hexagon:2611.00287v1

Higher Galois Recognition and Parameter Descent

Model groupoids and finite homotopy semantics, II

Zhenyu Sun

Abstract

Let $n \geq 0$ and let $\mathcal{X}$ be a bounded coherent ∞-topos. Suppose that finitely many noninitial coherent n-truncated objects cover its terminal object, and that every coherent n-truncated object in each of their slices has a finite covering family of sections. We identify $\mathcal{X}^{\mathrm{coh}}_{\leq n}$ with $\mathrm{Fun}(M, \mathcal{S}_{\pi,\leq n})$ for a π-finite $(n+1)$-type M, recovered by all finite models preserving finite limits, finite coproducts, and effective epimorphisms. Descent uses degrees through $n+2$ for objects and $n+1$ for maps; both bounds are sharp for finite homotopy actions.

Every exact functor F between recognized categories has a universal parameter $P_F = F_!(1)$ and a canonical factorization through its slice with fully faithful second functor. This parameter characterizes full faithfulness, conservativity, and all exact slice factorizations. For successive abelian Postnikov layers, we identify the space of marked exact lifts with a path space of Eilenberg–Mac Lane sections with local coefficients. The obstruction, classes of lifts, and relative automorphisms occur in degrees $r+1$, r, and $r-1$. We compute a twisted parity obstruction and, for odd primes, classify cyclic self-equivalence components by a p-adic valuation. A nonzero Postnikov class can persist in the identity component.

For continuous finite families over a profinite $(n+1)$-type, we prove that a uniform finite-section atlas exists precisely when the complete model space is π-finite. Exact functors from finite model semantics, together with their lift spaces, satisfy finite-stage descent, while explicit profinite examples separate objectwise trivialization from uniform parameter elimination.

Contents

1 Introduction . . . 2
2 Finite homotopy semantics and effective quotients . . . 4
3 Finite parameters on a coherent slice . . . 6
4 Recognition from a finite atlas . . . 8
5 Finite descent and sharpness . . . 12
6 A finite comparison criterion . . . 17
7 The homotopy type detected by finite models . . . 19
8 The universal parameter of an exact functor . . . 21
9 Relative Postnikov obstructions for exact functors . . . 25
10 Cyclic Postnikov families . . . 32
11 Continuous semantics and uniform finite parameters . . . 38
12 Higher families and truncation . . . 42
13 Finite two-types with the same homotopy groups . . . 46
14 Further questions . . . 48

1 Introduction

Finite homotopy semantics retains finite parameter spaces together with homotopies between their maps. Its recognition problem has a relative form: after identifying a category of finite families, one must determine which exact functors descend without parameters, which acquire a fully faithful factorization after adjoining a parameter, and which lift through the next homotopy layer. The answers involve a universal parameter object, relative Postnikov classes, and the monodromy of the space of lifts.

This is the second paper in a series on model groupoids and finite homotopy semantics. The first studies definability after restriction to a model subgroupoid and the extension of finite group actions [12]. Here the relative problem also retains higher transformations and successive Postnikov layers. The definitions and proofs needed below are given independently.

Fix $n \geq 0$ and write $\mathcal{F}_n = \mathcal{S}_{\pi,\leq n}$ for the ∞-category of π-finite n-types. Its mapping spaces are n-truncated, while its core is an $(n+1)$-type. Accordingly, finite n-type semantics detects an $(n+1)$-type of models. For a space M, put

$$\mathcal{L}_n(M) = \mathrm{Fun}(M, \mathcal{F}_n).$$

Throughout, an *exact functor* preserves finite limits, finite coproducts, and effective epimorphisms. Exact functor categories include all natural transformations and higher homotopies.

1.1 Recognition and bounded descent

Our categorical input is a bounded coherent ∞-topos $\mathcal{X}$ and $\mathcal{C} = \mathcal{X}^{\mathrm{coh}}_{\leq n}$. A finite local-section atlas is an effective epimorphism

$$\coprod_{i \in I} P_i \longrightarrow 1$$

with finitely many noninitial $P_i \in \mathcal{C}$, such that every object of $\mathcal{C}/P_i$ has an effective epimorphism from a finite coproduct of the terminal object of that slice. This is a condition inside $\mathcal{C}$ on its objects, sections, and covering maps.

Theorems 4.1 and 4.4, together with Proposition 4.5, prove that this condition is equivalent to

$$\mathcal{C} \simeq \mathcal{L}_n(M) \tag{1.1}$$

for a π-finite $(n+1)$-type M. The full category of exact models $\mathcal{C} \to \mathcal{F}_n$ is precisely M. The atlas objects represent the fibre functors, and Yoneda supplies all their transformations:

$$\mathrm{Nat}(\mathrm{Map}_{\mathcal{C}}(P_i, -), \mathrm{Map}_{\mathcal{C}}(P_j, -)) \simeq \mathrm{Map}_{\mathcal{C}}(P_j, P_i). \tag{1.2}$$

Every arrow between atlas objects is proved to be an equivalence. Their opposite groupoid therefore reconstructs M, independently of the atlas.

The proof first derives full faithfulness of constants from finite covering families of sections. An induction on truncation degree then reconstructs every local object from points and their path objects. Effective descent assembles the slices. For a degreewise finite Kan presentation $G_\bullet$ of M, the resulting formula is

$$\mathcal{C} \simeq \lim_{[q] \in \Delta_{\leq n+2}} \mathcal{F}_n^{G_q}. \tag{1.3}$$

Maps between objects carrying descent data require only degrees through $n+1$. Both bounds are optimal within the descent diagrams of finite C_2-actions (Proposition 5.5). The comparison criterion in Section 6 also applies when a point atlas has not been given.

1.2 Universal parameters of exact functors

Let $F : \mathcal{C} \to \mathcal{D}$ be exact between categories recognized at the same level. Theorem 8.2 constructs a left adjoint $F_!$ and the intrinsic object

$$P_F = F_!(1_{\mathcal{D}}).$$

For every $A \in \mathcal{C}$, its universal property is

$$\mathrm{Map}_{\mathcal{C}}(P_F, A) \simeq \mathrm{Map}_{\mathcal{D}}(1_{\mathcal{D}}, F(A)). \tag{1.4}$$

This space also classifies exact factorizations of F through $\mathcal{C}/A$, including the chosen identification of their composite with F. The distinguished section of $F(P_F)$ gives a canonical factorization

$$\mathcal{C} \longrightarrow \mathcal{C}/P_F \longrightarrow \mathcal{D} \tag{1.5}$$

whose second functor is fully faithful. It is the unique such slice factorization, up to contractible choice.

The object P_F tests two different properties:

$$\begin{aligned} F \text{ is fully faithful} &\iff P_F \simeq 1_{\mathcal{C}}, \\ F \text{ is conservative} &\iff P_F \to 1_{\mathcal{C}} \text{ is an effective epimorphism.} \end{aligned}$$

In the second case, the powers of this same intrinsic parameter give the finite descent formula with the sharp cutoffs $n+2$ and $n+1$. For $F = f^*$, its fibre at x is $\tau_{\le n}\,\mathrm{hofib}_x(f)$. Thus the construction is an internal form of relative truncation, together with a classification of all possible exact parameter extensions. Composition and change of parameters satisfy

$$P_{GF} \simeq F_! P_G, \qquad P_{F_A} \simeq (A \times P_F \to A).$$

1.3 Lifting through successive homotopy layers

Let $r \ge 2$, and let X, Y be π-finite r-types. Write $B_X = \tau_{\le r-1}X$, $B_Y = \tau_{\le r-1}Y$, and let A_X, A_Y be their top local coefficient systems. A lower exact functor is a map $f : B_Y \to B_X$. Prescribe also a homomorphism $\alpha : A_Y \to f^*A_X$. If κ_X, κ_Y are the Postnikov classes, Theorem 9.2 identifies the complete space of marked exact lifts with a path space between their classifying sections. In particular, the obstruction is

$$f^*\kappa_X - \alpha_*\kappa_Y \in H^{r+1}(B_Y; f^*A_X). \tag{1.6}$$

When it vanishes, the lift space is a torsor for the section space $\Gamma_{B_Y} K_{B_Y}(f^*A_X, r)$. Its components form an H^r-torsor, and its ith homotopy group is H^{r-i} for $1 \le i \le r$. The lower functor and its band map determine whether these lifts are fully faithful or invertible.

The functor moduli are recovered by retaining transport in the space of lower data. In the cyclic examples of Section 10, existence is an explicit scalar equation, and forgetting a lower marking produces a computable translation action. A separate example with the sign action on $\mathbb{Z}/4$ exhibits a parity obstruction: a fixed lower identity admits a fully faithful next-level lift precisely in the odd degrees (Proposition 9.7). The higher gauge calculation classifies cyclic self-equivalence components in every degree. For $r = 2j-1$, their full homotopy type depends on the p-adic valuation of j (Corollary 10.5). It distinguishes fundamental groups C_{p^2} and $C_p \times C_p$, and in degree $2p-1$ retains the original nonzero Postnikov class inside the identity component of the self-equivalence space (Example 10.6).

1.4 The finite boundary inside continuous semantics

For a profinite $(n+1)$-type $P = \{P_i\}$, define

$$\mathcal{C}_n(P) = \operatorname*{colim}_i \mathcal{L}_n(P_i).$$

These are continuous finite families. Higher Stone duality gives full duality with exact functors in this fixed range, and

$$\mathrm{Mod}_n(\mathcal{C}_n(P)) \simeq \lim_i P_i.$$

Theorem 11.5 proves an exact criterion for the uniformity of parameters: $\mathcal{C}_n(P)$ has a finite local-section atlas if and only if this whole model space is π-finite, equivalently if and only if P is represented by a single π-finite $(n+1)$-type. Every prescribed finite family still becomes constant on some finite coherent cover. Finite continuous $\mathbb{Z}_p$-sets show why these separate covers need not yield a common atlas.

1.5 Relation to existing theory

The identification of fibre-functor endomorphisms with loop spaces belongs to higher Galois theory; see Toën [13] and Hoyois [8]. The coherent and Stone frameworks are developed in Exodromy [3], and the finite homotopy closure properties used here are treated by Anel [1]. The finite totalization cutoffs follow from standard truncation arguments [1, 7]. The relative obstruction calculation uses the classical classification of Eilenberg–Mac Lane fibrations and their section spaces [10, 11].

The results below organize these constructions around an intrinsic finite-section condition. They derive its full model space, identify the universal parameter of every exact functor, and translate successive Postnikov compatibility into the full moduli of exact lifts. The explicit action examples establish the sharp descent bounds and calculate the distinction between marked and unmarked lifts. The uniform parameter theorem then identifies precisely when continuous semantics reduces to the finite recognition statement.

All recognition statements concern a prescribed truncation range. The local-section hypothesis quantifies over all coherent objects in that range. A finite simplicial window bounds the coherence data used for a given valid presentation; its underlying homotopy type is π-finite and can have nonzero cohomology in arbitrarily high degrees.

2 Finite homotopy semantics and effective quotients

All categories, limits, colimits, and mapping spaces are interpreted in the ∞-categorical sense unless stated otherwise. A space is π-finite if it has finitely many components, all its homotopy groups are finite, and it is truncated in some finite degree. We write $\mathcal{S}_\pi$ for these spaces and $\mathcal{F}_n = \mathcal{S}_{\pi,\le n}$. The empty space is allowed. A category is *locally n-truncated* when its mapping spaces are n-types. In particular, $\mathcal{F}_n$ is locally n-truncated, and its core is an $(n+1)$-type.

The category $\mathcal{S}_\pi$ is closed under finite limits and finite coproducts. Fibres over a π-finite base are π-finite exactly when their total space is π-finite. Mapping spaces between π-finite spaces are π-finite as well [1, Proposition 3.4.4]. A uniform truncation bound is imposed on the objects of $\mathcal{F}_n$, but no uniform bound is imposed on the orders of their homotopy groups.

For a bounded coherent ∞-topos $\mathcal{X}$, write

$$\mathcal{C} = \mathcal{X}^{\mathrm{coh}}_{\le n}.$$

We use the standard coherence convention in which the truncated coherent objects form a bounded ∞-pretopos [3, Section 3.8]. The full subcategory $\mathcal{C}$ is closed under finite limits and finite coproducts. Effective epimorphisms are computed in $\mathcal{X}$. The constant-object functor $\Delta : \mathcal{S} \to \mathcal{X}$ preserves finite limits and all colimits; its restriction to $\mathcal{F}_n$ takes values in $\mathcal{C}$. Indeed, the geometric morphism $\mathcal{X} \to \mathcal{S}$ is coherent: its target is a Stone ∞-topos [3, Corollary 4.4.15].

Fixed truncation requires care with quotients. For instance, the groupoid with one object and nontrivial finite automorphism group H has terms in $\mathcal{F}_0$, but its quotient BH is a one-type. Thus the fixed subcategory $\mathcal{F}_0$ is not closed under all effective groupoid quotients in spaces. The following restricted quotient operation is the one needed in the paper.

Definition 2.1. A groupoid object $R_\bullet$ in a category with finite limits is *n-efficient* if its arrow-to-endpoint map

$$R_1 \longrightarrow R_0 \times R_0$$

is $(n-1)$-truncated. For $n = 0$, this says that the relation is a monomorphism. Truncatedness of maps can be expressed by the iterated diagonal criterion, so it is preserved by finite-limit functors and reflected by conservative finite-limit functors.

Lemma 2.2 (Quotients in the fixed range)**.** *An n-efficient groupoid object in $\mathcal{C}$ has an effective quotient in $\mathcal{C}$, computed in $\mathcal{X}$. An n-efficient groupoid object in $\mathcal{F}_n$ has a π-finite n-truncated quotient, computed in spaces.*

Proof. Let $Q = |R_\bullet|$ in $\mathcal{X}$. The groupoid is effective, so $R_0 \to Q$ is an effective epimorphism and $R_1 \simeq R_0 \times_Q R_0$. The diagonal of Q, after pullback along the effective epimorphism $R_0 \times R_0 \to Q \times Q$, is the endpoint map. Truncatedness is local for effective epimorphisms [9, Proposition 6.2.3.17]; hence Q is n-truncated. Its coherence follows from closure under effective groupoid quotients of the bounded coherent category. These are the descent and exactness properties of coherent ∞-topoi; see [9, Sections 6.1–6.2] and [3, Section 3.8].

For the assertion in spaces, the same argument bounds the truncation degree. The set $\pi_0 Q$ is a quotient of $\pi_0 R_0$, so it is finite. Every component contains the image of some point of R_0. The fibre of $R_0 \to Q$ at this image is a fibre of $R_1 \to R_0$, hence is π-finite. Its long exact homotopy sequence shows that the homotopy groups of Q are finite. In degree one, the kernel is a quotient of $\pi_1 R_0$ and the remaining cosets are indexed by a subset of the finite component set of the fibre. This also proves finiteness in that degree. □

Remark 2.3 (The effective epimorphisms are intrinsic). For a map $A \to B$ in $\mathcal{C}$, its Čech nerve is n-efficient: its endpoint map is a pullback of the diagonal of B. By Lemma 2.2, its ambient quotient belongs to $\mathcal{C}$ and is also its quotient in that full subcategory. Thus a map in $\mathcal{C}$ is an ambient effective epimorphism exactly when it is the effective quotient map of its Čech nerve inside $\mathcal{C}$. In particular, equivalences of these categories preserve the class of effective epimorphisms.

Lemma 2.4 (Coherent descent along a finite atlas)**.** *Let $P = \coprod_{i \in I} P_i \to 1$ be an effective epimorphism with I finite and $P_i \in \mathcal{C}$. Then*

$$\mathcal{C} \simeq \lim_{[q] \in \Delta} \mathcal{C}/P^{q+1}.$$

For $P_i \in \mathcal{C}$, the category $\mathcal{C}/P_i$ identifies with the coherent n-truncated objects of the slice $\mathcal{X}/P_i$.

Proof. Effective descent gives the equivalence for the full topos and its slices. Truncatedness is local for effective epimorphisms. Since P_i is itself n-truncated, a map to P_i has an n-truncated total object exactly when it is n-truncated as an object of the slice; see also [9, Lemma 5.5.6.14].

Coherence over a coherent base is the same in the slice and in $\mathcal{X}$ [3, Paragraph 3.3.7]. It remains to check that an object Y reconstructed by descent is coherent when $Y \times P$ is coherent. The cover $Y \times P \to Y$ has coherent source, and its self-overlap is $(Y \times P) \times P$, which is coherent. The local criterion for m-coherence, first in degree zero and then in every positive degree, shows that Y is coherent [3, Lemmas 3.6.2–3.6.3]. Thus the full descent equivalence restricts to the displayed coherent subcategories. □

3 Finite parameters on a coherent slice

The local recognition argument first recovers maps between constants from components of their sections. Finite covers by sections then recover every coherent n-truncated object by induction on its truncation degree. Throughout this section, $n \geq 0$ and $\mathcal{F}_n = \mathcal{S}_{\pi,\leq n}$.

Let $\mathcal{Y}$ be a nondegenerate ∞-topos, with constant-object functor and global sections

$$c : \mathcal{S} \rightleftarrows \mathcal{Y} : \Gamma.$$

The functor c preserves finite limits and all colimits. In particular, it preserves the initial object. Nondegeneracy implies $\Gamma(0) = \varnothing$, since the initial object of an ∞-topos is strict.

Lemma 3.1 (Components of finite parameters)**.** *Suppose that, for every $K \in \mathcal{F}_n$, the unit induces a surjection*

$$\pi_0 K \longrightarrow \pi_0 \Gamma(cK). \tag{3.1}$$

Then the unit $K \to \Gamma(cK)$ is an equivalence for every $K \in \mathcal{F}_n$. Consequently

$$c : \mathcal{F}_n \longrightarrow \mathcal{Y}$$

is fully faithful.

Proof. First the map on components is injective. If $a, b \in K$ belong to different components, their path space is empty. Since c preserves finite limits and the initial object, the path object between the constant sections $a, b : 1 \to cK$ is $c(\varnothing) = 0$. Applying Γ gives an empty path space. Thus their images cannot belong to the same component of $\Gamma(cK)$.

For a point $a \in K$ and every $k \geq 1$, preservation of finite limits by both c and Γ gives

$$\Omega^k_{c(a)} \Gamma(cK) \simeq \Gamma\big(c(\Omega^k_a K)\big).$$

The space $\Omega^k_a K$ lies in $\mathcal{F}_n$. Applying (3.1) and the injectivity just proved to this loop space shows that the unit induces an isomorphism on π_k. The target $\Gamma(cK)$ is n-truncated, because cK is n-truncated and global sections preserve limits. The unit is therefore an equivalence. Finally, for $U, V \in \mathcal{F}_n$, adjunction gives

$$\mathrm{Map}_{\mathcal{Y}}(cU, cV) \simeq \mathrm{Map}_{\mathcal{S}}(U, \Gamma(cV)) \simeq \mathrm{Map}_{\mathcal{S}}(U, V).$$

□

The hypothesis concerns components of sections, but its test objects include finite homotopy types through degree n. It is therefore stronger than a condition imposed only on finite sets. Closure under loop spaces is what converts these component conditions into the comparison of all homotopy groups in the prescribed range.

We next recognize the whole category from local sections. Suppose that $\mathcal{Y}$ is coherent. Its constant-object functor preserves coherent objects, since the target of the geometric morphism $\mathcal{Y} \to \mathcal{S}$ is Stone; see [3, Corollary 4.4.15]. In particular, cK is coherent for $K \in \mathcal{F}_n$. Write $\mathcal{Y}^{\mathrm{coh}}_{\leq n}$ for its coherent n-truncated objects.

Proposition 3.2 (Local recognition by finite sections)**.** *The following conditions are equivalent.*

(i) *The constant-object functor induces an equivalence*

$$\mathcal{F}_n \xrightarrow{\sim} \mathcal{Y}^{\mathrm{coh}}_{\leq n}.$$

(ii) *Every $Y \in \mathcal{Y}^{\mathrm{coh}}_{\leq n}$ admits an effective epimorphism $\coprod_{a \in A} 1 \to Y$, where A is a finite set.*

(iii) *Every $Y \in \mathcal{Y}^{\mathrm{coh}}_{\leq n}$ admits an effective epimorphism $cK \to Y$ for some $K \in \mathcal{F}_n$.*

Under these conditions (3.1) *holds.*

Proof. If (i) holds, choose one point in each of the finitely many components of the space representing Y. This gives (ii). Condition (ii) implies (iii) by using a finite discrete K. Conversely, a finite set of points meeting every component of K gives an effective epimorphism from a finite discrete space to K. Applying c and composing effective epimorphisms proves (ii) from (iii).

