A MICHAEL SPACE IN THE LAVER MODEL

Abstract. A Michael space is a regular Lindelöf space whose product with the irrationals fails to be Lindelöf. Whether such a space exists in ZFC is a sixty-year-old problem of E. Michael. We show that one exists in the Laver model, where every previously known construction breaks down: there $\mathrm{cov}(\mathcal{M}) = \aleph_1 < \mathfrak{b} = \mathfrak{d} = \aleph_2$ and there are no selective ultrafilters. Two ingredients go into the proof. The first is a dichotomy for countable support iterations of Laver forcing of any length: each real in the extension is either bounded by a real of the ground model, or dominates every real of the ground model along one infinite set. The second is a ZFC theorem saying that, when $\mathfrak{b} = \mathfrak{d}$, a scale which satisfies this dichotomy at its points of uncountable cofinality gives rise to a Michael sequence in the sense of J. T. Moore.

hexagon:2611.00286v1

1. Introduction

In 1963 E. Michael [7] used the Continuum Hypothesis to build a Lindelöf space whose product with the space ω^ω of irrationals is not normal, and in particular not Lindelöf. The example showed how badly the Lindelöf property can behave under products, even when one factor is as tame as the irrationals. A regular Lindelöf space X for which $X \times \omega^\omega$ is not Lindelöf has since been called a *Michael space*, and *Michael's problem* asks whether one can be constructed without any hypothesis beyond ZFC. The problem is still open.

What is known is that a Michael space exists whenever the cardinal characteristics of the continuum are arranged favourably. This is the case if $\mathfrak{b} = \aleph_1$ (Lawrence [5]), under Martin's Axiom (Alster [1]), if $\mathfrak{d} = \mathrm{cov}(\mathcal{M})$ (Moore [10]), and if there is a selective ultrafilter and $\max\{\mathfrak{b}, \mathfrak{g}\} = \mathfrak{d}$ (Martínez-Celis [6]). Moore's paper also changed the shape of the problem. He showed that building a Michael space comes down to building a certain increasing sequence of sets of irrationals, a *Michael sequence* (Definition 2.1), so that the question is really one about the combinatorics of ω^ω.

A model of ZFC without a Michael space, if there is one, must defeat all of these constructions at once. The most natural place to look is the Laver model, obtained by iterating Laver forcing ω_2 times with countable support over a model of CH [4]. In that model $\mathrm{cov}(\mathcal{M}) = \aleph_1$ while $\mathfrak{b} = \mathfrak{d} = \aleph_2$, and by a theorem of Miller [8] there are no Q-points, hence no selective ultrafilters. So none of the results above applies, and to our knowledge it was not known whether the Laver model contains a Michael space; compare the discussion and questions in [6]. Our main result is that it does.

Theorem 1.1. *There is a Michael space in the Laver model.*

The proof separates cleanly into a forcing part and a combinatorial part, and each may be of interest on its own. Throughout, for $u, h \in \omega^\omega$ and $A \subseteq \omega$ we write $h \leq^* u{\restriction}A$ to mean that $h(k) \leq u(k)$ for all but finitely many $k \in A$.

The forcing part describes how a new real can sit relative to the old ones after a Laver iteration. One might expect many intermediate behaviours: a new real could be unbounded

Date: Draft of 10 November 2026.

2020 *Mathematics Subject Classification.* Primary 54D20, 03E35; Secondary 54B10, 03E17.

Key words and phrases. Michael space, Lindelöf space, Laver forcing, Michael sequence, dominating real.

over the ground model and yet exceed each ground model function only on a set that depends on the function. The theorem says that this does not happen.

Theorem 1.2 (Dichotomy)**.** *Let W be a model of ZFC, let $\alpha \geq 1$ be an ordinal, and let G be generic over W for the countable support iteration $\mathbb{P}_\alpha$ of Laver forcing of length α. For every $u \in \omega^\omega \cap W[G]$, exactly one of the following holds.*

(1) *$u \leq^* h^*$ for some $h^* \in \omega^\omega \cap W$.*
(2) *There is an infinite $A \subseteq \omega$ in $W[G]$ such that $h \leq^* u{\restriction}A$ for every $h \in \omega^\omega \cap W$.*

The combinatorial part is a theorem of ZFC. It says that the simplest conceivable candidate for a Michael sequence, the one read off directly from a scale, works as soon as the scale obeys the same dichotomy.

Theorem 1.3 (ZFC)**.** *Assume $\mathfrak{b} = \mathfrak{d} = \kappa$ and let $\langle g_\gamma : \gamma < \kappa \rangle$ be a scale. Suppose that for every $\alpha < \kappa$ of uncountable cofinality and every $u \in \omega^\omega$, either $u \leq^* g_\gamma$ for some $\gamma < \alpha$, or there is an infinite $A \subseteq \omega$ with $g_\gamma \leq^* u{\restriction}A$ for all $\gamma < \alpha$. Then there is a κ-Michael sequence, and therefore a Michael space.*

When $\kappa = \aleph_1$ there are no ordinals $\alpha < \kappa$ of uncountable cofinality, the hypothesis is empty, and Theorem 1.3 becomes the familiar fact that $\mathfrak{b} = \mathfrak{d} = \aleph_1$ yields a Michael space. The content of the theorem lies in the case $\kappa > \aleph_1$.

