Stochastic Connections on Surface Bundles

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Abstract

We construct local stochastic connection dynamics on arbitrary principal bundles over closed oriented Riemannian surfaces. For each smooth reference connection, the bare additive spatial heat approximations of the Yang–Mills–DeTurck equation converge to a maximal local strong Markov family in a separable negative Hölder space. A finite Gaussian enhancement defines the singular products, and cancellation of the total logarithmic mass identifies the limiting equation. Rough comparison gauges and predictable rotations of the Wiener noise identify the local orbit laws for different references. The finite quadratic tensor anomaly vanishes after the two-dimensional Yang–Mills contraction, with the finite normalisation fixed by the spatial heat approximation. Continuing through bounded representatives gives a reference-independent maximal local strong Markov family on the full rough orbit space.

Gaussian line primitives have every Hölder exponent below one on bounded regular curve families and define non-Abelian Young parallel transport. On a dense, gauge-invariant Borel full-heat trace domain, polygonal holonomies separate rough gauge orbits, admit Borel representative maps, and recover the bundle class. The orbit dynamics has continuous holonomy paths on this domain, and the regularised holonomies converge uniformly on bounded regular curve families and compact surviving positive-time intervals.

We also prove regularisation-independent convergence of first-order Gaussian currents after time integration, including a harmonic example of infinite variation and zero quadratic variation. At fixed regularisation scale, the field-dependent covariant heat approximation has classical solutions and an explicit Itô correction in Coulomb coordinates. In every Abelian sector, the quotient diffusion is global, reference independent, and reversible with an explicitly identified unique invariant law.

Keywords. Stochastic Yang–Mills; principal bundles; regularity structures; Gaussian currents; holonomy; reference connections.

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1 Introduction

Let Σ be a connected closed oriented Riemannian surface and let $P \to \Sigma$ be a smooth principal bundle with compact connected structure group G. Fix an Ad-invariant inner product on its Lie algebra. Connections on P form an affine space: the difference of two connections is a section of

$$E = T^*\Sigma \otimes \operatorname{ad} P.$$

Working with these sections retains the topology of P while placing the equation in a linear function space. The problem is to construct the singular stochastic equation in a reference gauge, compare solutions for different references, and recover their gauge orbits from parallel transport.

For the action

$$S(A) = \frac{1}{2}\int_\Sigma |F_A|^2 \operatorname{vol}_g,$$

the formal Langevin equation and its background DeTurck form are

$$dA = -d_A^* F_A\, dt + \sigma\, dW, \tag{1.1}$$

$$dA = \mathcal{V}_B(A)\, dt + \sigma\, dW, \qquad \mathcal{V}_B(A) = -d_A^* F_A - d_A d_B^*(A - B), \tag{1.2}$$

where B is any smooth reference connection and W is cylindrical Wiener noise on $L^2(E)$. The second equation has a Laplace-type principal part. Its coefficients include the curvature of B; flatness and triviality of the bundle are not assumed. In two spatial dimensions its linear stochastic convolution has regularity $\mathcal{C}^{-\kappa}$ for every $\kappa > 0$, so the quadratic derivative and cubic terms require a common enhanced interpretation.

1.1 The main results

Write $L_B = d_B^* d_B + d_B d_B^*$ on E and $\mathcal{S}_{B,\varepsilon} = e^{-\varepsilon(1+L_B)}$. The parameter ε is heat time. Let $\mathrm{c}^r(E)$ be the closure of smooth sections in the Hölder–Besov space $\mathcal{C}^r(E)$; this choice gives a separable state space and strong continuity of the heat semigroup.

Theorem A (Fixed-reference continuum dynamics)**.** *Fix a smooth B, $0 < \kappa < 1/16$, and $a_0 \in \mathrm{c}^{-\kappa}(E)$. The classical approximations*

$$da^\varepsilon = \mathcal{V}_B(B + a^\varepsilon)\, dt + \mathcal{S}_{B,\varepsilon}\, dW_t, \qquad a^\varepsilon(0) = \mathcal{S}_{B,\varepsilon} a_0 \tag{1.3}$$

converge locally in probability in $C_t\mathrm{c}^{-\kappa}(E)$ to a unique maximal solution in the enhanced solution class constructed below. The convergence has localised L^p bounds for every finite p. The solution map is locally Lipschitz in the initial state and the finite enhanced data.

These solutions form a maximal local strong Markov family on $\mathrm{c}^{-\kappa}(E)$. They are causal functions of the positive-time Wiener path, independent of the auxiliary negative-time noise used in the stationary enhancement. Their lifetime τ is a stopping time and satisfies

$$\tau < \infty \quad \Longrightarrow \quad \lim_{t \uparrow \tau} \|a_t\|_{\mathrm{c}^{-\kappa}} = \infty.$$

With $S_t = e^{-t(1+L_B)}$, the nonlinear response

$$a_t - S_t a_0 - \int_0^t S_{t-r}\, dW_r$$

has parabolic regularity $\mathcal{C}_{\mathfrak{s}}^{1-4\kappa}$ locally up to the initial surface and vanishes there. The construction is natural under simultaneous transport of the bundle, reference connection, initial state and noise, and is independent of the auxiliary charts, frames and kernel decompositions used in its proof.

The local convergence and its probability localisations are proved in Theorem 8.14. The causal restart and strong Markov statements are proved in Section 9; naturality is Proposition 21.2. The approximation (1.3) contains no added counterterm. In a subtracted-model representation its finite affine compensation must nevertheless be retained. The forest identity and cancellation of the logarithmic contributions identify this compensation.

Theorem B (Continuum reference comparison)**.** *Let B, C be arbitrary smooth reference connections on P, $A_0 \in \mathcal{A}^{-\kappa}(P)$, $0 < \kappa < 1/16$, and $u_0 \in \mathcal{G}^{1-\kappa}(P)$. Couple the B-reference continuum solution A^B to the unique rough comparison gauge starting at u_0, and set*

$$\widetilde{W}_t = \int_0^t \mathrm{Ad}_{u_s}\, dW_s.$$

This is a cylindrical Wiener process for the original filtration. The canonical C-reference solution driven by this noise satisfies

$$A^C(u_0 \cdot A_0, \widetilde{W})_t = u_t \cdot A^B(A_0, W)_t \tag{1.4}$$

on the common local existence interval, and throughout every common continuation on which the comparison is defined. The comparison gauges satisfy the strict reference cocycle, with the same composition law for the rotated noises.

For smooth initial gauges, the proof compares the transformed heat enhancement directly. General $\mathrm{c}^{1-\kappa}$ initial gauges follow by approximation, weighted gauge stability, and stability of the finite Itô enhancement under predictable noise rotations. The result fixes the finite normalisation of the bare spatial heat equation and proves reference independence of its local full-gauge orbit laws. It does not identify the maximal explosion times of different DeTurck representatives.

The deterministic gauge construction is given in Section 10. The finite noise comparison on the equation-generated enhancement is proved in Section 11; Section 12 proves the continuum identity, including rough initial gauges and its stopping-time form.

The reference comparison also has an exact geometric form at positive regularisation scale. For the covariant smoother $Q_\varepsilon(A) = e^{-\varepsilon\Delta_A}$, we construct maximal H^s strong solutions for every integer $s \geq 4$ and every $\varepsilon > 0$. A comparison gauge of finite time variation transports the solution between reference gauges, and its adjoint action rotates the cylindrical Wiener process. The comparison holds until the common lifetime of the representatives and hence on common orbit localisations. The main analytic estimate is

$$\|e^{-\varepsilon\Delta_A} - e^{-\varepsilon\Delta_C}\|_{\mathrm{HS}(L^2,H^s)} \leq C_R \varepsilon^{-s/2} \|A - C\|_{H^s} \quad (0 < \varepsilon \leq 1)$$

on bounded H^s sets. The nonlinear Coulomb reduction is derived with its oblique projection and Itô trace in Section 2; existence and stochastic reference comparison are established in Section 4.

The harmonic and weak formulations also involve time-integrated derivative currents. It applies to general surface bundles and to spatial smoothers which need not commute with the linear operator.

Theorem C (Integrated Gaussian currents)**.** *Let L be a nonnegative self-adjoint Laplace-type operator on a metric vector bundle over Σ, and let Π be an orthogonal projection commuting with its heat semigroup, including $\Pi = I$. Set $X_t = \int_0^t e^{-(t-r)L}\Pi\, dW_r$, and let D be a formally symmetric differential operator of order at most one. For the admissible spatial regularisations R_ε of Section 7, the centred currents*

$$H_D^\varepsilon(t) = \int_0^t \left(\langle R_\varepsilon X_r, D R_\varepsilon X_r\rangle - \mathbb{E}\langle R_\varepsilon X_r, D R_\varepsilon X_r\rangle\right) dr$$

converge in $L^p(\Omega; C^\gamma([0,T]))$ for every $p < \infty$ and $0 < \gamma < 1$. The limit is independent of R_ε, is natural under bundle isometries, and satisfies an explicit restart identity at every finite stopping time. There is also a distribution-valued version for first-order quadratic bundle nonlinearities.

For an explicit non-Abelian harmonic Yang–Mills current on the flat torus, the variance of small increments away from time zero is comparable to $h^2 \log(1/h)$. Its integrated limit has almost surely infinite variation on every nontrivial interval and zero quadratic variation. Its distributional time derivative has no almost surely locally integrable representative.

Theorem 7.3 proves the convergence; the subsequent restart and sharpness results prove the remaining claims. A critical logarithmic trace estimate in two spatial dimensions controls the time integral, even in the harmonic example where the current has no locally integrable time derivative.

For gauge-invariant states we use parallel transport on compact families of short geodesic polygons. Their uniform topology, followed by compact conjugation, gives a Polish holonomy completion which separates smooth full gauge orbits. A two-orbit infimum in a negative connection norm, by contrast, can identify smooth $SU(2)$ connections with different Wilson loops. In the Abelian theory, the holonomies determine the state and support its quotient dynamics.

Theorem D (Abelian dynamics and holonomy states)**.** *Let P be a principal $U(1)$ bundle of degree $m \in \mathbb{Z}$. For every $s > 0$, its stochastic dynamics with noise amplitude $\sqrt{2}$ is the global diffusion on the completed gauge quotient*

$$\mathsf{E}_{m,s} = H^{-s}_{\mathrm{coex}}(\Sigma; T^*\Sigma) \times \mathcal{J}_g, \qquad \mathcal{J}_g = \mathcal{H}^1(\Sigma)/\Lambda,$$

where $\mathcal{H}^1(\Sigma) = \ker \Delta_1$ and Λ is the lattice of harmonic forms with periods in $2\pi\mathbb{Z}$. In constant-curvature reference coordinates, the process consists of a coexact Ornstein–Uhlenbeck process and independent Brownian motion on the Jacobian. It is strong Markov, strong Feller and irreducible in this state topology, and its law is independent of the smooth reference connection under the corresponding affine coordinate change.

In the degree-m holonomy completion there is a dense Borel set $\mathscr{H}^{C^2}_{m,s}$, containing all smooth connection images, which is Borel isomorphic to a Borel subset of $\mathsf{E}_{m,s}$ and supports a continuous strong Markov family. For every initial state in $\mathsf{E}_{m,s}$, the associated holonomy process lies in this domain simultaneously at all positive times almost surely. Its completed holonomies recover the coexact state, the Jacobian coordinate and the integer m. The Wilson observables extend simultaneously to every piecewise regular C^2 closed curve. Their values are jointly continuous in time and on curve families with a fixed number of regular pieces, a positive lower speed bound, and bounded speed and acceleration. Geodesic interpolation with N equal subintervals per piece converges uniformly on these families and compact positive-time intervals in every finite L^p at rate $N^{-\theta}$ for every $0 < \theta < 1/2$.

The unique invariant law in each sector is the image of

$$\mu_{m,g} = \mathcal{N}(0, \Delta_1^{-1}|_{\mathrm{coex}}) \otimes m_{\mathcal{J}_g}.$$

It is reversible, and its L^2 spectral gap is $\min\{\lambda_{\mathrm{coex}}, \lambda_{\mathcal{J}_g}\}$, with the absent harmonic factor omitted in genus zero. Its curvature is

$$\left(\frac{2\pi m}{\mathrm{Vol}_g(\Sigma)} + \textit{mean-zero scalar white noise} \right) \mathrm{vol}_g.$$

The statements extend to compact torus groups and every Chern vector, with the covariance and harmonic metric determined by the chosen invariant fibre inner product.

The diffusion, reference translation and equilibrium formulas are proved in Section 13. Theorem 16.4 gives the Borel inverse, and Corollary 17.4 gives the stated invariant domain and simultaneous regular-curve observables. Strong Feller continuity above concerns the Sobolev state topology; the holonomy topology is used for continuous observable paths and the strong Markov realisation.

Theorem E (Non-Abelian holonomy and rough orbit recovery)**.** *Let $0 < \kappa < 1/16$ and $1/2 < \alpha < 1 - \kappa$. For every principal G-bundle $P \to \Sigma$ there is a dense Borel full-heat trace domain $\mathcal{D}_\kappa^\alpha(P) \subset \mathcal{A}^{-\kappa}(P)$, independent of the reference connection and of local bundle frames, with the following properties.*

1. *It carries Young parallel transport on every piecewise regular C^2 path, continuously on bounded families with a fixed number of pieces and uniform speed and acceleration bounds. It is invariant under $\mathcal{G}^{1-\kappa}(P)$, and the canonical transports obey the endpoint gauge identity.*

2. *Two domain connections have equal framed polygonal holonomies exactly when they differ by a unique based rough gauge; equality up to simultaneous conjugation is exactly full rough gauge equivalence. The realised holonomy images are Borel subsets of the intrinsic Polish completions, and admit Borel inverse representative maps. Equality of data on two bundles implies that the bundles are isomorphic.*

3. *The continuum solution belongs to this domain simultaneously at all surviving positive times. Its holonomy path is continuous, and the classical heat-approximation holonomies converge uniformly on bounded regular curve families and compact positive-time intervals, locally in every finite L^p. Geodesic interpolation with N equal subintervals per regular piece has error $O(N^{-\zeta})$ for every $0 < \zeta < 1/4$, with locally bounded moments. Holonomies at surviving stopping times determine the same rough gauge orbits and the same bundle class.*

If the initial connection lies in the trace domain, the limiting holonomy path is continuous from time zero as well.

The Gaussian short-arc estimates, including the entropy argument for entire regular curve families, are established in Section 18. Section 19 proves rough gauge covariance, orbit separation, closedness of the rough gauge relation, and the Borel identification. These results concern the realised domain inside the completion; the completion may also contain limits without a representative in this domain.

Theorem F (The maximal local orbit diffusion)**.** *The full rough orbit space*

$$\mathcal{Q}_\kappa(P) = \mathcal{A}^{-\kappa}(P)/\mathcal{G}^{1-\kappa}(P), \qquad 0 < \kappa < 1/16,$$

has a standard Borel structure and carries a time-homogeneous maximal local strong Markov family with absorbing cemetery state. Its local flow germs are those of the continuum equation above. Its law and its orbit lifetime ζ are independent of the smooth reference connection, the representative selections, and the equivalent norms used in its construction. Its surviving paths are continuous in the rough state quotient topology.

For every smooth B, put

$$r_B([A]) = \inf_{u \in \mathcal{G}^{1-\kappa}} \|u \cdot A - B\|_{\mathrm{c}^{-\kappa}}.$$

Almost surely, finite orbit lifetime implies

$$\limsup_{t \uparrow \zeta} r_B(Z_t) = \infty.$$

Consequently no surviving orbit path extends through its death as a continuous path in this rough quotient topology. The lifetime is defined by continuation of the orbit through successive bounded representatives; it is not defined by killing a chosen DeTurck representative at its explosion time.

The realised unframed holonomy image of Theorem E is invariant for this family, and every rough initial orbit enters it simultaneously at all surviving positive times. On that Borel image the family has continuous surviving holonomy paths and is strong Markov. It preserves the principal-bundle isomorphism class.

Section 20 constructs this continuation using Borel representatives with nearly minimal rough norm. A uniform conditional survival probability prevents switching times from accumulating at bounded orbit size. Predictable gauge rotations couple any two switching constructions, including a restart at an arbitrary stopping time. This proves reference independence and the strong Markov property on the rough orbit space.

1.2 Proof strategy

The covariant expansion of $\mathcal{V}_B$ separates the quadratic derivative term from the cubic bracket. Within the working homogeneities, these terms generate a finite collection of stochastic products, including finite first-chaos contractions. The heat kernel and its covariant derivative gain two and one analytic orders, respectively; treating them as distinct integration types gives a coupled fixed point for the connection and its derivative. We construct the Gaussian products with the reexpansion bounds this fixed point requires. The initial weight controls the singularity at time zero, and a $T^{1/4}$ factor makes the map contractive on a short interval. The forest reconstruction identity and the logarithmic coefficients $-1/4$, $-3/4$ and $+1$ then identify the original heat-cutoff equation. A simultaneous set of regular noise paths permits restart at every real time; prefix locality yields adaptedness and the strong Markov property.

The Abelian holonomy argument follows a different sequence. Gaussian line traces first construct the observables. Averaged small-triangle fluxes recover curvature, after which Hodge theory recovers the coexact connection and finitely many periods recover the Jacobian coordinate. Uniform estimates on regular curve families give simultaneous path observables and establish the invariant Borel domain. This reconstruction, valid along the entire stochastic trajectory, permits transfer of the Markov family to the holonomy state space.

The non-Abelian reference argument uses cancellation twice. In the gauge equation the derivatives of the rough connection cancel, giving a deterministic parabolic equation with meaningful products and a weighted rough initial trace. In the noise comparison a finite tensor contraction survives the heat limit, but the Yang–Mills combination annihilates it in dimension two. The fixed point is therefore restricted to its equation-generated finite enhancement before comparison. A triangular comparison of the transported models, their Gaussian cutoffs, and the varying rotated Brownian motions then identifies the continuum solutions.

For non-Abelian observables, the local small-arc entropy estimate gives Young exponents above $1/2$ uniformly over whole regular curve families. Heat product commutators prove invariance of the full-heat trace domain under rough gauges. Equal polygonal holonomies reconstruct a continuous gauge; short-geodesic difference quotients identify its distributional derivative and improve it to $\mathrm{c}^{1-\kappa}$. Compactness with two converging connection representatives gives closed orbits, from which an explicit Borel selector and the Lusin–Souslin theorem identify the realised holonomy space. These steps keep the topology of the original principal bundle throughout. The final probabilistic step chooses bounded representatives measurably, continues through their norm-exit times, and uses the conditional gauge comparison to prove choice-independent continuation. Its explosion criterion concerns the rough orbit size and is distinct from that of any fixed DeTurck representative.

1.3 Related work

Chandra–Chevyrev–Hairer–Shen construct the two-dimensional Yang–Mills Langevin dynamic on the torus, including a rough state space with gauge transformations and holonomy [7]. Bringmann–Cao give a paracontrolled construction and a detailed gauge-covariance analysis [4]. The bibliography records the published Memoirs version; numbered results from that work are identified below by their arXiv version. Chevyrev–Shen establish invariance of the Yang–Mills measure and universality for the trivial bundle on the two-dimensional torus [8].

Hairer–Singh develop regularity structures on manifolds and vector bundles [16]. We use their reconstruction, product and integration calculus, with the finite stochastic model and its precision estimates supplied here for the present equation. The general Euclidean regularity-structure and paracontrolled theories originate in [14, 15]; semigroup formulations are developed in [1, 2]. Time-integrated singular nonlinearities also occur in the energy-solution and controlled-process theory of Gubinelli–Jara [13]. Our current estimates concern first-order quadratic bundle operators, noncommuting regularisations, the stopping-time decomposition, and the sharp harmonic Yang–Mills example.

The surface Yang–Mills measure and its topological sectors have an established independent theory. Lévy treats nontrivial bundles [19]; Dang–Nohra construct a distributional surface measure and prove lattice universality [12]; Chhaibi–Dang–Guedes Bonthonneau–Rivière–Tô give a Morse-theoretic construction [11]. The November 2026 preprint of Bringmann–Cao–Hairer–Zhao constructs the non-Abelian Yang–Mills–Higgs measure on the two-dimensional torus by stochastic quantisation using a covariant non-Markovian approximation [5]. The finite gauge-normalisation question has also been studied in three-dimensional Yang–Mills–Higgs theory [9]. Chevyrev–Klose–Mohamed introduce rough additive functions, controlled gauge transformations and rough holonomy, and prove a rough Uhlenbeck compactness theorem through a singular elliptic Coulomb-gauge equation on the unit square [10]. Their work concerns a rough-path regime and an elliptic gauge construction; the Gaussian line primitives used here lie in the Young regime and are controlled uniformly on regular curve families on arbitrary surface bundles. Against this background, we construct local dynamics and compare references on arbitrary surface bundles, prove the integrated derivative-current estimates, and recover rough gauge orbits and bundle topology on the full-heat trace domain. The Abelian construction gives the invariant law and spectral gap in every sector. The rough holonomy and two-dimensional gauge cancellation used here build on the works cited above.

1.4 Notation and normalisation

All Laplacians are nonnegative. The derivative ∇_B on adjoint-valued tensors includes the Levi–Civita connection. All Hilbert pairings use the Riemannian volume and the fixed fibre metric. We write $\mathcal{C}^\alpha = B^\alpha_{\infty,\infty}$ and use c^α when a separable, strongly continuous state space is needed. Parabolic regularity uses time weight two and space weight one.

Geometric stochastic identities retain a general amplitude σ. The non-Abelian enhancement and the Gaussian-current estimates use unit noise; the Abelian equilibrium theory uses $\sigma = \sqrt{2}$, corresponding to the formal Gibbs factor $\exp(-S)$. The Gaussian field is started from zero in the integrated-current theorem and is massive stationary in the model construction; the initial-value identity in (8.41) relates the latter to the causal equation.

Our gauge convention is $u \cdot A = uAu^{-1} - (du)u^{-1}$, so local parallel transport is the path-ordered exponential of $-A$. The Abelian identification $\mathbb{R} \ni r \mapsto -ir \in \mathfrak{u}(1)$ gives real Wilson phases with sign $+\int a$. Gauge automorphisms cover the identity on Σ unless a bundle isomorphism over an isometry is specified. The bundle is fixed throughout each evolution.

2 Principal bundles and reference connections

Let $P \to \Sigma$ be a smooth principal bundle over a closed Riemannian surface, with compact connected structure group G. Fix an Ad-invariant inner product on its Lie algebra. The associated bundles $\operatorname{ad} P$ and $E_P = T^*\Sigma \otimes \operatorname{ad} P$ then have canonical fibre metrics. Write $\mathcal{A}^\infty(P)$ for the affine space of smooth connections and $\mathcal{G}^\infty(P)$ for the group of smooth gauge transformations. For $u \in \mathcal{G}^\infty(P)$, let U_u denote its pointwise adjoint action on adjoint-valued forms. In a local trivialisation our convention is

$$u \cdot A = uAu^{-1} - (\mathrm{d}u)u^{-1}. \tag{2.1}$$

The induced action on tangent vectors to $\mathcal{A}^\infty(P)$ is U_u. The formulas below therefore define global bundle operations.

Choose a smooth reference connection B. Every connection has the form $A = B + a$ with $a \in \Omega^1(\operatorname{ad} P)$. Define

$$\delta_B = d_B^*, \qquad \Delta_B = d_B\delta_B + \delta_B d_B, \qquad \chi_B(A) = \delta_B(A - B).$$

The degree of Δ_B is understood from its argument. If a is an adjoint-valued one-form, the notation $[a \lrcorner\, \cdot\,]$ means the formal adjoint of $[a \wedge \cdot\,]$. In particular, on one-forms b, $[a \lrcorner\, b] = -\sum_i [a_i, b_i]$ in an orthonormal frame. Set

$$\mathcal{V}_B(A) = -d_A^* F_A - d_A \chi_B(A). \tag{2.2}$$

Proposition 2.1 (The background DeTurck operator)**.** *For every smooth principal bundle P, reference connection B, and $a \in \Omega^1(\operatorname{ad} P)$,*

$$\begin{aligned}\mathcal{V}_B(B + a) = {} & -\Delta_B a - \delta_B F_B - [a \lrcorner\, F_B] - \tfrac{1}{2}\delta_B[a \wedge a] - [a \lrcorner\, d_B a] \\ & - [a, \delta_B a] - \tfrac{1}{2}[a \lrcorner\, [a \wedge a]].\end{aligned} \tag{2.3}$$

Its principal symbol is $-|\zeta|_g^2 \,\mathrm{id}_{E_P}$. For every smooth gauge transformation,

$$\chi_{u\cdot B}(u \cdot A) = U_u \chi_B(A), \qquad \mathcal{V}_{u\cdot B}(u \cdot A) = U_u \mathcal{V}_B(A). \tag{2.4}$$

If $C = B + b$ is another reference connection, then

$$\chi_C(A) - \chi_B(A) = -\delta_B b + [b \lrcorner\, (A - B)], \tag{2.5}$$

$$\mathcal{V}_C(A) - \mathcal{V}_B(A) = -d_A(\chi_C(A) - \chi_B(A)). \tag{2.6}$$

Proof. The identities

$$F_{B+a} = F_B + d_B a + \tfrac{1}{2}[a \wedge a], \qquad d_{B+a} = d_B + [a \wedge \cdot\,], \qquad d_{B+a}^* = \delta_B + [a \lrcorner\, \cdot\,]$$

give (2.3) by direct expansion. The terms with two derivatives of a are $-\delta_B d_B a - d_B \delta_B a$. The Hodge-type operator Δ_B is elliptic with scalar principal symbol $|\zeta|_g^2 \,\mathrm{id}$, irrespective of the curvature of B. The remaining terms in (2.3) contain at most one derivative of a. Gauge covariance of d_B, its adjoint, and curvature proves (2.4). Finally,

$$\chi_C(A) = (\delta_B + [b \lrcorner\, \cdot\,])(A - B - b),$$

and $[b \lrcorner\, b] = 0$. This proves (2.5); subtraction of (2.2) proves (2.6). □

Corollary 2.2 (Exact covariant polynomial form)**.** *Let ∇_B be the tensor-product connection induced by the Levi-Civita connection and B. In any orthonormal frame, define*

$$f_B = -\delta_B F_B, \qquad (V_B a)_i = \sum_j [a_j, (F_B)_{ji}], \tag{2.7}$$

$$Q_B(a,b)_i = \sum_j [a_j, 2(\nabla_B)_j b_i - (\nabla_B)_i b_j], \tag{2.8}$$

$$C(a,b,c)_i = \sum_j [a_j, [b_j, c_i]]. \tag{2.9}$$

These are global tensor operations, and

$$\mathcal{V}_B(B+a) = -\Delta_B a + f_B + V_B a + Q_B(a,a) + C(a,a,a). \tag{2.10}$$

The quadratic term consists entirely of the displayed parallel bracket contraction of a and $\nabla_B a$.

Proof. At a point choose a normal orthonormal frame. The three quadratic terms in (2.3) have components

$$\begin{aligned} -\tfrac{1}{2}\big(\delta_B[a \wedge a]\big)_i &= \sum_j \big([(\nabla_B)_j a_j, a_i] + [a_j, (\nabla_B)_j a_i]\big), \\ -[a \lrcorner\, d_B a]_i &= \sum_j [a_j, (\nabla_B)_j a_i - (\nabla_B)_i a_j], \\ -[a, \delta_B a]_i &= -\sum_j [(\nabla_B)_j a_j, a_i]. \end{aligned}$$

The first and third lines cancel their divergence terms, yielding (2.8). The adjoint contraction convention gives $-[a \lrcorner F_B]_i = (V_B a)_i$ and $-\frac{1}{2}[a \lrcorner [a \wedge a]]_i = C(a,a,a)_i$. The bracket is parallel and all spatial indices are contracted with the metric, so the calculation is independent of the chosen normal frame and proves the global identity. □

Remark 2.3. The operator Δ_B has a finite-dimensional smooth kernel and compact resolvent. When $F_B \neq 0$, the identity $d_B^2 = [F_B \wedge \cdot\,]$ prevents d_B from defining a complex in general. In particular, an orthogonal decomposition into d_B-exact, d_B^*-exact, and Δ_B-harmonic one-forms does not follow from ellipticity. Spectral decompositions of Δ_B remain available.

2.1 Relative Coulomb coordinates

Fix an integer $s \geq 3$. Sobolev spaces below are defined using any smooth auxiliary connection; different choices give equivalent norms. Let

$$\mathfrak{k}_B = \ker\big(d_B : H^{s+1}(\mathrm{ad}\, P) \longrightarrow H^s(E_P)\big), \qquad \mathcal{K}_B = \{u \in \mathcal{G}^\infty(P) : u \cdot B = B\}.$$

The space $\mathfrak{k}_B$ consists of smooth parallel sections. Removing these infinitesimal stabilisers makes the linearised Coulomb equation invertible. Orthogonal complements below are taken in L^2.

Proposition 2.4 (Local relative Coulomb reduction)**.** *There are neighbourhoods $\mathcal{U}_B$ of B in $B + H^s(E_P)$ and $\mathcal{W}_B$ of zero in $H^{s+1}(\mathrm{ad}\, P) \cap \mathfrak{k}_B^{\perp}$, and a smooth map*

$$\varphi_B : \mathcal{U}_B \longrightarrow \mathcal{W}_B, \qquad \varphi_B(B) = 0,$$

such that

$$R_B(A) = \exp(\varphi_B(A)) \cdot A, \qquad \delta_B(R_B(A) - B) = 0. \tag{2.11}$$

The solution $\varphi_B(A) \in \mathcal{W}_B$ is unique. The neighbourhoods may be chosen $\mathcal{K}_B$-invariant, and then

$$R_B(k \cdot A) = k \cdot R_B(A), \qquad k \in \mathcal{K}_B. \tag{2.12}$$

This reduction retains the action of $\mathcal{K}_B$ on the relative Coulomb slice.

Proof. Consider

$$\mathcal{F}(\varphi, a) = \delta_B(\exp(\varphi) \cdot (B + a) - B).$$

Its values lie in $H^{s-1}(\operatorname{ad} P) \cap \mathfrak{k}_B^\perp$, since $\langle \delta_B v, z\rangle = \langle v, d_B z\rangle = 0$ for $z \in \mathfrak{k}_B$. Sobolev multiplication and the smoothness of the pointwise exponential make $\mathcal{F}$ a smooth map between these Banach spaces. Its derivative in the first variable at $(0,0)$ is $-\delta_B d_B$. The nonnegative elliptic operator $\delta_B d_B$ has kernel $\mathfrak{k}_B$ and is an isomorphism

$$H^{s+1}(\operatorname{ad} P) \cap \mathfrak{k}_B^\perp \longrightarrow H^{s-1}(\operatorname{ad} P) \cap \mathfrak{k}_B^\perp.$$

Surjectivity follows from the spectral inverse on the orthogonal complement and elliptic regularity; the same estimate gives boundedness of the inverse. The Banach-space implicit function theorem proves (2.11) and uniqueness.

Evaluation of a parallel gauge transformation at one point embeds $\mathcal{K}_B$ into the compact group G as the centraliser of the holonomy group of B. Thus $\mathcal{K}_B$ is compact. It preserves $\mathfrak{k}_B^\perp$, and the equation defining $\mathcal{F}$ is $\mathcal{K}_B$-equivariant. Choose invariant small neighbourhoods by using an invariant norm, or averaging a norm over $\mathcal{K}_B$. Uniqueness gives $\varphi_B(k \cdot A) = U_k \varphi_B(A)$ and hence (2.12). □

For $A \in \mathcal{U}_B$ with $\delta_B(A - B) = 0$, define

$$M_{B,A} = \delta_B d_A\big|_{\mathfrak{k}_B^\perp}, \qquad Q_{B,A} = I - d_A M_{B,A}^{-1} \delta_B. \tag{2.13}$$

After shrinking $\mathcal{U}_B$, the first operator is an isomorphism between the same spaces as $\delta_B d_B$ in the preceding proof. Indeed, $M_{B,A} - \delta_B d_B = \delta_B[A - B, \cdot\,]$ has operator norm bounded by a constant times $\|A - B\|_{H^s}$ from H^{s+1} to H^{s-1}. The range remains in $\mathfrak{k}_B^\perp$.

Proposition 2.5 (First and second derivatives of the reduction)**.** *At a relative Coulomb point $A \in \mathcal{U}_B$, put $\eta_h = M_{B,A}^{-1} \delta_B h$ for $h \in H^s(E_P)$. Then*

$$DR_B(A)h = Q_{B,A} h, \tag{2.14}$$

$$D^2 R_B(A)[h,k] = Q_{B,A}\left([\eta_h, k] + [\eta_k, h] - \tfrac{1}{2}[\eta_h, d_A\eta_k] - \tfrac{1}{2}[\eta_k, d_A\eta_h]\right). \tag{2.15}$$

In particular, $Q_{B,A}$ is a bounded projection onto $\ker \delta_B$*. Its kernel is $d_A(H^{s+1} \cap \mathfrak{k}_B^\perp)$.*

Proof. At a slice point, uniqueness gives $\varphi_B(A) = 0$. Differentiating the Coulomb equation yields $D\varphi_B(A)h = M_{B,A}^{-1}\delta_B h$. Differentiating the gauge action gives $DR_B(A)h = h - d_A\eta_h$, proving (2.14). The identities $\delta_B Q_{B,A} = 0$ and $Q_{B,A}h = h$ for $\delta_B h = 0$ prove the projection statements.

For the second derivative, use the expansion

$$\exp(\varphi) \cdot (A + v) = A + v - d_A\varphi + [\varphi, v] - \tfrac{1}{2}[\varphi, d_A\varphi] + O_3(\varphi, v),$$

where the remainder has vanishing derivatives through order two at $(0,0)$ as a map to H^s. For $v = th$ write $\varphi_B(A + th) = t\eta_h + \frac{1}{2}t^2\zeta_h + o(t^2)$. The second derivative of the reduction is

$$-d_A\zeta_h + 2[\eta_h, h] - [\eta_h, d_A\eta_h].$$

Applying δ_B and using the Coulomb condition determines $\zeta_h = M_{B,A}^{-1}\delta_B(2[\eta_h, h] - [\eta_h, d_A\eta_h])$. This proves (2.15) on the diagonal; polarisation gives its stated bilinear form. □

2.2 Itô reduction and the framing coordinate

Let H be a separable real Hilbert space and W a cylindrical Brownian motion on H. Consider an $H^s(E_P)$-valued continuous semimartingale, stopped inside $\mathcal{U}_B$,

$$\mathrm{d}A_t = V_t\,\mathrm{d}t + S_t\,\mathrm{d}W_t, \tag{2.16}$$

where V is locally integrable in H^s and $S_t \in \mathrm{HS}(\mathsf{H}, H^s(E_P))$ is predictable and locally square integrable.

Proposition 2.6 (Itô formula in relative Coulomb coordinates)**.** *For any orthonormal basis (e_j) of H,*

$$\begin{aligned}\mathrm{d}R_B(A_t) &= DR_B(A_t)V_t\,\mathrm{d}t + DR_B(A_t)S_t\,\mathrm{d}W_t \\ &\quad + \tfrac{1}{2}\sum_j D^2R_B(A_t)[S_te_j, S_te_j]\,\mathrm{d}t.\end{aligned} \tag{2.17}$$

The series converges absolutely in H^s, locally uniformly under bounds for D^2R_B and the Hilbert–Schmidt norm of S. At a slice point A, the diffusion coefficient is $Q_{B,A}S$. Its Itô correction is

$$Q_{B,A}\sum_j \Big([\eta_j, Se_j] - \tfrac{1}{2}[\eta_j, d_A\eta_j]\Big), \qquad \eta_j = M_{B,A}^{-1}\delta_B(Se_j). \tag{2.18}$$

Proof. Apply the Hilbert-space Itô formula to the smooth map R_B. After stopping in a neighbourhood where its second derivative is bounded, the trace is bounded by

$$\sum_j \|D^2R_B(A)[Se_j, Se_j]\|_{H^s} \le \|D^2R_B(A)\|\,\|S\|^2_{\mathrm{HS}(\mathsf{H},H^s)}.$$

Finite-rank approximations to S therefore converge in the drift and martingale terms. The derivative formulas in Proposition 2.5 give (2.18). □

The distinction between the full and based gauge groups enters at this point. Fix a point $p_0 \in P$ and let $\mathcal{G}_*(P) = \{u : u(p_0) = e\}$. The reduction above normalises its infinitesimal gauge parameter by L^2-orthogonality to $\mathfrak{k}_B$. It imposes no condition at p_0. A description of the based quotient must therefore retain the value of the reducing gauge at p_0, together with the residual action of $\mathcal{K}_B$. In particular, R_B alone is not a coordinate on the based quotient. For a differentiable gauge path with $\vartheta_t = (\partial_t u_t)u_t^{-1}$,

$$\partial_t u_t(p_0) = \vartheta_t(p_0)u_t(p_0). \tag{2.19}$$

An initially based gauge path remains based exactly when its velocity vanishes at p_0 throughout the interval.

3 Covariant regularisation and changes of reference

Set $H = L^2(\Sigma; E_P)$. For a smooth connection A, let

$$Q_\varepsilon(A) = \exp(-\varepsilon\Delta_A), \qquad \varepsilon > 0, \tag{3.1}$$

where $\Delta_A = d_Ad_A^* + d_A^*d_A$ acts on one-forms. This operator smooths the noise using the evolving connection. For smooth A, it is Hilbert–Schmidt from H to $H^r(E_P)$ for every finite r.

Proposition 3.1 (Covariance of the smoothing operator)**.** *For every smooth gauge transformation u,*

$$Q_\varepsilon(u \cdot A) = U_u Q_\varepsilon(A) U_u^{-1}. \tag{3.2}$$

The same identity holds for any smooth bundle isomorphism covering an isometry of the base, with the induced unitary map between the two L^2 spaces.

Proof. The invariant fibre metric makes U_u unitary on each L^2 space of adjoint-valued forms. Covariance of the covariant differential gives $d_{u\cdot A} U_u = U_u d_A$. Taking adjoints and composing gives $\Delta_{u\cdot A} = U_u \Delta_A U_u^{-1}$ as self-adjoint operators. The spectral functional calculus proves (3.2). Bundle isomorphisms covering isometries also intertwine the covariant differential, its adjoint, and the L^2 inner product, so the same proof applies. □

We consider the regularised Itô equation

$$\mathrm{d}A_t = \left(-d^*_{A_t} F_{A_t} - d_{A_t} \chi_B(A_t) + J(A_t)\right) \mathrm{d}t + \sigma Q_\varepsilon(A_t)\, \mathrm{d}W_t. \tag{3.3}$$

Here $\sigma \in \mathbb{R}$ is fixed and J is a smooth drift satisfying

$$J(u \cdot A) = U_u J(A). \tag{3.4}$$

The choice $J = 0$ is included. The comparison identities below first apply to classical local solutions whose drift is integrable in H^s, whose noise coefficient is locally square integrable in $\mathrm{HS}(H, H^s)$, and whose spatial regularity permits the gauge operations; smooth spatial paths with these properties suffice. Section 4 constructs the maximal H^s solutions and extends the comparison through their strong integral equations in H^{s-2}.

3.1 The reference-change equation

Let B, C be smooth reference connections and let A_t be a given continuous path of sufficiently regular connections. For a gauge path u_t, write $\widetilde{A}_t = u_t \cdot A_t$. Consider

$$(\partial_t u_t) u_t^{-1} = \chi_C(\widetilde{A}_t) - U_{u_t} \chi_B(A_t). \tag{3.5}$$

The equation is intrinsic on P.

Lemma 3.2 (Classical local solvability of the comparison gauge)**.** *Suppose that A_t is continuous in time and smooth in space, with its spatial derivatives bounded on each compact time interval. For every smooth initial gauge u_0, equation (3.5) has a unique classical local solution. It depends only on the past of A, and is therefore adapted when A is adapted. Its paths have finite variation in every spatial norm of sufficiently low order relative to the derivatives imposed on A.*

Proof. Put $a = A - B$, $b = C - B$, and $k = (d_B u) u^{-1}$, where $d_B u = \mathrm{d}u + Bu - uB$ in a faithful matrix representation. Then

$$u \cdot A = B + U_u a - k.$$

The product rule gives

$$\delta_B(U_u a) - U_u \delta_B a = [k \lrcorner U_u a].$$

Consequently the right-hand side of (3.5) is

$$-\delta_B k - \delta_B b + [k \lrcorner U_u a] + [b \lrcorner U_u a] - [b \lrcorner k]. \tag{3.6}$$

In particular, no derivative of the evolving connection a occurs in this expression. The highest-order term in u is the connection Laplacian with heat sign. Write $u = \exp(\psi)u_0$ in a neighbourhood of the initial section. In these coordinates the equation takes the form

$$\partial_t \psi = -\Delta_B^{(0)} \psi + N(t, x, \psi, \nabla_B \psi), \qquad \psi(0) = 0,$$

with smooth coefficients in the spatial variables and local variables $(\psi, \nabla_B \psi)$, continuous time dependence, and at most first derivatives of ψ in N. The principal second derivatives of $\exp(\psi)$ are multiplied by the differential of the exponential on both sides of the equation; applying its local inverse leaves the scalar principal symbol of $\Delta_B^{(0)}$.

For a fixed nonintegral exponent $r > 1$, on bounded sets the map N is locally Lipschitz from C^r to C^{r-1}. The heat estimate

$$\left\| \int_0^t e^{-(t-v)\Delta_B^{(0)}} f(v)\, \mathrm{d}v \right\|_{C^r} \leq C T^{1/2} \sup_{v \leq T} \|f(v)\|_{C^{r-1}}$$

gives a contraction on a small time interval. Taking sufficiently many spatial derivatives and repeating the same estimate gives the required spatial regularity. The equation then gives a continuous time derivative in spatial norms two orders lower. Uniqueness of the contraction implies dependence only on the restricted input path, proving causality and adaptedness. The use of exponential coordinates ensures that the solution remains a section of the gauge group bundle. □

Theorem 3.3 (Exact comparison of regularised Itô solutions)**.** *Let A be a local solution of (3.3) with reference B, satisfying the regularity conditions above. Let u be an adapted finite-variation solution of (3.5) with prescribed smooth initial gauge u_0. On their common interval of existence, $\widetilde{A} = u \cdot A$ solves*

$$\mathrm{d}\widetilde{A}_t = \left(-d^*_{\widetilde{A}_t} F_{\widetilde{A}_t} - d_{\widetilde{A}_t} \chi_C(\widetilde{A}_t) + J(\widetilde{A}_t)\right) \mathrm{d}t + \sigma Q_\varepsilon(\widetilde{A}_t)\, \mathrm{d}\widetilde{W}_t, \tag{3.7}$$

where

$$\widetilde{W}_t = \int_0^t U_{u_v}\, \mathrm{d}W_v \tag{3.8}$$

is an H-cylindrical Brownian motion for the original filtration. The two paths lie in the same full gauge orbit at every time.

Proof. For a differentiable gauge path and a smooth connection path the derivative of the affine gauge action is

$$\partial_t(u \cdot A) = U_u \partial_t A - d_{u \cdot A}\big((\partial_t u) u^{-1}\big).$$

The same identity holds for the semimartingale A: the process u has finite variation, so its quadratic covariation with A vanishes. Thus, with $\vartheta = (\partial_t u)u^{-1}$,

$$\mathrm{d}\widetilde{A} = \left(-d^*_{\widetilde{A}} F_{\widetilde{A}} - d_{\widetilde{A}}(U_u \chi_B(A) + \vartheta) + J(\widetilde{A})\right) \mathrm{d}t + \sigma U_u Q_\varepsilon(A)\, \mathrm{d}W.$$

Equation (3.5) gives the required drift. Proposition 3.1 gives $U_u Q_\varepsilon(A) = Q_\varepsilon(\widetilde{A}) U_u$. For any $f, g \in H$, the real martingales defined by (3.8) have quadratic covariation

$$\int_0^t \langle U_{u_v}^{-1} f, U_{u_v}^{-1} g \rangle_H \, \mathrm{d}v = t \langle f, g \rangle_H.$$

The finite-dimensional Lévy characterisation, applied to every finite family of test vectors, proves that $\widetilde{W}$ is cylindrical Brownian motion. This proves (3.7). □

For the next corollary, equality of orbit laws means equality of the expectations of bounded measurable path functionals that are constant under pointwise full gauge transformations. Whenever a measurable orbit path space is specified, this is equality of the corresponding pushforward laws.

Corollary 3.4 (Reference independence under classical uniqueness)**.** *Fix $T > 0$. Suppose the regularised equations with references B and C are weakly well posed in the preceding regularity class on $[0, T]$, and the comparison gauge exists on this interval. Their laws on full gauge orbits, started from the same orbit, coincide. The assertion also holds for a common killed problem whose stopping rule is a measurable function of the orbit path, provided the comparison exists until that stopping time.*

Proof. Start the B equation from A_0 and choose u_0 with $u_0 \cdot A_0 = \widetilde{A}_0$. Theorem 3.3 constructs a weak solution of the C equation with initial condition $\widetilde{A}_0$. Weak uniqueness identifies its law with the given C-solution law. The constructed paths have identical full gauge orbits. An orbit-measurable stopping rule takes the same value on both paths, which proves the stopped assertion. □

Taking $B = C$ gives the corresponding local gauge covariance in law for a fixed reference. The comparison gauge is generally not based: its value at p_0 evolves according to (2.19). The corollary concerns the full gauge quotient.

Remark 3.5 (Counterterms with a vertical transformation defect). The comparison extends to reference-dependent additional drifts J^ε_B whenever there is a specified smooth adjoint-valued function $\rho^\varepsilon_{B,C,u}(A)$ satisfying

$$U_u J^\varepsilon_B(A) - J^\varepsilon_C(u \cdot A) = d_{u\cdot A}\rho^\varepsilon_{B,C,u}(A). \tag{3.9}$$

One adds $\rho^\varepsilon_{B,C,u}(A)$ to the right-hand side of (3.5). Substitution into the preceding proof gives exact cancellation of the defect. Local existence of the modified gauge equation is required separately if the defect changes its differential order. Thus (3.9) is a precise sufficient compatibility condition for additional drifts.

3.2 Generators and stochastic conventions

A smooth real function f on a Sobolev neighbourhood of connections is called gauge invariant if $f(u \cdot A) = f(A)$ whenever both sides are defined. Differentiation along gauge orbits gives $Df(A)d_A z = 0$ for smooth z. At a smooth connection where its derivatives are defined, the generator of (3.3) therefore acts by

$$\mathcal{L}_\varepsilon f(A) = Df(A)(-d^*_A F_A + J(A)) + \frac{\sigma^2}{2}\sum_j D^2 f(A)[Q_\varepsilon(A)e_j, Q_\varepsilon(A)e_j]. \tag{3.10}$$

The series is absolutely convergent whenever $D^2 f$ is bounded on a Sobolev space into which $Q_\varepsilon(A)$ is Hilbert–Schmidt. Formula (3.10) is independent of B. It is gauge invariant because the gauge action is affine, $-d^*_A F_A + J(A)$ is equivariant, and Q_ε satisfies (3.2). This gives the same reference-independent operator on invariant observables as the pathwise comparison theorem.

Equation (3.3) uses the Itô convention. A Stratonovich equation with the same displayed drift and noise coefficient has the additional Itô drift

$$\Gamma_\varepsilon(A) = \frac{\sigma^2}{2}\sum_j DQ_\varepsilon(A)[Q_\varepsilon(A)e_j]e_j. \tag{3.11}$$

For smooth A, this expression is defined as a nuclear contraction and is independent of the orthonormal basis. To see this, differentiate the heat semigroup along a smooth tangent vector v:

$$DQ_\varepsilon(A)[v] = -\int_0^\varepsilon e^{-(\varepsilon-r)\Delta_A}(D\Delta_A[v])e^{-r\Delta_A}\,\mathrm{d}r. \tag{3.12}$$

The operator $D\Delta_A[v]$ has order at most one in its argument and uses at most one derivative of v. Splitting the integral at $r = \varepsilon/2$ shows, for each integer $m \geq 0$, that $(v, h) \mapsto DQ_\varepsilon(A)[v]h$ is bounded from $C^{m+1}(E_P) \times H$ to $H^m(E_P)$, with a constant depending on the fixed smooth A and on ε. The operator $Q_\varepsilon(A) : H \to C^{m+1}(E_P)$ is nuclear by elliptic heat smoothing. If $Q_\varepsilon(A) = \sum_n \lambda_n v_n \otimes w_n$ is a nuclear representation, contraction of the bounded bilinear map gives

$$\sum_n \lambda_n DQ_\varepsilon(A)[v_n]w_n,$$

an absolutely convergent series in H^m. For finite-rank orthogonal projections $P_N \to I$ in H, the corresponding basis sums equal $\sum_n \lambda_n DQ_\varepsilon(A)[v_n]P_N w_n$ and converge to the same expression. This proves the asserted basis independence. Differentiation of (3.2) and invariance of this contraction show

$$\Gamma_\varepsilon(u \cdot A) = U_u \Gamma_\varepsilon(A).$$

Whenever Itô–Stratonovich conversion applies in the chosen solution class, the comparison theorem thus also applies with the appropriate drift $J + \Gamma_\varepsilon$.

4 Existence for the covariantly regularised equation

Fix a smooth reference connection B on an arbitrary principal bundle $P \to \Sigma$ as in Section 2. Write $E = T^*\Sigma \otimes \operatorname{ad} P$ and $H = L^2(\Sigma; E)$. Throughout this section $s \geq 4$ is an integer. Sobolev norms are defined using $1 + \Delta_B$, or equivalently a fixed smooth bundle atlas. Put $\mathcal{A}^s(P) = B + H^s(E)$.

For $A \in \mathcal{A}^s(P)$, the operator Δ_A is defined by the closed nonnegative quadratic form

$$\mathfrak{q}_A(v) = \|d_A v\|_{L^2}^2 + \|d_A^* v\|_{L^2}^2, \qquad v \in H^1(E). \tag{4.1}$$

Its principal symbol is fixed, and its lower-order coefficients contain at most one derivative of A. This permits uniform heat estimates on H^s balls by perturbing the smooth-reference operator. We write $S_A(t) = e^{-t\Delta_A}$ and $Q_\varepsilon(A) = S_A(\varepsilon)$, and use heat estimates only through Sobolev order s.

4.1 Uniform heat estimates on Sobolev balls

Lemma 4.1 (Lower-order operator bounds)**.** *For $R < \infty$, let $A, C \in \mathcal{A}^s(P)$ satisfy $\|A - B\|_{H^s} + \|C - B\|_{H^s} \leq R$. For $j = 0, \dots, s-1$,*

$$\|(\Delta_A - \Delta_B)v\|_{H^j} \leq C_R \|v\|_{H^{j+1}}, \tag{4.2}$$

$$\|(\Delta_A - \Delta_C)v\|_{H^j} \leq C_R \|A - C\|_{H^s} \|v\|_{H^{j+1}}. \tag{4.3}$$

In addition,

$$\|(\Delta_A - \Delta_C)v\|_{H^{-1}} \leq C_R \|A - C\|_{H^s} \|v\|_{L^2}. \tag{4.4}$$

The form in (4.1) is closed on $H^1(E)$ and has compact resolvent. Its operator domain on H is $H^2(E)$.

Proof. Set $a = A - B$. Expanding $d_A = d_B + [a \wedge \cdot\,]$ and its adjoint shows that $\Delta_A - \Delta_B$ is a sum of terms of the forms

$$a \nabla_B v, \qquad (\nabla_B a) v, \qquad a a v,$$

with fixed smooth contractions and lower-order coefficients. For $0 \le j \le s-1$, the coefficients $a \in H^s$ and $\nabla_B a \in H^{s-1}$ act as the required Sobolev multipliers on a surface. In particular H^{s-1} is an algebra. These product estimates give (4.2); polarisation gives (4.3). Integration by parts in the first-order terms gives (4.4), using $H^s \hookrightarrow W^{1,\infty}$.

The inequality $\|x+y\|^2 \ge \frac{1}{2}\|x\|^2 - \|y\|^2$ gives

$$\mathfrak{q}_A(v) \ge \tfrac{1}{2}\mathfrak{q}_B(v) - C_R \|v\|_{L^2}^2. \tag{4.5}$$

Together with the reverse bound and the elliptic estimate for the smooth operator Δ_B, this makes $\mathfrak{q}_A(v) + \|v\|_{L^2}^2$ equivalent to the H^1 norm squared, with constants depending only on R. Closedness and compactness of the form-domain embedding follow. If v belongs to the operator domain, then $v \in H^1$ and $(\Delta_A - \Delta_B)v \in L^2$ by (4.2) at $j = 0$. Hence $\Delta_B v \in L^2$ and $v \in H^2$ by elliptic regularity. The converse follows from the same operator bound. □

Lemma 4.2 (Heat and Hilbert–Schmidt bounds)**.** *For $A \in \mathcal{A}^s(P)$ with $\|A - B\|_{H^s} \le R$ and $0 < t \le 1$,*

$$\|S_A(t)\|_{H^j \to H^{j+1}} \le C_R t^{-1/2}, \qquad 0 \le j \le s-1, \tag{4.6}$$

$$\|S_A(t)\|_{H^s \to H^s} \le C_R, \tag{4.7}$$

$$\|S_A(t)\|_{\mathrm{HS}(H, H^s)} \le C_R t^{-(s+1)/2}, \tag{4.8}$$

$$\|S_A(t)\|_{\mathrm{HS}(H^{-1}, H^s)} \le C_R t^{-(s+2)/2}. \tag{4.9}$$

The Sobolev-space semigroups in these formulas agree with the self-adjoint L^2 semigroup on their common domains.

Proof. Let $R_A = \Delta_A - \Delta_B$. The Volterra equation

$$v(t) = S_B(t) f - \int_0^t S_B(t-r) R_A v(r)\, dr \tag{4.10}$$

can first be solved in $C([0,T]; H^{j+1})$ for $f \in H^{j+1}$. The smooth-background heat estimate and (4.2) make its integral operator locally Lipschitz with norm at most $C_R T^{1/2}$. This proves boundedness on H^{j+1} for small T, and iteration gives a uniform bound for $T \le 1$.

For $f \in H^j$, use the norm $\sup_{0<t\le T} t^{1/2}\|v(t)\|_{H^{j+1}}$, together with continuity in H^j. The weighted integral is controlled by

$$t^{1/2} \int_0^t (t-r)^{-1/2} r^{-1/2}\, dr = \pi t^{1/2}.$$

This proves (4.6); the preceding unweighted argument at $j = s-1$ proves (4.7). The Volterra solutions agree with the L^2 semigroup by uniqueness of the form evolution, first for smooth initial data and then by density. Thus the argument also proves strong continuity in the Sobolev spaces used here. Iterating (4.6) with the semigroup identity gives

$$\|S_A(t)\|_{H \to H^s} \le C_R t^{-s/2}. \tag{4.11}$$

Self-adjointness and the case $j = 0$ imply by duality that $\|S_A(t)\|_{H^{-1} \to H} \le C_R t^{-1/2}$.

By the min–max principle, (4.5) implies $\lambda_n(A) \ge \frac{1}{2}\lambda_n(B) - C_R$, with eigenvalues repeated according to multiplicity. The two-dimensional counting bound for the fixed smooth operator Δ_B therefore gives

$$\|S_A(t)\|_{\mathrm{HS}(H,H)}^2 = \operatorname{Tr} e^{-2t\Delta_A} \le C_R t^{-1}, \qquad 0 < t \le 1. \tag{4.12}$$

Factor $S_A(t) = S_A(t/2)S_A(t/2)$ and combine (4.11) with (4.12) to obtain (4.8). A second factorisation, using $S_A(t/2) : H^{-1} \to H$ and $S_A(t/2) \in \mathrm{HS}(H, H^s)$, proves (4.9). □

Theorem 4.3 (Local Lipschitz continuity of the covariant smoother)**.** *For $s \geq 4$ and every $\varepsilon > 0$, the map*

$$Q_\varepsilon : \mathcal{A}^s(P) \longrightarrow \mathrm{HS}(H, H^s(E))$$

is bounded and Lipschitz on every H^s ball. More precisely, for $0 < \varepsilon \leq 1$ and A, C in the ball of Lemma 4.1,

$$\|Q_\varepsilon(A)\|_{\mathrm{HS}(H,H^s)} \leq C_R \varepsilon^{-(s+1)/2}, \tag{4.13}$$

$$\|Q_\varepsilon(A) - Q_\varepsilon(C)\|_{\mathrm{HS}(H,H^s)} \leq C_R \varepsilon^{-s/2} \|A - C\|_{H^s}. \tag{4.14}$$

If $u \in \mathcal{G}^{s+1}(P)$, then

$$Q_\varepsilon(u \cdot A) = U_u Q_\varepsilon(A) U_u^{-1}. \tag{4.15}$$

Here $\mathcal{G}^{s+1}(P)$ is the Sobolev gauge group with one more derivative than the connections.

Proof. The size estimate is (4.8). For the difference, use

$$S_A(\varepsilon) - S_C(\varepsilon) = -\int_0^\varepsilon S_A(\varepsilon - r)(\Delta_A - \Delta_C) S_C(r)\, dr. \tag{4.16}$$

This identity follows by differentiating $S_A(\varepsilon - r)S_C(r)$ between positive endpoints and then passing to the endpoints in the weak formulation. The following bounds also show that its integral converges in $\mathrm{HS}(H, H^s)$.

For $0 < r \leq \varepsilon/2$, place the Hilbert–Schmidt factor on the left. Equations (4.4) and (4.9), and L^2 contractivity of $S_C(r)$, give

$$\|S_A(\varepsilon - r)(\Delta_A - \Delta_C) S_C(r)\|_{\mathrm{HS}(H,H^s)} \leq C_R \varepsilon^{-(s+2)/2} \|A - C\|_{H^s}.$$

For $\varepsilon/2 \leq r < \varepsilon$, use the factorisation through $H^s \to H^{s-1} \to H^s$. Equations (4.3), (4.6), and (4.8) give

$$\|S_A(\varepsilon - r)(\Delta_A - \Delta_C) S_C(r)\|_{\mathrm{HS}(H,H^s)} \leq C_R (\varepsilon - r)^{-1/2} \varepsilon^{-(s+1)/2} \|A - C\|_{H^s}.$$

Both endpoint bounds are integrable. Their integrals are bounded by $C_R \varepsilon^{-s/2} \|A - C\|_{H^s}$, which proves (4.14). In particular, the coefficient $\nabla A \in H^{s-1}$ costs one order in the second factorisation; the heat operator recovers that order with an integrable exponent.

For a fixed $\varepsilon > 1$, the same argument uses the Sobolev heat bounds on the finite interval $[0, \varepsilon]$. Iteration of the Volterra estimates supplies those bounds, and the same endpoint split proves local Lipschitz continuity with a constant depending on ε.

For the covariance identity, multiplication by U_u preserves the form domain H^1. The covariant differential and its adjoint intertwine under this unitary map on L^2. The closed quadratic forms, and hence their self-adjoint operators and functional calculi, are unitarily conjugate. This proves (4.15) at the stated Sobolev regularity. □

4.2 Strong solutions at every fixed regularisation scale

For the rest of the section fix $\varepsilon > 0$ and $\sigma \in \mathbb{R}$. The equation to be solved is

$$dA_t = \mathcal{V}_B(A_t)\, dt + \sigma Q_\varepsilon(A_t)\, dW_t. \tag{4.17}$$

The stochastic integral has the Itô convention. Write

$$N_B(a) = \mathcal{V}_B(B + a) + \Delta_B a.$$

The expansion in Proposition 2.1 and Sobolev multiplication show that $N_B : H^s \to H^{s-1}$ is bounded and Lipschitz on bounded sets.

We shall use the following maximal estimate. With the spectral H^s norm defined by $1+\Delta_B$, the semigroup S_B is a contraction on H^s. For predictable $G_t \in \mathrm{HS}(H, H^s)$ and $p \geq 2$,

$$\mathbb{E} \sup_{t \leq T} \left\| \int_0^t S_B(t-r) G_r \, dW_r \right\|_{H^s}^p \leq C_p \mathbb{E} \left(\int_0^T \|G_r\|_{\mathrm{HS}(H,H^s)}^2 \, dr \right)^{p/2}. \tag{4.18}$$

Apply Itô's formula and the Burkholder–Davis–Gundy inequality to the spectral Galerkin equation $dZ = -\Delta_B Z \, dt + G \, dW$. The generator contribution to the H^s energy is nonpositive. Young's inequality absorbs the martingale term containing $\sup \|Z\|_{H^s}$ and gives (4.18), with a constant independent of the Galerkin dimension. Apply the same estimate to differences of Galerkin approximations. Their diffusion coefficients converge in the Hilbert–Schmidt norm, so the stochastic convolution has a continuous H^s-valued version and the estimate passes to the limit.

Theorem 4.4 (Maximal strong covariantly regularised dynamics)**.** *For every $A_0 \in \mathcal{A}^s(P)$, equation (4.17) has a unique maximal local strong solution*

$$A \in C([0,\tau); \mathcal{A}^s(P)), \qquad \tau > 0 \quad \textit{almost surely.}$$

The lifetime is a stopping time. If $\tau < \infty$, then

$$\limsup_{t \uparrow \tau} \|A_t - B\|_{H^s} = \infty. \tag{4.19}$$

The solution is pathwise unique, its law is uniquely determined by A_0, and the resulting maximal family is strong Markov. On bounded sets and before exit, it depends locally Lipschitz continuously on the initial state in $L^p(\Omega; C_t H^s)$ for every finite $p \geq 2$.

In affine coordinates its mild equation is

$$\begin{aligned} a_t = S_B(t)a_0 &+ \int_0^t S_B(t-r) N_B(a_r) \, dr \\ &+ \sigma \int_0^t S_B(t-r) Q_\varepsilon(B + a_r) \, dW_r. \end{aligned} \tag{4.20}$$

It also satisfies the strong integral equation (4.17) in H^{s-2} on its stochastic interval of existence.

Proof. Choose a Lipschitz cutoff χ_R that is one on $\{\|a\|_{H^s} \leq R\}$ and zero on $\{\|a\|_{H^s} \geq 2R\}$. The maps

$$N_{B,R}(a) = \chi_R(a) N_B(a), \qquad G_{\varepsilon,R}(a) = \sigma \chi_R(a) Q_\varepsilon(B + a)$$

are globally Lipschitz and bounded into H^{s-1} and $\mathrm{HS}(H, H^s)$ respectively. The latter assertion is Theorem 4.3.

For the truncated mild equation use the Banach space of predictable continuous processes with norm $(\mathbb{E} \sup_{t \leq T} \|a_t\|_{H^s}^p)^{1/p}$. For two such processes, the deterministic convolution satisfies

$$\left\| \int_0^t S_B(t-r)(N_{B,R}(a_r) - N_{B,R}(b_r)) \, dr \right\|_{L^p(\Omega; C_T H^s)} \leq C_R T^{1/2} \|a - b\|_{L^p(\Omega; C_T H^s)}.$$

Equation (4.18) gives the analogous bound $C_{p,\varepsilon,R} T^{1/2}$ for the stochastic convolution. For small deterministic T their sum is less than one. Picard iteration proves existence, pathwise uniqueness and measurability with respect to the initial state and driving Wiener process. Iteration on successive deterministic intervals gives a global solution of the truncated equation.

