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POSITIVE BASES AND POLYTOPES FOR LAURENT PHENOMENON ALGEBRAS

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Abstract. We prove that the cluster monomials of every graph Laurent phenomenon algebra form an atomic positive basis. For every finite normalized Laurent phenomenon algebra over a unique factorization domain, we construct a simple lattice polytope with unimodular normal fan whose polar realizes the cluster complex. We also prove that cluster monomials form a basis of the upper algebra in the finite case. In particular, every linear Laurent phenomenon algebra admits such a polyhedral realization.

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1. Introduction

Fomin and Zelevinsky introduced cluster algebras to study total positivity and canonical bases [12]. Their generators occur in overlapping algebraically independent sets, called clusters. Mutation replaces one generator through a binomial exchange relation, and every resulting generator is a Laurent polynomial in every cluster. Their separate treatment of the Laurent phenomenon [13] established a caterpillar argument for more general systems of rational recurrences. Two questions accompanying these constructions concern the signs of Laurent coefficients and the combinatorics of the collection of clusters. For cluster algebras, Fomin and Zelevinsky classified finite type by finite root systems [14]. Chapoton, Fomin and Zelevinsky realized the corresponding cluster complexes as the boundaries of simplicial polytopes dual to generalized associahedra [6].

Lam and Pylyavskyy introduced Laurent phenomenon algebras, or LP algebras, in their 2012 preprints, published as [17, 18]. Their seeds permit arbitrary irreducible exchange polynomials over a unique factorization domain (UFD). The mutation rule first associates to an exchange polynomial F_i a Laurent polynomial $\widehat{F}_i$ by dividing by a monomial in the other cluster variables. It then updates the exchange polynomials by substitution and removal of specified polynomial factors. Lam and Pylyavskyy proved the Laurent phenomenon for this rule and classified finite-type LP algebras of rank two [17]. Their linear and graph examples [18] led to questions about positive bases and polytopality in higher rank. The present paper addresses their Conjectures 7.3(2) and 7.6.

1.1. **Graph LP algebras and positive bases.** Let Γ be a directed graph on $[n] = \{1, \ldots, n\}$ without loops or multiple edges. Its graph LP algebra $\mathcal{A}_\Gamma$ has coefficient ring $R = \mathbb{Z}[A_1, \ldots, A_n]$, initial cluster $(X_1, \ldots, X_n)$, and exchange polynomials

$$F_i = A_i + \sum_{i \to j} X_j. \tag{1.1}$$

The parameters $A_1, \ldots, A_n$ are algebraically independent. Lam and Pylyavskyy described the cluster variables by strongly connected vertex sets and the clusters by nested collections [18, Theorem 1.1]. This description proves finite type and gives canonical representatives of the variables by sums over acyclic functions. Their Conjecture 7.3 asserts that cluster monomials form an R-basis and that every monomial in cluster variables has a nonnegative expansion in that basis. A cluster monomial uses variables from a single cluster; an arbitrary monomial in cluster variables may use several clusters.

The positivity of a basis can be formulated through Laurent expansions in all clusters. Sherman and Zelevinsky characterized canonical bases in rank-two cluster algebras of finite and affine types by this property [19]. Cerulli Irelli proved that cluster monomials form the atomic basis in coefficient-free skew-symmetric cluster algebras of finite type [5]. His proof combines a

cluster-monomial basis, Laurent positivity, and the proper Laurent monomial property: a monomial from another cluster contributes no coefficient to a distinct cluster monomial in its own chart. Reading such a coefficient determines the corresponding basis coefficient.

For graph LP algebras, Banaian, Chepuri, Kelley and Zhang obtained positive Laurent expansions in rooted clusters of tree graphs and gave a hyper-T-path expansion [2]. Dantas e Moura, Telekicherla Kandalam and Woodruff proved the cluster-monomial basis assertion for every graph, as well as the nonnegative multiplication assertion for bidirected trees [8]. Glew proved Laurent positivity for every graph variable in every graph cluster, with coefficients in $\mathbb{Z}_{\geq 0}[A_1, \ldots, A_n]$ [15]. Cao established Newton-polytope containment and global-monomial avoidance for positive spaces [4]. In particular, his avoidance result supplies the monomial separation used in the coefficient argument above. Together, the graph basis theorem, Glew's positivity theorem, and Cao's separation theorem imply graph atomicity. We give a direct proof of the required separation for LP mutations over a UFD, and deduce atomicity over the original graph coefficient ring.

Put $R_+ = \mathbb{N}[A_1, \ldots, A_n]$, where $\mathbb{N} = \mathbb{Z}_{\geq 0}$, and let $\mathcal{B}_\Gamma$ be the set of distinct cluster monomials in the canonical graph normalization. For a cluster C, write $\mathbf{x}_C$ for any ordering of its coordinates, and set

$$\mathcal{A}_{\Gamma,+} = \mathcal{A}_\Gamma \cap \bigcap_C R_+[\mathbf{x}_C^{\pm 1}].$$

A basis is *atomic positive* if its finite R_+-linear combinations are exactly these universally Laurent-positive elements.

Theorem A. *For every finite directed graph without loops or multiple edges,*

$$\mathcal{A}_{\Gamma,+} = \bigoplus_{b \in \mathcal{B}_\Gamma} R_+ b.$$

In particular, for $b_1, b_2 \in \mathcal{B}_\Gamma$ there is a unique finite expansion

$$b_1 b_2 = \sum_{b \in \mathcal{B}_\Gamma} c_b(A) b, \qquad c_b(A) \in \mathbb{N}[A_1, \ldots, A_n].$$

Every monomial in cluster variables has such a nonnegative expansion.

Theorem 8.2 proves this statement and hence Conjecture 7.3(2) of [18]. The coefficients are nonnegative polynomials in the parameters A_i.

1.2. **Finite LP algebras and polytopes.** Lam and Pylyavskyy called an LP algebra *linear* if it has a seed whose exchange polynomials have total degree at most one in the mutable variables. They conjectured that every linear LP algebra is of finite type [18, Conjecture 7.5]. Their Conjecture 7.6 asks for a simple convex polytope with vertices indexed by clusters and facets indexed by cluster variables. The required face correspondence says that the facets containing a face are indexed by the variables common to all its vertex clusters. For graph LP algebras, their nested-collection description identifies the cluster complex with an extended nested complex. Dao, Meng, Wellman,

Xu, Yost-Wolff and Yu realized every extended nested complex of a building set as a simplicial-polytope boundary [9]. The dual extended nestohedron therefore gives the requested realization for graph LP algebras.

For finite-type cluster algebras, polyhedral realizations also reflect the geometry of Laurent exponents. Hohlweg, Pilaud and Stella constructed polytopes whose normal fans are the finite-type g-vector fans [16]. Bazier-Matte, Chapelier-Laget, Douville, Mousavand, Thomas and Yıldırım related generalized associahedra to Newton polytopes of F-polynomials in simply laced finite type [3]. For acyclic seeds of finite Dynkin type, Arkani-Hamed, He and Lam identified the normal fan of the Minkowski sum of the F-polynomial Newton polytopes with the g-vector fan for the dual exchange matrix [1, Theorem 2.4]. These works connect finite-type cluster fans with Newton polytopes. We construct a regular unimodular fan and a polytope for every finite LP mutation pattern.

Daisey proved that every linear LP algebra over a UFD is of finite type [7]. His theorem includes linear seeds with coefficient relations and vanishing constant terms. The resulting finite normalizations allow the construction below to apply to the full linear class. Here an LP pattern uses the Lam–Pylyavskyy mutation rule, and it is *normalized* if it is mutation closed and contains exactly one seed in each coefficient-unit equivalence class. The cluster complex has the cluster variables as vertices and the subsets of clusters as faces. Section 2 gives the precise conventions for arbitrary patterns.

Theorem B. *Let $\mathcal{A}$ be an LP algebra of rank n over a UFD R, with a finite normalized mutation pattern. There is an n-dimensional simple lattice polytope whose polar has boundary complex isomorphic to the cluster complex of $\mathcal{A}$. The normal fan of the simple polytope is unimodular. Its vertices correspond to clusters, its facets correspond to cluster variables, and vertex–facet containment corresponds to membership of a variable in a cluster.*

Theorem 6.3 constructs the polytope and proves the full face correspondence. Combining it with Daisey's theorem yields Corollary 6.5: every linear LP algebra admits this realization in a finite normalized pattern. Thus Conjecture 7.6 of [18] holds with a lattice polytope.

1.3. **The construction.** Two algebraic results underlie the argument. Fei proved that the intersection of the Laurent rings of a seed and its immediate neighbors equals the upper LP algebra $\mathcal{U}$, the intersection of all cluster Laurent rings [11, Theorem 1.3]. Du and Li proved that a cluster determines its seed up to the coefficient-unit equivalence [10, Theorem 3.1]. Section 2 includes proofs of these results from the LP caterpillar lemmas, together with the primality of cluster variables in the upper algebra.