Theorem 1.1 follows by combining the two: in the Laver model the Laver reals form a scale, and Theorem 1.2, applied over the intermediate models, shows that this scale satisfies the hypothesis of Theorem 1.3.

A word on the proof of Theorem 1.2. The obvious strategy is an induction on the length of the iteration, and it runs into trouble at limit stages of countable cofinality, where a condition can make demands at infinitely many coordinates simultaneously. We avoid the induction altogether. Laver showed [4, Lemma 14] that a name for a real can be approximated, at each node of the first coordinate's tree, by finitely many candidate reals from the ground model. We apply this lemma once, after allowing the candidates to take the value ∞. A candidate with an entry ∞ at position k records that $u(k)$ is being pushed to infinity as the first Laver real grows, and the dichotomy is decided by whether such candidates keep appearing along the generic branch. Only the first coordinate is examined, so the length of the iteration plays no role.

The paper is organised as follows. Section 2 fixes notation and recalls Michael sequences. Section 3 proves Theorem 1.3. Section 4 recalls what we need about Laver forcing, and Section 5 proves Theorem 1.2. Section 6 derives Theorem 1.1, and Section 7 collects remarks and questions.

2. Preliminaries

For $f, g \in \omega^\omega$, $f \leq^* g$ means $f(n) \leq g(n)$ for all but finitely many n, and $f <^* g$ means $f(n) < g(n)$ for all but finitely many n. The cardinals $\mathfrak{b}$ and $\mathfrak{d}$ are the least sizes of an unbounded and of a dominating subfamily of $(\omega^\omega, \leq^*)$. A *scale* is a $\leq^*$-dominating sequence $\langle g_\gamma : \gamma < \kappa \rangle$ with $g_\gamma <^* g_\delta$ for $\gamma < \delta$; a scale of length κ exists if and only if $\mathfrak{b} = \mathfrak{d} = \kappa$. For $s \in \omega^{<\omega}$ let $[s] = \{x \in \omega^\omega : s \subseteq x\}$. For closed $K \subseteq \omega^\omega$, the tree of K is $\{x{\restriction}n : x \in K,\ n < \omega\}$. For $g \in \omega^\omega$ and $m < \omega$ put

$$B_m(g) = \{x \in \omega^\omega :\ x(n) \leq g(n) \text{ for all } n \geq m\}.$$

Each $B_m(g)$ is closed, and the set of reals eventually below g is the increasing union $\bigcup_m B_m(g)$. A compact subset of ω^ω is always contained in some $B_0(g)$.

Definition 2.1 (Moore [10]; in this form [6])**.** Let θ be an ordinal of uncountable cofinality. A *θ-Michael sequence* is a strictly increasing sequence $\langle Y_\alpha : \alpha \leq \theta\rangle$ of subsets of ω^ω with $Y_\theta = \omega^\omega$ such that for every compact $K \subseteq \omega^\omega$ and every $\alpha \leq \theta$ of uncountable cofinality, if $K \subseteq Y_\alpha$ then $K \subseteq Y_\gamma$ for some $\gamma < \alpha$.

In words: the sets Y_α exhaust the irrationals, and a compact set can never be captured for the first time at a stage of uncountable cofinality. Moore's original formulation uses the complements of the sets Y_α inside a compactification of ω^ω with countable remainder; the two formulations are equivalent.

Theorem 2.2 (Moore [10, Corollary 1.4])**.** *If there is a θ-Michael sequence for some θ of uncountable cofinality, then there is a Michael space.*

The next proposition is not needed for the main theorem, but it explains where the difficulty lies. A Michael sequence of regular length must have length between $\mathfrak{b}$ and $\mathfrak{d}$. So once $\mathfrak{b} > \aleph_1$, the sequence has length at least ω_2 and the defining condition has to be verified at ordinals of cofinality ω_1 strictly inside it. Those are exactly the ordinals addressed by the hypothesis of Theorem 1.3.

Proposition 2.3. *If θ is regular uncountable and there is a θ-Michael sequence, then $\mathfrak{b} \leq \theta \leq \mathfrak{d}$.*

Proof. Let $\langle Y_\alpha : \alpha \leq \theta\rangle$ be a θ-Michael sequence. By the condition at $\alpha = \theta$, every compact set is contained in some Y_γ with $\gamma < \theta$, and since $\mathrm{cf}(\theta) > \omega$ the same holds for every σ-compact set.

Suppose $\theta < \mathfrak{b}$. For each $\xi < \theta$ choose $f_\xi \notin Y_\xi$ and let $h \in \omega^\omega$ with $f_\xi \leq^* h$ for all ξ. The set $\{f : f \leq^* h\}$ is σ-compact, so it is contained in some Y_γ, $\gamma < \theta$. But then $f_\gamma \in Y_\gamma$, a contradiction.

