POSITIVITY FOR WEIGHTED LINEAR LAURENT PHENOMENON ALGEBRAS

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Abstract. We prove Laurent positivity for ordinary Laurent phenomenon algebras whose initial exchange polynomials are nonconstant affine linear polynomials with positive constant terms and nonnegative numerical coefficients. The coefficient field may be any subfield of the real numbers, and the weights may have arbitrary zero patterns and algebraic relations. Every cluster variable has nonnegative Laurent coefficients in every cluster, after choosing positive representatives. The algebra coincides with its upper algebra, and its cluster monomials form an atomic positive basis with nonnegative multiplication constants. The proof uses the factorization of rooted-forest determinants, a finite system of positive exchange identities, and a refinement by prime factors that controls specialization to zero weights. A nonlinear example with all nonzero initial coefficients equal to one shows that the linearity hypothesis cannot be omitted.

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1. Introduction

The Laurent phenomenon asserts that variables obtained by repeated exchange are Laurent polynomials in the variables of an initial cluster. In the cluster algebras of Fomin and Zelevinsky, this phenomenon originally led to the stronger positivity conjecture: the Laurent coefficients should be nonnegative [3]. These two assertions are distinct. Even a rational expression obtained using only addition, multiplication, and division can simplify to a Laurent polynomial with negative coefficients. A proof of positivity must therefore control the divisions that occur during mutation.

Lam and Pylyavskyy introduced Laurent phenomenon algebras, or LP algebras, by allowing irreducible exchange polynomials in place of exchange binomials [5]. Their mutation rule removes common factors before defining the next exchange polynomial. It preserves the Laurent phenomenon, but the removal of factors makes coefficient positivity delicate. In their subsequent

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study of linear LP algebras, they described a family attached to directed graphs [6]. Its initial exchanges have the form

$$F_i = A_i + \sum_{i \to j} x_j,$$

with independent coefficient variables A_i. The cluster variables are given by sums over directed forests, and the clusters are described by nested collections of strongly connected vertex sets. This description gives positivity in the initial cluster. Their Conjecture 7.1 asks for positivity in every cluster; Conjecture 7.3 concerns a cluster-monomial basis and its multiplication constants.

Glew has proved positivity for graph LP algebras with respect to an arbitrary seed [4, Theorem 5.1]. Daisey has proved that every LP algebra admitting a linear seed is of finite type over an arbitrary unique factorization domain [1, Theorem 1.2]. The latter result allows coefficient relations and vanishing coefficients. Neither statement directly gives positivity for arbitrary numerical weights. Specializing coefficients can create new common factors and change the monomial denominators in the exchange Laurent polynomials; this obstruction is also emphasized in [1, Remark 1.1].

Here we consider ordinary LP algebras over a field $K \subseteq \mathbb{R}$. The initial exchange polynomials are

$$F_i = a_i + \sum_{j \neq i} c_{ij} x_j, \qquad a_i > 0, \quad c_{ij} \geq 0. \tag{1.1}$$

We require each F_i to be nonconstant. Over a field this is part of the seed axioms: a nonzero constant is a unit, whereas an exchange polynomial must be an irreducible nonunit. No further restriction is placed on the rank, the support of (c_{ij}), or the equalities and algebraic relations among its entries.

An LP variable is defined up to a unit of its coefficient ring. We fix one representative of each association class, positive on the positive orthant of the initial cluster. The proof constructs such representatives simultaneously. Multiplying any of them by a positive element of K preserves every positivity assertion below. A *cluster monomial* is a product of variables in a single cluster with nonnegative integer exponents, and equal products are counted only once.

Theorem 1.1. *Let $K \subseteq \mathbb{R}$ be a field and let $\mathcal{A}$ be an ordinary LP algebra with an initial seed of the form (1.1), with every F_i nonconstant. Then positive representatives can be chosen so that:*

(1) *every cluster variable has Laurent coefficients in $K_{\geq 0}$ with respect to every cluster;*
(2) *every exchange Laurent polynomial, and hence every ordinary exchange polynomial in the corresponding normalization, has nonnegative coefficients;*
(3) *if $\mathcal{U}$ is the upper LP algebra and $\mathscr{B}$ is the set of cluster monomials, then*

$$\mathcal{A} = \mathcal{U} = \bigoplus_{b \in \mathscr{B}} Kb;$$

(4) *the universally nonnegative elements are exactly*

$$\bigcap_{T} K_{\geq 0}[\mathbf{x}_T^{\pm 1}] = \bigoplus_{b \in \mathscr{B}} K_{\geq 0} b.$$

In particular, the cluster-monomial basis is atomic and all its multiplication constants are nonnegative.

The normalized mutation pattern is finite.

The coefficients in this theorem are numerical elements of the ordered field K. The assertion is not a statement about the individual coefficients of a polynomial in formal weight parameters. In particular, the theorem imposes no algebraic independence assumption on numerical weights. For primitive nonconstant initial exchanges over $\mathbb{Z}$, we also obtain nonnegative integer Laurent coefficients; see Corollary 9.1.

We briefly describe the proof. After dividing each initial exchange by its positive constant term, all roots are equal to one. The relevant determinants are then rooted-forest polynomials. Equalities among rows outside a vertex set define an outgoing module. Minimal modules and maximal cliques of pair modules determine a forest of blocks. The irreducible forest factors have canonical prime lifts to the upper algebra. Their coordinate denominators are computed by a signed Laplacian on each coordinate boundary.

For strictly positive weights, we construct a finite collection of candidate clusters. A candidate is described by retained initial variables, block cores, and an increasing chain of principal determinants. The principal step may involve a whole tree of blocks. Source determinants give simultaneous factorizations of the port factors and the previously omitted variables. These factorizations produce nonnegative identities for every direction of exchange. Primality in the upper algebra identifies each proposed partner with the actual LP mutation partner. A separate finite-pattern argument then converts exchange positivity into Laurent positivity in every cluster, and gives the assertions about the upper algebra and the basis.

Zero weights require a further argument. Uniformly completing the coefficient matrix by a positive parameter preserves all outgoing modules. The positive charts have limits, but their coordinates can become reducible. We prove that every prime factor is itself a coordinate of the limiting collection. Each limiting chart therefore refines by a unimodular monomial change into a genuine LP cluster. Completeness of the leading-exponent cones proves that no actual cluster is omitted. This is the step that allows arbitrary zero patterns without assuming that mutation commutes with specialization.

The final remark gives a nonlinear rank-three counterexample with positive initial coefficients and verifies its mutations explicitly. Thus the distinction between linear and nonlinear exchange polynomials is already visible after two mutations.

2. Ordinary LP mutation and prime elements of the upper algebra

Let K be a field, let $\mathcal{F} = K(x_1, \dots, x_n)$, and let $\mathbf{x} = (x_1, \dots, x_n)$ be a transcendence basis. A seed consists of $\mathbf{x}$ and polynomials $F_i \in K[\mathbf{x}]$ which are irreducible nonunits, are independent of x_i, and are not divisible by any x_j. For a Laurent polynomial G put

$$\operatorname{pol}(G) = G \prod_j x_j^{-m_j}, \qquad m_j = \min\{\alpha_j : [\mathbf{x}^\alpha]G \neq 0\}.$$

Thus $\operatorname{pol}(G)$ is a polynomial without coordinate factors.

The exchange Laurent polynomial is

$$\widehat{F}_i = F_i \prod_{j \neq i} x_j^{-e_{ij}}, \qquad e_{ij} \geq 0. \tag{2.1}$$

For $j \neq i$, its defining condition is that substituting $x_j = F_j/\xi$ gives a Laurent polynomial not divisible by F_j. Equivalently, e_{ij} is the F_j-adic order of $F_i|_{x_j = F_j/\xi}$ in the resulting Laurent ring. These exponents exist and are unique [5, Lemmas 2.3–2.4].

Mutation at k replaces x_k by $x_k' = \widehat{F}_k/x_k$. The exchange polynomial at x_k' is F_k. If $j \neq k$ and F_j does not involve x_k, its new exchange is F_j, up to a coefficient unit. Otherwise put

$$Z_k = \widehat{F}_k|_{x_j = 0}, \qquad G_j = F_j|_{x_k = Z_k/x_k'}. \tag{2.2}$$

Remove from G_j all irreducible factors which divide Z_k, with their entire multiplicity in G_j, in the Laurent ring in the coordinates other than x_j. Apply pol to the result. The resulting polynomial, up to a coefficient unit, is the new exchange at x_j. The specialization in (2.2) is defined: if a negative x_j-power occurred in $\widehat{F}_k$, then F_j would be independent of x_k. The resulting operation is reversible and gives another seed [5, Section 2.2].

The ordinary LP algebra $\mathcal{A}$ is generated over K by all the variables in the mutation pattern. Write

$$\mathcal{L}(T) = K[\mathbf{x}_T^{\pm 1}], \qquad \mathcal{U} = \bigcap_T \mathcal{L}(T).$$

The Laurent phenomenon gives $\mathcal{A} \subseteq \mathcal{U}$ [5, Theorem 5.1]. Two variables or seeds are identified only up to coefficient units when speaking about a normalized pattern; in all displayed identities we choose actual rational-function representatives.

We recall the local input from the proof of the Laurent phenomenon. For an alternating three-step path in two distinct directions, suppress unchanged coordinates and write

$$T_0 = (x, y), \quad T_1 = (z, y), \quad T_2 = (z, u), \quad T_3 = (v, u).$$

Then

$$u, v \in \mathcal{L}(T_0), \qquad \gcd_{\mathcal{L}(T_0)}(z, u) = \gcd_{\mathcal{L}(T_0)}(z, v) = 1. \tag{2.3}$$

These are [5, Lemmas 5.2–5.4]; proportional exchanges are included. We use this classical local Laurent statement and prove the upper-algebra consequences explicitly.

Lemma 2.1. *For every seed T,*

$$\mathcal{U} = \mathcal{L}(T) \cap \bigcap_i \mathcal{L}(\mu_i T). \tag{2.4}$$

Every cluster variable is a prime nonunit of $\mathcal{U}$, and $\mathcal{U}^\times = K^\times$.

Proof. Write $L_j = \mathcal{L}(T_j)$ for the alternating path above. The element z is an irreducible polynomial times a Laurent unit in the UFD L_0. For $h \in L_1 \cap L_3$, Laurentness in (2.3) allows expressions

$$h = a/z^r = b/(u^s v^t), \qquad a, b \in L_0, \quad r, s, t \geq 0.$$

The two coprimalities force $z^r \mid a$, so $h \in L_0$. Reversing the path gives

$$L_1 \cap L_3 \subseteq L_0, \qquad L_0 \cap L_2 \subseteq L_3.$$

Apply these inclusions to $\mu_j T, T, \mu_k T, \mu_j \mu_k T$, for each $j \neq k$. They show that the right-hand side of (2.4) is invariant under mutation. It is therefore contained in every Laurent ring in the pattern, proving (2.4). This finite-intersection result also appears in [2, Theorem 1.3].

Fix $T = (\mathbf{x}, \mathbf{F})$ and let $L_i = \mathcal{L}(\mu_i T)$. The polynomial F_i is prime in $K[x_j^{\pm 1} : j \neq i]$ and hence in L_i. The exchange identity makes x_i a Laurent unit times F_i in L_i. We claim

$$x_i \mathcal{U} = \mathcal{U} \cap F_i L_i. \tag{2.5}$$

If h belongs to the right side, then $h/x_i \in L_i$. It also belongs to $\mathcal{L}(T)$ and all $\mathcal{L}(\mu_j T)$ with $j \neq i$, because x_i is an invertible unchanged coordinate in these rings. Equation (2.4) proves the claim. The right side of (2.5) is a proper prime ideal, so x_i is a prime nonunit.

Let ν_i be the F_i-adic valuation in L_i. Then

$$\nu_i(x_j) = \delta_{ij}, \qquad \nu_i(h) \geq 0 \quad (0 \neq h \in \mathcal{U}), \qquad \nu_i(h) > 0 \iff h \in x_i \mathcal{U}. \tag{2.6}$$

A unit of $\mathcal{U}$ is a unit of $\mathcal{L}(T)$, so it has the form $c \prod_j x_j^{a_j}$. Applying ν_i to it and its inverse gives $a_i = 0$ for every i, proving $\mathcal{U}^\times = K^\times$. □

In particular, products of upper prime elements have unique prime multiplicities, even though we do not assume that $\mathcal{U}$ is a UFD. If a prime divides a product of primes, it divides one factor; these two primes are associates. Repeated cancellation proves the assertion.

Lemma 2.2. *Fix an initial seed with nonconstant affine linear exchanges. For an irreducible polynomial $P \in K[\mathbf{x}]$ having no coordinate factor, write*

$$Q_P = \frac{P}{\prod_i x_i^{\nu_i(P)}}. \tag{2.7}$$

Then Q_P is a prime nonunit of $\mathcal{U}$. If $P = \sum_{k\geq 0} x_i^k P_k$, its exponent is computed by

$$\nu_i(P) = \min_k\{k + \mathrm{ord}_{F_i}(P_k)\} = \mathrm{ord}_{(x_i,F_i)}\, P. \tag{2.8}$$

It is the unique upper-prime lift of the Laurent prime P, up to a coefficient unit and association in the initial Laurent ring.

Proof. Substitute $x_i = \widehat{F}_i/x_i'$ into an initial Laurent expression. Its only possible noncoordinate denominator is a power of F_i; different powers of x_i' cannot cancel. Consequently

$$\mathcal{U} = \{f \in K[\mathbf{x}^{\pm 1}] : \nu_i(f) \geq 0 \text{ for all } i\},$$

and (2.8) follows. Since $\nu_i(x_j) = \delta_{ij}$, all coordinate valuations of Q_P are zero. Thus $Q_P \in \mathcal{U}$. It is prime after inverting all initial coordinates. If Q_P divides ab there, it divides, say, a, and a/Q_P has every coordinate valuation nonnegative. Hence $a/Q_P \in \mathcal{U}$, which proves primality in $\mathcal{U}$. It is a nonunit since it is a nonunit in the initial Laurent ring. Any associated Laurent lift differs by a monomial in the x_i. A prime not associated to an x_i must have zero ν_i: otherwise (2.5) would make it divisible by x_i. This forces precisely the exponents in (2.7). □

Lemma 2.3 (Recognition of a mutation partner)**.** *Let $C = \{x, s_1, \ldots, s_{n-1}\}$ be an actual cluster and let $y \in \mathcal{U}$ be a prime nonunit. Suppose*

$$xy = E(\mathbf{s}), \qquad E \in K[\mathbf{s}^{\pm 1}] \setminus \{0\}.$$

Then y is associated to the actual mutation partner of x. If both representatives are positive on the initial positive orthant, the association scalar is positive.

Proof. Let x^* be the actual partner and write $xx^* = H(\mathbf{s})$. Since $y \in \mathcal{U}$, it is Laurent in the actual neighboring cluster and

$$y = (E/H)x^*.$$

Uniqueness of powers of the independent coordinate x^* gives $E/H \in K[\mathbf{s}^{\pm 1}]$. Clearing a shared monomial denominator gives $\mathbf{s}^\alpha y = R(\mathbf{s})x^*$, with $\alpha \geq 0$. The prime x^* is not associated to any shared coordinate, so it divides y in $\mathcal{U}$. The two prime nonunits are associates. Their values on the positive orthant determine the sign of the scalar. □

We also record why a finite set of variable classes gives a finite normalized pattern. If two seeds of the same pattern have the same cluster, their exchange polynomials agree up to units. Indeed, fix a coordinate x_i and invert all the shared coordinates $s_j = x_j$ with $j \neq i$. Localization commutes with the finite intersection in (2.4); all the other neighbor rings then contain the initial Laurent ring. Thus

$$\mathcal{U}[\mathbf{s}^{-1}] = \mathcal{L}(T) \cap \mathcal{L}(\mu_i T).$$

Their common upper algebra consequently determines the ideal

$$\{E \in K[x_j^{\pm 1} : j \neq i] : E/x_i \in \mathcal{U}[\mathbf{s}^{-1}]\} = F_i K[x_j^{\pm 1} : j \neq i].$$

The equality follows from (2.6) and the finite intersection. Since F_i has no coordinate factor, its ordinary polynomial representative is determined up to a coefficient unit. The hat rule then determines its exchange Laurent polynomial. Independent rescaling by coefficient units gives equivalent seeds.

3. Finite patterns: exchange positivity, bases, and the upper algebra

We first prove two results for an arbitrary finite ordinary LP pattern over a field K. Linearity is not assumed in this section. Choose one rational-function representative of each variable class and use those representatives in every cluster containing the class. Rescaling by coefficient units transports the seed structure, so an actual exchange between these charts has the form

$$xx' = E(s_1, \dots, s_{n-1}), \qquad E \in K[\mathbf{s}^{\pm 1}] \setminus \{0\}. \tag{3.1}$$

3.1. **Transport of Newton polytopes.** For $0 \neq f \in \mathcal{U}$ and a cluster C, denote its Laurent support and Newton polytope by $\operatorname{supp}_C(f)$ and $P_C(f)$, and put

$$h_{C,f}(u) = \max_{\alpha \in \operatorname{supp}_C(f)} \langle \alpha, u \rangle.$$

The product identity $P_C(fg) = P_C(f) + P_C(g)$ holds over a field without a sign hypothesis: a generic weight exposes a unique highest term of each factor, whose product cannot cancel, and continuity gives equality of support functions at all weights.

A weight $u \in \mathbb{R}^n$ is called rationally independent if $\langle m, u \rangle = 0$, $m \in \mathbb{Z}^n$, implies $m = 0$. Such weights are dense. For one of them, the largest exponent of a nonzero Laurent polynomial is unique, and

$$\nu_{C,u}(p/q) = h_{C,p}(u) - h_{C,q}(u)$$

is a well-defined maximum valuation on $\mathcal{F}$.

Lemma 3.1. *For any two clusters C, D there is a piecewise integral linear homeomorphism $\tau_{C,D} : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$h_{C,f}(u) = h_{D,f}(\tau_{C,D}(u)) \qquad (0 \neq f \in \mathcal{U}). \tag{3.2}$$

In particular, for $f, g \neq 0$ in $\mathcal{U}$,

$$P_C(f) \subseteq P_C(g) \iff P_D(f) \subseteq P_D(g). \tag{3.3}$$

Neither assertion requires finiteness or positivity.

Proof. For (3.1), with x in position i, define

$$(\tau_i(u))_j = u_j \ (j \neq i), \qquad (\tau_i(u))_i = -u_i + h_{C,E}(u).$$

Since E is independent of x, this is a continuous piecewise integral linear involution. At a rationally independent u, let β be the highest exponent of E. On its linearity region the map is an integral matrix of determinant -1, so $\tau_i(u)$ is again rationally independent. It is the vector of the $\nu_{C,u}$-values of the new coordinates. Distinct Laurent monomials in those coordinates therefore have distinct values; the largest one in f cannot cancel. This proves (3.2) for adjacent charts and generic u. Continuity gives it for every u, and composition along a mutation path proves the general assertion. Its coordinates are the support functions of the variables of D in C, so it is independent of the path. Finally, containment of polytopes is equivalent to an inequality of support functions at every weight. Surjectivity of $\tau_{C,D}$ gives (3.3) in both directions. □

Lemma 3.2. *If b, b' are cluster monomials and $P_C(b) \subseteq P_C(b')$ in one cluster, then $b = b'$. Consequently, if $b = \mathbf{x}_D^a$, the point a is outside $P_D(b')$ for every other cluster monomial b'.*

Proof. Transport containment to a cluster where b' is a monomial. Its Newton polytope is a point, so $b = cb'$ for $c \in K^\times$. All factors are prime nonunits of $\mathcal{U}$ by Lemma 2.1. Uniqueness of their multiplicities identifies the factors on both sides. Since representatives are fixed, the two products are equal. The last assertion follows by applying the first to $P_D(b) = \{a\}$. □

Fix an initial cluster C_0 and a rationally independent weight w. Let $g(f)$ be the exponent of the highest term of f in that cluster. It is additive on products. Distinct cluster monomials have distinct g-vectors: if $g(b) = g(b')$, transport the valuation to a cluster where $b' = \mathbf{x}_D^a$. Rational independence forces a to be the highest exponent of b there, and Lemma 3.2 then gives $b = b'$.

