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SMALE'S MEAN VALUE CONSTANT (1981) EQUALS 4, AND OTHER MISCELLANEA

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This is the expanded version of the paper D. Goel, Smale's mean value constant (1981) equals 4 [13].

Abstract. For a complex polynomial p of degree $d \geq 2$ with $p(0) = 0$ and $p'(0) \neq 0$, let $S(p)$ be the minimum of $|p(c)/(c\,p'(0))|$ over the critical points c of p, and let $K(d)$ be the supremum of $S(p)$ over all such p of degree d. In 1981 Smale proved that $K(d) \leq 4$ for every d, and asked whether 4 can be replaced by 1. We prove that Smale's mean value constant $\sup_d K(d)$ equals 4 and that $K(d) \to 4$ as $d \to \infty$, combining a reduction due to Jatar and Ng with an explicit construction. Each $K(d)$ is known to be below 4, so the supremum is not attained. We give three further proofs: by a variant of the main construction in which part of the boundary of an auxiliary domain is a level curve of an entire function; by a Riemann map onto a domain with thin hairs; and by a Riemann surface whose conformal radius at its base point is near 1. We also include a complete proof of the known fact that $K(d)$ does not decrease.[1,2]

AI Disclosure. This work was done overwhelmingly by AI models, steered by the author in a continuous critical dialogue, variously with Google's Gemini, Anthropic's Claude and OpenAI's ChatGPT. The author takes full responsibility for the paper.

1. Introduction

Let p be a complex polynomial of degree $d \geq 2$ with $p(0) = 0$ and $p'(0) \neq 0$. A *critical point* of p is a zero of p'; since $p'(0) \neq 0$, no critical point is 0. Compare, at each critical point c, the slope $p(c)/c$ of the chord from 0 to c with the slope $p'(0)$ at 0, and put

$$S(p) := \min_{p'(c)=0} \left| \frac{p(c)}{c\,p'(0)} \right|.$$

Any point where the derivative does not vanish can be moved to 0 by a translation, and subtracting a constant then gives $p(0) = 0$, so the normalization loses nothing. Smale's mean value theorem [24] says that $S(p) \leq 4$ for every such p. For $d \geq 2$ let

$$K(d) := \sup\{S(p) : \deg p = d\}, \qquad K^* := \sup_{d \geq 2} K(d).$$

Thus $K(d)$ is the best constant in Smale's theorem for degree d, and K^*, *Smale's mean value constant*, is the best constant for all degrees. Jatar and Ng [17] call it M. The polynomial $z + z^d$ has $S = 1 - 1/d$, so $K^* \geq 1$, and Smale asked whether $K^* = 1$ [24, 25]. Moreover $K(d)$ does not decrease with d [9]; Appendix A gives a full proof. The idea is that adding εz^{d+1} to p, with ε small, moves the old critical points and the quotients $|p(c)/(c\,p'(0))|$ there only slightly, and creates one new critical point, of modulus of order $1/\varepsilon$, where this quotient is large. So $\lim_d K(d)$ exists and equals K^*.

Theorem 1.1. $K^* = \lim_{d \to \infty} K(d) = 4$.

[1] This is an expanded version of the following focused version [13]: https://hexagonmath.org/2611.00282

[2] Additionally, Theorem 1.1 is proved in Lean 4 from Lean's three standard axioms alone, and checked with `lake comparator`:
https://github.com/david44hello/paper_smale-mean-value-constant-equals-4/tree/v2.50

So for every number $a < 4$ there is a polynomial with $|p(c)/(c\,p'(0))| > a$ at *every* critical point c.

The proof is not effective: for a given $a < 4$ it gives no bound on the least d with $K(d) > a$. The supremum is not attained: $K(d) \le 4^{(d-2)/(d-1)} < 4$ [3], and $K(d) < 4 - 2.263/\sqrt{d}$ for $d \ge 8$ [10].

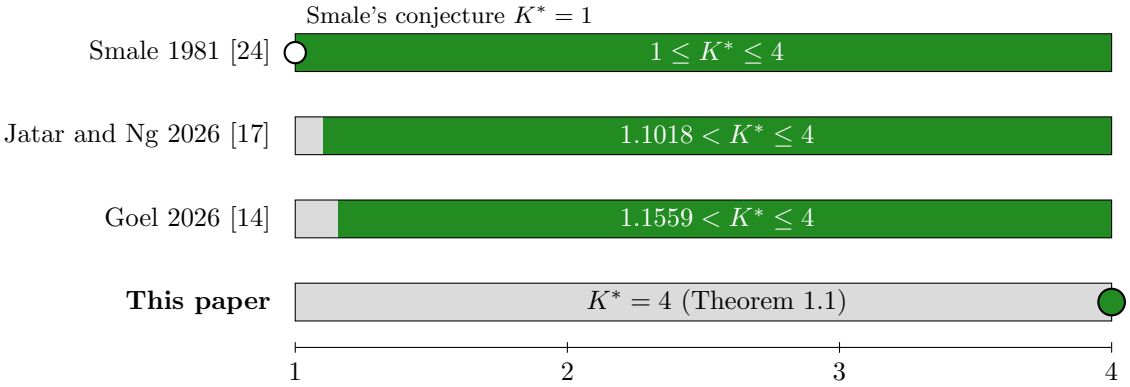


FIGURE 1. What is known about K^*, to scale. In each row the green part is where K^* may still lie; the grey part is ruled out. Smale proved $K^* \le 4$ in 1981 and conjectured $K^* = 1$ (white dot); that conjecture was disproved in 2026 [1, 2]. Theorem 1.1 shrinks the interval to the single point 4.

Notation. $\mathbb{D}$ is the open unit disc and $\mathbb{T}$ the unit circle. A *Jordan domain* is a bounded domain whose boundary is a Jordan curve, and it is *analytic* if that curve is analytic. A function is *univalent* if it is holomorphic and injective. Koebe's quarter theorem says that if g is univalent on $\mathbb{D}$ and $g(0) = 0$, then $g(\mathbb{D})$ contains the disc of radius $|g'(0)|/4$ about 0; the growth theorem gives the lower bound $|g(t)| \ge |g'(0)|\,|t|/(1+|t|)^2$ on $\mathbb{D}$. The *Koebe function* $k(t) = t/(1+t)^2$ is univalent on $\mathbb{D}$ and maps it onto $\mathbb{C} \setminus [1/4, \infty)$. The boundary point of $k(\mathbb{D})$ nearest 0 is $1/4 = \lim_{t\to 1} k(t)$.

Earlier work. The conjecture $K^* = 1$ was disproved in 2026 [1, 2]. Jatar and Ng then proved $K^* > 1.1018$ [17], and the author proved $K^* > 1.1559$ [14] (Figure 1).

Our proof rests on a reduction due to Jatar and Ng [17, Proposition 4.3, Theorem 4.4] (Proposition 2.1 below). The reduction replaces polynomials by functions with nonvanishing derivative near the closure of a domain; it goes back to the construction of [1, 2] (see [2, Proposition 6.4]). Jatar and Ng show that K^* equals the supremum A of $\inf_{\partial\Omega} |f(z)/z|$ over such functions f on analytic Jordan domains $\Omega \ni 0$, normalized by $f(0) = 0$ and $f'(0) = 1$. In their notation our construction gives $A \ge 4$, so $A = M = 4$; this answers their question [17, Section 9] whether "one can get a larger lower bound of M through A". Theorem 5.1 of [17] shows that a different constant, the one in Smale's Problem 1A, equals 4; see Remark 6.3.

The idea of the proof. By Proposition 2.1, it is enough to find, for each $a < 4$, an analytic Jordan domain $\Omega \ni 0$ and a function f, holomorphic near $\overline{\Omega}$, with $f(0) = 0$, $f'(0) = 1$ and $f' \ne 0$ on $\overline{\Omega}$ such that $|f(z)/z| > a$ on $\partial\Omega$. For $a > 1$ the function f must have zeros other than 0 on $\overline{\Omega}$, so it is not injective (Remark 6.2). We look for f of the form $f = (W'(0)/F'(0))\,F \circ W^{-1}$, where W is univalent on $\mathbb{D}$ with $W(0) = 0$ and $\Omega = W(\rho\mathbb{D})$ for some $\rho < 1$ close to 1. If t_0 is a point of the circle $\rho\mathbb{T}$ where $|F|$ is least, then the value of $|f(z)/z|$ at $z = W(t_0)$ is

$$\frac{|F(t_0)|}{|F'(0)|} \cdot \frac{|W'(0)|}{|W(t_0)|}.$$

The first factor is at most ρ, by Schwarz's lemma: the branch of F^{-1} with value 0 at 0 extends to the disc $\{|v| < |F(t_0)|\}$ with values in $\rho\mathbb{D}$ (by analytic continuation of the inverse branch, using $F' \ne 0$; see Remark 6.1), so $|F(t_0)| \le \rho|F'(0)|$. The second is at most $(1+\rho)^2/\rho$, by Koebe's

growth theorem. So this value is at most $(1+\rho)^2 < 4$. Our construction makes both bounds nearly sharp at the same point; this is possible because F and W can be chosen separately. The only link between them is a set of n points of $\mathbb{T}$, the n-th roots $b_1, \dots, b_n$ of -1: near them $|F|$ dips, and there $|W|$ must be close to its smallest possible size.

We take $W = g \circ \phi - g(\phi(0))$, where g is a function close to the Koebe function and ϕ is an explicit univalent self-map of $\mathbb{D}$, with $\phi(0)$ close to 0 and $\phi'(0)$ close to 1, whose limit at *each* of the points b_j is the single point 1. The image $\phi(\mathbb{D})$ is the disc with n thin walls removed, and the channels between the walls all lead to the point 1. Since g sends 1 to the boundary point nearest 0, the second factor is close to 4 at each b_j. We construct F with $F'(0)$ close to 1 and with $|F|$ large near the whole circle, except near the points b_j, where $|F|$ drops to about 1. Away from these points $|F|$ is so large that $|f(z)/z|$ is large. Our function F is built from an entire function E with $E'(\xi) = \exp(\xi^n)(1 - \xi^n/c_\sigma)$, where c_σ is a large constant (Section 4). Figure 2 shows the two pieces.

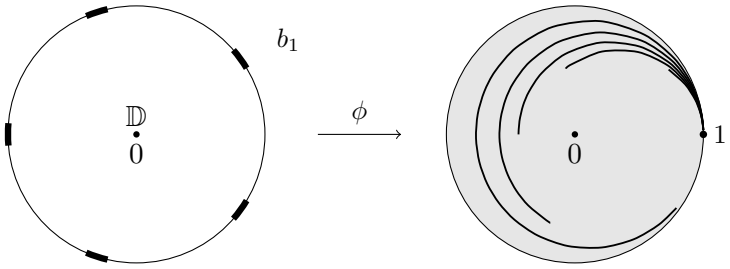


FIGURE 2. The construction for $n = 5$, not to scale; the walls are drawn far apart. On the left is the unit disc; the thick arcs mark the neighbourhoods of the points b_j. On the right is $\phi(\mathbb{D})$, the disc minus 5 arcs of circles tangent to $\mathbb{T}$ at the point 1. The map ϕ sends points near each b_j into a channel between neighbouring walls, or between the outermost wall and $\mathbb{T}$, close to the point 1. The function g, close to the Koebe function, then sends the point 1 to the boundary point nearest 0.

The main step is the following theorem, which we prove in Section 5. For a function g on $\mathbb{D}$ we write $g(1) := \lim_{t\to 1,\, t\in\mathbb{D}} g(t)$, when this limit exists. We need the unrestricted limit, not only the radial or nontangential one, because $\phi(t)$ tends to 1 tangentially to $\mathbb{T}$ as $t \to b_j$.

Theorem 1.2. *Let g be a bounded univalent function on $\mathbb{D}$ with $g(0) = 0$ and $g'(0) = 1$, and suppose that g has a limit $g(1)$ at the point 1. Then $K^* \geq 1/|g(1)|$.*

The point $g(1)$ is not in $g(\mathbb{D})$, since g is open and injective, so it lies on $\partial g(\mathbb{D})$; hence $|g(1)| \geq 1/4$ by Koebe's theorem, and the lower bound $1/|g(1)|$ never exceeds 4. Theorem 1.2 has the following consequence for Jordan domains; Section 8 gives a direct proof of it.

Corollary 1.3. *Let U be a bounded Jordan domain with $0 \in U$, and let $G\colon \mathbb{D} \to U$ be conformal with $G(0) = 0$. Then $K^* \geq |G'(0)|/\operatorname{dist}(0, \partial U)$.*

Proof. By Carathéodory's theorem, G extends to a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{U}$. Take $t_* \in \mathbb{T}$ with $|G(t_*)| = \operatorname{dist}(0, \partial U)$. The function $g(t) := G(t_* t)/(t_* G'(0))$ is bounded and univalent on $\mathbb{D}$, with $g(0) = 0$, $g'(0) = 1$ and the limit $g(1) = G(t_*)/(t_* G'(0))$ at 1. Theorem 1.2 gives $K^* \geq 1/|g(1)| = |G'(0)|/\operatorname{dist}(0, \partial U)$. □

Sections 1–6 of this paper are those of [13], with Corollary 1.3 added. Sections 7–9 give three further proofs of Theorem 1.1, Appendix A gives a full proof that $K(d)$ does not decrease, and Appendix B gives a rate for the convergence of the conformal radius in Lemma 9.5; the proofs do not need this rate.

2. From polynomials to pairs of functions on the disc

The following proposition is due to Jatar and Ng [17, Proposition 4.3]. For completeness we include the proof.

Proposition 2.1. *Let Ω be an analytic Jordan domain with $0 \in \Omega$. Let f be holomorphic on a neighbourhood of $\overline{\Omega}$, with $f(0) = 0$, $f'(0) = 1$ and $f' \neq 0$ on $\overline{\Omega}$. If $a > 0$ and $|f(z)/z| > a$ on $\partial\Omega$, then $K^* > a$.*

The proof uses the following form of Hilbert's lemniscate theorem [15, 19].

Lemma 2.2. *In a closed topological annulus there is a nonconstant polynomial q such that the whole set $\{|q| = 1\}$ is an analytic Jordan curve in the interior of the annulus, separating its two boundary components. For such a q, $|q| < 1$ inside this curve and $|q| > 1$ outside it, and all zeros and all critical points of q lie inside it.*

Proof. We derive this single-curve form from the classical proof. Take an analytic Jordan curve in the interior of the annulus that separates its boundary components, and let X be the closed region bounded by this curve. Let G_X be the Green function of $\mathbb{C} \setminus X$ with pole at ∞; it has no critical point, since the complement of X in the Riemann sphere is simply connected, so that G_X is the logarithm of the modulus of a conformal map of $\mathbb{C} \setminus X$ onto $\{|w| > 1\}$. Fix $\gamma > 0$ so small that the set $\{0 < G_X \leq 2\gamma\}$ lies in the interior of the annulus. Let Q_j be a monic Fekete polynomial of degree j of X, and put $q_j := (e^\gamma \operatorname{cap} X)^{-j} Q_j$, where $\operatorname{cap} X$ is the capacity of X. Put $u_j := j^{-1} \log|q_j| - (G_X - \gamma) = j^{-1} \log|Q_j| - \log \operatorname{cap} X - G_X$. The zeros of Q_j (the Fekete points) lie on ∂X, so each u_j is harmonic on $\mathbb{C} \setminus X$; it extends harmonically to ∞ with $u_j(\infty) = 0$, since Q_j is monic and $G_X(z) = \log|z| - \log \operatorname{cap} X + o(1)$. By the Bernstein–Walsh inequality, $u_j \leq \varepsilon_j := j^{-1} \log \|Q_j\|_X - \log \operatorname{cap} X$ on $\mathbb{C} \setminus X$, and $\varepsilon_j \to 0$ [23, Theorems 5.5.2, 5.5.4 and 5.5.7]. So $\varepsilon_j - u_j$ is a nonnegative harmonic function on $(\mathbb{C} \setminus X) \cup \{\infty\}$ with value ε_j at ∞, and Harnack's inequality on the compact set $\{G_X \geq \gamma/4\} \cup \{\infty\}$ gives $0 \leq \varepsilon_j - u_j \leq V_0 \varepsilon_j$ there, with V_0 independent of j. Hence $u_j \to 0$ uniformly on $\{G_X \geq \gamma/4\}$, including near ∞, and, by the interior estimates for harmonic functions, its first derivatives tend to 0 uniformly on $\{\gamma/2 \leq G_X \leq 2\gamma\}$. Consider three regions, for large j.

- On the collar $\{\gamma/2 \leq G_X \leq 2\gamma\}$ the gradient of G_X does not vanish. The gradient lines of G_X make the collar a product of $\{G_X = \gamma\}$ with an interval. Along each of them, parametrized by arc length, $j^{-1} \log|q_j|$ increases (its derivative tends to $|\nabla G_X| > 0$, uniformly on the compact collar) from about $-\gamma/2$ to about γ, so it vanishes exactly once. Hence $\{|q_j| = 1\}$ meets the collar in a graph over $\{G_X = \gamma\}$ along the gradient lines, that is, in one Jordan curve close to $\{G_X = \gamma\}$; it is analytic, since $q_j \neq 0$ and $\nabla \log|q_j| \neq 0$ in the collar.
- On $\{G_X \geq 2\gamma\}$, up to ∞, we have $j^{-1} \log|q_j| \geq \gamma/2$, so $|q_j| > 1$.
- On $\{G_X \leq \gamma/2\}$, which contains X, the Bernstein–Walsh inequality $|Q_j| \leq \|Q_j\|_X \, e^{jG_X}$ [23, Theorem 5.5.7(a)] and $\|Q_j\|_X^{1/j} \to \operatorname{cap} X$ [23, Theorems 5.5.2 and 5.5.4] give $|q_j| < 1$.

So $q := q_j$ has the required property.

For such a q, $|q| < 1$ inside this curve and $|q| > 1$ outside it, by the maximum principle and connectedness. Since $|q| > 1$ outside the curve, all $\deg q$ zeros of q lie inside it; so q maps the inside of the curve onto $\mathbb{D}$ as a proper map of degree $\deg q$, and by the Riemann–Hurwitz formula it has $\deg q - 1$ critical points there, and these are all of them. □

Proof of Proposition 2.1. Step 1: a polynomial close to f. By continuity we can find $h > 0$ and the closure N of a Jordan domain, with $\overline{\Omega} \subset \operatorname{int} N$, such that f is holomorphic near N, $f' \neq 0$ on N, and $|f(z)/z| > a + 2h$ on $N \setminus \Omega$. Since $\mathbb{C} \setminus N$ is connected, Runge's theorem gives polynomials converging to f uniformly near N, and by Cauchy's estimates their derivatives converge on N.

Subtracting the constant term, we obtain a polynomial p with $p(0) = 0$, with $p' \neq 0$ on N, with $|p(z)/z| > a + h$ on $N \setminus \Omega$, and with $|p'(0)| < (a + h)/(a + h/2)$.

Step 2: a lemniscate. Apply Lemma 2.2 to the closed annulus $N \setminus \Omega$. We obtain a polynomial q whose lemniscate $\{|q| = 1\}$ is a Jordan curve in the interior of this annulus. Then $\overline{\Omega}$ lies inside the curve, so $|q| < 1$ on $\overline{\Omega}$. Moreover $zq'(z) \neq 0$ on the curve and outside it, because 0 and all critical points of q lie inside. Fix $\nu \in (0, 1)$ so small that the compact set $A_\nu := \{1 - \nu \leq |q| \leq 1 + \nu\}$ lies in $\operatorname{int} N \setminus \overline{\Omega}$, and so that $\{|q| < 1 - \nu\}$ contains $\overline{\Omega}$ and every critical point of q.

Step 3: the polynomials H_m. Let $H_m := p + zq^m$, a polynomial of degree $m \deg q + 1$ once m is large. Then $H_m(0) = 0$, and

$$H_m' = p' + q^{m-1}\,(q + mzq').$$

We show that for all large m every critical point of H_m lies in A_ν.

(i) On $\{|q| \leq 1 - \nu\}$, a compact subset of N, we have $m|q|^{m-1} \to 0$ uniformly, so $H_m' \to p'$ uniformly. Since $p' \neq 0$ on N, the polynomial H_m has no critical point there.

(ii) On $\{|q| \geq 1 + \nu\}$ the polynomial zq' has no zero, and it has the same degree as q. Hence there are positive constants V_1, V_2, V_3 with $|zq'| \geq V_1(1 + |z|)^{\deg q}$, with $|q| \geq V_2(1 + |z|)^{\deg q}$ and with $|q/(zq')| \leq V_3$ on this set. So $|q + mzq'| \geq (m/2)|zq'|$ for large m. Let $d_1 := \deg p'$, so that $|p'| \leq V_4(1 + |z|)^{d_1}$ on this set for some V_4, and let $m - 1 \geq d_1$. Since $|q| \geq 1 + \nu$ there, $|q|^{m-1} \geq (1 + \nu)^{m-1-d_1}|q|^{d_1}$, and $|q|^{d_1} \geq V_2^{d_1}(1 + |z|)^{d_1 \deg q} \geq V_2^{d_1}(1 + |z|)^{d_1}$. With $|q + mzq'| \geq (m/2)V_1(1 + |z|)^{\deg q} \geq (m/2)V_1$ we get

$$\left|q^{m-1}(q + mzq')\right| \;\geq\; (1 + \nu)^{m-1-d_1}\, V_2^{d_1}(1 + |z|)^{d_1} \cdot \frac{m}{2}\, V_1 \;>\; V_4(1 + |z|)^{d_1} \;\geq\; |p'|$$

for large m, uniformly on this set. So there is no critical point there.

(iii) Therefore every critical point c of H_m lies in A_ν. On A_ν we have $|zq'| \geq V_5 > 0$, $|q| \leq 2$ and $|p'| \leq V_6$, so $H_m'(c) = 0$ gives

$$|q(c)|^{m-1} = \frac{|p'(c)|}{|q(c) + mcq'(c)|} \leq \frac{2V_6}{V_5 m}$$

for large m, and hence $|q(c)|^m = O(1/m)$. Since $H_m(c)/c = p(c)/c + q(c)^m$ and $c \in N \setminus \Omega$, we obtain $|H_m(c)/c| \geq a + h - O(1/m)$, uniformly over the critical points. Finally $H_m'(0) = p'(0) + q(0)^m \to p'(0)$, because $|q(0)| < 1$. Therefore

$$\liminf_{m\to\infty} S(H_m) \;\geq\; \frac{a + h}{|p'(0)|} \;>\; a + \frac{h}{2},$$

and so $K^* > a$. □

We use Proposition 2.1 in the following form.

Lemma 2.3. *Let W be univalent on $\mathbb{D}$ with $W(0) = 0$, and let F be holomorphic on $\mathbb{D}$ with $F(0) = 0$ and $F' \neq 0$ on $\mathbb{D}$. If $0 < \rho < 1$, $s > 0$, and*

$$|F(t)|\,|W'(0)| \;>\; s\,|F'(0)|\,|W(t)| \qquad \textit{for all } t \textit{ with } |t| = \rho,$$

then $K^* > s$.