Fix an initial cluster and a real weight vector whose coordinates are linearly independent over $\mathbb{Q}$. In each successive LP chart, mutation transforms the weight by an integral matrix of determinant -1. Distinct Laurent monomials then have distinct weights. The unique term of greatest weight gives transport

identities for Newton support functions, valid over the coefficient UFD. These identities transport Newton-polytope containment between charts and separate distinct cluster monomials. This extends the support argument used for positive spaces in [4] to the ordinary LP exchange rule.

For a cluster variable v, let $g(v)$ be the exponent of its largest-weight Laurent term in the initial cluster. For finite patterns we prove that the cluster cones generated by these vectors form a complete unimodular fan. For each cluster C, let $P_C(v)$ be the Newton polytope of v in its mutable coordinates. Define the integer height

$$H(v) = \sum_C \sum_{j=1}^{n} \left(\max_{a \in P_C(v)} a_j - \min_{a \in P_C(v)} a_j \right). \tag{1.2}$$

The transport identities give weak convexity at every wall. A nonmonomial exchange contributes a strictly positive integer from its own chart. In a monomial exchange, the ordinary LP axioms force all shared-coordinate exponents to be nonpositive. A uniform rescaling of H, followed by addition of 1 to each height, makes every wall inequality strict. The resulting support function defines the polytope of Theorem B.

The complete fan also gives a finite reduction of every element of the upper LP algebra to cluster monomials. Each subtraction coefficient is read in a cluster containing the chosen monomial, and the Newton polytope of every remainder stays inside the original one. This proves that cluster monomials form an R-basis and that $\mathcal{A} = \mathcal{U}$ for every finite normalized LP pattern. In the graph case, coefficient extraction and Glew's theorem determine the positive cone and give Theorem A.

Section 2 gives the LP and upper-algebra preliminaries. Section 3 proves Newton-support transport and monomial separation. Sections 4–6 construct the fan and polytope. Sections 7 and 8 prove the basis and atomicity results.

2. LP mutation and algebraic preliminaries

Throughout, R is a commutative unique factorization domain, K is its fraction field, and $\mathcal{F}$ is a rational function field of transcendence degree $n \geq 1$ over K. Two nonzero elements of $\mathcal{F}$ are *associated over* R if their quotient belongs to $R^\times$. All LP algebras below use the ordinary mutation of Lam and Pylyavskyy [17].

2.1. Seeds, mutation, and normalization.

A *seed* $T = (\mathbf{q}, \mathbf{F})$ consists of a cluster $\mathbf{q} = (q_1, \ldots, q_n)$ of algebraically independent elements with $\mathcal{F} = K(q_1, \ldots, q_n)$ and exchange polynomials $F_1, \ldots, F_n \in R[q_1, \ldots, q_n]$. Each F_i is nonzero, irreducible, and a nonunit; no coordinate q_j divides F_i, and F_i is independent of q_i. The indexing only records the correspondence between a variable and its exchange polynomial. In particular, an irreducible element of R is an allowed exchange polynomial.

For $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, write $\mathbf{q}^a = \prod_j q_j^{a_j}$, and let $[\mathbf{q}^a]G$ denote its coefficient in a Laurent polynomial G. Exponent and weight vectors

are columns, e_i is the ith standard basis vector, and $\langle a, u\rangle = \sum_j a_j u_j$ is the standard pairing. When coordinates are ordered, mutation retains the positions of the unchanged coordinates; reordering acts by the corresponding permutation matrix.

For a nonzero Laurent polynomial G in coordinates $\mathbf{q}$, put

$$\mathrm{pol}_{\mathbf{q}}(G) = G \prod_{j=1}^{n} q_j^{-m_j}, \qquad m_j = \min\{a_j : [\mathbf{q}^a]G \neq 0\}. \tag{2.1}$$

Thus $\mathrm{pol}_{\mathbf{q}}(G)$ is polynomial and has no coordinate factor. This operation retains the coefficient content.

The *exchange Laurent polynomial* at q_i is

$$\widehat{F}_i = F_i \prod_{j\neq i} q_j^{-e_{ij}}, \qquad e_{ij} \in \mathbb{Z}_{\geq 0}. \tag{2.2}$$

For each $j \neq i$ and a new indeterminate ξ, its defining requirement is that

$$\widehat{F}_i\Big|_{q_j = F_j/\xi} \in R[q_1^{\pm 1}, \ldots, q_{j-1}^{\pm 1}, \xi^{\pm 1}, q_{j+1}^{\pm 1}, \ldots, q_n^{\pm 1}]$$

and that this element is not divisible by F_j in the displayed ring. Equivalently, e_{ij} is the F_j-adic order of $F_i|_{q_j = F_j/\xi}$ in that Laurent ring. These conditions determine the exponents uniquely [17, Lemmas 2.3–2.4]. If $e_{ij} > 0$, then F_j is independent of q_i. Indeed, writing $F_i = \sum_{r\geq 0} b_r q_j^r$ gives $b_0 \neq 0$, and divisibility after substitution implies $F_j \mid b_0$. This divisibility holds already in the polynomial ring, since no coordinate divides F_j. As b_0 is independent of q_i, additivity of polynomial degree proves the assertion.

Mutation at k, denoted by μ_k, replaces q_k by q_k' according to

$$q_k q_k' = \widehat{F}_k \tag{2.3}$$

and retains the other coordinates. The new exchange polynomial at q_k' is exactly F_k. For $j \neq k$, if F_j is independent of q_k, its new exchange polynomial is an R-unit multiple of F_j. Otherwise set

$$Z = \widehat{F}_k\Big|_{q_j = 0}, \qquad G_j = F_j|_{q_k = Z/q_k'}.$$

The preceding observation ensures that this specialization of $\widehat{F}_k$ is defined; the absence of a q_j factor ensures $Z \neq 0$. In the Laurent UFD

$$R[q_\ell^{\pm 1} : \ell \neq j, k][(q_k')^{\pm 1}],$$

remove from G_j every prime factor that divides Z, with its full multiplicity in G_j. Apply pol in the new coordinates to the result. The resulting polynomial, up to a unit of R, is the new exchange polynomial at q_j. This rule produces a seed and is reversible with the permitted choices of units [17, Section 2.2].

Two seeds are *equivalent* if, after a permutation, their corresponding variables and exchange polynomials are associated over R as elements of $\mathcal{F}$. Mutation respects this equivalence [17, Lemma 3.1]. An LP mutation pattern is a connected collection of seeds closed under mutation with specified unit choices. Its LP algebra $\mathcal{A}$ is the R-subalgebra of $\mathcal{F}$ generated by all cluster

variables. A pattern is *normalized* if it contains exactly one seed in each of its seed-equivalence classes. A *normalization* is a mutation-compatible choice of equivalent representative seeds. The LP algebra is of *finite type* if it admits a normalization with finitely many seeds. A cluster monomial is a product of variables from one cluster with nonnegative integer exponents.

Write

$$\mathcal{L}(T) = R[q_1^{\pm 1}, \dots, q_n^{\pm 1}], \qquad \mathcal{U} = \bigcap_T \mathcal{L}(T),$$

where the intersection runs over the mutation pattern. This intersection is the *upper LP algebra*. The Laurent phenomenon [17, Theorem 5.1] says that every cluster variable belongs to every $\mathcal{L}(T)$; hence $\mathcal{A} \subseteq \mathcal{U}$. Passing to an equivalent seed does not change its Laurent ring.

2.2. **A finite intersection and prime cluster variables.** The following local result is the caterpillar lemma. Unchanged coordinates are omitted from its notation.

Lemma 2.1 (Lam–Pylyavskyy, Lemmas 5.2–5.4 [17])**.** *For a three-step mutation path in two distinct directions,*

$$T_0 = (x, y) \longleftrightarrow T_1 = (z, y) \longleftrightarrow T_2 = (z, u) \longleftrightarrow T_3 = (v, u),$$

one has $u, v \in \mathcal{L}(T_0)$ *and*

$$\gcd_{\mathcal{L}(T_0)}(z, u) = \gcd_{\mathcal{L}(T_0)}(z, v) = 1.$$

Here a greatest common divisor equal to 1 *means that all common divisors are units. The assertion includes associated exchange polynomials and irreducible constant exchange polynomials.*

Proposition 2.2 (Fei [11, Theorem 1.3])**.** *For every seed* T*, one has*

$$\mathcal{U} = \mathrm{UB}(T) := \mathcal{L}(T) \cap \bigcap_{i=1}^{n} \mathcal{L}(\mu_i T). \tag{2.4}$$

Proof. In the notation of Lemma 2.1, set $L_a = \mathcal{L}(T_a)$. The element z is an irreducible exchange polynomial times a Laurent unit in the UFD L_0, and therefore is prime in L_0. If $h \in L_1 \cap L_3$, Laurentness of u, v gives expressions

$$h = \frac{a}{z^r} = \frac{b}{u^s v^t}, \qquad a, b \in L_0, \quad r, s, t \in \mathbb{Z}_{\geq 0}.$$

Clearing denominators and using the two coprimalities gives $z^r \mid a$, whence $h \in L_0$. Reversing the path gives the second of the inclusions

$$L_1 \cap L_3 \subseteq L_0, \qquad L_0 \cap L_2 \subseteq L_3. \tag{2.5}$$

Let $T' = \mu_k T$. For $j \neq k$, apply (2.5) to the path

$$\mu_j T \longleftrightarrow T \longleftrightarrow T' \longleftrightarrow \mu_j T'.$$

It follows that

$$\mathcal{L}(\mu_j T) \cap \mathcal{L}(T') \subseteq \mathcal{L}(\mu_j T'), \qquad \mathcal{L}(T) \cap \mathcal{L}(\mu_j T') \subseteq \mathcal{L}(\mu_j T).$$

Together with the two rings $\mathcal{L}(T)$ and $\mathcal{L}(T')$ common to the upper bounds, these inclusions give $\mathrm{UB}(T) = \mathrm{UB}(T')$. For $n = 1$ the same equality follows directly from reversibility. Connectedness now shows that the common upper bound lies in every seed Laurent ring, proving (2.4). □

The next consequence is also recorded in [7, Section 3].