3.2. **The leading cones and a strictly convex support function.** For a cluster C put

$$G_C = (g(v))_{v\in C}, \qquad \sigma_C = \operatorname{cone}\{g(v) : v \in C\}.$$

These matrices are unimodular. Initially $G_{C_0} = I$; at an exchange (3.1) the unique leading exponent β_0 of E gives

$$g(x) + g(x') = \sum_j (\beta_0)_j g(s_j). \tag{3.4}$$

Replacing one column in this way changes the determinant by -1.

Lemma 3.3. *For any two actual clusters, even in an infinite pattern,*

$$\sigma_C \cap \sigma_D = \operatorname{cone}\{g(v) : v \in C \cap D\}. \tag{3.5}$$

For a finite pattern these cones and their faces form a finite complete simplicial unimodular fan. Its facets correspond to actual mutations.

Proof. A rational point of the intersection has nonnegative rational coordinates in both integral bases G_C, G_D. Clearing denominators gives two cluster monomials with the same leading vector. They are equal, and prime-factor uniqueness leaves only shared variables. This proves (3.5) for rational points. Both cones are rational polyhedral, so rational points are dense in their intersection, proving it in general.

Thus the cones form a fan. Across the facet omitting $g(x)$, mutation gives a second cone on the opposite side, by (3.4). There cannot be two distinct cones on the same side of that facet, since their interiors would overlap near a relative interior point. Every facet has exactly two incident cones.

For a finite collection their union is closed. If it had an exterior point, take a segment from an interior point to the complement, avoiding the finitely many faces of codimension at least two and crossing facet hyperplanes transversely. Such a choice is possible because the excluded endpoints lie in finitely many proper affine subspaces. At the first exit the segment meets the relative interior of a facet. The adjacent cone on the other side contains its continuation, a contradiction. For rank one the two half-lines give the same conclusion. □

For a variable v define

$$H(v) = \sum_T \sum_{j=1}^{n} \bigl(h_{T,v}(e_j) + h_{T,v}(-e_j)\bigr).$$

This is the sum of all coordinate widths of its Newton polytopes. The pattern is finite, so $H(v)$ is a nonnegative integer. Define

$$M = \max\left(0, \max_{C,i,\beta\in\operatorname{supp} E_{C,i}} \left(\sum_j \beta_j - 2\right)\right), \qquad \omega(v) = (M+1)H(v) + 1. \tag{3.6}$$

Lemma 3.4. *There is a bounded full-dimensional polytope P whose outer normal cones at the vertices are precisely the σ_C. If p_C is the corresponding vertex, adjacent clusters satisfy*

$$\langle g(s_j), p_{C'} - p_C\rangle = 0, \qquad \langle g(x), p_{C'} - p_C\rangle < 0 \tag{3.7}$$

for (3.1).

Proof. For a function a on variables write $\Delta a = a(x) + a(x') - \sum_j (\beta_0)_j a(s_j)$. Newton transport applied to the actual identity $xx' = E(\mathbf{s})$ gives, for every T and every weight u,

$$h_{T,x}(u) + h_{T,x'}(u) = \max_{\beta \in \operatorname{supp} E} \sum_j \beta_j h_{T,s_j}(u).$$

Hence the wall slack of each support function is nonnegative, and $\Delta H \geq 0$. In the exchange chart itself all old coordinate widths vanish, while the widths of $x' = E/x$ are those of E. Over a field an ordinary exchange is a nonconstant polynomial without coordinate factors, and therefore has at least two monomials. Consequently this chart contributes a positive integer, so $\Delta H \geq 1$. Equation (3.6) now gives $\Delta\omega \geq 1$.

Let p_C be the solution of $\langle g(v), p_C \rangle = \omega(v)$ for $v \in C$. Define ψ to be this linear function on σ_C. It is continuous on the fan because adjacent values agree on shared rays. The strict wall inequalities just proved say that the linear slopes of ψ increase across every wall. Restricting to a generic line gives a convex piecewise linear function; approximating nongeneric lines gives convexity on every line. Thus ψ is globally convex. Each of its linear pieces is a supporting linear function, whence

$$\psi(z) = \max_C \langle p_C, z \rangle.$$

It is the support function of $P = \operatorname{conv}\{p_C\}$. Strictness at every wall implies that its maximal linearity cones are exactly the σ_C, rather than unions of them. For instance, a segment from the interior of a cone to an exterior point crosses a first wall with strictly increasing slope, which excludes equality with the original linear piece beyond that wall. Since each ray height is positive, $\psi(z) > 0$ for $z \neq 0$. Its positive minimum on the unit sphere shows that P contains a ball about zero, so it is full-dimensional. It is bounded as the convex hull of finitely many points. The asserted normal cones and (3.7) follow: the new vertex remains on the shared facets and lies strictly below the old exchanged facet. □

3.3. **A basis without a positivity assumption.**

Theorem 3.5. *For a finite ordinary LP pattern over K, its cluster monomials form a K-basis of $\mathcal{U}$, and $\mathcal{A} = \mathcal{U}$. If $f = \sum_{b \in \mathscr{B}} c_b b$ and $b = \mathbf{x}_C^a$, then*

$$c_b = [\mathbf{x}_C^a] f. \tag{3.8}$$

Proof. Lemma 3.2 shows that the coefficient on the right of (3.8) receives no contribution from a different cluster monomial. This proves linear independence and the extraction formula whenever an expansion exists.

For spanning, take $0 \neq f \in \mathcal{U}$ and let $\gamma = g(f)$. Completeness gives a cluster C with $\gamma \in \sigma_C$. Unimodularity gives $a = G_C^{-1}\gamma \in \mathbb{Z}^n_{\geq 0}$. Set $b = \mathbf{x}_C^a$. Rational independence of the transported weight shows that a is the highest exponent of f in C. Let c be its coefficient there. Then $\{a\} \subseteq P_C(f)$; Newton transport implies $P_D(b) \subseteq P_D(f)$ in every cluster D. Subtracting cb cancels the leading term, lowers its initial w-weight, and creates no exponent outside $P_{C_0}(f)$. Repetition terminates because this fixed polytope contains only finitely many lattice points and the leading weights strictly decrease. It expresses f as a finite linear combination of cluster monomials. These monomials belong to $\mathcal{A}$, so $\mathcal{A} = \mathcal{U}$. □

3.4. **Exchange positivity implies Laurent positivity.**

Theorem 3.6. *Let a finite ordinary LP pattern over K have fixed representatives, and let $S \subseteq K$ be a subsemiring containing $0, 1$. If every actual exchange product (3.1) has coefficients in S, then all Laurent coefficients of every cluster monomial belong to S in every cluster. Moreover,*

$$\bigcap_C S[\mathbf{x}_C^{\pm 1}] = \bigoplus_{b \in \mathscr{B}} Sb, \tag{3.9}$$

and every multiplication constant of this basis belongs to S.

Proof. Fix a cluster monomial b. Suppose one of its Laurent coefficients in some cluster lies outside S. Among all such terms in the finitely many cluster expansions choose one maximizing $\langle w, \gamma\rangle$, where a term $\mathbf{x}_C^{\alpha}$ has common vector $\gamma = G_C\alpha$.

Write its expansion along an exchange as

$$b = \sum_k a_k(\mathbf{s})x^k = \sum_\ell b_\ell(\mathbf{s})(x')^\ell.$$

If the chosen exponent of x is $-r < 0$, uniqueness of powers of x' gives the polynomial identity

$$a_{-r} = b_r E^r. \tag{3.10}$$

All coefficients of E^r lie in S. A coefficient of a_{-r} outside S therefore requires a coefficient of b_r outside S at an exponent $\eta - \delta$, where η is its old shared exponent and $\delta \in \operatorname{supp}(E^r)$. If β_0 is the leading exponent of E, the new common vector is

$$\gamma' = \gamma + G_{\text{shared}}(r\beta_0 - \delta).$$

Its w-weight is at least that of γ. Maximality of the chosen bad coefficient forces equality. Rational independence then forces $\gamma' = \gamma$. Thus whenever a negative coordinate is present, a bad coefficient with exactly the same common vector occurs in the adjacent cluster.

Use the polytope in Lemma 3.4. For an exchange at a negative coordinate α_i, equation (3.7) gives

$$\langle \gamma, p_{C'} - p_C\rangle = \alpha_i\langle g(x), p_{C'} - p_C\rangle > 0.$$

The common vector remains fixed at every transfer. Hence this walk cannot cycle on the finite vertex set of P. It ends in a cluster where its exponent vector is nonnegative. That exponent is a cluster monomial. By Lemma 3.2, its coefficient in b is zero unless it is b itself, in which case the coefficient is one. Both belong to S, a contradiction. This proves the Laurent assertion. The argument never divides an element of S.

By Theorem 3.5, every $f \in \mathcal{U}$ has a unique cluster-monomial expansion, and each basis coefficient is extracted by (3.8). If f is universally S-positive, these coefficients belong to S. Conversely every S-linear combination of the basis is universally S-positive by the first part. This proves (3.9). Products of two basis monomials are universally S-positive, so their extracted structure constants belong to S as well. □

4. Forest determinants, modules, and canonical coordinates

From now on $K \subseteq \mathbb{R}$. Dividing F_i in (1.1) by its positive constant term is an allowed change of coefficient units. We may therefore write

$$F_i = 1 + \sum_{j\neq i} C_{ij}x_j. \tag{4.1}$$

Put $V = [n]$, set $C_{ii} = 0$, and define the target-weighted row Laplacian

$$(L_C)_{ii} = \sum_{j\neq i} C_{ij}x_j, \qquad (L_C)_{ij} = -C_{ij}x_j \quad (i \neq j).$$

For $J \subseteq V$ set

$$\begin{aligned} D_J(t,x) &= \det((tI + L_C(x))[J,J]),\\ Z &= \operatorname{diag}(F_i/x_i) - C, \qquad Y_J = \det Z[J,J],\\ D_J(1,x) &= Y_J\prod_{i\in J} x_i, \qquad H_C(t,x) = \det(tI + L_C(x))/t. \end{aligned} \tag{4.2}$$

Empty determinants are one. The last quotient is polynomial because $L_C\mathbf{1} = 0$. We abbreviate $Z[J,J]$ to Z_J; thus $Y_J = \det Z_J$.

We use the elementary directed-forest expansion. In the determinant of a row Laplacian plus diagonal roots, choose for each nonroot vertex one outgoing edge; the contributions with a directed cycle cancel against the corresponding permutation cycle. The remaining choices are rooted directed forests. Each contributes the product of its edge weights and its root weights, with sign $+1$. This argument also proves the principal-minor version: vertices outside J act as roots, and a chosen edge from J to such a vertex retains its weight in the diagonal. Thus D_J and H_C have nonnegative coefficients when $C_{ij} \geq 0$. The coefficient of the highest power of t is one. In particular, every dehomogenized polynomial used below has constant term one.

4.1. **An irreducibility criterion.**

Definition 4.1. A nonempty set $M \subseteq V$ is an *outgoing module* if

$$C_{ik} = C_{jk} \qquad (i, j \in M, k \notin M). \tag{4.3}$$

Singletons and V itself are modules. A nontrivial module has at least two vertices; a minimal nontrivial module is minimal among all such sets, including V when it has no smaller one.

We give an irreducibility argument that also applies to signed and zero coefficients. Its only dimension-theoretic input is the generic hyperplane formula for a finite family of nonzero real subspaces $V_i \subseteq E$:

$$\dim \sum_i (V_i \cap K_0) = \min_{\mathcal{P}} \sum_{B \in \mathcal{P}} \left(\dim \sum_{i \in B} V_i - 1 \right) \tag{4.4}$$

for a generic hyperplane $K_0 \subseteq E$, where $\mathcal{P}$ ranges over partitions of the index set. This is the classical generic Dilworth truncation formula; its generic characteristic-zero statement is given in [7, Theorems 5.2 and 7.1]. We apply it only on a nonempty Zariski-open set of real hyperplanes.

Lemma 4.2. *Let $U_i \subseteq W^*$, $i \in S$, be two-dimensional real subspaces, where $r = |S|$. Suppose*

$$\dim \sum_{i \in T} U_i \geq |T| + 2 \quad (T \subseteq S, |T| \geq 2).$$

For $r = 1$ suppose $\dim W \geq 3$. A nonzero homogeneous complex polynomial vanishing on all the projective spaces $\mathbb{P}(U_i^{\perp})$ has degree at least r.

Proof. Choose a generic real point $p \in \mathbb{P}(W)$ outside these spaces and put $K_0 = p^{\perp}$. The lines $\ell_i = U_i \cap K_0$ span a space R_p of dimension r: in (4.4), singleton blocks have total cost r, whereas every larger block has cost at least its cardinality plus one. Thus $T_p = R_p^{\perp}$ contains p and has dimension at least two. Every projective line through p in $\mathbb{P}(T_p)$ meets each $\mathbb{P}(U_i^{\perp})$, since the two forms in U_i restrict dependently and are not both zero at p.

These intersections can be chosen distinct. For $i \neq j$ put $Z_{ij} = (U_i + U_j)^{\perp}$ and $W_{ij} = \langle p, Z_{ij} \rangle$. A line through p meets both boundary spaces at the same point precisely when it lies in $\mathbb{P}(W_{ij})$; if $Z_{ij} = 0$ there are no such lines. Apply (4.4) to the family with U_i, U_j replaced by $U_i + U_j$. A partition block containing this merged space and k other members has cost at least $k + 3$, two more than its number of members; all other blocks have cost at least their number of members. Therefore

$$\dim \left((U_i + U_j) \cap K_0 + \sum_{k \neq i,j} \ell_k \right) \geq r + 1.$$

It follows that $T_p \not\subseteq W_{ij}$. Choose a second point in $\mathbb{P}(T_p)$ outside all these finitely many proper subspaces. The resulting line has r distinct prescribed zeros of the polynomial. A polynomial of degree less than r must vanish identically on that line, hence at p. Since such p form a nonempty real Zariski-open set, the polynomial is zero, even with complex coefficients. This is a contradiction. □

Proposition 4.3. *For a real off-diagonal matrix C on $n \geq 2$ vertices, $H_C(t,x)$ is absolutely irreducible if and only if there is no proper nontrivial outgoing module. Signs and zeros are allowed. The same statement holds when each target vertex carries several independent variables and each incoming weight is a real linear form in that vertex's variables.*

Proof. For the ordinary target variables put

$$\phi_i = t + \sum_{j \neq i} C_{ij} x_j, \qquad U_i = \operatorname{span}\{x_i, \phi_i\} \subseteq (\mathbb{R}^{n+1})^*.$$

For $T \subseteq V$, quotienting by the independent forms x_i, $i \in T$, gives

$$\dim \sum_{i \in T} U_i = |T| + \operatorname{rank}[(1, (C_{ij})_{j \notin T})]_{i \in T}. \tag{4.5}$$

For $2 \leq |T| < n$, the last rank is one exactly when T is a module. If there is no proper nontrivial module, Lemma 4.2 applies to every proper set of indices. A polynomial vanishing on the corresponding $|T|$ boundary spaces has degree at least $|T|$.

Expanding the determinant in column i at $x_i = 0$ shows

$$H_C|_{x_i=0} = \phi_i H_{C[V \setminus \{i\}]}. \tag{4.6}$$

Thus H_C, homogeneous of degree $n-1$, vanishes on every $\mathbb{P}(U_i^{\perp})$. Suppose it were a product of at least two nonconstant homogeneous complex factors. Each boundary space, being irreducible, lies in the zero set of one factor; assign it to such a factor. A factor assigned all n spaces has degree at least $n-1$, leaving no room for another factor. Otherwise every nonempty assigned set is proper, and the sum of the factor degrees is at least the sum of their assigned cardinalities, namely n. Both alternatives are impossible. For $n = 2$, H_C is monic linear.

Conversely, for a proper module M, vectors constant on M form an invariant subspace for all $L_C(x)$. Modulo the constant vector this subspace has dimension $n - |M|$, strictly between zero and $n-1$. An adapted basis makes the quotient pencil block triangular, so its determinant H_C factors into monic homogeneous polynomials of positive degrees $n - |M|$ and $|M| - 1$.

For the multiple-target statement, let the variables at vertex B be x_j, $j \in B$, and the edge from A to B be $\sum_{j \in B} \beta_{A,j} x_j$. A module requires equality of every coefficient $\beta_{A,j}$ toward outside vertices. If there is no module, specialize $x_j = \xi_j z_B$ with positive real ξ_j outside finitely many hyperplanes. Every failed module equality still fails, so the ordinary theorem gives an absolutely irreducible specialized polynomial. Any factorization before specialization would have homogeneous factors monic in t, each of positive t-degree; no such factor can become a unit under this specialization. This is impossible. The invariant subspace argument proves the converse without specialization. □

An affine change $t \mapsto t + \ell(x)$ preserves absolute irreducibility. Dehomogenizing an irreducible homogeneous polynomial monic in t preserves irreducibility unless that polynomial is t. Indeed, an affine factorization homogenizes to a homogeneous factorization after multiplication by a power of t.

4.2. **The forest of minimal blocks.** Distinct minimal nontrivial modules intersect in at most one vertex: the intersection and the union of two intersecting modules are modules. To see the intersection assertion, for a target outside the intersection use whichever of the two modules excludes the target. For the union assertion compare each row with a row in the intersection at every outside target.

Proposition 4.4. *Fuse the pair modules into their maximal pair cliques, and retain every minimal nontrivial module of cardinality at least three. The incidence graph of these blocks and their original vertices is a forest. Every subset of a pair clique is itself an outgoing module.*

Proof. Suppose a minimal module M with $|M| \geq 3$ lay on an incidence cycle. Choose a shortest path outside its block node joining two vertices $p, q \in M$. No other vertex of M belongs to a block on this path, since that would shorten it. For each target $k \in M \setminus \{p, q\}$, propagate equalities of rows along the path. For each target outside M, use the module M. Thus p, q agree toward every target outside $\{p, q\}$, so $\{p, q\}$ is a proper nontrivial submodule of M, a contradiction.

Now form the graph whose edges are pair modules. In a biconnected block, any two vertices admit a path avoiding a prescribed third target vertex. Propagating the edge-module equalities along that path shows that the two vertices form a pair module. Hence every biconnected block is a clique. Replacing these cliques by block nodes removes the pair cycles. Any remaining cycle would either expand to one of the excluded cycles through a larger minimal module or to a cycle in the block-incidence graph of the pair graph. Both are impossible. The last statement follows directly from the equalities for all pairs of a clique. □

Call the blocks of cardinality at least three which are minimal modules *proper-type blocks*; this describes their core type and does not exclude a block equal to V. Maximal pair cliques, including pairs not in a larger clique, are *clique-type blocks*. For a block B write

$$a_B = 1 + \sum_{j \notin B} C_{Bj} x_j,$$

where C_{Bj} is the common outside row. Define

$$\begin{aligned} P_B &= H_{C[B]}(a_B, x_B), \quad B \text{ of proper type}, \\ P_B &= a_B + \sum_{i \in B} d_i x_i, \quad B \text{ of clique type}, \\ q_B &= P_B / \prod_{i \in B} x_i. \end{aligned} \tag{4.7}$$

Here $C_{ji} = d_i$ for $j \in B \setminus \{i\}$ in a clique. In the first line $\deg P_B = |B| - 1$, and in the second it is one. The first numerator is absolutely irreducible by minimality and Proposition 4.3. The second is affine linear. All have constant term one and nonnegative coefficients.

Put $\delta_B = |B| - 1$ for proper-type blocks and $\delta_B = 1$ for clique-type blocks. A set J is *admissible* if

$$|J \cap B| \leq \delta_B \quad \text{for every block } B. \tag{4.8}$$

Equivalently, it contains no nontrivial outgoing module. In particular, V is never admissible for $n \geq 2$. For $h \notin J$ define

$$\begin{aligned} d_h(J) &= \#\{B \ni h : |J \cap B| = \delta_B\}, \qquad G(J) = \prod_{h \notin J} x_h^{\max(d_h(J)-1,0)}, \\ p_J &= Y_J / G(J), \qquad p_\varnothing = 1. \end{aligned} \tag{4.9}$$

The next boundary calculation proves that these are the required canonical coordinates when all weights are strictly positive.