Assume (ii). Every noninitial coherent n-truncated object has a global section: its covering finite sum of copies of 1 must have at least one summand. To prove (3.1), let $t : 1 \to cK$ be a section. Choose representatives $k_a \in K$ for the finite set $\pi_0 K$. Pulling the effective epimorphism $\coprod_a 1 \to cK$ back along t gives a finite cover of 1 by the coherent objects

$$E_a = 1 \times_{cK} 1,$$

where the two maps to cK are t and the constant section k_a. Each E_a is $(n-1)$-truncated. Since $\mathcal{Y}$ is nondegenerate, at least one of the E_a is noninitial. It has a section by (ii), and that section is a homotopy from t to k_a. Lemma 3.1 now proves full faithfulness of $c|_{\mathcal{F}_n}$.

We prove essential surjectivity by induction on the truncation degree k, starting with $k = -1$. A coherent subterminal object is either initial or has a section by (ii). In the second case its monomorphism to 1 is an equivalence. Thus every coherent (-1)-truncated object is constant.

Let $0 \leq k \leq n$ and let Y be coherent and k-truncated. Choose an effective epimorphism $cA = \coprod_{a \in A} 1 \to Y$ with A finite. For each ordered pair (a, b), its path object

$$R_{ab} = 1 \times_Y 1$$

is coherent and $(k-1)$-truncated. By induction it is constant on a π-finite $(k-1)$-type. The kernel pair of $cA \to Y$ is the finite sum $\coprod_{a,b} R_{ab}$, so it too is in the image of $c|_{\mathcal{F}_n}$. For $k = 0$ its summands are initial or terminal, and its total space is a finite discrete space.

The essential image of this fully faithful functor is closed under finite limits. Consequently the entire Čech nerve, including all its higher compatibilities, lifts to a groupoid object $R_\bullet$ in $\mathcal{F}_n$. Realize it in spaces, writing $S = |R_\bullet|$. Its object space is the finite discrete space A, and the fibre of $R_1 \to A \times A$ over (a, b) is the π-finite $(k-1)$-type representing R_{ab}. Effectivity in spaces gives $R_1 \simeq A \times_S A$. Thus the path spaces of S are $(k-1)$-truncated and π-finite; its components are a quotient of the finite set A. It follows that S is a π-finite k-type. Finally, preservation of realizations by c gives

$$cS \simeq |cR_\bullet| \simeq Y.$$

This completes the induction and the proof of (i). □

Finite constant covers supply parameter elimination and recover kernel presentations by induction. The decrease of truncation degree is explicit: path objects of a k-truncated object have degree $k-1$.

Example 3.3 (Parameter elimination alone does not present objects)**.** Let $\mathcal{Y} = \mathrm{Fun}([1], \mathcal{S})$. The global sections of a constant diagram are its constant value, so $\Gamma(cK) \simeq K$ for every space K. Nevertheless the coherent zero-truncated object $(\varnothing \to *)$ is noninitial and has no section. A map from a constant diagram $(K \to K)$ to it forces $K = \varnothing$, and cannot be an effective epimorphism. Thus even the full comparison on constant mapping spaces does not imply the finite-section hypothesis.

4 Recognition from a finite atlas

We apply the local recognition result to the slices over a finite cover. The resulting slice equivalences identify the overlaps and their transport maps; effective descent then recovers the category.

Let $\mathcal{X}$ be a bounded coherent ∞-topos and put

$$\mathcal{C} = \mathcal{X}^{\mathrm{coh}}_{\leq n}.$$

Let $\Delta : \mathcal{S} \to \mathcal{X}$ be its constant-object functor. As recalled in Section 2, it sends $\mathcal{F}_n$ into $\mathcal{C}$. Lemma 2.4 supplies effective descent in $\mathcal{C}$ and the identification $\mathcal{C}/P = (\mathcal{X}/P)^{\mathrm{coh}}_{\leq n}$ for $P \in \mathcal{C}$.

Theorem 4.1 (Recognition from finite local parameters)**.** *Suppose there are finitely many noninitial objects $P_i \in \mathcal{C}$, indexed by a finite set I, such that*

$$P = \coprod_{i \in I} P_i \longrightarrow 1$$

is an effective epimorphism. Suppose that, for each i, every $Y \in \mathcal{C}/P_i$ admits an effective epimorphism

$$\coprod_{a \in A} P_i \longrightarrow Y \quad \text{over } P_i,$$

where A is a finite set. Let $\mathcal{G}$ be the full subcategory of $\mathcal{C}$ on the objects P_i, and put $\mathcal{M} = \mathcal{G}^{\mathrm{op}}$. Then all morphisms of $\mathcal{G}$ are equivalences, and $\mathcal{M}$ is a π-finite $(n+1)$-type. The functor

$$\mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{M}, \mathcal{F}_n), \qquad A \longmapsto (i \longmapsto \mathrm{Map}_{\mathcal{C}}(P_i, A)) \tag{4.1}$$

is an equivalence. Under it, ΔK corresponds to the constant functor with value K.

Proof. If I is empty, the cover forces $\mathcal{X}$ to be degenerate; then $\mathcal{C}$ is terminal and $\mathcal{M} = \varnothing$, so the assertion holds. Otherwise apply Proposition 3.2 to $\mathcal{X}/P_i$. Its nondegeneracy follows from $P_i \neq 0$. We obtain equivalences

$$\Phi_i : \mathcal{F}_n \xrightarrow{\sim} \mathcal{C}/P_i, \qquad K \longmapsto P_i \times \Delta K. \tag{4.2}$$

The inverse is the restriction of global sections in the slice. In particular, for $A \in \mathcal{C}$,

$$\omega_i(A) := \mathrm{Map}_{\mathcal{C}}(P_i, A) \simeq \Phi_i^{-1}(P_i \times A) \quad \text{lies in } \mathcal{F}_n. \tag{4.3}$$

Consider a morphism $f : P_i \to P_j$. Under Φ_j, its object in $\mathcal{C}/P_j$ corresponds to some $F \in \mathcal{F}_n$. Taking slices of (4.2) gives

$$\mathcal{C}/P_i \simeq (\mathcal{C}/P_j)/f \simeq \mathcal{F}_n/F.$$

The identity $f^*\Phi_j = \Phi_i$ identifies the functor $F^* : \mathcal{F}_n \to \mathcal{F}_n/F$, $K \mapsto F \times K$, with an equivalence. Its full faithfulness, applied to 1 and F, says that the map

$$F = \mathrm{Map}(1, F) \longrightarrow \mathrm{Map}(F, F)$$

sending a point to the corresponding constant map is an equivalence. In particular, the identity of F is homotopic to a constant map. Thus F is contractible, and f is an equivalence. This proves that $\mathcal{G}$ is an ∞-groupoid. Its mapping spaces are π-finite n-types by (4.3). It has finitely many objects, so its space of components is finite and its loop spaces are π-finite n-types. Hence $\mathcal{M}$ is a π-finite $(n+1)$-type.

It remains to prove (4.1). By Lemma 2.4, descent along $P \to 1$ gives

$$\mathcal{C} \simeq \lim_{[q]\in\Delta} \mathcal{C}/P^{q+1}. \tag{4.4}$$

Finite coproducts are disjoint and universal. For a tuple $\mathbf{i} = (i_0, \dots, i_q)$, put $P_{\mathbf{i}} = P_{i_0} \times \cdots \times P_{i_q}$. In the slice over P_{i_0} this object corresponds to

$$F_{\mathbf{i}} = \prod_{k=1}^{q} \mathrm{Map}_{\mathcal{C}}(P_{i_0}, P_{i_k}).$$

Consequently

$$\mathcal{C}/P^{q+1} \simeq \prod_{\mathbf{i}} \mathcal{F}_n/F_{\mathbf{i}} \simeq \prod_{\mathbf{i}} \mathrm{Fun}(F_{\mathbf{i}}, \mathcal{F}_n). \tag{4.5}$$

The last equivalence is straightening for spaces. It restricts to π-finite n-types because both base and fibres are π-finite n-types.

The slice counits defining these equivalences are natural, and straightening is natural under base change. Thus the equivalences respect the face and degeneracy maps with their coherences. Explicitly, a point $u : P_i \to P_j$ of an overlap carries $\omega_j(A)$ to $\omega_i(A)$ by precomposition; the maps on multiple overlaps are composition and insertion of identities. Changing the distinguished index of a multiple overlap uses the inverse of u, which exists by the preceding argument. Thus the cosimplicial diagram in (4.5) is the descent diagram for a functor on $\mathcal{G}^{\mathrm{op}}$. Equivalently, it is obtained from the Čech nerve of the map of spaces $I \to \mathcal{M}$ by applying $\mathrm{Fun}(-, \mathcal{F}_n)$. The realization of that nerve is $\mathcal{M}$, and functor categories turn colimits in the source into limits. Equations (4.4)–(4.5) therefore prove (4.1). The unit equivalences in Lemma 3.1 identify each $\omega_i(\Delta K)$ with K, compatibly with transport, which proves the assertion about constants. □

Corollary 4.2 (Independence of the atlas)**.** *If two finite families satisfy the hypotheses of Theorem 4.1, their associated groupoids are equivalent. Both embed as full, essentially surjective subgroupoids of the groupoid associated with their union.*

Proof. Write the two families as (P_i) and (Q_j). Their union again satisfies the local hypotheses, so every morphism between its objects is an equivalence. Pull the cover $\coprod_i P_i \to 1$ back to Q_j. Under $\Phi_{Q_j}^{-1}$ the resulting effective epimorphism becomes

$$\coprod_i \mathrm{Map}_{\mathcal{C}}(Q_j, P_i) \longrightarrow 1.$$

Its source is nonempty. Thus Q_j admits a morphism to some P_i, and this morphism is an equivalence. The same argument with the families interchanged proves the assertion. □

4.1 The groupoid of models

Lemma 4.3 (Preservation of constants)**.** *Let $H : \mathcal{F}_n \to \mathcal{F}_n$ preserve finite limits, finite coproducts, and effective epimorphisms. Then the Yoneda transformation $\mathrm{id}_{\mathcal{F}_n} \to H$ is an equivalence, and its space of choices is contractible.*

Proof. The identity functor of $\mathcal{F}_n$, regarded as a space-valued functor, is represented by the point. Yoneda gives

$$\mathrm{Nat}(\mathrm{id}_{\mathcal{F}_n}, H) \simeq H(*) \simeq *.$$

Write η for its transformation. For $K \in \mathcal{F}_n$, choose a finite set of points meeting every component of K. The resulting effective epimorphism $A \to K$ remains an effective epimorphism under H. Since H preserves finite coproducts and the terminal object, $\eta_A : A \to H(A)$ is an equivalence. Naturality now shows that $\pi_0\eta_K$ is surjective.

If $a, b \in K$ belong to different components, their path space is empty. Preservation of finite limits and the initial object shows that the path space between $\eta_K(a)$ and $\eta_K(b)$ is also empty. This proves injectivity on components. For every basepoint a, finite-limit preservation identifies $\Omega^k\eta_K$ with $\eta_{\Omega^k_a K}$. The same component argument therefore gives isomorphisms on all homotopy groups. Both sides are n-types, so η_K is an equivalence. The contractible space of Yoneda transformations thus consists entirely of equivalences. □

Let $\mathrm{Mod}_n(\mathcal{C})$ be the full ∞-subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{F}_n)$ on the functors preserving finite limits, finite coproducts, and effective epimorphisms. Its morphisms are all natural transformations. For each such functor F, Lemma 4.3 applied to $F\Delta$ supplies a canonical equivalence $F\Delta \simeq \mathrm{id}_{\mathcal{F}_n}$.

Theorem 4.4 (Recognition of all models)**.** *Under the hypotheses of Theorem 4.1, each $\omega_i = \mathrm{Map}_{\mathcal{C}}(P_i, -)$ is a model, and evaluation induces an equivalence*

$$\mathcal{M} \xrightarrow{\sim} \mathrm{Mod}_n(\mathcal{C}).$$

For every i, j, its map on mapping spaces is the Yoneda equivalence

$$\mathrm{Map}_{\mathcal{M}}(i, j) = \mathrm{Map}_{\mathcal{C}}(P_j, P_i) \simeq \mathrm{Nat}(\omega_i, \omega_j).$$

In particular, every natural transformation between models is an equivalence, and $\mathrm{Mod}_n(\mathcal{C})$ is a π-finite $(n+1)$-groupoid.

Proof. The description of ω_i in (4.3) expresses it as pullback to P_i, followed by the equivalence Φ_i^{-1}. It therefore preserves finite limits, finite coproducts, and effective epimorphisms.

If the atlas is empty, $\mathcal{C}$ is degenerate. A model would have to send its initial and terminal object both to $\varnothing$ and to $*$, so there are no models, as required. Otherwise, let F be a model. Since F preserves the finite sum and the cover of the terminal object,

$$F(P) = \coprod_i F(P_i) \longrightarrow 1$$

is an effective epimorphism of spaces. Choose i and a point $a \in F(P_i)$. Yoneda associates to a a natural transformation $\omega_i \to F$. For every $A \in \mathcal{C}$, the counit of the slice equivalence gives a natural equivalence over P_i,

$$P_i \times \Delta\omega_i(A) \xrightarrow{\sim} P_i \times A.$$

Applying F, using preservation of products and the canonical identification of constants from Lemma 4.3, and taking the fibre at a gives an equivalence

$$\omega_i(A) \xrightarrow{\sim} F(A).$$

It is exactly the component of the Yoneda transformation and is natural in A. This proves essential surjectivity.

Yoneda gives $\mathrm{Nat}(\omega_i, \omega_j) \simeq \mathrm{Map}_{\mathcal{C}}(P_j, P_i)$. Every arrow on the right is an equivalence by Theorem 4.1, so every transformation on the left is an equivalence. Thus evaluation is fully faithful, and all natural transformations between arbitrary models are equivalences by the essential-surjectivity assertion. □

4.2 The converse construction

Proposition 4.5 (The converse construction)**.** *Let M be a π-finite $(n+1)$-type and put*

$$\mathcal{X}_M = \mathrm{Fun}(M, \mathcal{S}), \qquad \mathcal{C}_M = \mathrm{Fun}(M, \mathcal{F}_n).$$

Then $\mathcal{X}_M$ is a bounded coherent ∞-topos and $\mathcal{C}_M = (\mathcal{X}_M)^{\mathrm{coh}}_{\leq n}$. Choose one point x_i in each component of M and let

$$P_i = \mathrm{Map}_M(x_i, -) \in \mathcal{C}_M.$$

This finite family satisfies the hypotheses of Theorem 4.1. Its groupoid $\mathcal{M}$ is equivalent to M, compatibly with evaluation and the constant-object functors.

Proof. Straightening identifies $\mathrm{Fun}(M, \mathcal{S})$ with $\mathcal{S}/M$. A π-finite space is a truncated coherent object of $\mathcal{S}$, so the slice is bounded coherent; see [3, Examples 3.2.8 and 3.3.5, Paragraph 3.3.7]. In this slice, coherence of $Y \to M$ is equivalent to coherence of its total space, since the base is coherent. For an n-truncated map to the π-finite base M, its total space is π-finite if and only if its fibres are π-finite. One direction uses finite limits of π-finite spaces. For the other, use the long exact sequence of homotopy groups on each of the finitely many base components; the total space is truncated, and its homotopy groups and set of components are finite. This proves the description of $\mathcal{C}_M$.

The object P_i corresponds to the point $x_i : * \to M$. Its total space is coherent, and its fibres are path spaces in M, which are π-finite n-types. Thus P_i is coherent and n-truncated in $\mathcal{X}_M$. The map $\coprod_i P_i \to 1$ is an effective epimorphism because the chosen points meet every component of M.

Taking a slice at $x_i : * \to M$ identifies

$$\mathcal{X}_M/P_i \simeq \mathcal{S}, \qquad \mathcal{C}_M/P_i \simeq \mathcal{F}_n.$$

Under this equivalence $P_i \times \Delta K$ corresponds to K. Every $K \in \mathcal{F}_n$ has an effective epimorphism from a finite discrete space, obtained by choosing one point per component. The finite-section hypothesis therefore holds.

Finally, Yoneda gives

$$\mathrm{Map}_{\mathcal{C}_M}(P_i, P_j) \simeq \mathrm{Map}_M(x_j, x_i).$$

The chosen points meet all components, so the opposite of the full subgroupoid on the P_i is equivalent to M. The Yoneda equivalence identifies $\mathrm{Map}_{\mathcal{C}_M}(P_i, A)$ with $A(x_i)$, which proves the compatibility assertions. □

The standard facts about truncated coherent spaces and their slices in this proof can also be expressed intrinsically through the exact structure of π-finite spaces; see [1]. Theorem 4.1 and Proposition 4.5 give a recognition of finite homotopy semantics by the finite local-section condition.

5 Finite descent and sharpness

Truncation bounds the simplicial degrees needed for descent. Mapping spaces are n-types, whereas the core of a locally n-truncated category is an $(n+1)$-type; objects therefore require one additional degree. The totalization results used here are classical; compare [1, Lemmas 3.4.2–3.4.3] and [7, Proposition A.1(2)]. We give the space-valued argument to fix the indexing. The convention here uses n-truncated mapping spaces, whereas an r-category in the cited formulation has $(r-1)$-truncated mapping spaces.

Let $\Delta_{\leq m}$ be the full subcategory of Δ on $[0], \dots, [m]$. Limits below are homotopy limits.

Lemma 5.1 (Finite totalization)**.** *For a cosimplicial diagram of n-truncated spaces $Z^\bullet$, restriction induces an equivalence*

$$\lim_{\Delta} Z^\bullet \xrightarrow{\sim} \lim_{\Delta_{\leq n+1}} Z^\bullet.$$

Proof. Put $m = n+1$ and let $j : \Delta_{\leq m} \to \Delta$ be the inclusion. The left Kan extension of the constant point has value

$$(\mathrm{Lan}_j *)([q]) \simeq B(\Delta_{\leq m}/[q]).$$

Sending a map $[p] \to [q]$ to its image retracts this indexing category, up to homotopy, onto the poset of nonempty subsets of $\{0, \dots, q\}$ of cardinality at most $m+1$. Its classifying space is the barycentric subdivision of the m-skeleton of Δ^q. This skeleton is $(m-1)$-connected: the full simplex is obtained by adding cells of dimensions at least $m+1$. Hence the natural map $\mathrm{Lan}_j * \to *$ becomes an equivalence after objectwise n-truncation.

Since $Z^\bullet$ is objectwise n-truncated, mapping into it depends only on that objectwise truncation. Adjunction therefore gives

$$\begin{aligned} \lim_{\Delta} Z^\bullet &\simeq \mathrm{Map}_{\mathrm{Fun}(\Delta, \mathcal{S})}(*, Z^\bullet) \\ &\simeq \mathrm{Map}_{\mathrm{Fun}(\Delta, \mathcal{S})}(\mathrm{Lan}_j *, Z^\bullet) \\ &\simeq \lim_{\Delta_{\leq m}} Z^\bullet. \end{aligned}$$

□

Theorem 5.2 (Finite descent for objects and maps)**.** *Let $\mathcal{D}^\bullet$ be a cosimplicial diagram of locally n-truncated ∞-categories. Then*

$$\lim_{\Delta} \mathcal{D}^\bullet \xrightarrow{\sim} \lim_{\Delta_{\leq n+2}} \mathcal{D}^\bullet. \tag{5.1}$$

For objects A, B of the limit, their mapping space is

$$\mathrm{Map}(A, B) \simeq \lim_{[q] \in \Delta_{\leq n+1}} \mathrm{Map}_{\mathcal{D}^q}(A_q, B_q). \tag{5.2}$$

In particular, the reconstruction in Theorem 4.1 uses Čech degrees at most $n+2$ for objects and at most $n+1$ for maps.

Proof. Mapping spaces in a limit of ∞-categories are the limits of the mapping spaces of its compatible objects. These mapping spaces are n-truncated, so Lemma 5.1 proves (5.2).

