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BROUÉ'S ABELIAN DEFECT GROUP CONJECTURE FOR 2-BLOCKS

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Abstract. We prove that every 2-block of a finite group with an abelian defect group is splendidly Rickard equivalent to its Brauer correspondent.

1. Introduction

Let $(K, \mathcal{O}, k)$ be a splitting 2-modular system: $\mathcal{O}$ is a complete discrete valuation ring of characteristic zero, k is its algebraically closed residue field of characteristic two, and K is its field of fractions, sufficiently large for the finite groups considered below. A block of $\mathcal{O}G$ is a primitive central idempotent a; its block algebra is $\mathcal{O}Ga$. If P is a defect group of a, Brauer's first main theorem associates to a a block a_P of $\mathcal{O}N_G(P)$, called its Brauer correspondent. Broué's abelian defect group conjecture predicts an equivalence

$$D^{\mathrm{b}}(\mathcal{O}Ga\text{-mod}) \simeq D^{\mathrm{b}}(\mathcal{O}N_G(P)a_P\text{-mod})$$

when P is abelian. Here D^{b} denotes the bounded derived category of finitely generated modules.

The conjecture grew out of Broué's study of perfect isometries and their compatibility with local character theory. His 1990 paper [7] develops isotypies as compatible systems of perfect isometries associated with Brauer pairs. Conjecture 6.1 concerns such a local character correspondence; Question 6.2 asks for the corresponding equivalence of derived categories. Section 3 relates derived equivalences to perfect isometries, and the appendix treats principal 2-blocks of simple groups with abelian Sylow 2-subgroups. Thus the characteristic-two families already occur among the examples that motivated the conjecture.

Rickard's theory expresses a derived equivalence by a complex of bimodules with a two-sided tensor inverse. His splendid equivalences [25] impose a permutation condition on the terms. This condition permits Brauer constructions at subgroups of the defect group and connects the global equivalence with local block algebras. The conjecture is known for cyclic defect groups, a case discussed already in [7, Section 6]. Harris and Linckelmann [15] proved the splendid form for blocks of finite p-solvable groups with abelian defect groups. Source algebras and endopermutation modules provide the connection between the Morita equivalences arising in these reductions and the required complexes.

Two further developments are relevant to the present problem. Chuang and Rouquier [8] constructed derived equivalences by $\mathfrak{sl}_2$-categorification and proved Broué's conjecture for symmetric groups, with related results for general linear groups. For groups of Lie type, Rickard's equivariant cohomology construction [24] and Rouquier's étale chain complexes [27] associate permutation complexes to Deligne–Lusztig varieties. Bonnafé, Dat and Rouquier [2] established a splendid Jordan decomposition when the dual centralizer is contained in the chosen Levi subgroup. Ruhstorfer [28] developed equivariant and disconnected versions of this construction. These results supply both the component equivalences and the normalizer actions used below.

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For 2-blocks, Craven and Rouquier [10, Theorem 4.36] proved the principal-block case. The classification of Klein four source algebras by Craven, Eaton, Kessar and Linckelmann [9] identifies three source algebra types, represented by $\mathcal{O}V_4$, $\mathcal{O}A_4$, and the principal block of $\mathcal{O}A_5$. The nonprincipal elementary abelian block of Co_3 was treated by Koshitani, Müller and Noeske [18]. Eaton, Kessar, Külshammer and Sambale [11, Theorem 6.1] subsequently classified the 2-blocks of quasisimple groups with abelian defect groups up to Morita equivalence. Their classification separates the elementary and Klein four cases, certain rank-one Levi cases, and blocks covered by nilpotent blocks. Its proof also identifies the groups and central quotients that occur in the Levi cases.

This classification has led to results for arbitrary finite groups with prescribed defect groups. Eaton and Livesey [12], together with Wu, Zhang and Zhou [29], completed the classification for abelian 2-groups of rank at most three. Ardito and Sambale [1] proved Broué's conjecture for elementary abelian defect groups of order 32. Eaton and Livesey [13] classified the Morita equivalence classes for abelian defect groups of rank at most four and obtained the derived equivalence with the Brauer correspondent. The rank here is the minimal number of generators. Together these results cover all abelian defect groups of order dividing 32. The source and permutation structure of the equivalence is an additional issue for the splendid form.

There has also been recent progress without a bound on the rank. Zhang and Zhou [31] proved that blocks with defect group

$$C_{2^{n_1}} \times \cdots \times C_{2^{n_t}}, \qquad n_i \geq 2,$$

are inertial. Zhou and Zhang [32] classified 2-blocks with abelian defect groups whose inertial quotient has prime order, and deduced Broué's conjecture for those blocks. For a maximal Brauer pair (P, e_P), the inertial quotient is $N_G(P, e_P)/(PC_G(P))$, including its conjugation action on P. Zhang and Zhou's extension theorem [30] treats extensions of inertial blocks under a hypothesis on the block of PH. Miller [21] characterized endosplit permutation resolutions and studied their descent over coefficient fields. Huang [16] studies fusion-stable endosplit resolutions; the abelian case gives the source resolution used in Section 2.

Passing from a quasisimple component to a block of an arbitrary finite group requires compatibility with the automorphisms induced by the ambient group. Zhou's equivariant reduction [33] formulates this requirement for principal blocks. In the present argument the component complexes carry actions of their marked stabilizers, and their restrictions to the upper defect group come from specified upper complexes. This permits tensoring the components over their shared centers and inducing to the ambient group. The resulting theorem covers every abelian 2-group.

Theorem 1.1. *Let G be a finite group and let a be a block of $\mathcal{O}G$ with abelian defect group P. There is a bounded complex X of $(\mathcal{O}Ga, \mathcal{O}N_G(P)a_P)$-bimodules whose terms are 2-permutation modules, projective on both sides and relatively ΔP-projective, such that*

$$X \otimes_{\mathcal{O}N_G(P)a_P} X^\vee \simeq \mathcal{O}Ga, \tag{1.1}$$

$$X^\vee \otimes_{\mathcal{O}Ga} X \simeq \mathcal{O}N_G(P)a_P \tag{1.2}$$

in the respective homotopy categories of bimodule complexes. Here $X^\vee = \mathrm{Hom}_{\mathcal{O}}(X, \mathcal{O})$, with the dual differential. In particular, these blocks are splendidly Rickard equivalent.

Reduction modulo the maximal ideal of $\mathcal{O}$ gives the corresponding statement over every algebraically closed field of characteristic two. The theorem applies to arbitrary ranks and exponents of P.

We prove the theorem by induction on

$$\kappa(G) = |G : O_{2'}(Z(G))|, \tag{1.3}$$

where $O_{2'}(Z(G))$ is the odd-order subgroup of the center. Section 2 resolves Morita bimodules with endopermutation source and constructs their local return. Section 3 proves the central quotient and product statements needed for the inertial case. Sections 4–6 construct the component complexes, including their upper actions. Section 7 assembles one orbit of components using transporters that preserve the common defect group. Section 8 completes the reduction to a proper normalizer.

At each reduction, let $R : A \longrightarrow C$ be the chosen equivalence to a block of a smaller group, and let $R_P : B_A \longrightarrow B_C$ be its maximal Brauer companion. Induction gives $Z : C \longrightarrow B_C$. Its maximal companion Z_P is an autoequivalence of B_C, since the defect group is normal in the group underlying B_C. The return is a reduced integral representative of

$$R \otimes_C Z \otimes_{B_C} Z_P^{\vee} \otimes_{B_C} R_P^{\vee}. \tag{1.4}$$

The Brauer-pair correspondence of Breland and Miller [4] shows that $Z \otimes Z_P^{\vee}$ matches the chosen maximal roots, and that the final complex has the fixed diagonal vertex bound. In the inertial case, the component argument establishes inertia of the block of $PF^*(G)$, where $F^*(G)$ is the generalized Fitting subgroup, so the upper-block hypothesis of [30, Theorem 1.3] applies.

2. Sources and local complexes

2.1. **Conventions.** For a finite group G, let $O_2(G)$ and $O_{2'}(G)$ be its largest normal 2-subgroup and largest normal odd-order subgroup. A quasisimple group is perfect with simple central quotient; a component is a subnormal quasisimple subgroup. The layer $E(G)$ is the product of the components, the Fitting subgroup $F(G)$ is the largest nilpotent normal subgroup, and $F^*(G) = F(G)E(G)$. For a positive integer n, let n_2 be its largest power-of-two divisor. For a finite abelian group A, write A_2 for its unique Sylow 2-subgroup.

We regard an $(\mathcal{O}G, \mathcal{O}L)$-bimodule as a module for $G \times L$ by $(g, l)m = gml^{-1}$. For an isomorphism $\eta : P_C \to P$, put

$$\Delta(P, \eta, P_C) = \{(\eta(u), u) : u \in P_C\}.$$

For the identity isomorphism we write ΔP. A marking consists of the chosen maximal Brauer pairs and defect-group identification. Conjugation transports these data simultaneously.

Fix a uniformizer π of $\mathcal{O}$. We use the same letter for a block idempotent and its reduction modulo π, and interpret its Brauer image after reduction. For an algebra A, write $J(A)$ for its Jacobson radical and $K^{\mathrm{b}}(A\text{-mod})$ for the bounded homotopy category of finitely generated A-modules, abbreviated to $K^{\mathrm{b}}(A)$ when the module category is understood. We use analogous notation for bimodules.

A lattice is a finitely generated free $\mathcal{O}$-module. A 2-permutation lattice for a finite group is a direct summand of a permutation lattice; the same definition over k gives a 2-permutation module. If U is a kG-module and $Q \leq G$ is a 2-subgroup, its Brauer construction is

$$U(Q) = U^Q \Big/ \sum_{R<Q} \mathrm{Tr}_R^Q(U^R), \qquad \mathrm{Tr}_R^Q(u) = \sum_{x \in Q/R} xu.$$

For a lattice M, write $M(Q) = (k \otimes_{\mathcal{O}} M)(Q)$; for a complex apply this functor termwise. On a group algebra it gives the Brauer homomorphism $\mathrm{Br}_Q : (kG)^Q \to kC_G(Q)$, which deletes the coefficients of elements outside $C_G(Q)$. If Q acts on a normal subgroup H, the same formula defines the relative map $\mathrm{Br}_Q : (kH)^Q \to kC_H(Q)$. Integral local complexes are lifted from these constructions by Lemma 2.3.

An a-Brauer pair is a pair (Q, e) with e a block of $kC_G(Q)$ and $e \,\mathrm{Br}_Q(a) \neq 0$. We use the usual inclusion of Brauer pairs; beneath a fixed maximal pair (P, e_P) there is a unique pair (Q, e_Q) for each $Q \leq P$. Its block fusion system $\mathcal{F}$ has the subgroups of P as objects and the conjugations

in G that preserve these pairs as morphisms. A source idempotent is a primitive idempotent $i \in (kGa)^P$ with $\mathrm{Br}_P(i) \neq 0$; $i(kG)i$ is its source algebra. The integral terminology is defined by reduction. A local point of Q on an algebra is a conjugacy class, under its Q-fixed units, of primitive Q-fixed idempotents with nonzero Brauer image.

Write $A^e = A \otimes A^{\mathrm{op}}$, over the coefficient ring in use. A complex is split if it is homotopy equivalent to its homology with zero differential. An endosplit resolution of a module V is a bounded complex S with $H^0(S) = V$, other homology zero, and $S \otimes S^\vee$ split. An endopermutation P-module is a module whose endomorphism algebra, with the conjugation action of P, is a permutation module. It is capped if it has an indecomposable summand with vertex P; the isomorphism class of such a summand is uniquely determined and is called its cap. Here a vertex is a minimal subgroup for relative projectivity, and a source is an indecomposable module at that subgroup inducing the given summand. The Dade group records capped endopermutation modules up to isomorphism of their caps, with tensor product followed by taking the cap.

Write $D(P)$ for this group. For an injective homomorphism $\varphi : Q \to P$, ${}_\varphi M$ denotes restriction along φ, so that q acts as $\varphi(q)$. The notation applies termwise to complexes. A capped endopermutation module is $\mathcal{F}$-stable if restriction to Q and restriction along φ have the same class in $D(Q)$ for every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$. For an indecomposable capped module this implies ${}_\alpha V \cong V$ for every $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$.

For symmetric block algebras A, B, use the symmetrizing forms given by the coefficient of the group identity. The dual and these forms give the two adjunction evaluations

$$X \otimes_B X^\vee \longrightarrow A, \qquad X^\vee \otimes_A X \longrightarrow B. \tag{2.1}$$

Whenever we call X a Rickard complex, both maps are homotopy equivalences in the indicated bimodule categories. For complexes projective on both sides over symmetric algebras, $X^\vee$ is the tensor functor's left and right adjoint. If it is a two-sided inverse, the adjunction unit and counit are isomorphisms in the homotopy categories. Thus the inverse identities also give these canonical evaluations. Relative to a fixed defect-group isomorphism $\eta : P_C \to P$, a Rickard complex is splendid if its terms are 2-permutation, projective on both sides, and relatively $\Delta(P, \eta, P_C)$-projective. Tensor products use the Koszul sign rule. A complex with an additional group action is always a complex for that finite group, including its inner intersections and multiplication relations. More explicitly, if finite extensions $\widetilde{G}, \widetilde{L}$ of G, L have a common quotient Γ, the group of pairs is

$$\{(g, l) \in \widetilde{G} \times \widetilde{L} : gG = lL \text{ in } \Gamma\}.$$

For a tensor through a middle extension $\widetilde{H}$, an outer pair (g, l) acts by $(g, h)m \otimes (h, l)n$, where h has their common image in Γ. Changing h by an element of H leaves this tensor unchanged by the balanced relation. This defines the tensor functors in the operator categories used below.

2.2. **The source resolution.** We recall two source-theoretic inputs. If a Morita equivalence between blocks $A = kGa$ and $B = kLc$ is induced by an indecomposable bimodule M with endopermutation source and identified defect group P, the source theorem supplies source idempotents i, j, a common fusion system $\mathcal{F}$, and a capped $\mathcal{F}$-stable endopermutation kP-module V such that M is a summand of

$$U = Ai \otimes_{kP} T(V) \otimes_{kP} jB, \qquad T(V) = \mathrm{Ind}_{\Delta P}^{P \times P} V. \tag{2.2}$$

The graph of the given defect identification is understood. This is the source theorem recalled in [20, Section 5]; its use in this form is also the starting point of [19]. Moreover, every indecomposable summand of the source algebra jBj as a kP-kP-bimodule has the form

$$kP \otimes_{kQ} {}_\varphi kP, \qquad Q \leq P, \quad \varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P). \tag{2.3}$$

The integral version has the analogous permutation basis [19, Proposition 2.5].

Lemma 2.1. *Suppose that P is abelian and V is the capped source in (2.2). There is a bounded endosplit 2-permutation resolution S of V such that ${}_\alpha S \cong S$ for every $\alpha \in E = \mathrm{Aut}_{\mathcal{F}}(P)$.*

Proof. The extension axiom for a saturated fusion system on an abelian group extends every fusion morphism to P. The Sylow axiom gives $|E|$ odd, since $\mathrm{Aut}_P(P) = 1$. The Dade group of an abelian 2-group is generated by the inflated relative syzygies associated with the quotients P/Q for $Q < P$, including $Q = 1$. Its normal form has integer exponents for noncyclic quotients and exponents modulo two for cyclic quotients of order at least four. The quotient of order two contributes no nontrivial generator. The exponents of the class of V are constant on E-orbits.

Represent these factors by truncated permutation resolutions and use their duals for negative exponents. Their tensor is endosplit and is isomorphic to every E-transport. Its homology has V as its cap. For an endosplit resolution S_0, the canonical homology map gives

$$\mathrm{End}_{K^{\mathrm{b}}(kP)}(S_0) \cong \mathrm{End}_{kP}(H^0(S_0)).$$

Choose a primitive projector with image V on $H^0(S_0)$ and lift it through this isomorphism. Splitting it gives an endosplit resolution of V. For $\alpha \in E$, the transported projector, viewed on the same $H^0(S_0)$ through the chosen transport of S_0, also has image isomorphic to V. The two primitive projectors are conjugate in $\mathrm{End}_{kP}(H^0(S_0))$. Their lifted images are therefore homotopy equivalent. After removing contractible summands, a homotopy equivalence of these minimal complexes is a chain isomorphism. This proves the assertion and is the abelian construction of [16, Lemmas 4.1–4.2 and Proposition 4.3]. □

Proposition 2.2. *Let M be a Morita bimodule over k as in (2.2), with P abelian. There is a splendid Rickard complex X over k for the same primitive blocks and defect identification, with $H^0(X) \cong M$ and all other homology zero.*

Proof. Use Lemma 2.1 and form

$$Y = Ai \otimes_{kP} T(S) \otimes_{kP} jB. \tag{2.4}$$

Each term is a summand of an induced diagonal permutation module, projective on both sides. The outside factors are projective over kP, and $T(S)$ is free on either P-side. Thus $H^0(Y) = U$ and all other homology vanishes. The bimodule U is also projective on either side.

Decompose the middle factor jBj in $Y \otimes_B Y^\vee$ using (2.3). For its (Q, φ)-summand, the coefficient complex on the diagonal Q is

$$\mathrm{Res}^P_Q S \otimes_{k\,\varphi} \mathrm{Res}^P_{\varphi(Q)} S^\vee. \tag{2.5}$$

Extend φ to an element of E. The corresponding individual isomorphism from Lemma 2.1 identifies (2.5) with a restriction of $S \otimes S^\vee$, which is split. Inducing this splitting and tensoring with the outside Ai and iA factors gives an A-A-bimodule splitting. The same argument, with iAi in the middle, gives a B-B-bimodule splitting. Consequently

$$Y \otimes_B Y^\vee \simeq (U \otimes_B U^\vee)[0], \qquad Y^\vee \otimes_A Y \simeq (U^\vee \otimes_A U)[0]. \tag{2.6}$$

The splittings are in the full bimodule categories, since the outer factors carry the given group actions throughout.

Symmetric adjunction identifies $\mathrm{Hom}^\bullet_B(Y, Y)$ with $Y \otimes_B Y^\vee$. The invariants formula is

$$\mathrm{End}^\bullet_{A-B}(Y) = \mathrm{Hom}_{A^e}(A, \mathrm{Hom}^\bullet_B(Y, Y)). \tag{2.7}$$

The argument of this additive functor is already split by bimodule homotopies. Applying the functor preserves the given bimodule homotopies and hence the splitting. The canonical homology

map is an algebra isomorphism

$$\mathrm{End}_{K^{\mathrm{b}}(A-B)}(Y) \xrightarrow{\sim} \mathrm{End}_{A-B}(U). \tag{2.8}$$

Lift the idempotent whose image is the prescribed M and split it in the bounded homotopy category of 2-permutation modules. This category is idempotent complete. More concretely, remove contractible summands from Y; null-homotopic endomorphisms then lie in the radical of its finite chain-endomorphism algebra, so the idempotent has a strict lift. Let X be its termwise image.

