ be the set of $m \in \mathbb{Z}$ such that Π_m contains an infinite plus FK cluster at level 0 of the image, and $\mathcal{M}'$ the set of m such that Π'_m contains an infinite minus FK cluster at level M of the image, both computed for a representative of $\bar{\omega}$ modulo V_{2M}, which shifts them by 1. By Lemmas 4.3(b) and 4.12, $\mathcal{M}$ and $\mathcal{M}'$ are finite almost surely.

Case 1: $\mathbb{P}^\sharp(\mathcal{M} \neq \emptyset) > 0$ *or* $\mathbb{P}^\sharp(\mathcal{M}' \neq \emptyset) > 0$. Say the first. A gradient triple $(\nabla\psi_1, \nabla\psi_2, n)$ has $2M$ classes of honest lifts modulo V_{2M}, of which $\mathbb{P}^\sharp$ picks one uniformly. Let $\mathcal{E}_0$ be the event, of the gradient triple, that $\mathcal{M}$ is finite for every one of the $2M$ classes and nonempty for at least one; $\bar{\mu}(\mathcal{E}_0) > 0$. On $\mathcal{E}_0$, each class with $\mathcal{M} \neq \emptyset$ has exactly one honest lift D for which $\min \mathcal{M} = 0$; let $\mathfrak{d}$ be the smallest of these at most $2M$ lifts, which differ by constants. We apply Lemma 3.12 with $\mathcal{U}$ the space of n, $\varpi = \pi$ and $\mathcal{E} = \mathcal{E}_0$. Condition (a) holds since $\mathcal{E}_0$ is translation-invariant. For (b), let $\tilde{\psi}_2$ differ from ψ_2 on a finite set and fix an honest lift D; the lift $\tilde{D} := D + (\tilde{\psi}_2 - \psi_2)$ of the new triple still has $I^- = \mathbb{Z}$ (Lemma 3.10). Let S be a box containing all blocks that meet the changed set, and S^+ its $*$-neighbourhood.

(1) If jM exceeds D and $\tilde{D}$ on S^+, the minus graphs at level jM of the two triples coincide: all faces of S^+ are minus faces for both, and a jM-saddle cannot meet S, since it would lie in S^+ and contain faces with $D = jM$; so the bridges are the same.

(2) If jM is at most D and $\tilde{D}$ on S^+, there is no minus face at level jM in S^+ for either triple, no jM-saddle of sign $-$ meets S, and again the minus graphs coincide. In both cases T_j is unchanged.

(3) For the finitely many remaining j, T_j changes by finitely many faces: a face of the old T_j is in the new one unless it lies in S or in a finite component of its old cluster minus S, and symmetrically, as in Lemma 3.10.

So k, and hence $R(\bar{\omega})$, changes on finitely many faces and diagonals only. As in the proof of Lemma 3.10, an infinite plus cluster at level 0 of the old image contains an infinite connected piece that is unchanged, is contained in a cluster of the new image, and lies in the same Π_m, since old and new k agree there and the new cluster lies in one Π_m by Lemma 4.3(b); and symmetrically. So $\mathcal{M}$, computed in the lifts D and $\tilde{D}$, is the same set, the canonical lifts of corresponding classes differ by $\tilde{\psi}_2 - \psi_2$, and so do their minima: $\mathfrak{d}$ changes by $\tilde{\psi}_2 - \psi_2$. The same argument applies to finite changes of ψ_1, for which $\mathfrak{d}$ changes by $-(\tilde{\psi}_1 - \psi_1)$. Lemma 3.12, in both versions, shows that μ_1 and μ_2 are smooth.

Case 2: $\mathcal{M} = \emptyset = \mathcal{M}'$ almost surely. Then, with Lemma 4.3(a), $\bar{\rho}$-almost surely the image has no infinite FK cluster at level 0 and none at level M. By Proposition 4.10 and the proof of Lemma 3.9 applied to the probability measure $\bar{\rho}$ at level 0, $\bar{\rho}$ is invariant under the exchange $(\psi_1', \psi_2', n') \mapsto (\psi_2', \psi_1', n')$, so $D'(0)$ and $-D'(0)$ have the same law. The same argument at level M (swap all clusters at level M contained in Λ_N; in the limit $(\psi_1', \psi_2') \mapsto (\psi_2' - M, \psi_1' + M)$) shows that $D'(0)$ and $2M - D'(0)$ have the same law. Hence $D'(0)$ and $D'(0) + 2M$ have the same law, which is impossible for an integer-valued random variable. So Case 2 does not occur. □

5 The plane measure: proof of Theorem 1.5

Throughout this section $1 \le c \le 2$. The plane measure is a gradient Gibbs measure of slope 0 by [5] (Subsection 5.1). Its spin representation provides nested open circuits around every face, with a Markov property across each circuit (Subsection 5.2); these give tail triviality (Subsection 5.3) and roughness (Subsection 5.4, Figure 5).

Notation. We use the notation of [5, Section 14]. The faces split into even faces F°, forming a square lattice with the diagonal adjacency E°, and odd faces $F^\bullet$. For a height function h, the spins are $\sigma^\circ(u) = +$ if and only if $h(u) \in 4\mathbb{Z}$ for u even, and $\sigma^\bullet(v) = +$ if and only if $h(v) \in 1 + 4\mathbb{Z}$ for v odd; for NN-adjacent faces u even and v odd, $h(v) - h(u) = \sigma^\circ(u)\sigma^\bullet(v)$ [5, (247)]. An *even domain* $\mathcal{V}$ is bounded by a self-avoiding circuit $\partial\mathcal{V}$ of E°. For an even domain $\mathcal{V}$ and $\# \in \{+, -\}$, $\mu_{\mathcal{V}}^{\#}$ is the measure of [5, Definition 14.5] on triples $(\sigma^\circ, \omega, \sigma^\bullet)$, with $\omega \subset E^\circ(\mathcal{V})$ a set of open edges. Under $\mu_{\mathcal{V}}^{+}$ the height function with $h = 0$ on $\partial\mathcal{V}$ has law proportional to $\mathbf{1}\{h|_{\partial\mathcal{V}} \equiv 0\}\, c^{\#A(h)}$ [5, (248)], where $A(h)$ is the set of diagonals of $\mathcal{V}$ on which h is constant; this is the six-vertex measure on $\mathcal{V}$ with that boundary condition, up to the blocks that straddle $\partial\mathcal{V}$, which contribute a constant. Let $\mu_{\mathbb{Z}^2}$ be the full-plane measure of [5, Theorem 17.1]: the common limit of $\mu_{\mathcal{V}}^{+}$ and $\mu_{\mathcal{V}}^{-}$ as $\mathcal{V} \uparrow \mathbb{Z}^2$ [5, (306)].