Lemma 2.3. *Every cluster variable is a prime nonunit in $\mathcal{U}$, and $\mathcal{U}^\times = R^\times$. More precisely, for any cluster $\mathbf{q} = (q_1, \dots, q_n)$ there are discrete valuations $\nu_i : \mathcal{F}^\times \to \mathbb{Z}$ such that, for all i, j and every $0 \neq h \in \mathcal{U}$,*

$$\begin{aligned} \nu_i(q_j) = \delta_{ij}, \qquad & \nu_i(h) \geq 0, \\ \nu_i(h) > 0 \iff\ & h \in q_i\mathcal{U}. \end{aligned} \tag{2.6}$$

Proof. Fix a seed T with cluster $\mathbf{q}$ and put $L_i = \mathcal{L}(\mu_i T)$. Since F_i is independent of q_i and has no coordinate factor, it is a prime nonunit of $R[q_j^{\pm 1} : j \neq i]$, and remains so in $L_i = R[q_j^{\pm 1} : j \neq i][(q_i')^{\pm 1}]$. The exchange relation makes q_i a Laurent unit times F_i in L_i. We claim that

$$q_i\mathcal{U} = \mathcal{U} \cap F_i L_i. \tag{2.7}$$

One inclusion is immediate. If h belongs to the right hand side, then $h/q_i \in L_i$. It also belongs to $\mathcal{L}(T)$ and every $\mathcal{L}(\mu_j T)$ with $j \neq i$, because q_i is an unchanged invertible coordinate in those rings. Proposition 2.2 therefore gives $h/q_i \in \mathcal{U}$. This proves the claim. The right hand side of (2.7) is the contraction of a proper prime ideal, proving primality and noninvertibility of q_i in $\mathcal{U}$.

Take ν_i to be the F_i-adic valuation of the fraction field of the UFD L_i. Equation (2.7) and the exchange relation give all assertions in (2.6). Finally, a unit h of $\mathcal{U}$ is a unit of $\mathcal{L}(T)$, so

$$h = \varepsilon \prod_{j=1}^{n} q_j^{a_j}, \qquad \varepsilon \in R^\times, \quad a_j \in \mathbb{Z}.$$

Both h and h^{-1} belong to $\mathcal{U}$; applying ν_i gives $a_i \geq 0$ and $-a_i \geq 0$. Thus every a_i vanishes, and $h \in R^\times$. □

In particular, products of cluster variables have unique factorizations into their R-association classes: if two such products differ by a unit of R, then the multiplicity of each class is the same on both sides. Indeed, a prime dividing a product divides one factor; two prime nonunits related by divisibility are associated. Successive cancellation identifies every multiplicity.

Lemma 2.4 (Du–Li [10, Theorem 3.1])**.** *Two seeds in the same LP mutation pattern whose clusters agree up to coefficient units are equivalent. Consequently, a set of cluster-variable R-association classes determines at most one seed-equivalence class.*

Proof. Match the coordinates and express both seeds in common coordinates $\mathbf{q}$, allowing coefficient-unit rescaling. Fix i and write their two exchange relations in these coordinates as

$$q_i u_i = E_i, \qquad q_i u_i' = E_i', \qquad E_i, E_i' \in B_i := R[q_j^{\pm 1} : j \neq i].$$

Here u_i and u_i' are the respective mutation partners, and $\mathrm{pol}_{\mathbf{q}}(E_i)$ and $\mathrm{pol}_{\mathbf{q}}(E_i')$ are coefficient-unit multiples of the original exchange polynomials in these coordinates. The Laurent phenomenon puts $u_i = (E_i/E_i')u_i'$ in $B_i[(u_i')^{\pm 1}]$. Since u_i' is transcendental over the fraction field of B_i, uniqueness of Laurent coefficients gives $E_i/E_i' \in B_i$. Interchanging the seeds gives its reciprocal in B_i. Thus E_i/E_i' is a Laurent unit, namely a coefficient unit times a monomial in the shared coordinates. Applying $\mathrm{pol}_{\mathbf{q}}$ shows that the two exchange polynomials are associated over R. This holds for every i, and proves seed equivalence. □

2.3. **Representative coordinates and the finite-type input.** Choose a set $\mathscr{V}$ consisting of exactly one representative of each cluster-variable R-association class. In every cluster, replace its variables by these representatives, and write $\mathscr{C}$ for the resulting family of sets C. Such a set is still a transcendence basis, and Lemma 2.4 identifies $\mathscr{C}$ with the seed-equivalence classes of the pattern. We write $\mathbf{x}_C = (x_{1;C}, \dots, x_{n;C})$ for an ordering of C. The *cluster complex* is the abstract simplicial complex on $\mathscr{V}$ whose faces are the subsets contained in a cluster. The *exchange graph* has seed-equivalence classes as vertices and mutations as edges.

These replacements are changes of coordinates by units of R. They preserve all Laurent rings, $\mathcal{A}$, and $\mathcal{U}$. They also preserve every Laurent support: in the rescaled coordinates $\varepsilon_j q_j$, with $\varepsilon_j \in R^\times$, the coefficient at exponent a is multiplied by the unit $\prod_j \varepsilon_j^{-a_j}$. Multiplying the expanded element itself by an R-unit has the same property. An exchange relation in these coordinates still has the form

$$v v' = \varepsilon F \prod_{s \in C \setminus \{v\}} s^{-e_s}, \qquad \varepsilon \in R^\times, \quad e_s \geq 0, \tag{2.8}$$

where F is irreducible, a nonunit, and has no coordinate factor in the shared variables. In the following sections, cluster monomials are formed from $\mathscr{V}$. Proposition 5.2 will identify the actual variables of a finite normalized pattern with one such set of representatives.

We finally record the finite-type theorem used to pass from arbitrary finite patterns to linear initial seeds.

Theorem 2.5 (Daisey, Theorem 1.2 [7])**.** *Let an ordinary LP algebra over a UFD R have a seed with exchange polynomials*

$$F_i = a_{i0} + \sum_{j \neq i} a_{ij} q_j, \qquad a_{ij} \in R.$$

Then it has finitely many cluster-variable R-association classes and admits a finite normalization.

Here linear means total degree at most one in the mutable variables. Thus Theorem 2.5 permits irreducible constant exchanges, vanishing constant terms, relations among the coefficients, and arbitrary characteristic, subject to the seed axioms. The polyhedral construction below applies to every finite normalized ordinary LP pattern.

3. Newton polytopes and monomial separation

We use the representative charts $C \in \mathscr{C}$ fixed in Section 2.

3.1. **Support functions and weights.** For a chart C and a nonzero Laurent polynomial

$$f = \sum_{a \in \mathbb{Z}^n} c_a \mathbf{x}_C^a, \qquad c_a \in R,$$

define

$$\operatorname{supp}_C(f) = \{a : c_a \neq 0\}, \qquad P_C(f) = \operatorname{conv} \operatorname{supp}_C(f),$$
$$h_{C,f}(u) = \max_{a \in P_C(f)} \langle a, u \rangle \quad (u \in \mathbb{R}^n).$$

Here conv denotes the convex hull. For a nonzero coefficient $c \in R$ one has $P_C(cf) = P_C(f)$. We use the same support notation for Laurent polynomials with coefficients in K. Also, for nonzero Laurent polynomials f_1, f_2,

$$P_C(f_1 f_2) = P_C(f_1) + P_C(f_2). \tag{3.1}$$

Here $P + Q = \{p + q : p \in P,\ q \in Q\}$ is the Minkowski sum. To check this identity over any domain, take a weight exposing a single vertex of each factor. The two highest terms have nonzero product and cannot cancel. Their weights add. Such weights are dense, so equality of support functions holds at every weight, proving (3.1).