Both balanced squares of X are retracts of the split squares in (2.6). Their degree-zero homology is respectively $M \otimes_B M^\vee$ and $M^\vee \otimes_A M$. The evaluations (2.1) induce the Morita evaluations, which are isomorphisms onto A and B. A map between split complexes that induces an isomorphism on homology is a homotopy equivalence. This proves both required evaluations on the same complex X. □

Lemma 2.3. *Let X be a splendid Rickard complex between blocks $A = kGa$ and $B = kLc$, equipped with a compatible action of a finite group of pairs. Suppose that every term is 2-permutation for this entire finite action group and that the two evaluation cones are contractible in the corresponding full operator categories. Then X has an integral 2-permutation lift with those actions and evaluations. The same assertion holds in a specified degree-preserving category.*

Proof. Lift the terms to their integral 2-permutation lattices for the whole finite action group. Reduction on Hom spaces is surjective in that group-module category: orbit sums prove it for permutation lattices, and lifting summand projectors proves it for their summands. Without extra operators, tensoring with the Rickard inverse gives

$$\mathrm{Hom}_{K^{\mathrm{b}}(A-B)}(X, X[n]) \cong \mathrm{Hom}_{K^{\mathrm{b}}(A-A)}(A, A[n]) = 0 \qquad (n \neq 0). \tag{2.9}$$

With the prescribed operators, the tensor functors stay in the full pair-group categories. In a presentation with a common operator quotient, choose one middle-group lift of a quotient element and use it on both tensor factors. Changing that lift is canceled by the balanced relation, and multiplication retains the specified inner intersections. The stipulated equivariant evaluations make these functors mutually inverse. The inverse-tensor functor sends X to the regular degree-zero bimodule with its diagonal operator action, so the corresponding equivariant homotopy Hom groups also vanish in nonzero degree. In the special case of an odd operator quotient this follows equivalently by taking invariants, which is exact. This homotopy vanishing supplies the differential lifts. The whole-group permutation hypothesis supplies the termwise lifts and surjectivity on Hom spaces.

Suppose that a differential has been lifted modulo π^m. Its curvature modulo π^{m+1} is a degree-two cocycle in the same endomorphism complex. Equation (2.9) corrects it. Completeness gives a differential on the fixed finite lattices. The integral evaluation cones reduce to contractible complexes. Their terms are again permutation summands for the full pair groups: before block cuts, balanced permutation bases give balanced orbit sets, and tensor products of the summand projectors descend to the corresponding linearized orbit modules. The regular block terms are invariant summands of the regular permutation bimodules. Thus the same Hom-surjectivity applies to the cone terms. Lift a reduced contraction h and put $u = dh + hd$. Then u is a chain automorphism congruent to the identity modulo π, and hu^{-1} is an integral contraction.

In the graded applications, the grading is the group-coset grading: its identity component is a module for the corresponding group of pairs. Lift in that finite-group category and induce to recover the graded terms. For a group grading, project a map on a homogeneous input of degree g to its output of degree g. This projection is bimodule-linear: left and right multiplication of degrees h, k send g to hgk on both sides. It commutes with the Hom differential and applies

to the homotopies as well. The same lifting argument therefore stays in the degree-preserving category. □

2.3. **The local complex of an equivalence.** The anti-involution $*$ of a group algebra sends g to g^{-1}. For a complex U of 2-permutation bimodules between blocks A, B and an $A \otimes B^{\mathrm{op}}$-Brauer pair $\lambda = (\Delta(Q, \varphi, R), e \otimes f^*)$, put

$$U(\lambda) = eU(\Delta(Q, \varphi, R))f.$$

We say that U supports λ if this complex is noncontractible, and write $\mathrm{BP}(U)$ for its set of supported Brauer pairs. The centralizer block idempotents e, f are called the roots of the pair. The Lefschetz invariant is the alternating sum $\sum_n (-1)^n [U^n]$ in the split Grothendieck group of 2-permutation modules.

Proposition 2.4. *Let $R : A \longrightarrow C$ be an integral splendid Rickard complex between primitive blocks, with identified abelian defect groups. Fix a supported maximal pair*

$$\lambda_R = (\Delta(P, \eta, P_C), e \otimes f^*).$$

There is an integral maximal companion $R_P : B_A \longrightarrow B_C$ between the prescribed Brauer correspondents, supporting this same pair. If $Z : C \longrightarrow B_C$ is splendid Rickard, it has a maximal companion $Z_P : B_C \longrightarrow B_C$ such that a reduced integral representative

$$X \simeq R \otimes_C Z \otimes_{B_C} Z_P^\vee \otimes_{B_C} R_P^\vee \tag{2.10}$$

is splendid Rickard from A to B_A. Its terms are relatively ΔP-projective for the fixed diagonal, and $(\Delta P, e \otimes e^*)$ is a supported maximal pair.

Proof. Brauer constructions in this proof are over k; reduction of an integral complex is understood. Write $A = \mathcal{O}Ga$ and $C = \mathcal{O}G_C c$, and put

$$I = N_G(P, e), \qquad I_C = N_{G_C}(P_C, f), \qquad W = N_{I \times I_C}(\Delta(P, \eta, P_C)).$$

With $\widehat{e}, \widehat{f}$ the block idempotents of the full normalizers corresponding to the two maximal pairs, the modular companion is

$$\widehat{e}\, kN_G(P)e \otimes_{kI} \mathrm{Ind}_W^{I \times I_C}(eR(\Delta(P, \eta, P_C))f) \otimes_{kI_C} f\, kN_{G_C}(P_C)\widehat{f}. \tag{2.11}$$

Theorem 8.5 and Corollary 8.9 of [4] give its two Rickard evaluation identities. Its endpoints are the prescribed Brauer correspondents.

To identify its maximal pair, apply [4, Corollary 7.11(b)] to the Lefschetz invariant of R at the full defect groups. It says that the inertia-level complex inside (2.11) has λ_R as a maximal Brauer pair for its Lefschetz invariant. By [4, Corollary 8.8], this is a Brauer pair of that complex. The two Fong–Reynolds bimodules in (2.11) have, at their respective diagonal maximal pairs, the regular centralizer blocks:

$$e(\widehat{e}\, kN_G(P)e)(\Delta P)e \cong kC_G(P)e, \qquad f(f\, kN_{G_C}(P_C)\widehat{f})(\Delta P_C)f \cong kC_{G_C}(P_C)f.$$

Indeed the fixed group elements in the regular permutation bimodule are precisely the corresponding centralizer elements, and the displayed block cuts retain the indicated root. The blockwise Brauer–tensor formula [4, Theorem 6.7] therefore gives a noncontractible root-cut Brauer complex at λ_R for (2.11). Thus the full-normalizer companion supports the same λ_R, including its two roots and its graph. Remove its contractible summands in the ordinary bimodule category. By [4, Corollary 9.5], its terms have vertices below the graph in λ_R. Lift the resulting complex integrally by Lemma 2.3. The lift has the same Brauer support.

Fix the left root f for Z. By [4, Lemma 8.3 and Corollary 8.8(c)], it extends to a supported maximal pair

$$\lambda_Z = (\Delta(P_C, \theta, P_C), f \otimes g^*)$$

for some root g of B_C and some automorphism θ of P_C. The right defect group is the chosen subgroup P_C: it is normal in $N_{G_C}(P_C)$, so it is the unique defect subgroup of B_C. Apply the preceding companion construction to this pair. The left full-normalizer block is B_C. The right full normalizer is again $N_{G_C}(P_C)$, and B_C is its own Brauer correspondent there. Hence the resulting integral complex is an autoequivalence Z_P of B_C supporting λ_Z.

Let Y denote the four-factor product on the right of (2.10). The ordinary tensor-product assertion [4, Lemma 6.3] makes its terms permutation summands with twisted diagonal vertices, meaning graphs of isomorphisms between 2-subgroups of the endpoints. The terms are projective on both sides. Associativity and the four given Rickard evaluations show that the canonical evaluations of Y are homotopy equivalences: in the first square successively cancel $R_P^\vee R_P$, $Z_P^\vee Z_P$, $ZZ^\vee$, and $RR^\vee$; in the second square cancel $R^\vee R$, $Z^\vee Z$, $Z_P Z_P^\vee$, and $R_P R_P^\vee$.

It remains to recover the fixed diagonal and root. Put

$$E = kC_G(P)e, \qquad F = kC_{G_C}(P_C)f, \qquad F_g = kC_{G_C}(P_C)g.$$

By [4, Theorem 8.5], the local complexes of R, R_P at λ_R are Rickard equivalences from E to F, and those of Z, Z_P at λ_Z are Rickard equivalences from F to F_g. Consequently

$$R(\lambda_R) \otimes_F Z(\lambda_Z) \otimes_{F_g} Z_P(\lambda_Z)^\vee \otimes_F R_P(\lambda_R)^\vee$$

is a Rickard autoequivalence of E, so it is noncontractible. Iterating [4, Theorem 6.7 and Proposition 8.2] exhibits this complex as a direct summand of the restriction of $eY(\Delta P)e$ to its centralizer bimodule category: the chosen intermediate roots are f, g, f, and the graph maps compose as $\eta\theta\theta^{-1}\eta^{-1} = \mathrm{id}_P$. Thus $(\Delta P, e \otimes e^*)$ belongs to $\mathrm{BP}(Y)$. It is maximal by [4, Corollary 8.8(c)].

Finally choose a reduced integral representative. For permutation lattices and their summands, reduction on group-module Hom spaces is surjective, and their finite endomorphism algebras are complete. Therefore a summand projector of a term after reduction lifts integrally. If the reduction of the current complex has an invertible differential component between two summands, lift their projectors. The corresponding integral differential component is invertible by Nakayama's lemma, and Gaussian elimination splits off an integral contractible two-term complex. Repeating this finite process produces $X \simeq Y$ such that $k \otimes_{\mathcal{O}} X$ has no contractible direct summands. Its evaluations and Brauer support are unchanged. By [4, Proposition 5.4], its reduction is indecomposable as a complex. Corollary 9.5 of [4], applied to its supported maximal pair $(\Delta P, e \otimes e^*)$, now puts every term vertex below a conjugate of the original ΔP. Reduction of permutation lattices preserves vertices [4, Theorem 3.2(b)], so the same bound holds integrally. This proves the stated splendidness, with the fixed marking and the prescribed target block. □

3. Central quotients and inertial blocks

A basic Morita equivalence is induced by an indecomposable Morita bimodule whose vertex is the graph of the chosen defect-group isomorphism and whose source is endopermutation. For a maximal Brauer pair (P, e_P), put $I = N_G(P, e_P)$. The inertial block is the Fong–Reynolds correspondent in I of the Brauer correspondent in $N_G(P)$. A block is nilpotent if $N_G(Q, e)/C_G(Q)$ is a 2-group for each of its Brauer pairs (Q, e), and is inertial if it is basically Morita equivalent to its inertial block.

An interior P-algebra has a specified homomorphism $P \to A^\times$. A primitive Dade P-algebra is the endomorphism algebra of an indecomposable capped endopermutation lattice, with this

interior action and Brauer quotient k at P. It is E-stable if transport by every element of E gives an isomorphic interior P-algebra. A determinant-one interior structure means that every $p \in P$ has determinant one on the corresponding lattice. An inertial source algebra has the form $S \otimes_{\mathcal{O}} \mathcal{O}_\alpha(P \rtimes E)$, where S is a primitive E-stable Dade P-algebra and E is the odd inertial quotient. Here α is a normalized $\mathcal{O}^\times$-valued 2-cocycle on E, inflated to $P \rtimes E$.

We use two consequences of Puig's theory [22]: a normal subblock of an inertial block is inertial (Theorem 3.13), and the rank criterion in Proposition 3.11. The latter says that an E-stable Dade algebra S gives inertia if the local corner of $S^{\mathrm{op}} \otimes \mathcal{O}G$ at the product of the Dade point and the source point has rank $|P||E|$. Its proof shows that the canonical interior embedding into that corner remains injective after reduction; rank equality then makes it an isomorphism.

3.1. **Inertia and a central 2-quotient.**

Proposition 3.1. *Let a be a block with abelian defect group P, and let $Z \leq Z(G) \cap P$. The block a is inertial if and only if the dominated block of G/Z is inertial.*

Proof. Put $A = i\mathcal{O}Gi$ for a source idempotent, and let $E = \mathrm{Aut}_{\mathcal{F}}(P)$. The group E has odd order and fixes Z pointwise. The source-algebra permutation basis has stabilizers $\Delta_\varphi Q$, with φ a fusion morphism. Since $za = az$ for $z \in Z$, ΔZ fixes every basis element. Hence $Z \leq Q$ and $\varphi|_Z = 1$ in every orbit. Left multiplication by Z is free on that basis.

Suppose that conjugation by $r \in P$ fixes the Z-orbit of a basis element ξ. Then $r\xi r^{-1} = z\xi$ for some $z \in Z$. The graph equation gives $r \in Q$ and $\varphi(r) = rz^{\pm 1}$. Extend φ to $\alpha \in E$. On $\langle r, Z\rangle$, the endomorphism $\delta = \alpha - 1$ has image in Z, vanishes on Z, and satisfies $\delta^2 = 0$. If m is the odd order of this restriction, $1 = (1+\delta)^m = 1 + m\delta$; thus $\delta = 0$ and $z = 1$. Passage to Z-orbits creates no new conjugation stabilizers.

Write $I_Z = (z - 1 : z \in Z)\mathcal{O}Z$ and $\overline{A} = A/I_Z A$. Orbit sums therefore lift with coefficient one, giving

$$A^R \twoheadrightarrow \overline{A}^{RZ/Z} \qquad (R \leq P). \tag{3.1}$$

The kernel is radical: $I_Z \subseteq J(\mathcal{O}Z)$ and A is finite over the central local ring $\mathcal{O}Z$, so $I_Z A \subseteq J(A)$. For a fixed element, the inverse of a fixed unit is fixed; hence $J(A) \cap A^R \subseteq J(A^R)$. In particular $\overline{A}^{P/Z}$ is local. Its identity has nonzero Brauer image: an expression by proper quotient traces would lift, using (3.1), to proper traces and radical elements in A^P. The proper-trace ideal is contained in the proper Brauer kernel of the local algebra A^P, hence in its radical, contradicting $1 \notin J(A^P)$. Thus $\overline{A}$ is the source algebra of the dominated block.

Quotienting each graph gives its quotient fusion graph. Every maximal automorphism graph occurs in a source algebra, by [19, Proposition 2.5]. Since $Z \leq Q$, a quotient graph can be maximal only when $Q = P$. The downstairs inertial quotient is therefore the image of E. Its kernel is trivial: an automorphism fixing Z and P/Z lies in a 2-group automorphism kernel, whereas E is odd. Write this identification as $E \cong \overline{E}$.

Suppose that the dominated block is inertial. There is an $\overline{E}$-stable primitive Dade algebra $\overline{S}$ with

$$\overline{A} \cong \overline{S} \otimes_{\mathcal{O}} \mathcal{O}_\beta((P/Z) \rtimes \overline{E}).$$

Inflate $\overline{S}$ to S and put $T = S^{\mathrm{op}} \otimes A$. A permutation conjugation basis of S^{op}, together with the preceding graph basis, gives $T^P \twoheadrightarrow \overline{T}^{P/Z}$ with radical kernel. Cancel the downstairs Dade factors and choose the resulting primitive local idempotent $\overline{j}$, so that

$$\overline{j}\,\overline{T}\,\overline{j} \cong \mathcal{O}_\beta((P/Z) \rtimes \overline{E}).$$

Lift it to $j \in T^P$. Locality follows from the exact trace identity

$$\overline{\mathrm{Tr}_R^P(t)} = [RZ : R]\,\mathrm{Tr}_{RZ/Z}^{P/Z}(\overline{t}). \tag{3.2}$$

For $R < P$, the right side is either a proper trace or is divisible by two. Thus a nonlocal lift would have a nonlocal image.

The lifted idempotent belongs to the product local point. A primitive Dade algebra satisfies $S(P) = k$, so $T(P) \cong S^{\mathrm{op}}(P) \otimes A(P) \cong A(P)$. The last algebra is a nonzero quotient of the local algebra A^P and hence is local. There is a unique local P-point of T, namely the product of its Dade point and the chosen source point. The lifted j belongs to it. Now T is free over the central local ring $\mathcal{O}Z$, and $t \mapsto jtj$ is an $\mathcal{O}Z$-linear projector. Thus $C = jTj$ is free over $\mathcal{O}Z$ and

$$\operatorname{rank}_{\mathcal{O}} C = |Z| \operatorname{rank}_{\mathcal{O}}(C/I_Z C) = |Z|\,|P/Z|\,|\overline{E}| = |P|\,|E|.$$

Puig's rank criterion proves ascent of inertia.

Conversely, write an inertial source as $S \otimes \mathcal{O}_\beta(P \rtimes E)$, using the determinant-one interior structure on its Dade module. Centrality of Z forces its action on the Dade module to be scalar and forces E to fix Z. A scalar of 2-power order with determinant one in odd dimension is one. The source therefore descends to the indicated inertial source for P/Z. This proves descent. □

3.2. **A supported nilpotent cover.** We specify the classification input used in this subsection. In the residual nonnilpotent Lie-type case, there is a simple simply connected algebraic group $\mathbf{X}$ in odd characteristic, a Steinberg endomorphism F, and a central subgroup $Z \le X := \mathbf{X}^F$ such that $S = X/Z$. The block of X dominating the specified block of S has a semisimple $2'$-label s for which $C^\circ_{\mathbf{X}^*}(s)$ is a torus. This is the concrete case in the proof of [11, Theorem 6.1] to which [11, Lemma 4.2] is applied. The other case needed here is that the block of S itself is nilpotent. The toral input includes this nonexceptional covering realization; the regular, field, and graph automorphism construction below is made for this realization.

We first record two block-theoretic facts in the forms that we use. Suppose that $H = NP$, where $N \lhd H$ and P is a defect group of a block B of H. Then every block of N covered by B is P-stable. Indeed, the covered blocks form one H-orbit, hence one P-orbit. If b is one of them and its stabilizer P_b is proper in P, their sum is $\operatorname{Tr}^P_{P_b}(b)$ and has zero P-Brauer image. Multiplying this sum by B contradicts $\operatorname{Br}_P(B) \neq 0$. Thus the orbit has one member. Since H/N is a 2-group, b has a unique covering block in H, and, as central idempotents in $\mathcal{O}H$, this block is $B = b$. In addition, $P \cap N$ is a defect group of b. We use the usual normal-subgroup theorem for defect groups and compatible Brauer pairs, in the form recorded in [22, Sections 3.2–3.4].

Lemma 3.2. *Let $N \lhd H$, with H/N a 2-group, and let B be a block of H covering a block b of N.*

(1) If B is nilpotent, then b is nilpotent.

(2) Suppose that $H = NP$ and that P is an abelian defect group of B. If b is nilpotent, then B is nilpotent.

Proof. For the first assertion, take a b-Brauer pair (Q, d) and a compatible B-Brauer pair (Q, e), so that e covers d. Such pairs exist by the normal-subgroup Brauer-pair theorem, after conjugating a compatible choice of maximal pairs. The quotient $C_H(Q)/C_N(Q)$ is a 2-group. Consequently e is the unique block of $C_H(Q)$ covering d. Every element of $N_N(Q, d)$ therefore stabilizes e, and conjugation gives an injection

$$N_N(Q, d)/C_N(Q) \longrightarrow N_H(Q, e)/C_H(Q).$$

The latter group is a 2-group by nilpotence of B. This proves the nilpotence condition for every b-Brauer pair.

For the second assertion, put $D = P \cap N$ and choose a maximal B-Brauer pair (P, e_P). The compatible normal-subgroup pair theorem supplies a P-stable maximal b-Brauer pair (D, d) below it. Let (D, e_D) be the unique B-subpair of (P, e_P) with first component D; it covers d.

Since P is abelian and $H = NP$, we have

$$C_H(D) = C_N(D)P.$$

The P-stability of d therefore makes it invariant under all of $C_H(D)$. Its unique block extension to $C_H(D)$ is e_D; in particular, e_D covers only d. Now every $y \in N_H(P, e_P)$ normalizes D and stabilizes e_D, and hence stabilizes d.

Write $y = xp$ with $x \in N$ and $p \in P$. Both y and p stabilize (D, d), so $x \in N_N(D, d)$. Nilpotence of b and abelianness of its defect group D give $N_N(D, d) = C_N(D)$: its action on D is both a 2-group, by nilpotence, and a $2'$-group, by the maximal-pair inertial-quotient theorem. Thus y acts trivially on D. It also acts trivially on P/D, since this is the image of P in the abelian group $H/N = P/D$.