5.1 The plane measure is a gradient Gibbs measure of slope zero

Lemma 5.1 (Gibbs property, invariance, slope)**.**

(a) The gradient law of h under $\mu_{\mathbb{Z}^2}$ is $\mathbb{P}_c$, and it is a gradient Gibbs measure of the six-vertex model.

(b) $\mathbb{P}_c$ is invariant under the translations of $\mathbb{Z}^2$ and has slope 0.

(c) Under $\mu_{\mathcal{V}}^{+}$, conditionally on the spins, the edges $xy \in E^\circ(\mathcal{V}) \setminus \partial\mathcal{V}$ are independent, with $\omega(xy) = 0$ if $\sigma^\circ(x) \ne \sigma^\circ(y)$, $\omega(xy) = 1$ if $\sigma^\circ(x) = \sigma^\circ(y)$ and $\sigma^\bullet$ disagrees across xy, and $\omega(xy)$ Bernoulli$(1/c)$ if both agree. This rule depends on the spins only through ∇h, and it holds under $\mu_{\mathbb{Z}^2}$.

Proof. (a) By [5, (248)], for a finite set Λ of faces and an even domain $\mathcal{V}$ such that every block meeting Λ consists of faces of $\mathcal{V}$ not on $\partial\mathcal{V}$, the conditional law under $\mu_{\mathcal{V}}^{+}$ of h on Λ given h off Λ is γ_Λ. Both sides of the DLR equation for a local test function are local, so the equation passes to the limit $\mathcal{V} \uparrow \mathbb{Z}^2$. By [5, Lemma 18.7(iii)] the gradient law of h under $\mu_{\mathbb{Z}^2}$ is $\mathbb{P}_{\mathbb{Z}^2} = \mathbb{P}_c$. (b) Invariance is part of [5, Theorem 2.2]. The measure $\mu_{\mathcal{V}}^{+}$ is invariant under $h \mapsto -h$ by (248), so the limit is, and $\mu_{\mathbb{Z}^2}[h(x + e_i) - h(x)] = 0$. (c) This is [5, (238)–(240)]. The agreement pattern is $A(h)$, a function of ∇h; the rule is local and passes to the limit. At $c = 1$ the last case is deterministic. □

By (c), the pair $(\nabla h, \omega)$ can be realised from any random gradient field with law $\mathbb{P}_c$ by sampling ω with this rule, and then has the law it has under $\mu_{\mathbb{Z}^2}$. The measure $\mu_{\mathbb{Z}^2}$ is invariant under the

global flip of both spins, since $\mu_{\mathcal{V}}^{+}$ and $\mu_{\mathcal{V}}^{-}$ are exchanged by it and have the same limit; statements below about $(\nabla h, \omega)$ do not depend on the global sign.

5.2 Circuits and the Markov property

An *ω-circuit* is a self-avoiding circuit of E° all of whose edges are open. The height is constant on its faces, since σ° is constant along open edges and diagonally adjacent even faces with equal σ° have equal heights. Fix the odd face $o := (1, 0)$. An ω-circuit $\mathfrak{C}$ *surrounds* o if o is a face of the even domain $\mathcal{V}_{\mathfrak{C}}$ that it bounds.

Lemma 5.2 (Nested circuits)**.** *$\mu_{\mathbb{Z}^2}$-almost surely, o is surrounded by infinitely many ω-circuits on which $\sigma^\circ = +$ and infinitely many on which $\sigma^\circ = -$. Hence there is almost surely an infinite nested family of ω-circuits surrounding o in which consecutive circuits have different heights. The same holds with o replaced by any finite box.*

Proof. By [5, Corollary 16.6], for $r \geq 4$, a point x, and an even domain $\mathcal{V}$ whose boundary surrounds $\mathrm{Ball}_{2r}(x)$, the $\mu_{\mathcal{V}}^{+}$-probability that the annulus $\mathrm{Ball}_{2r}(x) \setminus \mathrm{Ball}_{r}(x)$ contains an ω-circuit with $\sigma^\circ = -$ around x is at least a constant $c_{\mathrm{circ}} > 0$, conditionally on any event measurable with respect to the even spins and the edges not meeting $\mathrm{Ball}_{2r}(x)$; the same holds for $\sigma^\circ = +$. The balls are ℓ^∞-boxes. Applying this to the annuli at scales $2^i r$ from the outside in, and passing to the limit with [5, Theorem 17.1], gives infinitely many circuits of each sign almost surely. Two ω-circuits surrounding o either share a face or are nested, because E° is planar; circuits with different σ° share no face and carry different heights. For a box, the same annulus argument applies around its centre. □

Outermost circuits. Consider a property of ω-circuits that is inherited by the circuit bounding the outer face of $\mathcal{V}_{\mathfrak{C}} \cup \mathcal{V}_{\mathfrak{C}'}$; we use "surrounds o and lies in a given domain", possibly together with "has height different from a given value", which is inherited because circuits that share a face have the same height. If two circuits with such a property surround o, the outer boundary of $\mathcal{V}_{\mathfrak{C}} \cup \mathcal{V}_{\mathfrak{C}'}$ is again a self-avoiding ω-circuit with the property: both domains contain o, so the union has no pinch point. Hence among finitely many candidates there is a unique outermost one, as in [5, proofs of Theorem 16.4 and Lemma 17.3].