For the assertion about categories, use the complete Segal nerve

$$N(\mathcal{D})_r = \mathrm{Fun}([r], \mathcal{D})^{\simeq}.$$

It preserves limits and detects equivalences. If $\mathcal{D}$ is locally n-truncated, so is its functor category on $[r]$, and the core of that functor category is an $(n+1)$-type. For each r, Lemma 5.1, with $n+1$ in place of n, therefore identifies the full cosimplicial limit of these nerve spaces with the limit through degree $n+2$. This proves (5.1) levelwise on the complete Segal nerve. □

5.1 Finite simplicial presentations

Theorem 5.3 (A finite simplicial presentation)**.** *Let M be a π-finite $(n+1)$-type. Choose a Kan simplicial set $G_\bullet$ with finite sets of simplices in every degree and an equivalence $|G_\bullet| \simeq M$. Such a model can be chosen to be $(n+2)$-coskeletal. Then*

$$\mathrm{Fun}(M, \mathcal{F}_n) \simeq \lim_{[q]\in\Delta_{\le n+2}} \mathcal{F}_n^{G_q}. \tag{5.3}$$

For $A, B \in \mathrm{Fun}(M, \mathcal{F}_n)$, write $A_q, B_q : G_q \to \mathcal{F}_n$ for their restrictions along the colimit cocone. There is a natural equivalence

$$\mathrm{Map}(A, B) \simeq \lim_{[q]\in\Delta_{\le n+1}} \prod_{g\in G_q} \mathrm{Map}_{\mathcal{F}_n}(A_q(g), B_q(g)). \tag{5.4}$$

Proof. The existence of a degreewise finite Kan presentation and its coskeletal truncation is the finite Kan model construction for π-finite types; see [1, Proposition 3.2.3]. Since M is the colimit of $G_\bullet$ in spaces,

$$\mathrm{Fun}(M, \mathcal{F}_n) \simeq \lim_{\Delta} \mathrm{Fun}(G_q, \mathcal{F}_n) = \lim_{\Delta} \mathcal{F}_n^{G_q}.$$

Each of the finite-product categories $\mathcal{F}_n^{G_q}$ is locally n-truncated. Applying Theorem 5.2 gives (5.3); its mapping-space formula gives (5.4). □

Here finiteness has two precise meanings: only the indicated simplicial degrees enter the limit, and each indexing set G_q in those degrees is finite. The categories $\mathcal{F}_n$ still contain all π-finite n-types. The recognition statement does not impose a bound on their cardinalities or on the orders of their homotopy groups.

Family level	Model-space bound	Maps through	Objects through
$n = 0$	1-type	degree 1	degree 2
$n = 1$	2-type	degree 2	degree 3
$n = 2$	3-type	degree 3	degree 4
n	$(n+1)$-type	degree $n+1$	degree $n+2$

5.2 Sharpness for truncated cosimplicial descent

Proposition 5.4 (Sharpness of the coherence degrees)**.** *For every $n \ge 0$, the degree $n+1$ in Lemma 5.1 cannot be replaced by n, even for cosimplicial π-finite n-types. The degree $n+2$ in (5.1) cannot be replaced by $n+1$, even for cosimplicial π-finite ∞-groupoids.*

Proof. Let $G = C_2 = \{1, g\}$ and let $A = \mathbb{F}_3$ with the action

$$ga = \varepsilon a, \qquad \varepsilon = (-1)^{n+1}.$$

Use the usual group-cochain cosimplicial space

$$Z^q = K(A, n)^{G^q}.$$

Its totalization is the homotopy fixed-point space $K(A,n)^{hG}$. The normalized cochain group in every degree is A: a normalized cochain is determined by its value on $(g,\dots,g)$. All internal terms in its coboundary vanish, because $g^2 = 1$ inserts an identity. Thus the differential on normalized cochains is multiplication by

$$d^q = \varepsilon + (-1)^{q+1}.$$

In particular, $d^{n-1} = 0$ for $n > 0$, whereas $d^n = 2\varepsilon$ is invertible in $\mathbb{F}_3$. The degree-n components of partial totalization are consequently

$$\pi_0 \operatorname{Tot}_n Z^\bullet \cong \mathbb{F}_3, \qquad \pi_0 \operatorname{Tot} Z^\bullet \cong H^n(G,A) = 0.$$

For $n = 0$, the first equality just says that $\operatorname{Tot}_0 = A$, whereas full totalization is the fixed-point set of the sign action, which contains only zero. For $n > 0$, normalization in the simplicial direction places A in homological degree n. Normalization in the cosimplicial direction then gives the group-cochain complex as a single row. The component group of partial totalization through degree n is therefore $C^n / \operatorname{im} d^{n-1}$; the component group of full totalization is $\ker d^n / \operatorname{im} d^{n-1}$. Thus through degree n there is no cocycle condition on the top cochain, and degree $n+1$ imposes $d^n = 0$.

For the categorical assertion, perform the same construction with $n+1$ in place of n and regard each resulting $(n+1)$-type as an ∞-groupoid. Its mapping spaces are n-truncated. Its limit through degree $n+1$ has three components, while the full limit has one component. Thus that finite limit is not equivalent to the full limit category. □

5.3 Sharpness for finite homotopy actions

The two cutoffs are optimal for the descent diagrams occurring in finite homotopy semantics. Put $G = C_2 = \{1, g\}$ and

$$\mathcal{D}^q = \mathcal{F}_n^{G^q}, \qquad \mathcal{D}_{\le m} = \lim_{\Delta_{\le m}} \mathcal{D}^\bullet.$$

The structure maps are pullback along the nerve of G. Thus $\lim_\Delta \mathcal{D}^\bullet \simeq \operatorname{Fun}(BG, \mathcal{F}_n)$. Write R_m for restriction from this full limit to $\mathcal{D}_{\le m}$.

Proposition 5.5 (Sharpness within finite homotopy actions)**.** *For every $n \ge 0$, the functor R_n is not full on homotopy categories. The functor R_{n+1} is not essentially surjective. Consequently neither the mapping cutoff $n+1$ nor the object cutoff $n+2$ can be decreased, even for the diagrams $\mathcal{F}_n^{C_2^q}$.*

Proof. We first describe precisely the partial descent spaces used below. Let $B_\bullet G$ be the nerve of G, and put $B_m = |\operatorname{sk}_m B_\bullet G|$. For every space T,

$$\lim_{\Delta_{\le m}} T^{G^q} \simeq \operatorname{Map}(B_m, T). \tag{5.5}$$

One way to obtain this equivalence is to represent T by a Kan complex. The cosimplicial space T^{G^q} is Reedy fibrant: its matching maps restrict functions to the degenerate simplices of $B_q G$, and hence are projections between products of Kan complexes. Its partial totalization is therefore the derived limit on the left. The end–coend adjunction identifies this partial totalization with maps from the realization of the simplicial m-skeleton, proving (5.5). Since passage to the core preserves limits, this also gives

$$(\mathcal{D}_{\le m})^\simeq \simeq \operatorname{Map}(B_m, \mathcal{F}_n^\simeq). \tag{5.6}$$

The standard bar model of BC_2 has one nondegenerate simplex in every degree. Its skeletal filtration is the usual $B_m \simeq \mathbb{RP}^m \subset \mathbb{RP}^\infty \simeq BC_2$.

Here is the cochain calculation we shall use twice. Let $A = \mathbb{F}_3$, and let g act on A by $\varepsilon \in \{1, -1\}$. A normalized degree-$q$ group cochain is determined by its value on $(g, \dots, g)$, so its group is A in every degree. All internal terms of the bar coboundary vanish: multiplying two adjacent g's inserts the identity. In these coordinates the differential is

$$d^q a = (\varepsilon + (-1)^{q+1})a. \tag{5.7}$$

This is also the cellular cochain complex of the skeletal bar model, with the corresponding choice of orientations and local coefficients. Truncating at B_m removes the outgoing differential in degree m.

Step 1: Maps. Take the terminal local system U, and let V have fibre $K(A, n)$ with the strict linear G-action $\varepsilon = (-1)^{n+1}$. These are genuine objects of $\mathrm{Fun}(BG, \mathcal{F}_n)$. Their mapping cosimplicial space is the group-cochain diagram $K(A, n)^{G^q}$, with the first coface twisted by this action. Its partial totalization computes sections of the associated $K(A, n)$-bundle on B_m. For $n \geq 1$, components of those sections at $m = n$ are

$$\pi_0 \operatorname{Map}_{\mathcal{D}_{\leq n}}(R_n U, R_n V) \cong H^n(B_n; A_\varepsilon) \cong A.$$

Indeed, (5.7) gives $d^{n-1} = 0$. For the full mapping space the next differential is present, and $d^n = 2\varepsilon$ is invertible in $\mathbb{F}_3$. Hence

$$\pi_0 \operatorname{Map}_{\mathrm{Fun}(BG, \mathcal{F}_n)}(U, V) \cong H^n(BG; A_\varepsilon) = 0.$$

The zero on the right denotes a single component. Restriction therefore misses both nonzero partial components. For $n = 0$, the partial mapping set is A, while the full mapping set is the fixed set of the sign action, namely $\{0\}$. This proves the assertion about R_n in all degrees.

Step 2: Objects for $n \geq 1$. Suppose $n \geq 1$ and put $r = n + 1 \geq 2$. We use the component of $K = K(A, n)$ in $\mathcal{F}_n^{\simeq}$, namely $B \operatorname{Aut}(K)$. There is an equivalence of ∞-groups

$$\operatorname{Aut}(K(A, n)) \simeq K(A, n) \rtimes \operatorname{Aut}(A). \tag{5.8}$$

For completeness, choose a simplicial abelian model of $K(A, n)$. Translations and linear automorphisms give the map in (5.8), compatibly with composition. Evaluation at zero identifies the fibre of the target over zero with pointed self-equivalences. The pointed mapping space has components $\mathrm{Hom}(A, A)$ and has no positive homotopy groups: its degree-j homotopy group is $\widetilde{H}^{n-j}(K(A, n); A) = 0$ for $1 \leq j \leq n$, and it is n-truncated. Restricting to self-equivalences gives the discrete fibre $\operatorname{Aut}(A)$. Thus the affine map is an equivalence on the base and on every fibre, proving (5.8).

Since $\operatorname{Aut}(A) = C_2$, delooping gives the split fibration

$$K(A, r) \longrightarrow E := B \operatorname{Aut}(K) \simeq K(A, r)//C_2 \xrightarrow{p} BC_2. \tag{5.9}$$

The zero of $K(A, r)$ supplies a section. We call the composite to BC_2 the *band*; it records the induced action on $A = \pi_n(K)$.

Choose $\rho : G \to C_2$ with $\rho(g) = \varepsilon = (-1)^{r+1} = (-1)^{n+2}$. Fix its restriction $b_m : B_m \to BC_2$. The space of lifts of b_m to E is the section space of the associated $K(A, r)$-bundle. Consequently its components are

$$\pi_0 \operatorname{Map}_{/BC_2}(b_m, E) \cong H^r(B_m; A_\varepsilon). \tag{5.10}$$

This is the usual cochain description of an Eilenberg–Mac Lane bundle: choices on r-cells are r-cochains, changes of choices on $(r-1)$-cells add coboundaries, and extension over $(r+1)$-cells imposes the cocycle condition. There are no additional obstruction groups, because the fibre has only the homotopy group A in degree r.

By (5.7),

$$d^{r-1} = 0, \qquad d^r = 2\varepsilon \in \mathbb{F}_3^\times.$$

It follows that the banded partial objects over B_r have component set A, whereas every lift over B_{r+1} restricts to the zero component. In particular, a nonzero class $a \in A$ does not extend with its band fixed.

To prove nonextendibility up to equivalence, we must also account for changes of band identification. For every $m \geq 2$, the inclusion $B_m \to BG$ induces an isomorphism on fundamental groups. Since the target BC_2 is a 1-type, its mapping space is the groupoid of homomorphisms $G \to C_2$ and their conjugacies. Thus the component containing b_m is

$$\mathrm{Map}(B_m, BC_2)_{b_m} \simeq BC_2.$$

Its loop group is the centralizer $C_{C_2}(\rho(G)) = C_2$. For the split fibration (5.9), a gauge element $h \in C_2$ acts on the lift space through the coefficient automorphism $a \mapsto ha$. In particular it sends the zero section to the zero section. Explicitly, applying the same h in every fibre of the associated bundle sends each cochain to h times that cochain and commutes with (5.7).

The elementary component sequence of this mapping-space fibration now says that the unbanded homotopy classes lying over the component of b_m are precisely

$$H^r(B_m; A_\varepsilon)/C_2. \tag{5.11}$$

Indeed, two points of the fibre become connected in the total space exactly when their components are related by the monodromy of a loop in the base. Restriction from B_{r+1} to B_r respects these actions. Its banded image is zero, hence its unbanded image is the orbit $\{0\}$. The nonzero classes form the different orbit $\{1, -1\}$. Moreover a different homomorphism ρ' cannot help: conjugacy in the abelian group C_2 does not change ρ, and all bands on B_r, for $r \geq 2$, already come from unique components of bands on BG. Thus the nonzero partial object is not equivalent to the restriction of any object over B_{r+1}, or over BG. Equation (5.6) proves the failure of essential surjectivity of R_{n+1} when $n \geq 1$.

Step 3: Objects for $n = 0$. For $n = 0$ one has $r = 1$, and the preceding argument using $\pi_1(B_r) = G$ is unavailable: $B_1 \simeq S^1$ has fundamental group $\mathbb{Z}$. We therefore handle this case directly. Let $K = A$ be the three-element set, so $\mathrm{Aut}(K) = S_3 = A \rtimes C_2$. The map

$$B_1 \simeq S^1 \longrightarrow BS_3$$

which sends its generator to the translation $x \mapsto x + 1$ is a degree-one partial descent object with trivial band in C_2, as required by $\varepsilon = (-1)^{r+1} = 1$. It cannot extend to B_2: the relation imposed there is $g^2 = 1$, whereas this translation has order three. Equivalence after forgetting the band only conjugates its monodromy in S_3, and conjugation preserves order. More explicitly, conjugation by translations fixes the translation subgroup, and conjugation by its linear C_2 factor sends 1 to -1. Its nonzero orbit still does not meet zero. Hence no change of band or gauge can turn this partial object into a genuine C_2-action. This proves the remaining case and completes the proof. □

Remark 5.6. The mapping counterexample uses two genuine full descent objects and a nonextendible partial map between them. The object counterexample uses an object of the partial limit itself,

and the gauge argument proves nonextendibility up to equivalence, without adding a chosen identification of its homotopy group. Every set G^q occurring here is finite. The examples therefore establish sharpness inside the same degreewise finite descent format as the recognition theorem.

6 A finite comparison criterion

The atlas theorem derives its mapping comparison from local sections. In other semantic presentations, one starts instead with a comparison functor and a component-lifting property. The following criterion separates that property from the presentations needed for objects.

6.1 From components to mapping spaces

We use the left exact full-faithfulness criterion of [5, Lemma 13.3]. The proof below also records how conservativity supplies injectivity on components. The argument uses based loop objects in slices, so it retains the basepoint information needed to compare homotopy groups.

Lemma 6.1 (Components of the space of sections)**.** *Let $p : E \to A$ be a morphism in an ∞-category $\mathcal{C}$. Write*

$$\mathrm{Sec}_{\mathcal{C}}(p) := \mathrm{Map}_{\mathcal{C}_{/A}}(\mathrm{id}_A, p).$$

Forgetting the specified homotopy $ps \simeq \mathrm{id}_A$ induces a natural bijection

$$\pi_0 \mathrm{Sec}_{\mathcal{C}}(p) \xrightarrow{\sim} \{[s] \in \mathrm{Hom}_{h\mathcal{C}}(A, E) : [p][s] = [\mathrm{id}_A]\}.$$

Proof. Put $G = \mathrm{Aut}_{\mathcal{C}}(A)$ and

$$T = \mathrm{Map}_{\mathcal{C}}(A, E) \times_{\mathrm{Map}_{\mathcal{C}}(A,A)} G.$$

Here G is the union of the equivalence components in the endomorphism space. Composition gives a map over G

$$\mathrm{Sec}_{\mathcal{C}}(p) \times G \longrightarrow T, \qquad (s, g) \longmapsto s \circ g,$$

with the homotopy $p(sg) \simeq g$ induced by the section data. On the fibre over each $g \in G$, this is an equivalence: its inverse is precomposition with g^{-1}. Hence it is an equivalence of spaces over G. Since T is the union of the components of $\mathrm{Map}_{\mathcal{C}}(A, E)$ whose composite with p is an equivalence, the fibre over $[\mathrm{id}_A]$ of $\pi_0 T \to \pi_0 G$ is exactly the set on the right. The product description identifies this fibre with $\pi_0 \mathrm{Sec}(p)$. □

Lemma 6.2 (Passing to slices)**.** *Let $\mathcal{C}, \mathcal{D}$ admit pullbacks. Suppose that $F : \mathcal{C} \to \mathcal{D}$ preserves them and $hF : h\mathcal{C} \to h\mathcal{D}$ is fully faithful. Then, for every $B \in \mathcal{C}$, the induced functor*

$$h(F_{/B}) : h(\mathcal{C}_{/B}) \longrightarrow h(\mathcal{D}_{/FB})$$

is fully faithful.

Proof. For $a : A \to B$ and $e : E \to B$, there is a natural equivalence

$$\mathrm{Map}_{\mathcal{C}_{/B}}(a, e) \simeq \mathrm{Sec}_{\mathcal{C}}(A \times_B E \to A).$$

Apply Lemma 6.1. Full faithfulness of hF identifies the two ordinary hom-sets defining the section condition, and preservation of pullbacks identifies their pullback objects. It therefore induces a bijection on the displayed mapping spaces' sets of components. □

Proposition 6.3 (Left exact full-faithfulness criterion)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ preserve finite limits, where $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories with finite limits. The following are equivalent:*

(i) *F is fully faithful.*

(ii) *hF is fully faithful.*

(iii) *F is conservative and hF is full.*

In particular, in (iii) no independent lifting or injectivity assumption on positive homotopy groups is required.

Proof. The implications (i)⇒(ii)⇒(iii) hold for arbitrary ∞-functors. For (iii)⇒(ii), consider $f, g : A \to B$ with $Ff \simeq Fg$, and form the homotopy equalizer

$$p : E := A \times_{B\times B} B \longrightarrow A,$$

where $A \to B \times B$ is (f, g) and $B \to B \times B$ is the diagonal. The chosen homotopy gives a section $t : FA \to FE$ of Fp. Fullness of hF supplies $s : A \to E$ with $Fs \simeq t$. Consequently $F(ps) \simeq \mathrm{id}_{FA}$. Conservativity implies that ps is an equivalence. Replacing s by $s(ps)^{-1}$ gives a section of p, and hence a homotopy $f \simeq g$. Thus hF is faithful as well as full.

It remains to prove (ii)⇒(i). Fix $A, B \in \mathcal{C}$ and a basepoint $f \in \mathrm{Map}_{\mathcal{C}}(A, B)$. In $\mathcal{C}_{/A}$, let $1_A = \mathrm{id}_A$, let $Q = (A \times B \to A)$, and point Q by the section $\sigma_f = (\mathrm{id}_A, f) : 1_A \to Q$. Define $\Omega_{\sigma_f} Q = 1_A \times_Q 1_A$ in the slice, with its canonical pointing, and iterate this construction. Finite limits give natural equivalences, for every $k \geq 1$,

$$\mathrm{Map}_{\mathcal{C}_{/A}}(1_A, \Omega^k_{\sigma_f} Q) \simeq \Omega^k_f \,\mathrm{Map}_{\mathcal{C}}(A, B).$$

The functor $F_{/A}$ preserves these pointed loop objects. By Lemma 6.2, its homotopy-category functor is fully faithful. Taking components therefore gives

$$\pi_k(\mathrm{Map}_{\mathcal{C}}(A, B), f) \xrightarrow{\sim} \pi_k(\mathrm{Map}_{\mathcal{D}}(FA, FB), Ff) \qquad (k \geq 1).$$

Hypothesis (ii) already gives a bijection on π_0. Thus the map of mapping spaces is an equivalence, by the homotopy-group criterion for spaces. This holds for all A, B. □

6.2 Effective presentations at a finite descent stage

Let $\mathcal{X}$ and $\mathcal{Z}$ be bounded coherent ∞-topoi and put

$$\mathcal{C} = \mathcal{X}^{\mathrm{coh}}_{\leq n}, \qquad \mathcal{D} = \mathcal{Z}^{\mathrm{coh}}_{\leq n}.$$

Suppose a cosimplicial diagram $\mathcal{D}^\bullet$ of locally n-truncated categories presents $\mathcal{D}$:

$$\mathcal{D} \simeq \lim_{\Delta} \mathcal{D}^\bullet.$$

For a functor $e : \mathcal{C} \to \mathcal{D}$, write e_q for its component in $\mathcal{D}^q$. For $A, B \in \mathcal{C}$, let

$$T_{A,B} = \lim_{[q]\in\Delta_{\leq n+1}} \mathrm{Map}_{\mathcal{D}^q}(e_q A, e_q B). \tag{6.1}$$

The compatible identifications in the cone defining e are part of this diagram of mapping spaces.

