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BROUÉ'S ABELIAN DEFECT GROUP CONJECTURE FOR 2-BLOCKS

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Abstract. We prove that every 2-block of a finite group with an abelian defect group is splendidly Rickard equivalent to its Brauer correspondent.

1. Introduction

Let $(K, \mathcal{O}, k)$ be a splitting 2-modular system: $\mathcal{O}$ is a complete discrete valuation ring of characteristic zero, k is its algebraically closed residue field of characteristic two, and K is its field of fractions, sufficiently large for the finite groups considered below. For a finite group G, a block of $\mathcal{O}G$ is a primitive central idempotent a; its block algebra is $\mathcal{O}Ga$. If P is a defect group of a, Brauer's first main theorem associates to a a block a_P of $\mathcal{O}N_G(P)$, called its Brauer correspondent. Broué's abelian defect group conjecture predicts an equivalence

$$D^{\mathrm{b}}(\mathcal{O}Ga\text{-mod}) \simeq D^{\mathrm{b}}(\mathcal{O}N_G(P)a_P\text{-mod})$$

when P is abelian. Here D^{b} denotes the bounded derived category of finitely generated modules.

The conjecture grew out of Broué's study of perfect isometries and their compatibility with local character theory. His 1990 paper [8] develops isotypies as compatible systems of perfect isometries associated with Brauer pairs. Conjecture 6.1 concerns such a local character correspondence; Question 6.2 asks for the corresponding equivalence of derived categories over $\mathcal{O}$. The original local block is written on $N_G(P, e_P)$, where e_P is the block of $\mathcal{O}C_G(P)$ in a maximal Brauer pair. Fong–Reynolds correspondence gives the formulation with the Brauer correspondent on $N_G(P)$. Section 3 relates derived equivalences to perfect isometries, and the appendix treats principal 2-blocks of simple groups with abelian Sylow 2-subgroups. These characteristic-two families formed part of the early evidence for the conjecture. Section 6 also proposes a geometric construction using Deligne–Lusztig cohomology, with the local block algebra as the endomorphism algebra of a tilting complex.

Rickard's tilting theory [31] characterizes derived equivalences by complexes of projective modules whose endomorphism algebras recover the other algebra. In the two-sided formulation, a complex of bimodules has an inverse for the derived tensor product. His splendid equivalences [33] impose a permutation condition on the terms. This condition permits Brauer constructions at subgroups of the defect group and connects the global equivalence with local block algebras. For cyclic defect groups, Broué's discussion following Question 6.2 credits Rickard with the equivalence over k and Linckelmann with its integral form [8, Section 6]. Harris and Linckelmann [19] proved the splendid form for blocks of finite p-solvable groups with abelian defect groups. Source algebras and endopermutation modules provide the connection between the Morita equivalences arising in these reductions and the required complexes. Dade introduced endopermutation modules and classified them for abelian p-groups in [13, 14]. In that classification, augmentation kernels for quotient groups generate the source classes. Rickard's endosplit resolutions [33] replace such

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modules by permutation complexes whose endomorphism complexes split. The source-algebra formulation in [24] relates these resolutions to block fusion and to stable equivalences with endopermutation source.

Külshammer and Puig [23] described extensions of nilpotent block algebras by a matrix factor arising from an endopermutation module and a twisted algebra of a local extension group. This construction reduces a block covering a nilpotent normal subblock to a block on a group controlled by the quotient and the lower defect group. Puig and Zhou [30, Section 3] give the graded source-corner construction used in this reduction. Coconeţ, Marcus and Todea [10, Theorem 1.1(b)] express this construction as a graded basic Morita equivalence and describe its correspondence of local pointed groups. Puig [29, Theorem 3.13] proved that a normal subblock of an inertial block is inertial. These results supply the nilpotent and inertial reductions in the proof below.

Chuang and Rouquier [9] constructed derived equivalences by $\mathfrak{sl}_2$-categorification and proved Broué's conjecture for symmetric groups, with related results for general linear groups. For groups of Lie type, Rickard's equivariant cohomology construction [32] and Rouquier's étale chain complexes [35] associate permutation complexes to Deligne–Lusztig varieties. Bonnafé, Dat and Rouquier [2] established a splendid Jordan decomposition when the dual centralizer is contained in the chosen Levi subgroup. Ruhstorfer [36] developed equivariant and disconnected versions of this construction. These results provide the geometric component equivalences. Section 6 constructs their actions by the full marked stabilizers and verifies the upper-group compatibility used in the assembly.

Marcus [27] developed the reduction of principal blocks with abelian Sylow subgroups to simple components. His construction treats extension of Rickard complexes to groups of prime-to-p index and complexes for wreath products. It makes the compatibility with the ambient automorphisms part of the reduction data. The principal 2-block case is due to Marcus; see [12, Theorem 4.36(a)]. Write C_n for a cyclic group of order n and $V_4 = C_2 \times C_2$. The classification of Klein four source algebras by Craven, Eaton, Kessar and Linckelmann [11, Theorem 1.1] identifies three source algebra types, represented by $\mathcal{O}V_4$, $\mathcal{O}A_4$, and the principal block of $\mathcal{O}A_5$. The nonprincipal elementary abelian block of Co_3 was treated by Koshitani, Müller and Noeske [22]. Eaton, Kessar, Külshammer and Sambale [15, Theorem 6.1] subsequently classified the 2-blocks of quasisimple groups with abelian defect groups up to Morita equivalence. Their classification separates the elementary and Klein four cases, certain rank-one Levi cases, and blocks covered by nilpotent blocks. Its proof also identifies the groups and central quotients that occur in the Levi cases.

This classification has led to results for arbitrary finite groups with prescribed defect groups. Eaton and Livesey [16], together with Wu, Zhang and Zhou [37], completed the classification for abelian 2-groups of rank at most three. Ardito and Sambale [1] proved Broué's conjecture for elementary abelian defect groups of order 32. Eaton and Livesey [17] classified the Morita equivalence classes for abelian defect groups of rank at most four and obtained the derived equivalence with the Brauer correspondent [17, Corollary 1.3]. The rank here is the minimal number of generators. Together these results cover all abelian defect groups of order dividing 32. The source and permutation structure of the equivalence is an additional issue for the splendid form [17, Remark 1.4].

There has also been recent progress without a bound on the rank. Zhang and Zhou [39, Theorem 1.1] proved that blocks with defect group

$$C_{2^{n_1}} \times \cdots \times C_{2^{n_t}}, \qquad n_i \geq 2,$$

are inertial. Zhou and Zhang [40, Theorem 1.1 and Corollary 1.2] classified 2-blocks with abelian defect groups whose inertial quotient has prime order, and deduced Broué's conjecture for

those blocks. For a maximal Brauer pair (P, e_P), the inertial quotient is $N_G(P, e_P)/(PC_G(P))$, including its conjugation action on P. Zhang and Zhou's extension theorem [38, Theorem 1.3] treats extensions of inertial blocks under an inertiality hypothesis after extension by the upper defect group. Miller [28] characterized endosplit permutation resolutions and studied their descent over coefficient fields. Huang [20, Theorems 1.1 and 1.2] obtains fusion-stable endosplit resolutions for endopermutation modules whose Dade classes are generated by relative syzygies, and applies them to Morita equivalences with such sources. For abelian defect groups, Dade's classification gives the source condition. The abelian resolution follows Linckelmann's construction [25, Theorem 7.11.9(i)], as developed in [20, Proposition 4.3]. Section 2 records this resolution and the resulting splendid Rickard complex with the primitive blocks and defect identification used in the induction.

Passing from a quasisimple component to a block of an arbitrary finite group requires compatibility with the automorphisms induced by the ambient group. Zhou's equivariant reduction [41, Theorem 1.1 of the preprint] transfers suitably stable principal-block complexes from nonabelian simple groups to normal subgroups of odd index with abelian Sylow 2-subgroups. His formulation distinguishes stability under automorphisms from extension to the corresponding group of operators. In the present argument the component complexes carry actions of their marked stabilizers, and their restrictions to the upper defect group come from specified upper complexes. This permits tensoring the components over their shared centers and inducing to the ambient group.

Theorem 1.1. *Let G be a finite group and let a be a block of $\mathcal{O}G$ with abelian defect group P. Put $\Delta P = \{(u, u) : u \in P\} \leq G \times N_G(P)$. There is a bounded complex X of $(\mathcal{O}Ga, \mathcal{O}N_G(P)a_P)$-bimodules whose terms are 2-permutation lattices, projective on both sides and relatively ΔP-projective, such that*

$$X \otimes_{\mathcal{O}N_G(P)a_P} X^\vee \simeq \mathcal{O}Ga, \tag{1.1}$$

$$X^\vee \otimes_{\mathcal{O}Ga} X \simeq \mathcal{O}N_G(P)a_P \tag{1.2}$$

in the respective homotopy categories of bimodule complexes. Here $X^\vee = \mathrm{Hom}_{\mathcal{O}}(X, \mathcal{O})$, with the dual differential. In particular, these blocks are splendidly Rickard equivalent.

Reduction modulo the maximal ideal of $\mathcal{O}$ gives the corresponding statement over every algebraically closed field of characteristic two.

We prove the theorem by induction on

$$\kappa(G) = |G : O_{2'}(Z(G))|, \tag{1.3}$$

where $O_{2'}(Z(G))$ is the odd-order subgroup of the center. Section 2 resolves Morita bimodules with endopermutation source and constructs their local return. Section 3 proves the central quotient and product statements needed for the inertial case. Sections 4–6 construct the component complexes, including their upper actions. Section 7 assembles one orbit of components using transporters that preserve the common defect group. Section 8 completes the reduction to a proper normalizer.

At each reduction, let $R : A \longrightarrow C$ be the chosen equivalence to a block of a smaller group. Let B_A, B_C be their Brauer correspondent block algebras at the chosen defect groups, and let $R_P : B_A \longrightarrow B_C$ be its maximal Brauer companion. Induction gives $Z : C \longrightarrow B_C$. Its maximal companion Z_P is an autoequivalence of B_C, since the defect group is normal in the group underlying B_C. The return is a reduced integral representative of

$$R \otimes_C Z \otimes_{B_C} Z_P^\vee \otimes_{B_C} R_P^\vee. \tag{1.4}$$

The Brauer-pair correspondence of Breland and Miller [4] shows that $Z \otimes Z_P^\vee$ matches the chosen maximal roots, and that the final complex has the fixed diagonal vertex bound. In the inertial case, the component argument establishes inertiality of the block of $PF^*(G)$, where $F^*(G)$ is the generalized Fitting subgroup, so the upper-block hypothesis of [38, Theorem 1.3] applies.

2. Sources and local complexes

2.1. **Conventions.** For a finite group G, let $O_2(G)$ and $O_{2'}(G)$ be its largest normal 2-subgroup and largest normal odd-order subgroup. A quasisimple group is perfect with simple central quotient; a component is a subnormal quasisimple subgroup. The layer $E(G)$ is the product of the components, the Fitting subgroup $F(G)$ is the largest nilpotent normal subgroup, and $F^*(G) = F(G)E(G)$. For a positive integer n, let n_2 be its largest power-of-two divisor. For a finite abelian group A, write A_2 for its unique Sylow 2-subgroup. Write $\mathrm{Syl}_2(G)$ for the set of Sylow 2-subgroups of G, and $\exp(G)$ for the exponent of G. We write A_n and S_n for the alternating and symmetric groups, and Q_8 and D_8 for the quaternion and dihedral groups of order eight.

We regard an $(\mathcal{O}G, \mathcal{O}L)$-bimodule as a module for $G \times L$ by $(g,l)m = gml^{-1}$. For an isomorphism $\eta : P_C \to P$, put

$$\Delta(P, \eta, P_C) = \{(\eta(u), u) : u \in P_C\}.$$

For the identity isomorphism we write ΔP. A marking consists of the chosen maximal Brauer pairs and defect-group identification. Conjugation transports these data simultaneously.

Fix a uniformizer π of $\mathcal{O}$. We use the same letter for a block idempotent and its reduction modulo π, and interpret its Brauer image after reduction. For an algebra A, write $J(A)$ for its Jacobson radical and $K^{\mathrm{b}}(A\text{-mod})$ for the bounded homotopy category of finitely generated A-modules, abbreviated to $K^{\mathrm{b}}(A)$ when the module category is understood. We use analogous notation for bimodules. Complexes are cohomologically graded. Over the coefficient ring $R \in \{\mathcal{O}, k\}$, the notation $X^\vee$ means $\mathrm{Hom}_R(X, R)$; thus $(X^\vee)^n = \mathrm{Hom}_R(X^{-n}, R)$ and its differential is $d^n_{X^\vee} = (-1)^{n+1}(d_X^{-n-1})^\vee$. A complex is reduced if it has no nonzero contractible direct summand. In the bounded categories of permutation summands used here, removing contractible summands makes every differential radical; a homotopy equivalence between reduced complexes is then a chain isomorphism.

A lattice is a finitely generated free $\mathcal{O}$-module. A 2-permutation lattice for a finite group is a direct summand of a permutation lattice; the same definition over k gives a 2-permutation module. If U is a kG-module and $Q \le G$ is a 2-subgroup, its Brauer construction is

$$U(Q) = U^Q \Big/ \sum_{R<Q} \mathrm{Tr}_R^Q(U^R), \qquad \mathrm{Tr}_R^Q(u) = \sum_{x \in Q/R} xu.$$

For a lattice M, write $M(Q) = (k \otimes_{\mathcal{O}} M)(Q)$; for a complex apply this functor termwise. On a group algebra it gives the Brauer homomorphism $\mathrm{Br}_Q : (kG)^Q \to kC_G(Q)$, which deletes the coefficients of elements outside $C_G(Q)$. If Q acts on a normal subgroup H, the same formula defines the relative map $\mathrm{Br}_Q : (kH)^Q \to kC_H(Q)$. Integral local complexes are lifted from these constructions by Lemma 2.3.

An a-Brauer pair is a pair (Q, e) with e a block of $kC_G(Q)$ and $e\,\mathrm{Br}_Q(a) \neq 0$. We use the usual inclusion of Brauer pairs; beneath a fixed maximal pair (P, e_P) there is a unique pair (Q, e_Q) for each $Q \le P$. We write $N_G(Q, e) = \{g \in N_G(Q) : {}^g e = e\}$; an idempotent subscript on an automorphism group likewise denotes its stabilizer. The block fusion system $\mathcal{F}$ has the subgroups of P as objects and the conjugations in G that preserve these pairs as morphisms. A source idempotent is a primitive idempotent $i \in (kGa)^P$ with $\mathrm{Br}_P(i) \neq 0$; $i(kG)i$ is its source algebra.

The integral terminology is defined by reduction. A local point of Q on an algebra is a conjugacy class, under its Q-fixed units, of primitive Q-fixed idempotents with nonzero Brauer image.

Write $A^e = A \otimes A^{\mathrm{op}}$, over the coefficient ring in use. A complex is split if it is homotopy equivalent to its homology with zero differential. An endosplit resolution of a module V is a bounded complex S with $H^0(S) = V$, other homology zero, and $S \otimes S^\vee$ split. An endopermutation P-module is a module whose endomorphism algebra, with the conjugation action of P, is a permutation module. It is capped if it has an indecomposable summand with vertex P; the isomorphism class of such a summand is uniquely determined and is called its cap. Here a vertex is a minimal subgroup for relative projectivity, and a source is an indecomposable module at that subgroup inducing the given summand. The Dade group records capped endopermutation modules up to isomorphism of their caps, with tensor product followed by taking the cap.

Write $D(P)$ for this group. For an injective homomorphism $\varphi : Q \to P$, ${}_\varphi M$ denotes restriction along φ, so that q acts as $\varphi(q)$. The notation applies termwise to complexes. A capped endopermutation module is $\mathcal{F}$-stable if restriction to Q and restriction along φ have the same class in $D(Q)$ for every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$. For an indecomposable capped module this implies ${}_\alpha V \cong V$ for every $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$.

For symmetric block algebras A, B, use the symmetrizing forms given by the coefficient of the group identity. The dual and these forms give the two adjunction evaluations

$$X \otimes_B X^\vee \longrightarrow A, \qquad X^\vee \otimes_A X \longrightarrow B. \tag{2.1}$$

Whenever we call X a Rickard complex, both maps are homotopy equivalences in the indicated bimodule categories. For complexes projective on both sides over symmetric algebras, $X^\vee$ is the tensor functor's left and right adjoint. If it is a two-sided inverse, the adjunction unit and counit are isomorphisms in the homotopy categories. Thus the inverse identities also give these canonical evaluations. Relative to a fixed defect-group isomorphism $\eta : P_C \to P$, a Rickard complex is splendid if its terms are 2-permutation, projective on both sides, and relatively $\Delta(P, \eta, P_C)$-projective. Tensor products use the Koszul sign rule. A complex with an additional group action is always a complex for that finite group, including its inner intersections and multiplication relations. More explicitly, if finite extensions $\widetilde{G}, \widetilde{L}$ of G, L have a common quotient Γ, the group of pairs is

$$\{(g, l) \in \widetilde{G} \times \widetilde{L} : gG = lL \text{ in } \Gamma\}.$$

For a tensor through a middle extension $\widetilde{H}$, an outer pair (g, l) acts by $(g, h)m \otimes (h, l)n$, where h has their common image in Γ. Changing h by an element of H leaves this tensor unchanged by the balanced relation. This defines the tensor functors in the operator categories used below.

2.2. **The source resolution.** We recall two source-theoretic inputs. If a Morita equivalence between blocks $A = kGa$ and $B = kLc$ is induced by an indecomposable bimodule M with endopermutation source and identified defect group P, the source theorem supplies source idempotents i, j, a common fusion system $\mathcal{F}$, and an indecomposable capped $\mathcal{F}$-stable endopermutation kP-module V such that M is a summand of

$$U = Ai \otimes_{kP} T(V) \otimes_{kP} jB, \qquad T(V) = \mathrm{Ind}_{\Delta P}^{P \times P} V. \tag{2.2}$$

The induction functor T is applied termwise to complexes. The graph of the given defect identification is understood. This is the source theorem recalled in [26, Section 5]; its use in this form is also the starting point of [24]. Moreover, every indecomposable summand of the source algebra jBj as a kP-kP-bimodule has the form

$$kP \otimes_{kQ} {}_\varphi kP, \qquad Q \leq P, \quad \varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P). \tag{2.3}$$

The integral version has the analogous permutation basis [24, Proposition 2.5(i)].

Lemma 2.1 ([20, Proposition 4.3]). *Suppose that P is abelian and V is the capped source in (2.2). There is a bounded endosplit 2-permutation resolution S of V such that ${}_\alpha S \cong S$ for every $\alpha \in E = \mathrm{Aut}_{\mathcal{F}}(P)$.*

Proof. The extension axiom for a saturated fusion system on an abelian group extends every fusion morphism to P. The Sylow axiom gives $|E|$ odd, since $\mathrm{Aut}_P(P) = 1$. For $Q < P$, put

$$\Omega_Q = \ker(k[P/Q] \xrightarrow{\varepsilon} k), \qquad \varepsilon\Big(\sum_x a_x x\Big) = \sum_x a_x.$$

Dade's classification [14], in the presentation of [5, Theorem 4.6], gives a unique normal form for $[V] \in D(P)$ in the classes $[\Omega_Q]$: the exponents are integers for noncyclic quotients and lie in $\mathbb{Z}/2\mathbb{Z}$ for cyclic quotients of order at least four. Quotients of order two contribute the identity class. Transport by E permutes these generators. Since $[V]$ is E-stable, its exponents are constant on E-orbits; for the generators of order two in $D(P)$ choose exponents in $\{0, 1\}$.

The complex

$$C_Q = [k[P/Q] \xrightarrow{\varepsilon} k], \qquad \deg k[P/Q] = 0, \quad \deg k = 1,$$

resolves Ω_Q. To verify that it is endosplit, work first over kR for $R = P/Q$. The complex $C_Q \otimes C_Q^\vee$ has three terms, in degrees $-1, 0, 1$, with both extreme terms isomorphic to kR. Its first differential is injective: its component in $kR \otimes_k kR$ is $x \mapsto x \otimes \sum_{r \in R} r$. Its last differential is surjective, since its restriction to that middle summand is $\varepsilon \otimes 1$, up to sign. The algebra kR is symmetric, so kR is both injective and projective. Split off the first injection and then the last surjection. What remains is the degree-zero homology $\Omega_Q \otimes \Omega_Q^\vee$. Inflating the splitting to P proves the assertion for C_Q.

Tensor the required powers of the C_Q, using $C_Q^\vee$ for negative exponents. Tensor products and duals preserve the displayed splittings, so the resulting permutation complex S_0 is endosplit. It has homology only in degree zero, whose cap is V. The constant exponents on E-orbits and the Koszul permutation isomorphisms give ${}_\alpha S_0 \cong S_0$ for each $\alpha \in E$.

The endomorphism complex $\mathrm{End}_k^\bullet(S_0)$ is split as a kP-complex, with degree-zero homology $\mathrm{End}_k(H^0(S_0))$. Taking P-fixed points preserves these explicit homotopies. Consequently the canonical homology map is an algebra isomorphism

$$\mathrm{End}_{K^\flat(kP)}(S_0) \cong \mathrm{End}_{kP}(H^0(S_0)).$$

Choose a primitive projector with image V on $H^0(S_0)$ and lift it through this isomorphism. The bounded homotopy category of permutation summands is idempotent complete, so its image is an endosplit resolution S of V. For $\alpha \in E$, use the chosen transport of S_0 to view the transported projector on the same homology module. Its image is isomorphic to V, since ${}_\alpha V \cong V$. The two primitive projectors are conjugate in $\mathrm{End}_{kP}(H^0(S_0))$, and their lifted images are therefore homotopy equivalent. Choosing S reduced turns these homotopy equivalences into the required chain isomorphisms. □

Proposition 2.2 ([20, Theorem 1.2]). *Let M be a Morita bimodule over k as in (2.2), with P abelian. There is a splendid Rickard complex X over k for the same primitive blocks and defect identification, with $H^0(X) \cong M$ and all other homology zero.*

Proof. Use Lemma 2.1 and form

$$Y = Ai \otimes_{kP} T(S) \otimes_{kP} jB. \tag{2.4}$$

Each term is a summand of an induced diagonal permutation module, projective on both sides. The outside factors are projective over kP, and $T(S)$ is free on either P-side. Thus $H^0(Y) = U$ and all other homology vanishes. The bimodule U is also projective on either side.

Decompose the middle factor jBj in $Y \otimes_B Y^\vee$ using (2.3). For its (Q, φ)-summand, the coefficient complex on the diagonal Q is

$$\mathrm{Res}^P_Q S \otimes_{k\,\varphi} \mathrm{Res}^P_{\varphi(Q)} S^\vee. \tag{2.5}$$

Extend φ to an element of E. The corresponding individual isomorphism from Lemma 2.1 identifies (2.5) with a restriction of $S \otimes S^\vee$, which is split. Inducing this splitting and tensoring with the outside Ai and iA factors gives an A-A-bimodule splitting. The same argument, with iAi in the middle, gives a B-B-bimodule splitting. Consequently

$$Y \otimes_B Y^\vee \simeq (U \otimes_B U^\vee)[0], \qquad Y^\vee \otimes_A Y \simeq (U^\vee \otimes_A U)[0]. \tag{2.6}$$

Symmetric adjunction identifies $\mathrm{Hom}^\bullet_B(Y, Y)$ with $Y \otimes_B Y^\vee$. The invariants formula is

$$\mathrm{End}^\bullet_{A-B}(Y) = \mathrm{Hom}_{A^e}(A, \mathrm{Hom}^\bullet_B(Y, Y)). \tag{2.7}$$

The argument of this additive functor is already split by bimodule homotopies. Applying the functor preserves the given bimodule homotopies and hence the splitting. The canonical homology map is an algebra isomorphism

$$\mathrm{End}_{K^\mathrm{b}(A-B)}(Y) \xrightarrow{\sim} \mathrm{End}_{A-B}(U). \tag{2.8}$$

Lift the idempotent whose image is the prescribed M and split it in the bounded homotopy category of 2-permutation modules. This category is idempotent complete. More concretely, remove contractible summands from Y; null-homotopic endomorphisms then lie in the radical of its finite chain-endomorphism algebra, so the idempotent has a strict lift. Let X be its termwise image.