Proof. The set $\Omega := W(\rho\mathbb{D})$ is an analytic Jordan domain whose closure lies in $W(\mathbb{D})$. The function $f := (W'(0)/F'(0))\, F \circ W^{-1}$ is holomorphic on $W(\mathbb{D})$, with $f(0) = 0$, $f'(0) = 1$ and $f' \neq 0$. On $\partial\Omega$, with $z = W(t)$ and $|t| = \rho$, we have $|f(z)/z| = |W'(0)/F'(0)|\,|F(t)/W(t)| > s$. Proposition 2.1 gives $K^* > s$. □

3. The map W

The map $\mu(t) := i(1+t)/(1-t)$ sends $\mathbb{D}$ conformally onto the upper half-plane $\mathbb{H}$, with $\mu(0) = i$ and $\mu'(0) = 2i$. Its inverse is $\mu^{-1}(w) = (w-i)/(w+i)$, so that $\mu^{-1}(w) - 1 = -2i/(w+i)$. Let $P \subset \mathbb{T} \setminus \{1\}$ be a finite nonempty set. For $b \in P$ the point $x_b := \mu(b)$ is real. For $\lambda > 0$ let

$$\Gamma_\lambda(z) := z + \lambda \sum_{b \in P} \log(z - x_b) \quad (z \in \mathbb{H}), \qquad \phi_\lambda := \mu^{-1} \circ \Gamma_\lambda \circ \mu,$$

with the principal branch of the logarithm.

Lemma 3.1. (a) *ϕ_λ is a univalent map of $\mathbb{D}$ into $\mathbb{D}$.*
(b) *For every $b \in P$, $\phi_\lambda(t) \to 1$ as $t \to b$ in $\mathbb{D}$.*
(c) *As $\lambda \to 0$, $\phi_\lambda(0) \to 0$ and $\phi_\lambda'(0) \to 1$.*

Proof. (a) We have $\operatorname{Im}\Gamma_\lambda(z) = \operatorname{Im} z + \lambda \sum_b \arg(z - x_b) > 0$, so Γ_λ maps $\mathbb{H}$ into $\mathbb{H}$. Further, $\Gamma_\lambda' = 1 + \lambda \sum_b 1/(z - x_b)$ has negative imaginary part on $\mathbb{H}$. Since $\mathbb{H}$ is convex, the difference quotient of Γ_λ between two points of $\mathbb{H}$, being the mean of Γ_λ' on the segment between them, has negative imaginary part, and so is not 0. Hence Γ_λ is injective.

(b) As $t \to b$, $z = \mu(t) \to x_b$ in $\mathbb{H}$. Then $\operatorname{Re}\Gamma_\lambda(z) = \lambda \log|z - x_b| + (\text{a term bounded near } x_b) \to -\infty$, so $\Gamma_\lambda(z) \to \infty$, and $\phi_\lambda(t) - 1 = -2i/(\Gamma_\lambda(z) + i) \to 0$. This holds however t approaches b.

(c) As $\lambda \to 0$, $\Gamma_\lambda(i) \to i$ and $\Gamma_\lambda'(i) \to 1$. So $\phi_\lambda(0) = \mu^{-1}(\Gamma_\lambda(i)) \to 0$, and $\phi_\lambda'(0) = -4\Gamma_\lambda'(i)/(\Gamma_\lambda(i) + i)^2 \to 1$. □

Figure 2 shows the shape of $\phi_\lambda(\mathbb{D})$. On each interval of the real axis whose right end is a point x_b, including the unbounded interval $(-\infty, \min_b x_b)$, the imaginary part of Γ_λ is constant and its real part runs from $-\infty$ up to a maximum and back. One checks that $\Gamma_\lambda(\mathbb{H})$ is $\mathbb{H}$ minus horizontal rays going to the left, and $\phi_\lambda(\mathbb{D})$ is $\mathbb{D}$ minus arcs of circles tangent to $\mathbb{T}$ at the point 1: the walls of Figure 2. This description is not used in the proofs.

Proposition 3.2. *Let g be as in Theorem 1.2, and let $B := \sup_{\mathbb{D}} |g|$. For every $\varepsilon \in (0,1)$ and every finite nonempty set $P \subset \mathbb{T} \setminus \{1\}$ we can find a univalent function W on $\mathbb{D}$ with $W(0) = 0$ and a number $\theta_0 > 0$ such that $|W'(0)| \geq 1 - \varepsilon$, $|W| \leq 2B$ on $\mathbb{D}$, and $|W(t)| < (1+\varepsilon)|g(1)|$ for every $t \in \mathbb{D}$ with $\operatorname{dist}(t, P) < \theta_0$.*

Note that the bound $2B$ does not depend on P. This is what allows us, in Section 5, to fix the size of $|F|$ before the number of points in P.

Proof. Let $W := g \circ \phi_\lambda - g(\phi_\lambda(0))$. This is univalent on $\mathbb{D}$ by Lemma 3.1(a), $W(0) = 0$, and $|W| \leq 2B$. By Lemma 3.1(c), as $\lambda \to 0$ we have $g(\phi_\lambda(0)) \to 0$ and $W'(0) = g'(\phi_\lambda(0))\phi_\lambda'(0) \to 1$. Fix λ so small that $|W'(0)| \geq 1 - \varepsilon$ and $|g(\phi_\lambda(0))| < (\varepsilon/2)|g(1)|$. By Lemma 3.1(b) and the limit of g at 1, for each $b \in P$ we have $|g(\phi_\lambda(t)) - g(1)| < (\varepsilon/2)|g(1)|$ for all $t \in \mathbb{D}$ near b. Since P is finite, this gives θ_0. □

4. The function F

Throughout this section $n \geq 8$ is an integer, $\sigma \geq 100$, $c_\sigma := 2\sigma$, and $R \geq 1$, with

$$200\,\sigma n \leq e^{(\sigma-1)/2}, \qquad R \leq e^{(\sigma-1)/2}. \tag{1}$$

Proposition 4.1. *For all $\varepsilon, \theta > 0$ we can find a function F, holomorphic on $\mathbb{D}$ with $F(0) = 0$ and $F' \neq 0$ on $\mathbb{D}$, and a number $r_0 \in (0,1)$, such that $0 < F'(0) \leq (2\sigma)^{1/n}$ and, for every t with $r_0 < |t| < 1$,*

$$|F(t)| \geq R, \qquad \text{or else both} \quad |F(t)| \geq \frac{n}{n+1} - \varepsilon \quad \text{and} \quad |t^n + 1| < \theta.$$

The plan of the proof is as follows (Figure 3). We put $F := E \circ \xi$, where E is the entire function of Step 1 and $\xi(t)^n = \eta(t^n)$, with η a conformal map of $\mathbb{D}$ onto the half strip $\Pi := \{y : \operatorname{Re} y < \sigma,\ |\operatorname{Im} y| < C\}$, where C is a large odd multiple of π, chosen last (Step 2; oddness is used in Step 4). Then $|F(t)| = |\Phi(y)|$ with $y = \eta(t^n)$, where $\Phi(y) := E(y^{1/n})$, and as $|t| \to 1$ the point y tends to the boundary of Π. In the band of width 1 along the right edge $\operatorname{Re} y = \sigma$ we show $|\Phi| \geq R$ (Step 3). In the bands of width $\frac{1}{2}$ along the two horizontal edges, and to the left of $\operatorname{Re} y = -C$, we show that $|\Phi|$ is at least about $n/(n+1)$ (Steps 4 and 5). The rest of Π is a compact set, so for $|t|$ close to 1 the point y lies in one of these regions; it lies outside the right band only when t^n is close to -1, that is, when t is close to one of the n-th roots of -1 (Step 6). On Π we have $\operatorname{Re} y < \sigma = c_\sigma/2$, so the factor $1 - y/c_\sigma$ (Step 1) does not vanish there, and it is bounded below in terms of $|y|$ (Step 3): $|1 - y/c_\sigma| \geq |y|/(6c_\sigma)$. This lower bound is what lets Step 3 bound $|\Phi|$ from below along the whole right edge, at every height up to C, although C is chosen last. The condition $n \geq 8$ is used in Step 4.

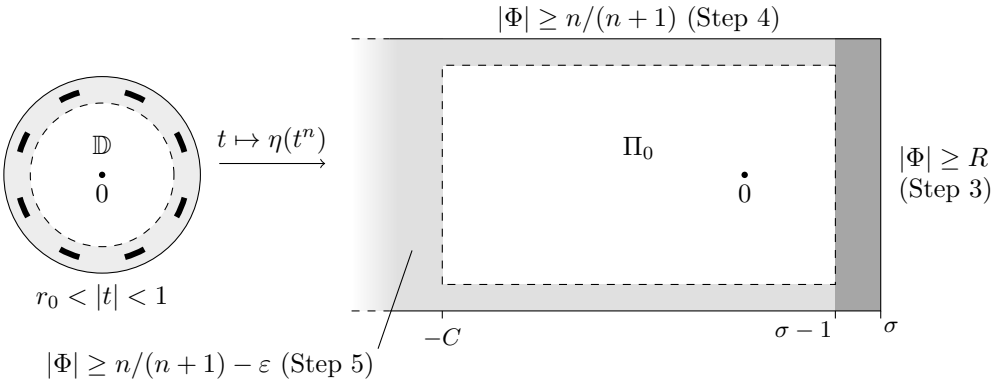


FIGURE 3. The plan of Section 4, for $n = 8$, not to scale. The map $t \mapsto \eta(t^n)$ sends $\mathbb{D}$ onto the half strip Π. Off the compact set Π_0, Steps 3–5 bound $|\Phi|$ from below (the band at the bottom edge is like the one at the top; in the far left, ε comes from the choice of C at the end of Step 6). For $r_0 < |t| < 1$ the point $\eta(t^n)$ lies outside Π_0, and it lies outside the dark band only when t is close to an n-th root of -1 (thick arcs; Step 6).

Proof. Step 1: the function E. Let

$$E(\xi) := \int_0^\xi \exp(\zeta^n)\Big(1 - \frac{\zeta^n}{c_\sigma}\Big)\, d\zeta.$$

This is an entire function with $E'(\xi) = \exp(\xi^n)(1 - \xi^n/c_\sigma)$, and $E(\omega\xi) = \omega E(\xi)$ whenever $\omega^n = 1$. Integrating by parts, with $E_0(\xi) := \int_0^\xi \exp(\zeta^n)\, d\zeta$ and $\kappa := 1/(c_\sigma n)$,

$$E(\xi) = (1+\kappa)E_0(\xi) - \kappa\, \xi \exp(\xi^n). \tag{2}$$

On $\mathbb{C} \setminus (-\infty, 0]$, with principal branches and $u = \zeta^n$, put

$$\Phi(y) := E(y^{1/n}) = \int_0^y J(u)\, du, \qquad J(u) := \frac{1}{n}\, e^u \chi(u), \qquad \chi(u) := u^{1/n-1}\Big(1 - \frac{u}{c_\sigma}\Big).$$

The function Φ itself jumps across the negative axis, but since $\Phi(\bar{y}) = \overline{\Phi(y)}$, its modulus $|\Phi|$ extends continuously to all of $\mathbb{C}$. Every ξ with ξ^n off the negative axis has the form $\omega\,(\xi^n)^{1/n}$ with $\omega^n = 1$, so the rotation symmetry of E and continuity give

$$|E(\xi)| = |\Phi(\xi^n)| \qquad (\xi \in \mathbb{C}). \tag{3}$$

So it suffices to bound $|\Phi|$ on the closed upper half-plane, with boundary values from above on the negative axis.

Step 2: the half strip and the definition of F. Let $C \geq 3\pi$ be an odd multiple of π, chosen last. Let $\beta := \pi/(2C)$, $\alpha := \sinh(\beta\sigma)$, and

$$\Pi := \{y : \operatorname{Re} y < \sigma,\ |\operatorname{Im} y| < C\}.$$

The function $w(y) := \sinh(\beta(y-\sigma))$ maps Π conformally onto the left half-plane: $y \mapsto \beta(y-\sigma)$ maps Π onto the half strip $\{\operatorname{Re} z < 0,\ |\operatorname{Im} z| < \pi/2\}$, which sinh maps conformally onto the left half-plane. So

$$\Theta(y) := \frac{\alpha + w(y)}{\alpha - w(y)}$$

maps Π conformally onto $\mathbb{D}$, with $\Theta(0) = 0$ since $w(0) = -\alpha$. Let $\eta := \Theta^{-1} : \mathbb{D} \to \Pi$. Then $\eta(0) = 0$, and from $w(\eta(t)) = -\alpha(1-t)/(1+t)$ we get

$$\eta'(0) = \frac{2\alpha}{\beta\cosh(\beta\sigma)} = \frac{2\tanh(\beta\sigma)}{\beta} \in (0, 2\sigma].$$

The quotient $\eta(t)/t$ has no zero on $\mathbb{D}$, so

$$\xi(t) := t\left(\frac{\eta(t^n)}{t^n}\right)^{1/n},$$

with the branch that is positive at $t = 0$, is holomorphic on $\mathbb{D}$. It satisfies $\xi(t)^n = \eta(t^n)$ and $\xi(\omega t) = \omega\xi(t)$ for $\omega^n = 1$. These two identities show that ξ is injective: if $\xi(t_1) = \xi(t_2)$, then $\eta(t_1^n) = \eta(t_2^n)$, so $t_1^n = t_2^n$ and $t_2 = \omega t_1$ with $\omega^n = 1$; then $\xi(t_1) = \xi(t_2) = \omega\xi(t_1)$, so $\omega = 1$ or $\xi(t_1) = 0$, that is, $t_1 = t_2 = 0$. Hence $\xi' \neq 0$. Now let $F := E \circ \xi$. Then $F(0) = 0$, and

$$F'(t) = \exp(\eta(t^n))\Big(1 - \frac{\eta(t^n)}{c_\sigma}\Big)\xi'(t) \neq 0,$$

because $\operatorname{Re}\eta < \sigma < c_\sigma$. Also $F'(0) = \eta'(0)^{1/n} \leq (2\sigma)^{1/n}$. By (3),

$$|F(t)| = |\Phi(\eta(t^n))| \qquad (t \in \mathbb{D}). \tag{4}$$

Step 3: the right band, $\sigma - 1 \leq \operatorname{Re} y < \sigma$: here $|\Phi(y)| \geq R$. Let $y = x + i\tau$ with $\tau \geq 0$ and $\sigma - 1 \leq x < \sigma$, so that $|y| \geq x \geq 99$. Since $\operatorname{Re} y \leq c_\sigma/2$, we have $1 + |y|/c_\sigma \leq 6|1 - y/c_\sigma|$: if $|y| \leq 2c_\sigma$, then $|1 - y/c_\sigma| \geq 1 - \operatorname{Re} y/c_\sigma \geq \frac{1}{2}$ and $1 + |y|/c_\sigma \leq 3$; if $|y| > 2c_\sigma$, then $|1 - y/c_\sigma| \geq |y|/c_\sigma - 1 \geq |y|/(2c_\sigma)$ and $1 + |y|/c_\sigma \leq 3|y|/(2c_\sigma)$. Hence $|1 - y/c_\sigma| \geq |y|/(6c_\sigma)$. Let $T_0 := (1/n)e^x|\chi(y)|$, the modulus of the main term $(1/n)e^y\chi(y)$. Then

$$T_0 \ \geq\ \frac{e^x|y|^{1/n}}{6c_\sigma n}.$$

To bound the rest of $\Phi(y)$, note that the function $\Phi(u) - (1/n)e^u\chi(u)$ has derivative $-(1/n)e^u\chi'(u)$, since $\Phi' = J = (1/n)e^u\chi$. Integrate this derivative along the horizontal segment from $y_0 := x/2 + i\tau$ to y. For u on this segment, $|y|/2 \leq |u| \leq |y|$ and $|c_\sigma - u| \geq \sigma$. So $|\chi(u)| \leq 2(1 + |y|/c_\sigma)|y|^{1/n-1} \leq 12|\chi(y)|$ (the first step uses $|y|/2 \leq |u| \leq |y|$, the second $1 + |y|/c_\sigma \leq 6|1 - y/c_\sigma|$), and $|\chi'(u)/\chi(u)| \leq 1/|u| + 1/|c_\sigma - u| \leq 3/(\sigma - 1)$. Therefore

$$\Big|\Phi(y) - \frac{1}{n}e^y\chi(y)\Big| \ \leq\ |\Phi(y_0)| + \frac{1}{n}e^{x/2}|\chi(y_0)| + \frac{1}{n}\int_{x/2}^{x} e^v\,\frac{36|\chi(y)|}{\sigma - 1}\,dv.$$

We bound the three terms on the right in turn.

(i) Put $\xi_0 := y_0^{1/n}$, so that $\Phi(y_0) = E(\xi_0)$. On the segment from 0 to ξ_0 we have $|\exp(\zeta^n)| \le e^{x/2}$, so $|E_0(\xi_0)| \le |\xi_0| e^{x/2}$; also $|\xi_0 \exp(\xi_0^n)| = |\xi_0| e^{x/2}$. By (2), $|y_0| \le |y|$ and the lower bound for T_0,

$$|\Phi(y_0)| \le (1+2\kappa)|y_0|^{1/n} e^{x/2} \le 2|y|^{1/n} e^{x/2} \le 12 c_\sigma n\, e^{-x/2}\, T_0 \le \frac{24}{200}\, T_0,$$

the last step because $12 c_\sigma n\, e^{-x/2} = 24\sigma n\, e^{-x/2} \le 24\sigma n\, e^{-(\sigma-1)/2} \le 24/200$ by (1).

(ii) Since $|\chi(y_0)| \le 12|\chi(y)|$, the second term is at most $12 e^{-x/2} T_0 \le 10^{-4}\, T_0$, as $x \ge 99$.

(iii) Since $\int_{x/2}^{x} e^v\, dv \le e^x$, the third term is at most $36 T_0/(\sigma-1) \le (36/99)\, T_0$.

Since $24/200 + 10^{-4} + 36/99 < 1/2$, the total is at most $T_0/2$, and

$$|\Phi(y)| \;\ge\; \frac{T_0}{2} \;\ge\; \frac{e^x}{24\sigma n} \;\ge\; \frac{e^{\sigma-1}}{24\sigma n} \;\ge\; \frac{200}{24}\, e^{(\sigma-1)/2} \;\ge\; R.$$

Step 4: the band $C - \frac12 \le \operatorname{Im} y < C$: here $|\Phi(y)| \ge n/(n+1)$, if $10 c_\sigma \le C - \frac12$. Fix τ in this range, and let $u = v + i\tau$ run along the horizontal line. Write

$$J(u) = \frac{1}{n c_\sigma}\, e^v \left(-e^{i\tau}\right) u^{1/n} \Big(1 - \frac{c_\sigma}{u}\Big).$$

Since C is an odd multiple of π, the factor $-e^{i\tau}$ has argument in $[-\frac12, 0]$. The factor $e^{-i\pi/n} u^{1/n}$ has argument in $(-\pi/n, 0)$. And $|c_\sigma/u| \le 1/10$, so $1 - c_\sigma/u$ has argument of modulus at most 0.11. Since $\frac12 + \pi/8 + 0.11 < \pi/2$, with $n \ge 8$ the total argument of $e^{-i\pi/n} J(u)$ has modulus below $\pi/2$, so $\operatorname{Re}(e^{-i\pi/n} J) > 0$ along the whole line, and $\operatorname{Re}(e^{-i\pi/n}\Phi)$ increases along it as v increases. As $v \to -\infty$, $\Phi(v+i\tau) - \Phi(v) \to 0$, where $\Phi(v)$ is the boundary value from above. Indeed, on the vertical segment from v to $v + i\tau$, of length less than C, we have $|J| \le (1/n) e^v (1 + (|v| + C)/c_\sigma)$ once $|v| \ge 1$, because then $|u|^{1/n - 1} \le 1$ and $|1 - u/c_\sigma| \le 1 + (|v| + C)/c_\sigma$. So the integral of J over the segment is at most C times this bound, which tends to 0. On the negative axis $\Phi(-r^n) = e^{i\pi/n} \int_0^r \exp(-\zeta^n)(1 + \zeta^n/c_\sigma)\, d\zeta$. So the limit of Φ at the far left of the line is $e^{i\pi/n}\Lambda_n$, where

$$\Lambda_n := \int_0^\infty \exp(-\zeta^n)\Big(1 + \frac{\zeta^n}{c_\sigma}\Big)\, d\zeta \;\ge\; \int_0^1 (1 - \zeta^n)\, d\zeta = \frac{n}{n+1}.$$

Hence $|\Phi(y)| \ge \operatorname{Re}(e^{-i\pi/n}\Phi(y)) \ge n/(n+1)$.

Step 5: the far left, $\operatorname{Re} y \le -C$, $0 \le \operatorname{Im} y < C$: here $|\Phi(y)| \ge n/(n+1) - (C/n) e^{-C}(1 + 2C/c_\sigma)$. Let $y = x + i\tau$. On the negative axis, $|\Phi(x)| \ge \int_0^1 (1 - \zeta^n)\, d\zeta = n/(n+1)$, because $|x|^{1/n} \ge 1$. On the vertical segment from x to y, of length at most C, we have $1 \le |u| \le |x| + C$, so $|J(u)| \le (1/n) e^x (1 + (|x| + C)/c_\sigma) \le (1/n) e^{-C}(1 + 2C/c_\sigma)$, the last step because the middle expression decreases as $|x|$ grows. Integrating J along this segment gives $|\Phi(y) - \Phi(x)| \le (C/n) e^{-C}(1 + 2C/c_\sigma)$, and the bound follows.

Step 6: where η goes, and the conclusion. Let

$$\Pi_0 := \{y : -C \le \operatorname{Re} y \le \sigma - 1,\ |\operatorname{Im} y| \le C - \tfrac12\},$$

a compact subset of Π, and let $r_0 := (\max_{\Pi_0} |\Theta|)^{1/n} < 1$. Let $r_0 < |t| < 1$ and $y := \eta(t^n)$. Then $|\Theta(y)| = |t|^n > r_0^n = \max_{\Pi_0} |\Theta|$, so $y \notin \Pi_0$. If $\operatorname{Re} y > \sigma - 1$, Step 3 and (4) give $|F(t)| \ge R$. Otherwise $\operatorname{Re} y < -C$ or $|\operatorname{Im} y| > C - \frac12$, and Steps 4 and 5 bound $|F(t)|$ from below. In this case t^n is close to -1. Indeed $|\sinh(a_1 + i a_2)|^2 = \sinh^2 a_1 + \sin^2 a_2$ shows that $|w(y)| \ge \sinh(\pi/2)$ if $\operatorname{Re} y < -C$, and $|w(y)| \ge \cos(\pi/(4C)) \ge \frac12$ if $|\operatorname{Im} y| > C - \frac12$. Since $1 + t^n = 2\alpha/(\alpha - w)$ and $|\alpha - w| \ge |w|$ for $\operatorname{Re} w < 0$, we get $|t^n + 1| \le 4\alpha$.