Lemma 5.3 (Strong Markov property)**.** *Let $\mathcal{C}$ be a random self-avoiding circuit of E° such that, for every deterministic circuit $\mathfrak{C}_0$, the event $\{\mathcal{C} = \mathfrak{C}_0\}$ is measurable with respect to the spins and edges on $\mathfrak{C}_0$ and outside $\mathcal{V}_{\mathfrak{C}_0}$ and is contained in $\{\mathfrak{C}_0 \subset \omega\}$. Then under $\mu_{\mathbb{Z}^2}$, conditionally on $\{\mathcal{C} = \mathfrak{C}_0\}$ and on the configuration on $\mathfrak{C}_0$ and outside $\mathcal{V}_{\mathfrak{C}_0}$, the configuration inside $\mathcal{V}_{\mathfrak{C}_0}$ has law $\mu^{\flat}_{\mathcal{V}_{\mathfrak{C}_0}}$, where $\flat$ is the value of σ° on $\mathfrak{C}_0$. In terms of heights, the field $h - h(\mathfrak{C}_0)$ inside $\mathcal{V}_{\mathfrak{C}_0}$ has the law of h under $\mu^{+}_{\mathcal{V}_{\mathfrak{C}_0}}$, and the joint law of $(h - h(\mathfrak{C}_0), \omega)$ inside is invariant under $h - h(\mathfrak{C}_0) \mapsto -(h - h(\mathfrak{C}_0))$ with ω unchanged.*

Proof. For even domains $\mathcal{V} \subset B$, local f of the configuration in $\mathcal{V}$ and local g of the configuration outside $\mathcal{V}$, [5, Lemma 14.12] gives $\mu_B^{\#}[f \cdot g \cdot \mathbf{1}\{\partial\mathcal{V} \subset \omega^\flat\}] = \mu_{\mathcal{V}}^{\flat}[f] \cdot \mu_B^{\#}[g \cdot \mathbf{1}\{\partial\mathcal{V} \subset \omega^\flat\}]$, where $\omega^\flat$ denotes open edges with $\sigma^\circ = \flat$. Both sides are limits of local quantities, so letting $B \uparrow \mathbb{Z}^2$ with [5, (306)] gives the same identity for $\mu_{\mathbb{Z}^2}$, and a monotone class argument extends it to bounded g measurable with respect to the configuration outside $\mathcal{V}$. Summing over the deterministic circuits $\mathfrak{C}_0$ gives the statement for $\mathcal{C}$, as in [5, (299) and (310)]. Since $\mu_{\mathcal{V}}^{+}$ and $\mu_{\mathcal{V}}^{-}$ are exchanged by the global flip and have the same gradient law, the field $h - h(\mathfrak{C}_0)$ inside has the law of h under $\mu^{+}_{\mathcal{V}_{\mathfrak{C}_0}}$, and (248) and Lemma 5.1(c) give the symmetry, with ω unchanged because the rule for ω depends on h only through $A(h)$, which is invariant under $h \mapsto -h$. □

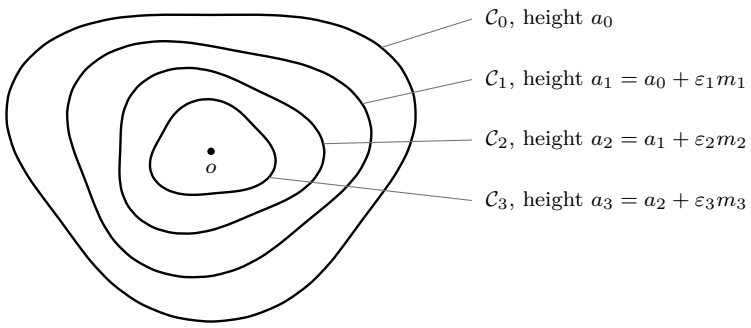


Figure 5: The circuit argument for roughness. Around a face o the plane measure has infinitely many nested open circuits, on each of which the height is constant; $\mathcal{C}_i$ is the outermost one inside $\mathcal{C}_{i-1}$ whose height differs from that of $\mathcal{C}_{i-1}$. By the Markov property and a reflection symmetry inside each circuit (Lemma 5.3), the signs ε_i of the height increments between consecutive circuits are independent fair coin flips given everything else. If $\mathbb{P}_c$ were the gradient of a Gibbs measure ν on heights, the height at o under ν would be the height of an outer circuit plus a sum of more and more independent symmetric increments, which cannot have a tight law (Proposition 5.5). The same random walk of circuit heights appears in [10, Section 4].

The circuits used below satisfy the hypothesis of Lemma 5.3: the outermost ω-circuit surrounding o in a box is determined by the edges outside it, and the outermost ω-circuit surrounding o strictly inside a given circuit with height different from that circuit is determined by the spins and edges outside it together with the sign on it.

5.3 Tail triviality

Lemma 5.4 (Tail triviality)**.** *$\mu_{\mathbb{Z}^2}$ is trivial on the tail σ-field of $(\sigma^\circ, \omega, \sigma^\bullet)$. Hence $\mathbb{P}_c$ is tail-trivial and ergodic under the translations of $\mathbb{Z}^2$.*