Both balanced squares of X are retracts of the split squares in (2.6). Their degree-zero homology is respectively $M \otimes_B M^\vee$ and $M^\vee \otimes_A M$. The evaluations (2.1) induce the Morita evaluations, which are isomorphisms onto A and B. A map between split complexes that induces an isomorphism on homology is a homotopy equivalence. □

Lemma 2.3. *Let X be a splendid Rickard complex between blocks $A = kGa$ and $B = kLc$, equipped with a compatible action of a finite group of pairs. Suppose that every term is 2-permutation for this entire finite action group and that the two evaluation cones are contractible in the corresponding full operator categories. Then X has an integral 2-permutation lift with those actions and evaluations. If X has a common group-coset grading and the evaluations and their contractions preserve this grading, the lift, evaluations and contractions can be chosen degree-preserving.*

Proof. Lift the terms to their integral 2-permutation lattices for the whole finite action group. Reduction on Hom spaces is surjective in that group-module category: orbit sums prove it for permutation lattices, and lifting summand projectors proves it for their summands. Without extra operators, tensoring with the Rickard inverse gives

$$\mathrm{Hom}_{K^\mathrm{b}(A-B)}(X, X[n]) \cong \mathrm{Hom}_{K^\mathrm{b}(A-A)}(A, A[n]) = 0 \qquad (n \neq 0). \tag{2.9}$$

With the prescribed operators, the tensor functors stay in the full pair-group categories. In a presentation with a common operator quotient, choose one middle-group lift of a quotient element and use it on both tensor factors. Changing that lift is canceled by the balanced relation, and multiplication retains the specified inner intersections. The stipulated equivariant evaluations make these functors mutually inverse. The inverse-tensor functor sends X to the regular degree-zero bimodule with its diagonal operator action, so the corresponding equivariant homotopy Hom groups also vanish in nonzero degree. In the special case of an odd operator quotient this follows equivalently by taking invariants, which is exact. This homotopy vanishing

supplies the differential lifts. The whole-group permutation hypothesis supplies the termwise lifts and surjectivity on Hom spaces.

Suppose that integral degree-one maps d_m lift the differential of X and satisfy $d_m^2 = \pi^m c_m$, with $m \geq 1$. The equality $d_m d_m^2 = d_m^2 d_m$ shows that the reduction $\bar{c}_m$ is a degree-two cocycle in the same endomorphism complex. By (2.9), there is a degree-one map $\bar{t}_m$ with $d\bar{t}_m + \bar{t}_m d = \bar{c}_m$. Lift it in the full operator category and set $d_{m+1} = d_m - \pi^m t_m$. Then

$$d_{m+1}^2 = \pi^m(c_m - d_m t_m - t_m d_m) + \pi^{2m} t_m^2 \equiv 0 \pmod{\pi^{m+1}}.$$

These successive corrections converge on the fixed finite lattices to a differential, by completeness. The integral evaluation cones reduce to contractible complexes. Their terms are again permutation summands for the full pair groups: before block cuts, balanced permutation bases give balanced orbit sets, and tensor products of the summand projectors descend to the corresponding linearized orbit modules. The regular block terms are invariant summands of the regular permutation bimodules. Thus the same Hom-surjectivity applies to the cone terms. Lift a contraction modulo π to a degree-minus-one map h and put $u = dh + hd$. Then u is a chain automorphism congruent to the identity modulo π, and hu^{-1} is an integral contraction.

In the graded applications, the grading is the group-coset grading: its identity component is a module for the corresponding group of pairs. Lift in that finite-group category and induce to recover the graded terms. For a group grading, project a map on a homogeneous input of degree g to its output of degree g. This projection is bimodule-linear: left and right multiplication of degrees h, k send g to hgk on both sides. It commutes with the Hom differential and applies to the homotopies as well. The same lifting argument therefore stays in the degree-preserving category. Reduction preserves vertices of permutation lattices [4, Theorem 3.2(b)]. Thus the lift retains the prescribed defect graph and is projective on both sides. □

2.3. **The local complex of an equivalence.** The anti-involution $*$ of a group algebra sends g to g^{-1}. For a complex U of 2-permutation bimodules between blocks A, B and an $A \otimes B^{\mathrm{op}}$-Brauer pair $\lambda = (\Delta(Q, \varphi, R), e \otimes f^*)$, put

$$U(\lambda) = eU(\Delta(Q, \varphi, R))f.$$

We say that U supports λ if this complex is noncontractible, and write $\mathrm{BP}(U)$ for its set of supported Brauer pairs. The centralizer block idempotents e, f are called the roots of the pair. The Lefschetz invariant is the alternating sum $\sum_n (-1)^n [U^n]$ in the split Grothendieck group of 2-permutation modules.

Proposition 2.4. *Let $R : A \longrightarrow C$ be an integral splendid Rickard complex between primitive blocks, with identified abelian defect groups. Fix a supported maximal pair*

$$\lambda_R = (\Delta(P, \eta, P_C), e \otimes f^*).$$

There is an integral maximal companion $R_P : B_A \longrightarrow B_C$ between the prescribed Brauer correspondents, supporting this same pair. If $Z : C \longrightarrow B_C$ is splendid Rickard, it has a maximal companion $Z_P : B_C \longrightarrow B_C$ such that a reduced integral representative

$$X \simeq R \otimes_C Z \otimes_{B_C} Z_P^\vee \otimes_{B_C} R_P^\vee \tag{2.10}$$

is splendid Rickard from A to B_A. Its terms are relatively ΔP-projective for the fixed diagonal, and $(\Delta P, e \otimes e^*)$ is a supported maximal pair.

Proof. Brauer constructions in this proof are over k; reduction of an integral complex is understood. Write $A = \mathcal{O}Ga$ and $C = \mathcal{O}G_C c$, and put

$$I = N_G(P, e), \qquad I_C = N_{G_C}(P_C, f), \qquad W = N_{I \times I_C}(\Delta(P, \eta, P_C)).$$

With $\widehat{e}, \widehat{f}$ the block idempotents of the full normalizers corresponding to the two maximal pairs, the modular companion is

$$\widehat{e}\,kN_G(P)e \otimes_{kI} \mathrm{Ind}_W^{I\times I_C}(eR(\Delta(P,\eta,P_C))f) \otimes_{kI_C} f\,kN_{G_C}(P_C)\widehat{f}. \quad (2.11)$$

Theorem 8.5 and Corollary 8.9 of [4] give its two Rickard evaluation identities. Its endpoints are the prescribed Brauer correspondents.

To identify its maximal pair, apply [4, Corollary 7.11(b)] to the Lefschetz invariant of R at the full defect groups. It says that the inertia-level complex inside (2.11) has λ_R as a maximal Brauer pair for its Lefschetz invariant. By [4, Corollary 8.8], this is a Brauer pair of that complex. The two Fong–Reynolds bimodules in (2.11) have, at their respective diagonal maximal pairs, the regular centralizer blocks:

$$e(\widehat{e}\,kN_G(P)e)(\Delta P)e \cong kC_G(P)e, \qquad f(f\,kN_{G_C}(P_C)\widehat{f})(\Delta P_C)f \cong kC_{G_C}(P_C)f.$$

Indeed the fixed group elements in the regular permutation bimodule are precisely the corresponding centralizer elements, and the displayed block cuts retain the indicated root. The blockwise Brauer–tensor formula [4, Theorem 6.7] therefore gives a noncontractible root-cut Brauer complex at λ_R for (2.11). Thus the full-normalizer companion supports the same λ_R, including its two roots and its graph. Remove its contractible summands in the ordinary bimodule category. By [4, Corollary 9.5], its terms have vertices below the graph in λ_R. Lift the resulting complex integrally by Lemma 2.3. The lift has the same Brauer support.

Fix the left root f for Z. By [4, Lemma 8.3 and Corollary 8.8(c)], it extends to a supported maximal pair

$$\lambda_Z = (\Delta(P_C,\theta,P_C), f\otimes g^*)$$

for some root g of B_C and some automorphism θ of P_C. The right defect group is the chosen subgroup P_C: it is normal in $N_{G_C}(P_C)$, so it is the unique defect subgroup of B_C. Apply the preceding companion construction to this pair. The left full-normalizer block is B_C. The right full normalizer is again $N_{G_C}(P_C)$, and B_C is its own Brauer correspondent there. Hence the resulting integral complex is an autoequivalence Z_P of B_C supporting λ_Z.

Let Y denote the four-factor product on the right of (2.10). The ordinary tensor-product assertion [4, Lemma 6.3] makes its terms permutation summands with twisted diagonal vertices, meaning graphs of isomorphisms between 2-subgroups of the endpoints. The terms are projective on both sides. Associativity and the four given Rickard evaluations show that the canonical evaluations of Y are homotopy equivalences: in the first square successively cancel $R_P^\vee R_P$, $Z_P^\vee Z_P$, $ZZ^\vee$, and $RR^\vee$; in the second square cancel $R^\vee R$, $Z^\vee Z$, $Z_P Z_P^\vee$, and $R_P R_P^\vee$.

It remains to recover the fixed diagonal and root. Put

$$E = kC_G(P)e, \qquad F = kC_{G_C}(P_C)f, \qquad F_g = kC_{G_C}(P_C)g.$$

By [4, Theorem 8.5], the local complexes of R, R_P at λ_R are Rickard equivalences from E to F, and those of Z, Z_P at λ_Z are Rickard equivalences from F to F_g. Consequently

$$R(\lambda_R)\otimes_F Z(\lambda_Z)\otimes_{F_g} Z_P(\lambda_Z)^\vee \otimes_F R_P(\lambda_R)^\vee$$

is a Rickard autoequivalence of E, so it is noncontractible. Iterating [4, Theorem 6.7 and Proposition 8.2] exhibits this complex as a direct summand of the restriction of $eY(\Delta P)e$ to its centralizer bimodule category: the chosen intermediate roots are f, g, f, and the graph maps compose as $\eta\theta\theta^{-1}\eta^{-1} = \mathrm{id}_P$. Thus $(\Delta P, e\otimes e^*)$ belongs to $\mathrm{BP}(Y)$. It is maximal by [4, Corollary 8.8(c)].

Finally choose a reduced integral representative. For permutation lattices and their summands, reduction on group-module Hom spaces is surjective, and their finite endomorphism algebras are complete. Therefore a summand projector of a term after reduction lifts integrally. If the reduction of the current complex has an invertible differential component between two summands, lift their projectors. The corresponding integral differential component is invertible by Nakayama's lemma, and Gaussian elimination splits off an integral contractible two-term complex. Repeating this finite process produces $X \simeq Y$ such that $k \otimes_{\mathcal{O}} X$ has no contractible direct summands. Its evaluations and Brauer support are unchanged. By [4, Proposition 5.4], its reduction is indecomposable as a complex. Corollary 9.5 of [4], applied to its supported maximal pair $(\Delta P, e \otimes e^*)$, now puts every term vertex below a conjugate of the original ΔP. Reduction of permutation lattices preserves vertices [4, Theorem 3.2(b)], so the same bound holds integrally. This proves the stated splendidness, with the fixed marking and the prescribed target block. □

3. Central quotients and inertial blocks

A basic Morita equivalence is induced by an indecomposable Morita bimodule whose vertex is the graph of the chosen defect-group isomorphism and whose source is endopermutation. For a maximal Brauer pair (P, e_P), put $I = N_G(P, e_P)$. The inertial block is the Fong–Reynolds correspondent in I of the Brauer correspondent in $N_G(P)$. A block is nilpotent if $N_G(Q, e)/C_G(Q)$ is a 2-group for each of its Brauer pairs (Q, e), and is inertial if it is basically Morita equivalent to its inertial block.

An interior P-algebra has a specified homomorphism $P \to A^\times$. A primitive Dade P-algebra is the endomorphism algebra of an indecomposable capped endopermutation lattice, with this interior action and Brauer quotient k at P. For $E \leq \mathrm{Aut}(P)$, it is E-stable if transport by every element of E gives an isomorphic interior P-algebra. A determinant-one interior structure means that every $p \in P$ has determinant one on the corresponding lattice. An inertial source algebra has the form $S \otimes_{\mathcal{O}} \mathcal{O}_\alpha(P \rtimes E)$, where S is a primitive E-stable Dade P-algebra and E is the odd inertial quotient. Here α is a normalized $\mathcal{O}^\times$-valued 2-cocycle on E, inflated to $P \rtimes E$.

We use two consequences of Puig's theory [29]: a normal subblock of an inertial block is inertial (Theorem 3.13), and the rank criterion in Proposition 3.11. The latter says that an E-stable Dade algebra S gives inertiality if the local corner of $S^{\mathrm{op}} \otimes \mathcal{O}G$ at the product of the Dade point and the source point has rank $|P||E|$. Its proof shows that the canonical interior embedding into that corner remains injective after reduction; rank equality then makes it an isomorphism.

3.1. **Inertia and a central 2-quotient.**

Proposition 3.1. *Let a be a block of $\mathcal{O}G$ with abelian defect group P, and let $Z \leq Z(G) \cap P$. The block a is inertial if and only if the dominated block of G/Z is inertial.*

Proof. Put $A = i\mathcal{O}Gi$ for a source idempotent, and let $E = \mathrm{Aut}_{\mathcal{F}}(P)$. The group E has odd order and fixes Z pointwise. The source-algebra permutation basis has stabilizers $\Delta(\varphi(Q), \varphi, Q)$, where $Q \leq P$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$. Since $Z \leq Z(G)$, conjugation by Z is trivial on A, so ΔZ fixes every basis element. Hence $Z \leq Q$ and $\varphi|_Z = 1$ in every orbit. Left multiplication by Z is free on that basis.

Suppose that conjugation by $r \in P$ fixes the Z-orbit of a basis element ξ. Then $r\xi r^{-1} = z\xi$ for some $z \in Z$. The element $(z^{-1}r, r)$ fixes ξ, so the graph equation gives $r \in Q$ and $\varphi(r) = rz^{-1}$. Extend φ to $\alpha \in E$ and write the group law on $\langle r, Z\rangle$ additively. The endomorphism $\delta = \alpha - 1$ has image in Z, vanishes on Z, and satisfies $\delta^2 = 0$. If m is the odd order of this restriction, $1 = (1+\delta)^m = 1 + m\delta$; thus $\delta = 0$ and $z = 1$. Passage to Z-orbits creates no new conjugation stabilizers.

Write $I_Z = (z-1 : z \in Z)\mathcal{O}Z$ and $\overline{A} = A/I_Z A$. Orbit sums therefore lift with coefficient one, giving

$$A^R \twoheadrightarrow \overline{A}^{RZ/Z} \qquad (R \leq P). \tag{3.1}$$

The kernel is radical: $I_Z \subseteq J(\mathcal{O}Z)$ and A is finite over the central local ring $\mathcal{O}Z$, so $I_Z A \subseteq J(A)$. For a fixed element, the inverse of a fixed unit is fixed; hence $J(A) \cap A^R \subseteq J(A^R)$. In particular $\overline{A}^{P/Z}$ is local. Its identity has nonzero Brauer image: an expression by proper quotient traces would lift, using (3.1), to proper traces and radical elements in A^P. The proper-trace ideal is contained in the proper Brauer kernel of the local algebra A^P, hence in its radical, contradicting $1 \notin J(A^P)$. Thus $\overline{A}$ is the source algebra of the dominated block.

Quotienting each graph gives its quotient fusion graph. Every maximal automorphism graph occurs in a source algebra, by [24, Proposition 2.5(ii)]. Since $Z \leq Q$, a quotient graph can be maximal only when $Q = P$. The downstairs inertial quotient is therefore the image of E. Its kernel is trivial: an automorphism fixing Z and P/Z lies in a 2-group automorphism kernel, whereas E is odd. Write this identification as $E \cong \overline{E}$.

Suppose that the dominated block is inertial. There is an $\overline{E}$-stable primitive Dade algebra $\overline{S}$ with

$$\overline{A} \cong \overline{S} \otimes_{\mathcal{O}} \mathcal{O}_\beta((P/Z) \rtimes \overline{E}).$$

Inflate $\overline{S}$ to S, put $T = S^{\mathrm{op}} \otimes A$ and $\overline{T} = T/I_Z T$. A permutation conjugation basis of S^{op}, together with the preceding graph basis, gives $T^P \twoheadrightarrow \overline{T}^{P/Z}$ with radical kernel. Cancel the downstairs Dade factors and choose the resulting primitive local idempotent $\overline{j}$, so that

$$\overline{j}\,\overline{T}\,\overline{j} \cong \mathcal{O}_\beta((P/Z) \rtimes \overline{E}).$$

Lift it to $j \in T^P$. For $R \leq P$ and $t \in T^R$, the exact trace identity is

$$\overline{\mathrm{Tr}_R^P(t)} = [RZ : R]\,\mathrm{Tr}_{RZ/Z}^{P/Z}(\bar{t}). \tag{3.2}$$

For $R < P$, the right side is either a proper trace or is divisible by two. Thus a nonlocal lift would have a nonlocal image.

The lifted idempotent belongs to the product local point. A primitive Dade algebra satisfies $S(P) = k$, so $T(P) \cong S^{\mathrm{op}}(P) \otimes A(P) \cong A(P)$. The last algebra is a nonzero quotient of the local algebra A^P and hence is local. There is a unique local P-point of T, namely the product of its Dade point and the chosen source point. The lifted j belongs to it. Now T is free over the central local ring $\mathcal{O}Z$, and $t \mapsto jtj$ is an $\mathcal{O}Z$-linear projector. Thus $C = jTj$ is finite projective, hence free, over $\mathcal{O}Z$. Reduction commutes with this projector, so $C/I_Z C \cong \overline{j}\,\overline{T}\,\overline{j}$ and

$$\mathrm{rank}_{\mathcal{O}} C = |Z| \,\mathrm{rank}_{\mathcal{O}}(C/I_Z C) = |Z|\,|P/Z|\,|\overline{E}| = |P|\,|E|.$$

Puig's rank criterion proves ascent of inertiality.

Conversely, write an inertial source as $S \otimes \mathcal{O}_\beta(P \rtimes E)$, using the determinant-one interior structure on its Dade module V. The conjugation permutation basis of S has one P-fixed element, since $S(P) = k$; all its other P-orbits have even size. Hence $\mathrm{rank}_{\mathcal{O}} S = (\mathrm{rank}_{\mathcal{O}} V)^2$ is odd, so $\mathrm{rank}_{\mathcal{O}} V$ is odd. Centrality of Z forces its action on V to be scalar and forces E to fix Z. A scalar of 2-power order with determinant one in odd dimension is one. The source therefore descends to the indicated inertial source for P/Z. This proves descent. □

3.2. **A supported nilpotent cover.** We specify the classification input used in this subsection. In the residual nonnilpotent Lie-type case, there is a simple simply connected algebraic group $\mathbf{X}$ in odd characteristic, a Steinberg endomorphism F, and a central subgroup $Z \leq X := \mathbf{X}^F$ such that $S = X/Z$. Write $\mathbf{X}^*$ for the dual connected reductive group; its corresponding Steinberg endomorphism is also denoted by F. We use this dual-group convention throughout. The block of X dominating the specified block of S has a semisimple $2'$-label s for which $C^\circ_{\mathbf{X}^*}(s)$ is a torus.

This is the concrete case in the proof of [15, Theorem 6.1] to which [15, Lemma 4.2] is applied. The other case needed here is that the block of S itself is nilpotent. The toral input includes this nonexceptional covering realization; the regular, field, and graph automorphism construction below is made for this realization.

We first record two block-theoretic facts in the forms that we use. Suppose that $H = NP$, where $N \lhd H$ and P is a defect group of a block B of H. Then every block of N covered by B is P-stable. Indeed, the covered blocks form one H-orbit, hence one P-orbit. If b is one of them and its stabilizer P_b is proper in P, their sum is $\mathrm{Tr}_{P_b}^{P}(b)$ and has zero P-Brauer image. Multiplying this sum by B contradicts $\mathrm{Br}_P(B) \neq 0$. Thus the orbit has one member. Since H/N is a 2-group, b has a unique covering block in H, and, as central idempotents in $\mathcal{O}H$, this block is $B = b$. In addition, $P \cap N$ is a defect group of b. We use the usual normal-subgroup theorem for defect groups and compatible Brauer pairs, in the form recorded in [29, Sections 3.2–3.4].

Lemma 3.2. *Let $N \lhd H$, with H/N a 2-group, and let B be a block of H covering a block b of N.*

(1) If B is nilpotent, then b is nilpotent.

(2) Suppose that $H = NP$ and that P is an abelian defect group of B. If b is nilpotent, then B is nilpotent.

Proof. For the first assertion, take a b-Brauer pair (Q, d) and a compatible B-Brauer pair (Q, e), so that e covers d. Such pairs exist by the normal-subgroup Brauer-pair theorem, after conjugating a compatible choice of maximal pairs. The quotient $C_H(Q)/C_N(Q)$ is a 2-group. Consequently e is the unique block of $C_H(Q)$ covering d. Every element of $N_N(Q, d)$ therefore stabilizes e, and conjugation gives an injection

$$N_N(Q, d)/C_N(Q) \longrightarrow N_H(Q, e)/C_H(Q).$$

The latter group is a 2-group by nilpotence of B. This proves the nilpotence condition for every b-Brauer pair.

For the second assertion, put $D = P \cap N$ and choose a maximal B-Brauer pair (P, e_P). The compatible normal-subgroup pair theorem supplies a P-stable maximal b-Brauer pair (D, d) below it. Let (D, e_D) be the unique B-subpair of (P, e_P) with first component D; it covers d. Since P is abelian and $H = NP$, we have

$$C_H(D) = C_N(D)P.$$

The P-stability of d therefore makes it invariant under all of $C_H(D)$. Its unique block extension to $C_H(D)$ is e_D; in particular, e_D covers only d. Now every $y \in N_H(P, e_P)$ normalizes D and stabilizes e_D, and hence stabilizes d.

Write $y = xp$ with $x \in N$ and $p \in P$. Both y and p stabilize (D, d), so $x \in N_N(D, d)$. Nilpotence of b and abelianness of its defect group D give $N_N(D, d) = C_N(D)$: its action on D is both a 2-group, by nilpotence, and a $2'$-group, by the maximal-pair inertial-quotient theorem. Thus y acts trivially on D. It also acts trivially on P/D, since this is the image of P in the abelian group $H/N = P/D$.

For an abelian 2-group P, the automorphisms that fix D and P/D pointwise form a 2-group. Indeed, writing the group law additively, every such automorphism is $1 + \delta$, where $\delta \in \mathrm{Hom}(P/D, D)$; these maps compose by addition because the product of any two such endomorphisms is zero. The inertial quotient of B is a $2'$-subgroup of $\mathrm{Aut}(P)$, so it is trivial. Fusion in a block with abelian defect group is controlled by its maximal Brauer pair. It follows that B is nilpotent. □

Lemma 3.3. *Let S be a quasisimple component, let b be its specified block, and suppose that b is in the toral case described above or is nilpotent. Let P be an abelian 2-subgroup normalizing*

S, and let c be a block of $G_0 = SP$ covering b whose defect group is the subgroup P. Then c is inertial.

Proof. Put $D = P \cap S$. The preceding observation shows that b is P-stable, that D is a defect group of b, and that $c = b$ as idempotents in $\mathcal{O}G_0$. In particular, $\mathrm{Br}_P(b) = \mathrm{Br}_P(c) \neq 0$, with the first expression interpreted as the relative Brauer map on $(\mathcal{O}S)^P$. If b is nilpotent, Lemma 3.2(2) proves that c is nilpotent. We henceforth treat the toral case.