Given $\varepsilon, \theta > 0$, choose C so large that $10 c_\sigma \le C - \frac12$, that $(C/n) e^{-C}(1 + 2C/c_\sigma) \le \varepsilon$, and that $4\alpha = 4 \sinh(\pi\sigma/(2C)) < \theta$. This proves the proposition. □

5. Proof of the theorems

Proof of Theorem 1.2. Let $B := \sup_{\mathbb{D}} |g|$, so that $B \geq |g(1)|$, and let $R := 2B/|g(1)| \geq 2$. Let $\varepsilon \in (0,1)$. For a large integer ℓ put $\sigma := \ell$ and $n := \ell^2$. Since R does not depend on ℓ, taking ℓ large we have $\sigma \geq 100$, $n \geq 8$, the conditions (1), the bound $(2\sigma)^{1/n} \leq 1+\varepsilon$ (by Bernoulli's inequality, $(1+\varepsilon)^{\ell^2} \geq 1+\ell^2\varepsilon \geq 2\ell$), and $n/(n+1) \geq 1-\varepsilon/2$.

Let $P := \{b_1, \dots, b_n\}$ be the set of n-th roots of -1; it does not contain 1. Let W and θ_0 be as in Proposition 3.2. By compactness there is $\theta > 0$ with $|t^n+1| \geq \theta$ for all $t \in \overline{\mathbb{D}}$ with $\operatorname{dist}(t,P) \geq \theta_0$. Let F and r_0 be as in Proposition 4.1, for $\varepsilon/2$ and this θ, and fix $\rho \in (r_0, 1)$. Let $|t| = \rho$.

- If $|t^n+1| < \theta$, then $\operatorname{dist}(t,P) < \theta_0$, so $|W(t)| < (1+\varepsilon)|g(1)|$. Also $|F(t)| \geq 1-\varepsilon$, because $R \geq 2$ and $n/(n+1) - \varepsilon/2 \geq 1-\varepsilon$. So $|F(t)|/|W(t)| > (1-\varepsilon)/((1+\varepsilon)|g(1)|)$.
- If $|t^n+1| \geq \theta$, then $|F(t)| \geq R$ by Proposition 4.1, since its other case needs $|t^n+1| < \theta$. Also $|W(t)| \leq 2B$, so $|F(t)|/|W(t)| \geq R/(2B) = 1/|g(1)| > (1-\varepsilon)/((1+\varepsilon)|g(1)|)$.

So Lemma 2.3 applies with $s = (1-\varepsilon)|W'(0)|/((1+\varepsilon)|g(1)|F'(0))$. Using $|W'(0)| \geq 1-\varepsilon$ and $F'(0) \leq 1+\varepsilon$,

$$K^* \;>\; \Big(\frac{1-\varepsilon}{1+\varepsilon}\Big)^2 \frac{1}{|g(1)|}.$$

Letting $\varepsilon \to 0$ gives the theorem. □

Proof of Theorem 1.1. Let k be the Koebe function, and for $0 < \delta < 1$ let $g(t) := k(\delta t)/\delta = t/(1+\delta t)^2$. Then g is univalent on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, $|g| \leq (1-\delta)^{-2}$, $g(0) = 0$, $g'(0) = 1$, and $g(1) = (1+\delta)^{-2}$. Theorem 1.2 gives $K^* \geq (1+\delta)^2$, and letting $\delta \to 1$ gives $K^* \geq 4$. Smale's theorem gives $K^* \leq 4$. So $K^* = 4$, and $\lim_{d\to\infty} K(d) = K^*$ because $K(d)$ does not decrease with d ([9], Appendix A). □

6. Remarks

Remark 6.1 (Why the method stops at 4). Let F, W, ρ, s be as in Lemma 2.3, normalized so that $F'(0) = W'(0) = 1$. Then $s < 4$. Let r be the largest radius for which the branch ψ of F^{-1} at 0 extends to the disc $\{|v| < r\}$ with values in $\rho\mathbb{D}$. The branch ψ cannot map $\{|v| < r\}$ into a disc $\{|t| \leq \rho'\}$ with $\rho' < \rho$. Indeed, otherwise, since $F' \neq 0$, the inverse function theorem and compactness give one radius $r_* > 0$ such that for every z_0 with $|z_0| \leq \rho'$, F has an inverse branch with values in $\rho\mathbb{D}$ on the disc of radius r_* about $F(z_0)$, taking $F(z_0)$ to z_0. Taking $z_0 = \psi(v_1)$ with $r - r_*/2 < |v_1| < r$, these branches continue ψ to discs that cover $\{r \leq |v| < r + r_*/2\}$; two of these discs that meet do so in a convex set that meets $\{|v| < r\}$ (it contains the midpoint of their centres), where both continuations equal ψ. So ψ would extend to $\{|v| < r + r_*/2\}$ with values in $\rho\mathbb{D}$, against the choice of r. Hence there are points v_k with $|\psi(v_k)| \to \rho$; then $|v_k| \to r$, since ψ maps each disc $\{|v| \leq r'\}$ with $r' < r$ onto a compact subset of $\rho\mathbb{D}$. Passing to a subsequence, $\psi(v_k)$ tends to a point t^* with $|t^*| = \rho$, and then $|F(t^*)| = \lim |v_k| = r$. In particular $r \geq \min_{|t|=\rho} |F(t)|$. Schwarz's lemma, applied to ψ, gives $r \leq \rho$, so $\min_{|t|=\rho} |F(t)| \leq \rho$: this is the bound on the first factor used in the introduction. The function $W \circ \psi$ is univalent on this disc, with derivative 1 at 0, and $W(t^*)$ is a boundary point of its image, so $|W(t^*)| \geq r/4$ by Koebe's theorem. The hypothesis gives $r = |F(t^*)| > s|W(t^*)| \geq sr/4$. This is Smale's argument, carried over to the pair (F, W). Jatar and Ng use the same two bounds in [17, Theorem 7.1, Remark 7.3].

Remark 6.2 (Why f cannot be injective). If f in Proposition 2.1 has no zero other than 0 on $\overline{\Omega}$, for instance if it is univalent, then z/f is holomorphic on $\overline{\Omega}$ with value 1 at 0, and the maximum principle gives $\min_{\partial\Omega} |f(z)/z| \leq 1$. So $\min_{\partial\Omega} |f(z)/z| > 1$ requires f to have zeros other than 0 on $\overline{\Omega}$. The function F of Section 4 has such zeros t; the points $\eta(t^n)$ lie inside the compact set Π_0, since Steps 3–5 bound $|\Phi|$ from below off Π_0. By Jensen's formula for $F(t)/t$, the mean of $\log|F|$ over $\rho\mathbb{T}$ equals $\log(\rho|F'(0)|) + \sum_k \log(\rho/|t_k|)$, summed over the zeros $t_k \neq 0$ of F in $\rho\mathbb{D}$. For ρ close

to 1, $|F|$ is large on most of $\rho\mathbb{T}$ and bounded below on the rest (Steps 3–6), so this mean is large, while $\log(\rho|F'(0)|)$ is near 0. The zeros supply the difference.

Remark 6.3 (The constant of Smale's Problem 1A). The constant of [17, Theorem 5.1], called K there, is the one in Smale's Problem 1A [24], in the form of [17, Section 5]:

$$K_{1\mathrm{A}} := \sup_{j\ge 2}\ \sup_{p}\ \min_{p'(c)=0} \Big|\frac{p^{(j)}(0)}{j!}\Big|^{1/(j-1)} |p(c)| \qquad (p(0)=0,\ p'(0)=1),$$

in which $|p(c)|$ is not divided by $|c|$. The proof of [17, Theorem 5.1] applies a variant of the reduction of Proposition 2.1 [17, Proposition 5.2] to the inverse of the Koebe function. That argument does not give K^*, because for that function $|f(z)/z|$ is small at boundary points far from 0.

7. A second proof: a domain bounded in part by a level curve

In this section we prove Proposition 7.6, a variant of Proposition 4.1, and we show that it can take the place of Proposition 4.1 in the proof of Theorem 1.2; this gives a second proof of Theorem 1.1. The function it constructs is E_0 composed with a Riemann map onto a domain that is bounded in part by a level curve of $|E_0|$. Proposition 2.1, Lemma 2.3 and Proposition 3.2 are used as they stand, and the deduction of Theorem 1.1 from Theorem 1.2 is unchanged.

Notation in this section. Throughout this section $n \ge 8$ is an integer, $R \ge 3$, and ω denotes an n-th root of unity. We write ζ for the variable of E_0 and y for the variable ζ^n. As in Section 4, u is a variable of integration in the y-plane, v denotes a real part and τ an imaginary part, and on the negative real axis we use boundary values from above. The letters n, R, ε, P, r_0, and in the last proof g, B, W, ρ, θ_0, s, have the roles they have in Sections 4 and 5; this section is a separate proof, so their values are chosen anew here. The function E_0 is that of Section 4. Objects with a tilde, and Φ_0, J_0, χ_0, are defined below; they replace the objects of Section 4 with the same letters.

The plan. In the proof of Proposition 4.1, as $|t| \to 1$ with t away from the n-th roots of -1, the point $\eta(t^n)$ approaches the right edge $\operatorname{Re} y = \sigma$ of the half strip Π, and $|\Phi| \ge R$ has to be proved near this straight line at every height up to C. This is why Section 4 needs the factor $1 - \zeta^n/c_\sigma$, the closed form (2), Step 3, and the choice of σ before C. Here this part of the boundary is the level curve $|\Phi_0| = R$ in the y-plane. A lower bound is still needed on one straight side, the right side of a rectangle (Lemma 7.2(a)), but that side lies at a real part that may grow with the height, so the factor $1 - \zeta^n/c_\sigma$ is not needed. The function $\tilde F$ is E_0 composed with a Riemann map onto the set of n-th roots of the points of a domain bounded in part by this level curve (Lemma 7.3). Koebe's quarter theorem at one boundary point bounds $\tilde F'(0)$, and this bound does not depend on the size of the domain (Lemma 7.3(d)). On the rest of the boundary $|\Phi_0|$ is at least nearly $n/(n+1)$, and the part where $|\Phi_0| < R - 1$ has harmonic measure at 0 that tends to 0 as the domain grows (Lemmas 7.4 and 7.5). A phase condition, (6) below, takes the place of the choice of C as an odd multiple of π in Section 4.

The functions E_0 and Φ_0. Recall from Step 1 of the proof of Proposition 4.1 the entire function E_0, with $E_0(0) = 0$ and $E_0'(\zeta) = \exp(\zeta^n)$. Its derivative has no zero, and

$$E_0(\omega\zeta) = \omega E_0(\zeta), \qquad E_0(\bar\zeta) = \overline{E_0(\zeta)}.$$

On $\mathbb{C} \setminus (-\infty, 0]$, with principal branches, put

$$\Phi_0(y) := E_0(y^{1/n}) = \int_0^y J_0(u)\,du, \qquad J_0(u) := \frac{1}{n}\,e^u \chi_0(u), \qquad \chi_0(u) := u^{1/n-1}.$$

These are the functions Φ, J, χ of Step 1 of Section 4 with E_0 in place of E, that is, without the factor $1 - u/c_\sigma$. We have $\Phi_0(\bar y) = \overline{\Phi_0(y)}$, and by the argument that gave (3), using the two symmetries of E_0 above, $|\Phi_0|$ extends continuously to all of $\mathbb{C}$ and

$$|E_0(\zeta)| = |\Phi_0(\zeta^n)| \qquad (\zeta \in \mathbb{C}). \tag{5}$$

Lemma 7.1. (a) *On $[0,\infty)$ the function Φ_0 is real and increasing, from 0 to ∞. Let y_R be the point of $(0,\infty)$ with $\Phi_0(y_R) = R$. Then $\Phi_0(x) \geq e^{x-1}/(nx)$ for $x \geq 1$, and $y_R \leq 2 + 2\log(nR)$.*

(b) *For $r > 0$, the boundary value of Φ_0 at $-r$ from above is*

$$\Phi_0(-r) = e^{i\pi/n} \int_0^{r^{1/n}} e^{-x^n}\, dx.$$

This integral lies in $(0,1]$, and in $[n/(n+1),1]$ if $r \geq 1$. In particular $|\Phi_0| \leq 1$ on the negative real axis.

Proof. (a) On $(0,\infty)$ the function J_0 is positive, and it is integrable at 0 since $1/n - 1 > -1$. So Φ_0 is real and increasing on $[0,\infty)$, with $\Phi_0(0) = 0$. For $x \geq 1$ and $x - 1 \leq u \leq x$ we have $e^u \geq e^{x-1}$ and $u^{1/n-1} \geq x^{1/n-1} \geq 1/x$, so

$$\Phi_0(x) \;\geq\; \frac{1}{n} \int_{x-1}^{x} e^u\, u^{1/n-1}\, du \;\geq\; \frac{e^{x-1}}{nx}.$$

In particular $\Phi_0(x) \to \infty$ as $x \to \infty$, so y_R is well defined. At $x = 2+2\log(nR)$ the last bound equals $e\,nR^2/(2 + 2\log(nR))$, which is at least R because $e\,nR \geq 2 + 2\log(nR)$. Since Φ_0 is increasing, $y_R \leq 2 + 2\log(nR)$.

(b) The boundary value from above is $E_0(e^{i\pi/n} r^{1/n})$, and on the segment from 0 to $e^{i\pi/n} r^{1/n}$ we have $\zeta^n = -x^n$ with $x = |\zeta|$; this gives the formula. The integrand is positive, and the integral of e^{-x^n} over $(0,\infty)$ is the value of Euler's gamma function at $1 + 1/n$, which is at most 1; so the integral lies in $(0,1]$. If $r \geq 1$, then $e^{-x^n} \geq 1 - x^n$ on $[0,1]$ gives the lower bound $\int_0^1 (1 - x^n)\, dx = n/(n+1)$. □

The parameters and the domains. Let $\varepsilon' \in (0, 1/8)$. Let $\tilde{C} > 0$ satisfy the *phase condition*

$$\tilde{C} \equiv \frac{\pi}{2} + \frac{\pi}{2n} \pmod{2\pi}, \tag{6}$$

and put

$$\Xi_- := \log \frac{\tilde{C}}{\varepsilon'}, \qquad \Xi := 2\log(8n\tilde{C}) + \log(R+1), \qquad \delta_R := \frac{1}{10\sqrt{R}}.$$

Since $R \geq 3$, we have $\delta_R < 0.06$, and so $\tan \delta_R \leq 2\delta_R$. Since Ξ_- and Ξ grow only logarithmically with $\tilde{C}$, every large $\tilde{C}$ with (6) also satisfies

$$\frac{\max(\Xi_-, \Xi)}{\tilde{C}} \leq \delta_R, \qquad \Xi \leq \tilde{C}, \qquad \tilde{C} \geq R, \qquad \Xi \geq 10, \qquad \Xi_- \geq 1, \tag{7}$$

and we assume (7) from now on. Then $\Xi \geq 2\log(8n) + 2\log R > 2 + 2\log(nR) \geq y_R$ by Lemma 7.1(a), so that

$$y_R < \Xi \leq \tilde{C}. \tag{8}$$

Let

$$\mathcal{Q} := \{y : -\tilde{C} < \operatorname{Re} y < \Xi,\ |\operatorname{Im} y| < \tilde{C}\},$$

a rectangle containing 0. Its *right side* is $\{\operatorname{Re} y = \Xi,\ |\operatorname{Im} y| \leq \tilde{C}\}$, its *top side* and *bottom side* are $\{\operatorname{Im} y = \pm\tilde{C},\ -\tilde{C} \leq \operatorname{Re} y \leq \Xi\}$, and its *left side* is $\{\operatorname{Re} y = -\tilde{C},\ |\operatorname{Im} y| \leq \tilde{C}\}$. Let $\mathcal{Q}_- := \{y \in \mathcal{Q} : \operatorname{Re} y \leq -\Xi_-\}$. Let $\mathcal{Y}$ be the component containing 0 of the open set $\{y \in \mathcal{Q} : |\Phi_0(y)| < R\}$, and let $\mathcal{Z}$ be the component containing 0 of the open set $\{\zeta : \zeta^n \in \mathcal{Q},\ |E_0(\zeta)| < R\}$.

Lemma 7.2. (a) *$|\Phi_0(\Xi + i\tau)| > R + 1$ for $0 \leq \tau \leq \tilde{C}$. Hence the right side of $\mathcal{Q}$ does not meet $\partial\mathcal{Y}$.*

(b) $|\Phi_0| \ge n/(n+1) - 2\varepsilon'$ *on the top side of* $\mathcal{Q}$ *and on* $\{\operatorname{Re} y = -\tilde{C},\ 0 \le \operatorname{Im} y \le \tilde{C}\}$. *Moreover, the function*

$$\mathcal{A}(v) := \operatorname{Re}\big(e^{-i\pi/n}\Phi_0(v+i\tilde{C})\big)$$

is strictly increasing on $[-\Xi_-, \Xi]$, *and* $n/(n+1) - \varepsilon' \le \mathcal{A}(v) \le |\Phi_0(v+i\tilde{C})|$ *there. If* $v \in [-\Xi_-, \Xi]$ *and* $\mathcal{A}(v) \le R-1$, *then* $|\Phi_0(v+i\tilde{C})| < R$.

(c) *If* $\operatorname{Re} y \le -\Xi_-$ *and* $|\operatorname{Im} y| \le \tilde{C}$, *then* $|\Phi_0(y)| \le 1 + \varepsilon' < R-1$. *The segment* $(-\tilde{C}, 0]$ *and the set* $\mathcal{Q}_-$ *lie in* $\mathcal{Y}$.

Since $|\Phi_0(\bar{y})| = |\Phi_0(y)|$, *the bounds on* $|\Phi_0|$ *in (a) and (b) hold also at the conjugate points; in particular they hold on the whole right side, on the bottom side, and on the whole left side.*

Proof. (a) Let $y := \Xi + i\tau$ and $y_0 := \Xi/2 + i\tau$, and let $u = v + i\tau$ run along the horizontal segment from y_0 to y, so that $J_0(u) = (1/n)e^{i\tau}e^v\chi_0(v+i\tau)$. Integrating by parts,

$$\int_{\Xi/2}^{\Xi} e^v\chi_0(v+i\tau)\,dv = e^{\Xi}\chi_0(y) - e^{\Xi/2}\chi_0(y_0) - \int_{\Xi/2}^{\Xi} e^v\chi_0'(v+i\tau)\,dv.$$

For $v \ge \Xi/2$ we have $|v+i\tau| \ge |y|/2$, so $|\chi_0(v+i\tau)| \le 2|\chi_0(y)|$, and $|\chi_0'(v+i\tau)| \le |\chi_0(v+i\tau)|/v \le 4|\chi_0(y)|/\Xi$. Hence, since $\Xi \ge 10$ by (7),

$$\Big|\int_{\Xi/2}^{\Xi} e^v\chi_0(v+i\tau)\,dv\Big| \ \ge\ |\chi_0(y)|\,e^{\Xi}\Big(1 - \frac{4}{\Xi} - 2e^{-\Xi/2}\Big) \ \ge\ \frac{|\chi_0(y)|\,e^{\Xi}}{2}.$$

On the segment from 0 to $y_0^{1/n}$ the point ζ^n runs along the segment from 0 to y_0, and $\operatorname{Re} y_0 = \Xi/2 > 0$; so $|\exp(\zeta^n)| \le e^{\Xi/2}$ there, and $|\Phi_0(y_0)| = |E_0(y_0^{1/n})| \le |y_0|^{1/n}e^{\Xi/2}$. Now $\Phi_0(y) = \Phi_0(y_0) + \int_{y_0}^{y} J_0(u)\,du$, $|\chi_0(y)| = |y|^{1/n-1}$, $|y_0| \le |y|$, and $1 \le |y| \le \Xi + \tilde{C} \le 2\tilde{C}$ by (7). Therefore

$$|\Phi_0(y)| \ \ge\ |y|^{1/n}\Big(\frac{e^{\Xi}}{2n|y|} - e^{\Xi/2}\Big) \ \ge\ \frac{e^{\Xi}}{8n\tilde{C}} = 8n\tilde{C}(R+1) \ >\ R+1,$$

because $e^{\Xi/2} = 8n\tilde{C}\sqrt{R+1} \ge 8n\tilde{C}$. Since $|\Phi_0| \le R$ on the closure of $\mathcal{Y}$, the right side does not meet $\partial\mathcal{Y}$.

(b) Let $v \in [-\Xi_-, \Xi]$. With principal arguments, an argument of $e^{-i\pi/n}J_0(v+i\tilde{C})$ is

$$\Big(\tilde{C} - \frac{\pi}{2} - \frac{\pi}{2n}\Big) + \Big(\frac{1}{n} - 1\Big)\Big(\arg(v+i\tilde{C}) - \frac{\pi}{2}\Big).$$

By the phase condition (6) the first bracket is a multiple of 2π. In the second term, $|\arg(v+i\tilde{C}) - \pi/2| = \arctan(|v|/\tilde{C}) \le \max(\Xi_-, \Xi)/\tilde{C} \le \delta_R$ by (7). So at the point $v + i\tilde{C}$

$$\operatorname{Re}\big(e^{-i\pi/n}J_0\big) \ge \cos(\delta_R)\,|J_0| > 0, \qquad \big|\operatorname{Im}\big(e^{-i\pi/n}J_0\big)\big| \le \tan(\delta_R)\operatorname{Re}\big(e^{-i\pi/n}J_0\big) \le 2\delta_R \operatorname{Re}\big(e^{-i\pi/n}J_0\big).$$

Since $\frac{d}{dv}\Phi_0(v+i\tilde{C}) = J_0(v+i\tilde{C})$, the function $\mathcal{A}$ is strictly increasing on $[-\Xi_-, \Xi]$.

To start, $\Phi_0(-\Xi_- + i\tilde{C}) = \Phi_0(-\Xi_-) + i\int_0^{\tilde{C}} J_0(-\Xi_- + i\tau)\,d\tau$, and $|J_0(-\Xi_- + i\tau)| \le e^{-\Xi_-}/n$ because $|-\Xi_- + i\tau| \ge 1$; so this integral has modulus at most $\tilde{C}e^{-\Xi_-}/n = \varepsilon'/n$. By Lemma 7.1(b), $e^{-i\pi/n}\Phi_0(-\Xi_-)$ is real and lies in $[n/(n+1), 1]$, since $\Xi_- \ge 1$. Hence $\mathcal{A}(-\Xi_-) \ge n/(n+1) - \varepsilon'$ and $|\operatorname{Im}(e^{-i\pi/n}\Phi_0(-\Xi_- + i\tilde{C}))| \le \varepsilon'$. Integrating the two inequalities above from $-\Xi_-$ to v, and using $\mathcal{A}(-\Xi_-) > 0$, we obtain, for $v \in [-\Xi_-, \Xi]$,

$$\mathcal{A}(v) \ge \frac{n}{n+1} - \varepsilon', \qquad \big|\operatorname{Im}\big(e^{-i\pi/n}\Phi_0(v+i\tilde{C})\big)\big| \le \varepsilon' + 2\delta_R\,\mathcal{A}(v).$$

Since $\mathcal{A}(v)$ is the real part of $e^{-i\pi/n}\Phi_0(v+i\tilde{C})$, we have $\mathcal{A}(v) \le |\Phi_0(v+i\tilde{C})|$. If moreover $\mathcal{A}(v) \le R-1$, then, since $0 < \mathcal{A}(v)$ and $\varepsilon' < 1/8$,

$$|\Phi_0(v+i\tilde{C})|^2 \ \le\ (R-1)^2 + \Big(\frac{1}{8} + 2\delta_R R\Big)^2 = (R-1)^2 + \Big(\frac{1}{8} + \frac{\sqrt{R}}{5}\Big)^2 \ <\ R^2,$$

because $2R - 1 > 1/64 + \sqrt{R}/20 + R/25$ for $R \geq 3$.