Proof. Let A be an event depending on the configuration in Ball_r, T a tail event, and $\varepsilon > 0$. By [5, Theorem 17.1] there is $r' > r$ with $|\mu^{\pm}_{\mathcal{V}}(A) - \mu_{\mathbb{Z}^2}(A)| \le \varepsilon$ for every even domain $\mathcal{V}$ whose boundary surrounds $\mathrm{Ball}_{r'}$. For $R > r'$ let $\mathcal{G}_R$ be the event that some ω-circuit in Ball_R surrounds $\mathrm{Ball}_{r'}$, and let $\mathcal{C}$ be the outermost one. The event T is measurable with respect to the configuration outside $\mathcal{V}_{\mathcal{C}}$, so by Lemma 5.3, $|\mu_{\mathbb{Z}^2}(A \cap T \cap \mathcal{G}_R) - \mu_{\mathbb{Z}^2}(A)\,\mu_{\mathbb{Z}^2}(T \cap \mathcal{G}_R)| \le \varepsilon$. Letting $R \to \infty$ ($\mu_{\mathbb{Z}^2}(\mathcal{G}_R) \to 1$ by Lemma 5.2) and $\varepsilon \to 0$ gives $\mu_{\mathbb{Z}^2}(A \cap T) = \mu_{\mathbb{Z}^2}(A)\mu_{\mathbb{Z}^2}(T)$, so $\mu_{\mathbb{Z}^2}(T) \in \{0, 1\}$. The gradient field is a function of the spins. For ergodicity, let I be a translation-invariant event and A a local event with $\mathbb{P}_c(I \triangle A) < \varepsilon$. By tail triviality and reverse martingale convergence, $\sup_B |\mathbb{P}_c(A \cap B) - \mathbb{P}_c(A)\mathbb{P}_c(B)| \to 0$ over events B measurable outside Λ_N as $N \to \infty$, so $\mathbb{P}_c(A \cap \theta_x A) \to \mathbb{P}_c(A)^2$ as $|x| \to \infty$, while $|\mathbb{P}_c(A \cap \theta_x A) - \mathbb{P}_c(I)| \le 2\varepsilon$. Hence $\mathbb{P}_c(I) = \mathbb{P}_c(I)^2$. □

Ergodicity and extremality of the full-plane spin measure are also proved in [10, Proposition 4.11], for $0 < a, b \le c \le a + b$; for $c \in [\sqrt{3}, 2]$, Lis proves that the torus measures converge to an ergodic, polynomially mixing limit [13, Theorem 1.1]. We included the argument above because it uses only the Markov property of Lemma 5.3, which is needed again below.

5.4 Roughness

Proposition 5.5 (Roughness)**.** *There is no Gibbs measure ν on heights, translation-invariant or not, whose gradient law is $\mathbb{P}_c$.*

Proof. Suppose that ν exists, and let h denote the honest height under ν. Sample ω given ∇h as in Lemma 5.1(c), and let Q be the joint law of (h, ω); then $(\nabla h, \omega)$ has its law under $\mu_{\mathbb{Z}^2}$.

Step 1: the honest exterior does not bias the interior. Let $\mathcal{C}$ be a circuit as in Lemma 5.3, and $\mathcal{H}_{\mathcal{C}}$ the σ-field generated by $\mathcal{C}$, by ω on $\mathcal{C}$ and outside $\mathcal{V}_{\mathcal{C}}$, and by the honest heights of the faces not strictly inside $\mathcal{C}$. We claim that, conditionally on $\mathcal{H}_{\mathcal{C}}$, the pair $(h - h(\mathcal{C}), \omega)$ inside $\mathcal{V}_{\mathcal{C}}$ has the law given by Lemma 5.3. Let Λ be a box containing the region where $\mathcal{C}$ lies (for the circuits used below, a fixed box) and b a face outside Λ. By the Gibbs property of ν and the covariance of γ_Λ under constant shifts, conditionally on the honest heights outside Λ the gradient field on Λ and across its boundary has a law that depends only on the gradient field outside Λ. Since the complement of Λ is connected, $h(b)$ is therefore conditionally independent of $(\nabla h, \omega)$ given the gradient field outside Λ; this uses neither translation invariance nor any other property of ν. The σ-field $\mathcal{H}_{\mathcal{C}}$ is generated by gradient information on $\mathcal{C}$ and outside it together with $h(b)$, so the conditional law of the interior given $\mathcal{H}_{\mathcal{C}}$ equals its conditional law given the gradient information alone, which is the one in Lemma 5.3.

Step 2: a random walk of circuit heights. This is the mechanism of [10, Section 4]. Fix R. Let $\mathcal{C}_0$ be the outermost ω-circuit surrounding o in Ball_R, and $a_0 := h(\mathcal{C}_0)$. Recursively, as long as one exists, let $\mathcal{C}_i$ be the outermost ω-circuit surrounding o strictly inside $\mathcal{C}_{i-1}$ with $h(\mathcal{C}_i) \neq h(\mathcal{C}_{i-1})$. Let $T = T_R$ be the number of steps, with $T_R := 0$ if $\mathcal{C}_0$ does not exist; $a_i := h(\mathcal{C}_i)$, $m_i := |a_i - a_{i-1}| \geq 2$, $\varepsilon_i := \mathrm{sign}(a_i - a_{i-1})$, and $Y := h(o) - a_T$. For $1 \leq i \leq T$ let Ξ_i be the map that reflects h inside $\mathcal{C}_{i-1}$ about a_{i-1} and then reflects the new h inside $\mathcal{C}_i$ about its new height, with ω unchanged. Each of the two reflections preserves Q, by Step 1 and the symmetry in Lemma 5.3. Both preserve T and the circuits $\mathcal{C}_0, \dots, \mathcal{C}_T$ as sets: a circuit that crosses $\mathcal{C}_{i-1}$ shares a face with it and so has height a_{i-1}, which the first reflection fixes, and the defining properties of the circuits are invariant under reflection of the interior. The composition changes ε_i to $-\varepsilon_i$ and leaves a_0, all m_j, all ε_j with $j \neq i$, and Y unchanged; inside $\mathcal{C}_i$ it is the shift $h \mapsto h - 2\varepsilon_i m_i$. Therefore, conditionally on $\mathcal{G} := \sigma(a_0, T, (\mathcal{C}_i), (m_i), Y)$, the signs $\varepsilon_1, \dots, \varepsilon_T$ are independent and uniform. Since $h(o) = a_0 + \sum_{i \leq T} \varepsilon_i m_i + Y$, Lemma A.2 gives, for every integer j,

$$\nu(h(o) = j) \leq \mathbb{E}_Q[\beta(T_R)], \qquad \beta(T) := \binom{T}{\lfloor T/2 \rfloor} 2^{-T}.$$

Step 3. By Lemma 5.2 there is almost surely an infinite nested family of ω-circuits around o in which consecutive circuits have different heights. If Ball_R contains $2m$ such circuits, then by induction $\mathcal{C}_i$ exists for $i \leq m$ and its domain contains that of the $2i$-th of them, so $T_R \geq m$. Hence $T_R \to \infty$ almost surely as $R \to \infty$, $\mathbb{E}_Q[\beta(T_R)] \to 0$, and $\nu(h(o) = j) = 0$ for every j, which is absurd. □

Proof of Theorem 1.5. Lemma 5.1, Lemma 5.4 and Proposition 5.5. □

The hypotheses $1 \leq c \leq 2$ enter through the results of [5] used here: Theorem 2.2, Corollary 16.6, Theorem 17.1 and Lemma 18.7, all of which rest on [5, Theorem 16.4]; and through the coupling of Lemma 5.1(c), which needs $c \geq 1$.