Suppose $\theta > \mathfrak{d}$ and let $\{h_i : i < \mathfrak{d}\}$ be dominating. Each $\{f : f \leq^* h_i\}$ is contained in some Y_{γ_i}, and $\gamma = \sup_i \gamma_i < \theta$. Then $Y_\gamma = \omega^\omega$, contradicting strict monotonicity. □

3. Michael sequences from scales

Given a scale $\langle g_\gamma : \gamma < \kappa\rangle$, the obvious candidate for a Michael sequence is

$$Y_\alpha = \{x \in \omega^\omega : \ x \leq^* g_\gamma \text{ for some } \gamma < \alpha\}, \qquad \alpha \leq \kappa,$$

the reals bounded by an initial segment of the scale. What could go wrong is the following. Take α of uncountable cofinality and a compact set K every point of which is bounded by some g_γ with $\gamma < \alpha$. We need a single $\gamma < \alpha$ that works for all of K, and there is no reason for one to exist: the bounds may climb cofinally in α as the point moves through K. The proof shows that if they do climb, then inside K there is a point that escapes every g_γ, $\gamma < \alpha$. The escaping point is built by walking up the tree of K and, at each step, taking a value as large as K allows at a well-chosen coordinate. The hypothesis of the theorem is what tells us which coordinates to choose.

Proof of Theorem 1.3. Define Y_α as above. The sequence is strictly increasing, since $g_\gamma \in Y_{\gamma+1} \setminus Y_\gamma$, and $Y_\kappa = \omega^\omega$ because the g_γ are dominating. A compact set lies in some $B_0(h)$, and $h \leq^* g_\gamma$ for some γ, so the condition of Definition 2.1 holds at $\alpha = \kappa$.

Now fix $\alpha < \kappa$ with $\mathrm{cf}(\alpha) > \omega$ and a compact $K \subseteq Y_\alpha$. The set K is covered by the closed sets $B_m(g_\gamma)$ with $\gamma < \alpha$ and $m < \omega$. We first discard the part of K where countably many of them suffice. Let O be the union of all relatively open subsets of K which are covered

by countably many sets $B_m(g_\gamma)$, $\gamma < \alpha$. Since K is second countable, O is itself covered by countably many such sets. Let $K' = K \setminus O$.

If $K' = \emptyset$, then K is covered by sets $B_{m_j}(g_{\gamma_j})$, $j < \omega$, with $\gamma_j < \alpha$. As $\mathrm{cf}(\alpha) > \omega$ there is $\gamma < \alpha$ above all γ_j, and $K \subseteq Y_\gamma$, which is what we want.

So suppose, towards a contradiction, that $K' \neq \emptyset$. Let T be the tree of K'. For $t \in T$ let $u_t \in \omega^\omega$ record the largest values that points of K' passing through t can take:

$$u_t(n) = \max\{x(n) :\ x \in K',\ t \subseteq x\}.$$

Claim. For no $t \in T$ and $\gamma < \alpha$ is $u_t \leq^* g_\gamma$.

Indeed, if $u_t \leq^* g_\gamma$ then $K' \cap [t] \subseteq B_m(g_\gamma)$ for some m. Then $K \cap [t] \subseteq O \cup B_m(g_\gamma)$ would be a relatively open subset of K covered by countably many of our closed sets, so $K \cap [t] \subseteq O$ and $K' \cap [t] = \emptyset$, contradicting $t \in T$.

So every u_t falls under the second alternative of the hypothesis: for each $t \in T$ there is an infinite $A_t \subseteq \omega$ along which u_t dominates the whole initial segment of the scale, that is, $g_\gamma \leq^* u_t{\restriction}A_t$ for all $\gamma < \alpha$. How far out one must go before u_t exceeds g_γ on A_t depends on γ; let $F_\gamma(t)$ be the least N such that $g_\gamma(n) \leq u_t(n)$ for all $n \in A_t$ with $n \geq N$. There are fewer than $\mathfrak{b}$ functions F_γ, all defined on the countable set T, so a single $e : T \to \omega$ dominates them: for every $\gamma < \alpha$, $F_\gamma(t) \leq e(t)$ for all but finitely many $t \in T$.

We now walk up the tree. Let $t_0 = \emptyset$. Given $t_k \in T$, pick a coordinate in A_{t_k} beyond both the length of t_k and the threshold $e(t_k)$,

$$n_k = \min\big(A_{t_k} \setminus \max(|t_k|, e(t_k))\big),$$

choose a point $x_k \in K' \cap [t_k]$ which attains the maximum there, $x_k(n_k) = u_{t_k}(n_k)$, and let $t_{k+1} = x_k{\restriction}(n_k + 1)$. The nodes t_k are strictly increasing, so they are pairwise distinct and $x = \bigcup_k t_k$ is a point of K' with $x(n_k) = u_{t_k}(n_k)$ for all k.