Stop this solution at the first exit from the radius-R ball. Pathwise uniqueness makes the stopped solutions for different cutoffs agree. Letting $R \uparrow \infty$ constructs the maximal solution and proves (4.19). Applying the same contraction estimate to two initial conditions gives the stated local stability. The Picard construction on any probability space carrying the Wiener process is the same measurable strong construction. Every other solution coincides with it after localisation by the same estimates, proving uniqueness in law.

The truncated solution maps satisfy the restart identity at deterministic times by pathwise uniqueness and independence of future Wiener increments. Their initial-state stability gives the Feller property on H^s. Approximate a bounded stopping time from above by dyadic stopping times. Path continuity and the Feller property pass the deterministic-time Markov identity to this limit for bounded continuous test functions; a monotone class argument gives bounded Borel functions. The consistent cutoff construction transfers this strong Markov property to the maximal family up to its lifetime.

Finally, $\Delta_B : H^s \to H^{s-2}$ is bounded. The drift is therefore locally integrable in H^{s-2}, and the diffusion is locally square integrable into that space. The semigroup identity in the deterministic convolution and stochastic Fubini in the stochastic convolution turn (4.20) into the strong integral equation in H^{s-2}. All these operations are justified after stopping on an H^s ball by the preceding bounds. □

Remark 4.5. The same existence theorem holds with an additional drift $J : \mathcal{A}^s(P) \to H^{s-1}(E)$ that is bounded and Lipschitz on bounded sets. For the reference comparison below, that drift must also satisfy $J(u \cdot A) = U_u J(A)$, or a specified vertical-defect identity as in (3.9).

4.3 Sobolev gauge comparison and common stopping times

Let C be another smooth reference connection. For a given path $A \in C([0,T]; \mathcal{A}^s(P))$, consider

$$(\partial_t u)u^{-1} = \chi_C(u \cdot A) - U_u \chi_B(A), \qquad u(0) = u_0 \in \mathcal{G}^{s+1}(P). \tag{4.21}$$

Lemma 4.6 (Sobolev comparison gauges)**.** *Equation* (4.21) *has a unique maximal local solution with*

$$u \in C_t \mathcal{G}^{s+1}(P) \cap C_t^1 H^{s-1}, \tag{4.22}$$

where the second space is taken in a faithful matrix representation. It is causal as a function of A. On a time interval on which $\|A - B\|_{H^s}$ remains bounded, continuation can fail only if $\|u\|_{H^{s+1}}$ becomes unbounded.

Moreover, if $\widetilde{A} = u \cdot A$ and both $\|A - B\|_{H^s}$ and $\|\widetilde{A} - B\|_{H^s}$ are bounded by R, then

$$\|u\|_{H^{s+1}} \le C_R. \tag{4.23}$$

Proof. The expansion (3.6) is valid at the present regularity. With $a = A - B$, $b = C - B$ and $k = (d_B u)u^{-1}$, its coefficients contain a and b, and no derivative of a. Choose a smooth gauge v such that $u_0 v^{-1}$ lies in a fixed pointwise exponential neighbourhood of the identity. Such a v exists by smooth approximation of Sobolev sections of the compact group bundle. Write $u = \exp(\psi)v$ in this neighbourhood. The equation has the form

$$\partial_t \psi = -\Delta_B^{(0)} \psi + \mathcal{N}_v(a_t, \psi, \nabla_B \psi). \tag{4.24}$$

Here $\psi_0 \in H^{s+1}$, and the nonlinear map on bounded sets is locally Lipschitz from $H^s \times H^{s+1}$ to H^s. Indeed its variable factors are $a \in H^s$, first derivatives of $\psi \in H^{s+1}$, and smooth pointwise functions of ψ; H^s is an algebra. Derivatives of the fixed smooth gauge v are fixed coefficients. The heat estimate from H^s to H^{s+1} has the integrable kernel $(t-r)^{-1/2}$. A pathwise contraction therefore solves (4.24) locally in $C_t H^{s+1}$. Its right-hand side is continuous in H^{s-1}, giving the

second regularity in (4.22). Uniqueness proves causality and allows successive charts to be joined.

We justify the asserted continuation criterion. On a bounded H^{s+1} set, the smooth centres v can be chosen with uniform coefficient bounds. Embed the group bundle in the endomorphism bundle of a faithful orthogonal representation. Smooth u by a fixed auxiliary heat operator for a sufficiently small time depending only on its H^{s+1} bound. The approximation is uniformly close in C^0, so fibrewise nearest-point projection in a fixed tubular neighbourhood gives a smooth gauge v. Its H^{s+2} norm is bounded by a constant depending only on that bound and the chosen smoothing time. The initial logarithm $\log(uv^{-1})$ has a bounded H^{s+1} norm and stays in a strictly smaller exponential neighbourhood. The linear heat evolution of this logarithm differs from its initial value in C^0 by at most $Ct\|\psi_0\|_{H^{s+1}}$: use $\Delta_B^{(0)} : H^{s+1} \to H^{s-1}$ and $H^{s-1} \hookrightarrow C^0$. The nonlinear convolution has H^{s+1} norm bounded by $C\sqrt{t}$ on the contraction ball. Consequently the solution stays in that chart for a positive time controlled by the two stated Sobolev bounds. This proves the continuation criterion.

For (4.23), put $\widetilde{a} = \widetilde{A} - B$. The gauge identity is

$$d_B u = ua - \widetilde{a}\, u. \tag{4.25}$$

The matrix-valued function u is pointwise bounded because G is compact. Equation (4.25) first bounds u in H^1. Differentiating once and using the L^∞ bounds of $a, \widetilde{a}$ bounds u in H^2. For $2 \le j \le s$, the algebra estimate in H^j gives

$$\|ua - \widetilde{a}u\|_{H^j} \le C_R \|u\|_{H^j}.$$

The fixed-connection estimate $\|u\|_{H^{j+1}} \le C(\|d_B u\|_{H^j} + \|u\|_{L^2})$ then proves (4.23) by induction. □

Theorem 4.7 (Comparison of the constructed regularised solutions)**.** *Fix smooth references B, C, an initial connection $A_0 \in \mathcal{A}^s(P)$ and an initial gauge $u_0 \in \mathcal{G}^{s+1}(P)$. The maximal solution with reference B and initial condition A_0 can be coupled to the maximal solution with reference C and initial condition $u_0 \cdot A_0$ so that*

$$A_t^C = u_t \cdot A_t^B, \qquad 0 \le t < \tau_B \wedge \tau_C, \tag{4.26}$$

where u satisfies (4.21) and (4.22). The Wiener process driving the second solution is the predictable orthogonal rotation

$$W_t^C = \int_0^t U_{u_r}\, dW_r^B \tag{4.27}$$

on the comparison interval. Consequently the full gauge orbits agree pathwise on every common stopped interval contained in both lifetimes.

Proof. Construct A^B by Theorem 4.4, and solve the pathwise gauge equation by Lemma 4.6 up to its maximal time τ_u. Its solution is adapted and has finite variation in H^{s-1}. The strong integral equation for A^B holds in H^{s-2}. Since $s \ge 4$, multiplication by H^{s-1} coefficients is bounded on H^{s-2}. The product rule for a finite-variation process and a semimartingale therefore applies in this latter Hilbert space to the affine gauge action:

$$d(u \cdot A^B) = U_u\, dA^B - d_{u\cdot A^B}\big((\partial_t u)u^{-1}\big)\, dt.$$

There is no quadratic-covariation term. Equations (4.21) and (4.15) turn this identity into the C-equation, with noise (4.27). The transformed path is continuous in H^s; the variation-of-constants identity in H^{s-2} recovers its mild equation, so the uniqueness statement of Theorem 4.4 applies to it. Extend the orthogonal integrand by the identity after τ_u. The

resulting cylindrical process is Brownian on the whole probability space by the finite-dimensional Lévy characterisation. Theorem 4.4 constructs its maximal C-solution. Pathwise uniqueness identifies that solution with $u \cdot A^B$ on the initial common interval.

The comparison gauge cannot terminate strictly before $\tau_B \wedge \tau_C$. If it did, both connection paths would have bounded H^s norms on a compact interval containing its terminal time. Equation (4.23) would bound the gauge in H^{s+1} there, contradicting its continuation criterion. This proves (4.26) throughout the asserted interval. Uniqueness in law gives the prescribed maximal C-solution law for the second marginal. □

For example, on this coupling one may use the explicit common stopping times

$$\sigma_R = T \wedge \inf\{t : \|A_t^B - B\|_{H^s} \geq R\} \wedge \inf\{t : \|A_t^C - C\|_{H^s} \geq R\}, \tag{4.28}$$

with R larger than both initial norms. The two orbit paths agree up to σ_R. These are common stopping times in the coupling; their definition does not provide an intrinsic killing rule on gauge orbits.

Corollary 4.8 (Reference-independent laws on common orbit localisations)**.** *Suppose a bounded stopping rule is a measurable function of the full gauge-orbit path and occurs within both lifetimes in the coupling of Theorem 4.7. The stopped orbit laws for the two reference equations coincide. The same conclusion applies to different initial representatives of one orbit. Bundle isomorphisms covering base isometries transport the constructed regularised laws to the corresponding laws on the target bundle.*

Proof. The stopped orbit paths agree under (4.26), and an orbit-measurable stopping rule has the same value on those paths. Initial representatives are handled by the prescribed u_0. For a bundle isomorphism, the drift, closed forms and heat operators intertwine with its unitary action on L^2. Transporting the Wiener process and using uniqueness in law proves the last assertion. □

Remark 4.9 (Localisation and the regularisation parameter). The comparison uses the maximal Sobolev solutions of Theorem 4.4. Their explosion times are defined in the Sobolev norm of the chosen representative. The fixed-scale reference comparison is therefore stated on common orbit localisations. Moreover, (4.13)–(4.14) have explicit negative powers of ε. A continuum limit requires estimates in the rough state and model norms, in addition to the fixed-scale existence proved here.

5 Parabolic analysis and the finite stochastic rule

Throughout this section B is a smooth connection on P and $E = T^*\Sigma \otimes \operatorname{ad} P$. Sobolev and Hölder–Besov spaces of sections are defined by a finite smooth trivialising cover and a subordinate partition of unity. Changing this auxiliary data gives equivalent norms. Equivalently, one can use the spectral resolution of $1 + L_B$, where

$$L_B = d_B^* d_B + d_B d_B^* \quad \text{on } \Omega^1(\operatorname{ad} P).$$

This is a nonnegative self-adjoint operator of Laplace type. Its heat-kernel and pseudodifferential estimates are the standard bundle estimates of [3, 20]. Its ellipticity does not imply that d_B is a differential: in general $d_B^2 = [F_B, \cdot]$. In particular, no exact/coexact Hodge decomposition for d_B is used in the non-Abelian theory.

5.1 Classical weighted local existence

Write the full DeTurck equation in affine coordinates as

$$(\partial_t + L_B)a = f_B + V_B a + Q_B(a,a) + C(a,a,a) + \xi, \tag{5.1}$$

with the coefficients of Corollary 2.2. In particular f_B is smooth, V_B has order zero, and $Q_B(a,a)$ is the parallel contraction in (2.8). The following classical estimate controls the first spatial derivative and the initial time singularity in the same weighted norm.

Theorem 5.1 (Weighted local equation with a smooth driving path)**.** *Let $s > 2$, $0 \le \theta < 1/2$, and $0 < T \le 1$. Suppose $a_0 \in H^{s-\theta}(E)$ and $z \in C([0,T]; H^{s-\theta}) \cap C((0,T]; H^s)$, with $z(0) = 0$ and*

$$\|z\|_{s,\theta;T} := \sup_{0<t\le T} t^{\theta/2}\|z(t)\|_{H^s} < \infty.$$

For some positive T_* depending on these norms and on B, there is a unique mild solution in the same weighted class to

$$a(t) = e^{-tL_B}a_0 + z(t) + \int_0^t e^{-(t-r)L_B}\{f_B + V_B a(r) + Q_B(a(r),a(r)) + C(a(r),a(r),a(r))\}\,dr. \tag{5.2}$$

It is continuous at zero in $H^{s-\theta}$ and depends locally Lipschitz continuously on (a_0, z), using the sum of the $C_T H^{s-\theta}$ norm and the weighted H^s norm for z. On a ball of weighted radius R, the contraction constant is bounded by

$$C_B(T^{1/2} + RT^{(1-\theta)/2} + R^2 T^{1/2-\theta}). \tag{5.3}$$

The same result holds for time-dependent smooth coefficients with bounded coefficient norms on the interval in question.

Proof. The product theorem on a two-dimensional compact manifold gives

$$\begin{aligned}
\|V_B u\|_{H^{s-1}} &\le C_B\|u\|_{H^s},\\
\|Q_B(u,v)\|_{H^{s-1}} &\le C_B\|u\|_{H^s}\|v\|_{H^s},\\
\|C(u,v,w)\|_{H^{s-1}} &\le C\|u\|_{H^s}\|v\|_{H^s}\|w\|_{H^s}.
\end{aligned}$$

For example H^{s-1} is an algebra because $s - 1 > 1$, so the derivative in Q_B costs exactly one Sobolev order. The heat estimate

$$\|e^{-rL_B}\|_{H^{s-1}\to H^s} \le C_B r^{-1/2} \qquad (0 < r \le 1)$$

therefore controls a degree-j term on a weighted ball by

$$t^{\theta/2}\int_0^t (t-r)^{-1/2} r^{-j\theta/2}\,dr = \mathrm{B}(1 - j\theta/2, 1/2)\,t^{1/2-(j-1)\theta/2}. \tag{5.4}$$

For $j = 1, 2, 3$ all beta integrals are finite, and all displayed time powers are positive. Polarisation gives exactly the difference estimate (5.3). The smooth forcing contributes $C_B T^{(1+\theta)/2}$. Finally,

$$\sup_{0<t\le T} t^{\theta/2}\|e^{-tL_B}a_0\|_{H^s} \le C_B\|a_0\|_{H^{s-\theta}}.$$

Choosing R larger than twice the linear-data bound and then choosing T_* so that the contraction constant is less than $1/2$ proves existence and uniqueness by the Banach theorem.

To check the initial condition, use the heat estimate from H^{s-1} to $H^{s-\theta}$ instead. For the worst, cubic term its bound is

$$CR^3 \int_0^t (t-r)^{-(1-\theta)/2} r^{-3\theta/2}\, dr = C' R^3 t^{1/2-\theta} \longrightarrow 0.$$

The other terms have larger time exponents. Continuity away from zero follows from the same integrable kernels. The contraction estimate applied to two data sets proves the stated Lipschitz bound. □

Corollary 5.2 (Fixed-cutoff additive dynamics)**.** *For every $\varepsilon > 0$, the equation*

$$da = (-L_B a + f_B + V_B a + Q_B(a,a) + C(a,a,a))\, dt + \sigma e^{-\varepsilon L_B}\, dW$$

has a unique maximal local mild solution for the initial data of Theorem 5.1. Its lifetime is a stopping time, and the solution is strong with respect to the given cylindrical Wiener process. The statement is local and contains no estimate uniform as $\varepsilon \downarrow 0$.

Proof. The stochastic convolution with $e^{-\varepsilon L_B}$ has continuous paths in H^r for every finite r: its covariance is trace class there, and the usual Gaussian increment estimate follows from the spectral theorem. Apply the pathwise contraction up to the first time that its norm exceeds a fixed level. Uniqueness makes these local solutions consistent as the levels increase. □

This cutoff is convenient for analytic existence. It is attached to B and does not itself commute with an arbitrary gauge transformation of A. The covariant comparison in Section 3 uses a different, explicitly stated equivariance hypothesis on the diffusion coefficient.

5.2 Reference changes in the Gaussian coordinate

The affine bundle of connections has a canonical translation space $\Omega^1(\mathrm{ad}\, P)$. In particular the L^2 space in which the white noise lives does not change when B is replaced by another connection C. This allows the Gaussian coordinates to be compared using one Wiener process.

Proposition 5.3 (A one-order improvement under a reference change)**.** *Let*

$$X_B(t) = \int_0^t e^{-(t-r)L_B}\, dW_r, \qquad X_C(t) = \int_0^t e^{-(t-r)L_C}\, dW_r.$$

For every $\kappa > 0$, these fields have versions in $C([0,T]; \mathcal{C}^{-\kappa}(E))$. For every $0 < \delta < 2$ their difference satisfies

$$X_B - X_C \in C([0,T]; \mathcal{C}^{1-\kappa-\delta}(E)), \qquad \|X_B - X_C\|_{C_T\mathcal{C}^{1-\kappa-\delta}} \le C_{B,C,\kappa,\delta} T^{\delta/2} \|X_C\|_{C_T\mathcal{C}^{-\kappa}} \quad (5.5)$$

for $0 < T \le 1$. Estimates in a lower target regularity follow by embedding from a choice $0 < \delta_0 < 2$, with time factor $T^{\delta_0/2}$.

Proof. For completeness, a dyadic spectral block at frequency 2^j of $X_B(t)$ has pointwise variance bounded uniformly in j, t, since a surface has $O(2^{2j})$ modes at that scale while the covariance of each mode is $O(2^{-2j})$. The local spectral-function estimate for Laplace-type operators gives this statement in any bundle chart. Gaussian moment estimates and the Bernstein inequality, with an arbitrarily small loss of spatial regularity, yield $X_B \in C_T\mathcal{C}^{-\kappa}$; time increments are treated with $|1 - e^{-h\lambda}| \lesssim (h\lambda)^\eta$ and then the Kolmogorov criterion. This argument also gives every finite moment of the displayed norm. It is unchanged by the finite-dimensional zero-mode part. Taking the spatial estimates first at a strictly larger

regularity and then embedding into $\mathcal{C}^{-\kappa}$ puts these paths in the closure of smooth sections and gives strong continuity in that norm.

The principal symbols of L_B and L_C coincide, so $D_{BC} = L_C - L_B$ has order at most one. Subtracting their linear equations, or first doing so after a common finite-dimensional noise cutoff, gives

$$X_B(t) - X_C(t) = \int_0^t e^{-(t-r)L_B} D_{BC} X_C(r)\, dr.$$

The operator D_{BC} maps $\mathcal{C}^{-\kappa}$ continuously to $\mathcal{C}^{-1-\kappa}$. The heat operator from the latter space to $\mathcal{C}^{1-\kappa-\delta}$ has norm at most $C(t-r)^{-1+\delta/2}$. Integration proves (5.5), including continuity at zero. Approximation and the same bound justify the distributional identity. □

The estimate gives almost one extra derivative for the linear stochastic convolution under a smooth reference change. The product $(X_B - X_C)\nabla X_C$ still has a negative sum of regularities; its use in a nonlinear reference comparison therefore requires a compatible enhanced product.

5.3 The finite stochastic rule

Use parabolic scaling $(2, 1, 1)$ and assign white noise homogeneity $|\Xi| = -2 - \kappa$, where $0 < \kappa < 1/8$. Write $\mathcal{I}$ for integration of order two. The stochastic solution trees are generated by

$$X = \mathcal{I}\Xi, \qquad \mathcal{I}Q(U, V), \qquad \mathcal{I}C(U, V, W), \tag{5.6}$$

where every Q vertex differentiates exactly one of its two inputs once, and every C vertex is trilinear without differentiation. There are finitely many tensor types at each vertex, determined by the rank of E. Linear zero-order vertices and nonnegative polynomial decorations may be included. In this rule neither a free derivative nor a product of arbitrary previously generated words is a new solution symbol.

Proposition 5.4 (Nominal degrees and the negative nonlinear trees)**.** *An undecorated solution tree with n noise leaves, b binary vertices, and c cubic vertices, and no zero-order linear vertices, satisfies*

$$n = 1 + b + 2c, \qquad |\tau| = n(1 - \kappa) - 1. \tag{5.7}$$

There are only finitely many such typed trees below any fixed homogeneity. The same is true after adjoining zero-order linear vertices and polynomial decorations. Among the purely stochastic nonlinear right-hand-side trees, the negative-homogeneity ones have two or three noise leaves. Up to tensor types and permutation of the inputs, they are

$$Q(X, X), \qquad C(X, X, X), \qquad Q(X, Y), \qquad Q(Y, X), \quad Y = \mathcal{I}Q(X, X). \tag{5.8}$$

Their homogeneities are $-1 - 2\kappa$ and -3κ, respectively. Partial products such as $X \otimes X$ and differentiated edges are retained when the corresponding algebra is closed under subtrees.

Proof. Counting edges in a rooted tree gives $n = 1 + b + 2c$. Each leaf X contributes $-\kappa$, each integrated binary derivative vertex contributes $2 - 1 = 1$, and each integrated cubic vertex contributes 2. This gives (5.7). A homogeneity bound bounds n, hence $b + c$, and there are only finitely many rooted typed trees with bounded numbers of vertices. A linear zero-order integrated vertex adds two to the degree, and a polynomial decoration adds its nonnegative parabolic degree. Their counts below a fixed threshold are therefore also bounded; the zero polynomial decoration is not counted as an extra vertex.

Removing the root integration subtracts two. Consequently a right-hand-side tree with n leaves has degree $n(1 - \kappa) - 3$. For $n \geq 4$ this is positive because $\kappa < 1/8$. A tree with two leaves is a binary vertex. A tree with three leaves is either a cubic vertex or two binary vertices. These are exactly (5.8). □

Symbol or expression	Homogeneity	Role
X	$-\kappa$	linear stochastic field
∇X	$-1-\kappa$	differentiated input
$X \otimes X$	-2κ	quadratic partial product
$Q(X,X)$	$-1-2\kappa$	quadratic derivative forcing
$Y = \mathcal{I}Q(X,X)$	$1-2\kappa$	first nonlinear response
$C(X,X,X), Q(X,Y), Q(Y,X)$	-3κ	cubic-order forcing
$Z = \mathcal{I}\{\text{cubic-order forcing}\}$	$2-3\kappa$	next response

Derivative-input jets are essential to the nonlinear construction. A derivative on the high-frequency input in $R \prec \nabla X$ leaves regularity $-1-\kappa$ even when R is bounded, and heat integration then gives only $1-\kappa$. The coupled modelled equation below retains the corresponding derivative-control symbols. Their stochastic realisations and convergence are established in Section 8.

These are nominal heat-order-two degrees. The analytic bundle model uses a slightly smaller nonresonant abstract kernel order, specified in Subsection 5.5. Its small downward degree shifts preserve finiteness and the list of singular stochastic products. The actual heat estimates and all physical contraction coefficients still use the heat operator of order two.

5.4 Bundle models and weighted norms

The model norms below distinguish a symbol's homogeneity from its reexpansion precision. We use the bundle calculus of [16], Sections 4–9.

On $\mathbb{R} \times \Sigma$ put

$$\rho((t,x),(s,y)) = |t-s|^{1/2} + d_g(x,y).$$

Local test sections φ_z^λ have support in a parabolic ball of radius λ, volume normalisation λ^{-4}, and a fixed sufficiently large rescaled C^r bound. Changes of charts and smooth bundle frames give equivalent testing families. We write $\mathcal{C}_{\mathfrak{s}}^a$ for the corresponding parabolic Hölder–Besov space.

Each output type ℓ has a finite collection of graded symbol bundles $\mathcal{T}_{\alpha,\delta}^\ell$. The index α is the homogeneity and δ is the precision; the latter may be infinite. The output bundle is denoted by E^ℓ. Jets of smooth sections form the polynomial part, and $Q_{<r}$ denotes projection onto homogeneities below r. A model consists of linear realisations

$$\Pi_z : \mathcal{T}_z^\ell \longrightarrow \mathcal{D}'(\mathbb{R} \times \Sigma; E^\ell)$$

and reexpansion maps $\Gamma_{z,w}$ between the corresponding fibres. For a unit vector in the indicated source fibre, the three local model bounds are

$$|\langle \Pi_z \tau, \varphi_z^\lambda \rangle| \lesssim \lambda^\alpha, \tag{5.9}$$

$$|\Gamma_{z,w}\tau|_a \lesssim \rho(z,w)^{(\alpha-a)\vee 0}, \tag{5.10}$$

$$|\langle (\Pi_z \Gamma_{z,w} - \Pi_w)\tau, \varphi_z^\lambda \rangle| \lesssim \lambda^\delta. \tag{5.11}$$

In the first estimate $\tau \in \mathcal{T}_{z,\alpha,\delta}$; in the other two $\tau \in \mathcal{T}_{w,\alpha,\delta}$. For $\delta = +\infty$ the last bound means an exactly zero defect. The constants are uniform on compact cylinders and the specified finite symbol collection. The model distance is obtained by subtracting two models in these three estimates and taking the corresponding normalised suprema. The graded product and integration rules determine the precision of each symbol.

Let V be a sector with lowest homogeneity α_0 and let $\eta < \gamma$, $\gamma > 0$, and $0 < T \le 1$. The space $\mathcal{D}_Z^{\gamma,\eta}(V)$ consists of sections with components of homogeneity $a < \gamma$ and precision greater than γ, for which the following norm is finite:

$$\|U\|_{\gamma,\eta;T} = \sup_{\substack{z=(t,x),\ 0<t\le T \\ a<\gamma}} \frac{|U(z)|_a}{t^{((\eta-a)\wedge 0)/2}} + \sup_{\substack{z=(t,x),\ w=(s,y),\ 0<s,t\le T \\ 0<\rho(z,w)\le 1\wedge\sqrt{s}\wedge\sqrt{t} \\ a<\gamma}} \frac{|U(z) - Q_{<\gamma}\Gamma_{z,w}U(w)|_a}{\rho(z,w)^{\gamma-a}(s\wedge t)^{(\eta-\gamma)/2}}. \tag{5.12}$$

Negative η allows the initial-time singularities occurring in this paper. For sections over two models, the two-model distance uses the difference of their first terms and the difference of their respective reexpansion increments in the second term. All norms below are taken over the displayed sectors; the smooth geometric data are fixed.

The reconstruction, product and integration estimates used below have the following parameters.

(i) If $\alpha_0 \le 0$ and $\alpha_0 \wedge \eta > -2$, weighted reconstruction gives

$$\mathcal{R}_Z U \in \mathcal{C}_{\mathfrak{s}}^{\alpha_0\wedge\eta}$$

across the initial surface. On cylinders away from that surface, the reconstruction is characterised by $\langle \mathcal{R}_Z U - \Pi_z U(z), \varphi_z^\lambda \rangle = O(\lambda^\gamma)$. The corresponding two-model reconstruction bound is locally Lipschitz. The lower threshold -2 is the negative parabolic codimension of $\{t = 0\}$.

(ii) A precise typed product of sectors with triples $(\alpha_i, \gamma_i, \eta_i)$ maps to indices

$$\begin{aligned} \alpha &= \alpha_1 + \alpha_2, \\ \gamma &= (\gamma_1 + \alpha_2) \wedge (\gamma_2 + \alpha_1), \\ \eta &= (\eta_1 + \alpha_2) \wedge (\eta_2 + \alpha_1) \wedge (\eta_1 + \eta_2). \end{aligned}$$

Here $\gamma_i > 0$, $\eta_i < \gamma_i$, and the displayed target γ is positive in every application below. The factors and their retained products have precision greater than γ, and the abstract product obeys

$$Q_{<\gamma}\Gamma(\tau_1 \star \tau_2) = Q_{<\gamma}(\Gamma\tau_1 \star \Gamma\tau_2).$$

The realisation of a singular product is part of the enhanced model. Products and their two-model differences satisfy the usual bilinear bounds with these indices.

(iii) Let a full causal kernel have abstract order $b > 0$, with an admissible integration map of precision δ_0. If the input sector has regularity $\alpha > -2$, $\alpha \wedge \eta > -2$, and $\gamma + b < \delta_0$, then, for a weight loss $0 < \nu \le 2$,

$$\|\mathscr{K}_Z \mathbf{1}_+ F\|_{\gamma+b,(\eta\wedge\alpha)+b-\nu;T} \lesssim T^{\nu/2}\|F\|_{\gamma,\eta;T}. \tag{5.13}$$

The full lift includes the smooth kernel remainder. Its reconstruction is the causal convolution of the reconstructed forcing. The two-model bound has the same time factor and an additional term controlled by the model distance. The kernel and its smooth remainder are both non-anticipative and satisfy the local dyadic derivative bounds of their assigned order. The input satisfies $\eta < \gamma$, and the shifted homogeneities, shifted precisions after truncation at the jet precision, $\gamma + b$, and $\eta + b$ avoid the integer resonances in the integration theorem. These are finitely many conditions in the working sectors.

These are the reconstruction, product and short-time integration estimates of [16], Proposition 8.5, Lemma 8.6, and Theorem 9.3 with Lemma 9.4, specialised to parabolic dimension four. The finite typed model in Subsection 8.9 supplies their algebraic and analytic hypotheses. The following subsection verifies the nonlinear indices and the two different kernel orders needed by the equation.

5.5 The singular contraction for an admissible bundle model

We specify the sectors and weights for the nonlinear analytic problem and prove local well-posedness for its admissible models. Section 8 constructs the stochastic model to which this theorem will be applied. We use the bundle model, jet, product and integration conventions of [16], Sections 6–9, and its separate kernel types for differentiated unknowns in Section 14, equation (14.5) and Propositions 14.7 and 14.9. In particular we do not assume that differentiation commutes exactly with a curved jet realisation. All regularities in this subsection are parabolic, with scaling $\mathfrak{s} = (2, 1, 1)$.

Fix $0 < \kappa < 1/16$ and choose a generic $\vartheta \in (0, \kappa)$. We use the abstract heat order $\beta = 2 - \vartheta$ and noise degree $|\Xi| = -\beta - \kappa < -2$, so $|X| = -\kappa$. The actual kernel still gains two analytic orders. Assigning it the slightly smaller abstract order is permitted by [16], Remark 7.5, and avoids integer resonances, including those from pure-jet inputs. The finitely many forbidden values of ϑ and of the precision parameters are excluded. Set

$$\begin{aligned} \alpha = -\kappa, \quad \gamma = 3/2, \quad \eta = -2\kappa, \\ \bar{\alpha} = -1 - 2\kappa, \quad \bar{\gamma} = 1/2 - \kappa, \quad \bar{\eta} = -1 - 4\kappa. \end{aligned} \tag{5.14}$$

Set $L_+ = 1 + L_B$ and $S_t = e^{-tL_+}$. Moving the added identity to the right-hand side replaces V_B by $V_B + I$ and changes none of the derivative estimates. The mass makes it possible to use a stationary linear field. It does not add a mass term to the underlying equation.

Jets in the rule. Alongside $X = \mathcal{I}\Xi$, include jet inputs, of nonnegative parabolic degree, at the solution slots of the quadratic and cubic vertices. The nominal heat-order-two degrees are those used in Proposition 5.4. In the working grading, integration of a quadratic vertex gains $1 - \vartheta$ and integration of a cubic or zero-order unary vertex gains $2 - \vartheta$. Multiplication by a coefficient is the prescribed smooth bundle morphism at that vertex; arbitrary repeated products are not additional rules. A degree-zero jet used as an input is counted as one leaf, and the unit is never duplicated as a new decoration.

Lemma 5.5 (Finiteness with solution jets)**.** *If an integrated connection-output tree has n noise leaves, p jet leaves, adjusted total jet degree j, b binary vertices, c cubic vertices, and ℓ zero-order unary vertices, then*

$$|\tau| = n(1 - \kappa) + p - 1 + j + 2\ell - \vartheta(b + c + \ell). \tag{5.15}$$

Here j is the sum of ordinary jet degrees plus one for every jet leaf in a first-derivative input slot. Equivalently, a degree-k derivative-input jet is counted as the derivative of a degree-$(k + 1)$ connection jet. For an output of the derivative kernel type, subtract one from the displayed degree. Consequently the typed rule, its necessary partial products, and its Taylor and contraction subexpressions have finitely many shapes in every fixed precision window. The solution sector below γ contains, in addition to jets and X, the quadratic response Y and the symbols obtained by inserting a degree-zero connection jet in the undifferentiated slot of $\mathcal{I}Q$. The latter include the derivative control $\mathcal{I}\nabla X$, of degree $1 - \kappa - \vartheta$.

Proof. If b and c count binary and cubic vertices, the leaf identity is $n + p = 1 + b + 2c$. With the derivative-slot convention, summing the nominal degrees gives $-\kappa n + j + b + 2c + 2\ell$. Each

non-noise integration then subtracts ϑ, proving (5.15). Since $b+c \le n+p-1$, its right side is at least

$$n(1-\kappa-\vartheta)+(1-\vartheta)p-1+\vartheta+j+(2-\vartheta)\ell.$$

All coefficients of the counting variables are positive. A degree bound bounds each of n, p, j, ℓ, and hence also b, c. The coefficient bundles have finite rank and only finitely many types occur. Passing to a partial product or a differentiated edge is a prescribed operation at one of these vertices, and involves at most the differentiation orders of that finite tree. Taylor decorations are bounded by the chosen precision. Extracting and contracting a subtree reduces the number of vertices. These observations prove finiteness also for the operations needed by the model. They do not allow free iteration of differentiation on the output sector.

For the last assertion, $|Y| = 1-2\kappa-\vartheta < \gamma$, and an integrated binary vertex with differentiated X and a degree-zero undifferentiated jet input has degree $1-\kappa-\vartheta<\gamma$. These inputs represent coefficients of the unknown solution. Omitting them would omit the response to a coefficient multiplying the differentiated Gaussian field. □

Let V be the connection sector of this jet-completed rule, of lowest degree α, let V_1 be the first-derivative kernel sector of lowest degree $\alpha-1$, and let $\bar V$ be the sector of the nonlinear right-hand side, of lowest degree $\bar\alpha$. Include all tensor types on E, $T^*\Sigma\otimes E$, and their prescribed products. Choose a common working precision above four, avoiding the finitely many integer resonances. Underlying jet-realisation precision is taken sufficiently large that the recursively assigned precisions of these sectors exceed the working precision; these are distinct parameters. Local finiteness and [16], Lemma 12.24, permit this choice. Denote the common working precision by δ_0. The two integration types realise the causal heat kernel K and its covariant derivative in the output variable, $K_1=\nabla_B K$, with abstract orders β and $\beta-1$. Each has its own bundle jet correction and admissible integration map. An *admissible typed model* below means a bundle model with these operations and their compatibility and analytic bounds. Its nonlinear products need not be pointwise products of distributions. This definition is a requirement on a model, not an assertion that a given stochastic regularisation converges to one. Only the model restriction generated by solution degrees below γ, derivative-sector degrees below $\gamma-1$, forcing degrees below $\bar\gamma$, their reexpansion descendants, and their typed heat integrals is used. A high jet precision does not require adding every nonlinear tree whose homogeneity is below that precision.

For such a model Z, write

$$\mathcal{F}_B(U,U_1)=\mathbf{f}_B+(V_B+I)U+\mathfrak{q}_B(U,U_1)+\mathcal{C}(U,U,U),$$

where $\mathfrak{q}_B$ is the parallel tensor contraction in (2.8), with U_1 in place of $\nabla_B a$. The bold forcing and all coefficient morphisms are lifted by jets and each product is truncated at its target degree. The additive noise is incorporated in the prescribed linear modelled paths, outside $\mathcal{F}_B$. Equip

$$\mathcal{X}_Z(T)=\mathcal{D}_Z^{\gamma,\eta}(V)\times\mathcal{D}_Z^{\gamma-1,\eta-1}(V_1)$$

with the sum norm.

Proposition 5.6 (All nonlinear weights and derivative losses)**.** *For the parameters in* (5.14)*,*

$$\mathcal{F}_B:\mathcal{X}_Z(T)\longrightarrow\mathcal{D}_Z^{\bar\gamma,\bar\eta}(\bar V) \tag{5.16}$$

is strongly locally Lipschitz. On a model-norm bound M and a solution-norm bound $R\ge1$, its Lipschitz constant is at most $C_{B,M}(1+R+R^2)$, uniformly for $0<T\le1$.

Proof. The second unknown is assigned the triple $(\alpha-1,\gamma-1,\eta-1)$, as required by the output-derivative kernel. This is a sector specification, not an assertion of an exact abstract

differential map on curved jets. Its relation to the derivative of the reconstructed first unknown will follow from the two convolution identities.

For two factors with triples $(\alpha_i, \gamma_i, \eta_i)$, the product estimate uses modelled regularity $\min(\gamma_1 + \alpha_2, \gamma_2 + \alpha_1)$ and weight $\min(\eta_1 + \alpha_2, \eta_2 + \alpha_1, \eta_1 + \eta_2)$; see [16], Lemma 8.6. Applied first to U and U_1, and then twice to U, this gives

Expression	Lowest degree	Modelled regularity	Boundary weight
U	$-\kappa$	$3/2$	-2κ
U_1	$-1-\kappa$	$1/2$	$-1-2\kappa$
UU_1	$-1-2\kappa$	$1/2-\kappa$	$-1-4\kappa$
U^2	-2κ	$3/2-\kappa$	-4κ
U^3	-3κ	$3/2-2\kappa$	-6κ

Smooth tensor contractions and their jets are bounded on these spaces. In particular the cubic weight is greater than $\bar{\eta}$, because $1-2\kappa > 0$, and its modelled regularity is greater than $\bar{\gamma}$. The linear and smooth terms satisfy the same target bounds by embedding. This proves (5.16). Polarisation replaces one factor at a time by a difference, giving the factors 1, R and R^2. The corresponding two-model product estimates give strong local Lipschitz continuity, including variation of Z. □

Let $\mathscr{K}_Z$ and $\mathscr{K}_{1,Z}$ denote the abstract lifts of the full causal kernels K and K_1, including their smooth remainders. Thus $K_1 = \nabla_B K$ is an identity for the full kernels even if their chosen local singular parts require different smooth corrections. Write $\mathbf{1}_+$ for restriction to positive times and extension across time zero in the weighted reconstruction sense.

Lemma 5.7 (A uniform small-time factor)**.** *For the spaces above and a bounded model norm,*

$$\|\mathscr{K}_Z \mathbf{1}_+ F\|_{\gamma,\eta;T} + \|\mathscr{K}_{1,Z} \mathbf{1}_+ F\|_{\gamma-1,\eta-1;T} \le C_{B,M} T^{1/4} \|F\|_{\bar{\gamma},\bar{\eta};T}. \tag{5.17}$$

The analogous estimate between two models has the same time factor and an additional term proportional to their model distance.

Proof. The forcing threshold is integrable at the initial surface:

$$\bar{\eta} \wedge \bar{\alpha} = -1 - 4\kappa > -2.$$

Apply the weighted short-time Schauder estimate of [16], Theorem 9.3, with abstract heat gain $\beta = 2 - \vartheta$ and a loss of $1/2$ in the output weight. Its output indices are

$$\gamma_* = \bar{\gamma} + \beta = 5/2 - \kappa - \vartheta,$$
$$\eta_* = (\bar{\eta} \wedge \bar{\alpha}) + \beta - 1/2 = 1/2 - 4\kappa - \vartheta,$$

and its time factor is $T^{(1/2)/2} = T^{1/4}$. The hypothesis on precision holds because $\bar{\gamma} + \beta < \delta_0$. Both $\gamma_* > \gamma$ and $\eta_* > \eta$, so projection and the weighted-space embedding are bounded uniformly for $T \le 1$. For K_1, apply the same theorem with gain $\beta - 1$ and the same weight loss $1/2$. The output indices are $\gamma_* - 1$ and $\eta_* - 1$, respectively greater than $\gamma - 1$ and $\eta - 1$, and its factor is again $T^{1/4}$. The smooth kernel remainder is causal and smooth across the time diagonal, hence flat there. Its weighted estimate has a factor T, which is at most $T^{1/4}$ on this interval even with the displayed negative source weight. The two-model versions of these same estimates prove the last assertion. □

Theorem 5.8 (Singular local solution map with explicit weights)**.** *Fix an admissible typed bundle model Z of the rule above. For every prescribed $\mathbf{v} = (v, v_1) \in \mathcal{X}_Z(T)$ there is a unique local solution of*

$$\begin{aligned} U &= v + \mathscr{K}_Z \mathbf{1}_+ \mathcal{F}_B(U, U_1), \\ U_1 &= v_1 + \mathscr{K}_{1,Z} \mathbf{1}_+ \mathcal{F}_B(U, U_1), \end{aligned} \qquad (U, U_1) \in \mathcal{X}_Z(T). \tag{5.18}$$

On a ball of radius R, its contraction constant is bounded by

$$C_{B,M} T^{1/4} (1 + R + R^2). \tag{5.19}$$

Solutions depend locally jointly Lipschitz continuously on the model and on (v, v_1), up to a common local lifetime. If $\mathcal{R}_Z v_1 = \nabla_B \mathcal{R}_Z v$, then $\mathcal{R}_Z U_1 = \nabla_B \mathcal{R}_Z U$ as distributions.

Suppose, in addition, that Z carries typed lifts $\mathbf{X}$ and $\mathbf{X}_1$ of the linear convolution X and $\nabla_B X$, with $X \in C([0,T]; \mathcal{C}^{-\kappa}(E))$, and that these lifts have the weighted modelled bounds of the rule. If $a_0 \in \mathcal{C}^{-\kappa}(E)$, one may take

$$v = \mathbf{X} + \mathscr{S}(a_0 - X(0)), \qquad \mathcal{R}_Z v(t) = X(t) + S_t(a_0 - X(0)), \tag{5.20}$$

and $v_1 = \mathbf{X}_1 + \mathscr{S}_1(a_0 - X(0))$, where $\mathscr{S}$ and $\mathscr{S}_1$ are the jet lifts of S_t and $\nabla_B S_t$, respectively. Then the reconstructed solution belongs to $C([0,T_*]; \mathcal{C}^{-2\kappa}(E))$, has initial value a_0, and its nonlinear response satisfies

$$\mathcal{R}_Z(U - v) \in \mathcal{C}_{\mathfrak{s}}^{1-4\kappa}([0,T_*] \times \Sigma), \tag{5.21}$$

with value zero at time zero. On compact subintervals of $(0,T_*]$ the exponent improves to $1 - 2\kappa$.

Proof. The product proposition and the short-time lemma give (5.19). On a bounded model set, $\|\mathcal{F}_B(U,U_1)\| \le C_{B,M}(1 + R + R^2 + R^3)$. Choose $R > 2\|\mathbf{v}\| + 1$ and then $T > 0$ so that both the ball invariance and contraction bounds hold. For example a sufficiently small multiple, depending on B, M, of R^{-8} suffices. The Banach fixed-point theorem gives existence and uniqueness. Applying the two-model estimates and absorbing the contraction term proves joint local Lipschitz continuity. Local uniqueness makes the constructions at different radii consistent and gives a maximal continuation with the usual norm blow-up alternative.

The heat lifts satisfy the required initial-time bounds. For $b \in \mathcal{C}^{-\kappa}$, the heat estimates give $\|\nabla^j S_t b\|_\infty \lesssim t^{(-\kappa - j)/2} \|b\|_{\mathcal{C}^{-\kappa}}$. Parabolic Taylor estimates give the corresponding jet reexpansion bound. They imply $\mathscr{S} b \in \mathcal{D}^{\gamma, -2\kappa}$, with a strictly positive κ margin at time zero. Taking one more covariant derivative in these classical heat estimates gives $\mathscr{S}_1 b \in \mathcal{D}^{\gamma - 1, -1 - 2\kappa}$. Moreover $S_t b \to b$ in $\mathcal{C}^{-2\kappa}$: the high-frequency tail is bounded by $2^{-j\kappa}$ times the $\mathcal{C}^{-\kappa}$ norm, and every finite set of frequency blocks converges strongly. The loss of regularity gives strong continuity without requiring it at the exact Hölder–Besov exponent.

The weighted reconstruction theorem extends $\mathbf{1}_+ \mathcal{F}_B(U, U_1)$ to a distribution of parabolic regularity $-1 - 4\kappa$ across time zero, because this index exceeds the negative time codimension -2. Admissibility identifies the reconstruction of its abstract integral with its causal heat convolution. The second kernel type similarly reconstructs to $K_1 * (\mathbf{1}_+ \mathcal{R}_Z \mathcal{F}_B(U, U_1))$. Since $K_1 = \nabla_B K$, these identities and the prescribed relation between v and v_1 prove the distributional derivative relation. They include the respective bundle jet corrections and do not require $\mathscr{D}\Gamma = \Gamma\mathscr{D}$ for a separately postulated abstract differential map. The first convolution is the actual heat kernel, so it gains two orders despite the smaller abstract grading. It vanishes for negative times, proving (5.21) and the zero initial value. Away from the initial surface the forcing has regularity $\bar{\alpha} = -1 - 2\kappa$, giving the improved exponent. Combining this response with (5.20) proves the stated continuous time trace and its initial condition. □

The theorem gives, in particular, convergence of solutions whenever their admissible models and initial data converge in the displayed norms. It also permits restart on a time interval on which the model is consistently restricted: replace the linear datum by $\mathbf{X}(t)+\mathscr{S}_{t-s}(a(s)-X(s))$, replace the second datum by its typed derivative lift, and use the causal heat identity. Section 9 proves causal measurability, independence of the auxiliary past and the strong Markov property for the stochastic model constructed below.

The explicit inequalities behind the construction are

$$\begin{aligned} \eta &= -2\kappa < 1 - 4\kappa - \vartheta = \bar{\eta} + \beta, \\ \gamma &= 3/2 < 5/2 - \kappa - \vartheta = \bar{\gamma} + \beta, \qquad \bar{\eta} > -2. \end{aligned} \tag{5.22}$$

The response in (5.21) has positive regularity approaching one, with a loss at time zero; no unstructured remainder of order $2 - 3\kappa$ is used. Its higher modelled regularity is carried by Y and the degree-$1 - \kappa - \vartheta$ derivative-control symbols already retained in the sector.

6 Local counterterms and covariance tests

This section distinguishes three questions that impose different conditions on a counterterm: whether it is a globally defined bundle map, whether it is produced by a stochastic contraction, and whether the resulting dynamics has gauge-covariant laws. The first condition alone does not imply either of the other two.

6.1 Global endomorphisms and universal geometric form

Proposition 6.1 (Equivariant endomorphisms of the adjoint bundle)**.** *A linear map $M : \mathfrak{g} \to \mathfrak{g}$ defines a canonical fibrewise endomorphism of* $\operatorname{ad} P$ *on every principal G-bundle if and only if $M \operatorname{Ad}_g = \operatorname{Ad}_g M$ for every $g \in G$. If G is connected and compact and*

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N$$

is its decomposition into its centre and compact simple ideals, then

$$\operatorname{End}_G(\mathfrak{g}) = \operatorname{End}(\mathfrak{z}) \oplus \bigoplus_{j=1}^{N} \mathbb{R} \operatorname{Id}_{\mathfrak{g}_j} . \tag{6.1}$$

For disconnected G, one takes the subalgebra in (6.1) fixed by the component-group action.

Proof. In two overlapping trivialisations, a section transforms by $a_i = \operatorname{Ad}_{g_{ij}} a_j$. Thus Ma_i transforms in the same way precisely when M commutes with these transition actions. Requiring the construction for all bundles and all changes of frame is equivalent to commutation with every element of G.

For a connected compact group the centre is the trivial summand. Each simple ideal is an irreducible real adjoint module whose complexification is the adjoint module of a complex simple Lie algebra. Its commutant is therefore real scalar. There are no intertwiners between different simple ideals: the action of one ideal on itself is nontrivial, while its action on the other ideals is zero. There are likewise none between a simple ideal and the centre. This proves (6.1). Commutation with the full group imposes exactly the remaining component-group invariance. □

On an oriented surface the rotation $J = * : \Omega^1 \to \Omega^1$ is also invariant under oriented orthonormal frame changes. Thus, in the restricted class of zero-order expressions linear in a,

at most linear in the Gaussian curvature K, and involving no other background tensor, the universal geometric form is

$$R_g(a) = M_0 a + N_0 Ja + K(M_1 a + N_1 Ja), \qquad M_i, N_i \in \mathrm{End}_G(\mathfrak{g}). \tag{6.2}$$

This follows from the commutant of the real standard $SO(2)$ module, which is spanned by Id and J. If covariance under orientation reversal is required, the J terms are absent. This is a classification within the specified geometric class, not a power-counting determination of the actual renormalisation group. For example extra dependence on F_B or its derivatives is outside that class. On a single constant-curvature surface the representation $M_0 + KM_1$ is not unique; uniqueness of its coefficients is a statement across metrics, not an inference from one fixed metric.

6.2 A computable Lie-algebra contraction

Let (e_α) be an orthonormal basis of $\mathfrak{g}$ and define the nonpositive adjoint Casimir

$$\mathcal{C}_{\mathrm{ad}} v = \sum_\alpha [e_\alpha, [e_\alpha, v]].$$

It is independent of the orthonormal basis, belongs to $\mathrm{End}_G(\mathfrak{g})$, and vanishes on the centre.

Lemma 6.2 (The cubic Wick contraction)**.** *Let $X_1, \dots, X_d$ be centred Gaussian $\mathfrak{g}$-valued variables with covariance $\mathbb{E}[X_i^\alpha X_j^\beta] = c\,\delta_{ij}\delta_{\alpha\beta}$. For $T_i(X) = \sum_j [X_j, [X_j, X_i]]$ one has*

$$T_i(X) =: T_i(X) : + (d-1) c \mathcal{C}_{\mathrm{ad}} X_i. \tag{6.3}$$

Proof. Contracting the first two entries gives $dc\mathcal{C}_{\mathrm{ad}}X_i$. Contracting the first and third entries forces $j = i$ and gives $c\sum_\alpha [e_\alpha, [X_i, e_\alpha]] = -c\mathcal{C}_{\mathrm{ad}}X_i$. Contracting the second and third entries gives $c[X_i, \sum_\alpha [e_\alpha, e_\alpha]] = 0$. The Wick expansion is the sum of these three contractions and the third-chaos component. □

For unprojected unit-amplitude white noise and heat regularisation $X^\varepsilon = e^{-\varepsilon L} X$, the covariance on the diagonal at any fixed $t > 0$ has leading term

$$\mathbb{E}[X^\varepsilon(t,x) \otimes X^\varepsilon(t,x)] = \frac{1}{8\pi} \log(1/\varepsilon)\, \mathrm{Id}_{E_x} + O_t(1). \tag{6.4}$$

Indeed it is the diagonal kernel of $\int_0^t e^{-2(r+\varepsilon)L}\, dr$; the leading heat coefficient on a surface is $(8\pi(r+\varepsilon))^{-1}\,\mathrm{Id}$. The bounded term depends on the geometry, on L and on the cutoff. Curvature enters later heat coefficients, so the leading logarithmic term in this particular covariance is independent of the reference connection. A noise amplitude σ multiplies it by σ^2.

The cubic contraction is only one diagram. Quadratic derivative vertices and their iterates produce other diagrams, and the sum can cancel a contribution found in (6.3). Consequently neither (6.2) nor the cubic Wick calculation selects the finite mass of a nonlinear equation. In particular, bracket contractions do not produce an arbitrary endomorphism on the Abelian centre. In the spatial regularisation treated by Bringmann–Cao [4], the prescribed torus equation needs no additional counterterm; this does not identify the answer for every heat-kernel or space–time regularisation on a curved bundle.

6.3 Affine reference consistency and two necessary tests

Proposition 6.3 (The affine consistency equation)**.** *Suppose an additional drift is written in a reference B as $R_B(a) = Ma + r_B$, where M is fixed and r_B is independent of a. The same*

drift is a reference-independent vector field on the affine space of connections if and only if, for $C = B + b$,

$$r_C = r_B + Mb. \tag{6.5}$$

In particular the prescription $R_B(a) = Ma$ *in every reference is reference independent only when* $M = 0$.

Proof. The coordinates of the same connection are $a = A - B$ and $a' = A - C = a - b$. Equality $R_C(a - b) = R_B(a)$ is exactly (6.5). The conclusion follows by allowing all smooth b. □

There is a related pointwise gauge test. A gauge transformation equal to the identity at x can have arbitrary first derivative at x. It changes the local coefficient of a connection there by an arbitrary $\mathfrak{g}$-valued covector. Hence a zeroth-jet function of the connection coefficient that transforms tensorially must be independent of that coefficient. A nonzero bare linear mass fails this test. This observation concerns a separately defined local vector field. It does not exclude a finite term in a renormalised DeTurck equation whose covariance is recovered only after combining the term with the limiting noise transformation.

Proposition 6.4 (Large gauge transformations rule out an Abelian OU mass)**.** *Suppose* Σ *has positive genus and let* Λ *be its integral harmonic lattice. The linear diffusion*

$$dh = -Mh\,dt + \sigma\,dB_t^{\mathcal{H}} \quad \text{on } \mathcal{H}^1(\Sigma;\mathbb{R})$$

descends through the quotient by Λ *for all initial data only if* $M = 0$. *More generally a spatially varying scalar mass* $k(x)a$ *fails even the small-gauge test when* $dk \neq 0$.

Proof. The quotient identifies h_0 and $h_0 + \lambda$ for every $\lambda \in \Lambda$. Their solutions under the same Brownian path differ by $e^{-tM}\lambda$. Equality of the projected transition laws also forces this difference to lie in Λ: the wrapped Gaussian has nonzero Fourier coefficients at every dual-lattice frequency, so none of its nonlattice translations preserves its law. The continuous map $t \mapsto e^{-tM}\lambda$ into the discrete lattice must be constant. Differentiating at zero gives $M\lambda = 0$. Because the lattice spans the harmonic space, $M = 0$.

For the second assertion choose a smooth real f with $dk \wedge df \not\equiv 0$. The initial one-forms 0 and df are gauge equivalent. A drift difference $k\,df$ has exterior derivative $dk \wedge df \neq 0$, so their curvature derivatives are different. Gauge equivalence cannot persist, even infinitesimally. □

These necessary tests constrain the counterterms through explicit observables. A universal finite-counterterm theorem for the non-Abelian singular equation also requires control of all contraction diagrams and of the regularised noise commutator in a time-dependent gauge.

6.4 Cancellation of the logarithmic mass in two dimensions

We fix a single approximation convention in this subsection: the noise is regularised in space by the scalar-principal-symbol heat operator $e^{-\varepsilon L_+}$, and every contraction uses this same regularisation. The stationary linear field has unit noise amplitude and covariance $(2L_+)^{-1}$. The parameter ε is heat time, consistently with (8.2). The conclusions in this subsection concern divergent contractions of the DeTurck nonlinearity. The finite contraction needed for predictable reference comparison is analysed separately in Section 11.

In an orthonormal frame the highest-order local terms are

$$q_i(a,b) = \sum_{j=1}^{2} [a_j, 2\partial_j b_i - \partial_i b_j], \qquad c_i(a) = \sum_{j=1}^{2} [a_j, [a_j, a_i]]. \tag{6.6}$$

The principal symbol of the heat operator is $|k|^2 I$. For a constant external one-form v, linearising q gives

$$(P_v(k))_{i\ell} = \mathrm{i}(2\delta_{i\ell}\,\mathrm{ad}(v\cdot k) - k_i\,\mathrm{ad}(v_\ell)), \qquad \Psi_v = K(P_v X). \tag{6.7}$$

The flat cancellation is consistent with the spatially regularised torus construction of [4]. We give its tensor coefficients explicitly in order to track the bundle remainders. The term $q(X, v)$ vanishes for constant v. Its nonconstant counterpart has one fewer singular derivative in its contraction and will be integrable below.

Lemma 6.5 (The three leading contractions)**.** *Let $m_\varepsilon(k)$ be a uniformly bounded real scalar spatial cutoff tending pointwise to one, with sufficient decay for the displayed integrals to be finite; the heat cutoff is the case used below. Set*

$$c_\varepsilon^{\mathrm{uv}} = \int_{|k|\geq 1} \frac{|m_\varepsilon(k)|^2}{2|k|^2}\frac{dk}{(2\pi)^2}.$$

For radial cutoffs, the logarithmic coefficients of the constant external-field contractions in dimension two are

$$\begin{aligned} \mathbb{E}q_i(X_\varepsilon, \Psi_{v,\varepsilon}) &= -\tfrac{1}{4}c_\varepsilon^{\mathrm{uv}}\mathcal{C}_{\mathrm{ad}}v_i + \textit{a convergent ultraviolet remainder},\\ \mathbb{E}q_i(\Psi_{v,\varepsilon}, X_\varepsilon) &= -\tfrac{3}{4}c_\varepsilon^{\mathrm{uv}}\mathcal{C}_{\mathrm{ad}}v_i + \textit{a convergent ultraviolet remainder}, \end{aligned} \tag{6.8}$$

$$\mathbb{E}\left[\textit{the part of } c(v+X_\varepsilon) \textit{ linear in } v\right]_i = c_\varepsilon^{\mathrm{uv}}\mathcal{C}_{\mathrm{ad}}v_i + \textit{a convergent ultraviolet remainder}.$$

Their logarithmic sum is zero. The cancellation in the sum of the two derivative terms takes place before angular integration.

Proof. The high-frequency covariance of $X(k)$ is $|m_\varepsilon(k)|^2/(2|k|^2)$. The equal-time cross covariance with its response Ψ_v has the additional integral $\int_0^\infty e^{-2r|k|^2}\,dr$, giving the factor $|m_\varepsilon(k)|^2/(4|k|^4)$ times $P_v(k)$. Replacing $|k|^2$ by $1+|k|^2$ changes the resulting integrands by an integrable order at infinity.

Use the two identities

$$\sum_\alpha [e_\alpha,[v,e_\alpha]] = -\mathcal{C}_{\mathrm{ad}}v, \qquad \sum_\alpha [[v,e_\alpha],e_\alpha] = \mathcal{C}_{\mathrm{ad}}v.$$

Substitution of (6.7) in the first two contractions gives, at frequency k, respectively

$$\begin{aligned} &-\frac{|m_\varepsilon(k)|^2}{4|k|^4}k_i\mathcal{C}_{\mathrm{ad}}(v\cdot k),\\ &\frac{|m_\varepsilon(k)|^2}{4|k|^4}\left\{k_i\mathcal{C}_{\mathrm{ad}}(v\cdot k) - 2|k|^2\mathcal{C}_{\mathrm{ad}}v_i\right\}. \end{aligned}$$

The $k_i(v\cdot k)$ contributions cancel exactly in their sum, leaving $-c_\varepsilon^{\mathrm{uv}}\mathcal{C}_{\mathrm{ad}}v_i$ after integration. For a radial cutoff the angular identity $\int k_i k_\ell f(|k|)\,dk = \frac{1}{2}\delta_{i\ell}\int |k|^2 f(|k|)\,dk$ gives the separate fractions $-1/4$ and $-3/4$. The cubic coefficient is Lemma 6.2 with $d=2$. Adding the three coefficients proves the cancellation. □

The external-field calculation also identifies the coefficient shared by the noise-leaf and solution-jet subtractions. In a cubic derivative tree, replacing the uncontracted noise leaf by v produces precisely the second-chaos response in (6.7). Thus its degree-zero subtraction has the same coefficient as the first-chaos contraction of the three-noise tree. The two coefficients cannot be chosen independently.

Proposition 6.6 (No curvature-dependent ultraviolet divergence)**.** *Let P be any smooth principal G-bundle on the closed surface and let B be smooth. Use the tensor-product connection induced by B and the Levi-Civita connection, which is the Bochner connection of the Hodge Laplacian L_B by its Weitzenböck formula. For the stationary L_+ heat regularisation specified above, the divergent zeroth-order contraction coefficients of the nonlinear DeTurck rule have the same leading terms as in Lemma 6.5. Their total logarithmic coefficient vanishes. Curvature, derivatives of the reference connection, and the topology of P introduce no additional divergent coefficient at these negative homogeneities.*

More precisely, after subtracting the displayed flat logarithms, each degree-zero contraction coefficient has a limit in C^r for every finite r. The quadratic one-vertex expectation also has a finite smooth limit. For the heat cutoff,

$$c_\varepsilon^{\mathrm{uv}} = \frac{1}{8\pi}\log(1/\varepsilon) + O(1). \tag{6.9}$$

The finite limits depend on the specified kernel decomposition and subtraction convention; this proposition does not identify them with a gauge-covariant finite normalisation.

Proof. Only the negative trees in the finite rule and their solution-jet replacements can require a subtraction. A complete Gaussian contraction of an odd number of noise leaves is zero. The cubic one-vertex contraction is the covariance on the diagonal. The two-vertex contractions with a constant external leaf have, after moving the outer derivative onto the heat kernel, the local kernel types

$$\nabla K(z,w)\nabla C_\varepsilon(z,w), \qquad K(z,w)\nabla_z\nabla_w C_\varepsilon(z,w).$$

Here C_ε is the covariance of the smoothed stationary linear field and K is the causal heat kernel, localised near the diagonal when defining the subtraction. Both displayed kernels have critical order $\rho(z,w)^{-4}$ in parabolic dimension four. A derivative on the external leaf, or its first Taylor remainder, improves this order by at least one. Products without the outer derivative have the integrable order $\rho^{-3}(1+|\log\rho|)$.

To identify the critical coefficients, choose normal coordinates and a frame obtained by radial parallel transport from the basepoint. In this frame the principal metric is Euclidean at the basepoint, its first derivatives vanish, and the connection form vanishes there. The heat parametrix has leading term the scalar Euclidean kernel times parallel transport; the remaining terms gain a strictly positive power of parabolic distance in each of the critical contractions above. The same statement follows directly for the covariance from

$$C_\varepsilon(t,x;s,y) = \frac{1}{2}\int_{|t-s|+2\varepsilon}^{\infty} e^{-rL_+}(x,y)\,dr.$$

To see the positive gain quantitatively, write the local heat parametrix as

$$H_r(x,y) = \frac{e^{-d(x,y)^2/(4r)}}{4\pi r}\bigl(a_0(x,y) + ra_1(x,y) + \cdots + r^N a_N(x,y)\bigr) + \mathcal{E}_N(r,x,y).$$

In the radial frame $a_0 = I + O(d(x,y)^2)$ and $\nabla a_0 = O(d(x,y))$. Gaussian polynomial factors can be absorbed by slightly weakening the exponential. Hence the differences from the frozen leading heat kernel at derivative orders zero and one are bounded by $Ce^{-d^2/(Cr)}$ and $Cr^{-1/2}e^{-d^2/(Cr)}$, respectively, up to the harmless smooth localisation terms. At order two the bound is $Cr^{-1}e^{-d^2/(Cr)}$. Integrating these bounds in the covariance formula shows that the residual first covariance derivative is bounded and the residual second derivative is at most logarithmic. Long-time parts are smooth. The same bounds hold with arbitrary fixed centre derivatives, after taking N large enough. It follows that the differences of the critical contraction kernels are bounded, uniformly in the cutoff, by

$$C_r\rho(z,w)^{-3}(1+|\log\rho(z,w)|)^N \tag{6.10}$$

in a fixed neighbourhood of the diagonal. The same estimate holds after r derivatives in the basepoint, when the freezing frame is transported with that basepoint. One can use the stronger second-order gain in normal frames, but the displayed one-order gain already suffices. All coefficients and their basepoint derivatives are uniformly bounded on the finite cover.

The bound (6.10) is integrable: an annulus of parabolic radius 2^{-n} contributes at most $C_r 2^{-n}(1+n)^N$. Off the diagonal the regularised heat kernels converge with all derivatives; the massive kernel controls large time separations. Dominated convergence therefore gives the asserted C^r limits of the residual two-vertex coefficients. This argument also covers the smooth cutoff used to localise the singular kernel and the parallel transports used for its tensor indices.

For the one-vertex covariance, the diagonal heat expansion is $(4\pi r)^{-1} I + O(1)$, with smooth uniform remainder after any number of basepoint derivatives. Its integral gives (6.9) and a smooth finite remainder. For the one-derivative diagonal covariance the covariant derivative of the leading term vanishes in the freezing frame. The next diagonal heat coefficients are bounded as $r \downarrow 0$ before integration, so the quadratic expectation has a smooth finite limit. These are the usual local bundle heat-parametrix bounds [3, 20].