On the part $-\tilde{C} \leq v \leq -\Xi_-$ of the top side we have $|J_0(v + i\tilde{C})| \leq e^v/n$, so $|\Phi_0(v + i\tilde{C}) - \Phi_0(-\Xi_- + i\tilde{C})| \leq e^{-\Xi_-}/n \leq \varepsilon'$, and hence $|\Phi_0(v + i\tilde{C})| \geq \mathcal{A}(-\Xi_-) - \varepsilon' \geq n/(n+1) - 2\varepsilon'$. On the rest of the top side, $|\Phi_0| \geq \mathcal{A} \geq n/(n+1) - \varepsilon'$. On the left side, for $0 \leq \tau \leq \tilde{C}$, we have $|\Phi_0(-\tilde{C} + i\tau) - \Phi_0(-\tilde{C})| \leq \tilde{C}e^{-\tilde{C}}/n \leq \varepsilon'$, since $\tilde{C} \geq \Xi_-$ by (7); together with Lemma 7.1(b) this gives $|\Phi_0(-\tilde{C} + i\tau)| \geq n/(n+1) - \varepsilon'$.

(c) Let $y = v + i\tau$ with $v \leq -\Xi_-$ and $0 \leq \tau \leq \tilde{C}$. On the vertical segment from v to y, of length at most $\tilde{C}$, we have $|u| \geq 1$ and so $|J_0(u)| \leq e^v/n$. Hence $|\Phi_0(y) - \Phi_0(v)| \leq \tilde{C}e^v/n \leq \tilde{C}e^{-\Xi_-}/n \leq \varepsilon'$, and by Lemma 7.1(b), $|\Phi_0(y)| \leq 1 + \varepsilon'$; this is less than $R - 1$, as $R \geq 3$ and $\varepsilon' < 1/8$. The lower half follows by conjugation. The segment $(-\tilde{C}, 0]$ lies in $\mathcal{Q}$, contains 0, and $|\Phi_0| \leq 1 < R$ on it by Lemma 7.1(b); so it lies in $\mathcal{Y}$. The set $\mathcal{Q}_-$ is connected, lies in $\{y \in \mathcal{Q} : |\Phi_0(y)| < R\}$, and meets this segment; so it lies in $\mathcal{Y}$ as well. □

Lemma 7.3. (a) *$\mathcal{Z} = \{\zeta : \zeta^n \in \mathcal{Y}\}$. The domain $\mathcal{Z}$ is invariant under $\zeta \mapsto \omega\zeta$ and under $\zeta \mapsto \bar{\zeta}$, and it is bounded and simply connected.*

(b) *The boundary of $\mathcal{Z}$ is a finite union of closed analytic arcs. Hence the Riemann map $\tilde{\xi}$ of $\mathbb{D}$ onto $\mathcal{Z}$, that is, the conformal map of $\mathbb{D}$ onto $\mathcal{Z}$ with $\tilde{\xi}(0) = 0$ and $\tilde{\xi}'(0) > 0$, which exists by (a), extends continuously to $\overline{\mathbb{D}}$.*

(c) *The function $\tilde{F} := E_0 \circ \tilde{\xi}$ is holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, with $\tilde{F}(0) = 0$ and $\tilde{F}' = \exp(\tilde{\xi}^n)\,\tilde{\xi}' \neq 0$ on $\mathbb{D}$.*

(d) *$\tilde{\xi}(\omega t) = \omega\tilde{\xi}(t)$. Hence $\tilde{\eta}(t) := \tilde{\xi}(t^{1/n})^n$ does not depend on the choice of the n-th root, and $\tilde{\eta}$ is a conformal map of $\mathbb{D}$ onto $\mathcal{Y}$, continuous on $\overline{\mathbb{D}}$, with $\tilde{\eta}(0) = 0$ and $\tilde{\eta}'(0) = \tilde{\xi}'(0)^n$. For $t \in \overline{\mathbb{D}}$,*

$$|\tilde{F}(t)| = |\Phi_0(\tilde{\eta}(t^n))|. \tag{9}$$

Finally,

$$0 < \tilde{F}'(0) = \tilde{\xi}'(0) \leq (4y_R)^{1/n} \leq \big(8 + 8\log(nR)\big)^{1/n}.$$

The bound on $\tilde{F}'(0)$ in (d) does not depend on $\tilde{C}$, and it tends to 1 as $n \to \infty$ with R fixed.

Proof. (a) The map $\zeta \mapsto \zeta^n$ is a covering map of $\mathbb{C} \setminus \{0\}$ onto itself. Let $\zeta \neq 0$ with $\zeta^n \in \mathcal{Y}$. A path in $\mathcal{Y}$ from ζ^n to 0 that reaches 0 only at its end point lifts, through this map, to a path from ζ that tends to 0 at the end, and along which the n-th power stays in $\mathcal{Y}$. Hence the set $\{\zeta : \zeta^n \in \mathcal{Y}\}$ is connected. It contains 0 and, by (5), lies in $\{\zeta : \zeta^n \in \mathcal{Q},\ |E_0(\zeta)| < R\}$; so it lies in $\mathcal{Z}$. Conversely, the image of $\mathcal{Z}$ under $\zeta \mapsto \zeta^n$ is connected, contains 0, and lies in $\{y \in \mathcal{Q} : |\Phi_0(y)| < R\}$; so it lies in $\mathcal{Y}$. This proves the first claim, and since $(\omega\zeta)^n = \zeta^n$, it shows that $\mathcal{Z}$ is invariant under $\zeta \mapsto \omega\zeta$. The open set $\{\zeta : \zeta^n \in \mathcal{Q},\ |E_0(\zeta)| < R\}$ is invariant under $\zeta \mapsto \bar{\zeta}$, because $\mathcal{Q}$ is symmetric under conjugation and $|E_0(\bar{\zeta})| = |E_0(\zeta)|$; this map fixes 0, so it maps $\mathcal{Z}$ onto itself. The domain $\mathcal{Z}$ is bounded because $\mathcal{Q}$ is.

The domain $\mathcal{Z}$ is a component of the intersection of $\{\zeta : \zeta^n \in \mathcal{Q}\}$ with the component of $\{|E_0| < R\}$ that contains $\mathcal{Z}$. The first of these sets is star-shaped about 0, because $\mathcal{Q}$ is convex and contains 0; so it is simply connected. Each component of $\{|E_0| < R\}$ is simply connected, by the maximum principle: if a Jordan curve lies in such a component, then $|E_0| < R$ inside the curve, so the inside of the curve lies in that component. Finally, a component of the intersection of two simply connected plane domains is simply connected: if a Jordan curve lies in the component, then its inside lies in both domains, hence in their intersection, and, being connected and meeting the component, it lies in the component. So $\mathcal{Z}$ is simply connected.

(b) The level set $\{|E_0| = R\}$ is a smooth curve, since $E_0' \neq 0$ and $E_0 \neq 0$ on it, and by the implicit function theorem it is locally one analytic arc, so, by compactness, only finitely many of its components meet a given compact set. It contains no arc of the preimage, under $\zeta \mapsto \zeta^n$, of a side

of $\mathcal{Q}$. Indeed, along each side of $\mathcal{Q}$ the function $|\Phi_0|^2$ is real-analytic; it is greater than $(R+1)^2$ on the right side, by Lemma 7.2(a), and at most $(1+\varepsilon')^2$ on the parts of the top, bottom and left sides with $\operatorname{Re} y \le -\Xi_-$, by Lemma 7.2(c); so it is not identically R^2 on any side. Hence the level set meets the preimages of the sides of $\mathcal{Q}$ at only finitely many points of the compact set $\{\zeta : \zeta^n \in \overline{\mathcal{Q}}\}$.

Every point of $\partial\mathcal{Z}$ lies on $\{|E_0| = R\}$ or on the preimage under $\zeta \mapsto \zeta^n$ of a side of $\mathcal{Q}$, since $\mathcal{Z}$ is a component of the open set $\{\zeta : \zeta^n \in \mathcal{Q},\ |E_0(\zeta)| < R\}$. Let ζ_1 be a point of the level set with $\zeta_1^n \in \mathcal{Q}$, and suppose that ζ_1 lies on $\partial\mathcal{Z}$. Near ζ_1 the set $\{|E_0| < R\}$ is one side of the level curve, a half disc; it meets $\mathcal{Z}$ and lies in $\{\zeta : \zeta^n \in \mathcal{Q},\ |E_0(\zeta)| < R\}$, so it lies in $\mathcal{Z}$, and the level arc through ζ_1 lies on $\partial\mathcal{Z}$ near ζ_1. So on each arc of the level set between two consecutive points where it meets the preimages of the sides, the part that lies on $\partial\mathcal{Z}$ is either the whole arc or empty; the same holds, by the same argument with $\{\zeta : \zeta^n \in \mathcal{Q}\}$ in place of $\{|E_0| < R\}$, on the preimages of the sides of $\mathcal{Q}$. Hence $\partial\mathcal{Z}$ is a finite union of closed analytic arcs, and so it is locally connected. By the Carathéodory–Torhorst theorem [21, Chapter 2], $\tilde{\xi}$ extends continuously to $\overline{\mathbb{D}}$.

(c) This follows from (b), since E_0 is entire and $\tilde{\xi}' \neq 0$ on $\mathbb{D}$.

(d) By (a), $t \mapsto \omega^{-1}\tilde{\xi}(\omega t)$ is also a conformal map of $\mathbb{D}$ onto $\mathcal{Z}$ with value 0 and positive derivative at 0, so by the uniqueness of the Riemann map, $\tilde{\xi}(\omega t) = \omega\tilde{\xi}(t)$. Hence $\tilde{\eta}$ is well defined, and by (a) it is a conformal map of $\mathbb{D}$ onto $\mathcal{Y}$ with $\tilde{\eta}(0) = 0$ and $\tilde{\eta}'(0) = \tilde{\xi}'(0)^n$; it is continuous on $\overline{\mathbb{D}}$ by (b). For $t \in \overline{\mathbb{D}}$ we have $\tilde{\xi}(t)^n = \tilde{\eta}(t^n)$, so (5) gives (9). Next, $\tilde{F}'(0) = \exp(0)\,\tilde{\xi}'(0) = \tilde{\xi}'(0) > 0$. The point y_R lies on $\partial\mathcal{Y}$: the segment $[0, y_R)$ lies in $\mathcal{Q}$ by (8), and in $\{|\Phi_0| < R\}$ by Lemma 7.1(a), so it lies in $\mathcal{Y}$; and $|\Phi_0(y_R)| = R$. By Koebe's quarter theorem (recalled in Section 1), $\mathcal{Y} = \tilde{\eta}(\mathbb{D})$ contains the disc of radius $\tilde{\eta}'(0)/4$ about 0, so $\tilde{\eta}'(0) \le 4\operatorname{dist}(0, \partial\mathcal{Y}) \le 4y_R$. Hence $\tilde{F}'(0) = \tilde{\eta}'(0)^{1/n} \le (4y_R)^{1/n}$, and Lemma 7.1(a) gives the last bound. □

Lemma 7.4. *There is exactly one $v_1 \in (-\Xi_-, \Xi)$ with $\mathcal{A}(v_1) = R - 1$. Let $\mathcal{U}$ be the path that runs from $v_1 - i\tilde{C}$ along the bottom side of $\mathcal{Q}$ to the corner $-\tilde{C} - i\tilde{C}$, up the left side to the corner $-\tilde{C} + i\tilde{C}$, and along the top side to $v_1 + i\tilde{C}$; the two corners belong to $\mathcal{U}$, and its two end points $v_1 \pm i\tilde{C}$ do not.*

(a) *Every point y of $\partial\mathcal{Y}$ with $|\Phi_0(y)| < R - 1$ lies on $\mathcal{U}$.*

(b) *$|\Phi_0| < R$ on $\mathcal{U}$, and $\mathcal{U} \subset \partial\mathcal{Y}$. More precisely, for every $y_1 \in \mathcal{U}$ there is $r_2 > 0$ with*

$$\mathcal{Y} \cap \{|y - y_1| < r_2\} = \mathcal{Q} \cap \{|y - y_1| < r_2\},$$

a half disc, or a quarter disc if y_1 is one of the two corners.

(c) *Every point of $\mathcal{U}$ is the image under $\tilde{\eta}$ of exactly one point of $\mathbb{T}$. The set I of these points of $\mathbb{T}$ is one open arc of $\mathbb{T}$; it is symmetric under conjugation and contains -1, and $\tilde{\eta}$ maps I homeomorphically onto $\mathcal{U}$.*

Proof. By Lemma 7.2(b), $\mathcal{A}$ is continuous and strictly increasing on $[-\Xi_-, \Xi]$. By Lemma 7.2(b) and (c), $\mathcal{A}(-\Xi_-) \le |\Phi_0(-\Xi_- + i\tilde{C})| \le 1 + \varepsilon' < R - 1$. And $\mathcal{A}(\Xi) > R - 1$: by Lemma 7.2(a), $|\Phi_0(\Xi + i\tilde{C})| > R + 1$, while $\mathcal{A}(\Xi) \le R - 1$ would give $|\Phi_0(\Xi + i\tilde{C})| < R$ by Lemma 7.2(b). So there is exactly one $v_1 \in (-\Xi_-, \Xi)$ with $\mathcal{A}(v_1) = R - 1$.

(a) Let $y \in \partial\mathcal{Y}$ with $|\Phi_0(y)| < R - 1$. Since $\mathcal{Y}$ is a component of the open set $\{y \in \mathcal{Q} : |\Phi_0(y)| < R\}$, a point of $\partial\mathcal{Y}$ that lies in $\mathcal{Q}$ lies on the level set $\{|\Phi_0| = R\}$; so y is not in $\mathcal{Q}$, and y lies on a side of $\mathcal{Q}$. It is not on the right side, by Lemma 7.2(a). The points of the left side, and the points of the top and bottom sides with real part at most $-\Xi_-$, lie on $\mathcal{U}$, since $v_1 > -\Xi_-$. If $y = v + i\tilde{C}$ with $v \in [-\Xi_-, \Xi]$, then $\mathcal{A}(v) \le |\Phi_0(y)| < R - 1 = \mathcal{A}(v_1)$, so $v < v_1$ and $y \in \mathcal{U}$. The same holds on the bottom side, by conjugation. So $y \in \mathcal{U}$.

(b) On $\mathcal{U}$ we have $|\Phi_0| < R$: on the left side and where $\operatorname{Re} y \le -\Xi_-$ by Lemma 7.2(c), and at the points $v \pm i\tilde{C}$ with $-\Xi_- \le v < v_1$ by Lemma 7.2(b), since $\mathcal{A}(v) < R - 1$ there. So every $y_1 \in \mathcal{U}$ has a neighbourhood $\mathcal{Q} \cap \{|y - y_1| < r_2\}$ in $\mathcal{Q}$ on which $|\Phi_0| < R$. These neighbourhoods, together with $\mathcal{Q}_-$, form a connected subset of $\{y \in \mathcal{Q} : |\Phi_0(y)| < R\}$ that meets $\mathcal{Y}$: the half discs

are chained along $\mathcal{U}$, since two overlapping discs centred on a line meet inside the open half-plane, and the chain meets $\mathcal{Q}_-$, which lies in $\mathcal{Y}$ by Lemma 7.2(c). So these neighbourhoods lie in $\mathcal{Y}$, and $\mathcal{U}$ lies on $\partial\mathcal{Y}$. Since $\mathcal{Y} \subset \mathcal{Q}$, this gives $\mathcal{Y} \cap \{|y - y_1| < r_2\} = \mathcal{Q} \cap \{|y - y_1| < r_2\}$.

(c) Fix $y_1 \in \mathcal{U}$ and r_2 as in (b). As y tends, inside $\mathcal{Q} \cap \{|y - y_1| < r_2\}$, to a point of a side of $\mathcal{Q}$, it tends to $\partial\mathcal{Y}$, so $|\tilde\eta^{-1}(y)| \to 1$. By the Schwarz reflection principle, applied after the change of variable $y \mapsto (y - y_1)^2$ if y_1 is a corner, $\tilde\eta^{-1}$ extends continuously to the side or sides of $\mathcal{Q}$ near y_1, and analytically at the points that are not corners. Suppose that $\tilde\eta(t_1) = \tilde\eta(t_2) = y_1$ with $t_1, t_2 \in \mathbb{T}$, and let t tend to t_j along the radius. Then $\tilde\eta(t) \to y_1$ from inside $\mathcal{Q} \cap \{|y - y_1| < r_2\}$, so $t = \tilde\eta^{-1}(\tilde\eta(t))$ tends to the value at y_1 of the extended $\tilde\eta^{-1}$. Hence $t_1 = t_2$. The extended $\tilde\eta^{-1}$ is continuous on $\mathcal{U}$, with values in $\mathbb{T}$; it is injective there, because $\tilde\eta$, being continuous on $\overline{\mathbb{D}}$, maps its value at y back to y. In particular every point of $\mathcal{U}$ has a preimage on $\mathbb{T}$, and by the above only one. Since $\mathcal{U}$ is homeomorphic to an open interval, invariance of domain in dimension one shows that the image I of $\mathcal{U}$ is one open arc of $\mathbb{T}$, and that $\tilde\eta^{-1} : \mathcal{U} \to I$ is a homeomorphism. Since $\mathcal{Y}$ is symmetric under conjugation, by Lemma 7.3(a), and $\tilde\eta'(0) > 0$, the map $\tilde\eta$ commutes with conjugation; as $\mathcal{U}$ is symmetric under conjugation, so is I. Finally, since $\tilde\eta$ commutes with conjugation, it maps the diameter $(-1, 1)$ onto $\mathcal{Y} \cap \mathbb{R} = (-\tilde C, y_R)$ (by Lemma 7.1(a), $\Phi_0 \geq R$ on $[y_R, \Xi)$), increasingly, as $\tilde\eta'(0) > 0$; so it maps the radius $(-1, 0]$ onto the segment $(-\tilde C, 0]$, which lies in $\mathcal{Y}$ by Lemma 7.2(c). Hence $\tilde\eta(-1) = -\tilde C$, and $-1 \in I$. □

Lemma 7.5. (a) *Let $|I|$ be the length of the arc I. Then*

$$\frac{|I|}{2\pi} \;\leq\; \frac{4}{\pi}\sqrt{\frac{y_R}{\tilde C - y_R}}.$$

Since y_R does not depend on $\tilde C$, the right-hand side tends to 0 as $\tilde C \to \infty$.

(b) *The set $\{t \in \mathbb{T} : t^n \in I\}$ is the union of n open arcs of length $|I|/n$, centred at the n-th roots of -1. For $t \in \mathbb{T}$ off these arcs, $|\tilde F(t)| \geq R - 1$; for t on them, $|\tilde F(t)| \geq n/(n+1) - 2\varepsilon'$.*

Proof. (a) Every point y of $\mathcal{U}$ satisfies $|y| \geq \tilde C$, since $\mathcal{U}$ lies on the left, top and bottom sides of $\mathcal{Q}$. By Lemma 7.1(a), $\Phi_0 \geq R$ on $[y_R, \infty)$, so $\mathcal{Y}$ lies in the slit plane $\mathbb{C} \setminus [y_R, \infty)$. Inside the disc $\{|y| < \tilde C\}$ the boundary of $\mathcal{Y}$ consists only of points of the level set $\{|\Phi_0| = R\}$, because the only side of $\mathcal{Q}$ that meets this disc is the right side, which does not meet $\partial\mathcal{Y}$ by Lemma 7.2(a). Let $\mathcal{Y}_0$ be the component containing 0 of $\mathcal{Y} \cap \{|y| < \tilde C\}$. Let ϖ be the harmonic function on $\mathcal{Y}$ such that $\varpi \circ \tilde\eta$ is the Poisson integral, on $\mathbb{D}$, of the indicator function of I. This is the harmonic measure of $\mathcal{U}$ in $\mathcal{Y}$; we have $0 \leq \varpi \leq 1$ and $\varpi(0) = |I|/(2\pi)$.

We find the boundary values of ϖ on $\mathcal{Y}_0$. A boundary point of $\mathcal{Y}_0$ inside the disc $\{|y| < \tilde C\}$ is a boundary point of $\mathcal{Y}$. Let $y_2 \in \partial\mathcal{Y}$ with $|y_2| < \tilde C$, and let $y \to y_2$ in $\mathcal{Y}$. Since $\tilde\eta$ is continuous on $\overline{\mathbb{D}}$, the limit points of $\tilde\eta^{-1}(y)$ lie in the set of points of $\mathbb{T}$ that $\tilde\eta$ maps to y_2. This set is closed, and it does not meet the closure $\overline{I}$ of I, because $\tilde\eta(\overline{I}) = \overline{\mathcal{U}}$ lies in $\{|y| \geq \tilde C\}$. Since the Poisson integral of the indicator function of I tends to 0 at the points of $\mathbb{T} \setminus \overline{I}$, we get $\varpi(y) \to 0$. On the circle $\{|y| = \tilde C\}$ we have $\varpi \leq 1$. The point $\tilde C$, where the slit $[y_R, \infty)$ meets this circle, is not in the closure of $\mathcal{Y}$, since $\Phi_0(\tilde C) > R$ by Lemma 7.1(a) and (8). Let ϖ_1 be the harmonic measure of the circle $\{|y| = \tilde C\}$ in the slit disc $\{|y| < \tilde C\} \setminus [y_R, \tilde C)$, that is, the bounded harmonic function there with boundary values 1 on the circle and 0 on the slit $[y_R, \tilde C)$. By the maximum principle, $\varpi \leq \varpi_1$ on $\mathcal{Y}_0$.