A vector $u \in \mathbb{R}^n$ will be called *rationally independent* if $\langle a, u \rangle = 0$ with $a \in \mathbb{Z}^n$ implies $a = 0$. These vectors are dense. Indeed, in a product of nonempty open intervals, choose the first coordinate nonzero and each subsequent coordinate outside the rational span of its predecessors. Each forbidden span is countable, so every choice is possible. For such u, every nonzero Laurent polynomial in $\mathbf{x}_C$ has a unique term of largest u-weight. For $p, q \in K[\mathbf{x}_C^{\pm 1}] \setminus \{0\}$ put

$$\nu_{C,u}(p/q) = h_{C,p}(u) - h_{C,q}(u). \tag{3.2}$$

If $p/q = r/s$, then $ps = rq$, and (3.1) gives $h_{C,p}(u) + h_{C,s}(u) = h_{C,r}(u) + h_{C,q}(u)$. Thus (3.2) is independent of the fraction. The product formula and passage to a common denominator give

$$\nu_{C,u}(fg) = \nu_{C,u}(f) + \nu_{C,u}(g), \qquad \nu_{C,u}(f+g) \leq \max\{\nu_{C,u}(f), \nu_{C,u}(g)\},$$

with $\nu_{C,u}(0) = -\infty$; equality holds in the second formula when the two values differ. Every nonzero element of K has value zero. We use this maximum convention throughout.

3.2. **Transport through a mutation.** Consider adjacent charts C, C' with shared coordinates $s_1, \dots, s_{n-1}$ and exchanged coordinates x, x', satisfying

$$xx' = E(s), \qquad 0 \neq E \in R[s_1^{\pm 1}, \dots, s_{n-1}^{\pm 1}]. \tag{3.3}$$

We write $\operatorname{supp} E$ for its support in the shared coordinates, and insert a zero in the exchanged position when regarding its exponents as vectors in $\mathbb{Z}^n$. In ordered coordinates where x has index i, define the map $\tau_i : \mathbb{R}^n \to \mathbb{R}^n$ by

$$(\tau_i(u))_j = u_j \quad (j \neq i), \qquad (\tau_i(u))_i = -u_i + h_{C,E}(u). \tag{3.4}$$

The polynomial E does not involve coordinate i, so this map is an involution. In particular, it is a continuous piecewise integral linear homeomorphism.

Suppose that u is rationally independent, and let $\alpha \in \mathbb{Z}^n$, with $\alpha_i = 0$, be the exponent of the unique highest-weight term of E. Then $\tau_i(u) = B_\alpha u$, where B_α is the identity matrix with row i replaced by $(\alpha - e_i)^\mathsf{T}$. Its determinant is -1, so $B_\alpha \in \mathrm{GL}_n(\mathbb{Z})$. If $\langle m, B_\alpha u\rangle = 0$ for $m \in \mathbb{Z}^n$, then $\langle B_\alpha^\mathsf{T} m, u\rangle = 0$; rational independence forces $B_\alpha^\mathsf{T} m = 0$ and hence $m = 0$. Thus $\tau_i(u)$ is again rationally independent. It is the vector of $\nu_{C,u}$-values of the coordinates of C'.

If $f \neq 0$ is Laurent in C', its distinct Laurent terms have distinct $\nu_{C,u}$-values. Its unique largest value cannot cancel in their sum. Consequently, whenever f is Laurent in both charts,

$$h_{C,f}(u) = h_{C',f}(\tau_i(u)) \tag{3.5}$$

for rationally independent u. Both sides are continuous, so the same identity holds for every real u.

Proposition 3.1 (Newton transport)**.** *For any two cluster charts C, D, a mutation path gives a piecewise integral linear homeomorphism $\tau_{C,D} : \mathbb{R}^n \to \mathbb{R}^n$ such that, for every $f \in \mathcal{U} \setminus \{0\}$,*

$$h_{C,f}(u) = h_{D,f}(\tau_{C,D}(u)) \qquad (u \in \mathbb{R}^n). \tag{3.6}$$

This map is independent of the chosen path between the ordered charts. For $f, g \in \mathcal{U} \setminus \{0\}$,

$$P_C(f) \subseteq P_C(g) \iff P_D(f) \subseteq P_D(g). \tag{3.7}$$

Proof. Compose (3.5) along the path. Every element of $\mathcal{U}$ is Laurent in all its charts, and every elementary map is a homeomorphism preserving rational independence. This proves (3.6). For a coordinate $x_{j;D}$ of D, the right side of that identity is the jth coordinate of $\tau_{C,D}(u)$. The left side depends only on the rational function $x_{j;D}$ and the initial chart C. These identities for all j determine the map, proving path independence.

Containment of compact convex polytopes is equivalent to inequality of their support functions at every real weight. Use (3.6) for both f and g and the surjectivity of $\tau_{C,D}$ to transport this inequality in either direction. This gives (3.7). □

Corollary 3.2 (Exchange support identity)**.** *For the exchange* (3.3)*, any chart D and any $u \in \mathbb{R}^n$ satisfy*

$$h_{D,x}(u) + h_{D,x'}(u) = \max_{\beta \in \operatorname{supp} E} \sum_{j=1}^{n-1} \beta_j h_{D,s_j}(u). \tag{3.8}$$

Proof. The rational function $xx' = E(s)$ belongs to $\mathcal{U}$. Transport its support function from D to the chart $C = \{x, s_1, \dots, s_{n-1}\}$. The coordinates of the transported weight are the support values of these variables in D, by the preceding proof. In C, the support function of E is the maximum on the right of (3.8). On the other hand, (3.1) gives the sum on the left. □

3.3. **Separation of monomials.**

Lemma 3.3 (Monomial avoidance)**.** *Let b, b' be cluster monomials in the fixed representatives. If $P_C(b) \subseteq P_C(b')$ for one chart C, then $b = b'$. In particular, if $b = \mathbf{x}_D^a$ and $b' \neq b$, then*

$$a \notin P_D(b').$$

Proof. Choose a chart D' in which b' is a monomial. By Proposition 3.1, $P_{D'}(b) \subseteq P_{D'}(b')$. The right side is a singleton, so $b = cb'$ for a nonzero $c \in R$. Choose a chart in which b is a monomial with coefficient one. The Laurent expansion of $b' = c^{-1}b$ in that chart has coefficient $c^{-1} \in R$. Thus c is a unit of R.

By Lemma 2.3, all factors of b and b' are prime nonunits in $\mathcal{U}$, whose units are $R^\times$. If a prime factor of b divides cb', it therefore divides one factor of b', and the two factors are associated. They are equal because both belong to $\mathscr{V}$. Successive cancellation identifies all factors with their multiplicities; a product of remaining prime nonunits cannot be a unit. Thus $b = b'$. Finally, $a \in P_D(b')$ would give $P_D(b) = \{a\} \subseteq P_D(b')$ and hence $b = b'$, proving the avoidance assertion. □

Cao's global monomial avoidance theorem for positive spaces [4, Theorem 2.5 and Corollary 2.7] uses the same passage from Newton containment to a monomial chart. Proposition 3.1 supplies the containment statement needed here over R directly.

3.4. **Leading exponents.** Fix a chart C_0 and a rationally independent weight $w \in \mathbb{R}^n$. For $f \in \mathcal{U} \setminus \{0\}$, let $g(f) \in \mathbb{Z}^n$ be the exponent of its unique largest-weight term in C_0. Thus

$$\nu_{C_0,w}(f) = \langle g(f), w \rangle, \qquad g(ff') = g(f) + g(f') \tag{3.9}$$

for nonzero $f, f' \in \mathcal{U}$. The notation g refers throughout to this fixed chart and weight.

Lemma 3.4. *Distinct cluster monomials in fixed representatives of the variables have distinct vectors $g(b)$.*

Proof. Suppose $g(b) = g(b')$ and choose a chart D containing the factors of b', so that $b' = \mathbf{x}_D^a$. The two functions have equal $\nu_{C_0,w}$-values. The transported weight $\tau_{C_0,D}(w)$ is rationally independent. If a' is the leading exponent of b in D, equality of the values gives

$$\langle a' - a, \tau_{C_0,D}(w)\rangle = 0,$$

and therefore $a' = a$. In particular $a \in P_D(b)$. Since $P_D(b') = \{a\} \subseteq P_D(b)$, Lemma 3.3 gives $b = b'$. □

4. The complete leading exponent fan

Assume that the ordinary LP pattern has finitely many seed equivalence classes. We retain the representatives $\mathscr{V}$, the charts $\mathscr{C}$, and the initial chart C_0 and weight w fixed in Section 3. An exchange between representative charts has the form

$$\begin{gathered} xx' = E(s_1, \dots, s_{n-1}), \\ C = \{x, s_1, \dots, s_{n-1}\}, \quad C' = \{x', s_1, \dots, s_{n-1}\}, \end{gathered} \tag{4.1}$$

where E has the form given in (2.8).

For a finite set $S \subset \mathbb{R}^n$, write $\operatorname{cone} S = \{\sum_{s\in S} a_s s : a_s \geq 0\}$. A polyhedral cone is *pointed* if it contains no nonzero linear subspace. Its faces are its intersections with supporting hyperplanes, together with the cone itself. A *fan* is a collection of pointed polyhedral cones closed under faces, with every intersection a face of both cones. Its *support* is the union of its cones, and it is *complete* if this union is $\mathbb{R}^n$. A cone is *simplicial* if its ray generators are linearly independent. A cone generated by integer vectors is *unimodular* if its primitive integral ray generators extend to a lattice basis; primitive means that the coordinates have greatest common divisor one. These adjectives apply to a fan when they apply to each cone. A ray has dimension one, and a facet has codimension one in its cone.