4.3. **Exact boundary valuations.** The following elementary fact prevents cancellations in the signed matrices which arise on an exchange hyperplane.

Lemma 4.5. *Let A have arbitrary real off-diagonal entries and give the edge $i \to j$ weight $A_{ij} x_j$. Over $\mathbb{R}(x)$, zero is a semisimple eigenvalue of L_A whose multiplicity is the number of terminal strongly connected components of its support. The assertion remains true with several independent target variables at each vertex.*

Proof. First consider a strongly connected support. The coefficient of t in $\det(tI + L_A)$ is the sum of all rooted spanning tree weights. This sum is nonzero even when the entries of A have different signs. Choose a vertex r reachable from every vertex, and order the targets by increasing shortest distance to r, breaking ties. Each vertex other than r chooses its earliest outgoing neighbor. Its distance strictly decreases, so these choices give a tree toward r.

This tree has a unique lexicographically maximal target-indegree vector among trees with every possible root. Indeed, process the targets in the chosen order. Before processing a target h, the previously forced edges form one component containing r, and all other vertices are isolated. The vertex h already belongs to the component of r, except at the first step. Every isolated vertex with an edge to h can attach to h simultaneously. The only vertex of the existing component without a chosen outgoing edge is r, and $r \to h$ would close a cycle. Consequently the greatest possible indegree at h forces precisely those attachments. Induction proves uniqueness of the exponent vector and of the tree with that vector. Its nonzero coefficient cannot cancel. Thus a terminal strongly connected block has algebraically simple zero; the one-vertex case gives the same conclusion directly.

A nonterminal strongly connected diagonal block is invertible. Choose an outside target k receiving an edge from it. The variable x_k occurs only on its diagonal. Its highest power in the determinant has coefficient equal to the product of the nonzero coefficients A_{ik} times the principal determinant obtained by removing all such rows and columns. The removed vertices are now outside targets. If any rows remain, strong connectivity supplies an edge from a remaining vertex to a removed vertex. Repeat the same highest-power extraction. It eventually produces a nonzero product of edge coefficients, proving invertibility.

Order the strongly connected components so that the whole matrix is block triangular. Its algebraic zero multiplicity is the number q of terminal components. A kernel vector may have an independent constant value on each terminal component; the invertible nonterminal blocks then determine all other values uniquely. Its nullity is therefore q, proving semisimplicity. For multiple target variables, specialize each target group to a generic positive ray. Avoiding finitely many nonzero linear forms preserves the support. The just-proved nonzero coefficients and rank minors after specialization are nonzero before specialization. The zero row sums and the block structure give the reverse rank bounds, so the conclusion is unchanged. □

Write $F_i^h = t + \sum_{j \neq i} C_{ij} x_j$. For an arbitrary nonempty $J \subseteq V$ and an arbitrary original index i, form a signed system on $J \cup \{*\}$. The target group at $*$ consists of all x_j with $j \notin J$. Put $r_j = C_{ij}$, with $r_i = 0$, and set

$$A_{vj} = C_{vj} - r_j \quad (v \in J,\ j \text{ outside the target block of } v), \qquad A_{*j} = -r_j \quad (j \in J). \tag{4.10}$$

Coefficients into a target group are summed against that group's independent variables. An empty target group is allowed.

Lemma 4.6. *With $s = F_i^h$, the preceding signed system satisfies*

$$D_J(t, x) = \frac{\det(sI_{|J|+1} + L_A(x))}{s}. \tag{4.11}$$

Let $q_i(J)$ be the number of terminal strongly connected components of this system after setting $x_i = 0$. Then

$$\nu_i(D_J(1, x)) = q_i(J) - 1 = \operatorname{ord}_{F_i^h}\big(D_J(t, x)|_{x_i=0}\big). \tag{4.12}$$

For every irreducible homogeneous factor P of $D_J(t, x)$,

$$\nu_i(P(1, x)) = \operatorname{ord}_{F_i^h}\big(P|_{x_i=0}\big). \tag{4.13}$$

Factors which become units after dehomogenization contribute zero. Here each initial F_i is assumed nonconstant.

Proof. Start with the system on $J \cup \{*\}$ in which $*$ has zero row and carries the outside target variables. Its full determinant divided by t is D_J. Adjoin a source p, with no incoming edges and with row weights $r_j x_j$. Its full determinant divided by t is sD_J. Alternatively pass to the quotient by the constant vector, using representatives with p-coordinate zero. Subtract the p-output from every other output. The off-diagonal entries become $-(C_{vj} - r_j)x_j$, while the diagonal entries increase by $r_v x_v$. The reduced pencil is exactly $sI + L_A$. This proves (4.11).

Treat s, x_i as independent normal variables, over the rational function field in the remaining x_j. Write $L_A = B_0 + x_i B_1$. By Lemma 4.5, B_0 has semisimple zero of nullity $q = q_i(J)$. Constant invertible row and column operations put B_0 into an invertible block and a zero block of size q. Every term of $\det(sI + B_0 + x_i B_1)$ has normal degree at least q. On $x_i = 0$ this determinant is $s^q g(s)$ with $g(0) \neq 0$. After division by s, the order is exactly $q - 1$, and equals the boundary multiplicity in (4.12). The affine chart $t \neq 0$ meets the generic boundary $x_i = F_i^h = 0$ because row i is nonzero. Localization at t therefore preserves this order, and Lemma 2.2 identifies it with ν_i.

For each factor P, the order along (x_i, F_i^h) is at most the F_i^h-order after setting $x_i = 0$. The restriction is nonzero, since D_J is monic in t and has no coordinate factor. Both orders are additive over factors. Their sums are equal for D_J, as just proved, so equality holds for each factor separately. This gives (4.13). □

For a full quotient determinant there is also the smaller identity

$$H_C(t, x) = \det\left(F_i^h I_{n-1} + L_{A'}(x_{V\setminus\{i\}}) + x_i \operatorname{diag}(C_{ji})_{j\neq i}\right), \qquad A'_{jk} = C_{jk} - C_{ik}. \tag{4.14}$$

It follows by passing to the constant-vector quotient using the i-coordinate. The same proof shows that its order along (x_i, F_i^h) is the number of terminal components of A'.

Proposition 4.7. *Suppose $C_{ij} > 0$ for $i \neq j$. For every nonempty admissible J, $D_J(1, x)$ is absolutely irreducible and*

$$\nu_i(D_J(1,x)) = \begin{cases} 1, & i \in J, \\ \max(d_i(J) - 1, 0), & i \notin J. \end{cases} \tag{4.15}$$

For every block B the numerator P_B is absolutely irreducible, and its valuations are one at the members of B and zero elsewhere. Consequently the functions p_J and q_B of (4.7) and (4.9) are prime elements of the upper algebra. They have nonnegative Laurent coefficients in the initial cluster and are strictly positive at every positive initial point.

Proof. Adjoin a zero-row vertex carrying all outside target variables to the system induced on J. Its full quotient determinant is $D_J(t, x)$. A proper nontrivial module avoiding the new vertex would be a global module contained in J, excluded by admissibility. A module containing the new vertex would contain a nonempty proper subset T of J and force $C_{vk} = 0$ for $v \in T$, $k \in J \setminus T$. Strict positivity excludes this. Proposition 4.3 and dehomogenization prove the asserted irreducibility.

For a fixed i, consider the difference graph on $V \setminus \{i\}$ with edge $v \to j$ exactly when $C_{vj} \neq C_{ij}$. Let $m_i(J)$ count its terminal components contained in $J \setminus \{i\}$. In the shifted system (4.10), the vertex $*$ reaches every retained original vertex because $-C_{ij} \neq 0$. A terminal component avoiding $*$ is exactly one of the $m_i(J)$ components just counted: an edge to an outside target appears as an edge to $*$, without cancellation among independent target variables. If $m_i(J) > 0$, these are all terminal components; if $m_i(J) = 0$, the unique terminal component contains $*$. When $i \in J$, the vertex i is additionally isolated after $x_i = 0$. Hence Lemma 4.6 gives

$$\nu_i(D_J) = \begin{cases} \max(1, m_i(J)), & i \in J, \\ \max(0, m_i(J) - 1), & i \notin J. \end{cases}$$

A terminal component T of the difference graph gives the module $T\cup\{i\}$. It is minimal among nontrivial modules containing i, since a smaller such module would give a proper nonempty outgoing-closed subset of the strongly connected set T. If $i\in J$, admissibility thus forces $m_i(J)=0$. If $i\notin J$ and $T\subseteq J$, any smaller module avoiding i would lie in J, also impossible. Thus $T\cup\{i\}$ is globally minimal. Proper-type blocks contribute exactly when their other $|B|-1$ vertices belong to J; a fused pair clique contributes exactly when its unique permissible member in J is present. Conversely each such ready block gives a terminal component. Therefore $m_i(J)=d_i(J)$ for $i\notin J$. This proves (4.15).

For a proper-type block, irreducibility follows from its minimality and Proposition 4.3, as already observed. The row-i difference graph on $B\setminus\{i\}$ is strongly connected: a proper terminal component would give a smaller nontrivial module of B, hence a global submodule. Formula (4.14) gives valuation one at $i\in B$. For an outside i, setting $x_i=0$ merely removes a term from the affine root a_B. The resulting homogeneous polynomial remains absolutely irreducible of degree $|B|-1\geq 2$, by an invertible translation of t. It cannot have the linear factor F_i^h. Since P_B divides D_B, equation (4.13) gives outside valuation zero.

For a clique, $P_B|_{x_i=0}=F_i$ at every member i, giving valuation one. At an outside vertex h, a positive valuation would require equality $P_B|_{x_h=0}=F_h$, since both affine polynomials have constant term one. Together with the clique equalities, this would make $\{h,i\}$ a pair module for every $i\in B$, contradicting maximality. Thus the outside valuations vanish. Lemma 2.2 now proves primality of the displayed canonical lifts. Their initial positivity follows from the forest expansion. □

5. Finite candidate charts for a positive coefficient matrix

In this section all off-diagonal entries of C are strictly positive. Let $\mathcal{B}$ be the block family of Proposition 4.4. Thus a block is either a minimal outgoing module of cardinality at least three, called a proper block, or a maximal clique of outgoing pairs. A two-element block is of the second kind. A proper block is allowed to be V. Set

$$\delta_B=\begin{cases}|B|-1, & B \text{ proper},\\ 1, & B \text{ a clique}.\end{cases}$$

We use the admissible sets, coordinates p_J,q_B, and gauges $G(J)$ of Proposition 4.7. A vertex $h\notin J$ is *ready at* J if $d_h(J)>0$. In particular readiness records all blocks, whether or not their cores occur in the chart under consideration.

Definition 5.1. A candidate consists of a set $O\subseteq V$, a set $\mathcal{E}\subseteq\mathcal{B}$, and a strictly increasing chain of nonempty admissible sets

$$I_1\subsetneq\cdots\subsetneq I_s.$$

Its proposed coordinates are

$$\{x_i : i\in O\}\ \cup\ \{q_B : B\in\mathcal{E}\}\ \cup\ \{p_{I_1},\dots,p_{I_s}\}. \tag{5.1}$$

The data must admit the following sequence of events. Initially put $R=O$ and $J=\varnothing$. At event j put $A=I_j\setminus J$ and choose a previously unused subset $\mathcal{E}_j\subseteq\mathcal{E}$. Require:

(1) $A\cap R=\varnothing$;
(2) every $B\in\mathcal{E}_j$ has exactly two members outside R, say a,h, with $a\in A$, $h\notin A$; if B is proper then $B\setminus\{a,h\}\subseteq J$, while if B is a clique then $B\setminus\{a,h\}\subseteq R\setminus J$;
(3) the pairs $\{a,h\}$, indexed by $B\in\mathcal{E}_j$, form a tree T whose one complete bipartition class is A; when $\mathcal{E}_j=\varnothing$, the tree is a singleton A;
(4) every vertex of T is unready at J.

Replace R by $R \cup V(T)$ and J by I_j. At the end require $R = V$ and $\mathcal{E} = \coprod_j \mathcal{E}_j$. The vertices in the other bipartition class of an event are called its omitted vertices. An omitted vertex of an earlier event is called an old omitted vertex. The members of O are retained originals.

This definition specifies finite combinatorial data. It does not assume that the proposed coordinates already form an LP cluster. That fact will follow from the identities in Section 7.

Lemma 5.2. *Every candidate has exactly n coordinate labels. At every intermediate stage every unrecovered vertex is unready. Every old omitted vertex is ready. There are finitely many candidates, and every coordinate p_J with J admissible, and every q_B, occurs in a candidate.*

Proof. An event with e edges recovers $e+1$ vertices and contributes e core coordinates and one principal coordinate. Summing this equality gives $|O| + |\mathcal{E}| + s = n$. For a vertex which is still unrecovered, its readiness count is nondecreasing as the admissible prefix increases. Were it ready now, it could not satisfy the unready condition at the later event which recovers it. This proves the assertion about unrecovered vertices. An omitted vertex is incident to an event edge. At the end of that event this block becomes ready there, and stays ready subsequently.

Finiteness follows from the finiteness of V, $\mathcal{B}$, and the strict chains of subsets of V. To obtain a candidate containing p_J, retain $V \setminus J$ and add the vertices of J one at a time as singleton events. A vertex of J cannot become ready before its own insertion: a ready proper block would be contained in J, and a ready clique would have two members in J, contrary to admissibility. For a proper core q_B, retain $V \setminus B$, choose two members $a, h \in B$, insert the members of $B \setminus \{a, h\}$ as singleton events, and finish with the residual edge ah of B, selecting a. Minimality of B, together with the incidence forest, verifies readiness and admissibility. For a clique core retain $V \setminus \{a, h\}$ and use its residual edge as the single event. □

5.1. **The gauge of an event.** Consider an event $T = A \sqcup B$ after a prefix J; here B denotes the omitted colour, not a block. Put

$$\Delta_A = \prod_{b \in B} x_b^{\deg_T(b)-1}.$$

For an old omitted vertex h, count each block which becomes ready at h during the event once, and let H_A be the product of the corresponding occurrences of x_h. Products count occurrences, including when two factors are equal as rational functions. Put

$$m_A = \prod_{z \in O} x_z^{\max(d_z(J \cup A)-1,0)-\max(d_z(J)-1,0)}.$$

Lemma 5.3. *Writing $K^J = Z/Z_J$ for the Schur complement, one has*

$$\frac{G(J \cup A)}{G(J)\Delta_A} = H_A m_A, \qquad U_A := \frac{\det K^J[A,A]}{\Delta_A} = H_A m_A \frac{p_{J \cup A}}{p_J}. \tag{5.2}$$

If two residual-disjoint events can be performed in either order, with selected sets A, D, then

$$X(J; A, D) = \frac{G(J)G(J \cup A \cup D)}{G(J \cup A)G(J \cup D)} \tag{5.3}$$

is a monomial with nonnegative exponents in retained originals and old omitted originals.

Proof. A current omitted vertex b is initially unready. Precisely its $\deg_T(b)$ incident event blocks become ready during this event. An additional block meeting the connected event in another vertex would make a cycle in the incidence forest. Its contribution to the gauge is therefore $x_b^{\deg_T(b)-1}$. Every old omitted vertex already has readiness count at least one, so each newly ready block adds exactly one to its gauge exponent. These are precisely the factors of H_A. Retained originals contribute m_A. No unrecovered exterior vertex can contribute, by Lemma 5.2. These

observations prove the first identity. The Schur determinant identity $Y_{J\cup A} = Y_J \det K^J[A, A]$ proves the second.

For the cross gauge fix an exterior vertex v and write the four readiness counts as

$$d_0, \quad d_0 + a, \quad d_0 + b, \quad d_0 + a + b + c, \qquad a, b, c \geq 0.$$

The last number includes blocks requiring selected vertices from both increments. No block is counted in both a and b: for proper blocks the last missing selected set is nonempty, and a clique cannot meet both increments in an admissible union. With $e(d) = \max(d - 1, 0)$, its exponent in X is

$$e(d_0 + a + b + c) + e(d_0) - e(d_0 + a) - e(d_0 + b) \geq 0,$$

by monotonicity and convexity of e. A vertex in $A \cup D$ has zero relevant readiness count before its insertion, since the final union is admissible. It contributes no negative exponent. A current omitted vertex cannot gain a block from the opposite event when the order is reversible. A block requiring vertices from both events and also meeting such an omitted vertex would meet its own event module twice, contradicting the incidence forest. Unrecovered exterior vertices again cannot become ready. Thus only the stated shared originals occur. □

5.2. **Local changes of candidates.** The union of the full blocks of one event is an outgoing module. For clique blocks their already recovered omitted members may be removed when forming the corresponding residual module. The incidence subtree of an event is connected. Distinct events have disjoint block nodes, and their full incidence subtrees meet in at most one original vertex. Indeed, two common vertices would force their entire joining path, and hence a block node, to be common. This observation also applies to connected branches cut from events.

Proposition 5.4. *In each coordinate direction of a candidate there is a local change to another candidate which keeps the other $n - 1$ coordinates. The changes are precisely the following ones.*

(1) *A selected core is removed by cutting its residual edge.*
(2) *Two consecutive residual-disjoint events are interchanged; if their reversed order is blocked by an unselected block, the two events are merged and the core of that block is inserted instead of their intermediate principal.*
(3) *Consecutive events meeting at a newly selected vertex undergo a proper-block three-port rotation.*
(4) *Consecutive events meeting at a newly omitted vertex undergo a clique three-port rotation.*
(5) *The last nontrivial event changes colour; a last singleton is replaced by its original variable.*
(6) *A retained original is inserted at its first readiness event, using an inverse edge cut or an inverse clique rotation; if it never becomes ready, it is inserted as a final singleton.*

All candidates are connected to the all-original candidate by these changes.

Proof. We describe the changes and check the conditions of Definition 5.1.

First cut an event edge ah, with a selected and h omitted, and let W be its h-side. Process W before its complement, using their inherited colours. If $I = A \cap W$ is nonempty, the new coordinate is $p_{J\cup I}$; if $I = \varnothing$, then $W = \{h\}$ and the new coordinate is x_h, which is retained from the outset. All unchanged proper blocks retain their earlier selected interiors; all unchanged cliques retain their earlier unselected interiors. The cut block cannot become ready on the second side at the intermediate prefix: its selected endpoint a has not yet entered. Another new obstruction between the two sides would give a second incidence path between them. Thus both events are valid. Treating h as retained in the singleton case changes no earlier used block, since h was unrecovered at every earlier stage.

Next consider two consecutive events T_1, T_2 after J. Suppose first that their residual vertex sets are disjoint and that the interiors of T_2 do not use a newly recovered vertex of T_1. If their reversed order satisfies readiness, it gives the new intermediate coordinate $p_{J\cup A_2}$. If readiness fails, some omitted vertex h of T_1 becomes ready through a selected vertex a of T_2. The vertex h cannot be selected, because the final selected union is admissible. There is exactly one obstructing block: two would give two paths between the event subtrees. For a proper block all its other members lie in J. For an unselected clique all other members have already been recovered unselected or are retained. Otherwise an additional unrecovered clique member would become ready now and could never be processed later. The block is therefore an eligible residual edge ah. Insert its core and merge the two trees. This is the inverse of the preceding cut. If the obstructing clique was already selected, it is used at the second event and belongs to the clique rotation described below; no additional copy of its core is inserted.

There remains the case in which an interior of T_2 uses a new vertex v of T_1. Their full modules have at most one common vertex, so there is only one such vertex and only one block of T_2 containing it. If v was selected, this block must be proper: a used clique has no earlier selected member. Before both events the central proper block has three residual ports v, ℓ, h, with v, ℓ selected and h omitted. Removing the central block leaves branches V, L, H at these ports. The old order is V, then $L \cup H$ joined by ℓh; the new order is L, then $V \cup H$ joined by vh. The new intermediate principal is the one for the selected colour of L. The final prefix and recovered union are unchanged. The central block retains the same earlier selected interior, and another readiness obstruction would create a second incidence connection between the old event subtrees. Both new events therefore satisfy the definition.