First we justify the central reductions. Let $R = O_2(G_0)$. Since R is contained in every defect group of c, we have $R \leq P$. Also $[R, S] \leq R \cap S \leq O_2(S) \leq Z(S)$. For a fixed element of R, the resulting central commutator map on the perfect group S is trivial. Thus R centralizes S, and it centralizes P because P is abelian. Conversely, $C_P(S)$ centralizes both P and S and is a normal 2-subgroup of G_0. Hence

$$R = C_P(S) \leq Z(G_0), \qquad R \cap S = O_2(Z(S)).$$

Apply Proposition 3.1 to c and pass to G_0/R. We retain the notation G_0, S, P, b, c. Now $O_2(G_0) = 1$ and $Z(S)$ has odd order. The group $C_{G_0}(S)/Z(S)$ is a 2-group and $Z(S)$ is central in $C_{G_0}(S)$. Schur–Zassenhaus gives a unique Sylow 2-subgroup of this centralizer. It is normal in G_0 and is therefore trivial. Consequently

$$C_{G_0}(S) = Z(S), \qquad C_P(S) = 1.$$

Let θ be the odd central character of b and put $K_0 = \ker(\theta) \leq Z(S)$. The P-stability of b makes K_0 normal in G_0. The averaging idempotent of K_0 acts as the identity on $c = b$, so quotienting by K_0 gives an isomorphism of block algebras, with the subgroup P identified with its image. This preserves the source algebra and inertiality. The centralizer is preserved: if an element centralizes S/K_0, its commutators with S lie in K_0, and perfection of S again makes that central commutator map trivial. Thus

$$C_{G_0/K_0}(S/K_0) = Z(S)/K_0.$$

In particular $O_2(G_0/K_0) = 1$. We may therefore retain the notation and assume that θ is faithful. Its invariance implies that P centralizes $Z(S)$. The same block of the simply connected group X dominates the successive central quotients of b, so the toral classification input is retained. Replace Z by the kernel of the resulting map $X \to S$.

Choose a regular embedding $\mathbf{X} \hookrightarrow \widetilde{\mathbf{X}}$ with connected centre, compatible with F, and put $\widetilde{X} = \widetilde{\mathbf{X}}^F$. Let Γ be the finite group of field and graph automorphisms in the regular-embedding realization of [21, proof of Proposition 4.2, preceding Lemma 4.5], acting on $\widetilde{X}$, and put $\Gamma_Z = \{\gamma \in \Gamma : \gamma(Z) = Z\}$. That realization gives $\mathrm{Aut}(S) \cong (\widetilde{X}/Z(\widetilde{X})) \rtimes \Gamma_Z$. Set

$$J_0 = Z\, O_2(Z(\widetilde{X})), \qquad \check{G} = \widetilde{X}/J_0, \qquad \check{A} = (\widetilde{X} \rtimes \Gamma_Z)/J_0.$$

Since $Z(S)$ has odd order, $O_2(Z(X)) \leq Z$, and hence $J_0 \cap X = Z$. The original S therefore embeds in $\check{G}$. Conjugation gives a homomorphism $\rho : \check{A} \longrightarrow \mathrm{Aut}(S)$ whose kernel is the odd-order group $Z(\widetilde{X})/J_0$.

Let $\tau : G_0 \longrightarrow \mathrm{Aut}(S)$ be the original action. The restriction $\tau|_P$ is injective and $\tau(P) \cap \mathrm{Inn}(S) = \tau(D)$. Choose a Sylow 2-subgroup $\check{P}$ of $\rho^{-1}(\tau(P))$ containing the already embedded D. Then $\rho|_{\check{P}}$ is an isomorphism onto $\tau(P)$. Thus

$$\lambda = (\rho|_{\check{P}})^{-1}\tau|_P : P \longrightarrow \check{P}$$

is an isomorphism, equals the identity on D, and satisfies $\check{P} \cap S = D$. Since $\lambda(x)$ induces the same automorphism of S as x, the map

$$G_0 \longrightarrow \check{A}, \qquad sx \longmapsto s\lambda(x) \quad (s \in S,\ x \in P) \tag{3.3}$$

is a well-defined homomorphism. If its value is 1, then $\lambda(x) \in S$, hence $x \in D$, and $s\lambda(x) = sx$. It is therefore injective and is the identity on S. We now identify G_0 and P with these embedded subgroups.

Every block of the regular subgroup $\check{G}$ covering b is nilpotent, as follows. Let b_X be the block of X dominating b. The proof of [15, Lemma 4.2] starts with an arbitrary block $\widetilde{B}$ of $\widetilde{X}$ covering b_X. Its semisimple label has toral centralizer in the dual regular group, and the Bonnafé–Rouquier equivalence in that proof makes $\widetilde{B}$ nilpotent. Thus every such $\widetilde{B}$ is nilpotent. The odd part of Z is in its kernel; quotienting that part and the central 2-group $O_2(Z(\widetilde{X}))$ preserves nilpotence. Conversely, every block of $\check{G}$ covering b lifts to one of these blocks of $\widetilde{X}$. Consequently every block of $\check{G}$ covering b is nilpotent.

The quotient $\check{G}/S$ is abelian. Let $\check{K}$ be the inverse image of its odd-order subgroup. Then $\check{K}/S$ is odd and abelian, $\check{G}/\check{K}$ is a 2-group, and P normalizes $\check{K}$. Every block of $\check{K}$ covering b is covered by a block of $\check{G}$ covering b, and is nilpotent by Lemma 3.2(1).

Let e be the sum of all blocks of $\check{K}$ covering b. Then e is P-stable and $eb = b$, so

$$\mathrm{Br}_P(e)\,\mathrm{Br}_P(b) = \mathrm{Br}_P(b) \neq 0. \tag{3.4}$$

The sum of a nonfixed P-orbit of covering blocks is a relative trace from its proper stabilizer, and its P-Brauer image is zero. Therefore there is a P-fixed nilpotent block f of $\check{K}$ such that $\mathrm{Br}_P(f)\,\mathrm{Br}_P(b) \neq 0$.

Put $U = P\check{K}$. The invariant block f is the unique block F of U covering it, with $F = f$ as an idempotent. Its P-Brauer image is nonzero, so some defect group of F contains P. Every defect group of f has order $|D|$: $\check{K}/S$ is odd, and the intersection theorem for defects applied to $S \triangleleft \check{K}$ gives this assertion. Also $P \cap \check{K} = D$. The normal-extension defect bound, for a defect group Q of F, therefore gives

$$|Q| \leq |D|\,|U : \check{K}| = |D|\,|P : D| = |P|.$$

It follows that the subgroup P is a defect group of F. Lemma 3.2(2), with $N = \check{K}$, now proves that F is nilpotent.

Choose a block e_P of $kC_U(P)$ satisfying

$$e_P\,\mathrm{Br}_P(F) = e_P, \qquad e_P\,\mathrm{Br}_P(b) \neq 0.$$

This is possible by the chosen nonzero product of Brauer images, and (P, e_P) is a maximal F-Brauer pair. Put $N = N_U(P)$ and $L = SN$. Brauer's first main theorem first gives the correspondent F_N of F in N, and then gives the unique block E_L of L with defect P whose correspondent in N is F_N. Indeed,

$$N_L(P) = N_U(P) = N, \qquad C_L(P) = C_U(P).$$

All three blocks use the same maximal pair (P, e_P). Their inertial quotients are therefore equal. The inertial quotient of F is trivial; hence so is that of E_L, and the abelianness of P makes E_L nilpotent.

The same maximal pair also retains the original covering relation. Namely

$$e_P\,\mathrm{Br}_P(E_L c) = e_P\,\mathrm{Br}_P(c) = e_P\,\mathrm{Br}_P(b) \neq 0,$$

so $E_L c \neq 0$. Finally, $G_0 = SP$ is normal in L: the subgroup S is normal and every element of N normalizes both S and P. Thus c is a normal subblock of the nilpotent block E_L. Puig's theorem [29, Theorem 3.13], including the stabilizer reduction in its Section 3.2, proves that c is inertial. The normal odd-kernel block isomorphism and Proposition 3.1 restore the original group and original block. □

3.3. **Products of the component extensions.**

Lemma 3.4. *Let $G = PE$, where P is abelian and $E = S_1 \cdots S_m$ is a central product of normal quasisimple components of G. Let a be a block of G with defect group P, and let b_i be the component blocks covered by a. Suppose that, for each i, the block c_i of PS_i covering b_i is inertial. Then a is inertial.*

Proof. We first establish stability and the defects of the blocks in this statement. Let b be a block of E covered by a. The support observation above, applied to $G = EP$, shows that b is P-stable, is the unique block of E covered by a, and satisfies $a = b$ as an idempotent. The block b covers a unique block b_i of each component S_i, and

$$b = b_1 \cdots b_m$$

inside $\mathcal{O}E$. This is the block correspondence obtained by inflating to $\prod_i S_i$ and then taking the central character sector of the central product. Since each S_i is normal in G, the P-stability of b makes each b_i P-stable. The normal-subgroup defect theorem gives $D_i := P \cap S_i$ as a defect group of b_i.

Brauer multiplicativity gives

$$0 \neq \mathrm{Br}_P(a) = \mathrm{Br}_P(b) = \prod_i \mathrm{Br}_P(b_i).$$

Thus every $\mathrm{Br}_P(b_i)$ is nonzero. The unique block c_i of $G_i := PS_i$ covering b_i is the idempotent b_i in $\mathcal{O}G_i$. A defect group of c_i contains P, while its order is at most $|D_i|\,|G_i : S_i| = |D_i|\,|P : D_i| = |P|$. Consequently P is a defect group of every c_i.

Let $R = O_2(G)$. We have $R \leq P$. For every i, $[R, S_i] \leq O_2(S_i) \leq Z(S_i)$, so perfection of S_i makes R centralize S_i. It also centralizes P. Hence

$$R = C_P(E) \leq Z(G).$$

Apply Proposition 3.1 simultaneously to a and to all c_i and quotient by R. The component images remain quasisimple, $O_2(G) = 1$, and $Z(E)$ has odd order. The last assertion follows since $O_2(Z(E)) \leq R$ before quotienting and a central quotient of a perfect group does not enlarge its centre.

Let θ be the character of $Z(E)$ belonging to b and let $K = \ker(\theta)$. Stability of b makes K normal in G, and the averaging idempotent of K acts as the identity on $a = b$. Thus quotienting by K gives the corresponding block-algebra isomorphism, identifying P with its image. For each i, the kernel of the map $G_i \to G/K$ is $K \cap S_i$: it is odd and G_i/S_i is a 2-group. It is contained in the kernel of c_i, so the same quotient preserves the source algebra and inertiality of every c_i.

The quotient also has trivial 2-core. Before taking it, $C_G(E)/Z(E)$ is a 2-group with central odd kernel $Z(E)$. Its unique Sylow 2-subgroup would be normal in G, so $C_G(E) = Z(E)$. Since E is perfect and $K \leq Z(E)$, an element centralizing E/K already centralizes E; hence

$$C_{G/K}(E/K) = Z(E)/K.$$

A normal 2-subgroup of G/K centralizes E/K, since $O_2(E/K) = 1$. The displayed centralizer has odd order, so that normal 2-subgroup is trivial. We retain the notation and assume from now on that

$$O_2(G) = 1, \qquad C_G(E) = Z(E), \qquad \theta \text{ is faithful.}$$

The P-invariance of the faithful character θ forces P to centralize $Z(E)$, and in particular every $Z(S_i)$.

For $G_i = PS_i$ put $R_i = O_2(G_i)$. The same commutator argument gives $R_i = C_P(S_i) \leq Z(G_i)$. Since $Z(S_i)$ has odd order, $R_i \cap S_i = 1$. Moreover, $C_{G_i}(S_i)/Z(S_i)$ is a 2-group with central odd kernel. Its unique Sylow 2-subgroup is normal in G_i, and is exactly R_i. Thus

$$C_{G_i}(S_i) = Z(S_i) \times R_i. \tag{3.5}$$

Put $\overline{G}_i = G_i/R_i$. The original S_i embeds elementwise in $\overline{G}_i$, its centralizer is $Z(S_i)$, and the block $\bar{c}_i$ dominated by c_i is inertial by Proposition 3.1. Its defect group is $P_i = P/R_i$, with $P_i \cap S_i = D_i$.

Let

$$A = \ker(\prod_i S_i \longrightarrow E).$$

This is an odd subgroup of $\prod_i Z(S_i)$. Because P centralizes each component centre, A is central in $\prod_i \overline{G}_i$. Form

$$\widehat{G} = (\prod_i \overline{G}_i)/A.$$

It contains the original E, and

$$\widehat{G}/E \cong \prod_i (\overline{G}_i/S_i)$$

is an abelian 2-group.

We now construct an embedding of the original G which is the identity on E. Let $\tau : G \to \prod_i \mathrm{Aut}(S_i)$ be its componentwise action, and let $\rho : \widehat{G} \to \prod_i \mathrm{Aut}(S_i)$ be the componentwise action of the constructed product. The image of ρ contains $\tau(G)$, its restriction to E is $\tau|_E$, and

$$\ker(\tau) = Z(E), \qquad \ker(\rho) = Z(E).$$

The latter kernel is central and has odd order. The actions are well defined on the central product because every factor fixes the shared centres pointwise.

Choose a Sylow 2-subgroup T of G containing P. The group $\rho^{-1}(\tau(T))$ is a central extension of a 2-group by the odd group $Z(E)$ and therefore has a unique Sylow 2-subgroup $\widehat{T}$. It contains the already embedded $T \cap E$. The map

$$\lambda_T = (\rho|_{\widehat{T}})^{-1}\tau|_T : T \longrightarrow \widehat{T}$$

is an isomorphism and is the identity on $T \cap E$. Also $G = ET$, and $\lambda_T(t)$ induces the original action of t on E. Hence

$$\Phi : G \longrightarrow \widehat{G}, \qquad et \longmapsto e\lambda_T(t)$$

is a well-defined homomorphism. If $\lambda_T(t) \in E$, then $\tau(t) \in \tau(E)$, so $t \in E\ker(\tau) = E$ and $\lambda_T(t) = t$. This proves that Φ is injective. In particular it is the identity on E. For $p \in P$, its image is the class of the tuple $(pR_i)_i$: that tuple is a 2-element with action $\tau(p)$, and belongs to the unique Sylow 2-subgroup of $\rho^{-1}(\tau(\langle p \rangle))$.

It remains to keep the specified block and central character. On $Z(S_i)$ the block $\bar{c}_i$ has character $\theta_i = \theta|_{Z(S_i)}$. Thus the tensor block $\widetilde{a} = \bigotimes_i \bar{c}_i$ of $\prod_i \overline{G}_i$ has character

$$(z_i)_i \longmapsto \prod_i \theta_i(z_i) = \theta(\prod_i z_i),$$

which is trivial on A. It therefore descends to a block $\widehat{a}$ of $\widehat{G}$ via the central odd-kernel block-algebra isomorphism.

The tensor block is inertial. Explicitly, if the source of $\bar{c}_i$ is $\mathcal{D}_i \otimes_{\mathcal{O}} \mathcal{O}_{\alpha_i}(P_i \rtimes I_i)$, where I_i is its inertial quotient and α_i the corresponding normalized cocycle, then its product source is

$$\Big(\bigotimes_i \mathcal{D}_i\Big) \otimes_{\mathcal{O}} \mathcal{O}_{\prod_i \alpha_i}\Big((\prod_i P_i) \rtimes (\prod_i I_i)\Big).$$

The product Dade algebra is primitive and stable under the product inertial quotient. The odd central quotient by A preserves this source with its defect-group identification, so $\widehat{a}$ is inertial. In fact $\overline{c}_i = b_i$ as idempotents in $\mathcal{O}\overline{G}_i$, because $c_i = b_i$ and the quotient by R_i embeds S_i identically. The product and the compatible central character therefore give $\widehat{a} = b$ as idempotents in $\mathcal{O}\widehat{G}$.

Finally, G is normal in $\widehat{G}$ because it contains E and $\widehat{G}/E$ is abelian. The block of G covering b is uniquely $a = b$. Hence $\widehat{a}$ covers the block a. Puig's normal-subblock theorem [29, Theorem 3.13] now proves that a is inertial. The preceding odd-kernel block isomorphism and Proposition 3.1 restore all the given groups and blocks. □

3.4. **Extension of inertial blocks.** The following extension theorem applies when the indicated block of PH is inertial.

Theorem 3.5 (Zhang–Zhou, [38, Theorem 1.3])**.** *Let $H \triangleleft G$, let b be a G-stable block of $\mathcal{O}H$, and let P be a maximal 2-subgroup with nonzero relative Brauer support for b. Put $Q = P \cap H$. Suppose that P is abelian and that the unique block of PH covering b is inertial. Put $J = N_G(Q)$, $H_0 = N_H(Q)$, and let b_0 be the Brauer correspondent of b in H_0. There is a module for $(H \times H_0)\Delta J$ whose lower restriction is a basic Morita bimodule and whose induction to $G \times J$ gives a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Jb_0$. Every corresponding primitive summand is a basic Morita equivalence with the chosen defect identification.*

We describe the graded source comparison and its consequence for the diagonal action. Put

$$A = \mathcal{O}Gb, \quad B = \mathcal{O}Hb, \quad A' = \mathcal{O}Jb_0, \quad B' = \mathcal{O}H_0b_0, \qquad \Gamma = G/H \cong J/H_0.$$

The last identification is the pointed Frattini equality. The algebras A, A' have their group-coset Γ-gradings, with identity components B, B'.

Inertiality of the PH-block supplies a determinant-one Dade P-algebra. The extension construction of [30], followed by the cancellation and comparison in [38, Lemmas 5.12 and 5.15, Theorem 5.16 and Section 5.17], gives idempotents $i \in B^P$, $i' \in (B')^P$ and compatible graded interior-P isomorphisms, with V, W the $\mathcal{O}P$-lattices of the two Dade factors,

$$\begin{aligned} iAi &\cong \mathrm{End}_{\mathcal{O}}(V) \otimes_{\mathcal{O}} R, & iBi &\cong \mathrm{End}_{\mathcal{O}}(V) \otimes_{\mathcal{O}} R_1, \\ i'A'i' &\cong \mathrm{End}_{\mathcal{O}}(W) \otimes_{\mathcal{O}} R, & i'B'i' &\cong \mathrm{End}_{\mathcal{O}}(W) \otimes_{\mathcal{O}} R_1. \end{aligned} \tag{3.6}$$

Here R_1 is the identity component of the Γ-grading on R. The idempotents contain source idempotents for the lower blocks and are full in B, B'. The matrix factors have degree 1. The algebra R is the twisted group algebra of the local extension group, graded by the original quotient Γ; it contains a unit in every homogeneous component. Both occurrences use the same central extension and the same embedding of P. In [38, Section 3.9 and Lemma 5.15], the central extensions are identified over the identity of the local extension group. Their splittings over the 2-subgroup agree by uniqueness: the central kernel $k^\times$ has no 2-torsion, and powering by a power of two is an automorphism of $k^\times$. Equivalently, choose a finite odd central realization of the cocycle and its unique 2-subgroup splitting. This supplies the stated interior identification.

The fixed-algebra condition used in this comparison follows from the upper inertiality hypothesis. Representatives of its odd inertial quotient E belong to PH, whose quotient $PH/H \cong P/Q$ is abelian. Thus E acts trivially on P/Q. Choose n with $n|E| \equiv 1 \pmod{\exp(P)}$. For $p \in P$, the element

$$c = \left(\prod_{\alpha \in E} \alpha(p) \right)^n$$

belongs to $C_P(E)$ and satisfies $cQ = pQ$. Thus $P = QC_P(E)$. After the Dade cancellation, $R_1 = \mathcal{O}_\beta(Q \rtimes E)$. The chosen 2-subgroup splitting shows that $C_P(E)$ commutes with Q and with the lifted elements of E. Conjugation by $p = qc$ on R_1 is consequently conjugation by q,

and is trivial on $(R_1)^Q$. This is the fixed-algebra condition in [38, Sections 5.9–5.15], under the hypothesis of its Theorem 1.3.

Lemma 3.6. *The common graded source corners (3.6) give a Morita bimodule D for A, A' whose identity component is a Morita bimodule for B, B'. This component is a module for $(H \times H_0)\Delta J$, and its induction to $G \times J$ is D.*

Proof. Put $S = \mathrm{End}_{\mathcal{O}}(V)$ and $S' = \mathrm{End}_{\mathcal{O}}(W)$ and form

$$F = Ai \otimes_{S\otimes R} (V \otimes R), \qquad F' = (R \otimes W^\vee) \otimes_{S'\otimes R} i'A', \qquad D = F \otimes_R F'.$$

Fullness of i, i' gives the full-idempotent Morita equivalences; the natural column and row modules give the two matrix-algebra Morita equivalences. Their composition is D. All tensor products have the displayed Γ-gradings.

Multiplication gives graded isomorphisms

$$F \cong F_1 \otimes_{R_1} R, \qquad F' \cong R \otimes_{R_1} F_1'.$$

Indeed, choose a unit $r_t \in R_t$. Its inverse image $u_t \in iA_ti$, under the corner isomorphism applied to $1 \otimes r_t$, is a unit of iAi. For $a \in A_ti$, $au_t^{-1} \in Bi$, which proves the first formula in degree t; the second follows by right multiplication at the other endpoint. The inverses use r_t^{-1}, so the formulas are independent of the chosen homogeneous units. Hence

$$D \cong F_1 \otimes_{R_1} R \otimes_{R_1} F_1', \qquad M := D_1 = F_1 \otimes_{R_1} F_1'.$$

The two identity-component corner isomorphisms in (3.6) show that F_1, F_1' are Morita bimodules. Thus M gives the lower Morita equivalence.

The existing $G \times J$ action on D is $(g, j)d = gdj^{-1}$. It sends degree t to $(gH)t(jH)^{-1}$. Its subgroup preserving degree 1 is

$$\{(g, j) : gH = jH\} = (H \times H_0)\Delta J.$$

Restricting this action to M gives all multiplication and intersection relations. For each $t \in \Gamma$, choose $g_t \in G$ above t. Then $D_t = g_tM$, since g_t is invertible on D. The map

$$\mathrm{Ind}_{(H\times H_0)\Delta J}^{G\times J} M \longrightarrow D, \qquad \sigma \otimes m \longmapsto \sigma m,$$

is an isomorphism: its coset summands map bijectively to the homogeneous components D_t. □

In the application below, the selected primitive block has defect group P. Its correspondence is basic by Theorem 3.5; the source theorem in Section 2 supplies its capped, fusion-stable endopermutation P-source with the specified defect identification. Reducing the primitive bimodule modulo the maximal ideal, applying Proposition 2.2, and then Lemma 2.3 gives the integral splendid equivalence used below.

4. Quasisimple blocks and the upper defect action

Let S be a quasisimple component of a finite group G, let $P \leq G$ normalize S, and suppose that P is the abelian defect group of the specified block of SP. Let b be the block of S that it covers. Its defect group $D = P \cap S$ is assumed noncentral. Automorphisms are induced by the ambient stabilizers.

4.1. **The classification in the form used here.** The classification [15, Theorem 6.1 and Propositions 3.4, 5.3 and 5.4], together with its proof, gives the following cases. In the Lie-type cases, $\mathbf{X}$ denotes the simply connected algebraic realization with Frobenius F, $\mathbf{X}^*$ its dual group with the corresponding Frobenius, and $s \in (\mathbf{X}^*)^F$ the semisimple odd-order block label. We first assign all nilpotent blocks, and all Lie-type cases in which $C^\circ_{\mathbf{X}^*}(s)$ is a torus, to (iv). Among the remaining cases, the type-A blocks in (ii) are precisely those of [15, Proposition 5.3(iii)], in both linear and unitary forms. The cases of [15, Proposition 5.3(v)] are assigned to (iii), including their Klein-defect overlaps.

(i) The principal blocks of $\mathrm{SL}_2(2^a)$, with $a \geq 2$, the Ree groups ${}^2G_2(3^{2a+1})$, with $a \geq 1$, and J_1, and the nonprincipal block of Co_3 with defect C_2^3.

(ii) The nonnilpotent Klein four blocks in the specified nontoral type-A cases, in alternating groups, in the nonprincipal Ree case, and in the sporadic and central-cover cases described below. Their possibilities are retained from [11, Sections 8 and 12] and the proof in [15]; the classified D_{2t}/E_7 Levi cases are assigned to (iii).