For an abelian 2-group P, the automorphisms that fix D and P/D pointwise form a 2-group. Indeed, writing the group law additively, every such automorphism is $1 + \delta$, where $\delta \in \mathrm{Hom}(P/D, D)$; these maps compose by addition because the product of any two such endomorphisms is zero. The inertial quotient of B is a $2'$-subgroup of $\mathrm{Aut}(P)$, so it is trivial. Fusion in a block with abelian defect group is controlled by its maximal Brauer pair. It follows that B is nilpotent. □

Lemma 3.3. *Let S be a quasisimple component, let b be its specified block, and suppose that b is in the toral case described above or is nilpotent. Let P be an abelian 2-subgroup normalizing S, and let c be a block of $G_0 = SP$ covering b whose defect group is the subgroup P. Then c is inertial.*

Proof. Put $D = P \cap S$. The preceding observation shows that b is P-stable, that D is a defect group of b, and that $c = b$ as idempotents in $\mathcal{O}G_0$. In particular, $\mathrm{Br}_P(b) = \mathrm{Br}_P(c) \neq 0$, with the first expression interpreted as the relative Brauer map on $(\mathcal{O}S)^P$. If b is nilpotent, Lemma 3.2(2) proves that c is nilpotent. We henceforth treat the toral case.

First we justify the central reductions. Let $R = O_2(G_0)$. Since R is contained in every defect group of c, we have $R \leq P$. Also $[R, S] \leq R \cap S \leq O_2(S) \leq Z(S)$. For a fixed element of R, the resulting central commutator map on the perfect group S is trivial. Thus R centralizes S, and it centralizes P because P is abelian. Conversely, $C_P(S)$ centralizes both P and S and is a normal 2-subgroup of G_0. Hence

$$R = C_P(S) \leq Z(G_0), \qquad R \cap S = O_2(Z(S)).$$

Apply Proposition 3.1 to c and pass to G_0/R. We retain the notation G_0, S, P, b, c. Now $O_2(G_0) = 1$ and $Z(S)$ has odd order. The group $C_{G_0}(S)/Z(S)$ is a 2-group and $Z(S)$ is central in $C_{G_0}(S)$. Schur–Zassenhaus gives a unique Sylow 2-subgroup of this centralizer. It is normal in G_0 and is therefore trivial. Consequently

$$C_{G_0}(S) = Z(S), \qquad C_P(S) = 1.$$

Let θ be the odd central character of b and put $K_0 = \ker(\theta) \leq Z(S)$. The P-stability of b makes K_0 normal in G_0. The averaging idempotent of K_0 acts as the identity on $c = b$, so quotienting by K_0 gives an isomorphism of block algebras, with the subgroup P identified with its image. This preserves the source algebra and inertia. The centralizer is preserved: if an element centralizes S/K_0, its commutators with S lie in K_0, and perfection of S again makes that central commutator map trivial. Thus

$$C_{G_0/K_0}(S/K_0) = Z(S)/K_0.$$

In particular $O_2(G_0/K_0) = 1$. We may therefore retain the notation and assume that θ is faithful. Its invariance implies that P centralizes $Z(S)$. The same block of the simply connected group X dominates the successive central quotients of b, so the toral classification input is retained.

Choose a regular embedding $\mathbf{X} \hookrightarrow \widetilde{\mathbf{X}}$ with connected centre, compatible with F, and put $\widetilde{X} = \widetilde{\mathbf{X}}^F$. Let Γ_Z denote the suitable field and graph automorphisms stabilizing Z, so that $\widetilde{X} \rtimes \Gamma_Z$ induces the automorphisms of $S = X/Z$. This is the regular automorphism realization used in the proof of [17, Proposition 4.2 and Lemma 4.5]. Set

$$J_0 = Z\,O_2(Z(\widetilde{X})), \qquad \check{G} = \widetilde{X}/J_0, \qquad \check{A} = (\widetilde{X} \rtimes \Gamma_Z)/J_0.$$

Since $Z(S)$ has odd order, $O_2(Z(X)) \le Z$, and hence $J_0 \cap X = Z$. The original S therefore embeds in $\check{G}$. Conjugation gives a homomorphism $\rho : \check{A} \longrightarrow \mathrm{Aut}(S)$ whose kernel is the odd-order group $Z(\widetilde{X})/J_0$.

Let $\tau : G_0 \longrightarrow \mathrm{Aut}(S)$ be the original action. The restriction $\tau|_P$ is injective and $\tau(P) \cap \mathrm{Inn}(S) = \tau(D)$. Choose a Sylow 2-subgroup $\check{P}$ of $\rho^{-1}(\tau(P))$ containing the already embedded D. Then $\rho|_{\check{P}}$ is an isomorphism onto $\tau(P)$. Thus

$$\lambda = (\rho|_{\check{P}})^{-1}\tau|_P : P \longrightarrow \check{P}$$

is an isomorphism, equals the identity on D, and satisfies $\check{P} \cap S = D$. Since $\lambda(x)$ induces the same automorphism of S as x, the map

$$G_0 \longrightarrow \check{A}, \qquad sx \longmapsto s\lambda(x) \quad (s \in S,\ x \in P) \tag{3.3}$$

is a well-defined homomorphism. If its value is 1, then $\lambda(x) \in S$, hence $x \in D$, and $s\lambda(x) = sx$. It is therefore injective and is the identity on S. We now identify G_0 and P with these embedded subgroups.

Every block of the regular subgroup $\check{G}$ covering b is nilpotent, as follows. Let b_X be the block of X dominating b. The proof of [11, Lemma 4.2] starts with an arbitrary block $\widetilde{B}$ of $\widetilde{X}$ covering b_X. Its semisimple label has toral centralizer in the dual regular group, and the Bonnafé–Rouquier equivalence in that proof makes $\widetilde{B}$ nilpotent. Thus every such $\widetilde{B}$ is nilpotent. The odd part of Z is in its kernel; quotienting that part and the central 2-group $O_2(Z(\widetilde{X}))$ preserves nilpotence. Conversely, every block of $\check{G}$ covering b lifts to one of these blocks of $\widetilde{X}$. Consequently every block of $\check{G}$ covering b is nilpotent.

The quotient $\check{G}/S$ is abelian. Let $\check{K}$ be the inverse image of its odd-order subgroup. Then $\check{K}/S$ is odd and abelian, $\check{G}/\check{K}$ is a 2-group, and P normalizes $\check{K}$. Every block of $\check{K}$ covering b is covered by a block of $\check{G}$ covering b, and is nilpotent by Lemma 3.2(1).

Let e be the sum of all blocks of $\check{K}$ covering b. Then e is P-stable and $eb = b$, so

$$\mathrm{Br}_P(e)\,\mathrm{Br}_P(b) = \mathrm{Br}_P(b) \neq 0. \tag{3.4}$$

The sum of a nonfixed P-orbit of covering blocks is a relative trace from its proper stabilizer, and its P-Brauer image is zero. Therefore there is a P-fixed nilpotent block f of $\check{K}$ such that $\mathrm{Br}_P(f)\,\mathrm{Br}_P(b) \neq 0$.

Put $U = P\check{K}$. The invariant block f is the unique block F of U covering it, with $F = f$ as an idempotent. Its P-Brauer image is nonzero, so some defect group of F contains P. Every defect group of f has order $|D|$: $\check{K}/S$ is odd, and the intersection theorem for defects applied to $S \lhd \check{K}$ gives this assertion. Also $P \cap \check{K} = D$. The normal-extension defect bound, for a defect group Q of F, therefore gives

$$|Q| \le |D|\,|U : \check{K}| = |D|\,|P : D| = |P|.$$

It follows that the subgroup P is a defect group of F. Lemma 3.2(2), with $N = \check{K}$, now proves that F is nilpotent.

Choose a block e_P of $kC_U(P)$ satisfying

$$e_P\,\mathrm{Br}_P(F) = e_P, \qquad e_P\,\mathrm{Br}_P(b) \neq 0.$$

This is possible by the chosen nonzero product of Brauer images, and (P, e_P) is a maximal F-Brauer pair. Put $N = N_U(P)$ and $L = SN$. Brauer's first main theorem first gives the correspondent F_N of F in N, and then gives the unique block E_L of L with defect P whose correspondent in N is F_N. Indeed,

$$N_L(P) = N_U(P) = N, \qquad C_L(P) = C_U(P).$$

All three blocks use the same maximal pair (P, e_P). Their inertial quotients are therefore equal. The inertial quotient of F is trivial; hence so is that of E_L, and the abelianness of P makes E_L nilpotent.

The same maximal pair also retains the original covering relation. Namely

$$e_P \operatorname{Br}_P(E_L c) = e_P \operatorname{Br}_P(c) = e_P \operatorname{Br}_P(b) \neq 0,$$

so $E_L c \neq 0$. Finally, $G_0 = SP$ is normal in L: the subgroup S is normal and every element of N normalizes both S and P. Thus c is a normal subblock of the nilpotent block E_L. Puig's theorem [22, Theorem 3.13], including the stabilizer reduction in its Section 3.2, proves that c is inertial. The normal odd-kernel block isomorphism and Proposition 3.1 restore the original group and original block. □

3.3. **Products of the component extensions.**

Lemma 3.4. *Let $G = PE$, where P is abelian and $E = S_1 \cdots S_m$ is a central product of normal quasisimple components of G. Let a be a block of G with defect group P, and let b_i be the component blocks covered by a. Suppose that, for each i, the block c_i of PS_i covering b_i is inertial. Then a is inertial.*

Proof. We first establish stability and the defects of the blocks in this statement. Let b be a block of E covered by a. The support observation above, applied to $G = EP$, shows that b is P-stable, is the unique block of E covered by a, and satisfies $a = b$ as an idempotent. The block b covers a unique block b_i of each component S_i, and

$$b = b_1 \cdots b_m$$

inside $\mathcal{O}E$. This is the block correspondence obtained by inflating to $\prod_i S_i$ and then taking the central character sector of the central product. Since each S_i is normal in G, the P-stability of b makes each b_i P-stable. The normal-subgroup defect theorem gives $D_i := P \cap S_i$ as a defect group of b_i.

Brauer multiplicativity gives

$$0 \neq \operatorname{Br}_P(a) = \operatorname{Br}_P(b) = \prod_i \operatorname{Br}_P(b_i).$$

Thus every $\operatorname{Br}_P(b_i)$ is nonzero. The unique block c_i of $G_i := PS_i$ covering b_i is the idempotent b_i in $\mathcal{O}G_i$. A defect group of c_i contains P, while its order is at most $|D_i|\,|G_i : S_i| = |D_i|\,|P : D_i| = |P|$. Consequently P is a defect group of every c_i.

Let $R = O_2(G)$. We have $R \leq P$. For every i, $[R, S_i] \leq O_2(S_i) \leq Z(S_i)$, so perfection of S_i makes R centralize S_i. It also centralizes P. Hence

$$R = C_P(E) \leq Z(G).$$

Apply Proposition 3.1 simultaneously to a and to all c_i and quotient by R. The component images remain quasisimple, $O_2(G) = 1$, and $Z(E)$ has odd order. The last assertion follows since $O_2(Z(E)) \leq R$ before quotienting and a central quotient of a perfect group does not enlarge its centre.

Let θ be the character of $Z(E)$ belonging to b and let $K = \ker(\theta)$. Stability of b makes K normal in G, and the averaging idempotent of K acts as the identity on $a = b$. Thus quotienting

by K gives the corresponding block-algebra isomorphism, identifying P with its image. For each i, the kernel of the map $G_i \to G/K$ is $K \cap S_i$: it is odd and G_i/S_i is a 2-group. It is contained in the kernel of c_i, so the same quotient preserves the source algebra and inertia of every c_i.

The quotient also has trivial 2-core. Before taking it, $C_G(E)/Z(E)$ is a 2-group with central odd kernel $Z(E)$. Its unique Sylow 2-subgroup would be normal in G, so $C_G(E) = Z(E)$. Since E is perfect and $K \leq Z(E)$, an element centralizing E/K already centralizes E; hence

$$C_{G/K}(E/K) = Z(E)/K.$$

A normal 2-subgroup of G/K centralizes E/K, since $O_2(E/K) = 1$. The displayed centralizer has odd order, so that normal 2-subgroup is trivial. We retain the notation and assume from now on that

$$O_2(G) = 1, \qquad C_G(E) = Z(E), \qquad \theta \text{ is faithful.}$$

The P-invariance of the faithful character θ forces P to centralize $Z(E)$, and in particular every $Z(S_i)$.

For $G_i = PS_i$ put $R_i = O_2(G_i)$. The same commutator argument gives $R_i = C_P(S_i) \leq Z(G_i)$. Since $Z(S_i)$ has odd order, $R_i \cap S_i = 1$. Moreover, $C_{G_i}(S_i)/Z(S_i)$ is a 2-group with central odd kernel. Its unique Sylow 2-subgroup is normal in G_i, and is exactly R_i. Thus

$$C_{G_i}(S_i) = Z(S_i) \times R_i. \tag{3.5}$$

Put $\overline{G}_i = G_i/R_i$. The original S_i embeds elementwise in $\overline{G}_i$, its centralizer is $Z(S_i)$, and the block $\bar{c}_i$ dominated by c_i is inertial by Proposition 3.1. Its defect group is $P_i = P/R_i$, with $P_i \cap S_i = D_i$.

Let

$$A = \ker(\prod_i S_i \longrightarrow E).$$

This is an odd subgroup of $\prod_i Z(S_i)$. Because P centralizes each component centre, A is central in $\prod_i \overline{G}_i$. Form

$$\widehat{G} = (\prod_i \overline{G}_i)/A.$$

It contains the original E, and

$$\widehat{G}/E \cong \prod_i (\overline{G}_i/S_i)$$

is an abelian 2-group.

We now construct an embedding of the original G which is the identity on E. Let $\tau : G \to \prod_i \mathrm{Aut}(S_i)$ be its componentwise action, and let $\rho : \widehat{G} \to \prod_i \mathrm{Aut}(S_i)$ be the componentwise action of the constructed product. The image of ρ contains $\tau(G)$, its restriction to E is $\tau|_E$, and

$$\ker(\tau) = Z(E), \qquad \ker(\rho) = Z(E).$$

The latter kernel is central and has odd order. The actions are well defined on the central product because every factor fixes the shared centres pointwise.

Choose a Sylow 2-subgroup T of G containing P. The group $\rho^{-1}(\tau(T))$ is a central extension of a 2-group by the odd group $Z(E)$ and therefore has a unique Sylow 2-subgroup $\widehat{T}$. It contains the already embedded $T \cap E$. The map

$$\lambda_T = (\rho|_{\widehat{T}})^{-1}\tau|_T : T \longrightarrow \widehat{T}$$

is an isomorphism and is the identity on $T \cap E$. Also $G = ET$, and $\lambda_T(t)$ induces the original action of t on E. Hence

$$\Phi : G \longrightarrow \widehat{G}, \qquad et \longmapsto e\lambda_T(t)$$

is a well-defined homomorphism. If $\lambda_T(t) \in E$, then $\tau(t) \in \tau(E)$, so $t \in E\ker(\tau) = E$ and $\lambda_T(t) = t$. This proves that Φ is injective. In particular it is the identity on E. For $p \in P$, its

image is the class of the tuple $(pR_i)_i$: that tuple is a 2-element with action $\tau(p)$, and belongs to the unique Sylow 2-subgroup of $\rho^{-1}(\tau(\langle p\rangle))$.

It remains to keep the specified block and central character. On $Z(S_i)$ the block $\overline{c}_i$ has character $\theta_i = \theta|_{Z(S_i)}$. Thus the tensor block $\widetilde{a} = \bigotimes_i \overline{c}_i$ of $\prod_i \overline{G}_i$ has character

$$(z_i)_i \longmapsto \prod_i \theta_i(z_i) = \theta(\prod_i z_i),$$

which is trivial on A. It therefore descends to a block $\widehat{a}$ of $\widehat{G}$ via the central odd-kernel block-algebra isomorphism.

The tensor block is inertial. Explicitly, if the source of $\overline{c}_i$ is $\mathcal{D}_i \otimes_{\mathcal{O}} \mathcal{O}_{\alpha_i}(P_i \rtimes I_i)$, then its product source is

$$\Big(\bigotimes_i \mathcal{D}_i\Big) \otimes_{\mathcal{O}} \mathcal{O}_{\prod_i \alpha_i}\Big((\prod_i P_i) \rtimes (\prod_i I_i)\Big).$$

The product Dade algebra is primitive and stable under the product inertial quotient. The odd central quotient by A preserves this source with its defect-group identification, so $\widehat{a}$ is inertial. In fact $\overline{c}_i = b_i$ as idempotents in $\mathcal{O}\overline{G}_i$, because $c_i = b_i$ and the quotient by R_i embeds S_i identically. The product and the compatible central character therefore give $\widehat{a} = b$ as idempotents in $\mathcal{O}\widehat{G}$.

Finally, G is normal in $\widehat{G}$ because it contains E and $\widehat{G}/E$ is abelian. The block of G covering b is uniquely $a = b$. Hence $\widehat{a}$ covers the block a. Puig's normal-subblock theorem [22, Theorem 3.13] now proves that a is inertial. The preceding odd-kernel block isomorphism and Proposition 3.1 restore all the given groups and blocks. □

3.4. **The strong inertial extension theorem.** The following extension theorem applies when the indicated block of PH is inertial.

Theorem 3.5 (Zhang–Zhou, strong form)**.** *Let $H \triangleleft G$, let b be a G-stable block of $\mathcal{O}H$, and let P be a maximal 2-subgroup with nonzero relative Brauer support for b. Put $Q = P \cap H$. Suppose that P is abelian and that the unique block of PH covering b is inertial. Put $J = N_G(Q)$, $H_0 = N_H(Q)$, and let b_0 be the Brauer correspondent of b in H_0. There is a module for $(H \times H_0)\Delta J$ whose lower restriction is a basic Morita bimodule and whose induction to $G \times J$ gives a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Jb_0$. Every corresponding primitive summand is a basic Morita equivalence with the chosen defect identification.*

This is [30, Theorem 1.3]. We describe the graded source comparison and its consequence for the diagonal action. Put

$$A = \mathcal{O}Gb, \quad B = \mathcal{O}Hb, \quad A' = \mathcal{O}Jb_0, \quad B' = \mathcal{O}H_0b_0, \qquad \Gamma = G/H \cong J/H_0.$$

The last identification is the pointed Frattini equality. The algebras A, A' have their group-coset Γ-gradings, with identity components B, B'.

Inertiality of the PH-block supplies a determinant-one Dade P-algebra. The extension construction of [23], followed by the cancellation and comparison in [30, Sections 3–5], gives idempotents $i \in B^P$, $i' \in (B')^P$ and compatible graded interior-P isomorphisms, with V, W the $\mathcal{O}P$-lattices of the two Dade factors,

$$\begin{aligned} iAi &\cong \mathrm{End}_{\mathcal{O}}(V) \otimes_{\mathcal{O}} R, & iBi &\cong \mathrm{End}_{\mathcal{O}}(V) \otimes_{\mathcal{O}} R_1, \\ i'A'i' &\cong \mathrm{End}_{\mathcal{O}}(W) \otimes_{\mathcal{O}} R, & i'B'i' &\cong \mathrm{End}_{\mathcal{O}}(W) \otimes_{\mathcal{O}} R_1. \end{aligned} \tag{3.6}$$

The idempotents contain source idempotents for the lower blocks and are full in B, B'. The matrix factors have degree 1. The algebra R is the twisted group algebra of the local extension group, graded by the original quotient Γ; it contains a unit in every homogeneous component. Both occurrences use the same central extension and the same embedding of P. In [30, Section 3.9 and Lemma 5.15], the central extensions are identified over the identity of the local extension

group. Their splittings over the 2-subgroup agree by uniqueness: the central kernel $k^\times$ has no 2-torsion, and powering by a power of two is an automorphism of $k^\times$. Equivalently, choose a finite odd central realization of the cocycle and its unique 2-subgroup splitting. This supplies the stated interior identification.