This x escapes the initial segment. Let $\gamma < \alpha$; then also $\gamma + 1 < \alpha$. For all but finitely many k we have $F_{\gamma+1}(t_k) \leq e(t_k) \leq n_k$ and $n_k \in A_{t_k}$, hence

$$x(n_k) = u_{t_k}(n_k) \geq g_{\gamma+1}(n_k) > g_\gamma(n_k),$$

where the strict inequality holds for all large k because $g_\gamma <^* g_{\gamma+1}$ and $n_k \to \infty$. So $x \not\leq^* g_\gamma$. Since $\gamma < \alpha$ was arbitrary, $x \notin Y_\alpha$, contradicting $x \in K' \subseteq K \subseteq Y_\alpha$.

Hence $\langle Y_\alpha : \alpha \leq \kappa \rangle$ is a κ-Michael sequence, and Theorem 2.2 gives a Michael space. □

Remark 3.1. The hypothesis of Theorem 1.3 can be read as a statement about towers. For u not below any g_γ, $\gamma < \alpha$, the sets $\{k : u(k) \geq g_\gamma(k)\}$ form a $\subseteq^*$-decreasing sequence of infinite sets, and the set A we ask for is an infinite pseudo-intersection. If $\mathfrak{t} = \mathfrak{d}$ the sequence is shorter than $\mathfrak{t}$ and a pseudo-intersection exists for free, so every scale satisfies the hypothesis. Since $\mathfrak{t} \leq \mathrm{cov}(\mathcal{M})$, that case is already covered by Moore's theorem for $\mathfrak{d} = \mathrm{cov}(\mathcal{M})$. In the Laver model $\mathfrak{t} = \aleph_1 < \mathfrak{d}$, towers of length ω_1 exist, and the hypothesis has to be verified by other means. That is the purpose of Theorem 1.2.

4. Laver forcing

We use the notation of [4]. A *Laver tree* is a tree $p \subseteq \omega^{<\omega}$ with a node $\mathrm{stem}(p)$ such that every $s \in p$ is comparable with $\mathrm{stem}(p)$ and every $s \in p$ with $s \supseteq \mathrm{stem}(p)$ has infinitely many immediate successors in p. Laver forcing consists of all Laver trees ordered by inclusion. For $t \in p$ let $p_t = \{s \in p : s \subseteq t \text{ or } t \subseteq s\}$. We write $q \leq^0 p$ if $q \leq p$ and $\mathrm{stem}(q) = \mathrm{stem}(p)$. For $s \in p$, $s \supseteq \mathrm{stem}(p)$, let $\mathrm{succ}_p(s) = \{i : s^\frown i \in p\}$.

Let $\mathbb{P}_\alpha$ denote the countable support iteration of Laver forcing of length α, and ℓ_β the generic real added at coordinate β. For $p \in \mathbb{P}_\alpha$ and $t \in p(0)$ with $t \supseteq \mathrm{stem}(p(0))$ let

$$p|t = p(0)_t{}^\frown p\restriction[1,\alpha),$$

the condition obtained from p by restricting its first coordinate to the part of the tree through t and leaving everything else alone. A generic filter G contains $p|t$ if and only if $p \in G$ and $t \subseteq \ell_0$.

The tool we need is a lemma of Laver. It says that although a name for a real may depend on the whole iteration, at each node of the first tree its value is determined, up to any prescribed precision and up to finitely many possibilities, by all but finitely many choices of the next value of ℓ_0. Laver states the lemma for $\mathbb{P}_{\omega_2}$ over a model of CH and for names of real numbers; the proof works for any length, over any ground model, and for names of elements of a compact metrizable space. The formulation for 2^ω is [9, Lemma 3.3]; see also [11].

Lemma 4.1 (Laver [4, Lemma 14])**.** *Let $\dot a$ be a $\mathbb{P}_\alpha$-name for an element of 2^ω and $p \in \mathbb{P}_\alpha$. There are $p' \le p$ with $p'(0) \le^0 p(0)$ and finite sets $U'_s \subseteq 2^\omega$ for all $s \in p'(0)$ with $s \supseteq \mathrm{stem}(p'(0))$, such that for every such s and every $n < \omega$, for all but finitely many $i \in \mathrm{succ}_{p'(0)}(s)$,*

$$p'|s^\frown i \ \Vdash\ \exists y \in U'_s\ (\dot a\restriction n = y\restriction n).$$

The lemma requires a compact space, and ω^ω is not compact. The remedy is to work in the compactification $\mathbb{K} = (\omega+1)^\omega$, where $\omega+1$ carries the order topology; $\mathbb{K}$ is compact, metrizable and zero-dimensional, and contains ω^ω. We write ∞ for the top point of $\omega+1$. The price is that candidates may now have infinite entries, and this turns out to be exactly the information we want. For $x \in \mathbb{K}$ and $n < \omega$ let

$$N_n(x) = \big\{y \in \mathbb{K} :\ \forall k < n\ \big[(x(k) < n \Rightarrow y(k) = x(k)) \wedge (x(k) \ge n \Rightarrow y(k) \ge n)\big]\big\}.$$

The sets $N_n(x)$, $n < \omega$, form a decreasing neighbourhood base of x in $\mathbb{K}$. For $u \in \omega^\omega$ we write $u \approx_n x$ if $u \in N_n(x)$, and say that x approximates u to precision n. Explicitly: on the first n coordinates, u agrees with x wherever x is small (below n), and u is large (at least n) wherever x is large or infinite.