Recall that $g(v)$ is the exponent of the unique w-highest term of v in the C_0-chart. Let G_C have columns $g(v)$ in the chosen ordering of C, and put

$$G_C = (g(v))_{v\in C}, \qquad \sigma_C = \operatorname{cone}\{g(v) : v \in C\}.$$

Lemma 4.1. *For every $C \in \mathscr{C}$, the matrix G_C belongs to $\mathrm{GL}_n(\mathbb{Z})$. For the exchange (4.1), there is an exponent $\alpha \in \operatorname{supp} E \subset \mathbb{Z}^{n-1}$ such that*

$$g(x) + g(x') = \sum_{j=1}^{n-1} \alpha_j g(s_j). \tag{4.2}$$

It is the unique exponent maximizing $\sum_j \alpha_j \langle w, g(s_j)\rangle$ on $\operatorname{supp} E$.

Proof. In the initial chart $G_{C_0} = I$. Suppose that G_C is unimodular. The coordinates of $G_C^{\mathsf{T}} w$ are then linearly independent over $\mathbb{Q}$. Consequently, the Laurent polynomial E has a unique highest term for these coordinate weights. Apply the maximum valuation $\nu_{C_0,w}$ to (4.1). The distinct terms of E have distinct values, so its largest value cannot cancel, even if some

shared-coordinate exponents are negative. The resulting scalar equality, together with the rational independence of w, gives (4.2). Order C and C' so that shared variables retain their positions and the exchanged variables occupy the same position. Then $G_{C'}$ is obtained from G_C by replacing the column $g(x)$ by $-g(x)+\sum_j \alpha_j g(s_j)$. This operation has determinant -1. Induction along a mutation path proves the assertion. In particular, every $g(v)$ is a nonzero primitive integral vector. □

Lemma 4.2. *For all $C, D \in \mathscr{C}$,*

$$\sigma_C \cap \sigma_D = \operatorname{cone}\{g(v) : v \in C \cap D\}. \tag{4.3}$$

Proof. Let q be a rational point of the intersection. Since G_C and G_D are integral invertible matrices, q has nonnegative rational coordinates in both bases. Clearing denominators gives

$$\sum_{v\in C} a_v g(v) = \sum_{v\in D} b_v g(v), \qquad a_v, b_v \in \mathbb{Z}_{\geq 0}.$$

The two cluster monomials $\prod_{v\in C} v^{a_v}$ and $\prod_{v\in D} v^{b_v}$ have the same leading exponent, and hence are equal by Lemma 3.4. By Lemma 2.3, their factors are prime in $\mathcal{U}$, and two members of $\mathscr{V}$ are associated in $\mathcal{U}$ only when they are equal. Cancellation of prime factors therefore gives $a_v = 0$ for $v \in C \setminus D$ and $b_v = 0$ for $v \in D \setminus C$. Thus q belongs to the cone on the right of (4.3). To pass to real points, the intersection is defined by the rational inequalities $G_C^{-1}q \geq 0$ and $G_D^{-1}q \geq 0$. At a point of this intersection, the equalities among these inequalities define a subspace with a rational basis, by Gaussian elimination over $\mathbb{Q}$. Approximate the point by rational linear combinations of that basis. Sufficiently close approximations preserve the remaining, strict inequalities, and therefore stay in the intersection. Its rational points are thus dense. The right-hand cone of (4.3) is closed, so the proved inclusion holds for every real point. The other inclusion is immediate. □

Theorem 4.3. *The cones σ_C, for $C \in \mathscr{C}$, together with all their faces form a finite complete simplicial unimodular fan Σ_w in $\mathbb{R}^n$. Its rays are in bijection with $\mathscr{V}$, its maximal cones are in bijection with $\mathscr{C}$, and $G_C \in \mathrm{GL}_n(\mathbb{Z})$ for every C. Two maximal cones share a facet if and only if their clusters are related by one mutation.*

Proof. Unimodularity follows from Lemma 4.1. The intersection in (4.3) is a face of each simplicial cone, so the cones and their faces form a fan. If two of its rays coincide, their primitive integral generators coincide. Lemma 3.4, applied to single variables, identifies their labels. The rays of a maximal cone therefore recover its cluster, which proves both stated bijections.

Consider a facet of σ_C, obtained by omitting $g(x)$. Mutation at x supplies the cone $\sigma_{C'}$ in (4.1). The variables x and x' cannot be coefficient-unit associates: otherwise the exchange equation would express x^2 as an element of $K(s_1, \ldots, s_{n-1})$, contradicting the algebraic independence of the coordinates of C. Therefore (4.3) identifies the common face with precisely the given

facet. Relation (4.2) puts $g(x)$ and $g(x')$ on opposite sides of its linear span. A facet of a geometric fan has at most one full-dimensional cone on either side: two such cones on the same side would have intersecting interiors near a relative interior point of the facet. Every facet thus has exactly two incident maximal cones. This also proves the assertion about mutation adjacency.

It remains to prove completeness. The support of Σ_w is closed, because the fan is finite. If its complement were nonempty, choose a segment from the interior of one maximal cone to this open complement. For $n \geq 2$, fix its initial point a outside the finitely many facet hyperplanes. For a face F of codimension at least two, endpoints whose segment from a meets F lie in $a + \operatorname{span}(F - a)$, which has dimension at most $n - 1$. Avoiding these finitely many sets gives a segment meeting only interiors of maximal cones and relative interiors of facets. Its facet crossings are transverse. At its first exit from the support it therefore meets the relative interior of a facet. The second cone across that facet contains the continuation of the segment for a positive distance, a contradiction. When $n = 1$, mutation already supplies both rays of $\mathbb{R}$. This proves completeness. □

5. Representatives in a normalized pattern

The complete fan supplies connectivity among seeds containing the same variable class. This identifies their actual representatives in a finite normalized pattern.

Lemma 5.1. *Let Σ be a finite complete simplicial fan in a real vector space V, and let ρ be one of its rays. The images of the cones containing ρ under $\pi : V \longrightarrow V/\operatorname{span}(\rho)$ form a complete fan. The maximal cones of Σ containing ρ can be joined by a sequence of maximal cones in which successive intersections are facets containing ρ.*

Proof. Choose $0 \neq r \in \rho$. A simplicial cone containing ρ projects to a pointed simplicial cone; its faces are the projections of the faces containing ρ. For cones σ, τ containing ρ, suppose that $a \in \sigma$ and $b \in \tau$ have the same image. Then $a - b$ is a real multiple of r. Adding a nonnegative multiple of r to one of a, b gives a common lift in $\sigma \cap \tau$. Hence

$$\pi(\sigma) \cap \pi(\tau) = \pi(\sigma \cap \tau).$$

The face functionals for faces containing ρ vanish on r and descend to the quotient. These observations prove the fan property. Moreover, maximal cones with the same projection coincide: their intersection contains ρ and projects to a full-dimensional quotient cone, so it is a full-dimensional common face.

For completeness, let $\bar{u}$ be a quotient vector and choose a lift $u \in V$. Each point $r + \varepsilon u$, with $\varepsilon > 0$, belongs to a maximal cone of Σ. Along a sequence $\varepsilon \downarrow 0$, finiteness supplies one fixed cone σ containing a subsequence of these points. By closure $r \in \sigma$, so ρ is a face of σ. Projecting and rescaling shows that $\bar{u} \in \pi(\sigma)$.

Finally, the maximal cones of a finite complete fan are connected through facets. For dimension at least two, choose interior points in the two cones outside the facet hyperplanes, and join them through an intermediate point avoiding the finitely many sets of dimension at most $\dim V - 1$ described in the proof of Theorem 4.3. The two segments cross only facets, giving the required sequence. In dimension one the two maximal cones meet at the origin, and dimension zero is immediate. Apply this observation to the quotient fan. Its maximal cones have unique lifts. Their adjacent intersections lift to common faces containing ρ of dimension $\dim V - 1$, since projection lowers their dimension by one. Thus the lifted sequence has the required facet intersections. □

Proposition 5.2. *In a finite normalized ordinary LP pattern, any two cluster variables that differ by a unit of R are the same element of the ambient field. Consequently, its actual clusters and variables label the maximal cones and rays of Σ_w.*

Proof. By Lemma 2.4, a cluster of variable classes determines the seed equivalence class. Normalization therefore makes the original seeds correspond bijectively to the maximal cones of Σ_w. Fix a variable class and its ray ρ. By Lemma 5.1, any two maximal cones containing ρ can be joined by facet crossings that all contain ρ. By Theorem 4.3, each crossing is the mutation at the variable omitted from the shared facet. Seed rigidity and normalization identify the seed obtained by that actual mutation with the original seed labeling the next cone. Since the omitted variable does not label ρ, mutation leaves the coordinate labeling ρ unchanged as a rational function. This holds at every crossing, so all occurrences of the chosen variable class agree exactly. □

6. Newton widths and the polytope

For $v \in \mathscr{V}$, $T \in \mathscr{C}$, and $1 \le j \le n$, define

$$\begin{aligned} \operatorname{width}_{T,j}(v) &= \max_{a \in P_T(v)} a_j - \min_{a \in P_T(v)} a_j \\ &= h_{T,v}(e_j) + h_{T,v}(-e_j), \end{aligned} \tag{6.1}$$

$$H(v) = \sum_{T \in \mathscr{C}} \sum_{j=1}^{n} \operatorname{width}_{T,j}(v). \tag{6.2}$$

The sum is finite and $H(v) \in \mathbb{Z}_{\ge 0}$. For an ordering $C = (x_1, \dots, x_n)$, let x_i' be the exchanged coordinate in the adjacent chart and put $E_{C,i} = x_i x_i'$. Its exponent vectors in the C-coordinates have zero ith entry. Set

$$M = \max\left(0, \max_{\substack{C \in \mathscr{C},\ 1 \le i \le n \\ \beta \in \operatorname{supp}_C E_{C,i}}} \left(\sum_{j=1}^{n} \beta_j - 2 \right)\right), \tag{6.3}$$

$$\omega(v) = (M+1)H(v) + 1. \tag{6.4}$$

Thus $\omega(v)$ is a positive integer. The definitions of H and M depend only on Laurent supports and are independent of w.