The fixed-algebra condition used in this comparison follows from the upper inertia hypothesis. If E is its odd inertial quotient, then E acts trivially on P/Q. Choose n with $n|E| \equiv 1 \pmod{\exp(P)}$. For $p \in P$, the element

$$c = \left(\prod_{\alpha \in E} \alpha(p) \right)^n$$

belongs to $C_P(E)$ and satisfies $cQ = pQ$. Thus $P = QC_P(E)$. After the Dade cancellation, $R_1 = \mathcal{O}_\beta(Q \rtimes E)$. The chosen 2-subgroup splitting shows that $C_P(E)$ commutes with Q and with the lifted elements of E. Conjugation by $p = qc$ on R_1 is consequently conjugation by q, and is trivial on $(R_1)^Q$. This is the fixed-algebra condition in [30, Sections 5.9–5.15], under the hypothesis of its Theorem 1.3.

Lemma 3.6. *The common graded source corners (3.6) give a Morita bimodule D for A, A' whose identity component is a Morita bimodule for B, B'. This component is a module for $(H \times H_0)\Delta J$, and its induction to $G \times J$ is D.*

Proof. Put $S = \mathrm{End}_{\mathcal{O}}(V)$ and $S' = \mathrm{End}_{\mathcal{O}}(W)$ and form

$$F = Ai \otimes_{S \otimes R} (V \otimes R), \qquad F' = (R \otimes W^\vee) \otimes_{S' \otimes R} i'A', \qquad D = F \otimes_R F'.$$

Fullness of i, i' gives the full-idempotent Morita equivalences; the natural column and row modules give the two matrix-algebra Morita equivalences. Their composition is D. All tensor products have the displayed Γ-gradings.

Multiplication gives graded isomorphisms

$$F \cong F_1 \otimes_{R_1} R, \qquad F' \cong R \otimes_{R_1} F_1'.$$

Indeed, choose a unit $r_t \in R_t$. Its inverse image $u_t \in iA_t i$, under the corner isomorphism applied to $1 \otimes r_t$, is a unit of iAi. For $a \in A_t i$, $au_t^{-1} \in Bi$, which proves the first formula in degree t; the second follows by right multiplication at the other endpoint. The inverses use r_t^{-1}, so the formulas are independent of the chosen homogeneous units. Hence

$$D \cong F_1 \otimes_{R_1} R \otimes_{R_1} F_1', \qquad M := D_1 = F_1 \otimes_{R_1} F_1'.$$

The two identity-component corner isomorphisms in (3.6) show that F_1, F_1' are Morita bimodules. Thus M gives the lower Morita equivalence.

The existing $G \times J$ action on D is $(g, j)d = gdj^{-1}$. It sends degree t to $(gH)t(jH)^{-1}$. Its subgroup preserving degree 1 is

$$\{(g, j) : gH = jH\} = (H \times H_0)\Delta J.$$

Restricting this action to M gives all multiplication and intersection relations. For each $t \in \Gamma$, choose $g_t \in G$ above t. Then $D_t = g_t M$, since g_t is invertible on D. The map

$$\mathrm{Ind}_{(H \times H_0)\Delta J}^{G \times J} M \longrightarrow D, \qquad \sigma \otimes m \longmapsto \sigma m,$$

is an isomorphism: its coset summands map bijectively to the homogeneous components D_t. □

For a primitive correspondence with defect subgroup $P_a \leq P$, the interior-P comparison gives the capped summand of $\mathrm{Res}^P_{P_a}(V \otimes W^\vee)$ as its endopermutation source. In the application below, the selected primitive block has $P_a = P$. Reducing the primitive bimodule modulo the maximal ideal, applying Proposition 2.2, and then Lemma 2.3 gives the integral splendid equivalence used below.

4. Quasisimple blocks and the upper defect action

Let S be a quasisimple component of the ambient finite group G, let P be the abelian defect group of the specified block of SP, and let b be the block of S that it covers. Its defect group $D = P \cap S$ is assumed noncentral. Automorphisms are induced by the ambient stabilizers.

4.1. **The classification in the form used here.** The classification [11, Theorem 6.1 and Propositions 3.4, 5.3 and 5.4], together with its proof, gives the following cases. We first assign all nilpotent blocks, and all Lie-type cases in which $C^\circ_{\mathbf{X}^*}(s)$ is a torus, to (iv). Among the remaining cases, the type-A blocks in (ii) are precisely those of [11, Proposition 5.3(iii)], in both linear and unitary forms. The cases of [11, Proposition 5.3(v)] are assigned to (iii), including their Klein-defect overlaps.

(i) The principal blocks of $\mathrm{SL}_2(2^a)$, with $a \geq 2$, the Ree groups ${}^2G_2(3^{2a+1})$, with $a \geq 1$, and J_1, and the nonprincipal block of Co_3 with defect C_2^3.

(ii) The nonnilpotent Klein four blocks in the specified nontoral type-A cases, in alternating groups, and in the sporadic and central-cover cases described below. Their possibilities are retained from [9, Section 8] and the proof in [11]; the classified D_{2t}/E_7 Levi cases are assigned to (iii).

(iii) The rank-one Levi cases in split type D_{2t}, with $t \geq 3$ odd, and in type E_7, with $t = 3$. Here q is an odd prime power with $q \equiv \pm 3 \pmod 8$. The full centralizer of the semisimple odd-order label is the classification Levi; its derived root system is a single Frobenius orbit of t factors of type A_1. After the prescribed central quotient, its finite group has the form $\mathrm{PSL}_2(q^t) \times A_0$ with A_0 abelian.

(iv) The nilpotent blocks and the cases in which the connected component $C^\circ_{\mathbf{X}^*}(s)$ of the dual centralizer is a torus. The latter have nilpotent regular-embedding covers by [11, Lemma 4.2].

The central-defect components will be removed before this list is applied. The classification branches enter the list as follows. In defining characteristic two, Proposition 8.7 and Lemma 8.8 of [9], together with the proof of [11, Theorem 6.1], leave the SL_2 family and the listed small exceptions after the central-defect cases are removed. For alternating and sporadic groups, [9, Section 8] gives (i) and (ii). In odd characteristic, the toral case of [11, Lemma 4.2] belongs to (iv). In [11, Proposition 5.3], cases (i) and (iv) give nilpotent blocks, as observed in the proof of [11, Theorem 6.1], and thus belong to (iv) here; case (iii) gives the type A quaternion preimage treated in (ii); case (v) gives precisely the D_{2t} and E_7 families in (iii). The remaining quasi-isolated E_7 case with dihedral defect D_8 before a central involution quotient belongs to (iv). An element normalizing the quotient defect lifts to an element normalizing its full preimage D_8. Its action on V_4 therefore lies in the image of $\mathrm{Aut}(D_8)$, a 2-group. The odd inertial quotient is trivial, and abelian defect then makes the quotient block nilpotent. These are all the alternatives in the proof of [11, Theorem 6.1].

The central-cover cases are as follows. In odd characteristic, the proof of [11, Theorem 6.1] excludes faithful blocks with noncentral abelian defect on exceptional covers. A nonnilpotent block with toral connected dual centralizer can have disconnected full centralizer only in type A or E_6, by [11, Proposition 5.4]. In odd characteristic the exceptional multipliers in type A occur for $\mathrm{PSL}_2(9)$ and $\mathrm{PSU}_4(3)$ [9, Sections 9–10]. Their standard simply connected centers are 2-groups. An odd-order semisimple label therefore has connected centralizer: lift it with odd order through the central 2-isogeny; a conjugate differing by a central 2-element must be equal, and the centralizer upstairs is connected. A toral case for these groups is consequently nilpotent by [11, Lemma 4.2]. The odd-characteristic E_6 and ${}^2E_6$ cases have their standard centers of orders $\gcd(3, q-1)$ and $\gcd(3, q+1)$, respectively [9, Lemmas 12.5 and 12.7]. A

nonfaithful odd central cover has the same block algebra after quotienting its block kernel. A central 2-quotient preserves the inertial quotient for abelian defect, as proved in Proposition 3.1; it therefore preserves nilpotence in this situation. Thus each nonnilpotent case in (iv) has the nonexceptional simply connected realization used in Lemma 3.3.

In defining characteristic two, Proposition 8.7 of [9] gives full defect after the central reductions. The abelian Sylow case is the SL_2 family. Its exceptional double cover at $\mathrm{SL}_2(4)$ has quaternion Sylow preimage and gives no further abelian-defect case. The Tits group has defect-zero blocks and a principal block of nonabelian defect; the remaining small groups are $\mathrm{PSp}_4(2)' \cong A_6$ and $G_2(2)' \cong \mathrm{PSU}_3(3)$, covered by the alternating and odd-characteristic unitary cases [9, Lemma 8.8]. The exceptional odd covers in the alternating and sporadic Klein cases are treated below.

Cases (i)–(iii) provide the component complexes used in the assembly. In case (iv), Lemma 3.3 proves that the block of SP is inertial.

4.2. **An inner-block test for upper 2-elements.** For an algebra A, $\mathrm{Pic}(A)$ is the group of isomorphism classes of invertible A-A-bimodules under tensor product. For an interior D-algebra A, $\mathrm{Out}_D(A)$ is the quotient of the algebra automorphisms fixing the specified image of D pointwise by conjugation with units in $(A^D)^\times$.

Lemma 4.1. *Let a finite 2-group T act on L by automorphisms. Suppose that it stabilizes a block of L, centralizes its defect group D, and preserves a supported maximal Brauer pair. Then its image in the Picard group of that block is trivial. In particular it fixes every ordinary irreducible character of the block.*

Proof. The twist of the regular block bimodule by an element of T is an invertible trivial-source bimodule with diagonal vertex ΔD. Its fusion automorphism is the identity. By [3, Theorem 1.1(ii) and Remark 1.2], its Picard class belongs to Out_D of a source algebra, which embeds in $\mathrm{Hom}(E, k^\times)$ for the inertial quotient E. This is an odd group. The homomorphism from T is therefore trivial. An inner algebra automorphism fixes all ordinary characters. □

The stable root required here is available from the given upper defect pair. Indeed $\mathrm{Br}_T(b) \neq 0$. In $\mathrm{Br}_D(b)$ a nontrivial T-orbit of primitive root idempotents is a proper relative trace and has zero T-Brauer image. Choose a fixed summand supported by the original pair. This gives the required supported maximal pair.

4.3. **Klein four blocks and upper 2-actions.**

Proposition 4.2. *In case (ii), the original abelian upper defect group P acts on S by inner automorphisms. Moreover $O_2(Z(S)) = 1$.*

Proof. The inertial quotient of a nonnilpotent Klein block is C_3, acting transitively on $D \setminus \{1\}$. A central involution would be fixed by this action, so $O_2(Z(S)) = 1$.

First consider the type-A cases of [11, Proposition 5.3(iii)]. Here q is an odd prime power, $\varepsilon \in \{1, -1\}$, and $\mathrm{GL}_n^+(q) = \mathrm{GL}_n(q)$, $\mathrm{GL}_n^-(q) = \mathrm{GU}_n(q)$, $\mathrm{SL}_n^+(q) = \mathrm{SL}_n(q)$, $\mathrm{SL}_n^-(q) = \mathrm{SU}_n(q)$. The natural dimension is $2t$, with t odd, and $q \equiv \pm 3 \pmod 8$. An odd-order lift of the semisimple label in $\mathrm{GL}_{2t}^\varepsilon(q)$ has t distinct eigenvalues, each of multiplicity two, forming one orbit of $f : \lambda \mapsto \lambda^{\varepsilon q}$. If an inner-diagonal graph automorphism stabilizes the label, then

$$j : \lambda \longmapsto c\lambda^{-1}$$

permutes these eigenvalues for an f-fixed scalar c. It is an involution commuting with the t-cycle f. The centralizer of that cycle is cyclic of odd order, so j is the identity. Every eigenvalue then has square c. Squaring is injective on their odd-order group, so $t = 1$. For $t > 1$ the graph component is excluded. For $t = 1$ the group is rank one and has the same conclusion below.

This also treats the graph component in the unitary outer group. Since $q \equiv 3$ or $5 \pmod 8$, the exponent of its defining prime is odd; there is no remaining field 2-part.

It remains to exclude the diagonal 2-part. The inverse image of D in $\mathrm{SL}^{\varepsilon}_{2t}(q)$ has Sylow 2-subgroup Q_8. Its natural module is the sum of t copies of the faithful two-dimensional quaternion module. If a general linear or unitary element induces an automorphism centralizing D, it normalizes this Q_8: the central 2-part consists of $\{\pm 1\}$, and an odd scalar cannot occur in the conjugate of an element of order at most four. The kernel of $\mathrm{Aut}(Q_8) \to \mathrm{Aut}(Q_8/Z(Q_8))$ is inner. After multiplication by an element of Q_8, our element centralizes Q_8. Its centralizer on the natural module is $\mathrm{GL}_t(q)$ or $\mathrm{GU}_t(q)$ on the multiplicity space, and

$$\det_{2t}(u) = \det_{t}(u)^2. \tag{4.1}$$

The diagonal outer quotient has 2-part of order two. A square determinant has trivial image there. Hence the upper 2-action is inner. The rank-one cases are included in this calculation.

For alternating groups, [9, Lemma 8.5] makes the Klein block S_n-stable. Its unique cover has defect of order eight and weight two, hence has five ordinary characters. If the four characters of the lower block comprise a fixed characters and b_0 exchanged pairs, Clifford theory gives

$$a + 2b_0 = 4, \qquad 2a + b_0 = 5.$$

Thus $b_0 = 1$: the outer involution moves a character and is excluded by Lemma 4.1. The exceptional odd covers give no additional faithful Klein case: A_6 and the faithful blocks of $3.A_6$ have no Klein defect, and the faithful blocks of $3.A_7$ have defect groups of order eight or one. These are the cover cases in the classification and its modular tables [9, 5].

The sporadic Klein four cases in (ii) are $M_{12}, J_2, HS, \mathrm{Fi}'_{24}$ [9, Section 8]. In the first three, the outer involution exchanges, respectively, the following two character rows in the Klein block:

S	rows	common degree
M_{12}	χ_4, χ_5	16
J_2	χ_{16}, χ_{17}	224
HS	χ_{14}, χ_{15}	896.

The upper character tables identify their induced characters of twice those degrees using the conjugate-pair notation in [5]. Apply Lemma 4.1.

For $S = \mathrm{Fi}'_{24}$, choose $u \in D \setminus \{1\}$. The classification gives $C = C_S(u) \cong 2.\mathrm{Fi}_{22}.2$ with a unique block c of defect D [9, Lemma 8.3]. Put $H_0 \cong 2.\mathrm{Fi}_{22} \lhd C$ and $z = u$. The block c contains two extensions each of the characters χ_{56} and χ_{104} of H_0, all of degree 1441792; the latter pair is faithful on z [5]. The full automorphism centralizer is

$$C_{\mathrm{Aut}(S)}(u) \cong (2 \times 2.\mathrm{Fi}_{22}) : 2,$$

with a normal $V_4 = \langle a, z \rangle$, where a, az are outside S [6, Section 2.4-10]. The perfect subgroup H_0 centralizes V_4, whereas $v \in C \setminus H_0$ satisfies $a^{-1}va = zv$. Conjugation by a therefore exchanges the two extensions of χ_{104}: on the v-coset it changes their representations by the sign of the faithful central involution. Every element of the outer coset has this same character action. The original P centralizes u and stabilizes the supported block c with defect D, so Lemma 4.1, now applied to C, excludes that outer coset.

Faithful $3.\mathrm{Fi}'_{24}$ blocks contribute no Klein case [9, Lemma 8.3]. Nonfaithful blocks of odd central covers are handled in their chosen character sector. Finally an automorphism of a perfect central cover inducing an inner automorphism of its simple quotient is itself inner: after correcting by the inner automorphism, the central correction is a homomorphism from a perfect group to its center and vanishes. □

Lemma 4.3. *Suppose that $O_2(Z(S)) = 1$ and that the original abelian P acts on S by inner automorphisms. Then, inside the given group,*

$$SP = S \times R, \qquad P = D \times R, \qquad R = P \cap C_G(S). \tag{4.2}$$

Proof. The central odd extension of the abelian 2-image in $\mathrm{Inn}(S)$ has a unique Sylow 2-subgroup. Hence the inner action has a unique homomorphic lift $\delta : P \to S$ with 2-group image and $\delta|_D = 1_D$. Set $r(p) = \delta(p)^{-1}p$. This is a homomorphism into $C_{SP}(S)$, with kernel D. Its image R_0 meets S trivially, and $SP = S \times R_0$.

The specified upper block is the tensor of the lower block with $\mathcal{O}R_0$, so its defect groups have form $D' \times R_0$, where D' is an S-conjugate of D. Apply this to P and use $P \cap S = D$. Then $P = D \times R_0$, giving $R_0 = R$. Uniqueness of δ makes this construction natural under further marked operators. □

5. Klein four and elementary abelian component complexes

5.1. **The relative operator group.** Let $N = N_S(D)$, let c be the original Brauer correspondent of b, and let $\mathcal{A} \leq \mathrm{Aut}(S)_b$ contain $\mathrm{Inn}(S)$. Suppose that $\mathcal{A}/\mathrm{Inn}(S)$ is odd and $O_2(Z(S)) = 1$. Write $\mathcal{A}_D = N_{\mathcal{A}}(D)$ and $\delta(n) = (n, n)$, and let $c_n(x) = nxn^{-1}$ denote conjugation by n. The finite relative group is

$$\mathcal{T} = ((S \times N) \rtimes \mathcal{A}_D)/\{(\delta(n)^{-1}, c_n) : n \in N\}. \tag{5.1}$$

The denominator centralizes the base and is normal. Its intersection with $S \times N$ is $\Delta Z(S)$; the resulting base is $(S \times N)/\Delta Z(S)$ and its quotient is $\mathcal{A}/\mathrm{Inn}(S)$, by the marked Frattini equality. The corner $b\mathcal{O}Sc$ is a module for (5.1), since c_n acts there by the inner diagonal action. The original central character is the same at both ends. An ambient stabilizer acts by pullback; its centralizer kernel acts as the identity on this factor, while remaining in the ambient group.

Proposition 5.1. *For a Klein four defect block under the preceding relative-operator hypotheses, the block and its Brauer correspondent have an integral splendid Rickard complex carrying the action (5.1). Its inner operators are the literal diagonal operators. In the situation of Lemma 4.3, it has an upper complex*

$$C^+ = C \otimes_{\mathcal{O}} \mathcal{O}R \tag{5.2}$$

for the given groups SP and NP, with the same lower identity component and the same P-action.

Proof. Let M be the unique marked full-ΔD-vertex summand of $b\mathcal{O}Sc$. Green correspondence gives this summand with multiplicity one; all complementary summands have proper vertices. Any competing vertex of full order in the restriction of the regular block has right projection contained in the normal defect group D of the right block. That projection has order $|D|$, so it equals D; its right transporter already belongs to N. Thus this vertex is conjugate to the marked ΔD inside $S \times N$, and Green uniqueness excludes a second such summand. It is invariant under the relative group. In the endomorphism algebra of the corner, its isotype projector modulo the radical is invariant. Odd averaging lifts the projector to an invariant element, and iteration of $t \mapsto 3t^2 - 2t^3$ gives an invariant idempotent. Thus the action on M is extracted from the original corner.