6 Proofs of Theorem 1.2 and Corollaries 1.3 and 1.6

Theorem 1.2 is now a combination of Theorems 1.4 and 1.5. The variational characterisation adds the ergodic decomposition and the strict convexity of the surface tension, and the identification is the argument of [15, Section 4] with the characterisation in place of the remark it cites. Corollary 1.7 is proved in Section 7.

Proof of Theorem 1.2. Let $1 < c \le 2$ and let μ be a translation-ergodic gradient Gibbs measure of slope 0. By Theorem 1.5, $\mathbb{P}_c$ is also one. By Lemma 2.4(iii) both are equilibrium states. If $\mu \neq \mathbb{P}_c$, Theorem 1.4 shows that $\mathbb{P}_c$ is smooth, which contradicts Theorem 1.5. □

Remark 6.1 (The case $c = 1$). At $c = 1$ the model is a Lipschitz simply attractive potential (Proposition 2.3), so [17, Theorem 9.1.1] gives at most one *minimal gradient phase* of slope 0, an interior slope $((\frac{1}{2}, \frac{1}{2})$ in the coordinates ψ, inside $U_\Phi = (0,1)^2)$, for the translation group, that is, one ergodic gradient measure of slope 0 that minimises the specific free energy among measures of its slope. By [17, Theorem 6.3.1] every ergodic gradient Gibbs measure of slope 0 is such a phase, and $\mathbb{P}_1$ is an ergodic gradient Gibbs measure of slope 0 by Theorem 1.5. So the conclusion of Theorem 1.2 holds at $c = 1$ as well.

Proof of Corollary 1.3. By Theorem 1.5 and Lemma 2.4, $\mathbb{P}_c$ minimises the specific free energy. Let μ be a translation-invariant measure that does. The specific free energy is affine on translation-invariant measures and respects the ergodic decomposition [17, Lemma 3.3.1 and Theorem 3.3.2], so almost every ergodic component μ' of μ minimises it, that is $\mathrm{SFE}(\mu') = \sigma_c(0)$. By definition of the surface tension, $\mathrm{SFE}(\mu') \ge \sigma_c(S(\mu'))$, where $S(\mu')$ is the slope.

The slopes of finite surface tension form the square $[-1,1]^2$ in height coordinates: random height functions $h(x) = f_1(x_1) + f_2(x_2)$, with f_i having independent unit steps of mean u_i, realise every slope $u \in [-1,1]^2$ by translation-invariant measures, and every function on the faces with unit steps between NN-adjacent faces is the height function of an arrow configuration. So 0 is an interior point. For $c \ge 1$ the surface tension is strictly convex on the interior of the set of slopes [12, Theorem 13.16]; the condition there, $c_+c_- \ge \max\{a_+a_-, b_+b_-\}$ in their notation, reads $c \ge 1$ when $a = b = 1$, and their surface tension agrees with σ_c up to an additive constant and the change of coordinates of Section 2, both being defined from the energy $-\log w$ per vertex. The function σ_c is even (Lemma 2.4). If it were minimal at some $u \neq 0$, it would be constant on the segment from $-u$ to u, contradicting strict convexity near the interior point 0. So $\sigma_c(u) > \sigma_c(0)$ for $u \neq 0$, and $S(\mu') = 0$.

A translation-invariant measure of finite slope that minimises the specific free energy among measures of its slope is a gradient Gibbs measure [17, Theorem 2.5.2], which applies by Proposition 2.2. So almost every ergodic component of μ is a translation-ergodic gradient Gibbs measure of slope 0, hence equal to $\mathbb{P}_c$ by Theorem 1.2. Therefore $\mu = \mathbb{P}_c$. □

Proof of Corollary 1.6. The sector event is invariant under the translations of the torus, so each $\mathbb{P}_{N,L}^{(n_L)}$ is, and so is the local limit ρ_c, which exists by (O1). The functional minimised in (O2), $\mu \mapsto -\mathsf{h}(\mu) - \mathbb{E}_\mu \log w_0$ over translation-invariant measures, is $-P(\mu)$, which differs from SFE by a constant (Lemma 2.4). So ρ_c is a translation-invariant minimiser of the specific free energy, and $\rho_c = \mathbb{P}_c$ by Corollary 1.3. □

We have used (O1) and (O2) as stated in [15] and have not checked their proofs. The Gibbs property of ρ_c, which is not needed here, also follows directly: for a box B and N large, the sector event is determined by any column that does not meet B, so conditionally on the configuration outside B the measure $\mathbb{P}_{N,L}^{(n)}$ restricted to B is the six-vertex specification, and the property passes to the local limit.

7 The limit in domains

This section proves Corollary 1.7. Subsection 7.1 checks that [15], [5] and [1] define the same full-plane measure and observables. Subsection 7.2 shows that the full-plane statement of [15] gives

membership in I_{CI}, whatever convention is used for points on grid lines. Subsection 7.3 writes out a harmonicity statement at positive mesh used in the proof of [1, Theorem 1.3].

Throughout this section $c \in [1,2]$ and $\Delta = 1 - c^2/2$. For a mesh $\delta > 0$ and a point $u \in \mathbb{R}^2$ off the grid lines of $\delta\mathbb{Z}^2$, $h_\delta(u)$ is the height of the face of $\delta\mathbb{Z}^2$ containing u, a face being identified with its lower left corner. A *grid-line rule* assigns to each δ and each point p on a grid line a face of $\delta\mathbb{Z}^2$ whose closure contains p, without reference to the configuration, and $h_\delta(p)$ is the height of that face. The *floor rule* takes the face with lower left corner $(\delta\lfloor p_1/\delta\rfloor, \delta\lfloor p_2/\delta\rfloor)$; it is the convention of [5, (7)] and [15, Section 1.1].