Theorem 6.4 (Finite comparison by maps and presentations)**.** *Assume that e preserves finite limits, finite coproducts, and effective epimorphisms, and is conservative. Suppose:*

(i) *For all $A, B \in \mathcal{C}$, the natural map*

$$\pi_0 \operatorname{Map}_{\mathcal{C}}(A, B) \longrightarrow \pi_0 T_{A,B}$$

is surjective.

(ii) *Every $Y \in \mathcal{D}$ has an effective epimorphism $eU \to Y$ with $U \in \mathcal{C}$, whose kernel-pair object is equivalent to eV for some $V \in \mathcal{C}$.*

Then e is an equivalence. The semantic target can be computed through degree $n+2$, and the mapping test (i) *uses only degrees through $n+1$.*

Proof. Theorem 5.2 identifies $T_{A,B}$ with $\operatorname{Map}_{\mathcal{D}}(eA, eB)$. Thus (i) says that *he* is full. Since e is conservative and left exact, Proposition 6.3 shows that e is fully faithful.

Let $eU \to Y$ be a presentation in (ii). Its entire Čech nerve has terms in the essential image of e: all terms are finite iterated fibre products of eU and eV. Full faithfulness therefore lifts the diagram, with its face maps, degeneracies, and coherences, to a groupoid object $R_\bullet$ of $\mathcal{C}$. The map $eV \to eU \times eU$ is a pullback of the diagonal of Y, so it is $(n-1)$-truncated. The iterated diagonal criterion and conservativity of e imply that $R_1 \to R_0 \times R_0$ is $(n-1)$-truncated in $\mathcal{C}$.

Lemma 2.2 gives its effective quotient $Q \in \mathcal{C}$. The map $U \to Q$ is an effective epimorphism. Since e preserves this epimorphism and its Čech nerve, eQ is the effective quotient of $eR_\bullet$. The same nerve has quotient Y. Thus $eQ \simeq Y$. This proves essential surjectivity; the bound on the target follows from Theorem 5.2. □

The object condition specifies only a covering object and its kernel-pair object. It does not postulate that the transition maps are definable. Those maps lift only after the mapping comparison has been proved. Likewise, a groupoid quotient is used only after its efficiency has been checked. These two steps are necessary when working at a fixed truncation level.

Example 6.5 (Conservativity does not supply the mapping test)**.** For a nontrivial finite group H, the forgetful functor from finite H-sets to finite sets is conservative and preserves finite limits, finite coproducts, and effective epimorphisms. It is essentially surjective, by giving any finite set the trivial action. It is not full: a map from the one-point trivial H-set to the regular H-set cannot be equivariant. Thus conservativity cannot replace (i), even at $n = 0$.

7 The homotopy type detected by finite models

For a π-finite space X and $n \geq 0$, write

$$\mathcal{L}_n(X) = \operatorname{Fun}(X, \mathcal{F}_n).$$

Equivalently, this is the category of families over X with π-finite n-truncated homotopy fibres. Truncation is relative to X. The total space of such a family need not be an n-type.

Lemma 7.1 (Dependence on the base truncation)**.** *For any space X, the truncation map induces an equivalence*

$$\operatorname{Fun}(\tau_{\leq n+1} X, \mathcal{F}_n) \xrightarrow{\sim} \operatorname{Fun}(X, \mathcal{F}_n).$$

More generally, the same assertion holds with $\mathcal{F}_n$ replaced by any locally n-truncated ∞-category.

Proof. Let $\mathcal{E}$ be locally n-truncated. The rth space of its complete Segal nerve is $\mathrm{Fun}([r],\mathcal{E})^{\simeq}$, an $(n+1)$-type. Exponential adjunction and the fact that X is a groupoid identify the rth space of the complete Segal nerve of $\mathrm{Fun}(X,\mathcal{E})$ with

$$\mathrm{Map}(X,\mathrm{Fun}([r],\mathcal{E})^{\simeq}).$$

Mapping into this $(n+1)$-type depends only on $\tau_{\leq n+1}X$. Thus the comparison is an equivalence at every level of the complete Segal nerve. □

Theorem 7.2 (Finite higher Galois duality)**.** *Let X, Y be π-finite $(n+1)$-types. Pullback gives an equivalence*

$$\mathrm{Map}(Y,X) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lex},\amalg,\mathrm{epi}}(\mathcal{L}_n(X),\mathcal{L}_n(Y)). \tag{7.1}$$

The right side is the full category of functors preserving finite limits, finite coproducts, and effective epimorphisms, with all natural transformations. In particular, it is a groupoid.

Proof. Proposition 4.5 supplies the finite point atlas of $\mathcal{L}_n(X)$. Theorem 4.4 therefore gives

$$\mathrm{Fun}^{\mathrm{lex},\amalg,\mathrm{epi}}(\mathcal{L}_n(X),\mathcal{F}_n) \simeq X.$$

Finite limits, finite coproducts, and effective epimorphisms in $\mathcal{L}_n(Y)$ are computed pointwise. Consequently exponential adjunction restricts to

$$\begin{aligned}\mathrm{Fun}^{\mathrm{lex},\amalg,\mathrm{epi}}(\mathcal{L}_n(X),\mathrm{Fun}(Y,\mathcal{F}_n)) &\simeq \mathrm{Fun}(Y,\mathrm{Fun}^{\mathrm{lex},\amalg,\mathrm{epi}}(\mathcal{L}_n(X),\mathcal{F}_n))\\ &\simeq \mathrm{Fun}(Y,X).\end{aligned}$$

The last category is the mapping space $\mathrm{Map}(Y,X)$. Under these equivalences a map $f: Y \to X$ yields the family of models $y \mapsto \omega_{f(y)}$, which is precisely pullback by f. □

Corollary 7.3 (Exact detection of the visible homotopy type)**.** *For a map $f: Y \to X$ between π-finite spaces, the following conditions are equivalent:*

(i) *$f^*: \mathcal{L}_n(X) \to \mathcal{L}_n(Y)$ is an equivalence;*

(ii) *$\tau_{\leq n+1}f$ is an equivalence.*

Moreover, for every π-finite X,

$$\mathrm{Mod}_n(\mathcal{L}_n(X)) \simeq \tau_{\leq n+1}X.$$

Proof. Lemma 7.1 reduces the question to $(n+1)$-types. For those types, precomposition by f^* on model categories identifies with $f: Y \to X$, since it takes evaluation at y to evaluation at $f(y)$. An equivalence f^* induces an equivalence of these full model categories, proving (i)⇒(ii). The converse is immediate from the lemma. The model-space formula follows from the same reduction and Theorem 4.4. □

This result concerns the entire finite model category, including its higher transformations. Taking only isomorphism classes of models, or replacing its mapping spaces by sets of components, can remove information inside the displayed $(n+1)$-type. The examples below locate that loss explicitly.

Remark 7.4 (The ambient topos need not be recovered). Take $\mathcal{X} = \mathrm{Fun}(K(A, n+2), \mathcal{S})$ for a nonzero finite abelian group A. Lemma 7.1 gives $\mathcal{X}^{\mathrm{coh}}_{\leq n} \simeq \mathcal{F}_n$, so the singleton atlas $P = 1$ satisfies the theorem and its recovered model space is the point. In the ambient topos, however, the point object over $K(A, n+2)$ has fibre $K(A, n+1)$ and retains the group A. This object lies outside the fixed n-truncation range. To see directly that this topos differs from $\mathcal{S}$, put $K = K(A, n+2)$. Then $\Gamma_{\mathcal{X}}\Delta K \simeq \mathrm{Map}(K, K)$ has distinct components containing the zero and identity maps, whereas $\Gamma_{\mathcal{S}}\Delta K \simeq K$ is connected. The conclusion of the theorem therefore concerns the fixed range, not the whole topos.

8 The universal parameter of an exact functor

Write $\mathrm{Fun}^{\mathrm{ex}}$ for functors preserving finite limits, finite coproducts, and effective epimorphisms. Throughout this section, $\mathcal{C}$ and $\mathcal{D}$ are categories satisfying the recognition theorem at the same level n. Thus

$$\mathcal{C} \simeq \mathcal{L}_n(X), \qquad \mathcal{D} \simeq \mathcal{L}_n(Y)$$

for π-finite $(n+1)$-types X and Y. The following result associates a single intrinsic parameter object to every exact functor. It also determines all its factorizations through a slice.

For $A \in \mathcal{C}$, let

$$\pi_A^* : \mathcal{C} \longrightarrow \mathcal{C}/A, \qquad B \longmapsto (B \times A \longrightarrow A).$$

The slice $\mathcal{C}/A$ again satisfies the recognition theorem: if $Z_A \to X$ is the finite n-truncated family represented by A, then Z_A is a π-finite $(n+1)$-type and

$$\mathcal{C}/A \simeq \mathcal{L}_n(Z_A). \tag{8.1}$$

Indeed, a family with n-truncated fibres over Z_A has n-truncated fibres over X, since $Z_A \to X$ is itself n-truncated. The converse follows by taking fibres of a map between n-types.

Lemma 8.1 (Finite relative left adjoints)**.** *Every exact functor $F : \mathcal{C} \to \mathcal{D}$ admits a left adjoint $F_!$. This adjoint satisfies the projection formula*

$$F_!(B \times F(A)) \simeq F_!(B) \times A, \qquad A \in \mathcal{C},\ B \in \mathcal{D}. \tag{8.2}$$

Put $P_F = F_!(1_{\mathcal{D}})$. Under Theorem 7.2, if $F = f^*$ for $f : Y \to X$, then P_F is the relative n-truncation of $Y \to X$:

$$(P_F)_x \simeq \tau_{\leq n}\,\mathrm{hofib}_x(f). \tag{8.3}$$

Proof. Use the equivalence $F \simeq f^*$ of Theorem 7.2. An object $B \in \mathcal{L}_n(Y)$ is a family $V \to Y$. Compose it with f and take relative n-truncation over X. This gives $F_!B$. Its fibre over x is

$$\tau_{\leq n}(V \times_X \{x\}).$$

The space before truncation is π-finite: both the relevant base fibre and the fibres of $V \to Y$ are π-finite, and the assertion follows from the homotopy exact sequence. Hence the construction remains inside $\mathcal{L}_n(X)$.

Composition of families is left adjoint to pullback in spaces, and relative truncation is left adjoint to the inclusion of n-truncated families. Their composite is therefore left adjoint to f^* on these finite categories. On the fibre over x, the left side of (8.2) is

$$\tau_{\leq n}\big((V \times_X \{x\}) \times A_x\big) \simeq \tau_{\leq n}(V \times_X \{x\}) \times A_x.$$

This equivalence uses preservation of finite products by truncation and $A_x \in \mathcal{F}_n$. It is natural in x, so proves the projection formula. Taking $B = 1_{\mathcal{D}}$ gives (8.3). □

Let $u_F : 1_{\mathcal{D}} \to F(P_F)$ denote the unit of this adjunction at $1_{\mathcal{D}}$. For $A \in \mathcal{C}$, define the space of slice factorizations by

$$\mathcal{R}_A(F) = \mathrm{hofib}_F \left(\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}/A, \mathcal{D}) \xrightarrow{J \mapsto J\pi_A^*} \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D}) \right).$$

Thus a point includes a specified equivalence $J\pi_A^* \simeq F$. The functor categories here are groupoids by finite higher Galois duality.

Theorem 8.2 (Universal parameter and exact slice factorization)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be exact.*

(i) *The pair (P_F, u_F) represents all parameters which acquire a chosen section after applying F:*

$$\mathcal{R}_A(F) \simeq \mathrm{Map}_{\mathcal{D}}(1_{\mathcal{D}}, F(A)) \simeq \mathrm{Map}_{\mathcal{C}}(P_F, A). \tag{8.4}$$

For a section $a : 1_{\mathcal{D}} \to F(A)$, the corresponding functor is

$$J_a(B \longrightarrow A) = 1_{\mathcal{D}} \times_{F(A)} F(B). \tag{8.5}$$

(ii) *There is a canonical factorization*

$$\mathcal{C} \xrightarrow{\pi^*_{P_F}} \mathcal{C}/P_F \xrightarrow{J_F} \mathcal{D}, \qquad F \simeq J_F \pi^*_{P_F}, \tag{8.6}$$

where $J_F = J_{u_F}$ is fully faithful and exact. More generally, J_a is fully faithful if and only if the map $P_F \to A$ corresponding to a is an equivalence. Consequently (8.6) is, up to contractible choice, the unique slice factorization whose second functor is fully faithful.

(iii) *The following conditions are equivalent:*

(a) *F is fully faithful;*

(b) *$P_F \simeq 1_{\mathcal{C}}$;*

(c) *u_F is homotopic to $F(s)$ for a section $s : 1_{\mathcal{C}} \to P_F$;*

(d) *for every $A \in \mathcal{C}$, every section of $F(A)$ is homotopic to the image of a section of A.*

Under these conditions the comparison of section spaces is an equivalence, including all higher homotopies.

(iv) *F is conservative if and only if $P_F \to 1_{\mathcal{C}}$ is an effective epimorphism.*

Proof. The second equivalence in (8.4) is adjunction. For the first, a factorization $J\pi_A^* \simeq F$ takes the diagonal

$$1_{\mathcal{C}/A} \longrightarrow \pi_A^* A$$

to a section $a : 1_{\mathcal{D}} \to F(A)$. Every object $B \to A$ is the pullback in $\mathcal{C}/A$ of this diagonal along $\pi_A^*(B \to A)$. Left exactness therefore forces (8.5). Conversely, that formula defines an exact functor: finite limits are preserved, and finite coproducts and effective epimorphisms are stable under pullback. It comes with $J_a\pi_A^* \simeq F$. The two constructions are mutually inverse naturally in the section and in its higher homotopies; thus they give an equivalence of spaces, not just a bijection of components.

For any exact functor T between categories of the present form, Lemma 8.1 gives

$$T_! T(A) \simeq A \times T_!(1).$$

The counit is the projection to A. A right adjoint is fully faithful precisely when its counit is an equivalence. Consequently

$$T \text{ is fully faithful} \quad \Longleftrightarrow \quad T_!(1) \simeq 1. \tag{8.7}$$

For J_a, its left adjoint sends $D \in \mathcal{D}$ to $F_!D \to A$, where the structure map is adjoint to $D \to 1_{\mathcal{D}} \xrightarrow{a} F(A)$. This follows directly by taking the fibre of the adjunction equivalence over the specified structure map. In particular,

$$(J_a)_!(1_{\mathcal{D}}) = (P_F \longrightarrow A) \quad \text{in } \mathcal{C}/A.$$

By (8.7), J_a is fully faithful exactly when this object is terminal, equivalently when $P_F \to A$ is an equivalence. Taking $a = u_F$ gives the identity of P_F, and proves (ii). The space of equivalences out of a fixed object, with their targets allowed to vary, is contractible; this also gives the asserted uniqueness.

The equivalence of (iii)(a) and (iii)(b) is (8.7). If $u_F \simeq F(s)$, adjunction identifies this equality with

$$\mathrm{id}_{P_F} \simeq (P_F \longrightarrow 1_{\mathcal{C}} \xrightarrow{s} P_F).$$

Hence $P_F \to 1_{\mathcal{C}}$ is an equivalence. Conversely, if $P_F \simeq 1_{\mathcal{C}}$, adjunction identifies $\mathrm{Map}_{\mathcal{D}}(1, F(A))$ with $\mathrm{Map}_{\mathcal{C}}(1, A)$ naturally in A. This proves (iii)(b)⇒(iii)(d), including the assertion about higher section spaces. Applying (iii)(d) to $A = P_F$ and the distinguished section u_F gives (iii)(c), completing the equivalence of the four conditions.

Finally write $F = f^*$. It is conservative if $\pi_0(f)$ is surjective, since equivalences of families can then be tested over points in the image. If a component of X is missed, the map of families which is $\varnothing \to *$ on that component and an equivalence on the others disproves conservativity. On the other hand, (8.3) says that $P_F \to 1$ is an effective epimorphism exactly when every homotopy fibre of f is nonempty. This is again surjectivity of $\pi_0(f)$. □

The parameter P_F is initial among objects with a chosen F-section: the category of pairs (A, a) is equivalent to the undercategory $\mathcal{C}_{P_F/}$. Part (iii) shows that the single distinguished section u_F is a complete test for eliminating these relative parameters. Merely asking whether P_F has some section would be weaker; the section must induce u_F after applying F.

The topological interpretation of (8.6) is the relative truncation factorization

$$Y \xrightarrow{q} Z \xrightarrow{p} X, \qquad Z = (\tau^X_{\leq n} Y)_{\mathrm{total}}. \tag{8.8}$$

Here p has n-truncated fibres and the fibres of q have contractible n-truncations. In the convention that a space is n-connected when its n-truncation is contractible, the fibres of q are n-connected. This is the standard connected–truncated factorization of spaces; see [9, Sections 5.5.6 and 6.5.1]. The theorem identifies it entirely in terms of the exact functor and its possible parameter extensions.

Proposition 8.3 (Composition and change of parameters)**.** *For exact functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, there is a natural equivalence*

$$P_{GF} \simeq F_!(P_G). \tag{8.9}$$

For $A \in \mathcal{C}$, the induced exact functor $F_A : \mathcal{C}/A \to \mathcal{D}/F(A)$ has universal parameter

$$P_{F_A} \simeq (A \times P_F \longrightarrow A). \tag{8.10}$$

Consequently full faithfulness can be checked after a finite effective-epimorphic parameter extension: if $A \to 1$ is an effective epimorphism, then F_A is fully faithful if and only if F is fully faithful.

Proof. The composite has left adjoint $F_!G_!$, which proves (8.9). The left adjoint of F_A sends $D \to F(A)$ to $F_!D \to A$, using the counit. On the terminal object this gives $F_!F(A) \to A$. The projection formula identifies it with $A \times P_F \to A$, proving (8.10). An equivalence between families is detected after pullback along an effective epimorphism. Apply this to $P_F \to 1$ and use Theorem 8.2(iii). □

Corollary 8.4 (Finite descent from the universal parameter)**.** *For an exact functor $F : \mathcal{C} \to \mathcal{D}$, the following are equivalent:*

(i) *F is conservative;*

(ii) *$P_F \to 1_{\mathcal{C}}$ is an effective epimorphism;*

(iii) *restriction along the powers of P_F induces an equivalence*

$$\mathcal{C} \simeq \lim_{[q] \in \Delta_{\leq n+2}} \mathcal{C}/P_F^{q+1}. \tag{8.11}$$

Mapping spaces between compatible objects in this limit are determined through degree $n+1$. Both degree bounds are optimal among conservative exact functors.

Proof. The first equivalence is Theorem 8.2. If P_F covers 1, let $Z \to X$ be its total family. The augmented Čech nerve of $Z \to X$ has colimit X in spaces. Taking functor categories into $\mathcal{F}_n$ gives

$$\mathcal{C} \simeq \lim_{\Delta} \mathcal{C}/P_F^{q+1}.$$

Here (8.1) identifies the individual terms. Theorem 5.2 gives the two finite cutoffs. If P_F does not cover 1, all its powers miss a component of X. The non-equivalence used in the proof of conservativity above becomes an equivalence in every slice term, so (iii) cannot hold.

For sharpness, take the evaluation functor $\mathcal{L}_n(BC_2) \to \mathcal{F}_n$. Its universal parameter is the regular C_2-torsor, and the displayed diagram has terms $\mathcal{F}_n^{C_2^q}$. Thus Proposition 5.5 supplies both strict lower bounds within this class. □

Corollary 8.5 (Homotopy criterion for full faithfulness)**.** *Let $f : Y \to X$ be a map of π-finite $(n+1)$-types. The exact functor $f^* : \mathcal{L}_n(X) \to \mathcal{L}_n(Y)$ is fully faithful if and only if every homotopy fibre of f is n-connected. Equivalently, f is bijective on components, induces isomorphisms on π_i for $1 \leq i \leq n$, and induces surjections on π_{n+1} at every base point.*

Proof. The first assertion follows from (8.3) and Theorem 8.2(iii). The second is the homotopy exact sequence of each fibre. For $n = 0$ it says precisely that f is bijective on components and surjective on fundamental groups. □

Example 8.6 (Finite group images)**.** Let $\varphi : H \to G$ be a homomorphism of finite groups, and take $n = 0$. Restriction $F : \operatorname{Fin} G\text{-Set} \to \operatorname{Fin} H\text{-Set}$ has

$$P_F = G/\operatorname{im}\varphi.$$

Its canonical slice factorization is restriction to $\operatorname{im}\varphi$ followed by inflation along $H \twoheadrightarrow \operatorname{im}\varphi$. The latter is fully faithful. A parameter A and a chosen H-fixed point of its restriction give a fully faithful slice factorization precisely when the induced map $G/\operatorname{im}\varphi \to A$ is an isomorphism. Thus the ordinary orbit parameter is the degree-zero instance of the universal parameter theorem.