The integral source algebra classification [9] identifies the source algebra of the left block with $\mathcal{O}D$, $\mathcal{O}A_4$, or the principal $\mathcal{O}A_5$ block. These are interior-D identifications. The normal-defect right block has source $\mathcal{O}D$ or $\mathcal{O}A_4$: normal D acts trivially on every simple right module, excluding the nontrivial two-dimensional simple sources in the A_5 case. The induced Puig tensor functors preserve projective covers, indecomposability, permutation summands, and the marked full vertex. For the last assertion, a proper vertex after transport would remain proper after transport back, contradicting the original full vertex.

To identify the underlying bimodule M, use the source models. For the A_5–A_4 and A_4–A_4 models,

$$N_{A_5\times A_4}(\Delta D)/\Delta D \cong A_4, \qquad N_{A_4\times A_4}(\Delta D)/\Delta D \cong A_4.$$

The Brauer image of the standard Green module is $kD \cong \mathrm{Ind}_{C_3}^{A_4} k$, the indecomposable projective cover of the trivial character. Its three inertial-character twists exhaust the projective indecomposables of kA_4. The permutation-module Brauer correspondence therefore shows that the transported M is exactly one of these three twists. In the D–D case there is one possibility. Consequently M is already a Morita equivalence in the D and A_4 cases; in the A_5 case it is the standard Green stable equivalence with its specified twist.

In that last case $\mathrm{Pic}(\mathcal{O}A_5b_0) \cong C_2$, where b_0 is the principal block idempotent of $\mathcal{O}A_5$ [3, Theorem 1.5], so the odd operator quotient fixes the three left simple classes. The Green equivalence intrinsically identifies the distinguished right simple whose image is simple. It is fixed, and an odd group cannot exchange the other two. In the A_5–A_4 model, label the two nontrivial left simples U_1, U_2 over k. Each $(k \otimes_{\mathcal{O}} M)^\vee \otimes_{kA_5b_0} U_i$ is an indecomposable module of dimension two for the right block; let V_i be its unique simple submodule. The standard correction of [26, Sections 5.1 and 6.3.5] uses the two projective bimodules

$$P(U_1) \otimes_{\mathcal{O}} P(V_1)^\vee, \qquad P(U_2) \otimes_{\mathcal{O}} P(V_2)^\vee,$$

where $P(U)$ denotes the projective $\mathcal{O}$-lattice lifting the projective cover of the simple k-module U. The retained inertial twist transports these pairings together. Thus the two required isotypes of the projective cover of M are fixed by the odd operator quotient.

Take a projective cover $q : P_M \twoheadrightarrow M$ in the actual relative group. Its restriction is the lower projective cover. Here the normal base is $K_0 = (S \times N)/\Delta Z(S)$ and $\Lambda = \mathcal{T}$. For a normal subgroup $K_0 \triangleleft \Lambda$ of odd index,

$$J(\mathcal{O}\Lambda) = J(\mathcal{O}K_0)\mathcal{O}\Lambda : \tag{5.3}$$

the right side is radical modulo the uniformizer, and its quotient is a crossed product of a semisimple algebra by a group of invertible order, hence semisimple by averaging splittings. The two selected isotypes give an invariant idempotent modulo the radical of $\mathrm{End}_{\mathcal{O}K_0}(P_M)$. Lift an invariant element e_0 by odd averaging, and iterate

$$e_{n+1} = 3e_n^2 - 2e_n^3.$$

The error satisfies

$$e_{n+1}^2 - e_{n+1} = -(e_n^2 - e_n)^2(3 - 2e_n)(1 + 2e_n).$$

Completeness gives an invariant idempotent e. The required complex is

$$C = [eP_M \xrightarrow{q|_{eP_M}} M], \qquad \deg(eP_M) = -1, \quad \deg M = 0. \tag{5.4}$$

Under the Puig recognition this is the standard correction with the retained inertial twist. Projective-cover uniqueness and conjugacy of idempotent lifts preserve its underlying Rickard property. Its terms are permutation summands for the actual relative group: the correction term is projective there, and the Green term is a summand of the permutation corner $b\mathcal{O}Sc$. Alternatively, the odd base index places every 2-subgroup in the base, where the restriction is 2-permutation. The evaluations are equivariant, and their lower contractions become equivariant after averaging over the odd quotient. This proves both evaluations for (5.4).

Finally (4.2) makes (5.2) an upper splendid complex. Its identity component is C, with the fixed diagonal P-action. The regular factor retains the whole centralizer kernel and its group multiplication. □

Lemma 5.2. *The full maximal companion of the complex constructed in Proposition 5.1 is the regular right block in degree zero, with its operator action.*

Proof. If $D \triangleleft N$ and c is a central block idempotent of kN, then $\mathrm{Br}_D(c) = c$. Indeed the quotient map to $k(N/D)$ kills each nontrivial D-orbit sum. Thus c and $\mathrm{Br}_D(c)$ have the same image modulo the nilpotent ideal $J(kD)kN$; both are central idempotents and are equal. The maximal roots of $\mathrm{Br}_D(b)$ form the same full N-orbit as those of c. Consequently

$$\mathrm{Br}_D(b) = c, \qquad M(\Delta D) = kC_S(D)c.$$

The proper-vertex complementary terms and the projective correction vanish at ΔD. Moreover

$$N_{S\times N}(\Delta D) = (C_S(D) \times C_S(D))\Delta N.$$

The full companion is identified with kNc by the multiplication map $[(x, y), m] \mapsto xmy^{-1}$. It is a bijection on the original coset decomposition and commutes with all original operators. Uniqueness of the integral lift of a permutation summand in the full finite action group gives $\mathcal{O}Nc$ with this action. □

5.2. **The other elementary cases.**

Proposition 5.3. *For a relative operator group with odd image modulo* $\mathrm{Inn}(S)$*, case (i) has the same component data as Proposition 5.1: a full relative-operator splendid complex and an upper complex on* P *with the same lower identity component.*

Proof. For $S = \mathrm{SL}_2(2^a)$, the construction in [10, Sections 4.2.6 and 6.3] is in the full relative automorphism module category. We restrict it to the stipulated odd outer image. In the global application, oddness of the entire relevant image follows from the full relative-defect equality proved in Section 8. To see the exclusion of an upper outer 2-action, write the Sylow unipotents as $u(x)$, $x \in \mathbb{F}_{2^a}$. Let F square matrix entries. An element gF^r normalizing this Sylow subgroup has g in its Borel normalizer; write t, t^{-1} for its diagonal entries. If it centralizes every $u(x)$, its action on parameters is $x \mapsto t^2x^{2^r}$. Evaluating at 1 gives $t = 1$, and evaluating at all x gives $r = 0$ modulo a. The remaining representative is inner. Lemma 4.3 supplies the upper regular factor. Restriction to an odd outer quotient preserves the permutation property and allows averaging of both contractions. Apply Lemma 2.3 if the initial representative is modular.

For Ree groups, the lower integral splendid kernel is the one recorded in [18, Theorem 1.6] and [10, Section 6.2]. The Picard computation [14, Proposition 5.4] gives $\mathrm{Pic}(\mathcal{O}Sb) \cong C_3$; its nonidentity elements cycle three unit rows of the decomposition matrix, including the ordinary trivial character. Every actual group automorphism fixes that character, so its Picard class is trivial. The outer group is odd. The following construction gives its action on the chosen kernel.

Work first over k. A block-inner automorphism gives an isomorphism between the regular block bimodule and its twist. Their full vertices are the diagonal of D and the graph of that automorphism on D. Conjugacy of these vertices shows that the latter action belongs to the normalizer fusion. Put $\Omega = \mathcal{A}/\mathrm{Inn}(S)$ and choose representatives $a_g \in \mathcal{A}_D$, $g \in \Omega$. Correct them by inner normalizer elements so that they fix D pointwise. Since S is centerless, there are unique $n(g, h) \in C_S(D)$ satisfying

$$a_g a_h = c_{n(g,h)} a_{gh}.$$

Choose units $u_g \in (kSb)^D$ implementing a_g. Then

$$u_g u_h = \alpha(g, h)n(g, h)u_{gh}, \qquad \alpha(g, h) \in Z(kSb)^\times.$$

The center is a finite-dimensional local commutative k-algebra, and

$$Z(kSb)^\times = k^\times \times (1 + J(Z(kSb))).$$

If $J(Z(kSb))^m = 0$ and $2^r \geq m$, then $(1 + x)^{2^r} = 1$ for every x in this radical. The principal-unit group therefore has bounded 2-power exponent. The carry relation for the representatives, and the fact that u_g implements their action, give the cocycle identity for α. Inner automorphisms fix

the center. Multiplication by the odd order of the outer group is invertible on its principal-unit subgroup, so its positive-degree cohomology vanishes. Rescale the u_g by central principal units to obtain $\alpha(g,h) \in k^\times$. Set $v_g = \mathrm{Br}_D(u_g)$ in the right block. These are units, because $\mathrm{Br}_D(b) = c$ and the Brauer images of the inverses are their inverses. They implement the automorphism on the full right normalizer block: for $n \in N_S(D)$, the element $a_g(n)n^{-1}$ centralizes D, and applying the Brauer map to $u_g({}^n u_g^{-1}) = a_g(n)n^{-1}b$ gives the corresponding identity for v_g. They therefore satisfy the same product equation with the same scalar $\alpha(g,h)$. Hence

$$\tau_g(x) = u_g x v_g^{-1}, \qquad \tau_g\tau_h = \delta_{n(g,h)}\tau_{gh}. \tag{5.5}$$

Here $\delta_n(x) = nxn^{-1}$ is the diagonal bimodule action. On the dual use $y \mapsto v_g y u_g^{-1}$. The middle factors cancel in both balanced products. Every bimodule differential, evaluation, and contraction commutes with this action. It therefore gives the finite relative action with its inner intersections. Every 2-subgroup of the relative group is conjugate into the base, since the quotient is odd. Restriction to the base is 2-permutation, so the terms are 2-permutation for the whole relative group. The two contractions are equivariant by the displayed unit action. Lift the resulting complex integrally using Lemma 2.3.

For J_1, use the explicit splendid construction in [10, Section 6.2.3]. For the nonprincipal C_2^3-block of Co_3, [18, Theorems 1.3, 1.5–1.6 and Section 7] uses the original marked Green module for $(\mathrm{SL}_2(8) \rtimes C_3) \times S_3$. Its Puig equivalence, defect-zero factor and Brauer-normalizer identification are integral and retain the original defect group. Both sporadic groups have trivial outer automorphism group. Their relative action is just the prescribed inner action, pulled back with the ambient centralizer acting trivially.

The Ree groups and these sporadic groups are centerless, and the Ree outer group is odd. Thus the given upper P has inner image in every case. Lemma 4.3 gives the upper complex by the same regular tensor factor. Proposition 2.4 supplies its local return. □

6. The rank-one Levi components

We treat case (iii). We construct a complex for S and $N_S(D)$ with the marked component stabilizer action, as the identity component of a splendid complex for SP and $N_S(D)P$. The operator groups are the marked stabilizers in the reduced ambient situation of Section 8. In particular, the image of P is Sylow at the 2-part of their quotient by the lower inner operators. This condition will be used for the rank-one factor, whose outer operator image must be odd. Let $\mathcal{A}$ be the automorphism image on S of the group generated by S and the ambient marked component stabilizer, and put $\mathcal{A}_D = N_{\mathcal{A}}(D)$.

6.1. **The full-centralizer Levi and field operators.** Let $\mathbf{G}$ be the simply connected algebraic group giving the component before its central quotient, with Frobenius F. Write $S_{\mathrm{sc}} = \mathbf{G}^F$ and $S = S_{\mathrm{sc}}/Z_0$, where Z_0 is central; $Z_0 = 1$ is allowed. A regular embedding is an inclusion $\mathbf{G} \le \widetilde{\mathbf{G}}$ with connected center and $\widetilde{\mathbf{G}} = \mathbf{G}Z(\widetilde{\mathbf{G}})$. The proof of [11, Proposition 5.3(v)] gives, in the present D_{2t} and E_7 branches,

$$C_{\mathbf{G}^*}(s) = C^\circ_{\mathbf{G}^*}(s) = \mathbf{L}^* \tag{6.1}$$

for the chosen odd-order semisimple label s and the classification Levi.

Here is also a direct reason for connectedness in this setting. The simply connected covering of the dual group has a central 2-kernel. An odd-order s has an odd-order lift $\widehat{s}$. If $\widehat{g}\widehat{s}\widehat{g}^{-1} = z\widehat{s}$ for a central 2-element z, taking the odd order of $\widehat{s}$ gives $z = 1$. The connected centralizer upstairs therefore maps onto the full centralizer downstairs.

For type D_{2t}, the two possible A_1^t root systems can be written as paired difference roots, or as the same pairs with one pair replaced by a sum root. The parity of the number of sum pairs is

invariant under $W(D_{2t})$, whose signed permutations change an even number of coordinate signs. The nontrivial graph automorphism exchanges these two classes. An automorphism stabilizing the block stabilizes its rational-series idempotent and hence the full-centralizer Levi class. Thus its graph component is trivial. There is no triality case, since $t \geq 3$ is odd. Type E_7 has no nontrivial diagram automorphism. Also $q \equiv \pm 3 \pmod 8$ forces its field exponent to be odd. The given upper 2-group consequently acts by inner-diagonal automorphisms.

We choose the Levi compatibly with the field part. Let F_0 be the pure field root representing the relevant cyclic field subgroup, so $F = F_0^r$ with r odd. Its geometric semisimple class is stable. Lang's theorem gives an F_0-fixed representative of this class. Connectedness of its centralizer implies that the geometric class contains a single rational F-class, so this representative is $(\mathbf{G}^*)^F$-conjugate to s. Transport the label, the dual Levi, and all block and defect markings by that rational conjugation, retaining the notation $s, \mathbf{L}^*$. Choose the dual primal Levi $\mathbf{L}$ and put $L_{\mathrm{sc}} = \mathbf{L}^F$, $L = L_{\mathrm{sc}}/Z_0$.

Choose a parabolic radical $\mathbf{U}$ with Levi $\mathbf{L}$, and put

$$\mathbb{G} = \mathbf{G}^r, \quad \mathbb{L} = \mathbf{L}^r, \quad \mathbb{U} = \mathbf{U}^r, \quad \mathbb{F} = \tau(F_0, \dots, F_0), \tag{6.2}$$

where τ is cyclic permutation. First-coordinate projection identifies $\mathbb{G}^{\mathbb{F}}$ with $\mathbf{G}^F$; its inverse is $\iota(g) = (g, F_0^{r-1}g, \dots, F_0 g)$. The finite-order operator τ is F_0^{-1} on these finite points and preserves $\mathbb{U}$ literally. The dual tuple uses the reverse cycle. The rational-series comparison in [28, Section 5] retains the chosen label. The Deligne–Lusztig variety is

$$Y_{\mathbb{U}}^{\mathbb{G}} = \{g\mathbb{U} \in \mathbb{G}/\mathbb{U} : g^{-1}\mathbb{F}(g) \in \mathbb{U}\mathbb{F}(\mathbb{U})\}.$$

Its left and right actions are multiplication by $\mathbb{G}^{\mathbb{F}}$ and $\mathbb{L}^{\mathbb{F}}$. The same definition is used for the disconnected upper group [2, Section 2.E].

We use the point-stabilizer form of Rickard's cochain construction. Let Λ be the whole finite group acting on the chosen variety: it includes the left and right finite groups and all the stipulated regular-embedding and finite field operators. If a primitive block cut is not Λ-invariant, first restrict to its inertia group; full block orbits are recovered by finite induction. For a Λ-variety Z put

$$\mathscr{P}_\Lambda(Z) = \mathrm{add}\{\mathcal{O}[\Lambda/\Lambda_z] : z \in Z(\overline{\mathbb{F}}_q)\}.$$

Here add denotes finite direct sums and direct summands in the category of $\mathcal{O}\Lambda$-lattices. We write $\Omega_c(Z, \Lambda, \mathcal{O})$ for the resulting compact-support cochain object in that category; the group parameter is omitted when its action has been fixed. There are finitely many subgroup types in the displayed expression. Choose the bounded cochain complex in $K^b(\mathscr{P}_\Lambda(Z))$ supplied by [24, Theorems 3.2 and 3.5]; the integral point-stabilizer form is also explicit in [27, Section 2.5.1]. Apply the prescribed block projectors to this complex. This specifies the terms, differential, and finite operator action at the initial choice.

The integral lower equivalence is supplied by [2, Theorem 7.7], equivalently [28, Theorem 2.28] in the full-centralizer case. Our specified complex represents the same geometric object in the homotopy category of the base double group. Thus its canonical evaluations are homotopy equivalences there as well, by naturality of duality and evaluation under a homotopy equivalence. Denote the chosen lower block complex by C. For a central quotient, this choice is made on the quotient variety below.

Two consequences will be used. For a subgroup $B \leq \Lambda$, Mackey restriction of $\mathcal{O}[\Lambda/\Lambda_z]$ is a sum of $\mathcal{O}[B/B_{\lambda z}]$. Thus restriction to a smaller action group still has a point-stabilizer term bound. The same statement holds for pullback through a homomorphism of finite action groups. Also, every operator acting pointwise as the identity on Z acts as the identity on every term of this representative: it belongs to every point stabilizer. In particular, the diagonal central kernels and the inner-overlap relations hold on the chosen terms.

6.2. **Finite induction of the chosen cochain model.**

Lemma 6.1. *Let $T \leq \Lambda$ be finite groups, and let Y be a quasi-projective variety with T-action. With coefficients of characteristic different from the defining characteristic,*

$$\Omega_c(\Lambda \times_T Y, \Lambda, \mathcal{O}) \simeq \mathrm{Ind}_T^{\Lambda} \Omega_c(Y, T, \mathcal{O}) \tag{6.3}$$

in the bounded 2-permutation homotopy category. The comparison is natural for additional finite operators preserving the data.

Proof. The induced variety is a finite disjoint union of open-and-closed copies of Y. Godement stalk products, and subsequently compact-support sections, decompose into the corresponding finite sums. The action of Λ permutes them with the original coset carries. This gives induction at the cochain-construction level. The bounded permutation replacement is homotopy equivalent to this object [27, Section 2.5.1].

Equivalently, put $B = Y/T = (\Lambda \times_T Y)/\Lambda$. The finite quotient pushforwards satisfy the constructible-sheaf identity

$$(\pi_{\Lambda \times_T Y})_* \mathcal{O} \cong \mathcal{O}\Lambda \otimes_{\mathcal{O}T} (\pi_Y)_* \mathcal{O}.$$

On each geometric stalk this is the permutation-module identity for the induced fiber, compatible with restriction maps. Rickard's additive-functor compatibility [24, Lemma 2.8 and Theorem 3.5] gives (6.3), including the integral limit. In particular, restrict a representative chosen first for the whole lower finite action, and then induce that representative. It realizes the upper object in the homotopy category. □

6.3. **The disconnected upper group.** First suppose that the component is not centrally quotiented. Let $D = S \cap P$. Use the algebraic inner action of P in the restriction-of-scalars group and form

$$\widehat{\mathbb{G}} = (\mathbb{G} \rtimes P)/\{(\iota(d)^{-1}, d) : d \in D\}. \tag{6.4}$$

The graph subgroup centralizes $\mathbb{G}$ and is normalized by P, so it is normal. The Frobenius acts as $\mathbb{F}$ on $\mathbb{G}$ and trivially on the finite factor. If a class of (g, p) is fixed, then

$$(\mathbb{F}(g), p) = (g, p)(\iota(d)^{-1}, d)$$

for some $d \in D$. Its P coordinate forces $d = 1$. Thus the finite fixed-point group is exactly $(S \rtimes P)/\Delta(D^{-1}, D) \cong SP$, with the original multiplication and powers.