(iii) The rank-one Levi cases in split type D_{2t}, with $t \geq 3$ odd, and in type E_7, with $t = 3$. Here q is an odd prime power with $q \equiv \pm 3 \pmod 8$. The full centralizer of the semisimple odd-order label is the classification Levi; its derived root system is a single Frobenius orbit of t factors of type A_1. After the prescribed central quotient, its finite group has the form $\mathrm{PSL}_2(q^t) \times A_0$ with A_0 abelian.

(iv) The nilpotent blocks and the cases in which the connected component $C^\circ_{\mathbf{X}^*}(s)$ of the dual centralizer is a torus. The latter have nilpotent regular-embedding covers by [15, Lemma 4.2].

The central-defect components will be removed before this list is applied. The classification branches enter the list as follows. In defining characteristic two, Proposition 8.7 and Lemma 8.8 of [11], together with the proof of [15, Theorem 6.1], leave the SL_2 family and the listed small exceptions after the central-defect cases are removed. For alternating and sporadic groups, [11, Section 8] gives (i) and (ii). For $S = {}^2G_2(3^{2a+1})$, $a \geq 1$, the central cover is trivial, a Sylow 2-subgroup is C_2^3, and the principal block is the unique block of full defect [15, proof of Theorem 6.1]. A nonprincipal block therefore has defect of order at most four. Its defect is a subgroup of C_2^3, so a nonnilpotent such block has Klein four defect and belongs to (ii); the other blocks belong to (iv). In odd characteristic, the toral case of [15, Lemma 4.2] belongs to (iv). In [15, Proposition 5.3], cases (i) and (iv) give nilpotent blocks, as observed in the proof of [15, Theorem 6.1], and thus belong to (iv) here; case (iii) gives the type A quaternion preimage treated in (ii); case (v) gives precisely the D_{2t} and E_7 families in (iii). The remaining quasi-isolated E_7 case with dihedral defect D_8 before a central involution quotient belongs to (iv). The full preimage of the quotient defect V_4 under the central quotient is D_8. An element normalizing V_4 therefore lifts to an element normalizing D_8. Its action on V_4 lies in the image of $\mathrm{Aut}(D_8)$, a 2-group. The odd inertial quotient is trivial. Fusion on an abelian defect group is controlled by the maximal Brauer-pair normalizer, so the quotient block is nilpotent. These are all the alternatives in the proof of [15, Theorem 6.1].

The central-cover cases are as follows. After cyclic, Klein four and $C_4 \times C_2$ defect groups have been separated, the odd-characteristic part of the proof of [15, Theorem 6.1] excludes faithful exceptional-cover blocks among the remaining noncentral abelian-defect cases. A nonnilpotent block with toral connected dual centralizer can have disconnected full centralizer only in type A or E_6, by [15, Proposition 5.4]. In odd characteristic the exceptional multipliers in type A occur for $\mathrm{PSL}_2(9)$ and $\mathrm{PSU}_4(3)$ [11, Sections 9–10]. Their standard simply connected centers are 2-groups. An odd-order semisimple label therefore has connected centralizer: lift it with odd order through the central 2-isogeny; a conjugate differing by a central 2-element must be equal,

and the centralizer upstairs is connected. A toral case for these groups is consequently nilpotent by [15, Lemma 4.2]. The odd-center covers of $\mathrm{PSU}_4(3)$ have no blocks with defect groups of order two or four [11, proof of Lemma 10.1]; the $3.A_6$ case is treated below. The faithful blocks of $3.\mathrm{P}\Omega_7(3)$ have defect groups of order 2^3 or 2^9, and $3.G_2(3)$ has no Klein four block [11, proofs of Proposition 11.1 and Lemma 12.2]. The odd-characteristic E_6 and ${}^2E_6$ cases have their standard centers of orders $\gcd(3, q-1)$ and $\gcd(3, q+1)$, respectively [11, Lemmas 12.5 and 12.7]. A nonfaithful odd central cover has the same block algebra after quotienting its block kernel. A central 2-quotient preserves the inertial quotient for abelian defect, as proved in Proposition 3.1; it therefore preserves nilpotence in this situation. Thus each nonnilpotent case in (iv) has the nonexceptional simply connected realization used in Lemma 3.3.

In defining characteristic two, Proposition 8.7 of [11] gives full defect after the central reductions. The abelian Sylow case is the SL_2 family. Its exceptional double cover at $\mathrm{SL}_2(4)$ has quaternion Sylow preimage and gives no further abelian-defect case. The Tits group has defect-zero blocks and a principal block of nonabelian defect; the remaining small groups are $\mathrm{PSp}_4(2)' \cong A_6$ and $G_2(2)' \cong \mathrm{PSU}_3(3)$, covered by the alternating and odd-characteristic unitary cases [11, Lemma 8.8]. The exceptional odd covers in the alternating and sporadic Klein cases are treated below.

Cases (i)–(iii) provide the component complexes used in the assembly. In case (iv), Lemma 3.3 proves that the block of SP is inertial.

4.2. **Automorphisms centralizing a defect group.** For an algebra A, $\mathrm{Pic}(A)$ is the group of isomorphism classes of invertible A-A-bimodules under tensor product. For an interior D-algebra A, $\mathrm{Out}_D(A)$ is the quotient of the algebra automorphisms fixing the specified image of D pointwise by conjugation with units in $(A^D)^\times$.

Lemma 4.1. *Let a finite 2-group T act on a finite group L by automorphisms. Suppose that it stabilizes a block of L, centralizes its defect group D, and preserves a supported maximal Brauer pair. Then its image in the Picard group of that block is trivial. In particular it fixes every ordinary irreducible character of the block.*

Proof. The twist of the regular block bimodule by an element of T is an invertible trivial-source bimodule with diagonal vertex ΔD. Its fusion automorphism is the identity. By [3, Theorem 1.1(ii) and Remark 1.2], its Picard class belongs to Out_D of a source algebra, which embeds in $\mathrm{Hom}(E, k^\times)$ for the inertial quotient E. This is an odd group. The homomorphism from T is therefore trivial. An inner algebra automorphism fixes all ordinary characters. □

For the original component block, let c be the specified block of SP with defect group P. The normal 2-extension support argument preceding Lemma 3.2 shows that b is P-stable and $c = b$. Thus $\mathrm{Br}_P(b) \neq 0$. Since P centralizes D, coefficient deletion gives

$$\mathrm{Br}_P(\mathrm{Br}_D(b)) = \mathrm{Br}_P(b).$$

In $\mathrm{Br}_D(b)$, a nontrivial P-orbit of primitive root idempotents is a proper relative trace and has zero P-Brauer image. If e_P is the root of the original upper maximal pair, some P-fixed root e_D therefore satisfies $\mathrm{Br}_P(e_D)e_P \neq 0$. The pair (D, e_D) is the required supported maximal pair of b.

4.3. **Klein four blocks and upper 2-actions.**

Proposition 4.2. *In case (ii), the original abelian upper defect group P acts on S by inner automorphisms. Moreover $O_2(Z(S)) = 1$.*

Proof. The inertial quotient of a nonnilpotent Klein block is C_3, acting transitively on $D \setminus \{1\}$. A central involution would be fixed by this action, so $O_2(Z(S)) = 1$.

For $S = {}^2G_2(3^{2a+1})$, the outer automorphism group has odd order [11, proof of Lemma 12.3]. The image of the 2-group P in this quotient is trivial, so its action on S is inner.

Consider the type-A cases of [15, Proposition 5.3(iii)]. Here q is an odd prime power, $\varepsilon \in \{1, -1\}$, and $\mathrm{GL}_n^+(q) = \mathrm{GL}_n(q)$, $\mathrm{GL}_n^-(q) = \mathrm{GU}_n(q)$, $\mathrm{SL}_n^+(q) = \mathrm{SL}_n(q)$, $\mathrm{SL}_n^-(q) = \mathrm{SU}_n(q)$. The natural dimension is $2t$, with t odd, and $q \equiv \pm 3 \pmod 8$. An odd-order lift of the semisimple label in $\mathrm{GL}_{2t}^{\varepsilon}(q)$ has t distinct eigenvalues, each of multiplicity two, forming one orbit of $f : \lambda \mapsto \lambda^{\varepsilon q}$ [15, Proposition 3.4(i),(a)]. If an inner-diagonal graph automorphism stabilizes the label, then

$$j : \lambda \longmapsto c\lambda^{-1}$$

permutes these eigenvalues for an f-fixed scalar c. It is an involution commuting with the t-cycle f. The centralizer of that cycle is cyclic of odd order, so j is the identity. Every eigenvalue then has square c. Squaring is injective on their odd-order group, so $t = 1$. For $t > 1$ the graph component is excluded. For $t = 1$ the group is rank one, where the graph automorphism is inner. In the unitary case the q-power field map is inverse-transpose up to an inner automorphism, so the same calculation treats its order-two part. Since $q \equiv 3$ or $5 \pmod 8$, the exponent of its defining prime is odd. The remaining field subgroup has odd order.

It remains to exclude the diagonal 2-part. The inverse image of D in $\mathrm{SL}_{2t}^{\varepsilon}(q)$ has Sylow 2-subgroup Q_8. Its natural module is the sum of t copies of the faithful two-dimensional quaternion module. If a general linear or unitary element induces an automorphism centralizing D, it normalizes this Q_8: the central 2-part consists of $\{\pm 1\}$, and an odd scalar cannot occur in the conjugate of an element of order at most four. The kernel of $\mathrm{Aut}(Q_8) \to \mathrm{Aut}(Q_8/Z(Q_8))$ is inner. After multiplication by an element of Q_8, our element centralizes Q_8. The faithful two-dimensional quaternion module is absolutely irreducible in odd characteristic. Its commuting endomorphisms on the t-fold isotypic sum act on the multiplicity space; in the unitary case this space carries the induced hermitian form. The centralizer is therefore $\mathrm{GL}_t(q)$ or $\mathrm{GU}_t(q)$, and

$$\det_{2t}(u) = \det_t(u)^2. \tag{4.1}$$

The diagonal outer quotient has 2-part of order two. A square determinant has trivial image there. Hence the upper 2-action is inner. The rank-one cases are included in this calculation.

For alternating groups, [11, Lemma 8.5] makes the Klein block S_n-stable. Its unique cover has defect of order eight and weight two, hence has five ordinary characters. If the four characters of the lower block comprise a fixed characters and b_0 exchanged pairs, Clifford theory gives

$$a + 2b_0 = 4, \qquad 2a + b_0 = 5.$$

Thus $b_0 = 1$: the outer involution moves a character and is excluded by Lemma 4.1. The exceptional odd covers give no additional faithful Klein case: A_6 and the faithful blocks of $3.A_6$ have no Klein defect, and the faithful blocks of $3.A_7$ have defect groups of order eight or one. These are the cover cases in the classification and its modular tables [11, 6].

The sporadic Klein four cases in (ii) are $M_{12}, J_2, HS, \mathrm{Fi}'_{24}$ [11, Section 8]. In the first three, the outer involution exchanges, respectively, the following two character rows in the Klein block. The indices are those of the ordinary-character rows in the corresponding characteristic-two tables of [6]:

S	rows	common degree
M_{12}	χ_4, χ_5	16
J_2	χ_{16}, χ_{17}	224
HS	χ_{14}, χ_{15}	896.

In Block 2 of the characteristic-two tables for $M_{12}.2$, $J_2.2$ and $HS.2$, their induced characters have degrees 32, 448 and 1792 and labels χ_{4+}, χ_{16+} and χ_{14+}, respectively. The plus sign denotes the sum over the exchanged pair upon restriction [6]. Apply Lemma 4.1.

For $S = \mathrm{Fi}'_{24}$, choose $u \in D \setminus \{1\}$. The classification gives $C = C_S(u) \cong 2.\mathrm{Fi}_{22}.2$ with a unique block c of defect D [11, Lemma 8.3]. Let H_0 be the normal subgroup of C isomorphic to $2.\mathrm{Fi}_{22}$. The block c contains two extensions of each of the characters χ_{56} and χ_{104} of H_0, all of degree 1441792. These are the rows $\chi_{56,0}$, $\chi_{56,1}$, $\chi_{104,0}$ and $\chi_{104,1}$ in Block 2 of the characteristic-two $\mathrm{Fi}_{22}.2$ table, in its displayed double-cover sector. The latter pair is faithful on u [6]. The full automorphism centralizer is

$$C_{\mathrm{Aut}(S)}(u) \cong (2 \times 2.\mathrm{Fi}_{22}) : 2,$$

with a normal $V_4 = \langle a, u\rangle$, where a, au are outside S [7, Section 2.4-10]. The perfect subgroup H_0 centralizes V_4, whereas $v \in C \setminus H_0$ satisfies $a^{-1}va = uv$. Let ρ be a representation of C whose character extends χ_{104}, and write $\rho^a(x) = \rho(a^{-1}xa)$ for $x \in C$. Faithfulness gives $\rho(u) = -I$. Hence

$$\rho^a|_{H_0} = \rho|_{H_0}, \qquad \rho^a(v) = \rho(uv) = -\rho(v).$$

Thus the character of ρ^a is the other extension of χ_{104}. All elements of the outer coset differ by elements of C, whose action on its characters is inner, so they induce the same character permutation. The original P centralizes u and stabilizes the supported block c with defect D, so Lemma 4.1, now applied to C, excludes that outer coset.

Faithful $3.\mathrm{Fi}'_{24}$ blocks contribute no Klein case [11, Lemma 8.3]. Nonfaithful blocks of odd central covers are handled in their chosen character sector. Finally an automorphism of a perfect central cover inducing an inner automorphism of its simple quotient is itself inner: after correcting by the inner automorphism, the central correction is a homomorphism from a perfect group to its center and vanishes. □

Lemma 4.3. *Let S be quasisimple with $O_2(Z(S)) = 1$, and let P normalize S. Suppose that P is an abelian defect group of a block of SP. Put $D = P \cap S$. If P acts on S by inner automorphisms, then, inside the given group,*

$$SP = S \times R, \qquad P = D \times R, \qquad R = P \cap C_{SP}(S). \tag{4.2}$$

Proof. Let T be the inverse image in S of the 2-image of P in $\mathrm{Inn}(S)$. Its central subgroup $Z(S)$ has odd order and $T/Z(S)$ is a 2-group. Schur–Zassenhaus gives a complement to $Z(S)$; all complements are conjugate by $Z(S)$ and are therefore equal, since $Z(S)$ is central. Thus T has a unique Sylow 2-subgroup. The inner action consequently has a unique homomorphic lift $\delta : P \to S$ with 2-group image and $\delta|_D = \mathrm{id}_D$. Set $r(p) = \delta(p)^{-1}p$. This is a homomorphism into $C_{SP}(S)$, with kernel D. Its image R_0 meets S trivially, and $SP = S \times R_0$.

The specified upper block is the tensor of the lower block with $\mathcal{O}R_0$, so its defect groups have form $D' \times R_0$, where D' is an S-conjugate of D. Apply this to P and use $P \cap S = D$. Then $P = D \times R_0$, giving $R_0 = R$. Uniqueness of δ makes this construction natural under further marked operators. □

5. Klein four and elementary abelian component complexes

5.1. **The relative operator group.** Let $N = N_S(D)$, let c be the original Brauer correspondent of b, and let $\mathcal{A} \le \mathrm{Aut}(S)_b$ contain $\mathrm{Inn}(S)$. Suppose that $\mathcal{A}/\mathrm{Inn}(S)$ is odd and $O_2(Z(S)) = 1$. Write $\mathcal{A}_D = N_{\mathcal{A}}(D)$ and use $\delta(x) = (x, x)$ for a diagonal pair. Let $c_n(x) = nxn^{-1}$ denote conjugation by n. The finite relative group is

$$\mathcal{T} = \big((S \times N) \rtimes \mathcal{A}_D\big)/\{(\delta(n)^{-1}, c_n) : n \in N\}. \tag{5.1}$$

For $f(n) = (\delta(n)^{-1}, c_n)$, multiplication in the semidirect product gives $f(n)f(m) = f(nm)$. The denominator centralizes the base and is normal. Its intersection with $S \times N$ is $\Delta Z(S)$; the resulting base is $(S \times N)/\Delta Z(S)$ and its quotient is $\mathcal{A}/\mathrm{Inn}(S)$, by the marked Frattini equality. The corner $b\mathcal{O}Sc$ is a module for (5.1), since c_n acts there by the inner diagonal action. The

original central character is the same at both ends. An ambient stabilizer acts by pullback; its centralizer kernel acts as the identity on this factor, while remaining in the ambient group.

Proposition 5.1. *For a Klein four defect block under the preceding relative-operator hypotheses, the block and its Brauer correspondent have an integral splendid Rickard complex C carrying the action (5.1). Its inner operators are the diagonal operators. In the situation of Lemma 4.3, it has an upper complex*

$$C^+ = C \otimes_{\mathcal{O}} \mathcal{O}R \tag{5.2}$$

for the given groups SP and NP, with the same lower identity component and the same P-action.

Proof. The regular block $\mathcal{O}Sb$ is an indecomposable $\mathcal{O}(S \times S)$-lattice with vertex ΔD, and

$$N_{S\times S}(\Delta D) = (C_S(D) \times C_S(D))\Delta N \leq S \times N.$$

Green correspondence gives a unique summand M with vertex ΔD in its restriction to $S \times N$, with multiplicity one. The marked root lies in c, so the right block cut retains M. Any full-order vertex of a summand of $b\mathcal{O}Sc$ is conjugate to ΔD in $S \times S$ and projects on the right into a defect group of c. Since $D \lhd N$, that projection equals D. Its right conjugator therefore belongs to N. Such a vertex is already conjugate to ΔD inside $S \times N$, so Green uniqueness excludes a second full-order summand. All complementary summands have proper vertices. It is invariant under the relative group. In the endomorphism algebra of the corner, its isotype projector modulo the radical is invariant. Odd averaging lifts the projector to an invariant element, and iteration of $t \mapsto 3t^2 - 2t^3$ gives an invariant idempotent. Thus the action on M is extracted from the original corner.

The integral source algebra classification [11] identifies the source algebra of the left block with $\mathcal{O}D$, $\mathcal{O}A_4$, or the principal $\mathcal{O}A_5$ block. These are interior-D identifications. The normal-defect right block has source $\mathcal{O}D$ or $\mathcal{O}A_4$: normal D acts trivially on every simple right module, excluding the nontrivial two-dimensional simple sources in the A_5 case. The induced Puig tensor functors preserve projective covers, indecomposability, permutation summands, and the marked full vertex. For the last assertion, a proper vertex after transport would remain proper after transport back, contradicting the original full vertex.

To identify the underlying bimodule M, use the source models. For the A_5–A_4 and A_4–A_4 models,

$$N_{A_5\times A_4}(\Delta D)/\Delta D \cong A_4, \qquad N_{A_4\times A_4}(\Delta D)/\Delta D \cong A_4.$$

The Brauer image of the standard Green module is $kD \cong \mathrm{Ind}_{C_3}^{A_4} k$, the indecomposable projective cover of the trivial character. Its three inertial-character twists exhaust the projective indecomposables of kA_4. The permutation-module Brauer correspondence therefore shows that the transported M is exactly one of these three twists. In the D–D case there is one possibility. Consequently M is already a Morita equivalence in the D and A_4 cases; in the A_5 case it is the standard Green stable equivalence with its specified twist. In the first two cases take $C = M[0]$.

In that last case $\mathrm{Pic}(\mathcal{O}A_5b_0) \cong C_2$, where b_0 is the principal block idempotent of $\mathcal{O}A_5$ [3, Theorem 1.5], so the odd operator quotient fixes the three left simple classes. The Green equivalence intrinsically identifies the distinguished right simple whose image is simple. It is fixed, and an odd group cannot exchange the other two. In the A_5–A_4 model, label the two nontrivial left simples U_1, U_2 over k. Each $(k \otimes_{\mathcal{O}} M)^\vee \otimes_{kA_5b_0} U_i$ is an indecomposable module of dimension two for the right block; let V_i be its unique simple submodule. For the modular model algebras $\overline{A} = kA_5b_0$ and $\overline{B} = kA_4$, write $\overline{M} = k \otimes_{\mathcal{O}} M$. For simple modules U, V on the respective sides, adjunction gives

$$\mathrm{Hom}_{\overline{A}\otimes_k \overline{B}^{\mathrm{op}}}(\overline{M}, U \otimes_k V^\vee) \cong \mathrm{Hom}_{\overline{B}}(V, \overline{M}^\vee \otimes_{\overline{A}} U).$$

Thus the projective-cover multiplicity is the multiplicity of V in the socle on the right. The correction of [34, Lemma 5.1 and Proposition 5.2] uses

$$P(U_1)\otimes_{\mathcal{O}} P(V_1)^{\vee}, \qquad P(U_2)\otimes_{\mathcal{O}} P(V_2)^{\vee},$$

where $P(U)$ denotes the projective $\mathcal{O}$-lattice lifting the projective cover of the simple k-module U. The retained inertial twist transports these pairings together. Thus the two required isotypes of the projective cover of M are fixed by the odd operator quotient.

In the untwisted A_5–A_4 source model, take $A=\mathcal{O}A_5b_0$, $B=\mathcal{O}A_4$ and its Green bimodule M. Put $P_i=P(U_i)$, $Q_i=P(V_i)$ and $W=\bigoplus_{i=1}^{2} P_i\otimes_{\mathcal{O}} Q_i^{\vee}$. The projective decompositions in [34, Proposition 5.2] are

$$W\otimes_B M^{\vee}\cong M\otimes_B W^{\vee}\cong \bigoplus_{i,j=1}^{2} P_i\otimes_{\mathcal{O}} P_j^{\vee},$$

$$W\otimes_B W^{\vee}\cong \bigoplus_{i=1}^{2} P_i\otimes_{\mathcal{O}} P_i^{\vee}\oplus \bigoplus_{i,j=1}^{2} P_i\otimes_{\mathcal{O}} P_j^{\vee},$$

$$M\otimes_B M^{\vee}\cong A\oplus \bigoplus_{i\neq j} P_i\otimes_{\mathcal{O}} P_j^{\vee}.$$

The opposite-side identities replace A, P_i, W, M by $B, Q_i, W^{\vee}, M^{\vee}$: the Cartan submatrices on the two nontrivial projectives are both $\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)$, and the Green functors satisfy

$$M\otimes_B Q_i\cong P_1\oplus P_2, \qquad M^{\vee}\otimes_A P_i\cong Q_1\oplus Q_2 \quad (i=1,2).$$

For a projective-cover map $q: P_M\to M$, set $\delta=q|_W$. The omitted summand is $W_0=P(k_A)\otimes_{\mathcal{O}} P(k_B)^{\vee}$, where k_A, k_B denote the trivial simple modules; thus $P_M=W\oplus W_0$. Its tensor with $W^{\vee}$ has left projective factor $P(k_A)$, whereas $M\otimes_B W^{\vee}$ has only P_1, P_2. Thus the omitted summand contributes a radical map to the section-composite of the split surjection $q\otimes 1$. Put $T=M\otimes_B W^{\vee}$ and write this surjection as

$$f=(f_W, f_0): (W\otimes_B W^{\vee})\oplus (W_0\otimes_B W^{\vee})\longrightarrow T.$$

Let $s=(s_W, s_0)$ be a section, with components from T to the indicated summands. Then

$$f_W s_W = 1_T - r, \qquad r=f_0 s_0\in J(\mathrm{End}_{A\text{-}A}(T)).$$

The map $s_W(1-r)^{-1}$ is a section of $f_W=\delta\otimes 1$. Dually $1\otimes\delta^{\vee}$ is split injective. Cancel these two contractible pairs in the total tensor square. The displayed decompositions leave A in degree zero by Krull–Schmidt cancellation. The opposite-side calculation leaves B. Source-algebra tensor equivalences and the inertial twist preserve these contractions.