Finally the Lie bracket and the Riemannian contractions are parallel tensor operations. At the freezing point they are exactly (6.6); any variation contributes to the integrable remainders already estimated. Lemma 6.5 therefore cancels the entire logarithmic part on every chart. Transition functions conjugate these tensors, so the calculation glues on $\operatorname{ad} P$. Global topology affects the finite heat kernel and cannot alter its local leading singularity. □

This calculation fixes the divergent part for the stated spatial heat scheme. The separate diagrams retain finite terms, whose compatibility with a change of reference and a time-dependent random gauge requires the transformed-noise comparison.

7 Time-integrated quadratic currents on vector bundles

Let $E \to \Sigma$ be a smooth real metric vector bundle over a closed Riemannian surface. Write $\mathcal{H} = L^2(\Sigma; E)$ and let L be a nonnegative self-adjoint operator of Laplace type. Set

$$P_t = e^{-tL}, \qquad A = 1 + L.$$

The Sobolev spaces defined by A agree, with equivalent norms, with the usual bundle Sobolev spaces. We allow a fixed orthogonal projection Π on $\mathcal{H}$ which commutes with P_t. The choice $\Pi = 1$ applies to an arbitrary bundle. On the trivial bundle, Π may also be the coexact Hodge projection.

Let W be a cylindrical Brownian motion on $\mathcal{H}$ and define

$$X_t = \int_0^t P_{t-r} \Pi \, \mathrm{d}W_r. \tag{7.1}$$

For $\beta > 0$, this process has a continuous $H^{-\beta}(E)$-valued version. Its covariance operators are

$$Q_t = \int_0^t P_{2r} \Pi \, \mathrm{d}r, \qquad C_{t,s} = \mathbb{E}[X_t \otimes X_s] = P_{|t-s|} Q_{t \wedge s}. \tag{7.2}$$

At a zero eigenvalue, the scalar function defining Q_t takes the value t. Thus these formulas include the finite-dimensional kernel of L.

7.1 Regularisation and the covariance estimate

Definition 7.1 (Spatial regularisations)**.** A family $(S_\varepsilon)_{0<\varepsilon\le 1}$ is admissible if each S_ε has a smooth real Schwartz kernel, the family is uniformly bounded on $L^2(E)$ and $H^1(E)$, and

$$S_\varepsilon v \longrightarrow v \quad \text{in } H^1(E), \qquad S_\varepsilon^* w \longrightarrow w \quad \text{in } L^2(E)$$

for every $v \in H^1(E)$ and $w \in L^2(E)$.

Heat regularisations, smooth spectral cutoffs, and finite spectral projections satisfy this definition. Local convolution regularisations assembled with smooth bundle trivialisations and a partition of unity also qualify when their adjoints form an approximate identity. The definition allows S_ε to fail to commute with L or Π.

Let D be a smooth first-order differential operator on E. Replacing D by $(D + D^*)/2$ leaves $\langle v, Dv\rangle_{L^2}$ unchanged. We therefore assume throughout that D is formally symmetric and put

$$m_D = \|DA^{-1/2}\|_{\mathcal{H}\to\mathcal{H}}.$$

For an admissible regularisation define

$$D_\varepsilon = S_\varepsilon^* D S_\varepsilon, \tag{7.3}$$

$$J_D^\varepsilon(t) = \langle S_\varepsilon X_t, DS_\varepsilon X_t\rangle_{L^2} - \mathbb{E}\langle S_\varepsilon X_t, DS_\varepsilon X_t\rangle_{L^2}, \tag{7.4}$$

$$H_D^\varepsilon(t) = \int_0^t J_D^\varepsilon(r)\,\mathrm{d}r. \tag{7.5}$$

The expectation in (7.4) specifies the Wick ordering relative to the covariance (7.2).

Lemma 7.2 (A logarithmic covariance bound)**.** *Fix $T < \infty$. For two formally symmetric first-order operators D, F and two admissible regularisation families, there is a constant C_T such that, for $0 \le s, t \le T$ with $s \ne t$,*

$$\left|\mathbb{E}[J_D^\varepsilon(t) J_F^\delta(s)]\right| \le C_T m_D m_F \left(1 + \log_+ \frac{1}{|t-s|}\right). \tag{7.6}$$

The constant may depend on the uniform L^2 and H^1 bounds of the regularisations. It is independent of $\varepsilon, \delta, s, t$.

Proof. The eigenvalue counting function of a Laplace-type operator on a closed surface satisfies $N_L(\lambda) \le C(1+\lambda)$. Moreover,

$$0 \le Q_r \le c_T A^{-1}, \qquad \|A^{1/2} C_{t,s}^{1/2}\|^2 \le c_T, \tag{7.7}$$

and, with $a = |t-s| > 0$,

$$\operatorname{Tr} C_{t,s} \le c_T \operatorname{Tr}(A^{-1} e^{-aL}) \le C_T \left(1 + \log_+ \frac{1}{a}\right). \tag{7.8}$$

Indeed, the last estimate follows by integrating $(1+\lambda)^{-1} e^{-a\lambda}$ against the counting function. Its contribution from $1 \le \lambda \le a^{-1}$ is bounded by a constant times $\log(a^{-1})$, and the remaining contributions are bounded.

The regularisation assumptions and elliptic norm equivalence imply

$$\sup_\varepsilon \|D_\varepsilon A^{-1/2}\| \le C m_D. \tag{7.9}$$

For $C = C_{t,s}$, which is positive and self-adjoint, this gives

$$\begin{aligned}\|C^{1/2} D_\varepsilon C^{1/2}\|_{\mathrm{HS}} &\le \|C^{1/2}\|_{\mathrm{HS}} \|D_\varepsilon A^{-1/2}\| \|A^{1/2} C^{1/2}\| \\ &\le C_T m_D \left(1 + \log_+ \frac{1}{|t-s|}\right)^{1/2}. \end{aligned} \tag{7.10}$$

Here HS denotes the Hilbert–Schmidt norm.

The Gaussian fourth-moment formula yields

$$\mathbb{E}[J_D^\varepsilon(t) J_F^\delta(s)] = 2 \operatorname{Tr}(D_\varepsilon C_{t,s} F_\delta C_{s,t}). \tag{7.11}$$

This identity follows first in a finite eigenbasis from the two cross-pairings of four centred Gaussian coordinates; approximation then gives the displayed trace formula. Since $C_{s,t} = C_{t,s}$, the right-hand side is twice the Hilbert–Schmidt inner product of $C^{1/2} D_\varepsilon C^{1/2}$ and $C^{1/2} F_\delta C^{1/2}$. Cauchy–Schwarz and (7.10) prove the result. □

Theorem 7.3 (Canonical integrated currents)**.** *For every $T < \infty$, $1 \le p < \infty$, and $0 < \gamma < 1$, there is a centred second-Wiener-chaos process H_D such that*

$$H_D^\varepsilon \longrightarrow H_D \quad \text{in } L^p\big(\Omega; C^\gamma([0,T])\big). \tag{7.12}$$

The limit is independent of the admissible spatial regularisation. It is adapted, starts from zero, depends linearly on D, and satisfies

$$\|H_D(t) - H_D(s)\|_{L^p(\Omega)} \le C_{p,T} m_D |t-s| \left(1 + \log_+ \frac{1}{|t-s|}\right)^{1/2}, \tag{7.13}$$

$$\|H_D\|_{L^p(\Omega; C^\gamma([0,T]))} \le C_{p,T,\gamma} m_D. \tag{7.14}$$

The same statements hold jointly for every finite collection of operators and regularisations.

Proof. For a fixed t, the Gaussian fourth-moment formula gives

$$\mathbb{E}|H_D^\varepsilon(t) - H_D^\delta(t)|^2 = 2 \int_0^t \int_0^t \|C_{r,q}^{1/2} (D_\varepsilon - D_\delta) C_{r,q}^{1/2}\|_{\mathrm{HS}}^2 \, \mathrm{d}r \, \mathrm{d}q. \tag{7.15}$$

The two regularisations in this formula may belong to different admissible families. Their relative operators converge strongly:

$$D_\varepsilon A^{-1/2} \longrightarrow D A^{-1/2} \quad \text{strongly on } \mathcal{H}. \tag{7.16}$$

To see this, apply S_ε to $A^{-1/2} v$ in H^1, then D into L^2, and finally S_ε^* in L^2.

When $r \ne q$, the operator $A^{1/2} C_{r,q}^{1/2}$ is Hilbert–Schmidt. Strong convergence in (7.16), together with uniform boundedness, therefore implies

$$\|C_{r,q}^{1/2} (D_\varepsilon - D_\delta) C_{r,q}^{1/2}\|_{\mathrm{HS}} \longrightarrow 0.$$

One can verify this implication by approximating the rightmost Hilbert–Schmidt operator by finite-rank operators. Estimate (7.10) supplies the integrable majorant $C(1 + \log_+(|r-q|^{-1}))$ for the squared norm. Dominated convergence in (7.15) proves L^2 convergence at every fixed time and independence of the regularisation.

For an interval of length $h \le 1$, (7.6) gives

$$\begin{aligned}\mathbb{E}|H_D^\varepsilon(t+h) - H_D^\varepsilon(t)|^2 &\le C_T m_D^2 \int_0^h \int_0^h \left(1 + \log_+ \frac{1}{|r-q|}\right) \mathrm{d}r \, \mathrm{d}q \\ &\le C_T m_D^2 h^2 (1 + |\log h|). \end{aligned} \tag{7.17}$$

All these random variables lie in the second homogeneous Wiener chaos. For each finite p, their L^p norms are bounded by a constant depending only on p times their L^2 norms. For completeness, this moment bound follows by diagonalising a finite-rank quadratic form as $\sum_n a_n(g_n^2-1)$ and expanding

$$\mathbb{E}\exp\left(z\sum_n a_n(g_n^2-1)\right)=\prod_n e^{-za_n}(1-2za_n)^{-1/2}.$$

For $|z|(\sum_n a_n^2)^{1/2}$ sufficiently small the logarithm is bounded by $Cz^2\sum_n a_n^2$. Integration of the resulting tail bound proves the moment estimate; finite-rank approximation gives it for every second-chaos variable.

It follows from (7.17) that, for every $\eta<1$ and every finite p,

$$\|H_D^\varepsilon(t)-H_D^\varepsilon(s)\|_{L^p}\le C_{p,T,\eta}m_D|t-s|^\eta.$$

Choose $\gamma<\gamma'<\eta<1$ and then a moment order so large that $\gamma'<\eta-1/p$. The quantitative Kolmogorov criterion gives uniform moments of the $C^{\gamma'}$ norms. Fixed-time L^p convergence on a finite time grid, followed by the uniform $C^{\gamma'}$ bound, gives convergence in $L^p(C^0)$. Finally use

$$\|f\|_{C^\gamma}\le C\|f\|_{C^0}^{1-\gamma/\gamma'}\|f\|_{C^{\gamma'}}^{\gamma/\gamma'}$$

and higher moments to obtain (7.12) for the prescribed p. The estimates pass to the limit and give (7.13)–(7.14). Linearity and adaptedness pass from the regularised processes to the limit. □

7.2 Bundle-valued nonlinearities and Yang–Mills currents

Let $F\to\Sigma$ be another metric vector bundle and let

$$q:E\otimes(T^*\Sigma\otimes E)\longrightarrow F,\qquad q_0:E\otimes E\longrightarrow F$$

be smooth bilinear bundle maps. Put $\mathcal{Q}(v)=q(v,\nabla v)+q_0(v,v)$. For every $h\in C^\infty(F)$ there is a formally symmetric first-order operator D_h such that

$$\int_\Sigma\langle h,\mathcal{Q}(v)\rangle\,\mathrm{vol}_g=\langle v,D_hv\rangle_{L^2}. \tag{7.18}$$

Indeed, express the first term as $\langle v,a_h^j\nabla_j v\rangle$ and take the formal symmetric part. The resulting zero-order coefficients include derivatives of h. In particular,

$$\|D_hA^{-1/2}\|\le C\|h\|_{C^1}. \tag{7.19}$$

Corollary 7.4 (Distribution-valued integrated fields)**.** *For every $s>3$, $\gamma<1$, and finite p, the fields*

$$\mathcal{K}^\varepsilon(t)=\int_0^t\left(\mathcal{Q}(S_\varepsilon X_r)-\mathbb{E}\mathcal{Q}(S_\varepsilon X_r)\right)\mathrm{d}r$$

converge in $L^p(\Omega;C^\gamma([0,T];H^{-s}(F)))$ to a limit $\mathcal{K}$. It is independent of the admissible regularisation and satisfies $\langle\mathcal{K}(t),h\rangle=H_{D_h}(t)$.

Proof. Take an orthonormal eigensection basis (f_n) of a positive Laplace-type operator on F, with eigenvalues μ_n. Sobolev embedding and elliptic regularity give, for any $\eta>0$,

$$\|f_n\|_{C^1}\le C_\eta(1+\mu_n)^{1+\eta/2}.$$

Choose $0 < \eta < s - 3$. Weyl's bound implies

$$\sum_n (1+\mu_n)^{-s} \|f_n\|_{C^1}^2 < \infty.$$

Apply (7.14) and (7.19) to each coefficient, and sum the squares with weights $(1+\mu_n)^{-s}$. Minkowski's inequality proves the assertion for $p \geq 2$; smaller moments follow by inclusion. Dominated convergence in this summation proves convergence and regularisation independence. □

For a principal bundle $P \to \Sigma$ with compact structure group and a smooth reference connection B, take

$$E = T^*\Sigma \otimes \operatorname{ad} P, \qquad L = d_B^* d_B + d_B d_B^*, \qquad \Pi = 1.$$

This is a nonnegative self-adjoint operator of Laplace type even when $d_B^2 \neq 0$. The quadratic part of the augmented Yang–Mills operator is

$$\mathcal{Q}_B(a) = -\tfrac{1}{2} d_B^* [a \wedge a] - [a \lrcorner d_B a] - [a, d_B^* a]. \tag{7.20}$$

It has the form used in (7.18). Consequently every smooth test one-form, including any element of $\ker L$, determines a canonical integrated quadratic current. This construction does not require a trivialisation of P or a coexact decomposition.

Proposition 7.5 (Naturality)**.** *Let U be an orthogonal bundle isomorphism, covering a Riemannian isometry, and transport the data by*

$$\widetilde{L} = ULU^{-1}, \quad \widetilde{\Pi} = U\Pi U^{-1}, \quad \widetilde{D} = UDU^{-1}, \quad \widetilde{W} = UW.$$

Then $H_{\widetilde{D}}(\widetilde{W}) = H_D(W)$ under this coupling. For (7.20), this includes simultaneous gauge transformation of the reference connection, the noise and the test one-form.

Proof. The convolution transforms as $\widetilde{X} = UX$. With transported regularisations $\widetilde{S}_\varepsilon = US_\varepsilon U^{-1}$, the quadratic forms and their expectations agree at every positive ε. Pass to the limit using Theorem 7.3; regularisation independence removes the restriction to transported regularisations. □

7.3 Restarting the integrated current

Restarting an integrated current produces an old-field term and a mixed term. The following identity holds at stopping times.

Theorem 7.6 (Restart formula)**.** *Fix $S, T < \infty$ and let $\sigma \leq S$ be a stopping time for the completed Brownian filtration. Write*

$$W_r^{(\sigma)} = W_{\sigma+r} - W_\sigma, \qquad Z_r^{(\sigma)} = \int_0^r P_{r-v} \Pi \, \mathrm{d}W_v^{(\sigma)}.$$

Let $H_D^{(\sigma)}$ be the current of Theorem 7.3 constructed from $Z^{(\sigma)}$. It is independent of the past and has the same law as H_D on $[0,T]$. Almost surely, for $0 \leq t \leq T$,

$$H_D(\sigma+t) - H_D(\sigma) = \mathcal{A}_D(\sigma, X_\sigma; t) + \mathcal{M}_D(X_\sigma, W^{(\sigma)}; t) + H_D^{(\sigma)}(t), \tag{7.21}$$

where

$$\mathcal{A}_D(s,x;t) = \int_0^t \left(\langle P_r x, DP_r x \rangle - \operatorname{Tr}(DP_{2r}Q_s) \right) \mathrm{d}r, \tag{7.22}$$

$$\mathcal{M}_D(x, W'; t) = 2 \int_0^t \langle DP_r x, Z_r' \rangle \, \mathrm{d}r. \tag{7.23}$$

Both integrals exist as ordinary time integrals and define paths of finite variation. In (7.23), the inner product is the distributional pairing with the smooth field $DP_r x$ for $r > 0$. For a stopping time, the trace in (7.22) is evaluated at the random parameter $s = \sigma$; it is the prescribed Wick centring. In particular,

$$\mathbb{E}\big[H_D(\sigma + t) - H_D(\sigma) \mid \mathcal{F}_\sigma\big] = \mathcal{A}_D(\sigma, X_\sigma; t). \tag{7.24}$$

The trace in (7.22) does not assert a conditional Gaussian law for X_σ.

Proof. Choose $0 < \beta < 1/2$. The eigenfunction expansion of (7.1) gives

$$\mathbb{E}\|X_t - X_s\|_{H^{-\beta}}^2 \le C_T|t - s|^\beta \quad (0 < \beta < 1).$$

Gaussian moment bounds and the Kolmogorov criterion give a continuous $H^{-\beta}$-valued version and finite moments of its supremum on compact time intervals. Thus $X_\sigma \in H^{-\beta}$ almost surely.

For deterministic $x \in H^{-\beta}$, semigroup smoothing yields

$$|\langle P_r x, DP_r x\rangle| \le Cr^{-1/2-\beta}\|x\|_{H^{-\beta}}^2, \qquad 0 < r \le 1. \tag{7.25}$$

Indeed, the left-hand side is bounded by $m_D\|P_r x\|_{L^2}\|P_r x\|_{H^1}$, up to a fixed Sobolev norm equivalence constant. The same argument holds uniformly for D_ε, using (7.9). The exponent is integrable because $\beta < 1/2$. Using an eigenbasis of L, the relative bound on D, and $Q_s \le c_S A^{-1}$ gives

$$|\operatorname{Tr}(DP_{2r}Q_s)| \le C_S \sum_n e^{-2r\lambda_n}(1 + \lambda_n)^{-1/2} \le C_S(1 + r^{-1/2}). \tag{7.26}$$

The possible projection Π only decreases this bound. This proves absolute integrability in (7.22).

Conditionally on x, the integrand in (7.23) is Gaussian and

$$\begin{aligned}\mathbb{E}|\langle DP_r x, Z_r'\rangle|^2 &= \langle DP_r x, Q_r DP_r x\rangle \\ &\le C_T\|A^{-1/2}DP_r x\|_{L^2}^2 \le C_T r^{-\beta}\|x\|_{H^{-\beta}}^2.\end{aligned} \tag{7.27}$$

We used formal symmetry to obtain boundedness of $A^{-1/2}D$. Integration of the square root proves that the mixed term has an L^1 time integrand, conditionally almost surely, with finite moments after averaging over X_σ.

At every regularisation level the exact identities

$$X_{\sigma+r} = P_r X_\sigma + Z_r^{(\sigma)}, \qquad Q_{s+r} = P_r Q_s P_r + Q_r$$

give (7.21) with D replaced by D_ε. There is no mixed expectation term, since $W^{(\sigma)}$ is independent of the past. In the deterministic centring, the covariance identity is simply evaluated at $s = \sigma$.

For $r > 0$, relative strong convergence (7.16) implies convergence of the old and mixed integrands. Estimates (7.25)– (7.27) hold uniformly with D_ε in place of D. They therefore give convergence of the corresponding primitives, uniformly in t, by dominated convergence. For the trace, use the absolutely summable bound at each $r > 0$ and then (7.26). Conditional Gaussian moments handle the mixed term. The full current converges uniformly in time by Theorem 7.3, so it may be evaluated at $\sigma + t$ and σ. The fresh current converges by the same theorem and the Brownian strong Markov property. Passing to the limit proves (7.21). The mixed term and the fresh current have zero conditional mean given $\mathcal{F}_\sigma$, whereas the old term is $\mathcal{F}_\sigma$-measurable. Their integrability, proved above, therefore also gives (7.24). □

Remark 7.7. The fresh current in (7.21) is an independent copy. The mixed term depends on both the initial field and the new noise. These two contributions have different conditioning properties and both enter a restarted enhanced equation.

7.4 Sharp harmonic currents on the flat torus

A quadratic Yang–Mills current attains the logarithm in (7.13). Let $\Sigma = (\mathbb{R}/2\pi\mathbb{Z})^2$ with normalised area measure. For this subsection take the Lie algebra to be compact, simple and three-dimensional, and choose an orthonormal basis T_1, T_2, T_3 with $[T_1, T_2] = cT_3$, $c \neq 0$. Thus the structure group may be taken to be $\mathrm{SU}(2)$. Let X be the coexact convolution, so $\Pi = P_{\mathrm{coex}}$ and L is the ordinary Hodge Laplacian. Choose a half lattice $\mathbb{Z}^2_+ \subset \mathbb{Z}^2 \setminus \{0\}$ containing one element from each nonzero pair $\{k, -k\}$, put $\lambda_k = |k|^2$, and set $e(k) = k^\perp/|k|$.

For $r = 1, 2$, the Lie algebra components of X have the expansion

$$X^r_i(t,x) = \sqrt{2} \sum_{k \in \mathbb{Z}^2_+} e_i(k) \left(a^r_k(t) \cos(k \cdot x) + b^r_k(t) \sin(k \cdot x) \right), \tag{7.28}$$

where all the real Ornstein–Uhlenbeck coordinates are independent and

$$\mathbb{E}[a^r_k(t) a^r_k(s)] = \mathbb{E}[b^r_k(t) b^r_k(s)] = q_k(t,s) = \frac{e^{-\lambda_k |t-s|} - e^{-\lambda_k (t+s)}}{2\lambda_k}. \tag{7.29}$$

In particular $v_k(t) = q_k(t,t) = (1 - e^{-2\lambda_k t})/(2\lambda_k)$.

The quadratic Yang–Mills expression is

$$\mathfrak{B}_j(X, \nabla X) = 2 \sum_i [X_i, \partial_i X_j] - \sum_i [X_i, \partial_j X_i].$$

Its harmonic coefficient in the direction $T_3 \, \mathrm{d}x_j$, after heat regularisation, equals

$$J^\varepsilon_j(t) = -2c \sum_{k \in \mathbb{Z}^2_+} e^{-2\varepsilon\lambda_k} k_j \left(a^1_k(t) b^2_k(t) - b^1_k(t) a^2_k(t) \right). \tag{7.30}$$

Indeed, the spatial integral of the first term in $\mathfrak{B}_j$ vanishes by $\operatorname{div} X = 0$, and integration by parts turns the remaining bracket into $-2c \sum_i \int X^1_i \partial_j X^2_i$. Orthogonality of the trigonometric functions gives (7.30). Its expectation is zero.

Theorem 7.8 (Sharp regularity and infinite variation)**.** *Let H_j be the limit of $\int_0^\cdot J^\varepsilon_j(r)\,\mathrm{d}r$ given by Theorem 7.3. Then the following hold.*

1. *For every fixed $t > 0$,*

$$\operatorname{Var} J^\varepsilon_j(t) = 8c^2 \sum_{k \in \mathbb{Z}^2_+} e^{-4\varepsilon\lambda_k} k_j^2 v_k(t)^2 \asymp_t \log(1/\varepsilon). \tag{7.31}$$

   The family $(J^\varepsilon_j(t))_\varepsilon$ is not tight.

2. *For every $0 < a < b < T$, there are $h_0, c_1, c_2 > 0$ such that*

$$c_1 h^2 \log(1/h) \le \mathbb{E}|H_j(t+h) - H_j(t)|^2 \le c_2 h^2 \log(1/h) \tag{7.32}$$

   for $a \le t \le b$ and $0 < h < h_0$.

3. *Almost surely $H_j \in C^\gamma([0,T])$ for every $\gamma < 1$, and H_j has infinite total variation on every nontrivial time interval.*

Proof. Set $d_k(t) = a^1_k(t) b^2_k(t) - b^1_k(t) a^2_k(t)$. The variables indexed by distinct k are independent, and

$$\mathbb{E}[d_k(t) d_k(s)] = 2 q_k(t,s)^2.$$

Formula (7.31) follows immediately. For fixed $t > 0$, its large-frequency summands are comparable to $e^{-4\varepsilon|k|^2} k_j^2/|k|^4$. Their sum is comparable to $\log(1/\varepsilon)$.

Every $J_j^\varepsilon(t)$ belongs to the second homogeneous Wiener chaos. A centred quadratic Gaussian variable Z satisfies $\mathbb{E}Z^4 \le 15(\mathbb{E}Z^2)^2$: diagonalisation reduces the assertion to

$$\mathbb{E}\left(\sum_n a_n(g_n^2 - 1)\right)^4 = 12\left(\sum_n a_n^2\right)^2 + 48\sum_n a_n^4.$$

Paley–Zygmund applied to Z^2 consequently gives

$$\mathbb{P}(|Z| \ge \|Z\|_{L^2}/\sqrt{2}) \ge 1/60.$$

This and (7.31) prove lack of tightness.

For $t \in [a, b]$ define

$$Y_h(t) = \frac{H_j(t+h) - H_j(t)}{h}.$$

The covariance formula and L^2 convergence of the integrated series give

$$\mathbb{E}|Y_h(t)|^2 = \frac{8c^2}{h^2} \sum_{k \in \mathbb{Z}^2_+} k_j^2 \int_t^{t+h} \int_t^{t+h} q_k(r,s)^2 \,\mathrm{d}r \,\mathrm{d}s. \tag{7.33}$$

If $\lambda_k a$ is sufficiently large and $\lambda_k h \le 1$, then $q_k(r,s) \ge c_a/\lambda_k$ throughout the integration square. Consequently the terms with $R_a \le |k| \le h^{-1/2}$ yield a lower bound $c\sum k_j^2/|k|^4 \ge c' \log(1/h)$. The upper bound follows from (7.17). This proves (7.32) uniformly on $[a, b]$.

It remains to prove infinite variation. Fix a closed interval $I \subset (0, T)$ of positive length and a smaller closed interval I_0 contained in its interior. For sufficiently small h, set

$$V_h = \int_{I_0} |Y_h(t)| \,\mathrm{d}t.$$

The second-chaos fourth-moment bound and interpolation imply $\mathbb{E}|Y_h(t)| \ge 15^{-1/2}\|Y_h(t)\|_{L^2}$. Together with (7.33), this gives

$$\mathbb{E}V_h \ge c|I_0|\sqrt{\log(1/h)}, \qquad \mathbb{E}V_h^2 \le C|I_0|^2 \log(1/h).$$

Paley–Zygmund applied to V_h shows that the family $(V_h)_{h>0}$ is not tight. If H_j had bounded variation on I almost surely, then Fubini's inequality for its variation measure would give

$$V_h \le \frac{1}{h}\int_{I_0} |\mathrm{d}H_j|([t, t+h]) \,\mathrm{d}t \le |\mathrm{d}H_j|(I)$$

for all sufficiently small h. This would make (V_h) tight. Thus the event of bounded variation on I has probability less than one.

To sharpen this conclusion, use finite Fourier cutoffs, which are admissible by Definition 7.1. The integrated series converges in $L^p(C^\gamma)$, and deleting any finite set of Fourier modes changes its limit by

$$-2c\sum_{k \in F} k_j \int_0^t d_k(r) \,\mathrm{d}r,$$

a C^1 function. Enumerate the independent mode Brownian paths. For every N, the limit of the integrated series over modes with index greater than N is measurable with respect to those modes and agrees almost surely with H_j minus the finite C^1 sum above. The bounded-variation subset of $C(I)$ is Borel, since total variation is the supremum over a countable collection of

rational partitions. Consequently its event agrees, up to a null set, with an event in each tail sigma-field. Kolmogorov's zero–one law and the preceding paragraph show that its probability is zero. Apply this argument to the countable collection of rational intervals contained in $(0, T)$. Every nontrivial interval contains one of them. Finally, the asserted Hölder regularity is Theorem 7.3, applied to a countable sequence of exponents tending to one. □

Corollary 7.9 (Zero quadratic variation and failure of the semimartingale property)**.** *The process H_j has zero quadratic variation along every sequence of partitions whose mesh tends to zero. It is not a semimartingale with respect to any filtration satisfying the usual conditions to which it is adapted. In particular its distributional time derivative has no almost surely locally integrable representative.*

Proof. Choose $1/2 < \gamma < 1$. On every compact interval, every partition $\mathcal{P}$ satisfies

$$\sum_{[u,v]\in\mathcal{P}} |H_j(v) - H_j(u)|^2 \le \|H_j\|_{C^\gamma}^2 |\mathcal{P}|^{2\gamma-1} \sum_{[u,v]\in\mathcal{P}} (v-u),$$

which tends to zero. A continuous semimartingale with zero quadratic variation has a constant local-martingale part and therefore finite variation. This contradicts Theorem 7.8. A locally integrable derivative would make H_j locally absolutely continuous and hence locally of finite variation, giving the same contradiction. □

Remark 7.10 (Relation to the nonlinear equation). The field (7.20) is the integrated Gaussian quadratic component of the nonlinear enhancement. The remaining products and their contraction identities are constructed in Section 8. The restart identity above retains the old-field and mixed contributions explicitly, even though the new-noise current has no finite-variation drift density in the harmonic example.

8 A finite Gaussian enhancement for the derivative nonlinearity

The nonlinear equation requires additional distributional products beyond the integrated Gaussian current. We construct them with explicit local counterterms using the bundle heat kernel, without assuming spatial translation invariance. Global distributions and their pointed versions are treated separately because their regularities differ.

Let E be a real metric vector bundle over a closed Riemannian surface, let $L \ge 0$ be a self-adjoint operator of Laplace type, and put $A = 1 + L$. Write $K(z, w)$ for the causal kernel of $\partial_t + A$ on $\mathbb{R} \times \Sigma$, where $z = (t, x)$ and $w = (s, y)$, and set

$$X(t) = \int_{-\infty}^{t} e^{-(t-r)A}\, \mathrm{d}W_r. \tag{8.1}$$

All tensor contractions below are smooth bundle morphisms. We fix a metric connection ∇ for their derivatives. It need not be flat. The parabolic distance and homogeneous dimension are

$$\rho(z, w) = |t - s|^{1/2} + d(x, y), \qquad |\mathfrak{s}| = 4.$$

The notation $\mathcal{C}^\alpha_{\mathfrak{s}}$ denotes the local parabolic Besov space $B^\alpha_{\infty,\infty}$. Statements on a compact time interval refer to restriction from a slightly larger open interval. Every convergence assertion below holds in L^p of the indicated local norm, for every finite p.

8.1 The cutoff class and covariance bounds

We use spatial heat cutoffs

$$S_\varepsilon = \int_{a_0}^{a_1} e^{-a\varepsilon A}\,\mu_\varepsilon(\mathrm{d}a), \qquad X_\varepsilon = S_\varepsilon X, \qquad 0 < a_0 < a_1 < \infty, \tag{8.2}$$

where μ_ε is a probability measure. The measures may depend on ε and need not converge. A single heat cutoff is included by taking a point mass. The parameter ε is heat time, and the associated length scale is $\sqrt{\varepsilon}$. The wider class in Definition 7.1 suffices for weak integrated currents. Local enhanced distributions require the covariance-kernel control supplied by (8.2).

For two such families on the same probability space, let $C_{\varepsilon,\delta}(z,w) = \mathbb{E}[X_\varepsilon(z) \otimes X_\delta(w)]$. Functional calculus gives

$$C_{\varepsilon,\delta}(t,x;s,y) = \frac{1}{2}\iint \left[A^{-1}e^{-(|t-s|+a\varepsilon+b\delta)A}\right](x,y)\,\mu_\varepsilon(\mathrm{d}a)\nu_\delta(\mathrm{d}b). \tag{8.3}$$

In particular, all cutoff comparisons below are coupled through the same white noise.

Lemma 8.1 (Uniform local covariance estimates)**.** *For $\rho(z,w) \le 1$, uniformly over (8.2),*

$$|C_{\varepsilon,\delta}(z,w)| \le C(1 + \log_+(\rho(z,w) + \sqrt{\varepsilon} + \sqrt{\delta})^{-1}), \tag{8.4}$$

$$|\nabla_x^a \nabla_y^b C_{\varepsilon,\delta}(z,w)| \le C_{a,b}(\rho(z,w) + \sqrt{\varepsilon} + \sqrt{\delta})^{-a-b}, \qquad a+b \ge 1. \tag{8.5}$$

The kernels and their spatial derivatives converge off the parabolic diagonal to those of $C(z,w) = \frac{1}{2}[A^{-1}e^{-|t-s|A}](x,y)$. On the causal region $t > s$, the corresponding bounds also hold for time derivatives, a time derivative counting as two orders. In coordinates $(x;r,v)$ with $r = t-s$ and $y = x - v$, derivatives in the centre variable x do not increase the singular order. The same assertions hold for the heat kernel, with singular order 2 in place of 0, and order 3 for its spatial derivative.

Proof. The bundle heat-kernel parametrix, with any prescribed finite number of derivatives, gives

$$|\nabla_x^a \nabla_y^b e^{-uA}(x,y)| \le C_{a,b} u^{-1-(a+b)/2} e^{-d(x,y)^2/(Cu)} \qquad (0 < u \le 1).$$

For $u \ge 1$ all these norms decrease exponentially. Integrate this bound over $u \ge |t-s| + a\varepsilon + b\delta$ in (8.3). At order zero the integral is logarithmic, and at positive order m it is bounded by $C(\sqrt{|t-s| + a\varepsilon + b\delta} + d(x,y))^{-m}$. This proves the first two assertions. Off-diagonal convergence follows from the same integral formula and dominated convergence.

For the centre-derivative assertion, write the leading parametrix in a fixed chart as a Gaussian in v, with covariance given by the metric at x, times a smooth transport amplitude. A centre derivative of its exponent produces a factor bounded by a polynomial in $|v|/\sqrt{u}$, rather than an extra power of $u^{-1/2}$. The same is true of every parametrix coefficient and of its remainder, after increasing the expansion order. This proves the centre derivative assertion, including the volume-density and bundle-frame factors. Differentiation in $r > 0$ has parabolic order two. No claim of smoothness across $t = s$ is needed for the cutoff covariance; in the causal kernels used below its multiplication by K is smooth away from the parabolic diagonal. □

We will repeatedly use the following elementary Gaussian criterion. If random distributions in a fixed finite sum of Wiener chaoses have mixed covariance kernels bounded by $C\rho^{-2a}$, with $2a < 4$, and those kernels converge off the diagonal, then they converge locally in $L^p(\mathcal{C}_{\mathfrak{s}}^{-a-\eta})$ for every $\eta > 0$. Indeed a test at parabolic scale λ has second moment at most $C\lambda^{-2a}$. Gaussian chaos moment equivalence gives the same scaling in every finite moment. A parabolic wavelet grid has $O(2^{4n})$ elements on a compact set at scale 2^{-n}; using a moment order greater than $4/\eta$

makes the weighted supremum summable. Mixed covariance convergence gives convergence of each of the finitely many wavelet coefficients below a fixed scale. The uniformly summable tail then gives norm convergence, using a slightly smaller regularity loss for the tail bound. This argument also shows that the versions take values in the closure of smooth functions in the stated, slightly weaker, Besov norms.

8.2 The first two nonlinear levels

Write $P_\varepsilon(z) = C_{\varepsilon,\varepsilon}(z,z)$ and define the tensor Wick powers

$$W_{2,\varepsilon} = X_\varepsilon^{\otimes 2} - P_\varepsilon, \qquad W_{3,\varepsilon} = X_\varepsilon^{\otimes 3} - \mathrm{Sym}(P_\varepsilon \otimes X_\varepsilon),$$

where Sym denotes the sum over the three pair contractions. Let $q : E \otimes (T^*\Sigma \otimes E) \to E$ be smooth, and put

$$Q_{q,\varepsilon} = q(X_\varepsilon, \nabla X_\varepsilon) - \mathbb{E}q(X_\varepsilon, \nabla X_\varepsilon), \tag{8.6}$$
$$Y_{q,\varepsilon} = KQ_{q,\varepsilon}. \tag{8.7}$$

The time convolution in (8.7) extends over the past. It is well defined by the exponential decrease of the massive kernel.

Proposition 8.2 (Quadratic and cubic fields)**.** *For $0 < \kappa < 1/16$, jointly on every compact cylinder,*

$$\begin{aligned}
X_\varepsilon &\longrightarrow X && \text{in } \mathcal{C}_{\mathfrak{s}}^{-\kappa},\\
W_{2,\varepsilon} &\longrightarrow W_2 && \text{in } \mathcal{C}_{\mathfrak{s}}^{-2\kappa},\\
Q_{q,\varepsilon} &\longrightarrow Q_q && \text{in } \mathcal{C}_{\mathfrak{s}}^{-1-2\kappa},\\
Y_{q,\varepsilon} &\longrightarrow Y_q = KQ_q && \text{in } \mathcal{C}_{\mathfrak{s}}^{1-2\kappa},\\
W_{3,\varepsilon} &\longrightarrow W_3 && \text{in } \mathcal{C}_{\mathfrak{s}}^{-3\kappa}.
\end{aligned} \tag{8.8}$$

The limits are independent of the cutoff family (8.2). The conclusions hold simultaneously for any finite list of tensor contractions q, and for zeroth-order quadratic contractions as well.

Proof. Wick's formula gives the following bounds for mixed covariances:

$$\begin{aligned}
|\mathbb{E}[W_{2,\varepsilon}(z)W_{2,\delta}(w)]| &\lesssim (1 + \log_+ \rho^{-1})^2,\\
|\mathbb{E}[W_{3,\varepsilon}(z)W_{3,\delta}(w)]| &\lesssim (1 + \log_+ \rho^{-1})^3,\\
|\mathbb{E}[Q_{q,\varepsilon}(z)Q_{q,\delta}(w)]| &\lesssim \rho^{-2}(1 + \log_+ \rho^{-1}).
\end{aligned}$$

Tensor norms are understood, and $\rho = \rho(z,w) \le 1$. The last line consists of $C\nabla\nabla C$ and $\nabla C \nabla C$. Each bound is locally integrable in parabolic dimension four. Replace each logarithm by an arbitrarily small power, apply the Gaussian criterion above, and use the mixed covariance formula to obtain cutoff-independent convergence. Parabolic Schauder estimates give the assertion for Y_q. For this last application, split the time convolution into a compact past interval and its complement; the latter has summable exponential bounds in every derivative and every finite moment. □

Remark 8.3. The assertion for Q_q is a spacetime assertion. It does not give a random spatial distribution $Q_q(t)$ at each time. The harmonic counterexample in Theorem 7.8 rules out that interpretation for the Yang–Mills quadratic term.

8.3 A compensated critical-kernel estimate

The next deterministic estimate is used for the first-chaos contraction of a third-order tree. Its conclusion includes an arbitrarily small loss; an unqualified order-zero bound would not follow from the hypotheses.

Lemma 8.4 (Logarithmic-order compensated operators)**.** *Let $B_\varepsilon(z,w)$ be a causal bundle kernel invariant under time translations, smooth away from the parabolic diagonal and integrable for each $\varepsilon > 0$. Suppose, in centre and difference coordinates, that*

$$|\partial_x^m \partial_{(r,v)}^\ell B_\varepsilon(x;r,v)| \le C_{m,\ell}\rho(r,v)^{-4-|\ell|_{\mathfrak{s}}} \tag{8.9}$$

locally, uniformly in ε, and that these derivatives converge away from the diagonal. Assume smooth integrable bounds for the remaining kernel. Choose a smooth cutoff $\chi(z,w)$ equal to one near the diagonal and supported where spatial parallel transport $\tau_{y\leftarrow x} : E_x \to E_y$ is defined. Set

$$M_\varepsilon(z) = \int B_\varepsilon(z,w)\chi(z,w)\tau_{y\leftarrow x}\,\mathrm{d}w, \tag{8.10}$$

$$R_\varepsilon f(z) = \int B_\varepsilon(z,w)f(w)\,\mathrm{d}w - M_\varepsilon(z)f(z). \tag{8.11}$$

After localisation of both variables to fixed compact cylinders, for every finite α and every $\eta > 0$, the operators R_ε converge in operator norm

$$\mathcal{C}^\alpha_{\mathfrak{s}} \longrightarrow \mathcal{C}^{\alpha-\eta}_{\mathfrak{s}}. \tag{8.12}$$

Massive tails can be added when the input is controlled in a corresponding weighted norm; in the stochastic applications their moments are estimated separately. The limiting operator depends only on the off-diagonal limit kernel and the chosen χ, τ, not on its approximation.

For a kernel A_ε invariant under time translations, replace the right-hand bound by

$$\rho^{-3-|\ell|_{\mathfrak{s}}}(1+\log_+\rho^{-1}).$$

Then A_ε converges locally in operator norm from $\mathcal{C}^{\alpha-1}_{\mathfrak{s}}$ to $\mathcal{C}^{\alpha-\eta}_{\mathfrak{s}}$.

Proof. Partition the diagonal kernel into annuli $\rho \asymp 2^{-k}$, $k \ge 0$. In a local frame, replacing parallel transport in the subtracted term by the identity changes the operator by a smooth coefficient of uniformly finite size: $\tau - I = O(\rho)$ makes this correction integrable, with all centre derivatives. It suffices to consider the symbol of the kth compensated piece,

$$b_{\varepsilon,k}(x;\omega,\xi) = \int B_{\varepsilon,k}(x;r,v)(e^{-i(\omega r+\xi\cdot v)} - 1)\,\mathrm{d}r\,\mathrm{d}v.$$

For parabolic frequency $|\omega|^{1/2} + |\xi| \asymp 2^n$, its centre derivatives satisfy

$$|\partial_x^m b_{\varepsilon,k}| \le C_m \min\{2^{n-k}, 1\}. \tag{8.13}$$

When $n \le k$, use $|e^{-i(\omega r+\xi\cdot v)} - 1| \lesssim 2^{n-k} + 2^{2(n-k)}$. When $n > k$, the uniform L^1 bound of the annular kernel suffices. After a frequency derivative of parabolic order $j \ge 1$, the corresponding bound is $C2^{-nj}\min\{2^{n-k}, 1\}$: for $n \le k$ this follows from the annular moments, and for $n > k$ it follows by integrating the smooth annular kernel by parts.

These estimates also give, for local parabolic Littlewood–Paley blocks and arbitrary N,

$$\|\Delta_m R_{\varepsilon,k}\Delta_n\|_{L^\infty\to L^\infty} \le C_N 2^{-N|m-n|}\min\{2^{n-k}, 1\}. \tag{8.14}$$

One way to verify this estimate is to insert the two block kernels, rescale the frequency annulus to unit size, and integrate by parts in the frequency variable to bound the resulting kernel in L^1. Off-diagonal block interactions require centre frequencies of size $2^{\max(m,n)}$; repeated centre differentiation in (8.13) gives the displayed decay. A partition of unity adds only uniformly smoothing terms.

Summing over the annuli gives

$$\sum_{k\geq 0} \min\{2^{n-k}, 1\} \leq n+2.$$

Consequently

$$\|\Delta_m R_\varepsilon f\|_\infty \leq C_N \sum_{n\geq 0} 2^{-N|m-n|}(n+2)\|\Delta_n f\|_\infty. \tag{8.15}$$

Since $\sup_n (n+2)2^{-\eta n} < \infty$, this proves (8.12). It also proves operator-norm convergence. First discard input frequencies $n > N$, using the small factor $(n+2)2^{-\eta n}$. For the finitely many remaining frequencies, the annuli $k > K$ contribute at most $C2^{N-K}$. On the finitely many annuli left over, the kernel and its needed derivatives converge uniformly. Taking these limits in this order proves convergence using only the stated kernel bounds.

For A_ε the annular L^1 size is $C(k+1)2^{-k}$. Integration by parts when $n > k$ and direct summation when $n \leq k$ give an operator symbol bounded at frequency 2^n by $C(n+2)2^{-n}$, with the same off-diagonal estimates. This proves its stated one-order gain with an arbitrary small loss and the same convergence argument applies. □

8.4 The third-chaos estimate

Let $p : E \otimes (T^*\Sigma \otimes E) \to E$ be another smooth contraction. Denote by

$$T_{p,q,\varepsilon} = \mathrm{Proj}_3\, p(X_\varepsilon, \nabla Y_{q,\varepsilon})$$

the projection onto the third homogeneous Wiener chaos. The full renormalised product also contains the first-chaos terms identified below.

Lemma 8.5 (The two-vertex contraction graphs)**.** *For every $\eta > 0$, $T_{p,q,\varepsilon}$ converges locally in $L^p(\mathcal{C}_{\mathfrak{s}}^{-\eta})$, for all finite p. Its limit is the same for all cutoff families* (8.2)*.*

Proof. Fix a sufficiently small $d > 0$. In all estimates below replace $1 + \log_+ \rho^{-1}$ by $C_d\rho^{-d}$. The covariance of two third-chaos trees is a sum of the six pairings joining their three noise leaves. Let z, z' be the roots and u, v their respective inner vertices. Each differentiated heat kernel has order 3.

If the two outer leaves are paired, the other two pairings have absolute value at most

$$\rho(z,z')^{-d} \iint \rho(z,u)^{-3}\rho(z',v)^{-3}\rho(u,v)^{-2-2d}\,\mathrm{d}u\,\mathrm{d}v \lesssim \rho(z,z')^{-3d}.$$

For the inequality, twice use the parabolic convolution bound $\rho^{-a} * \rho^{-b} \lesssim \rho^{-(a+b-4)}$ when $a, b < 4 < a+b$, with smooth tail terms understood.

In each remaining pairing one outer leaf is paired with a leaf at the opposite inner vertex. The bound is

$$\iint \frac{\mathrm{d}u\,\mathrm{d}v}{\rho(z,u)^3\rho(z',v)^3\rho(z,v)^a\rho(z',u)^b\rho(u,v)^c}, \tag{8.16}$$

where, including the case with a, b exchanged,

$$(a,b,c) = (d,d,2+d), \quad (d,1+d,1+d), \quad (1+d,1+d,d).$$

The two derivatives at the inner vertices give exactly this list. In every case $a+b+c = 2+3d$. The integral in (8.16) is at most $C_d\rho(z,z')^{-3d}$.

Here are the integrability checks for this last assertion. Rescale by $R = \rho(z,z')$ so that the roots have distance one. A single integration vertex has exponent 3, a, b or c, all less than 4. A two-vertex cluster at either root has exponent $3+a+c$ or $3+b+c$, both less than 8. A collision $u = v$ away from the roots has exponent $c < 4$. At infinity the simultaneous scaling exponent is $8+3d > 8$; when only u or only v escapes, the respective exponent is $3+b+c > 4$ or $3+a+c > 4$. For the remaining region u, v far from the roots but close to each other, integrating their difference up to the distance to the roots again gives the simultaneous exponent $8+3d$. Splitting these regions into dyadic annuli gives convergent geometric series. The rescaling contributes R^{-3d}. These checks work in every chart, while the massive tails are smooth and integrable.

The bounds are uniform for mixed cutoff covariances. All proper collisions are integrable for fixed $z \neq z'$, so dominated convergence proves off-diagonal covariance convergence. The Gaussian criterion with $a = 3d/2$, and then sufficiently small d, proves the assertion. □

8.5 Keeping the first chaos and subtracting a local mass

The first-chaos terms are determined by two contraction kernels. In local orthonormal frames write

$$p(X,\nabla Y)^r = p^r_{aid}X^a\nabla_i Y^d, \qquad q(X,\nabla X)^e = q^e_{bjc}X^b\nabla_j X^c.$$

With $C_\varepsilon = C_{\varepsilon,\varepsilon}$, put

$$A^r_{p,q,\varepsilon jc}(z,w) = p^r_{aid}(x)(\nabla_{x_i}K)^d{}_e(z,w)q^e_{bjc}(y)C^{ab}_\varepsilon(z,w), \tag{8.17}$$

$$B^r_{p,q,\varepsilon b}(z,w) = p^r_{aid}(x)(\nabla_{x_i}K)^d{}_e(z,w)q^e_{bjc}(y)(\nabla_{y_j}C_\varepsilon)^{ac}(z,w). \tag{8.18}$$

All indices are contracted tensorially, so these formulas define global bundle kernels. Define $M_{p,q,\varepsilon}$ from $B_{p,q,\varepsilon}$ by (8.10), with the same fixed cutoff and transport for every occurrence of this contraction.

Theorem 8.6 (The renormalised derivative response)**.** *Let $0 < \kappa < 1/16$. The distributions*

$$Z_{p,q,\varepsilon} = p(X_\varepsilon,\nabla Y_{q,\varepsilon}) - M_{p,q,\varepsilon}X_\varepsilon \tag{8.19}$$

converge, jointly with (8.8), in $L^p(\mathcal{C}^{-3\kappa}_{\mathfrak{s}})$ on compact cylinders, for all finite p. Their limits are independent of (8.2). The counterterm is multiplication by a smooth endomorphism, and

$$\|M_{p,q,\varepsilon}\|_{C^m(\Sigma)} \le C_m(1+\log(1/\varepsilon)), \qquad 0 < \varepsilon \le 1/2. \tag{8.20}$$

Changing the fixed transport or cutoff changes the limiting product by a finite smooth endomorphism times X.

Proof. Wick's formula, remembering the centring already present in Y_q, gives the exact decomposition

$$p(X_\varepsilon,\nabla Y_{q,\varepsilon}) = T_{p,q,\varepsilon} + A_{p,q,\varepsilon}\nabla X_\varepsilon + B_{p,q,\varepsilon}X_\varepsilon. \tag{8.21}$$

The first contraction pairs the outer X with the undifferentiated inner X; the second pairs it with the differentiated inner X. There are no other contractions, since the inner pair has already been removed by (8.6).

Lemma 8.1 gives all the hypotheses of Lemma 8.4: $A_{p,q,\varepsilon}$ has order 3 with a logarithm, and $B_{p,q,\varepsilon}$ has order 4. For fixed ε, the latter is integrable at the diagonal, because ∇C_ε is spatially smooth and ∇K is locally integrable. The same lemma therefore gives

$$A_{p,q,\varepsilon}\nabla X_\varepsilon \longrightarrow A_{p,q}\nabla X, \qquad (B_{p,q,\varepsilon} - M_{p,q,\varepsilon})X_\varepsilon \longrightarrow R_{p,q}X$$

in any negative regularity, after choosing sufficiently small regularity losses for X and for the operators. Localisation of the infinite past is justified by the exponential tail estimates as before. Combine this with Lemma 8.5 in (8.21).

For (8.20), use the sharper bound $|B_{p,q,\varepsilon}(z,w)| \lesssim \rho(z,w)^{-3}(\rho(z,w)+\sqrt{\varepsilon})^{-1}$. Its integral over $\rho \le 1$ is at most $C\int_0^1 (r+\sqrt{\varepsilon})^{-1}\,\mathrm{d}r$. Centre derivatives satisfy the same estimate. Finally, the difference between two choices of cutoff and transport vanishes to positive order at the diagonal. Its contraction against the limiting order-four kernel is therefore integrable, with all centre derivatives. This proves the final assertion. □

Remark 8.7. The limit in Theorem 8.6 contains the finite first-chaos terms $A_{p,q}\nabla X + R_{p,q}X$. Deleting the entire first chaos would produce a different, generally nonlocal modification of the equation. The counterterm used here is local in the evolving field. No assertion is made at this stage that its finite part is a finite-jet function of the background geometry, or that it is the finite part selected by gauge covariance of a nonlinear equation.

8.6 Solution jets and compatible two-noise products

Let v be a smooth, possibly time-dependent section of the pullback of E to $\mathbb{R}\times\Sigma$, with bounded derivatives, and set

$$\Psi_{q,v,\varepsilon} = K\{q(v,\nabla X_\varepsilon) + q(X_\varepsilon,\nabla v)\}. \tag{8.22}$$

Either of the two terms can be retained separately. In particular the first term gives the symbol $\mathcal{I}(\mathbf{1}\,\nabla X)$ when v realises a degree-zero solution jet. For jet families on a fixed working cylinder, extend the local jet realisations to the past using fixed smooth cutoffs. Their mixed parameter and spacetime derivatives are then bounded uniformly on compact parameter sets. The massive kernel makes the corresponding tail integrals finite.

Proposition 8.8 (The jet response uses the same counterterm)**.** *For $0<\kappa<1/16$,*

$$\begin{aligned} \Psi_{q,v,\varepsilon} &\longrightarrow \Psi_{q,v} && \text{in } \mathcal{C}^{1-\kappa}_{\mathfrak{s}},\\ U_{p,q,v,\varepsilon} := p(X_\varepsilon,\nabla\Psi_{q,v,\varepsilon}) - M_{p,q,\varepsilon}v &\longrightarrow U_{p,q,v} && \text{in } \mathcal{C}^{-2\kappa}_{\mathfrak{s}}. \end{aligned} \tag{8.23}$$

The conclusions hold in every finite moment, independently of (8.2), and locally uniformly with all parameter derivatives for smooth finite-dimensional families v_λ whose mixed parameter and spacetime derivatives are uniformly bounded on $K\times\mathbb{R}\times\Sigma$ for every compact parameter set K.

Proof. Schauder estimates give the first assertion. More explicitly, the noise kernel of $\nabla\Psi_{q,v}$ is a convolution of an order-three kernel with an order-three kernel, plus a smoother term. The parabolic convolution bound makes it an order-two kernel, with smooth massive tails. Its covariance with itself or with X is therefore bounded by $C(1+\log_+\rho^{-1})$. Wick's formula shows that the centred product

$$p(X_\varepsilon,\nabla\Psi_{q,v,\varepsilon}) - \mathbb{E}p(X_\varepsilon,\nabla\Psi_{q,v,\varepsilon})$$

converges in every strictly negative parabolic regularity, by the same mixed-covariance argument as for W_2.

The contraction formulas identify its expectation as

$$\mathbb{E}p(X_\varepsilon,\nabla\Psi_{q,v,\varepsilon}) = A_{p,q,\varepsilon}\nabla v + B_{p,q,\varepsilon}v. \tag{8.24}$$

Subtracting the same $M_{p,q,\varepsilon}v$ as in (8.19) leaves a quantity converging to $A_{p,q}\nabla v + R_{p,q}v$ by Lemma 8.4. This proves (8.23), including compatibility of the counterterms under replacement

of an inner noise leaf by a solution jet. Parameter derivatives act only on v. The assumed bounds on mixed derivatives make the local estimates and the exponentially decaying past-tail bounds uniform on compact parameter sets. Applying these estimates at arbitrarily high parameter derivative orders, followed by local Sobolev embedding in the parameters, proves the final assertion. □

8.7 Derivative placement and pointed bounds

Products with the derivative on the rough factor are defined by the covariant Leibniz rule. Suppress tensor contractions for a moment. Since $X \in \mathcal{C}_{\mathfrak{s}}^{-\kappa}$ and $Y_q \in \mathcal{C}_{\mathfrak{s}}^{1-2\kappa}$, the ordinary product $X \otimes Y_q$ exists. Define

$$((\nabla X) \otimes Y_q)^{\mathrm{ren}} = \nabla(X \otimes Y_q) - (X \otimes \nabla Y_q)^{\mathrm{ren}}, \tag{8.25}$$

where the second term is constructed componentwise by Theorem 8.6. Smooth coefficient derivatives in a contracted Leibniz rule multiply the already defined product $X \otimes Y_q$. At cutoff level (8.25) adds the opposite local mass counterterm to the reversed derivative product. The two derivative placements therefore use one local renormalisation convention.

Proposition 8.9 (Pointed derivative estimates)**.** *The global distribution in* (8.25) *converges in* $\mathcal{C}_{\mathfrak{s}}^{-1-\kappa}$*, with an arbitrarily small additional loss if needed. For a base point* z*, however, put* $Y_{q,z}(w) = Y_q(w) - \tau_{y \leftarrow x} Y_q(z)$ *near* z*. Then the renormalised pointed product satisfies*

$$\left| \left\langle ((\nabla X) \otimes Y_{q,z})^{\mathrm{ren}}, \varphi_z^\lambda \right\rangle \right| \le C\lambda^{-3\kappa}, \qquad 0 < \lambda \le 1, \tag{8.26}$$

uniformly over bounded test families and base points in compact sets. The random constant has every finite moment. The pointed fields converge in this bound if the exponent is weakened by any positive amount. The analogous statement for $\Psi_{q,v}$ *has exponent* -2κ*.*

Proof. Continuity of multiplication $\mathcal{C}^{-\kappa} \times \mathcal{C}^{1-2\kappa} \to \mathcal{C}^{-\kappa}$ proves the global assertion from (8.25). We record the elementary local version of that product estimate: if $f \in \mathcal{C}^{-a}$, $g \in \mathcal{C}^b$, $0 < a < b < 1$, and $g(z) = 0$, then

$$|\langle fg, \varphi_z^\lambda \rangle| \le C\|f\|_{\mathcal{C}^{-a}} \|g\|_{\mathcal{C}^b} \lambda^{b-a}.$$

For blocks below scale λ this uses $\sup_{B(z,\lambda)} |g| \lesssim \lambda^b$. For finer blocks the resonant sum is

$$\sum_{2^{-n} \le \lambda} 2^{-n(b-a)} \lesssim \lambda^{b-a}.$$

The other two paraproduct sums obey the same bound. This also proves the estimate with a derivative on the test function, with one power of λ lost.

Apply this with $a = \kappa$, $b = 1 - 2\kappa$ and $g = Y_{q,z}$. The derivative of $X \otimes Y_{q,z}$ has the bound $\lambda^{-3\kappa}$. The other term in the Leibniz identity is $X \otimes \nabla Y_q$, already in $\mathcal{C}_{\mathfrak{s}}^{-3\kappa}$, together with the smooth transport correction. Parallel transport has vanishing covariant first derivative on the diagonal, so this correction has a better bound; mere smooth boundedness would also suffice for the negative target exponent. This proves (8.26). Difference estimates follow from the same bilinear bounds and the convergence already proved. Replacing $1 - 2\kappa$ by $1 - \kappa$ and using Proposition 8.8 proves the jet case. □

8.8 The resulting finite analytic data and its scope

For a finite list of the tensor contractions in the Yang–Mills quadratic and cubic terms, the preceding results give a joint, cutoff-independent enhancement consisting of

$$(X, W_2, Q_q, Y_q, W_3, Z_{p,q}, \Psi_{q,v}, U_{p,q,v}), \tag{8.27}$$

with the derivative placements supplied by (8.25) and the pointed bounds (8.26). The smooth jet families v represent the coefficient multiplying ∇X in the singular fixed point and are therefore included in the enhancement. All estimates use finitely many coefficient and geometric seminorms, and are uniform on bounded smooth families of these data.

For $E = T^*\Sigma \otimes \operatorname{ad} P$ and $L = d_B^* d_B + d_B d_B^*$, the tensor contractions are global on any principal bundle. No topological trivialisation of P and no coexact decomposition is used. A simultaneous bundle isometry of A, ∇, p, q, the noise and the chosen transport sends every formula above to the corresponding formula for the transformed data. This gives joint gauge naturality under transformation of the reference connection. It does not identify the laws produced by two different references held fixed in two nonlinear equations.

Two separate mathematical steps connect (8.27) with dynamics. Theorem 8.11 embeds these products and their jet replacements into an admissible bundle model on the equation's working sectors, including its reexpansion precision defects and the identities for both heat kernel types. Together with the singular fixed point and the affine counterterm calculation, this gives the fixed-reference continuum equation in Theorem 8.14. Sections 11–12 identify its finite normalisation under a predictable nonlinear gauge and a change of reference. That argument uses the equation-generated enhancement after Yang–Mills contraction; centering the third chaos alone does not give this comparison.

This distinction is also necessary when invoking general theorems. The analytic BPHZ theorem of Chandra–Hairer [6] uses convolution kernels and spatially stationary noise in its stated hypotheses (Definition 2.7 and Definition 2.17 of their paper). The Hairer–Steele formulation [17] likewise assumes translation-stationary random models in Definition 2.27 of arXiv:2301.10081v3. Hairer–Singh [16] develop the bundle algebra and deterministic analysis, but explicitly describe a general stochastic convergence theorem of that scope as a desired extension before their equation-specific stochastic estimates in Section 16.2.1. The estimates above prove convergence of the listed enhanced distributions directly on the bundle.

Finally, the stationary choice is deliberate. A convolution started from zero has covariance

$$\tfrac{1}{2} A^{-1} (e^{-|t-s|A} - e^{-(t+s)A}).$$

On every time interval bounded away from zero, the second term is smooth and all arguments above apply. The initial surface is handled by the stationary decomposition $X(t) + e^{-tA}(a_0 - X(0))$ and the weighted bounds in Theorems 5.8 and 8.14; interior enhancement estimates alone would not give that boundary statement.

8.9 Assembly of the model on the equation's working sectors

The local equation uses a restricted model whose retained degrees are distinct from the jet precision. We retain the solution sector below $\gamma = 3/2$, the nonlinear forcing sector below $\bar{\gamma} = 1/2 - \kappa$, their reexpansion descendants and jet decorations, and the planted integrations of those forcing symbols. We use separate planted kernel types K and ∇K of abstract orders β and $\beta - 1$, as in Section 14 of [16]. Their solution and derivative components are retained below γ and $\gamma - 1$, respectively. No exact abstract differentiation map on curved jet realisations is assumed. The planted outputs above those thresholds are not fed back into new

nonlinear products. This is the same distinction between an equation's working sectors and an unnecessarily large model that appears in [16], Proposition 16.10.

Use the nonresonant abstract heat order $\beta = 2 - \vartheta$, where $0 < \vartheta < \kappa < 1/16$, and assign $|\Xi| = -\beta - \kappa$. The actual kernel still gains two analytic orders. Choose ϑ outside the finitely many resonances of the retained degrees. A sufficiently large nonintegral jet precision, for example $D = 15/2$ with an arbitrarily small generic perturbation if necessary, is used in the construction. We can then restrict to a common working precision greater than 4. The precision of a product is understood recursively, as in Section 12.6 of [16]; it is not identified with the precision of a single jet.

The noise is intrinsic and additive. Its abstract generator is a trivial scalar line, realised as the E-valued distribution Ξ, with $\Pi_p \Xi = \Xi$ and $\Gamma_{q,p} \Xi = \Xi$. The rule contains the source Ξ itself, without an arbitrary coefficient-jet edge at that node; the noise is outside the nonlinear map. Normalisation of this additive rule does not add such an edge. Both $K\Xi$ and $\nabla K\Xi$ have negative working degree, no J or E correction, and a basepoint-independent realisation. This convention does not claim a model theorem for an arbitrary multiplicative noise coefficient.

Tensor response families. All constructions in Section 8 are polarised before contracting tensor indices. On each member of a finite trivialising cover choose a frame, extend its elements smoothly using a cutoff, and construct the response and its enhanced product for each such element. Scalar coefficients of a general smooth section can then be pulled through a kernel by a commutator. This gives a finite list of response tensors. It does not require a global frame of E, or commutation of a matrix-valued coefficient with the bundle heat kernel.

The following commutator estimate controls the reexpansion precision defects.