Let $\mathbf{Z}_1 = C_{\widehat{\mathbb{G}}}(\mathbb{G})$. Algebraic innerness gives $\widehat{\mathbb{G}} = \mathbb{G}\mathbf{Z}_1$. Put $\widehat{\mathbb{L}} = \mathbb{L}\mathbf{Z}_1$ and extend the parabolic by $\mathbf{Z}_1$. Define the upper finite Levi by $L^+ = \widehat{\mathbb{L}}^{\mathbb{F}}$. Lang's theorem in the connected Levi implies that L^+ meets every component, so $SP = SL^+$. Each component preserves the chosen rational series. For example, if an algebraically inner representative g commutes with F, then $g^{-1}F(g)$ is central. In a rational torus solve $a^{-1}F(a) = g^{-1}F(g)$; the rational element ga^{-1} transports the same torus-character pair as g. The same argument applies inside the Levi. This gives the series-inertia equality required in [28, Corollary 3.8].

The full-centralizer hypothesis gives almost-superregularity. Propositions 3.6–3.7 and Corollary 3.8 of [28] therefore give the upper splendid complex and its full bimodule-homotopy endomorphism splitting. Those statements allow disconnected groups with even component group. The upper splendid correspondence puts a conjugate of the marked P inside L^+. Since $SP = SL^+$, a conjugation by an element of S suffices. Conjugate the Levi, radical, field operators and all markings together. We now have $L^+ = LP$ and the original P preserves the same lower variety.

6.4. **The central quotient and its coarse identity component.** For a component $S = S_{\mathrm{sc}}/Z_0$, let $\alpha : SP \to \mathrm{Aut}(S)$ be conjugation and let $\rho : \widetilde{\mathbf{G}}^F \to \mathrm{Aut}(S)$ be the regular-embedding action. Form the finite fiber product

$$\{(u, g) \in SP \times \widetilde{\mathbf{G}}^F : \alpha(u) = \rho(g)\}.$$

The kernel is central: commutators of an element inducing the identity on S with S_{sc} lie in Z_0 and define a homomorphism from a perfect group to a central group, so vanish. Choose a Sylow 2-subgroup T_0 of the inverse image of P in this fiber product, and let

$$E_0 = S_{\mathrm{sc}} T_0 \twoheadrightarrow SP$$

be the resulting finite extension, with central kernel A and $A \cap S_{\mathrm{sc}} = Z_0$. The construction (6.4) applies to this inner-diagonal extension as well. Denote its finite upper Levi by L_0^+. The central kernel satisfies $A \leq L_0^+$; put $L^+ = L_0^+/A$. The same correspondence and simultaneous conjugation place the marked P in L^+.

Let $Y_0 = Y_{\mathbb{U}}^{\mathbb{G}}$ be the variety for the connected lower group with the chosen radical, and let Y_0^+ be the corresponding variety for the upper group. Before quotienting, coarsen the component grading to

$$E_0/(S_{\mathrm{sc}} A) \cong SP/S. \tag{6.5}$$

Its identity piece is AY_0, and the quotient map gives

$$(AY_0)/A \cong Y_0/Z_0 =: \overline{Y}_0. \tag{6.6}$$

An A-orbit meeting Y_0 meets it in exactly one Z_0-orbit. The finite quotient universal property gives the inverse as a map of varieties, which proves the variety isomorphism. Two lifts of the same original $p \in P$ differ by A, whose diagonal conjugation on the variety is pointwise trivial. Thus (6.6) is equivariant for the original P, with all its power relations.

After placing P in the upper Levi, the stabilizer of the identity component consists of pairs with equal quotient degrees. Multiplying by one common original P-lift turns them into lower pairs. Consequently, with $K = SP$,

$$\overline{Y}_0^+ \cong (K \times L^+) \times_{(S\times L)\Delta P} \overline{Y}_0. \tag{6.7}$$

The map is $[(g, l), [y]] \mapsto [gyl^{-1}]$; the same degree calculation proves its well-definedness, injectivity and surjectivity. The original stable primitive block idempotents lie in identity degree: a stable block in a normal 2-extension has a unique covering block, represented by that same idempotent. This applies at both endpoints and preserves the chosen cuts in (6.7).

Choose the lower point-stabilizer cochain representative directly on $\overline{Y}_0$ with its whole field and regular-embedding action; these operators preserve the central quotient. The action of $(S \times L)\Delta P$ on the lower cochains is obtained from its base action and the conjugation action of P; their intersection acts diagonally. Restriction to this action is denoted by $\mathrm{Res}\, C$. Lemma 6.1 gives the representative

$$C^+ = \mathrm{Ind}_{(S\times L)\Delta P}^{K\times L^+} \mathrm{Res}\, C, \qquad (C^+)_1 = C. \tag{6.8}$$

This equality is literal for the displayed induced model.

Put $T = (S \times L)\Delta P$ and $\Gamma = K \times L^+$. After restricting each of the specified point-stabilizer summands and then inducing, its permutation overmodule is $\mathcal{O}[\Gamma/T_z]$. This is the permutation module of a point stabilizer on $\Gamma \times_T \overline{Y}_0$. Thus the chosen induced complex has the point-stabilizer term bound for the upper variety.

We verify that the two indicated P-actions are equal. A nontrivial field power cannot be inner-diagonal on the finite central quotient. If it were, its implementing regular-embedding element, preserving the opposite standard Borels, would be toral. Since a field map fixes every $x_\alpha(1)$,

all root multipliers would be one, contradicting its nontrivial action on some $x_\alpha(t)$. A central quotient does not identify distinct root-group elements. Thus the kernel of the regular/field finite operator group over its automorphism image has no field part. The remaining kernel consists of central regular-embedding elements and acts pointwise trivially on the variety. The two lifts of an original P-operator in (6.8) consequently induce the same cochain action. The kernel of $P \to \mathrm{Aut}(S)$ acts through its homogeneous units in the upper grading.

Lemma 6.2. *Let K_0 contain L_0, let $A \le Z(K_0) \cap L_0$, and let Z be a (K_0, L_0)-variety on which the left and right actions of A agree and the resulting deck action of A is free. Put $K = K_0/A$, $L = L_0/A$, and $\overline{Z} = Z/A$. Suppose every 2-subgroup of a point stabilizer on Z is $(K_0 \times 1)$-conjugate to a subgroup of ΔL_0. Then every 2-subgroup of a point stabilizer on $\overline{Z}$ is $(K \times 1)$-conjugate to a subgroup of ΔL. If $P \in \mathrm{Syl}_2(L)$, it is $(K \times L)$-conjugate to a subgroup of the same literal ΔP.*

Proof. Let $T \le (K \times L)_{\bar z}$ be a 2-subgroup and choose $z \in Z$ above $\bar z$. For the inverse image $\widetilde{T}$ in $K_0 \times L_0$ there is an exact sequence

$$1 \longrightarrow A \times A \longrightarrow \widetilde{T} \longrightarrow T \longrightarrow 1.$$

Choose a Sylow 2-subgroup Q of $\widetilde{T}$. Since $A \times A$ is central, Q contains $A_2 \times A_2$ and maps onto T.

For each $u \in Q$, freeness supplies a unique $\chi(u) \in A$ with $uz = \chi(u)z$. Centrality of the deck action implies $\chi(uv) = \chi(u)\chi(v)$, so $\chi : Q \to A_2$ is a homomorphism. With the bimodule convention $(a, b)z = azb^{-1}$, it satisfies

$$\chi(a, b) = ab^{-1} \qquad (a, b \in A_2).$$

Every u can therefore be multiplied by $(\chi(u)^{-1}, 1)$, without changing its image in T, to obtain an element fixing z. Hence $Q_z = \ker \chi$ is a 2-subgroup of the upstairs point stabilizer and

$$1 \longrightarrow \Delta A_2 \longrightarrow Q_z \longrightarrow T \longrightarrow 1$$

is exact. The stipulated upstairs conjugacy applies to Q_z. Its image gives the first assertion. Finally conjugate the right projection inside L into P and use the same element diagonally in $K \times L$. □

In the present application $Z = Y_0^+$ and $(K_0, L_0) = (E_0, L_0^+)$. The deck kernel lies in the central part used to extend the Levi, so $A \le L_0^+$. It acts freely: an element of this central deck group fixing $y\mathbb{U}$ would lie in the unipotent radical, whose intersection with the central deck group is trivial. The upstairs point-stabilizer assertion is [2, Lemma 3.7]. That lemma is stated in the section allowing disconnected reductive groups, and its conjugator belongs to the finite left group. For the extended parabolic and Levi used here their simultaneous normalizer is the extended Levi, so its diagonal is the stated ΔL_0^+. The lemma therefore applies to the central quotient, including its additional fixed-point components.

Combining Lemma 6.2 with the initial point-stabilizer choice proves the following precise term assertion once $P \in \mathrm{Syl}_2(L^+)$ has been established:

$$(C^+)^i \in \mathrm{add}\{\mathcal{O}[(K \times L^+)/\Delta R] : R \le P\} \quad \text{for every } i. \tag{6.9}$$

Indeed, for a point stabilizer B and $R \in \mathrm{Syl}_2(B)$, $\mathcal{O}[\Gamma/B]$ is a direct summand of $\mathcal{O}[\Gamma/R]$: the coset projection and transfer have composite $[B : R]$, a unit of $\mathcal{O}$. The preceding lemma conjugates R into ΔP. Block cuts preserve this term assertion. The conjugations establish relative projectivity upon restriction to the base double group and preserve the chosen operator action.

Lemma 6.3. *The central quotient above preserves the two upper evaluations, their homotopies, and the relative dual on the specified geometric complexes.*

Proof. The central deck action is free. Use Rickard's point-stabilizer representative: its terms are direct summands of permutation modules whose restriction to the deck group is free. In particular the terms, including their block cuts, are finite projective over $\mathcal{O}A$. On each free orbit the norm identifies the basis of coinvariants with the orbit-sum basis of invariants with coefficient one. Thus coinvariants and invariants are isomorphic on these free terms and on their direct summands, without dividing by $|A|$. The quotient theorem [24, Theorem 4.1], or [27, Theorem 2.28], identifies these invariants with quotient cochains in the homotopy category, retaining the remaining finite operators.

Central balance is termwise: the left and right central operators agree. Tensoring the quotient complexes over the quotient middle algebra is therefore the quotient of the original balanced tensor. For a finite projective $\mathcal{O}A$-module V, the coefficient-of-identity Frobenius trace gives

$$\mathrm{Hom}_{\mathcal{O}A}(V, \mathcal{O}A) \cong \mathrm{Hom}_{\mathcal{O}}(V, \mathcal{O}). \tag{6.10}$$

The inverse sends $f \in \mathrm{Hom}_{\mathcal{O}}(V, \mathcal{O})$ to

$$v \longmapsto \sum_{a \in A} f(a^{-1}v)a.$$

To check the evaluations, put $R = \mathcal{O}A$ and let B_0 be an upstairs middle block algebra. Its relative symmetrizing trace is

$$t^R_{B_0}\Big(\sum_g b_g g\Big) = \sum_{a \in A} b_a a,$$

restricted to the block. Composing with the coefficient of 1 in R gives the ordinary symmetrizing trace; composing with augmentation $R \to \mathcal{O}$ gives the symmetrizing trace of the quotient block. Since V is finite projective over R,

$$\mathcal{O} \otimes_R \mathrm{Hom}_R(V, R) \cong \mathrm{Hom}_{\mathcal{O}}(\mathcal{O} \otimes_R V, \mathcal{O}).$$

The relative tensor-dual adjunction and these trace identities identify the quotient of each evaluation with the downstairs evaluation. Their coefficients are unchanged. Homotopies descend under the additive quotient functor. The upper contractions supplied by [28, Propositions 3.6–3.7] therefore descend to the specified quotient complexes. When A is a 2-group, the finite projective R-terms are free because R is local. □

6.5. **Rank one and the upper direct product.** The lower quotient Levi is $L = S_0 \times A_0$ with $S_0 = \mathrm{PSL}_2(q^t)$, A_0 central, and a principal Klein block on S_0. Here $t \geq 3$, also in the E_7 case where $t = 3$; thus $q^t \geq 27$ and S_0 is perfect and centerless. Let $V \cong V_4$ be its marked defect group. The projective centralizer of V in the algebraic PGL_2 is V: diagonalize one involution and impose commutation with a second swapping involution; precisely four projective matrices remain.

The original abelian P centralizes $D = V \times (A_0)_2$ and acts algebraically inner on the Levi. It therefore acts on S_0 by conjugation with a unique $v(p) \in V$. Uniqueness makes $v : P \to V$ a homomorphism, identical on V. Put $P_0 = \ker v$. Then $P = V \times P_0$. Algebraic innerness fixes the connected central torus, so A_0 centralizes P. Hence, inside the given upper Levi,

$$L^+ = S_0 \times A', \qquad A' = A_0 P_0, \qquad P = V \times A'_2. \tag{6.11}$$

The intersection $A' \cap S_0$ is trivial by centerlessness. The factors are the derived subgroup and center and are preserved by the relevant operators.

Lemma 6.4. *In the decomposition*

$$L^+ = S_0 \times A', \qquad P = V \times A'_2, \qquad D = V \times (A_0)_2,$$

where $V \in \mathrm{Syl}_2(S_0)$ *and* A' *is central abelian, put*

$$H^+ = N_{L^+}(D) = N_{S_0}(V) \times A'.$$

Then

$$H^+ = N_{L^+}(P), \qquad \{l \in L^+ : l^{-1}Dl \le P\} = H^+.$$

Moreover H^+ *controls the fusion of all subgroups of the abelian Sylow subgroup* P *in* L^+.

Proof. Write $l = (s, a)$. The condition $l^{-1}Dl \le P$ says $s^{-1}Vs \le V$, hence $s^{-1}Vs = V$. All central factors are fixed under conjugation. This proves the two normalizer equalities and the transporter assertion.

For the fusion assertion, let $Q \le P$ and $lQl^{-1} \le P$. Both P and lPl^{-1} centralize lQl^{-1}, because P is abelian. They are Sylow 2-subgroups of $C_{L^+}(lQl^{-1})$. Choose c in that centralizer with $c(lPl^{-1})c^{-1} = P$. Then $h = cl \in H^+$ and $hqh^{-1} = lql^{-1}$ for every $q \in Q$. □

Here $q^t \equiv \pm 3 \pmod 8$ gives $|S_0|_2 = 4$, so the marked V is a Sylow subgroup and the displayed P is indeed a Sylow 2-subgroup of L^+.

The Klein four complex of Proposition 5.1 on S_0 has its odd outer action and the literal inner restriction $v(p)$. In the global application the relevant outer image is odd: the given upper P represents its Sylow 2-image and has just been shown inner. Tensor that complex with the original regular $\mathcal{O}A'$ block in its original odd central character sector. This gives Y^+, whose identity part is the corresponding complex Y on L. All central powers in A' are retained.

6.6. **The companion after quotienting and the full overlap.** Write

$$N = N_S(D), \quad H = N_L(D), \quad N^+ = N_K(D) = NP, \quad H^+ = N_{L^+}(D) = HP.$$

Choose a maximal supported pair $(\Delta P, e_P^+ \otimes (f_P^+)^*)$ of $k \otimes C^+$, and let (D, e_D^+) and (D, f_D^+) be its two subpairs at D. The block e_D^+ of $C_K(D) = C_S(D)P$ covers a single P-orbit of blocks of $C_S(D)$ and is their orbit sum, by the normal 2-extension block theorem and Fong–Reynolds correspondence. Since

$$\mathrm{Br}_P(e_D^+)e_P^+ = e_P^+ \neq 0,$$

this orbit has size one: a nontrivial orbit sum is a proper relative trace from a subgroup of P and has zero Brauer image at P. Write its unique member as e; thus $e_D^+ = e$. The same argument gives $f_D^+ = f$ on the right. In particular e, f are P-stable lower roots and are also the upper roots. We use the same symbols for their integral lifts.

These lower roots are supported by $k \otimes C$. Indeed, downward closure of the supported-pair set [4, Corollary 8.8] gives the supported upper pair at ΔD. Its cut is a Rickard equivalence of nonzero centralizer blocks by [4, Theorem 8.5], and hence has nonzero ordinary homology. In the induced model (6.8), its homogeneous pieces are the p-translates of $eC(\Delta D)f$, for $pD \in P/D$: p centralizes D and fixes e, f. Nonzero homology of the upper cut therefore implies nonzero homology of this identity piece. The lower pair is supported and maximal, since D is a defect group at both lower endpoints.

Let $\widetilde{E}$ be the finite group of regular-embedding Levi and field operators preserving the block, the chosen geometric data, and D, whose automorphism image lies in $\mathcal{A}_D$. Denote this image by E. We keep its finite action on the cochains. Set

$$\begin{aligned} I = N_N(e), \qquad & T = N_H(f), \qquad & J_0 = N_{I\times T}(\Delta D), \\ I^+ = IP, \qquad & T^+ = TP, \qquad & J^+_{e,f} = N_{I^+\times T^+}(\Delta D), \\ W_0 = N \times H, \quad & W^+ = N^+ \times H^+, \quad & J = N_{K\times L^+}(\Delta D). \end{aligned}$$

Since P centralizes D and fixes e, f, direct multiplication in the given groups gives

$$J^+_{e,f} = J_0(P \times P), \qquad (J^+_{e,f})_{\text{degree}\,1} = J_0 \Delta P. \tag{6.12}$$

Put $\mathcal{J} = N_{S\times L}(\Delta D)$ and choose

$$\Omega_D = \Omega_c(\overline{Y}_0^{\Delta D}, \mathcal{O})$$

with the point-stabilizer construction for the whole finite $\mathcal{J} \rtimes \widetilde{E}$ action. All supported root pairs at ΔD form one $\mathcal{J}$-orbit, by maximal-pair conjugacy in [4, Corollary 8.8]. This joint orbit is $\widetilde{E}$-stable. Consequently

$$\pi_D = \sum_{(e',f')\in\mathcal{J}(e,f)} e' \otimes (f')^*, \qquad V_D = e\Omega_D f$$

give an invariant complex $\pi_D\Omega_D$ and an isomorphism

$$\operatorname{Ind}_{J_0}^{\mathcal{J}} V_D \xrightarrow{\sim} \pi_D\Omega_D, \qquad [j, v] \longmapsto jv. \tag{6.13}$$

Distinct cosets give orthogonal root summands; this proves injectivity and surjectivity term by term and commutes with the differential. The selected V_D carries the action of $J_0\Delta P$, because P fixes the roots.

Define, at this initial choice,

$$V_D^+ = \operatorname{Ind}_{J_0\Delta P}^{J^+_{e,f}} V_D. \tag{6.14}$$

This is the fixed-point cochain object for $(\overline{Y}_0^+)^{\Delta D}$ with the specified cuts. Indeed, the cosets of the component grading have representatives $(p, 1)$ with $p \in P$, and these representatives centralize ΔD. Thus taking fixed points in the induced variety gives the same finite induction. This also follows modulo the maximal ideal from [24, Theorem 4.2], or [27, Theorem 2.29], including the normalizer action.

Let

$$\widehat{e} = \operatorname{Tr}_I^N(e), \qquad \widehat{f} = \operatorname{Tr}_T^H(f).$$

These are the blocks on the full lower normalizers. The same sums, in degree one, are the upper block idempotents: coset representatives for N/I and H/T also represent NP/IP and HP/TP. The complete companions are the following complexes:

$$U_D = \widehat{e}\,\mathcal{O}Ne \otimes_{\mathcal{O}I} \operatorname{Ind}_{J_0}^{I\times T}(\operatorname{Res}_{J_0} V_D) \otimes_{\mathcal{O}T} f\,\mathcal{O}H\widehat{f}, \tag{6.15}$$

$$U_D^+ = \widehat{e}\,\mathcal{O}N^+e \otimes_{\mathcal{O}I^+} \operatorname{Ind}_{J^+_{e,f}}^{I^+\times T^+}(V_D^+) \otimes_{\mathcal{O}T^+} f\,\mathcal{O}H^+\widehat{f}. \tag{6.16}$$

These are precisely the two Fong–Reynolds factors and the graph-normalizer induction of [4, Theorem 8.5 and Corollary 8.9]. Their subpair form applies also when $D < P$. The lower formula also equals

$$\widehat{e}\,\operatorname{Ind}_{\mathcal{J}}^{N\times H}(\pi_D\Omega_D)\,\widehat{f}$$

by (6.13) and induction transitivity. This expression supplies its $\widetilde{E}$-action; the selected root formula supplies its compatible P-action.