Lemma 4.2. *Let W be the ground model, $\dot u$ a $\mathbb{P}_\alpha$-name for an element of ω^ω, and $p \in \mathbb{P}_\alpha$. There are $p' \le p$ with $p'(0) \le^0 p(0)$ and finite sets $U_s \subseteq \mathbb{K}$ in W, for $s \in p'(0)$ with $s \supseteq \mathrm{stem}(p'(0))$, such that for every such s and every $n < \omega$, for all but finitely many $i \in \mathrm{succ}_{p'(0)}(s)$,*

$$p'|s^\frown i \ \Vdash\ \exists x \in U_s\ (\dot u \approx_n x).$$

Proof. We transfer Lemma 4.1 along an embedding. Fix in W a homeomorphism e of $\mathbb{K}$ onto a closed set $C \subseteq 2^\omega$; its definition is absolute, and in the extension $e(\dot u) \in C$. Apply Lemma 4.1 to a name $\dot a$ for $e(\dot u)$ to obtain p' and finite sets U'_s, and let $U_s = e^{-1}[U'_s \cap C]$. Candidates outside C are simply dropped; we check that nothing is lost.

Fix s and n. For each $y \in U'_s \setminus C$ choose m_y such that no element of C extends $y\restriction m_y$; this is possible because C is closed. For each $x \in U_s$, by continuity of e^{-1} at $e(x)$, choose m_x such that every $z \in C$ with $z\restriction m_x = e(x)\restriction m_x$ satisfies $e^{-1}(z) \in N_n(x)$. Let m be the maximum of these finitely many numbers. For all but finitely many $i \in \mathrm{succ}_{p'(0)}(s)$, the condition $p'|s^\frown i$ forces that $\dot a\restriction m = y\restriction m$ for some $y \in U'_s$. Since $\dot a \in C$ is forced, such a y cannot lie outside C. So $y = e(x)$ for some $x \in U_s$, and then $\dot u = e^{-1}(\dot a) \in N_n(x)$. $\square$

The second fact we need is the familiar strong form of the statement that a Laver real is dominating: it eventually exceeds any ground model function of its own past.

Fact 4.3. *Let $p' \in \mathbb{P}_\alpha$ and let $H \in W$ be a function from $\{s \in p'(0) : s \supseteq \mathrm{stem}(p'(0))\}$ to ω. Then p' forces that $\ell_0(m) \geq H(\ell_0{\restriction}m)$ for all but finitely many m.*

Proof. Let $r \leq p'$. Let $r'(0)$ consist of all $t \in r(0)$ such that $t(m) \geq H(t{\restriction}m)$ whenever $|\mathrm{stem}(r(0))| \leq m < |t|$, and let $r'(\beta) = r(\beta)$ for $\beta \geq 1$. Each node above the stem loses only finitely many successors, so $r'(0)$ is a Laver tree with the same stem as $r(0)$. Thus $r' \leq r$, and r' forces $\ell_0(m) \geq H(\ell_0{\restriction}m)$ for all $m \geq |\mathrm{stem}(r(0))|$. □

5. The dichotomy

Here is the idea. Fix a real u of the extension and a condition p' in the generic filter as in Lemma 4.2. As the first Laver real ℓ_0 is revealed one value at a time, each new value $\ell_0(m)$ tells us more about u: it pins u down, to a precision n_m which tends to infinity, to one of the finitely many candidates attached to the node $\ell_0{\restriction}m$. A candidate is a point of $(\omega+1)^\omega$, and an entry ∞ at a position k means that $u(k)$ has not been decided yet and is known only to be large, at least n_m.

Either candidates with infinite entries keep being used for ever, or from some stage on they do not. In the first case each use hands us a position k with $u(k) \geq n_m$. Since ℓ_0 outgrows every ground model function of its past, n_m can be made to exceed $h(k)$ for any ground model h, and the positions collected in this way form the set A. In the second case every value $u(k)$ is eventually read off from a candidate with finite entries, and there are only finitely many candidates at each node; taking the maximum over the ones that matter produces a ground model bound.

The point which makes this work for iterations, and not only for a single Laver real, is that we never need to know *which* candidate is the right one. That choice is made by the later coordinates, about which we assume nothing.

Proof of Theorem 1.2. The two alternatives exclude each other, since (2) applied to $h = h^*+1$ contradicts (1). We show that one of them holds.

Let $\dot u$ be a name for u. By Lemma 4.2 and genericity there is $p' \in G$ with finite sets $U_s \subseteq \mathbb{K}$, $s \in p'(0)$, $s \supseteq \mathrm{stem}(p'(0))$, as in that lemma. All objects defined below from p', $\langle U_s\rangle$ and the forcing relation belong to W. By a *node* we mean an $s \in p'(0)$ extending the stem.