Lemma 8.10 (Jet commutators and their pointed remainder)**.** *For $\sigma \in \{1, 2\}$ put $T_1 = \nabla K$ and $T_2 = K$, and let $f \in \mathcal{C}_{\mathfrak{s}}^{-1-a}$, where $0 < a < 1$. Localise all kernels to a fixed compact cylinder; smooth tails can be added separately. For a smooth scalar coefficient v, the commutator*

$$[T_\sigma, v] f(z) = \int T_\sigma(z, w)(v(w) - v(z)) f(w)\, \mathrm{d}w$$

is interpreted by the kernel acting on the distribution f and belongs to $\mathcal{C}_{\mathfrak{s}}^{\sigma-a}$. Its norm is bounded by finitely many smooth norms of v times $\|f\|_{\mathcal{C}^{-1-a}}$.

Suppose r has zero parabolic jets of orders less than the positive integer N at q. Then $[T_\sigma, r]f$ has pointed Taylor coefficients of integer orders less than $N + \sigma - 1$ at q, defined by convergent weighted kernel integrals and, when $\sigma = 2$, the value $(Kf)(q)$. Write

$$h_\sigma = [T_\sigma, r] f - \mathcal{T}_q^{<N+\sigma-1} [T_\sigma, r] f.$$

Let $\|\cdot\|_{\sigma-a;q,\lambda}$ be the usual $\mathcal{C}_{\mathfrak{s}}^{\sigma-a}$ norm after rescaling $B_{\mathfrak{s}}(q, \lambda)$ to a unit ball. Thus, for $\sigma = 1$, it is equivalent to $\|h\|_\infty + \lambda^{1-a}[h]_{\mathcal{C}^{1-a}}$; for $\sigma = 2$ it also includes $\lambda\|\nabla h\|_\infty$ and the corresponding top Hölder seminorm. Then

$$\|h_\sigma\|_{\sigma-a;q,\lambda} \le C\lambda^{N+\sigma-1-a} \|f\|_{\mathcal{C}^{-1-a}}, \qquad 0 < \lambda \le 1. \tag{8.28}$$

The constant is uniform for bounded smooth families of r. This statement concerns pointed Taylor coefficients; it does not assert higher classical regularity in a neighbourhood of q.

If $G_\sigma \in \mathcal{C}_{\mathfrak{s}}^{1-\sigma-b}$ with $a + b < 1$, then $G_\sigma h_\sigma$ is a classical product and

$$|\langle G_\sigma h_\sigma, \varphi_q^\lambda \rangle| \le C\lambda^{N-a-b} \|G_\sigma\|_{\mathcal{C}^{1-\sigma-b}} \|f\|_{\mathcal{C}^{-1-a}}. \tag{8.29}$$

All statements are locally Lipschitz in the displayed data.

Proof. The difference $v(w)-v(z)$ improves the kernel of T_σ by one order. Its centre derivatives obey the same improvement. The resulting kernel gains $\sigma+1$ orders and maps $\mathcal{C}^{-1-a}$ to $\mathcal{C}^{\sigma-a}$ by the dyadic Schauder estimate. This proves the first assertion.

We detail the stronger pointed statement. At an annulus of radius R centred at q, differentiating the kernel to parabolic order $j<N+\sigma-1$ and multiplying by $r(w)$ gives, when paired with f, a bound $CR^{N+\sigma-1-j-a}$. Its sum as $R\downarrow 0$ converges. In differentiating $rT_\sigma f$, all terms vanish for $\sigma=1$. For $\sigma=2$ the only additional terms, at order N, are derivatives of r of that order times the well-defined continuous value $(Kf)(q)$. These formulas define the claimed Taylor coefficients.

For $z\in B_{\mathfrak{s}}(q,\lambda)$, split the kernel at radius 2λ. In the near part, $|r(w)-r(z)|\lesssim\lambda^{N-1}\rho(z,w)$, and the order-$(\sigma+1)$ Schauder estimate, rescaled to radius λ, gives the right side of (8.28). In the far part subtract the Taylor polynomial in z through order $N+\sigma-2$. An annulus of radius $R\geq\lambda$ contributes at most $C\lambda^{N+\sigma-1}R^{-a}$; thus

$$\lambda^{N+\sigma-1}\sum_{R=2^{-j}\geq\lambda}R^{-a}\lesssim\lambda^{N+\sigma-1-a}.$$

Differentiating on that far part and interpolating to order $\sigma-a$ gives its scaled norm in (8.28). The same bounds apply to the subtracted near-scale Taylor coefficients. A formulation using parabolic Taylor monomials gives the same calculation: the ℓth term has kernel $(w-z)^\ell T_\sigma(z,w)$, gains $\sigma+|\ell|_{\mathfrak{s}}$ orders, and is multiplied by a derivative of r vanishing to order $N-|\ell|_{\mathfrak{s}}$. The Taylor-remainder kernel gains $N+\sigma$ orders. These descriptions also show that smooth bundle transport and coordinate corrections have the same or better estimates.

The local paraproduct estimate gives

$$|\langle G_\sigma h_\sigma,\varphi_q^\lambda\rangle|\leq C\lambda^{1-\sigma-b}\|G_\sigma\|_{\mathcal{C}^{1-\sigma-b}}\|h_\sigma\|_{\sigma-a;q,\lambda}.$$

Insert (8.28) to get (8.29). Applying the same linear and bilinear estimates to differences proves the final assertion. □

Why the lemma covers an inner jet. For a fixed first-order response $\Psi=Kf$ and its enhanced product $U=(X\nabla\Psi)^{\mathrm{ren}}$, use the same local counterterm for the scalar coefficient v. At each cutoff, and therefore in the limit,

$$(X\nabla K(vf))^{\mathrm{ren}}=vU+X[\nabla K,v]f. \tag{8.30}$$

The last product is classical. For the Gaussian jet response $f=DX$, take $a=\kappa$, $b=\kappa$; for an inner variation of $Y=KQ$, take $f=Q$, $a=2\kappa$, $b=\kappa$. Thus a jet remainder vanishing to order N produces respective precision errors $N-2\kappa$ and $N-3\kappa$, after the corresponding far-response Taylor jets are included in the reexpansion. For the reversed placement write $X_1=\nabla X$. The exact coefficient identity is instead

$$(X_1K(vf))^{\mathrm{ren}}=v(X_1Kf)^{\mathrm{ren}}+X_1[K,v]f. \tag{8.31}$$

The last product is classical by the $\sigma=2$ case of the lemma: its exponents are $-1-b$ and $2-a$, whose sum is $1-a-b>0$. This proves the reverse-placement precision estimate without differentiating an abstract reexpansion map.

One additional Taylor coefficient is essential in this case. With $Y=Kf$, the identity

$$\mathcal{T}_q^{<N+1}K(rf)=\mathcal{T}_q^{<N+1}[K,r]f+\mathcal{T}_q^{<N+1}(r\,\tau_{\cdot\leftarrow q}Y(q)) \tag{8.32}$$

follows because $r(Y-\tau Y(q))$ has pointed order $N+1-a>N$. The last term can contain a degree-N polynomial. After its subtraction, the first term in (8.31) uses the pointed product

$X_1(Y - \tau Y(q))$, whose bound is λ^{-a-b} by Proposition 8.9. Multiplication by r and the $\sigma = 2$ commutator remainder therefore both give λ^{N-a-b}. The top polynomial in (8.32) must be retained whenever the prescribed jet truncation includes degree N.

In a frame, derivatives of the coefficient or of the frame add zeroth-order sources to f. Their heat responses gain an extra order and their products with X are classical. These additional sources extend the scalar commutator identities to tensor coefficients. Root coefficient remainders multiply X, W_2, Q, W_3 or Z directly, giving the corresponding pointed power by multiplication by a smooth function vanishing to order N.

For completeness, moving a smooth scalar multiplier through a noise convolution has the analytic identity

$$K(r\Xi) = rX - K\big([\partial_t + A, r]X\big). \tag{8.33}$$

For a scalar coefficient, or componentwise in a finite tensor frame, $[\partial_t + A, r]$ has spatial order at most one. Its coefficients contain at most two derivatives of r. Formula (8.33) follows by applying $\partial_t + A$; the massive past convolution, or a compact localisation with a smooth boundary term, fixes the inverse. This identity is useful for localisation but does not introduce independent weighted-noise generators into our rule. If it is used inside a stochastic product, all induced Wick and local contractions must be retained; the zero-order subtraction table below is not a prescription for arbitrary multiplicative-noise equations. Bundle chart changes do not create this issue: the transformation of the noise at the inner endpoint cancels the inverse transformation of the bundle kernel at that endpoint. Localisation cutoffs may be kept in the test functions and kernels.

A coefficient-compatible subtraction convention. When a nonlinear contraction is regarded as a variable tensor coefficient, its value inside (8.10) is transported from the root before defining the local subtraction tensor. This is the coefficient-frozen convention. Its difference from the convention with the actual inner coefficient at y has a factor $O(d(x, y))$ against an order-four kernel. The difference therefore has a cutoff-independent limit in every C^m norm. Replacing the corresponding finite mass term in Theorem 8.6 implements this convention without changing any estimate. With it, multiplication of an inner coefficient by v multiplies its local subtraction by $v(z)$, so (8.30) is an exact identity. For the parallel bracket and metric contractions of Yang–Mills the two conventions agree under the compatible transports.

Theorem 8.11 (A convergent admissible model on the working sectors)**.** *For the intrinsic additive-noise rule, the massive bundle Laplacian and the heat cutoffs of Section 8, the finite analytic enhancement, polarised over the tensor response families and with the coefficient-compatible local subtractions just described, generates admissible typed bundle models on the working sectors specified above. These restricted models converge locally in every finite moment in the model norms required by the singular fixed point. Their limits are independent of the heat-mixture cutoffs* (8.2)*.*

The assertion includes the bundle precision defect $\Pi_q \Gamma_{q,p} - \Pi_p$, not an exact flat-space cocycle identity. The two kernel types have their own canonical jet subtractions. In particular this theorem does not assert an abstract operator $\mathcal{D}$ commuting exactly with Γ and satisfying $\Pi\mathcal{D} = \nabla\Pi$; this extra identity is unnecessary in the coupled formulation of Section 14 of [16]. It asserts no convergence of unused higher nonlinear products introduced only by expanding the entire rule to the jet precision.

Proof. We specify the algebra and the estimates separately.

The finite list. Besides jets, the solution sector below $3/2$ contains X, Y and Ψ_v. At forcing precision $1/2 - \kappa$, the only products requiring stochastic construction are, up to tensor types,

differentiation placement and jet multiplication,

$$X^{\otimes 2}, \quad X \otimes \nabla X, \quad X^{\otimes 3}, \quad X \otimes \nabla Y, \quad (\nabla X) \otimes Y, \quad X \otimes \nabla \Psi_v, \quad (\nabla X) \otimes \Psi_v.$$

The single-factor terms are X, ∇X, ∇Y, $\nabla \Psi_v$, and smooth jets. A further nontrivial nonlinear generation has forcing degree at least $1 - 4\kappa - 2\vartheta > 1/2 - \kappa$. Jet decorations and reexpansion descendants of the displayed shapes are retained up to the chosen jet precision. They do not introduce a different nonlinear shape.

Smooth models and the forest prescription. Temporarily smooth also in time. On the complete smooth typed rule, prescribe the connected negative counterterms by their realised insertion in the forest formula. Tensor indices are left free until the nonlinear contraction is applied. The insertion table is

contracted component	inserted local operator
$X \otimes X$	$-P_\varepsilon$
$X \otimes \nabla X$	$-\mathbb{E}[X_\varepsilon \otimes \nabla X_\varepsilon]$
$X \otimes \nabla K(\mathbf{1} \nabla X)$	$-M_\varepsilon$
$(\nabla X) \otimes K(\mathbf{1} \nabla X)$	$+M_\varepsilon$ with the corresponding tensor ordering.

The table specifies the realised counterterm signs. Use the identification of each contracted component with its local differential operator in (13.5) of [16] to choose $g_\varepsilon(T, 0)$ with the displayed insertion. This identification is made with free tensor indices and arbitrary smooth sections in all surviving slots.

On every fully contracted connected coloured tree set $g_\varepsilon(T, \widehat{T}) = \mathrm{id}$ as required by the unit prescription. On every other connected generator not listed above, including every proper partially contracted coloured tree, set its value to zero. Extend these connected values multiplicatively over disjoint forests, with the canonical tensor symmetrisation. Thus the prescription covers the coloured domain of the renormalisation group, not just its uncoloured trees. The inserted operators are smooth and of order zero and satisfy the defining forest multiplicativity in Section 13.4 of [16]. The global products with different derivative placements are prescribed by the covariant Leibniz identity. In particular the two derivative covariances add to ∇P_ε, and the two placements in (8.25) have opposite mass insertions. These identities fix the global counterterm convention; they are not imposed as an exact differentiation identity between basepoint-dependent jet subtractions.

The forest formula gives exactly the Wick quadratic and cubic fields, the centred Q inside Y, and (8.21) with only its local $M_\varepsilon X_\varepsilon$ removed. Replacing the surviving inner leaf by a solution jet gives (8.24) with the same $M_\varepsilon v$. The contraction with kernel A_ε is retained: its two-noise subtree has positive degree and its first-chaos contribution converges. There is no independent subtraction of that contribution. With three noise leaves no two overlapping pair contractions form a disjoint forest, which gives exactly the pair sums in these formulas and no additional term.

Propositions 13.26 and 13.40 of [16] now give admissible smooth bundle models. Their reexpansion is the construction of Section 13.6, with the integration correction J_p of Section 7. In particular its Taylor order on a symbol of degree α is below $\alpha + \beta$, as in Lemma 7.6. We use neither spatial translation invariance nor a stochastic BPHZ theorem in this step.

Uniform bounds and precision defects. Invoke the estimates of Section 8 with an auxiliary loss $0 < \rho < \kappa - \vartheta$, reduced further if needed for a finite collection of products. In particular, $X \in \mathcal{C}^{-\rho}$ and $\Xi \in \mathcal{C}^{-2-\rho}$ satisfy the working degrees with positive slack. The local bounds on the displayed negative products follow from that section; the use of $\beta < 2$ leaves further slack in the working degrees. Positive integrated symbols are centred by their own kernel type's prescribed jet correction. Differentiating the realisation of a K-symbol and realising the

corresponding ∇K-symbol can differ by a smooth jet correction, coming from derivatives of the jet transport. Such terms are smooth coefficients times the already constructed fields and their response tensors. No equality of these two procedures is used. The pointed estimates for their derivative products are Proposition 8.9. Multiplication by a jet of positive degree uses its vanished Taylor coefficients and gives the corresponding additional power of the testing scale.

The white-noise leaf itself also converges at its assigned degree. Indeed, L^2 boundedness of the spatial heat cutoff gives

$$\mathbb{E}|\langle \Xi_\varepsilon, \varphi_z^\lambda\rangle|^2 = \|S_\varepsilon^* \varphi_z^\lambda\|_{L^2}^2 \lesssim \lambda^{-4}.$$

Strong convergence of the cutoff gives convergence for fixed tests. The Gaussian wavelet estimate then gives convergence in $\mathcal{C}_{\mathfrak{s}}^{-2-\eta}$ for every $\eta > 0$. Taking $\eta = \rho < \kappa - \vartheta$ verifies the assigned noise degree $-\beta - \kappa$.

For the precision defect, a jet realisation at p minus its reexpansion at q has vanishing jets through the retained precision at q, uniformly on a compact set of (p, q). At a root it multiplies one of the already bounded distributions. At an inner heat vertex it is handled by Lemma 8.10 and (8.30). To identify the far-response correction, write the bundle formula (7.4) of [16]:

$$E_q \tau_p = \sum_n Q_{<(\delta+\beta)\wedge D}\, j_q K_n (\Pi_p - \Pi_q Q_{<D-\beta} \Gamma_{q,p}) \tau_p. \tag{8.34}$$

For an inner jet remainder the expression in parentheses is rf, together with already controlled lower terms. For the kernel $T_1 = \nabla K$, replace β by $\beta - 1$. The distribution $rT_1 f$ has pointed order $N - a > N - 1$, so its Taylor coefficients of integer orders less than N vanish. Those of $T_1(rf)$ are therefore the coefficients of $[T_1, r]f$ from Lemma 8.10. For the kernel K, the appropriate formula is (8.32), including its degree-N polynomial when that degree is retained. These are exactly the two E_q corrections in (8.34), with its stated sign. The additive noise leaf and its negative-degree planted responses have zero precision defect, as specified above. Root and inner nonlinear vertices are therefore all the required coefficient positions in this rule. Telescoping one coefficient position at a time proves the precision estimate for a product with several jet inputs.

The recursive rules give the following precision bounds. Let J denote a coefficient-jet factor, of precision D, and allow any of the retained jet degrees at each indicated coefficient slot. Set $Y_1 = \mathcal{I}_1 Q$ and $\Psi_1 = \mathcal{I}_1(JX_1)$, where $\mathcal{I}_1$ has order $\beta - 1$. The worst precisions for the displayed shapes, including their coefficient-jet slots and root decorations, are

shape	precision	shape	precision
JX	$D - \kappa$	JX_1	$D - 1 - \kappa$
JW_2	$D - 2\kappa$	JQ	$D - 1 - 2\kappa$
JW_3	$D - 3\kappa$	$Y,\ \Psi$	D
Y_1	$D - 2\kappa - \vartheta$	Ψ_1	$D - \kappa - \vartheta$
XY_1	$D - 3\kappa - \vartheta$	$X\Psi_1$	$D - 2\kappa - \vartheta$
$X_1 Y$	$D - 1 - \kappa$	$X_1\Psi$	$D - 1 - \kappa$.

(8.35)

The noise and its negative-degree planted responses have exact compatibility and precision $+\infty$. A product without any coefficient-jet slot can likewise have exact compatibility; the table records the least precision over the corresponding jet-decorated family. To check the table, use the product rule $\min(\delta_1 + \alpha_2, \delta_2 + \alpha_1)$ and the integration rule $\min(\delta + \beta_i, D)$, and include the contracted-pair terms in the minimum in Section 12.6 of [16]. For example, a critical pair has degree $-2\kappa - \vartheta$ and a surviving X costs another κ, giving $D - 3\kappa - \vartheta$ for XY_1. For $X_1 Y$, the cut replacing Y by its value instead gives $D - 1 - \kappa$, which is smaller and is the required precision.

These are the precisions actually estimated, not a replacement of the recursive grading by a single number. For $D = 15/2$, coefficient-jet reexpansion remainders vanish to order 8. The root estimates and the two commutator estimates therefore have strictly greater powers than the corresponding entries of (8.35), using the auxiliary loss $\rho < \kappa - \vartheta$. The input truncation $Q_{<D-\beta}$ for K can also discard a coefficient jet of degree 7 multiplying Q or X_1. For this term $N = 7$, and the output correction includes its degree-7 Taylor coefficient. Formula (8.32) removes precisely the additional $r\tau Y(q)$ term. The remaining reversed critical product has bound $\lambda^{7-3\rho}$ (or $\lambda^{7-2\rho}$), which is stronger than its required $\lambda^{D-1-\kappa}$ bound. For ∇K the input truncation is $Q_{<D-\beta+1}$; it retains all coefficient jets of degrees at most 7 in these sources. Higher-degree root products that are omitted by a truncation have their displayed homogeneity as a pointed gain and are controlled by the same estimates.

Terminal integrations use the integration precision rule and Theorem 7.19. All precisions in the table exceed $9/2$; the fixed point can consequently use a common lower working threshold without changing the model's assigned precisions. This establishes the realisation and precision-defect bounds in Definition 4.13 of [16]. The reexpansion norm is obtained in the following order; a bound on the individual far-response jets alone would not establish its required powers of the distance between the two basepoints.

The staged integration and reexpansion construction. Start with the jet model and the noise, then add X by Theorem 7.19 of [16]. Add the root products W_2, Q, W_3, with all retained jet decorations. Their reexpansions are induced by those of the factors, and their defects are the root-remainder estimates above. Apply the same theorem to construct Y after Q and the first-order response family Ψ_v after its one-noise forcing. Lemma 7.18 in the same paper supplies the cancellations among $J_p\Gamma - \Gamma J_q$ and E_q and thus the required distance powers in Γ for these integrated symbols. The formula (8.34) and the commutator estimate verify precisely its defect input, including positive jet decorations. No divergent derivative of a rough field is used as an ordinary pointwise Taylor coefficient.

Next add the critical products containing Y or Ψ_v. Their realisation bounds are the pointed estimates of Section 8, and their precision defects are the commutator estimates just proved. Their reexpansion is multiplicative, by Remark 13.33 of [16], so its distance powers follow from those of the already constructed factors and the recursive graded product. Finally apply Theorem 7.19 to the terminal planted integrations needed by the Schauder map. This establishes all three model bounds without using an integration extension before its input model is known.

Each such integration extension, for both orders β and $\beta - 1$, is locally Lipschitz. The generic choice of β and the nonintegral precisions verify its nonresonance hypotheses, including on pure jet inputs. The normal, equation-like completion of the smooth rule supplies the ambient spaces; only the restrictions just estimated are used in this construction. No higher nonlinear product is inferred merely from a positive formal homogeneity.

On this restricted collection, the construction is a finite composition of the locally Lipschitz operations proved above, jet reexpansion and the deterministic integration extension. Its model norm is bounded by a polynomial in the finite enhancement norms, and the corresponding two-model estimate holds. The L^p convergence of Section 8 therefore gives convergence in the restricted model norms. This is the deterministic argument used after the equation-specific estimates in the proof of Proposition 16.2 of [16], with the precision-defect estimates here supplied by the commutator calculation.

Finally remove the auxiliary time smoothing at fixed spatial ε. The field and every fixed spatial derivative then have time regularity C^a for $a < 1/2$ on compact intervals, with all finite moments. Every operation in the finite list consequently converges, as do its jet and integration corrections. This defines the spatially smoothed model used above, without calling a temporally white noise a smooth spacetime noise. □

Remark 8.12 (Local kernels and the full massive heat kernel). In applying the deterministic bundle theorems, split the full massive kernel into a singular kernel supported near the diagonal and a smooth remainder. The latter is lifted by smooth jets in the abstract convolution. Replacing a heat response by its singular-kernel response changes it by a smooth random section; its outer products with X or ∇X are classical. The corresponding counterterm difference is a finite smooth endomorphism. If the same split is made in the Gaussian field, the change is a smooth Gaussian field. Expanding the Wick polynomials and the derivative products then uses only smooth multiplication, the tensor response families Ψ_v, U_v, and their commutator identities. These identities hold pathwise also for a random smooth coefficient, since they were proved as identities continuous in its smooth norms. Thus the full-kernel convention in the enhancement and the local-kernel convention in the model construction describe equivalent finite data, with all finite corrections kept explicit.

Proposition 8.13 (The affine correction in the cutoff equation)**.** *For the prescribed forest subtractions, the equation reconstructed from a smooth cutoff model has the original DeTurck polynomial drift minus a smooth affine correction*

$$\mathfrak{C}_B^\varepsilon(a) = b_B^\varepsilon + M_B^\varepsilon a. \tag{8.36}$$

Its three contributions are the quadratic one-vertex expectation, the cubic pair contractions, and the two placements of the two-vertex critical contraction. The last coefficient acts on the full solution, not just on its Gaussian coordinate. No derivative of the evolving field is subtracted by this scheme. Adding a smooth affine map $\mathfrak{D}(a) = d + Na$ to the abstract drift leaves the subtraction (8.36) unchanged: its reconstructed classical drift is $F_B + \mathfrak{D} - \mathfrak{C}_B^\varepsilon$. In particular, taking $\mathfrak{D} = \mathfrak{C}_B^\varepsilon$ represents the uncorrected classical approximation in the same renormalised model. The Leibniz prescription for the reversed derivative placement differs from its direct zeroth-moment subtraction by a coefficient converging in every smooth norm. Hence the convergence and cancellation of the total coefficient can be tested with either of these two conventions.

Proof. Apply Proposition 14.9 of [16] to the smooth model, at sufficiently large precision for its algebraic identification. Equivalently one may read its elementary differentials directly from the preceding forest prescription. A pair contraction at the quadratic vertex consumes both field inputs and gives b_B^ε. A pair contraction at the cubic vertex leaves one input and gives a linear endomorphism. At a two-vertex critical pair contraction the surviving input is a degree-zero solution slot. The equality of the mass coefficient in (8.19) and (8.23) makes its elementary differential $M_B^\varepsilon a$. The reversed derivative placement uses the corresponding transposed contraction dictated by Leibniz. Complete Gaussian contractions of an odd number of noise leaves vanish. Every potential contraction with four or more noise leaves has positive working degree in this rule, and a derivative on the surviving input belongs to the convergent $A_\varepsilon \nabla a$ contraction, which has not been removed. This proves both the affine form and the exhaustive list.

The assertion about adding $\mathfrak{D}$ also follows from this list, and is needed when identifying the uncorrected approximation. The root pair at the quadratic vertex and the pair at the cubic vertex use second derivatives of the nonlinearity, which an affine addition does not change. At a critical two-vertex contraction the inner contracted input is the differentiated field; differentiation of the nonlinearity in this input gives $q_B(a, \cdot)$, also unchanged by $\mathfrak{D}$. A contraction using the added unary vertex instead has one fewer singular derivative and degree at least $1 - 2\kappa - \vartheta > 0$; its assigned counterterm is zero. A constant addition supplies no contracted noise input. Thus the forest correction is identical, rather than merely having the same divergent part.

To compare the two reversed-derivative conventions, add the two derivative placements before contracting. Their sum is the covariant derivative of the corresponding undifferentiated pair response. The latter contraction has local kernel of order at most $\rho^{-3}(1+|\log\rho|)$. Its centre derivatives obey the same order, so its coefficient and all those derivatives converge by dominated convergence. Transport and coefficient derivatives add only terms of this integrable order. This proves the claimed finite smooth difference.

The large precision used for this smooth algebraic identity does not assert uniform stochastic bounds at that precision. At each fixed spatial cutoff, remove the auxiliary time smoothing. The classical spatially smooth equations and their finite enhanced models converge, preserving the equation identity. Only the restricted model convergence of Theorem 8.11 is used when the spatial cutoff is subsequently removed. □

8.10 The fixed-reference local continuum equation

We combine the model construction, the explicit contraction, and the cancellation of divergent coefficients. The approximation in this subsection is the additive, fixed-reference heat scheme. Write ε for its heat-time parameter and put $\mathcal{S}_\varepsilon = e^{-\varepsilon L_+}$. For a real exponent r, let $\mathrm{c}^r(E)$ denote the closure of smooth sections in $\mathcal{C}^r(E)$.

Theorem 8.14 (Local convergence of the uncorrected heat approximation)**.** *Let B be any smooth connection on an arbitrary principal bundle P over the closed surface. Fix $0<\kappa<1/16$ and $a_0\in \mathrm{c}^{-\kappa}(E)$, and set $a_0^\varepsilon=\mathcal{S}_\varepsilon a_0$. Consider the classical local equations with unit noise amplitude*

$$\begin{aligned} da^\varepsilon &= \{-L_B a^\varepsilon + f_B + V_B a^\varepsilon + Q_B(a^\varepsilon,a^\varepsilon) + C(a^\varepsilon,a^\varepsilon,a^\varepsilon)\}\,dt + \mathcal{S}_\varepsilon\,dW_t,\\ a^\varepsilon(0) &= a_0^\varepsilon. \end{aligned} \tag{8.37}$$

There is a unique local limit in the solution class of Theorem 5.8. It has continuous paths in $\mathrm{c}^{-\kappa}(E)$ and initial value a_0. The convergence is in probability in $C([0,T],\mathrm{c}^{-\kappa})$ on common local intervals of existence. More precisely, there are data sizes D, D_ε with uniformly bounded moments of every finite order, and deterministic times $T_R>0$, decreasing in R, such that, for $\Omega_R^\varepsilon=\{D\le R,\ D_\varepsilon\le R\}$, both solutions exist on $[0,T_R]$ and

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_R^\varepsilon}\|a^\varepsilon-a\|^p_{C([0,T_R],\mathrm{c}^{-\kappa})}\Big]\longrightarrow 0,\qquad 1\le p<\infty. \tag{8.38}$$

The norm inside the indicator is evaluated only on this common existence event. The limiting-data events $\{D\le R\}$ increase to the whole probability space and supply a positive local lifetime. Moreover, for every finite $q\ge 1$, $\sup_{0<\varepsilon\le\varepsilon_0}\mathbb{P}((\Omega_R^\varepsilon)^c)\le C_qR^{-q}$. The solution map is locally Lipschitz in the initial and enhanced data in the norms of Theorem 5.8.

For the subtraction convention of Proposition 8.13, there are smooth limits

$$b_B^\varepsilon\longrightarrow b_B,\qquad M_B^\varepsilon\longrightarrow M_B\quad \text{in every } C^j \text{ norm.} \tag{8.39}$$

If $\widehat{Z}_B$ is its limiting model, the limit of (8.37) is the reconstruction of the first component of the paired equation

$$\begin{aligned} U &= v + \mathscr{K}_{\widehat{Z}_B}\mathbf{1}_+\{\mathcal{F}_B(U,U_1)+\mathbf{b}_B+M_BU\},\\ U_1 &= v_1 + \mathscr{K}_{1,\widehat{Z}_B}\mathbf{1}_+\{\mathcal{F}_B(U,U_1)+\mathbf{b}_B+M_BU\}. \end{aligned} \tag{8.40}$$

Here the linear data satisfy $\mathcal{R}v_1=\nabla_B\mathcal{R}v$; the full-kernel identities then give $\mathcal{R}U_1=\nabla_B\mathcal{R}U$. The finite affine terms in this representation compensate the chosen model subtractions. They are not extra terms added to (8.37).

Proof. The initial approximants are smooth, and the noise is spatially smooth at each fixed ε. The classical local existence result in Section 5 therefore applies to (8.37). Moving the mass between the two sides writes its linear part as $\partial_t + L_+$ without changing the equation.

Theorem 8.11 gives the convergent restricted models $\widehat{Z}_B^\varepsilon \to \widehat{Z}_B$. The affine equation identity of Proposition 8.13 says that using $\mathcal{F}_B$ alone in the renormalised model subtracts $\mathfrak{C}_B^\varepsilon(a) = b_B^\varepsilon + M_B^\varepsilon a$ from the classical polynomial drift. Consequently the uncorrected classical equation is represented by the same model with $\mathcal{F}_B(U, U_1) + \mathbf{b}_B^\varepsilon + M_B^\varepsilon U$ as the common forcing in both kernel types. This identity concerns the full solution coefficient in the two-vertex diagrams; replacing it by the Gaussian coordinate alone would not produce the classical equation.

The coefficient list in that proposition is precisely the list estimated in Proposition 6.6. The quadratic expectation has a smooth finite limit. The three linear logarithmic contributions have coefficients $-1/4$, $-3/4$ and $+1$ and cancel in their sum. Each residual coefficient converges smoothly by the integrable heat-parametrix bound. This proves (8.39). In particular the extra affine maps and their Lipschitz constants are bounded uniformly as $\varepsilon \downarrow 0$.

Extend the driving Wiener process to negative times and use its massive stationary convolution. With $S_t = e^{-tL_+}$, the common noise coupling gives $X^\varepsilon = \mathcal{S}_\varepsilon X$ and the exact identity

$$X^\varepsilon(t) - S_t X^\varepsilon(0) = \int_0^t S_{t-r}\mathcal{S}_\varepsilon \, dW_r. \tag{8.41}$$

Thus the stationary decomposition

$$\begin{aligned} v^\varepsilon &= \mathbf{X}^\varepsilon + \mathscr{S}(a_0^\varepsilon - X^\varepsilon(0)), \\ v_1^\varepsilon &= \mathbf{X}_1^\varepsilon + \mathscr{S}_1(a_0^\varepsilon - X^\varepsilon(0)) \end{aligned}$$

has precisely the linear solution of the stated initial-value problem and its covariant derivative as its reconstructions. The derivative is realised by the output-derivative kernel; no commutation of ∇_B with S_t is used. The auxiliary past does not alter the cutoff equation.

For $0 < \kappa' < \kappa$, the Gaussian spectral argument of Proposition 5.3, applied to the stationary covariance, gives $X \in L^p(\Omega; C([0,1], \mathrm{c}^{-\kappa'}))$ for every finite p. The heat estimate consequently gives

$$\|X^\varepsilon - X\|_{L^p(C_1\mathrm{c}^{-\kappa})} \lesssim \varepsilon^{(\kappa-\kappa')/2}\|X\|_{L^p(C_1\mathcal{C}^{-\kappa'})}.$$

The heat semigroup is strongly continuous on the closure of smooth sections: its uniform boundedness and its convergence on smooth sections imply $a_0^\varepsilon \to a_0$ in $\mathrm{c}^{-\kappa}$.

Here convergence of the abstract linear lifts also uses their canonical construction, not just convergence of their reconstructions. The degree-$-\kappa$ and degree-$-1-\kappa$ noise-integration symbols have coefficient one; the smooth jet corrections that replace the local kernel by the full massive kernel converge in every required smooth norm, as in Remark 8.12. Together with model convergence this proves $\|\mathbf{X}^\varepsilon; \mathbf{X}\|_{\gamma,\eta;1} + \|\mathbf{X}_1^\varepsilon; \mathbf{X}_1\|_{\gamma-1,\eta-1;1} \to 0$ in every finite moment. The pure-jet heat-lift estimate in Theorem 5.8 yields, on bounded model sets,

$$\begin{aligned} \|(v^\varepsilon, v_1^\varepsilon); (v, v_1)\|_{\mathcal{X};1} \lesssim_R \|\mathbf{X}^\varepsilon; \mathbf{X}\|_{\gamma,\eta;1} \\ + \|\mathbf{X}_1^\varepsilon; \mathbf{X}_1\|_{\gamma-1,\eta-1;1} + \|a_0^\varepsilon - a_0\|_{\mathcal{C}^{-\kappa}} \\ + \|X^\varepsilon(0) - X(0)\|_{\mathcal{C}^{-\kappa}}. \end{aligned}$$

The margins $\eta = -2\kappa < -\kappa$ and $\eta - 1 < -1 - \kappa$ make this bound uniform down to time zero. This verifies the required two-model convergence of the prescribed linear data.

Fix a compact model cylinder containing $[0,1] \times \Sigma$. Let D be one plus the limiting model norm, the $\mathcal{X}$ norm of (v, v_1) on this cylinder, the Gaussian time-trace norm, and the finitely many smooth coefficient norms used by the contraction; define D_ε from the approximating

data in the same way. The initial norm is included in these sizes. The preceding estimates and model construction give

$$\mathbb{E}D^q + \sup_{0<\varepsilon\le\varepsilon_0} \mathbb{E}D_\varepsilon^q < \infty \qquad (q < \infty).$$

On Ω_R^ε, the contraction estimate (5.19) holds uniformly in the cutoff and in the limiting model, including the added affine terms. Choose its deterministic local time T_R, decreasing with R. At fixed cutoff, spatial parabolic bootstrapping gives a classical reconstructed solution: the linear field is spatially smooth and the nonlinear response starts above regularity $1/2$. The affine reconstruction identity and classical local uniqueness therefore identify the reconstructed cutoff solution with a^ε in (8.37) on this interval. Compare the two paired fixed-point systems, use the two-model Schauder and product bounds, and absorb their contraction term. This bounds the solution difference by a constant depending on R times the sum of the model, initial-data and smooth coefficient differences. All of those differences tend to zero in every finite moment on the common bounded-data event. The same argument gives local uniqueness and stability.

Reconstruction of the nonlinear response is controlled in $\mathcal{C}_{\mathfrak{s}}^{1-4\kappa}$ by the weighted reconstruction and the actual order-two heat kernel, as in (5.21). Its difference has the same bound in any fixed slightly weaker positive exponent. Such a parabolic Hölder bound embeds into $C([0,T_R],\mathrm{c}^{-\kappa})$; its causal response vanishes at time zero. Together with the preceding linear convergence, this proves (8.38) and the claimed initial trace.

The increasing events $\Omega_R = \{D \le R\}$ exhaust the probability space. Uniqueness identifies the limiting constructions on their overlapping intervals, yielding a positive local lifetime; the same causal equation gives maximal continuation in the modelled solution class. Markov's inequality and the uniform moment bound give

$$\sup_{0<\varepsilon\le\varepsilon_0} \mathbb{P}((\Omega_R^\varepsilon)^c) \le C_q R^{-q}.$$

This proves the stated localisation in probability. Section 9 gives the causal continuation and identifies its lifetime as the state-norm explosion time. □

The same argument permits a prescribed finite smooth affine modification of the limiting model equation. Such a modification specifies a different finite renormalisation unless compensated at the approximation level. In particular, the above local existence theorem first supplies a fixed-reference continuum process. Sections 10–12 then compare its actual bare heat normalisation under changes of reference, including the finite noise contractions.

9 Causality, restart, and the maximal Markov family

Throughout this section the reference connection B is fixed. The equation is the limit of the uncorrected additive heat approximations in Theorem 8.14. In particular its modelled representation includes the finite affine compensation in (8.40). Set

$$\mathsf{E} = \mathrm{c}^{-\kappa}(E), \qquad A = L_+ = 1 + L_B, \qquad S_t = e^{-tA}.$$

The Banach space E is separable: a countable set of smooth sections is dense in its defining closure. Throughout, solutions are reconstructions of the paired modelled equation (8.40).

Choose the auxiliary time cutoffs and jet realisations to commute with translations of the time variable. This is possible on the product $\mathbb{R}\times\Sigma$: the reference geometry is independent of time, and time Taylor polynomials and their cutoffs can be centred at the base time. The stationary enhancement, its counterterms and the paired kernel construction then commute with time translation.

9.1 Uniform restart and a state-space explosion criterion

Write $\mathbf{z}$ for the enhanced data consisting of the restricted model, its stationary linear fields and their time traces, and the finite affine coefficients. Use slightly weaker exponents than those in the stochastic estimates, with sufficient margin for the working degrees. The data can be placed in a separable complete metric space $\mathfrak{Z}$: take the closure of compatible smooth enhanced tuples in the finite enhancement spaces on an increasing sequence of compact time cylinders, including $C_{\mathrm{loc}}(\mathbb{R}, \mathsf{E})$ for the linear trace. The model assembly and the linear lifts extend as locally Lipschitz maps of these data. The local seminorms and the resulting solution maps are consequently Borel measurable.

Let $\theta_s\mathbf{z}$ denote time translation by s. On every bounded set of shifts, its local seminorms are controlled by those of $\mathbf{z}$ on a larger compact cylinder. Translation is continuous in the slightly weaker local norms: first check this for smooth data and then use their density and the uniform bounds for translations.

Lemma 9.1 (Uniform restart in the rough state norm)**.** *For admissible enhanced data* $\mathbf{z}$*, the paired equation* (8.40) *defines a maximal reconstructed path*

$$\mathsf{S}(t; x, \mathbf{z}) \in \mathsf{E}, \qquad 0 \le t < \zeta(x, \mathbf{z}), \qquad x \in \mathsf{E}.$$

It has the following properties.

(i) For every finite T and R, a bound M for the enhanced data on a fixed larger cylinder containing $[-2, T+2] \times \Sigma$ gives a positive time $h(R, M)$ on which every restart at $s \in [0, T]$, with initial state of norm at most R, exists. The time and the local stability constants are uniform in s.

(ii) For every $s < \zeta(x, \mathbf{z})$, the restart identity holds throughout the common interval of existence:

$$\mathsf{S}(s+t; x, \mathbf{z}) = \mathsf{S}(t; \mathsf{S}(s; x, \mathbf{z}), \theta_s\mathbf{z}). \tag{9.1}$$

The remaining maximal lifetimes are equal.

(iii) If $\zeta(x, \mathbf{z}) < \infty$, then

$$\lim_{t \uparrow \zeta(x,\mathbf{z})} \|\mathsf{S}(t; x, \mathbf{z})\|_{\mathsf{E}} = \infty. \tag{9.2}$$

(iv) The lifetime and the solution are jointly Borel measurable in the initial and enhanced data. On each compact time interval strictly before the lifetime, the solution is continuous in these data in $C_t\mathsf{E}$, and the lifetime is lower semicontinuous.

Proof. At a new initial time s, the prescribed linear pair is the lift of

$$X(s+t) + S_t\{x - X(s)\}, \qquad \nabla_B X(s+t) + \nabla_B S_t\{x - X(s)\}.$$

The pure-jet heat estimates in Theorem 5.8 bound this pair in $\mathcal{D}^{\gamma,\eta} \times \mathcal{D}^{\gamma-1,\eta-1}$ by a constant depending only on R and M. The time-trace bound on $X(s)$ and the separate gradient heat lift give these estimates for rough initial states. The strict weight margins $\eta < -\kappa$ and $\eta - 1 < -1 - \kappa$ are uniform under time translation. The contraction with factor $T^{1/4}$ therefore gives the time $h(R, M)$ and the corresponding stability bound.

To prove restart, denote the common nonlinear forcing of the paired equation by F and split it at time s into past and future restrictions. These are weighted restrictions in the reconstruction sense: their boundary regularity exceeds -2, the negative time codimension. For $t > s$, the abstract integral of the past restriction has zero nonjet component, because that

restriction vanishes at t. Its remaining component is the canonical jet lift of its reconstructed heat convolution. The causal heat identity gives

$$K * (\mathbf{1}_{(0,s]}\mathcal{R}F)(s+t) = S_t\big(K * (\mathbf{1}_{(0,s]}\mathcal{R}F)(s)\big).$$

The value at s exists since the nonlinear response has positive regularity. The second full kernel gives the covariant derivative of the same identity. Combining these past contributions with the old initial heat extension gives exactly the linear pair displayed above with $x = \mathsf{S}(s; x_0, \mathbf{z})$. The future contributions are the two integrals in the restarted equation. Its uniqueness proves (9.1). This deterministic argument applies simultaneously to every real s in the interval. Maximality then identifies the remaining lifetimes.

Suppose a finite maximal lifetime admits times s_n increasing to it with $\|\mathsf{S}(s_n; x, \mathbf{z})\|_{\mathsf{E}} \le R$. The enhanced data have a finite bound on a compact cylinder containing that lifetime. The first part supplies a common positive extension time $h(R, M)$ from every s_n. Restart and uniqueness would extend the path beyond its lifetime, a contradiction. This proves the limit in (9.2).

The local contraction is jointly continuous in initial and enhanced data on each of its bounded-data domains. Picard iteration also gives measurability there. For a compact interval before a given lifetime, finitely many of the preceding uniform restart intervals suffice. Iterating local stability proves continuity on that interval and existence there for all sufficiently close data. This proves lower semicontinuity of the lifetime. A countable cover by bounded-data domains, integer radii and rational time intervals gives the joint Borel assertions for the maximal continuation. □

The bounded-data times provide uniform analytic continuation intervals and may depend on future enhanced norms. Stochastic localisation below uses the state exit times of the reconstructed E-valued path, whose limit is its maximal lifetime.

9.2 A simultaneous set of regular noise paths

Choose an integer $m > 2$ and put $\mathsf{U} = H^{-m}(E)$. The embedding $L^2(E) \hookrightarrow \mathsf{U}$ is Hilbert–Schmidt. Consequently the cylindrical Wiener process can be realised as a continuous U-valued Wiener process; its distributional time derivative is the given L^2 space–time white noise. Use the canonical space of two-sided continuous paths anchored at zero. A path is called polynomially bounded if, for some finite C, k,

$$\|w(t)\|_{\mathsf{U}} \le C(1 + |t|)^k, \qquad t \in \mathbb{R}.$$

These paths form a Borel, time-shift-invariant set $\mathcal{W}_{\mathrm{pol}}$. Two-sided Wiener measure gives it full mass: moment bounds for the supremum on consecutive unit intervals and the Borel–Cantelli lemma give such a polynomial bound on both half-lines. For $s \in \mathbb{R}$ set

$$(\theta_s w)(t) = w(s+t) - w(s).$$

At every fixed spatial heat scale, all approximating objects can be defined as Borel functions of $w \in \mathcal{W}_{\mathrm{pol}}$. For example, with $\mathcal{S}_\varepsilon = e^{-\varepsilon A}$,

$$X^\varepsilon[w](t) = \mathcal{S}_\varepsilon w(t) - \int_{-\infty}^{t} A S_{t-r}\mathcal{S}_\varepsilon w(r)\, dr. \tag{9.3}$$

The integral is a classical integral of smooth sections. Spatial smoothing controls its upper endpoint, and the mass controls the infinite past. On Wiener paths it equals the stationary Itô convolution by integration by parts. Its time-translation identity holds for every polynomially

bounded path. The cutoff Wick products, their causal responses and the finite subtraction tensors are treated in the same way. At fixed ε the noise itself is the distribution $\mathcal{S}_\varepsilon \partial_t w$; auxiliary temporal smoothing defines the same model as its limit. Thus the approximating enhanced data $\mathbf{z}^\varepsilon[w]$ are Borel and satisfy

$$\mathbf{z}^\varepsilon[\theta_s w] = \theta_s \mathbf{z}^\varepsilon[w]. \tag{9.4}$$

Lemma 9.2 (A universal regular set and pathwise cutoff convergence)**.** *There is a deterministic sequence $\varepsilon_n \downarrow 0$ and a Borel set $\mathcal{G} \subset \mathcal{W}_{\mathrm{pol}}$ with full two-sided Wiener measure such that:*

(i) $\mathbf{z}^{\varepsilon_n}[w]$ *converges in all required local enhancement and time-trace norms for every $w \in \mathcal{G}$; write $\mathbf{z}[w]$ for its limit.*

(ii) For every real s, $\theta_s \mathcal{G} = \mathcal{G}$ and $\mathbf{z}[\theta_s w] = \theta_s \mathbf{z}[w]$.

(iii) For every $w \in \mathcal{G}$, every $x \in \mathsf{E}$, and every sequence of smooth initial states $x_n \to x$ in E, the uncorrected classical solutions with data (ε_n, x_n, w) converge in $C([0,T], \mathsf{E})$ whenever $T < \zeta(x, \mathbf{z}[w])$. In particular, they exist on that interval for all sufficiently large n.

The exceptional set is independent of the initial state.

Proof. The enhancement convergence in every finite moment and the Gaussian time-trace convergence permit a deterministic subsequence which converges almost surely on every integer compact cylinder. Choose it by a diagonal summability argument. Define $\mathcal{G}$ as the set of polynomially bounded paths for which this sequence converges in the separable complete local data space. The Cauchy criterion makes this a Borel set, and the stochastic estimates give full measure.

Equation (9.4) and the bounds for translation on compact cylinders show that convergence for one path implies convergence for each of its real shifts. This is a deterministic implication: a bounded range of shifted cylinders is contained in one larger cylinder. Applying the same argument to the inverse shift proves invariance, including simultaneous invariance under all real shifts.

Fix a good path. Its approximating data are bounded on every compact cylinder and converge there. On the common contraction interval, deterministic stability gives convergence for any smooth sequence $x_n \to x$. Smoothness identifies the approximating equation with its classical solution. On a compact interval before the limiting lifetime, the limiting state path is bounded. Lemma 9.1 gives a uniform restart length there. Finitely many restarts, using the convergent approximating states at their left endpoints, extend the convergence to the whole interval. Once the enhanced data converge, this deterministic stability argument applies to all initial states and all convergent initial sequences on the same set $\mathcal{G}$. □

9.3 Prefix locality and independence of the auxiliary past

Introduce a cemetery state † for finite explosion. We use a Polish topology in which norm explosion converges to this state. For $x, y \in \mathsf{E}$ put

$$d_0(x,y) = 1 \wedge \|x - y\|_{\mathsf{E}}, \qquad h(x) = (1 + \|x\|_{\mathsf{E}})^{-1},$$

and define a metric on $\mathsf{E}_\dagger = \mathsf{E} \cup \{\dagger\}$ by

$$d_\dagger(x,y) = d_0(x,y) \wedge \{h(x) + h(y)\}, \qquad d_\dagger(x,\dagger) = h(x). \tag{9.5}$$

Since h is 1-Lipschitz for d_0, the triangle inequality follows by allowing a path through †. A Cauchy sequence either has h tending to zero and converges to †, or is eventually Cauchy for

d_0. Thus the metric is complete and separable, and it induces the original topology on E. Its Borel sets are those of E together with the added point. By (9.2), a maximal path extended by † at and after its lifetime is continuous in this metric. We write $\overline{\mathsf{S}}$ for this extension.

Proposition 9.3 (Pathwise prefix locality)**.** *Let $w, \widetilde{w} \in \mathcal{G}$ and suppose that their increments agree on $[0,T]$. For every $x \in \mathsf{E}$,*

$$\overline{\mathsf{S}}(\cdot; x, \mathbf{z}[w]) = \overline{\mathsf{S}}(\cdot; x, \mathbf{z}[\widetilde{w}]) \quad \text{on } [0,T]. \tag{9.6}$$

In particular the two survival events at T coincide. If the increments agree on the whole positive half-line, their maximal killed paths and lifetimes are identical.

Proof. Use the same smooth initial approximants $\mathcal{S}_{\varepsilon_n} x$. At every cutoff, the uncorrected initial-value equations have identical driving increments on $[0,T]$. Classical pathwise uniqueness therefore identifies their solutions on the common interval in $[0,T]$. The identity (8.41) shows that their stationary pasts do not enter these equations.

Lemma 9.2 passes this equality to the limits on every compact interval before both lifetimes. Suppose one lifetime were at most T and strictly smaller than the other. The second path would be bounded and continuous on a compact interval containing the first lifetime. Equality before that time would contradict (9.2) for the first path. Thus differing survival is impossible, including at the endpoint T. Extending by the absorbing state proves (9.6). Applying the argument to all integer T proves the assertion for a common future. □

9.4 A jointly measurable causal solution map

Let $\Omega_+ = C_0([0,\infty), \mathsf{U})$ with its Wiener law $\mathbb{P}_+$, and let Ω_- be the analogous space for the negative half-line. Pasting two independent paths gives the two-sided Wiener law. By Fubini's theorem, one can fix a single past path w_*^- such that

$$\Omega_* := \{w^+ : (w_*^- \sqcup w^+) \in \mathcal{G}\} \tag{9.7}$$

is Borel and has full $\mathbb{P}_+$ measure. The choice is made once, independently of the initial state. Define

$$\Phi(x, w^+) = \overline{\mathsf{S}}(\cdot; x, \mathbf{z}[w_*^- \sqcup w^+]) \in C([0,\infty), \mathsf{E}_\dagger) \tag{9.8}$$

on Ω_*, and define it to be the constant cemetery path on the null complement. Set $\Phi(\dagger, w^+)$ equal to that constant path as well. The preceding measurability statements make Φ jointly Borel; Borel measurability into the continuous path space can also be checked at rational evaluation times.

Proposition 9.4 (Causal measurability and stopping lifetimes)**.** *For every $x \in \mathsf{E}$, the path $a^x = \Phi(x, W)$ is adapted to the usual augmented filtration of the positive-time Wiener process, is continuous in E before its lifetime, and has initial value x. The exceptional set can be chosen independently of x. The same statement holds for an $\mathcal{F}_0$-measurable initial state when W is Wiener with respect to that filtration.*

Its lifetime τ^x is a stopping time. With $\| \dagger \|_{\mathsf{E}} = \infty$, set

$$\tau_R^x = \inf\{t \geq 0 : \|a_t^x\|_{\mathsf{E}} \geq R\}. \tag{9.9}$$

Then τ_R^x are stopping times and $\tau_R^x \uparrow \tau^x$. On finite explosion, $\tau_R^x < \tau^x$ for $R > \|x\|_{\mathsf{E}}$.

Proof. Measurable factorisation of prefixes proves adaptation. Fix a rational $t \geq 0$. A Wiener path decomposes into its prefix up to t and an independent future increment path. Fubini supplies one fixed future tail for which pasting with almost every prefix gives a member of

Ω_*. Evaluate (9.8) on this pasted path. This is a jointly Borel function of x and the prefix. Proposition 9.3 says that it equals the original killed solution through time t for almost every Wiener path, simultaneously for all x. There are only countably many rational t, so these statements hold on a common full-measure set. Continuity of the killed paths and right continuity of the augmented filtration give adaptation at every real time. The same joint factorisation allows an $\mathcal{F}_0$-measurable random initial state.

The lifetime is the first hitting time of †, which is closed in (9.5); it is absorbing. The exit times from the open sets $\{x \in \mathsf{E} : \|x\|_{\mathsf{E}} < R\}$ are likewise stopping times. The explosion criterion and continuity on each compact pre-explosion interval prove all assertions about (9.9). □

The fixed auxiliary past in (9.8) provides a measurable representative. For every other good two-sided extension with the same positive-time increments, Proposition 9.3 identifies its solution with Φ for every initial state. In particular an independent random stationary past gives exactly the same maximal process and lifetime.

9.5 Restart at stopping times and the strong Markov property

Theorem 9.5 (The maximal fixed-reference strong Markov family)**.** *The processes $\{a^x : x \in \mathsf{E}_\dagger\}$ form a time-homogeneous strong Markov family on $\mathsf{E}_\dagger$. For every finite stopping time σ of the Wiener filtration,*

$$a^x_{\sigma+t} = \Phi_t(a^x_\sigma, \theta_\sigma W), \qquad t \geq 0, \tag{9.10}$$

almost surely as an identity of killed continuous paths. On $\{\sigma < \tau^x\}$ the lifetime on the right is $\tau^x - \sigma$. The same assertion holds, in particular, for deterministic times and for the state exit times in (9.9) when finite.

For bounded Borel $f : \mathsf{E}_\dagger \to \mathbb{R}$, define

$$P_t f(x) = \int_{\Omega_+} f(\Phi_t(x, w))\, \mathbb{P}_+(dw).$$

Then P_t is a Borel transition semigroup, † is absorbing, and

$$\mathbb{E}[f(a^x_{\sigma+t}) \mid \mathcal{F}_\sigma] = P_t f(a^x_\sigma). \tag{9.11}$$

It also holds for the usual augmented natural filtration of the state process. Restricted to E, this is the sub-Markov family killed at its state-norm explosion time.

Proof. Paste the fixed past w^-_* to the positive-time Wiener path. This two-sided path belongs to $\mathcal{G}$ almost surely. The invariance in Lemma 9.2 is under every real shift; hence its shift by the random value σ is still in $\mathcal{G}$. Deterministic restart, valid for every real starting time for this fixed good path, describes the future solution using this shifted enhanced model and the state a^x_σ.

By the strong Markov property of the driving Hilbert-space Wiener process, the future increment process $\theta_\sigma W$ has Wiener law and is independent of $\mathcal{F}_\sigma$. This property follows directly by approximating bounded σ from above by dyadic stopping times, using independence at those times and continuity of Wiener paths for bounded cylindrical future functions; a monotone-class argument gives the whole future path, and localisation removes boundedness of σ. In particular $\theta_\sigma W \in \Omega_*$ almost surely.

There are now two good two-sided paths with this same future: the shift of the original good path, and $w^-_* \sqcup (\theta_\sigma W)$. The pathwise assertion in Proposition 9.3, valid for every initial state, identifies their maximal solutions starting at the random state a^x_σ. Together with deterministic

restart, this proves (9.10) and the lifetime identity on $\{\sigma < \tau^x\}$. On its complement both killed paths are the constant cemetery path.

Joint Borel measurability of Φ and independence of the future Wiener path now give (9.11) by conditioning. Deterministic stopping times give the semigroup identity. A stopping time of the state filtration is also a stopping time of the Wiener filtration; conditioning once more gives the claimed natural-filtration statement. □

Remark 9.6 (The fixed-reference Markov state). For the autonomous fixed-reference equation on $B + \mathsf{E}$, the current state and future Wiener increments determine the maximal continuation. Proposition 9.3 makes this continuation independent of the auxiliary stationary past, including its survival events. The state exit times (9.9) localise the process up to its explosion time.

The subsequence in Lemma 9.2 constructs a simultaneous pathwise version. For each initial state, Φ is a version of the full heat-family local probability limit in Theorem 8.14, with the common bounded-data existence intervals specified there. It supplies the causal maximal continuation of that limit.

10 Comparison gauges for distributional connections

Cancellation of the derivatives of the evolving connection extends the reference-change equation to the rough state space. We construct the comparison gauge from rough initial data and prove stability of its action and of the rotated noise. On the flat torus, a separate gauge equation also enters the covariance argument in [4, Section 7]. Here the construction allows arbitrary smooth reference connections on any surface bundle.

Let $P \to \Sigma$ be a smooth principal bundle with compact structure group G, and put $E = T^*\Sigma \otimes \operatorname{ad} P$. Fix smooth connections B, C, write $b = C - B$, and use the convention $u \cdot A = uAu^{-1} - (du)u^{-1}$. Fix a faithful unitary representation of G and its associated vector bundle. Gauge transformations are sections of the corresponding group subbundle of its endomorphism bundle. All norms below use this embedding and a fixed smooth connection; different such choices give equivalent local norms. For $r > 0$, let $\mathcal{G}^r(P)$ denote the gauge sections of regularity c^r, the closure of smooth sections in $\mathcal{C}^r = B^r_{\infty,\infty}$. In particular these are continuous sections with values in G.

10.1 Products and the endpoint heat estimate

If $0 < \kappa < \rho$ and v, f are sections of suitable tensor bundles, then every fixed smooth fibrewise bilinear contraction satisfies

$$\|vf\|_{\mathcal{C}^{-\kappa}} \le C\|v\|_{\mathcal{C}^{\rho}}\|f\|_{\mathcal{C}^{-\kappa}}. \tag{10.1}$$

The product extends continuously to the corresponding little spaces. To see the estimate in a bundle chart, write $vf = v \prec f + v \circ f + v \succ f$. The first paraproduct has regularity $-\kappa$; the other two terms have regularity $\rho - \kappa > 0$. A finite partition of unity gives the bundle assertion. Approximation by smooth sections gives the little-space assertion and the usual distributional Leibniz rule whenever the products on its two sides have these exponents.

Let L be a fixed nonnegative smooth connection Laplacian, on any of the bundles used below, and set

$$H_t f = e^{-tL} f, \qquad (\mathcal{K} f)(t) = \int_0^t H_{t-s} f(s)\, ds.$$

Adding a smooth zeroth-order term, or adding a positive mass, changes only the constants on bounded time intervals.

Lemma 10.1 (Heat integration in little Hölder spaces)**.** *For $0 < T \le 1$ and $0 \le r + \kappa < 2$,*

$$\|\mathcal{K}f\|_{C_T\mathrm{c}^r} \le CT^{1-(r+\kappa)/2}\|f\|_{C_T\mathrm{c}^{-\kappa}}. \tag{10.2}$$

At the endpoint, $f \in C([0,T];\mathrm{c}^{-\kappa})$ implies

$$\mathcal{K}f \in C([0,T];\mathrm{c}^{2-\kappa}) \cap C^1([0,T];\mathrm{c}^{-\kappa}), \qquad \|\mathcal{K}f\|_{C_T\mathrm{c}^{2-\kappa}} \le C\|f\|_{C_T\mathrm{c}^{-\kappa}}. \tag{10.3}$$

Here $\mathcal{K}f(0) = 0$ in the endpoint space. If instead f is continuous for positive time and $\|f(t)\|_{\mathrm{c}^{-\kappa}} \le Mt^{-\beta}$ with $0 \le \beta < 1$, then

$$\|\mathcal{K}f(t)\|_{\mathrm{c}^r} \le CMt^{1-\beta-(r+\kappa)/2}, \qquad 0 \le r + \kappa < 2, \tag{10.4}$$

$$\|\mathcal{K}f(t)\|_{\mathrm{c}^{2-\kappa}} + \|\partial_t\mathcal{K}f(t)\|_{\mathrm{c}^{-\kappa}} \le CMt^{-\beta}, \qquad t > 0. \tag{10.5}$$

Proof. Use a smooth dyadic spectral decomposition $(\Delta_j)_{j\ge -1}$ of $1 + L$. Its frequency-localised heat kernels satisfy, for $j \ge 0$,

$$\|H_t\Delta_j\|_{L^\infty\to L^\infty} \le Ce^{-ct2^{2j}}.$$

One may enlarge each spectral annulus by a fixed factor in this estimate. The bound follows by integrating the frequency-localised heat kernel; its kernel bounds follow from the smooth elliptic parametrix. Consequently

$$\|H_t\|_{\mathcal{C}^{-\kappa}\to\mathcal{C}^r} \le Ct^{-(r+\kappa)/2}$$

when $r + \kappa > 0$, with the usual bounded version for nonpositive gain. Integration gives (10.2) and, by the Beta integral, (10.4).

For the endpoint, estimate each block before taking the supremum:

$$2^{(2-\kappa)j}\|\Delta_j\mathcal{K}f(t)\|_\infty \le C\sup_{s\le t} 2^{-\kappa j}\|\Delta_j f(s)\|_\infty. \tag{10.6}$$

The low block has a bound $CT\|f\|_{C_T\mathrm{c}^{-\kappa}}$. The image of a continuous path in $\mathrm{c}^{-\kappa}$ on a compact interval is compact. Its little-Besov tails therefore vanish uniformly in time. Equation (10.6) gives the same uniform tail property in $\mathrm{c}^{2-\kappa}$. Each finite set of blocks is continuous and vanishes at time zero. This proves endpoint continuity, including the initial trace. The identity $\partial_t\mathcal{K}f = -L\mathcal{K}f + f$, first tested against smooth sections, now has a continuous right-hand side in $\mathrm{c}^{-\kappa}$ and proves the asserted time differentiability.

For (10.5), split the integral at $t/2$. The first part has norm at most $Ct^{-1}\int_0^{t/2} Ms^{-\beta}\,ds \le CMt^{-\beta}$. For the second part use (10.6), with $s^{-\beta} \le 2^\beta t^{-\beta}$. The derivative identity gives the remaining estimate. The same argument on compact positive-time intervals proves continuity there. All estimates pass to little spaces by the uniform-tail argument or by smooth approximation. □

10.2 The cancelled equation and its weighted solution

For a gauge section u and a connection $A = B + a$, set

$$k = (d_B u)u^{-1}, \qquad U = \mathrm{Ad}_u, \qquad \widetilde{a} = Ua - k, \qquad \widetilde{A} = B + \widetilde{a}.$$

Here $d_B u = du + Bu - uB$ in a local unitary frame. The right logarithmic derivative k is an intrinsic adjoint-valued one-form. Write $\delta_B = d_B^*$, and use the contraction convention $[v \lrcorner w] = -\sum_i [v_i, w_i]$ on one-forms.

Proposition 10.2 (Cancellation of derivatives of the connection)**.** *For smooth sections the reference-change equation is exactly*

$$(\partial_t u)u^{-1} = -\delta_B k - \delta_B b + [k \lrcorner Ua] + [b \lrcorner Ua] - [b \lrcorner k]. \tag{10.7}$$

If $0 < \kappa < 1/2$, $a \in \mathrm{c}^{-\kappa}$ and $u \in \mathcal{G}^{2-\kappa}(P)$, all terms on this right-hand side are defined in $\mathrm{c}^{-\kappa}$. In distributions it also equals

$$\chi_C(u \cdot (B + a)) - U\chi_B(B + a), \qquad \chi_D(A) = \delta_D(A - D).$$

Proof. The product rule gives

$$\delta_B(Ua) - U\delta_B a = [k \lrcorner Ua].$$

Substitute $u \cdot (B + a) = B + Ua - k$ and $\delta_C = \delta_B + [b \lrcorner \cdot]$. The term $[b \lrcorner b]$ vanishes, giving (10.7). For the asserted rough inputs, $k \in \mathrm{c}^{1-\kappa}$ and $1 - \kappa > \kappa$, so (10.1) defines the products in the expanded formula. The individual product $U\delta_B a$ is also defined, because $(2 - \kappa) + (-1 - \kappa) = 1 - 2\kappa > 0$. Smooth approximation extends the identity, and hence the cancellation, to distributions. □

Set

$$r_0 = 1 - \kappa, \qquad r_1 = 3/2, \qquad \alpha = \frac{1}{4} + \frac{\kappa}{2}, \qquad \theta = \frac{1}{4} - \frac{\kappa}{2} > 0. \tag{10.8}$$

For a time-dependent section v define

$$\|v\|_{\mathcal{X}_T} = \sup_{0 \le t \le T} \|v(t)\|_{\mathrm{c}^{r_0}} + \sup_{0 < t \le T} t^{\alpha} \|v(t)\|_{\mathrm{c}^{r_1}}.$$

The space $\mathcal{X}_T$ consists of sections continuous in the first norm, continuous in the second norm for positive time, and satisfying $t^{\alpha}\|v(t)\|_{\mathrm{c}^{r_1}} \to 0$ at zero. In exponential coordinates this is a Banach space.