Ordinary induction in the two factors and transitivity identify (6.16) with

$$U_D^+ \cong \widehat{e}\,\operatorname{Ind}_{J^+_{e,f}}^{W^+}(V_D^+)\,\widehat{f} = \widehat{e}\,\operatorname{Ind}_{J_0\Delta P}^{W^+}(V_D)\,\widehat{f}. \tag{6.17}$$

For example, the adjacent e and f can be removed from the outer tensor factors because they act as the identity on V_D^+; the remaining outer factors are the ordinary induction bimodules. This proves the formula as a complex, with its given differential and coset carries.

Let $Q = P/D$. The original quotient maps identify K/S and L^+/L with Q; the degree of a pair is the left quotient degree times the inverse right quotient degree. The subgroup of W^+ preserving degree one is $W_0\Delta P$. Restricting the degree-one part of (6.17) gives

$$\widehat{e}\,\mathrm{Ind}_{J_0\Delta P}^{W_0\Delta P} V_D\,\widehat{f} \cong \widehat{e}\,\mathrm{Ind}_{J_0}^{W_0}\mathrm{Res}_{J_0} V_D\,\widehat{f} = U_D.$$

The middle isomorphism sends $[w\delta(p), v]$ to $[w, \delta(p)v]$; its ambiguity is exactly $\Delta(P\cap S) = \Delta D \leq J_0$. Using these coset normal forms in the definitions gives the identity of complexes with their specified P-actions

$$(U_D^+)_1 = U_D. \tag{6.18}$$

All block idempotents used here have degree one.

Lemma 6.5. *Every term of the specified integral complex U_D^+ belongs to*

$$\mathrm{add}\{\mathcal{O}[(N^+\times H^+)/\Delta R] : R \leq P\}.$$

Proof. First note the useful finite-set calculation

$$((K\times L^+)/\Delta R)^{\Delta D} = \begin{cases} J/\Delta R, & D \leq R, \\ \varnothing, & D \not\leq R, \end{cases} \qquad R \leq P. \tag{6.19}$$

If $(g,l)\Delta R$ is fixed, then

$$(g,l)^{-1}\Delta D(g,l) \leq \Delta R, \qquad l^{-1}Dl \leq P.$$

Lemma 6.4 gives $l \in H^+$ and $l^{-1}Dl = D$. The same displayed inclusion then gives $g \in N^+$, $D \leq R$, and $g^{-1}dg = l^{-1}dl$ for every $d \in D$. Thus $(g,l) \in J$. Conversely P is abelian, so $\Delta R \leq J$, and every coset of $J/\Delta R$ is fixed when $D \leq R$. This proves (6.19). It explains directly why full normalizer induction has right endpoint H^+ and preserves the literal diagonal in the modular Brauer model.

For the preselected integral fixed-point model a separate point-stabilizer argument gives the same term assertion. Let $z \in (\overline{Y}_0^+)^{\Delta D}$ and let R be a Sylow 2-subgroup of its stabilizer in $J_{e,f}^+$. The normal subgroup ΔD acts trivially on this variety, so $\Delta D \leq R$. The preceding global point-stabilizer bound supplies $x = (g,l) \in K\times L^+$ with $x^{-1}Rx \leq \Delta P$. Applying the transporter argument just given to $x^{-1}\Delta D x \leq \Delta P$ shows that $x \in J \leq W^+$. Hence R is already W^+-conjugate to a subgroup of ΔP.

Restrict the lower point-stabilizer model to $J_0\Delta P$ and then induce as in (6.14). A permutation overmodule $\mathcal{O}[(J_0\Delta P)/B_z]$ becomes $\mathcal{O}[J_{e,f}^+/B_z]$, the module of the corresponding upper fixed-variety point stabilizer. Thus V_D^+ has terms in the additive category of these permutation modules. Passing to a Sylow 2-subgroup is a split operation on their permutation overmodules, as above. Formula (6.17) then proves the bound after both Fong–Reynolds factors and full normalizer induction. Primitive root cuts and full orbit induction preserve the term bound. □

The reductions of these particular fixed-point complexes are the Brauer cochain objects, so the modular local evaluations come from [4, Theorem 8.5 and Corollary 8.9]. For these integral models lift the contractions of their evaluation cones. The Hom-surjectivity for integral permutation summands and the correction $h \mapsto h(dh+hd)^{-1}$ in Lemma 2.3 apply. The grading projector preserves the bimodule equations, so these contractions can be taken degree-preserving.

For a variety $\mathcal{Z}$ with free central 2-deck group A, let R be the full inverse image of a quotient subgroup. Its quotient fixed-point variety has the decomposition

$$(\mathcal{Z}/A)^{\Delta(R/A)} = \coprod_{a\in\mathrm{Hom}(R/A,A)} \mathcal{Z}^{\{(r,a(rA)r):r\in R\}}/A,$$

with empty components omitted. Freeness supplies a unique deck change for each fixed orbit; centrality makes this change a homomorphism. The displayed fixed loci are disjoint and closed, and there are finitely many, so they are open and closed in their union. The maps $r \mapsto a(rA)r$ are automorphisms: they fix A, induce the identity on R/A, and have inverses $r \mapsto a(rA)^{-1}r$. Each component contributes to the quotient Brauer construction.

Define

$$X_S = C \otimes Y \otimes U_D^{\vee}, \qquad X_S^+ = C^+ \otimes Y^+ \otimes (U_D^+)^{\vee}, \tag{6.20}$$

with the forced intermediate block algebras. The endpoints are the block of S and its block in $N_S(D)$, and their given upper extensions.

Lemma 6.6. *For the specified models, every term of*

$$C^+ \otimes_{\mathcal{O}L^+} Y^+ \otimes_{\mathcal{O}H^+} (U_D^+)^{\vee}$$

is 2-permutation and relatively ΔP-projective as an $(\mathcal{O}K, \mathcal{O}N^+)$-bimodule. The assertion includes the indicated block cuts.

Proof. The terms of C^+ satisfy (6.9). The Green term in the rank-one complex has vertex ΔV, and its correction terms are projective. Tensoring with the regular central $\mathcal{O}A'$ block therefore puts each term of Y^+ in

$$\mathrm{add}\{\mathcal{O}[(L^+ \times H^+)/\Delta T] : T \le P\}.$$

Lemma 6.5 gives the same bound for $(U_D^+)^{\vee}$, with its factors reversed.

It suffices to tensor the displayed permutation overmodules: split inclusions and retractions remain split after balanced tensor product. For the first tensor, the double cosets $R\backslash L^+/T$ give stabilizers

$$\{(a, l^{-1}al) : a \in R \cap lTl^{-1}\}, \qquad l \in L^+.$$

Both projections lie in P. By the fusion assertion of Lemma 6.4 there is $h \in H^+$ with $h^{-1}ah = l^{-1}al$ on this intersection. Conjugation by $(1, h)$ puts the displayed graph in the literal ΔP. Thus the first tensor has the required (K, H^+) term bound.

For the second tensor the double-coset representative belongs to H^+, which is a subgroup of the right endpoint N^+. Its graph stabilizer is

$$\{(a, h^{-1}ah) : a \in R' \cap hT'h^{-1}\}, \qquad h \in H^+, \quad R', T' \le P.$$

Now $(1, h) \in K \times N^+$ itself absorbs the graph. Its resulting diagonal subgroup is contained in P.

Linearization of a balanced product of finite bisets is a permutation module. Block cuts and the given summand projectors give its direct summands. Tensoring over a block algebra is the same balanced tensor as over its group algebra on these block-supported modules. This proves the assertion for every term of the total complex. The conjugations witness relative projectivity and preserve the chosen complex and its operator action. □

The same proof applies to the lower complex, with (K, L^+, P, N^+, H^+) replaced by (S, L, D, N, H). Here D is Sylow in $L = S_0 \times A_0$ and $N_L(D)$ controls its fusion by the same centralizer argument.

For the graded tensor identity, let $B = \bigoplus_{g \in Q} B_g$ be an intermediate upper block algebra, with $Q = P/D$, and choose a homogeneous unit $u_g \in B_g$. For the adjacent graded bimodule complexes $M = \bigoplus M_g$ and $N = \bigoplus N_g$, multiplication by these units gives

$$(M \otimes_B N)_1 \xrightarrow{\sim} M_1 \otimes_{B_1} N_1, \qquad x_g \otimes y_{g^{-1}} \longmapsto x_g u_g^{-1} \otimes u_g y_{g^{-1}}. \tag{6.21}$$

Its inverse is induced by the inclusions of M_1, N_1. For $b_h \in B_h$, the relation $u_g b_h u_{gh}^{-1} \in B_1$ verifies balancing and independence of the chosen units. These units are group elements multiplied by the relevant block idempotent, so their inverses retain the group multiplication relations.

Equations (6.8) and (6.18) give $(X_S^+)_1 = X_S$, with the same operators and both evaluations. Degree-preserving projection of the upper homotopies is bimodule-linear even for a nonabelian grading; it retains the specified contractions.

Write c_n for conjugation by n on S, and, for $U \le S$, put $c(U) = \{c_u : u \in U\} \le \mathrm{Aut}(S)$. The kernel calculation following (6.8) shows that the finite operator presentation $\widetilde{E}$ has a pointwise-trivial kernel over E consisting of central regular-embedding elements. The point-stabilizer cochains, their fixed-point models and inductions kill this kernel termwise. Thus the action factors to $\rho : E \to \mathrm{Aut}_{\mathcal{O}}(X_S)$.

With $N = N_S(D)$ and $H = N_L(D)$ the precise overlap is

$$\mathcal{A}_D = c(N)E, \qquad E \cap c(N) = c(H). \tag{6.22}$$

For the intersection, an inner representative in S preserving the chosen Levi and its block preserves the unique rational-series idempotent containing that block. Its dual relative normalizer takes s to an $\mathbf{L}^*$-conjugate. Correcting by that conjugator puts it in $C_{\mathbf{G}^*}(s) = \mathbf{L}^*$, so its relative Weyl element is trivial and its representative lies in L. This is the normalizer calculation [2, Section 7.A, equation (7.1)], applied with $C_{\mathbf{G}^*}(s) = \mathbf{L}^*$. The same statement descends through the central quotient, whose kernel acts trivially. Imposing the marking D gives H. Conversely every $h \in H$ supplies this inner operator.

For the product equality, choose a regular embedding $\widetilde{\mathbf{G}} = \mathbf{G}Z(\widetilde{\mathbf{G}})$ and put $\widetilde{\mathbf{L}} = \mathbf{L}Z(\widetilde{\mathbf{G}})$. Then $\widetilde{\mathbf{L}}/\mathbf{L} \cong \widetilde{\mathbf{G}}/\mathbf{G}$. Lang's theorem for the connected groups $\mathbf{L}$ and $\mathbf{G}$ gives

$$\widetilde{\mathbf{L}}^F/\mathbf{L}^F \cong \widetilde{\mathbf{G}}^F/\mathbf{G}^F, \qquad \widetilde{\mathbf{G}}^F = \mathbf{G}^F\widetilde{\mathbf{L}}^F.$$

Together with the finite field operator τ, these regular Levi elements represent every relevant inner-diagonal-field outer class and preserve $\mathbb{U}$ in (6.2). The same assertions hold after the simultaneous conjugation used to place P in the Levi. The equivariance and the identification of the block labels are given by [28, Corollary 4.2, Proposition 4.9, Corollary 5.5 and Lemma 5.6]; comparison of block bijections uses [2, Theorem 7.2]. A representative in the same outer class as a block stabilizer fixes that block, since inner automorphisms fix blocks. Its geometric action then fixes the corresponding Levi block. Conjugating by an element of L returns its defect group to D. The resulting representative belongs to E, and differs from the original marked automorphism by c_n with $n \in N$. This proves (6.22).

On $c(H)$ the constructed action is the literal diagonal action: the geometric action of h is conjugation on the varieties, the rank-one action has this property by construction, and the induction and Fong–Reynolds formulas preserve it. Also $\delta_z = 1$ for every $z \in Z(S)$, because the original complexes are strictly balanced over that center. Therefore

$$\rho_{\mathrm{full}}(c_n e) = \delta_n \rho(e), \qquad n \in N,\ e \in E, \tag{6.23}$$

is well-defined even when S has a nontrivial center. Indeed, two factorizations differ by c_h with $h \in H$; the corresponding representatives in N differ by hz with $z \in Z(S)$, and $\delta_z = 1$. It is multiplicative because

$$\rho(e)\delta_n\rho(e)^{-1} = \delta_{e(n)}.$$

Equivalently X_S is a module complex for the finite relative group

$$\mathcal{T}_S = \big((S \times N) \rtimes \mathcal{A}_D\big) \Big/ \{(\delta(n)^{-1}, c_n) : n \in N\}.$$

The displayed normal subgroup centralizes the base, and its intersection with $S \times N$ is exactly $\Delta Z(S)$. Thus the base of $\mathcal{T}_S$ is $(S \times N)/\Delta Z(S)$. More precisely, the pre-existing Levi-operator group

$$\mathcal{T}_E = ((S \times N) \rtimes E)/\{(\delta(h)^{-1}, c_h) : h \in H\}$$

maps isomorphically to $\mathcal{T}_S$. Surjectivity follows from $\mathcal{A}_D = c(N)E$, since an inner operator is identified with its base diagonal operator. For the kernel, if an E-operator represents c_n, then $c_n \in E \cap c(N) = c(H)$; write $n = hz$ with $h \in H$ and $z \in Z(S) \leq H$. Its relation is already in the displayed denominator of $\mathcal{T}_E$. The isomorphism of relative groups transfers the chosen permutation terms and the action on their differentials. Pull back this action through the original ambient automorphism map. Its kernel $C_{\text{ambient}}(S)$ acts as the identity on this lower component. On P, the action agrees with the geometric action used in the identity-component inductions, because the two regular/field lifts differ only by the pointwise central kernel already identified. This assertion concerns the diagonal operator on the lower complex; the original kernel elements of P remain as homogeneous units with their powers in the upper groups and upper complexes.

Proposition 6.7. *For case (iii) with the ambient marked stabilizers specified above, there is an integral splendid component complex X_S with its genuine full marked stabilizer action and literal inner restriction. It is the identity component of the upper splendid complex X_S^+ for SP and $N_S(D)P$. Both evaluations admit homogeneous upper contractions, and the complexes are strictly balanced over their shared centers.*

7. Assembly of component complexes

Let $H = S_1 \cdots S_m C \lhd G$ be a central product, let $b = b_1 \cdots b_m b_C$ be a G-stable block of $\mathcal{O}H$, and let a be a block of $\mathcal{O}G$ covering b, with abelian defect group P. Assume

$$D = P \cap H, \qquad |P : D| = |G : H|_2. \tag{7.1}$$

The factors S_i constitute one chosen G-orbit of components in cases (i)–(iii); the remaining factors form C. Assume that P fixes each S_i and that $N_G(P)$ is transitive on this component orbit. Put $D_i = P \cap S_i$ and

$$Q = P \cap (S_1 \cdots S_m), \quad J = N_G(Q), \quad H' = H \cap J. \tag{7.2}$$

The factor blocks and defect intersections give

$$H' = N_{S_1}(D_1) \cdots N_{S_m}(D_m)C.$$

Let c_i be the Brauer correspondent of b_i in $N_{S_i}(D_i)$, and write $b' = c_1 \cdots c_m b_C$ for the resulting block of H'. The marked Frattini argument on the normal component product gives $G = HJ$. Hence $\Gamma = G/H \cong J/H'$, $P \leq J$, and $P \cap H' = D$.

For every chosen factor use the data established in Sections 5 and 6: an integral splendid complex X_i with the marked stabilizer action and diagonal inner action; an upper complex X_i^+ for S_iP and $N_{S_i}(D_i)P$ whose identity component is X_i with that P-action; degree-preserving evaluations and contractions; and balance over the shared centers in the chosen central-character summands. On C use the identity complex $\mathcal{O}Cb_C$. Choose the data on one representative S_1. For each i, choose $g_i \in N_G(P)$ with ${}^{g_i}S_1 = S_i$, taking $g_1 = 1$, and transport its blocks, Brauer pairs, lower and upper complexes, evaluations, and contractions by this same g_i. Since the component product is normal, $N_G(P) \leq J$. Thus every transported upper complex still uses the specified group P.

Proposition 7.1. *These data give a splendid Rickard complex between the block-sum algebras on G and J. A primitive cut at the block gives a primitive target block with the same marked defect group P.*

Proof. First construct the full action. With the transporters g_i fixed above, for $g \in J$ the operator on the ith factor is the given operator of

$$h(g, i) = g_{g(i)}^{-1} g g_i.$$

The equality

$$h(g_2g_1, i) = h(g_2, g_1(i))h(g_1, i) \tag{7.3}$$

proves the group law. For $p \in P$, the equality $p(i) = i$ gives $h(p, i) = g_i^{-1}pg_i \in P$. Hence the action on each transported lower complex is the identity-component action of its specified upper complex. Tensor these maps with the Koszul permutation signs and with the original conjugation action on the regular C factor. Inner elements of H' act by the literal diagonal bimodule action. Balance over the shared centers and retain the chosen odd character sector. This gives a lower complex X for

$$T = (H \times H')\Delta J, \qquad T_P = (H \times H')\Delta P. \tag{7.4}$$

We prove that its restriction to T_P is a permutation summand with the required vertex bound. Temporarily keep P as a separate acting group:

$$\widetilde{T}_P = \left(\prod_i (S_i \times N_{S_i}(D_i)) \times (C \times C)\right) \rtimes P.$$

For the ith factor the inverse image of the original ΔP is

$$J_i = \Delta D_i \times P. \tag{7.5}$$

Indeed (sp, tp) is diagonal on P precisely when $s = t \in D_i$. The product is direct here because P is abelian. Restrict the upper complex X_i^+ to its identity stabilizer and take its identity summand. Mackey intersections remain below the diagonal bound: an upper pair is a base pair times $(p, q) \in P \times P$, and the latter centralizes ΔP. Thus the pulled-back X_i is 2-permutation and relatively J_i-projective.

The corresponding witness for the remainder is the regular block of CP. Its original P-Brauer support is nonzero, and the normal-extension defect bound is $|P \cap C|\,|P : P \cap C| = |P|$; hence its defect is P. It has identity component $\mathcal{O}Cb_C$ and relative inverse-image bound $\Delta(P \cap C) \times P$.

Permutation summands relatively projective for J_i are summands of sums of permutation modules with stabilizers $Q_i \leq J_i$. The stabilizers in their external tensor are intersections of conjugates of the inverse-image Q_i. Remove the P-coordinate from each conjugator, since P normalizes J_i. The remaining conjugators belong to independent base factors and combine into one base conjugation. Every stabilizer is consequently conjugate into

$$\left(\prod_i \Delta D_i \times \Delta(P \cap C)\right) \times P. \tag{7.6}$$

Now impose the central-product relations. Strict central balance makes the left relation and the corresponding right relation act identically. For a quotient kernel K, the coinvariants of $\mathcal{O}[\widetilde{T}_P/Q_0]$ are the permutation module on the orbit set, with stabilizer Q_0K/K. Summand projectors descend. Next impose the original graph identifying $d \in D$ with its inner diagonal action. This graph already acts trivially, by the literal inner restriction and the identity action of off-factor centralizers. Under these two quotients the image of (7.6) is first $\Delta D \cdot P$ and then exactly the original ΔP. This proves the whole T_P permutation and relative-vertex assertions. One-sided projectivity is preserved by the same balanced tensor: tensoring summands of free factor modules gives summands of free modules for the central-product algebra.