For a node s and $i \in \mathrm{succ}_{p'(0)}(s)$, let $n_s(i)$ be the precision that the successor i guarantees:

$$n_s(i) = \max\{n \leq i :\ p'|s^\frown i \Vdash \exists x \in U_s\ (\dot u \approx_n x)\}.$$

Since the relations $\approx_n$ get stronger as n grows, Lemma 4.2 gives $n_s(i) \to \infty$ as $i \to \infty$ in $\mathrm{succ}_{p'(0)}(s)$.

Work in $W[G]$. Let $m_0 = |\mathrm{stem}(p'(0))|$, and for $m \geq m_0$ let $s_m = \ell_0{\restriction}m$ and $n_m = n_{s_m}(\ell_0(m))$. Applying Fact 4.3 to the functions $s \mapsto \min\{i_0 : \forall i \geq i_0\ n_s(i) \geq N\}$, $N < \omega$, we get $n_m \to \infty$. Since $p'|s_{m+1} \in G$, for every $m \geq m_0$ there is $x \in U_{s_m}$ with $u \approx_{n_m} x$; call such an x a *witness at* m. Let

$$\Gamma = \{m \geq m_0 :\ \text{some witness at } m \text{ has an entry equal to } \infty\}.$$

Case 1: Γ is infinite. We collect the first infinite entries of the witnesses. For $x \in \mathbb{K} \setminus \omega^\omega$ let $k_x = \min\{k : x(k) = \infty\}$, and let

$$A = \{k_x :\ m \in \Gamma,\ x \text{ a witness at } m,\ x \notin \omega^\omega,\ k_x < n_m\}.$$

If $k = k_x$ arises from m in this way, then $k < n_m$ and $x(k) \geq n_m$, so $u \approx_{n_m} x$ gives

$$u(k) \geq n_m. \tag{1}$$

We must show that u beats every ground model function on A. Let $h \in \omega^\omega \cap W$. The precision n_m is controlled by $\ell_0(m)$, so it is enough to ask $\ell_0(m)$ to be large. In W define, for a node s, $H_h(s)$ to be the least i_0 such that for all $i \geq i_0$ in $\mathrm{succ}_{p'(0)}(s)$ and all $x \in U_s \setminus \omega^\omega$,

$$n_s(i) > \max\big(k_x, h(k_x)\big).$$

This is well defined because U_s is finite and $n_s(i) \to \infty$. By Fact 4.3 there is m_h such that $\ell_0(m) \geq H_h(s_m)$ for all $m \geq m_h$. Hence for $m \in \Gamma$, $m \geq m_h$, and every witness $x \notin \omega^\omega$ at m we have $k_x < n_m$, so $k_x \in A$, and by (1)

$$u(k_x) \geq n_m > h(k_x).$$

The finitely many $m < m_h$ contribute only finitely many elements to A, because each U_{s_m} is finite. Therefore $h(k) \leq u(k)$ for all but finitely many $k \in A$.

It remains to see that A is infinite. By the above with $h = 0$, every sufficiently large $m \in \Gamma$ contributes at least one element to A. By (1) a given k can arise only from those m with $n_m \leq u(k)$, and there are finitely many such m because $n_m \to \infty$. Since Γ is infinite, so is A. Thus (2) holds.

Case 2: Γ is finite. Fix $m_1 \geq m_0$ such that for $m \geq m_1$ every witness at m lies in ω^ω, and for each $m \geq m_1$ fix a witness $x^m \in U_{s_m} \cap \omega^\omega$. We look for $h^* \in \omega^\omega \cap W$ with $u \leq^* h^*$.

Say that a position k is *resolved at* m if $k < n_m$ and $x^m(k) < n_m$. When this happens, $u \approx_{n_m} x^m$ gives $u(k) = x^m(k)$: the value of $u(k)$ has been read off. Every k is resolved at all sufficiently large m, for once $n_m > \max(k, u(k))$ the alternative $x^m(k) \geq n_m$ would give $u(k) \geq n_m$. The plan is to bound $u(k)$ by looking at the stage at which k is first resolved. At the stage just before, k was still unresolved, and for a fixed node only finitely many successors leave a given k unresolved. That finiteness is what gives a bound in the ground model.