Take a projective cover $q: P_M\twoheadrightarrow M$ in the relative group. Its restriction is the lower projective cover. Here the normal base is $K_0=(S\times N)/\Delta Z(S)$ and $K_0\triangleleft\mathcal{T}$ has odd index. Hence

$$J(\mathcal{O}\mathcal{T})=J(\mathcal{O}K_0)\mathcal{O}\mathcal{T}: \tag{5.3}$$

the right side is radical modulo the uniformizer, and its quotient is a crossed product of a semisimple algebra by a group of invertible order, hence semisimple by averaging splittings. The two selected isotypes give an invariant idempotent modulo the radical of $\mathrm{End}_{\mathcal{O}K_0}(P_M)$. Lift an invariant element e_0 by odd averaging, and iterate

$$e_{n+1}=3e_n^2-2e_n^3.$$

The error satisfies

$$e_{n+1}^2-e_{n+1}=-(e_n^2-e_n)^2(3-2e_n)(1+2e_n).$$

Completeness gives an invariant idempotent e. The required complex is

$$C=[eP_M\xrightarrow{q|_{eP_M}} M], \qquad \deg(eP_M)=-1, \quad \deg M=0. \tag{5.4}$$

Projective-cover uniqueness and conjugacy of idempotent lifts identify its source model with the correction of [34, Proposition 5.2] and its retained inertial twist. Its terms are permutation summands for the relative group: the correction term is projective there, and the Green term is a summand of the permutation corner $b\mathcal{O}Sc$. Alternatively, the odd base index places every 2-subgroup in the base, where the restriction is 2-permutation. The evaluations are equivariant, and their lower contractions become equivariant after averaging over the odd quotient. This proves both evaluations for (5.4).

Finally (4.2) makes (5.2) an upper splendid complex. Grade $\mathcal{O}R$ by $R \cong P/D$; its identity component is $\mathcal{O}$. The identity component of C^+ is therefore C, with the fixed diagonal P-action. The regular factor retains the whole centralizer kernel and its group multiplication. □

Lemma 5.2. *The full maximal companion of the complex constructed in Proposition 5.1 is the regular right block in degree zero, with its operator action.*

Proof. If $D \triangleleft N$ and c is a central block idempotent of kN, then $\mathrm{Br}_D(c) = c$. Since $D \triangleleft N$, equivariance of the Brauer map makes $\mathrm{Br}_D(c)$ central in kN. The quotient map to $k(N/D)$ kills each nontrivial D-orbit sum. Thus c and $\mathrm{Br}_D(c)$ have the same image modulo the nilpotent ideal $J(kD)kN$; both are central idempotents and are equal. The maximal roots of $\mathrm{Br}_D(b)$ form the same full N-orbit as those of c. Consequently

$$\mathrm{Br}_D(b) = c, \qquad M(\Delta D) = kC_S(D)c.$$

The proper-vertex complementary terms and the projective correction vanish at ΔD. Moreover

$$N_{S\times N}(\Delta D) = (C_S(D) \times C_S(D))\Delta N.$$

The full companion is identified with kNc by the multiplication map $[(x, y), m] \mapsto xmy^{-1}$. It is a bijection on the original coset decomposition and commutes with all original operators. Uniqueness of the integral lift of a permutation summand in the full finite action group gives $\mathcal{O}Nc$ with this action. □

5.2. **The other elementary cases.**

Proposition 5.3. *For a relative operator group with odd image modulo* $\mathrm{Inn}(S)$*, case (i) has the same component data as Proposition 5.1: a full relative-operator splendid complex and an upper complex on P with the same lower identity component.*

Proof. For $S = \mathrm{SL}_2(2^a)$, the construction in [12, Sections 4.2.6 and 6.3] is in the full relative automorphism module category. We restrict it to the stipulated odd outer image. In the global application, oddness of the entire relevant image follows from the full relative-defect equality proved in Section 8. To see the exclusion of an upper outer 2-action, write the Sylow unipotents as $u(x)$, $x \in \mathbb{F}_{2^a}$. Let F square matrix entries. An element gF^r normalizing this Sylow subgroup has g in its Borel normalizer; write t, t^{-1} for its diagonal entries. If it centralizes every $u(x)$, its action on parameters is $x \mapsto t^2x^{2^r}$. Evaluating at 1 gives $t = 1$, and evaluating at all x gives $r = 0$ modulo a. The remaining representative is inner. Lemma 4.3 supplies the upper regular factor. Restriction to an odd outer quotient preserves the permutation property and allows averaging of both contractions. Apply Lemma 2.3 if the initial representative is modular.

For Ree groups, the lower integral splendid kernel is the one recorded in [22, Theorem 1.6] and [12, Section 6.2]. The Picard computation [18, Proposition 5.4] gives $\mathrm{Pic}(\mathcal{O}Sb) \cong C_3$ and shows that a Picard element fixing all ordinary characters is the identity. Under the Puig equivalence with the principal block of $\mathrm{Aut}(\mathrm{SL}_2(8))$, the decomposition matrix has exactly three unit rows and two other columns; its eight rows are distinct. An element of order three fixing the unit rows fixes their columns and also the remaining two columns. It then fixes every row, and is the identity. Thus every nonidentity Picard element cycles the three unit rows. The ordinary trivial

character reduces to a simple module and gives one of these rows. Every group automorphism fixes it, so its Picard class is trivial. The outer group is odd. The following construction gives its action on the chosen kernel.

Work first over k. A block-inner automorphism gives an isomorphism between the regular block bimodule and its twist. Their full vertices are the diagonal of D and the graph of that automorphism on D. Conjugacy of these vertices shows that the latter action belongs to the normalizer fusion. Put $\Omega = \mathcal{A}/\operatorname{Inn}(S)$ and choose representatives $a_g \in \mathcal{A}_D$, $g \in \Omega$. Correct them by inner normalizer elements so that they fix D pointwise. Since S is centerless, there are unique $n(g,h) \in C_S(D)$ satisfying

$$a_g a_h = c_{n(g,h)} a_{gh}.$$

Associativity and centerlessness give, for $g, h, l \in \Omega$,

$$n(g,h)n(gh,l) = a_g(n(h,l))n(g,hl). \tag{5.5}$$

Choose units $u_g \in (kSb)^D$ implementing a_g. Then

$$u_g u_h = \alpha(g,h)n(g,h)u_{gh}, \qquad \alpha(g,h) \in Z(kSb)^\times.$$

The center is a finite-dimensional local commutative k-algebra, and

$$Z(kSb)^\times = k^\times \times (1 + J(Z(kSb))).$$

If $J(Z(kSb))^m = 0$ and $2^r \geq m$, then $(1+x)^{2^r} = 1$ for every x in this radical. The principal-unit group therefore has bounded 2-power exponent. Compare $(u_g u_h)u_l$ with $u_g(u_h u_l)$, use $u_g n u_g^{-1} = a_g(n)$, and cancel (5.5). Since the implemented automorphisms fix the center, the result is

$$\alpha(g,h)\alpha(gh,l) = \alpha(h,l)\alpha(g,hl). \tag{5.6}$$

Let $\alpha_2(g,h)$ be the principal-unit component of $\alpha(g,h)$ and put $n = |\Omega|$. The nth-power map is an automorphism of this abelian group of bounded 2-power exponent. Define

$$\beta_g = \left(\prod_{h \in \Omega} \alpha_2(g,h) \right)^{1/n}.$$

Multiplying (5.6) over its third argument gives $\alpha_2(g,h)^n = \beta_g^n \beta_h^n \beta_{gh}^{-n}$. Uniqueness of the nth root yields $\alpha_2(g,h) = \beta_g \beta_h \beta_{gh}^{-1}$. Replace u_g by $\beta_g^{-1} u_g$ to obtain $\alpha(g,h) \in k^\times$. Here $C_S(D) = D$ and $N_S(D) = D : (C_7 : C_3)$ [12, Section 6.2]. The normalizer has one block, and $\operatorname{Br}_D(b) = c = 1$ in $kN_S(D)$. Set $v_g = \operatorname{Br}_D(u_g)$ in the right block. These are units, because $\operatorname{Br}_D(b) = c$ and the Brauer images of the inverses are their inverses. They implement the automorphism on the full right normalizer block: for $n \in N_S(D)$, the element $a_g(n)n^{-1}$ centralizes D, and applying the Brauer map to $u_g({}^n u_g^{-1}) = a_g(n)n^{-1}b$ gives the corresponding identity for v_g. They therefore satisfy the same product equation with the same scalar $\alpha(g,h)$. For x in any term of the chosen modular lower complex, put

$$\tau_g(x) = u_g x v_g^{-1}, \qquad \tau_g \tau_h = \delta_{n(g,h)} \tau_{gh}. \tag{5.7}$$

Here $\delta_n(x) = nxn^{-1}$ is the diagonal bimodule action. For $r \in kSb$, $t \in kN_S(D)c$ and $n \in N_S(D)$, these maps satisfy

$$\tau_g(rxt) = a_g(r)\tau_g(x)a_g(t), \qquad \tau_g \delta_n \tau_g^{-1} = \delta_{a_g(n)}.$$

Together with (5.7), these are the defining multiplication and semilinearity relations for the relative group. On the dual use $y \mapsto v_g y u_g^{-1}$. The middle factors cancel in both balanced products, so the operators there are

$$\tau_g(x \otimes y) = u_g(x \otimes y)u_g^{-1}, \qquad \tau_g(y \otimes x) = v_g(y \otimes x)v_g^{-1}.$$

Every bimodule differential, evaluation and contraction commutes with these operators. This gives the finite relative action with its inner intersections and both full-operator contractions. Every 2-subgroup of the relative group lies in the base, since the quotient is odd. Restriction to the base is 2-permutation, so the terms are 2-permutation for the whole relative group. With the preceding full-operator contractions, lift the resulting complex integrally using Lemma 2.3.

For J_1, use the explicit splendid construction in [12, Section 6.2.3]. For the nonprincipal C_2^3-block of Co_3, [22, Theorems 1.3, 1.5–1.6 and Section 7] uses the original marked Green module for $(\mathrm{SL}_2(8) \rtimes C_3) \times S_3$. Its Puig equivalence, defect-zero factor and Brauer-normalizer identification are integral and retain the original defect group. Both sporadic groups have trivial outer automorphism group. Their relative action is just the prescribed inner action, pulled back with the ambient centralizer acting trivially.

The Ree groups and these sporadic groups are centerless, and the Ree outer group is odd. Thus the given upper P has inner image in every case. Lemma 4.3 gives the upper complex by the same regular tensor factor. Proposition 2.4 supplies its local return. □

6. The rank-one Levi components

We treat case (iii). We construct a complex for S and $N_S(D)$ with the marked component stabilizer action, as the identity component of a splendid complex for SP and $N_S(D)P$. The operator groups are the marked stabilizers in the reduced ambient situation of Section 8. Lemma 8.2 makes the image of P a Sylow 2-subgroup of the quotient by the lower inner operators. This will give the odd outer image on the rank-one factor. Let $\mathcal{A}$ be the automorphism image on S of the group generated by S and the ambient marked component stabilizer, and put $\mathcal{A}_D = N_{\mathcal{A}}(D)$.

6.1. **The full-centralizer Levi and field operators.** Let $\mathbf{G}$ be the simply connected algebraic group giving the component before its central quotient, with Frobenius F. Write $S_{\mathrm{sc}} = \mathbf{G}^F$ and $S = S_{\mathrm{sc}}/Z_0$, where Z_0 is central; $Z_0 = 1$ is allowed. A regular embedding is an inclusion $\mathbf{G} \le \widetilde{\mathbf{G}}$ with connected center and $\widetilde{\mathbf{G}} = \mathbf{G}Z(\widetilde{\mathbf{G}})$. The proof of [15, Proposition 5.3(v)] gives, in the present D_{2t} and E_7 branches,

$$C_{\mathbf{G}^*}(s) = C^\circ_{\mathbf{G}^*}(s) = \mathbf{L}^* \tag{6.1}$$

for the chosen odd-order semisimple label s and the classification Levi.

Connectedness also follows directly. The simply connected covering of the dual group has a central 2-kernel. An odd-order s has an odd-order lift $\widehat{s}$. If $\widehat{g}\widehat{s}\widehat{g}^{-1} = z\widehat{s}$ for a central 2-element z, taking the odd order of $\widehat{s}$ gives $z = 1$. The connected centralizer upstairs therefore maps onto the full centralizer downstairs.

For type D_{2t}, the two possible A_1^t root systems can be written as paired difference roots, or as the same pairs with one pair replaced by a sum root. The parity of the number of sum pairs is invariant under $W(D_{2t})$, whose signed permutations change an even number of coordinate signs. The nontrivial graph automorphism exchanges these two classes. An automorphism stabilizing the block stabilizes its rational-series idempotent and hence the full-centralizer Levi class. Thus its graph component is trivial. There is no triality case, since $t \ge 3$ is odd. Type E_7 has no nontrivial diagram automorphism. Also $q \equiv \pm 3 \pmod 8$ forces its field exponent to be odd. The given upper 2-group consequently acts by inner-diagonal automorphisms.

We choose the Levi compatibly with the field part. Let F_0 be the pure field root representing the relevant cyclic field subgroup, so $F = F_0^r$ with r odd. Its geometric semisimple class is stable. Lang's theorem gives an F_0-fixed representative of this class [36, Lemma 4.6]. Connectedness of its centralizer implies that the geometric class contains a single rational F-class, so this representative is $(\mathbf{G}^*)^F$-conjugate to s. Use this representative and its centralizer as $s, \mathbf{L}^*$. The

rational series idempotent is determined by that rational F-class, so it contains the original block with its marked defect pair. By F_0-duality choose an F_0-stable Levi $\mathbf{L}$ dual to $\mathbf{L}^*$; this duality also holds for $F = F_0^r$ [36, Remark 4.7]. Put $L_{\mathrm{sc}} = \mathbf{L}^F$, $L = L_{\mathrm{sc}}/Z_0$.

Choose a parabolic radical $\mathbf{U}$ with Levi $\mathbf{L}$, and put

$$\mathbb{G} = \mathbf{G}^r, \quad \mathbb{L} = \mathbf{L}^r, \quad \mathbb{U} = \mathbf{U}^r, \quad \mathbb{F} = \tau(F_0, \dots, F_0), \tag{6.2}$$

where τ is cyclic permutation. First-coordinate projection identifies $\mathbb{G}^{\mathbb{F}}$ with $\mathbf{G}^F$; its inverse is $\iota(g) = (g, F_0^{r-1}g, \dots, F_0 g)$. The finite-order operator τ is F_0^{-1} on these finite points and preserves $\mathbb{U}$. The dual tuple uses the reverse cycle. The rational-series comparison in [36, Section 5] retains the chosen label. The Deligne–Lusztig variety is

$$Y_{\mathbb{U}}^{\mathbb{G}} = \{g\mathbb{U} \in \mathbb{G}/\mathbb{U} : g^{-1}\mathbb{F}(g) \in \mathbb{U}\mathbb{F}(\mathbb{U})\}.$$

Its left and right actions are multiplication by $\mathbb{G}^{\mathbb{F}}$ and $\mathbb{L}^{\mathbb{F}}$. The same definition is used for the disconnected upper group [2, Section 2.E].

We use the point-stabilizer form of Rickard's cochain construction. For its coefficients, choose a finite extension K_{fin} of $\mathbb{Q}_2$ inside K containing the roots of unity needed to split the finite groups in the construction, and let $\mathcal{O}_{\mathrm{fin}}$ be its valuation ring. The canonical embedding $\mathbb{Z}_2 \subseteq \mathcal{O}$ extends to $\mathcal{O}_{\mathrm{fin}} \subseteq \mathcal{O}$. The block idempotents have coefficients in $\mathcal{O}_{\mathrm{fin}}$ by their ordinary-character formula and integrality. Construct the finite-action cochains over $\mathcal{O}_{\mathrm{fin}}$ and extend scalars to $\mathcal{O}$. The extension is flat, since $\mathcal{O}$ is torsion-free over the discrete valuation ring $\mathcal{O}_{\mathrm{fin}}$. It preserves permutation lattices, summand projectors, block cuts, balanced tensor products and the cochain differentials. Duals commute with this extension on finite free terms. Extending each chain map and homotopy preserves its defining equations, including the two evaluation contractions.

Let Λ be the whole finite group acting on the chosen variety: it includes the left and right finite groups and all the stipulated regular-embedding and finite field operators. If a primitive block cut is not Λ-invariant, first restrict to its inertia group; full block orbits are recovered by finite induction. For a Λ-variety Z and a geometric point z, let $\Lambda_z = \{\lambda \in \Lambda : \lambda z = z\}$; for $B \leq \Lambda$, write $B_z = B \cap \Lambda_z$. Put

$$\mathscr{P}_\Lambda(Z) = \mathrm{add}\{\mathcal{O}[\Lambda/\Lambda_z] : z \in Z(\overline{\mathbb{F}}_q)\}.$$

Here add denotes finite direct sums and direct summands in the category of $\mathcal{O}\Lambda$-lattices. We write $\Omega_c(Z, \Lambda, \mathcal{O})$ for the resulting compactly supported cochain object in that category; the group parameter is omitted when its action has been fixed. There are finitely many subgroup types in the displayed expression. Choose the bounded cochain complex in $K^b(\mathscr{P}_\Lambda(Z))$ supplied by [32, Theorems 3.2 and 3.5]; the integral point-stabilizer form is also explicit in [35, Section 2.5.1]. Apply the prescribed block projectors to this complex. The integral lower equivalence is supplied by [2, Theorem 7.7], equivalently [36, Theorem 2.28] in the full-centralizer case. The comparison of the Godement and Rickard constructions in [35, Section 2.5.1], also for compact supports, identifies the chosen model with this complex in the homotopy category of the base double group. The comparison commutes with restriction of the finite action group. Naturality of duality and evaluation under a homotopy equivalence therefore gives the two evaluation homotopies for the chosen model. Denote the chosen lower block complex by C. For a central quotient, this choice is made on the quotient variety below.

Two consequences will be used. For a subgroup $B \leq \Lambda$, Mackey restriction of $\mathcal{O}[\Lambda/\Lambda_z]$ is a sum of $\mathcal{O}[B/B_{\lambda z}]$. Thus restriction to a smaller action group still has a point-stabilizer term bound. The same statement holds for pullback through a homomorphism of finite action groups. Also, every operator acting pointwise as the identity on Z acts as the identity on every term of this representative: it belongs to every point stabilizer. In particular, the diagonal central kernels and the inner-overlap relations hold on the chosen terms.

6.2. **Finite induction of the chosen cochain model.**

Lemma 6.1. *Let $T \leq \Lambda$ be finite groups, and let Y be a quasi-projective variety over $\overline{\mathbb{F}}_q$, where q is odd, with a T-action. Then*

$$\Omega_c(\Lambda \times_T Y, \Lambda, \mathcal{O}) \simeq \mathrm{Ind}_T^{\Lambda} \Omega_c(Y, T, \mathcal{O}) \tag{6.3}$$

in the bounded 2-permutation homotopy category. The comparison is natural for additional finite operators preserving the data.

Proof. The induced variety is a finite disjoint union of open-and-closed copies of Y. Godement stalk products, and subsequently compactly supported sections, decompose into the corresponding finite sums. The action of Λ permutes them with the original coset carries. This gives induction at the cochain-construction level. The bounded permutation replacement is homotopy equivalent to this object [35, Section 2.5.1].

Equivalently, put $B = Y/T = (\Lambda \times_T Y)/\Lambda$ and let $\pi_Y : Y \to B$ and $\pi_{\Lambda \times_T Y} : \Lambda \times_T Y \to B$ be the quotient maps. The finite quotient pushforwards satisfy the constructible-sheaf identity

$$(\pi_{\Lambda \times_T Y})_* \mathcal{O} \cong \mathcal{O}\Lambda \otimes_{\mathcal{O}T} (\pi_Y)_* \mathcal{O}.$$

On each geometric stalk this is the permutation-module identity for the induced fiber, compatible with restriction maps. Rickard's additive-functor compatibility [32, Lemma 2.8 and Theorem 3.5] gives (6.3), including the integral limit. In particular, restrict a representative chosen first for the whole lower finite action, and then induce that representative. It realizes the upper object in the homotopy category. □

6.3. **The disconnected upper group.** First take $S = S_{\mathrm{sc}}$. Put $\overline{\mathbf{G}} = \mathbf{G}/Z(\mathbf{G})$. The inner-diagonal action gives a homomorphism $\eta : P \to \overline{\mathbf{G}}^F$. Extend the coordinate formula for ι to $\overline{\mathbf{G}}^F$. For $p \in P$, conjugation by

$$\bar{\iota}(\eta(p)) = \big(\eta(p), F_0^{r-1}\eta(p), \ldots, F_0\eta(p)\big)$$

defines an algebraic automorphism of $\mathbb{G}$ commuting with $\mathbb{F}$. On $\mathbb{G}^{\mathbb{F}}$ it is the given action of p, and for $d \in D$ it is conjugation by $\iota(d)$. With this action form

$$\widehat{\mathbb{G}} = (\mathbb{G} \rtimes P)/\{(\iota(d)^{-1}, d) : d \in D\}. \tag{6.4}$$

The graph subgroup centralizes $\mathbb{G}$ and is normalized by P, so it is normal. The Frobenius acts as $\mathbb{F}$ on $\mathbb{G}$ and trivially on the finite factor. If a class of (g, p) is fixed, then

$$(\mathbb{F}(g), p) = (g, p)(\iota(d)^{-1}, d)$$

for some $d \in D$. Its P coordinate forces $d = 1$. Thus the finite fixed-point group is exactly $(S \rtimes P)/\{(d^{-1}, d) : d \in D\} \cong SP$, with the original multiplication and powers.

Let $\mathbf{Z}_1 = C_{\widehat{\mathbb{G}}}(\mathbb{G})$. Algebraic innerness gives $\widehat{\mathbb{G}} = \mathbb{G}\mathbf{Z}_1$. Put $\widehat{\mathbb{L}} = \mathbb{L}\mathbf{Z}_1$ and extend the parabolic by $\mathbf{Z}_1$. Their simultaneous normalizer in $\widehat{\mathbb{G}}$ is $\widehat{\mathbb{L}}$: writing an element as gz with $g \in \mathbb{G}$ and $z \in \mathbf{Z}_1$ reduces the condition to normalizing the connected parabolic and its Levi. A parabolic is self-normalizing and its Levi has trivial normalizer in the unipotent radical, so $g \in \mathbb{L}$. Define the upper finite Levi by $L^+ = \widehat{\mathbb{L}}^{\mathbb{F}}$. Lang's theorem in the connected Levi implies that L^+ meets every component, so $SP = SL^+$. Conjugation by a rational element of any component is induced on $\mathbb{G}$ by an element g whose conjugation commutes with $\mathbb{F}$. Thus $g^{-1}\mathbb{F}(g)$ is central. For a chosen rational torus–character pair, let $\mathbb{T}$ be its $\mathbb{F}$-stable torus. Lang's theorem in $\mathbb{T}$ gives $a \in \mathbb{T}$ with $a^{-1}\mathbb{F}(a) = g^{-1}\mathbb{F}(g)$. Then $\mathbb{F}(ga^{-1}) = ga^{-1}$. Since a centralizes $\mathbb{T}$, the rational element ga^{-1} induces the same torus and character transport as g. Each component therefore preserves the rational series. The same argument applies inside the Levi. Let e_S and e_L be the central

idempotents of the rational 2-series determined by s on S and L. Every component fixes these idempotents, so the series-inertia condition in [36, Corollary 3.8] is

$$N_{L^+}(e_L)S = L^+S = SP = N_{SP}(e_S). \tag{6.5}$$

The full-centralizer equality implies

$$C^\circ_{\mathbf{G}^*}(s)\, C_{(\mathbf{G}^*)^F}(s) \le \mathbf{L}^*.$$

The product label in the restriction-of-scalars group satisfies the same containment. This is the almost-superregularity condition of [36, Definition 2.21]. Proposition 3.6 of [36] applies to $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{L}}$, including these disconnected groups. The endomorphism homotopy splittings and Proposition 3.7, together with the inertia equality (6.5) and Corollary 3.8, give the upper splendid complex on SP and L^+ with both bimodule evaluation homotopies. The upper splendid correspondence puts a conjugate of the marked P inside L^+. Since $SP = SL^+$, a conjugation by an element of S suffices. Choose $s_0 \in S$ with $P \le {}^{s_0}L^+$, and transport the Levi and radical by $\iota(s_0)$. Transport every field operator as well; in particular, replace τ by $c_{\iota(s_0)}\tau c^{-1}_{\iota(s_0)}$, where c_x denotes conjugation by x. These transported operators preserve the transported variety and retain their multiplication relations. We now have $L^+ = LP$ and the original P preserves the same lower variety.