Theorem 10.3 (Local comparison gauges in the rough state topology)**.** *Let $0 < \kappa < 1/2$, let $a \in C([0, T_a); \mathrm{c}^{-\kappa}(E))$, and let $u_0 \in \mathcal{G}^{1-\kappa}(P)$. Equation (10.7), with $u(0) = u_0$, has a unique maximal local solution in the class $\mathcal{X}$. Write $\tau_u \in (0, T_a]$ for its lifetime. For $T < \tau_u$,*

$$\begin{aligned} u &\in C([0, T]; \mathcal{G}^{1-\kappa}(P)) \cap C((0, T]; \mathcal{G}^{2-\kappa}(P)), \\ \partial_t u &\in C((0, T]; \mathrm{c}^{-\kappa}), \\ \sup_{0 < t \le T} t^{1/2} \left(\|u(t)\|_{\mathrm{c}^{2-\kappa}} + \|\partial_t u(t)\|_{\mathrm{c}^{-\kappa}} \right) &< \infty. \end{aligned} \tag{10.9}$$

In particular u has finite variation in $\mathrm{c}^{-\kappa}$ on every such interval, including its initial endpoint. The solution is causal and locally Lipschitz in $(u_0, a) \in \mathcal{G}^{1-\kappa} \times C_T\mathrm{c}^{-\kappa}$, both in $\mathcal{X}_T$ and in the weighted norms in (10.9), on common local existence intervals.

For bounded initial $\mathrm{c}^{1-\kappa}$ norm and bounded input $C_T\mathrm{c}^{-\kappa}$ norm there is a uniform positive local existence time. Consequently, if $\tau_u < T_a$ is finite, then

$$\lim_{t \uparrow \tau_u} \|u(t)\|_{\mathrm{c}^{1-\kappa}} = \infty. \tag{10.10}$$

If $u_0 \in \mathcal{G}^{2-\kappa}(P)$, the stronger conclusion is

$$u \in C([0, T]; \mathcal{G}^{2-\kappa}(P)) \cap C^1([0, T]; \mathrm{c}^{-\kappa}), \tag{10.11}$$

with local Lipschitz dependence in these unweighted norms when the initial gauge is varied in $\mathrm{c}^{2-\kappa}$.

Proof. Coordinates on the group bundle. Choose a smooth gauge u_* uniformly close to u_0 and write $u = \exp(\psi)u_*$ in a fixed fibrewise exponential neighbourhood. This choice is available on every bundle: smooth the ambient endomorphism section and project to the group subbundle inside a conjugation-invariant tubular neighbourhood. For initial norm bounded by R, a smoothing scale depending only on R makes the uniform error smaller than a fixed exponential radius. The smooth centre u_* then has bounds in every fixed smooth norm depending only on R, since

$$\|e^{-hL}u_0 - u_0\|_\infty \le Ch^{(1-\kappa)/2}\|u_0\|_{\mathcal{C}^{1-\kappa}}.$$

The logarithm ψ_0 has bounded $\mathrm{c}^{1-\kappa}$ norm and lies strictly inside that exponential neighbourhood. For local stability, one fixed smooth centre works for all sufficiently close initial gauges.

In these coordinates the equation becomes

$$\partial_t\psi + \Delta_B^{(0)}\psi = N(\psi, \nabla_B\psi; a), \qquad \psi(0) = \psi_0. \tag{10.12}$$

The coefficients of N depend smoothly on B, b, u_* and on ψ. It contains at most two first derivatives of ψ, is affine in a, and contains at most one first derivative of ψ in each term involving a. This description follows directly from (10.7). To check its principal part, the top second derivatives of $\exp(\psi)$ in $-\delta_B((d_Bu)u^{-1})$ are multiplied by the same differential of the exponential as $\partial_t\psi$ on the left. Inverting that differential leaves $-\Delta_B^{(0)}\psi$; every remaining term has the derivative structure just stated. The construction is equivariant under a change of bundle frame, so ψ is a global section of $\operatorname{ad} P$.

The tame bound. On a bounded c^{r_0} set contained in the coordinate neighbourhood, smooth functions of ψ have uniformly bounded c^{r_0} norms. Since $r_0 > \kappa$ and $r_1 - 1 = 1/2 > \kappa$, (10.1) gives

$$\|\nabla_B\psi\,\nabla_B\psi\|_{\mathrm{c}^{-\kappa}} \le C\|\psi\|_{\mathrm{c}^{r_0}}\|\psi\|_{\mathrm{c}^{r_1}}. \tag{10.13}$$

Here one derivative is measured in $\mathrm{c}^{-\kappa}$ and the other in $\mathrm{c}^{1/2}$. The same product estimate gives $\|a\nabla_B\psi\|_{\mathrm{c}^{-\kappa}} \le C\|a\|_{\mathrm{c}^{-\kappa}}\|\psi\|_{\mathrm{c}^{r_1}}$. It follows that, for $\|a\|_{C_T\mathrm{c}^{-\kappa}} \le M$,

$$\|N(\psi, \nabla_B\psi; a)(t)\|_{\mathrm{c}^{-\kappa}} \le C_{R,M}(1 + \|\psi(t)\|_{\mathrm{c}^{r_1}}). \tag{10.14}$$

Polarising the gradient square and taking differences of its smooth coefficients proves, for ψ, φ in a bounded $\mathcal{X}_T$ ball,

$$\begin{aligned}&\|N(\psi, \nabla_B\psi; a)(t) - N(\varphi, \nabla_B\varphi; \bar a)(t)\|_{\mathrm{c}^{-\kappa}}\\ &\qquad \le C_{R,M}t^{-\alpha}\left(\|\psi - \varphi\|_{\mathcal{X}_T} + \|a - \bar a\|_{C_T\mathrm{c}^{-\kappa}}\right).\end{aligned} \tag{10.15}$$

For example, the difference of two gradient squares is bounded by the sum of a low norm times a high difference and a high norm times a low difference. This proves the time weight in (10.15) without squaring $t^{-\alpha}$.

Contraction and regularity. The heat extension of ψ_0 belongs to $\mathcal{X}_T$. The vanishing weighted high norm at zero follows by smooth density in c^{r_0}. If $\|f(t)\|_{\mathrm{c}^{-\kappa}} \le Mt^{-\alpha}$, then (10.4) gives

$$\|\mathcal{K}f\|_{\mathcal{X}_T} \le CMT^\theta. \tag{10.16}$$

Indeed the low-norm integral has exponent $1-\alpha-(r_0+\kappa)/2 = 1/2-\alpha = \theta$; after multiplication by t^α, the high-norm integral has exponent $1-(r_1+\kappa)/2 = \theta$. Both Beta integrals are finite. Equations (10.14)– (10.16) give a contraction in a bounded $\mathcal{X}_T$ ball for sufficiently small T. The nonlinear increment is small in c^{r_0}, and the heat extension converges uniformly to ψ_0, so the coordinate neighbourhood is preserved. Exponentiation keeps the solution in the group bundle.

The forcing is continuous for positive time with norm bounded by $C_{R,M}t^{-\alpha}$. The initial heat extension satisfies

$$\|H_t\psi_0\|_{\mathrm{c}^{2-\kappa}} + \|\partial_t H_t\psi_0\|_{\mathrm{c}^{-\kappa}} \le Ct^{-1/2}\|\psi_0\|_{\mathrm{c}^{1-\kappa}}.$$

Lemma 10.1 bounds the nonlinear contribution by $Ct^{-\alpha}$ in the two endpoint norms. Since $\alpha < 1/2$, this proves (10.9). Composition with the smooth exponential preserves the spatial estimates. Its time derivative is the corresponding smooth multiplier applied to $\partial_t\psi$, a valid $\mathrm{c}^{-\kappa}$ product. This chain rule follows at positive time by a Taylor expansion: the remainder is bounded by a $\mathrm{c}^{-\kappa}$ increment times a positive-regularity increment tending to zero. Its integrable $t^{-1/2}$ bound then extends the fundamental theorem of calculus to time zero. Thus u is absolutely continuous in $\mathrm{c}^{-\kappa}$.

Taking differences in the contraction proves local Lipschitz dependence in $\mathcal{X}_T$. Applying (10.5) to (10.15) proves it in the endpoint weighted norms as well. If the initial gauge belongs to $\mathrm{c}^{2-\kappa}$, perform the same contraction first in $C_T\mathrm{c}^{3/2}$. Its forcing is then continuous in $\mathrm{c}^{-\kappa}$, and (10.3) gives (10.11) and the stated unweighted stability. Uniqueness identifies this solution with the weighted one.

Continuation and causality. The coordinate choice and all contraction constants above are uniform on bounded initial $\mathrm{c}^{1-\kappa}$ sets and bounded input sets. Restart therefore gives a maximal solution. If its norm stayed bounded along a sequence increasing to a finite time strictly below T_a, the input would also be bounded on a larger compact interval. The uniform restart time would extend the solution through that time. This proves the full limit in (10.10). The Volterra equation and uniqueness identify solutions driven by equal input prefixes. The same continuation argument identifies their survival through the endpoint of a prefix. This proves causality of the maximal solution and its lifetime. □

10.3 Rough gauge action, continuation, and parallel transport

Proposition 10.4 (The action and a bound from two representatives)**.** *For $0 < \kappa < 1/2$ the map*

$$\mathcal{G}^{1-\kappa}(P) \times \big(B + \mathrm{c}^{-\kappa}(E)\big) \longrightarrow B + \mathrm{c}^{-\kappa}(E), \qquad (u, B+a) \longmapsto B + Ua - (d_B u)u^{-1}, \tag{10.17}$$

is locally Lipschitz and is a group action. Moreover

$$\|u\|_{\mathcal{C}^{1-\kappa}} \le C_B\left(1 + \big(\|a\|_{\mathcal{C}^{-\kappa}} + \|u\cdot(B+a) - B\|_{\mathcal{C}^{-\kappa}}\big)^p\right), \qquad p = \frac{2(1-\kappa)}{1-2\kappa}. \tag{10.18}$$

The norms of gauge sections include their ambient zeroth-order norms. In particular a comparison gauge cannot terminate while both connection representatives remain bounded in the rough state norm. If $\tau_u < T_a$ in Theorem 10.3, then

$$\lim_{t\uparrow\tau_u} \|u_t\cdot(B+a_t) - B\|_{\mathrm{c}^{-\kappa}} = \infty.$$

Proof. The derivative $d_B u$ has regularity $-\kappa$, and $1-\kappa > \kappa$. Thus (10.1) defines each term and gives its local Lipschitz bound. Smooth density, using the tubular projection for gauges, passes the classical composition law to these spaces.

Put $\widetilde{a} = u\cdot(B+a) - B$. The action identity gives

$$d_B u = ua - \widetilde{a}\,u.$$

The elliptic estimate for the full covariant gradient and (10.1), used with $\rho = 1/2$, give

$$M := \|u\|_{\mathcal{C}^{1-\kappa}} \le C_B(1 + R\|u\|_{\mathcal{C}^{1/2}}), \qquad R = \|a\|_{\mathcal{C}^{-\kappa}} + \|\widetilde{a}\|_{\mathcal{C}^{-\kappa}}.$$

The L^∞ norm of a G-valued section is uniformly bounded. Interpolation yields $\|u\|_{\mathcal{C}^{1/2}} \le CM^{1/(2(1-\kappa))}$. Absorbing this sublinear power proves (10.18). The continuation conclusions follow from (10.10) and local boundedness of the input path. □

Corollary 10.5 (Composition of reference changes)**.** *Let B, C, D be smooth references. Suppose u is the comparison gauge from B to C along A, and v is the comparison gauge from C to D along $u \cdot A$. Then vu is the comparison gauge from B to D along A, with initial value $v_0 u_0$, throughout their common interval. The inverse u^{-1} is the reverse comparison from C to B along $u \cdot A$.*

Proof. At positive times the gauges have spatial regularity $2-\kappa$, and their time derivatives have regularity $-\kappa$. The product rule is therefore valid in $\mathrm{c}^{-\kappa}$ and gives

$$(\partial_t(vu))(vu)^{-1} = (\partial_t v)v^{-1} + \mathrm{Ad}_v((\partial_t u)u^{-1}).$$

Use the distributional χ expression from Proposition 10.2 on the right. The two $\mathrm{Ad}_v\,\chi_C(u \cdot A)$ terms cancel, leaving $\chi_D(vu \cdot A) - \mathrm{Ad}_{vu}\,\chi_B(A)$. The product vu belongs to the same weighted solution class, and its initial value follows by continuity in $\mathcal{G}^{1-\kappa}$. Uniqueness proves the composition assertion. Differentiating u^{-1} gives the reverse equation, or equivalently one may apply the same cancellation to the identity gauge. □

Corollary 10.6 (Convergence of comparison gauges)**.** *Suppose $a_n \to a$ in $C_T\mathrm{c}^{-\kappa}$ and $u_{0,n} \to u_0$ in $\mathcal{G}^{1-\kappa}$. On every compact interval before the limiting gauge lifetime, the comparison gauges exist for all sufficiently large n and converge in the local weighted norms of Theorem 10.3. Their transformed connections converge in $C_t\mathrm{c}^{-\kappa}$. With initial convergence in $\mathcal{G}^{2-\kappa}$, the gauge convergence is also in $C_t\mathrm{c}^{2-\kappa}$ and $C_t^1\mathrm{c}^{-\kappa}$ on common initial intervals. The same conclusions hold in probability under the corresponding common bounded-data localisations.*

Proof. Local stability proves convergence on the first common contraction interval. A compact limiting interval is covered by finitely many uniform restart intervals, so induction proves the first assertion. Proposition 10.4 then gives convergence of the transformed connections. The probabilistic assertion follows by restricting to bounded data sets, applying the deterministic estimates, and exhausting these sets. This argument concerns compact intervals before the limiting lifetime and does not require convergence of explosion times. □

Proposition 10.7 (Covariance of existing path observables)**.** *Let $A_n \to A$ in $\mathrm{c}^{-\kappa}$ and $u_n \to u$ in $\mathcal{G}^{1-\kappa}$ be smooth spatial approximations, and let $\mathcal{K}$ be any family of piecewise smooth paths for which their parallel transports $T_{A_n}(\gamma)$ converge uniformly. Then the parallel transports of $u_n \cdot A_n$ converge uniformly and their limit is*

$$T_{u\cdot A}(\gamma) = u(\gamma(1))\, T_A(\gamma)\, u(\gamma(0))^{-1}. \tag{10.19}$$

The formula is intrinsic as a map between endpoint fibres. For closed paths it is conjugation at the base point, so every conjugation-invariant holonomy observable has the same value.

Proof. At the smooth level (10.19) follows by differentiating $u_n(\gamma(s))T_{A_n}(\gamma|_{[0,s]})$. Use a bi-invariant metric on the compact group G. The distance between the two sides at different approximation indices is bounded by the transport distance and the two endpoint gauge distances. Taking a supremum over $\mathcal{K}$ proves the assertion. The same argument applies uniformly in time whenever the hypotheses hold uniformly in time. □

The proposition applies to any independently constructed trace space or holonomy completion. It does not define line integrals for every element of $\mathrm{c}^{-\kappa}$. The comparison equation evolves a full gauge transformation; its value at a chosen point is generally free to move. Thus

its orbit interpretation uses the full gauge group. For example, with $u_0 = 1$ and $a_0 = 0$, its initial velocity for smooth data is $-\delta_B b$, which need not vanish at the chosen point.

10.4 Time regularity and the leading controlled term

Proposition 10.8 (Pointwise time regularity)**.** *On an interval where the norms in* (10.11) *are bounded, the comparison gauge satisfies*

$$\|u(t) - u(s)\|_\infty \le C|t-s|^{1-\kappa/2}, \tag{10.20}$$

$$\|\nabla_B u(t) - \nabla_B u(s)\|_\infty \le C|t-s|^{(1-\kappa)/2}. \tag{10.21}$$

Consequently u has parabolic regularity $\mathcal{C}^{2-\kappa}_{\mathfrak{s}}$ there, and $k = (d_B u)u^{-1}$ has regularity $\mathcal{C}^{1-\kappa}_{\mathfrak{s}}$. For initial gauges in $\mathcal{G}^{1-\kappa}$ these conclusions hold on every compact positive-time interval. Their spatial gradient also obeys

$$\|\nabla_B u(t)\|_\infty \le C t^{-\kappa/2}, \qquad t > 0, \tag{10.22}$$

on the local weighted existence interval.

Proof. For $h = |t-s| \le 1$, the time derivative and spatial endpoint bounds give

$$\|\Delta_j(u(t) - u(s))\|_\infty \le C \min\{h 2^{\kappa j}, 2^{-(2-\kappa)j}\}.$$

Sum the first bound over $2^j \le h^{-1/2}$ and the second over the remaining blocks. Both geometric sums are bounded by $Ch^{1-\kappa/2}$. This proves the L^∞ bound, including the low block. Multiplying each block estimate by 2^j before summation proves (10.21). Together with the spatial $\mathcal{C}^{2-\kappa}$ bound these are the parabolic Hölder estimates. Smooth composition and the ordinary positive-regularity product give the assertion for k. Finally, the two spatial bounds in Theorem 10.3 give

$$\|\Delta_j \nabla_B u(t)\|_\infty \le C \min\{2^{\kappa j}, t^{-1/2} 2^{-(1-\kappa)j}\}.$$

Splitting at $2^j = t^{-1/2}$ and summing proves (10.22). □

The following controlled expansion identifies the additional stochastic products in a reference comparison. It holds on interior cylinders; the initial heat extension retains its boundary weights.

Lemma 10.9 (The leading term of the adjoint gauge)**.** *Suppose $0 < \kappa < 1/3$ and, on a compact cylinder strictly after the initial time, write $a = X + r$ with $X \in \mathcal{C}^{-\kappa}_{\mathfrak{s}}(E)$ and $r \in \mathcal{C}^{1-2\kappa}_{\mathfrak{s}}(E)$. Let $U = \mathrm{Ad}_u$, and let Δ_B^{End} denote the connection Laplacian on $\mathrm{End}(\mathrm{ad}\, P)$. Define the linear tensor map $H : E \to \mathrm{End}(\mathrm{ad}\, P)$ by*

$$H(v) = -\sum_j \mathrm{ad}_{[k_j + b_j, U v_j]}\, U. \tag{10.23}$$

Then $H \in \mathcal{C}^{1-\kappa}_{\mathfrak{s}}$ and

$$(\partial_t + \Delta_B^{\mathrm{End}})U = H(a) + R_0, \qquad R_0 = \mathrm{ad}_{-\delta_B b + \sum_j [b_j, k_j]}\, U - \sum_j \mathrm{ad}^2_{k_j}\, U. \tag{10.24}$$

For a space–time paraproduct in smooth bundle charts and the massive heat integration K_E on E, this gives the local expansion

$$U = H \prec (K_E X) + U^\sharp, \qquad U^\sharp \in \mathcal{C}^{3-2\kappa}_{\mathfrak{s}}. \tag{10.25}$$

Its estimates, on a smaller compact cylinder, are locally Lipschitz in the displayed fields and their norms.

Proof. The identity $\nabla_B U = \mathrm{ad}_k U$ gives

$$\Delta_B^{\mathrm{End}} U = \mathrm{ad}_{\delta_B k} U - \sum_j \mathrm{ad}_{k_j}^2 U.$$

Also $\partial_t U = \mathrm{ad}_{(\partial_t u)u^{-1}} U$. Substitution of (10.7) proves (10.24); the products have positive total regularity by Proposition 10.8. Thus R_0 has regularity $1-\kappa$, and $H(r)$ has regularity $1-2\kappa$. Adding the mass puts U itself into this regular remainder. After a smooth time and space localisation, the heat representation therefore writes $U = K_{\mathrm{End}}(HX)$ plus a term of regularity $3-2\kappa$. Localisation commutators contain at most one derivative of U and have sufficient regularity for the same conclusion.

Decompose $HX = H \prec X + H \circ X + H \succ X$. The last two terms have regularity $1-2\kappa > 0$. The heat paraproduct commutator satisfies

$$K_{\mathrm{End}}(H \prec X) - H \prec (K_E X) \in \mathcal{C}_{\mathfrak{s}}^{3-2\kappa}.$$

At parabolic frequency 2^j, its kernel receives a gain 2^{-2j} from heat integration and a gain $2^{-(1-\kappa)j}$ from the increment of H, whereas the X block is bounded by $2^{\kappa j}$. Their product is $2^{-(3-2\kappa)j}$; off-diagonal frequency blocks have arbitrarily fast decay. In bundle charts the two heat kernels have the same leading scalar metric kernel. Their differing parallel transports and lower-order terms supply at least one additional order in their kernel difference, which is stronger than the required $1-\kappa$ gain. These block estimates prove the commutator bound on a smaller localisation. Polarisation proves its difference bound. Combining the three terms proves (10.25). □

For the fixed-reference stochastic solution, the stationary linear field and the interior response in Theorem 5.8 meet the hypotheses of this lemma. The expansion identifies a leading coefficient for random-gauge noise products; it does not assign their stochastic contractions. In particular, multiplying the leading $K_E X$ term by space–time white noise still requires a joint construction.

10.5 Predictable rotation and the linear heat-noise defect

Proposition 10.10 (Rotated Wiener noise)**.** *Let the input path a be adapted and continuous in $\mathrm{c}^{-\kappa}$, globally or up to a stopping lifetime, and let u_0 be measurable at the initial time. The comparison gauge is adapted and its lifetime is a stopping time. The multiplication operator $U_t = \mathrm{Ad}_{u_t}$ on $H = L^2(E)$ is predictable and orthogonal. After extending it by the identity beyond its local lifetime,*

$$\widetilde{W}_t = \int_0^t U_s \, dW_s \tag{10.26}$$

is an H-cylindrical Wiener process for the original filtration.

Suppose a sequence of such gauges converges in probability uniformly in space and time on a common stopped interval, with all noise rotations taken against the same W. For every $m > 1$, the corresponding rotated noises converge in $L^2(\Omega; C_t H^{-m}(E))$ on that interval.

Proof. The Volterra Picard iterates are measurable functions of the input prefix. Uniform local existence on bounded data sets and restart show that the local solution is adapted. Its norm exit times in $\mathrm{c}^{1-\kappa}$ are stopping times; their increasing limit is the lifetime by (10.10), with the input lifetime included. One can equivalently first stop the input at a norm exit, apply the deterministic Picard construction on its uniform interval, and iterate this procedure. Continuity of u in the uniform spatial norm implies predictability of its multiplication operator. The invariant fibre metric makes this operator orthogonal. An identity extension after the lifetime gives a predictable version modulo $dt\, d\mathbb{P}$ null sets.

For any $f, g \in H$, the real martingales in (10.26) have covariation

$$\int_0^t \langle U_s^{-1} f, U_s^{-1} g \rangle_H \, ds = t \langle f, g \rangle_H .$$

The finite-dimensional Lévy characterisation proves the Wiener assertion from the predictability and orthogonality of U.

Let $I_m : H \to H^{-m}(E)$ be the natural embedding. It is Hilbert–Schmidt in dimension two when $m > 1$. For a common stopping time $\sigma \le T$, the Hilbert-space martingale inequality gives

$$\begin{aligned} \mathbb{E} \sup_{t \le T} \left\| \int_0^{t \wedge \sigma} I_m (U_s^n - U_s) \, dW_s \right\|_{H^{-m}}^2 \\ \le 4 \| I_m \|_{\mathrm{HS}}^2 \mathbb{E} \int_0^{T \wedge \sigma} \| U_s^n - U_s \|_{H \to H}^2 \, ds . \end{aligned}$$

Uniform gauge convergence implies convergence in probability of the integrand to zero. It is bounded by four, so the expectation tends to zero. This proves the convergence assertion. □

We next estimate the linear defect produced by rotating an additive heat cutoff. Put $S_{D,\varepsilon} = e^{-\varepsilon(1+L_D)}$ for a smooth reference D, where L_D acts on adjoint-valued one-forms.

Lemma 10.11 (Operator bound for the conjugated heat cutoff)**.** *For a C^1 gauge u and $0 < \varepsilon \le 1$,*

$$\| U S_{B,\varepsilon} U^{-1} - S_{C,\varepsilon} \|_{H \to H} \le C_{B,C} \varepsilon^{1/2} (1 + \| \nabla_B U \|_\infty). \tag{10.27}$$

Proof. Write $L_B = \nabla_B^* \nabla_B + V_B^{\mathrm{W}}$ with smooth Weitzenböck potential. The weak commutator form obeys

$$\begin{aligned} |\langle g, [L_B, U] f \rangle| \le C \| \nabla_B U \|_\infty (\| \nabla_B g \|_2 \| f \|_2 + \| g \|_2 \| \nabla_B f \|_2) \\ + C_B \| g \|_2 \| f \|_2 . \end{aligned}$$

This follows by expanding the two gradient forms; only first derivatives of U occur. Insert $f = S_{B,s} f_0$ and $g = S_{B,\varepsilon - s} g_0$ into the Duhamel formula for $[S_{B,\varepsilon}, U]$. The gradient heat estimate is $\| \nabla_B S_{B,s} \|_{H \to H} \le C(1 + s^{-1/2})$. Integration of $s^{-1/2} + (\varepsilon - s)^{-1/2}$ gives $C \varepsilon^{1/2}$, proving the required commutator bound. Smooth approximation gives the formula for C^1 multipliers.

The difference $L_B - L_C$ is a first-order operator with smooth coefficients, and hence is bounded from H^1 to H. Its Duhamel formula gives $\| S_{B,\varepsilon} - S_{C,\varepsilon} \|_{H \to H} \le C_{B,C} \varepsilon^{1/2}$. Finally write

$$U S_{B,\varepsilon} U^{-1} - S_{C,\varepsilon} = [U, S_{B,\varepsilon}] U^{-1} + S_{B,\varepsilon} - S_{C,\varepsilon} .$$

Since U^{-1} is unitary, (10.27) follows. □

Lemma 10.12 (A stochastic heat estimate for small operator noise)**.** *Let D_t be a predictable operator on $H = L^2(E)$ with $\| D_t \|_{H \to H} \le d$ almost surely, and let*

$$Z_t = \int_0^t e^{-(t-s)(1+L_C)} D_s \, dW_s .$$

For every $q > 0$, $p < \infty$ and finite T,

$$\| Z \|_{L^p(\Omega; C([0,T]; \mathrm{c}^{-q}))} \le C_{p,q,T} d. \tag{10.28}$$

The same statement holds after stopping the integrand at any stopping time.

Proof. Let Δ_j be spectral blocks for $1 + L_C$. The squared L^2 norm of a point-evaluation row of $\Delta_j e^{-r(1+L_C)}$ is bounded by $C2^{2j}e^{-cr2^{2j}}$. For the row difference with an additional time increment h, the bound acquires the factor $\min\{1, h2^{2j}\}^2$. The scalar martingale moment inequality and $\|D_s^* f\|_2 \le d\|f\|_2$ therefore give, for $p \ge 2$,

$$\mathbb{E}\|\Delta_j Z_t\|_{L_x^p}^p \le C_p d^p,$$
$$\mathbb{E}\|\Delta_j (Z_t - Z_s)\|_{L_x^p}^p \le C_p d^p \min\{1, (|t-s|2^{2j})^{p/2}\}.$$

No independence between D and the noise is used. Choose $1/p < \gamma < 1/2$. Integrating the second estimate against $|t-s|^{-1-\gamma p}$ and using the one-dimensional fractional Sobolev embedding gives

$$\mathbb{E}\sup_{t\le T}\|\Delta_j Z_t\|_{L_x^p}^p \le C_{p,\gamma,T} d^p 2^{2\gamma p j}.$$

The spatial Bernstein inequality contributes $2^{2j/p}$ on passing from L^p to L^∞. Choose p large enough and $\gamma > 1/p$ close enough to $1/p$ that $2\gamma + 2/p < q$. The sum

$$\sum_{j\ge 0} 2^{-qpj}\mathbb{E}\sup_{t\le T}\|\Delta_j Z_t\|_\infty^p$$

then converges and is bounded by Cd^p. The low block has the same bound by the elementary martingale estimate. The summable tails prove continuity in the little space, since the finite-frequency paths are continuous. Larger moments followed by Jensen's inequality give every specified finite p. A stopped integrand has the same operator bound, so the proof is unchanged. □

Corollary 10.13 (Vanishing of the linear rotated-noise defect)**.** *Let u^ε be predictable comparison gauges, localised on a common interval where $\sup_t \|\nabla_B \mathrm{Ad}_{u_t^\varepsilon}\|_\infty \le R$ uniformly in ε. Put*

$$D_t^\varepsilon = \mathrm{Ad}_{u_t^\varepsilon} S_{B,\varepsilon} \mathrm{Ad}_{u_t^\varepsilon}^{-1} - S_{C,\varepsilon}.$$

For the rotated Wiener processes $\widetilde{W}^\varepsilon$, every $q > 0$ and every finite p,

$$\left\| \int_0^t e^{-(t-s)(1+L_C)} D_s^\varepsilon \, d\widetilde{W}_s^\varepsilon \right\|_{L^p(\Omega; C_T \mathrm{c}^{-q})} \le C_{p,q,T,B,C}(1+R)\varepsilon^{1/2}. \tag{10.29}$$

In particular such a localisation is supplied on initial intervals by bounded data in Theorem 10.3 with $u_0 \in \mathcal{G}^{2-\kappa}$, including $u_0 = 1$. For rough initial gauges it is available on compact positive-time intervals; the initial gradient has the explicit weight (10.22).

Proof. Combine Lemmas 10.11 and 10.12. When the interval is specified by a stopping time, first stop the defect coefficient and extend the orthogonal noise integrand as in Proposition 10.10. On the common interval the resulting integrals equal the original ones. □

Remark 10.14 (The nonlinear comparison). Applied to convergent fixed-reference heat approximations, Corollary 10.6 gives the limiting gauge and transformed connection. Proposition 10.10 identifies its rotated Wiener noise, and (10.29) removes the diffusion defect after one heat integration. Identifying that transformed connection with the prescribed C-reference continuum solution also requires comparison of the critical enhanced products and their finite contractions.

The time regularity proved here explains this remaining step. Although u has finite variation in $\mathrm{c}^{-\kappa}$, the formal differential of the affine gauge action contains $(\partial_t u)a$. Both factors can have spatial regularity $-\kappa$, so their ordinary product is not defined by (10.1). Low-topology finite variation therefore does not supply a classical Itô change-of-variables formula for the singular equation. The controlled expansion (10.25) identifies the stochastic products involved. Section 11 constructs the required comparison directly from predictable heat-kernel remainders and their finite contractions.

11 Predictable gauge transport of the heat enhancement

A change of reference can produce a finite correction in the quadratic tensor data. Its Yang–Mills contraction vanishes. We prove the transport estimate for predictable gauges with smooth initial value. The additional spatial derivative supplied by this initial regularity permits estimates of the stochastic remainder by Itô isometry and Burkholder–Davis–Gundy inequalities.

Throughout this section B, C are smooth reference connections on the same principal bundle over a closed Riemannian surface, $E = T^*\Sigma \otimes \operatorname{ad} P$, and $A_C = 1 + \Delta_C$. We write $S_{C,\epsilon} = e^{-\epsilon A_C}$ and $\mathcal{K}_C = (\partial_t + A_C)^{-1}$ for the causal massive heat integration operator. All comparisons use the same heat time ϵ. Spatial Hölder spaces are denoted by $\mathcal{C}^a_x$; $C_T\mathcal{C}^a_x$ denotes $C([0,T], \mathcal{C}^a_x)$, with an arbitrarily small loss in the exponent when strong continuity is needed. The paraproduct $f \prec g$ is spatial. A finite collection of coordinate charts, bundle frames, and smooth Littlewood–Paley operators may be fixed to define it. Changes of this collection give the usual smoother commutators and do not alter the limits proved below.

11.1 The one finite quadratic contraction

Proposition 11.1 (Change of the connection in the spatial heat cutoff)**.** *Let $X^\epsilon_{B,C}$ be the stationary solution of*

$$dX^\epsilon_{B,C} = -A_B X^\epsilon_{B,C}\, dt + S_{C,\epsilon}\, dW_t,$$

and set $b = C - B$. In a normal orthonormal frame and a radial B-frame, the difference of the equal-time differentiated covariance has the following limit, intrinsically interpreted on the diagonal:

$$\mathbb{E}\big[(X^\epsilon_{B,C})^a_i (\nabla_B)_\ell (X^\epsilon_{B,C})^b_j\big] - \mathbb{E}\big[(X^\epsilon_{B,B})^a_i (\nabla_B)_\ell (X^\epsilon_{B,B})^b_j\big] \longrightarrow \frac{\delta_{ij}}{8\pi} (\operatorname{ad} b_\ell)^{ab}. \tag{11.1}$$

Convergence holds in every smooth diagonal coefficient norm. In particular, with $\mathcal{C}_{\mathrm{ad}} = -\sum_a \operatorname{ad}(e_a)^2$,

$$\Delta\mathbb{E}[X^\epsilon_i, (\nabla_B)_\ell X^\epsilon_j] \longrightarrow -\frac{\delta_{ij}}{8\pi} \mathcal{C}_{\mathrm{ad}} b_\ell. \tag{11.2}$$

Its Yang–Mills contraction is zero:

$$\Delta\mathbb{E}\, Q_B(X^\epsilon, X^\epsilon)_i \longrightarrow -\frac{2 - \dim \Sigma}{8\pi} \mathcal{C}_{\mathrm{ad}} b_i = 0. \tag{11.3}$$

Proof. The equal-time covariance operator is

$$R^\epsilon_{B,C} = \int_0^\infty e^{-rA_B} e^{-2\epsilon A_C} e^{-rA_B}\, dr.$$

At the centre of a normal radial frame, $A_C - A_B = -2 \operatorname{ad}(b_j)\partial_j + V$, where V has order zero; variations of the principal coefficients are common to the two operators. Duhamel's formula for the middle heat operator shows that the only term contributing to a differentiated diagonal limit is

$$4\epsilon\, \operatorname{ad}(b_j(x)) \int_0^\infty \partial_j H_{2(r+\epsilon)}(x - y)\, dr,$$

where $H_s(z) = (4\pi s)^{-1} \exp(-|z|^2/(4s))$ is the Euclidean two-dimensional heat kernel. Consequently its y_ℓ derivative on the diagonal equals

$$4\epsilon \operatorname{ad}(b_j(x)) \int_0^\infty \frac{\delta_{j\ell}}{32\pi(r+\epsilon)^2}\, dr = \frac{1}{8\pi} \operatorname{ad}(b_\ell(x)).$$

The identity tensor on the one-form slot gives δ_{ij}.

The omitted terms vanish on the diagonal in the limit. An order-zero insertion, a first-order coefficient increment, or the first omitted heat-parametrix term gains at least one power of the heat length. After the one diagonal derivative its absolute contribution, including any fixed number of derivatives along the diagonal, is bounded by

$$C_m\epsilon\left(1+\int_0^1 (r+\epsilon)^{-3/2}\,dr\right)\le C_m\sqrt{\epsilon}.$$

Gaussian off-diagonal bounds treat the complements of the normal neighbourhoods, and the massive large-time part is $O(\epsilon)$ in every smooth norm. These estimates also justify Duhamel integration and diagonal differentiation before taking the limit.

Finally, $\sum_{a,b}[e_a,e_b](\operatorname{ad} v)^{ab}=\sum_b[[v,e_b],e_b]=-\mathcal{C}_{\mathrm{ad}}v$. Applying $2\delta_{ji}b_j-\delta_{jj}b_i=(2-d)b_i$ proves the last two identities. In particular the tensor limit in (11.1) must not be discarded before this contraction is performed. □

11.2 A kernel estimate with predictable coefficients

Fix $0<\kappa<1/16$ and put $\alpha=2-\kappa$. Let u be a predictable gauge, $U=\mathrm{Ad}_u$, and assume on a stopped time interval that

$$\|U\|_{L_T^\infty\mathcal{C}_x^\alpha}+\|U^{-1}\|_{L_T^\infty\mathcal{C}_x^\alpha}+\|\partial_t U\|_{L_T^\infty\mathcal{C}_x^{\alpha-2}}\le R. \tag{11.4}$$

The time derivative is a Bochner derivative in the displayed negative space; in particular every spatially smoothed coefficient has finite variation in time. We take $u(0)$ smooth and deterministic and extend u constantly to negative times. This convention makes stationary linear objects available without an anticipative extension. The same proof with zero-start linear objects is described below. When (11.4) is obtained up to a stopping time σ, use $U_{t\wedge\sigma}$ thereafter. This terminal-value extension preserves the spatial bound and the almost-everywhere weak time-derivative bound. An extension by the identity with a jump at σ, although harmless for merely identifying a rotated Wiener process, must not be used in the commutator argument.

Let $h=u\cdot B-C$. The transformation formula for a connection gives

$$\|\operatorname{ad} h\|_{L_T^\infty\mathcal{C}_x^{1-\kappa}}+\|\partial_t\operatorname{ad} h\|_{L_T^\infty\mathcal{C}_x^{-1-\kappa}}\le C_{B,C,R}. \tag{11.5}$$

Indeed, $\operatorname{ad} h$ contains one spatial derivative of U, and its time derivative is a sum of products of $\nabla\partial_t U$ with U^{-1} and of ∇U with $\partial_t U^{-1}$. Both products are defined by the ordinary Hölder product theorem, with $\partial_t\operatorname{ad} h$ in the negative space displayed above. Only this adjoint coefficient is used below. A central component of h is invisible to both the commutator and $\mathcal{C}_{\mathrm{ad}}$ and need not be controlled by a norm of U. In operator and paraproduct estimates below, writing h as a multiplier abbreviates this adjoint coefficient; Lie-algebra-valued expressions such as $\mathcal{C}_{\mathrm{ad}}h$ retain their literal meaning.

Lemma 11.2 (The remainder of the spatial heat commutator)**.** *Define the predictable operator*

$$D_\epsilon(t)=U_tS_{B,\epsilon}U_t^{-1}-S_{C,\epsilon}.$$

There is a decomposition, interpreted as an identity of operators on the noise space,

$$D_\epsilon\,dW=\sum_j\operatorname{ad}(h_j)\prec(2\epsilon\nabla_jS_{C,\epsilon}\,dW)+E_\epsilon\,dW. \tag{11.6}$$

For a spatial dyadic block Δ_m, $m\ge0$, its remainder satisfies the Hilbert–Schmidt evaluation bound

$$\sup_x\|\Delta_mE_\epsilon(t;x,\cdot)\|_{L^2(\Sigma)}\le C_{B,C,R}\min\{\epsilon^{(\alpha-1)/2},2^{-m(\alpha-1)}\}. \tag{11.7}$$

All bundle indices are included in the Hilbert–Schmidt norm. The bound is pathwise and uniform over predictable U satisfying (11.4).

Proof. In a radial C-frame, compare the two heat kernels before applying them to a distribution. Their scalar principal Gaussian and one-form identity are the same. Taylor expansion of their relative parallel transport gives

$$U(x)H_\epsilon^B(x,y)U^{-1}(y) - H_\epsilon^C(x,y) = -\operatorname{ad}(h_j(x))(x-y)^j H_\epsilon(x-y) + \mathscr{R}_\epsilon(x,y).$$

The leading term is $2\epsilon \operatorname{ad}(h_j)\nabla_j H_\epsilon^C$, up to a remainder of the same admissible size. A Taylor remainder of U is $O(d(x,y)^\alpha)$, and the first heat coefficient remainder is $O(\epsilon)$. Gaussian moment estimates therefore give

$$\sup_x \|\mathscr{R}_\epsilon(x,\cdot)\|_{L^2_y} \le C_R \epsilon^{(\alpha-1)/2}, \qquad \|\mathscr{R}_\epsilon\|_{\mathcal{C}_x^{\alpha-1}(L^2_y)} \le C_R. \tag{11.8}$$

We prove the second estimate by taking difference quotients of U. For $d(x,x') \ge \sqrt{\epsilon}$ it follows from the first estimate. For $d(x,x') < \sqrt{\epsilon}$, subtract the two first-order Taylor remainders. The increment of ∇U is $O(d(x,x')^{\alpha-1})$; the differentiated Gaussian terms are bounded using $d(x,x')\epsilon^{(\alpha-2)/2} \le d(x,x')^{\alpha-1}$. The term involving the increment of h has the same bound since $\|2\epsilon\nabla H_\epsilon^C(x,\cdot)\|_{L^2_y} \le C$. This proves (11.8) in each chart. Parallel transport and changes of frames add only bounded smooth coefficients.

Replacing the full product $\operatorname{ad}(h_j)(2\epsilon\nabla_j S_{C,\epsilon}\, dW)$ by its low–high paraproduct produces the same two bounds. To see the small bound explicitly, put $N = 2^n$. The high coefficient block has size $N^{-(\alpha-1)}$, whereas the Hilbert–Schmidt evaluation norm of the corresponding differentiated noise block is bounded by $C\epsilon N^2 e^{-c\epsilon N^2}$. Their sum is at most

$$C_R \sum_{N \text{ dyadic}} \epsilon N^{3-\alpha} e^{-c\epsilon N^2} \le C_R \epsilon^{(\alpha-1)/2}.$$

The high–low terms are bounded by summing the coefficient tail first. The $\mathcal{C}_x^{\alpha-1}(L^2_y)$ bound follows from the same dyadic sum without the small factor. Smooth localizations have rapidly decreasing off-diagonal block tails, which satisfy these bounds as well. Applying a dyadic block to the two bounds in (11.8) proves (11.7). □

Lemma 11.3 (A stochastic convolution of the kernel remainder)**.** *Under the preceding assumptions, for every $p < \infty$ and $0 < \beta < \alpha - 1$, with an arbitrarily small $\nu > 0$ satisfying $\beta + \nu < \alpha - 1$,*

$$\left(\mathbb{E}\,\|\mathcal{K}_C(E_\epsilon\, dW)\|^p_{C_T\mathcal{C}_x^{1+\beta}}\right)^{1/p} \le C_{p,R,T,\beta,\nu}\epsilon^{(\alpha-1-\beta-\nu)/2}. \tag{11.9}$$

The same process tends to zero in parabolic $\mathcal{C}_{\mathfrak{s}}^{1-\kappa}$, after a further arbitrarily small loss if necessary. No Gaussian assumption on E_ϵ is made. In addition, for $0 < \gamma < 1$ and every sufficiently small $\nu > 0$, the same proof gives the faster low-norm estimate

$$\left(\mathbb{E}\,\|\mathcal{K}_C(E_\epsilon\, dW)\|^p_{C_T\mathcal{C}_x^{\gamma}}\right)^{1/p} \le C_{p,R,T,\gamma,\nu}\epsilon^{(\alpha-1-\nu)/2}. \tag{11.10}$$

Proof. Use a spectral dyadic resolution for A_C to commute the output block with the heat flow; changing the resolution in (11.7) preserves that bound. Its frequency-localized heat kernel has L^1 norm at most $Ce^{-c2^{2m}r}$. Itô isometry and BDG therefore give, at a fixed point and time,

$$\|\Delta_m\mathcal{K}_C(E_\epsilon\, dW)(t,x)\|_{L^p(\Omega)} \le C_p R\, 2^{-m}\min\{\epsilon^{(\alpha-1)/2}, 2^{-m(\alpha-1)}\}.$$

Time increments follow by splitting the stochastic integral at the earlier time and using the difference of the two heat semigroups in the remaining integral. Spatial increments follow from

localized kernel derivatives. Kolmogorov's estimate, with the moment order chosen sufficiently large, costs at most $2^{\nu m}$ in taking the space–time supremum. For $2^m \le \epsilon^{-1/2}$ the weighted bound is $C\epsilon^{(\alpha-1)/2}2^{(\beta+\nu)m}$; for $2^m \ge \epsilon^{-1/2}$ it is $C2^{-(\alpha-1-\beta-\nu)m}$. Summation proves (11.9). The temporal estimate with exponent less than $1/2$, together with the spatial estimate, gives the stated parabolic conclusion. It is not asserted that the stochastic remainder has parabolic regularity greater than one. For (11.10), use the first term of the minimum in (11.7); the sum of $2^{-(1-\gamma-\nu')m}$ is finite when the Kolmogorov loss $\nu' > 0$ is sufficiently small. This proves even the rate without ν; the displayed weakened rate is convenient for subsequent localization estimates. □

11.3 Moving the low–high coefficient through the heat equation

Let $X_\epsilon = \mathcal{K}_C S_{C,\epsilon}\xi$ be stationary and put $\eta_\epsilon^j = 2\epsilon\nabla_j X_\epsilon$. For every $\rho > 0$, standard heat covariance estimates give a positive power of ϵ in

$$\|\eta_\epsilon\|_{C_T\mathcal{C}_x^{1-\rho}} + \|\eta_\epsilon\|_{\mathcal{C}_{\mathfrak{s}}^{1-\rho}} \longrightarrow 0 \quad \text{in every finite } L^p. \tag{11.11}$$

More precisely, if $X = \mathcal{K}_C\xi$ is bounded in $C_T\mathcal{C}_x^{-\rho_0}$, heat smoothing gives

$$\|\eta_\epsilon\|_{C_T\mathcal{C}_x^s} \le C\epsilon^{(1-s-\rho_0)/2}\|X\|_{C_T\mathcal{C}_x^{-\rho_0}}, \qquad -1 < s < 1-\rho_0. \tag{11.12}$$

Here $\rho_0 > 0$ can be chosen as small as needed independently of the regularity loss κ in the gauge norm. One may first solve the differentiated heat equation on $T^*\Sigma \otimes E$. Its difference from $2\epsilon\nabla X_\epsilon$ has the form $2\epsilon\mathcal{K}^{(1)}[\nabla, A_C]X_\epsilon$. The commutator has order at most one when the tensor-product connection is used. Heat estimates show that this difference tends to zero in $C_T\mathcal{C}_x^{2-2\kappa}$; it can be included in the remainder below.

Proposition 11.4 (Expansion for a predictable smooth-initial gauge)**.** *Define*

$$Z_\epsilon = \mathcal{K}_C(D_\epsilon\, dW), \qquad \widetilde{X}_\epsilon = X_\epsilon + Z_\epsilon.$$

Then

$$Z_\epsilon = \sum_j \mathrm{ad}(h_j) \prec \eta_\epsilon^j + R_\epsilon, \qquad R_\epsilon \longrightarrow 0 \quad \textit{in } C_T\mathcal{C}_x^{2-2\kappa}, \tag{11.13}$$

locally in probability and in every finite L^p under stopped bounds. Also $Z_\epsilon \to 0$ in $\mathcal{C}_{\mathfrak{s}}^{1-\kappa}$ and in $C_T\mathcal{C}_x^{1-\kappa}$. The convergences have a positive power of ϵ, with constants depending on the stopped gauge norm. The following additional low-norm rate is part of the assertion: for $0 < \nu < \kappa/4$,

$$\|R_\epsilon\|_{C_T\mathcal{C}_x^{1/2}} + \|Z_\epsilon\|_{C_TL_x^\infty} = O_{L^p}\big(\epsilon^{(1-\kappa-\nu)/2}\big). \tag{11.14}$$

Proof. Apply Lemma 11.3 with $1+\beta = 2-2\kappa$ and $0 < \nu < \kappa$. It remains to consider the low–high term in (11.6). Every low spatial block of h has finite variation in time. Itô's product formula for that block and the stochastic heat equation for η_ϵ therefore has no quadratic covariation term. It gives

$$(\partial_t + A_C)(h \prec \eta_\epsilon) = h \prec (2\epsilon\nabla S_{C,\epsilon}\xi) + (\partial_t h) \prec \eta_\epsilon + \mathcal{C}_A(h, \eta_\epsilon),$$

where $\mathcal{C}_A$ is the spatial commutator between the product and the two compatible Laplace-type operators. Its leading terms are $(Ah) \prec \eta_\epsilon$ and $-2\nabla h \prec \nabla\eta_\epsilon$; chart and bundle commutators have no worse regularity.

Take $\rho > 0$ in (11.11) much smaller than κ. The temporal derivative term is a paraproduct, not a full product. Indeed,

$$\|S_{m-2}\partial_t h\|_\infty \le C_R 2^{(1+\kappa)m}, \qquad \|\Delta_m\eta_\epsilon\|_\infty \le o(1)2^{-(1-\rho)m}.$$

Its mth block is bounded by $o(1)2^{(\kappa+\rho)m}$, and heat integration contributes 2^{-2m}. The terms involving Ah and ∇h have the same bound. A small exponent loss in the deterministic Schauder estimate puts their integrals in $C_T\mathcal{C}_x^{2-2\kappa}$, with convergence to zero. This is the specific frequency summation that permits $\partial_t h \in \mathcal{C}_x^{-1-\kappa}$.

For the faster low norm, use (11.12) with $s = -\rho$, where $\rho, \rho_0 > 0$ are much smaller than κ. The same commutator forcing now belongs to $\mathcal{C}_x^{-1-\kappa-\rho}$ with size $O_{L^p}(\epsilon^{(1+\rho-\rho_0)/2})$. Heat integration places it in $C_T\mathcal{C}_x^{1/2}$, with that rate, since $1-\kappa-\rho > 1/2$. Combine this with (11.10). Finally use (11.12) at a sufficiently small positive s to bound $h \prec \eta_\epsilon$ in L^∞. This proves (11.14). This rate controls the low–high product of R_ϵ with ∇X_ϵ in $\mathcal{C}_x^{-2\kappa}$.

The constant extension to negative times makes the product identity valid for the whole causal massive integral without an initial boundary term. Alternatively use zero-start X_ϵ^0 and $\eta_\epsilon^0 = 2\epsilon\nabla X_\epsilon^0$; then $\eta_\epsilon^0(0) = 0$ and again no initial product occurs. If one uses a stationary η_ϵ together with a zero-start defect, the term $-e^{-tA_C}(h(0) \prec \eta_\epsilon(0))$ must be retained. The first two conventions eliminate this initial boundary term. □

11.4 The Gaussian resonance and the contracted enhancement

Lemma 11.5 (A disappearing resonance with a nonzero mean)**.** *For stationary X_ϵ and $\eta_\epsilon^\ell = 2\epsilon\nabla_\ell X_\epsilon$, the centred products*

$$X^a_{\epsilon,i}\,\nabla_k\eta^{\ell,b}_{\epsilon,j} - \mathbb{E}[X^a_{\epsilon,i}\,\nabla_k\eta^{\ell,b}_{\epsilon,j}], \qquad \eta^{\ell,a}_{\epsilon,i}\,\nabla_k X^b_{\epsilon,j} - \mathbb{E}[\eta^{\ell,a}_{\epsilon,i}\,\nabla_k X^b_{\epsilon,j}]$$

tend to zero in $C_T\mathcal{C}_x^{-\sigma}$ for every $\sigma > 0$, in every finite L^p, with a positive power of ϵ. Their means satisfy

$$\mathbb{E}[X^a_{\epsilon,i}\,\nabla_k\eta^{\ell,b}_{\epsilon,j}] = -\frac{1}{16\pi}\delta_{ij}\delta_{k\ell}\delta^{ab} + o(1), \tag{11.15}$$

$$\mathbb{E}[\eta^{\ell,a}_{\epsilon,i}\,\nabla_k X^b_{\epsilon,j}] = \frac{1}{16\pi}\delta_{ij}\delta_{k\ell}\delta^{ab} + o(1). \tag{11.16}$$

The remainders tend to zero in every smooth spatial coefficient norm. For the zero-start η_ϵ^0, paired with stationary X_ϵ, replace $1/(16\pi)$ by $t/(16\pi(t+\epsilon))$; the spatial remainder still tends uniformly to zero on bounded time intervals.

Proof. Wick's formula bounds the equal-time covariance of either centred product, at spatial distance r, by

$$C\epsilon^2(r+\sqrt{\epsilon})^{-4}(1+|\log(r+\sqrt{\epsilon})|).$$

This function has spatial L^1 norm at most $C\epsilon(1+|\log\epsilon|)$, and the pointwise variance is at most $C(1+|\log\epsilon|)$. Consequently a spatial block of scale 2^m has pointwise second moment bounded by

$$C\min\{\epsilon 2^{2m}, 1\}(1+|\log\epsilon|).$$

The two terms in Wick's formula use respectively $C_\epsilon\,\epsilon^2\nabla^4 C_\epsilon$ and $\epsilon^2(\nabla^2 C_\epsilon)^2$. These bounds follow from the heat covariance representation and its Gaussian derivative estimates, so they remain valid in local bundle frames. The smooth off-diagonal terms are smaller.

Gaussian hypercontractivity supplies all finite moments. Time increments follow from the same heat covariance formula, at a cost of an arbitrarily small negative power of ϵ in the temporal supremum estimate; spatial Bernstein and a sufficiently large moment order have an arbitrarily small dyadic cost. Multiplication by $2^{-\sigma m}$ and summation, splitting at $2^m = \epsilon^{-1/2}$, proves the claimed convergence. This argument also covers the zero-start covariance, whose extra heat factors only improve the bounds.

For the means, the covariance is $\frac{1}{2}\int_{2\epsilon}^\infty H_r\,dr$. At a normal radial diagonal, two derivatives on the same variable of the principal heat kernel give $-\delta_{k\ell}/(8\pi r^2)$, whereas one derivative

on each variable gives the opposite sign. Multiplication by 2ϵ yields (11.15)–(11.16). The first omitted diagonal term is $O(r^{-1})$, hence contributes $O(\epsilon(1+|\log\epsilon|))$. For zero start the integral ends at $2\epsilon+2t$, giving the displayed factor. The statement about smooth coefficient derivatives follows by diagonal differentiation of the heat parametrix. □

Lemma 11.6 (The equation-generated finite model)**.** *The coupled Yang–Mills fixed point factors through the following equation-generated finite model. Retain the noise and jet sectors, X, $X_1=\nabla\mathcal{K}_C\xi$, the Wick-square and Wick-cube sectors, and the contracted forcing symbol*

$$\mathsf{Q}=\mathfrak{q}_C(X,X_1),\qquad Y=\mathcal{K}_C\mathsf{Q},\qquad Y_1=\nabla\mathcal{K}_C\mathsf{Q}.$$

Here $\mathfrak{q}_C$ is the algebraic contraction of a solution slot and a derivative slot in (2.8). Retain also the one-noise responses of $\mathfrak{q}_C(v,X_1)$ and $\mathfrak{q}_C(X,w)$ for smooth solution and derivative jets v,w, their required planted responses Ψ, and the four critical contractions

$$\mathfrak{q}_C(X,Y_1),\quad \mathfrak{q}_C(Y,X_1),\quad \mathfrak{q}_C(X,\Psi_1),\quad \mathfrak{q}_C(\Psi,X_1).$$

Include their jet multiples, reexpansion descendants, and the two planted kernel types in the finite forcing and solution windows of the singular fixed-point theorem. Denote this collection by $\mathfrak{T}_{\mathrm{YM}}$. The outer contractions and one-noise responses are polarised over the finite tensor coefficient families of solution and derivative jets. In particular an endomorphism acting on a Y or Ψ slot is retained. The inner pure two-noise vertex remains the fixed contraction $\mathsf{Q}=\mathfrak{q}_C(X,X_1)$.

This collection is closed under reexpansion and under every truncated product and integration used by the fixed point on its closed affine Picard subspace: the coefficients of X in U and of X_1 in U_1 are fixed to the identity. The restricted model and the solution map on this subspace are locally Lipschitz functions of this contracted finite enhancement. In particular, the free tensor component $X\otimes\nabla X$ is not needed separately. A finite shift

$$(X\otimes\nabla X)^{ab}_{ij,k}\longmapsto(X\otimes\nabla X)^{ab}_{ij,k}+\delta_{ij}(\operatorname{ad}v_k)^{ab} \tag{11.17}$$

does not change any equation-generated component, provided its descendants are formed after the Yang–Mills contraction.

Proof. We restrict the finite construction to the equation-generated symbols. The bracket and metric contractions in Q_C are parallel for the chosen tensor-product connections. They commute with bundle transport and its covariant jet operations. In this construction one first contracts to the E-valued forcing slot and then applies the original heat kernel on E, or its differentiated kernel. No commutation of a contraction with heat kernels on different tensor bundles is asserted. The precision corrections are those of this integration operator on contracted inputs. The contraction of (11.17) is zero pointwise by $2\delta_{ji}v_j-\delta_{jj}v_i=0$; any outer jet or bundle endomorphism multiplies this zero after contraction.

First consider reexpansion without jet decorations. The negative linear symbols X,X_1 have no jet subtraction. In the paired solution sectors, reexpansion of higher planted symbols contributes only jets and higher planted descendants. Thus the coefficients of X in U and of X_1 in U_1 remain the identity. Reexpanding Y or a one-noise response Ψ_v adds its canonical jet and, for Ψ_v, reexpands the coefficient jet v. Hence reexpanding $\mathfrak{q}_C(Y,X_1)$ adds only $\mathfrak{q}_C(j,X_1)$, while reexpanding $\mathfrak{q}_C(X,Y_1)$ adds only $\mathfrak{q}_C(X,j_1)$. These are precisely the one-noise jet-labelled terms already retained. The same argument applies to Ψ,Ψ_1. Reexpanding the cubic contraction adds only its retained Wick-square and one-noise jet-labelled terms. For decorated symbols the jet product rule gives this same finite list with lower jet degrees. The bundle precision defect has the same form, since the two integration extensions act linearly on that contracted E-valued input. Thus the staged integration and reexpansion proof of

the model-construction theorem restricts to $\mathfrak{T}_{\mathrm{YM}}$, with the same precision bounds and local Lipschitz estimates.

It remains to check that these components suffice for the iteration. The coefficient of the additive noise response X is the identity at every Picard iterate; it is not an arbitrary matrix-valued solution jet. Thus its two-noise derivative product occurs only as Q. Its planted response is Y, and any next critical multiplication is one of the four displayed contractions. Replacing an input by a solution jet gives the retained Ψ families. Cubic terms require the Wick powers and their jet multiples. A further nontrivial generation is above the forcing window, by the original finite-tree count. Every Picard iterate and its reconstruction therefore use only the restricted data. The product, reconstruction, and two-kernel Schauder estimates give the same contraction and two-model estimate on this closed affine subspace. The restriction does not define arbitrary products of two independently matrix-weighted leading noise symbols; those products are unnecessary for the additive equation. This proves the factorization and the assertion about (11.17). □

Theorem 11.7 (Predictable transport of the Yang–Mills enhancement)**.** *Let u satisfy (11.4), with smooth deterministic initial value. Let X_ϵ be the stationary linear object driven by $S_{C,\epsilon}\,dW$, and let $\widetilde{X}_\epsilon$ be driven by $US_{B,\epsilon}U^{-1}\,dW$, using the constant past extension specified above. Then, locally in probability,*

$$Q_C(\widetilde{X}_\epsilon,\widetilde{X}_\epsilon)-Q_C(X_\epsilon,X_\epsilon)\longrightarrow 0\quad \text{in } C_T\mathcal{C}_x^{-2\kappa}. \tag{11.18}$$

Its heat integral converges to zero in $\mathcal{C}_{\mathfrak{s}}^{2-3\kappa}$ and in $C_T\mathcal{C}_x^{2-3\kappa}$, with the usual initial lift when stationary objects are used. The Wick powers, jet-response resonances, and the two three-noise derivative resonances needed for the Yang–Mills fixed point have the same limits for the two noises. The corresponding models on $\mathfrak{T}_{\mathrm{YM}}$ converge to each other in their working model norms with any positive homogeneity slack. The same deterministic local subtraction coefficients are used on both sides. The assertion holds uniformly under stopped bounds for families $u=u_\epsilon$ satisfying (11.4).

This is a statement for the contracted Yang–Mills equation and its required jet responses. It does not assert equality of the limits of all uncontracted quadratic tensor components.

Proof. Insert (11.13). First check the low–high term which has the least favourable output regularity. Heat smoothing and the uncut Gaussian bound give

$$\|\nabla X_\epsilon\|_{C_T\mathcal{C}_x^{-2\kappa}}\le C\epsilon^{-(1-2\kappa+\rho_0)/2}\|X\|_{C_T\mathcal{C}_x^{-\rho_0}}.$$

Consequently (11.14) implies

$$\|Z_\epsilon\prec\nabla X_\epsilon\|_{C_T\mathcal{C}_x^{-2\kappa}}=O_{\mathbb{P}}\big(\epsilon^{(\kappa-\nu-\rho_0)/2}\big)\longrightarrow 0, \tag{11.19}$$

where $\nu+\rho_0<\kappa$. In particular this treats $R_\epsilon\prec\nabla X_\epsilon$. The other two Bony pieces of $R_\epsilon\nabla X_\epsilon$ have positive regularity sum, because $(2-2\kappa)+(-1-\rho_0)>0$, and tend to zero using the high norm of R_ϵ. In $X_\epsilon\nabla Z_\epsilon$, both nonresonant paraproducts tend to zero in $\mathcal{C}_x^{-2\kappa}$ directly from $\nabla Z_\epsilon\to 0$ in $\mathcal{C}_x^{-\kappa}$ and the Gaussian loss $\rho_0<\kappa$. Terms with two copies of Z_ϵ, or with a spatial derivative falling on the low coefficient h, are treated by these same Bony bounds and the positive-sum commutator estimates.

The only remaining resonances are the two in Lemma 11.5, multiplied by $\mathrm{ad}(h_\ell)$. The standard paraproduct commutator permits this coefficient to be moved outside the resonance: the sum of its regularity $1-\kappa$ and those of the two other factors is positive. Its commutator tends to zero by (11.11). The centred Gaussian part tends to zero by Lemma 11.5; multiplication by the random h uses its pathwise Hölder bound and requires no independence.

The sum of the two finite means, before Yang–Mills contraction, is

$$-\frac{\delta_{ij}}{8\pi}\mathcal{C}_{\mathrm{ad}}h_k + o(1).$$

In the zero-start convention, the coefficient $1/(8\pi)$ is replaced by $t/(8\pi(t+\epsilon))$. In either case its contraction is proportional to $2\delta_{ji}h_j - \delta_{jj}h_i = 0$. This proves (11.18), including at the initial time, and the deterministic heat estimate proves the assertion about its integral δY_ϵ.