If v was omitted, the central block must be a clique, because a proper used block has only selected earlier interior members. Write its three residual ports as a, b, u, with a selected and b, u omitted, and let A, B, U be the three attached branches. The old order is B, then $A \cup U$ joined by au; the new order is U, then $A \cup B$ joined by ab. The clique core is shared. The new intermediate coordinate is the U-branch principal, or x_u if $U = \{u\}$. The final selected union is unchanged, and the same incidence-path argument verifies readiness. If both omitted branches are singletons this is a change between the retained originals x_b and x_u. Empty selected branches are treated as originals, not as a coordinate p_J.

For the last nontrivial event interchange its two colours. An exterior block made inadmissible by the new colour could meet the connected event at only one vertex; it would already have made that vertex ready before the event, a contradiction. Proper earlier interiors and clique earlier interiors are unchanged. A last singleton simply returns its original variable.

Finally let $u \in O$. If it is never ready, remove it from O and append it as a singleton. Otherwise consider its first ready event. Exactly one completing block joins u to a selected terminal of that event; two would form a cycle. A proper such block is not already selected, since a selected proper block has no retained-original interior. An unselected clique is treated in the same way. Insert its core and attach u as an omitted leaf, using the inverse cut. If the clique is already selected, keep u as its third residual port and use the inverse clique rotation. Before this first ready event no used block contains u, so removing u from O leaves all earlier events valid. These cases cover every coordinate of (5.1).

To prove connectedness, cut selected cores successively. Each cut reduces their number by one and preserves the candidate conditions. When none remains, all events are singletons. Successive last-singleton removals give the all-original candidate. □

6. Source functions and positive elimination

Throughout this section all off-diagonal entries of C are strictly positive. We fix a candidate of Definition 5.1, and process its events in their prescribed order. At a given stage J is the

selected prefix. An original variable recovered at an earlier event but not selected is called an *old omitted variable*; we write its value as $h = x_h$ when no confusion with its index is possible. The retained original variables are invertible coordinates throughout the calculation. All statements of positivity below mean coefficientwise Laurent nonnegativity over K.

More precisely, after the first ℓ events let $\mathcal{P}_\ell$ be the semiring generated by $K_{\geq 0}$, the retained original coordinates and their inverses, the selected core coordinates and their inverses, and $p_{I_1}^{\pm 1}, \dots, p_{I_\ell}^{\pm 1}$. It is harmless to include the cores of events not yet processed in this definition; the expressions constructed before such an event do not use those cores. An assertion that f has a positive quotient by a product H means that an equality $f = Hf_0$ with $f_0 \in \mathcal{P}_\ell$ is constructed. It does not mean only that H divides a positive expression for f.

6.1. **Decorated source modules.** Delete every block which is ready at J, whether or not its core occurs in the fixed candidate. An unready proper block is kept with all its vertices. An unready clique has no selected vertex. Delete its retained original members and its old omitted members, but keep its block node whenever at least one residual member remains. For each deleted old omitted member h, put one mark with origin h at that clique node. If an old omitted vertex belongs to a surviving proper block, keep a mark at that vertex. Isolated marked vertices and isolated selected vertices may also be kept. Retained original members contribute invertible monomial factors, not marks.

The resulting decorated incidence graph is a forest. Two marks with the same origin lie in different connected components: their unique path in the original incidence forest used one of the incidences deleted when that origin was removed from a clique. Further deletion of ready blocks cannot reconnect them. In particular, a connected decorated subtree contains at most one mark with any specified origin.

A *source* is the vertex set N of a connected decorated subtree made of whole proper blocks and arbitrary nonempty residual subcliques. A singleton residual member may carry its decorated clique node. Singleton vertices, including vertices of J, are also sources. Every source is an outgoing module: subcliques are outgoing, and unions of intersecting outgoing modules are outgoing. The decoration is part of the source data, although two decorations can have the same vertex set and hence the same rational source functions. Let $\mathfrak{m}(N)$ be the product of its marked original values. For an old omitted target h, let $\mathfrak{m}_h(N)$ be the same product with the occurrence of origin h removed, if that occurrence is present.

For a source N put $R_N = J \setminus N$, and define

$$q_N^{\mathrm{src}} = \det(Z_{R_N} + \mathbf{1}C_{N,R_N}), \tag{6.1}$$

$$P_{Nj}^{\mathrm{src}} = C_{Nj}Y_{R_N} + C_{N,R_N}\operatorname{adj}(Z_{R_N})C_{R_N,j}, \qquad j \notin N \cup J, \tag{6.2}$$

$$r_N = q_N^{\mathrm{src}}/Y_J, \qquad b_{Nj} = P_{Nj}^{\mathrm{src}}/Y_J. \tag{6.3}$$

The common row $C_{N,k}$ is well defined for $k \notin N$. Empty determinants are 1. The target restriction in (6.2) permits all live exterior targets and all old omitted targets, including a deleted clique member whose mark occurs in the decoration of N. For nested decorated sources $N \subseteq H$ set

$$\Phi_N^H = \det(Z_{J\cap(H\setminus N)} + \mathbf{1}C_{N,J\cap(H\setminus N)}). \tag{6.4}$$

Containment here includes the decorations. Adjoining a decorated singleton node can change the required mark product without changing the determinant in (6.4). We write z_i^J for the ith diagonal entry of $K^J = Z/Z_J$; its off-diagonal (i,j) entry is $-k_{ij}$.

Lemma 6.1 (Source factorization)**.** *For nested sources $N \subseteq H$ one has*

$$r_N = \Phi_N^H r_H, \qquad b_{Nj} = \Phi_N^H b_{Hj} \quad (j \notin H \cup J). \tag{6.5}$$

If $i \in H \setminus J$ and

$$f_i^H = \det(Z_{J\cap H} + \mathbf{1}C_{i,J\cap H}),$$

then the effective root and off-diagonal coefficients after eliminating J satisfy

$$a_i := 1 + C_{i,J}Z_J^{-1}\mathbf{1} = f_i^H r_H, \qquad k_{ij} := C_{ij} + C_{i,J}Z_J^{-1}C_{J,j} = f_i^H b_{Hj} \quad (j \notin H \cup J). \tag{6.6}$$

These are factorizations of whole products. In particular, all factors coming from several attached source modules can be extracted in one application of the same identity.

Proof. Order R_N as $S \sqcup R_H$, where $S = J \cap (H \setminus N)$. In $Z_{R_N} + \mathbf{1}C_{N,R_N}$ the block from S to R_H is zero: its (s,k) entry is $-C_{sk} + C_{Nk} = 0$ because $s \in H$ and $k \notin H$. The two diagonal blocks are the matrix defining Φ_N^H and the matrix defining q_H^{src}. Taking determinants proves $q_N^{\mathrm{src}} = \Phi_N^H q_H^{\mathrm{src}}$.

For the target identity use the bordered determinant

$$P_{Nj}^{\mathrm{src}} = \det \begin{pmatrix} Z_{R_N} & -C_{R_N,j} \\ C_{N,R_N} & C_{Nj} \end{pmatrix}. \tag{6.7}$$

Add the last row to each row indexed by S. The S-to-R_H block and the target-column entries in those rows become zero, again by the outgoing-module equalities. The remaining determinant is block triangular and gives $P_{Nj}^{\mathrm{src}} = \Phi_N^H P_{Hj}^{\mathrm{src}}$. Dividing by Y_J proves (6.5). Taking $N = \{i\}$ proves (6.6); its numerator is the determinant lemma applied to Z_J, or the corresponding bordered determinant. If several modules are attached in an incidence subtree, order their selected interiors from the leaves toward the source. The same row additions produce all their diagonal factors at once. This proves the last assertion, with multiplicity even when different labeled factors happen to be equal as functions. □

For an earlier selected singleton $i \in J$ the definitions have the useful interpretation

$$r_{\{i\}} = e_i^{\mathsf{T}} Z_J^{-1}\mathbf{1}, \qquad b_{\{i\},j} = e_i^{\mathsf{T}} Z_J^{-1} C_{J,j} \quad (j \notin J). \tag{6.8}$$

Indeed, eliminate $J \setminus \{i\}$ and apply the one-by-one Schur complement formula. These are the root and exterior-target coefficients in the recovery of x_i. Thus selected singleton sources require no duplicate matrix row.

6.2. **The algebra of a tree event.** We first record an elementary identity without a selected prefix. Suppose a tree $T = A \sqcup B$ has an outgoing pair module at every edge. Write its coefficient matrix as c, let ρ be a common affine root, and put

$$z_i = \frac{\rho + \sum_{j\neq i} c_{ij}x_j}{x_i}, \qquad M = \mathrm{diag}(z_i) - c.$$

Exterior variables, if present, are included in ρ and remain formal. For an edge $e = ab$ its pair core is

$$s_e = \frac{\rho + \sum_{i\in A} c_{bi}x_i + \sum_{j\in B} c_{aj}x_j}{x_a x_b}.$$

Choose monomials v_i in the $s_e^{\pm 1}$ with $s_e v_a v_b = 1$ and one selected-root value $v_a = 1$. Put

$$\ell_i = \sum_{j \text{ of opposite colour}} c_{ij}v_j, \quad R_0 = \sum_{a\in A, b\in B} c_{ab}c_{ba}v_a v_b, \quad \kappa_A = \frac{1}{\prod_{a\in A} v_a \prod_{b\in B} v_b^{\deg_T(b)-1}}.$$

Lemma 6.2 (Pair-tree identity)**.** *There is a rational function θ such that*

$$x_b = v_b(\theta + \ell_b), \qquad b \in B, \tag{6.9}$$

$$x_a = v_a\left(\frac{\rho + R_0}{\theta} + \ell_a\right), \qquad a \in A, \tag{6.10}$$

$$z_a = \frac{\theta + \sum_{c\in A\setminus\{a\}} c_{ac}v_c}{v_a}, \qquad a \in A. \tag{6.11}$$

Moreover

$$\frac{\det M[A,A]}{\prod_{b\in B} x_b^{\deg_T(b)-1}} = \kappa_A \theta. \tag{6.12}$$

All identities remain valid when the tree has a single selected vertex and no edge, with $B = \varnothing$, $v = \kappa_A = 1$, *and* $\theta = z_a$.

Proof. Set $\xi_i = x_i/v_i$. If $a, a' \in A$ are both adjacent to b, subtract the core equations for ab and $a'b$. For every target $j \in B \setminus \{b\}$, the equalities along $a - b - a'$ give $c_{aj} = c_{a'j}$. After cancellation of the nonzero ξ_b, the difference is

$$\xi_a - \xi_{a'} = (c_{ab} - c_{a'b})v_b = \ell_a - \ell_{a'}.$$

The distance-two graphs on the two colour classes are connected. Consequently $\xi_a - \ell_a = t_A$ for all $a \in A$ and $\xi_b - \ell_b = t_B$ for all $b \in B$.

For a fixed edge ab, substitute these expressions in its core equation. The coefficients of t_A, t_B cancel. For $i \in A, j \in B$, the remaining coefficient identity is

$$(c_{bi} - c_{ji})(c_{ij} - c_{aj}) = 0. \tag{6.13}$$

To see it, if deleting i does not separate b from j, propagate the pair-module equalities along their path to obtain $c_{bi} = c_{ji}$. If deleting i does separate them, the path from a to i avoids j (with $i = a$ giving the equality directly), so $c_{ij} = c_{aj}$. Thus $t_A t_B = \rho + R_0$. Take $\theta = t_B$ to obtain (6.9)–(6.10). Dividing the core equation at ab by x_a gives $z_a = s_e x_b - c_{ba}$, which is (6.11).

Let L_A be the row Laplacian on A with target weights $c_{ac}v_c$. Equation (6.11) says $M[A,A]\,\mathrm{diag}(v_A) = \theta I + L_A$. For each $b \in B$, all rows indexed by neighbours of b agree off their own diagonals with the row $(-c_{bc}v_c)_{c\in A}$. Hence, for neighbours a, a' of b,

$$(e_a - e_{a'})^{\mathsf{T}} L_A = \ell_b (e_a - e_{a'})^{\mathsf{T}}.$$

For each b choose one neighbour as base and take the $\deg_T(b) - 1$ resulting differences. These differences are linearly independent: contract, for each b, its edge to the chosen base. The remaining edges form a tree on A, and their incidence vectors are precisely these differences. Deleting leaves of that tree proves their independence. Their total number is $|A| - 1$, since $\sum_{b\in B}(\deg_T(b) - 1) = |A| - 1$. They span the row subspace annihilating $\mathbf{1}$. As $L_A\mathbf{1} = 0$, an adapted basis gives

$$\det(\theta I + L_A) = \theta \prod_{b\in B} (\theta + \ell_b)^{\deg_T(b)-1}.$$

Divide by $\prod_{a\in A} v_a$, and use (6.9). This proves (6.12). The singleton assertions follow directly from $z_a x_a = \rho$. □

Return to an event after J. Let its residual tree have colours A, B. For a proper edge e put $S_e = J \cap B_e$ and

$$f_{e,i} = \det(Z_{S_e} + \mathbf{1}C_{i,S_e}) \quad (i \text{ a residual endpoint of } e).$$

For a clique edge put $f_{e,i} = 1$. The proper sets S_e are disjoint; an intersection would create a second path between two block nodes. Write $\pi_e(i)$ for the endpoint of edge e nearest i. Let T_i be the product of the ports nearest i over all event edges, and Λ_i its subproduct for which that nearest endpoint has the colour of i. The physical edge core is

$$s_e = q_{B_e} \quad \text{for a proper edge,} \qquad s_e = \gamma_e H_e q_{B_e} \quad \text{for a clique edge,} \tag{6.14}$$

where γ_e is the product of its deleted retained members and H_e is the product of its old omitted members. The latter equality is simply the common affine clique numerator divided by its two remaining original variables.

Let U be the source consisting of all proper event blocks and the residual pairs of the clique blocks, retaining their node decorations. It contains no old omitted vertex. Put $\delta = r_U$. Source factorization, ordering edge interiors away from the source vertex, gives

$$a_i = \delta T_i, \qquad T_i = f_{e,i} T_e \quad (i \in e), \tag{6.15}$$

where T_e is the product of the ports of all other edges pointing toward e. Indeed, the shifted matrix on J has diagonal blocks $Z_{S_e} + \mathbf{1}C_{\pi_e(i),S_e}$ and $Z_{J\setminus U} + \mathbf{1}C_{U,J\setminus U}$; its blocks pointing away from the source vanish. Thus its determinant is precisely $q_U^{\mathrm{src}} T_i$.

Lemma 6.3 (Residual-block scaling)**.** *Let a proper block D have any number of residual ports after eliminating J. Normalize the effective roots by $y_i = x_i/a_i$. The core of the normalized residual block is*

$$q_D^{\mathrm{norm}} = \frac{\det(Z_{J\setminus D} + \mathbf{1}C_{D,J\setminus D})}{Y_J} q_D. \tag{6.16}$$

For an edge of a two-port event this becomes $s_e^{\mathrm{norm}} = \delta T_e s_e$, also for a physical clique pair core.

Proof. Before elimination, write $E_D = 1 + \sum_{j\notin D} C_{Dj} x_j$. The equations $Z_D x_D = E_D \mathbf{1}$ and $\det Z_D = E_D q_D$ give $\operatorname{adj}(Z_D)\mathbf{1} = q_D x_D$. A rank-one shift therefore satisfies

$$\det(Z_D - \mathbf{1}h^{\mathsf{T}}) = q_D(E_D - h^{\mathsf{T}} x_D).$$

Eliminate $R = J \setminus D$ first. Because the rows of D agree toward R, its Schur correction is just such a rank-one shift. Its common root coefficient is $a_0 = \det(Z_R + \mathbf{1}C_{D,R})/Y_R$, and its core remains q_D. Write $Z_D^R = Z_D - C_{D,R} Z_R^{-1} C_{R,D}$ for this matrix. Its equation is $Z_D^R x_D = E'\mathbf{1}$, where

$$E' = a_0 + \sum_{j\notin D\cup J} (C_{Dj} + C_{D,R} Z_R^{-1} C_{R,j}) x_j.$$

Indeed, substituting the equations for x_R gives $E' = E_D - C_{D,R} Z_R^{-1} C_{R,D} x_D$. Now put $S = J \cap D$, $P = D \setminus J$, and $Q = Z_D^R[S,S]$. If $\widetilde{C}_{P,S} = -Z_D^R[P,S]$, the port matrix and its root multiplier are

$$N = Z_D^R[P,P] - Z_D^R[P,S] Q^{-1} Z_D^R[S,P], \qquad d = \mathbf{1} + \widetilde{C}_{P,S} Q^{-1} \mathbf{1}.$$

Block elimination gives $N x_P = E' d$ and

$$\det N = \frac{E' q_D}{\det Q}.$$

In particular $a_i = a_0 d_i$. The substitution $y_i = x_i/a_i$ changes the matrix to $\operatorname{diag}(a_P)^{-1} N \operatorname{diag}(a_P)$ and changes its equation to $\operatorname{diag}(a_P)^{-1} N \operatorname{diag}(a_P) y_P = (E'/a_0)\mathbf{1}$. Its determinant is still $\det N$, so its normalized core is $a_0 q_D / \det Q$. Since $Y_J = Y_R \det Q$, this is (6.16). In a two-port event the numerator in (6.16) factors as $q_U^{\mathrm{src}} T_e$ by the same triangularization as (6.15). A residual clique pair is itself an outgoing pair with core s_e, so the two-port argument applies to it as well. □

Proposition 6.4 (Event identities)**.** *Choose v_i with $s_e v_i v_j = 1$ on event edges and a selected root $v_i = 1$, and define*

$$\Delta_A = \prod_{b\in B} x_b^{\deg_T(b)-1}, \qquad \kappa_A = \frac{1}{\prod_{a\in A} v_a \prod_{b\in B} v_b^{\deg_T(b)-1}}, \qquad U_A = \frac{\det((Z/Z_J)[A,A])}{\Delta_A}.$$

Here Z/Z_J denotes the Schur complement. Then $\theta = U_A/\kappa_A$ satisfies

$$z_a^J = \frac{T_a}{\Lambda_a}\frac{\theta}{v_a} + \sum_{c\in A\setminus\{a\}} k_{ac}\frac{\Lambda_c}{\Lambda_a}\frac{v_c}{v_a}, \tag{6.17}$$

$$x_b = v_b\left[\Lambda_b\theta + \sum_{a\in A}\frac{\Lambda_b\Lambda_a}{T_b}k_{ba}v_a\right], \tag{6.18}$$

$$x_a = v_a\left[\frac{\Lambda_a(R_U+R_0)}{\theta} + \sum_{b\in B}\frac{\Lambda_a\Lambda_b}{T_a}k_{ab}v_b\right], \tag{6.19}$$

$$R_0 = \sum_{a\in A,b\in B} k_{ab}k_{ba}\frac{\Lambda_a\Lambda_b}{T_aT_b}v_av_b, \tag{6.20}$$

$$R_U = r_U + \sum_{j\notin J\cup U} b_{Uj}x_j. \tag{6.21}$$

In particular, untouched exterior variables remain in the affine root. If $K_A = (Z/Z_J)[A,A]$, then

$$K_A^{-1}a_A = r_U(\Lambda_av_a/\theta)_{a\in A}, \qquad K_A^{-1}k_{Aj} = b_{Uj}(\Lambda_av_a/\theta)_{a\in A} \quad (j\notin J\cup U). \tag{6.22}$$

Proof. Put $w_a = \delta\Lambda_a$, $w_b = \Lambda_b$, and $g_i = w_i/a_i$. For an edge $e = ab$, $\Lambda_a\Lambda_b = f_{e,a}f_{e,b}T_e$ and $T_aT_b = f_{e,a}f_{e,b}T_e^2$. Hence $g_ag_b\delta T_e = 1$. By Lemma 6.3, the normalized pair-tree monomials are g_iv_i. The normalized off-diagonal coefficients are $k_{ij}a_j/a_i$, and the normalized common affine root is R_U/δ.