To carry this out, work in W. For a node s' and $i' \in \mathrm{succ}_{p'(0)}(s')$ let $Q_{s',i'}$ be the set of positions that the successor i' can leave unresolved:

$$Q_{s',i'} = \big\{k < \omega :\ k \geq n_{s'}(i') \ \text{ or } \ x'(k) \geq n_{s'}(i') \text{ for some } x' \in U_{s'} \cap \omega^\omega\big\}.$$

Since $U_{s'}$ is finite and $n_{s'}(i') \to \infty$, each k belongs to $Q_{s',i'}$ for only finitely many i'. Hence

$$H_{s'}(k) = \max\big\{x(k) :\ i' \in \mathrm{succ}_{p'(0)}(s'),\ k \in Q_{s',i'},\ x \in U_{s'^\frown i'} \cap \omega^\omega\big\}$$

(with $\max\emptyset = 0$) defines a function $H_{s'} \in \omega^\omega \cap W$: it bounds the value that k can receive at the next node if k was unresolved at s'. There are countably many nodes, so in W we can choose $h^* \in \omega^\omega$ and numbers $N(s')$ such that $H_{s'}(k) \leq h^*(k)$ for all $k \geq N(s')$. The positions below $N(s')$ are not covered by this, and we deal with them by making sure they are resolved early. Let $H(s')$ be the least i_0 such that for all $i' \geq i_0$ in $\mathrm{succ}_{p'(0)}(s')$,

$$n_{s'}(i') > N(s') \quad \text{and} \quad n_{s'}(i') > x'(k) \ \text{ for all } k < N(s'),\ x' \in U_{s'} \cap \omega^\omega.$$

By Fact 4.3 fix $m_2 \geq m_1$ such that $\ell_0(m) \geq H(s_m)$ for all $m \geq m_2$.

Back in $W[G]$, let $m(k)$ be the least $m \geq m_2$ at which k is resolved. If $m(k) = m_2$ then $k < n_{m_2}$, so this happens for only finitely many k.

Let k be such that $m = m(k) > m_2$, and put $s' = s_{m-1}$ and $i' = \ell_0(m-1)$, so that $s_m = s'^\frown i'$ and $n_{m-1} = n_{s'}(i')$. By minimality, k is not resolved at $m-1$, that is, $k \geq n_{s'}(i')$ or $x^{m-1}(k) \geq n_{s'}(i')$. As $x^{m-1} \in U_{s'} \cap \omega^\omega$, this means $k \in Q_{s',i'}$. As $x^m \in U_{s'^\frown i'} \cap \omega^\omega$, we get

$$u(k) = x^m(k) \leq H_{s'}(k).$$

Moreover $k \geq N(s')$. Otherwise, since $m-1 \geq m_2$ gives $i' \geq H(s')$, we would have $k < N(s') < n_{s'}(i')$ and $x^{m-1}(k) < n_{s'}(i')$, so k would already be resolved at $m-1$. Hence $u(k) \leq H_{s'}(k) \leq h^*(k)$.

Thus $u(k) \leq h^*(k)$ for all but finitely many k, and (1) holds. □

Remark 5.1. It is worth isolating why the later coordinates of the iteration cause no trouble. In Case 1 we use only that *some* witness has an infinite entry, and in Case 2 the bound $H_{s'}$ is a maximum over *all* candidates. A condition below p' can restrict coordinate 0 only above its stem, which affects finitely many stages along the generic branch, and whatever it does at coordinates $\beta \geq 1$ cannot affect statements forced by the conditions $p'|t$. This is the feature that an induction on the length of the iteration lacks at limit stages of countable cofinality.

In the language of towers, the theorem says the following.

Corollary 5.2. *Let W, α and G be as in Theorem 1.2 and let $\mathcal{H} \subseteq \omega^\omega \cap W$ be $\leq^*$-cofinal in $\omega^\omega \cap W$. For every $u \in \omega^\omega \cap W[G]$ which is not $\leq^*$-below any member of $\mathcal{H}$, the family $\{\{k : u(k) \geq h(k)\} : h \in \mathcal{H}\}$ has an infinite pseudo-intersection in $W[G]$.*

6. Proof of the main theorem

Let V be a model of CH, let $\mathbb{P} = \mathbb{P}_{\omega_2}$ and let G be $\mathbb{P}$-generic over V. For $\alpha \leq \omega_2$ let $V_\alpha = V[G \cap \mathbb{P}_\alpha]$. We use the following standard facts; see [4], [2] and [3].

- (F1) $\mathbb{P}$ is proper and has the $\aleph_2$-chain condition; every real of V_{ω_2} belongs to V_β for some $\beta < \omega_2$, and $\mathfrak{c} = \aleph_2$ in V_{ω_2}.
- (F2) If $\alpha \leq \omega_2$ has uncountable cofinality, then $\omega^\omega \cap V_\alpha = \bigcup_{\gamma<\alpha} \omega^\omega \cap V_\gamma$.
- (F3) For $\alpha < \beta \leq \omega_2$, the model V_β is a generic extension of V_α by the countable support iteration of Laver forcing of length $\beta - \alpha$ defined in V_α, whose generic reals are $\ell_{\alpha+\xi}$, $\xi < \beta - \alpha$.
- (F4) For every $\gamma < \omega_2$, $f \leq^* \ell_\gamma$ for all $f \in \omega^\omega \cap V_\gamma$.

With these in hand the main theorem is a matter of putting the pieces together. The Laver reals form a scale. The reals bounded by the first α of them are, for α of cofinality ω_1, exactly the reals bounded by the intermediate model V_α. And the rest of the iteration is again a Laver iteration over V_α, so Theorem 1.2 applies with V_α as the ground model.