To compare the remaining singular products, write $\delta X = Z_\epsilon$ and δY for the quadratic response difference. The factors in their differences have the following spatial regularity sums, using an arbitrarily small Gaussian loss $\rho > 0$:

product	sum of spatial regularities
$\delta X\, \nabla Y$	$(1-\kappa) + (-2\rho) > 0$
$X\, \nabla \delta Y$	$-\rho + (1-3\kappa) > 0$
$\delta Y\, \nabla X$	$(2-3\kappa) + (-1-\rho) > 0$
$Y\, \nabla \delta X$	$(1-2\rho) + (-\kappa) > 0$
$:X^2:\, \delta X$	$-2\rho + (1-\kappa) > 0.$

Thus the differences are defined by classical spatial products. These estimates are uniform over each retained finite tensor polarisation of the outer coefficient slots; only the inner two-noise vertex required the Yang–Mills Ward identity. The reverse derivative product still has its low–high component of regularity $-1-\rho$; in the model its basepoint value is subtracted, exactly as in the original pointed construction. The increment of δY has an extra positive power, so the pointed bound is stronger than its required working degree. No claim of improved global regularity of $Y\nabla X$ is being made. More explicitly, $\delta Y \in \mathcal{C}_{\mathfrak{s}}^{2-3\kappa}$ and $Z \in \mathcal{C}_{\mathfrak{s}}^{1-\kappa}$ accompany the spatial bounds. On a parabolic ball of radius λ, subtracting the value of δY leaves an $O(\lambda)$ spatial increment and an $O(\lambda^{2-3\kappa})$ temporal increment. Pairing this with ∇X costs $\lambda^{-1-\rho}$ and hence gives $O(\lambda^{-\rho})$, stronger than the critical degree -3κ. The estimates with Z, Ψ and their jet increments follow in the same way.

For a smooth solution-jet coefficient v, the difference of its response is $\mathcal{K}_C\{Q_C(v,\delta X) + Q_C(\delta X, v)\}$ and has regularity $2-\kappa$ up to an arbitrarily small loss. The same product check treats both jet resonances. The deterministic logarithmic subtractions multiply either the same surviving jet on both sides or δX; the latter contribution tends to zero because (11.13) has a positive power rate. Wick-square corrections in the cubic difference are handled in the same way. This covers the finite forcing and solution window in Lemma 11.6, including its pointed and jet decorations. That lemma shows that these restricted data determine the model and solution map. The argument uses only this restriction of the free-tensor construction. All estimates are pathwise in the coefficient norms and use BDG only for predictable stochastic integrals, which proves the uniform stopped-family assertion.

It remains to verify convergence in the model norms. For the noise leaf, the distributional identity

$$D_\epsilon \xi = (\partial_t + A_C)Z_\epsilon \longrightarrow 0 \quad \text{in } \mathcal{C}_{\mathfrak{s}}^{-1-\kappa}$$

follows from Proposition 11.4; this is stronger than its assigned white-noise degree. The linear planted leaf and differentiated leaf follow in the same way.

For the precision defects, apply Lemma 8.10 to differences. An inner quadratic source difference is in $\mathcal{C}_{\mathfrak{s}}^{-2\kappa}$, one order better than the original quadratic source; an inner one-noise response source difference is in $\mathcal{C}_{\mathfrak{s}}^{-\kappa}$, again one order better. Multiplication by a coefficient remainder vanishing to parabolic order N, followed by either of the two heat kernels and its prescribed jet subtraction, therefore obeys the original precision table with at least its original slack. At a critical outer vertex the differences are the classical products in the displayed

table; the pointed value subtraction was checked explicitly above. Root coefficient remainders and several coefficient slots are handled by the same finite telescoping identity as in the model-assembly proof. The local logarithmic terms cause no loss because $|\log \epsilon| Z_\epsilon \to 0$ in its stated norms.

Thus both the realization bounds and the $\Pi_q \Gamma_{q,p} - \Pi_p$ precision bounds converge on $\mathfrak{T}_{\mathrm{YM}}$. The staged deterministic integration extension first adds Y, Ψ and their differentiated kernel types; its local Lipschitz estimate gives their Γ bounds. Multiplicative reexpansion then gives the Γ bounds for the retained critical products, and the last integration extension gives the terminal responses. This proves convergence in all three model seminorms. □

Corollary 11.8 (The fixed drift is independent of the smooth heat cutoff connection)**.** *Fix the DeTurck drift with reference B and initial data in the class of the fixed-reference local existence theorem. For any smooth connection C, replacing $S_{B,\epsilon}\, dW$ by $S_{C,\epsilon}\, dW$ in its bare approximation gives the same local limit. The convergence is in the local topologies of that theorem.*

Proof. Apply Theorem 11.7 with constant gauge $u = 1$ and interchange the names of the drift and cutoff references. The contracted quadratic difference tends to zero, and the proof checks every remaining forcing symbol. Continuity of the coupled singular fixed point and its initial lift gives the conclusion. Equivalently, in the forest description the only potentially nonvanishing new local finite coefficient is (11.2); it vanishes after the Yang–Mills contraction. Every logarithmically divergent mass graph containing a cutoff difference gains one heat length and hence has vanishing difference. This corollary concerns spatial heat regularization; it does not identify schemes with a temporal mollifier. □

11.5 Stability when the rotated Brownian motion also varies

Lemma 11.9 (Finite Itô enhancements of converging rotations)**.** *Let V_n, V be predictable orthogonal bundle multipliers, on the same filtered probability space, and set $dW^{(n)} = V_n\, dW$, $dW^{(\infty)} = V\, dW$. Suppose that $V_n \to V$ locally in probability in the operator supremum norm on space and time. Every fixed finite collection of deterministic L^2-kernel, time-ordered iterated Itô integrals of these Brownian motions converges jointly in probability. If the corresponding canonical enhancement has uniform bounds in a strictly stronger Hölder model norm, its finite working model converges in the weaker norm as well.*

Proof. Start with finite-rank deterministic step kernels, and telescope a time-ordered integral of order n by replacing one noise differential at a time. On a stopping interval where $\|V_n - V\|_\infty \le \delta$, the layer containing the difference has an L^2 operator factor at most δ. At every other layer, conditional Itô isometry and orthogonality preserve the Hilbert norm. Iterating the isometry gives an L^2 bound at most $n\delta \|F\|_{L^2}$ for the difference, up to the convention for symmetrization. Density extends this conclusion to arbitrary deterministic L^2 kernels. Removing the stopping and then letting $\delta \downarrow 0$ proves convergence in probability.

Each $W^{(n)}$ is a cylindrical Brownian motion in the given filtration. Thus each of its deterministic-kernel enhancements has the usual Gaussian-chaos moment bounds, even though the collection $(W^{(n)}, W^{(\infty)})$ need not be jointly Gaussian. The renormalized finite diagrams decompose into finitely many such iterated integrals, including their integrable first-chaos contractions. Convergence for a countable determining family of tests, uniform stronger Hölder bounds, and compact embedding with the stated exponent slack give convergence in the weaker model norm. Massive tails of kernels on the whole time line are removed in L^2 before applying the finite-time argument. □

The finite anomaly in Proposition 11.1 is the heat-kernel version of the tensor anomaly proved by Bringmann and Cao [4]. The internal numbering used here is that of arXiv:2305.07197v2, dated 13 August 2025: Lemma 7.60 and Proposition 7.58; the bibliography records the published Memoirs version. The present argument uses the stronger regularity furnished by a smooth initial comparison gauge. For rough initial gauges the direct Hilbert–Schmidt summation in Lemma 11.3 does not give the required remainder regularity; the additional causal product analysed in Section 7.2 of that arXiv version then becomes relevant. Spatial, predictable regularization is essential to the finite-variation Itô calculation above. Space–time regularizations have additional finite covariance corrections, as demonstrated in [7, Section 7].

12 Reference covariance of the local continuum flow

We combine the comparison-gauge equation with the contracted enhancement estimate. The solutions are the bare spatial-heat limits of Theorem 8.14, with the finite normalization fixed there. Write $A_B(t; A_0, W)$ for the solution with reference B, initial connection A_0, and cylindrical Brownian motion W. The gauge convention is $u \cdot A = uAu^{-1} - (du)u^{-1}$.

Theorem 12.1 (Pathwise comparison with a smooth initial gauge)**.** *Let $0 < \kappa < 1/16$, let B, C be smooth connections on a fixed principal bundle over the closed surface, and suppose $A_0 - B \in \mathrm{c}^{-\kappa}(E)$. Let u_0 be a smooth deterministic gauge. On the filtered probability space of $A_B = A_B(\,\cdot\,; A_0, W)$, solve the comparison equation (10.7) with initial gauge u_0, and write $U = \mathrm{Ad}_u$. Define the predictable rotation*

$$d\widetilde{W}_t = U_t \, dW_t. \tag{12.1}$$

Then $\widetilde{W}$ is a cylindrical Brownian motion for the original filtration, after a predictable orthogonal extension beyond the comparison lifetime. On each bounded stopping interval use the terminal-value extension of U; an arbitrary predictable orthogonal extension beyond the maximal lifetime supplies a full driving Brownian motion. Almost surely,

$$A_C(t; u_0 \cdot A_0, \widetilde{W}) = u(t) \cdot A_B(t; A_0, W) \tag{12.2}$$

until $\tau_B \wedge \tau_C$, the minimum of the two canonical representative lifetimes. The comparison gauge cannot terminate strictly earlier while both representatives are alive. The approximation and identification hold in probability in the local state topology, and in every finite moment on the bounded localizations of the fixed-point theorem.

No equality of the maximal lifetimes of different DeTurck representatives is part of this statement.

Proof. The exact approximation identity. Set $A_0^\epsilon = B + S_{B,\epsilon}(A_0 - B)$, as in the fixed-reference approximation, and let A_B^ϵ be the bare cutoff solution with that initial value. Let u^ϵ solve the comparison equation along this path, with initial value u_0. The fixed-reference convergence and Theorem 10.3 imply, on every common bounded localization,

$$\begin{aligned} A_B^\epsilon &\longrightarrow A_B && \text{in } C_T\mathrm{c}^{-\kappa}, \\ u^\epsilon &\longrightarrow u && \text{in } C_T\mathrm{c}^{2-\kappa}, \\ \partial_t u^\epsilon &\longrightarrow \partial_t u && \text{in } C_T\mathrm{c}^{-\kappa}. \end{aligned} \tag{12.3}$$

The gauges have finite variation in the last spatial space. At positive cutoff and away from the initial time the fields are spatially smooth, so the classical time-dependent gauge identity and

Itô's product rule apply. The gauge factor has no quadratic covariation with the driving noise. Passing to the initial time by continuity in $\mathrm{c}^{-\kappa}$ gives

$$\begin{aligned} \widehat{A}^\epsilon &= u^\epsilon \cdot A^\epsilon_B, & d\widetilde{W}^\epsilon &= U^\epsilon\, dW, \\ d\widehat{A}^\epsilon &= \mathcal{V}_C(\widehat{A}^\epsilon)\, dt + U^\epsilon S_{B,\epsilon}(U^\epsilon)^{-1} d\widetilde{W}^\epsilon, & \widehat{A}^\epsilon(0) &= u_0 \cdot A^\epsilon_0. \end{aligned} \tag{12.4}$$

This equation has the bare drift $\mathcal{V}_C$. Its initial connection converges to $u_0 \cdot A_0$ in $\mathrm{c}^{-\kappa}$. The initial-data continuity estimate therefore applies to these transformed initial values as well as to the C-heat smoothing of $u_0 \cdot A_0$.

Convergence of the noise models. Each $\widetilde{W}^\epsilon$ is a Brownian motion for the original filtration. The family U^ϵ satisfies the stopped bounds (11.4) by (12.3). Extend a stopped gauge by its terminal value, as required in Section 11. Theorem 11.7, applied with noise $\widetilde{W}^\epsilon$, shows that the equation-generated model for the noise in (12.4) differs by $o_{\mathbb{P}}(1)$ from the model for $S_{C,\epsilon}\, d\widetilde{W}^\epsilon$. The same deterministic forest subtractions are used in the two models. In particular this step does not appeal to Gaussianity of the joint pair $(U^\epsilon, \widetilde{W}^\epsilon)$.

Second, the model for $S_{C,\epsilon}\, d\widetilde{W}^\epsilon$ converges to the canonical model built from $\widetilde{W}^\epsilon$. Although this Brownian motion varies with ϵ, the convergence estimate is uniform: the pair of canonical cutoff and limiting models is a fixed measurable function of its white-noise argument, whose law is the same for every ϵ.

Third, (12.3) implies convergence of U^ϵ to U in the operator supremum norm on space and time. Lemma 11.9 therefore shows convergence of the canonical models of $\widetilde{W}^\epsilon$ to that of $\widetilde{W}$. This is the required enhancement stability; convergence of the rotated noises as distributions alone would not suffice.

Identification of the nonlinear limit. The forest identity of the fixed-reference theorem is algebraic and applies to every smooth input, including the adapted input in (12.4). A model with the prescribed subtractions reconstructs the drift $\mathcal{V}_C - \mathfrak{C}^\epsilon_C$; adding $\mathfrak{C}^\epsilon_C$ to the modelled equation reconstructs exactly the bare drift in (12.4). These deterministic affine coefficients converge, and no extra finite term survives the Ward cancellation. Lemma 11.6 and the local two-model fixed-point estimate now identify the limit of $\widehat{A}^\epsilon$ as $A_C(\,\cdot\,; u_0 \cdot A_0, \widetilde{W})$.

On the other hand, continuity of the rough gauge action and (12.3) identify the same limit as $u \cdot A_B$. This proves (12.2) on the first common local interval. Uniform local estimates and restart prove it on each compact common interval. Exhausting bounded localizations and rational terminal times makes the identity simultaneous almost surely. The Brownian assertion follows from predictability, orthogonality, and the cylindrical Lévy characterization, as in Proposition 10.10.

To verify the asserted lifetime range, suppose the comparison gauge ended before both canonical representatives. Those two paths are bounded in the state norm on a compact neighbourhood of that time. Their equality up to the gauge lifetime and (10.18) bound the gauge in $\mathcal{G}^{1-\kappa}$. This contradicts (10.10). The identity therefore holds up to $\tau_B \wedge \tau_C$, without any assertion that these two lifetimes are equal. □

Theorem 12.2 (Comparison for a rough initial gauge)**.** *Theorem 12.1 remains valid for $u_0 \in \mathcal{G}^{1-\kappa}(P)$, with the weighted comparison gauge of Theorem 10.3. In particular the identity* (12.2) *includes the initial time in the state topology $\mathrm{c}^{-\kappa}$.*

Proof. Choose smooth gauges $u_{0,n} \to u_0$ in $\mathcal{G}^{1-\kappa}$. Such approximation follows from the definition of the little Hölder gauge group, using ambient smoothing followed by a fibrewise tubular retraction. Solve the comparison equations u_n along the same limiting path A_B. The rough comparison theorem gives a uniform initial existence interval on bounded data sets and

$$u_n \longrightarrow u \quad \text{in } C_T\mathcal{G}^{1-\kappa}, \qquad u_n \cdot A_B \longrightarrow u \cdot A_B \quad \text{in } C_T\mathrm{c}^{-\kappa},$$

with the weighted stronger convergence at positive times. Moreover $u_{0,n} \cdot A_0 \to u_0 \cdot A_0$ in $\mathrm{c}^{-\kappa}$. For each fixed n, the smooth-initial identity first holds locally. It extends throughout a chosen compact interval on which A_B and u_n are bounded: their product is bounded in the state norm, so the C-solution cannot terminate there by (9.2). The uniform rough-gauge bounds therefore supply a common interval for these identities even when $\|u_{0,n}\|_{\mathrm{c}^{2-\kappa}}$ tends to infinity.

Put $d\widetilde{W}^n = \mathrm{Ad}_{u_n}\, dW$ and $d\widetilde{W} = \mathrm{Ad}_u\, dW$. The rotations converge uniformly in space and time in probability on the common localizations. Lemma 11.9 gives convergence of their canonical enhancements. The initial-data and two-model fixed-point estimates therefore yield

$$A_C(\,\cdot\,; u_{0,n} \cdot A_0, \widetilde{W}^n) \longrightarrow A_C(\,\cdot\,; u_0 \cdot A_0, \widetilde{W}) \quad \text{locally in probability in } C_T\mathrm{c}^{-\kappa}.$$

For every fixed n the left-hand side equals $u_n \cdot A_B$ by Theorem 12.1. Taking $n \to \infty$ proves the result. The two-representative argument at the end of the smooth-initial proof gives the same lifetime range. For each fixed smooth initial gauge we first take the heat-cutoff limit. The subsequent rough-initial limit uses the uniform $\mathrm{c}^{1-\kappa}$ comparison theory and Brownian-model stability, so it requires no uniform $\mathrm{c}^{2-\kappa}$ bound on the approximants. □

Proposition 12.3 (Composition of gauges and rotated noises)**.** *Let u compare references B and C along A_B, and let v compare C and D along $u \cdot A_B$. On a common existence interval, their composition vu is the comparison gauge from B to D with initial value v_0u_0. The corresponding Brownian rotations satisfy*

$$\int_0^t \mathrm{Ad}_{v_s}\, d\widetilde{W}_s = \int_0^t \mathrm{Ad}_{v_s u_s}\, dW_s. \tag{12.5}$$

The inverse gauge gives the inverse rotation.

Proof. The deterministic comparison identity telescopes:

$$\begin{aligned}(\partial_t(vu))(vu)^{-1} &= (\partial_t v)v^{-1} + \mathrm{Ad}_v((\partial_t u)u^{-1}) \\ &= \chi_D(vu \cdot A_B) - \mathrm{Ad}_{vu}\,\chi_B(A_B).\end{aligned}$$

The weighted rough products are justified by the comparison theorem, or by smooth approximation. Its uniqueness proves the gauge assertion. Equation (12.5) is the associativity of Itô integration for bounded predictable orthogonal multipliers. Applying the same argument to u^{-1} proves the inverse statement. □

Corollary 12.4 (Reference-independent local orbit laws)**.** *Let π denote the rough gauge-orbit map on $\mathcal{A}_\kappa(P)$. If $A_0' = u_0 \cdot A_0$ with $u_0 \in \mathcal{G}^{1-\kappa}(P)$, then for any smooth references B, C the two local solutions starting from A_0, A_0' admit a coupling by cylindrical Brownian motions such that*

$$\pi A_B(t; A_0, W) = \pi A_C(t; A_0', \widetilde{W})$$

on their common comparison interval, almost surely. In particular their local orbit-path laws are consistent under restriction to common coupled stopping intervals and depend only on the initial rough gauge orbit. The composition of these local comparisons is consistent for three or more references.

Proof. Use Theorem 12.2 and apply π to (12.2). The rotated driving motion has the same white-noise law, so each marginal is the prescribed fixed-reference continuum solution law. The local cocycle in Proposition 12.3 gives consistency of repeated reference changes. The statement identifies local orbit germs, equivalently these common stopped comparisons. It does not compare laws killed at reference-specific explosion times. Section 20 continues them by measurable changes of representative to obtain the maximal local orbit process and its intrinsic lifetime. □

All assertions have a conditional version at a stopping time. The gauge at that time is measurable with respect to the past, and the future Brownian increments are independent of that past. An independent auxiliary negative-time noise can be used for the stationary model at the restart; the causal solution map removes dependence on this auxiliary choice. Conditioning on the initial sigma-field and applying the same stopped estimates then gives the local comparison and noise cocycle after restart.

13 Abelian dynamics in every topological sector

Let Σ be a connected, closed, oriented Riemannian surface, and put $V = \mathrm{Vol}(\Sigma)$. We identify $\mathfrak{u}(1)$ with $\mathbb{R}$ by $r \mapsto -ir$. Thus a real local connection form a represents the unitary connection form $-ia$, its real curvature is da, and parallel transport contributes $\exp(i\int a)$. In this convention the degree of a Hermitian line bundle is $(2\pi)^{-1}\int_\Sigma F_A$. All Laplacians below are nonnegative.

Fix an integer m and a Hermitian line bundle $L_m \to \Sigma$ of degree m. There is a smooth unitary connection A_m with real curvature

$$F_{A_m} = \frac{2\pi m}{V}\,\mathrm{vol}_g. \tag{13.1}$$

Indeed, if B is any smooth connection, the two-form $F_B - (2\pi m/V)\mathrm{vol}_g$ has zero integral. Hodge theory gives a smooth one-form b with $db = F_B - (2\pi m/V)\mathrm{vol}_g$, and $B - b$ has the required curvature. The difference of any two connections on L_m is a globally defined real one-form.

13.1 A complete state space for the Abelian gauge quotient

Write Δ_q for the Hodge Laplacian on real q-forms and $\mathcal{H} = \ker \Delta_1$. Define the lattice

$$\Lambda = \left\{ h \in \mathcal{H} : \int_\gamma h \in 2\pi\mathbb{Z} \text{ for every smooth closed curve } \gamma \right\}, \qquad J_g = \mathcal{H}/\Lambda. \tag{13.2}$$

The quotient J_g carries the flat metric induced by the L^2 inner product on $\mathcal{H}$. In genus zero, J_g is a single point.

For $s > 0$, let H^{-s}_{coex} be the coexact subspace of $H^{-s}(\Sigma; T^*\Sigma)$. Our state space in degree m is

$$\mathcal{E}_{m,s} = H^{-s}_{\mathrm{coex}} \times J_g. \tag{13.3}$$

It is Polish for the product of the Sobolev and flat torus metrics. The Hodge decomposition identifies the exact component with gauge translations and the harmonic ambiguity with the period lattice.

Proposition 13.1 (Completion of the smooth Abelian gauge quotient)**.** *Every $a \in H^{-s}(\Sigma; T^*\Sigma)$ has a unique decomposition*

$$a = d\phi + a_{\mathrm{c}} + h, \qquad \int_\Sigma \phi\,\mathrm{vol}_g = 0, \quad \phi \in H^{1-s}, \quad a_{\mathrm{c}} \in H^{-s}_{\mathrm{coex}}, \quad h \in \mathcal{H}. \tag{13.4}$$

The translation subgroup

$$\mathcal{K}_s = dH^{1-s}(\Sigma; \mathbb{R}) + \Lambda \subset H^{-s}(\Sigma; T^*\Sigma)$$

is closed. The map

$$[A_m + a] \longmapsto (P_{\mathrm{coex}} a, [P_{\mathcal{H}} a]) \tag{13.5}$$

is a homeomorphism from the quotient by $\mathcal{K}_s$ onto $\mathcal{E}_{m,s}$. The subgroup $\mathcal{K}_s$ is the closure, in H^{-s}, of the translations induced by smooth $U(1)$ gauge transformations. For smooth connections, the equivalence relation in (13.5) is precisely smooth gauge equivalence.

Proof. Let G_0 be the inverse of Δ_0 on functions of mean zero. The formula $\phi = G_0 d^* a$ gives the exact component, and the remaining components are the bounded Hodge projections. Elliptic regularity gives $\phi \in H^{1-s}$, with continuous dependence on a.

The exact subspace is the range of the bounded projection dG_0d^*. It is closed. If $d\phi_n + \lambda_n$ converges in H^{-s}, its harmonic projections λ_n converge in the finite-dimensional space $\mathcal{H}$. The lattice Λ is discrete, so this sequence is eventually constant. This proves that $\mathcal{K}_s$ is closed. The continuous Hodge projections identify its quotient with (13.3); the inverse sends $(a_c, [h])$ to $[A_m + a_c + h]$. This inverse is continuous by the quotient topology of the finite-dimensional lattice action.

The adjoint action of $U(1)$ is trivial, so gauge transformations of any L_m are canonically maps $u : \Sigma \to U(1)$. For a smooth gauge, its real logarithmic derivative has periods in $2\pi\mathbb{Z}$ and is closed. It therefore has the form $d\phi + \lambda$, with ϕ smooth and $\lambda \in \Lambda$. Conversely, any such one-form is a logarithmic derivative: integrate it from a fixed base point and exponentiate; the period condition makes the resulting circle-valued function single-valued. Smooth functions are dense in H^{1-s}, which proves the closure assertion. Finally, if a smooth one-form equals $d\phi + \lambda$ with $\phi \in H^{1-s}$, then $\Delta_0\phi = d^*(a - \lambda)$ is smooth. The mean-zero potential ϕ is smooth by elliptic regularity. Thus no additional identifications of smooth connections have been introduced. □

Smooth connections modulo smooth gauges are dense in (13.3). To define line observables for the diffusion on this completion, we first construct Gaussian traces and then identify their full-measure reconstructible domain in Section 16.

13.2 The global diffusion and its invariant measure

Let $(e_j)_{j\geq 1}$ be an L^2-orthonormal eigenbasis of coexact real one-forms, with $\Delta_1 e_j = \lambda_j e_j$ and $\lambda_j > 0$. The positive coexact eigenvalues are the positive scalar eigenvalues, with multiplicities: if f is a scalar eigenfunction of eigenvalue $\lambda > 0$, then $\lambda^{-1/2} * df$ is a normalized coexact eigenform. In particular, Weyl's law on a surface gives $\lambda_j \asymp j$ for large j.

Let $(\beta_j)_{j\geq 1}$ be independent real Brownian motions, and let $B^{\mathcal{H}}$ be an independent Brownian motion on $\mathcal{H}$ whose covariance is the L^2 inner product. For $x = (a_0, [h_0]) \in \mathcal{E}_{m,s}$, define

$$a_t = e^{-t\Delta_1} a_0 + X_t, \qquad X_t = \sqrt{2} \sum_{j\geq 1} \left(\int_0^t e^{-\lambda_j (t-r)}\, d\beta_j(r) \right) e_j, \tag{13.6}$$

$$[h_t] = [h_0 + \sqrt{2} B_t^{\mathcal{H}}]. \tag{13.7}$$

These are the gauge quotient equations for the Abelian Yang–Mills Langevin dynamics with action $\frac{1}{2}\|F_A\|_{L^2}^2$ and noise amplitude $\sqrt{2}$. Indeed, the constant-curvature reference satisfies $d^* F_{A_m} = 0$, and on coexact forms $d^* d = \Delta_1$. The exact noise is removed by the completed gauge quotient; the harmonic noise remains as Brownian motion on J_g.

Theorem 13.2 (Global dynamics in a fixed degree)**.** *For every $m \in \mathbb{Z}$ and $s > 0$, equations (13.6)–(13.7) define a continuous, global, strong Markov process on $\mathcal{E}_{m,s}$. Its transition semigroup is strong Feller and topologically irreducible. Its unique invariant probability measure is*

$$\mu_{m,s} = \nu_s \otimes \mathrm{Haar}_{J_g}, \qquad \nu_s = \mathrm{Law}\left(\sum_{j\geq 1} \lambda_j^{-1/2} \zeta_j e_j \right), \tag{13.8}$$

where the ζ_j are independent standard normal random variables. The process is reversible with respect to $\mu_{m,s}$, and its law from every initial state converges weakly to $\mu_{m,s}$ as $t \to \infty$.

Proof. The covariance of the jth coordinate of X_t is

$$q_j(t) = \frac{1 - e^{-2t\lambda_j}}{\lambda_j}. \tag{13.9}$$

Consequently,

$$\mathbb{E}\|X_t\|_{H^{-s}}^2 = \sum_{j\geq 1}(1+\lambda_j)^{-s} q_j(t) < \infty.$$

For $0 \leq u \leq t \leq T$, independence of Brownian increments gives

$$\mathbb{E}\|X_t - X_u\|_{H^{-s}}^2 \leq C \sum_{j\geq 1}(1+\lambda_j)^{-s}\lambda_j^{-1}\min\{1, \lambda_j(t-u)\}.$$

For every $0 < \theta < \min\{s, 1\}$ the right-hand side is at most $C_{s,\theta,T}|t-u|^\theta$. The Gaussian moment inequality and Kolmogorov's continuity theorem give a continuous H^{-s}-valued version. The deterministic semigroup is strongly continuous on H^{-s}, so the full process is continuous at zero as well.

The explicit formulas give pathwise uniqueness and the restart identity at every deterministic time. They also give it at stopping times: the countable family of Brownian motions together with $B^{\mathcal{H}}$ has independent future increments after a stopping time, and the semigroup identity rewrites the continuation using these increments and the current state. This proves the strong Markov property. The construction is defined for every $t \geq 0$, so there is no finite lifetime.

For the strong Feller property, the Cameron–Martin norm of the change in the mean of the coexact transition law is

$$\sum_{j\geq 1} \frac{e^{-2t\lambda_j}(a_j - b_j)^2}{q_j(t)} \leq C_{s,t}\|a-b\|_{H^{-s}}^2,$$

because

$$\sup_{\lambda \geq \lambda_1} \frac{\lambda(1+\lambda)^s e^{-2t\lambda}}{1 - e^{-2t\lambda}} < \infty.$$

The Cameron–Martin change-of-mean formula implies continuity in total variation as $a \to b$. More explicitly, for a Cameron–Martin shift v, the density with respect to the unshifted Gaussian measure has second moment $\exp(\|v\|_{\mathrm{CM}}^2)$; Cauchy–Schwarz therefore bounds the total variation difference by a constant times $(\exp(\|v\|_{\mathrm{CM}}^2) - 1)^{1/2}$. Brownian motion on the finite-dimensional torus has a smooth heat kernel, and its translated transition densities vary continuously in L^1. The product kernel therefore maps bounded Borel functions to continuous functions.

Each coexact coordinate has positive variance at $t > 0$, and the Gaussian covariance is trace class in H^{-s}. Its support is all of H^{-s}_{coex}: a finite-dimensional projection can enter any prescribed ball with positive probability, while an independent tail has arbitrarily small second moment. The torus heat kernel is strictly positive. This proves irreducibility.

The series in (13.8) converges in $L^2(\Omega; H^{-s})$. Each of its coordinates is the stationary law of the corresponding Ornstein–Uhlenbeck equation, and Haar measure is stationary for torus Brownian motion. Independence proves invariance. The one-dimensional Ornstein–Uhlenbeck kernels are symmetric with respect to their centered Gaussian laws, and the torus heat kernel is symmetric with respect to Haar measure. Finite products are reversible; cylinder approximation gives reversibility of the full process.

Finally, $e^{-t\Delta_1}a_0 \to 0$ in H^{-s}. On a common sequence of normal variables, the centered transition law and ν_s can be coupled by coefficients $q_j(t)^{1/2}$ and $\lambda_j^{-1/2}$. Dominated convergence of the weighted square sum shows convergence in $L^2(\Omega; H^{-s})$. The torus heat kernel converges to Haar measure. Thus every pointwise transition law converges weakly to $\mu_{m,s}$. If $\widetilde{\mu}$ were another invariant probability measure, apply this convergence to any bounded continuous function and integrate against $\widetilde{\mu}$. Dominated convergence yields $\widetilde{\mu} = \mu_{m,s}$. □

Proposition 13.3 (The sharp spectral gap)**.** *Let λ_{c} be the first positive coexact Hodge eigenvalue. Let*

$$\Lambda^* = \{k \in \mathcal{H} : \langle k, \lambda\rangle_{L^2} \in 2\pi\mathbb{Z} \text{ for every } \lambda \in \Lambda\}, \qquad \lambda_J = \min_{k\in\Lambda^*\setminus\{0\}} \|k\|_{L^2}^2,$$

with $\lambda_J = +\infty$ if $\mathcal{H} = \{0\}$. The $L^2(\mu_{m,s})$ spectral gap of the process is

$$\rho_g = \min\{\lambda_{\mathrm{c}}, \lambda_J\}. \tag{13.10}$$

In particular, if $\int F\, d\mu_{m,s} = 0$, then

$$\|P_t F\|_{L^2(\mu_{m,s})} \le e^{-\rho_g t}\|F\|_{L^2(\mu_{m,s})}.$$

Proof. On a function depending on finitely many coexact coordinates and on the harmonic torus, the generator is

$$\mathcal{L} = \sum_j (\partial_{a_j}^2 - \lambda_j a_j \partial_{a_j}) - \Delta_{J_g}.$$

Products of Hermite polynomials in the normalized Gaussian coordinates and Fourier characters on J_g form a complete orthogonal basis in $L^2(\mu_{m,s})$. Their nonnegative decay rates are

$$\sum_j n_j\lambda_j + \|k\|_{L^2}^2, \qquad n_j \in \mathbb{N}_0 \text{ finitely supported}, \quad k \in \Lambda^*.$$

The smallest nonzero rate is (13.10). It is attained by a first coexact coordinate, or by a first nonconstant torus character, according to which term is smaller. □

Remark 13.4 (Topological sectors). The integer m is conserved by the dynamics. The uniqueness in Theorem 13.2 is a statement in a fixed sector. On the disjoint union over all degrees, every probability mixture $\sum_m p_m\mu_{m,s}$ is invariant. A choice of weights between topological sectors is additional data and is not selected by this diffusion.

13.3 Independence of the reference connection

Proposition 13.5 (Affine change of reference)**.** *Let $B = A_m + b$ be any smooth reference connection on L_m, and write $b = d\phi + b_{\mathrm{c}} + b_{\mathcal{H}}$. The change of coordinates*

$$T_b(a, [h]) = (a - b_{\mathrm{c}}, [h - b_{\mathcal{H}}]) \tag{13.11}$$

is an affine homeomorphism of the completed gauge quotients. In B-coordinates the process satisfies

$$d\widetilde{a}_t = \left(-\Delta_1\widetilde{a}_t - d^*F_B\right)dt + \sqrt{2}P_{\mathrm{coex}}\, dW_t, \tag{13.12}$$

$$[\widetilde{h}_t] = [\widetilde{h}_0 + \sqrt{2}B_t^{\mathcal{H}}]. \tag{13.13}$$

Its invariant coexact law has mean $-b_{\mathrm{c}}$ and covariance Δ_1^{-1} on the coexact subspace. The transition semigroups and invariant measures in the two references are conjugate by T_b.

Proof. The exact term $d\phi$ is a gauge translation. Since $F_B = F_{A_m} + db_{\mathrm{c}}$ and b_{c} is coexact,

$$d^* F_B = d^* db_{\mathrm{c}} = \Delta_1 b_{\mathrm{c}}.$$

Subtracting the fixed vector b_{c} from (13.6) gives (13.12). Subtracting $[b_{\mathcal{H}}]$ translates the harmonic torus. The stated assertions follow directly from these identities. In particular, two constant-curvature references differ by a closed one-form; their coexact difference is zero. □

13.4 Gaussian line traces and Wilson loops

We use closed curves whose nonconstant pieces are regular C^2 arcs, with finitely many pieces. Self-intersections and repeated traversal are allowed. Their one-currents are defined by

$$\langle J_\gamma, \omega \rangle = \int_\gamma \omega, \qquad \omega \in C^\infty(\Sigma; T^*\Sigma).$$

Lemma 13.6 (Sobolev regularity of a curve current)**.** *For every such curve γ and every $r \in (1/2, 1)$, $J_\gamma \in H^{-r}(\Sigma; T^*\Sigma)$. Consequently,*

$$Q(\gamma, \eta) := \left\langle \Delta_1^{-1/2} P_{\mathrm{coex}} J_\gamma, \Delta_1^{-1/2} P_{\mathrm{coex}} J_\eta \right\rangle_{L^2} \tag{13.14}$$

is finite for every pair of such curves.

Proof. Cover the finitely many arcs by finitely many coordinate neighborhoods in which each arc is a C^2 graph. A C^2 change of coordinates straightens the graph and acts boundedly on H^r for $0 < r < 1$. On a straight segment, the trace estimate $\|v(\cdot, 0)\|_{L^2} \le C_r \|v\|_{H^r(\mathbb{R}^2)}$ holds for $r > 1/2$. One proves it by Fourier transformation in the tangential coordinate and Cauchy–Schwarz in the normal frequency, using the integrability of $(1 + \eta^2)^{-r}$. Multiplication by the smooth localization functions and by the C^1 tangent coefficients preserves this estimate. Summing over the finite cover proves the current bound. The operator $\Delta_1^{-1/2} P_{\mathrm{coex}}$ has order -1, so it maps H^{-r} into $H^{1-r} \subset L^2$. □

Set $P_\varepsilon = e^{-\varepsilon \Delta_1}$. For $t > 0$, define the regularized coexact trace

$$I_{\gamma,\varepsilon}(t) = \langle J_\gamma, P_\varepsilon a_t \rangle. \tag{13.15}$$

Theorem 13.7 (Positive-time Wilson observables)**.** *Fix $x \in \mathcal{E}_{m,s}$ and a curve γ as above. For every $0 < \delta < T < \infty$ and $p < \infty$, the traces (13.15) converge in $L^p(\Omega; C([\delta, T]; \mathbb{R}))$ as $\varepsilon \downarrow 0$. Denote the limit by $I_\gamma(t)$. It has a version locally θ-Hölder in positive time for every $\theta < 1/4$. The random variables*

$$\mathcal{W}_{\gamma,n}(t) = \mathrm{Hol}_\gamma(A_m)^n \exp\left(i n I_\gamma(t) + i n \int_\gamma h_t \right), \qquad n \in \mathbb{Z}, \tag{13.16}$$

are well defined, independent of the representative of $[h_t]$, and independent of the chosen reference connection under (13.11). For any fixed countable family of curves, the construction can be made simultaneously on a single event of probability one. If the initial state has law $\mu_{m,s}$, the same convergence and time continuity hold on $[0, T]$.

Proof. Put $c_{\gamma,j} = \langle J_\gamma, e_j \rangle$. For each $r \in (1/2, 1)$, Lemma 13.6 gives

$$\sum_j (1 + \lambda_j)^{-r} |c_{\gamma,j}|^2 < \infty. \tag{13.17}$$

The deterministic part $\langle J_\gamma, e^{-t\Delta_1} a_0 \rangle$ is smooth in $t > 0$, and its heat regularizations converge uniformly with all time derivatives on $[\delta, T]$.

For the centered stochastic part, write $Y_\gamma(t) = \sum_j c_{\gamma,j} X_{t,j}$. The series converges in $L^2(\Omega)$ at each time because

$$\sum_j \frac{|c_{\gamma,j}|^2}{\lambda_j} < \infty.$$

The scalar Ornstein–Uhlenbeck covariance gives

$$\mathbb{E}|Y_\gamma(t) - Y_\gamma(u)|^2 \le C \sum_j |c_{\gamma,j}|^2 \lambda_j^{-1} \min\{1, \lambda_j |t-u|\}. \tag{13.18}$$

For every $0 < \beta < 1/2$, choose $r > 1/2$ with $\beta < 1 - r$. Using $\min\{1, z\} \le z^\beta$ and (13.17), the right-hand side is bounded by $C_{\gamma,\beta}|t-u|^\beta$. Gaussian moments and Kolmogorov's theorem give every time exponent below $1/4$.

Uniform convergence follows from a quantitative bound on the regularization error. Choose $q > 0$ and $\beta > 0$ with $2q + \beta < 1/2$, and then choose $r \in (1/2, 1 - 2q - \beta)$. The multiplier bound $|1 - e^{-\varepsilon\lambda}| \le C_q \varepsilon^q \lambda^q$ and the preceding covariance calculation imply

$$\mathbb{E}\,|(Y_{\gamma,\varepsilon} - Y_\gamma)(t) - (Y_{\gamma,\varepsilon} - Y_\gamma)(u)|^2 \le C\varepsilon^{2q}|t-u|^\beta,$$

as well as $\mathbb{E}|(Y_{\gamma,\varepsilon} - Y_\gamma)(t)|^2 \le C_T \varepsilon^{2q}$. Indeed the relevant series is bounded by $\sum_j |c_{\gamma,j}|^2 \lambda_j^{2q+\beta-1}$, which converges by (13.17). Apply the Gaussian moment inequality with a moment order larger than $2/\beta$, followed by the quantitative Kolmogorov estimate. This proves convergence in every finite $L^p(\Omega; C([0,T]))$ for the centered stochastic part. The stationary centered process satisfies the same estimates, since its increment variance in coordinate j is $2\lambda_j^{-1}(1 - e^{-\lambda_j|t-u|})$.

For $\lambda \in \Lambda$, the number $\int_\gamma \lambda$ is in $2\pi\mathbb{Z}$. Hence the harmonic exponential in (13.16) is a character of J_g. Its sample paths are continuous. The relation between the classical holonomies of two references is

$$\mathrm{Hol}_\gamma(B) = \mathrm{Hol}_\gamma(A_m) \exp\left(i \int_\gamma b \right).$$

Subtracting b_{c} and $b_{\mathcal{H}}$ from the random coordinates cancels this factor; the exact part of b integrates to zero. This proves reference independence.

The regularization bounds imply almost sure uniform convergence along $\varepsilon = 2^{-k}$ by Markov's inequality and Borel–Cantelli. Intersect the corresponding events over a countable family of curves and intervals $[1/N, N]$. This gives the simultaneous assertion. The stationary version includes time zero by the same argument on $[0, N]$. □

Remark 13.8 (Measurability and the scope of the trace construction). For a fixed curve, each heat-regularized Wilson expression is a continuous function on $\mathcal{E}_{m,s}$. Its limit along $\varepsilon = 2^{-k}$, assigned an arbitrary fixed value when the limit fails to exist, is therefore a bounded Borel function. Theorem 13.7 identifies this Borel function with the Gaussian Wilson observable almost surely under every positive-time transition law and under the invariant law. This first construction applies to each specified finite or countable family of curves. Section 17 proves a simultaneous extension to all piecewise regular C^2 loops on an explicitly specified, pathwise invariant Borel domain, using uniform estimates on bounded regular curve families. No line trace is assigned to every negative Sobolev connection.

13.5 Exact joint Wilson formulas

For a finite integral combination of closed curves, write

$$J = \sum_{\ell=1}^{N} n_\ell J_{\gamma_\ell}, \qquad n_\ell \in \mathbb{Z}, \qquad \ell_J(h) = \langle J, h \rangle \quad (h \in \mathcal{H}).$$

Let $v_J \in \mathcal{H}$ be the L^2 Riesz representative of ℓ_J. The period condition implies $v_J \in \Lambda^*$.

Proposition 13.9 (Joint characters at finite time and at equilibrium)**.** *For deterministic initial state* $(a_0, [h_0])$ *and* $t > 0$*,*

$$\mathbb{E}_{(a_0,[h_0])} \prod_{\ell=1}^{N} \mathcal{W}_{\gamma_\ell,n_\ell}(t) = \left(\prod_{\ell=1}^{N} \mathrm{Hol}_{\gamma_\ell}(A_m)^{n_\ell}\right) \exp\Big(i\langle J, e^{-t\Delta_1} a_0\rangle + i\ell_J(h_0)\Big) \times \exp\left(-\frac{1}{2}\left\langle J, \Delta_1^{-1}(I - e^{-2t\Delta_1})P_{\mathrm{coex}} J\right\rangle - t\|v_J\|_{L^2}^2\right). \tag{13.19}$$

The quadratic pairing is interpreted through (13.14)*. At equilibrium,*

$$\mathbb{E}_{\mu_{m,s}} \prod_{\ell=1}^{N} \mathcal{W}_{\gamma_\ell,n_\ell} = \left(\prod_{\ell=1}^{N} \mathrm{Hol}_{\gamma_\ell}(A_m)^{n_\ell}\right) \mathbf{1}_{\{v_J=0\}} \exp\left(-\frac{1}{2}\left\langle J, \Delta_1^{-1} P_{\mathrm{coex}} J\right\rangle\right). \tag{13.20}$$

The condition $v_J = 0$ *is equivalent to* $\sum_\ell n_\ell[\gamma_\ell] = 0$ *in* $H_1(\Sigma; \mathbb{Z})$*.*

Proof. The coexact trace is Gaussian with the mean and variance displayed in (13.19). It is independent of the harmonic Brownian motion. The torus character $[h] \mapsto e^{i\ell_J(h)}$ has generator eigenvalue $-\|v_J\|_{L^2}^2$. Multiplying its expectation by the Gaussian characteristic function proves the finite-time formula. Integration against Haar measure annihilates every nontrivial torus character, and the stationary coexact covariance is Δ_1^{-1}, which gives (13.20). Harmonic one-forms separate real homology classes, and the integral first homology of a closed oriented surface is torsion free. Thus $v_J = 0$ is equivalent to the stated integral homology relation. □

Corollary 13.10 (Fixed-flux Gaussian curvature)**.** *Under* $\mu_{m,s}$*, the distributional curvature has law*

$$F_A = \left(\frac{2\pi m}{V} + \eta_0\right) \mathrm{vol}_g, \qquad \eta_0 = \eta - \frac{\langle \eta, 1\rangle}{V}, \tag{13.21}$$

where η *is real scalar white noise on* Σ*. This curvature is independent of the harmonic coordinate, which has Haar law on* J_g*. In particular, the total curvature is almost surely* $2\pi m$*.*

Proof. Choose an orthonormal basis (f_j) of mean-zero scalar eigenfunctions, $\Delta_0 f_j = \lambda_j f_j$, and choose the coexact eigenbasis $e_j = -\lambda_j^{-1/2} * df_j$. Then $de_j = \lambda_j^{1/2} f_j \,\mathrm{vol}_g$. Applying d to the stationary Gaussian series gives

$$d\left(\sum_j \lambda_j^{-1/2} \zeta_j e_j\right) = \left(\sum_j \zeta_j f_j\right) \mathrm{vol}_g.$$

The equality holds in H^{-1-s}, and the scalar series is mean-zero white noise. Add (13.1). Independence follows from the product invariant law, and pairing with the constant function gives the total curvature. □

Corollary 13.11 (The area covariance and the degree-dependent Wilson law)**.** *Let* $D, E \subset \Sigma$ *be regions with piecewise* C^2 *boundaries, oriented by* Σ*. For a boundary with several components,* $\mathcal{W}_{\partial D,n}$ *denotes the product of their oriented Wilson characters. For the centered coexact process started at zero,*

$$\mathbb{E}[I_{\partial D}(t) I_{\partial E}(t)] = \left\langle \mathbf{1}_D, (I - e^{-2t\Delta_0})\mathbf{1}_E\right\rangle_{L^2}. \tag{13.22}$$

At equilibrium this covariance is

$$Q(\partial D, \partial E) = \mathrm{Area}(D \cap E) - \frac{\mathrm{Area}(D)\,\mathrm{Area}(E)}{V}. \tag{13.23}$$

In particular, for $n \in \mathbb{Z}$ and $d = \mathrm{Area}(D)$,

$$\mathbb{E}_{\mu_{m,s}} \mathcal{W}_{\partial D, n} = \exp\left(i \frac{2\pi n m d}{V} - \frac{n^2}{2} \left(d - \frac{d^2}{V} \right) \right). \tag{13.24}$$

More generally, for an integral linear combination of region indicators $f = \sum_\ell n_\ell \mathbf{1}_{D_\ell}$,

$$\mathbb{E}_{\mu_{m,s}} \prod_\ell \mathcal{W}_{\partial D_\ell, n_\ell} = \exp\left(i \frac{2\pi m}{V} \int_\Sigma f \,\mathrm{vol}_g - \frac{1}{2} \| f - \overline{f} \|_{L^2}^2 \right), \qquad \overline{f} = V^{-1} \int_\Sigma f \,\mathrm{vol}_g. \tag{13.25}$$

Proof. With the orientation and current convention above, Stokes' theorem gives $J_{\partial D} = -{*}d\mathbf{1}_D$, where the current is identified with a distributional one-form by the L^2 pairing. The two minus signs cancel in each covariance. The identities

$$e^{-t\Delta_1} * d = {*}d\, e^{-t\Delta_0}, \qquad \Delta_1^{-1} * d = {*}d\, \Delta_0^{-1}$$

hold on the mean-zero scalar subspace. Therefore

$$\left\langle J_{\partial D}, \Delta_1^{-1}(I - e^{-2t\Delta_1}) J_{\partial E} \right\rangle = \left\langle \mathbf{1}_D, (I - e^{-2t\Delta_0}) \mathbf{1}_E \right\rangle.$$

The computation can first be made with smooth approximations to the indicators; convergence follows from their L^2 convergence and from the boundedness of $I - e^{-2t\Delta_0}$. Letting $t \to \infty$ gives (13.23). The boundary current has zero harmonic projection. Moreover, the Abelian Stokes formula for the reference connection gives

$$\mathrm{Hol}_{\partial D}(A_m) = \exp\left(i \int_D F_{A_m} \right) = \exp\left(i \frac{2\pi m d}{V} \right).$$

Substitution in (13.20) proves (13.24). Bilinearity gives (13.25). Adding an integer constant to f leaves the final expression unchanged: the variance is unchanged and the phase changes by an integer multiple of $2\pi i m$. □

13.6 Compact torus structure groups

Corollary 13.12 (Every principal torus bundle)**.** *Let $G = \mathbb{R}^r/(2\pi\mathbb{Z}^r)$ and equip its Lie algebra with the positive definite inner product matrix M. For every Chern vector $c \in \mathbb{Z}^r$, the preceding construction gives a global strong Feller, irreducible, reversible diffusion on*

$$H^{-s}_{\mathrm{coex}}(\Sigma; \mathbb{R}^r) \times \frac{\mathcal{H} \otimes \mathbb{R}^r}{2\pi H^1(\Sigma; \mathbb{Z}) \otimes \mathbb{Z}^r},$$

where integral cohomology is represented harmonically. Its invariant coexact covariance, in the displayed real coordinates, is $\Delta_1^{-1} \otimes M^{-1}$, and its harmonic invariant law is Haar measure. It is the unique invariant law in Chern sector c. Equip the harmonic torus $J_{g,M}$ in the displayed state space with the flat metric induced by $L^2 \otimes M$, and let $\lambda_{J_{g,M}}$ be its first positive Laplace eigenvalue, with value $+\infty$ when the harmonic space is zero. The sharp L^2 spectral gap is

$$\rho_{g,M} = \min\{\lambda_{\mathrm{c}}, \lambda_{J_{g,M}}\}. \tag{13.26}$$

For a character $n \in \mathbb{Z}^r$ and $d = \mathrm{Area}(D)$, the equilibrium boundary Wilson expectation is

$$\exp\left(i \frac{2\pi (n \cdot c) d}{V} - \frac{n^{\mathsf{T}} M^{-1} n}{2} \left(d - \frac{d^2}{V} \right) \right). \tag{13.27}$$

Proof. Every principal T^r-bundle is the fiber product of its coordinate circle bundles. Their degrees are the entries of c, and their constant-curvature connections combine to a reference with curvature $(2\pi/V)c\,\mathrm{vol}_g$. Apply Hodge decomposition to $\mathbb{R}^r$-valued forms, with identity-covariance noise for the Lie algebra metric M. In the displayed real coordinates its covariance is M^{-1}, giving the stated coexact covariance and the dual character variance $n^{\mathsf{T}}M^{-1}n$. The Hodge projections act only on the surface factor, and the harmonic Brownian motion descends to the period lattice of smooth torus gauges. The Gaussian and finite-dimensional torus arguments above prove the Markov, support, invariance and trace assertions. The coexact drift eigenvalues remain λ_j in an orthonormal Lie algebra basis, while the harmonic characters have the decay rates of $\Delta_{J_{g,M}}$. The Hermite–Fourier argument of Proposition 13.3 gives (13.26). □

14 Separation of gauge orbits by observables

An unrestricted infimum over two non-Abelian gauge orbits can vanish for connections with different Wilson loops. The following example uses unbounded sequences of representatives.

Proposition 14.1 (Collapse of the double-infimum formula)**.** *On the flat torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ with structure group $SU(2)$, let $0 < \alpha < 1$ and $\kappa > 0$. Equip smooth Lie-algebra-valued one-forms with the sum of the $\mathcal{C}^{-\kappa}$ norm and the norm*

$$\sup_{\ell} \frac{|\int_\ell A|}{|\ell|^\alpha} + \sup_{R} \frac{|\int_{\partial R} A|}{|R|^{\alpha/2}},$$

where ℓ runs over short straight segments and R over coordinate rectangles in a fixed finite flat atlas. Denote this sum by $\|\cdot\|_{\mathcal{X}}$. There are smooth connections A, B in different based gauge orbits, with different Wilson loops, such that

$$\inf_{u,v \in C^1_*(\mathbb{T}^2, SU(2))} \|u \cdot A - v \cdot B\|_{\mathcal{X}} = 0. \tag{14.1}$$

The same example applies if all sufficiently small flat triangles are included among the circulation tests.

Proof. Let $K = i\sigma_3$ and $H = i\sigma_1$, with σ_1, σ_3 the Pauli matrices. Set $A = \frac{1}{4}H\,dx$, $B = 0$, and $u_N(x, y) = \exp(NxK)$ for positive integers N. These are smooth periodic gauges based at $(0, 0)$. Under the convention $u \cdot A = uAu^{-1} - (du)u^{-1}$,

$$u_N \cdot A - u_N \cdot B = \tfrac{1}{4}\,\mathrm{Ad}_{u_N} H\,dx = \tfrac{1}{4}\big(\cos(2Nx)H + \sin(2Nx)J\big)\,dx$$

for a fixed $J \in \mathfrak{su}(2)$. The final expression is the exterior derivative of a periodic function of size $O(N^{-1})$. Its circulation around every contractible closed curve is zero, and its integral along a straight segment ℓ is bounded by $C\min\{|\ell|, N^{-1}\}$. Its segment norm is therefore at most $CN^{\alpha-1}$. The single-frequency Besov estimate gives a $\mathcal{C}^{-\kappa}$ norm at most $CN^{-\kappa}$. Taking $u = v = u_N$ proves (14.1).

Along the fundamental x-circle based at $(0, 0)$, the holonomies of A and B are respectively $\exp(-\frac{\pi}{2}i\sigma_1)$ and I. Their traces in the defining representation are 0 and 2. Based gauge transformations preserve these holonomies, and all gauge transformations preserve their traces. Thus the two connections belong to different gauge orbits. □

The two transformed representatives escape every bounded set: $u_N \cdot 0 = -NK\,dx$. Thus the unrestricted double infimum can identify smooth gauge classes with different Wilson values; estimates confined to bounded sets of representatives do not exclude this mechanism.

15 An intrinsic completion of polygonal holonomies

Fix a base point $o \in \Sigma$ and a number $\rho > 0$ such that every geodesic ball of radius ρ is strongly convex. Throughout this section $0 < r < \rho/4$. For $m \geq 1$, let

$$K_m^r = \left\{ (x_1, \dots, x_{m-1}) \in \Sigma^{m-1} : d(x_{j-1}, x_j) \leq r \ (1 \leq j \leq m), \quad x_0 = x_m = o \right\}. \tag{15.1}$$

The space K_m^r is compact. A point $p \in K_m^r$ specifies a based loop γ_p obtained by joining successive vertices by their unique shortest geodesics. Repeated vertices give constant edges.

Let $P \to \Sigma$ be a smooth principal bundle with connected compact structure group G. Choose a bi-invariant Riemannian distance d_G on G. If A is a smooth connection and $p_o \in P_o$ is a frame, write $h_m(A, p_o) : K_m^r \to G$ for the holonomy of γ_p expressed in that frame. Parallel transport depends smoothly on the endpoints of a short geodesic, including at a constant edge. Thus $h_m(A, p_o)$ is continuous, and is the restriction of a smooth function defined on a neighborhood of K_m^r in the vertex manifold.

Definition 15.1 (The polygonal holonomy completion)**.** Equip

$$\mathcal{C}_r = \prod_{m \geq 1} C(K_m^r, G)$$

with the metric

$$d_r(h, k) = \sum_{m \geq 1} 2^{-m} \min \left\{ 1, \sup_{p \in K_m^r} d_G(h_m(p), k_m(p)) \right\}. \tag{15.2}$$

Define $\mathscr{H}_r(P)$ as the closure in $\mathcal{C}_r$ of

$$\{ (h_m(A, p_o))_{m \geq 1} : A \text{ a smooth connection on } P, \quad p_o \in P_o \}.$$

The structure group acts on this space by simultaneous conjugation. The quotient is denoted by $\mathscr{U}_r(P) = \mathscr{H}_r(P)/G$.

The isomorphism type of P indexes the completion. On the Abelian reconstruction domains below, the bundle degree is itself recovered from the holonomy family (Corollary 16.5).

Theorem 15.2 (Intrinsic Polish spaces of holonomy data)**.** *The following assertions hold.*

1. *The space $\mathscr{H}_r(P)$ is Polish. Every element satisfies the holonomy identities for reversal, concatenation, subdivision of an edge, insertion of a constant edge, and insertion of an immediately retraced path.*

2. *The formula*
$$D_r([h], [k]) = \min_{g \in G} d_r(h, g^{-1} k g) \tag{15.3}$$
is a complete separable metric on $\mathscr{U}_r(P)$ and induces its quotient topology.

3. *Smooth connections with a frame at o, modulo frame-preserving bundle isomorphisms covering the identity of Σ, inject into $\mathscr{H}_r(P)$. Smooth connections modulo smooth bundle automorphisms covering the identity of Σ inject into $\mathscr{U}_r(P)$. Their maps from the smooth connection space are continuous.*

4. *For any other $0 < R < \rho/4$, subdivision gives canonical homeomorphisms $\mathscr{H}_r(P) \simeq \mathscr{H}_R(P)$ and $\mathscr{U}_r(P) \simeq \mathscr{U}_R(P)$, agreeing with the identity on the holonomy data of smooth connections.*

These constructions use the metric, the bundle and the base point, and require no coordinate atlas or reference connection.

Proof. For each compact metric space K_m^r, the space $C(K_m^r, G)$ is complete and separable in the uniform metric. To see separability, embed the compact Lie group as a closed submanifold of a Euclidean space and use separability of the corresponding vector-valued continuous-function space; a subspace of a separable metric space is separable. Completeness follows from uniform convergence and closedness of G. The countable product with (15.2) is therefore Polish, and its closed subspace $\mathscr{H}_r(P)$ is Polish.

Each holonomy identity is a closed equality between evaluations of finitely many components, composed with continuous vertex maps and group operations. For example, subdividing the jth edge at its geodesic parameter $s \in [0,1]$ gives a continuous map $K_m^r \times [0,1] \to K_{m+1}^r$; the values of h_m and h_{m+1} are equal on the associated pair of polygons. Concatenation gives a continuous map $K_m^r \times K_n^r \to K_{m+n}^r$. The equalities hold for every smooth connection and therefore on their closure. This also proves the assertions for reversal and retracing.

Simultaneous conjugation acts by isometries for d_r. Compactness of G shows that the minimum in (15.3) is attained. In particular, distance zero means equality of actual conjugacy orbits. The triangle inequality follows by composing conjugations and using their isometric action. The quotient projection is continuous. Conversely, if a saturated open set contains the compact orbit of h, it contains a positive-radius metric neighborhood of that orbit: otherwise a sequence of points in its closed complement approaches the orbit, and compactness produces a limiting orbit point in that complement. This proves that D_r induces the quotient topology.

For completeness, take a D_r-Cauchy sequence and a subsequence with successive quotient distances below 2^{-j}. Choose representatives successively, using the minimizing conjugation, so that the d_r-distance between successive representatives is below 2^{-j}. These representatives converge in $\mathscr{H}_r(P)$. Their orbits give a limit of the subsequence, and the original Cauchy sequence has the same limit. The image of a countable dense subset is dense, so the quotient is separable.

We prove the injectivity assertion for two arbitrary smooth principal G-bundles P, Q. Suppose framed smooth connections (A, p_o) and (B, q_o) have the same based polygonal holonomies. For each $x \in \Sigma$, choose a piecewise short-geodesic path c from o to x. Send the A-parallel transport of p_o along c to the B-parallel transport of q_o along c, and extend this map by G-equivariance on the fiber. A second such path differs from the first by a based polygonal loop. Equality of its holonomies proves path independence. Locally, one may take a fixed path to a nearby center followed by a radial geodesic. Smooth dependence of parallel transport then proves that the resulting bundle map is smooth. These local maps agree on overlaps by path independence and therefore define a global bundle isomorphism. It preserves parallel transport along short geodesics, hence the connections, and takes p_o to q_o. Equality up to simultaneous conjugation gives the unframed assertion. On polygons with at most m edges, continuous dependence of the transport ODE is uniform over paths of length at most mr. Hence the map from smooth connections is continuous in every factor and in the product topology.

Assume now $r < R$. Choose an integer N with $R/N \le r$. Subdivide every edge of a polygon in K_m^R into N equal geodesic segments. This gives a continuous map $S_{m,N} : K_m^R \to K_{Nm}^r$. The rule

$$(Sh)_m = h_{Nm} \circ S_{m,N}$$

is continuous for the product uniform topologies. On smooth holonomy data it is the identity expressed at the larger radius, so it sends $\mathscr{H}_r(P)$ into $\mathscr{H}_R(P)$. Restriction to $K_m^r \subset K_m^R$ defines a continuous map in the reverse direction. These maps are inverse on smooth data and thus on the closures. The same observation shows independence of the integer N. They

commute with conjugation, giving the quotient homeomorphism. The case $R < r$ follows by interchanging the radii. □

Corollary 15.3 (Change of base point)**.** *Write $\mathscr{H}_{r,o}(P)$ and $\mathscr{U}_{r,o}(P)$ to display the base point. A fixed short-geodesic polygonal path c from o to o' induces a homeomorphism $\mathscr{H}_{r,o}(P) \to \mathscr{H}_{r,o'}(P)$. On unframed quotients it is independent of c.*

Proof. If c has n edges, insert c before a based polygon at o' and c^{-1} after it. This is a continuous vertex map $K^r_k(o') \to K^r_{k+2n}(o)$. Precomposition defines a continuous map T_c of holonomy families. On smooth connections it is the change to the frame at o' obtained by parallel transport along c, so it maps one holonomy completion into the other. Its inverse is $T_{c^{-1}}$, by the retracing identities. The maps commute with simultaneous conjugation and therefore descend to the quotients. For another path d from o to o', the families $T_c h$ and $T_d h$ are simultaneously conjugate by the value of h on the fixed loop dc^{-1}. This identity holds for smooth connections and extends to the completion by continuity of evaluation and group operations. It proves independence of the connecting path on the unframed quotient. □

A point of $\mathscr{H}_r(P)$ is a compatible family of continuous polygonal observables. The compact group in (15.3) changes the frame in one fiber. Smooth gauge classes are represented faithfully by these data. Section 16 reconstructs distributional Abelian states, and Section 19 identifies the rough non-Abelian orbits on the full-heat trace domain. The topology controls each fixed-edge family K^r_k separately. Section 17 supplies the additional uniform subdivision estimates that extend the Abelian stochastic observables to piecewise regular C^2 loops.

15.1 Uniform estimates for polygon currents

For the Abelian analysis, let J_p be the real one-current of γ_p, and set

$$d_m(p,q) = \max_{1\le j\le m-1} d(x_j, y_j), \qquad p = (x_j), \quad q = (y_j).$$

For the one-point space K^r_1, set $d_1 = 0$. The constants in the next lemma may depend on m, r and the metric.

Lemma 15.4 (Uniform current and filling estimates)**.** *For every $a \in (1/2, 1)$,*

$$\sup_{p\in K^r_m} \|J_p\|_{H^{-a}} < \infty. \tag{15.4}$$

There are $\delta_0, C_m > 0$ such that whenever $d_m(p,q) \le \delta_0$, the difference current is the boundary of an integer-valued two-chain with density $w_{p,q} \in L^2(\Sigma)$ and

$$J_p - J_q = \partial w_{p,q}, \qquad \|w_{p,q}\|_{L^\infty} \le 2m, \qquad \|w_{p,q}\|_{L^1} \le C_m d_m(p,q). \tag{15.5}$$

In particular,

$$\left\langle J_p - J_q, \Delta_1^{-1} P_{\mathrm{coex}}(J_p - J_q) \right\rangle \le C_m d_m(p,q). \tag{15.6}$$

Proof. Any geodesic segment of length at most r extends to a geodesic arc of a fixed, slightly larger length contained in a normal neighborhood. Uniform local graph representations follow from compactness of the unit tangent bundle and the positive convexity radius. The trace estimate used in Lemma 13.6 is therefore uniform over all these extended arcs. Integration over a subsegment is bounded by its length to the power $1/2$ times the L^2 trace norm. Constant segments contribute zero. Summing over the m edges proves (15.4).