For the wall relation (4.2) and any function a on $\mathscr{V}$, write

$$\Delta_\alpha a = a(x) + a(x') - \sum_{j=1}^{n-1} \alpha_j a(s_j).$$

Lemma 6.1. *For every exchange wall, $\Delta_\alpha \omega > 0$. More precisely,*

$$\Delta_\alpha \omega \geq \begin{cases} 1, & \text{if } E \text{ has at least two Laurent monomials,} \\ 2, & \text{if } E \text{ has one Laurent monomial.} \end{cases}$$

Proof. Fix a chart T and a weight $u \in \mathbb{R}^n$, and put $a(v) = h_{T,v}(u)$. Support-function transport (Proposition 3.1) applied to (4.1) gives

$$a(x) + a(x') = \max_{\beta \in \operatorname{supp} E} \sum_j \beta_j a(s_j) \geq \sum_j \alpha_j a(s_j). \tag{6.5}$$

In particular, each summand in (6.1) has nonnegative wall slack, and hence $\Delta_\alpha H \geq 0$.

In the exchange chart $C = \{x, s_1, \ldots, s_{n-1}\}$, every old coordinate has width zero. Since $x' = E/x$, its widths are those of E, with zero width in the x-coordinate. The slack contributed by this chart is therefore

$$\sum_{j=1}^{n-1} \left(\max_{\beta \in \operatorname{supp} E} \beta_j - \min_{\beta \in \operatorname{supp} E} \beta_j \right). \tag{6.6}$$

If E has at least two monomials, this is a positive integer. All other chart contributions are nonnegative by (6.5), so $\Delta_\alpha H \geq 1$. Since $\sum_j \alpha_j - 2 \leq M$, formula (6.4) yields

$$\Delta_\alpha \omega = (M+1)\Delta_\alpha H + 2 - \sum_j \alpha_j \geq (M+1) - M = 1.$$

Suppose instead that E is a single Laurent monomial. Its ordinary exchange polynomial is then a monomial. No mutable variable divides an ordinary exchange polynomial, so this polynomial is a constant irreducible nonunit of R. The Laurent-hat correction has only nonpositive exponents. Consequently,

$$E = c \prod_j s_j^{-d_j}, \qquad c \in R \setminus \{0\}, \quad d_j \in \mathbb{Z}_{\geq 0}, \quad \alpha_j = -d_j \leq 0.$$

Equality holds in (6.5) for every T, u, because the maximum has one term. Thus $\Delta_\alpha H = 0$ and

$$\Delta_\alpha \omega = 2 - \sum_j \alpha_j \geq 2.$$

□

A polytope is the convex hull of finitely many points. Its faces are the sets on which a linear functional attains its maximum, together with the empty face. Vertices have dimension zero, and facets have codimension one. An n-dimensional polytope is *simple* if exactly n facets meet at each vertex, and it is a *lattice polytope* if its vertices belong to the ambient lattice. For a nonempty face F of a full-dimensional polytope $P \subset (\mathbb{R}^n)^*$, its outer normal cone is

$$N_P(F) = \{z \in \mathbb{R}^n : \langle p, z\rangle = \max_{q\in P}\langle q, z\rangle \text{ for every } p \in F\}.$$

These cones form the *normal fan* of P; a fan realized in this way is called *regular*. For $0 \in \operatorname{int} P$, the polar is

$$P^\circ = \{z \in \mathbb{R}^n : \langle p, z\rangle \le 1 \text{ for every } p \in P\}.$$

For each cluster C, let $p_C \in (\mathbb{R}^n)^*$ be the unique solution of

$$G_C^{\mathsf{T}} p_C = \omega_C, \qquad \omega_C = (\omega(v))_{v\in C}. \tag{6.7}$$

The conventions identify dual vectors with column coordinate vectors in this equation. By unimodularity, $p_C \in (\mathbb{Z}^n)^*$. On σ_C, define

$$\psi(z) = \langle p_C, z\rangle. \tag{6.8}$$

The exact intersection formula makes this a well-defined continuous, positively homogeneous piecewise-linear function on $\mathbb{R}^n$.

Lemma 6.2. *The function ψ is convex, and its full-dimensional maximal cones of linearity are precisely the σ_C. More precisely,*

$$\psi(z) \ge \langle p_C, z\rangle \quad \textit{for all } z, \qquad \{z : \psi(z) = \langle p_C, z\rangle\} = \sigma_C. \tag{6.9}$$

Proof. Across the wall (4.2), the two linear pieces agree on the shared facet, while

$$\begin{aligned}\langle p_{C'}, g(x')\rangle - \langle p_C, g(x')\rangle &= \omega(x') - \left(\sum_j \alpha_j \omega(s_j) - \omega(x)\right)\\ &= \Delta_\alpha \omega > 0.\end{aligned}$$

Thus the new linear piece is strictly larger on the side of the wall containing $\sigma_{C'}$, and strictly smaller on the other side. Along any line meeting only relative interiors of maximal cones and facets, the slope of ψ increases at each facet crossing. Its restriction to that line is convex. The exceptional endpoints lie in the finite union of sets of dimension at most $n-1$ used in Theorem 4.3. Thus such segments approximate every segment, and continuity proves convexity on all of $\mathbb{R}^n$.

Choose z_0 in the interior of σ_C. The function is linear in a neighborhood of z_0 with slope p_C. Convexity on the segment from z_0 to any z gives $\psi(z) \ge \psi(z_0) + \langle p_C, z - z_0\rangle = \langle p_C, z\rangle$. The equality set is a convex cone containing σ_C. It cannot contain a point $y \notin \sigma_C$. Indeed, choose $z_0 \in \operatorname{int}\sigma_C$ so that the segment from z_0 to y first exits σ_C through the relative interior of a facet. For a face F of codimension at least two, the choices whose segment

meets F before reaching y lie in $y+\operatorname{span}(F-y)$, of dimension at most $n-1$. Their finite union has empty interior, so such a choice of z_0 exists. Convexity would force equality with $\langle p_C, \cdot\rangle$ along the whole segment, whereas the strict wall inequality forces a strictly larger slope immediately after that exit. This contradiction proves (6.9) and the statement about linearity cones. □

Theorem 6.3. *For a finite normalized ordinary LP pattern of rank n, the set*

$$P_w(\mathcal{A}) = \{p \in (\mathbb{R}^n)^* : \langle p, g(v)\rangle \le \omega(v) \text{ for every } v \in \mathscr{V}\} \tag{6.10}$$

is a full-dimensional simple lattice polytope with normal fan Σ_w. Its vertices are p_C, its facets are

$$F_v = \{p \in P_w(\mathcal{A}) : \langle p, g(v)\rangle = \omega(v)\},$$

and

$$p_C \in F_v \quad \Longleftrightarrow \quad v \in C. \tag{6.11}$$

For every set $S \subseteq \mathscr{V}$, the intersection $\bigcap_{v\in S} F_v$ is nonempty if and only if S is contained in a cluster. In that case it has codimension $|S|$ and vertices

$$\{p_C : S \subseteq C\}. \tag{6.12}$$

These intersections give all nonempty faces, with the empty set S corresponding to the whole polytope. In particular, the cluster complex is the boundary complex of the polar polytope, and the edge graph of $P_w(\mathcal{A})$ is the exchange graph.

Proof. By Lemma 6.2, every p_C satisfies all the inequalities in (6.10), and

$$\psi(z) = \max_{C\in\mathscr{C}} \langle p_C, z\rangle. \tag{6.13}$$

Conversely, if p satisfies these inequalities, write an arbitrary $z \in \mathbb{R}^n$ in a cone σ_C as $z = \sum_{v\in C} a_v g(v)$, with $a_v \ge 0$. Then

$$\langle p, z\rangle \le \sum_{v\in C} a_v \omega(v) = \psi(z).$$

If p were outside $\operatorname{conv}\{p_C : C \in \mathscr{C}\}$, a separating linear functional would contradict this inequality and (6.13). Therefore

$$P_w(\mathcal{A}) = \operatorname{conv}\{p_C : C \in \mathscr{C}\}. \tag{6.14}$$

It is bounded, and 0 satisfies all finitely many defining inequalities strictly because $\omega(v) > 0$. Hence it is full-dimensional.