Let $G(x,y) := -\frac{1}{2\pi}\log|x-y|$, let Γ be the full-plane Gaussian free field with covariance kernel G, and for points u_1, u_1', u_2, u_2' put

$$\mathcal{K}(u_1,u_1';u_2,u_2') := G(u_1',u_2') - G(u_1',u_2) - G(u_1,u_2') + G(u_1,u_2), \tag{7.1}$$

the covariance of the increments $\Gamma(u_1') - \Gamma(u_1)$ and $\Gamma(u_2') - \Gamma(u_2)$.

Definition 7.1 (The set I_{CI})**.** Fix a grid-line rule. Following [1, (11)], I_{CI} is the set of $\Delta \in [-1, \frac{1}{2}]$ such that, for every $k \geq 1$ and all distinct points $u_1, u_1', \dots, u_k, u_k' \in \mathbb{R}^2$,

$$\mathbb{E}_{\mathbb{P}_c}\Big[\prod_{i=1}^{k}(h_\delta(u_i) - h_\delta(u_i'))\Big] \xrightarrow[\delta\to 0]{} v^{k/2}\,\mathbb{E}\Big[\prod_{i=1}^{k}(\Gamma(u_i) - \Gamma(u_i'))\Big], \qquad v = \frac{2}{\arccos\Delta}. \tag{7.2}$$

In [1, footnote 2] the ambiguity on grid lines is left open, with the observation that any deterministic rule fixed in advance resolves it and that it has no effect on their results. The results below hold for every grid-line rule.

Definition 7.2 (The full-plane statement at c)**.** Following [15, Theorem 1.1(3)], the *full-plane statement holds at c* if, for every $k \geq 1$ and with h_δ defined by the floor rule,

$$\mathbb{E}_{\mathbb{P}_c}\Big[\prod_{i=1}^{k}(h_\delta(u_i') - h_\delta(u_i))\Big] \xrightarrow[\delta\to 0]{} v(c)^{k/2}\sum_{\mathfrak{p}}\prod_{\{i,j\}\in\mathfrak{p}}\mathcal{K}(u_i,u_i';u_j,u_j'), \qquad v(c) = \frac{1}{\arcsin(c/2)}, \tag{7.3}$$

uniformly on compact sets of tuples $(u_i,u_i')_{i\leq k}$ with $\{u_i,u_i'\}\cap\{u_j,u_j'\} = \emptyset$ for $i \neq j$, where the sum is over the pairings $\mathfrak{p}$ of $\{1,\dots,k\}$ and is zero for odd k.

Since $\arccos(1-c^2/2) = 2\arcsin(c/2)$, the two variances agree, $v(c) = 2/\arccos\Delta$, and we write v for both. The differences in (7.2) and (7.3) are taken in opposite orders, each as in its source.

7.1 The two works concern the same measure

Proposition 7.3. *The plane law of [15, Section 1.1] is the measure $\mathbb{P}_c$ of Definition 1.1, and the height functions and their rescalings in [15], [5] and [1] agree off the grid lines. In particular (7.3) and (7.2) concern the same expectations, up to the grid-line rule.*

Proof. Weights. In [5] the vertex types 1–2, 3–4 and 5–6 have weights a, b and c, and types 5 and 6 are the two vertices whose horizontal arrows both point in or both point out. In [15] those two vertices have weight c and the other four weight 1. With $a = b = 1$ the finite-volume measures on a torus coincide. *Conditioning and limits.* Both condition on the balanced sector with L even, and both let $N \to \infty$ first and then $L \to \infty$ in the local weak topology; the limit exists by [5, Theorem 2.2]. *Heights.* In all three works the face to the left of an arrow is one unit higher, a point off the grid lines takes the value of the face containing it, and $h_\delta(u) = h(u/\delta)$. The kernel G is the

same. *The measure in [1, (11)].* The condition there is formulated for "the unique flip-symmetric Gibbs measure", with a reference to [5] for its definition. The measure defined in [5] is the limit of Definition 1.1, and the proofs of [1] use that limit: their full-plane measure is the one of [5, Sections 14–17], whose gradient marginal is $\mathbb{P}_c$ by [5, Lemma 18.7], and it is reached from balanced cylinders in [1, Theorem 6.3]. For Read as in the remark after [5, (6)], as the unique flip-symmetric gradient Gibbs measure that is ergodic under translations, the phrase designates $\mathbb{P}_c$ for $1 \le c \le 2$ (Theorem 1.2 and Remark 6.1); without ergodicity it designates no single measure, since a phase of nonzero slope averaged with its flip is flip-symmetric; and no Gibbs measure on heights has gradient law $\mathbb{P}_c$ (Theorem 1.5). □

Remark 7.4 (The construction of [15] for $1 \le c < \sqrt{2}$). For $c < \sqrt{2}$ the construction of [15] uses the balanced sector, so the state it studies is the limit of Definition 1.1. Sections 4 and 8 of [15] describe the change of regime at $c = 1$ and its Section 3 at $c = \sqrt{2}$; for $1 \le c < \sqrt{2}$ the direct argument given there for $c < 1$ applies, with one exception. At $c = 1$ a step that compares the states obtained with the roles of the two axes exchanged uses a matrix that is singular there. At $c = 1$ the comparison follows instead from the invariance of $\mathbb{P}_c$ under the automorphisms of $\mathbb{Z}^2$ [5, Theorem 2.2], on the assumption, which we have not verified, that the state obtained in [15] with the axes exchanged is the limit of Definition 1.1 with the axes exchanged.

7.2 Membership

Proposition 7.5 (Independence of the grid-line rule)**.** *If the full-plane statement holds at c, then* (7.2) *holds for all distinct points and every grid-line rule; in particular* $\Delta \in I_{\mathrm{CI}}$.