Put $\delta = d_m(p,q)$ and join each x_j to y_j by its short geodesic. For each pair of corresponding edges, fill the quadrilateral with vertices $x_{j-1}, x_j, y_j, y_{j-1}$ by the two oriented geodesic triangles

obtained by inserting the diagonal from x_{j-1} to y_j. When δ_0 is sufficiently small, all these triangles lie in strongly convex balls. A geodesic triangle in such a ball is an embedded disk, or is degenerate; its oriented indicator has absolute value at most one. Each of the two triangles has an edge of length at most δ and its other edges have uniformly bounded length. Its area is at most $C\delta$. One direct verification is to parametrize it by the geodesics from the opposite vertex to that short edge. Bounds on the differential of the exponential map on the relevant compact set bound the area Jacobian by a constant times the speed along the short edge.

Sum the $2m$ oriented triangle indicators. The inserted diagonals and the short joining edges cancel in the boundary, leaving $J_p - J_q$. At every point the sum has absolute value at most $2m$, and its L^1 norm is bounded by the sum of the triangle areas. This proves (15.5). The Hodge identity for a boundary current gives

$$\left\langle \partial w, \Delta_1^{-1} P_{\mathrm{coex}} \partial w \right\rangle = \|w - \overline{w}\|_{L^2}^2.$$

Thus the left side of (15.6) is at most $\|w_{p,q}\|_{L^\infty} \|w_{p,q}\|_{L^1}$, as required. □

Lemma 15.5 (A Gaussian continuity estimate)**.** *Let (T, d) be a compact metric space with covering numbers $N(T, d, \varepsilon) \le C\varepsilon^{-D}$ for $0 < \varepsilon \le 1$. Suppose a centered real Gaussian field $(Z_z)_{z \in T}$ satisfies*

$$\mathbb{E}|Z_z - Z_w|^2 \le L d(z, w), \qquad \mathbb{E}|Z_{z_0}|^2 \le L$$

at some fixed $z_0 \in T$. For every $\eta < 1/2$ it has an η-Hölder version. For every $p < \infty$, its supremum and η-Hölder seminorm have L^p norm at most $C_{T,D,C,\eta,p} L^{1/2}$.

Proof. Choose finite 2^{-n}-nets of cardinality at most $C2^{Dn}$. For the union of the nets at levels n and $n+1$, consider every pair whose distance is at most $6 \cdot 2^{-n}$. There are at most $C' 2^{2D(n+1)}$ such pairs. Gaussian moments imply that the L^q norm of the maximum of their increments is at most

$$C_q L^{1/2} 2^{-n/2} 2^{2D(n+1)/q}.$$

Choose $q \ge p$ so large that $\eta + 2D/q < 1/2$. The sum over n of these maxima, multiplied by $2^{n\eta}$, has finite L^q norm, bounded by a constant times $L^{1/2}$. Approximate every point by successive net points. Their telescoping increments converge uniformly. For two points at distance between 2^{-n-1} and 2^{-n}, compare their level-n approximations and then the two tails. The resulting bound is a constant times $d(z, w)^\eta$. The variance bound makes this continuous extension a modification of the original field. The value at z_0 and the same estimates control the supremum. Enlarging constants covers the finitely many scales above the diameter normalization. □

15.2 Abelian paths in the intrinsic holonomy completion

Return to the degree-m line bundle L_m and the $\sqrt{2}$-noise diffusion of Theorem 13.2. To avoid confusing the bundle degree with the number of polygon edges, use k for the latter. Let

$$A_t^\varepsilon = A_m + e^{-\varepsilon \Delta_1} a_t + h_t,$$

where h_t is any continuous lift of its harmonic torus Brownian motion. Each A_t^ε is a smooth connection in space, and its closed-loop holonomies do not depend on the chosen lift.

Theorem 15.6 (Uniform polygonal observables)**.** *For every initial state in $\mathcal{E}_{m,s}$, the holonomy families of A_t^ε converge, as $\varepsilon \downarrow 0$, in*

$$L^p(\Omega; C([\delta, T]; \mathscr{H}_r(L_m))), \qquad 0 < \delta < T < \infty, \quad p < \infty, \tag{15.7}$$

where convergence is measured with d_r. Their limit has a version with continuous paths for $t > 0$. It is defined simultaneously for all based short-geodesic polygons, satisfies the holonomy identities, and agrees with the Gaussian trace observables of Theorem 13.7 on each fixed polygon. For every fixed k, its k-edge component is locally η-Hölder in the vertices and θ-Hölder in time for all $\eta < 1/2$ and $\theta < 1/4$.

If the initial connection is smooth, or if the initial state has the invariant law $\mu_{m,s}$, convergence and continuity also hold on $[0, T]$. The limiting holonomy family is independent of the reference connection, and the homeomorphisms for different radii in Theorem 15.2 identify the resulting limits. The analogous statements hold for principal compact torus bundles and for $\mathscr{U}_r$.

Proof. We first treat the centered coexact process X_t started at zero. For each $p \in K_k^r$, the current trace $Y_t(p) = \langle J_p, X_t \rangle$ is the Gaussian variable defined in Theorem 13.7. The covariance operator

$$Q_t = \Delta_1^{-1}(I - e^{-2t\Delta_1})P_{\mathrm{coex}}$$

is bounded above by $Q_\infty = \Delta_1^{-1}P_{\mathrm{coex}}$. Hence Lemma 15.4 gives

$$\mathbb{E}|Y_t(p) - Y_t(q)|^2 \le C_k d_k(p, q) \tag{15.8}$$

for nearby vertices, uniformly for $t \ge 0$. For vertex distances bounded below by δ_0, the uniform current bound absorbs the individual variances and proves the same inequality after increasing the constant.

Uniformly over $p \in K_k^r$, the proof of (13.18) gives, for every $\beta < 1/2$,

$$\mathbb{E}|Y_t(p) - Y_u(p)|^2 \le C_{k,\beta}|t - u|^\beta. \tag{15.9}$$

Indeed choose $a \in (1/2, 1 - \beta)$ in (15.4) and sum the resulting spectral bound. The same two estimates hold for the stationary coexact process.

The vertex space K_k^r is a subset of the compact manifold Σ^{k-1}, so it has covering numbers at most $C\varepsilon^{-2(k-1)}$. On $[0, T] \times K_k^r$ the metric

$$d_k(p, q) + |t - u|^\beta$$

therefore has covering numbers bounded by $C\varepsilon^{-2(k-1)-1/\beta}$. Apply Lemma 15.5 to (15.8) and (15.9). It gives a jointly continuous Gaussian version, with any vertex exponent below $1/2$ and any time exponent below $1/4$. To obtain a specified pair of exponents, choose a Hölder exponent $\eta' < 1/2$ larger than both the desired vertex exponent and twice the desired time exponent, and then choose $\beta < 1/2$ with $\beta\eta'$ larger than the desired time exponent.

We next prove uniform convergence of the heat regularizations. Put $Z_t^\varepsilon(p) = \langle J_p, (e^{-\varepsilon\Delta_1} - I)X_t \rangle$. Choose $q > 0$ and $\beta > 0$ with $2q + \beta < 1/2$. The spectral estimates in Theorem 13.7, now using (15.4), give

$$\sup_{t,p} \mathbb{E}|Z_t^\varepsilon(p)|^2 \le C\varepsilon^{2q}, \tag{15.10}$$

$$\mathbb{E}|Z_t^\varepsilon(p) - Z_u^\varepsilon(p)|^2 \le C\varepsilon^{2q}|t - u|^\beta. \tag{15.11}$$

The spectral multiplier $e^{-\varepsilon\lambda} - 1$ has absolute value at most one. Consequently the spatial increment also obeys the bound in (15.8), while (15.10) bounds it by $C\varepsilon^{2q}$. For any $\vartheta \in (0, 1)$, interpolation of these two bounds yields

$$\mathbb{E}|Z_t^\varepsilon(p) - Z_t^\varepsilon(q')|^2 \le C\varepsilon^{2q\vartheta} d_k(p, q')^{1-\vartheta}.$$

Here q' denotes a vertex tuple, to distinguish it from the exponent q. Together with (15.11), this is an increment estimate with coefficient $C\varepsilon^{2q\vartheta}$ for the metric

$$d_k(p,q')^{1-\vartheta} + |t-u|^{\beta}.$$

Its covering numbers are polynomial as well. Applying Lemma 15.5 to this error field gives

$$\left\| \sup_{0\le t\le T,\, p\in K_k^r} |Z_t^{\varepsilon}(p)| \right\|_{L^p(\Omega)} \le C\varepsilon^{q\vartheta} \tag{15.12}$$

for every finite moment order p. The stationary process satisfies the same estimate.

For deterministic initial data in H^{-s}, the form $e^{-t\Delta_1}a_0$ is smooth uniformly on $[\delta,T]$. Its heat-regularization error has C^0 norm $O(\varepsilon)$ there, and every polygon has length at most kr. Thus its trace errors converge uniformly in the vertices and time. Its trace as a function of the vertices and positive time is smooth. The holonomy family of the smooth reference A_m is smooth in the vertices. The harmonic factor is a finite-dimensional continuous Gaussian exponential, and integration of a fixed harmonic form along the polygons is a smooth vertex function. It follows that

$$h_k(A_t^{\varepsilon})(p) = h_k(A_m)(p)\exp\left(i\langle J_p, e^{-\varepsilon\Delta_1}a_t\rangle + i\int_{\gamma_p} h_t \right)$$

converges in every finite moment in the uniform norm on $[\delta,T]\times K_k^r$, with the asserted regularity of its limit. The harmonic factor has every time exponent below $1/2$ and does not lower the exponents already proved.

For each $\varepsilon > 0$ the regularized coexact field has a continuous version in every spatial Sobolev space, as follows from the exponentially decaying spectral coefficients. It therefore defines a continuous path of smooth connections and a continuous path in $\mathscr{H}_r(L_m)$. The uniform estimates just proved hold for every k. The summable weights in (15.2), followed by a finite-factor approximation, give (15.7). Along $\varepsilon = 2^{-n}$, the moment estimates and Borel–Cantelli give almost sure uniform convergence in every factor on every interval $[1/N, N]$, simultaneously. Since $\mathscr{H}_r(L_m)$ is closed, the limiting family belongs to that space at every positive time and satisfies all its holonomy identities. Uniform convergence in the product metric proves continuity of its paths.

For a smooth initial connection the deterministic part is smooth down to time zero. For a stationary initial state, the stationary Gaussian estimates above apply down to time zero. These establish the corresponding assertions on $[0,T]$. Reference independence follows from Proposition 13.5 and the cancellation in (13.16). Each radius homeomorphism agrees on the smooth approximants, so its continuity identifies the limits. The compact torus case follows componentwise, and projection to $\mathscr{U}_r$ is continuous. □

Theorem 15.6 gives continuous holonomy observables of the strong Markov process on $\mathcal{E}_{m,s}$ from Theorem 13.2. To use these observables as states, Section 16 recovers the coexact and harmonic coordinates by a Borel inverse on a dense Borel domain. Pathwise invariance of that domain then transfers the strong Markov family to its holonomy image.

16 Reconstruction and the Abelian holonomy Markov process

Fix a degree m, an exponent $s > 0$, and the constant-curvature reference A_m. Throughout this section the structure group is $U(1)$. Conjugation is therefore trivial. Put $E = \mathcal{E}_{m,s}$ and $\mathscr{H} = \mathscr{H}_r(L_m)$, and define the continuous maps

$$\mathfrak{H}_{\varepsilon} : E \longrightarrow \mathscr{H}, \qquad \mathfrak{H}_{\varepsilon}(a,[h]) = \mathbf{h}(A_m + e^{-\varepsilon\Delta_1}a + h), \qquad \varepsilon > 0. \tag{16.1}$$

They are independent of the representative of $[h]$. Continuity follows from heat smoothing, the uniform length bound on each K_k^r, and continuity of the harmonic period characters.

Definition 16.1 (The reconstructible domain)**.** Let

$$E^{\mathrm{hol}} = \{x \in E : \mathfrak{H}_\varepsilon(x) \text{ converges in } \mathscr{H} \text{ as } \varepsilon \downarrow 0\}. \tag{16.2}$$

For $x \in E^{\mathrm{hol}}$, write $\mathfrak{H}(x) = \lim_{\varepsilon\downarrow 0} \mathfrak{H}_\varepsilon(x)$ and set $\mathscr{H}^{\mathrm{rec}}_{m,s} = \mathfrak{H}(E^{\mathrm{hol}})$.

The limit in this definition is the full heat limit. It can be tested using only rational positive ε: the map $\varepsilon \mapsto \mathfrak{H}_\varepsilon(x)$ is continuous away from zero. The Cauchy criterion over rational values therefore shows that E^{hol} is Borel, and its limit map is Borel. The domain contains all smooth coexact coordinates and all harmonic coordinates.

16.1 Local fluxes and smooth averaging kernels

We reconstruct the distributional curvature by smoothly averaging the triangle fluxes encoded by holonomy. These averages produce smooth test functions for pairing with the curvature.

Choose finitely many strongly convex normal coordinate balls U_i, with oriented coordinates, relatively compact concentric subneighborhoods V_i, and a smooth partition of unity (χ_i) with $\operatorname{supp}\chi_i \subset V_i$. Take their radii sufficiently small that any two points in U_i are joined inside U_i by a unique geodesic of length less than r. Fix a point $c_i \in U_i$ and a short-geodesic polygonal path from o to c_i. For $z \in \overline{V}_i$, append the geodesic from c_i to z to obtain a stem $c_i(z)$. The number of edges in these stems is uniformly bounded, and their vertices depend smoothly on z.

Let $E_{i,1}, E_{i,2}$ be the positively oriented coordinate vector fields. For all sufficiently small $\delta > 0$, let $D_{i,z,\delta}$ be the positively oriented small geodesic triangle with vertices

$$z, \qquad \exp_z(\delta E_{i,1}(z)), \qquad \exp_z(\delta E_{i,2}(z)).$$

Write $b_{i,\delta}(z)$ for its area and $\ell_{i,z,\delta}$ for the based polygon obtained by following $c_i(z)$, traversing the triangle boundary, and returning along $c_i(z)^{-1}$. The map $(z,\delta) \mapsto \ell_{i,z,\delta}$ is a continuous map into one fixed polygon space $K^r_{k_i}$, including $\delta = 0$.

Lemma 16.2 (Averaging over moving geodesic triangles)**.** *For $\varphi \in C_c^\infty(V_i)$ define*

$$(T_{i,\delta}\varphi)(y) = \int_{V_i} \varphi(z) \frac{\mathbf{1}_{D_{i,z,\delta}}(y)}{b_{i,\delta}(z)} \operatorname{vol}_g(dz). \tag{16.3}$$

For small δ, this is a smooth function compactly supported in U_i, and

$$T_{i,\delta}\varphi \longrightarrow \varphi \quad \text{in } C_c^\infty(U_i) \quad \text{as } \delta \downarrow 0. \tag{16.4}$$

Proof. We work in the oriented coordinate chart, suppressing the index i. Parametrize the triangle by the geodesic fan

$$\Phi_\delta(z,u,t) = \exp_z\Big(t \exp_z^{-1} q_\delta(z,u)\Big), \qquad 0 \le u,t \le 1,$$

where $q_\delta(z,u)$ is the point at parameter u on the geodesic joining the two noncentral vertices. For $t > 0$ this parametrizes the triangle once, apart from its boundary. Its area Jacobian has the form

$$J_\delta(z,u,t) = \delta^2 t\, j_\delta(z,u,t),$$

where j_δ extends smoothly to $\delta = 0$ and $t = 0$ and is strictly positive for small δ. This follows directly from smoothness of the exponential map and its derivative at the zero section. In coordinates,

$$\Phi_\delta(z,u,t) = z + \delta t\big((1-u)e_1 + ue_2\big) + O(\delta^2),$$

with the remainder controlled in every derivative with respect to z, uniformly in u, t. Consequently

$$b_\delta(z) = \int_0^1 \int_0^1 J_\delta(z,u,t)\,du\,dt, \qquad \frac{J_\delta(z,u,t)}{b_\delta(z)} \longrightarrow 2t$$

smoothly in z, uniformly on the parameter square.

For each fixed (u,t), the map $z \mapsto \Phi_\delta(z,u,t)$ is C^1-close to the identity on a convex coordinate neighborhood of the support of φ. It is therefore injective there for small δ and has a smooth inverse $z = z_\delta(y,u,t)$. A support margin permits a smooth extension of this near-identity change of variables outside that neighborhood. Let $g_0(z)$ be the coordinate density of vol_g. Changing variables from z to y in the adjoint of the fan average gives

$$T_\delta\varphi(y) = \int_0^1 \int_0^1 \varphi(z_\delta(y,u,t)) \frac{J_\delta(z_\delta(y,u,t),u,t)}{b_\delta(z_\delta(y,u,t))} \times \frac{g_0(z_\delta(y,u,t))}{g_0(y)} |\det D_y z_\delta(y,u,t)|\,du\,dt.$$

This expression is smooth in y. The inverse maps tend to the identity smoothly, the density and determinant factors tend to one, and the normalized area factor tends to $2t$. Differentiation under the fixed parameter integral proves $T_\delta\varphi \to \varphi$ in every C^k norm. The supports remain in a fixed compact subset of U_i, which proves (16.4). □

Let $\mathrm{Arg} : U(1) \to (-\pi, \pi]$ denote the principal argument. For a limiting holonomy family $H \in \mathscr{H}$, set

$$L_{i,\delta}(H,\varphi) = \int_{V_i} \varphi(z) \frac{\mathrm{Arg}\, H(\ell_{i,z,\delta})}{b_{i,\delta}(z)}\, \mathrm{vol}_g(dz). \tag{16.5}$$

These are Borel functionals. They can also be specified using fixed coordinate Riemann sums. On every sufficiently small scale, the arguments in (16.5) lie in a common open arc and the integrands are continuous. Thus only countably many evaluations and limits are needed in the reconstruction below. Values on the finitely many larger scales do not affect that reconstruction.

Lemma 16.3 (Recovery of distributional curvature)**.** *Let $x = (a,[h]) \in E^{\mathrm{hol}}$ and $H = \mathfrak{H}(x)$. Write*

$$f_x = \frac{2\pi m}{V} + *da$$

for the scalar distributional curvature. For every $\varphi \in C_c^\infty(V_i)$ and all sufficiently small δ,

$$L_{i,\delta}(H,\varphi) = \langle f_x, T_{i,\delta}\varphi \rangle. \tag{16.6}$$

In particular,

$$\lim_{\delta \downarrow 0} L_{i,\delta}(H,\varphi) = \langle f_x, \varphi \rangle. \tag{16.7}$$

Proof. At $\delta = 0$ the lasso $\ell_{i,z,0}$ consists of a stem followed by its reverse. Every element of $\mathscr{H}$ takes value one on that loop. Compactness of $\overline{V}_i$, continuity of H on $K^r_{k_i}$, and continuity of the lasso family imply

$$\sup_{z \in \overline{V}_i,\, 0 \le \rho \le \delta} d_{U(1)}(H(\ell_{i,z,\rho}), 1) \longrightarrow 0 \quad \text{as } \delta \downarrow 0.$$

Choose δ so that all these values lie in the arc with $|\mathrm{Arg}| < \pi/4$. Uniform convergence of $\mathfrak{H}_\varepsilon(x)$ on the same polygon space puts their values in $|\mathrm{Arg}| < \pi/2$, simultaneously in z and $0 \le \rho \le \delta$, for every sufficiently small ε.

Let $f_{x,\varepsilon}$ be the smooth scalar curvature of $A_m + e^{-\varepsilon\Delta_1}a + h$. Abelian Stokes' theorem gives

$$\mathfrak{H}_\varepsilon(x)(\ell_{i,z,\rho}) = \exp\left(i \int_{D_{i,z,\rho}} f_{x,\varepsilon}\, \mathrm{vol}_g \right).$$

For fixed z, ε, the real integral is continuous in ρ, equals zero at $\rho = 0$, and its exponential stays in the displayed arc. It is therefore the principal argument throughout $0 \le \rho \le \delta$. This excludes any undetected multiple of 2π. Averaging at $\rho = \delta$ now gives

$$L_{i,\delta}(\mathfrak{H}_\varepsilon(x), \varphi) = \langle f_{x,\varepsilon}, T_{i,\delta}\varphi \rangle.$$

The left side converges to $L_{i,\delta}(H, \varphi)$ uniformly under the integral. On the right, $f_{x,\varepsilon} \to f_x$ distributionally and $T_{i,\delta}\varphi$ is smooth by Lemma 16.2. This proves (16.6). The approximation identity in that lemma proves (16.7). □

16.2 A countable Borel inverse

Choose an orthonormal basis $(f_j)_{j\ge1}$ of mean-zero scalar eigenfunctions, with $\Delta_0 f_j = \lambda_j f_j$, and use $e_j = -\lambda_j^{-1/2} * df_j$ as the coexact eigenbasis. Then $de_j = \lambda_j^{1/2} f_j\, \mathrm{vol}_g$. Choose also $2g$ based short-geodesic polygonal loops $\gamma_1, \dots, \gamma_{2g}$ representing an integral homology basis. Their period characters give a torus isomorphism

$$\Pi : J_g \longrightarrow U(1)^{2g}, \qquad \Pi([h])_\ell = \exp\left(i \int_{\gamma_\ell} h \right). \tag{16.8}$$

In genus zero this is the unique map between one-point spaces.

Theorem 16.4 (Deterministic reconstruction from completed holonomies)**.** *The map $\mathfrak{H} : E^{\mathrm{hol}} \to \mathscr{H}$ is injective. Its image $\mathscr{H}^{\mathrm{rec}}_{m,s}$ is a Borel subset of $\mathscr{H}$, and it has a Borel inverse*

$$R : \mathscr{H}^{\mathrm{rec}}_{m,s} \longrightarrow E^{\mathrm{hol}}. \tag{16.9}$$

The reconstruction uses countably many curvature coefficients, triangle scales and period characters. The image is dense in $\mathscr{H}$.

Proof. Fix a sequence $\delta_n \downarrow 0$ small enough for all the local triangle constructions. For $H \in \mathscr{H}$, define the following coefficients whenever the limits exist:

$$b_j(H) = \lim_{n\to\infty} \sum_i L_{i,\delta_n}(H, \chi_i f_j), \qquad a_j(H) = \lambda_j^{-1/2} b_j(H). \tag{16.10}$$

The set on which all these real limits exist is Borel. Impose also the Borel condition

$$\sum_j (1+\lambda_j)^{-s} |a_j(H)|^2 < \infty.$$

On this domain the convergent series $a(H) = \sum_j a_j(H) e_j$ is a Borel H^{-s}_{coex}-valued map. If $H = \mathfrak{H}(a, [h])$, Lemma 16.3 shows that $b_j(H) = \sqrt{\lambda_j} a_j$; the constant reference curvature has zero pairing with f_j. Thus $a(H) = a$.

For each basis loop, define the following circle-valued quantity whenever its limit exists:

$$v_\ell(H) = \lim_{n\to\infty} \exp\Big(i \langle J_{\gamma_\ell}, e^{-2^{-n}\Delta_1} a(H) \rangle \Big). \tag{16.11}$$

Existence of these finitely many circle-valued limits is again a Borel condition. When $H = \mathfrak{H}(a,[h])$, they exist because the full regularized holonomies converge, while the reference and harmonic factors are constant in the regularization parameter. Define

$$z_\ell(H) = \frac{H(\gamma_\ell)}{\mathrm{Hol}_{\gamma_\ell}(A_m)v_\ell(H)}, \qquad [h(H)] = \Pi^{-1}((z_\ell(H))_{\ell=1}^{2g}). \tag{16.12}$$

The partial map $R_0(H) = (a(H),[h(H)])$ is Borel on its Borel domain and satisfies $R_0(\mathfrak{H}(x)) = x$ for every $x \in E^{\mathrm{hol}}$. This proves injectivity.

To show that the image is Borel, extend the partial maps by fixed values outside their domains. Then

$$\mathscr{H}_{m,s}^{\mathrm{rec}} = \{H :\ R_0(H) \text{ is defined}, \quad R_0(H) \in E^{\mathrm{hol}}, \quad \mathfrak{H}(R_0(H)) = H\}$$

is Borel. Restricting R_0 gives (16.9). Finally, all smooth connections admit smooth coexact–harmonic coordinates modulo an exact gauge. Their holonomy families lie in the displayed image, and are dense in $\mathscr{H}$ by its definition. □

Corollary 16.5 (Recovery of degree and change of reference)**.** *In the common ambient polygon-family space, every reconstructed Abelian holonomy family determines its degree by*

$$\deg_{\mathrm{hol}}(H) = \frac{1}{2\pi}\lim_{n\to\infty}\sum_i L_{i,\delta_n}(H,\chi_i). \tag{16.13}$$

Consequently the reconstructed images belonging to different degrees are disjoint. The degree is a Borel function on their union. Under an arbitrary smooth change of reference, the affine coordinate transformation of Proposition 13.5 preserves the full-heat reconstructible domain and its physical holonomy map.

Proof. Apply Lemma 16.3 with the test functions χ_i and sum over i. The limit is $\langle f_x, 1\rangle = 2\pi m$, since the distributional exact curvature da pairs to zero with the constant function. The formula is the same in every sector and consists of Borel limits, which proves disjointness and measurability.

For a smooth reference change $B = A_m + b$, write $b = d\phi + b_{\mathrm{c}} + b_{\mathcal{H}}$. The regularized connections in the two coordinate systems differ, modulo the exact gauge $d\phi$, by $(I - e^{-\varepsilon\Delta_1})b_{\mathrm{c}}$. Its C^0 norm is $O(\varepsilon)$. On K_k^r the change of each holonomy is thus at most $kr\|(I - e^{-\varepsilon\Delta_1})b_{\mathrm{c}}\|_{C^0}$. It tends to zero in every factor and in the product metric. Hence either full heat limit exists exactly when the other does, and the two limits coincide. The same estimate, with the uniform length bound in place of kr, applies to the regular curve families used below. □

16.3 Full heat limits along the entire stochastic path

Lemma 16.6 (Uniform heat limits over dyadic parameter blocks)**.** *Let X_t be the centered coexact process started at zero, or the stationary centered coexact process. For every finite T and every k, the Gaussian traces satisfy, almost surely,*

$$\sup_{0\le t\le T,\ p\in K_k^r} |\langle J_p, (e^{-\varepsilon\Delta_1} - I)X_t\rangle| \longrightarrow 0 \quad \text{as } \varepsilon \downarrow 0. \tag{16.14}$$

The event can be chosen simultaneously over all positive integers k, T.

Proof. Choose $q, \beta > 0$ with $2q + \beta < 1/2$, and $\vartheta \in (0,1)$. On a dyadic block write $\varepsilon = 2^{-n}u$, $1 \le u \le 2$, and denote the corresponding error field by $Z_n(t,p,u)$. The proof of (15.12) gives

the point, space and time variance bounds, uniformly over this block. The extra parameter has the estimate

$$\mathbb{E}|Z_n(t,p,u) - Z_n(t,p,v)|^2 \le C_k 2^{-2qn}|u-v|^{2q}.$$

Indeed,

$$|e^{-2^{-n}u\lambda} - e^{-2^{-n}v\lambda}| \le C_q 2^{-qn}\lambda^q |u-v|^q,$$

and the uniform current bound controls the sum $\sum_j |\langle J_p, e_j\rangle|^2 \lambda_j^{2q-1}$. After the spatial interpolation used previously, all increments are bounded by $C2^{-2q\vartheta n}$ times the metric

$$d_k(p,p')^{1-\vartheta} + |t-t'|^\beta + |u-v|^{2q}.$$

This compact parameter space has polynomial covering numbers. Lemma 15.5 consequently gives

$$\left\| \sup_{t\le T,\ p\in K_k^r,\ 1\le u\le 2} |Z_n(t,p,u)| \right\|_{L^p} \le C_{k,T,p} 2^{-q\vartheta n}$$

for every finite moment order. The expectations of these suprema are summable in n. Their sum is therefore finite almost surely, which proves (16.14). Intersecting countably many events gives the final assertion. □

Proposition 16.7 (Pathwise invariance of the reconstructible domain)**.** *The invariant measure $\mu_{m,s}$ gives E^{hol} full measure. From every initial $x \in E$, the Abelian diffusion X^x satisfies*

$$X_t^x \in E^{\mathrm{hol}} \quad \text{for every } t > 0 \quad \text{almost surely.} \tag{16.15}$$

For every $x \in E^{\mathrm{hol}}$ this holds also at $t = 0$, and $t \mapsto \mathfrak{H}(X_t^x)$ is continuous on $[0,\infty)$ in the ambient holonomy metric. For arbitrary $x \in E$ the same path is continuous on $(0,\infty)$.

Proof. The stationary assertion at time zero follows from Lemma 16.6. For a deterministic initial coordinate a, the part $e^{-t\Delta_1}a$ is smooth uniformly on compact intervals in positive time. Its full heat limits are uniform on every fixed polygon family. The same lemma handles the stochastic convolution simultaneously at every time, and the harmonic factor is smooth in space. This proves (16.15) simultaneously at every positive time on one event of probability one.

For continuity at zero, use the exact Abelian factorization. Let $\mathcal{N}_t$ be the holonomy factor contributed by the centered coexact noise and the harmonic Brownian increment, with the reference factor omitted. The previous estimates give a continuous path of circle-valued polygon families with $\mathcal{N}_0 = 1$. For $t > 0$,

$$\mathfrak{H}(X_t^x) = \mathfrak{H}_t(x)\,\mathcal{N}_t, \tag{16.16}$$

where multiplication is componentwise. To verify this identity, first regularize the random connection and use $e^{-\varepsilon\Delta_1}e^{-t\Delta_1} = e^{-(\varepsilon+t)\Delta_1}$, and then take its full heat limit. If $x \in E^{\mathrm{hol}}$, $\mathfrak{H}_t(x) \to \mathfrak{H}(x)$ as $t \downarrow 0$ by the definition of the domain. Together with $\mathcal{N}_t \to 1$, this proves continuity at zero. Continuity at positive times follows from the same factorization and the previous uniform estimates. □

Theorem 16.8 (The reconstructed holonomy strong Markov family)**.** *On the Borel state space $\mathscr{H}_{m,s}^{\mathrm{rec}}$ there is a global strong Markov family with continuous paths in $\mathscr{H}_r(L_m)$, given by*

$$Y_t^H = \mathfrak{H}(X_t^{R(H)}), \qquad H \in \mathscr{H}_{m,s}^{\mathrm{rec}}. \tag{16.17}$$

It takes values in $\mathscr{H}_{m,s}^{\mathrm{rec}}$ at every time. For bounded Borel f on this state space its transition semigroup is

$$(Q_t f)(H) = P_t(f \circ \mathfrak{H})(R(H)), \tag{16.18}$$

where $f \circ \mathfrak{H}$ is extended arbitrarily outside E^{hol}. Its unique invariant probability measure is

$$\widehat{\mu}_{m,s} = \mathfrak{H}_* \mu_{m,s}. \tag{16.19}$$

It is reversible, and its $L^2(\widehat{\mu}_{m,s})$ spectral gap is the same sharp constant ρ_g as in Proposition 13.3. These conclusions hold componentwise for compact torus groups in each fixed Chern sector, with the gap $\rho_{g,M}$ of (13.26) when the Lie algebra has metric M.

Proof. The Borel inverse in Theorem 16.4 and the pathwise invariance in Proposition 16.7 make (16.17) well defined at all times, with continuous paths. These path laws are Borel in H: their finite-dimensional laws are Borel by the explicit diffusion and the Borel maps $R, \mathfrak{H}$, and rational-time evaluations generate the Borel sigma field of each continuous ambient path space. Formula (16.18) defines Borel kernels and immediately gives their semigroup property. The extension outside the reconstructible domain has no effect, because all positive-time transition laws are supported on that domain.

Moreover, along the entire path, $X_t^{R(H)} = R(Y_t^H)$. The natural filtrations of the two processes are therefore identical, as are their usual augmentations. A stopping time for the holonomy process is a stopping time for the original diffusion. Applying its strong Markov property and then (16.18) proves the strong Markov property of Y. The statement remains valid at arbitrary finite stopping times, since the pathwise membership in the reconstructible domain holds simultaneously at all times.

The full-measure assertion and the inverse map identify $L^2(\widehat{\mu}_{m,s})$ unitarily with $L^2(\mu_{m,s})$ by composition with $\mathfrak{H}$. Invariance, reversibility and the sharp spectral gap follow from this unitary intertwining. If a second invariant probability measure on $\mathscr{H}_{m,s}^{\mathrm{rec}}$ existed, its pullback by R would be an invariant probability measure for the original diffusion, contradicting its uniqueness. The vector-valued Abelian construction uses the same reconstruction in each component and the period isomorphism for the corresponding torus. □

The transition state space in Theorem 16.8 is the dense Borel domain $\mathscr{H}_{m,s}^{\mathrm{rec}}$, with paths continuous in the ambient Polish completion. From an arbitrary Sobolev initial state, the process enters this domain simultaneously at all positive times, including strictly positive stopping times.

17 Uniform extension from polygons to regular curves

We next establish a samplewise extension to all piecewise regular C^2 loops. Uniform statements are formulated on families with specified regularity bounds. Fix an integer $b \geq 1$ and constants $0 < v < L$ and $0 \leq M < \infty$. Let $\mathcal{K}_{b,v,L,M}$ be the space of based loops which are $C^{1,1}$ on each interval $[(j-1)/b, j/b]$, $1 \leq j \leq b$, and satisfy, on each such interval,

$$v \leq |\dot{\gamma}| \leq L, \qquad |\nabla_t \dot{\gamma}| \leq M \quad \text{almost everywhere.} \tag{17.1}$$

One-sided derivatives are used at the division points. We give this space the uniform distance $d_0(\gamma, \eta) = \sup_{t \in [0,1]} d(\gamma(t), \eta(t))$. Arzelà–Ascoli on the finitely many intervals, including the uniformly Lipschitz derivatives in local coordinates, shows that this is a compact metric space. Its topology agrees with piecewise C^1 convergence on this bounded family.

Let $\Pi_N \gamma$ be the polygon obtained by subdividing each of the b intervals into N equal pieces and joining successive values of γ by shortest geodesics. For N sufficiently large these are polygons in K_{bN}^r. All constants below may depend on b, v, L, M, the metric and r.

Lemma 17.1 (Uniform geometry of regular curve families)**.** *For $\mathcal{K} = \mathcal{K}_{b,v,L,M}$ the following estimates hold.*

1. *For every* $a \in (1/2, 1)$,

$$\sup_{\gamma \in \mathcal{K}} \|J_\gamma\|_{H^{-a}} + \sup_{N \geq N_0, \gamma \in \mathcal{K}} \|J_{\Pi_N \gamma}\|_{H^{-a}} < \infty \tag{17.2}$$

for a sufficiently large fixed N_0.

2. *Put* $Q(J) = \langle J, \Delta_1^{-1} P_{\mathrm{coex}} J \rangle$. *Then, uniformly for* $N \geq N_0$,

$$Q(J_\gamma - J_\eta) \leq C d_0(\gamma, \eta), \tag{17.3}$$

$$Q(J_{\Pi_N \gamma} - J_{\Pi_N \eta}) \leq C d_0(\gamma, \eta), \tag{17.4}$$

$$Q(J_{\Pi_N \gamma} - J_\gamma) \leq C N^{-2}. \tag{17.5}$$

3. *The covering numbers satisfy*

$$\log N(\mathcal{K}, d_0, \delta) \leq C \delta^{-1/2} \log(C/\delta), \qquad 0 < \delta < 1. \tag{17.6}$$

Proof. On each of the b parameter intervals, subdivide further into a fixed number of intervals, using the same parameter subdivision for every curve, whose lengths are small in terms of v, L, M and the coordinate bounds for the metric. Every resulting arc is contained in a normal chart. In that chart its velocity changes by less than a fixed small fraction of v. Its projection onto the direction of its initial velocity is consequently strictly increasing, with derivative bounded below. Each arc is therefore a graph with a uniform Lipschitz constant, and the total number of these graph arcs is uniformly bounded.

The same assertion holds for $\Pi_N \gamma$, for all $N \geq N_0$. The elementary interpolation estimates in normal coordinates give

$$d_0(\Pi_N \gamma, \gamma) \leq C N^{-2}, \qquad |(\Pi_N \gamma)'(t) - \gamma'(t)| \leq C N^{-1} \tag{17.7}$$

away from the subdivision vertices, with derivatives compared in the chosen chart. Choose N_0 so that the second error is small relative to v. The polygon arcs on the fixed larger parameter intervals are then uniformly Lipschitz graphs as well.

Flattening a Lipschitz graph acts boundedly on H^a for $0 < a < 1$. This follows directly from the Gagliardo seminorm under a bi-Lipschitz change of variables. The flat trace theorem for $a > 1/2$, followed by Cauchy–Schwarz along a graph, bounds its current in H^{-a}. The uniform graph constants and the fixed number of arcs prove (17.2).

For the covariance estimates, fill the difference of the two loop currents. Suppose two loops of the types under consideration are δ-close in the uniform distance and that the corresponding arcs are increasing graphs in the same direction. Close each pair of corresponding graph arcs by short connectors between their endpoints. In the common coordinate chart the two graph functions differ by at most $C\delta$, because their horizontal velocities are bounded below. Their graph intervals have endpoints within $C\delta$. Extend both graph functions constantly over the short missing endpoint intervals to obtain a common horizontal domain. The signed region between the extended graphs has multiplicity at most one and area at most $C\delta$, even if the graphs intersect. Its boundary differs from the two original arcs and the straight endpoint connectors only by polygons of a bounded number of sides, each contained in a ball of radius $C\delta$. Triangulating these endpoint caps adds bounded multiplicity and area $O(\delta^2)$. Finally, a straight connector and the short geodesic with the same endpoints form, in a coordinate direction along that connector, a graph lens of bounded multiplicity and area $O(\delta^2)$. This constructs an integer-valued filling of the desired boundary with a multiplicity bound independent of the polygon subdivision.

Sum these local fillings over the fixed graph decomposition. The joining connectors cancel, and the resulting density w satisfies

$$J_1 - J_2 = \partial w, \qquad \|w\|_{L^\infty} \le C, \qquad \|w\|_{L^1} \le C\delta. \tag{17.8}$$

For two curves $\gamma, \eta \in \mathcal{K}$, closeness in C^0 and their uniform second-derivative bound imply closeness of their velocities in each piece, with bound $C\delta^{1/2}$. This is the usual interpolation inequality, obtained by comparing a derivative with a difference quotient of step $\delta^{1/2}$; one-sided quotients handle endpoints. Hence, for small δ, the two curves use the same graph directions. Their polygonal approximations do also, by (17.7) and the choice of N_0. Moreover geodesic interpolation is uniformly Lipschitz in its endpoints, so $d_0(\Pi_N\gamma, \Pi_N\eta) \le C d_0(\gamma, \eta)$. This proves (17.8) for the first two pairs in the statement. Applying it to $\gamma, \Pi_N\gamma$ gives $\|w\|_{L^1} \le CN^{-2}$. Since $Q(\partial w) = \|w - \overline{w}\|_{L^2}^2$, these estimates give (17.3)– (17.5) for nearby curves. For distances bounded below, the uniform current bound absorbs the individual variances and gives the stated global bounds.

Finally, sample a curve at parameter spacing comparable with $\delta^{1/2}$, including all piece endpoints. Geodesic interpolation changes the curve by at most $C\delta$. Approximate each sampled point by a δ-net of the surface, of cardinality at most $C\delta^{-2}$. The interpolation remains within $C\delta$ of the original curve. There are at most $C\delta^{-1/2}$ sample points, so the logarithm of the number of possible approximants is at most the right-hand side of (17.6). Selecting one curve from each occupied approximation class produces a net with centers in $\mathcal{K}$ and proves the estimate. □

Lemma 17.2 (Gaussian continuity with curve-family entropy)**.** *Let (T, d) be compact and suppose, for some $0 < a < 1$,*

$$\log N(T, d, \varepsilon) \le C\varepsilon^{-a} \log(C/\varepsilon).$$

If a centered Gaussian field satisfies $\mathbb{E}|Z_z - Z_w|^2 \le L d(z, w)$ and $\mathbb{E}|Z_{z_0}|^2 \le L$, it has a continuous version. For every $\eta < (1 - a)/2$ and every finite p, its supremum and η-Hölder seminorm have L^p norm at most $C_p L^{1/2}$.

Proof. Use dyadic nets and neighboring pairs as in Lemma 15.5. If M centered Gaussian variables have variances at most σ^2, their maximum absolute value has L^p norm at most $C\sigma\sqrt{\log(2M) + p}$. This follows from the Gaussian tail bound, a union bound, and integration of that bound. At level n the increment variance is at most $CL2^{-n}$, while the logarithm of the number of pairs is at most $Cn2^{an}$. The maximum increment therefore has L^p norm at most

$$C_p L^{1/2} \sqrt{n + 1}\, 2^{-(1-a)n/2}.$$

Multiplication by $2^{\eta n}$ leaves a summable series when $\eta < (1 - a)/2$. The same telescoping construction and comparison of neighboring net points prove the assertion. □

Theorem 17.3 (Uniform curve traces and polygon approximation)**.** *Fix $\mathcal{K} = \mathcal{K}_{b,v,L,M}$. For every initial state $x \in E$, the heat-regularized Wilson holonomies converge, almost surely and in every finite moment, uniformly on $[\delta, T] \times \mathcal{K}$, for $0 < \delta < T < \infty$. Their limit $W_t(\gamma)$ is jointly continuous in positive time and in the uniform curve topology. It agrees with the fixed-curve Gaussian trace and with the polygonal holonomies already constructed. If the initial state is smooth or stationary, the same statements hold on $[0, T]$.*

For every $0 < \theta < 1/2$ and every finite p,

$$\left\| \sup_{\delta \le t \le T,\, \gamma \in \mathcal{K}} d_{U(1)}\big(W_t(\Pi_N\gamma), W_t(\gamma)\big) \right\|_{L^p} \le C_{x,\delta,T,p,\theta} N^{-\theta}. \tag{17.9}$$

The supremum in (17.9) tends to zero almost surely as $N \to \infty$. For any deterministic sequence of partitions respecting the piece endpoints, the same convergence holds in every finite moment as the maximum interval length tends to zero.

Proof. Write $Y_t(\gamma) = \langle J_\gamma, X_t \rangle$ for the centered Gaussian trace. The uniform current bound and the filling estimates give

$$\mathbb{E}|Y_t(\gamma) - Y_t(\eta)|^2 \le C d_0(\gamma, \eta), \qquad \mathbb{E}|Y_t(\gamma) - Y_u(\gamma)|^2 \le C_\beta |t-u|^\beta$$

for every $\beta < 1/2$. The first inequality uses $Q_t \le Q_\infty$; the second is the spectral time-increment estimate with the uniform H^{-a} current norm. The metric $d_0(\gamma,\eta) + |t-u|^\beta$ on $[0,T] \times \mathcal{K}$ has the entropy bound in Lemma 17.2 with $a = 1/2$. The time coordinate adds only a logarithmic term to the logarithm of its covering number. That lemma gives a jointly continuous version, with all finite moments of its supremum.

For heat convergence choose $q, \beta > 0$ with $2q + \beta < 1/2$, and $0 < \vartheta < 1/2$. The error $Z_t^\varepsilon(\gamma) = \langle J_\gamma, (e^{-\varepsilon\Delta_1} - I)X_t \rangle$ has point variance at most $C\varepsilon^{2q}$, spatial increment variance at most $C \min\{d_0(\gamma,\eta), \varepsilon^{2q}\}$, and time increment variance at most $C\varepsilon^{2q}|t-u|^\beta$. Consequently its increments are bounded by $C\varepsilon^{2q\vartheta}$ times

$$d_0(\gamma,\eta)^{1-\vartheta} + |t-u|^\beta.$$

For this metric the entropy exponent is $a = 1/(2(1-\vartheta)) < 1$. The Gaussian entropy lemma yields

$$\left\| \sup_{t \le T,\, \gamma \in \mathcal{K}} |Z_t^\varepsilon(\gamma)| \right\|_{L^p} \le C_p \varepsilon^{q\vartheta}.$$

Include $\varepsilon = 2^{-n}u$, $1 \le u \le 2$, as an additional parameter. Its increment estimate is the one in Lemma 16.6; it contributes only a polynomial factor to covering numbers, and hence does not change the entropy exponent $a < 1$. The expectations of the resulting block suprema are summable. Thus the convergence is almost sure for the full heat parameter, uniformly in time and curve.

The deterministic positive-time part and the smooth reference are uniformly continuous on $\mathcal{K}$. More precisely, the filling in (17.8) bounds the change in the integral of a smooth one-form by its curvature supremum times $C d_0(\gamma,\eta)$. It gives the same bound for the change of the reference holonomy. Harmonic periods are locally constant in this curve topology and give continuous circle-valued factors. These observations, with the Gaussian convergence, prove the first assertion. The stationary Gaussian estimates or a smooth initial connection include time zero.

For polygon approximation set

$$E_N(t,\gamma) = \langle J_{\Pi_N\gamma} - J_\gamma, X_t \rangle.$$

Lemma 17.1 gives point variance CN^{-2} and curve-increment variance $C d_0(\gamma,\eta)$, uniformly in N. The uniform current bound gives time-increment variance $C|t-u|^\beta$. Interpolation with the point variance bounds all increments by $CN^{-2\theta}$ times

$$d_0(\gamma,\eta)^{1-\theta} + |t-u|^{\beta(1-\theta)}.$$

Its entropy exponent is $1/(2(1-\theta)) < 1$ when $\theta < 1/2$. The entropy lemma proves an L^p supremum bound $C_p N^{-\theta}$. The smooth reference and deterministic field contribute at most CN^{-2} by the L^1 filling estimate. The harmonic contribution cancels exactly, because the curve and its fine polygon are homologous. The circle distance is bounded by the real phase

difference, so (17.9) follows. Its almost sure conclusion follows by Markov's inequality with a moment order large enough to be summable in N.

For a general partition of maximum interval length h, the interpolation bounds become Ch^2 in position and Ch in velocity. The graph decomposition and current estimates are unchanged. Replacing N^{-2} by h^2 in the preceding argument gives the asserted moment convergence for every deterministic partition sequence. In every case the limit is the same heat trace, which proves its independence of the approximation. □

Corollary 17.4 (Simultaneous regular-loop observables at stopping times)**.** *There is a dense Borel subset*

$$\mathscr{H}_{m,s}^{C^2} \subset \mathscr{H}_{m,s}^{\mathrm{rec}}$$

of full $\widehat{\mu}_{m,s}$ measure on which all based piecewise regular C^2 loop holonomies are defined simultaneously. The strong Markov family in Theorem 16.8 preserves this subset along its entire path when started there. From every Sobolev initial state it enters this subset at all positive times almost surely. When started in this domain, the observables have continuous paths in every bounded-family function space $C(\mathcal{K}_{b,v,L,M}, U(1))$, including at time zero. These loop observables satisfy concatenation, reversal and reparametrization identities. Along the process, the uniform approximation statement of Theorem 17.3 holds on compact positive-time intervals; it includes time zero for smooth or stationary initial states. The restricted family retains its unique invariant law, reversibility and sharp spectral gap. These observables are available at every strictly positive stopping time of the original process.

Proof. Use the countable collection $\mathcal{K}_{b,1/n,n,n}$, with b, n positive integers and the vacuous or empty cases omitted. On the reconstructible state space require, in addition, existence of the full heat limit in each $C(\mathcal{K}_{b,1/n,n,n}, U(1))$. Each of these spaces is Polish, and the rational-parameter Cauchy criterion makes the additional domain Borel. Pulling the regularized connection back through the Borel inverse R makes each heat map Borel. Every smooth connection belongs to this domain, since heat convergence in C^0 controls its line integrals uniformly on each bounded-length family. Smooth holonomy data are dense in the completion, proving the density assertion.

The full heat convergence proved in Theorem 17.3, intersected over this countable collection and over compact positive-time intervals, gives simultaneous membership at every positive time. The stationary version gives full invariant measure. If the initial state belongs to this domain, the Abelian factorization (16.16), now in each curve-family function space, also gives continuity at zero and preservation for the whole path. The transition kernels therefore restrict to this domain, and the strong Markov identity remains valid at stopping times. Its invariant law has full mass on this domain; pulling back any other invariant measure through R proves uniqueness, and the same L^2 intertwining proves reversibility and the gap.

Every piecewise regular C^2 loop can be reparametrized so that its finitely many regular pieces occupy the equal intervals for some b. Its speed has a positive lower bound on each closed piece, and its speed and covariant acceleration have finite upper bounds. It consequently belongs to one of the displayed families after reparametrization. Constant pieces can be removed. At positive heat scale, holonomy is invariant under these reparametrizations and satisfies the path identities. Taking the simultaneous uniform limits preserves all these identities and makes the definitions consistent on overlapping families. □

18 Non-Abelian parallel transport on regular curve families

Young integration gives parallel transport when the primitive of a connection along a curve has Hölder exponent greater than 1/2. This approach is used for the torus in [7]. We establish

the estimates needed on an arbitrary surface bundle, uniformly on bounded families of regular curves, and apply them to the local solutions of Theorem 8.14. The construction also gives a Borel observable of the distributional connection.

Throughout, G is compact and connected, $P \to \Sigma$ is an arbitrary smooth principal G-bundle, and $E = T^*\Sigma \otimes \operatorname{ad} P$. Fix a smooth reference connection B and put $A = 1 + L_B$, $S_\varepsilon = e^{-\varepsilon A}$. We use the unit-amplitude stationary Gaussian field

$$X_t = \int_{-\infty}^{t} e^{-(t-s)A}\, dW_s.$$

Thus its covariance is $(2A)^{-1}$. Multiplying the noise by σ multiplies all covariance bounds below by σ^2. All constants may depend on a fixed compact time interval I.

Choose a finite family of smooth bundle charts with compact subcharts whose interiors cover Σ. In one such chart we write $X = X_i\, dx^i$, with values in a fixed copy of $\mathfrak{g}$. For a curve contained in the chart set

$$Z^\varepsilon_\gamma(t,u) = \int_0^u (S_\varepsilon X_t)_i(\gamma(s))\dot\gamma^i(s)\, ds, \qquad 0 \le u \le 1. \tag{18.1}$$

The value at $\varepsilon = 0$ will be constructed as a Gaussian trace. Component currents are interpreted in this fixed local frame. In particular a closed scalar current in that frame is the boundary of an ordinary scalar two-chain; no parallel trivialisation is used in the filling estimates.

18.1 Gaussian primitives and the entropy of short arcs

We use Sobolev norms defined by any fixed smooth positive Laplace-type operator. These norms are equivalent to those defined by A. For $1/2 < a < 1$, functional calculus gives

$$\mathbb{E}|\langle S_\varepsilon X_t, J\rangle|^2 \le C\|J\|^2_{H^{-1}}, \tag{18.2}$$

$$\mathbb{E}|\langle S_\varepsilon X_t - S_\delta X_s, J\rangle|^2 \le C\big(|t-s|^{1-a} + |\varepsilon-\delta|^{1-a}\big)\|J\|^2_{H^{-a}}, \tag{18.3}$$

$$\mathbb{E}|\langle (S_\varepsilon - I)X_t, J\rangle|^2 \le C\varepsilon^{1-a}\|J\|^2_{H^{-a}}. \tag{18.4}$$

These estimates hold for $0 \le \varepsilon, \delta \le 1$, with $S_0 = I$. To verify them, use

$$\mathbb{E}[X_t \otimes X_s] = \tfrac{1}{2}A^{-1}e^{-|t-s|A}, \qquad 1 - e^{-r\lambda} \le C(r\lambda)^{1-a},$$

and $(1-e^{-r\lambda})^2 \le C(r\lambda)^{1-a}$. The estimates are valid first for smooth currents and then by completion. In particular every current in H^{-a} has a canonical Gaussian pairing, without a pathwise restriction of an arbitrary distribution to a curve.

Here and below a bounded regular curve family means the compact piecewise $C^{1,1}$ family of Lemma 17.1, with a fixed number of pieces, a positive lower speed, an upper speed, and an upper bound on covariant acceleration. We permit variable endpoints when working in a chart. Its compactness in the uniform distance and the equivalence with piecewise C^1 convergence follow by Arzelà–Ascoli. Denote its uniform distance by d_0 and let $\Pi_N\gamma$ be geodesic interpolation on N equal subintervals of each piece.

Lemma 18.1 (Open currents and short-arc fillings)**.** *The first two assertions hold uniformly on such a family $\mathcal{K}$ and on its interpolants for $N \ge N_0$. The third assertion concerns the original regular arcs.*

1. *Every prefix current has bounded H^{-a} norm for $1/2 < a < 1$. A subarc of parameter length h, contained in one of a fixed number of graph charts, satisfies*

$$\|J\|^2_{H^{-a}} \le Ch^{2a}, \qquad \|J\|^2_{H^{-1}} \le Ch^2\log(e/h). \tag{18.5}$$

A short geodesic of length ℓ satisfies the same estimates with h replaced by ℓ.

2. *For prefixes with the same parameter endpoint u,*

$$\|J_{\gamma|[0,u]} - J_{\eta|[0,u]}\|_{H^{-1}}^2 \le C d_0(\gamma,\eta), \tag{18.6}$$

$$\|J_{\Pi_N\gamma|[0,u]} - J_{\Pi_N\eta|[0,u]}\|_{H^{-1}}^2 \le C d_0(\gamma,\eta), \tag{18.7}$$

$$\|J_{\Pi_N\gamma|[0,u]} - J_{\gamma|[0,u]}\|_{H^{-1}}^2 \le C N^{-2}. \tag{18.8}$$

3. *Let $\mathcal{A}_h$ consist of regular arcs on an interval of length between $h/2$ and h, rescaled to $[0,1]$, with the same speed and acceleration bounds in the original parameter. Work in a fixed compact subchart and take h sufficiently small. Let c_γ be the geodesic with the same endpoints and put $R_\gamma = J_\gamma - J_{c_\gamma}$. Then*

$$\|R_\gamma\|_{H^{-1}}^2 \le C h^3, \qquad \|R_\gamma - R_\eta\|_{H^{-1}}^2 \le C h\, d_0(\gamma,\eta). \tag{18.9}$$

For $0 < \delta \le h^2$,

$$\log N(\mathcal{A}_h, d_0, \delta) \le C(1 + h\delta^{-1/2}) \log(C/\delta). \tag{18.10}$$

Proof. Use the fixed graph decomposition in Lemma 17.1. Its graph constants are uniform for the interpolants, including intervals crossing their vertices. On each graph the distance between two points is bounded below by a constant times their parameter distance. The kernel of a Sobolev operator of order $-2a$ is bounded by $C d(x,y)^{2a-2}$, and the kernel at order -2 is bounded by $C \log(e/d(x,y))$. These bounds follow by integrating the bundle heat-kernel estimate. Integration on a graph gives (18.5). Splitting at the fixed graph division points gives the asserted uniform prefix bounds. The factor ℓ^{2a}, respectively $\ell^2 \log(e/\ell)$, also proves the short-geodesic assertion at a zero-length edge by continuity.

For a locally supported filling w, Stokes' formula in the chosen scalar component gives

$$\|\partial w\|_{H^{-1}} \le C \|w\|_{L^2}. \tag{18.11}$$

Indeed its pairing with a smooth test one-form is the integral of w times a first derivative of that component. The local frame and metric introduce only bounded smooth coefficients. Close two prefixes by short connectors at their endpoints. The fixed graph-strip construction in Lemma 17.1 produces a filling with bounded multiplicity and area $C\delta$, where $\delta = d_0(\gamma,\eta)$. The connectors have length $O(\delta)$ and hence squared H^{-1} norm $O(\delta^2 \log(e/\delta))$. Equation (18.11) proves (18.6) and (18.7). For a curve and its interpolant, the position error is CN^{-2}; the same construction, including the connector at a partial edge, proves (18.8).

For a short arc, its distance from its chord is Ch^2 and its length is Ch. The signed graph region between them has bounded multiplicity and area Ch^3. If two such arcs are δ-close with $\delta \le ch^2$, their derivatives after rescaling to $[0,1]$ differ by at most $C(\delta + h^2)$, by the difference-quotient estimate. These rescaled curves have speed at least ch and second derivative bounded by Ch^2. They and their chords consequently use a common increasing graph direction. The regions between the two arcs and between their two chords have total area $Ch\delta$. The endpoint caps have area $C\delta^2$ and bounded multiplicity. They identify the difference of the two arc–chord fillings: two compactly supported fillings with the same boundary in this coordinate disc agree. This proves (18.9) by (18.11). For $\delta \ge ch^2$, the first bound in (18.9) proves the second one directly.

Finally sample an arc at original-parameter spacing $c\sqrt{\delta}$. It has at most $C(1 + h/\sqrt{\delta})$ sample points, and its geodesic interpolant has uniform error $C\delta$. Quantising these points in a surface δ-net gives (18.10), after adjusting the covering radius by a constant. Choosing a representative from each occupied class puts the centres back in $\mathcal{A}_h$. □

If a centred Gaussian field has point variances at most σ^2 and its canonical metric has covering entropy $H(q) \le Cq^{-1}\log(C/q)$, then, for every finite p,

$$\|\sup|Y|\|_{L^p} \le C_p\sigma^{1/2}\sqrt{\log(C/\sigma)} \quad (0<\sigma\le 1/2). \tag{18.12}$$

For polynomial covering numbers the corresponding bound is $C_p\sigma\sqrt{\log(C/\sigma)}$. To prove these statements, take nets at canonical scales $\sigma 2^{-j}$. The L^p norm of the largest of M Gaussian increments of variance q^2 is at most $Cq\sqrt{\log(2M)+p}$. Summing this bound on successive nets gives the assertions and a continuous version. This is the same chaining argument as in Lemma 17.2; it also applies with the scale-dependent entropy in (18.10).

Theorem 18.2 (Uniform Gaussian Young traces)**.** *For every bounded regular curve family in a chart, every compact time interval I, every $0<\alpha<1$, and every finite p, the primitives (18.1) have versions such that*

$$\sup_{t\in I,\ 0\le\varepsilon\le 1,\ \gamma\in\mathcal{K}} \|Z^\varepsilon_\gamma(t,\cdot)\|_{C^\alpha} \in L^p. \tag{18.13}$$

They depend jointly continuously on (t,ε,γ) with values in the little α-Hölder space. The full heat limit, rather than just a subsequential limit, satisfies

$$\left\| \sup_{\substack{t\in I,\ \gamma\in\mathcal{K}\\ 0\le\rho\le\varepsilon}} \|Z^\rho_\gamma(t) - Z^0_\gamma(t)\|_{C^\alpha} \right\|_{L^p} \le C\varepsilon^\eta, \qquad 0<\eta<(1-\alpha)/8. \tag{18.14}$$

The same construction on geodesic interpolants satisfies

$$\left\| \sup_{\substack{t\in I,\gamma\in\mathcal{K}\\ 0\le\varepsilon\le 1}} \|Z^\varepsilon_{\Pi_N\gamma}(t) - Z^\varepsilon_\gamma(t)\|_{C^\alpha} \right\|_{L^p} \le CN^{-\zeta}, \qquad 0<\zeta<(1-\alpha)/2. \tag{18.15}$$

The analogous assertions for a fixed finite-dimensional compact family of short geodesics include constant edges. Uniformly over short geodesic segments and short coordinate segments in compact subcharts, their limiting integrals obey

$$|Z(\gamma)| \le M_\alpha\ \mathrm{length}(\gamma)^\alpha, \qquad M_\alpha \in \bigcap_{p<\infty} L^p. \tag{18.16}$$

The random constants can be chosen uniformly in $t\in I$ and $0\le\varepsilon\le 1$.

Proof. The covariance estimates and (18.6) give a canonical metric bounded by

$$C\big(d_0(\gamma,\eta)^{1/2} + |u-v|^a + |t-s|^{(1-a)/2} + |\varepsilon-\delta|^{(1-a)/2}\big)$$

for the primitive field. The curve entropy from Lemma 17.1 therefore gives $H(q) \le Cq^{-1}\log(C/q)$; the three interval parameters add only logarithmic terms. Chaining yields joint continuity in the supremum norm, including $\varepsilon=0$.

To obtain an exponent above $1/2$ uniformly on the whole curve family, use the short-arc estimate before taking the supremum. For $Y_\gamma = \langle S_\varepsilon X_t, R_\gamma\rangle$ on $\mathcal{A}_h$, (18.9) gives point variance Ch^3 and canonical distance $C(hd_0)^{1/2}$. At canonical scale $q \le Ch^{3/2}$, (18.10) gives

$$H_h(q) \le C(1+h^{3/2}/q)\log(Ch/q^2).$$

Time and heat-scale parameters still add only logarithmic terms: by (18.5) and (18.3), their canonical modulus is bounded by $Ch^a(|t-s|+|\varepsilon-\delta|)^{(1-a)/2}$. The dyadic-net sum consequently gives

$$\left\| \sup_{\gamma\in\mathcal{A}_h,\ t\in I,\ 0\le\varepsilon\le 1} |\langle S_\varepsilon X_t, R_\gamma\rangle| \right\|_{L^p} \le C_p h^{3/2}\sqrt{\log(e/h)}. \tag{18.17}$$

For example at scales $q_j = Ch^{3/2}2^{-j}$ the summands are bounded by

$$C_p h^{3/2} 2^{-j/2} \sqrt{1 + |\log h| + j}.$$

Their sum is finite.

The chords of length at most Ch form a finite-dimensional compact family. Thin quadrilaterals and their short endpoint connectors give a canonical modulus $C\delta^{1/2}$ in their endpoints. Their point variance is $Ch^2 \log(e/h)$ by (18.5). The polynomial-entropy version of (18.12) bounds their supremum in L^p by $C_p h \log(e/h)$. The same argument works for coordinate segments. Adding (18.17) gives this bound for all regular subarcs of parameter length between $h/2$ and h. Summing over $h = 2^{-n}$ with weight $2^{\beta n}$ proves (18.13) for every $\beta < 1$ and proves (18.16). Subarcs crossing one of the finitely many regular-piece boundaries are split there. Interpolation between uniform convergence and this C^β bound gives joint continuity with values in c^α for $\alpha < \beta$.

For the heat error, (18.4) bounds the point standard deviation, uniformly for $0 \le \rho \le \varepsilon$, by $C\varepsilon^{(1-a)/2}$. Its canonical entropy has the same bound $Cq^{-1}\log(C/q)$. Equation (18.12) therefore bounds its supremum norm in L^p by $C\varepsilon^\theta$ for every $\theta < (1-a)/4$. Interpolate with the uniform C^β bounds. Since a can decrease to $1/2$ and β increase to one, this proves (18.14).

For polygon approximation, (18.8) bounds the point standard deviation by CN^{-1}, while (18.7) and the uniform prefix estimates give the same entropy bound, uniformly in N. Chaining gives a supremum-norm error $C_p N^{-\theta}$ for every $\theta < 1/2$. To bound the stronger norm, apply the uniform short-geodesic C^β estimate on each edge. Splitting an interval at its polygon vertices and using $\sum_{j=1}^n \ell_j^\beta \le n^{1-\beta}(\sum_j \ell_j)^\beta$ gives

$$\sup_{t,\varepsilon,\gamma} \|Z^\varepsilon_{\Pi_N\gamma}(t)\|_{C^\beta} \le CM_\beta N^{1-\beta}.$$

Hölder interpolation gives an error exponent $\theta(1-\alpha/\beta) - (1-\beta)\alpha/\beta$. Letting $\theta \uparrow 1/2$ and $\beta \uparrow 1$ proves (18.15). Hölder's inequality in probability justifies all the interpolated L^p bounds, because the input estimates hold at every finite moment order.