Example 8.7 (Higher image parameters)**.** Let $n \geq 1$, let A, B be finite abelian groups, and let $\alpha : B \to A$. For the exact functor induced by

$$f = K(\alpha, n+1) : K(B, n+1) \longrightarrow K(A, n+1),$$

the universal parameter is the family

$$K(\operatorname{im}\alpha, n+1) \longrightarrow K(A, n+1). \tag{8.12}$$

Its fibre is $K(\operatorname{coker}\alpha, n)$. Indeed, the fibre of f has $\pi_{n+1} = \ker\alpha$, $\pi_n = \operatorname{coker}\alpha$, and no other nonzero homotopy groups, while the factorization through $K(\operatorname{im}\alpha, n+1)$ has respectively n-connected and n-truncated fibres. In particular, f^* is fully faithful exactly when α is surjective.

For $B = 0$ and $A \neq 0$, the universal parameter has connected fibres $K(A, n)$, but it is not terminal. Thus replacing a higher parameter by its set of components can remove the complete obstruction to full faithfulness. Concretely, the family $* \to K(A, n+1)$ has no section, whereas its evaluation at the base point has sections.

Proposition 8.8 (The first failure in a Postnikov projection)**.** *Let $r \geq 1$ and let $p : E \to B$ be a Postnikov projection from a connected π-finite r-type to its $(r-1)$-truncation. Assume $\pi_r(E)$ is nonzero. Then pullback has the following three regimes:*

Semantic level	*Behaviour of p^**
$0 \leq n \leq r-2$	*equivalence*
$n = r-1$	*fully faithful and not essentially surjective*
$n \geq r$	*conservative and not fully faithful*

At the final range, the induced functor on homotopy categories is not full. The first range is empty when $r = 1$.

Proof. For $n \leq r-2$, the two bases have the same $(n+1)$-truncation, so Corollary 7.3 applies. The fibres of p are $K(\pi_r(E), r)$, with their natural monodromy. They have contractible $(r-1)$-truncations but noncontractible r-truncations. The universal-parameter criterion therefore gives full faithfulness precisely through level $r-1$. At level $r-1$ an equivalence would, by the same recognition corollary, force p itself to be an equivalence, contrary to the nonzero top group. Conservativity holds at every level because p is surjective on components. If the homotopy-category functor were full in the last range, the conservative left exact full-faithfulness criterion (Proposition 6.3) would make p^* fully faithful, a contradiction. □

9 Relative Postnikov obstructions for exact functors

Finite higher Galois duality identifies a tower of semantic categories with the Postnikov tower of its model space. We now determine the obstruction to extending an exact functor by one level and the full space of extensions when that obstruction vanishes. The underlying classification of Eilenberg–Mac Lane fibrations and their section spaces is classical; see Møller [10, Theorem 3.1] and Møller–Scherer [11, Lemma 2.1]. Here it supplies intrinsic invariants of exact functors, including their relative natural automorphisms.

9.1 Local coefficients and affine maps

Let B be a space and let $A : B \to \mathbf{Ab}_{\mathrm{fin}}$ be a local system of finite abelian groups. Thus A factors through the fundamental groupoid of B, and transport acts by group isomorphisms. Write $K_B(A, q) \to B$ for its Eilenberg–Mac Lane family, with its zero section, and put

$$\mathcal{H}^q(B; A) = \Gamma_B K_B(A, q) = \mathrm{Map}_{\mathcal{S}/B}(B, K_B(A, q)).$$

For $q \geq 0$ this is an abelian group object in spaces. In particular, $\pi_0\mathcal{H}^q(B; A) = H^q(B; A)$, and

$$\pi_i\mathcal{H}^q(B; A) = H^{q-i}(B; A) \quad (0 \leq i \leq q), \qquad \pi_i\mathcal{H}^q(B; A) = 0 \quad (i > q). \tag{9.1}$$

The homotopy groups in positive degree are taken at the zero section. Translation gives the corresponding identifications at any other section. These formulas follow by looping the section space: limits commute with loops and $\Omega_B K_B(A, q) = K_B(A, q - 1)$.

A *band identification* for a fibration with fibres $K(A_b, r)$ is a specified identification of the local system of rth homotopy groups with A. Band identifications will be retained unless their removal is stated explicitly.

Lemma 9.1 (Affine maps and banded extensions)**.** *Let $r \geq 1$ and let D, C be abelian groups. The map*

$$K(C, r) \times \mathrm{Hom}(D, C) \longrightarrow \mathrm{Map}(K(D, r), K(C, r)), \qquad (c, \alpha) \longmapsto c + K(\alpha, r), \tag{9.2}$$

is an equivalence. These equivalences respect composition in the form

$$(c, \alpha) \circ (d, \beta) = (c + \alpha(d), \alpha\beta).$$

Consequently, for a local system A on B, the space of fibrations over B with Eilenberg–Mac Lane fibres in degree r and band identification A is

$$\mathcal{H}^{r+1}(B; A). \tag{9.3}$$

It is equivalently the space of torsors under the group object $K_B(A, r)$ in $\mathcal{S}/B$.

Proof. The pointed mapping space in (9.2) is discrete. Indeed, its components are

$$\widetilde{H}^r(K(D, r); C) = \mathrm{Hom}(D, C),$$

by the Hurewicz theorem and the universal coefficient sequence. Its positive homotopy groups are the reduced cohomology groups in degrees below r, which vanish. Evaluation at the base point of $K(D, r)$ splits because the target is an abelian group object: subtracting that value makes a map pointed. This proves the equivalence. Choosing the functorial simplicial abelian models for $K(-, r)$ gives the displayed composition formula.

In particular, the automorphism group object of $K(A, r)$ is $K(A, r) \rtimes \mathrm{Aut}(A)$. Its classifying space fits into the split fibration [11, Lemma 2.1]

$$K(A, r + 1) \longrightarrow B\,\mathrm{Aut}(K(A, r)) \longrightarrow B\,\mathrm{Aut}(A).$$

Fixing the map $B \to B\,\mathrm{Aut}(A)$ that describes a local system leaves precisely the space of sections of its associated $K(A, r + 1)$-family. This proves (9.3), component by component when the fibres of A have different isomorphism types. The same description classifies $K_B(A, r)$-torsors: the universal torsor is the fibrewise path fibration from the zero section of $K_B(A, r + 1)$. □

Choose a classifying section $\kappa \in \mathcal{H}^{r+1}(B; A)$. Its torsor can be represented as the family of paths from the zero section to κ. Thus its cohomology class is $[\kappa] \in H^{r+1}(B; A)$. Distinguishing a section from its cohomology class is necessary when the entire mapping space is to be retained.

9.2 The obstruction and the full space of lifts

Fix $r \geq 2$, and let X, Y be π-finite r-types. Put

$$B_X = \tau_{\leq r-1} X, \qquad B_Y = \tau_{\leq r-1} Y.$$

The top homotopy groups give local systems A_X on B_X and A_Y on B_Y. The Postnikov maps are banded torsors with classifying sections

$$\kappa_X \in \mathcal{H}^{r+1}(B_X; A_X), \qquad \kappa_Y \in \mathcal{H}^{r+1}(B_Y; A_Y).$$

For a map $f : B_Y \to B_X$, let

$$\alpha : A_Y \longrightarrow f^* A_X$$

be a homomorphism of local systems. It need not be invertible. The homomorphism α determines a pushforward of torsors, whose classifying section is $\alpha_* \kappa_Y$.

Let $\mathsf{D}_r(Y, X)$ be the space of pairs (f, α). More precisely, it is the total space over $\mathrm{Map}(B_Y, B_X)$ with discrete fibre

$$\mathrm{Hom}_{\mathrm{LocAb}(B_Y)}(A_Y, f^* A_X)$$

over f, with the transport induced by homotopies of f. Taking the lower Postnikov map and the top homotopy homomorphism defines

$$\mathrm{Map}(Y, X) \longrightarrow \mathsf{D}_r(Y, X). \tag{9.4}$$

Write $\mathsf{Lift}_r(f, \alpha)$ for its homotopy fibre over (f, α). A point of this fibre includes a map $Y \to X$ and the specified identification of its lower map with f.

The lower semantic category is intrinsic:

$$\big(\mathcal{L}_{r-1}(X)\big)_{\leq r-2} = \mathcal{L}_{r-2}(X) \simeq \mathcal{L}_{r-2}(B_X).$$

Indeed, the point objects of X belong to $\mathcal{L}_{r-1}(X)$ and mapping out of them is evaluation. Thus internal truncation of an object is equivalent to pointwise truncation. A finite-limit-preserving functor preserves truncated objects, so restriction gives a map from the exact functor space in degree $r-1$ to that in degree $r-2$. Under Theorem 7.2, this restriction is precisely $\mathrm{Map}(Y, X) \to \mathrm{Map}(B_Y, B_X)$. Accordingly, $\mathsf{Lift}_r(f, \alpha)$ is also the space of exact functors

$$\mathcal{L}_{r-1}(X) \longrightarrow \mathcal{L}_{r-1}(Y)$$

with a specified identification of their restriction with $f^* : \mathcal{L}_{r-2}(B_X) \to \mathcal{L}_{r-2}(B_Y)$ and specified induced homomorphism α.

Theorem 9.2 (Relative Postnikov lifting for exact functors)**.** *With the preceding notation, there is an equivalence*

$$\mathsf{Lift}_r(f, \alpha) \simeq \mathrm{Path}_{\mathcal{H}^{r+1}(B_Y; f^* A_X)}(\alpha_* \kappa_Y, f^* \kappa_X). \tag{9.5}$$

It has the following consequences.

(i) *A lift exists if and only if*

$$o_r(f, \alpha) = f^*[\kappa_X] - \alpha_*[\kappa_Y] = 0 \quad \text{in } H^{r+1}(B_Y; f^* A_X). \tag{9.6}$$

(ii) *When a lift exists, its entire space is a torsor for*

$$\mathcal{H}^r(B_Y; f^*A_X). \tag{9.7}$$

In particular, its components form a torsor for $H^r(B_Y; f^*A_X)$. At any chosen lift L,

$$\pi_i(\mathsf{Lift}_r(f,\alpha), L) \cong H^{r-i}(B_Y; f^*A_X) \quad (1 \le i \le r), \tag{9.8}$$

and all higher homotopy groups vanish.

(iii) *The space of natural automorphisms of L relative to the prescribed lower functor and band map is $\Omega_L\mathsf{Lift}_r(f,\alpha)$. Its components are $H^{r-1}(B_Y; f^*A_X)$ and its jth homotopy group is $H^{r-1-j}(B_Y; f^*A_X)$ for $1 \le j \le r-1$.*

All these spaces are π-finite. The equivalences are natural in the Postnikov data and hence apply without choosing an atlas for the semantic categories.

Proof. The homotopy fibre of $\mathrm{Map}(Y,X) \to \mathrm{Map}(B_Y, B_X)$ over f is $\mathrm{Map}_{\mathcal{S}/B_Y}(Y, f^*X)$. Indeed, composition with $X \to B_X$ has target $\mathrm{Map}(Y, B_X)$, and the truncation universal property identifies this space with $\mathrm{Map}(B_Y, B_X)$. We must take the part inducing α on the top local systems.

Regard Y and f^*X as torsors under $K_{B_Y}(A_Y, r)$ and $K_{B_Y}(f^*A_X, r)$, respectively. Every fibre map inducing α is, with a contractible space of choices, $K(\alpha, r)$-equivariant. To verify the assertion, trivialize the two torsors and their bands locally. It then becomes the equivalence (9.2) on the component labelled by α. The affine composition formula shows that these equivalences respect the transition maps. Descent therefore identifies the full mapping spaces, including their higher homotopies.

Extending the structure group along $K(\alpha, r)$ gives a $K_{B_Y}(f^*A_X, r)$-torsor α_*Y. Equivariant maps from Y to f^*X identify with equivariant maps $\alpha_*Y \to f^*X$. Every such map is an equivalence of torsors. Their classifying sections are $\alpha_*\kappa_Y$ and $f^*\kappa_X$. The classification in Lemma 9.1 now identifies the space of these equivalences with the path space in (9.5).

A path exists exactly when its endpoints define the same component, proving (9.6). A nonempty path space in an abelian group object is a torsor for its loop space, and

$$\Omega\mathcal{H}^{r+1}(B_Y; f^*A_X) = \mathcal{H}^r(B_Y; f^*A_X).$$

This proves the torsor assertion and all the homotopy-group formulas. The natural-automorphism assertion is the definition of the loop space of the relative functor groupoid.

For finiteness, a π-finite space has a simplicial model with finitely many simplices in each degree. One constructs such models inductively using the finite bar construction for finite fundamental groups, the degreewise finite simplicial abelian models for $K(A, q)$, and their twisted path pullbacks. Normalized cochains with a finite local system are therefore finite in each degree, so each group in (9.8) is finite. The number of components is finite as well. Finally, the models of a recognized category recover X, its intrinsic truncation recovers B_X, and the homotopy groups and Postnikov class of this map recover A_X and $[\kappa_X]$. Naturality of finite duality proves the last assertion. □

The theorem distinguishes three successive degrees: the obstruction lies in degree $r+1$, equivalence classes of relative lifts in degree r, and relative natural automorphisms in degree $r-1$. For $r = 2$, these are respectively H^3, H^2, and H^1, while modifications of those natural automorphisms are recorded by H^0.

Corollary 9.3 (Fully faithful and invertible lifts)**.** *If $o_r(f,\alpha)=0$, every lift with these data is fully faithful exactly when f is an equivalence and α is surjective on every fibre. Every lift is an equivalence exactly when f and α are equivalences. Hence the nonempty spaces of fully faithful lifts, and of equivalence lifts, have the entire torsor and automorphism structure of Theorem 9.2.*

Proof. By Corollary 8.5, pullback along a map of r-types is fully faithful at semantic level $r-1$ if and only if its homotopy fibres have contractible $(r-1)$-truncations. For r-types this is equivalent to an equivalence on their $(r-1)$-truncations and a surjection on their rth homotopy groups. These are exactly the two stated conditions. Equivalences of r-types are detected by the same lower equivalence and an isomorphism on the top homotopy groups; apply finite duality again. □

Corollary 9.4 (Finite cochain description)**.** *Choose a degreewise finite simplicial model for B_Y and normalized cocycles representing the two Postnikov sections. Put $A=f^*A_X$ and $\delta=f^*\kappa_X-\alpha_*\kappa_Y\in Z^{r+1}(B_Y;A)$. The relative lift space is the space of cochain solutions*

$$db=\delta,\qquad b\in C^r(B_Y;A), \tag{9.9}$$

with all cochain homotopies retained. If a solution b_0 exists, translation by b_0 identifies this space with the Dold–Kan space of the nonnegative chain complex

$$D_i=\begin{cases} Z^r(B_Y;A), & i=0,\\ C^{r-i}(B_Y;A), & 1\le i\le r,\\ 0, & i>r, \end{cases}\qquad \partial=d. \tag{9.10}$$

Every group involved is finite. In particular, when the lift space is nonempty its homotopy cardinality is

$$|\mathsf{Lift}_r(f,\alpha)|=\prod_{q=0}^{r}|H^q(B_Y;A)|^{(-1)^{r-q}}. \tag{9.11}$$

Proof. Use the cocycle model for the local Eilenberg–Mac Lane family. A path between the classifying cocycles is a cochain b as in (9.9). A homotopy from b to b' is a cochain $c\in C^{r-1}$ with $dc=b'-b$; higher homotopies are given successively by cochains in degrees $r-2,\dots,0$. This is precisely the Dold–Kan construction on the good truncation in (9.10). Its homology is $H^{r-i}(B_Y;A)$ in degree i. The formula for homotopy cardinality is therefore the alternating product of the orders of the homotopy groups, including components. □

9.3 Removing identifications and the first stage

Proposition 9.5 (Homotopy transport of the lift space)**.** *Let $d=(f,\alpha)$ and let $\mathsf{D}_r(Y,X)_d$ be its connected component. The space of exact functors whose Postnikov data lie in this component is*

$$\mathsf{Lift}_r(f,\alpha)//\Omega_d\mathsf{D}_r(Y,X), \tag{9.12}$$

where the action is homotopy transport in (9.4). Its set of components is the orbit set of $\pi_0\mathsf{Lift}_r(f,\alpha)$ under $\pi_1(\mathsf{D}_r(Y,X),d)$.

Assume that the lift space is nonempty. Suppose that a loop γ in the data space has trivial coefficient transport, and choose the resulting identification of the pulled-back coefficient system on $B_Y\times S^1$ with the pullback of $A=f^*A_X$, restricting to the identity over $B_Y\times\{*\}$. Let

$e_\gamma : B_Y \times S^1 \to B_X$ be the corresponding evaluation map, and let α_γ be the corresponding band map on this product. Then its action on the torsor of components is translation by

$$\left(e_\gamma^*[\kappa_X] - (\alpha_\gamma)_* \operatorname{pr}_{B_Y}^*[\kappa_Y]\right) / [S^1] \in H^r(B_Y; A). \tag{9.13}$$

The sign convention follows the path orientation from $\alpha_*\kappa_Y$ to $f^*\kappa_X$ in (9.5).

Proof. A space over a connected base is the homotopy quotient of its homotopy fibre by the loop space of that base. Apply this fact to (9.4). Passing to components gives the orbit statement. For the final formula, transport a path from $\alpha_*\kappa_Y$ to $f^*\kappa_X$ around γ. Its endpoint differs from its starting point by the target endpoint loop minus the source endpoint loop. Under the loop–cohomology identification in (9.1), this difference is exactly slant product with the oriented circle. If coefficient transport is nontrivial, the same construction is an affine action whose linear part is that transport; a translation formula requires an identification of the coefficient systems. □

There is a different operation in which the band identification itself is forgotten. For a fixed base B and an isomorphism class of local systems represented by A, the moduli space of Postnikov extensions with that unmarked band is

$$\mathcal{H}^{r+1}(B; A) // \mathrm{Aut}_{\mathrm{LocAb}(B)}(A). \tag{9.14}$$

This follows directly from Lemma 9.1. It changes components by the action on cohomology and retains all stabilizers. In contrast, merely removing the chosen identification of a lower functor in (9.12) does not remove the band identifications of the source and target.

The restriction $r \geq 2$ in Theorem 9.2 is required for a general Postnikov tower: its first homotopy groups need not be abelian. The initial stage has a direct finite-groupoid description. Write

$$X_1 = \coprod_{i \in I} BH_i, \qquad Y_1 = \coprod_{j \in J} BG_j$$

for finite sets I, J and finite groups H_i, G_j. Then

$$\mathrm{Fun}^{\mathrm{lex},\amalg,\mathrm{epi}}(\mathcal{L}_0(X_1), \mathcal{L}_0(Y_1)) \simeq \coprod_{\sigma : J \to I} \prod_{j \in J} (\mathrm{Hom}(G_j, H_{\sigma(j)}) // H_{\sigma(j)}). \tag{9.15}$$

Indeed, a functor of one-object groupoids is a group homomorphism and a natural transformation is an element conjugating one homomorphism to the other. The automorphism group at $\varphi : G \to H$ is $C_H(\varphi(G))$. This proves the mapping-space formula, and finite duality gives the semantic statement. No abelian cohomology formula is imposed on this first stage.

Corollary 9.6 (The finite tower of exact-functor lifts)**.** *For $R \geq 1$ and π-finite R-types X, Y, the full exact functor space $\mathrm{Fun}^{\mathrm{lex},\amalg,\mathrm{epi}}(\mathcal{L}_{R-1}(X), \mathcal{L}_{R-1}(Y))$ is obtained from (9.15) by $R-1$ successive applications of Theorem 9.2. At stage r, a chosen lower map $f_{r-1} : Y_{r-1} \to X_{r-1}$ and band map α_r have obstruction*

$$f_{r-1}^*[\kappa_{X,r}] - (\alpha_r)_*[\kappa_{Y,r}] \in H^{r+1}(Y_{r-1}; f_{r-1}^*\pi_r X),$$

and a nonempty homotopy fibre is a torsor for $\mathcal{H}^r(Y_{r-1}; f_{r-1}^*\pi_r X)$. The data spaces and their loop-space actions are retained at each step.