Proof of Theorem 1.1. Work in V_{ω_2}. By (F4), $\ell_\gamma <^* \ell_\delta$ for $\gamma < \delta < \omega_2$, since $\ell_\gamma + 1 \in V_\delta$. By (F1) and (F4), $\langle \ell_\gamma : \gamma < \omega_2 \rangle$ is dominating, and every subset of ω^ω of size $\aleph_1$ is contained in some V_γ and hence bounded. So $\langle \ell_\gamma : \gamma < \omega_2 \rangle$ is a scale and $\mathfrak{b} = \mathfrak{d} = \aleph_2$.

We verify the hypothesis of Theorem 1.3 for this scale. Let $\alpha < \omega_2$ with $\mathrm{cf}(\alpha) = \omega_1$ and let $u \in \omega^\omega$. By (F2) and (F4), $\{\ell_\gamma : \gamma < \alpha\}$ is a $\leq^*$-cofinal subset of $\omega^\omega \cap V_\alpha$. By (F1), $u \in V_\beta$ for some $\beta < \omega_2$, and we may assume $\beta > \alpha$. By (F3) and Theorem 1.2, applied with $W = V_\alpha$, one of two things happens. Either $u \leq^* h^*$ for some $h^* \in \omega^\omega \cap V_\alpha$, in which case $u \leq^* \ell_\gamma$ for some $\gamma < \alpha$. Or there is an infinite A with $h \leq^* u{\restriction}A$ for all $h \in \omega^\omega \cap V_\alpha$, and in particular $\ell_\gamma \leq^* u{\restriction}A$ for all $\gamma < \alpha$.

By Theorem 1.3 there is an ω_2-Michael sequence and a Michael space. □

Remark 6.1. For the record, here is why none of the previously known sufficient conditions holds in the Laver model. There $\mathfrak{b} = \aleph_2$. Countable support iterations of Laver forcing have the Laver property and therefore add no Cohen reals, so the ground model meagre sets cover the reals and $\mathrm{cov}(\mathcal{M}) = \aleph_1 < \mathfrak{d}$; in particular Martin's Axiom fails. By Miller's theorem [8] there are no Q-points, hence no selective ultrafilters.

7. Remarks and questions

Theorem 1.1 does not settle Michael's problem, but it narrows the search for a counterexample. By Proposition 2.3 and Moore's results [10], when $\mathfrak{c} < \aleph_\omega$ a Michael space exists if and only if there is a κ-Michael sequence for some regular κ with $\mathfrak{b} \leq \kappa \leq \mathfrak{d}$. If $\mathfrak{c} = \aleph_2$, a model with no Michael space must therefore satisfy $\mathrm{cov}(\mathcal{M}) = \aleph_1 < \mathfrak{b} = \mathfrak{d} = \aleph_2$ and have no selective ultrafilters. Among the standard models of $\mathfrak{c} = \aleph_2$, the Laver and Mathias models are the ones satisfying these requirements that we are aware of. Theorem 1.1 disposes of the first.

Question 7.1. Is there a Michael space in the Mathias model?

There is some reason for optimism. For a selective ultrafilter $\mathcal{U}$, Mathias forcing relative to $\mathcal{U}$ is forcing equivalent to Laver forcing relative to $\mathcal{U}$, which suggests that an analogue of Theorem 1.2 may hold for the Mathias iteration.

The second question is whether the forcing can be dispensed with.

Question 7.2. Does $\mathfrak{b} = \mathfrak{d}$ imply that there is a Michael space? More specifically, if $\mathfrak{b} = \mathfrak{d}$, is there a scale satisfying the hypothesis of Theorem 1.3?

Not every scale does. If $\mathrm{cov}(\mathcal{M}) = \aleph_1$, one can construct a $\leq^*$-increasing sequence $\langle g_i : i < \omega_1 \rangle$ and a perfect compact set $K \subseteq \bigcup_i \{f : f \leq^* g_i\}$ such that each $\{f \in K : f \leq^* g_i\}$ is meagre in K; when $\mathfrak{b} = \mathfrak{d} > \aleph_1$, any scale extending such a sequence fails the conclusion of Theorem 1.3 at $\alpha = \omega_1$. So a positive answer would require choosing the scale with care, as the Laver reals are chosen for us in the Laver model.

Question 7.3. Is the dichotomy of Theorem 1.2 preserved by countable support iterations of proper forcings in some generality? Precisely: for which classes of forcings does every new real either lie below a ground model real or dominate the ground model reals on an infinite set?

A positive answer for a class containing forcings that do not add dominating reals cannot be expected in this form, but a suitable variant might produce Michael sequences from other scales, or from unbounded families when $\mathfrak{b} < \mathfrak{d}$.

Acknowledgments. The statements and proofs in this paper were developed in a working session with Claude, an AI system made by Anthropic, which proposed the reduction to the Laver model, found the proofs of Theorems 1.2 and 1.3, and drafted the text.

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