It remains to bound $\varpi_1(0)$. The map $z = \sqrt{y - y_R}$, with values in the upper half-plane, sends 0 to $z_* := i\sqrt{y_R}$, the slit to the real axis, and $\{|y| \geq \tilde C\}$ into $\{|z| \geq r_{\tilde C}\}$, where $r_{\tilde C} := \sqrt{\tilde C - y_R}$. So the image of this slit disc contains the half disc $\{|z| < r_{\tilde C},\ \operatorname{Im} z > 0\}$. By (7) and (8), $\tilde C \geq 10\sqrt{R}\,\Xi > 2\Xi > 2y_R$, so $|z_*| = \sqrt{y_R} < r_{\tilde C}$ and z_* lies in the half disc. Hence $\varpi_1(0)$ is at most the harmonic measure at z_* of the semicircle $\{|z| = r_{\tilde C},\ \operatorname{Im} z > 0\}$ in this half disc. The

map $z \mapsto (r_{\tilde C}+z)/(r_{\tilde C}-z)$ sends this half disc onto the open first quadrant, the semicircle onto the positive imaginary axis, and the diameter onto the positive real axis; in the quadrant the harmonic measure of the positive imaginary axis is $2/\pi$ times the argument, and the image of z_* has argument $2\arctan(\sqrt{y_R}/r_{\tilde C})$. Hence

$$\frac{|I|}{2\pi} = \varpi(0) \;\le\; \varpi_1(0) \;\le\; \frac{4}{\pi}\arctan\frac{\sqrt{y_R}}{r_{\tilde C}} \;\le\; \frac{4}{\pi}\sqrt{\frac{y_R}{\tilde C - y_R}}.$$

(b) The map $\tilde\eta$, continuous on $\overline{\mathbb{D}}$, maps $\mathbb{T}$ into $\partial\mathcal{Y}$. Let $t \in \mathbb{T}$. If $t^n \notin I$, then $\tilde\eta(t^n) \notin \mathcal{U}$, because by Lemma 7.4(c) the only preimage on $\mathbb{T}$ of a point of $\mathcal{U}$ lies in I; so $|\Phi_0(\tilde\eta(t^n))| \ge R-1$ by Lemma 7.4(a), since $\tilde\eta(t^n) \in \partial\mathcal{Y}$. If $t^n \in I$, then $\tilde\eta(t^n) \in \mathcal{U}$, and $|\Phi_0| \ge n/(n+1) - 2\varepsilon'$ on $\mathcal{U}$ by Lemma 7.2(b) and conjugation. By (9) this gives the two bounds for $|\tilde F(t)|$. Since I is an open arc, symmetric under conjugation and containing -1, it is centred at -1; so $\{t \in \mathbb{T} : t^n \in I\}$ consists of n open arcs of length $|I|/n$, centred at the n-th roots of -1. □

Proposition 7.6. *Let $R \ge 3$, $\varepsilon \in (0,1)$, and let $n \ge 8$ be an integer with*

$$\bigl(8 + 8\log(nR)\bigr)^{1/n} \le 1+\varepsilon, \qquad \frac{n}{n+1} \ge 1 - \frac{\varepsilon}{2}.$$

Let P be the set of n-th roots of -1. For every $\theta > 0$ there are a function $\tilde F$, holomorphic on $\mathbb{D}$ with $\tilde F(0) = 0$ and $\tilde F' \ne 0$ on $\mathbb{D}$, and a number $r_0 \in (0,1)$, such that $0 < \tilde F'(0) \le 1+\varepsilon$ and, for every t with $r_0 < |t| < 1$,

$$|\tilde F(t)| \ge R-2, \qquad \text{or else both} \quad |\tilde F(t)| \ge 1-\varepsilon \quad \text{and} \quad \operatorname{dist}(t,P) < \theta.$$

Proof. Let $\varepsilon' := \varepsilon/8$, so that $\varepsilon' \in (0, 1/8)$. By Lemma 7.5(a) we can take $\tilde C$ so large that it satisfies (6) and (7), and that $|I|/n < \theta$. Let $\tilde F$ be the function of Lemma 7.3(c). By Lemma 7.3(c) and (d) and the hypothesis on n, it is holomorphic on $\mathbb{D}$, $\tilde F(0) = 0$, $\tilde F' \ne 0$ on $\mathbb{D}$, and $0 < \tilde F'(0) \le (8+8\log(nR))^{1/n} \le 1+\varepsilon$. By Lemma 7.5(b), on $\mathbb{T}$ we have $|\tilde F| \ge R-1$ off n open arcs, and $|\tilde F| \ge n/(n+1) - 2\varepsilon' \ge 1 - \varepsilon/2 - \varepsilon/4$ on them. Each of these arcs has length less than θ, so each of its points is within distance $\theta/2$ of its centre, which is a point of P. Since $\tilde F$ is uniformly continuous on $\overline{\mathbb{D}}$, we can choose $r_0 \in (0,1)$ such that $|\tilde F(t) - \tilde F(t/|t|)| < \varepsilon/4$ and $|t - t/|t|| < \theta/2$ for $r_0 < |t| < 1$. Let $r_0 < |t| < 1$. If $t/|t|$ lies off the n arcs, then $|\tilde F(t)| > R - 1 - \varepsilon/4 \ge R-2$. Otherwise $|\tilde F(t)| > 1-\varepsilon$ and $\operatorname{dist}(t,P) < \theta/2 + \theta/2 = \theta$. □

Proof of Theorem 1.2 from Proposition 7.6. Let g be as in Theorem 1.2, let $B := \sup_{\mathbb{D}} |g|$, so that $B \ge |g(1)|$, and let $R := 2 + 2B/|g(1)| \ge 4$. Let $\varepsilon \in (0,1)$. Choose an integer $n \ge 8$ with $(8+8\log(nR))^{1/n} \le 1+\varepsilon$ and $n/(n+1) \ge 1-\varepsilon/2$; this is possible because both left-hand sides tend to 1 as $n \to \infty$ with R fixed. Let P be the set of n-th roots of -1; it does not contain 1. Proposition 3.2, with this ε and this P, gives a univalent function W on $\mathbb{D}$ with $W(0) = 0$, $|W'(0)| \ge 1-\varepsilon$ and $|W| \le 2B$ on $\mathbb{D}$, and a number $\theta_0 > 0$ such that $|W(t)| < (1+\varepsilon)|g(1)|$ for every $t \in \mathbb{D}$ with $\operatorname{dist}(t,P) < \theta_0$. Proposition 7.6, with these R, ε and n and with $\theta := \theta_0$, gives $\tilde F$ and r_0. Fix $\rho \in (r_0, 1)$ and let $|t| = \rho$.

- If $\operatorname{dist}(t,P) < \theta_0$, then $|W(t)| < (1+\varepsilon)|g(1)|$, and $|\tilde F(t)| \ge \min(R-2, 1-\varepsilon) = 1-\varepsilon$, as $R-2 \ge 2$. So $|\tilde F(t)|/|W(t)| > (1-\varepsilon)/((1+\varepsilon)|g(1)|)$.
- If $\operatorname{dist}(t,P) \ge \theta_0$, then the second alternative of Proposition 7.6 is excluded, so $|\tilde F(t)| \ge R-2 = 2B/|g(1)|$, while $|W(t)| \le 2B$. So $|\tilde F(t)|/|W(t)| \ge 1/|g(1)| > (1-\varepsilon)/((1+\varepsilon)|g(1)|)$.

So Lemma 2.3 applies to $\tilde F$ and W with $s := (1-\varepsilon)|W'(0)|/((1+\varepsilon)|g(1)|\tilde F'(0))$: for all t with $|t| = \rho$ we have $|\tilde F(t)|\,|W'(0)| > s\,\tilde F'(0)\,|W(t)|$. Using $|W'(0)| \ge 1-\varepsilon$ and $\tilde F'(0) \le 1+\varepsilon$,

$$K^* \;>\; s \;\ge\; \Bigl(\frac{1-\varepsilon}{1+\varepsilon}\Bigr)^2 \frac{1}{|g(1)|}.$$

Letting $\varepsilon \to 0$ gives $K^* \ge 1/|g(1)|$. □

The choices are made in this order: g (and so B and R), ε, n, P, then W and θ_0, then $\theta := \theta_0$, then $\tilde{C}$ inside the proof of Proposition 7.6, and finally r_0 and ρ. None of them depends on a later one.

8. A third proof: a Riemann map with hairs

In this section we prove, for every bounded Jordan domain U with $0 \in U$, the inequality $K^* \geq |G'(0)|/\operatorname{dist}(0, \partial U)$, where G is a conformal map of $\mathbb{D}$ onto U with $G(0) = 0$ (Theorem 8.10); a suitable choice of U then gives Theorem 1.1 again. This inequality is Corollary 1.3, a consequence of Theorem 1.2. What is new here is the proof, which uses neither Theorem 1.2 nor Propositions 3.2 and 4.1. The function F is built from the entire function E_0 of Section 4 after an n-th root change of variable onto a domain Y_L, and W is the Riemann map of U with n thin hairs attached. We reuse Proposition 2.1 through Lemma 2.3, and the injectivity argument of Step 2 in the proof of Proposition 4.1.

Notation in this section. Throughout, $n \geq 8$ is an integer and $b_1, \dots, b_n$ are the n-th roots of -1. As in Section 4, E_0 is the entire function with $E_0(0) = 0$ and $E_0'(\xi) = \exp(\xi^n)$. U is a bounded Jordan domain with $0 \in U$, $G : \mathbb{D} \to U$ is conformal with $G(0) = 0$, and $d_U := \operatorname{dist}(0, \partial U)$; k and δ are as in Section 5. The letter s has the meaning of Lemmas 2.3 and 8.1. The functions Φ_0 and Φ_1 below, in the plane of the variable $y = \xi^n$, replace Φ of Section 4, and the radius L is a large parameter, chosen last by a condition on a phase. The letters n and R have the roles they have in Sections 4 and 5, with values chosen anew here, and λ is the angular half-width of the hairs. We write $o(1)$ for a quantity that tends to 0 as $L \to \infty$, with n and R fixed, uniformly in all other variables; from the end of the proof of Lemma 8.5 on, L runs along the sequence chosen there.

Reduction to a pair of functions continuous on the closed disc. The following variant of Lemma 2.3 asks for the inequality on the unit circle itself.

Lemma 8.1. *Let F and W be continuous on $\overline{\mathbb{D}}$ and holomorphic on $\mathbb{D}$, with $F(0) = W(0) = 0$ and $F' \neq 0$ on $\mathbb{D}$, and suppose that W is a homeomorphism of $\overline{\mathbb{D}}$ onto the closure of a Jordan domain. If $s > 0$ and*

$$|F(t)|\,|W'(0)| \;\geq\; s\,|F'(0)|\,|W(t)| \qquad \text{for all } t \in \mathbb{T},$$

then $K^* \geq s$.

Proof. The map W is holomorphic and injective on $\mathbb{D}$, so it is univalent there, with $W(0) = 0$; and $F'(0) \neq 0$. For $0 < \rho < 1$ the set $\Omega_\rho := W(\rho\mathbb{D})$ is an analytic Jordan domain, and $f := (W'(0)/F'(0))\, F \circ W^{-1}$ is holomorphic near $\overline{\Omega_\rho}$, with $f' \neq 0$, $f(0) = 0$ and $f'(0) = 1$, as in the proof of Lemma 2.3. At $z = W(t) \in \partial\Omega_\rho$, $|t| = \rho$, we have

$$\Big|\frac{f(z)}{z}\Big| = \Big|\frac{W'(0)}{F'(0)}\Big|\,\frac{|F(t)|}{|W(t)|}.$$

Since W is injective on $\overline{\mathbb{D}}$ and $W(0) = 0$, W has no zero on $\overline{\mathbb{D}} \setminus \{0\}$. So the right-hand side is a continuous function of t on the closed annulus $\{\frac{1}{2} \leq |t| \leq 1\}$, hence uniformly continuous there, and its values on the circle $|t| = \rho$ tend uniformly to its values on $\mathbb{T}$ as $\rho \to 1$. On $\mathbb{T}$ these values are at least s, by hypothesis. Let $0 < s' < s$. Then for ρ close to 1 the right-hand side exceeds s' on $|t| = \rho$, and Lemma 2.3 gives $K^* > s'$. Since $s' < s$ was arbitrary, $K^* \geq s$. □

The function E_0 in the variable $y = \xi^n$. The function E_0 has $E_0' \neq 0$ everywhere, and $E_0(\omega\xi) = \omega E_0(\xi)$ whenever $\omega^n = 1$. On $\mathbb{C} \setminus (-\infty, 0]$, with principal branches and the substitution $\xi = u^{1/n}$, put, as in Section 7,

$$\Phi_0(y) := E_0(y^{1/n}) = \frac{1}{n}\int_0^y e^u\, u^{1/n-1}\, du.$$

The function Φ_0 jumps across the negative axis, but since $\Phi_0(\bar y) = \overline{\Phi_0(y)}$, its modulus $|\Phi_0|$ extends continuously to all of $\mathbb{C}$. On the negative axis we use the boundary values of Φ_0 from above. Let

$$v_+ := e^{i\pi/n}\int_0^\infty \exp(-u^n)\,du, \qquad \text{so that} \quad |v_+| = \int_0^\infty \exp(-u^n)\,du \le 1.$$

Here $|v_+| \le 1$ because, with $s = u^n$ and an integration by parts, the integral is $\int_0^\infty s^{1/n}e^{-s}\,ds$, which is at most $\bigl(\int_0^\infty s\,e^{-s}\,ds\bigr)^{1/n} = 1$ by Jensen's inequality.

Lemma 8.2. (a) *As $y \to -\infty$ along the negative axis, $\Phi_0(y) \to v_+$. For $\operatorname{Im} y \ge 0$ the function $\Phi_1 := \Phi_0 - v_+$ is given by*

$$\Phi_1(y) = \frac{1}{n}\int_{-\infty}^{y} e^u\, u^{1/n-1}\,du,$$

the integral being taken along the horizontal half-line that ends at y; and

$$\Phi_1(y) = \frac{1}{n}\,e^y\,y^{1/n-1}\Bigl(1 + O\bigl(\tfrac{1}{|y|}\bigr)\Bigr) \qquad \text{for } \operatorname{Im} y \ge \tfrac{|y|}{2},$$

where the term $O(1/|y|)$ has modulus at most $4/|y|$. For $\operatorname{Im} y \le 0$ the same statements hold after complex conjugation.

(b) $|\Phi_0(y)| \le 2$ *for every y with $\operatorname{Re} y \le 1$ and $\operatorname{Im} y \ge 0$.*

Proof. (a) For $y < 0$ the boundary value from above is $\Phi_0(y) = E_0(|y|^{1/n}e^{i\pi/n})$, a value of E_0 on the ray $\arg\xi = \pi/n$, on which $\xi^n = -|\xi|^n$. So

$$\Phi_0(y) = e^{i\pi/n}\int_0^{|y|^{1/n}} \exp(-u^n)\,du \;\longrightarrow\; v_+ \qquad (y \to -\infty).$$

Next let $x < 0$ and $\tau > 0$. The difference $\Phi_0(x+i\tau) - \Phi_0(x)$, with the boundary value from above at x, is $\frac{1}{n}$ times the integral of $e^u u^{1/n-1}$ along the vertical segment from x to $x+i\tau$. On this segment $|e^u| = e^x$ and $|u| \ge |x|$, so the difference has modulus at most $\tau\,e^x|x|^{1/n-1}/n$, which tends to 0 as $x \to -\infty$. With the limit on the negative axis, this gives $\Phi_0(x+i\tau) \to v_+$ as $x \to -\infty$, for every $\tau > 0$. For $\tau = \operatorname{Im} y > 0$ the integral along the horizontal half-line $\{\operatorname{Im} u = \tau,\ \operatorname{Re} u \le \operatorname{Re} y\}$ converges absolutely, since $|e^u u^{1/n-1}| \le e^{\operatorname{Re} u}\tau^{1/n-1}$ there; for $\operatorname{Im} y = 0$ we use boundary values from above, and the integrand is integrable at $u = 0$. For $\operatorname{Im} y \ge 0$ this gives the stated integral for $\Phi_1(y) = \Phi_0(y) - v_+$, along the horizontal half-line ending at y. Integrating by parts, since $\frac{d}{du}\bigl(e^u u^{1/n-1}\bigr) = e^u u^{1/n-1} - (1-\frac{1}{n})e^u u^{1/n-2}$ and $e^u u^{1/n-1} \to 0$ at the left end,

$$\Phi_1(y) = \frac{1}{n}\,e^y\,y^{1/n-1} + \frac{1}{n}\Bigl(1-\frac{1}{n}\Bigr)\int_{-\infty}^{y} e^u\,u^{1/n-2}\,du.$$

If $\operatorname{Im} y \ge |y|/2$, every point u of the half-line has $|u| \ge \operatorname{Im} u = \operatorname{Im} y \ge |y|/2$, so $|u|^{1/n-2} \le (|y|/2)^{1/n-2} \le 4|y|^{1/n-2}$; and the integral of $|e^u|$ along the half-line is $e^{\operatorname{Re} y}$. So the last integral has modulus at most $4|y|^{1/n-2}e^{\operatorname{Re} y}$, which is $4/|y|$ times $|e^y\,y^{1/n-1}|$. Since the last integral carries the factor $\frac{1}{n}(1-\frac{1}{n})$ and the main term the factor $\frac{1}{n}$, this is the stated formula, with an O-term of modulus at most $(1-\frac{1}{n})\frac{4}{|y|} \le \frac{4}{|y|}$. For $\operatorname{Im} y \le 0$ we use $\Phi_0(\bar y) = \overline{\Phi_0(y)}$.

(b) For $|y| \le 2$ we use the series

$$\Phi_0(y) = \sum_{\nu\ge 0} \frac{y^{\nu+1/n}}{\nu!\,(n\nu+1)},$$

obtained by integrating $\exp(\xi^n) = \sum_\nu \xi^{n\nu}/\nu!$ term by term. It gives

$$|\Phi_0(y)| \le 2^{1/n}\Bigl(1 + \sum_{\nu\ge 1}\frac{2^\nu}{\nu!\,(n+1)}\Bigr) \le 2^{1/8}\Bigl(1 + \frac{e^2-1}{9}\Bigr) < 2.$$

For $|y| > 2$ the horizontal half-line ending at y has $|u| \geq 1$ at all its points: either $\operatorname{Im} y \geq 1$, and then $|u| \geq \operatorname{Im} u \geq 1$; or $\operatorname{Im} y < 1$, and then $(\operatorname{Re} y)^2 > 3$ and $\operatorname{Re} y \leq 1$ give $\operatorname{Re} y < -\sqrt{3}$, so $|u| \geq |\operatorname{Re} u| > \sqrt{3}$. Hence, by (a), $|\Phi_1(y)| \leq \frac{1}{n} e^{\operatorname{Re} y} \leq e/n$, and $|\Phi_0(y)| \leq |v_+| + e/n \leq 1 + e/8 < 2$. □

The domain Y_L and the function F_L. Fix $R > 3$, let $x_0 := \log(2nR)$, and let

$$Y := \{y \in \mathbb{C} : \ \operatorname{Re} y < x_0 + \log(1 + |y|)\}.$$

On each ray from 0, the function $\operatorname{Re} y - \log(1 + |y|) - x_0$ is a convex function of $|y|$ (a linear function minus a concave one), and it is negative at $y = 0$. So on each ray it is negative on an initial segment, possibly the whole ray, and positive beyond it. Hence Y is star-shaped about 0; it is also symmetric in the real axis. For a point $\hat{y}$ with $|\hat{y}| = 1$ let $\operatorname{rad}_Y(\hat{y}) \in (0, \infty]$ be the length of that initial segment on the ray through $\hat{y}$. Let x_+ be the boundary point of Y on the positive axis, so that

$$x_+ = x_0 + \log(1 + x_+).$$

For $L > x_+$ let $Y_L := Y \cap \{|y| < L\}$. It is a Jordan domain, since its radial function $\min(L, \operatorname{rad}_Y)$ is continuous. Indeed, let $|\hat{y}| = 1$. If $\operatorname{Re} \hat{y} > 0$, the function $\ell \mapsto \ell \operatorname{Re} \hat{y} - \log(1 + \ell) - x_0$ on $[0, \infty)$ is convex, equals $-x_0$ at $\ell = 0$ and tends to ∞; so it has exactly one zero, $\operatorname{rad}_Y(\hat{y})$, and by convexity its derivative there is at least $x_0 / \operatorname{rad}_Y(\hat{y}) > 0$. By the implicit function theorem, rad_Y is continuous on $\{\operatorname{Re} \hat{y} > 0\}$. Moreover $\operatorname{rad}_Y(\hat{y}) \geq x_0 / \operatorname{Re} \hat{y}$, which tends to ∞ as $\operatorname{Re} \hat{y}$ decreases to 0, and $\operatorname{rad}_Y = \infty$ where $\operatorname{Re} \hat{y} \leq 0$. So $\min(L, \operatorname{rad}_Y)$ is continuous. The boundary of Y_L consists of two arcs: the *exit curve* $\{\operatorname{Re} y = x_0 + \log(1 + |y|), \ |y| \leq L\}$ and the *outer arc* $\{|y| = L, \ \operatorname{Re} y \leq x_0 + \log(1 + L)\}$. Both are symmetric in the real axis; the outer arc contains $-L$, and the two arcs meet at the two *corners*, the points with $|y| = L$ and $\operatorname{Re} y = x_0 + \log(1 + L)$.

Let $\eta_L : \mathbb{D} \to Y_L$ be the conformal map with $\eta_L(0) = 0$ and $\eta_L'(0) > 0$. By the symmetry of Y_L and the uniqueness of this map, η_L is real on $(-1, 1)$. By Carathéodory's theorem [7] it extends to a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{Y_L}$; it maps $[-1, 1]$ increasingly onto $\overline{Y_L} \cap \mathbb{R} = [-L, x_+]$, so $\eta_L(-1) = -L$. Let also $\eta_Y : \mathbb{D} \to Y$ be the conformal map with $\eta_Y(0) = 0$ and $\eta_Y'(0) > 0$. Since $Y_L \subset Y$, Schwarz's lemma applied to $\eta_Y^{-1} \circ \eta_L$ gives $\eta_L'(0) \leq \eta_Y'(0)$; and since $x_+ \notin Y$, Koebe's theorem gives $\eta_Y'(0) \leq 4x_+$. So

$$0 < \eta_L'(0) \leq 4x_+. \tag{10}$$

The quotient $\eta_L(t)/t$ is continuous and has no zero on $\overline{\mathbb{D}}$ (its value at 0 is $\eta_L'(0)$), so we can define the n-th root transform

$$\xi_L(t) := t \left(\frac{\eta_L(t^n)}{t^n} \right)^{1/n} \qquad (t \in \overline{\mathbb{D}}),$$

with the branch that is positive at $t = 0$, and put

$$F_L := E_0 \circ \xi_L.$$

Lemma 8.3. *The function ξ_L is continuous on $\overline{\mathbb{D}}$, holomorphic and injective on $\mathbb{D}$, and satisfies $\xi_L(t)^n = \eta_L(t^n)$ and $\xi_L(\omega t) = \omega\, \xi_L(t)$ for $\omega^n = 1$. The function F_L is continuous on $\overline{\mathbb{D}}$ and holomorphic on $\mathbb{D}$, and*

$$F_L' = \exp(\xi_L^n)\, \xi_L' \neq 0 \quad \text{on } \mathbb{D}, \qquad F_L(0) = 0, \qquad 0 < F_L'(0) = \eta_L'(0)^{1/n} \leq (4x_+)^{1/n},$$

$$|F_L(t)| = |\Phi_0(\eta_L(t^n))| \qquad (t \in \overline{\mathbb{D}}). \tag{11}$$

Proof. The two identities for ξ_L follow from its definition. Holomorphy and injectivity on $\mathbb{D}$ follow from them and from the injectivity of η_L, exactly as for ξ in Step 2 of the proof of Proposition 4.1; in particular $\xi_L' \neq 0$ on $\mathbb{D}$. Continuity on $\overline{\mathbb{D}}$ holds because η_L is continuous there. Hence F_L is continuous on $\overline{\mathbb{D}}$ and holomorphic on $\mathbb{D}$, $F_L(0) = E_0(0) = 0$, $F_L' = E_0'(\xi_L)\, \xi_L' = \exp(\xi_L^n)\, \xi_L' \neq 0$, and $F_L'(0) = \xi_L'(0) = \eta_L'(0)^{1/n}$, which is at most $(4x_+)^{1/n}$ by (10).