We next retain both evaluations. The homogeneous identity comparison (6.21) gives P-linear contractions for the chosen lower factor evaluations. Tensor them and the regular remainder identity, and descend through the central relations. The duality here is the relative Frobenius dual. For a common central subgroup Z, put $R = \mathcal{O}Z$. Finite projective R-modules satisfy

$$\begin{aligned} \mathrm{Hom}_R(U, R) &\cong \mathrm{Hom}_{\mathcal{O}}(U, \mathcal{O}), \\ \mathrm{Hom}_R(U \otimes_R V, R) &\cong \mathrm{Hom}_R(U, R) \otimes_R \mathrm{Hom}_R(V, R). \end{aligned}$$

The first map composes with the coefficient-of-identity trace $R \to \mathcal{O}$. Its inverse sends $f \in \mathrm{Hom}_{\mathcal{O}}(U, \mathcal{O})$ to

$$u \longmapsto \sum_{z \in Z} f(z^{-1}u)z.$$

The second is the ordinary dual-tensor isomorphism for finite projective modules over the commutative ring R. These maps are natural under automorphisms of Z. Thus for $A = A_1 \otimes_R A_2$, $B = B_1 \otimes_R B_2$ and $X = X_1 \otimes_R X_2$,

$$X \otimes_B X^\vee \cong (X_1 \otimes_{B_1} X_1^\vee) \otimes_R (X_2 \otimes_{B_2} X_2^\vee), \tag{7.7}$$

and likewise on the other side. The resulting evaluations are the quotient-algebra evaluations. On a coset basis the displayed inverse has coefficient one at the identity, including when Z has even order.

By (7.1),

$$[T : T_P] = [\Gamma : PH/H] \tag{7.8}$$

is odd. Restriction and induction across this index show that the already constructed T-module terms are permutation summands with vertices below ΔP. For either evaluation, let $\mathcal{U}$ be its operator group, $(H \times H)\Delta J$ or $(H' \times H')\Delta J$, and let $\mathcal{U}_P$ be the subgroup obtained by replacing J with P. If h_P is its supplied $\mathcal{U}_P$-linear contraction, put

$$h_{\mathcal{U}} = \frac{1}{[\mathcal{U} : \mathcal{U}_P]} \sum_{u \in \mathcal{R}} u h_P u^{-1}, \tag{7.9}$$

where $\mathcal{R}$ represents the left cosets of $\mathcal{U}_P$ in $\mathcal{U}$. This index is odd by (7.1). Changing a representative by a right element of $\mathcal{U}_P$ leaves its summand unchanged. Left multiplication permutes the cosets with these relations, proving equivariance. Linearity of $dh + hd = 1$ proves that $h_{\mathcal{U}}$ is a contraction.

Write τ_j for the diagonal action of $j \in J$ on X and $\tau_j^\vee(f) = f \circ \tau_{j^{-1}}$ for its action on the dual. Finally induce X from T to $G \times J$. Equivalently use $\mathcal{O}Gb \otimes_{\mathcal{O}Hb} X$, with right action

$$(a \otimes x)j = aj \otimes \tau_{j^{-1}}(x). \tag{7.10}$$

The inner intersection relation makes this well-defined. Put $A = \mathcal{O}Hb$, $A^+ = \mathcal{O}Gb$, $B = \mathcal{O}H'b'$, and $B^+ = \mathcal{O}Jb'$, where b' is the retained block of H'. Write $Y = A^+ \otimes_A X$. Its dual is $X^\vee \otimes_A A^+$, with

$$j(f \otimes a) = \tau_j^\vee(f) \otimes ja \qquad (j \in J).$$

Choose representatives $j \in J$ for G/H. Every homogeneous element of A^+ has the form ja_0 with $a_0 \in A$. The two evaluation identifications are

$$Y \otimes_{B^+} Y^\vee \longrightarrow A^+ \otimes_A (X \otimes_B X^\vee),$$
$$(a \otimes x) \otimes (f \otimes ja_0) \longmapsto aj \otimes (\tau_{j^{-1}}x \otimes \tau_{j^{-1}}^\vee f)a_0,$$

and

$$X^\vee \otimes_A A^+ \otimes_A X \longrightarrow B^+ \otimes_B (X^\vee \otimes_A X),$$
$$f \otimes ja_0 \otimes x \longmapsto j \otimes (\tau_{j^{-1}}^\vee f \otimes a_0 x).$$

On the first target put $Z = X \otimes_B X^\vee$ and $\tau_j^Z = \tau_j \otimes \tau_j^\vee$. Its upper right action is

$$(a \otimes z)j = aj \otimes \tau_{j^{-1}}^Z(z).$$

The second target has the analogous action, with $Z = X^\vee \otimes_A X$ and $\tau_j^Z = \tau_j^\vee \otimes \tau_j$. Their inverses send $a \otimes (x \otimes f)$ to $(a \otimes x) \otimes (f \otimes 1)$ and $j \otimes (f \otimes x)$ to $\tau_j^\vee f \otimes j \otimes x$, respectively. The relation $\tau_j(hxh') = jhj^{-1}\tau_j(x)jh'j^{-1}$ verifies balancing; changing a coset representative uses this same

relation. The maps intertwine both endpoint actions and carry the upper evaluations to the induced lower evaluations. The contractions from (7.9) therefore give both upper contractions.

The primitive-block correspondence [4, Theorem 8.4] gives a unique primitive idempotent c of B^+ such that aYc is noncontractible. For $c' \neq c$, the complex aYc' is contractible, and the two evaluations restrict to the regular blocks at a, c. Apply [4, Lemma 8.3] to the chosen left maximal pair (P, e_P). It gives a supported graph pair for aYc; [4, Corollary 8.8(c)] makes its right pair maximal, of order $|P|$. Every term already has vertices below conjugates of ΔP. A supported graph of this full order is therefore conjugate to ΔP in $G \times J$. Its right projection is a J-conjugate of P, so conjugating the right pair gives the chosen defect subgroup P without changing c. Since $N_G(P) \leq J$, a further right normalizer conjugation also matches the graph with the chosen left marking. Thus this primitive cut has the required defect group and diagonal bound. □

8. Reduction and proof of the theorem

8.1. **Strict reductions.** We use induction on the positive integer $\kappa(G)$ in (1.3). Every equivalence used in a reduction is an actual splendid complex with its specified defect graph, so Proposition 2.4 returns the inductive result to the original Brauer correspondent.

If a block d of a normal subgroup is covered and its inertia group $I = I_G(d)$ is proper, Fong–Reynolds gives a source-algebra equivalence to the corresponding primitive block of I. Transport the maximal pair simultaneously so its defect is the selected $P \leq I$. Put $Z_0 = O_{2'}(Z(G))$. Since $Z_0 \leq I \cap Z(I)$,

$$\kappa(I) \leq |I : Z_0| < \kappa(G). \tag{8.1}$$

Thus we may assume quasiprimitivity: every covered block of every normal subgroup is G-stable.

Lemma 8.1. *Suppose that a covered block of $N \lhd G$ is nilpotent and $N \not\subseteq Z(G)O_2(G)$. There is an actual splendid equivalence to a primitive block of a group G_1 with $\kappa(G_1) < \kappa(G)$.*

Proof. Put $M_0 = Z(G)N$. Its covered block is nilpotent and stable. Its defect is $Q = P \cap M_0$. The nilpotent-extension theorem [23, Theorem 3.5 and Corollary 3.15], with the source-corner correction in its Section 3.16, gives a model group L, a finite central odd cover $\widetilde{L} \to L$, and basic Morita correspondences on primitive summands, with

$$|L| = |G : M_0|\,|Q|. \tag{8.2}$$

Choose a maximal relative 2-subgroup $R \geq P$ for the covered M_0-block. Normal defect intersection gives $R \cap M_0 = Q$, since $P \cap M_0$ is already a defect group of that block. The construction has an exact sequence

$$1 \longrightarrow Q \longrightarrow L \longrightarrow G/M_0 \longrightarrow 1.$$

It embeds R with its original intersection Q and identifies the defect of the selected primitive model block with the given subgroup $P \leq R$. The source resolution is applied to this primitive correspondence and hence to its abelian defect group P.

Here is the source-corner passage in this correspondence. In [23, Section 3.1] the chosen source idempotent j of the lower block satisfies $j \leq i$. Since j is full in that lower block algebra, i is a full idempotent in the upper block-sum algebra. Corollary 3.15 identifies its corner, as an R-interior graded algebra, with a matrix factor given by a Dade R-algebra tensored with the local-extension algebra. The full-idempotent equivalence followed by matrix cancellation is therefore a Morita equivalence of the block-sum algebras. The induced center isomorphism selects the corresponding primitive block. Under the compatible pointed-subgroup embeddings, its source at the defect $P \leq R$ is the capped summand of the restriction of the Dade R-source. This is an endopermutation P-source with the given defect identification, so the primitive correspondence

is basic. The central $k^\times$-extension has a finite odd central realization in its prescribed character sector, giving the finite group $\widetilde{L}$ used above.

Apply Proposition 2.2 to this primitive Morita bimodule, whose capped endopermutation source and fusion stability are supplied by the source theorem. Lift integrally by Lemma 2.3. The central odd cover gives

$$\kappa(\widetilde{L}) \le |L| = |G : M_0|\,|Q| \le |G : Z_0|. \tag{8.3}$$

Equality in the last inequality would imply $M_0 = Z_0Q$. This is an internal direct product; Q is its unique Sylow 2-subgroup and is characteristic in M_0, hence $Q \le O_2(G)$. It would follow that $N \le Z(G)O_2(G)$, contrary to the hypothesis on N. The inequality is therefore strict. Take $G_1 = \widetilde{L}$ with the selected primitive block and its chosen central-character summand. □

After these reductions, for every $N \lhd G$ we have

$$\begin{gathered}\text{if a block of } \mathcal{O}N \text{ covered by } a \text{ is nilpotent, then}\\ N \le Z(G)O_2(G).\end{gathered} \tag{8.4}$$

8.2. **The generalized Fitting subgroup and relative defect.**

Lemma 8.2. *Let a be quasiprimitive and satisfy (8.4). Put $H = F^*(G)$. Every component has noncentral defect, P fixes each component, and, for $D = P \cap H$,*

$$|P : D| = |G : H|_2. \tag{8.5}$$

In particular, P is maximal among 2-subgroups with nonzero relative Brauer support for the covered H-block.

Proof. Every block of $O_{2'}(G)$ has defect zero and is nilpotent, so (8.4) puts $O_{2'}(G)$ in $Z(G)$. The normal 2-subgroup lies in abelian P. Thus H is the central product of its components and its original abelian factor $O_2(G)Z_0$.

We justify the component intersections before using them. Apply the normal defect-intersection theorem first to $E(G) \lhd G$, then to $S_i \lhd E(G)$. It gives that $P \cap E(G)$ is a defect group of the covered layer block and $D_i = P \cap S_i$ is a defect group of its component block. For a central product these factor defects give its defect by the group multiplication map, with the shared centers retained.

If a whole orbit of components had central-defect blocks, its covered central-product block would be nilpotent. Its normal perfect group would, by (8.4), lie in the abelian group $Z(G)O_2(G)$, a contradiction. Thus every component defect is noncentral. An element of P centralizes each D_i. If it moved S_i to a different component, D_i would lie in their central intersection, again a contradiction. Hence P fixes every component.

We also verify the upper-defect hypothesis of the component constructions. Write b_E for the covered $E(G)$-block and b_i for its retained component block. Since P fixes S_i and stabilizes b_E, uniqueness of the factor blocks makes b_i P-stable. In the group algebra, $ab_E = a$ and $b_Eb_i = b_E$, so $ab_i = a$. Applying the Brauer homomorphism at P gives $\mathrm{Br}_P(a)\,\mathrm{Br}_P(b_i) = \mathrm{Br}_P(a) \ne 0$. The unique covering block of the normal 2-extension PS_i/S_i is represented by b_i and has nonzero P-support. Its defect order is at most $|D_i|\,|P : D_i| = |P|$ by the normal-extension bound. It therefore has P as a defect group, as required in all four component cases.

Let K_0 be the kernel of the component permutation action, intersected with $C_G(O_2(G))$. Then

$$P \le K_0, \qquad C_G(P) \le K_0, \qquad H \le K_0.$$

The middle inclusion uses the same noncentral-defect argument and $O_2(G) \le P$. The covered K_0-block d has defect P. Since $ad = a$ and $C_{K_0}(P) = C_G(P)$, the original root satisfies

$$e_P\,\mathrm{Br}_P(d) = e_P\,\mathrm{Br}_P(a)\,\mathrm{Br}_P(d) = e_P.$$

It is therefore the same maximal root for d. Quasiprimitivity and conjugacy of maximal pairs therefore give

$$G = K_0 N_G(P, e_P). \tag{8.6}$$

The quotient G/K_0 is an image of the odd inertial quotient $N_G(P, e_P)/(PC_G(P))$, hence is odd. The componentwise outer-automorphism map has kernel H on K_0. Indeed an element in its kernel can be multiplied by elements of the components so as to centralize every component. It already centralizes $O_2(G)$ and Z_0, and hence centralizes H. The generalized-Fitting centralizer theorem gives $C_G(H) \leq H$. Conversely every element of H induces inner automorphisms on the components. Thus K_0/H embeds in the product of their outer automorphism groups and is solvable by the Schreier theorem. Consequently G/H is 2-solvable.

Pull back a $2/2'$ series of normal subgroups to $H = H_0 \lhd H_1 \lhd \cdots \lhd H_t = G$ with every $H_j \lhd G$, and retain the covered blocks. They are stable under the next group, by quasiprimitivity, and have defect groups $P_j = P \cap H_j$. If H_j/H_{j-1} is a 2-group, the normal-extension defect theorem gives $|P_j : P_{j-1}| = |H_j : H_{j-1}|$. If it has odd order, the defect groups have equal order and $P_j = P_{j-1}$. Multiplication gives (8.5). If $R \geq P$ has nonzero relative support, then $|R \cap H| \leq |D|$ and $|R : R \cap H| \leq |G : H|_2$. The equality for P forces $R = P$. □

The lemma also gives the odd outer image used in the component constructions. For the partial normalizer J in (7.2), the stabilizer J_i of a component contains P and H'. The image PH'/H' is a Sylow 2-subgroup of J/H', hence also of J_i/H'. Its image in the outer automorphism group is trivial for the elementary and direct Klein cases, by Propositions 4.2 and 5.3. Therefore the entire relevant outer image is odd. For the rank-one factor inside the geometric Levi, let E be its marked Levi/block stabilizer after the simultaneous conjugation, so that $P \leq E \leq J_i$. The projection $E \to J_i/H'$ has kernel $E \cap H'$. By the full-centralizer inner-overlap calculation, this kernel acts on S_0 through $N_L(D)$, with the other component factors acting trivially; its action is therefore inner. The image of P is a Sylow 2-subgroup of $E/(E \cap H')$, because it is already Sylow in J_i/H'. Its outer action on S_0 is trivial by (6.11). Consequently the entire relevant outer image of E on S_0 is odd.

8.3. **Completion of the induction.**

Proof of Theorem 1.1. Induct on $\kappa(G)$. A proper Fong–Reynolds reduction lowers it by (8.1); Lemma 8.1 lowers it whenever a covered nilpotent normal block satisfies that lemma's hypothesis. For either reduction, use induction on the actual primitive target and return by Proposition 2.4. We may therefore assume the hypotheses of Lemma 8.2.

If there are no components, $H = O_2(G)Z_0$ is centralized by P. The generalized-Fitting centralizer theorem gives $P \leq C_G(H) \leq H$, so $P = O_2(G)$ is normal. Its normalizer is G and the identity bimodule is the required splendid equivalence.

Suppose there are components and at least one active case (i), (ii), or (iii) from Section 4. Take its complete original G-orbit and set Q, J, H' as in (7.2). Equation (8.6) shows that $N_G(P, e_P)$ is transitive on the chosen component orbit, because K_0 fixes each component. Its elements belong to J and normalize the same P. Choose the transporters and all transported component data as in Section 7. The component propositions provide the full action, common-P identity component, central balance, and upper evaluations required by Proposition 7.1. The complementary layer keeps its identity bimodule. Hence there is a splendid Rickard complex

$$R : \mathcal{O}Ga \longrightarrow \mathcal{O}Jc$$

to a primitive block with the original defect P. The subgroup J is proper: if Q were normal in G, its intersection with each component of the orbit would be an abelian normal subgroup of

that quasisimple group, hence central, contradicting Lemma 8.2. Also $Z_0 \leq Z(J)$, so

$$\kappa(J) \leq |J : Z_0| < \kappa(G).$$

Because the component product is normal in G, every element of $N_G(P)$ normalizes its intersection Q with P. Thus $N_G(P) \leq J$. Apply induction to the selected block of J and return using the maximal companion of this same R.

Finally suppose that all components use case (iv). Lemma 3.3 makes every original PS_i-block inertial. Write b_E for the covered block of $E(G)$. The equality $ab_E = a$ gives $\mathrm{Br}_P(b_E) \neq 0$. Its defect is $P \cap E(G)$, and the unique block of $PE(G)$ covering b_E is represented by that idempotent. Its defect contains P and has order at most $|P \cap E(G)|\,|P : P \cap E(G)| = |P|$, so its defect is P. Lemma 3.4 therefore makes this $PE(G)$-block inertial. The remaining $O_2(G)$ is already in P. To include Z_0, put $K = PE(G)$, $I = K \cap Z_0$, let d be its covered block, and let λ be the central character of the block on Z_0. Put

$$e_\lambda = |Z_0|^{-1} \sum_{z \in Z_0} \lambda(z^{-1})z.$$

The matching character on I gives the interior-P algebra isomorphism

$$\mathcal{O}Kd \otimes_{\mathcal{O}I} \mathcal{O}Z_0 e_\lambda \xrightarrow{\sim} \mathcal{O}Kd, \qquad x \otimes ze_\lambda \longmapsto \lambda(z)x.$$

Consequently the block of the original PH, $H = F^*(G)$, is inertial. Lemma 8.2 gives the relative maximality of this same P. The strong hypotheses of Theorem 3.5 are now satisfied.

Take its primitive basic Morita bimodule from $\mathcal{O}Ga$ to the corresponding block of $J = N_G(D)$, with $D = P \cap H$. Proposition 2.2 and Lemma 2.3 give a splendid Rickard complex R with the original defect graph. There is a component, and its defect is noncentral; therefore D is not normal in G, so $J < G$. Again $Z_0 \leq Z(J)$, $N_G(P) \leq J$, and $\kappa(J) < \kappa(G)$. Apply induction and Proposition 2.4.

Every branch either ends with the identity complex or strictly lowers the same positive integer κ. Formula (2.10) and its two evaluation identities complete every return. All complexes remain bounded, integral, 2-permutation and projective on both sides, with vertices below the fixed defect diagonal. This proves (1.1) and (1.2). □

Corollary 8.3. *Let k be an algebraically closed field of characteristic two. Every block kGa with abelian defect group P has a splendid Rickard equivalence with its Brauer correspondent in $kN_G(P)$. In particular, Broué's abelian defect group conjecture holds at the prime two.*

Proof. Choose a sufficiently large splitting 2-modular system with residue field k, and lift the block idempotent to $\mathcal{O}G$. Reduce the complex and both integral contractions of Theorem 1.1 modulo the maximal ideal of $\mathcal{O}$. Reduction preserves the two identities, the contractions, and the permutation summands. □

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