6.4. **Central quotients and the identity component.** For a component $S = S_{\mathrm{sc}}/Z_0$, let $\alpha : SP \to \mathrm{Aut}(S)$ be conjugation and let $\rho : \widetilde{\mathbf{G}}^F \to \mathrm{Aut}(S)$ be the regular-embedding action. Form the finite fiber product

$$\{(u, g) \in SP \times \widetilde{\mathbf{G}}^F : \alpha(u) = \rho(g)\}.$$

The kernel of the projection to SP is $\{1\} \times \ker\rho$. Commutators of an element of $\ker\rho$ with S_{sc} lie in Z_0 and define a homomorphism from a perfect group to a central group, so vanish. If $g \in \widetilde{\mathbf{G}}^F$ centralizes S_{sc}, then it normalizes the unipotent radicals of two opposite rational Borels. Their normalizers are the corresponding finite Borels by the split BN-pair, so g lies in their common torus. For each root α, write $x_\alpha(t)$ for the root-group parametrization. Commutation with $x_\alpha(1)$ gives $\alpha(g) = 1$ for every α, whence $g \in Z(\widetilde{\mathbf{G}})^F$. Consequently, $C_{\widetilde{\mathbf{G}}^F}(S_{\mathrm{sc}}) = Z(\widetilde{\mathbf{G}}^F)$. Thus the projection kernel is central in the fiber product. Choose a Sylow 2-subgroup T_0 of the inverse image of P in this fiber product, and let

$$E_0 = S_{\mathrm{sc}}T_0 \twoheadrightarrow SP$$

be the resulting finite extension, with central kernel A and $A \cap S_{\mathrm{sc}} = Z_0$. In these D_{2t} and E_7 cases, $Z(S_{\mathrm{sc}})$ is a 2-group, so Z_0 is a 2-group. Also E_0/S_{sc} is a 2-group and A/Z_0 embeds in it. Hence A is a 2-group. Apply (6.4) with the inner-diagonal action of T_0 and intersection $T_0 \cap S_{\mathrm{sc}}$. Its graph normality and fixed-point calculation give the finite upper group E_0. Denote its finite upper Levi by L_0^+. The kernel A acts trivially on the connected algebraic group, since its regular-embedding image is central. It also centralizes E_0, and each algebraic component has a rational representative in E_0. Thus A centralizes the whole upper algebraic group, so $A \le L_0^+$ and its diagonal action on the upper variety is pointwise trivial. Its deck action is free: a deck stabilizer lies in a unipotent radical, whose elements have odd order. Put $L^+ = L_0^+/A$. Take the unique block of E_0 dominating the specified block of SP, and its upper Jordan correspondent in L_0^+. The latter dominates the target block of L^+. The central 2-quotient gives both block bijections. The same correspondence and simultaneous conjugation place the marked P in L^+.

Let $Y_0 = Y_{\mathbb{U}}^{\mathbb{G}}$ be the variety for the connected lower group with the chosen radical, and let Y_0^+ be the corresponding variety for the upper group. Put $\overline{Y}_0^+ = Y_0^+/A$. Before quotienting, coarsen the component grading to

$$E_0/(S_{\mathrm{sc}}A) \cong SP/S. \tag{6.6}$$

Its identity piece is AY_0, and the quotient map gives

$$(AY_0)/A \cong Y_0/Z_0 =: \overline{Y}_0. \tag{6.7}$$

An A-orbit meeting Y_0 meets it in exactly one Z_0-orbit. The finite quotient universal property gives the inverse as a map of varieties, which proves the variety isomorphism. Two lifts of the same original $p \in P$ differ by A, whose diagonal conjugation on the variety is pointwise trivial. Thus (6.7) is equivariant for the original P, with all its power relations.

After placing P in the upper Levi, the stabilizer of the identity component consists of pairs with equal quotient degrees. Multiplying by one common original P-lift turns them into lower pairs. Consequently, with $K = SP$,

$$\overline{Y}_0^+ \cong (K \times L^+) \times_{(S\times L)\Delta P} \overline{Y}_0. \tag{6.8}$$

The map is $[(g,l),[y]] \mapsto [gyl^{-1}]$; the same degree calculation proves its well-definedness, injectivity and surjectivity. The original stable primitive block idempotents lie in identity degree: a stable block in a normal 2-extension has a unique covering block, represented by that same idempotent. This applies at both endpoints and preserves the chosen cuts in (6.8).

Choose the lower point-stabilizer cochain representative directly on $\overline{Y}_0$ with its whole field and regular-embedding action; these operators preserve the central quotient. The action of $(S \times L)\Delta P$ on the lower cochains is obtained from its base action and the conjugation action of P; their intersection acts diagonally. Restriction to this action is denoted by $\operatorname{Res} C$. Lemma 6.1 gives the representative

$$C^+ = \operatorname{Ind}_{(S\times L)\Delta P}^{K\times L^+} \operatorname{Res} C, \qquad (C^+)_1 = C. \tag{6.9}$$

Put $T = (S \times L)\Delta P$ and $\Gamma = K \times L^+$. After restricting each of the specified point-stabilizer summands and then inducing, its permutation overmodule is $\mathcal{O}[\Gamma/T_z]$. This is the permutation module of a point stabilizer on $\Gamma \times_T \overline{Y}_0$. Thus the chosen induced complex has the point-stabilizer term bound for the upper variety.

We verify that the two indicated P-actions are equal. A nontrivial field power cannot be inner-diagonal on the finite central quotient. If it were, its implementing regular-embedding element, preserving the opposite standard Borels, would be toral. Since a field map fixes every $x_\alpha(1)$, all root multipliers would be one, contradicting its nontrivial action on some $x_\alpha(t)$. A central quotient does not identify distinct root-group elements. Thus the kernel of the regular/field finite operator group over its automorphism image has no field part. The remaining kernel consists of central regular-embedding elements and acts pointwise trivially on the variety. The two lifts of an original P-operator in (6.9) consequently induce the same cochain action. The kernel of $P \to \operatorname{Aut}(S)$ acts through its homogeneous units in the upper grading.

Lemma 6.2. *Let K_0 be a finite group, let $L_0 \le K_0$, let $A \le Z(K_0) \cap L_0$, and let Z be a (K_0, L_0)-variety on which the left and right actions of A agree and the resulting deck action of A is free. Put $K = K_0/A$, $L = L_0/A$, and $\overline{Z} = Z/A$. Suppose every 2-subgroup of a point stabilizer on Z is $(K_0 \times 1)$-conjugate to a subgroup of ΔL_0. Then every 2-subgroup of a point stabilizer on $\overline{Z}$ is $(K \times 1)$-conjugate to a subgroup of ΔL. If $P \in \operatorname{Syl}_2(L)$, every such stabilizer subgroup is $(K \times L)$-conjugate to a subgroup of ΔP.*

Proof. Let $T \le (K \times L)_{\bar z}$ be a 2-subgroup and choose $z \in Z$ above $\bar z$. For the inverse image $\widetilde{T}$ in $K_0 \times L_0$ there is an exact sequence

$$1 \longrightarrow A \times A \longrightarrow \widetilde{T} \longrightarrow T \longrightarrow 1.$$

Choose a Sylow 2-subgroup Q of $\widetilde{T}$. Since $A \times A$ is central, Q contains $A_2 \times A_2$ and maps onto T.

For each $u \in Q$, freeness supplies a unique $\chi(u) \in A$ with $uz = \chi(u)z$. Centrality of the deck action implies $\chi(uv) = \chi(u)\chi(v)$, so $\chi : Q \to A_2$ is a homomorphism. With the bimodule convention $(a, b)z = azb^{-1}$, it satisfies

$$\chi(a, b) = ab^{-1} \qquad (a, b \in A_2).$$

Every u can therefore be multiplied by $(\chi(u)^{-1}, 1)$, without changing its image in T, to obtain an element fixing z. Hence $Q_z = \ker \chi$ is a 2-subgroup of the upstairs point stabilizer and

$$1 \longrightarrow \Delta A_2 \longrightarrow Q_z \longrightarrow T \longrightarrow 1$$

is exact. The stipulated upstairs conjugacy applies to Q_z. Its image gives the first assertion. Finally conjugate the right projection inside L into P and use the same element diagonally in $K \times L$. □

In the present application $Z = Y_0^+$ and $(K_0, L_0) = (E_0, L_0^+)$. The deck kernel lies in the central part used to extend the Levi, so $A \leq L_0^+$. It acts freely: an element of this central deck group fixing $y\mathbb{U}$ would lie in the unipotent radical, whose intersection with the central deck group is trivial. The upstairs point-stabilizer assertion is [2, Lemma 3.7]. That lemma is stated in the section allowing disconnected reductive groups, and its conjugator belongs to the finite left group. For the extended parabolic and Levi used here their simultaneous normalizer is the extended Levi, so its diagonal is the stated ΔL_0^+. The lemma therefore applies to the central quotient, including its additional fixed-point components.

Combining Lemma 6.2 with the initial point-stabilizer choice gives, once $P \in \mathrm{Syl}_2(L^+)$ has been established,

$$(C^+)^i \in \mathrm{add}\{\mathcal{O}[(K \times L^+)/\Delta R] : R \leq P\} \quad \text{for every } i. \tag{6.10}$$

Indeed, for a point stabilizer B and $R \in \mathrm{Syl}_2(B)$, $\mathcal{O}[\Gamma/B]$ is a direct summand of $\mathcal{O}[\Gamma/R]$: the coset projection and transfer have composite $[B : R]$, a unit of $\mathcal{O}$. The preceding lemma conjugates R into ΔP. Block cuts preserve this term assertion. The conjugations establish relative projectivity upon restriction to the base double group and preserve the chosen operator action.

Lemma 6.3. *The central quotient above preserves the two upper evaluations, their homotopies, and the relative dual on the specified geometric complexes.*

Proof. The central deck action is free. Use Rickard's point-stabilizer representative: its terms are direct summands of permutation modules whose restriction to the deck group is free. In particular the terms, including their block cuts, are finite projective over $\mathcal{O}A$. On each free orbit the norm sends the coinvariant basis element to the orbit sum with coefficient one. It therefore identifies coinvariants with invariants on these terms and their direct summands. The quotient theorem [32, Theorem 4.1], or [35, Theorem 2.28], identifies these invariants with quotient cochains in the homotopy category, retaining the remaining finite operators.

Central balance is termwise: the left and right central operators agree. Tensoring the quotient complexes over the quotient middle algebra is therefore the quotient of the original balanced tensor. For a finite projective $\mathcal{O}A$-module V, the coefficient-of-identity Frobenius trace gives

$$\mathrm{Hom}_{\mathcal{O}A}(V, \mathcal{O}A) \cong \mathrm{Hom}_{\mathcal{O}}(V, \mathcal{O}). \tag{6.11}$$

The inverse sends $f \in \mathrm{Hom}_{\mathcal{O}}(V, \mathcal{O})$ to

$$v \longmapsto \sum_{a \in A} f(a^{-1}v)a.$$

To check the evaluations, put $R = \mathcal{O}A$ and let B_0 be an upstairs middle block algebra. Its relative symmetrizing trace is

$$t^R_{B_0}\Big(\sum_g b_g g\Big) = \sum_{a\in A} b_a a,$$

restricted to the block. Composing with the coefficient of 1 in R gives the ordinary symmetrizing trace; composing with augmentation $R \to \mathcal{O}$ gives the symmetrizing trace of the quotient block. Since V is finite projective over R,

$$\mathcal{O} \otimes_R \operatorname{Hom}_R(V, R) \cong \operatorname{Hom}_{\mathcal{O}}(\mathcal{O} \otimes_R V, \mathcal{O}).$$

The relative tensor-dual adjunction and these trace identities identify the quotient of each evaluation with the downstairs evaluation. Explicitly, let G_0 be an upstairs endpoint group, choose representatives $\mathcal{R}$ for G_0/A, and put $\phi(v) = \sum_{a\in A} f(a^{-1}v)a$ for $f \in \operatorname{Hom}_{\mathcal{O}}(V, \mathcal{O})$. Then

$$\sum_{t\in\mathcal{R}} \phi(t^{-1}v)t = \sum_{t\in\mathcal{R}}\sum_{a\in A} f(a^{-1}t^{-1}v)\, at = \sum_{g\in G_0} f(g^{-1}v)g.$$

The last expression is the ordinary group-basis evaluation. Under base change, the class of ϕ corresponds to the functional $\bar f$ defined by $\bar f(\bar v) = \epsilon(\phi(v))$, where $\epsilon : R \to \mathcal{O}$ is augmentation. The quotient of the displayed evaluation is therefore

$$\sum_{\bar t\in G_0/A} \bar f(\bar t^{-1}\bar v)\bar t,$$

the downstairs evaluation with coefficient one. Block cuts restrict these formulas to their direct summands, and the same calculation applies at the other endpoint. Homotopies descend under the additive quotient functor. The upper contractions supplied by [36, Propositions 3.6–3.7] therefore descend to the quotient complexes. The quotient comparison retains the normalizer of $A \times 1$ in $E_0 \times L_0^+$, which is the whole upper double group. The diagonal A already acts trivially, so the remaining action factors through $\Gamma = (E_0/A) \times (L_0^+/A)$. The Γ-linear homotopy comparison in Lemma 6.1 transfers these contractions to the specified C^+. Since A is a 2-group, the finite projective R-terms are free because R is local. □

6.5. **Rank one and the upper direct product.** The lower quotient Levi is $L = S_0 \times A_0$ with $S_0 = \mathrm{PSL}_2(q^t)$, A_0 central, and a principal Klein block on S_0. Here $t \geq 3$, also in the E_7 case where $t = 3$; thus $q^t \geq 27$ and S_0 is perfect and centerless. Let $V \cong V_4$ be its marked defect group. The projective centralizer of V in the algebraic PGL_2 is V: diagonalize one involution and impose commutation with a second swapping involution; precisely four projective matrices remain.

The original abelian P centralizes $D = V \times (A_0)_2$ and acts algebraically inner on the Levi. It therefore acts on S_0 by conjugation with a unique $v(p) \in V$. Uniqueness makes $v : P \to V$ a homomorphism, identical on V. Put $P_0 = \ker v$. Then $P = V \times P_0$. Algebraic innerness fixes the connected central torus, so A_0 centralizes P. Since $(A_0)_2 \leq D$ centralizes S_0, it lies in P_0; therefore $(A_0P_0)_2 = P_0$. Hence, inside the given upper Levi,

$$L^+ = S_0 \times A', \qquad A' = A_0P_0, \qquad P = V \times A'_2. \tag{6.12}$$

The intersection $A' \cap S_0$ is trivial by centerlessness. The factors are the derived subgroup and center and are preserved by the relevant operators. Since $q^t \equiv \pm 3 \pmod 8$, we have $|S_0|_2 = 4$. Thus $V \in \mathrm{Syl}_2(S_0)$ and $P \in \mathrm{Syl}_2(L^+)$.

Lemma 6.4. *In the decomposition*

$$L^+ = S_0 \times A', \qquad P = V \times A'_2, \qquad D = V \times (A_0)_2,$$

where $V \cong V_4$ is a Sylow 2-subgroup of S_0, A' is central abelian, and $A_0 \leq A'$, put

$$H^+ = N_{L^+}(D) = N_{S_0}(V) \times A'.$$

Then

$$H^+ = N_{L^+}(P), \qquad \{l \in L^+ : l^{-1}Dl \leq P\} = H^+.$$

Moreover H^+ controls the fusion of all subgroups of the abelian Sylow subgroup P in L^+.

Proof. Write $l = (s, a)$. The condition $l^{-1}Dl \leq P$ says $s^{-1}Vs \leq V$, hence $s^{-1}Vs = V$. All central factors are fixed under conjugation. This proves the two normalizer equalities and the transporter assertion.

For the fusion assertion, let $Q \leq P$ and $lQl^{-1} \leq P$. Both P and lPl^{-1} centralize lQl^{-1}, because P is abelian. They are Sylow 2-subgroups of $C_{L^+}(lQl^{-1})$. Choose c in that centralizer with $c(lPl^{-1})c^{-1} = P$. Then $h = cl \in H^+$ and $hqh^{-1} = lql^{-1}$ for every $q \in Q$. □

6.6. **The marked Levi operators.** Put $N = N_S(D)$ and $H = N_L(D)$. Write c_n for conjugation by n on S, and, for $U \leq S$, put $c(U) = \{c_u : u \in U\} \leq \mathrm{Aut}(S)$. Let $\widetilde{E}$ be the finite group of regular-embedding Levi and field operators preserving the block, the chosen geometric data, and D, whose automorphism image lies in $\mathcal{A}_D$. Denote this image by E. We keep its finite action on the cochains.

The overlap is

$$\mathcal{A}_D = c(N)E, \qquad E \cap c(N) = c(H). \tag{6.13}$$

For the intersection, an inner representative in S preserving the chosen Levi and its block preserves the unique rational-series idempotent containing that block. Its dual relative normalizer takes s to an $\mathbf{L}^*$-conjugate. Correcting by that conjugator puts it in $C_{\mathbf{G}^*}(s) = \mathbf{L}^*$, so its relative Weyl element is trivial and its representative lies in L. This is the normalizer calculation [2, Section 7.A, equation (7.1)], applied with $C_{\mathbf{G}^*}(s) = \mathbf{L}^*$. The same statement descends through the central quotient, whose kernel acts trivially. Imposing the marking D gives H. Conversely every $h \in H$ supplies this inner operator.

For the product equality, choose a regular embedding $\widetilde{\mathbf{G}} = \mathbf{G}Z(\widetilde{\mathbf{G}})$ and put $\widetilde{\mathbf{L}} = \mathbf{L}Z(\widetilde{\mathbf{G}})$. Then $\widetilde{\mathbf{L}}/\mathbf{L} \cong \widetilde{\mathbf{G}}/\mathbf{G}$. Lang's theorem for the connected groups $\mathbf{L}$ and $\mathbf{G}$ gives

$$\widetilde{\mathbf{L}}^F/\mathbf{L}^F \cong \widetilde{\mathbf{G}}^F/\mathbf{G}^F, \qquad \widetilde{\mathbf{G}}^F = \mathbf{G}^F\widetilde{\mathbf{L}}^F.$$

Together with the finite field operator τ, these regular Levi elements represent every relevant inner-diagonal-field outer class and preserve $\mathbb{U}$ in (6.2). The same assertions hold after the simultaneous conjugation used to place P in the Levi. The cochain action comes from the whole finite action on the chosen $\mathbb{U}$-variety. The identification of its block labels with the middle-cohomology correspondence is given by [36, Corollary 4.2, Proposition 4.9, Corollary 5.5 and Lemma 5.6]; comparison of block bijections uses [2, Theorem 7.2]. A representative in the same outer class as a block stabilizer fixes that block, since inner automorphisms fix blocks. Its geometric action then fixes the corresponding Levi block. Conjugating by an element of L returns its defect group to D. The resulting representative belongs to E, and differs from the original marked automorphism by c_n with $n \in N$. This proves (6.13).

In the global application, take the inverse image of E in the marked component stabilizer. Its intersection with the lower central product acts on S_0 through $c(H)$ by (6.13), and therefore acts innerly. Lemma 8.2 gives that the image of P is a Sylow 2-subgroup of the quotient by this intersection. Equation (6.12) shows that its action on S_0 is inner. The full outer image on S_0 consequently has odd order.

Apply Proposition 5.1 to S_0 with the automorphism group generated by $\mathrm{Inn}(S_0)$ and this operator image; its outer quotient still has odd order. Restrict the marked action back to E, with inner restriction $v(p)$. Tensor its complex with the regular $\mathcal{O}A'$ block in the prescribed odd

central character sector. The result is Y^+; its identity component is the corresponding complex Y on L. The group multiplication in A' retains the specified powers of the elements of P_0.

6.7. **The full normalizer companion.** Write

$$N^+ = N_K(D) = NP, \qquad H^+ = N_{L^+}(D) = HP.$$

Choose a maximal supported pair $(\Delta P, e_P^+ \otimes (f_P^+)^*)$ of $k \otimes C^+$, and let (D, e_D^+) and (D, f_D^+) be its two subpairs at D. The block e_D^+ of $C_K(D) = C_S(D)P$ covers a single P-orbit of blocks of $C_S(D)$ and is their orbit sum, by the normal 2-extension block theorem and Fong–Reynolds correspondence. Since

$$\mathrm{Br}_P(e_D^+)e_P^+ = e_P^+ \neq 0,$$

this orbit has size one: a nontrivial orbit sum is a proper relative trace from a subgroup of P and has zero Brauer image at P. Write its unique member as e; thus $e_D^+ = e$. The same argument gives $f_D^+ = f$ on the right. In particular e, f are P-stable lower roots and are also the upper roots. We use the same symbols for their integral lifts.

These lower roots are supported by $k \otimes C$. Indeed, downward closure of the supported-pair set [4, Corollary 8.8] gives the supported upper pair at ΔD. Its cut is a Rickard equivalence of nonzero centralizer blocks by [4, Theorem 8.5], and hence has nonzero ordinary homology. In the induced model (6.9), its homogeneous pieces are the p-translates of $eC(\Delta D)f$, for $pD \in P/D$: p centralizes D and fixes e, f. Nonzero homology of the upper cut therefore implies nonzero homology of this identity piece. The lower pair is supported and maximal, since D is a defect group at both lower endpoints.

Set

$$\begin{aligned} I = N_N(e), \qquad & T = N_H(f), \qquad & J_0 = N_{I\times T}(\Delta D), \\ I^+ = IP, \qquad & T^+ = TP, \qquad & J_{e,f}^+ = N_{I^+\times T^+}(\Delta D), \\ W_0 = N \times H, \quad & W^+ = N^+ \times H^+, \quad & J = N_{K\times L^+}(\Delta D). \end{aligned}$$

Since P centralizes D and fixes e, f, direct multiplication in the given groups gives

$$J_{e,f}^+ = J_0(P \times P), \qquad (J_{e,f}^+)_{\text{degree }1} = J_0\Delta P. \tag{6.14}$$

Put $\mathcal{J} = N_{S\times L}(\Delta D)$ and choose

$$\Omega_D = \Omega_c(\overline{Y}_0^{\Delta D}, \mathcal{O})$$

with the point-stabilizer construction for the whole finite $\mathcal{J} \rtimes \widetilde{E}$ action. All supported root pairs at ΔD form one $\mathcal{J}$-orbit, by maximal-pair conjugacy in [4, Corollary 8.8]. This joint orbit is $\widetilde{E}$-stable. Consequently

$$\pi_D = \sum_{(e',f')\in\mathcal{J}(e,f)} e' \otimes (f')^*, \qquad V_D = e\Omega_D f$$

give an invariant complex $\pi_D\Omega_D$ and an isomorphism

$$\mathrm{Ind}_{J_0}^{\mathcal{J}} V_D \xrightarrow{\sim} \pi_D\Omega_D, \qquad [j, v] \longmapsto jv. \tag{6.15}$$

Distinct cosets give orthogonal root summands; this proves injectivity and surjectivity term by term and commutes with the differential. The selected V_D carries the action of $J_0\Delta P$, because P fixes the roots.

Define, at this initial choice,

$$V_D^+ = \mathrm{Ind}_{J_0\Delta P}^{J_{e,f}^+} V_D. \tag{6.16}$$

This is the fixed-point cochain object for $(\overline{Y}_0^+)^{\Delta D}$ with the specified cuts. Indeed, the cosets of the component grading have representatives $(p, 1)$ with $p \in P$, and these representatives centralize ΔD. Thus taking fixed points in the induced variety gives the same finite induction. This also

follows modulo the maximal ideal from [32, Theorem 4.2], or [35, Theorem 2.29], including the normalizer action.