For (11), let first $t \in \overline{\mathbb{D}}$, $t \neq 0$, with $|\arg t| < \pi/n$. Then t^n lies off $[-1,0]$, so $\eta_L(t^n)$ lies off $[-L,0]$, because η_L is injective on $\overline{\mathbb{D}}$ and maps $[-1,0]$ onto $[-L,0]$; and $|\eta_L(t^n)| \le L$. So $\eta_L(t^n)$ lies off $(-\infty,0]$, and the principal n-th root of $\eta_L(t^n)$ is continuous in this sector (and holomorphic in its interior). It equals $\xi_L(t)$: both are n-th roots of $\eta_L(t^n)$, both are continuous on the sector, and both are positive for $0 < t \le 1$. Hence $F_L(t) = E_0\big(\eta_L(t^n)^{1/n}\big) = \Phi_0(\eta_L(t^n))$ there. Further $\xi_L(\omega t) = \omega\,\xi_L(t)$ and $E_0(\omega\xi) = \omega E_0(\xi)$ for $\omega^n = 1$, so $|F_L(\omega t)| = |F_L(t)|$, and every t reduces to this sector, or to its boundary rays, where (11) holds by continuity, since $|\Phi_0|$ is continuous on $\mathbb{C}$. □

The boundary values of $|F_L|$. By (11), for $t \in \mathbb{T}$ the number $|F_L(t)|$ is the value of $|\Phi_0|$ at the boundary point $\eta_L(t^n)$ of Y_L. We bound $|\Phi_0|$ from below on the exit curve and on the outer arc.

Lemma 8.4. *If $x_0 \ge 24$, then $|\Phi_0(y)| \ge R - 3$ for every y on the exit curve.*

Proof. By conjugation we may assume $\operatorname{Im} y \ge 0$. Let $x := \operatorname{Re} y$; then $x = x_0 + \log(1+|y|) \ge x_0$, and $|y| \ge x \ge x_0$. Let $y_1 := 1 + i \operatorname{Im} y$. Integrating $\frac{1}{n} e^u u^{1/n-1}$ along the horizontal segment from y_1 to y, and integrating by parts as in the proof of Lemma 8.2(a),

$$\Phi_0(y) = \Phi_0(y_1) + \frac{1}{n}\Big[e^u\, u^{1/n-1}\Big]_{y_1}^{y} + \frac{1}{n}\Big(1 - \frac{1}{n}\Big)\int_{y_1}^{y} e^u\, u^{1/n-2}\, du.$$

We bound the last integral. If $\operatorname{Im} y \ge |y|/2$, then $|u| \ge |y|/2$ on the segment, and the integral is at most $4|y|^{-2+1/n} e^x$. Otherwise $x \ge (\sqrt{3}/2)|y|$. We then split the segment at the point with real part $x/2$: on the right part $|u| \ge x/2 \ge (\sqrt{3}/4)|y|$, so that $|u|^{1/n-2} \le (16/3)|y|^{1/n-2}$, and on the left part $|u| \ge 1$; so the integral is at most $6|y|^{-2+1/n} e^x + e^{x/2}$.

Let

$$T_1 := \frac{1}{n}\, e^x\, |y|^{1/n-1}.$$

On the exit curve $e^x = e_0^x(1+|y|) = 2nR\,(1+|y|)$, so $T_1 = 2R\,(1+|y|)\,|y|^{1/n-1} \ge 2R$. The bracket at the upper end has modulus T_1. The term at y_1 is at most e/n in modulus, because $|y_1| \ge 1$, and $|\Phi_0(y_1)| \le 2$ by Lemma 8.2(b). The last term is at most $T_1 \cdot 6/|y|$ in the first case, and at most $T_1 \cdot 6/|y| + \frac{1}{n} e^{x/2} \le T_1\big(6/|y| + e^{-x/2}|y|\big)$ in the second. Hence

$$|\Phi_0(y)| \;\ge\; T_1\Big(1 - \frac{6}{|y|} - e^{-x/2}|y|\Big) - 2 - \frac{e}{n},$$

where the term $e^{-x/2}|y|$ occurs only in the second case. There $|y| \le 2x/\sqrt{3}$ and $x \ge x_0 \ge 24$, so $e^{-x/2}|y| \le 28\, e^{-12}$. Also $6/|y| \le 6/24$. So the factor in parentheses is at least $\frac{1}{2}$, and $|\Phi_0(y)| \ge R - 2 - e/8 \ge R - 3$. □

Since $x_0 = \log(2nR)$, the condition $x_0 \ge 24$ holds for all large n when R is fixed.

Lemma 8.5. *Fix $n \ge 8$ and $R > 3$. There is a sequence of radii $L \to \infty$ along which*

$$|\Phi_0(y)| \;\ge\; |v_+|\big(1 - o(1)\big) \qquad \textit{for every } y \textit{ on the outer arc of } Y_L.$$

Proof. By conjugation it suffices to treat the part of the outer arc in the closed upper half-plane. On this part put $x := \operatorname{Re} y$ and $\Phi_* := \Phi_1/v_+$, so that $\Phi_0 = v_+(1+\Phi_*)$ and $|\Phi_0| = |v_+|\,|1+\Phi_*|$. Where $\operatorname{Im} y \ge L/2$, Lemma 8.2(a) gives

$$|\Phi_*| = \frac{e^x}{n\,|v_+|\,L^{1-1/n}}\Big(1 + O\big(\tfrac{1}{L}\big)\Big), \qquad \arg\Phi_* = \operatorname{Im} y + \Big(\frac{1}{n} - 1\Big)\arg y - \frac{\pi}{n} + O\big(\tfrac{1}{L}\big) \pmod{2\pi}.$$

Where $\operatorname{Im} y < L/2$, we have $|x| > (\sqrt{3}/2)L$, and $x \le x_0 + \log(1+L) < L/2$ for large L, so $x < -L/2$; there the horizontal half-line ending at y has $|u| \ge L/2 \ge 1$, so $|\Phi_1(y)| \le e^x/n$ as in the proof of

Lemma 8.2(b), and $|\Phi_*| \le e^x/(n\,|v_+|) < e^{-L/2}$. Put

$$x_1 := \log\bigl(n\,|v_+|\bigr) + \Bigl(1 - \frac{1}{n}\Bigr)\log L, \qquad y_* := x_1 + i\bigl(L^2 - x_1^2\bigr)^{1/2}.$$

Since $|y| = L$ on the outer arc, the formula above gives $|\Phi_*| = e^{x-x_1}\bigl(1 + O(\frac{1}{L})\bigr)$ where $\operatorname{Im} y \ge L/2$; in particular $|\Phi_*| = 1 + O(\frac{1}{L})$ at y_*, which lies on the outer arc (see below). Hence $|\Phi_*| \le (1 + o(1))/\log L$ where $x \le x_1 - \log\log L$, and $|\Phi_*| \ge (1 + o(1))\log L$ where $x \ge x_1 + \log\log L$, up to the exit curve. At the far end, the corner, where $x = x_e := x_0 + \log(1 + L)$, the formula gives $|\Phi_*|$ about $2R\,L^{1/n}/|v_+|$, and

$$x_e - x_1 = \log\frac{2R}{|v_+|} + \frac{\log L}{n} + o(1),$$

which exceeds $\log\log L$ for large L at fixed n. So the window $|x - x_1| \le \log\log L$ lies inside the outer arc.

On the window $|x| \le |x_1| + \log\log L = O(\log L)$, so for large L, $|\operatorname{Im} y - \operatorname{Im} y_*| \le 2(\log\log L)(|x_1| + \log\log L)/L$ and $|\arg y - \arg y_*| \le 2(\log\log L)/L$; both are $o(1)$. Let

$$\theta_*(L) := \operatorname{Im} y_* + \Bigl(\frac{1}{n} - 1\Bigr)\arg y_* - \frac{\pi}{n},$$

the main term of $\arg\Phi_*$ at y_*. By the formula for $\arg\Phi_*$, $\arg\Phi_* = \theta_*(L) + o(1) \pmod{2\pi}$ on the window. The function θ_* is continuous in L, and $\theta_*'(L) \to 1$ as $L \to \infty$, since $x_1 = O(\log L)$ and $\frac{d}{dL}\arg y_* = O(\log L/L^2)$. By the intermediate value theorem there is a sequence $L_m \to \infty$ with $\theta_*(L_m) = 2\pi m$ for all large m. We take L along this sequence. Then $\arg\Phi_* = o(1) \pmod{2\pi}$ on the window, so $\operatorname{Re}\Phi_* \ge 0$ there, for large L. Therefore $|1 + \Phi_*| \ge 1 - (1 + o(1))/\log L$ on the whole part of the outer arc that we consider: to the left of the window because $|\Phi_*| \le (1 + o(1))/\log L$ there; on the window because $\operatorname{Re}\Phi_* \ge 0$; and to the right of it because $|\Phi_*| \ge (1 + o(1))\log L \ge 2$ there. Since $1/\log L = o(1)$, we get $|\Phi_0| \ge |v_+|(1 - o(1))$ on the outer arc. The part below the real axis follows by conjugation. □

Lemma 8.6. *The set $I_L := \eta_L^{-1}(\text{outer arc})$ is an arc of $\mathbb{T}$ about -1, symmetric under conjugation, and I_L shrinks to the point -1 as $L \to \infty$.*

Proof. Since η_L is a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{Y_L}$ and the outer arc is an arc of ∂Y_L containing $-L = \eta_L(-1)$, the set I_L is an arc of $\mathbb{T}$ containing -1; it is symmetric because $\eta_L(\bar t) = \overline{\eta_L(t)}$.

Let ϖ_L be the harmonic measure of the outer arc in Y_L, that is, the bounded harmonic function on Y_L such that $\varpi_L \circ \eta_L$ is the Poisson integral, on $\mathbb{D}$, of the indicator function of I_L. Since $\eta_L(0) = 0$, we have $\varpi_L(0) = |I_L|/(2\pi)$, where $|I_L|$ is the length of I_L. Let $x_e := x_0 + \log(1 + L)$, as in the proof of Lemma 8.5, and let L be so large that $x_e \le L/2$. Since $x_0 + \log(1 + |y|) < x_e$ for $|y| < L$, the domain Y_L lies in the half-plane $\{\operatorname{Re} y < x_e\}$. Let $\mathcal{N}_L := \{y : \operatorname{Re} y = x_e,\ |\operatorname{Im} y| \ge L/2\}$, let $\varpi_{\mathcal{N}}(y)$ be the harmonic measure at y of $\mathcal{N}_L$ in this half-plane, and put

$$\varpi_L^+(y) := 2\,\varpi_{\mathcal{N}}(y) + \frac{2\,(x_e - \operatorname{Re} y)}{L} \qquad (\operatorname{Re} y < x_e).$$

The function ϖ_L^+ is harmonic, and it is ≥ 0 on $\{\operatorname{Re} y \le x_e\}$, hence on the exit curve. It is ≥ 1 on the outer arc. Where $\operatorname{Re} y \le -L/2$, the second term is at least 1. Elsewhere on the outer arc, $|\operatorname{Re} y| \le L/2$ (recall $\operatorname{Re} y \le x_e \le L/2$) and $|y| = L$, so $|\operatorname{Im} y| \ge (\sqrt{3}/2)L > L/2$. Then the foot $x_e + i\operatorname{Im} y$ of the perpendicular from y to the line $\{\operatorname{Re} u = x_e\}$ lies in $\mathcal{N}_L$, and so does the half-line of that line that runs from the foot to infinity away from the real axis. This half-line is seen from y at the angle $\pi/2$, so its harmonic measure at y is $\frac{1}{2}$, and the first term is at least $2 \cdot \frac{1}{2} = 1$. The two end points $x_e \pm iL/2$ of $\mathcal{N}_L$ are not in $\overline{Y_L}$, since there $\operatorname{Re} y = x_e > x_0 + \log(1 + |y|)$; so ϖ_L^+ is continuous on $\overline{Y_L}$. Hence $\varpi_L^+ \circ \eta_L$ is harmonic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, so it is the Poisson integral of its boundary values. These are at least the indicator function of I_L, except at the two

end points of I_L, so $\varpi_L \le \varpi_L^+$ on Y_L. The segment of the line $\{\operatorname{Re} u = x_e\}$ between the two end points is seen from 0 at the angle $2\arctan(L/(2x_e))$, so $\varpi_{\mathcal{N}}(0) = 1 - \frac{2}{\pi}\arctan\frac{L}{2x_e} = \frac{2}{\pi}\arctan\frac{2x_e}{L}$. Hence

$$\frac{|I_L|}{2\pi} = \varpi_L(0) \;\le\; \varpi_L^+(0) = \frac{4}{\pi}\arctan\frac{2x_e}{L} + \frac{2x_e}{L} \;\le\; \Big(\frac{8}{\pi}+2\Big)\frac{x_e}{L} \;\le\; \frac{5x_e}{L},$$

which tends to 0 as $L \to \infty$, since $x_e = x_0 + \log(1+L)$. As I_L is an arc about -1, it shrinks to the point -1. □

Proposition 8.7. *Let $n \ge 8$ and $R > 3$, and suppose that $x_0 = \log(2nR) \ge 24$. Let $L \to \infty$ along the sequence of Lemma 8.5. Then F_L is continuous on $\overline{\mathbb{D}}$ and holomorphic on $\mathbb{D}$, $F_L(0) = 0$, $F_L' \ne 0$ on $\mathbb{D}$, $0 < F_L'(0) \le (4x_+)^{1/n}$, and for $t \in \mathbb{T}$*

$$|F_L(t)| \ge |v_+|\big(1 - o(1)\big) \quad \text{if } t^n \in I_L, \qquad |F_L(t)| \ge R - 3 \quad \text{otherwise.}$$

The set $\{t \in \mathbb{T} : t^n \in I_L\}$ consists of n arcs, one about each point b_j, and it shrinks to $\{b_1, \dots, b_n\}$ as $L \to \infty$.

Proof. The first assertions are Lemma 8.3. Let $t \in \mathbb{T}$. By (11), $|F_L(t)| = |\Phi_0(\eta_L(t^n))|$, and $\eta_L(t^n)$ lies on the outer arc if $t^n \in I_L$, and on the exit curve otherwise. Lemmas 8.5 and 8.4 give the two bounds. The set $\{t \in \mathbb{T} : t^n \in I_L\}$ is the preimage of the arc I_L about -1 under $t \mapsto t^n$, so it consists of n arcs about the n-th roots of -1, and it shrinks to them by Lemma 8.6. □

The Riemann map with hairs. Let U, G and d_U be as in the notation above. By Carathéodory's theorem [7], G extends to a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{U}$, so G^{-1} is defined on ∂U. A *crosscut* of $\mathbb{D}$ is an open Jordan arc in $\mathbb{D}$ whose two endpoints lie on $\mathbb{T}$.

Let $d_+ > d_U$. We construct, for each n, a domain $\hat{U}$, namely U with n thin hairs attached, whose conformal map W satisfies $|W(b_j)| < d_+$ for all j (Lemma 8.9 below). We first make the choices that do not depend on n; they end with the choice of $z_{\max}$.

The exterior domain. Let $U^* := (\mathbb{C} \cup \{\infty\}) \setminus \overline{U}$, the exterior of U on the sphere. Pick $z_1 \in U^*$ with $|z_1| < d_+$; it exists, since there are points of U^* near a point of ∂U of modulus $d_U < d_+$. Next we take a ray Υ from a point $z_\infty \in \partial U$ to ∞, lying in U^* apart from z_∞, in a direction other than that of z_1, and such that $G^{-1}(z_\infty)$ is not a root of unity. Such a ray exists: for each direction, take the last point of $\overline{U}$ on the ray from 0 in that direction; the part of that ray beyond this point lies in U^*. As $0 \in U$, these last points are distinct for distinct directions, so there are uncountably many of them, and only countably many points of ∂U are images under G of roots of unity. Let $\Psi : \mathbb{D} \to U^* \setminus (\Upsilon \cup \{\infty\})$ be conformal with $\Psi(0) = z_1$. It extends continuously to $\overline{\mathbb{D}}$, with values in the sphere, and it is injective on an open arc $\mathcal{J}$ of $\mathbb{T}$, which it maps onto $\partial U \setminus \{z_\infty\}$. To see this, note that U^* is a Jordan domain of the sphere, bounded by ∂U. Let $\Psi_0 : \mathbb{D} \to U^*$ be conformal with $\Psi_0(0) = z_1$; by Carathéodory's theorem [7] it extends to a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{U^*}$. Then $\Psi_0^{-1}(\Upsilon \cup \{\infty\})$ is a Jordan arc from the point $\Psi_0^{-1}(z_\infty)$ of $\mathbb{T}$ to the interior point $\Psi_0^{-1}(\infty)$, and $\Psi = \Psi_0 \circ \Psi_1$, where $\Psi_1 := \Psi_0^{-1} \circ \Psi$ maps $\mathbb{D}$ conformally onto the slit disc $\mathbb{D} \setminus \Psi_0^{-1}(\Upsilon \cup \{\infty\})$, with $\Psi_1(0) = 0$. The boundary of the slit disc is locally connected, so Ψ_1 extends continuously to $\overline{\mathbb{D}}$, by the continuity theorem [21, Chapter 2]. A point ζ_* of $\mathbb{T}$ other than $\Psi_0^{-1}(z_\infty)$ has exactly one preimage under Ψ_1. Indeed, suppose that $\Psi_1(\zeta_1) = \Psi_1(\zeta_2) = \zeta_*$ with $\zeta_1 \ne \zeta_2$. Then the images under Ψ_1 of the radii to ζ_1 and ζ_2, together with ζ_*, form a Jordan curve that lies in the slit disc apart from ζ_*. The connected set $(\mathbb{T} \setminus \{\zeta_*\}) \cup \Psi_0^{-1}(\Upsilon \cup \{\infty\})$ misses this curve, so one component of the complement of the curve lies in the slit disc. That component is the image of one of the two sectors between the radii, so $\Psi_1 = \zeta_*$ on an arc of $\mathbb{T}$, which is impossible by the reflection principle. These preimages form an open arc $\mathcal{J}$ of $\mathbb{T}$ (the set of these preimages is open in $\mathbb{T}$, and Ψ_1 maps each of its components homeomorphically onto an open subarc of $\mathbb{T} \setminus \{\Psi_0^{-1}(z_\infty)\}$; these subarcs are disjoint and cover that arc, so there is only one component), and Ψ_1 maps the complementary closed arc onto the two sides of the slit. So Ψ extends continuously to $\overline{\mathbb{D}}$, with values in the sphere,

and it is injective on $\mathcal{J}$, which it maps onto $\Psi_0(\mathbb{T}\setminus\{\Psi_0^{-1}(z_\infty)\}) = \partial U\setminus\{z_\infty\}$. Take $\varepsilon_1\in(0,1)$ with $\Psi(\varepsilon_1\overline{\mathbb{D}})\subset\{|z|<d_+\}$, which is possible since $|\Psi(0)|<d_+$, and let $z_{\max}$ be the supremum of $|z|$ over $\overline{U}$ and over the set $\Psi\big(\{\zeta:\varepsilon_1\le|\zeta|\le 1,\ \zeta/|\zeta|\in\overline{\mathcal{J}}\}\big)$. It is finite, since the closure of this set of ζ does not contain the point $\Psi^{-1}(\infty)$ of $\mathbb{T}$. The number $z_{\max}$ depends only on U and d_+ (through the choices just made), and not on n.

The domains with hairs. Let $\vartheta=(\vartheta_1,\dots,\vartheta_n)$ be angles with $e^{i\vartheta_j}\in\mathcal{J}$, and let $\lambda>0$ be small. Put

$$\Sigma_j := \{\zeta:\ \varepsilon_1<|\zeta|<1,\ |\arg(\zeta e^{-i\vartheta_j})|<\lambda\},\qquad \mathcal{J}_j := \{\zeta\in\mathbb{T}:\ |\arg(\zeta e^{-i\vartheta_j})|<\lambda\},$$

and suppose that the sectors Σ_j are disjoint and that $\overline{\mathcal{J}_j}\subset\mathcal{J}$. Let

$$\hat U=\hat U(\vartheta,\lambda) := U\cup\bigcup_j\Psi(\Sigma_j)\cup\bigcup_j\Psi(\mathcal{J}_j).$$

This is a Jordan domain: U with n hairs $\Psi(\Sigma_j)$ attached along the open *mouths* $\mathcal{O}_j:=\Psi(\mathcal{J}_j)\subset\partial U$. Indeed, Ψ is continuous and injective on $\mathbb{D}\cup\mathcal{J}$, so it is a homeomorphism of each compact set $\overline{\Sigma_j}\subset\mathbb{D}\cup\mathcal{J}$ onto its image. So the outer boundary of each hair, $\Psi(\partial\Sigma_j\setminus\mathcal{J}_j)$, is a Jordan arc. It joins the two end points of the mouth, lies in $U^*\setminus(\Upsilon\cup\{\infty\})$ apart from those two points, and misses the arcs of the other hairs. Replacing each mouth of the Jordan curve ∂U by this arc gives a Jordan curve, and that curve is $\partial\hat U$. By the choice of $z_{\max}$, $U\subset\hat U\subset\{|z|\le z_{\max}\}$. The *tip* $\hat z_j:=\Psi(\varepsilon_1 e^{i\vartheta_j})$ lies on $\partial\hat U$, and $|\hat z_j|<d_+$. Let W be the conformal map of $\mathbb{D}$ onto $\hat U$ with $W(0)=0$ and $W'(0)/G'(0)>0$, extended to $\overline{\mathbb{D}}$, and let $\tilde b_j=\tilde b_j(\vartheta,\lambda):=W^{-1}(\hat z_j)\in\mathbb{T}$. By the theorems of Radó [22] and Carathéodory [7], $\tilde b_j$ depends continuously on ϑ. Indeed, for the angle vectors near a given one there is one compact set of points ζ, with $\varepsilon_1\le|\zeta|\le 1$ and $\zeta/|\zeta|$ in a fixed compact subarc of $\mathcal{J}$, that contains the closures of all the sectors Σ_j and of all the arcs $\mathcal{J}_j$. Outside its image under Ψ, every boundary curve $\partial\hat U(\vartheta,\lambda)$ is the same arc of ∂U. Inside, the curve is the image under Ψ of a curve in this compact set, made of arcs of the circles $|\zeta|=1$ and $|\zeta|=\varepsilon_1$ and of the radial sides of the sectors, and this curve turns continuously with ϑ (the mouths move along ∂U with it). Since Ψ is uniformly continuous on the compact set, the boundary curves of $\hat U(\vartheta,\lambda)$ move uniformly with ϑ; by Radó's theorem the maps W converge uniformly on the closed disc, and the tips $\hat z_j$ move continuously; as W is injective on the closed disc, $\tilde b_j=W^{-1}(\hat z_j)$ moves continuously.