The selected matrix changes by diagonal similarity. If $\zeta_A = \prod_{b\in B} a_b^{\deg_T(b)-1}$, then $U_A^{\mathrm{norm}} = \zeta_AU_A$. Also $\kappa_A^{\mathrm{norm}} = \zeta_A\kappa_A$: on deleting any edge, count the selected vertices and the omitted degrees on each side, using $\sum_b(\deg(b)-1) = |A|-1$ for a tree. Equivalently, each labeled port has exponent zero in $\prod_{a\in A}g_a\prod_{b\in B}(g_ba_b)^{\deg(b)-1}$. Thus the phase in Lemma 6.2 is unchanged. Undoing $y_i = x_i/a_i$ in that lemma gives (6.17)–(6.19). For example the normalized pair sum is R_0/δ, while $a_ag_a = \delta\Lambda_a$ and $a_bg_b = \Lambda_b$; these three equalities cancel every root-normalizing divisor.

Finally (6.17) implies

$$K_A(\Lambda_av_a)_{a\in A} = \theta(T_a)_{a\in A}.$$

Use $a_a = T_ar_U$ and $k_{aj} = T_ab_{Uj}$ from (6.6) to obtain (6.22). □

6.3. **Exact updates of source functions.** A prime on a source function will mean that J is replaced by $J\cup A$. A source surviving the event is viewed before folding any new omitted members. Its intersection with the residual event is empty, one selected port, or one omitted port. Two intersections would force a processed block to survive, or create an incidence cycle. Intersection at an old selected articulation belongs to the first, residual-disjoint case.

Lemma 6.5 (Source updates)**.** *If $M\cap(U\setminus J) = \varnothing$, then*

$$r'_M = r_M + \frac{r_U}{\theta}\sum_{a\in A} b_{Ma}\Lambda_av_a, \tag{6.23}$$

$$b'_{Mj} = b_{Mj} + \frac{b_{Uj}}{\theta}\sum_{a\in A} b_{Ma}\Lambda_av_a \quad (j\notin M\cup U\cup J). \tag{6.24}$$

If $M\cap(U\setminus J) = \{v\}$ with $v\in A$, put $N = M\cup U$. Then

$$r'_M = \frac{v_v\Lambda_v}{\theta}r_N, \qquad b'_{Mj} = \frac{v_v\Lambda_v}{\theta}b_{Nj} \quad (j\notin N\cup J). \tag{6.25}$$

If the residual intersection is $\{b\}$ with $b\in B$, then

$$r'_M = \frac{T_b}{\Lambda_b}\frac{x_b}{v_b\theta}r_N, \qquad b'_{Mj} = \frac{T_b}{\Lambda_b}\frac{x_b}{v_b\theta}b_{Nj} \quad (j\notin N\cup J). \tag{6.26}$$

For nested post-event sources $N \subseteq H$, Φ_N^H is unchanged unless the event selects a vertex $v \in H\setminus N$; in that case

$$(\Phi_N^H)' = \Phi_N^H z_v^J + f_v^H b_{Nv}. \tag{6.27}$$

Proof. In the residual-disjoint case, take the Schur complement on the new selected block K_A. The common root and target row of M change by $b_{MA}K_A^{-1}a_A$ and $b_{MA}K_A^{-1}k_{Aj}$, respectively. Equation (6.22) gives (6.23)–(6.24). For a selected singleton already in J, the same calculation follows from its inverse-row interpretation (6.8).

If the intersection is a selected port v, first separate $J \cap M$ by the row additions in (6.7), and then eliminate $A \setminus \{v\}$. Cramer's rule for the remaining row gives

$$r'_M = \frac{(K_A^{-1}a_A)_v}{f_v^M}, \qquad b'_{Mj} = \frac{(K_A^{-1}k_{Aj})_v}{f_v^M}.$$

Concretely, their numerator determinants are obtained by replacing column v of K_A by a_A or k_{Aj}, and their denominator is $f_v^M \det K_A$. Source factorization gives $r_U = f_v^M r_N$ and $b_{Uj} = f_v^M b_{Nj}$, proving (6.25).

If the intersection is an omitted port b, source factorization gives $r_M = T_b r_N$, $b_{Mj} = T_b b_{Nj}$, and $b_{Ma} = k_{ba}/f_b^M$, whereas $r_U = f_b^M r_N$. The residual-disjoint Schur calculation is applicable since $M \cap A = \varnothing$. Thus

$$r'_M = r_N \left(T_b + \theta^{-1} \sum_{a\in A} k_{ba}\Lambda_a v_a \right).$$

Use (6.18) to obtain (6.26); the target calculation is identical with r_N replaced by b_{Nj}.

For the nested update only a single newly selected vertex can lie in $H \setminus N$, by the surviving-source intersection property. Put $L = H \cup U$. Before the event, factorization gives

$$r_H = T_v r_L, \qquad r_U = f_v^H r_L, \qquad f_v^H b_{Na} = \Phi_N^H k_{va} \quad (a \in A \setminus \{v\}).$$

Divide the residual-disjoint update for r_N by the selected-port update for r_H. The quotient is

$$\Phi_N^H \frac{T_v\theta + \sum_{a\neq v} k_{va}\Lambda_a v_a}{v_v\Lambda_v} + f_v^H b_{Nv}.$$

Its fraction is z_v^J by (6.17). This proves (6.27), including the singleton event. □

The following identities are needed because an unweighted coefficient toward an old omitted target need not be Laurent in the earlier prefix.

Lemma 6.6 (Weighted target columns)**.** *For a newly omitted $b \in B$, put $W_{ab} = (K_A^{-1}k_{Ab})_a x_b$. Then*

$$W_{ab} = v_a v_b \left[\frac{\Lambda_a\Lambda_b}{T_a} k_{ab} + \frac{\Lambda_a\Lambda_b}{T_b\theta} \sum_{c\in A} k_{bc}k_{cb}\frac{\Lambda_c}{T_c} v_c \right]. \tag{6.28}$$

For distinct $b_0, b \in B$ one has

$$k'_{b_0 b}x_b = x_{b_0}\frac{v_b}{v_{b_0}\Lambda_{b_0}} \left[\Lambda_b k_{b_0 b} + \frac{\Lambda_b T_{b_0}}{T_b\theta} \sum_{c\in A} k_{bc}k_{cb}\frac{\Lambda_c}{T_c} v_c \right]. \tag{6.29}$$

If a source M can be enlarged to a surviving source meeting the event only at b, then

$$b'_{Mb} = b_{Mb} + \frac{1}{\theta}\sum_{a\in A} b_{Ma}k_{ab}\frac{\Lambda_a}{T_a}v_a. \tag{6.30}$$

In particular the last expression has no inverse x_b.

Proof. In the normalized pair-tree system let $f_a = c_{ab}$, $\ell_b = \sum_c c_{bc}v_c$, and $S_b = \sum_c c_{bc}c_{cb}v_c$. The vector f is constant on each component of $T \setminus \{b\}$. For vertices a, c in different components, pair-module equality gives $c_{ac} = c_{bc}$. Thus, term by term,

$$L_A f = \ell_b f - S_b \mathbf{1}, \qquad L_A \mathbf{1} = 0, \qquad (\theta I + L_A)^{-1} f = \frac{f + (S_b/\theta)\mathbf{1}}{\theta + \ell_b}.$$

Multiplication by $x_b = v_b(\theta + \ell_b)$ gives $W_{ab} = v_a v_b(c_{ab} + S_b/\theta)$ in the normalized system. For $b_0 \neq b$, tree separation also gives, for every $a \in A$,

$$(c_{b_0 b} - c_{ab})(c_{ba} - c_{b_0 a}) = 0.$$

Insert the weighted inverse column in $k'_{b_0 b} = c_{b_0 b} + c_{b_0 A} K_A^{-1} c_{Ab}$, and sum this last identity with weights v_a. The result is $k'_{b_0 b} x_b = x_{b_0}(v_b/v_{b_0})(c_{b_0 b} + S_b/\theta)$. Undo the diagonal normalization used in Proposition 6.4, namely $c_{ij} = k_{ij}a_j/a_i$ and $v_i^{\mathrm{norm}} = g_i v_i$. Substitution of $g_a = \Lambda_a/T_a$ and $g_b = \Lambda_b/(\delta T_b)$ gives exactly (6.28) and (6.29); all δ factors cancel.

For the last assertion enlarge M to H with $H \cap (U \setminus J) = \{b\}$. Factorization makes the row b_{MA} proportional to k_{bA}. The preceding inverse-column computation gives

$$k_{bA} K_A^{-1} k_{Ab} = \theta^{-1} \sum_{a \in A} k_{ba} k_{ab} (\Lambda_a/T_a) v_a.$$

Multiply by that proportionality factor before expanding, and add the unchanged coefficient b_{Mb}. This gives (6.30) without dividing by an omitted variable. □

6.4. **Simultaneous port and mark extraction.** Here are the precise source branches used in applying Lemma 6.1. To separate a source from a target j, remove every proper block containing j as a whole. For a clique remove only its target vertex, keeping the decorated clique node attached to its other residual members. Take the source-side component of the event incidence tree with this convention, and include its whole proper blocks and its residual subcliques. Call the resulting source E_{ij}. Its selected interiors are precisely the interiors of the proper event blocks on that side. Ordering these interiors away from i in the shifted matrix gives

$$k_{ij} = \left(\prod_{e \text{ in the proper source branch}} f_{e,\pi_e(i)} \right) b_{E_{ij},j}. \tag{6.31}$$

In particular a final clique edge keeps its marked node even when j is its other endpoint. A clique contributes no port determinant, because none of its vertices is in J.

If source attachments enlarge E_{ij} to L while keeping the target outside, (6.31) becomes the single identity

$$k_{ij} = \left(\prod_{e \text{ in the proper source branch}} f_{e,\pi_e(i)} \right) \Phi^L_{E_{ij}} b_{Lj}. \tag{6.32}$$

This identity extracts port factors before using the entire marked source factorization of b_{Lj}. If an additional source is attached at an event port v, its selected interior is disjoint from all event interiors. Its factor f_v^M therefore occurs as an additional diagonal block in the same triangularization. No coprimality of these factors is required.

For clarity, the elementary port count used below is as follows. On an odd path from $a \in A$ to $b \in B$, a negative port exponent in $\Lambda_a\Lambda_b/T_a$ occurs only at an even-position edge. It is internal to the a-source branch and is supplied by (6.31). On a same-colour path, a negative port in Λ_j/Λ_i occurs only at an odd-position edge strictly before the target and is supplied by the i-source branch. For

$$k_{ab}k_{ba}\frac{\Lambda_a\Lambda_b}{T_aT_b}, \tag{6.33}$$

an off-path edge contributes at most its nearest endpoint port to the denominator, and that port occurs in one source branch. An even-position path edge contributes both endpoint ports,

one in each of the opposite source branches. The remaining port exponents are nonnegative. These assertions follow by considering the two components after deleting the edge: its port is in T_i exactly on the component containing its endpoint, and is in Λ_i exactly when that endpoint has the colour of i.

Opposite source branches cover every event vertex and every clique node. A clique node on the path occurs in both branches; an off-path node occurs in at least one. Assign each required exterior attachment to a branch containing its attachment vertex, and at an endpoint use the branch starting at that endpoint. All assignments can be made simultaneously. Two attachments meeting again outside the event would create a cycle in the block incidence forest. Equation (6.32) then extracts the full assigned product, including repeated numerical factors, from each source coefficient.

We next count the powers introduced by physical clique cores. Fix one mark h at a clique edge e, root the event at a selected vertex, and let L, R be the root and other side after deleting e. Write $v_i = v_i^0 h^{s_i}$ and $\kappa_A = \kappa_A^0 h^{\eta}$. The relation $s_e v_i v_j = 1$ and propagation along the tree give

edge orientation	s_i $(i \in L)$	s_a $(a \in A \cap R)$	s_b $(b \in B \cap R)$	η	$\deg_h \theta$
$A_L - B_R$	0	1	-1	0	1
$B_L - A_R$	0	-1	1	1	0

(6.34)

The entry for κ_A follows by summing these exponents in its definition; the entry for θ uses the canonical gauge

$$\theta = \frac{H_A}{\kappa_A}\tau_A, \qquad \tau_A = m_A p_{J\cup A}/p_J, \tag{6.35}$$

from Lemma 5.3. Every internal clique mark occurs once in H_A.

It follows that $\kappa_A v_a$ and κ_A/v_b have h-degree 0 or 1, while θ/v_a and $v_b\theta$ have nonnegative h-degree. A negative exponent in v_i/v_j for equal colours or in $v_i v_j$ for opposite colours occurs only when the path crosses e, and has size at most one. The source on that path retains the decorated clique node and supplies one h. Opposite coefficients $k_{ab}k_{ba}$ supply two occurrences when their path crosses e and at least one otherwise. Consequently each summand of R_0 retains an entire occurrence of every internal clique mark after its $v_a v_b$ factor is included. These counts are per labeled occurrence; several marks at one node satisfy the same count simultaneously.

6.5. **The joint positivity induction.**

Proposition 6.7. *After every prefix the following statements hold for all surviving decorated sources and nested source pairs:*

(1) $\Phi_N^H \in \mathcal{P}_\ell$;
(2) $r_N/\mathfrak{m}(N) \in \mathcal{P}_\ell$;
(3) $b_{Nj}/\mathfrak{m}(N) \in \mathcal{P}_\ell$ *for each live exterior target* j;
(4) $x_h b_{Nh}/\mathfrak{m}(N) \in \mathcal{P}_\ell$ *for each old omitted exterior target* h;
(5) *if the component of* N *contains an occurrence of origin* h, *then* $b_{Nh}/\mathfrak{m}_h(N) \in \mathcal{P}_\ell$.

Every quotient has a displayed positive expression. The selected Schur diagonals and the omitted recovery formulas are positive Laurent in the extended prefix. Selected originals have positive affine recovery in the still-live exterior originals. At the end of the events all original variables have positive Laurent expressions in the candidate coordinates.

Proof. At $J = \varnothing$ there are no marks, every nested determinant is 1, and the source functions are 1 or positive initial coefficients. Assume the five statements for the old prefix.

Source enlargements and the phase product. An old omitted vertex gains a new canonical-gauge occurrence precisely when one of its formerly unready blocks becomes ready during this event. A proper such attachment has residual pair $\{a, h\}$, where $a \in A$ and h is old omitted. A clique attachment has one residual selected attachment vertex and a node carrying its deleted old

members. It cannot have a second unrecovered member: that member would become ready now and could not be processed at a later eligible event. Internal clique-edge marks are already in the event source. These are exactly the factors of H_A.

Adjoin all the attachments to the event source, or distribute them between opposite branches as above. The resulting source is a permitted old source and its mark product contains the whole H_A. Equations (6.5) and (6.32), followed by the old states, give a positive quotient by this entire product after all specified port factors have been extracted. With a live target no attachment absorbs that target. With an old target h which is itself one of the phase marks, there are two cases. For a proper attachment leave the target-containing attachment out; source and target remain in the same old component, so state 5 supplies the unweighted coefficient with every other required mark. For a clique keep its node but apply state 5 with the origin-h exception. In either case the explicit factor x_h in the weighted coefficient supplies exactly the missing occurrence. It supplies that occurrence once, not once for each residual member of the clique.

Diagonals, omitted values, and nested factors. In (6.17), T_a/Λ_a is a product of ports and θ/v_a has nonnegative internal-mark exponents by (6.34). Each remaining term has its port denominators supplied by its source branch; a negative mark exponent can only cross the corresponding clique edge and is supplied by that same branch. Thus the diagonal is positive in the enlarged prefix semiring. The identical count applies to (6.18); its first term $v_b\Lambda_b\theta$ has no negative internal-mark exponent, and its other terms use the opposite-colour source quotient. In particular every new omitted value x_b is available as a positive expression before it is installed as a new mark.

For nested post-event sources use (6.27). Its diagonal has just been proved positive, while Φ_N^H, f_v^H, and the live-target coefficient b_{Nv} are old states. The other nested determinants do not change. This proves state 1 without assuming positivity of arbitrary raw principal minors.

Root and live-target states. Use the three updates in Lemma 6.5. In the residual-disjoint case the event factor supplies H_A, and the separate factor b_{Ma} retains all old marks of M. After substitution of (6.35), the remaining internal clique power is that of $\kappa_A v_a$, hence is nonnegative. All port factors are extracted together with the phase marks by (6.32).

In a selected-port intersection the same computation uses $\kappa_A v_v$, and in an omitted-port intersection it uses κ_A/v_b, followed by the explicitly positive new factor x_b. The factor T_b/Λ_b is a product of ports. A mark required by the surviving source cannot also be consumed as a phase occurrence in either intersection case: a phase clique node is deleted when ready, and a phase proper block is deleted and separates its old marked vertex. A second connecting path would be an incidence cycle. Thus the surviving mark product and the consumed phase product are both present with their required multiplicities. In the residual-disjoint case equal origins may occur in distinct components, but their two factors already occur in separate displayed source factors. This proves states 2 and 3 before folding, with the new omitted mark in the omitted-port case.

Persistence of old targets. Fix an old omitted target h. Multiply every exact source update by x_h. Phase extraction for its event factor uses the target-origin rule in the first paragraph of the induction. The other source factor retains its source marks, as in the root and live-target calculation. This proves state 4.

For state 5 require an occurrence of origin h in the *post-event* component of M. In an intersection update a consumed occurrence cannot be the surviving occurrence: phase clique nodes disappear, and phase proper attachments are cut. Uniqueness of an origin within a pre-event component rules out another copy there. Consequently the unweighted intersection formula extracts the phase product while retaining precisely $\mathfrak{m}_h(M)$.

In a residual-disjoint update the term to consider is

$$\frac{b_{Uh}}{\theta}\, b_{Ma}\Lambda_a v_a.$$

If U and the surviving occurrence are in different old components, enlarge M in its component to that occurrence, avoiding a. This gives an explicit x_h in b_{Ma} and permits the rewrite $b_{Ma}b_{Uh} = (b_{Ma}/x_h)(x_h b_{Uh})$. If the occurrence was already in M, its removal is exactly the exception in $\mathfrak{m}_h(M)$; otherwise it is an extra factor and all source marks remain. Any occurrence of the same origin in U is a different labeled occurrence and is supplied by its weighted state before the phase is divided out.

If U lies in the same old component, but the origin-h occurrence is not a phase mark, enlarge its unweighted source by all phase attachments, using state 5. If that occurrence is a phase mark and nevertheless survives in the component of M, it is a proper marked vertex beyond a newly ready attachment; a marked phase clique node would have disappeared. The surviving source component avoids every selected event vertex, since a path to one of them together with the phase attachment and the event path would make a cycle. Enlarge M to that component to obtain the extra x_h from b_{Ma}, and enlarge U by the other phase attachments to obtain H_A/x_h from b_{Uh}. The target vertex is outside M, so this extra cancellation removes no required source mark. These cases prove state 5 for old targets.

Birth of new weighted targets. Let $b \in B$. For a residual-disjoint source the new weighted target is

$$x_b b'_{Mb} = b_{Mb}x_b + \sum_{a\in A} b_{Ma}W_{ab}. \tag{6.36}$$

For a source attached at a selected port v it is W_{vb}/f_v^M; for a source attached at an omitted port $b_0 \neq b$ it is $k'_{b_0 b}x_b/f_{b_0}^M$. These follow from the same Schur and Cramer calculations used in Lemma 6.5. The direct terms in (6.28) have the opposite-colour port quotient. Its pair terms have the quotient (6.33), multiplied by a product of ports. For (6.29) the direct term has the same-colour port quotient. The extra divisor f_v^M or $f_{b_0}^M$ is extracted by including M in the source branch containing its attachment. That is the same simultaneous factorization which supplies the retained source marks.