Proof. Let $u_1, u_1', \dots, u_k, u_k'$ be distinct; they satisfy the disjointness condition of Definition 7.2. By Wick's formula, $\mathbb{E}[\prod_{i=1}^{k} v^{1/2}(\Gamma(u_i') - \Gamma(u_i))] = v^{k/2} \sum_{\mathfrak{p}} \prod_{\{i,j\} \in \mathfrak{p}} \mathcal{K}(u_i, u_i'; u_j, u_j')$, the right side of (7.3). The products in (7.2) and (7.3) differ by the factor $(-1)^k$, which is 1 for even k, and for odd k both limits vanish. Hence (7.2) holds under the floor rule.

For another rule and a point p, let $\zeta_\delta(p)$ be the centre of the face whose height defines $h_\delta(p)$. Then $h_\delta(p)$ is the floor-rule value at $\zeta_\delta(p)$, which lies off the grid lines, and $|\zeta_\delta(p) - p| \le \delta/\sqrt{2}$. Let $d_0 > 0$ be the smallest distance between two of the points and $\mathcal{Q}$ the set of tuples within distance $d_0/4$ of $(u_i, u_i')_{i \le k}$ in each coordinate. It is compact and consists of tuples of distinct points, so the convergence in (7.3) is uniform on $\mathcal{Q}$ and the limit is continuous there. For $\delta < d_0/4$ the tuple of centres lies in $\mathcal{Q}$ and tends to $(u_i, u_i')_{i \le k}$, so the expectation in (7.2), the floor-rule expectation at the tuple of centres, converges to the value of the limit at $(u_i, u_i')_{i \le k}$. □

A rule whose choice of face depends on the configuration, such as taking the largest adjacent value, is not covered: it changes $h_\delta(p)$ by a bounded local random variable whose correlation with a distant increment is not controlled by (7.3).

7.3 Discrete harmonicity

The proof of [1, Theorem 1.3] uses, besides membership in I_{CI} and inputs available for every $c \in [1, 2]$, one statement about the full-plane measure at positive mesh: the correlation between a circle-average defect of the height function and that of its reflection in a line vanishes as $\delta \to 0$ [1, (113)]. The reference given there is a proposition of [5] on the harmonicity of the limiting correlation functions [5, Proposition 7.2]. The analogous step for $-1 \le \Delta \le -\frac{1}{2}$ is stated in one sentence in [5, after (143)], by uniform convergence and harmonicity of the limiting covariance. We write the argument out at positive mesh, uniformly over the lines involved, from membership in I_{CI} alone.

Lemma 7.6 (Discrete harmonicity)**.** *Let $\Delta \in I_{\mathrm{CI}}$ for some grid-line rule, and let h_δ be defined with any grid-line rule. Fix $r > 0$, $\eta > 0$, $C_0 \geq 0$ and a compact set $\mathcal{L}$ of pairs (x, ℓ) with $x \in \mathbb{R}^2$ and ℓ a line at distance at least $r + \eta$ from x; write $p \mapsto p^\dagger$ for the reflection in ℓ. For each δ and each (x, ℓ) let $f \mapsto f^\dagger$ be a map on the faces of $\delta\mathbb{Z}^2$ such that the centre of $f^\dagger$ is within $C_0\delta$ of the reflection of the centre of f, and put $h^\dagger(f) := h(f^\dagger)$. With $\mathcal{D}_r F(x) := F(x) - \fint_{\partial B_r(x)} F(s)\,\mathrm{d}s$,*

$$\sup_{(x,\ell)\in\mathcal{L}} \left| \mathbb{E}_{\mathbb{P}_c}\left[\mathcal{D}_r h_\delta(x) \cdot \mathcal{D}_r h_\delta^\dagger(x)\right] \right| \xrightarrow[\delta\to 0]{} 0, \tag{7.4}$$

the supremum being also over all such face maps.

Proof. Let $\mathcal{F}_2^{(\delta)}(u_1, u_1'; u_2, u_2') := \mathbb{E}_{\mathbb{P}_c}[(h_\delta(u_1') - h_\delta(u_1))(h_\delta(u_2') - h_\delta(u_2))]$ with h_δ defined by the floor rule, and let D_2 be the set of tuples with $\{u_1, u_1'\} \cap \{u_2, u_2'\} = \emptyset$.

Step 1: uniform convergence. We claim that $\mathcal{F}_2^{(\delta)} \to v\,\mathcal{K}$ uniformly on compact subsets of D_2. If the convergence failed on some compact set along $\delta_n \to 0$, choose a countable dense set of tuples of distinct points with no coordinate in $\bigcup_n \delta_n\mathbb{Z}$. At these tuples no grid-line rule is involved, so (7.2) with $k = 2$ gives pointwise convergence along δ_n, and [5, Corollary 4.6], valid for $c \in [1, 2]$, upgrades pointwise convergence on a countable dense set to uniform convergence on compact sets, a contradiction.

Step 2: face centres. For a point p let $\zeta(p)$ be the centre of the face that defines $h_\delta(p)$ under the rule of the lemma, and $\zeta^\dagger(p)$ the centre of the image of that face under $f \mapsto f^\dagger$. Then $h_\delta(p)$ and $h_\delta^\dagger(p)$ are the floor-rule values at $\zeta(p)$ and $\zeta^\dagger(p)$, with $|\zeta(p) - p| \leq \delta/\sqrt{2}$ and $|\zeta^\dagger(p) - p^\dagger| \leq (C_0 + 1)\delta$, the second because the reflection is an isometry.

Step 3: an integral formula. $\mathcal{D}_r h_\delta(x)$ is the average over $s \in \partial B_r(x)$ of $h_\delta(x) - h_\delta(s)$, and similarly for $h_\delta^\dagger$. All integrands are bounded for fixed δ, so by Fubini's theorem

$$\mathbb{E}_{\mathbb{P}_c}\left[\mathcal{D}_r h_\delta(x)\,\mathcal{D}_r h_\delta^\dagger(x)\right] = \fint_{\partial B_r(x)} \fint_{\partial B_r(x)} \mathcal{F}_2^{(\delta)}\left(\zeta(s), \zeta(x); \zeta^\dagger(t), \zeta^\dagger(x)\right) \mathrm{d}s\,\mathrm{d}t,$$

in which every argument of $\mathcal{F}_2^{(\delta)}$ is a face centre.