The map $\tilde\phi$. Let $\tilde\phi:=W^{-1}\circ G$. It is continuous and injective on $\overline{\mathbb{D}}$, and it maps $\mathbb{D}$ onto $\mathbb{D}$ minus the n disjoint *hulls* $\mathcal{H}_j:=W^{-1}\big(\Psi(\Sigma_j\cup\mathcal{J}_j)\big)$; each $\mathcal{H}_j$ is cut off from $\mathbb{D}$ by the crosscut $W^{-1}(\mathcal{O}_j)$, the image of a mouth, and $\tilde b_j$ lies on the arc of $\mathbb{T}$ that bounds $\mathcal{H}_j$. We have $\tilde\phi(0)=0$ and $\tilde\phi'(0)=G'(0)/W'(0)>0$. Schwarz's lemma, applied to $\tilde\phi$ (as $U\subset\hat U$) and to $W/z_{\max}$ (as $\hat U\subset\{|z|\le z_{\max}\}$), gives

$$\frac{|G'(0)|}{z_{\max}}\ \le\ \tilde\phi'(0)\ \le\ 1. \tag{12}$$

Let $\mathcal{O}_j':=G^{-1}(\mathcal{O}_j)$, an open arc of $\mathbb{T}$, the *mouth arc*; then $\tilde\phi(\mathcal{O}_j')=W^{-1}(\mathcal{O}_j)$.

Lemma 8.8. *Fix n and a closed box of angle vectors ϑ such that the points $e^{i\vartheta_j}$ are distinct and lie in a fixed compact subset of the open arc $\mathcal{J}$. Then, as $\lambda\to0$, $\tilde\phi\to\mathrm{id}$ uniformly on $\overline{\mathbb{D}}$ and $\max_j\operatorname{diam}\mathcal{H}_j\to0$, uniformly over ϑ in the box.*

Proof. The map $\tilde\phi$ sends $\mathbb{T}\setminus\bigcup_j\overline{\mathcal{O}_j'}$ into $\mathbb{T}$. So, by the reflection principle, it extends to an injective meromorphic map on the sphere minus the closed mouth arcs, $(\mathbb{C}\cup\{\infty\})\setminus\bigcup_j\overline{\mathcal{O}_j'}$, by $\tilde\phi(t):=1/\overline{\tilde\phi(1/\bar t)}$ for $|t|>1$, with $\tilde\phi(\infty)=\infty$.

Suppose that $\lambda_m\to0$ and that $\vartheta^{(m)}$ lies in the box. By the uniform continuity of $G^{-1}\circ\Psi$ on $\overline{\mathcal{J}}$, and after passing to a subsequence, the arcs $\mathcal{O}_j'$ shrink to points $t_j\in\mathbb{T}$. Pick $t_0\in\mathcal{O}_1'$. The

value $\tilde\phi(t_0)$ is omitted by $\tilde\phi$ on this set: on $\overline{\mathbb{D}}\setminus\{t_0\}$ because $\tilde\phi$ is injective on $\overline{\mathbb{D}}$, and off $\overline{\mathbb{D}}$ because $|\tilde\phi|>1$ there while $|\tilde\phi(t_0)|\le 1$. Moreover $\tilde\phi(t_0)$ is not in $\tilde\phi(\mathbb{D})$, so Koebe's theorem and (12) give $|G'(0)|/(4z_{\max})\le\tilde\phi'(0)/4\le|\tilde\phi(t_0)|\le 1$. So on this set minus the points 0 and ∞ the maps omit 0, ∞ and a third value that stays away from both, and by Montel's theorem [18], applied to the maps divided by their third omitted value, which stays in a compact subset of $\mathbb{C}\setminus\{0\}$, the family is normal off the points t_j; it is normal near 0 because $|\tilde\phi|<1$ on $\mathbb{D}$, and near ∞ by reflection. Let $\tilde\phi_\infty$ be a limit, locally uniform off the points t_j. By Hurwitz's theorem [16] it is injective, since $\tilde\phi'(0)\ge|G'(0)|/z_{\max}$ keeps it nonconstant. An injective meromorphic map has removable singularities at the isolated points t_j, so $\tilde\phi_\infty$ is a Möbius transformation. It fixes 0 and ∞, maps $\mathbb{T}$ into $\mathbb{T}$, and has positive derivative at 0; so it is the identity. Hence $\tilde\phi\to\mathrm{id}$ locally uniformly off the points t_j.

Near t_j, take a small circle about t_j, of radius ε_2, with $\mathcal{O}_j'$ inside it (for large m). The image under $\tilde\phi$ of the part of this circle in $\overline{\mathbb{D}}$ is a crosscut of $\mathbb{D}$ close to the part of the circle in $\mathbb{D}$. As $\tilde\phi$ is injective on $\overline{\mathbb{D}}$, both $\tilde\phi(\mathbb{D}\cap\{|t-t_j|<\varepsilon_2\})$ and $\mathcal{H}_j$ lie in the piece of $\mathbb{D}$ that this crosscut cuts off from 0, and that piece lies in the disc of radius $2\varepsilon_2$ about t_j. So $|\tilde\phi-\mathrm{id}|$ is small on the part of $\overline{\mathbb{D}}$ inside the circle as well, and $\operatorname{diam}\mathcal{H}_j$ is small. Since every sequence $(\lambda_m,\vartheta^{(m)})$ with $\lambda_m\to 0$ has such a subsequence, the convergence is uniform over the box. □

Lemma 8.9. *Let $d_+>d_U$. There is a number $z_{\max}=z_{\max}(U,d_+)$ with the following property. For each n there is a Jordan domain $\hat U$ with $U\subset\hat U\subset\{|z|\le z_{\max}\}$ whose conformal map $W:\mathbb{D}\to\hat U$, normalized by $W(0)=0$ and $W'(0)/G'(0)>0$, satisfies $|W(b_j)|<d_+$ for $j=1,\dots,n$.*

Here W is extended to $\overline{\mathbb{D}}$ by Carathéodory's theorem.

Proof of Lemma 8.9. Fix n. For ϑ in a box as in Lemma 8.8, the point $G^{-1}(\Psi(e^{i\vartheta_j}))$ lies on $\mathcal{O}_j'$; its image under $\tilde\phi$ lies in $\overline{\mathcal{H}_j}$, and so does $\tilde b_j$. By Lemma 8.8,

$$\tilde b_j(\vartheta,\lambda)\ \to\ G^{-1}\big(\Psi(e^{i\vartheta_j})\big)\qquad(\lambda\to 0),\ \text{uniformly over the box.}$$

Since b_j is a root of unity, $G(b_j)\ne z_\infty$, so $G(b_j)\in\Psi(\mathcal{J})$; let $\vartheta^0=(\vartheta_j^0)$ be angles with $\Psi(e^{i\vartheta_j^0})=G(b_j)$. The points $e^{i\vartheta_j^0}$ are distinct. Let $\hat\vartheta_\lambda(\vartheta)$ be the vector of angles, in the chart Ψ, of the points $G(\tilde b_j(\vartheta,\lambda))$, that is, $\Psi(e^{i\hat\vartheta_{\lambda,j}(\vartheta)})=G(\tilde b_j(\vartheta,\lambda))$. By the limit above, $\hat\vartheta_\lambda(\vartheta)$ is uniformly close to ϑ when λ is small. Take a small closed box about ϑ^0 of the kind in Lemma 8.8, with disjoint sectors. Then the map $\vartheta\mapsto\vartheta-\hat\vartheta_\lambda(\vartheta)+\vartheta^0$ is continuous and sends the box into itself, for λ small. By Brouwer's fixed point theorem [5] it has a fixed point ϑ. There $\hat\vartheta_\lambda(\vartheta)=\vartheta^0$, so $G(\tilde b_j)=G(b_j)$ and $\tilde b_j=b_j$ for every j. Hence $W(b_j)=\hat z_j$ and $|W(b_j)|<d_+$. □

The inequality for Jordan domains, and Theorem 1.1.

Theorem 8.10. *Let U be a bounded Jordan domain with $0\in U$, let $G:\mathbb{D}\to U$ be conformal with $G(0)=0$, and let $d_U:=\operatorname{dist}(0,\partial U)$. Then*

$$K^*\ \ge\ \frac{|G'(0)|}{d_U}.$$

Proof. The choices are made in the order d_+; $z_{\max}$; R; n; λ; L. Let $d_+>d_U$, let $z_{\max}=z_{\max}(U,d_+)$ be as in Lemma 8.9, and put

$$R:=3+\frac{z_{\max}}{d_+}.$$

Let $n\ge 8$ be so large that $x_0=\log(2nR)\ge 24$. Let W be the map given by Lemma 8.9 for this n (with λ chosen there); it is a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{\hat U}$. Since W is continuous and $|W(b_j)|<d_+$, there is $\theta>0$ such that $|W(t)|<d_+$ for every $t\in\mathbb{T}$ within distance θ of one of the

points b_j. By Proposition 8.7, for L large along the sequence of Lemma 8.5, every point of the set $\{t \in \mathbb{T} : t^n \in I_L\}$ lies within distance θ of one of the points b_j. Let $t \in \mathbb{T}$.

- If $t^n \in I_L$, then t lies within distance θ of one of the points b_j, so $|W(t)| < d_+$, and $|F_L(t)| \geq |v_+|(1 - o(1))$. So $|F_L(t)|/|W(t)| \geq |v_+|(1 - o(1))/d_+$.
- Otherwise $|F_L(t)| \geq R - 3 = z_{\max}/d_+$ and $|W(t)| \leq z_{\max}$, so $|F_L(t)|/|W(t)| \geq 1/d_+ \geq |v_+|(1 - o(1))/d_+$, because $|v_+| \leq 1$.

The functions F_L and W satisfy the hypotheses of Lemma 8.1, by Proposition 8.7 and Lemma 8.9. So Lemma 8.1 applies with

$$s = |v_+|\big(1 - o(1)\big)\,\frac{|W'(0)|}{d_+\,F_L'(0)}.$$

By (12), $|W'(0)| = |G'(0)|/\tilde{\phi}'(0) \geq |G'(0)|$ (Schwarz's lemma applied to $\tilde{\phi}$), and $F_L'(0) \leq (4x_+)^{1/n}$ by Proposition 8.7. Hence

$$K^* \;\geq\; \frac{|v_+|\,(1 - o(1))}{(4x_+)^{1/n}} \cdot \frac{|G'(0)|}{d_+},$$

and letting $L \to \infty$ along the sequence removes the factor $1 - o(1)$. From $x_+ = x_0 + \log(1 + x_+)$ we get $x_+ \leq 2x_0$, since $\log(1 + 2x_0) \leq x_0$ for $x_0 \geq 3$; so $(4x_+)^{1/n} \leq (8\log(2nR))^{1/n} \to 1$; also $|v_+| \to 1$, by dominated convergence. So the factor in front of $|G'(0)|/d_+$ tends to 1 as $n \to \infty$, and $K^* \geq |G'(0)|/d_+$. Finally let $d_+ \to d_U$. □

Third proof of Theorem 1.1. For $0 < \delta < 1$ let $G(t) := k(\delta t)/\delta = t/(1 + \delta t)^2$ and $U := G(\mathbb{D})$. Since k is univalent on $\mathbb{D}$ and continuous on $\delta\overline{\mathbb{D}}$, U is a bounded Jordan domain containing 0; moreover $G'(0) = 1$ and $d_U = \operatorname{dist}(0, \partial U) = (1 + \delta)^{-2}$. Theorem 8.10 gives $K^* \geq (1 + \delta)^2$, and letting $\delta \to 1$ gives $K^* \geq 4$. Smale's theorem gives $K^* \leq 4$, so $K^* = 4$, and the rest of Theorem 1.1 follows as in Section 5, since $K(d)$ does not decrease with d (Appendix A). □

9. A fourth proof: a surface with conformal radius near 1

This section gives a fourth proof of Theorem 1.1. It does not use the map W, the function F or Theorem 1.2; from the earlier sections it reuses Proposition 2.1, the construction of Ψ in Section 8, Smale's bound $K^* \leq 4$, and the monotonicity of $K(d)$ (Appendix A).

The idea is as follows. We build a simply connected surface $\mathcal{S}$ over the plane, with no branch point, on whose boundary the modulus of the projection is large except on n short arcs, where it is about 1. Its conformal radius at the base point is at most $1 + o(1)$ as $n \to \infty$ (Lemma 9.5). We map $\mathcal{S}$ onto a domain Ω, a Jordan domain U with thin channels attached, by composing two Riemann maps, so that the short arcs are sent down the channels to points almost as close to 0 as ∂U is (Lemmas 9.6 and 9.7). The resulting function f on Ω then has $|f(z)/z|/|f'(0)|$ at least nearly 4 on $\partial\Omega$ when $U = k(\delta\mathbb{D})$ with δ close to 1, and Proposition 2.1 turns such an f into polynomials.

Notation in this section. The letters Ω, z, f, ε (a small positive number), k (the Koebe function) and δ keep their earlier meanings, and t is a point of $\overline{\mathbb{D}}$. U is a Jordan domain containing 0 and $G\colon \mathbb{D} \to U$ is conformal with $G(0) = 0$; we put $d_U := \operatorname{dist}(0, \partial U)$. The number $R \geq 2$ has the role it has in Section 4: away from a few small arcs, the modulus of the function we build is at least about R. The surface $\mathcal{S}$ lies over a plane; we write $\tilde{p}$ both for the coordinate of this plane and for the projection of $\mathcal{S}$ to it. The integer n is the number of tracts of $\mathcal{S}$, and $\Delta_n := 2\pi/n$. The integer n and the half-width λ of the channels (Section 9.4) are chosen anew here. The map $\zeta\colon \mathcal{S} \to \mathbb{D}$ is a Riemann map, and $r_{\mathcal{S}}$, r_Ω are conformal radii. Harmonic measure is written ϖ, with the domain or surface as a subscript: $\varpi_U(z, \cdot)$ is the harmonic measure in U at the point z. The letters ζ, τ, u, x, y do not have their meanings of Section 4.

Theorem 9.1. *Let U be a bounded analytic Jordan domain with $0 \in U$, let $G\colon \mathbb{D} \to U$ be the conformal map with $G(0) = 0$ and $G'(0) > 0$, and put $d_U := \operatorname{dist}(0, \partial U)$. Then for every $\varepsilon \in (0, \frac{1}{4})$*

there are a Jordan domain Ω with $0 \in \Omega$ and a function f, holomorphic and locally univalent on Ω and continuous on $\overline{\Omega}$, with $f(0) = 0$ and $f'(0) = 1$, such that

$$\Big|\frac{f(z)}{z}\Big| \;\geq\; \frac{G'(0)}{d_U}\,\frac{1-\varepsilon}{(1+\varepsilon)^2} \qquad \textit{for all } z \in \partial\Omega.$$

The following proposition is also the case of analytic U in Theorem 8.10, and so follows from Theorem 1.2; here we derive it from Theorem 9.1.

Proposition 9.2. *Let U be as in Theorem 9.1. Then $K^* \geq G'(0)/d_U$. In particular $K^* \geq (1+\delta)^2$ for every $\delta \in (0,1)$.*

Proof. Let $\varepsilon \in (0, \frac{1}{4})$, and let Ω and f be as in Theorem 9.1. Let $Z\colon \mathbb{D} \to \Omega$ be the Riemann map with $Z(0) = 0$. For $0 < \rho < 1$ the domain $Z(\rho\mathbb{D})$ is an analytic Jordan domain containing 0, the function f is holomorphic and locally univalent near its closure, and $|f(Z(\rho t))/Z(\rho t)|$ tends to $|f(Z(t))/Z(t)|$ uniformly in $t \in \mathbb{T}$ as $\rho \to 1$. Hence, for every number below $(G'(0)/d_U)(1-\varepsilon)/(1+\varepsilon)^2$, the function $|f(z)/z|$ exceeds it on $\partial Z(\rho\mathbb{D})$ once ρ is close to 1, and Proposition 2.1 shows that K^* exceeds it too. So $K^* \geq (G'(0)/d_U)(1-\varepsilon)/(1+\varepsilon)^2$, and letting $\varepsilon \to 0$ gives $K^* \geq G'(0)/d_U$.

Now let $0 < \delta < 1$, let k be the Koebe function, $U := k(\delta\mathbb{D})$ and $G(t) := k(\delta t)$. Then U is a bounded analytic Jordan domain with $0 \in U$, G maps $\mathbb{D}$ conformally onto U with $G(0) = 0$ and $G'(0) = \delta$, and the point of ∂U nearest 0 is $k(\delta)$, so that $d_U = \delta/(1+\delta)^2$. Thus $G'(0)/d_U = (1+\delta)^2$. □

Fourth proof of Theorem 1.1. By Proposition 9.2, $K^* \geq (1+\delta)^2$ for every $\delta < 1$, so $K^* \geq 4$. Smale's theorem [24] gives $K^* \leq 4$. Since $K(d)$ does not decrease with d (Appendix A), $\lim_{d\to\infty} K(d) = \sup_{d\geq 2} K(d) = K^* = 4$. □

The rest of this section proves Theorem 9.1. Throughout, U, G, d_U and ε are as in its statement.

9.1. **The surface.** In words, the surface $\mathcal{S}$ built below is the disc $\{|\tilde{p}| < 1+R\}$ cut along n radial slits $\mathcal{L}_j$, from the points v_j outward. Along the two sides of each slit a logarithmic end over v_j is glued in: sheets that wind around v_j over the disc $|\tilde{p} - v_j| < R$, cut off at $|u| = H$. Near v_j, the base disc is one sheet of this end.

Let n be a large integer, put $\Delta_n := 2\pi/n$, and for j modulo n let

$$\vartheta_j := j\Delta_n + \vartheta_*, \qquad v_j := e^{i\vartheta_j}.$$

The real number ϑ_* is fixed in Section 9.5. It only rotates the $\tilde{p}$-plane, $\tilde{p} \mapsto e^{i\vartheta_*}\tilde{p}$, so no modulus below depends on it. Let $R \geq 2$, put $\Xi_n := \log(4R/\Delta_n)$, and let H be an odd multiple of π with $H \geq 2\Xi_n^2$.

Let $\mathcal{L}_j := [v_j, (1+R)v_j]$ be the radial segment from v_j to $(1+R)v_j$ (the j-th slit), and

$$\mathcal{D} := \{|\tilde{p}| < 1+R\} \setminus \bigcup_j \mathcal{L}_j;$$

this domain is simply connected. Let $\mathcal{P} := \mathbb{C} \setminus \{u : \operatorname{Re} u \geq 0,\ |\operatorname{Im} u| \leq \pi\}$. The j-th *tract* is a copy of

$$\mathcal{T}_j := \{u \in \mathcal{P} :\ \operatorname{Re} u < \Xi_n,\ |u| < H\},$$

on which the projection is

$$\tilde{p} = v_j + \tilde{q}, \qquad \tilde{q} := \frac{\Delta_n}{4}\, e^{i(\vartheta_j+\pi)}\, e^u.$$

Thus $\arg \tilde{q} = \vartheta_j$ (modulo 2π) exactly where $\operatorname{Im} u$ is an odd multiple of π, in particular on the lines $\operatorname{Im} u = \pm\pi$.

The tracts are glued to $\mathcal{D}$ by $\tilde{p}$, as follows (Figure 4). The strip $\{\operatorname{Re} u < 0,\ |\operatorname{Im} u| < \pi\}$ is $\mathcal{D} \cap \{|\tilde{q}| < \Delta_n/4\}$, that is, the disc $|\tilde{q}| < \Delta_n/4$ off the slit: the map $\tilde{p}$ sends the strip bijectively onto this set, and we identify the two. The lines $\operatorname{Im} u = \pm\pi$, $\operatorname{Re} u < 0$, are the two sides of the

slit inside this disc. The window $\{\operatorname{Re} u = 0,\ |\operatorname{Im} u| < \pi\}$ opens onto the rest of $\mathcal{D}$. The edges $\{\operatorname{Im} u = \pm\pi,\ 0 < \operatorname{Re} u < \Xi_n\}$ of $\mathcal{T}_j$ are glued to the two sides of $\mathcal{L}_j$ where $|\tilde{q}| > \Delta_n/4$. The points $u = \pm i\pi$ lie over the point $(1 + \Delta_n/4)v_j$ of $\mathcal{L}_j$, one on each side of the slit; we add them to $\mathcal{S}$ as interior points (near each, three quadrants of the u-chart lie in $\mathcal{T}_j$ and one in $\mathcal{D}$). Finally, we remove from $\mathcal{D}$ the caps beyond $|u| = H$ in the strip, that is, the points of $\mathcal{D} \cap \{|\tilde{q}| < \Delta_n/4\}$ whose coordinate u satisfies $|u| \geq H$. The surface $\mathcal{S}$ is $\mathcal{D}$ together with all the tracts $\mathcal{T}_j$, glued in this way. Its *base point* o is the point of $\mathcal{D}$ over $\tilde{p} = 0$.

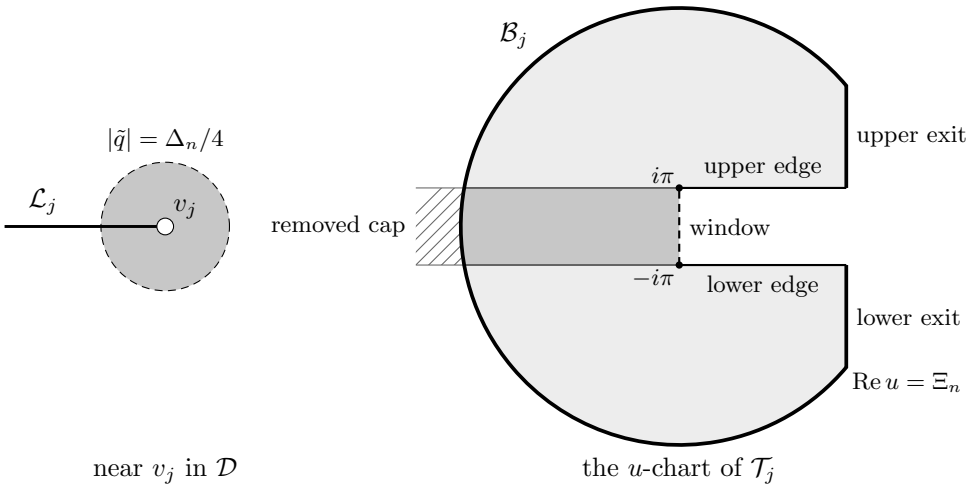


FIGURE 4. The j-th tract, not to scale. Right: the u-chart. The tract $\mathcal{T}_j$ is shaded; the white rectangle $\{0 \leq \operatorname{Re} u < \Xi_n,\ |\operatorname{Im} u| \leq \pi\}$ is not part of it. The dark strip $\{\operatorname{Re} u < 0,\ |\operatorname{Im} u| < \pi\}$ is identified with the dark disc on the left, the disc $|\tilde{q}| < \Delta_n/4$ off the slit $\mathcal{L}_j$, drawn so that $\mathcal{L}_j$ points left, away from o. The window (dashed) opens onto the rest of $\mathcal{D}$, and the upper and lower edges are glued to the upper and lower sides of $\mathcal{L}_j$ outside the dashed circle. The thick lines, the two exits and the end $\mathcal{B}_j$, are part of $\partial\mathcal{S}$. The cap beyond $|u| = H$ in the strip (hatched; on the left, the small white set around v_j) is removed.