We give the mark count for the pair terms explicitly. For one internal mark use s_i, η from (6.34) and factor out the phase product H_A by the assigned branches. Before the opposite coefficient product $k_{bc}k_{cb}$ is included, the exponent in the second term of (6.28) is $s_a + s_b + s_c + \eta - 1$. In the two orientations this is

$$\mathbf{1}_{a\in R} - \mathbf{1}_{b\in R} + \mathbf{1}_{c\in R} - 1, \qquad -\mathbf{1}_{a\in R} + \mathbf{1}_{b\in R} - \mathbf{1}_{c\in R}.$$

If b, c are on the same side, the deficit is at most one; otherwise it is at most two. Their two source branches supply one occurrence off their path and two on their path. For the second term of (6.29), after setting aside its new factor x_{b_0}, the corresponding exponent is $s_b - s_{b_0} + s_c + \eta - 1$, namely

$$-\mathbf{1}_{b\in R} + \mathbf{1}_{b_0\in R} + \mathbf{1}_{c\in R} - 1, \qquad \mathbf{1}_{b\in R} - \mathbf{1}_{b_0\in R} - \mathbf{1}_{c\in R}.$$

The same one-or-two occurrence count applies. Direct terms have a deficit only on their source-to-target path and are covered by that source. Exterior phase attachments are allocated to the same whole branches. Hence the displayed weighted births are positive with all old source marks retained; an omitted attachment additionally has the explicit new mark x_{b_0}.

Birth of the target-origin exception. Suppose the post-event component of M contains a mark with origin b. If it is a proper marked vertex, enlarge the source to that surviving component. If it is a clique-node mark, temporarily adjoin the deleted vertex b through that clique. In either case the enlarged module meets the current event only at b; two intersections would create an incidence cycle. Thus (6.30) applies. In each of its summands the M-to-a branch and the

a-to-b branch cover the phase attachments. For the first branch, source factorization through the enlarged module supplies the same port product as the b-to-a branch, without dividing by a virtual port. Extract both whole products before applying the old states. Equation (6.34) shows that v_a/θ needs at most one occurrence at each internal clique mark, which these branches supply. There is no inverse x_b in the formula. The unweighted state requires no new origin-b mark; multiplying it by x_b supplies exactly that occurrence in the weighted state.

Folding newly omitted clique members. Let a newly omitted b belong to a surviving clique. For a post-source N carrying that clique node, adjoin b to form N^+. Before folding, the omitted-port update has supplied one explicit x_b to its root and exterior live-target frames. As $b \notin J \cup A$,

$$q_N^{\mathrm{src}} = q_{N^+}^{\mathrm{src}}, \qquad P_{Nj}^{\mathrm{src}} = P_{N^+j}^{\mathrm{src}} \quad (j \notin N^+ \cup J \cup A) :$$

the determinant index sets are the same, and the source rows agree toward those sets and the exterior target. Thus the factor migrates to the clique node. Repeating this equality for several omitted members multiplies all their factors. The excluded target $j = b$ is exactly the separate exception handled in the preceding paragraph. Proper marked vertices do not migrate. Prune every newly ready block. The component argument at the beginning of the section remains valid, so all five states have now been proved at the new prefix.

The complete affine recovery. Split (6.21) as

$$R_U = r_U + \sum_{h \text{ old omitted}} x_h b_{Uh} + \sum_{j \text{ other exterior}} b_{Uj} x_j. \tag{6.37}$$

The first paragraph of the induction supplies its whole phase product, with the target-origin exception when a target is itself a required mark. The paired-source argument supplies the same product in every summand of R_0; internal marks retain at least one occurrence after the $v_a v_b$ factors, as established after (6.34). In (6.19), the remaining factors $\kappa_A v_a$ have nonnegative internal-mark exponents. Its cross terms have their opposite-colour source quotients. This proves positive affine recovery in the exterior live originals. Omitted originals have already been recovered using (6.18), which does not involve the full affine root. At the final event there are no unrecovered exterior originals except retained coordinates. Recover selected originals in reverse event order to obtain positive Laurent expressions for all originals in the candidate coordinates.

The order of the proof is noncircular: old source states give the selected diagonal and omitted recovery, then nested factors, root and live states, old targets, new targets, and finally the folded state. The selected affine recovery uses the old full root and the already extracted source products. No new state is used to justify its own positivity. □

Lemma 6.8 (A retained coordinate kept free)**.** *Let u be a retained original. Until the first event making u ready, the preceding phase factors, nested factors, source coefficients, constant root coefficients, and recovered omitted values admit expressions independent of the coordinate x_u.*

Proof. No preceding readiness gauge contains x_u. No preceding selected proper core contains u, and a preceding selected clique containing u would already make it ready. Thus none of the physical core factors used before that event contains x_u as a retained-member factor. Start with the constant source states. The phase formula, selected diagonal, nested update, and source updates use only preceding states and the new prefix/core coordinates. Omitted recovery (6.18) uses no full affine root. Induction therefore gives the stated independence. The full affine root can contain $b_{Uu}x_u$; it is kept separate and is not used in any of the preceding independence statements. □

7. Positive identities in every direction

We now identify the changes of Proposition 5.4 with ordinary LP mutation. All coefficients in this section belong to $K_{\geq 0}$. Positivity of a rational function will always mean a Laurent expression in the indicated shared coordinates, rather than positivity of its values alone.

We retain the event notation of Section 6. In particular, s_e is the physical two-port core, $q_e = q_{B_e}$ is the actual core coordinate, and

$$s_e = \begin{cases} q_e, & B_e \text{ proper}, \\ \gamma_e H_e q_e, & B_e \text{ a clique}. \end{cases}$$

Here γ_e is a monomial in retained originals, and H_e is the product of its old omitted members. The latter factors need not be monomials in the current coordinates. They are cancelled using Lemma 6.1 and Proposition 6.7, before any Laurent-positivity conclusion is drawn.

For reference, the exponent calculation used below has a particularly short form. Root an event at a selected vertex, and fix an occurrence of $h = x_h$ at a clique edge d. Deleting d gives the root side L and the other side R. If ϵ_i is its exponent in v_i and ϵ_κ its exponent in κ_A, then

d	ϵ_i $(i \in L)$	ϵ_a $(a \in A \cap R)$	ϵ_b $(b \in B \cap R)$	ϵ_κ
$A_L B_R$	0	1	-1	0
$B_L A_R$	0	-1	1	1

(7.1)

It follows directly from $s_e v_i v_j = 1$ along the unique paths of the tree. For the last column use $|A \cap R| - \sum_{b \in B \cap R} (\deg_T(b) - 1) = 0$ or 1, according to which endpoint of d lies in R. Thus $\kappa_A v_a$ and κ_A / v_b have exponent zero or one. A negative exponent in v_i / v_j for equal colours, or in $v_i v_j$ for opposite colours, has size at most one and occurs only if the path from i to j crosses d. The source factorization retains the clique node after deleting its target vertex. Consequently a coefficient in such a crossing direction supplies that occurrence of h. The product of the two opposite source coefficients supplies it twice on the path, and at least once off the path. These statements count labelled occurrences; they remain valid when several occurrences represent the same function.

7.1. Cutting a core.

Let $T = A \sqcup B$ be an event after J. Cut the edge $e = a_0 b_0$ with $a_0 \in A$, $b_0 \in B$, and let W be its b_0-side. Put $I = A \cap W$. For $I \neq \varnothing$ set

$$\Delta_I = \prod_{b \in B \cap W} x_b^{\deg_W(b) - 1}, \qquad U_I = \frac{\det K^J[I, I]}{\Delta_I}, \qquad \kappa_I = \frac{1}{\prod_{a \in I} v_a \prod_{b \in B \cap W} v_b^{\deg_W(b) - 1}}.$$

For an edge d not internal to W, the whole of W lies on one side of $T - d$. Let β_W be the product of the port factors $f_{d, b(d)}$ for which this side contains the omitted endpoint $b(d)$. Clique port factors are one. In particular, the factor of the cut edge is f_{e, b_0}.

Lemma 7.1. *One has*

$$U_I = \kappa_I \beta_W \left(\theta + \sum_{a \in A \setminus I} \frac{k_{b_0 a} \Lambda_a}{T_{b_0}} v_a \right). \tag{7.2}$$

After the canonical gauges are inserted, this gives the positive shared-coordinate identity

$$\begin{aligned} q_e p_{J \cup I} = {} & \frac{q_e \kappa_I \beta_W}{\kappa_A} \frac{H_A}{H_I} \frac{m_A}{m_I} p_{J \cup A} \\ & + \frac{p_J}{m_I} q_e \kappa_I \sum_{a \in A \setminus I} \frac{\beta_W k_{b_0 a} \Lambda_a}{T_{b_0} H_I} v_a. \end{aligned} \tag{7.3}$$

If $I = \varnothing$, then $W = \{b_0\}$, and the corresponding identity is

$$q_e x_{b_0} = q_e v_{b_0} \Lambda_{b_0} \left(\frac{H_A m_A}{\kappa_A} \frac{p_{J\cup A}}{p_J} + \sum_{a\in A} \frac{k_{b_0 a}\Lambda_a}{T_{b_0}} v_a \right). \tag{7.4}$$

Proof. First normalize the effective row roots as in Proposition 6.4. In these normalized coordinates the selected matrix, after its columns are multiplied by the tree monomials, is a rooted Laplacian with common root θ. Every row in the branch W has the same coefficients toward $A \setminus I$, namely those of its port b_0. Its restricted matrix therefore has common root $\theta + \sum_{a\notin I} c_{b_0 a}\widetilde{v}_a$. The tree determinant formula of Lemma 6.2 applied to W gives its principal determinant as this root times

$$\prod_{b\in B\cap W} \left(\frac{y_b}{\widetilde{v}_b}\right)^{\deg_W(b)-1}.$$

Dividing by the column monomials and the internal gauge proves the normalized branch identity.

We give the factors in pulling it back, since they determine the canonical core exchange. Write the original effective roots as $a_i = \delta T_i$. The normalized tree monomials are

$$\widetilde{v}_i = g_i v_i, \qquad g_a = \Lambda_a/T_a \ (a \in A), \qquad g_b = \Lambda_b/(\delta T_b) \ (b \in B).$$

Put $R_I = \prod_{b\in B\cap W} a_b^{\deg_W(b)-1}$. The normalized principal is $R_I U_I$. In $\widetilde{\kappa}_I/R_I$, all ports on edges internal to W cancel. On each external edge, the port facing a selected endpoint cancels, while the port facing an omitted endpoint occurs once. This follows by counting vertices on its W-side; for an internal edge the needed equality is $\sum_{b\in B\cap W}(\deg_W(b) - 1) = |I| - 1$. Thus

$$\widetilde{\kappa}_I/R_I = \kappa_I \beta_W.$$

The normalized shifted root pulls back to the parentheses in (7.2), proving that identity. Substitution of (5.2) gives (7.3).

We verify positivity of its quotients. Every old boundary occurrence in H_I is also in H_A: an exterior attachment meets the connected event at only one current vertex. For $a \notin I$ the source branch from b_0 towards a contains all of W. Enlarge it by all the H_I attachments. The target stays outside it, and Lemma 6.1 simultaneously extracts H_I and the required port factors. Specifically, an uncancelled negative port in $\beta_W \Lambda_a/T_{b_0}$ is either on an internal edge of W, at the endpoint nearest b_0, or on the path towards a at a selected endpoint facing b_0. The latter cannot be the final edge incident to a. Both kinds belong to this source branch; all off-path ports cancel in the displayed quotient.

It remains to check marks in physical clique cores. Root the tree at a_0. For a mark on the cut edge, κ_I has exponent -1, whereas κ_A and all v_a outside W have exponent zero. Its occurrence in H_A, absent from H_I, cancels the first term; the source towards a supplies it in the second. For a mark on an edge strictly inside W, κ_I/κ_A has exponent zero and H_A/H_I has exponent zero. In the second term κ_I has exponent zero or one, and the source supplies the required occurrence of H_I. For an edge outside W, H_I has exponent zero and H_A has exponent one. In the first row of (7.1), κ_I has exponent -1 when $W \subset R$ and zero otherwise. In the second row it has exponent 1 when $W \subset R$ and zero otherwise, while κ_A has exponent one. The first term therefore has nonnegative exponent. A deficit in the second term occurs only when W and its target are separated by that edge; the source factor supplies it. All these extractions are made in the same enlarged source factor as the port extractions, so their multiplicities are retained.

With root a_0, the exponent of q_e in κ_I is -1, and its exponents in κ_A and in all v_a, $a \notin W$, are zero. Earlier source functions contain no current core coordinate. The right-hand side is therefore independent of q_e and of its proposed partner.

If $I = \varnothing$, use the omitted recovery formula directly. It gives (7.4). Now $q_e v_{b_0} = 1$ for a proper edge, and $q_e v_{b_0} = 1/(\gamma_e H_e)$ for a clique edge. The first term supplies H_e through H_A, and each

source in the second supplies it through the central clique node. The preceding port and mark checks apply to the other edges. □

7.2. **Interchanging events and changing the last colour.** For residual-disjoint events C, D with selected colours A_C, A_D, let b_{Cj} denote the source coefficient of the full residual module of C. Define

$$\lambda_{CD} = \sum_{j\in A_D} b_{Cj}\Lambda_j^D v_j, \qquad \lambda_{DC} = \sum_{i\in A_C} b_{Di}\Lambda_i^C v_i.$$

Lemma 7.2. *When both event orders are allowed,*

$$p_{J\cup A_C}p_{J\cup A_D} = X(J;A_C,A_D)p_J p_{J\cup A_C\cup A_D} + \frac{p_J^2\kappa_C\kappa_D}{m_C m_D}\frac{\lambda_{CD}\lambda_{DC}}{H_C H_D} \tag{7.5}$$

is positive Laurent in the shared coordinates. A blocked-order merger through an unselected block has the inverse exchange of Lemma 7.1.

Proof. The source factorization gives $k_{ij} = T_i^C b_{Cj}$ for $i \in A_C, j \in A_D$. The selected row-sum identity gives

$$(K^J[A_C,A_C])^{-1}T^C = (\Lambda_i^C v_i/\theta_C)_{i\in A_C}.$$

The two cross blocks consequently have rank one. The determinant lemma applied to these two blocks gives

$$U_C U_D = U_{CD} + \kappa_C\kappa_D\lambda_{CD}\lambda_{DC}, \qquad U_{CD} = \frac{\det K^J[A_C\cup A_D, A_C\cup A_D]}{\Delta_C\Delta_D}.$$

Inserting the three gauges gives exactly (7.5); in particular the first coefficient is the cross gauge of Lemma 5.3.

Enlarge the full C source by its old phase attachments before taking the coefficient towards $j \in A_D$. Its factorization supplies the whole product H_C. The reverse source supplies H_D. After these products are removed, a summand of the second term contains $\kappa_C v_i$ and $\kappa_D v_j$. Their mark exponents are nonnegative by (7.1). If the same old original occurs in both products, the two source factors supply both occurrences. The port factors are extracted in those same factorizations. Every earlier principal, every current core, and the later union principal are shared. The formula contains neither intermediate principal. The merger statement follows from the inverse cut and Proposition 5.4. □

For a residual event module M, its complete affine root is

$$R_M = r_M + \sum_{h \text{ old omitted},\ h\notin M} x_h b_{Mh} + \sum_{j \text{ other exterior}} b_{Mj}x_j. \tag{7.6}$$

The second sum includes every untouched exterior original. Terms whose target is in the selected prefix are already included in the source functions and are not repeated.

Lemma 7.3. *For a nontrivial last event $T = A \sqcup B$,*

$$p_{J\cup A}p_{J\cup B} = \frac{p_J^2}{m_A m_B}\left(\prod_{e\in T} s_e\right)\frac{R_M + \sum_{a\in A, b\in B}\dfrac{k_{ab}k_{ba}\Lambda_a\Lambda_b}{T_aT_b}v_a v_b}{H_A H_B} \tag{7.7}$$

is a positive shared Laurent identity. For a last singleton v, the identity is

$$x_v p_{J\cup\{v\}} = \frac{p_J}{m_v}\frac{r_{\{v\}} + \sum_{j\notin J\cup\{v\}} k_{vj}x_j}{H_v}. \tag{7.8}$$

Proof. Write the sum in (7.7) as R_0. The two recovery formulas in Proposition 6.4 show that changing the colours replaces the phase θ by $(R_M + R_0)/\theta$. Also

$$\kappa_A \kappa_B = \prod_{i \in T} v_i^{-\deg_T(i)} = \prod_{e \in T} s_e.$$

Thus the raw identity is $U_A U_B = (\prod_e s_e)(R_M + R_0)$, and the canonical gauges give (7.7).

The port denominators in a pair term are supplied by its two opposite source branches. An off-path port lies in at least one branch. On the endpoint path the uncancelled pair of port denominators lies on an interior edge and is supplied by the two branches separately. Enlarge those same branches by the H_A, H_B boundary attachments. Such an attachment cannot meet both colours of the connected event without giving another incidence path. An internal clique mark occurs once in $\prod_e s_e$ and twice in $H_A H_B$. Its opposite source product supplies two copies if the endpoint path crosses the marked edge, and at least one otherwise. The exponent in $v_a v_b$ is at worst -1 in the first case, and zero in the second. Thus each pair term retains the one additional occurrence required to cancel the denominator. For the root term Proposition 6.7 gives the same product extraction from (7.6). If its target is itself a mark, the unweighted target exception omits precisely that occurrence and the explicit factor x_h restores it. This proves the stated Laurent positivity.

No phase remains on the right. Because this is the last event, all exterior originals are retained or previously recovered omitted originals, hence have positive expressions in shared coordinates. For a singleton the Schur row equation, multiplied by $p_J/(H_v m_v)$, is (7.8); the same root argument cancels its whole mark product. □

7.3. **A proper block with three residual ports.** Consider the proper-block rotation of Proposition 5.4. The central block has residual ports h, v, ℓ. The three attached trees are denoted by H, V', L, to distinguish the second branch from the full vertex set. Their selected colours are $A_H, A_{V'}, A_L$, where $h \in B_H$, $v \in A_{V'}$, and $\ell \in A_L$. First assume $A_H \neq \varnothing$. Their port products, monomials and internal gauges carry branch superscripts. Write

$$T_h = T_h^H, \quad T_v = T_v^{V'}, \quad T_\ell = T_\ell^L, \quad S = A_{V'} \cup A_L, \quad K_* = S \cup A_H, \quad U_* = \frac{\det K^J[K_*, K_*]}{\Delta_{V'} \Delta_L \Delta_H x_h}.$$

Let $f_{0,h}, f_{0,v}, f_{0,\ell}$ be the three central port factors, and let q_0 be its proper core. Define

$$\lambda_{V'L} = \sum_{j \in A_L} (k_{vj}/T_v) \Lambda_j^L v_j, \qquad \lambda_{LV'} = \sum_{i \in A_{V'}} (k_{\ell i}/T_\ell) \Lambda_i^{V'} v_i,$$

$$L^*_{ha} = \frac{\Lambda_h^H \Lambda_a^H k_{ha}}{T_h f_{0,h} T_v T_\ell} \qquad (a \in A_H).$$

Lemma 7.4. *The raw identity for this rotation is*

$$U_{V'} U_L = \frac{\Lambda_h^H v_h}{\kappa_H} U_* + \kappa_{V'} \kappa_L \left(\lambda_{V'L} \lambda_{LV'} + \Lambda_v^{V'} \Lambda_\ell^L v_h v_v v_\ell q_0 \sum_{a \in A_H} L^*_{ha} v_a \right). \tag{7.9}$$

Its canonical form is positive Laurent in the shared coordinates. If $H = \{h\}$, put $U_* = \det K^J[S, S]/(\Delta_{V'} \Delta_L)$ instead; the identity becomes

$$U_{V'} U_L = U_* + \kappa_{V'} \kappa_L \lambda_{V'L} \lambda_{LV'}. \tag{7.10}$$

Proof. We first prove the identity with normalized effective roots, so that all port factors are one. Denote the corresponding matrix by M, its coefficients by c_{ij}, and the tree monomials by w_i. The central three-vertex proper core q_0 satisfies

$$\operatorname{adj}(M[\{h, v, \ell\}, \{h, v, \ell\}])\mathbf{1} = q_0 (x_h, x_v, x_\ell)^{\mathsf{T}}. \tag{7.11}$$

Indeed, the product of this matrix with the original coordinate column is the common external root times $\mathbf{1}$; its determinant is that root times q_0. Multiplication by the adjugate proves (7.11) as a polynomial identity.