Step 4: passage to the limit. Let $\mathcal{Q}$ be the set of tuples $(s, x; t^\dagger, x^\dagger)$ with $(x, \ell) \in \mathcal{L}$ and $s, t \in \partial B_r(x)$; it is compact. The discs $\bar{B}_r(x)$ and $\bar{B}_r(x^\dagger)$ lie on opposite sides of ℓ at distance at least η from it, so every point of the first pair is at distance at least 2η from every point of the second. Let $\mathcal{Q}'$ be the closed $\eta/2$-neighbourhood of $\mathcal{Q}$, a compact subset of D_2 on which $\mathcal{K}$ is uniformly continuous. For $(C_0 + 1)\delta \leq \eta/2$ the argument of $\mathcal{F}_2^{(\delta)}$ above lies in $\mathcal{Q}'$ within $(C_0 + 1)\delta$ of $(s, x; t^\dagger, x^\dagger)$, so Step 1 gives

$$\sup_{(x,\ell)\in\mathcal{L}} \left| \mathbb{E}_{\mathbb{P}_c}\left[\mathcal{D}_r h_\delta(x)\,\mathcal{D}_r h_\delta^\dagger(x)\right] - v \fint_{\partial B_r(x)} \fint_{\partial B_r(x)} \mathcal{K}(s, x; t^\dagger, x^\dagger)\,\mathrm{d}s\,\mathrm{d}t \right| \longrightarrow 0.$$

Step 5: the limit vanishes. $\mathcal{K}(s, x; t^\dagger, x^\dagger) = G(x, x^\dagger) - G(x, t^\dagger) - G(s, x^\dagger) + G(s, t^\dagger)$. For $y \in \{x^\dagger, t^\dagger\}$ the function $G(\cdot, y)$ is harmonic near $\bar{B}_r(x)$, so its average over $\partial B_r(x)$ is $G(x, y)$, and the average of $\mathcal{K}$ over s vanishes for every t. □

Remark 7.7 (Use in [1])*.* In the proof of [1, Proposition 7.1] the statement is needed at the origin, for reflections in horizontal lines that separate the support of the circle average from a family of rectangles, at a distance from the origin bounded below by r plus a positive margin and bounded above in terms of the data of the proposition. The reflection of faces in [1] is defined through face centres; it maps faces to faces, so that $C_0 = 0$, when the line is a lattice translate of the lines of [1,

(107)], and in general any choice of a face whose closure contains the reflected centre has $C_0 \le 1$. Lemma 7.6 gives the required bound uniformly over such lines. It is a statement at fixed Δ, which is why δ_0 may depend on Δ in Corollary 1.7.

Proof of Corollary 1.7. Proposition 7.5 gives $\Delta \in I_{\mathrm{CI}}$ for every grid-line rule, and the theorems of [1] are stated for every $\Delta \in I_{\mathrm{CI}}$; Remark 7.7 explains the role of Lemma 7.6 in their proof. Apart from this step, the inputs of [1] for $\Delta \in I_{\mathrm{CI}}$ that we have traced are stated in their sources for all $c \in [1, 2]$ or all $c \ge 1$; we have not traced the proofs of [1, Proposition 3.5 and Corollary 3.6] and of the diagonal case of [1, Theorem 6.3]. □

A Two tools

A.1 The Lanford–Ruelle theorem for subshifts of finite type

Let $\mathcal{A}$ be a finite alphabet and $\mathcal{X} \subset \mathcal{A}^{\mathbb{Z}^2}$ a subshift of finite type. An interaction $\Phi = (\Phi_A)_{A \subset \mathbb{Z}^2 \text{ finite}}$ assigns to each finite A a function of the configuration on A; it is translation-invariant and absolutely summable if $\Phi_{A+x} = \Phi_A \circ \theta_x$ and $\sum_{A \ni 0} \|\Phi_A\|_\infty < \infty$. Put $f_\Phi := \sum_{A \ni 0} \Phi_A/|A|$. A translation-invariant probability measure μ on $\mathcal{X}$ is an *equilibrium measure* for f_Φ if it maximises $\mathsf{h}(\mu) - \mu(f_\Phi)$ over translation-invariant probability measures on $\mathcal{X}$ (for the six-vertex potential on $\mathbb{X}$ this is an equilibrium state in the sense of Section 2.3); it is a *Gibbs measure* for Φ if, for every finite A, the conditional law of the configuration on A given the configuration y off A is proportional to $\exp(-\sum_{B \cap A \neq \emptyset} \Phi_B(u \vee y))$ on the patterns u with $u \vee y \in \mathcal{X}$ [2, (7)].

Theorem A.1 ([2, Theorem 1.1(b)])**.** *Let $\mathcal{X}$ be a subshift of finite type over a finite alphabet on $\mathbb{Z}^2$ and Φ a translation-invariant absolutely summable interaction. Then every equilibrium measure for f_Φ is a Gibbs measure for Φ.*

The theorem is proved in [2] for subshifts with the topological Markov property on countable amenable groups, and every subshift of finite type has that property [2, p. 11]. It is used in Lemma 4.9 for the nearest-neighbour subshift $\mathbb{Y}_N$ and a bounded interaction of finite range.

A.2 Anti-concentration

Lemma A.2 (Erdős [8])**.** *Let $m_1, \dots, m_T$ be nonzero integers and $\varepsilon_1, \dots, \varepsilon_T$ independent uniform signs. Then $\sup_x \mathbb{P}(\sum_i \varepsilon_i m_i = x) \le \binom{T}{\lfloor T/2 \rfloor} 2^{-T}$, which tends to 0 as $T \to \infty$.*

Proof. Replacing ε_i by $-\varepsilon_i$ we may assume $m_i > 0$. For fixed x, the sets $\{i : \varepsilon_i = +1\}$ of the solutions form an antichain, since enlarging the set strictly increases the sum, so there are at most $\binom{T}{\lfloor T/2 \rfloor}$ of them by Sperner's theorem. □

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