Proof. Apply (9.4) to the successive pairs $(\tau_{\leq r}Y, \tau_{\leq r}X)$ for $2 \leq r \leq R$, and use finite duality at each stage. The final mapping space is $\mathrm{Map}(Y, X)$ because X, Y are R-truncated. □

The construction is a tower of spaces. Its later obstruction classes depend on the lower lifts already chosen, and its transport actions account for identifications between those choices. A product of the cohomology groups alone would discard this dependence.

9.4 A twisted obstruction with a parity effect

Let $B = BC_2$, and let $A = \mathbb{Z}/4_-$ be the local system on which the nonidentity element acts by $a \mapsto -a$. Put $C = \mathbb{F}_2$ with trivial action. Reduction modulo 2 defines a surjective homomorphism of local systems

$$q : A \longrightarrow C.$$

The normalized C_2-cochain complex with coefficients in A is

$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{0} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{0} \cdots . \tag{9.16}$$

To see the differentials directly, a normalized d-cochain is determined by its value on the tuple of d nonidentity elements. Every middle term of its coboundary contains an identity argument and vanishes. The remaining coefficient is $-1 + (-1)^{d+1}$, which is 2 modulo 4 for even d and 0 for odd d. Consequently,

$$H^0(B; A) = \{0, 2\}, \qquad H^{2j+1}(B; A) = A/2A \cong C_2, \qquad H^{2j+2}(B; A) = \{0, 2\} \cong C_2.$$

All differentials with coefficients in C are zero. Reduction q therefore induces an isomorphism in odd positive degrees and the zero map in even positive degrees.

Proposition 9.7 (Parity-dependent fully faithful lifting)**.** *For each $r \geq 2$, let $Y_r \to B$ be the Postnikov extension with band A and the nonzero class $[\kappa_Y] \in H^{r+1}(B; A)$. Let $X_r = B \times K(C, r)$, with its trivial Postnikov class and band C. Their lower semantic categories are both $\mathcal{L}_{r-2}(B)$. The identity functor on this lower category admits an exact lift*

$$\mathcal{L}_{r-1}(X_r) \longrightarrow \mathcal{L}_{r-1}(Y_r) \tag{9.17}$$

inducing q on top homotopy groups if and only if r is odd. Whenever such lifts exist, they are fully faithful, their marked equivalence classes form a two-element torsor, and at each lift

$$\pi_i \mathsf{Lift}_r(\mathrm{id}_B, q) = C_2 \quad (1 \leq i \leq r).$$

For even r there is no fully faithful lift of the lower identity, even if the top band map is not prescribed.

Proof. The obstruction is $-q_*[\kappa_Y] \in H^{r+1}(B; C)$. By (9.16), it vanishes precisely when $r + 1$ is even. For odd r, the lower map is the identity and the band map q is surjective, so Corollary 9.3 gives full faithfulness. Since $H^d(BC_2; \mathbb{F}_2) = \mathbb{F}_2$ for every $d \geq 0$, the lift-space formulas give exactly the asserted components and higher homotopy groups. Finally, q is the only surjective homomorphism from $\mathbb{Z}/4_-$ to $\mathbb{F}_2$. Full faithfulness would force that homomorphism; for even r its obstruction is nonzero. □

The spaces in the proposition are π-finite r-types. They have fundamental group C_2, top homotopy groups $\mathbb{Z}/4$ and C_2, and the indicated monodromy. Thus the parity obstruction concerns finite models and finite bands throughout. It arises one level above the common lower semantic category, and its vanishing controls the existence of a fully faithful functor at that next level.

10 Cyclic Postnikov families

The relative lifting theorem gives a family in every higher degree for which existence, equivalence, and the loss of marked distinctions can all be calculated. Let p be an odd prime and put $k = \mathbb{F}_p$. The usual periodic resolution of C_p, together with its cup product, gives

$$H^*(BC_p; k) = k[u] \otimes \Lambda(v), \qquad |v| = 1, \quad |u| = 2, \qquad u = \beta v.$$

Here β is the mod-p Bockstein. For the ring calculation and its Bockstein description, see [6, Examples 3.41 and 3E.2]; the periodic resolution is recalled in [4, Example 3.2]. Define a nonzero class in every degree by

$$\xi_{2j} = u^j, \qquad \xi_{2j+1} = vu^j.$$

For $r \geq 2$ and $c \in k$, let $X_c^{(r)}$ be the r-type with Postnikov projection to BC_p, constant band k in degree r, and invariant $c\xi_{r+1}$. Thus

$$X_c^{(r)} = \mathrm{hofib}\big(c\xi_{r+1} : BC_p \longrightarrow K(k, r+1)\big).$$

The specified projection and the identification of its band are part of this notation.

For $a, b \in k$, write $f_a : BC_p \to BC_p$ for the homomorphism $g \mapsto ag$ and $\alpha_b : k \to k$ for multiplication by b. Let $\mathcal{L}_{a,b}(c_Y, c_X)$ be the space of maps $X_{c_Y}^{(r)} \to X_{c_X}^{(r)}$ supplied with an identification of their lower map with f_a and with prescribed band map α_b. By Theorem 7.2, this is also the space of the corresponding marked exact functors

$$\mathcal{L}_{r-1}(X_{c_X}^{(r)}) \longrightarrow \mathcal{L}_{r-1}(X_{c_Y}^{(r)}).$$

Theorem 10.1 (Uniform cyclic lifting calculation)**.** *Put $e_r = \lceil (r+1)/2 \rceil$. The space $\mathcal{L}_{a,b}(c_Y, c_X)$ is nonempty if and only if*

$$a^{e_r} c_X = b c_Y \quad \text{in } k. \tag{10.1}$$

When nonempty, its set of components is a k-torsor and, at every point,

$$\pi_i \mathcal{L}_{a,b}(c_Y, c_X) \cong k \quad (1 \leq i \leq r), \qquad \pi_i = 0 \quad (i > r).$$

After choices there is an equivalence

$$\mathcal{L}_{a,b}(c_Y, c_X) \simeq \prod_{i=0}^{r} K(k, i). \tag{10.2}$$

For $c_X = c_Y = 1$, every $a \neq 0$ permits exactly the band map $b = a^{e_r}$, and every resulting functor is an equivalence.

Proof. Naturality of the Bockstein gives $f_a^* v = av$ and $f_a^* u = au$. Consequently $f_a^* \xi_{r+1} = a^{e_r} \xi_{r+1}$. The obstruction of Theorem 9.2 is therefore $(a^{e_r} c_X - b c_Y)\xi_{r+1}$, which proves the criterion. The same theorem identifies a nonempty lift space as a torsor for $\mathrm{Map}(BC_p, K(k, r))$. Its homotopy groups are $H^{r-i}(BC_p; k) = k$ for $0 \leq i \leq r$. For completeness, a connective chain complex presenting this mapping space is the truncation of the cochain complex $C^*(BC_p; k)$, with degrees reversed about r. A chain complex of vector spaces splits as its homology and a contractible complex. Dold–Kan then supplies the noncanonical product in (10.2). If $a \neq 0$ and $c_X = c_Y = 1$, equation (10.1) forces $b = a^{e_r} \neq 0$. The induced maps on π_1 and π_r are isomorphisms; these are all the nonzero homotopy groups. □

10.1 The monodromy of the space of lifts

The p marked components in the theorem need not remain distinct when the identification of the lower functor is forgotten. The band identifications on the source and target are kept fixed throughout. Let $\mathcal{U}_{a,b}(c_Y, c_X)$ denote the space of maps with induced homomorphisms (a, b), without a specified homotopy identifying the lower map with the chosen representative f_a. There is a fibre sequence

$$\mathcal{L}_{a,b}(c_Y, c_X) \longrightarrow \mathcal{U}_{a,b}(c_Y, c_X) \longrightarrow \mathrm{Map}(BC_p, BC_p)_{f_a} \simeq BC_p. \tag{10.3}$$

In particular, $\mathcal{U}$ is the homotopy quotient of the marked space by the loop group C_p of the lower mapping component.

Proposition 10.2 (Parity of the gauge translation)**.** *Assume* (10.1)*. Under the path-transport convention that a loop of the target invariant adds its slant class, $t \in C_p = k$ acts on the torsor of marked components by translation by*

$$\begin{cases} c_X a^{r/2} t\, u^{r/2}, & r \text{ even}, \\ 0, & r \text{ odd}. \end{cases}$$

Thus, if r is even and $c_X a \neq 0$, all p marked components belong to one component of $\mathcal{U}$. If r is odd, or if r is even and $c_X a = 0$, $\mathcal{U}$ has p components. Reversing the path-transport convention changes the displayed translation by a sign and leaves these orbit statements unchanged.

Proof. A loop t at f_a is represented by the evaluation map $F_t : BC_p \times S^1 \to BC_p$ whose induced homomorphism on fundamental groups is $(g, m) \mapsto ag + tm$. If z denotes the generator of $H^1(S^1; k)$, then

$$F_t^* v = av + tz, \qquad F_t^* u = au.$$

The second identity follows from $u = \beta v$ and $\beta z = 0$. Transport in the path-space description of Theorem 9.2 acts by the coefficient of z in $F_t^*(c_X \xi_{r+1})$. For $r = 2j$, this coefficient is $c_X a^j t u^j$. For $r = 2j - 1$, the invariant is $c_X u^j$, whose pullback has no z term. This proves the formula. The components of the total space of (10.3) are precisely the orbits of this action on the components of its fibre. A nonzero translation homomorphism $k \to k$ is transitive; the zero homomorphism fixes every component. □

Corollary 10.3 (The group of components of cyclic automorphisms)**.** *Let p be odd, let $k = \mathbb{F}_p$, and let $X = X_1^{(r)}$ with $r \geq 2$. The group of unmarked homotopy classes of self-equivalences is*

$$\pi_0 \mathrm{Aut}(X) \cong \begin{cases} k^\times, & r \text{ even}, \\ k \times k^\times, & r \text{ odd}. \end{cases} \tag{10.4}$$

The same groups are obtained from exact self-equivalences of $\mathcal{L}_{r-1}(X)$. In the odd case, the first factor is the group of vertical translations classified by $H^r(BC_p; k)$.

Proof. A self-equivalence induces multiplication by some $a \in k^\times$ on $\pi_1(X)$ and by some $b \in k^\times$ on $\pi_r(X)$. Theorem 10.1 forces $b = a^{e_r}$, and supplies such an equivalence for every a. Thus the induced action on the fundamental group defines a surjective group homomorphism

$$\rho : \pi_0 \mathrm{Aut}(X) \longrightarrow k^\times.$$

Its kernel consists of maps with lower and band maps both the identity, after forgetting the choice of a lower homotopy. Before that choice is forgotten, these components form $H^r(BC_p;k)$, with the identity self-map as zero. They are represented by fibrewise translations of the $K(k,r)$-torsor. Composition of translations adds their cohomology classes.

If r is even, Proposition 10.2 says that the lower loop group acts transitively on these marked components. Hence $\ker\rho$ is trivial, giving the first case of (10.4).

Suppose $r=2j-1$. The gauge action on components is trivial, so $\ker\rho = H^r(BC_p;k)\cong k$ as groups. Let T be a self-equivalence with lower map f_a and band map α_b. Conjugating a vertical translation by T changes its class γ to

$$b\,(f_a^{-1})^*\gamma.$$

This follows on each fibre from the affine formula: applying T^{-1} changes the base by f_a^{-1}, and applying T multiplies translation vectors by b. The group $H^r(BC_p;k)$ is generated by vu^{j-1}; f_a^* acts on it by a^j, while $b=a^{e_r}=a^j$. The displayed conjugation is therefore the identity. We have obtained a central extension

$$0\longrightarrow k\longrightarrow \pi_0\operatorname{Aut}(X)\xrightarrow{\rho} k^\times\longrightarrow 1.$$

It splits by an elementary coprime-order argument. Choose a generator of the cyclic group $k^\times$ and a lift t. Then t^{p-1} lies in the central subgroup k. Since multiplication by $p-1$ is invertible on k, multiply t by the unique element of k which makes its $(p-1)$st power equal to the identity. The resulting lift generates a complement of order $p-1$, proving the second case of (10.4).

Finite higher Galois duality identifies self-equivalences of X contravariantly with exact self-equivalences of its semantic category. Thus their component groups are opposite groups. The groups displayed above are abelian, so the same formulas hold for the semantic category. □

Proposition 10.4 (Higher homotopy of unmarked cyclic lifts)**.** *Keep the source and target band identifications fixed, and suppose (a,b) satisfies (10.1). Put $B=BC_p$ and $d=r+1$. There is a homotopy-fibre description*

$$\mathcal{U}_{a,b}(c_Y,c_X)\simeq \operatorname{hofib}_0\left(\Psi_{a,b}: BC_p\longrightarrow \operatorname{Map}(B,K(k,d))_0\right). \tag{10.5}$$

Here the subscript 0 denotes the component of the zero cohomology class. If $\mu_a: B\times BC_p\to B$ is induced by $(g,t)\mapsto ag+t$, then $\Psi_{a,b}$ is adjoint to the class

$$c_X\mu_a^*\xi_d - bc_Y\,\mathrm{pr}_1^*\,\xi_d\in H^d(B\times BC_p;k). \tag{10.6}$$

In particular:

(i) *If r is even and $c_Xa\neq 0$, the entire unmarked lift space has a noncanonical equivalence*

$$\mathcal{U}_{a,b}(c_Y,c_X)\simeq\prod_{i=1}^{r}K(k,i). \tag{10.7}$$

(ii) *If $r=2j-1$ is odd, so $j\geq 2$, put $m=c_Xja^{j-1}\in k$. At each point L of the unmarked lift space,*

$$\pi_1(\mathcal{U}_{a,b}(c_Y,c_X),L)\cong\begin{cases}C_{p^2}, & m\neq 0,\\ C_p\times C_p, & m=0.\end{cases} \tag{10.8}$$

For $2\leq i\leq r$ its ith homotopy group is k, and all groups above r vanish.

Consequently the identity component of the self-equivalence space of $X = X_1^{(r)}$ has fundamental group

$$\pi_1(\mathrm{Aut}(X), \mathrm{id}_X) \cong \begin{cases} C_p, & r \text{ even}, \\ C_{p^2}, & r \text{ odd and } p \nmid (r+1)/2, \\ C_p \times C_p, & r \text{ odd and } p \mid (r+1)/2. \end{cases} \tag{10.9}$$

The same formula describes the group of homotopy classes of natural automorphisms of the identity exact functor of $\mathcal{L}_{r-1}(X)$.

Proof. Over the component $\mathrm{Map}(B, B)_{f_a} \simeq BC_p$, the fixed band map is the constant homomorphism α_b. The path-space description of relative lifts therefore gives a point of $\mathcal{U}$ by choosing a lower map f in this component and a path from $bc_Y\xi_d$ to $f^*(c_X\xi_d)$ in $\mathrm{Map}(B, K(k,d))$. Subtracting the constant endpoint gives (10.5). The evaluation map on this entire mapping component is μ_a, so its adjoint class is (10.6). Compatibility makes the component in $H^d(B;k)$ zero, as required for the indicated target component.

Choose a splitting in simplicial k-modules of the truncated cochain complex which presents this target. It gives a noncanonical equivalence

$$\mathrm{Map}(B, K(k,d))_0 \simeq \prod_{i=1}^{d} K(k,i),$$

where the coefficient of the ith factor is $H^{d-i}(B;k)$, identified with k using ξ_{d-i} (and $\xi_0 = 1$). The Künneth components of (10.6) give the corresponding cohomology classes of $\Psi_{a,b}$. Thus the splitting of the coefficient space retains the coherent gauge action through these possibly nonzero coordinate maps.

Write u_s, v_s for the classes on the first, source copy of B, and u_g, v_g for the classes on the gauge copy. Naturality gives

$$\mu_a^* v = a v_s + v_g, \qquad \mu_a^* u = a u_s + u_g.$$

For $r = 2j-1$, the universal class becomes

$$c_X \sum_{\ell=1}^{j} \binom{j}{\ell} a^{j-\ell} u_s^{j-\ell} u_g^{\ell}. \tag{10.10}$$

Hence all odd coordinates of $\Psi_{a,b}$ are null, and its coordinate in $K(k, 2\ell)$ is represented by

$$c_X \binom{j}{\ell} a^{j-\ell} u_g^{\ell} \in H^{2\ell}(BC_p; k). \tag{10.11}$$

In particular its $K(k,1)$ coordinate is zero and its first potentially nonzero Postnikov coordinate is $mu_g \in H^2(BC_p; k)$.

This last class determines the fundamental group of each component of its homotopy fibre. Indeed, after lifting to the universal cover of the coefficient space, passage to its 2-truncation gives $K(k,2)$. Higher target coordinates do not change the fundamental group of the fibre. The coefficient space is an abelian group object, so its fundamental group acts trivially on higher homotopy groups; the class mu_g is independent of the choices in this lifting and in the splitting. It classifies the central extension

$$0 \longrightarrow k \longrightarrow \pi_1(\mathcal{U}, L) \longrightarrow C_p \longrightarrow 1.$$

Using the carry cocycle for u_g, this group is represented on pairs $(x, s) \in \{0, \dots, p-1\} \times k$ by

$$(x, s)(y, t) = \left(x + y \bmod p, s + t + m \left\lfloor \frac{x+y}{p} \right\rfloor \right).$$

If $m = 0$ it is $C_p \times C_p$. If $m \neq 0$, the pth power of $(1, 0)$ is $(0, m)$, so $(1, 0)$ has order p^2 and the group is cyclic. This proves (10.8).

For $r = 2j$, the two sorts of coordinates obtained from $(av_s + v_g)(au_s + u_g)^j$ are

$$\Psi_{2\ell} : c_X \binom{j}{\ell} a^{j-\ell+1} u_g^\ell, \qquad 1 \leq \ell \leq j,$$

$$\Psi_{2\ell+1} : c_X \binom{j}{\ell} a^{j-\ell} v_g u_g^\ell, \qquad 0 \leq \ell \leq j.$$

In particular, Ψ_1 is the map $BC_p \to K(k, 1)$ given by multiplication by $c_X a^j$. Under the assumption of (i) this is an equivalence. For any map $(h, g) : T \to T' \times W$ with h an equivalence and W connected, its homotopy fibre at a chosen point is equivalent to ΩW. Applying this with $W = \prod_{i=2}^{r+1} K(k, i)$ proves (10.7).

The higher homotopy groups in (ii) follow directly from (10.5), since the domain is a 1-type and the target has $\pi_{i+1} = H^{r-i}(B; k) = k$ for $2 \leq i \leq r$. For self-equivalences, take $c_X = c_Y = 1$ and $a = b = 1$. Then $m = j$, which gives (10.9). Finally the mapping-space equivalence in finite higher Galois duality takes the identity map to the identity exact functor and identifies their based loop spaces. □

Corollary 10.5 (The homotopy type of a cyclic automorphism component)**.** *Let p be odd, $k = \mathbb{F}_p$, and $r = 2j - 1 \geq 3$. Write $j = p^s q$ with $p \nmid q$, and put $h = 2p^s - 1$. Define the connected space*

$$E_s = \begin{cases} BC_{p^2}, & s = 0, \\ X_1^{(h)}, & s \geq 1. \end{cases}$$

For $X = X_1^{(r)}$, there is a noncanonical equivalence

$$\mathrm{Aut}(X)_{\mathrm{id}_X} \simeq E_s \times \prod_{\substack{1 \leq i \leq r \\ i \neq h}} K(k, i). \tag{10.12}$$

More generally, for every $a \in k^\times$, the full unmarked lift space with compatible band map $b = a^j$ is

$$\mathcal{U}_{a,a^j}(1, 1) \simeq k \times E_s \times \prod_{\substack{1 \leq i \leq r \\ i \neq h}} K(k, i),$$

where the first factor is discrete. These are equivalences of spaces; no splitting of the composition law is asserted.