Put $Q_S = \det(M[S,S] + \mathbf{1}c_{h,S})$. Multiply its column i by w_i. Inside the V' branch the row at v has zero entries in every nonport column, since $c_{hj} = c_{vj}$ for j outside the central block. The same holds at ℓ. Cross blocks between the two branches likewise have nonzero entries only in their port columns. After these two ports are put first, the matrix is block lower triangular. Its port block is

$$\begin{pmatrix} w_v(z_v + c_{hv}) & w_\ell(c_{h\ell} - c_{v\ell}) \\ w_v(c_{hv} - c_{\ell v}) & w_\ell(z_\ell + c_{h\ell}) \end{pmatrix}.$$

Its determinant is $w_v w_\ell x_h q_0$, by the h component of (7.11). For a branch R the remaining deleted-port determinant is

$$D_R = \prod_{b \in B_R} (\theta_R + l_b)^{\deg_R(b)-1}, \qquad l_b = \sum_{a \in A_R} c_{ba} w_a.$$

To verify this last identification directly, the full selected branch matrix is $\theta_R I + L_R$, with $L_R \mathbf{1} = 0$ and $\det(\theta_R I + L_R) = \theta_R D_R$. For every row vector a^{T} the determinant lemma gives

$$\det(\theta_R I + L_R + \mathbf{1}a^{\mathsf{T}}) = D_R(\theta_R + \sum_i a_i).$$

In the above triangular matrix its isolated port entry is exactly the final linear factor. Cancelling it proves the deleted-port assertion. Dividing the column scalings and the branch gauges now gives

$$\frac{Q_S}{\Delta_{V'}\Delta_L} = \kappa_{V'}\kappa_L w_v w_\ell x_h q_0. \tag{7.12}$$

Let $Y_S = \det M[S,S]$ and $A_S = c_{h,S}\,\mathrm{adj}(M[S,S])\mathbf{1}$, so $Q_S = Y_S + A_S$. The cross blocks between S and A_H have constant source rows. The determinant lemma and the selected row-sum identity give

$$\frac{\det M[K_*,K_*]}{\Delta_{V'}\Delta_L\Delta_H} = \kappa_H \left(\theta_H \frac{Y_S}{\Delta_{V'}\Delta_L} - l_h \frac{A_S}{\Delta_{V'}\Delta_L} \right).$$

Substitute $x_h = w_h(\theta_H + l_h)$ and (7.12), and divide the exact additional factor x_h in the union gauge. The result is

$$\frac{Y_S}{\Delta_{V'}\Delta_L} = \frac{w_h}{\kappa_H} U_* + w_h l_h \kappa_{V'}\kappa_L w_v w_\ell q_0.$$

The two-branch determinant lemma gives

$$U_{V'}U_L = \frac{Y_S}{\Delta_{V'}\Delta_L} + \kappa_{V'}\kappa_L \left(\sum_{j \in A_L} c_{vj} w_j \right) \left(\sum_{i \in A_{V'}} c_{\ell i} w_i \right).$$

Combining these two equations proves the normalized rotation. If $A_H = \varnothing$, the first elimination is absent and the second equation alone is the required identity, with no additional x_h in its gauge.

We record the complete change of scale. For the full local configuration write

$$\delta = \frac{q_M^{\mathrm{src}}}{Y_J}, \quad \Omega_R = f_{0,p_R} \prod_{D \neq R} T_{p_D}, \quad \delta_R = \delta\Omega_R,$$

where $p_H = h, p_{V'} = v, p_L = \ell$. The effective root at $i \in R$ is $\delta_R T_i^R$. The normalized monomials are

$$\widetilde{v}_i = \frac{\Lambda_i^R}{T_i^R} v_i \quad (i \in A_R), \qquad \widetilde{v}_i = \frac{\Lambda_i^R}{\delta_R T_i^R} v_i \quad (i \in B_R).$$

If $R_R = \prod_{b \in B_R} (\delta_R T_b^R)^{\deg_R(b)-1}$, then $\widetilde{U}_R = R_R U_R$ and $\widetilde{\kappa}_R = R_R \kappa_R$. The normalized central proper core is $\delta T_h T_v T_\ell q_0$, by the residual-block scaling in Lemma 6.3. Finally $\widetilde{U}_* =$

$R_{V'}R_LR_Ha_hU_*$ when H is nontrivial. Inserting these expressions in the normalized identity cancels every δ and gives precisely (7.9). For example, the first coefficient becomes $a_h(\widetilde{v}_h/v_h)v_h/\kappa_H = \Lambda_h^H v_h/\kappa_H$; the two cross coefficients acquire the reciprocal factors $\delta_L/\delta_{V'}$ and $\delta_{V'}/\delta_L$, so their product becomes the displayed $\lambda_{V'L}\lambda_{LV'}$; the central-core coefficient becomes L^*_{ha}. The singleton case gives (7.10) without that additional scale.

We finish with the canonical divisors. For old boundary marks, connectedness of the whole configuration gives $H_* = H_{V'}H_LH_H$. This refers to old marks only; its extra current x_h was included in U_*. Thus the canonical first coefficient is

$$\frac{\Lambda_h^H v_h}{\kappa_H} H_H \frac{m_*}{m_{V'}m_L} p_J p_{J\cup K_*}.$$

Its internal H marks are nonnegative because κ_H/v_h has exponent zero or one. The cross sources in the second term supply $H_{V'}$ and H_L. In the central-core term, the source from h towards $a \in A_H$ contains the central proper block and all of the V', L branches. One application of Lemma 6.1 extracts $f_{0,h}T_vT_\ell H_{V'}H_L$ and the remaining source-side H-branch port factors. Any negative internal H mark in v_hv_a lies on the endpoint path and is supplied there. The remaining V', L monomials are $\kappa_{V'}v_v$ and $\kappa_L v_\ell$, which have nonnegative exponents. This proves positivity after all gauge and port cancellations. The same cross-source extraction proves the singleton case. Every factor used lies in the shared earlier coordinates, cores, or the later union principal. □

7.4. **An affine clique with three residual ports.** For the clique rotation denote the three branches by A', B', U', with central ports a, b, u, where a is selected and b, u are omitted. Branch notation is otherwise unchanged. The central physical core is

$$r_0 = \gamma_0 H_0 q_0 = \frac{P_{B_0}}{x_a x_b x_u}.$$

Here γ_0 and H_0 contain the retained and old omitted members of the clique other than these three ports. Abbreviate the branch port products at the ports by $T_a, T_b, T_u, \Lambda_a, \Lambda_b, \Lambda_u$, and put

$$D_b = \sum_{i\in A_{B'}} k_{bi}\Lambda_i^{B'} v_i, \qquad D_u = \sum_{i\in A_{U'}} k_{ui}\Lambda_i^{U'} v_i,$$
$$D_a = k_{ba}\Lambda_a v_a + \sum_{i\in A_{A'}\setminus\{a\}} k_{ai}\Lambda_i^{A'} v_i.$$

The central clique equalities give $k_{ba} = k_{ua}$. When both omitted branches are nontrivial, let W be the selected union principal divided by $\Delta_{A'}\Delta_{B'}\Delta_{U'}x_bx_u$.

Lemma 7.5. *For two nontrivial omitted branches one has*

$$U_{B'}U_{U'} = \frac{T_a}{\Lambda_a r_0 \kappa_{A'} v_a} W + \kappa_{B'}\kappa_{U'}\frac{D_bD_u}{T_bT_u} + \frac{\kappa_{B'}}{v_b}\frac{\kappa_{U'}}{v_u}\frac{D_a}{r_0 v_a \Lambda_a\Lambda_b\Lambda_u}. \tag{7.13}$$

If $U' = \{u\}$, omit its x_u from the union gauge. Then

$$U_{B'}x_u = \frac{T_a}{\Lambda_a r_0 \kappa_{A'} v_a} W + \frac{\kappa_{B'}}{v_b}\frac{D_a}{r_0 v_a \Lambda_a\Lambda_b}. \tag{7.14}$$

If $B' = \{b\}$ and $U' = \{u\}$, the identity is

$$x_bx_u = \frac{T_a}{\Lambda_a r_0 \kappa_{A'} v_a} U_{A'} + \frac{D_a}{r_0 v_a \Lambda_a}. \tag{7.15}$$

The case in which only B' is a singleton is obtained by interchanging b and u. All these identities become positive shared Laurent identities after their canonical gauges are inserted.

Proof. Begin with normalized effective row roots. Let t_a, t_b, t_u be the three branch phases, and let L_a, L_b, L_u be their cross-row sums. More explicitly,

$$L_b = \sum_{i \in A_{B'}} c_{bi} w_i, \quad L_u = \sum_{i \in A_{U'}} c_{ui} w_i, \quad L_a = c_{ba} w_a + \sum_{i \in A_{A'} \setminus \{a\}} c_{ai} w_i.$$

The outgoing pair equalities make each cross block have constant rows. The selected row-sum identity and the determinant lemma reduce the selected union determinant, after removing its branch factors, to

$$Q = \operatorname{diag}(t_a + L_a, t_b + L_b, t_u + L_u) - \mathbf{1}(L_a, L_b, L_u). \tag{7.16}$$

Set $K_0 = \widetilde{r}_0 w_a w_b w_u$. The endpoint equations and the affine clique numerator give

$$y_b = w_b(t_b + L_b), \qquad y_u = w_u(t_u + L_u), \qquad t_a + L_a = K_0(t_b + L_b)(t_u + L_u).$$

Indeed, $z_a + c_{ba}$ is the affine clique numerator divided by y_a, so multiplication by w_a gives the last equation. Expanding the rank-one determinant in (7.16) and substituting this equation yields

$$\det Q = (t_b + L_b)(t_u + L_u)(K_0 t_b t_u - K_0 L_b L_u - L_a).$$

The additional union gauge is exactly $y_b y_u$. Consequently

$$\widetilde{W} = \frac{\widetilde{\kappa}_{A'} \widetilde{\kappa}_{B'} \widetilde{\kappa}_{U'}}{w_b w_u} (K_0 t_b t_u - K_0 L_b L_u - L_a),$$

which rearranges to

$$\widetilde{U}_{B'} \widetilde{U}_{U'} = \frac{w_b w_u}{\widetilde{\kappa}_{A'} K_0} \widetilde{W} + \widetilde{\kappa}_{B'} \widetilde{\kappa}_{U'} L_b L_u + \frac{\widetilde{\kappa}_{B'} \widetilde{\kappa}_{U'} L_a}{K_0}. \tag{7.17}$$

This calculation uses the entire determinant; no sign is assigned to its individual subtracted intermediate terms.

The common effective root at a, b, u is α, because the central clique has no selected member in the old prefix. Its affine numerator scales by α^{-1} under root normalization, and its three-variable denominator by α^{-3}. Hence

$$\widetilde{r}_0 = \alpha^2 r_0,$$

where the square is essential. The normalized endpoint monomials and cross-row sums are

$$w_a = (\Lambda_a / T_a) v_a, \qquad w_b = (\Lambda_b / \alpha) v_b, \qquad w_u = (\Lambda_u / \alpha) v_u,$$
$$L_a = D_a / T_a, \qquad L_b = D_b / T_b, \qquad L_u = D_u / T_u.$$

Thus $K_0 = r_0 v_a v_b v_u \Lambda_a \Lambda_b \Lambda_u / T_a$. For $R \in \{A', B', U'\}$ put $\zeta_R = \prod_{i \in B_R} a_i^{\deg_R(i) - 1}$. The selected determinant and core-monomial scalings are

$$\widetilde{U}_R = \zeta_R U_R, \quad \widetilde{\kappa}_R = \zeta_R \kappa_R, \quad \widetilde{W} = \zeta_{A'} \zeta_{B'} \zeta_{U'} \alpha^2 W.$$

Substitution in (7.17) cancels α and all ζ factors and gives (7.13).

For a singleton omitted branch, use the two-by-two version of (7.16). Equivalently in the preceding determinant calculation put $w_u = \widetilde{\kappa}_{U'} = 1$, $t_u = y_u$, and $L_u = 0$, and divide the union determinant by the additional factor y_b only. It gives

$$\widetilde{U}_{B'} y_u = \frac{w_b}{\widetilde{\kappa}_{A'} K_0} \widetilde{W} + \frac{\widetilde{\kappa}_{B'} L_a}{K_0}.$$

Here the singleton gauge is $w_u = 1$, rather than the gauge of a nonempty selected branch. Thus

$$K_0 = \frac{\alpha r_0 v_a v_b \Lambda_a \Lambda_b}{T_a}, \qquad \widetilde{W} = \zeta_{A'} \zeta_{B'} \alpha W, \qquad y_u = x_u / \alpha.$$

Together with the unchanged A', B' scalings these equations give (7.14). When both branches are singletons use the central relation itself: $\widetilde{r}_0 y_b y_u = (t_a + L_a)/w_a$. It gives (7.15). This derivation never replaces an empty selected branch by a principal coordinate.

We check the quotients in the final formulas. The old marks satisfy

$$H_W = H_{A'}H_{B'}H_{U'}, \qquad H_0 \mid H_{A'},$$

with each occurrence of H_0 present exactly once in $H_{A'}$. The central marks do not occur in $H_{B'}$ or $H_{U'}$. After canonicalization, the first coefficient, apart from shared retained-original monomials, is

$$\frac{H_{A'}T_a}{\Lambda_a r_0 \kappa_{A'} v_a}.$$

The quotient T_a/Λ_a is a product of port factors. The occurrence H_0 cancels that in r_0. For every internal A' mark, $\kappa_{A'}v_a$ has exponent zero or one, and $H_{A'}$ supplies one. Thus this whole coefficient has a positive Laurent expression.

In a term of D_bD_u, with targets $i \in A_{B'}$ and $j \in A_{U'}$, the source from b towards i contains the central clique and the entire U' branch. The source from u towards j contains the entire B' branch. Lemma 6.1 therefore extracts $T_uH_{U'}$ and $T_bH_{B'}$ from these two sources, respectively. The remaining internal monomials are $\kappa_{B'}v_i$ and $\kappa_{U'}v_j$, with nonnegative mark exponents. For the last term every source in D_a contains both opposite branches and the central clique node. For k_{ba} delete the target a and retain the subclique $\{b, u\}$; for k_{ai} delete i on the A' branch. One source factorization thus extracts $H_{B'}H_{U'}H_0$ together with the required opposite branch ports. The remaining monomials $\kappa_{B'}/v_b$ and $\kappa_{U'}/v_u$ have nonnegative exponents. Within A', the k_{ba} summand cancels v_a; in another summand a negative exponent in v_i/v_a lies on the a–i path and is supplied by that same source. The required port quotient inside A' is the equal-colour path quotient described immediately before (6.33) and supplied by Lemma 6.1; all its negative port occurrences lie strictly before the target. This checks the ports and marks in a single extraction, including coincident factors. The singleton formulas use the corresponding first and last terms of this argument and introduce no extra divisor. All earlier coordinates, cores and the union principal are shared. □

7.5. **Original variables and recognition of the native charts.**

Lemma 7.6. *The identities above in a direction which exchanges a retained original u are Laurent expressions in the genuinely shared coordinates, with no occurrence of u hidden in their coefficients.*

Proof. Keep u a free indeterminate while processing all events before its first readiness event. Its earlier readiness counts are zero. No earlier used core contains it: a proper core has no retained original member, and a used clique containing u would make it ready at that event. The earlier canonical gauges and physical core products are therefore independent of u. The source recurrences start with numerical coefficients and use only source coefficients, nested factors, and phases from these earlier coordinates. The omitted recovery formula uses the constant root and the selected phase, and has no full affine-root term. Induction through these displayed recurrences proves that all preceding source states, nested ports, phases, and recovered omitted values are independent of u. Weighted target states are products of these same functions and retain that independence.

At the first readiness event through an unselected block use (7.4) in reverse, with u a residual endpoint; in particular u is excluded from γ_e. This formula contains no complete affine root. If the completing clique is already selected, keep u as its third central port rather than as a member of γ_0, and use (7.14) or (7.15). The readiness count at u changes from zero to one, so its canonical gauge exponent is still zero. These right-hand sides are independent of u. If u never becomes ready, use (7.8); its summation excludes u, and all its earlier source functions are independent of it. This also proves the independence needed for the inverse singleton exit. All other retained originals remain shared Laurent units. □

Theorem 7.7. *Suppose that $F_i = 1 + \sum_{j \neq i} C_{ij} x_j$ with every $C_{ij} > 0$. All candidates of Definition 5.1 are ordinary LP charts, up to positive coefficient rescaling, and every ordinary LP chart occurs among them. The normalized pattern is finite. With the displayed positive representatives, every actual exchanged product has a nonnegative Laurent expansion in its shared coordinates. Consequently all cluster variables are Laurent-positive in every cluster; moreover $\mathcal{A} = \mathcal{U}$, the cluster monomials form an atomic positive basis, and their multiplication structure constants are nonnegative.*

Proof. Every proposed coordinate is a prime nonunit of $\mathcal{U}$, by Lemma 2.1 and Proposition 4.7. It is positive on the initial positive orthant. Proposition 5.4 lists a proposed partner in every coordinate direction. Lemmas 7.1–7.5 and 7.6 give a nonzero positive Laurent identity for its product with that coordinate, involving only the shared coordinates. The initial candidate is an actual LP chart. If a candidate is actual, Lemma 2.3 identifies each proposed prime partner with the actual mutation partner up to a coefficient unit. Choose the positive associate; this unit is positive, as is seen by evaluating the product identity on the positive initial orthant. Induction on the length of a mutation word therefore puts every actual chart in the candidate collection. Conversely the connectedness statement in Proposition 5.4 connects every candidate to the initial one by these same identities, and gives its actual realization by induction.

There are finitely many candidate coordinate sets by Lemma 5.2. In each set the exchange polynomial in a given direction is recovered, up to a coefficient unit, by removing the coordinate monomial from its exchanged product. Thus the normalized seed pattern is finite as well. All its actual exchanges have the positivity just proved. Apply Theorems 3.6 and 3.5 with the semiring $K_{\geq 0}$ to obtain the final assertions. □

8. Zero weights and refinement by prime factors

We now allow $C_{ij} \geq 0$, retaining the assumption that every F_i is nonconstant. A specialization of coefficients need not commute with ordinary LP mutation: an exchange polynomial may factor, and its exchange Laurent monomial may change. The following argument constructs the limiting charts first and then identifies their prime refinements with the actual sparse mutation pattern.

8.1. A finite positive limiting atlas.

Let K_0 be the subfield of $\mathbb{R}$ generated by the finitely many entries of C. For an indeterminate ε put

$$C_{ij}(\varepsilon) = C_{ij} + \varepsilon \qquad (i \neq j). \tag{8.1}$$

Every difference of two rows toward an outside target is unchanged. Thus all outgoing modules, minimal proper-type blocks, and maximal pair cliques are exactly the same before and after completion. In particular admissibility, the gauges $G(J)$, and the finite candidate data are unchanged.

Apply Theorem 7.7 to a positive real ε transcendental over the countable field K_0. Evaluation identifies $K_0(\varepsilon)$ with its real image. The resulting rational-function identities therefore define one finite atlas over $K_0(\varepsilon)$, with the coordinates

$$x_i, \qquad p_J(\varepsilon), \qquad q_B(\varepsilon).$$

Each is a forest numerator of constant term one divided by a fixed initial-coordinate monomial. Every coordinate has a nonzero limit at $\varepsilon = 0$, and its value at a positive initial point has a finite strictly positive limit. Write $\mathcal{G}$ for the finite set of these limiting coordinates, identifying duplicates.