For finite-dimensional geodesic families the same proof uses polynomial entropy. A zero-length edge has zero current, and (18.16) makes its inclusion continuous. Countably many compact time intervals and curve families can be treated on one event of probability one. All versions agree with the classical primitives at positive heat scale; additivity and restriction identities therefore persist in the full heat limit. □

18.2 Young transport and deterministic trace domains

Fix a faithful unitary representation of G. In a local frame we use the convention $\nabla = d + \mathcal{A}$; smooth parallel transport solves $dU = -dZ\,U$, where $Z = \int \mathcal{A}$. For a line bundle represented by $\mathcal{A} = -ia$, this gives $U = \exp(i\int a)$, agreeing with the Abelian convention above.

Lemma 18.3 (Compact-group Young stability)**.** *Let $1/2 < \alpha < 1$ and let Z, W be $\mathfrak{g}$-valued α-Hölder paths starting at zero. The Young equation*

$$U_Z(u) = 1 - \int_0^u dZ(s)\, U_Z(s) \tag{18.18}$$

has a unique solution taking values in G. With a fixed bi-invariant Riemannian distance on G,

$$[U_Z]_\alpha \le C_\alpha [Z]_\alpha, \tag{18.19}$$

$$\sup_u d_G(U_Z(u), U_W(u)) \le C_\alpha (1 + [Z]_\alpha + [W]_\alpha)[Z - W]_\alpha. \tag{18.20}$$

Proof. The Young estimate

$$\left|\int_s^t f\,dg - f(s)g_{s,t}\right| \le C_\alpha[f]_\alpha[g]_\alpha|t-s|^{2\alpha}$$

follows by comparing successive dyadic Riemann sums; the sum of the errors is geometric since $2\alpha > 1$ [21]. It gives a contraction for (18.18) on intervals with $[Z]_\alpha|t-s|^\alpha$ sufficiently small. Concatenation gives the unique global solution. Smooth approximation gives values in the closed group G.

On each sufficiently short interval the same estimate and the uniform bound on the unitary matrices give $[U_Z]_{\alpha;[s,t]} \le C[Z]_\alpha$ by absorption. For longer increments, compactness of G gives the same bound, proving (18.19). For smooth drivers interpolate $Z_\lambda = (1-\lambda)Z + \lambda W$. The trivialised variation is

$$U_{Z_\lambda}^{-1}\partial_\lambda U_{Z_\lambda} = -\int \mathrm{Ad}_{U_{Z_\lambda}^{-1}}\, d(W-Z).$$

The Young estimate and (18.19) bound its supremum by the right-hand side of (18.20). Integrate in λ and approximate the drivers to obtain the assertion. In particular the stability constant is polynomial in the driver norms. □

We define a domain by requiring full-parameter convergence of the heat-regularised line primitives. Fix $0 < \kappa < 1$ and $1/2 < \alpha < 1$. In each compact subchart take a countable collection of bounded regular arc families that exhausts all piecewise regular C^2 arcs after subdivision and affine reparametrisation of their regular pieces. One may use integer upper bounds for speed and acceleration, reciprocal-integer lower bounds for speed, and an integer number of pieces. Include fixed-length coordinate segments and fixed-length geodesics starting in smaller compact subcharts. Their subsegments include all sufficiently short segments, including the zero-length limit.

Definition 18.4 (The Young trace domain)**.** Let $\mathcal{D}_\kappa^\alpha(P)$ consist of connections $\mathcal{A} = B + a$, $a \in \mathrm{c}^{-\kappa}(E)$, for which the local primitives of $B + S_\varepsilon a$ converge, as the full parameter $\varepsilon \downarrow 0$, in $C(\mathcal{K};\mathrm{c}^\alpha)$ on every family in this countable collection. On a piecewise family use the finite product of the little Hölder spaces on its pieces.

For based holonomies fix a frame $p_o \in P_o$. Changing this frame acts by simultaneous conjugation; the unframed values are independent of that choice.

Proposition 18.5 (Intrinsic Borel traces and distributional averages)**.** *The domain $\mathcal{D}_\kappa^\alpha(P)$ is dense and Borel in the affine space of distributional connections. Its definition and its limiting parallel transports are independent of the smooth reference connection and of the finite smooth bundle atlas used above. Parallel transport along each regular curve, and its continuous family of values on each bounded regular curve family, are Borel maps on this domain. The based polygonal holonomies give a Borel map*

$$\mathfrak{H} : \mathcal{D}_\kappa^\alpha(P) \longrightarrow \mathscr{H}_r(P). \tag{18.21}$$

They satisfy reversal, concatenation, subdivision, and smooth change-of-frame identities.

In a coordinate rectangle, let e_i be a coordinate vector and $\gamma_{x,h}(s) = x + se_i$, $0 \le s \le h$. Its limiting local primitive $z_{i,h}(x)$ is continuous in x and satisfies

$$\frac{z_{i,h}}{h} = \int_0^1 \mathcal{A}_i(\,\cdot + \theta h e_i)\,d\theta \longrightarrow \mathcal{A}_i \quad \text{in } \mathrm{c}_{\mathrm{loc}}^{-\kappa}. \tag{18.22}$$

There is also a geodesic version. Put $F_s(x) = \exp_x(se_i(x))$. Then

$$\frac{1}{h}\int_{\gamma_{x,h}^{\mathrm{geo}}} \mathcal{A} = \frac{1}{h}\int_0^h \mathcal{A}(F_s(x))(\partial_s F_s(x))\,ds \longrightarrow \mathcal{A}_i \quad \text{in } \mathrm{c}_{\mathrm{loc}}^{-\kappa}. \tag{18.23}$$

Both equalities are distributional identities for the continuous trace on the left. On compact subcharts these short primitives are bounded by $C|h|^{\alpha}$, with a constant depending on the connection in its trace domain.

Proof. For fixed $\varepsilon > 0$, heat smoothing followed by local integration is a continuous map from $\mathrm{c}^{-\kappa}$ into $C(\mathcal{K}; \mathrm{c}^{\alpha})$. The target is a separable Banach space. The rational-parameter Cauchy criterion, together with continuity in positive heat time, proves Borelness. Smooth connections belong to the domain and are dense.

Let A_0, A_1 be the massive Laplace-type operators of two smooth references. Their difference has order at most one. Duhamel's formula and the heat bound from $\mathcal{C}^{-\kappa}$ to C^1 give

$$\|(e^{-\varepsilon A_1} - e^{-\varepsilon A_0})a\|_{C^0} \leq C \int_0^{\varepsilon} r^{-(1+\kappa)/2}\, dr\, \|a\|_{\mathcal{C}^{-\kappa}} \leq C\varepsilon^{(1-\kappa)/2}\|a\|_{\mathcal{C}^{-\kappa}}. \tag{18.24}$$

The physical approximants for different reference coordinates also differ by the heat error of a smooth one-form. Thus their difference tends to zero in C^0. Its primitive tends to zero in the Lipschitz norm on every bounded-length, bounded-speed curve family. This proves reference independence of the domain and of its traces.

For a smooth change of local frame, the primitive is transformed by a Young integral with the smooth matrix coefficient evaluated along the curve, together with the smooth inhomogeneous connection term. Young multiplication is continuous in these C^{α} norms, uniformly on a bounded regular family. The smooth cutoff identities therefore pass to the limit. Subdividing curves into chart pieces proves atlas independence and defines bundle parallel transport. Lemma 18.3 proves its continuity in each family. The reversal, concatenation, reparametrisation, and retracing identities follow from the corresponding smooth identities. Uniform short-arc bounds include degenerate geodesic edges, so the convergence is uniform on each K_m^r. Hence the polygonal data belong to the closed space $\mathscr{H}_r(P)$. The same rational-heat limit proves all the Borel assertions.

For fixed nonzero h, the smooth coordinate-segment identities converge uniformly in x by the trace-domain definition and converge distributionally by heat convergence. Their limits are equal, proving (18.22). Translations are strongly continuous on local little Hölder–Besov spaces, as follows by smooth approximation and their uniform boundedness. For (18.23), F_s is a smooth diffeomorphism onto its image for small s, on a slightly larger coordinate rectangle. Pullback by F_s, contraction with the smooth vector $\partial_s F_s$, and their localised operator norms are uniformly bounded on $\mathcal{C}^{-\kappa}$ and converge on smooth forms as $s \to 0$. Smooth approximation proves strong convergence on $\mathrm{c}^{-\kappa}$; averaging proves the limit. The full-heat trace identity is justified in exactly the same way as for coordinate segments. Finally restriction of the uniform C^{α} primitive on fixed-length segment families gives the bound $C|h|^{\alpha}$. □

The reference comparison in (18.24) concerns observables of a given distributional connection. Nonlinear processes constructed with different reference gauges are compared in Section 12, using the rough comparison gauges of Section 10.

18.3 Application to the non-Abelian continuum flow

Theorem 18.6 (Non-Abelian observable limits on arbitrary bundles)**.** *Let a be the maximal fixed-reference solution of Theorem 8.14, with arbitrary $a_0 \in \mathrm{c}^{-\kappa}(E)$, $0 < \kappa < 1/16$, and lifetime τ. The following assertions hold almost surely.*

1. *Simultaneously for every $0 < t < \tau$ and every $1/2 < \alpha < 1$, the connection $B + a_t$ belongs to $\mathcal{D}_{\kappa}^{\alpha}(P)$. Its parallel transports are defined on all piecewise regular C^2 paths. On every bounded regular curve family they are jointly continuous in the curve and in $0 < t < \tau$.*

2. *The based polygonal holonomies form a continuous path*

$$t \longmapsto \mathfrak{H}(B + a_t) \in \mathscr{H}_r(P), \qquad 0 < t < \tau, \tag{18.25}$$

and hence a continuous path in $\mathscr{U}_r(P)$. They have the intrinsic radius and basepoint identifications of Theorem 15.2 and Corollary 15.3.

3. *Let a^ε be the classical bare heat approximations. On the common bounded-data event Ω_R^ε of Theorem 8.14, for $0 < t_0 < T_R$ and every bounded regular family $\mathcal{K}$ of based loops,*

$$\mathbb{E}\left[\mathbf{1}_{\Omega_R^\varepsilon} \sup_{\substack{t_0 \le t \le T_R \\ \gamma \in \mathcal{K}}} d_G\big(\mathrm{Hol}(B + a_t^\varepsilon, \gamma), \mathrm{Hol}(B + a_t, \gamma)\big)^p\right] \longrightarrow 0 \quad (p < \infty). \tag{18.26}$$

The same holds on every fixed K_m^r, and in the bounded metric d_r of the holonomy completion.

4. *For every $0 < \zeta < 1/4$, geodesic interpolation on N equal subintervals of each regular piece satisfies, on the limiting bounded-data event $\Omega_R = \{D \le R\}$,*

$$\left\| \mathbf{1}_{\Omega_R} \sup_{\substack{t_0 \le t \le T_R \\ \gamma \in \mathcal{K}}} d_G\big(\mathrm{Hol}(B + a_t, \Pi_N \gamma), \mathrm{Hol}(B + a_t, \gamma)\big) \right\|_{L^p} \le C_{R, t_0, p, \mathcal{K}, \zeta} N^{-\zeta}. \tag{18.27}$$

In particular the interpolation converges almost surely along the full sequence, uniformly on these families and compact time intervals before τ.

All these transports commute with smooth bundle isomorphisms covering surface isometries, when the reference, noise and connection are transported together. Smooth gauge transformations give the usual endpoint conjugation law. If, for some $1/2 < \alpha < 1$, the initial connection $B + a_0$ belongs to $\mathcal{D}_\kappa^\alpha(P)$, the observable time-continuity and local convergence statements include $t = 0$. This includes every smooth initial connection.

Proof. The paired fixed point and its reconstruction give

$$a_t = X_t + R_t, \qquad R_t = S_t(a_0 - X_0) + \mathcal{R}(U - v)(t). \tag{18.28}$$

On compact subintervals of $(0, \tau)$ the remainder is a continuous function with spatial regularity $1 - 2\kappa$. For the comparison with approximants, any fixed $1/2 < \sigma < 1 - 4\kappa$ suffices. The two-model reconstruction bound used in the proof of Theorem 8.14 gives

$$\mathbb{E}\Big[\mathbf{1}_{\Omega_R^\varepsilon} \|R^\varepsilon - R\|^p_{C([t_0, T_R], \mathcal{C}^\sigma)}\Big] \longrightarrow 0, \tag{18.29}$$

with uniform bounds at every finite moment order. The initial heat terms are smooth for $t \ge t_0$ and obey the same estimate.

The primitives of $B + R$ are Lipschitz in the path parameter, uniformly on a bounded regular family; they are jointly continuous in time and the curve. Their heat errors tend to zero in this Lipschitz norm, since $\|S_\varepsilon R - R\|_{C^0} \le C\varepsilon^{\sigma/2}\|R\|_{\mathcal{C}^\sigma}$. Add Theorem 18.2 and choose $1/2 < \alpha < 1$. Lemma 18.3 constructs the local transports and proves their uniform full-heat convergence. Proposition 18.5 glues them on the arbitrary principal bundle and identifies them as observables of $B + a_t$. Countably many time intervals, chart families, and exponents increasing to one give simultaneous membership and continuity at all positive times. This proves the first two assertions.

For the classical approximants the Gaussian coordinate is exactly $S_\varepsilon X$. Equations (18.14) and (18.29) give convergence of the drivers in the uniform C^α norm on the common existence event. The polynomial stability estimate (18.20), with Hölder's inequality in probability, proves (18.26). For a general bundle, choose a finite cover of the compact curve family so that, on each set in the cover, a fixed parameter subdivision places every subarc inside a bundle chart with a positive margin. The transition functions are smooth, and the number of factors is bounded. This transfers the same estimate to the global transport. The identical argument on the finitely many edge charts proves convergence on K^r_m, including its constant edges. A finite-factor approximation in the bounded product metric d_r proves the completion claim.

For polygon approximation, Theorem 18.2 gives every driver error exponent $\zeta < (1-\alpha)/2$. In a chart the position and velocity errors of interpolation are CN^{-2} and CN^{-1}. Consequently the primitive error of a C^σ one-form is bounded in the Lipschitz norm by $C\|B+R\|_{\mathcal{C}^\sigma}(N^{-2\sigma}+N^{-1})$. Since $\sigma > 1/2$, this contribution is $O(N^{-1})$. The transition-function errors at the fixed chart division points are $O(N^{-2})$. Apply (18.20) and let $\alpha \downarrow 1/2$ to obtain (18.27). Taking a sufficiently large moment and using Borel–Cantelli gives the full-sequence almost-sure convergence. Localisation and a countable exhaustion of the interval before τ give the pathwise assertion.

The smooth bundle-isomorphism identities hold at every cutoff and therefore pass to these uniform limits. Suppose now that $B + a_0 \in \mathcal{D}^\alpha_\kappa(P)$ for some $\alpha > 1/2$. Write the linear part as $(X_t - S_t X_0) + S_t a_0$. The primitives of the first term vanish continuously at zero by Theorem 18.2. The primitives of $S_t a_0$ extend continuously to zero in the uniform c^α norm by the definition of the trace domain. The nonlinear response has positive regularity and vanishes uniformly at zero, so its primitive vanishes in the Lipschitz norm. This proves observable continuity at the initial time. For the approximants, the initial heat term is $S_{t+\varepsilon}a_0$. Uniform continuity of the just-extended heat-trace path on a compact interval gives convergence uniformly for $0 \le t \le T_R$. The two-model positive-response estimate is also valid down to zero. Young stability therefore proves the same local observable convergence on $[0, T_R]$. □

Corollary 18.7 (Observables at stopping times)**.** *The maps in* (18.21)*, and the regular-curve transport maps above, may be evaluated at every stopping time σ with $0 < \sigma < \tau$. They agree almost surely with the value at σ of their continuous observable paths and are measurable with respect to the stopped solution filtration.*

Proof. The domain is Borel and is visited simultaneously at all positive times. Extend each observable arbitrarily off that domain. Composition with the adapted continuous connection process is Borel and agrees, at every surviving positive time on one event, with the continuous path constructed above. Evaluation at a stopping time gives the assertion. □

Theorem 18.6 constructs non-Abelian holonomies of the local continuum process on every surface bundle. Section 19 identifies the unframed holonomy fibres as full rough gauge orbits and gives a Borel identification of their quotient. Section 20 combines this identification with continuum gauge comparison to construct the strong Markov law on the realised unframed image.

19 Rough gauge orbits and the realised holonomy space

The full rough gauge group acts on the Young trace domain, and unframed holonomies distinguish its orbits. We prove this invariance and separation, construct a standard Borel space of realised holonomies, and recover the isomorphism class of the principal bundle.

Throughout this section choose

$$0 < \kappa < \frac{1}{2}, \qquad \frac{1}{2} < \alpha < \rho := 1 - \kappa, \tag{19.1}$$

and use the domain $\mathcal{D}^{\alpha}_{\kappa}(P)$ of Definition 18.4. The rough connection space is $\mathcal{A}_{\kappa}(P) = B + \mathrm{c}^{-\kappa}(E)$, with its separable Banach topology. The smooth reference B identifies this affine space with a vector space; changing B is an affine homeomorphism. Write $\mathcal{G}^{\rho}(P)$ for the c^{ρ} gauge group and $\mathcal{G}^{\rho}_{*}(P)$ for its closed subgroup equal to the identity on the fibre at the chosen base point. Their actions are the continuous actions of Proposition 10.4.

19.1 Heat approximation and rough gauge covariance

Fix a faithful unitary representation and its associated vector bundle V. Extend the B-covariant Hodge operators to $\mathrm{End}(V)$-valued forms. Their restriction to $\mathrm{ad}\, P$ agrees with the operators already used. Let $T_{\varepsilon} = e^{-\varepsilon(1+\Delta_B^{(0)})}$ on endomorphism sections, and let $S_{\varepsilon} = e^{-\varepsilon(1+\Delta_B^{(1)})}$ on endomorphism-valued one-forms.

Lemma 19.1 (A heat product commutator)**.** *Let $0 < \kappa < \rho < 1$ and let $f \in \mathcal{C}^{\rho}$, $a \in \mathcal{C}^{-\kappa}$ be sections with a fixed smooth fibrewise bilinear product. Heat operators with the same scalar principal symbol, on the corresponding bundles, satisfy*

$$\|S_{\varepsilon}(fa) - fS_{\varepsilon}a\|_{C^0} \le C\varepsilon^{(\rho-\kappa)/2}\|f\|_{\mathcal{C}^{\rho}}\|a\|_{\mathcal{C}^{-\kappa}}, \qquad 0 < \varepsilon \le 1. \tag{19.2}$$

The right and left multiplication versions have the same estimate. For the compatible Hodge operators above,

$$\|(S_{\varepsilon}d_B - d_BT_{\varepsilon})f\|_{C^0} \le C\varepsilon^{(1+\rho)/2}\|f\|_{\mathcal{C}^{\rho}}. \tag{19.3}$$

Proof. Write $\ell = \sqrt{\varepsilon}$. After smooth localisation the leading heat kernel has the form $\ell^{-2}K_x((x-y)/\ell)$, with Gaussian bounds for every derivative of K_x. The commutator pairs a against this kernel times $f(y) - f(x)$. Littlewood–Paley duality bounds the pairing by $\|a\|_{\mathcal{C}^{-\kappa}}$ times the $B^{\kappa}_{1,1}$ norm of that test function. This norm is at most $C\ell^{\rho-\kappa}\|f\|_{\mathcal{C}^{\rho}}$. Indeed, write this test function as $g_{x,\ell}$. Its L^1 norm is at most $C\ell^{\rho}\|f\|_{\mathcal{C}^{\rho}}$. For a translation z with $|z| \le \ell$, splitting the difference between the kernel and coefficient factors bounds its L^1 modulus by $C\|f\|_{\mathcal{C}^{\rho}}(|z|\ell^{\rho-1} + |z|^{\rho}) \le C\|f\|_{\mathcal{C}^{\rho}}|z|^{\rho}$. For $|z| \ge \ell$, use twice the L^1 norm instead. The first-difference characterisation of $B^{\kappa}_{1,1}$ therefore gives

$$\begin{aligned}\|g_{x,\ell}\|_{B^{\kappa}_{1,1}} &\le C\|f\|_{\mathcal{C}^{\rho}}\left(\ell^{\rho} + \int_0^{\ell} r^{\rho-\kappa}\frac{dr}{r} + \ell^{\rho}\int_{\ell}^{1} r^{-\kappa}\frac{dr}{r}\right)\\ &\le C\ell^{\rho-\kappa}\|f\|_{\mathcal{C}^{\rho}}.\end{aligned}$$

Gaussian annuli are summed with their rapidly decreasing weights. Smooth approximation of f in an exponent strictly between κ and ρ identifies this pairing with the distributional product, while its uniform $\mathcal{C}^{\rho}$ bounds preserve the displayed estimate.

In bundle frames, parallel transport in the heat kernel differs from the identity by $O(d(x,y))$. The product compatibility error between the two kernels is of the same order, or of order ε. The identical test-function argument bounds these errors by $C\ell^{1-\kappa}$ times the displayed norms; this is no larger than $C\ell^{\rho-\kappa}$. A finite partition of unity proves (19.2) for the indicated bundle products.

For the derivative, the exact identity on endomorphism sections is

$$\Delta_B^{(1)}d_B - d_B\Delta_B^{(0)} = \delta_B d_B^2 = \delta_B[F_B, \cdot\,].$$

It has order one with smooth coefficients. Duhamel's formula, boundedness of S_t on C^0, and $\|T_r f\|_{C^1} \le C r^{-(1-\rho)/2}\|f\|_{\mathcal{C}^\rho}$ give

$$\|(S_\varepsilon d_B - d_B T_\varepsilon) f\|_{C^0} \le C\|f\|_{\mathcal{C}^\rho} \int_0^\varepsilon r^{-(1-\rho)/2}\, dr.$$

This is (19.3). The mass terms commute with d_B and make no change to the identity. □

Proposition 19.2 (Invariance of the full heat trace domain)**.** *Under* (19.1)*,* $\mathcal{D}^\alpha_\kappa(P)$ *is invariant under* $\mathcal{G}^{1-\kappa}(P)$*. Its canonical full-heat parallel transports satisfy*

$$T_{u\cdot A}(\gamma) = u(\gamma(1))\, T_A(\gamma)\, u(\gamma(0))^{-1} \tag{19.4}$$

for every piecewise regular C^2 *path. The identity is uniform on each bounded regular curve family. In particular the unframed holonomy map is constant on the full rough gauge orbits in this domain.*

Proof. Write $A = B + a$, $U = \mathrm{Ad}_u$, and $k = (d_B u)u^{-1}$. Then $u \cdot A = B + Ua - k$. Let $v_\varepsilon = T_\varepsilon u$. This is a smooth section of $\mathrm{End}(V)$, converges to u in c^ρ, and is invertible for small ε because it converges uniformly to a unitary section. It is used as an auxiliary $GL(V)$ bundle automorphism; membership of v_ε in the original compact group is not needed. Heat estimates give

$$\begin{aligned} &\|v_\varepsilon - u\|_\infty \le C\varepsilon^{\rho/2}\|u\|_{\mathcal{C}^\rho}, \\ &\|d_B v_\varepsilon\|_\infty \le C\varepsilon^{-\kappa/2}\|u\|_{\mathcal{C}^\rho}, \qquad \|S_\varepsilon a\|_\infty \le C\varepsilon^{-\kappa/2}\|a\|_{\mathcal{C}^{-\kappa}}. \end{aligned}$$

The inverse sections are uniformly bounded and converge in c^ρ. Apply Lemma 19.1 to the two matrix multiplications in $Ua = uau^{-1}$. It follows that

$$\|S_\varepsilon(Ua) - \mathrm{Ad}_{v_\varepsilon} S_\varepsilon a\|_\infty \le C_{u,a}\varepsilon^{(1-2\kappa)/2}.$$

For the derivative term use the exact decomposition

$$\begin{aligned} S_\varepsilon k - (d_B v_\varepsilon)v_\varepsilon^{-1} = {} & S_\varepsilon((d_B u)u^{-1}) - (S_\varepsilon d_B u)u^{-1} \\ & + (S_\varepsilon d_B u - d_B v_\varepsilon)u^{-1} + (d_B v_\varepsilon)(u^{-1} - v_\varepsilon^{-1}). \end{aligned}$$

Its first term has size $O(\varepsilon^{(1-2\kappa)/2})$ by (19.2), the second has the better size $O(\varepsilon^{1-\kappa/2})$ by (19.3), and the last has size $O(\varepsilon^{(1-2\kappa)/2})$ by the heat estimates above. Consequently the physical smooth approximations obey

$$\|B + S_\varepsilon(u\cdot A - B) - v_\varepsilon \cdot (B + S_\varepsilon a)\|_{C^0} \le C_{u,a}\varepsilon^{(1-2\kappa)/2}. \tag{19.5}$$

This comparison takes place in the ambient endomorphism-valued one-forms and is independent of a bundle frame.

On a bounded regular curve family, restriction of $v_\varepsilon \to u$ is uniform in c^ρ. If Z_ε is the primitive of $B + S_\varepsilon a$ in a local frame, the primitive after the auxiliary smooth gauge is

$$\int \mathrm{Ad}_{v_\varepsilon\circ\gamma}\, dZ_\varepsilon - \int d(v_\varepsilon\circ\gamma)(v_\varepsilon\circ\gamma)^{-1}. \tag{19.6}$$

Young continuity applies because $\rho+\alpha > 1$, $2\rho > 1$, and $\alpha < \rho$. Thus these primitives converge uniformly in c^α on every such family. The error in (19.5) has primitive tending to zero in the Lipschitz norm, since the speeds are bounded. This proves membership of $u \cdot A$ in the canonical full-heat domain, not merely existence of one chosen approximating trace. Applying the same reasoning to u^{-1} proves invariance.

The classical endpoint formula holds for the auxiliary smooth $GL(V)$ gauges. The Young integral equation is continuous on bounded sets of matrix-valued drivers, by its local contraction and finite concatenation. Passing to the limit gives (19.4); the canonical limiting transports are in G because their actual heat approximants are G connections. Smooth transition laws identify the local statements on overlaps. All estimates were uniform on the chosen family, which proves the uniform assertion. □

19.2 Reconstruction of a rough gauge from holonomy

Theorem 19.3 (Holonomy separation in the rough trace domain)**.** *Let $A, A' \in \mathcal{D}^{\alpha}_{\kappa}(P)$. Their framed based polygonal holonomies are equal if and only if $A' = u \cdot A$ for a unique $u \in \mathcal{G}^{1-\kappa}_{*}(P)$. Their unframed polygonal holonomies are equal if and only if they belong to the same $\mathcal{G}^{1-\kappa}(P)$ orbit.*

More generally, for connections in these domains on two smooth principal G-bundles P, Q, equality of the framed polygonal holonomies gives a unique frame-preserving $\mathrm{c}^{1-\kappa}$ bundle isomorphism taking one connection to the other. Equality up to simultaneous conjugation gives an unframed isomorphism. In particular the two smooth bundles have the same topological isomorphism class.

Proof. The implication from a rough gauge is Proposition 19.2. For the converse we work first on one bundle with fixed frame at o. If c is any piecewise short-geodesic path from o to x, define an equivariant automorphism of P_x by

$$u(x) = T_{A'}(c)\, T_A(c)^{-1}. \tag{19.7}$$

Here the expression means the composition of the indicated maps between endpoint fibres. If c is replaced by a second path, the two compositions differ by the holonomy of a based polygon. Equality of those holonomies proves path independence. Reversal and concatenation hold on the trace domain, so this reasoning uses only the established limiting transports. It gives $u(o) = 1$ and, for every short geodesic γ from x to y,

$$u(y) T_A(\gamma) = T_{A'}(\gamma) u(x). \tag{19.8}$$

Uniform short-segment trace bounds on a compact chart and Young stability imply $\|T_A(\gamma) - 1\| + \|T_{A'}(\gamma) - 1\| \le C\,\mathrm{length}(\gamma)^{\alpha}$ in its local frame. Equation (19.8) therefore gives $\|u(y) - u(x)\| \le C d(x,y)^{\alpha}$. The locally defined matrices are compatible on overlaps by path independence, so u is a global $\mathcal{C}^{\alpha}$ gauge section.

To recover the distributional connection, apply the short-geodesic transport identity to difference quotients. In a smaller coordinate rectangle let e_i be a coordinate vector field, $F_h(x) = \exp_x(h e_i(x))$, and let $z_{i,h}, z'_{i,h}$ be the canonical primitives on this short geodesic for A, A'. The Young equation and its sewing estimate give

$$T_A(\gamma_{x,h}) = 1 - z_{i,h}(x) + O(|h|^{2\alpha}), \qquad T_{A'}(\gamma_{x,h}) = 1 - z'_{i,h}(x) + O(|h|^{2\alpha}),$$

uniformly in x. Substitute in (19.8) and use the α-Hölder bound on u once to replace $u(F_h(x))$ by $u(x)$ in the product with $z_{i,h}$. This yields

$$\frac{u(F_h(x)) - u(x)}{h} = u(x)\frac{z_{i,h}(x)}{h} - \frac{z'_{i,h}(x)}{h} u(x) + O(|h|^{2\alpha-1}). \tag{19.9}$$

By (18.23), the two averaged primitives converge in $\mathrm{c}^{-\kappa}_{\mathrm{loc}}$ to the local connection coefficients A_i, A'_i. Multiplication by the fixed $\mathcal{C}^{\alpha}$ function u is continuous on this space, since $\alpha > \kappa$. The error tends to zero because $2\alpha > 1$. On the left, $F_h(x) = x + h e_i + O(h^2)$; its replacement

by $x + he_i$ changes the quotient by $O(|h|^{2\alpha-1})$. The remaining coordinate difference quotient converges distributionally to $\partial_i u$. Consequently

$$\begin{aligned} du &= uA - A'u, \qquad d_B u = ua - a'u, \\ A &= B + a, \qquad A' = B + a'. \end{aligned} \tag{19.10}$$

The right side of the second identity belongs to $\mathrm{c}^{-\kappa}$. To check the little-space assertion, choose $\kappa < \alpha' < \alpha$, approximate u in $\mathrm{c}^{\alpha'}$ and a, a' in $\mathrm{c}^{-\kappa}$ by smooth sections, and use the continuous product estimate. The elliptic estimate for the full covariant gradient now gives $u \in \mathrm{c}^{1-\kappa}$: apply $(1 + d_B^* d_B)^{-1}$ to $u + d_B^*(ua - a'u)$, whose terms have regularity at least $-1 - \kappa$. Thus (19.10) is exactly the rough gauge-action identity $A' = u \cdot A$. The path formula (19.7) proves uniqueness among based gauges. If holonomies are simultaneously conjugate, insert the corresponding fibre automorphism at o in that formula.

For two bundles the same construction uses $T_A : P_o \to P_x$, $T_{A'} : Q_o \to Q_x$ and the prescribed equivariant identification $P_o \to Q_o$. It produces a $\mathcal{C}^\alpha$ section of the bundle of equivariant fibre isomorphisms. In local unitary frames the same difference quotient proof gives $du = uA - A'u$. The covariant gradient on $\mathrm{Hom}(V(P), V(Q))$ improves it to $\mathrm{c}^{1-\kappa}$. In particular it is a continuous bundle isomorphism. A uniformly close smooth approximation in the ambient homomorphism bundle, projected onto the fibre-isomorphism subbundle in a $G \times G$-invariant tubular neighbourhood, is a smooth bundle isomorphism. This proves equality of the smooth topological classes and completes the proof. □

Corollary 19.4 (Stabilisers)**.** *For $A \in \mathcal{D}^\alpha_\kappa(P)$, evaluation at the base point identifies its rough gauge stabiliser with the compact centraliser in G of all its based polygonal holonomies. The based rough gauge action on this domain is free.*

Proof. If $u \cdot A = A$, (19.4) identifies $u(o)$ with a centralising element and determines u everywhere. Conversely, each such element can be inserted at o in (19.7), now with $A' = A$. The proof of Theorem 19.3 gives the resulting rough gauge. These maps preserve multiplication. Evaluation is continuous. For continuity of the inverse, two reconstructed sections have difference v with $\|v\|_\infty$ bounded by the distance of their initial matrices and $d_B v = va - av$. The elliptic estimate and interpolation, with $\kappa < \beta < 1 - \kappa$, give

$$\|v\|_{\mathcal{C}^{1-\kappa}} \le C_A(\|v\|_\infty + \|v\|_{\mathcal{C}^\beta}) \le \tfrac{1}{2}\|v\|_{\mathcal{C}^{1-\kappa}} + C'_A\|v\|_\infty.$$

Thus the inverse is continuous. The centraliser is a closed subgroup of the compact group G, and hence is compact. Taking $u(o) = 1$ gives the last assertion. □

19.3 Closed orbits and a Borel identification

Proposition 19.5 (Compactness with two converging representatives)**.** *Suppose $A_n \to A$ and $u_n \cdot A_n \to A'$ in $\mathcal{A}_\kappa(P)$, with $u_n \in \mathcal{G}^{1-\kappa}(P)$. Some subsequence of u_n converges in $\mathcal{G}^{1-\kappa}(P)$, and its limit u satisfies $u \cdot A = A'$. The same holds for based gauges. Consequently both orbit equivalence relations are closed subsets of $\mathcal{A}_\kappa(P)^2$, and all their orbits are closed.*

Proof. Write $A_n = B + a_n$ and $u_n \cdot A_n = B + a'_n$. The bound (10.18) gives a uniform $\mathcal{C}^{1-\kappa}$ norm for u_n. Compact embedding on the closed surface gives a subsequence converging in $\mathcal{C}^\beta$ for $\kappa < \beta < 1 - \kappa$, with a G-valued limit u. The action identity yields

$$d_B u_n = u_n a_n - a'_n u_n \longrightarrow ua - a'u \quad \text{in } \mathrm{c}^{-\kappa}.$$

Together with uniform convergence of u_n, the covariant elliptic gradient estimate implies that this subsequence is Cauchy in $\mathrm{c}^{1-\kappa}$. Its limit is therefore a member of that little space, and

passage to the action identity proves the claim. A prescribed base-point value passes to the uniform limit. Sequential closedness is closedness in these metric spaces. □

The compactness argument assumes that both representatives converge. In Proposition 14.1, both escape to infinity. Closedness of the orbits therefore does not prevent collapse of the unrestricted double infimum.

Lemma 19.6 (A Borel selector for the rough orbits)**.** *For either the full or the based rough gauge group there is a Borel map $s : \mathcal{A}_\kappa(P) \to \mathcal{A}_\kappa(P)$ such that $s(A)$ belongs to the orbit of A, and $s(A) = s(A')$ exactly when the two orbits agree. Its range is Borel. For every invariant Borel subset D of the connection space, the range of its restriction to D is Borel as well.*

Proof. Give the affine connection space a complete separable metric from its Banach norm and choose a countable basis of open balls with dense centres and rational radii. For a basis ball V, the set of states whose orbit meets V is the saturation $\bigcup_u u \cdot V$, an open set because each gauge acts by a homeomorphism.

For each orbit, choose the first basis ball of diameter at most $1/2$ that meets it. After a ball has been selected, choose the first basis ball of diameter at most 2^{-n} whose closure is contained in the previous ball and which meets the same orbit. At each stage such a ball exists: choose a point of the orbit in the previous open ball, then a sufficiently small basis ball near that point. Containment can, equivalently, be imposed by the strict centre-and-radius inequality, so it is a fixed countable relation between the basis balls. Each selected index is a Borel function of the input state, by the open saturation property. The selected centres form a Cauchy sequence. Completeness gives their limit $s(A)$, which is Borel as a pointwise limit of Borel maps. Its distance from the orbit tends to zero, and the orbit is closed by Proposition 19.5, so it lies in that orbit. All choices depend only on the orbit, proving the asserted equivalence. In particular $s \circ s = s$, and

$$s(\mathcal{A}_\kappa(P)) = \{A : s(A) = A\}$$

is Borel. If D is invariant, $s(D)$ is the intersection of this range with D, which proves the last assertion. □

Theorem 19.7 (The realised Borel holonomy space)**.** *Under (19.1), the subsets*

$$\begin{aligned} \mathscr{H}^{\alpha,\kappa}_{\mathrm{real}}(P) &= \mathfrak{H}(\mathcal{D}^\alpha_\kappa(P)) \subset \mathscr{H}_r(P), \\ \mathscr{U}^{\alpha,\kappa}_{\mathrm{real}}(P) &= [\mathfrak{H}](\mathcal{D}^\alpha_\kappa(P)) \subset \mathscr{U}_r(P) \end{aligned} \tag{19.11}$$

are Borel. They identify, as standard Borel spaces, respectively the based and full rough gauge orbits in the trace domain. There are Borel maps assigning to each realised framed or unframed holonomy a representative connection with those data.

For any countable list of pairwise nonisomorphic surface bundles with the same compact connected structure group, the corresponding realised holonomy images in the common polygonal data space are pairwise disjoint Borel sets. Their union therefore carries a Borel map recovering the bundle isomorphism class.

Proof. Apply Lemma 19.6 to the two groups, and let S_*, S be their Borel ranges intersected with $\mathcal{D}^\alpha_\kappa(P)$. These are Borel because the domain is Borel and invariant by Proposition 19.2. The Borel holonomy map is injective on S_* by Theorem 19.3; its unframed version is injective on S. The Lusin–Souslin theorem for injective Borel maps between standard Borel spaces gives Borel images and Borel inverse maps [18, Corollary 15.2]. The images are exactly (19.11), because the selected representative lies in the same orbit. Composition with the selectors

shows that the induced identifications and their inverses are measurable for the quotient Borel structures. The inverse maps give the claimed representative selections.

For distinct bundles, use the common ambient space $\mathcal{C}_r$ and its compact simultaneous-conjugation quotient. The same injective Borel maps have Borel images there. The two-bundle part of Theorem 19.3 proves that the images are disjoint. A map constant on each of countably many disjoint Borel pieces is Borel. In particular all topological sectors may be included: over a finite two-dimensional CW complex, the obstruction classes for a connected compact Lie group take values in its countable group $\pi_1(G)$, so there are only countably many principal-bundle isomorphism classes. □

The theorem identifies the realised domain inside the intrinsic Polish completion. It does not equate that domain with the entire completion, or identify the latter's topology with the quotient of the negative Hölder norm. It gives the measurable state identification needed to formulate transition laws on actual non-Abelian holonomies. Such laws must in addition respect the rough gauge fibres; that is a property of the stochastic dynamics.

Corollary 19.8 (Simultaneous identification along local solutions)**.** *For the local non-Abelian solution of Theorem 8.14, with $0 < \kappa < 1/16$, every surviving positive-time holonomy lies simultaneously in the realised spaces* (19.11)*. Its continuous unframed holonomy path determines the full rough gauge orbit at each such time, including every surviving positive stopping time, and recovers the fixed topological class of P.*

Proof. Choose $1/2 < \alpha < 1 - \kappa$ and apply Theorem 18.6, Corollary 18.7, and Theorem 19.7. The first result gives positive-time membership on one event for the whole path, so the identification applies at every surviving positive stopping time. □

20 The maximal local orbit process and its holonomies

Fix

$$0 < \kappa < 1/16, \qquad 1/2 < \alpha < 1 - \kappa,$$

and a smooth principal bundle $P \to \Sigma$. Write

$$\mathcal{A} = \mathcal{A}_\kappa(P), \quad \mathcal{G} = \mathcal{G}^{1-\kappa}(P), \quad D = \mathcal{D}^\alpha_\kappa(P), \quad \mathscr{S} = \mathscr{U}^{\alpha,\kappa}_{\mathrm{real}}(P).$$

Put $\mathcal{Q} = \mathcal{A}/\mathcal{G}$. Give this set the standard Borel structure furnished by the representative range in Lemma 19.6, and also retain its topological quotient of the little negative Hölder topology. The orbit map $\pi : \mathcal{A} \to \mathcal{Q}$ is Borel. The subset $\mathcal{Q}_D = \pi(D)$ is Borel, and Theorem 19.7 gives a Borel isomorphism

$$\Theta : \mathcal{Q}_D \longrightarrow \mathscr{S}.$$

We construct the process on all of $\mathcal{Q}$ and then give its holonomy realisation on this invariant subset.

The construction continues local orbit paths by choosing new representatives before their fixed-reference norms become large. Its lifetime records continuation of the orbit across these changes of DeTurck representative. Statements about maximal continuation refer to the state-space quotient topology on $\mathcal{Q}$. No equality between that topology on $\mathcal{Q}_D$ and the subspace topology of $\mathscr{S} \subset \mathscr{U}_r(P)$ is needed.

20.1 Measurable representatives and comparison gauges

Choose a smooth reference B and a Banach norm $\|\cdot\|_\kappa$ on $\mathrm{c}^{-\kappa}(E)$. Define the rough orbit size

$$r_B([A]) = \inf_{u\in\mathcal{G}} \|u\cdot A - B\|_\kappa. \tag{20.1}$$

It is finite and gauge invariant.

Lemma 20.1 (Small representatives and measurable gauge links)**.** *The function r_B is upper semicontinuous in the quotient topology and Borel in the specified standard Borel structure. There is a Borel representative map*

$$s_B : \mathcal{Q} \longrightarrow \mathcal{A}, \qquad \pi(s_B(q)) = q, \qquad \|s_B(q) - B\|_\kappa < r_B(q) + \tfrac{1}{2}. \tag{20.2}$$

There is also a Borel map on the closed orbit relation,

$$\ell(A, A') \in \mathcal{G}, \qquad \ell(A, A')\cdot A = A'. \tag{20.3}$$

Different smooth references and equivalent Banach norms give orbit sizes satisfying affine bounds in both directions.

Proof. Let (g_j) be dense in the separable gauge group. Continuity of the action gives $r_B([A]) = \inf_j \|g_j\cdot A - B\|_\kappa$. Its strict sublevel sets in $\mathcal{A}$ are open and invariant, so its descent to the topological quotient is upper semicontinuous. Compose this Borel function with the Borel representative from Lemma 19.6 to obtain its version on the standard Borel quotient. Starting with that representative, choose the first g_j whose norm is less than $r_B + 1/2$. This is a Borel choice and proves (20.2).

To select a comparison gauge measurably, give $\mathcal{G}$ the complete separable metric induced by its little Hölder norm in the ambient endomorphism bundle. For a closed metric ball $K \subset \mathcal{G}$, the set

$$\{(A, A') : \text{there is } u \in K \text{ with } u\cdot A = A'\}$$

is closed. Indeed, for converging pairs choose corresponding $u_n \in K$ and apply Proposition 19.5; a subsequence converges in $\mathcal{G}$ to a gauge still in K. The analogous set for an open ball is Borel, since that ball is a countable union of closed basis balls contained in it. For each equivalent pair, its set of linking gauges is nonempty and closed. Apply the nested-basis-ball construction of Lemma 19.6 to this closed set, using the just-proved Borel hitting tests. Completeness gives a Borel limit in the fibre, which is (20.3).

Finally, if the same norm is used for two references, the triangle inequality gives $|r_B - r_C| \le \|B - C\|_\kappa$. If the norms differ, their equivalence gives affine upper bounds for r_B in terms of r_C and conversely. □

20.2 Uniformly positive local segments

The bounded-data local existence theorem supplies the following uniform local segments. Their norm thresholds are chosen large enough to cover its reconstructed state bounds.

Lemma 20.2 (A uniform conditional survival bound)**.** *There are deterministic nondecreasing thresholds $K_B(R) > R + 1$ and numbers $\delta_R \in (0, 1]$, $p_R > 0$, for integers $R \ge 2$, with the following property. For a fixed-reference solution starting from $A = B + a$, $\|a\|_\kappa \le R$, define*

$$d_B(A, W; R) = 1 \wedge \inf\{t \ge 0 : \|A_B(t; A, W) - B\|_\kappa \ge K_B(R)\}. \tag{20.4}$$

The threshold is reached before the representative lifetime whenever that lifetime is at most one, by the state-norm explosion criterion. Thus this is a strictly positive stopping time, its endpoint is a finite state, and it precedes the representative lifetime.

It satisfies

$$\inf_{\|A-B\|_\kappa \le R} \mathbb{P}\{d_B(A, W; R) \ge \delta_R\} \ge p_R. \tag{20.5}$$

The same bound holds conditionally at every stopping time, for a random initial state with the indicated norm bound and independent future Brownian increments.

Proof. Use the common bounded-data construction of Theorem 8.14 and the causal map of Theorem 9.5. The random model and Gaussian bounds can be restricted to an event of probability at least $3/4$, independently of the initial state in this deterministic ball. On that event the heat-lift estimate for the initial datum and the fixed-point theorem supply a common positive time $\delta_R \le 1$ and a deterministic reconstructed state bound C_R throughout that interval. Choose

$$K_B(R) = 2 + \max_{2 \le j \le R} (2C_j + 2j).$$

There is no threshold crossing by δ_R on the indicated event, proving (20.5), with $p_R = 1/2$ if desired. The conditional assertion follows from the same uniform construction and the Brownian strong Markov property. □

The stopped segment, its endpoint, and its length are jointly Borel in the initial state and the Brownian path. This follows from the joint Borel causal solution map and the hitting-time functional for a closed norm threshold on a continuous state path. The cemetery construction in Section 9 may be used for that functional; the finite threshold is always reached before the cemetery value.

20.3 Pasting the local orbit paths

For $q_0 \in \mathcal{Q}$ set $T_0 = 0$. At step n, given the endpoint q_n, choose

$$A_n = s_B(q_n), \qquad R_n = 2 + \lceil r_B(q_n) \rceil.$$

Drive the fixed-reference solution from A_n by the Brownian increments after T_n, and run it for $d_n = d_B(A_n, W^{T_n}; R_n)$. Here $W_t^{T_n} = W_{T_n+t} - W_{T_n}$. Put $T_{n+1} = T_n + d_n$ and define, for $T_n \le t \le T_{n+1}$,

$$Z_t = \pi(A_B(t - T_n; A_n, W^{T_n})), \qquad q_{n+1} = Z_{T_{n+1}}. \tag{20.6}$$

There is no jump in the quotient topology at T_n, since the new representative belongs to the same orbit as the preceding endpoint. Each segment has a continuous representative, so the pasted $\mathcal{Q}$-path is continuous before any accumulation of switching times.

Theorem 18.6 puts every surviving positive-time state of the first segment in D, simultaneously in time. Its endpoints and all subsequent selected states therefore belong to D as well. Thus $Z_t \in \mathcal{Q}_D$ at every surviving $t > 0$. If $q_0 \in \mathcal{Q}_D$, this holds also at zero. In that case the initial-time assertion of the same theorem gives continuity of $H_t = \Theta(Z_t)$, including at every switching time.

Every T_n is a finite stopping time and $0 < d_n \le 1$. Let

$$\zeta = \lim_{n \to \infty} T_n \in (0, \infty]. \tag{20.7}$$

For $t \ge \zeta$ give the process the absorbing cemetery value †. The resulting process is jointly measurable in (t, q_0, W) and adapted. At each surviving time it takes values in $\mathcal{Q}$. All these assertions follow by countable iteration of the Borel stopped-segment map.

Lemma 20.3 (No accumulation at bounded orbit size)**.** *Almost surely,*

$$\zeta < \infty \quad \Longrightarrow \quad \limsup_{t \uparrow \zeta} r_B(Z_t) = \infty. \tag{20.8}$$

In particular only finitely many switches occur on a compact time interval on which the orbit size is bounded.

Proof. Fix an integer L and restrict attention to the event $R_n \le L$ for every n. Put $\delta = \min_{2 \le j \le L} \delta_j > 0$ and $p = \min_{2 \le j \le L} p_j > 0$. Conditionally on the past at T_n, whenever $R_n \le L$,

$$\mathbb{P}(d_n \ge \delta \mid \mathcal{F}_{T_n}) \ge p.$$

Conditional Borel–Cantelli implies that $d_n \ge \delta$ infinitely often on the event under consideration. Explicitly, the conditional probability of k successive failures while all the corresponding R_n remain at most L is bounded by $(1-p)^k$. Taking $k \to \infty$ and then a countable union over the possible last success proves the assertion without an independence assumption. Hence $\sum_n d_n = \infty$ there.

If $r_B(Z_{T_n})$ were bounded on an outcome with finite ζ, all its R_n would be bounded by some integer L, which is impossible by the first part. This proves (20.8). The same argument after discarding finitely many initial steps gives the compact-interval statement. □

20.4 Uniqueness of the continuation law

Lemma 20.4 (Coupling different switching constructions)**.** *The law of the killed path* (20.6)*, including its lifetime, is independent of the following choices: the smooth reference, the representative selector satisfying* (20.2)*, the equivalent norm, and the particular Borel, nonanticipative cap and norm-threshold rules used for the local segments, provided every switch is a stopping time, the segments have bounded representative norms, and the rules have the uniform positive-duration property on bounded sets of r_B. The same assertion permits a first partial segment of a previously started construction, followed by the usual rule.*

Proof. To compare the killed laws and their lifetimes, couple the constructions through every switching time. Take two switching constructions with references B, C and the same initial orbit. Choose their initial comparison gauge by (20.3). Drive the first representative by a Brownian motion and the second by its predictable rotation from Theorem 12.2. Their orbit paths agree until the first switching time of either construction.

The comparison cannot terminate earlier. Up to that switching time both representatives obey their finite norm thresholds. Their equality under the comparison gauge and (10.18) keep the gauge bounded in $\mathrm{c}^{1-\kappa}$. The restart and explosion assertions of Theorem 10.3 consequently extend the gauge to the switching time. A representative cannot reach its own explosion time before its finite threshold has been reached.

At that time carry out whichever switches are prescribed, keep the other representative and its remaining cap and threshold, and choose a new linking gauge using (20.3). This gauge is measurable with respect to the joint past. The conditional version of Theorem 12.2 again couples the future local solutions. A continuing representative may be restarted at this common stopping time by the fixed-reference strong Markov and causal restart identities; its remaining cap and threshold are retained as past-measurable parameters. Thus its marginal law is exactly that of the original switching rule.

Iterate on the union of the two sequences of switching times. The relative rotations are predictable orthogonal operators on successive stochastic intervals. Their stochastic integral against the first Brownian motion, extended by the identity after a possible accumulation

time, is a Brownian motion for the joint filtration. Hence both full marginal constructions retain their prescribed Brownian-driven laws. They agree pathwise as orbits up to the smaller of their two lifetimes. The composition law in Proposition 12.3 ensures compatibility of the comparisons on consecutive pieces.

Suppose their lifetimes satisfy $\zeta_1 < \zeta_2$ on some outcome with ζ_1 finite. The second construction has only finitely many segments on a compact interval containing $[0, \zeta_1]$ and lying before ζ_2. Its representative norms, and hence its orbit size, are bounded on that interval. The affine comparison in Lemma 20.1 gives a bound for r_B on the common orbit path before ζ_1. This contradicts Lemma 20.3 for the first construction. Interchanging the constructions proves equality of their lifetimes. Their killed orbit paths therefore agree under the coupling at every time, proving equality in law.

A first partial segment changes none of this reasoning: it has a positive remaining cap and a finite threshold, after which the standard rule applies. It does not affect the nonaccumulation argument. The proof also works conditionally on an initial sigma-field, because all gauge links and stopping rules are Borel and each new Brownian future is independent of the past. □

Theorem 20.5 (The reference-independent maximal local orbit process)**.** *There is a time-homogeneous strong Markov family on the standard Borel space $\mathcal{Q} \cup \{\dagger\}$ with the following properties.*

1. *From every $q \in \mathcal{Q}$ its lifetime ζ is strictly positive. Its surviving path is continuous in the state-space quotient topology, including at time zero. It belongs to $\mathcal{Q}_D$ simultaneously at every surviving positive time. If its initial state belongs to $\mathcal{Q}_D$, it remains there until death.*

2. *Every sufficiently short local segment, represented in any smooth reference gauge, has the orbit law of the bare continuum flow starting from a representative of the current orbit. The construction and the law of ζ are independent of the smooth reference and of the auxiliary switching choices.*

3. *A finite lifetime satisfies (20.8) for every smooth reference and every equivalent rough state norm. The path has no extension through that lifetime which is continuous in $\mathcal{Q}$. Thus the family is maximal among continuations of these local flow germs with rough-quotient-continuous paths.*

If P_t denotes its transition semigroup and σ is any finite stopping time for its usual augmented natural filtration, then for every bounded Borel f and $t \geq 0$,

$$\mathbb{E}_q[f(Z_{\sigma+t}) \mid \mathcal{F}^Z_\sigma] = P_t f(Z_\sigma). \tag{20.9}$$

The cemetery point is absorbing.

Proof. The Borel construction above defines Borel kernels $P_t(q, F) = \mathbb{P}(Z^q_t \in F)$, including the cemetery mass. Continuity on each segment and equality of the two orbit values at a switch prove the path assertions before ζ. There are only finitely many switches on each compact interval strictly before ζ, by the definition of the increasing switching sequence. Lemma 20.4 proves choice independence and the asserted local segment laws.

Let σ be a stopping time for the driving Brownian filtration. On $\{\sigma < \zeta\}$ choose the unique half-open segment $T_n \leq \sigma < T_{n+1}$. At an exact switching time this convention uses the newly selected representative. Thus the remaining cap is strictly positive and the current representative is strictly below its exit threshold. Condition on the past at σ. The current representative, its threshold, and the remaining cap are then fixed past-measurable data. For the displayed construction that cap is $1 - (\sigma - T_n)$; more generally it is the cap chosen at the

segment's start minus the elapsed time. It is not the unknown remaining time until the actual threshold crossing. The fixed-reference strong Markov property identifies its future up to the next switch with a newly driven local solution from that representative. Brownian increments after σ are independent of the past. After the next switch the construction uses its usual rule. This is the partial first-segment construction allowed in Lemma 20.4; its conditional law is therefore exactly the law started afresh from Z_σ. On $\{\sigma \geq \zeta\}$ the assertion is immediate from absorption. This proves the conditional identity for the Brownian filtration. It implies the Chapman–Kolmogorov identity by taking deterministic σ, and implies (20.9) for the usual augmented natural filtration, which is contained in that filtration. With an isolated cemetery point the path is right-continuous in the quotient topology; a left limit at ζ is not needed for this conditional argument.

Lemma 20.3 gives the explosion assertion, and Lemma 20.1 transfers it between references. If a path had a $\mathcal{Q}$-continuous extension through a finite ζ, upper semicontinuity and finiteness of r_B at the extended state would make r_B bounded in a neighbourhood of that time. This contradicts (20.8). The maximality statement is in this rough quotient topology; it imposes no assertion about possible additional boundary limits in the larger holonomy completion. □

Corollary 20.6 (The continuous holonomy realisation)**.** *The restriction of the preceding family to $\mathcal{Q}_D \cup \{\dagger\}$, transported by the Borel isomorphism Θ and $\Theta(\dagger) = \dagger$, is a reference-independent local strong Markov family on $\mathscr{S} \cup \{\dagger\}$. From every realised initial holonomy its surviving paths are continuous in $\mathscr{U}_r(P)$, including at zero. The process takes values in $\mathscr{S}$ simultaneously at all surviving times, and its lifetime is the maximal orbit lifetime in Theorem 20.5. Starting from any orbit in $\mathcal{Q}$, the same holonomy realisation exists continuously at all surviving positive times.*

Proof. The pathwise entrance and invariance established above make $\mathcal{Q}_D \cup \{\dagger\}$ invariant for the transition kernels. The Borel isomorphism identifies the full state at each time, so it also identifies the two natural filtrations. Transporting (20.9) proves the strong Markov assertion. Holonomy continuity on each segment is Theorem 18.6, including its initial-domain assertion. Gauge invariance identifies the values at switching times. There are finitely many switches on compact surviving intervals, which proves the claimed continuity. For an arbitrary initial orbit, the first segment enters D simultaneously at every positive time, and the same argument applies on compact positive-time intervals. □

The realised images of different bundle isomorphism classes are disjoint Borel sets by Theorem 19.7. The preceding kernels may therefore be combined over the countably many surface-bundle classes. They preserve the initial topological class up to the lifetime. Smooth bundle isomorphisms covering isometries transport this Markov family, by the local naturality theorem and the independence of representative and switching choices. No assertion of global nonexplosion or of an invariant non-Abelian measure is required for this local realisation.

21 Naturality and further directions

The geometric objects, the regularised equations and the continuum limits can be transported between isomorphic surface bundles. We establish naturality and independence from auxiliary analytic choices, and then describe the questions beyond the local continuum and observable constructions.

21.1 Intrinsic noise and the affine space of states

For $r \in \mathbb{R}$, the affine space

$$\mathcal{A}^r(P) = B + \mathrm{c}^r(T^*\Sigma \otimes \mathrm{ad}\, P)$$

is independent of the smooth origin B: replacing B by $B + b$ changes the coordinate a to $a - b$. Smooth translation is a homeomorphism, and the bundle Besov norms obtained from different smooth auxiliary connections are equivalent. In particular, $\mathcal{A}^{-\kappa}(P)$ is a separable affine Banach space.

Proposition 21.1 (Global noise and transition compatibility)**.** *Let P have smooth transition functions g_{ij}. The local representatives of a reference connection, its affine coordinate and the white noise satisfy*

$$\begin{aligned} B_i &= \mathrm{Ad}_{g_{ij}} B_j - (dg_{ij})g_{ij}^{-1}, \\ a_i &= \mathrm{Ad}_{g_{ij}} a_j, \qquad \xi_i = \mathrm{Ad}_{g_{ij}} \xi_j. \end{aligned} \tag{21.1}$$

The noise law depends only on the fibre metric and the surface volume. The DeTurck operator, the heat regularisations, and the integrated Gaussian currents satisfy the induced transition laws. Every connection path and comparison gauge constructed here lies on the original bundle P.

Proof. Let $H = L^2(T^*\Sigma \otimes \mathrm{ad}\, P)$. White noise is the isonormal Gaussian family on $L^2(\mathbb{R}; H)$, with covariance given by that Hilbert inner product. For a local construction, take smooth functions ρ_i supported in orthonormal bundle charts with $\sum_i \rho_i^2 = 1$, and independent chart noises ξ_i^{aux}. Extension by zero gives $\xi = \sum_i \rho_i \xi_i^{\mathrm{aux}}$. For test sections f, h,

$$\mathbb{E}[\xi(f)\xi(h)] = \sum_i \int \rho_i^2 \langle f, h\rangle = \int \langle f, h\rangle.$$

The resulting global Gaussian law is independent of the charts. Its restrictions transform as in (21.1). The transformation rules for d_B, its adjoint, and the bracket prove compatibility of the drift. Functional calculus gives compatibility of the heat kernels, and Proposition 7.5 gives that of the currents. Reference changes are affine translations on the same bundle; gauge transformations are its automorphisms over the identity. □

21.2 Naturality of the local continuum solution

Proposition 21.2 (Transport and auxiliary-choice independence)**.** *Let $\Phi : P \to P'$ be a principal-bundle isomorphism covering an orientation-preserving Riemannian isometry of closed surfaces. Write Φ_* for transport of connections and U_Φ for its linear action on adjoint-valued one-forms. Set*

$$B' = \Phi_* B, \qquad A_0' = \Phi_* A_0, \qquad W' = U_\Phi W.$$

The fixed-reference continuum solutions of Theorem 8.14 satisfy

$$A^{B'}(A_0', W') = \Phi_* A^B(A_0, W) \tag{21.2}$$

throughout their maximal lifetimes, which agree under this transport. In particular their transported laws agree.

For fixed B and the specified additive heat approximation, the reconstructed solution is independent of the choices of local frames, partitions, jet realisations, and singular/smooth kernel decompositions used in its construction, provided the finite affine compensation of Proposition 8.13 is included. Successive bundle transports compose exactly.

Proof. The map U_Φ is unitary on the noise Hilbert spaces. It intertwines the covariant derivatives, their adjoints and the Lie brackets, and therefore

$$U_\Phi L_B = L_{B'} U_\Phi, \qquad U_\Phi e^{-\varepsilon(1+L_B)} = e^{-\varepsilon(1+L_{B'})} U_\Phi.$$

The transported cylindrical Wiener process has the stated law. Equation (2.4), expressed under the bundle isomorphism, and classical pathwise uniqueness imply at every positive cutoff

$$A^{B',\varepsilon}(A'_0, W') = \Phi_* A^{B,\varepsilon}(A_0, W).$$

The initial heat approximations also intertwine. Transport is continuous in the local solution topology. On common bounded data events the two sides converge by Theorem 8.14. Uniqueness of limits in probability gives (21.2) there. The data-tail bounds exhaust the local solutions, and uniqueness extends the equality over overlapping continuation intervals. The transport is an isomorphism of the rough state spaces. If one lifetime were strictly smaller, the state on the other side would stay bounded through that time, contradicting the explosion criterion (9.2). Thus the maximal lifetimes agree.

For auxiliary choices, construct both enhanced representations using the same Wiener process and the same heat-cutoff initial data. The affine reconstruction identity identifies both with the identical classical cutoff solution. Localise by both data sizes and apply the same uniqueness-of-limit argument. The finite subtraction coefficients may change with the representation; the compensated reconstructed solution is unchanged. Finally the equality for successive transports follows at each cutoff and passes to the limit in the same way. □

For a smooth gauge transformation over the identity, this result simultaneously transports A, B and the noise. A comparison that changes B while keeping the geometric initial orbit fixed is governed by the reference-change equation of Section 3.

21.3 The scope of the non-Abelian comparison

The continuum comparison of Section 12 uses the bare additive spatial heat approximation. It relates local solutions in different reference gauges through a $\mathrm{c}^{1-\kappa}$ gauge and a predictable rotation of Wiener noise. Its finite quadratic cancellation is specific to the Yang–Mills contraction in two dimensions. It does not assert that every free tensor enhancement, or every temporal smoothing convention, has the same finite limit.

The realised state spaces in Theorem 19.7 identify the rough gauge fibres of the positive-time connection process. Section 20 constructs the corresponding reference-independent maximal local orbit family and its strong Markov holonomy realisation. Their ambient completions are intrinsic Polish spaces, but the realised Borel images need not fill those completions. Neither completeness of the ambient metric nor static orbit reconstruction proves global existence of a non-Abelian diffusion.

21.4 Global dynamics and covariant approximations

Global non-Abelian dynamics on an arbitrary surface bundle requires estimates preventing escape of the evolving gauge orbit. Identifying an invariant measure is a further problem. Neither is resolved by the local construction or the Abelian equilibrium calculation. The torus invariant-measure and universality results of [8], and the covariant non-Markovian Yang–Mills–Higgs approximation of [5], provide relevant precedents, with different geometric and approximation hypotheses.

For the field-dependent smoother $e^{-\varepsilon\Delta_A}$, Section 4 supplies maximal classical solutions and exact reference comparison at fixed $\varepsilon > 0$. A uniform singular limit for this family would

require stochastic estimates with the evolving distributional connection inside its heat kernel. This problem is separate from the additive-heat reference comparison proved here. Similarly, classifying finite normalisations for all space–time mollifiers requires the corresponding causal covariance calculations, beyond the spatial predictable convention used in the present argument.

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