Proof. In characteristic p, the odd-degree gauge polynomial satisfies

$$(au_s + u_g)^j - a^j u_s^j = \sum_{\ell=1}^{q} \binom{q}{\ell} a^{j-p^s\ell} u_s^{j-p^s\ell} u_g^{p^s\ell}.$$

Thus its first nonzero coordinate is in $K(k, 2p^s)$ and is represented by

$$\lambda u_g^{p^s}, \qquad \lambda = qa^{j-p^s} \neq 0.$$

Every remaining nonzero coordinate is a scalar multiple of a power of $u_g^{p^s}$. The $K(k, 1)$ coordinate is zero, so its loop factor contributes the discrete copy of k.

Put $E = \mathrm{hofib}(\lambda u^{p^s})$ for the map $BC_p \to K(k, 2p^s)$. Split this coordinate off from the product model of $\mathrm{Map}(BC_p, K(k, r+1))_0$. The remaining fibre calculation is the homotopy fibre of a map

$$E \longrightarrow W = \prod_{\substack{2 \leq t \leq r+1 \\ t \neq 2p^s}} K(k, t).$$

The class u^{p^s} pulls back to zero on E, since λ is invertible and λu^{p^s} has its tautological nullhomotopy there. Therefore every coordinate of this map to W is null. The target is a finite product of Eilenberg–Mac Lane spaces, so choosing these nullhomotopies gives a nullhomotopy of the whole map. Its fibre is consequently $E \times \Omega W$.

If $s = 0$, the space E is the classifying space of the nontrivial central extension of C_p by k, hence $E \simeq BC_{p^2}$. If $s \geq 1$, rescaling the coefficient group identifies E with $X_1^{(h)}$. This proves the formula for the full unmarked lift space. Set $a = 1$ and take its component containing the identity to obtain (10.12). □

Example 10.6 (The degree $2p-1$ retains the original Postnikov class)**.** Put $r = 2p-1$ and $X = X_1^{(r)}$. For every $a \in k^\times$ the compatible band map is $b = a^p = a$. In (10.10), all intermediate binomial coefficients vanish in k. Thus the only nonzero coordinate of $\Psi_{a,b}$ is the top coordinate $u_g^p \in H^{2p}(BC_p; k)$. The homotopy-fibre description therefore gives a noncanonical equivalence

$$\mathcal{U}_{a,a}(1,1) \simeq k \times X \times \prod_{i=1}^{2p-2} K(k, i),$$

where the first factor is discrete. In particular,

$$\mathrm{Aut}(X)_{\mathrm{id}_X} \simeq X \times \prod_{i=1}^{2p-2} K(k, i).$$

The original nonzero Postnikov class u^p thus occurs inside the identity component of the exact self-equivalence space of $\mathcal{L}_{2p-2}(X)$, while its fundamental group is the split group $C_p \times C_p$.

The calculation distinguishes three pieces of data: the lower exact functor, the homomorphism of bands, and the chosen coherence between Postnikov classes. The first two determine existence through (10.1); the third has a full r-type of choices. Forgetting a lower marking can identify distinct choices by the explicit action of Proposition 10.2.

10.2 Finite cochain representatives

These examples have normalized finite representatives. Write elements of C_p as integers in $\{0, \dots, p-1\}$ and set

$$v(g) = g \bmod p, \qquad u(g,h) = \left\lfloor \frac{g+h}{p} \right\rfloor \bmod p.$$

The identity $du = 0$ expresses the independence of carries from the parenthesization of three summands. The connecting homomorphism obtained by lifting v to $\mathbb{Z}/p^2$ gives this u, so it represents βv. Cup products of these cochains represent the classes ξ_j. For example, the $r = 2$ invariant is

$$\kappa_c(g, h, \ell) = cg \left\lfloor \frac{h + \ell}{p} \right\rfloor \bmod p.$$

The supplementary script `verify_relative_examples.py` checks cocycle identities, nontriviality, and the lift equation by exact row reduction in small normalized bar complexes. Its checked primes and degrees are recorded with the output. The periodic resolution and the proofs above establish the formulas in every degree.

11 Continuous semantics and uniform finite parameters

Let $n \geq 0$ and let

$$P = \{P_i\}_{i \in I} \in \mathrm{Pro}(\mathcal{S}_{\pi, \leq n+1})$$

be a cofiltered system of π-finite $(n+1)$-types. Its category of continuous finite n-type families is

$$\mathcal{C}_n(P) = \operatorname*{colim}_{i \in I^{\mathrm{op}}} \mathcal{L}_n(P_i). \tag{11.1}$$

The transition functors are pullbacks, and the colimit is taken in Cat_∞. In particular, an object is represented at a finite stage, while maps and their homotopies are identified after passage to a common later stage. Write

$$\mathrm{mat}(P) = \lim_{i \in I} P_i$$

for the materialization in spaces. The continuous family category depends on the pro-object, and not on its chosen presentation. We continue to call a functor exact when it preserves finite limits, finite coproducts, and effective epimorphisms.

We use two established facts about Stone ∞-topoi: the functor

$$P \longmapsto \mathcal{X}_P = \lim_i \mathrm{Fun}(P_i, \mathcal{S}),$$

with the limit taken in the ∞-category of ∞-topoi and geometric morphisms, is fully faithful on profinite spaces, and materialization on profinite spaces is conservative [3, Theorems 4.4.3 and 4.4.8]. The fixed truncation range permits a direct comparison between continuous exact functors and these geometric morphisms.

Lemma 11.1 (A coherent site in the fixed range)**.** *The ∞-topos $\mathcal{X}_P$ is bounded coherent and $(n+1)$-localic. There are natural equivalences*

$$\mathcal{C}_n(P) \simeq (\mathcal{X}_P)^{\mathrm{coh}}_{\leq n}, \qquad \mathrm{Sh}_{\mathrm{eff}}(\mathcal{C}_n(P)) \simeq \mathcal{X}_P, \tag{11.2}$$

where the topology is generated by finite jointly effective epimorphic families.

Proof. Each $\mathrm{Fun}(P_i, \mathcal{S}) \simeq \mathcal{S}/P_i$ is bounded coherent. Continuity for coherent objects gives

$$(\mathcal{X}_P)^{\mathrm{coh}}_{<\infty} \simeq \operatorname*{colim}_i \mathrm{Fun}(P_i, \mathcal{S}_\pi);$$

see [3, Propositions 3.9.1–3.9.2]. This equivalence restricts to the n-truncated objects. Indeed, truncatedness is expressed by a finite iteration of diagonals ending in an equivalence. Finite limits

commute with the displayed filtered colimit, and an equivalence in a filtered colimit is witnessed at a later stage. An object that becomes n-truncated therefore has an n-truncated representative after refinement. This proves the first equivalence in (11.2).

The ∞-topos $\mathcal{S}/P_i$ is $(n+1)$-localic because P_i is an $(n+1)$-type. The inclusion of $(n+1)$-localic ∞-topoi preserves limits, so $\mathcal{X}_P$ is also $(n+1)$-localic; these facts are recalled in [3, Paragraph 3.2.3 and Example 3.2.8]. To verify the site assertion, let A be a truncated coherent object of $\mathcal{X}_P$ and represent it by a finite family $A_i \to P_i$. Its total space is π-finite. Choose one point in each component of this total space. The resulting finite effective cover of A_i has domains of the form $(* \to P_i)$. Their homotopy fibres are path spaces in P_i, and hence are n-truncated. Pulling the cover to $\mathcal{X}_P$ gives a finite cover of A by objects of $\mathcal{C}_n(P)$.

Thus $\mathcal{C}_n(P)$ is a basis for the effective-epimorphism topology on $(\mathcal{X}_P)^{\mathrm{coh}}_{<\infty}$. It is closed under finite limits and is locally n-truncated. The associated sheaf ∞-topos is therefore $(n+1)$-localic. The basis comparison in this fixed localic range gives the second equivalence in (11.2); see [3, Corollary 3.12.14]. □

Theorem 11.2 (Continuous duality in a finite homotopy range)**.** *Let $P, Q \in \mathrm{Pro}(\mathcal{S}_{\pi,\le n+1})$. Pullback induces an equivalence*

$$\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_{\pi,\le n+1})}(P,Q) \simeq \mathrm{Fun}^{\mathrm{lex,\amalg,epi}}(\mathcal{C}_n(Q), \mathcal{C}_n(P)). \tag{11.3}$$

The right side contains all natural transformations; every one of them is invertible. In particular,

$$\mathrm{Mod}_n(\mathcal{C}_n(P)) \simeq \mathrm{mat}(P). \tag{11.4}$$

For a pro-morphism $f : P \to Q$, the following conditions are equivalent: f is an equivalence; f^* is an equivalence; and the induced map on the complete spaces of finite models is an equivalence.

Proof. Let $H : \mathcal{C}_n(Q) \to \mathcal{C}_n(P)$ be exact. Compose with the inclusion into $\mathcal{X}_P$. Since the source has finite limits, the colimit-preserving extension from its presheaf category is left exact [9, Proposition 6.1.5.2]. Preservation of finite coproducts and effective epimorphisms says that H carries every covering family to a covering family. Its extension therefore factors through the sheaf category in Lemma 11.1, giving an inverse-image functor

$$\widetilde{H} : \mathcal{X}_Q \longrightarrow \mathcal{X}_P.$$

This extension is unique, with its entire space of natural transformations, since the site objects generate the sheaf category under colimits.

Conversely, every geometric morphism $\mathcal{X}_P \to \mathcal{X}_Q$ is coherent because its target is Stone [3, Corollary 4.4.15]. Its inverse-image functor consequently restricts to an exact functor on the coherent n-truncated objects. The two constructions are inverse. Stone duality identifies these geometric morphisms with $\mathrm{Map}(P,Q)$, and their natural transformations are invertible [3, Proposition 4.4.6]. This proves (11.3).

Take the source prospace to be the point, so that its family category is $\mathcal{F}_n$. Then

$$\mathrm{Mod}_n(\mathcal{C}_n(P)) \simeq \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(*, P) \simeq \lim_i P_i,$$

which proves (11.4). Under this identification, precomposition with f^* induces $\mathrm{mat}(f)$. An equivalence f^* therefore gives an equivalence $\mathrm{mat}(f)$; profinite Whitehead then implies that f is an equivalence. The other implications follow from pullback and (11.4). □

The duality in Theorem 11.2 is the fixed-range form of higher Stone duality. Its use here is to give an exact boundary for the uniform parameter condition in Theorem 4.1.

Corollary 11.3 (Finite-stage descent for exact functors and lifts)**.** *For a π-finite $(n+1)$-type X and a prospace $P = \{P_i\}_i$ as above, the natural comparison*

$$\operatorname*{colim}_i \mathrm{Fun}^{\mathrm{lex},\amalg,\mathrm{epi}}(\mathcal{L}_n(X), \mathcal{L}_n(P_i)) \longrightarrow \mathrm{Fun}^{\mathrm{lex},\amalg,\mathrm{epi}}(\mathcal{L}_n(X), \mathcal{C}_n(P)) \tag{11.5}$$

is an equivalence of spaces.

More generally, let $u : X \to B$ be a map of π-finite $(n+1)$-types. Let $F : \mathcal{L}_n(B) \to \mathcal{C}_n(P)$ be exact, represented by F_i at a finite stage P_i. Let $\mathrm{Lift}_F(u^*)$ denote the space of pairs (H, η), where H is exact and

$$H : \mathcal{L}_n(X) \to \mathcal{C}_n(P), \qquad \eta : H \circ u^* \simeq F.$$

Write F_j for the pullback of F_i to P_j. Then

$$\mathrm{Lift}_F(u^*) \simeq \operatorname*{colim}_{(j\to i)\in (I_{/i})^{\mathrm{op}}} \mathrm{Lift}_{F_j}(u^*). \tag{11.6}$$

Consequently an extension exists precisely when it exists after one finite-stage refinement. Each specified equivalence or higher homotopy between such extensions also descends after refinement.

Proof. By Theorem 11.2, the right side of (11.5) is $\mathrm{Map}(P, X)$. The defining mapping formula for pro-objects identifies it with $\mathrm{colim}_i \mathrm{Map}(P_i, X)$. Theorem 7.2 identifies each term with the corresponding exact-functor space, proving (11.5).

The chosen representation of F corresponds to a map $f_i : P_i \to B$. Replace the index category by $I_{/i}$. The extension space is the homotopy fibre at the induced map $f : P \to B$ of

$$\mathrm{Map}(P, X) \longrightarrow \mathrm{Map}(P, B).$$

Apply the pro-mapping formula to both terms. Filtered colimits of spaces commute with finite limits, so taking this fibre gives the filtered colimit of the fibres over $f_i|_{P_j}$. These are exactly the finite-stage extension spaces in (11.6). □

Corollary 11.4 (Continuous obstruction and the space of sections)**.** *Let $1 \le r \le n+1$, let B be a π-finite $(n+1)$-type, and let A be a local system of finite abelian groups on B. Let $u : X \to B$ be a torsor under $K_B(A, r)$, with class $[\kappa] \in H^{r+1}(B; A)$. Thus X is also a π-finite $(n+1)$-type.*

Let $F : \mathcal{L}_n(B) \to \mathcal{C}_n(P)$ correspond to $f : P \to B$, represented by $f_i : P_i \to B$. Write $f_j : P_j \to B$ for its pullback to a refinement $j \to i$, and define

$$H^q_{\mathrm{cont}}(P; f^*A) = \operatorname*{colim}_{(j\to i)\in (I_{/i})^{\mathrm{op}}} H^q(P_j; f_j^*A) \qquad (q \ge 0). \tag{11.7}$$

The obstruction to extending F along $u^* : \mathcal{L}_n(B) \to \mathcal{L}_n(X)$ is

$$o(F, u) = [f_i^*\kappa]_{\mathrm{cont}} \in H^{r+1}_{\mathrm{cont}}(P; f^*A). \tag{11.8}$$

The extension space is nonempty if and only if $o(F, u) = 0$, and this is equivalent to $f_j^*[\kappa] = 0$ at some finite-stage refinement $j \to i$.

When nonempty, the full extension space $\mathrm{Lift}_F(u^*)$ is a torsor under the abelian group object

$$\mathcal{H}^r_{\mathrm{cont}}(P; f^*A) = \operatorname*{colim}_{(j\to i)\in (I_{/i})^{\mathrm{op}}} \Gamma_{P_j} K_{P_j}(f_j^*A, r). \tag{11.9}$$

Its components form a torsor for $H^r_{\mathrm{cont}}(P; f^*A)$, and at any chosen extension L,

$$\pi_\ell(\mathrm{Lift}_F(u^*), L) \cong H^{r-\ell}_{\mathrm{cont}}(P; f^*A) \quad (1 \le \ell \le r), \qquad \pi_\ell = 0 \quad (\ell > r). \tag{11.10}$$

No $(r-1)$-truncation hypothesis is imposed on the source stages P_j.

Proof. At a finite stage P_j, the lift space is the space of sections of $f_j^*X \to P_j$. The classification of banded Eilenberg–Mac Lane torsors in Lemma 9.1 identifies this space with the path space from the zero section to $f_j^*\kappa$ in $\Gamma_{P_j}K_{P_j}(f_j^*A, r+1)$. It is nonempty precisely when $f_j^*[\kappa] = 0$; when nonempty it is a torsor for $\Gamma_{P_j}K_{P_j}(f_j^*A, r)$.

Corollary 11.3 expresses the continuous extension space as the filtered colimit of these section spaces. A class is zero in the filtered colimit of the cohomology groups precisely when it is zero after some refinement. This proves the obstruction criterion.

If a section exists at one stage, pull it back over the cofinal system of subsequent refinements. Translation by these compatible sections identifies the lift spaces with their section group objects. Taking the filtered colimit gives the torsor in (11.9). Filtered colimits of spaces commute with finite limits and homotopy groups. Since

$$\pi_\ell\Gamma_{P_j}K_{P_j}(f_j^*A, r) = H^{r-\ell}(P_j; f_j^*A) \qquad (0 \le \ell \le r),$$

the component assertion and (11.10) follow. This argument concerns sections of the target torsor over the full source P_j, so it applies in every source truncation degree allowed above. □

Theorem 11.5 (Uniform finite parameter criterion)**.** *For $P \in \mathrm{Pro}(\mathcal{S}_{\pi,\le n+1})$, the following conditions are equivalent.*

(i) *There are finitely many noninitial $U_a \in \mathcal{C}_n(P)$ covering the terminal object such that every object of $\mathcal{C}_n(P)/U_a$ has an effective epimorphism from a finite coproduct of copies of U_a.*

(ii) *$\mathrm{Mod}_n(\mathcal{C}_n(P))$ is π-finite.*

(iii) *P is equivalent to the constant pro-object associated to a π-finite $(n+1)$-type M.*

(iv) *$\mathcal{C}_n(P) \simeq \mathcal{L}_n(M)$ for such an M.*

If these conditions hold, M is canonically identified with the complete finite model space of $\mathcal{C}_n(P)$.

For arbitrary P, every finite family of objects of $\mathcal{C}_n(P)$ becomes constant on some finite coherent cover of the terminal object.

Proof. Lemma 11.1 places $\mathcal{C}_n(P)$ in the setting of Theorem 4.1. That theorem and Theorem 4.4 show that (i) implies (ii). Suppose (ii) holds, and put $M = \mathrm{mat}(P)$ using (11.4). As a limit of $(n+1)$-types, M is an $(n+1)$-type. By assumption it is π-finite. The limit cone determines a canonical morphism of profinite spaces

$$\varepsilon : \mathrm{const}(M) \longrightarrow P.$$

Its materialization is the identity of M: under $\mathrm{Map}(\mathrm{const}(M), P) \simeq \mathrm{Map}(M, \mathrm{mat}(P))$, the morphism ε corresponds to id_M. The conservative materialization functor therefore makes ε an equivalence. This proves (iii). Condition (iii) immediately implies (iv), and Proposition 4.5 gives (iv)⇒(i). The assertion about M follows from the same construction. The empty case is included by allowing an empty covering family.

Finally, represent any given finite family of objects at one stage P_i. Choose one point in each component of P_i and pull its point atlas back to $\mathcal{X}_P$. Over each member of this finite cover, every selected object is constant with value its fibre at the selected point. Initial members may be discarded. This proves the last assertion. □

Corollary 11.6 (The visible pro-homotopy type)**.** *Let $P \in \mathrm{Pro}(\mathcal{S}_\pi)$ have no prescribed uniform truncation bound, and define $\mathcal{C}_n(P)$ by (11.1). Put $P^{(n+1)} = \{\tau_{\le n+1} P_i\}_i$. Then*

$$\mathcal{C}_n(P) \simeq \mathcal{C}_n(P^{(n+1)}), \qquad \mathrm{Mod}_n(\mathcal{C}_n(P)) \simeq \mathrm{mat}(P^{(n+1)}).$$

The uniform finite parameter condition holds precisely when $P^{(n+1)}$ is a constant π-finite prospace.

Proof. Apply Lemma 7.1 at each finite stage and then Theorems 11.2 and 11.5. □

Example 11.7 (Objectwise elimination without a uniform atlas)**.** Let $P = \{B(\mathbb{Z}/p^r)\}_{r\ge 1}$. For $n = 0$, $\mathcal{C}_0(P)$ is the category of finite continuous $\mathbb{Z}_p$-sets. Every finite family of such sets is trivialized by the finite cover $\mathbb{Z}_p/p^r\mathbb{Z}_p$ for sufficiently large r.

No finite local-section atlas exists. Indeed, a proposed finite atlas is a finite collection of finite continuous $\mathbb{Z}_p$-sets. There is an r such that $p^r\mathbb{Z}_p$ fixes every point of all these sets. Put $A = \mathbb{Z}_p/p^{r+1}\mathbb{Z}_p$. For any nonempty member U of the proposed atlas, the object $A \times U \to U$ has no section: the stabilizer of each point of U contains $p^r\mathbb{Z}_p$, and this subgroup has no fixed point in A. This noninitial slice object consequently cannot be covered by copies of the slice terminal object.

For every $n \ge 0$, the same absence of a uniform atlas also follows from Theorem 11.5, since

$$\mathrm{Mod}_n(\mathcal{C}_n(P)) \simeq B(\mathbb{Z}_p^{\mathrm{disc}})$$

has an infinite fundamental group. Here the quotient homomorphisms are surjective, so their inverse system has trivial $\lim^1$ and the homotopy limit is the classifying space of the abstract inverse limit group. Thus finite elimination for each prescribed family does not supply a common system of parameters for all finite semantics.