For the outer-normal convention, the normal cone at p_C is the set of z for which $\langle p_C, z\rangle$ is the maximum in (6.13). By (6.9), this cone is exactly σ_C. In particular p_C is a vertex. Distinct clusters have distinct cones and therefore distinct vertices, and (6.14) lists all the vertices. The maximal normal cones are precisely the maximal cones of Σ_w, so the full normal fan is Σ_w. Consequently its rays correspond to facets, and all the inequalities in (6.10) define distinct facets.

The rays of Σ_w contained in σ_C are precisely those labeled by C. Thus (6.9), evaluated on a ray generator, gives (6.11). Exactly the n independent facets labeled by C meet at p_C, proving simplicity. Formula (6.7) shows that every vertex is integral, so the polytope is a lattice polytope.

If $S \subseteq C$, then p_C lies in every F_v with $v \in S$. The independent active normals at p_C show that their intersection has codimension $|S|$. More explicitly, near this vertex the n active inequalities give independent local coordinates; setting the coordinates indexed by S to zero and the others to small positive values leaves all other inequalities strict. Conversely, a nonempty intersection of facets is a face and has a vertex p_C; by (6.11) its labels are contained in C. A face is the convex hull of its vertices, so its vertex set is exactly (6.12). Every face of a polytope is the intersection of the facets containing it. Simplicity shows that a codimension-k face is contained in exactly k facets, giving the claimed bijection and the entire reverse face order.

The polar has the explicit description

$$P_w(\mathcal{A})^\circ = \operatorname{conv}\{g(v)/\omega(v) : v \in \mathscr{V}\}.$$

The defining inequalities give the inclusion from right to left. For the converse, a separating functional for a point outside this hull has a positive maximum on the hull: completeness supplies a ray on which the nonzero functional is positive. Dividing by that maximum gives a point of $P_w(\mathcal{A})$ pairing to more than one with the separated point, so that point is outside the polar. By polarity, the facet exposed by p_C has exactly the vertices $g(v)/\omega(v)$ with $v \in C$. They are affinely independent because the vectors $g(v)$ are linearly independent. These simplex facets and their faces therefore give exactly the cluster complex. Finally, edges correspond to shared facets of maximal normal cones, which correspond to mutations by Theorem 4.3. Proposition 5.2 identifies all these labels with the actual variables and clusters of the original normalized pattern. □

Example 6.4 (Constant exchanges). Let π be an irreducible element of R, and take the initial cluster $(x_1, \dots, x_n)$ with $F_i = \pi$ for $1 \le i \le n$. Put $y = \pi/(x_1 \cdots x_n)$. The defining orders in (2.2) are $e_{ij} = 1$ for $i \neq j$, since substitution leaves the constant π unchanged. Thus

$$\widehat{F}_i = \frac{\pi}{\prod_{j \neq i} x_j}, \qquad x_i' = y.$$

Every ordinary exchange polynomial remains π under mutation, with unit factor one. The relation $x_1 \cdots x_n y = \pi$ shows that mutation in any n-element subset of $\{x_1, \dots, x_n, y\}$ replaces the chosen coordinate by the omitted one. These are exactly the clusters: each is obtained from the initial cluster by at most one mutation, and the collection is closed under mutation. Each subset generates the ambient field, by solving the product relation for its missing variable, and therefore is a transcendence basis. The initial Laurent exponents

$e_1, \dots, e_n, -\sum_i e_i$ are distinct, so the variables have distinct coefficient-unit classes. Hence the seeds form a finite normalized pattern.

Every variable is a Laurent monomial in every cluster, so $H = 0$. The shared-coordinate exponents of each exchange are all -1, giving $M = 0$ and $\omega = 1$. The leading vectors are $g(x_i) = e_i$ and $g(y) = -\sum_i e_i$. Consequently the polytope is

$$\{p \in (\mathbb{R}^n)^* : p_i \le 1 \ (1 \le i \le n),\ -\textstyle\sum_i p_i \le 1\}.$$

At the initial-cluster vertex all p_i equal one. At the vertex for the cluster omitting x_i, the active equations give $p_j = 1$ for $j \neq i$ and $-\sum_j p_j = 1$, hence $p_i = -n$. Its vertices are therefore $\mathbf{1}$ and $\mathbf{1} - (n+1)e_i$ for $1 \le i \le n$, where $\mathbf{1} = (1, \dots, 1)$. Their differences from $\mathbf{1}$ are linearly independent, so the polytope is an n-simplex. For $n = 1$ it is $[-1, 1]$.

Corollary 6.5. *Every linear ordinary LP algebra over a UFD admits a finite normalized pattern, and the cluster complex of every such pattern has the polyhedral realization of Theorem 6.3. For a face F of the realizing polytope and a cluster variable v, one has $F \subseteq F_v$ if and only if every cluster labeling a vertex of F contains v. Thus Conjecture 7.6 of* [18] *holds with a simple lattice polytope.*

Proof. Theorem 2.5 supplies a finite normalization. Apply Theorem 6.3. Its vertex–facet incidence formula gives the stated criterion, since a face is the convex hull of its vertices and F_v is a face. □

7. Cluster-monomial bases

Throughout this section, the normalized LP pattern is finite. Let $\mathcal{B}$ be the set of its cluster monomials, formed from the fixed representatives of cluster variables. Thus every element of $\mathcal{B}$ is a coefficient-one monomial $\mathbf{x}_C^a$, with $a \in \mathbb{Z}_{\ge 0}^n$, in at least one cluster C. Repeated occurrences of the same element in different clusters give a single element of $\mathcal{B}$.

Proposition 7.1 (Coefficient extraction)**.** *Let $b \in \mathcal{B}$, and choose a cluster C in which $b = \mathbf{x}_C^a$. For every $b' \in \mathcal{B}$, one has*

$$[\mathbf{x}_C^a]b' = \begin{cases} 1, & b' = b, \\ 0, & b' \neq b. \end{cases} \tag{7.1}$$

In particular, $\mathcal{B}$ is R-linearly independent. If $f = \sum_{b' \in \mathcal{B}} c_{b'} b'$ is a finite R-linear combination, then

$$c_b = [\mathbf{x}_C^a] f. \tag{7.2}$$

The right-hand side is the Laurent coefficient in R, for any cluster containing b as a monomial.

Proof. Suppose that $[\mathbf{x}_C^a]b' \neq 0$. Since $P_C(b) = \{a\}$, this implies

$$P_C(b) \subseteq P_C(b').$$

Lemma 3.3 gives $b = b'$. The coefficient of $\mathbf{x}_C^a$ in b is one, which proves (7.1). Taking this coefficient in a finite linear combination proves (7.2). Applied to a relation with sum zero, the same identity makes every coefficient of the relation zero. □

Theorem 7.2. *For a finite normalized ordinary LP pattern over a UFD R, the cluster monomials form an R-basis of the upper algebra $\mathcal{U}$, and*

$$\mathcal{U} = \mathcal{A}.$$

For every $f \in \mathcal{U}$, its coefficient at a cluster monomial b is given by (7.2).

Proof. Linear independence follows from Proposition 7.1. We prove spanning by a finite reduction.

Use the fixed initial cluster C_0 and weight w. For a nonzero $f \in \mathcal{U}$, let $\gamma \in \mathbb{Z}^n$ be its leading exponent in C_0; thus

$$h_{C_0,f}(w) = \langle \gamma, w \rangle.$$

By Theorem 4.3, there is a cluster C such that $\gamma \in \sigma_C$. Since $\sigma_C = G_C \mathbb{R}^n_{\geq 0}$ and $G_C \in \mathrm{GL}_n(\mathbb{Z})$, the vector

$$a = G_C^{-1}\gamma$$

belongs to $\mathbb{Z}^n_{\geq 0}$. Put $b = \mathbf{x}_C^a$. Multiplicativity of leading terms gives $g(b) = G_C a = \gamma$. If γ lies in several cluster cones, Lemma 3.4 shows that the resulting cluster monomial is the same.

For each $v \in C$, the v-coordinate of the transported weight $\tau_{C_0,C}(w)$ is

$$h_{C_0,v}(w) = \langle g(v), w \rangle.$$

Thus this weight is $w_C = G_C^{\mathsf{T}} w$. Its coordinates are $\mathbb{Q}$-linearly independent because G_C is unimodular. By Proposition 3.1,

$$h_{C,f}(w_C) = h_{C_0,f}(w) = \langle \gamma, w \rangle = \langle a, w_C \rangle. \tag{7.3}$$

There is a unique leading exponent of f in the cluster C. Equation (7.3) and the $\mathbb{Q}$-linear independence of the coordinates of w_C force that exponent to be a. Consequently

$$c = [\mathbf{x}_C^a] f \in R \setminus \{0\}$$

is its leading coefficient, which lies in R since $f \in \mathcal{U} \subseteq R[\mathbf{x}_C^{\pm 1}]$. The coefficient of $\mathbf{x}_C^a$ in b is one, so subtracting cb cancels this term.