Lemma 8.1. *Each limiting chart is a transcendence basis of $K(x)$. Every member of $\mathcal{G}$ is a Laurent polynomial with nonnegative coefficients in every limiting chart and belongs to the upper*

algebra of the actual sparse LP seed. Adjacent limiting charts satisfy a nonzero positive Laurent exchange in their shared coordinates.

Proof. In a fixed generic chart, each Laurent coefficient of a fixed generic coordinate is a rational function of ε. The strict-positive theorem makes it nonnegative at every positive transcendental value over K_0. These values are dense. Each coefficient is consequently nonnegative on a sufficiently small positive interval on which it has no pole.

Evaluate the Laurent expression at one fixed positive initial point. Its left side remains bounded, and all its finitely many terms are nonnegative. The chart coordinates are bounded away from zero and infinity. Each coefficient is therefore bounded as $\varepsilon \to 0^+$, so its rational function has a finite nonnegative limit. The same argument applies to the shared-coordinate expression for each adjacent exchange product. The product of its two limiting coordinates is strictly positive at the chosen point; hence the limiting exchange is not zero.

Starting at the initial chart and using connectedness of the candidate graph, each adjacent limiting substitution $f \mapsto E(\mathbf{s})/f$, with $E \neq 0$, is birational. Thus all limiting charts remain transcendence bases, and the coefficient limits just obtained are their actual Laurent expansions.

In the complete pattern the first neighbor at i has coordinate $F_i(\varepsilon)/x_i$: all the complete initial hats are trivial. Indeed a divisibility by $F_j(\varepsilon)$ would force a polynomial independent of x_i to be divisible by a polynomial which involves x_i. Its limit is F_i/x_i. The actual sparse first neighbor $\widehat{F}_i/x_i$ differs from it by a Laurent monomial in the shared initial coordinates. Their Laurent rings are therefore equal. Every $g \in \mathcal{G}$ is Laurent in the initial chart and these first-neighbor charts. The starfish identity, Lemma 2.1, gives $g \in \mathcal{U}$. □

8.2. **Prime factors remain in the limiting set.** For an irreducible polynomial P with no coordinate factor write

$$Q_P = P \prod_i x_i^{-\nu_i(P)}.$$

By Lemma 2.2, this is a prime element of the actual sparse upper algebra, with zero valuation at every initial coordinate prime.

Lemma 8.2. *Every limiting core q_B is a prime element of $\mathcal{U}$. Its numerator has valuation one at each member of B and zero at every outside vertex.*

Proof. For a proper-type minimal block, Proposition 4.3 still gives absolute irreducibility: its statement allows arbitrary real coefficients. At an inside vertex i, a proper terminal component of the row-i difference support on $B \setminus \{i\}$ would give a proper nontrivial submodule. That support is therefore strongly connected, and (4.14) gives valuation one. At an outside vertex h, setting $x_h = 0$ merely changes the affine root. The homogeneous restriction is still an invertible t-translation of an absolutely irreducible polynomial of degree $|B| - 1 \geq 2$. It cannot be divisible by the linear F_h^h. Equation (4.13), applied to the core factor of D_B, gives valuation zero.

For a clique, its numerator P_B is affine, and $P_B|_{x_i=0} = F_i$ for every $i \in B$. These polynomials are nonconstant by hypothesis, so P_B is a nonconstant irreducible affine polynomial. An outside positive valuation would imply $P_B|_{x_h=0} = F_h$ and would join h to every pair in the clique, contrary to maximality. These arguments do not require positive internal coefficients. The asserted primality follows from the canonical-lift lemma. □

Let J be admissible. If $\mathrm{SCC}(J)$ denotes the strongly connected components of the support induced by $C[J]$, block triangularity gives

$$D_J(t,x) = \prod_{S \in \mathrm{SCC}(J)} D_S(t,x). \tag{8.2}$$

The diagonal entries in every factor retain all original outside-target contributions.

Lemma 8.3. *For each strongly connected admissible S, the polynomial $D_S(1,x)$ is absolutely irreducible. Its canonical prime lift is a member of $\mathcal{G}$: it is either the limiting principal coordinate p_S or a limiting core q_B.*

Proof. Adjoin the zero-row vertex carrying all targets outside S. A proper nontrivial module avoiding this vertex would be a global module contained in S, which admissibility forbids. A proper module containing it would contain a nonempty proper $T \subset S$ and force $C_{ij} = 0$ for $i \in T$, $j \in S \setminus T$. This contradicts strong connectivity. Proposition 4.3 gives absolute irreducibility of $D_S(t,x)$ and hence of its dehomogenization. For a singleton this is directly the valid nonconstant affine exchange F_i.

We compute the canonical denominator. Fix $h \notin S$ and use the shifted system on $S \cup \{*\}$ after $x_h = 0$. Its internal edges have coefficients $C_{vj} - C_{hj}$, its edges from $*$ have coefficients $-C_{hj}$, and an edge from v to $*$ survives precisely when $C_{vk} \neq C_{hk}$ for some $k \notin S \cup \{h\}$. A terminal component T avoiding $*$ gives the module $T \cup \{h\}$. It is globally minimal: a smaller module avoiding h lies in S, while a smaller one containing h contradicts strong connectivity of T in the difference graph. Conversely each ready minimal block at h gives one such component. The fused-clique convention counts it once, since S contains at most one member of each clique. Their number is $d_h(S)$.

Suppose first that $C_{hj} > 0$ for some $j \in S$. Then $*$ reaches every vertex of S. To see this, follow an original directed path in $C[S]$. An edge $v \to k$ either survives as a difference edge, or satisfies $C_{vk} = C_{hk} > 0$, in which case there is a direct edge $* \to k$. Strong connectivity completes the reachability argument. If $d_h(S) > 0$, the terminal components avoiding $*$ are all terminal components; otherwise the unique terminal component contains $*$. Lemma 4.6 therefore gives

$$\nu_h(D_S) = \max(d_h(S) - 1, 0). \tag{8.3}$$

Suppose instead that $C_{hj} = 0$ for every $j \in S$. Now $*$ is terminal and the internal graph is exactly the strongly connected graph $C[S]$. The latter is an additional terminal component exactly when $S \cup \{h\}$ is an outgoing module. Consequently

$$\nu_h(D_S) = \begin{cases} 1, & S \cup \{h\} \text{ is an outgoing module,} \\ 0, & \text{otherwise.} \end{cases} \tag{8.4}$$

In the exceptional case $M = S \cup \{h\}$ is globally minimal. A smaller module avoiding h violates admissibility; one containing h would give a proper nonempty outgoing-closed subset of $C[S]$, because row h is zero toward S. The internal h-row in M is zero, so its quotient determinant satisfies

$$P_M = H_{C[M]}(a_M, x_M) = D_S(1,x). \tag{8.5}$$

If $|S| \geq 2$, this is a proper-type core. There cannot be two different exceptional vertices: the intersection of their two minimal modules would be the nontrivial module S, contrary to admissibility. If $S = \{i\}$, all exceptional pairs belong to one maximal pair clique B. Indeed two exceptional vertices have equal rows toward i, both coefficients being zero; toward every other outside target their equalities propagate through i. They therefore also form a pair module. The clique's target-i coefficient is zero, and its affine core numerator is precisely $F_i = D_S$. Thus in either case Lemma 8.2 identifies Q_{D_S} with the corresponding q_B.

Finally let $i \in S$. After $x_i = 0$, the shifted system has the isolated vertex i. Removing it, $*$ reaches every remaining vertex: follow a simple original path from i and use the same direct-jump argument when a difference edge vanishes. A terminal component avoiding $*$ would give a nontrivial module contained in S. It is therefore impossible. There are exactly two terminal components before removing i, giving

$$\nu_i(D_S) = 1. \tag{8.6}$$

The singleton case follows directly from $D_S = F_i$. If no exception in (8.4) occurs, the valuations (8.3) and (8.6) are exactly the completed principal gauge, so $Q_{D_S} = p_S$. This proves the lemma. □

Proposition 8.4. *Every $g \in \mathcal{G}$ is a positive constant times a monomial with nonnegative exponents in prime elements of $\mathcal{U}$ which themselves belong to $\mathcal{G}$.*

Proof. This is immediate for initial coordinates and for cores. For p_J, factor its numerator by (8.2) and replace each irreducible factor by its canonical prime lift, using Lemma 8.3. The remaining factor is a monomial in initial coordinates:

$$p_J = \prod_i x_i^{e_i} \prod_{S \in \mathrm{SCC}(J)} Q_{D_S}.$$

Every canonical lift has zero valuation at each initial coordinate prime. Since $p_J \in \mathcal{U}$ by Lemma 8.1, applying ν_i to this identity gives $e_i = \nu_i(p_J) \geq 0$. All chosen normalizations are positive. This proves the assertion, including repeated factors and coincidences among labels. □

8.3. **From limiting charts to actual LP clusters.** We give the refinement argument explicitly. Let $\mathbf{f} = (f_1, \dots, f_n)$ be a limiting chart and let $Q(\mathbf{f})$ be the set of distinct prime factors of its coordinates, as supplied by Proposition 8.4. Every $q \in Q(\mathbf{f})$ belongs to $\mathcal{G}$, hence to $K[\mathbf{f}^{\pm 1}]$. Since it divides the unit f_i in this Laurent ring, it is itself a unit there. Thus

$$f_i = c_i \prod_{q \in Q(\mathbf{f})} q^{A_{qi}}, \qquad q = d_q \prod_i f_i^{B_{iq}}, \qquad c_i, d_q > 0, \quad A_{qi} \in \mathbb{N},\ B_{iq} \in \mathbb{Z}. \tag{8.7}$$

Distinct prime elements of $\mathcal{U}$ are multiplicatively independent modulo $K^\times$, by primality and $\mathcal{U}^\times = K^\times$. Their exponent vectors in the Laurent ring of $\mathbf{f}$ are therefore linearly independent, so $|Q(\mathbf{f})| \leq n$. On the other hand, the first expressions in (8.7) and $\operatorname{trdeg} K(\mathbf{f}) = n$ give $|Q(\mathbf{f})| \geq n$. Hence there are exactly n prime coordinates, they form a transcendence basis, and the two integer matrices are inverse:

$$A \in \mathrm{GL}_n(\mathbb{Z}), \qquad A \geq 0. \tag{8.8}$$

Call $Q(\mathbf{f})$ the *prime chart* of $\mathbf{f}$. Every member of $\mathcal{G}$ has a nonnegative Laurent expansion in this prime chart, by substituting the positive monomials (8.7).

Lemma 8.5. *Prime charts of adjacent limiting charts either coincide or differ by an actual ordinary LP mutation. Every prime chart is an actual cluster of the sparse seed.*

Proof. The $n-1$ shared limiting coordinates are algebraically independent. The union S of their prime supports has cardinality at least $n-1$, and belongs to both prime charts, each of cardinality n. If $|S| = n$, the prime charts coincide. Otherwise $|S| = n-1$. Let p, p' denote their remaining primes. If these agree the charts again coincide.

Suppose they differ. In the exponent matrix A, the row of p vanishes on every shared column. Expanding $\det A = \pm 1$ in that row shows that the exponent of p in the unshared coordinate is one and that the shared $(n-1) \times (n-1)$ exponent matrix is unimodular. The same holds for p'. Consequently

$$f = c\, p M(\mathbf{q}), \qquad f' = c'\, p' M'(\mathbf{q}) \qquad (\mathbf{q} = S),$$

where M, M' are monomials. The inherited positive exchange $f f' = E(\text{shared } \mathbf{f})$ becomes

$$pp' = \frac{E(\text{shared } \mathbf{f})}{cc' M(\mathbf{q}) M'(\mathbf{q})} \in K_{\geq 0}[\mathbf{q}^{\pm 1}] \setminus \{0\}.$$

Starting with an actual prime chart, Lemma 2.3 identifies the prime $p' \in \mathcal{U}$ with its actual mutation partner, up to a positive coefficient unit. The prime chart of the initial limiting chart is the initial cluster. Connectedness of the finite limiting atlas now proves the second assertion and, along the way, the first. □

It remains to show that these finitely many native prime charts include every actual sparse cluster. This last step cannot be inferred merely from connectedness. Fix a rationally independent weight w on the initial coordinates. Every limiting coordinate has a well-defined leading exponent $g(f) \in \mathbb{Z}^n$. Along an adjacent positive exchange, the new leading vector equals the negative of the removed one plus an integral combination of the shared leading vectors. Beginning with the initial basis shows that every limiting chart has a unimodular leading matrix. Its cone

$$\sigma_{\mathbf{f}} = \operatorname{cone}\{g(f_1), \dots, g(f_n)\}$$

is therefore full-dimensional and simplicial. Across each facet its adjacent chart lies on the opposite side of the facet hyperplane and covers the other half of a neighborhood of every relative interior point of that facet.

The union of these finitely many cones is all of $\mathbb{R}^n$. For completeness, if it were not, join a generic interior point of one cone to a point outside their closed union. Choose the endpoint so the segment misses every face of codimension at least two. At its first exit it lies in the relative interior of a facet of some cone. The adjacent cone covers the other side locally, contradicting the definition of first exit. This argument uses no fan property for the limiting cones.

For the prime chart $\mathbf{q} = Q(\mathbf{f})$, (8.7) gives $G_{\mathbf{f}} = G_{\mathbf{q}}A$. Since A is invertible and nonnegative,

$$\sigma_{\mathbf{f}} \subseteq \sigma_{\mathbf{q}}, \qquad \operatorname{int}\sigma_{\mathbf{f}} \subseteq \operatorname{int}\sigma_{\mathbf{q}}. \tag{8.9}$$

Indeed every row of A has a positive entry, so it sends a vector with all positive entries to another such vector. Let T now be any actual sparse cluster. Choose a point in the interior of its leading cone avoiding the finitely many boundaries of the limiting cones. Completeness places it in the interior of some $\sigma_{\mathbf{f}}$, and then (8.9) places it in the interior of a known native prime-chart cone. The exact intersection property of Lemma 3.3, which does not assume finiteness, forces these two native clusters to be identical up to coefficient units. Thus every actual cluster is a prime chart.

Theorem 8.6. *For every valid numerical linear seed $F_i = 1 + \sum_{j\neq i} C_{ij}x_j$ with $C_{ij} \geq 0$, the normalized ordinary LP pattern is finite. Every cluster variable, and every actual exchange product, has nonnegative Laurent coefficients in every cluster in which it is expressed. The algebra equals its upper algebra; cluster monomials form an atomic positive basis, with nonnegative multiplication constants.*

Proof. Every actual cluster is one of the finitely many prime charts. Each of its coordinates is a member of $\mathcal{G}$, and the paragraph following (8.8) gives its nonnegative Laurent expansion in every prime chart. The same holds for products, in particular actual exchange products. Theorems 3.5 and 3.6 give all remaining assertions. □

9. The main theorem and the necessity of linearity

Proof of Theorem 1.1. Replace each initial exchange polynomial by F_i/a_i. This is an allowed positive coefficient-unit normalization, and gives (4.1) with $C_{ij} = c_{ij}/a_i \geq 0$. Theorem 8.6 applies, including arbitrary equalities among the weights. It provides positive representatives, finiteness of the normalized pattern, and Laurent positivity in every cluster. For a mutation $xx' = \widehat{F}_x$, the positive Laurent expansion of x' in the cluster containing x is $\widehat{F}_x/x$. Thus $\widehat{F}_x$ has nonnegative coefficients. Multiplication by the coordinate monomial which removes its negative exponents gives the ordinary exchange polynomial in a positive normalization, with the same coefficient signs.

Theorem 3.5 gives $\mathcal{A} = \mathcal{U} = \bigoplus_{b\in\mathscr{B}} Kb$. The coefficient-extraction formula (3.8) shows that a universally nonnegative element has nonnegative basis coefficients. Conversely every nonnegative linear combination of cluster monomials is universally nonnegative, by the Laurent positivity just proved. This is the claimed atomicity. A product of two cluster monomials is

universally nonnegative; applying the same extraction formula gives nonnegative multiplication constants. □

Corollary 9.1. *Let an ordinary LP seed over $\mathbb{Z}$ have primitive nonconstant linear exchanges*

$$F_i = a_i + \sum_{j \neq i} c_{ij} x_j, \qquad a_i \in \mathbb{Z}_{>0}, \quad c_{ij} \in \mathbb{Z}_{\geq 0}.$$

After choosing signs consistently, every cluster variable has Laurent coefficients in $\mathbb{Z}_{\geq 0}$ with respect to every cluster.

Proof. Localize the coefficient ring to $\mathbb{Q}$. Gauss's lemma preserves irreducibility of each primitive nonconstant initial exchange. At each mutation, localization preserves the multiplicity of every nonconstant irreducible factor: the factors removed in the slice rule are the same up to coefficient units. Polynomial parts differ only by rational units. Induction therefore identifies the localized pattern with the ordinary field pattern. No constant exchange can appear in the integral pattern during this induction, because it would become a unit, whereas the corresponding field mutation is a valid seed with nonunit exchanges. Choose the sign of each integral representative so it is positive on the initial positive orthant.

Theorem 1.1 gives nonnegative rational Laurent coefficients for these representatives, since positive rational rescalings preserve the conclusion. The ordinary Laurent phenomenon over $\mathbb{Z}$ gives integral coefficients for the very same expansion in any integral cluster. Uniqueness of Laurent coefficients makes them nonnegative integers. □

Remark 9.2 (The linearity assumption)**.** Nonnegative initial exchange coefficients do not imply Laurent positivity for nonlinear LP algebras, even over $\mathbb{Z}$. Let $m \geq 3$ be odd and consider the rank-three seed

$$F_a = 1 + b + c, \qquad F_b = 1 + a + c^m, \qquad F_c = 1 + a.$$

These exchange polynomials satisfy the LP axioms: each is primitive and linear in a variable with coefficient 1, omits its own variable, and has constant term 1. Put

$$Q_m(c) = \sum_{j=0}^{m-1} (-c)^j, \qquad (1 + c)Q_m(c) = 1 + c^m.$$

The exchange Laurent polynomials of the initial seed are

$$\widehat{F}_a = F_a, \qquad \widehat{F}_b = F_b/c, \qquad \widehat{F}_c = F_c.$$

Indeed, the only possible denominator is c in $\widehat{F}_b$: all other pairs of variables occurring in the exchange polynomials occur in both directions. Substituting $c = (1 + a)/z$ gives

$$F_b\big|_{c=(1+a)/z} = (1 + a)\big(1 + (1 + a)^{m-1} z^{-m}\big),$$

which has exactly one factor $1 + a$.

Mutate first at a and write $u = (1 + b + c)/a$. To update F_b, the mutation slice is $1 + c$, and the substituted expression is

$$1 + \frac{1 + c}{u} + c^m = \frac{(1 + c)(1 + uQ_m(c))}{u}.$$

Removing the slice factor and clearing the mutable monomial gives $F_b' = 1 + uQ_m(c)$. There is no further slice factor, since $Q_m(-1) = m$ and $1 + mu \neq 0$. The update of F_c similarly gives $F_c' = 1 + u + b$, with no slice factor removed. Thus the new exchange polynomials, in the coordinates (u, b, c), are

$$F_u = 1 + b + c, \qquad F_b' = 1 + uQ_m(c), \qquad F_c' = 1 + u + b.$$

All pairs now occur in both directions, so all three exchange Laurent polynomials equal their exchange polynomials. Mutation at b therefore produces the actual cluster variable

$$v = \frac{1 + uQ_m(c)}{b} = \frac{1 + a + c^m + bQ_m(c)}{ab}.$$

The coefficient of $a^{-1}c$ in this Laurent expansion is -1, whereas the coefficient of $a^{-1}b^{-1}$ is 1. These are distinct monomials, and multiplying v by either unit of $\mathbb{Z}$ cannot make all its coefficients nonnegative. The same two signs rule out every nonzero scalar multiple over a real coefficient field. For example, $m = 3$ gives

$$v = \frac{1 + a + b - bc + bc^2 + c^3}{ab}.$$

Thus Laurent positivity can fail after two ordinary LP mutations from a nonlinear seed whose initial nonzero coefficients are all 1.

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