The same example marks a boundary for the universal-parameter theorem. The forgetful exact functor $\omega : \mathcal{C}_0(P) \to \mathrm{FinSet}$ has no left adjoint in finite continuous sets. Such an adjoint would give a finite object $Q = \omega_!(1)$ representing $A \mapsto \omega(A)$. The object Q must be nonempty, by testing $A = \varnothing$. Choose r such that $p^r\mathbb{Z}_p$ fixes every point of Q. Then $\mathrm{Hom}_{\mathbb{Z}_p}(Q, \mathbb{Z}_p/p^{r+1}\mathbb{Z}_p)$ is empty, although the underlying target set is nonempty, a contradiction.

Example 11.8 (The ordinary model space does not retain continuity)**.** Let $K = \{0,1\}^{\mathbb{N}}$ be the Cantor profinite set and let $K' = K \amalg \{*\}$. For every $n \ge 0$, the model spaces of $\mathcal{C}_n(K)$ and $\mathcal{C}_n(K')$ are the underlying discrete sets K and K', respectively. These sets have the same cardinality, so the model spaces are equivalent as unstructured spaces.

The semantic categories are not equivalent. Their posets of subterminal objects are the Boolean algebras of clopen subsets of K and K'. The first is atomless: every nonempty clopen subset of the Cantor set contains two disjoint nonempty clopen subsets. The second has the atom $\{*\}$. An equivalence of categories preserves terminal objects, monomorphisms, and their subobject posets, so it cannot identify these two categories.

Theorem 11.2 retains continuity through the pro-object. The conservativity assertion in that theorem applies to a specified pro-morphism; it does not identify an arbitrary equivalence of underlying model spaces with a continuous morphism.

12 Higher families and truncation

For a π-finite space X, put

$$\mathcal{C}_X = (\mathcal{S}_\pi)_{/X}, \qquad \mathcal{L}_m(X) = \mathrm{Fun}(X, \mathcal{S}_{\pi,\le m}), \qquad m \ge 0.$$

The second category is identified with the full subcategory of $\mathcal{C}_X$ consisting of maps whose homotopy fibres are m-truncated. The truncation is relative to X.

Lemma 12.1. *If X is π-finite, a map $E \to X$ has π-finite homotopy fibres if and only if E is π-finite. Consequently*

$$\mathcal{C}_X \simeq \mathrm{Fun}(X, \mathcal{S}_\pi).$$

Proof. For a homotopy fibre F over a point of X, the long exact homotopy sequence shows that finite homotopy groups and a truncation bound for E and X imply the same properties for F. Its components are finite as well: they lie over the finitely many components of E, and the possible components over any one of them form a quotient of the finite group $\pi_1(X)$. Conversely, choose one point in every component of X. There are only finitely many chosen fibres. Their homotopy groups are finite and their truncation bounds have a finite maximum. The homotopy sequence gives these properties for E, whose components form a quotient of the disjoint union of the component sets of the chosen fibres. The displayed equivalence is the usual classification of families of spaces over an ∞-groupoid by functors from it. □

The category $\mathcal{C}_X$ is the bounded coherent part of $\mathcal{S}_{/X}$. Fixed relative truncation bounds give the categories $\mathcal{L}_m(X)$ used below. Quotients in the full bounded coherent category can increase a truncation bound, so these two categories have different closure properties.

Theorem 12.2 (Sharpness of the family truncation bound)**.** *Let A be a nonzero finite abelian group, let $r \geq 1$, and set $X = K(A, r+1)$, with its standard base point x. Then the following statements hold.*

(i) *For $0 \leq m \leq r-1$, evaluation at x induces an equivalence*

$$\mathcal{L}_m(X) \xrightarrow{\sim} \mathcal{S}_{\pi,\leq m}.$$

An inverse sends a space to its constant family.

(ii) *The point object $P_x = (* \xrightarrow{x} X)$ belongs to $\mathcal{L}_r(X)$ and is not a constant family.*

(iii) *Its endomorphism space in the slice is*

$$\mathrm{Map}_{\mathcal{C}_X}(P_x, P_x) \simeq \Omega_x X \simeq K(A, r).$$

Every endomorphism in this space is an equivalence.

Proof. An object F of $\mathcal{S}_{\pi,\leq m}$ has an m-truncated self-equivalence space. The component of F in the core $(\mathcal{S}_{\pi,\leq m})^{\simeq}$ is therefore the $(m+1)$-type $B\,\mathrm{Aut}(F)$. Since X is connected, every functor from X lands in one such component. Since X is r-connected and $m+1 \leq r$, evaluation induces an equivalence

$$\mathrm{Map}(X, B\,\mathrm{Aut}(F)) \simeq B\,\mathrm{Aut}(F).$$

Thus every object of $\mathcal{L}_m(X)$ is a constant family. For constant families F, G, their mapping space is

$$\mathrm{Map}_{\mathcal{L}_m(X)}(F_X, G_X) \simeq \mathrm{Map}(X, \mathrm{Map}(F, G)).$$

The inner mapping space is m-truncated, so evaluation is again an equivalence. This proves the first statement.

The homotopy fibre of P_x is $\Omega X \simeq K(A, r)$, which is π-finite and r-truncated. A constant family with that fibre has a section. A section of P_x would contract X, which is impossible because

$A \neq 0$. Finally, the slice mapping-space formula identifies $\mathrm{Map}_{/X}(P_x, P_x)$ with the homotopy fibre of the base-point map $* \to X$, namely ΩX. All these maps have an equivalence as their underlying map of total spaces, so they are equivalences in the slice. □

In particular, for every finite abelian A,

$$\mathcal{L}_0(K(A,2)) \simeq \mathbf{FinSet}.$$

The ordinary fundamental groupoid of $K(A,2)$ is terminal. The first relative truncation level already contains the point object, with self-equivalence space BA. Thus both finite-set semantics and the ordinary fundamental groupoid lose information that is present in the next family level. More generally, Theorem 12.2 shows that r-truncated families can require the $(r+1)$-type of the base.

Corollary 12.3. *For every finite abelian A, the zero-truncated part of $\mathcal{C}_{K(A,2)}$ is* **FinSet***, and its groupoid of set-valued models preserving finite limits, finite coproducts, and effective epimorphisms, with natural isomorphisms as arrows, is terminal. These data are independent of A, while the point object's higher self-equivalence space is BA.*

Proof. The category assertion is Theorem 12.2. A functor $\mathbf{FinSet} \to \mathbf{Set}$ with these preservation properties sends the terminal object to a singleton and sends every finite set, expressed as a finite coproduct of singletons, to that same finite set up to a unique compatible isomorphism. These isomorphisms commute with every function. Every natural automorphism is the identity, since it is the identity on the terminal object and commutes with all point maps. □

12.1 A computable family of higher torsors

For a finite abelian group B, the space $K(B,r)$ is a group object whose delooping is $K(B,r+1)$. A principal $K(B,r)$-bundle over X is classified by a map $X \to K(B,r+1)$. The group object and its action are retained in this classification. For $r=1$ these are gerbes with fixed constant abelian band B.

Proposition 12.4. *Let A, B be finite abelian groups and let $r \geq 1$. The moduli space of principal $K(B,r)$-bundles over $K(A,r+1)$ is*

$$\mathrm{Map}(K(A,r+1), K(B,r+1)) \simeq \mathrm{Hom}(A,B) \times K(B,r+1).$$

In particular, its equivalence classes are naturally indexed by homomorphisms $\alpha : A \to B$. The family corresponding to α can be represented by

$$E_\alpha = \mathrm{hofib}\big(K(\alpha, r+1)\big) \longrightarrow K(A, r+1).$$

Its total space is connected and π-finite, with

$$\pi_{r+1}(E_\alpha) \cong \ker \alpha, \qquad \pi_r(E_\alpha) \cong \mathrm{coker}\, \alpha,$$

and with all other positive-degree homotopy groups zero. The family has a section exactly when $\alpha = 0$.

Proof. Eilenberg–Mac Lane representability and the Hurewicz theorem give

$$[K(A,r+1), K(B,r+1)]_* = H^{r+1}(K(A,r+1); B) \cong \mathrm{Hom}(A,B).$$

The universal coefficient exact sequence has no Ext term in this degree, since $H_r(K(A, r+1); \mathbb{Z}) = 0$. The positive homotopy groups of the pointed mapping space are the reduced cohomology groups in degrees below $r+1$, all of which vanish by r-connectedness. Hence this pointed mapping space is the discrete space $\mathrm{Hom}(A, B)$. The abelian group structure on $K(B, r+1)$ splits evaluation at the base point: subtracting the value at that point identifies the unpointed mapping space with the product of the pointed mapping space and $K(B, r+1)$.

Pulling back the path fibration of $K(B, r+1)$ along $K(\alpha, r+1)$ gives the displayed family. Its homotopy sequence reduces in positive degrees to

$$0 \longrightarrow \pi_{r+1}(E_\alpha) \longrightarrow A \xrightarrow{\alpha} B \longrightarrow \pi_r(E_\alpha) \longrightarrow 0.$$

Its fibre is the connected space $K(B, r)$, so its total space is connected. The sequence proves π-finiteness and the homotopy-group assertions. A section is a nullhomotopy of the classifying map. By the first calculation this exists precisely when $\alpha = 0$. □

For $A = B$ and $\alpha = \mathrm{id}$, the total space E_α is contractible and the family is the point object of Theorem 12.2. For $\alpha = 0$ it is the product $K(A, r+1) \times K(B, r)$. Already $A = B = C_2$ and $r = 1$ give two inequivalent finite-band families over a base whose finite-set semantics is **FinSet**. As B varies, the pointed classes give the functor $B \mapsto \mathrm{Hom}(A, B)$ on finite abelian groups, which determines A by Yoneda.

12.2 An explicit finite simplicial presentation

For a finite abelian group A, use the standard simplicial abelian group whose q-simplices are normalized 2-cocycles on Δ^q with coefficients in A. Such a simplex is a collection

$$(a_{ijk})_{0 \le i < j < k \le q}, \qquad a_{jk\ell} - a_{ik\ell} + a_{ij\ell} - a_{ijk} = 0 \quad (i < j < k < \ell).$$

Faces restrict the collection, and degeneracies pull it back with repeated-index entries set to zero. The independent entries are a_{0ij} for $1 \le i < j \le q$: all remaining entries are

$$a_{ijk} = a_{0jk} - a_{0ik} + a_{0ij} \qquad (0 < i < j < k).$$

Thus this simplicial set is degreewise finite, with

$$|G_q| = |A|^{\binom{q}{2}}.$$

Its normalized chain complex is A in degree two and zero in every other degree. The normalized-chain calculation for simplicial abelian groups identifies its realization with $K(A, 2)$. For $A = C_2$ the first levels are

Simplicial degree q	0	1	2	3	4
Number of simplices $\lvert G_q \rvert$	1	1	2	8	64

Consequently the category $\mathcal{L}_1(K(C_2, 2))$ is recovered from the finite diagram of groupoid-valued families on the first four displayed levels, through degree three. Maps between specified families use degrees through two. The degree-four entries are not additional descent data in this calculation; their compatibility already belongs to the specified simplicial abelian group.

13 Finite two-types with the same homotopy groups

The next example keeps the fundamental group, the second homotopy group, and their action fixed. The remaining difference is an associativity class. The classification of coherent 2-groups by a group, a module, and a degree-three cohomology class is classical; see [2, Section 8.3]. We give the complete finite calculation needed here.

Proposition 13.1. *There are connected π-finite two-types X_0, X_1 with*

$$\pi_1(X_i) \cong C_2, \qquad \pi_2(X_i) \cong C_2,$$

and trivial fundamental-group action, such that X_0 and X_1 are not equivalent. Their zero-truncated family categories are both equivalent to $\mathbf{FinSet}^{C_2}$, whereas their one-truncated family categories are not equivalent.

Proof. Write $C_2 = \{0, 1\}$ additively and identify the coefficient group with $\mathbb{F}_2$, with trivial action. Define the normalized three-cochain

$$k(g, h, \ell) = gh\ell \in \mathbb{F}_2.$$

The group-cohomology differential is

$$\begin{aligned}(\delta k)(g, h, \ell, t) = {} & k(h, \ell, t) + k(g + h, \ell, t) + k(g, h + \ell, t) \\ & + k(g, h, \ell + t) + k(g, h, \ell).\end{aligned}$$

Expanding the products shows that every monomial occurs twice; hence $\delta k = 0$. A normalized two-cochain c has only one possibly nonzero value, namely $c(1, 1)$. Its coboundary satisfies

$$(\delta c)(1, 1, 1) = c(1, 1) + c(0, 1) + c(1, 0) + c(1, 1) = 0.$$

Since $k(1, 1, 1) = 1$, its cohomology class is nonzero.

Choose a representative map $k : BC_2 \to K(C_2, 3)$, and define

$$X_1 = \mathrm{hofib}(k), \qquad X_0 = BC_2 \times K(C_2, 2).$$

The homotopy-fibre sequence gives the asserted homotopy groups and vanishing in all higher degrees. The fibrations over BC_2 are principal $K(C_2, 2)$-bundles, so the action on π_2 is trivial. In each case the projection to BC_2 is the first Postnikov truncation. The projection for X_0 has a section. The projection for X_1 has a section exactly when k is nullhomotopic, which the cohomology calculation excludes. An equivalence of the two spaces would preserve their Postnikov truncations and the existence of such a section, a contradiction.

Zero-truncated families depend only on the fundamental groupoid. Both spaces therefore have finite-set families equivalent to the category of finite C_2-sets. If their one-truncated family categories were equivalent, their full model categories would be equivalent as well, since the preservation properties defining models are intrinsic. By Corollary 7.3, those model categories recover X_0 and X_1. This contradicts the distinction just proved. □

13.1 A finite crossed-module model

The nonzero associativity class has a finite strict model. Consider

$$\partial : C_4 \longrightarrow C_4, \qquad \partial(u) = 2u, \qquad t \cdot u = (-1)^t u. \tag{13.1}$$

The crossed-module identities can be checked directly. Equivariance holds because $2(-1)^t u = 2u$ in C_4, and the target group is abelian. The Peiffer identity holds because $\partial(u)$ is even and therefore acts trivially, as does conjugation by u in the abelian source group. Its kernel is $\{0, 2\} \cong C_2$ and its cokernel is C_2. The induced action on the kernel is trivial.

To compute its associativity class, choose the representatives $s(0) = 0, s(1) = 1$ in the target C_4. The multiplication defect has the lift

$$a(g, h) = gh \in C_4, \qquad \partial a(g, h) = s(g) + s(h) - s(g + h).$$

The resulting kernel-valued three-cocycle is

$$\kappa(g, h, \ell) = s(g) \cdot a(h, \ell) + a(g, h + \ell) - a(g + h, \ell) - a(g, h).$$

It vanishes whenever one argument is zero, while

$$\kappa(1, 1, 1) = -1 - 1 = 2 \in \ker \partial.$$

Identifying $2 \in C_4$ with $1 \in C_2$ recovers the cocycle of Proposition 13.1. In comparison, the crossed module $0 : C_2 \to C_2$ with trivial action gives X_0.

Equivalently, use the skeletal monoidal groupoid with objects $g \in C_2$, with $\mathrm{Hom}(g, g) = C_2$, and with no morphisms between distinct objects. Both tensor operations are addition, and the unit constraints are identities. Giving its associator the value $k(g, h, \ell)$ produces the nontrivial two-type; taking the zero associator produces the trivial one. The pentagon is exactly $\delta k = 0$. Thus the two structures have the same objects, morphisms, composition, and tensor functor, but different coherent associators.

Remark 13.2 (The pentagon is an independent condition). For $G = C_3$ and trivial coefficients C_2, take the normalized three-cochain b that is 1 only on $(1, 1, 1)$ and is zero elsewhere. The same underlying skeletal groupoid, tensor functor, and proposed associators are defined, but

$$(\delta b)(1, 1, 1, 2) = 1.$$

Hence these data do not satisfy the pentagon. This finite example separates the choice of associators from their four-input compatibility. For $n = 1$, the descent formula uses simplicial degrees through three of an already valid presentation. It does not declare an arbitrary three-dimensional specification of a two-group to be coherent: the pentagon must still hold. There is no conflict between this requirement and the finite descent bound.

13.2 Exact finite checks

The accompanying script `verify_higher_examples.py` uses integer arithmetic and exhaustive enumeration. It checks all four-tuples in the C_2 cocycle equation, both normalized two-cochains for the non-boundary assertion, all crossed-module axioms for (13.1), and the failed pentagons for the C_3 example. It also enumerates the cocycles on Δ^q for $K(C_2, 2)$ through $q = 4$, and verifies the $C_2/\mathbb{F}_3$ sharpness differentials for truncation levels 0 through 8. The general statements are established by the proofs above; these finite calculations provide reproducible checks of the displayed examples.

14 Further questions

The universal parameter is an actual object when the source and target semantics are represented by π-finite model spaces. Example 11.7 shows that an exact functor between continuous finite-family categories can fail to have a left adjoint. A relative extension of the theory should describe the pro-object representing its chosen sections and characterize when that pro-object is represented by a single coherent parameter. The composition and change-of-parameter formulas of Proposition 8.3 provide the finite case of this problem.

The Postnikov lifting theorem gives a finite tower of obstruction spaces for every prescribed finite homotopy range. The successive obstructions depend on the choices made in earlier layers. Explicit families with several nonzero higher homotopy groups would allow a systematic study of how these dependencies constrain the moduli of fully faithful exact functors. The cyclic calculation isolates one such effect already at a single layer: forgetting lower markings changes the group of components of self-equivalences according to the parity of the Postnikov degree.

For syntactic applications, the finite comparison criterion asks for component lifting and effective presentations derived from stated formation and elimination rules. A concrete syntax should also express the section u_F of the universal parameter and the cochains representing the relative obstruction. This would connect the intrinsic categorical statements with explicit rules for adjoining and eliminating higher parameters.

Finally, fixed truncation bounds control the number of coherence degrees without bounding the size of a finite presentation. It is natural to ask for quantitative bounds on the finite stages and cochain groups required by a specified family of theories. The finite-stage descent statements in Section 11 give existence after refinement; explicit complexity estimates require additional structure on the system of finite approximations.

References

[1] M. Anel, *The category of π-finite spaces*, arXiv:2107.02082, version 4, 2025. https://arxiv.org/abs/2107.02082v4.

[2] J. C. Baez and A. D. Lauda, Higher-dimensional algebra V: 2-groups, *Theory and Applications of Categories* **12** (2004), no. 14, 423–491. https://math.ucr.edu/home/baez/hda5.pdf.

[3] C. Barwick, S. Glasman, and P. J. Haine, *Exodromy*, arXiv:1807.03281, version of 21 August 2020. https://clarkbar.github.io/papers/exodromy_book.pdf.

[4] K. S. Brown, Lectures on the cohomology of groups, in *Cohomology of Groups and Algebraic K-theory*, Advanced Lectures in Mathematics, vol. 12, International Press, 2010, pp. 131–166. https://pi.math.cornell.edu/~kbrown/papers/cohomology_hangzhou.pdf.

[5] N. Gambino, S. Henry, C. Sattler, and K. Szumiło, The effective model structure and ∞-groupoid objects, *Forum of Mathematics, Sigma* **10** (2022), e34, 1–59. doi:10.1017/fms.2022.13.

[6] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002. https://pi.math.cornell.edu/~hatcher/AT/AT.pdf.

[7] L. Hesselholt and P. Pstrągowski, Dirac geometry I: Commutative algebra, *Peking Mathematical Journal* **8** (2025), 405–480. doi:10.1007/s42543-023-00072-6.

[8] M. Hoyois, Higher Galois theory, *Journal of Pure and Applied Algebra* **222** (2018), no. 7, 1859–1877. doi:10.1016/j.jpaa.2017.08.010.

[9] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, 2009. https://www.math.ias.edu/~lurie/papers/HTT.pdf.

[10] J. M. Møller, Spaces of sections of Eilenberg–Mac Lane fibrations, *Pacific Journal of Mathematics* **130** (1987), no. 1, 171–186. https://msp.org/pjm/1987/130-1/pjm-v130-n1-p10-s.pdf.

[11] J. M. Møller and J. Scherer, Can one classify finite Postnikov pieces?, *Bulletin of the London Mathematical Society* **42** (2010), no. 4, 661–672. doi:10.1112/blms/bdq027.

[12] Z. Sun, *Definability and Lifting in Model Subgroupoids*, manuscript, 2026.

[13] B. Toën, Vers une interprétation galoisienne de la théorie de l'homotopie, *Cahiers de Topologie et Géométrie Différentielle Catégoriques* **43** (2002), no. 4, 257–312. https://www.numdam.org/item/CTGDC_2002__43_4_257_0/.