Let

$$\widehat{e} = \mathrm{Tr}_I^N(e), \qquad \widehat{f} = \mathrm{Tr}_T^H(f).$$

These are the blocks on the full lower normalizers. The same sums, in degree one, are the upper block idempotents: coset representatives for N/I and H/T also represent NP/IP and HP/TP. The complete companions are the following complexes:

$$U_D = \widehat{e}\,\mathcal{O}Ne \otimes_{\mathcal{O}I} \mathrm{Ind}_{J_0}^{I\times T}(\mathrm{Res}_{J_0} V_D) \otimes_{\mathcal{O}T} f\,\mathcal{O}H\widehat{f}, \tag{6.17}$$

$$U_D^+ = \widehat{e}\,\mathcal{O}N^+e \otimes_{\mathcal{O}I^+} \mathrm{Ind}_{J_{e,f}^+}^{I^+\times T^+}(V_D^+) \otimes_{\mathcal{O}T^+} f\,\mathcal{O}H^+\widehat{f}. \tag{6.18}$$

These are precisely the two Fong–Reynolds factors and the graph-normalizer induction of [4, Theorem 8.5 and Corollary 8.9]. Their subpair form applies also when $D < P$. The lower formula also equals

$$\widehat{e}\,\mathrm{Ind}_{\mathcal{J}}^{N\times H}(\pi_D\Omega_D)\,\widehat{f}$$

by (6.15) and induction transitivity. This expression supplies its $\widetilde{E}$-action; the selected root formula supplies its compatible P-action.

Ordinary induction in the two factors and transitivity identify (6.18) with

$$U_D^+ \cong \widehat{e}\,\mathrm{Ind}_{J_{e,f}^+}^{W^+}(V_D^+)\,\widehat{f} = \widehat{e}\,\mathrm{Ind}_{J_0\Delta P}^{W^+}(V_D)\,\widehat{f}. \tag{6.19}$$

For example, the adjacent e and f can be removed from the outer tensor factors because they act as the identity on V_D^+; the remaining outer factors are the ordinary induction bimodules. This proves the formula as a complex, with its given differential and coset carries.

Let $Q = P/D$. The original quotient maps identify K/S and L^+/L with Q; the degree of a pair is the left quotient degree times the inverse right quotient degree. The subgroup of W^+ preserving degree one is $W_0\Delta P$. Restricting the degree-one part of (6.19) gives

$$\widehat{e}\,\mathrm{Ind}_{J_0\Delta P}^{W_0\Delta P} V_D\,\widehat{f} \cong \widehat{e}\,\mathrm{Ind}_{J_0}^{W_0}\mathrm{Res}_{J_0} V_D\,\widehat{f} = U_D.$$

The middle isomorphism sends $[w\delta(p), v]$ to $[w, \delta(p)v]$; its ambiguity is exactly $\Delta(P\cap S) = \Delta D \le J_0$. Using these coset normal forms in the definitions gives the identity of complexes with their specified P-actions

$$(U_D^+)_1 = U_D. \tag{6.20}$$

All block idempotents used here have degree one.

Lemma 6.5. *Every term of the specified integral complex U_D^+ belongs to*

$$\mathrm{add}\{\mathcal{O}[(N^+\times H^+)/\Delta R] : R \le P\}.$$

Proof. The fixed-point cosets satisfy

$$((K\times L^+)/\Delta R)^{\Delta D} = \begin{cases} J/\Delta R, & D \le R, \\ \varnothing, & D \not\le R, \end{cases} \qquad R \le P. \tag{6.21}$$

If $(g,l)\Delta R$ is fixed, then

$$(g,l)^{-1}\Delta D(g,l) \le \Delta R, \qquad l^{-1}Dl \le P.$$

Lemma 6.4 gives $l \in H^+$ and $l^{-1}Dl = D$. The same displayed inclusion then gives $g \in N^+$, $D \le R$, and $g^{-1}dg = l^{-1}dl$ for every $d \in D$. Thus $(g,l) \in J$. Conversely P is abelian, so $\Delta R \le J$, and every coset of $J/\Delta R$ is fixed when $D \le R$. This proves (6.21). For the integral fixed-point model, let $z \in (\overline{Y}_0^+)^{\Delta D}$ and let Q_z be a Sylow 2-subgroup of its stabilizer in $J_{e,f}^+$. The normal subgroup ΔD acts trivially on this variety, so $\Delta D \le Q_z$. The preceding global point-stabilizer

bound supplies $x = (g, l) \in K \times L^+$ with $x^{-1}Q_z x \leq \Delta P$. Applying the transporter argument just given to $x^{-1}\Delta D x \leq \Delta P$ shows that $x \in J \leq W^+$. Hence Q_z is W^+-conjugate to a subgroup of ΔP.

Restrict the lower point-stabilizer model to $J_0\Delta P$ and then induce as in (6.16). A permutation overmodule $\mathcal{O}[(J_0\Delta P)/(J_0\Delta P)_z]$ becomes $\mathcal{O}[J^+_{e,f}/(J_0\Delta P)_z]$, the module of the corresponding upper fixed-variety point stabilizer. Thus V_D^+ has terms in the additive category of these permutation modules. Passing to a Sylow 2-subgroup is a split operation on their permutation overmodules, as above. Formula (6.19) then proves the bound after both Fong–Reynolds factors and full normalizer induction. Primitive root cuts and full orbit induction preserve the term bound. □

The reductions of these particular fixed-point complexes are the Brauer cochain objects, so the modular local evaluations come from [4, Theorem 8.5 and Corollary 8.9]. For these integral models lift the contractions of their evaluation cones. The Hom-surjectivity for integral permutation summands and the correction $h \mapsto h(dh + hd)^{-1}$ in Lemma 2.3 apply. The grading projector preserves the bimodule equations, so these contractions can be taken degree-preserving.

Define

$$\begin{aligned} X_S &= C \otimes_{\mathcal{O}L} Y \otimes_{\mathcal{O}H} U_D^\vee, \\ X_S^+ &= C^+ \otimes_{\mathcal{O}L^+} Y^+ \otimes_{\mathcal{O}H^+} (U_D^+)^\vee. \end{aligned} \tag{6.22}$$

The endpoints are the block of S and its block in $N_S(D)$, and their given upper extensions.

Lemma 6.6. *For the specified models, every term of*

$$C^+ \otimes_{\mathcal{O}L^+} Y^+ \otimes_{\mathcal{O}H^+} (U_D^+)^\vee$$

is 2-permutation and relatively ΔP-projective as an $(\mathcal{O}K, \mathcal{O}N^+)$-bimodule. The assertion includes the indicated block cuts.

Proof. The terms of C^+ satisfy (6.10). The Green term in the rank-one complex has vertex ΔV, and its correction terms are projective. Tensoring with the regular central $\mathcal{O}A'$ block therefore puts each term of Y^+ in

$$\operatorname{add}\{\mathcal{O}[(L^+ \times H^+)/\Delta T] : T \leq P\}.$$

Lemma 6.5 gives the same bound for $(U_D^+)^\vee$, with its factors reversed.

It suffices to tensor the displayed permutation overmodules: split inclusions and retractions remain split after balanced tensor product. For the first tensor, the double cosets $R\backslash L^+/T$ give stabilizers

$$\{(a, l^{-1}al) : a \in R \cap lTl^{-1}\}, \qquad l \in L^+.$$

Both projections lie in P. By the fusion assertion of Lemma 6.4 there is $h \in H^+$ with $h^{-1}ah = l^{-1}al$ on this intersection. Conjugation by $(1, h)$ puts the displayed graph in ΔP. Thus the first tensor has the required (K, H^+) term bound.

For the second tensor the double-coset representative belongs to H^+, which is a subgroup of the right endpoint N^+. Its graph stabilizer is

$$\{(a, h^{-1}ah) : a \in R' \cap hT'h^{-1}\}, \qquad h \in H^+, \quad R', T' \leq P.$$

Now $(1, h) \in K \times N^+$ itself absorbs the graph. Its resulting diagonal subgroup is contained in P. Each graph stabilizer meets either endpoint factor trivially. The associated permutation bimodule is therefore free on each side, and its direct summands are projective on each side.

Linearization of a balanced product of finite bisets is a permutation module. Block cuts and the given summand projectors give its direct summands. Tensoring over a block algebra is the same balanced tensor as over its group algebra on these block-supported modules. This proves

the assertion for every term of the total complex. The conjugations witness relative projectivity and preserve the chosen complex and its operator action. □

The same proof applies to the lower complex, with (K, L^+, P, N^+, H^+) replaced by (S, L, D, N, H). Here D is Sylow in $L = S_0 \times A_0$ and $N_L(D)$ controls its fusion by the same centralizer argument.

For the graded tensor identity, let $B = \bigoplus_{g\in Q} B_g$ be an intermediate upper block algebra, with $Q = P/D$, and choose a homogeneous unit $u_g \in B_g$. For the adjacent graded bimodule complexes $M = \bigoplus M_g$ and $M' = \bigoplus M'_g$, multiplication by these units gives

$$(M \otimes_B M')_1 \xrightarrow{\sim} M_1 \otimes_{B_1} M'_1, \qquad x_g \otimes y_{g^{-1}} \longmapsto x_g u_g^{-1} \otimes u_g y_{g^{-1}}. \tag{6.23}$$

Its inverse is induced by the inclusions of M_1, M'_1. For $b_h \in B_h$, the relation $u_g b_h u_{gh}^{-1} \in B_1$ verifies balancing and independence of the chosen units. These units are group elements multiplied by the relevant block idempotent, so their inverses retain the group multiplication relations. Equations (6.9) and (6.20) give $(X_S^+)_1 = X_S$, with the same operators and both evaluations. Degree-preserving projection of the upper homotopies is bimodule-linear even for a nonabelian grading; it retains the specified contractions.

The kernel calculation following (6.9) shows that the finite operator presentation $\widetilde{E}$ has a pointwise-trivial kernel over E consisting of central regular-embedding elements. The point-stabilizer cochains, their fixed-point models and inductions kill this kernel termwise. Thus the action factors to $\rho : E \to \mathrm{Aut}_{\mathcal{O}}(X_S)$.

Write δ_n for the action of the diagonal pair (n, n) on X_S. On $c(H)$ the constructed action satisfies $\rho(c_h) = \delta_h$: the geometric action of h is conjugation on the varieties, the rank-one action has this property by construction, and the induction and Fong–Reynolds formulas preserve it. Also $\delta_z = 1$ for every $z \in Z(S)$, because the original complexes are strictly balanced over that center. Therefore

$$\rho_{\mathrm{full}}(c_n e) = \delta_n \rho(e), \qquad n \in N,\ e \in E, \tag{6.24}$$

is well-defined even when S has a nontrivial center. Indeed, two factorizations differ by c_h with $h \in H$; the corresponding representatives in N differ by hz with $z \in Z(S)$, and $\delta_z = 1$. It is multiplicative because

$$\rho(e)\delta_n \rho(e)^{-1} = \delta_{e(n)}.$$

Equivalently X_S is a module complex for the finite relative group

$$\mathcal{T}_S = \big((S \times N) \rtimes \mathcal{A}_D\big) \Big/ \{(\delta(n)^{-1}, c_n) : n \in N\}.$$

The displayed normal subgroup centralizes the base, and its intersection with $S \times N$ is exactly $\Delta Z(S)$. Thus the base of $\mathcal{T}_S$ is $(S \times N)/\Delta Z(S)$. The Levi-operator group

$$\mathcal{T}_E = ((S \times N) \rtimes E)/\{(\delta(h)^{-1}, c_h) : h \in H\}$$

maps isomorphically to $\mathcal{T}_S$. Surjectivity follows from $\mathcal{A}_D = c(N)E$, since an inner operator is identified with its base diagonal operator. For the kernel, if an E-operator represents c_n, then $c_n \in E \cap c(N) = c(H)$; write $n = hz$ with $h \in H$ and $z \in Z(S) \le H$. Its relation is already in the displayed denominator of $\mathcal{T}_E$. The isomorphism of relative groups transfers the chosen permutation terms and the action on their differentials. Pull back this action through the original ambient automorphism map. Its kernel $C_{\mathrm{ambient}}(S)$ acts as the identity on this lower component. On P, the action agrees with the geometric action used in the identity-component inductions, because the two regular/field lifts differ only by the pointwise central kernel already identified. This assertion concerns the diagonal operator on the lower complex; the original kernel elements of P remain as homogeneous units with their powers in the upper groups and upper complexes.

Proposition 6.7. *For case (iii) with the ambient marked stabilizers specified above, there is an integral splendid component complex X_S with its full marked stabilizer action and specified inner restriction. It is the identity component of the upper splendid complex X_S^+ for SP and $N_S(D)P$. Both evaluations admit homogeneous upper contractions, and the complexes are strictly balanced over their shared centers.*

Proof. Take the complexes in (6.22). Lemmas 6.3, 6.5 and 6.6 give the evaluations and the splendid term bounds. Equation (6.23) identifies the lower complex and its contractions with the identity components. The action is (6.24), and each factor is balanced over the specified centers. □

The homogeneous upper contractions restrict to P-linear contractions on the lower complexes. The full ambient operator contractions after assembly are obtained by the relative trace (7.9).

7. Assembly of component complexes

Let $H = S_1 \cdots S_m C \triangleleft G$ be a central product, let $b = b_1 \cdots b_m b_C$ be a G-stable block of $\mathcal{O}H$, and let a be a block of $\mathcal{O}G$ covering b, with abelian defect group P. Assume

$$D = P \cap H, \qquad |P : D| = |G : H|_2. \tag{7.1}$$

The factors S_i constitute one chosen G-orbit of components in cases (i)–(iii). The product C of the remaining component orbits and the central factor is normal in G. Uniqueness of the factor blocks of the stable block b makes b_C G-stable. Assume that P fixes each S_i and that $N_G(P)$ is transitive on this component orbit. Put $D_i = P \cap S_i$ and

$$Q = P \cap (S_1 \cdots S_m), \quad J = N_G(Q), \quad H' = H \cap J. \tag{7.2}$$

The defect group of a central-product block is the product of its factor defects. The subgroups D_i and $P \cap C$ are those factor defects and lie in the specified P. Comparing orders therefore gives the subgroup equalities

$$Q = D_1 \cdots D_m, \qquad D = Q(P \cap C).$$

An element of H normalizes Q precisely when its S_i-factor normalizes D_i for every i, since C centralizes the selected components. Hence

$$H' = N_{S_1}(D_1) \cdots N_{S_m}(D_m)C.$$

Let c_i be the Brauer correspondent of b_i in $N_{S_i}(D_i)$, and write $b' = c_1 \cdots c_m b_C$ for the resulting block of H'. Conjugation by J permutes the marked factors; Brauer correspondence commutes with this conjugation and b_C is stable. Thus b' is J-stable. The block $b_1 \cdots b_m$ of the normal subgroup $S_1 \cdots S_m$ is G-stable and has defect group Q. For $g \in G$, defect-group conjugacy in this subgroup sends ${}^gQ$ to Q. Thus $G = (S_1 \cdots S_m)N_G(Q) = HJ$. Hence $\Gamma = G/H \cong J/H'$, $P \leq J$, and $P \cap H' = D$.

For every chosen factor use the data established in Sections 5 and 6: an integral splendid complex X_i with the marked stabilizer action and diagonal inner action; an upper complex X_i^+ for S_iP and $N_{S_i}(D_i)P$ whose identity component is X_i with that P-action; degree-preserving evaluations and contractions; and balance over the shared centers in the chosen central-character summands. On C use the identity complex $\mathcal{O}Cb_C$. Choose the data on one representative S_1. For each i, choose $g_i \in N_G(P)$ with ${}^{g_i}S_1 = S_i$, taking $g_1 = 1$, and transport its blocks, Brauer pairs, lower and upper complexes, evaluations, and contractions by this same g_i. Since the component product is normal, $N_G(P) \leq J$. Thus every transported upper complex still uses the specified group P.

Proposition 7.1. *These data give a splendid Rickard complex between $\mathcal{O}Gb$ and $\mathcal{O}Jb'$. Its primitive cut at a gives a primitive target block c of $\mathcal{O}Jb'$ with the same marked defect group P.*

Proof. First construct the full action. Define the permutation $i \mapsto g(i)$ by $gS_ig^{-1} = S_{g(i)}$. With the transporters g_i fixed above, for $g \in J$ the operator on the ith factor is the given operator of

$$h(g,i) = g_{g(i)}^{-1} g g_i.$$

The equality

$$h(uv,i) = h(u,v(i))h(v,i) \qquad (u,v \in J) \tag{7.3}$$

proves the group law. For $p \in P$, the equality $p(i) = i$ gives $h(p,i) = g_i^{-1} p g_i \in P$. Hence the action on each transported lower complex is the identity-component action of its specified upper complex. Since P fixes every factor, its action introduces no Koszul permutation sign. Tensor these maps with the Koszul permutation signs and with the original conjugation action on the regular C factor. Inner elements of H' act by the diagonal bimodule action. Balance over the shared centers and retain the chosen odd character sector. This gives a lower complex X for

$$T = (H \times H')\Delta J, \qquad T_P = (H \times H')\Delta P. \tag{7.4}$$

We prove that its restriction to T_P is a permutation summand with the required vertex bound. Temporarily keep P as a separate acting group:

$$\widetilde{T}_P = \left(\prod_i (S_i \times N_{S_i}(D_i)) \times (C \times C) \right) \rtimes P.$$

For the ith factor the inverse image of the original ΔP is

$$J_i = \Delta D_i \times P. \tag{7.5}$$

Indeed (sp, tp) is diagonal on P precisely when $s = t \in D_i$. The product is direct here because P is abelian. Restrict the upper complex X_i^+ to its identity stabilizer and take its identity summand. Mackey intersections remain below the diagonal bound: an upper pair is a base pair times $(p,q) \in P \times P$, and the latter centralizes ΔP. Thus the pulled-back X_i is 2-permutation and relatively J_i-projective.

The corresponding witness for the remainder is the regular block of CP. Since $ab_C = a$, Brauer multiplicativity gives $\mathrm{Br}_P(b_C)\,\mathrm{Br}_P(a) = \mathrm{Br}_P(a) \neq 0$. The normal-extension defect bound is $|P \cap C|\,|P : P \cap C| = |P|$; hence its defect is P. It has identity component $\mathcal{O}Cb_C$ and relative inverse-image bound $\Delta(P \cap C) \times P$.

Permutation summands relatively projective for J_i are summands of sums of permutation modules with stabilizers $Q_i \leq J_i$. The stabilizers in their external tensor are intersections of conjugates of the inverse-image Q_i. Remove the P-coordinate from each conjugator, since P normalizes J_i. The remaining conjugators belong to independent base factors and combine into one base conjugation. Every stabilizer is consequently conjugate into

$$\left(\prod_i \Delta D_i \times \Delta(P \cap C) \right) \times P. \tag{7.6}$$

Now impose the central-product relations. Strict central balance makes the left relation and the corresponding right relation act identically. For a quotient kernel K, the coinvariants of $\mathcal{O}[\widetilde{T}_P/Q_0]$ are the permutation module on the orbit set, with stabilizer Q_0K/K. Summand projectors descend. Next impose the original graph identifying $d \in D$ with its inner diagonal action. This graph already acts trivially, by the specified inner restriction and the identity action of off-factor centralizers. Under these two quotients the image of (7.6) is first $\Delta D \cdot P$ and then exactly the original ΔP. This proves the whole T_P permutation and relative-vertex assertions.

One-sided projectivity is preserved by the same balanced tensor: tensoring summands of free factor modules gives summands of free modules for the central-product algebra.

We next retain both evaluations. The homogeneous identity comparison (6.23) gives P-linear contractions for the chosen lower factor evaluations. Tensor them and the regular remainder identity, and descend through the central relations. The duality here is the relative Frobenius dual. For a common central subgroup Z, put $R = \mathcal{O}Z$. Finite projective R-modules satisfy

$$\operatorname{Hom}_R(U, R) \cong \operatorname{Hom}_{\mathcal{O}}(U, \mathcal{O}),$$
$$\operatorname{Hom}_R(U \otimes_R V, R) \cong \operatorname{Hom}_R(U, R) \otimes_R \operatorname{Hom}_R(V, R).$$

The first map composes with the coefficient-of-identity trace $R \to \mathcal{O}$. Its inverse sends $f \in \operatorname{Hom}_{\mathcal{O}}(U, \mathcal{O})$ to

$$u \longmapsto \sum_{z \in Z} f(z^{-1}u)z.$$

The second is the ordinary dual-tensor isomorphism for finite projective modules over the commutative ring R. These maps are natural under automorphisms of Z. Thus for $A = A_1 \otimes_R A_2$, $B = B_1 \otimes_R B_2$ and $X = X_1 \otimes_R X_2$,

$$X \otimes_B X^\vee \cong (X_1 \otimes_{B_1} X_1^\vee) \otimes_R (X_2 \otimes_{B_2} X_2^\vee), \tag{7.7}$$

and likewise on the other side. The resulting evaluations are the quotient-algebra evaluations. On a coset basis the displayed inverse has coefficient one at the identity, including when Z has even order.

By (7.1),

$$[T : T_P] = [\Gamma : PH/H] \tag{7.8}$$

is odd. Put $n = [T : T_P]$ and choose left coset representatives $\mathcal{C}$ for T_P in T. For any term M of X, the maps

$$s : M \longrightarrow \mathcal{O}T \otimes_{\mathcal{O}T_P} \operatorname{Res}^T_{T_P} M, \qquad s(m) = n^{-1} \sum_{t \in \mathcal{C}} t \otimes t^{-1}m,$$
$$\varepsilon : \mathcal{O}T \otimes_{\mathcal{O}T_P} \operatorname{Res}^T_{T_P} M \longrightarrow M, \qquad \varepsilon(t \otimes m) = tm$$

are T-linear and satisfy $\varepsilon s = 1$. Replacing t by tv, with $v \in T_P$, leaves its summand unchanged by the balanced relation; left multiplication then proves the claimed T-linearity. Thus M is a summand of the induced restriction. The permutation property and the relative ΔP bound proved for that restriction pass to M. For either evaluation, let $\mathcal{U}$ be its operator group, $(H \times H)\Delta J$ or $(H' \times H')\Delta J$, and let $\mathcal{U}_P$ be the subgroup obtained by replacing J with P. If h_P is its supplied $\mathcal{U}_P$-linear contraction, put

$$h_{\mathcal{U}} = \frac{1}{[\mathcal{U} : \mathcal{U}_P]} \sum_{u \in \mathcal{R}} u h_P u^{-1}, \tag{7.9}$$

where $\mathcal{R}$ represents the left cosets of $\mathcal{U}_P$ in $\mathcal{U}$. This index is odd by (7.1). Changing a representative by a right element of $\mathcal{U}_P$ leaves its summand unchanged. Left multiplication permutes the cosets with these relations, proving equivariance. Linearity of $dh + hd = 1$ proves that $h_{\mathcal{U}}$ is a contraction.

Write τ_j for the diagonal action of $j \in J$ on X and $\tau_j^\vee(f) = f \circ \tau_{j^{-1}}$ for its action on the dual. Finally induce X from T to $G \times J$. Equivalently use $\mathcal{O}Gb \otimes_{\mathcal{O}Hb} X$, with right action, for $r \in \mathcal{O}Gb$ and $j \in J$,

$$(r \otimes x)j = rj \otimes \tau_{j^{-1}}(x). \tag{7.10}$$

The inner intersection relation makes this well-defined. Put $A = \mathcal{O}Hb$, $A^+ = \mathcal{O}Gb$, $B = \mathcal{O}H'b'$, and $B^+ = \mathcal{O}Jb'$, where b' is the retained block of H'. Write $Y = A^+ \otimes_A X$. Its dual is $X^\vee \otimes_A A^+$, with

$$j(f \otimes r) = \tau_j^\vee(f) \otimes jr \qquad (j \in J).$$

Choose representatives $j \in J$ for G/H. Every homogeneous element of A^+ has the form jr_0 with $r_0 \in A$. The two evaluation identifications are

$$Y \otimes_{B^+} Y^\vee \longrightarrow A^+ \otimes_A (X \otimes_B X^\vee),$$
$$(r \otimes x) \otimes (f \otimes jr_0) \longmapsto rj \otimes (\tau_{j^{-1}} x \otimes \tau^\vee_{j^{-1}} f) r_0,$$

and

$$X^\vee \otimes_A A^+ \otimes_A X \longrightarrow B^+ \otimes_B (X^\vee \otimes_A X),$$
$$f \otimes jr_0 \otimes x \longmapsto j \otimes (\tau^\vee_{j^{-1}} f \otimes r_0 x).$$

On the first target put $Z = X \otimes_B X^\vee$ and $\tau^Z_j = \tau_j \otimes \tau^\vee_j$. Its upper right action is

$$(r \otimes z)j = rj \otimes \tau^Z_{j^{-1}}(z).$$

The second target has the analogous action, with $Z = X^\vee \otimes_A X$ and $\tau^Z_j = \tau^\vee_j \otimes \tau_j$. Their inverses send $r \otimes (x \otimes f)$ to $(r \otimes x) \otimes (f \otimes 1)$ and $j \otimes (f \otimes x)$ to $\tau^\vee_j f \otimes j \otimes x$, respectively. The relation $\tau_j(hxh') = jhj^{-1}\tau_j(x)jh'j^{-1}$ verifies balancing; changing a coset representative uses this same relation. The maps intertwine both endpoint actions and carry the upper evaluations to the induced lower evaluations. The contractions from (7.9) therefore give both upper contractions.