The occurrence of this term implies $P_C(b) = \{a\} \subseteq P_C(f)$. Applying Proposition 3.1 yields

$$P_D(b) \subseteq P_D(f) \qquad \text{for every cluster } D. \tag{7.4}$$

Set $f_1 = f - cb$. This remains in $\mathcal{U}$. If it is nonzero, cancellation of the leading term in the cluster C gives

$$h_{C,f_1}(w_C) < h_{C,f}(w_C), \qquad h_{C_0,f_1}(w) < h_{C_0,f}(w),$$

where the second inequality is again transported by Proposition 3.1. Moreover, (7.4) implies

$$P_D(f_1) \subseteq P_D(f) \qquad \text{for every cluster } D \tag{7.5}$$

whenever $f_1 \neq 0$.

Repeat the construction for each nonzero remainder. By (7.5), all its leading exponents in C_0 belong to the fixed finite set

$$P_{C_0}(f) \cap \mathbb{Z}^n.$$

Their scalar products with w strictly decrease at every step. The procedure therefore ends after at most $\#(P_{C_0}(f) \cap \mathbb{Z}^n)$ subtraction steps, and expresses f as a finite R-linear combination of elements of $\mathcal{B}$.

Every cluster monomial belongs to $\mathcal{A}$, so spanning gives $\mathcal{U} \subseteq \mathcal{A}$. The Laurent phenomenon gives the reverse inclusion. Finally, Proposition 7.1 identifies each coefficient of the resulting unique expansion. □

For graph LP algebras, the basis assertion is already the theorem of Dantas e Moura, Telekicherla Kandalam, and Woodruff [8, Theorem 1.1]. Theorem 7.2 obtains that assertion from the finite leading fan and extends it to all finite normalized ordinary LP patterns. The coefficient formula (7.2) will also determine the positive cone in the graph case.

8. Atomicity for graph LP algebras

8.1. **Graph seeds and their canonical variables.** Let Γ be a finite directed graph on $[n] = \{1, \dots, n\}$, without loops or multiple edges. Set

$$R = \mathbb{Z}[A_1, \dots, A_n], \qquad R_+ = \mathbb{N}[A_1, \dots, A_n],$$

where the A_i are algebraically independent and $\mathbb{N} = \mathbb{Z}_{\geq 0}$. The standard graph LP algebra $\mathcal{A}_\Gamma$ has initial cluster $(X_1, \dots, X_n)$ and exchange polynomials

$$F_i = A_i + \sum_{i \longrightarrow j} X_j \qquad (i \in [n]). \tag{8.1}$$

We use the canonical graph normalization, described next.

For $I \subseteq [n]$, an acyclic function on I is a map $f : I \longrightarrow [n]$ such that, for each $i \in I$, either $f(i) = i$ or $i \longrightarrow f(i)$ is an edge of Γ, and the arrows $i \longrightarrow f(i)$ have no directed cycle of length greater than one. The fixed points of f mark roots. Define

$$Y_I = \frac{\displaystyle\sum_{f \text{ acyclic on } I} \prod_{i \in I} \mathrm{wt}_i(f)}{\displaystyle\prod_{i \in I} X_i}, \qquad \mathrm{wt}_i(f) = \begin{cases} A_i, & f(i) = i, \\ X_{f(i)}, & f(i) \neq i, \end{cases} \tag{8.2}$$

and put $Y_\varnothing = 1$. In particular,

$$Y_{\{i\}} = \frac{A_i + \sum_{i \longrightarrow j} X_j}{X_i}.$$

The all-root function occurs in the numerator of (8.2); hence each Y_I is nonzero. These formulas fix the canonical representatives of the graph variables.

We recall the cluster description that will be used. A nonempty subset of $[n]$ is called strongly connected when its induced subgraph is strongly

connected; singletons are included. A collection $\mathcal{N}$ of nonempty strongly connected subsets is nested if the following conditions hold:

(i) Any two members of $\mathcal{N}$ are either contained one in the other or disjoint.
(ii) For every pairwise disjoint subcollection $I_1, \dots, I_k$ of $\mathcal{N}$, the sets $I_1, \dots, I_k$ are precisely the strongly connected components of the induced subgraph on $I_1 \cup \cdots \cup I_k$.

Its support is $S = \operatorname{supp}\mathcal{N} = \bigcup_{I \in \mathcal{N}} I$. It is maximal on its support if no further nonempty strongly connected subset of S can be adjoined while preserving nestedness. The empty collection is maximal on its empty support.

The graph classification of Lam and Pylyavskyy [18, Theorem 1.1] states that the cluster variables are exactly

$$X_i \quad (i \in [n]), \qquad Y_I \quad (I \neq \varnothing \text{ strongly connected}),$$

and the clusters are exactly

$$C_{\mathcal{N}} = \{Y_I : I \in \mathcal{N}\} \cup \{X_i : i \notin \operatorname{supp}\mathcal{N}\}, \tag{8.3}$$

where $\mathcal{N}$ ranges over nested collections maximal on their support. Such a collection satisfies $|\mathcal{N}| = |\operatorname{supp}\mathcal{N}|$, so (8.3) has n variables. In particular, the graph LP pattern is finite. Thus Theorem 7.2 applies to the canonical cluster monomials, whose set we denote by $\mathcal{B}_\Gamma$.

8.2. **The positive cone and multiplication.** We use Glew's Laurent-positivity theorem.

Theorem 8.1 (Glew [15, Theorem 5.1])**.** *For every cluster $C_{\mathcal{N}}$ of the standard graph LP algebra and every $T \subseteq [n]$, one has*

$$Y_T \in R_+\big[Y_I^{\pm 1} \ (I \in \mathcal{N}), X_i^{\pm 1} \ (i \notin \operatorname{supp}\mathcal{N})\big].$$

The same inclusion holds for every initial variable X_j. Consequently every graph cluster variable is a Laurent polynomial with coefficients in $\mathbb{N}[A_1, \dots, A_n]$ in every cluster.

Let $\mathscr{C}_\Gamma$ be the set of graph clusters. Define the universally positive cone by

$$\mathcal{A}_{\Gamma,+} = \mathcal{A}_\Gamma \cap \bigcap_{C \in \mathscr{C}_\Gamma} R_+[\mathbf{x}_C^{\pm 1}]. \tag{8.4}$$

An R-basis is atomic for this cone if its finite R_+-linear combinations are exactly the elements of the cone.

Theorem 8.2. *The canonical cluster-monomial basis of a graph LP algebra is atomic:*

$$\mathcal{A}_{\Gamma,+} = \bigoplus_{b \in \mathcal{B}_\Gamma} R_+ b. \tag{8.5}$$

Here the direct sum means that every element has a unique finite expansion with coefficients in R_+. For any $b_1, b_2 \in \mathcal{B}_\Gamma$, the structure constants in

$$b_1 b_2 = \sum_{b \in \mathcal{B}_\Gamma} m^b_{b_1,b_2} b$$

belong to $\mathbb{N}[A_1, \ldots, A_n]$. More generally, every monomial in arbitrary cluster variables is an $\mathbb{N}[A_1, \ldots, A_n]$-linear combination of cluster monomials. This proves [18, Conjecture 7.3(2)].

Proof. By the graph classification and Theorem 7.2, every $f \in \mathcal{A}_\Gamma$ has a unique finite expansion

$$f = \sum_{b \in \mathcal{B}_\Gamma} c_b b, \qquad c_b \in R. \tag{8.6}$$

Suppose first that $f \in \mathcal{A}_{\Gamma,+}$. For each $b \in \mathcal{B}_\Gamma$, choose a cluster C containing it as a monomial, say $b = \mathbf{x}_C^a$. By Proposition 7.1,

$$c_b = [\mathbf{x}_C^a] f.$$

Since $f \in R_+[\mathbf{x}_C^{\pm 1}]$, this coefficient belongs to R_+. This proves that the left-hand side of (8.5) is contained in the right-hand side.

Conversely, Theorem 8.1 makes every cluster variable universally positive. In every cluster, Laurent polynomials with coefficients in R_+ are closed under addition and multiplication. Hence every cluster monomial, and every finite R_+-linear combination of cluster monomials, lies in $\mathcal{A}_{\Gamma,+}$. This proves the reverse inclusion in (8.5). Uniqueness follows from the R-basis property.

Finally, a product of any finite collection of cluster variables is an element of $\mathcal{A}_\Gamma$ and is universally positive by Theorem 8.1 and closure under multiplication. Apply (8.5) to this product. All its cluster-monomial coefficients lie in R_+, including the structure constants of the products $b_1 b_2$. □

For example, let Γ be the directed cycle $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 1$, and write $Y_i = Y_{\{i\}}$. The product of the three singleton forest sums enumerates all maps $f : \{1,2,3\} \to \{1,2,3\}$ satisfying $f(i) = i$ or $i \longrightarrow f(i)$. Exactly one of these functions has a nontrivial directed cycle, namely the cycle itself, and its weight after division by $X_1 X_2 X_3$ is one. Formula (8.2) therefore gives

$$Y_1 Y_2 Y_3 = Y_{\{1,2,3\}} + 1.$$

Both terms on the right are cluster monomials.

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