Apply the primitive-block correspondence [4, Theorem 8.4] to $k \otimes_{\mathcal{O}} Y$. Primitive central idempotents of the integral group algebras correspond bijectively to their reductions. Thus there is a unique primitive central idempotent c of B^+ for which $k \otimes_{\mathcal{O}} aYc$ is noncontractible. For $c' \neq c$, lift a contraction of $k \otimes_{\mathcal{O}} aYc'$ through the Hom-surjection for integral permutation summands. The correction $h \mapsto h(dh + hd)^{-1}$ in Lemma 2.3 gives an integral contraction. Hence the two integral evaluations restrict to the regular blocks at a, c. Apply [4, Lemma 8.3] to the chosen left maximal pair (P, e_P). It gives a supported graph pair for aYc; [4, Corollary 8.8(c)] makes its right pair maximal, of order $|P|$. Every term already has vertices below conjugates of ΔP. A supported graph of this full order is therefore conjugate to ΔP in $G \times J$. Write this graph as ${}^{(g,j)}\Delta P$, with $g \in G$ and $j \in J$. Its left projection is the prescribed P, so $g \in N_G(P) \leq J$. Conjugation by $(1, gj^{-1})$ sends it to ${}^{(g,g)}\Delta P = \Delta P$, leaves the left root unchanged, and transports the right root inside the same block c. Retain that transported root as the target marking. Thus the primitive cut has the required defect group and diagonal bound. □

8. Reduction and proof of the theorem

8.1. **Reductions.** We use induction on the positive integer $\kappa(G)$ in (1.3). Every equivalence used in a reduction is a splendid complex with its specified defect graph, so Proposition 2.4 returns the inductive result to the original Brauer correspondent.

If a block d of a normal subgroup is covered and its inertia group $I = \{g \in G : {}^g d = d\}$ is proper, Fong–Reynolds gives a source-algebra equivalence to the corresponding primitive block of I. Transport the maximal pair simultaneously so its defect is the selected $P \leq I$. Put $Z_0 = O_{2'}(Z(G))$. Since $Z_0 \leq I \cap Z(I)$,

$$\kappa(I) \leq |I : Z_0| < \kappa(G). \tag{8.1}$$

Thus we may assume quasiprimitivity: every covered block of every normal subgroup is G-stable.

Lemma 8.1. *Let a be a quasiprimitive block of $\mathcal{O}G$ with abelian defect group P. Suppose that a covers a nilpotent block of $N \triangleleft G$ and $N \not\subseteq Z(G)O_2(G)$. Then $\mathcal{O}Ga$ is splendidly Rickard equivalent to a primitive block of a finite group G_1 with $\kappa(G_1) < \kappa(G)$.*

Proof. Put $M_0 = Z(G)N$, and let b be the block of M_0 covered by a. It is obtained from the nilpotent N-block and the matching block of $Z(G)$ by their central-product map. Conjugation by the central factor is trivial, so the resulting block fusion is nilpotent. Quasiprimitivity makes

the block stable. Its defect is $Q = P \cap M_0$ by the normal defect-intersection theorem. Fix the maximal a-Brauer pair (P, e_P). Since $ab = a$, we have $e_P \operatorname{Br}_P(b) = e_P$. Choose a local point ε of P on $\mathcal{O}M_0b$, represented by u, with $e_P \operatorname{Br}_P(u) \neq 0$. Extend P_ε to a maximal local pointed group R_γ for the G-action on $\mathcal{O}M_0b$, and choose $i \in \gamma$ and $u \in \varepsilon$ with $u \leq i$. Thus $R \geq P$ is maximal among relative 2-subgroups with $\operatorname{Br}_R(b) \neq 0$. Coefficient deletion gives $\operatorname{Br}_{R\cap M_0}(b) \neq 0$. Consequently

$$Q = P \cap M_0 \leq R \cap M_0, \qquad |R \cap M_0| \leq |Q|,$$

and $R \cap M_0 = Q$. The nilpotent-extension theorem [30, Theorem 3.5 and Corollary 3.15], with the source-corner correction in its Section 3.16, gives a model group L and a finite central odd cover $\widetilde{L} \to L$, with

$$|L| = |G : M_0|\,|Q|. \tag{8.2}$$

The construction has an exact sequence

$$1 \longrightarrow Q \longrightarrow L \longrightarrow G/M_0 \longrightarrow 1.$$

The embedding $R \hookrightarrow L$ supplied by the model theorem has Sylow image; see the proof of [30, Theorem 3.5]. The inverse image of R in $\widetilde{L}$ has a unique Sylow 2-subgroup: Schur–Zassenhaus gives a complement to the central odd kernel, and centrality makes its conjugates identical. This gives a lift $\tau : R \hookrightarrow \widetilde{L}$. Thus $\tau(R) \in \operatorname{Syl}_2(\widetilde{L})$. In this proof write $\Delta R = \{(r, \tau(r)) : r \in R\}$.

The chosen $i \in (\mathcal{O}M_0b)^R$ represents the relative local point of [30, Section 3.1]. Choose its compatible source idempotent $j \in (\mathcal{O}M_0b)^Q$ with $j \leq i$. Since j is full in $\mathcal{O}M_0b$, the idempotent i is full in $\mathcal{O}Gb$. Corollary 3.15 of [30] gives an R-interior graded algebra isomorphism

$$\phi : i(\mathcal{O}Gb)i \xrightarrow{\sim} S_R \otimes_{\mathcal{O}} C, \qquad S_R = \operatorname{End}_{\mathcal{O}}(V_R),$$

where C is realized in the prescribed central-character summand of $\mathcal{O}\widetilde{L}$. The algebra S_R is a Dade R-algebra and $n = \operatorname{rank}_{\mathcal{O}}(V_R)$ is odd [30, Section 3.10]. The column module

$$M = \mathcal{O}Gbi \otimes_{S_R} V_R$$

with its right C-action from ϕ induces a Morita equivalence; this is the construction in [30, Section 7.4], also described by [10, Theorem 7.5]. Let c correspond to a under its center isomorphism. Then $\phi(ia) = 1 \otimes c$, and $M_a = aMc$ is an indecomposable Morita bimodule between $\mathcal{O}Ga$ and Cc.

On $\mathcal{O}Gbi \otimes_{\mathcal{O}} V_R$ use the action

$$(g, r)(d \otimes v) = gdr^{-1} \otimes rv \qquad (g \in G,\ r \in R,\ d \in \mathcal{O}Gbi,\ v \in V_R).$$

The R-interior property of ϕ makes the quotient $\mathcal{O}Gbi \otimes_{\mathcal{O}} V_R \to \mathcal{O}Gbi \otimes_{S_R} V_R$ equivariant for $G \times R$, where the second factor acts on M through its right $\tau(R)$-action: writing $\rho(r)$ for the action on V_R, we have $\phi(ri) = \rho(r) \otimes \tau(r)$ and hence

$$(1, \tau(r))(d \otimes_{S_R} v) = dr^{-1} \otimes_{S_R} \rho(r)v.$$

Choose matrix units $E_{\mu\nu} \in S_R$ for $1 \leq \mu, \nu \leq n$. The quotient has the section

$$\sigma(d \otimes_{S_R} v) = \frac{1}{n} \sum_{\mu,\nu=1}^{n} dE_{\mu\nu} \otimes E_{\nu\mu}v.$$

Indeed, the tensor $n^{-1} \sum_{\mu,\nu} E_{\mu\nu} \otimes E_{\nu\mu}$ centralizes the S_R-balancing and its product is 1. It is invariant under simultaneous conjugation by $\mathrm{GL}(V_R)$, and hence under R. These identities prove both well-definedness and $G \times R$-equivariance of σ.

Right multiplication by i makes $\mathcal{O}Gbi$ a $G \times R$-summand of $\mathcal{O}G$. Moreover, with the displayed action,

$$\operatorname{Ind}_{\Delta R}^{G\times\tau(R)} V_R \xrightarrow{\sim} \mathcal{O}G \otimes_{\mathcal{O}} V_R, \qquad (g, \tau(r)) \otimes v \longmapsto gr^{-1} \otimes rv.$$

Thus $\mathrm{Res}^{G\times\widetilde{L}}_{G\times\tau(R)} M$ is a summand of this induced module. Since $[\widetilde{L}:\tau(R)]$ is odd, the induction counit from this restriction to M splits: for representatives t of the left cosets of $\tau(R)$ in $\widetilde{L}$, a section is

$$m \longmapsto \frac{1}{[\widetilde{L}:\tau(R)]}\sum_t (1,t)\otimes(1,t)^{-1}m.$$

Induction transitivity therefore shows that M, and hence M_a, is a direct summand of $\mathrm{Ind}^{G\times\widetilde{L}}_{\Delta R} V_R$.

Choose a vertex $\Gamma \le \Delta R$ of M_a and a source W at Γ, so W is a full-vertex summand of $\mathrm{Res}^{G\times\widetilde{L}}_{\Gamma} M_a$. The Mackey decomposition of $\mathrm{Res}^{G\times\widetilde{L}}_{\Gamma}\mathrm{Ind}^{G\times\widetilde{L}}_{\Delta R} V_R$ has terms induced from $\Gamma\cap {}^x\Delta R$, where x runs through representatives of $\Gamma\backslash(G\times\widetilde{L})/\Delta R$. Terms for which this intersection is proper in Γ cannot contain W. The remaining terms are restrictions of conjugates of V_R along isomorphisms $\Gamma\cong T$ with $T\le R$. Each $\mathrm{Res}^R_T V_R$ is capped endopermutation, and every indecomposable summand of vertex T is its cap; the cap may occur with multiplicity. Krull–Schmidt now shows that W is an endopermutation module.

Apply the ordinary source theorem for Morita equivalences with endopermutation source, in the form recalled at the start of [26, Section 5]; see also [25, Theorem 9.11.2]. Applied to M_a, it identifies the two coordinate projections of Γ with defect groups of a and c, and supplies source idempotents determining corresponding maximal Brauer pairs, a common fusion system, and a fusion-stable capped source. Conjugate the left maximal pair to the fixed (P,e_P), transporting the source data and Γ simultaneously. Write (P_c,f_{P_c}) for the resulting right maximal pair and $\eta:P_c\to P$ for the isomorphism whose graph is the transported vertex.

Since P is abelian, Proposition 2.2, applied after reduction modulo π, gives a splendid Rickard complex for these same primitive blocks and defect identification. Lemma 2.3, in its ordinary bimodule case, gives an integral lift. For the return in Proposition 2.4, mark this complex by a supported maximal pair of its modular reduction with left pair (P,e_P); retain the corresponding right pair and graph throughout the return. The central odd cover gives

$$\kappa(\widetilde{L}) \le |L| = |G:M_0|\,|Q| \le |G:Z_0|. \tag{8.3}$$

Equality in the last inequality would imply $M_0 = Z_0Q$. This is an internal direct product; Q is its unique Sylow 2-subgroup and is characteristic in M_0, hence $Q\le O_2(G)$. It would follow that $N\le Z(G)O_2(G)$, contrary to the hypothesis on N. The inequality is therefore strict. Take $G_1=\widetilde{L}$ with the selected primitive block and its chosen central-character summand. □

After these reductions, for every $N\lhd G$ we have

$$\begin{gathered}\text{if a block of } \mathcal{O}N \text{ covered by } a \text{ is nilpotent, then}\\ N\le Z(G)O_2(G).\end{gathered} \tag{8.4}$$

8.2. **The generalized Fitting subgroup and relative defect.**

Lemma 8.2. *Let a be a quasiprimitive block of $\mathcal{O}G$ with abelian defect group P, satisfying (8.4). Put $H=F^*(G)$. Every component has noncentral defect, P fixes each component, and, for $D=P\cap H$,*

$$|P:D| = |G:H|_2. \tag{8.5}$$

In particular, P is maximal among 2-subgroups with nonzero relative Brauer support for the covered H-block.

Proof. The normal 2-subgroup $O_2(G)$ lies in the abelian group P, so $Z(G)O_2(G)$ is abelian. Every block of $O_{2'}(G)$ has defect zero and is nilpotent. Equation (8.4) therefore gives $O_{2'}(G)\le Z(G)O_2(G)$. The odd-order subgroup of the latter abelian group is Z_0, and hence $O_{2'}(G)\le Z_0\le Z(G)$. Thus H is the central product of its components and its original abelian factor $O_2(G)Z_0$.

Write $S_1, \dots, S_m$ for the components of G. Apply the normal defect-intersection theorem first to $E(G) \lhd G$, then to $S_i \lhd E(G)$. It gives that $P \cap E(G)$ is a defect group of the covered layer block and $D_i = P \cap S_i$ is a defect group of its component block. For a central product these factor defects give its defect by the group multiplication map, with the shared centers retained.

Centrality of a component defect group is constant on a G-orbit. If a whole orbit of components had central-defect blocks, its covered central-product block would be nilpotent. Its normal perfect group would, by (8.4), lie in the abelian group $Z(G)O_2(G)$, a contradiction. Thus every component defect is noncentral. An element of P centralizes each D_i. If it moved S_i to a different component, D_i would lie in their central intersection, again a contradiction. Hence P fixes every component.

We also verify the upper-defect hypothesis of the component constructions. Write b_E for the covered $E(G)$-block and b_i for its retained component block. Since P fixes S_i and stabilizes b_E, uniqueness of the factor blocks makes b_i P-stable. In the group algebra, $ab_E = a$ and $b_E b_i = b_E$, so $ab_i = a$. Applying the Brauer homomorphism at P gives $\mathrm{Br}_P(a)\,\mathrm{Br}_P(b_i) = \mathrm{Br}_P(a) \neq 0$. The unique covering block of the normal 2-extension PS_i/S_i is represented by b_i and has nonzero P-support. Its defect order is at most $|D_i|\,|P : D_i| = |P|$ by the normal-extension bound. It therefore has P as a defect group, as required in all four component cases.

Let K_0 be the kernel of the component permutation action, intersected with $C_G(O_2(G))$. Then

$$P \leq K_0, \qquad C_G(P) \leq K_0, \qquad H \leq K_0.$$

The middle inclusion uses the same noncentral-defect argument and $O_2(G) \leq P$. Choose a maximal a-Brauer pair (P, e_P). The covered K_0-block d has defect P. Since $ad = a$ and $C_{K_0}(P) = C_G(P)$, the original root satisfies

$$e_P\,\mathrm{Br}_P(d) = e_P\,\mathrm{Br}_P(a)\,\mathrm{Br}_P(d) = e_P.$$

It is therefore the same maximal root for d. Quasiprimitivity and conjugacy of maximal pairs therefore give

$$G = K_0 N_G(P, e_P). \tag{8.6}$$

The quotient G/K_0 is an image of the odd inertial quotient $N_G(P, e_P)/(PC_G(P))$, hence is odd. The componentwise outer-automorphism map has kernel H on K_0. Indeed an element in its kernel can be multiplied by elements of the components so as to centralize every component. It already centralizes $O_2(G)$ and Z_0, and hence centralizes H. The generalized-Fitting centralizer theorem gives $C_G(H) \leq H$. Conversely every element of H induces inner automorphisms on the components. Thus K_0/H embeds in the product of their outer automorphism groups and is solvable by the Schreier theorem. Consequently G/H is 2-solvable.

Pull back a $2/2'$ series of normal subgroups to $H = H_0 \lhd H_1 \lhd \cdots \lhd H_t = G$ with every $H_j \lhd G$, and retain the covered blocks. They are stable under the next group, by quasiprimitivity, and have defect groups $P_j = P \cap H_j$. If H_j/H_{j-1} is a 2-group, the normal-extension defect theorem gives $|P_j : P_{j-1}| = |H_j : H_{j-1}|$. If it has odd order, the defect groups have equal order and $P_j = P_{j-1}$. Multiplication gives (8.5). For the covered H-block b_H, the identity $ab_H = a$ implies $\mathrm{Br}_P(a)\,\mathrm{Br}_P(b_H) = \mathrm{Br}_P(a) \neq 0$, so P has nonzero relative support. For any 2-subgroup R with nonzero relative support, $\mathrm{Br}_R(b_H) \neq 0$ implies $\mathrm{Br}_{R\cap H}(b_H) \neq 0$ by coefficient deletion. Hence $|R \cap H| \leq |D|$ and $|R : R \cap H| \leq |G : H|_2$. It follows that

$$|R| = |R \cap H|\,|R : R \cap H| \leq |D|\,|G : H|_2 = |P|.$$

Thus P is maximal among all 2-subgroups with nonzero relative support. □

The lemma also gives the odd outer image used in the component constructions. For the partial normalizer J in (7.2), the stabilizer $N_J(S_i)$ of a component S_i contains P and H'. The image PH'/H' is a Sylow 2-subgroup of J/H', hence also of $N_J(S_i)/H'$. Its image in the outer

automorphism group is trivial for the elementary and direct Klein cases, by Proposition 4.2 and the group-action calculations in the proof of Proposition 5.3. Those calculations of the P-action precede the extension of the complex to the odd outer image. Therefore the entire relevant outer image is odd. For the rank-one factor inside the geometric Levi of $S = S_i$, let $E \leq \mathrm{Aut}(S)$ be the marked Levi/block stabilizer constructed in Section 6 with component defect $D_i = P \cap S_i$, and let $\widehat{E}$ be its inverse image under the conjugation map $N_J(S_i) \to \mathrm{Aut}(S)$. The marked automorphism group $\mathcal{A}_{D_i}$ is already the image of $N_J(S_i)$: if $c_s\alpha$ preserves D_i and α is induced by $N_J(S_i)$, then $s \in N_S(D_i) \leq H' \leq N_J(S_i)$. Thus this inverse image maps onto E. Then $P \leq \widehat{E} \leq N_J(S_i)$, and the projection $\widehat{E} \to N_J(S_i)/H'$ has kernel $\widehat{E} \cap H'$. By the full-centralizer inner-overlap calculation, this kernel acts on S_0 through $N_L(D_i)$, with the other component factors acting trivially; its action is therefore inner. The image of P is a Sylow 2-subgroup of $\widehat{E}/(\widehat{E} \cap H')$, because it is already Sylow in $N_J(S_i)/H'$. Its outer action on S_0 is trivial by (6.12). Consequently the entire relevant outer image of $\widehat{E}$ on S_0 is odd.

8.3. **Completion of the induction.**

Proof of Theorem 1.1. Induct on $\kappa(G)$. A proper Fong–Reynolds reduction lowers it by (8.1); Lemma 8.1 lowers it whenever a covered nilpotent normal block satisfies that lemma's hypothesis. For either reduction, use induction on the primitive target and return by Proposition 2.4. We may therefore assume the hypotheses of Lemma 8.2.

If there are no components, $H = O_2(G)Z_0$ is centralized by P. The generalized-Fitting centralizer theorem gives $P \leq C_G(H) \leq H$, so $P = O_2(G)$ is normal. Its normalizer is G and the identity bimodule is the required splendid equivalence.

Suppose there is a component in case (i), (ii), or (iii) from Section 4. Take its complete original G-orbit and set Q, J, H' as in (7.2). Equation (8.6) shows that $N_G(P, e_P)$ is transitive on the chosen component orbit, because K_0 fixes each component. Its elements belong to J and normalize the same P. Choose the transporters and all transported component data as in Section 7. The component propositions provide the full action, common-P identity component, central balance, and upper evaluations required by Proposition 7.1. The complementary layer keeps its identity bimodule. Hence there is a splendid Rickard complex

$$R : \mathcal{O}Ga \longrightarrow \mathcal{O}Jc$$

to a primitive block with the original defect P. The subgroup J is proper: if Q were normal in G, its intersection with each component of the orbit would be an abelian normal subgroup of that quasisimple group, hence central, contradicting Lemma 8.2. Also $Z_0 \leq Z(J)$, so

$$\kappa(J) \leq |J : Z_0| < \kappa(G).$$

Because the component product is normal in G, every element of $N_G(P)$ normalizes its intersection Q with P. Thus $N_G(P) \leq J$. Apply induction to the selected block of J and return using the maximal companion of this same R.

Finally suppose that all components use case (iv). Lemma 3.3 makes every original PS_i-block inertial. Write b_E for the covered block of $E(G)$. The equality $ab_E = a$ gives $\mathrm{Br}_P(b_E) \neq 0$. Its defect is $P \cap E(G)$, and the unique block of $PE(G)$ covering b_E is represented by that idempotent. Its defect contains P and has order at most $|P \cap E(G)|\,|P : P \cap E(G)| = |P|$, so its defect is P. Lemma 3.4 therefore makes this $PE(G)$-block inertial. The remaining $O_2(G)$ is already in P. To include Z_0, put $K = PE(G)$ and $I = K \cap Z_0$. Let d be the block of K covered by a, and let λ be the central character of a on Z_0. Put

$$e_\lambda = |Z_0|^{-1} \sum_{z \in Z_0} \lambda(z^{-1})z.$$

The matching character on I gives the interior-P algebra isomorphism

$$\mathcal{O}Kd \otimes_{\mathcal{O}I} \mathcal{O}Z_0 e_\lambda \xrightarrow{\sim} \mathcal{O}Kd, \qquad x \otimes z e_\lambda \longmapsto \lambda(z)x.$$

Its inverse is $x \mapsto x \otimes e_\lambda$; both maps respect the balancing over $\mathcal{O}I$ because I acts on d through $\lambda|_I$. Multiplication identifies the left-hand side with the block algebra of $KZ_0 = PH$, where $H = F^*(G)$, covering d and e_λ. Consequently this block of PH is inertial. Lemma 8.2 gives the relative maximality of this same P. The hypotheses of Theorem 3.5 are now satisfied.

Take its primitive basic Morita bimodule from $\mathcal{O}Ga$ to the corresponding block of $J = N_G(D)$, with $D = P \cap H$. Proposition 2.2 and Lemma 2.3 give a splendid Rickard complex R with the original defect graph. There is a component, and its defect is noncentral; therefore D is not normal in G, so $J < G$. Again $Z_0 \leq Z(J)$, $N_G(P) \leq J$, and $\kappa(J) < \kappa(G)$. Apply induction and Proposition 2.4.

Each reduction strictly decreases $\kappa(G)$. Proposition 2.4, through (2.10), supplies both evaluation identities and the fixed diagonal vertex bound on returning from each reduction. All constructions use bounded integral 2-permutation complexes projective on both sides. This proves (1.1) and (1.2) and completes the induction. □

Corollary 8.3. *Let G be a finite group and let k be an algebraically closed field of characteristic two. Every block kGa with abelian defect group P has a splendid Rickard equivalence with its Brauer correspondent in $kN_G(P)$. In particular, Broué's abelian defect group conjecture holds at the prime two.*

Proof. Choose a sufficiently large splitting 2-modular system with residue field k, and lift the block idempotent to $\mathcal{O}G$. Reduce the complex and both integral contractions of Theorem 1.1 modulo the maximal ideal of $\mathcal{O}$. Reduction preserves the two identities, the contractions, and the permutation summands. □

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