Lemma 9.3. (a) *$\mathcal{S}$ is simply connected, and $\tilde{p}\colon \mathcal{S} \to \mathbb{C}$ is locally univalent. The zeros of $\tilde{p}$ are o and, on each tract, the points of $\mathcal{T}_j$ with $u \in \log(4/\Delta_n) + 2\pi i(\mathbb{Z} \setminus \{0\})$.*

(b) *The rotation $\tilde{p} \mapsto e^{i\Delta_n}\tilde{p}$ and, for each j, the reflection $\tilde{p} \mapsto v_j^2\,\overline{\tilde{p}}$ (on $\mathcal{T}_j$ it is $u \mapsto \bar{u}$) are symmetries of $\mathcal{S}$.*

(c) *The boundary of $\mathcal{S}$ is one Jordan curve. Over each j it consists, in cyclic order, of the* upper exit $\{\operatorname{Re} u = \Xi_n,\ \operatorname{Im} u > \pi,\ |u| < H\}$ *of $\mathcal{T}_j$, the* end $\mathcal{B}_j := \{|u| = H,\ \operatorname{Re} u < \Xi_n\}$ *of $\mathcal{T}_j$, the* lower exit $\{\operatorname{Re} u = \Xi_n,\ \operatorname{Im} u < -\pi,\ |u| < H\}$, *and the* top $\{|\tilde{p}| = 1 + R,\ \vartheta_j < \arg\tilde{p} < \vartheta_j + \Delta_n\}$. *On the top $|\tilde{p}| = 1 + R$; on the exits $|\tilde{p}| \geq R - 1$; and on $\mathcal{B}_j$*

$$|\tilde{p}| \ \geq\ 1 - e^{-\sqrt{H}/2},$$

provided H is large enough (depending on n and R).

Proof. (a) We add the tracts to the base one at a time, and we use this form of the Seifert–van Kampen theorem: *if two simply connected open subsets of a surface have a nonempty connected intersection, then their union is simply connected.* Let $\mathcal{D}_0$ be $\mathcal{D}$ without the caps. It is a plane domain, and its complement in the Riemann sphere is connected, because each slit $\mathcal{L}_j$, together with the cap at its inner end, is a connected set that meets the circle $|\tilde{p}| = 1 + R$. So $\mathcal{D}_0$ is

simply connected. Fix a small $\varsigma > 0$, and let $\hat{\mathcal{T}}_j$ be the tract $\mathcal{T}_j$ together with its two glued edges, the two points $u = \pm i\pi$, and the two thin bands $\{0 \le \operatorname{Re} u < \Xi_n,\ \pi - \varsigma < |\operatorname{Im} u| < \pi\}$ of $\mathcal{D}$ next to the two sides of $\mathcal{L}_j$ (the coordinate u of the strip continues to them across the window). Then $\hat{\mathcal{T}}_j$ is an open subset of $\mathcal{S}$, and u maps it conformally onto the plane domain $\{\operatorname{Re} u < \Xi_n,\ |u| < H\} \setminus \{\operatorname{Re} u \ge 0,\ |\operatorname{Im} u| \le \pi - \varsigma\}$, which is simply connected. The set $\hat{\mathcal{T}}_j$ meets $\mathcal{D}_0$ in the strip $\{\operatorname{Re} u < 0,\ |\operatorname{Im} u| < \pi,\ |u| < H\}$ together with the two bands; this set is connected. If ς is small enough, the sets $\hat{\mathcal{T}}_j$ are pairwise disjoint. So $\mathcal{D}_0 \cup \hat{\mathcal{T}}_1$ is simply connected, then $\mathcal{D}_0 \cup \hat{\mathcal{T}}_1 \cup \hat{\mathcal{T}}_2$, and so on; the last of these sets is $\mathcal{S}$. On both sides of each gluing $\tilde{p}$ is one and the same analytic function, $v_j + \tilde{q}$; so $\tilde{p}$ is locally univalent on $\mathcal{S}$. The zeros are found by solving $\tilde{q} = -v_j$ on each tract.

(b) This is immediate from the construction: the rotation maps $\mathcal{T}_k$ onto $\mathcal{T}_{k+1}$, and the reflection maps v_k to v_{2j-k} and $\mathcal{T}_k$ onto $\mathcal{T}_{2j-k}$.

(c) By construction the boundary consists of the pieces listed, and they form one Jordan curve. Indeed, the top over $(\vartheta_{j-1}, \vartheta_j)$ ends at $(1+R)v_j$, on the side of $\mathcal{L}_j$ that is glued to the upper edge of $\mathcal{T}_j$; there it meets the upper exit, at $u = \Xi_n + i\pi$. The upper exit runs to $|u| = H$, the end $\mathcal{B}_j$ runs round to the lower exit, and the lower exit returns to $u = \Xi_n - i\pi$, at $(1+R)v_j$ on the other side of $\mathcal{L}_j$, where the top over $(\vartheta_j, \vartheta_{j+1})$ begins. These arcs meet only at their ends. The top and the exit meet tangentially on the ray over v_j, because the circles $|\tilde{q}| = R$ and $|\tilde{p}| = 1 + R$ are tangent there; the exit and the end meet at a corner. There are no other boundary pieces. On the top $|\tilde{p}| = 1 + R$ by definition. On the exits $|\tilde{q}| = (\Delta_n/4)e^{\Xi_n} = R$, so $|\tilde{p}| \ge R - 1$.

It remains to bound $|\tilde{p}|$ on $\mathcal{B}_j$; let $H \to \infty$ below. If $\operatorname{Re} u \le \log(4/\Delta_n) - \sqrt{H}/2$, then $|\tilde{q}| \le e^{-\sqrt{H}/2}$, so $|\tilde{p}| \ge 1 - e^{-\sqrt{H}/2}$. Otherwise $\operatorname{Re} u$ lies in $(\log(4/\Delta_n) - \sqrt{H}/2,\ \Xi_n]$, so that $|\operatorname{Re} u| \le \sqrt{H}/2 + \Xi_n$, and $\operatorname{Im} u = \pm(H - s)$ with

$$0 \ \le\ s \ \le\ \frac{(\operatorname{Re} u)^2}{H} \ \le\ \frac{(\sqrt{H}/2 + \Xi_n)^2}{H}.$$

For H large this is less than $\frac{1}{2}$, uniformly (it tends to $\frac{1}{4}$). Since H is an odd multiple of π, we have $\tilde{q} = |\tilde{q}|\, v_j\, e^{\mp is}$. So $|\tilde{p}| = \bigl|1 + |\tilde{q}| e^{\mp is}\bigr| \ge 1$, because $\cos s > 0$. □

Let $\zeta \colon \mathcal{S} \to \mathbb{D}$ be the Riemann map with $\zeta(o) = 0$, normalized so that near o

$$\tilde{p} = r_{\mathcal{S}}\, \zeta + O(\zeta^2), \qquad r_{\mathcal{S}} > 0.$$

The number $r_{\mathcal{S}}$ is the conformal radius of $\mathcal{S}$ at o.

Lemma 9.4. (a) *For every j and every H, $\zeta(\mathcal{B}_j)$ is an arc of $\mathbb{T}$ whose centre is exactly $e^{i\vartheta_j}$ (here ζ is extended to the boundary, as in Section 9.6).*

(b) *Fix R, and fix n so large that $\Xi_n \ge 12$. Then $\varpi_{\mathcal{S}}(o, \mathcal{B}_j) \to 0$ as $H \to \infty$.*

Proof. (a) Both symmetries of Lemma 9.3(b) pass to ζ. The fixed line of the reflection $\tilde{p} \mapsto v_j^2\, \overline{\tilde{p}}$ is the radius of $\mathcal{D}$ from 0 towards v_j, followed by the segment $\{\operatorname{Im} u = 0,\ -H < \operatorname{Re} u < 0\}$ in $\mathcal{T}_j$. Since $r_{\mathcal{S}} > 0$, ζ maps this line onto the radius of $\mathbb{D}$ through $e^{i\vartheta_j}$, and the line ends at the midpoint $u = -H$ of $\mathcal{B}_j$. So $\zeta(\mathcal{B}_j)$ is an arc of $\mathbb{T}$ centred exactly at $e^{i\vartheta_j}$, for every H.

(b) Put $H' := 2\Xi_n$. As $\Xi_n \ge 12$, the window and the edges of $\mathcal{T}_j$ lie in $\{|u| \le (\Xi_n^2 + \pi^2)^{1/2}\}$, and $(\Xi_n^2 + \pi^2)^{1/2} < H'$. So, for $H' \le r < H$, the arc

$$\mathcal{X}_r := \{|u| = r,\ \operatorname{Re} u < \Xi_n\}$$

lies in $\mathcal{T}_j$ and separates o from $\mathcal{B}_j$. The part of $\mathcal{S}$ beyond $\mathcal{X}_{H'}$ is

$$\mathcal{T}_j^+ := \{H' < |u| < H,\ \operatorname{Re} u < \Xi_n\},$$

and it meets the rest of $\mathcal{S}$ only along $\mathcal{X}_{H'}$. On $|u| = r \ge 2\Xi_n$, the condition $\operatorname{Re} u < \Xi_n$ means that $\arg u \in \bigl(\arccos(\Xi_n/r),\ 2\pi - \arccos(\Xi_n/r)\bigr)$, and this interval lies in $(\pi/3, 5\pi/3)$. So, in the

coordinates $(\log|u|, \arg u)$, the set $\mathcal{T}_j^+$ lies in a strip of width $4\pi/3$, from $\log H'$ to $\log H$, with $\mathcal{B}_j$ at the far end; the rest of its boundary lies on the two exits, where the harmonic measure of $\mathcal{B}_j$ is 0.

Let $\mathcal{X}^+ := \mathcal{X}_{e^{4\pi/3}H'}$ be the arc at distance $4\pi/3$ in $\log|u|$ beyond $\mathcal{X}_{H'}$ (as $H \to \infty$, we may take $H > e^{4\pi/3}H'$), and put

$$m^+ := \sup_{\mathcal{X}^+} \varpi_{\mathcal{S}}(\cdot, \mathcal{B}_j).$$

The function $\varpi_{\mathcal{S}}(\cdot, \mathcal{B}_j)$ is harmonic on the part of $\mathcal{S}$ inside $\mathcal{X}^+$, the component of $\mathcal{S} \setminus \mathcal{X}^+$ that contains o. It vanishes on the boundary of $\mathcal{S}$ there ($\mathcal{B}_j$ lies beyond $\mathcal{X}^+$), and it is at most m^+ on $\mathcal{X}^+$. So, by the maximum principle, $\varpi_{\mathcal{S}}(\cdot, \mathcal{B}_j) \le m^+$ on $\mathcal{X}_{H'}$ and at o. On $\mathcal{T}_j^+$ the maximum principle gives, on $\mathcal{X}^+$,

$$\varpi_{\mathcal{S}}(\cdot, \mathcal{B}_j) \;\le\; \varpi_{\mathcal{T}_j^+}(\cdot, \mathcal{B}_j) + m^+ \, \varpi_{\mathcal{T}_j^+}(\cdot, \mathcal{X}_{H'}).$$

In the half-strip $\{\hat{x} + i\hat{y} : \; \hat{x} < \hat{L},\; 0 < \hat{y} < \hat{w}\}$, the harmonic measure of the end $\{\hat{x} = \hat{L}\}$ at the point $\hat{x} + i\hat{y}$ is

$$\frac{2}{\pi} \arctan \frac{\sin(\pi\hat{y}/\hat{w})}{\sinh\bigl(\pi(\hat{L} - \hat{x})/\hat{w}\bigr)} \;\le\; \frac{4}{\pi}\, e^{-\pi(\hat{L}-\hat{x})/\hat{w}}$$

(since $\sin \le 1$ and $\arctan(1/\sinh\hat{v}) = 2\arctan(e^{-\hat{v}}) \le 2e^{-\hat{v}}$, the left side is at most $\frac{4}{\pi}e^{-\hat{v}}$, with $\hat{v} := \pi(\hat{L} - \hat{x})/\hat{w}$, for every $\hat{L} - \hat{x} > 0$). By domain monotonicity, with $\hat{w} = 4\pi/3$ and so $\pi/\hat{w} = 3/4$, this gives on $\mathcal{X}^+$

$$\varpi_{\mathcal{T}_j^+}(\cdot, \mathcal{B}_j) \;\le\; \frac{4}{\pi}\, e^{\pi} \Bigl(\frac{H'}{H}\Bigr)^{3/4}, \qquad \varpi_{\mathcal{T}_j^+}(\cdot, \mathcal{X}_{H'}) \;\le\; \frac{4}{\pi}\, e^{-\pi} \;\le\; 0.0552.$$

So $m^+ \le \frac{4}{\pi} e^{\pi} (H'/H)^{3/4} + 0.0552\, m^+$, and

$$\varpi_{\mathcal{S}}(o, \mathcal{B}_j) \;\le\; 32 \Bigl(\frac{2\Xi_n}{H}\Bigr)^{3/4},$$

which tends to 0 as $H \to \infty$. □

9.2. **A formula for $|f(z)/z|$.** Let Ω be a Jordan domain with $0 \in \Omega$, and let $Z \colon \mathbb{D} \to \Omega$ be the Riemann map with $Z(0) = 0$ and $Z'(0) =: r_\Omega > 0$. Then $Z \circ \zeta \colon \mathcal{S} \to \Omega$ is conformal, and

$$f_{\mathcal{S}} := \tilde{p} \circ (Z \circ \zeta)^{-1}$$

is holomorphic and locally univalent on Ω, with $f_{\mathcal{S}}(0) = 0$ and $f'_{\mathcal{S}}(0) = r_{\mathcal{S}}/r_\Omega$, since $\tilde{p} = r_{\mathcal{S}}\zeta + O(\zeta^2)$ and $Z \circ \zeta = r_\Omega \zeta + O(\zeta^2)$ near o. So on $\partial\Omega$, at the point $z = Z(t)$ with $t \in \mathbb{T}$,

$$\frac{|f_{\mathcal{S}}(z)/z|}{|f'_{\mathcal{S}}(0)|} \;=\; \frac{r_\Omega}{r_{\mathcal{S}}} \cdot \frac{|\tilde{p}(\zeta^{-1}(t))|}{|Z(t)|}, \tag{13}$$

where the right side is taken in the sense of boundary values (they exist; see Section 9.6).

9.3. **The conformal radius of the surface.**

Lemma 9.5. *Fix R. Then $\limsup r_{\mathcal{S}} \le 1$ as $n \to \infty$, uniformly in H: for every $\varepsilon > 0$ there is n_0 such that $r_{\mathcal{S}} \le 1 + \varepsilon$ for all $n \ge n_0$ and all H.*

Proof. The argument is that of the kernel theorem [6]; for surfaces over a bounded region see [26]. Suppose that the claim fails. Then there are $\varepsilon > 0$ and surfaces $\mathcal{S}_\ell$, with parameters (n_ℓ, H_ℓ) and $n_\ell \to \infty$, such that $r_{\mathcal{S}_\ell} > 1 + \varepsilon$ for every ℓ. All the $\mathcal{S}_\ell$ lie over $\{|\tilde{p}| < 1 + R\}$, so the functions $f_\ell := \tilde{p} \circ \zeta_\ell^{-1}$, where ζ_ℓ is the Riemann map of $\mathcal{S}_\ell$, form a normal family on $\mathbb{D}$, and $r_{\mathcal{S}_\ell} = f'_\ell(0)$. Every subsequence has a further subsequence along which f_ℓ converges locally uniformly, so we may assume that f_ℓ converges to a limit f_∞; then $f_\infty(0) = 0$ and $f'_\infty(0) \ge 1 + \varepsilon$. So f_∞ is not constant, and $f'_\infty \ne 0$, by Hurwitz's theorem [16]. If $|f_\infty| \le 1$ on $\mathbb{D}$, Schwarz's lemma gives $f'_\infty(0) \le 1$, a

contradiction. If not, follow a segment from 0 to a point of $\mathbb{D}$ where $|f_\infty| > 1$, and let t' be its first point with $|f_\infty(t')| = 1$. For ℓ large, f_ℓ is univalent near t', with an image that contains a fixed disc about $f_\infty(t')$. For t'' on the segment just before t', f_ℓ maps the segment up to t'' into $\{|\tilde p| < 1\}$, and $f_\ell(t'')$ lies in that disc, which contains some v_j once $\Delta_{n_\ell} = 2\pi/n_\ell$ is small. The chord from $f_\ell(t'')$ to v_j lies in $\{|\tilde p| < 1\}$ up to its end, and its lift in the univalent piece ends at an interior point of $\mathcal{S}_\ell$. A path over $\mathbb{D}$ that starts in the base disc cannot leave it: the window lies in the base disc, and the tracts are glued to it along the slits, where $|\tilde p| = 1 + |\tilde q| > 1$. So the component of $\tilde p^{-1}(\mathbb{D})$ that contains o is the plain disc minus the caps. The caps contain the points of the base disc with $0 < |\tilde q| \le (\Delta_{n_\ell}/4)e^{-H_\ell}$, a punctured neighbourhood of v_j in $\mathbb{D}$. The path made of $\zeta_\ell^{-1}([0,t''])$ and the lift of the chord starts at o and lies over $\{|\tilde p| < 1\}$ until its end. So it lies in this component, where $\tilde p$ is injective, and the lift is the chord itself, in the base disc. Hence the chord meets a removed cap before it reaches v_j, and it cannot end at an interior point of $\mathcal{S}_\ell$. This is a contradiction. (The value $\tilde p = v_j$ is attained on $\mathcal{S}_\ell$, but only on the sheets of other tracts; the lift of the chord starts on the base disc and must stay there.) □

9.4. **The domain with channels.** Fix $\tau_\infty \in \mathbb{R}$ and put $z_\infty := G(e^{i\tau_\infty})$. Let $U^* := (\mathbb{C} \cup \{\infty\}) \setminus \overline{U}$; it is simply connected and contains ∞. Let Υ be an analytic arc in U^* from z_∞ to ∞, and let $z_1 \in U^* \setminus \Upsilon$ with $|z_1| < d_U(1+\varepsilon/2)$. Let $\Psi\colon \mathbb{D} \to U^* \setminus \Upsilon$ be conformal with $\Psi(0) = z_1$; it is continuous on $\overline{\mathbb{D}}$, with values in the sphere. Choose $\varepsilon_1 \in (0,1)$ so small that $\Psi(\{|t| \le \varepsilon_1\}) \subset \{|z| < d_U(1+\varepsilon)\}$. Let $\mathcal{J} \subset \mathbb{T}$ be the arc that Ψ maps onto $\partial U \setminus \{z_\infty\}$ (this is shown as in the construction of Ψ in Section 8; there the arc is a ray, here the analytic arc Υ). The radial segments from the points of $\overline{\mathcal{J}}$ to the circle $|t| = \varepsilon_1$ stay at positive distance from the point of $\mathbb{T}$ that Ψ sends to ∞. So the supremum $z_{\max}$ of $|z|$ over their images under Ψ and over U is finite, and it depends only on U, ε, Υ, z_1 and ε_1. It does not depend on n.

Let $\hat\tau_1, \dots, \hat\tau_n$ be angles with $e^{i\hat\tau_j} \in \mathcal{J}$, and let $\lambda > 0$ be small. Define the *channel*, the *mouth* and the *end*

$$\mathcal{C}_j := \Psi\big(\{t :\ \varepsilon_1 < |t| < 1,\ |\arg t - \hat\tau_j| < \lambda\}\big), \qquad \mathcal{O}_j := \Psi\big(\{t \in \mathbb{T} :\ |\arg t - \hat\tau_j| < \lambda\}\big),$$
$$\mathcal{E}_j := \Psi\big(\{t :\ |t| = \varepsilon_1,\ |\arg t - \hat\tau_j| \le \lambda\}\big),$$

with arguments taken modulo 2π, and put

$$\Omega := U \cup \bigcup_j (\mathcal{O}_j \cup \mathcal{C}_j).$$

Here λ is so small that the closed sectors are disjoint and their arcs on $\mathbb{T}$ lie in $\mathcal{J}$.

Lemma 9.6. *Ω is a bounded Jordan domain with $U \subset \Omega$; $|z| \le z_{\max}$ on Ω; $|z| < d_U(1+\varepsilon)$ on each end $\mathcal{E}_j$; and the Riemann map $Z\colon \mathbb{D} \to \Omega$ with $Z(0) = 0$ and $Z'(0) > 0$ satisfies $r_\Omega := Z'(0) \ge G'(0)$.*

Proof. Ω is U with the disjoint channels attached along the mouths, so it is a bounded Jordan domain containing U. The bound $|z| \le z_{\max}$ holds by the definition of $z_{\max}$, and the bound on $\mathcal{E}_j$ by the choice of ε_1. Since $U \subset \Omega$, Schwarz's lemma gives $r_\Omega \ge G'(0)$. □

9.5. **Placing the channels.** We now choose $\vartheta_* := \tau_\infty - \Delta_n/2$ in Section 9.1, so that $\tau_\infty = \vartheta_0 + \Delta_n/2$. Since τ_∞ and G are fixed before n, we rotate the surface, not U. For $\tau = (\tau_1, \dots, \tau_n)$ in the cube

$$\prod_{j=1}^{n} [\vartheta_j - \Delta_n/4,\ \vartheta_j + \Delta_n/4], \tag{14}$$

let $\hat\tau_j$ be the argument of $\Psi^{-1}(G(e^{i\tau_j})) \in \mathcal{J}$; let $\Omega(\tau)$ be the domain of Section 9.4 built on these angles, with ends $\mathcal{E}_j(\tau)$; let Z_τ be its Riemann map with $Z_\tau(0) = 0$ and $Z_\tau'(0) > 0$; and let $\hat\vartheta_j(\tau)$ be the centre of the arc $Z_\tau^{-1}(\mathcal{E}_j(\tau))$ of $\mathbb{T}$. All the intervals $[\vartheta_j - \Delta_n/4, \vartheta_j + \Delta_n/4]$ stay at distance at least $\Delta_n/4$ from τ_∞. We want τ with $\hat\vartheta_j(\tau) = \vartheta_j$ for all j.

Lemma 9.7. *If $\lambda > 0$ is small enough, depending on U, Υ, z_1, ε_1 and n, then there is τ in the cube* (14) *with $\hat{\vartheta}_j(\tau) = \vartheta_j$ for every j.*

Proof. The displacement. Fix τ in the cube and write $Z = Z_\tau$. The map $\hat{Z} := Z^{-1} \circ G \colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is continuous and injective, with $\hat{Z}(0) = 0$ and $\hat{Z}'(0) > 0$. Let $\mathcal{O}'_j := G^{-1}(\mathcal{O}_j) \subset \mathbb{T}$ be the *mouth arcs*, $\mathcal{O} := \bigcup_j \mathcal{O}_j$, and

$$m_{\mathcal{O}} := \sum_j \varpi_U(0, \mathcal{O}_j) = \varpi_U(0, \mathcal{O}),$$

the total normalized length of the mouth arcs. As ∂U is analytic and the cube stays away from τ_∞, the map $\Psi^{-1} \circ G$ is an analytic diffeomorphism on the union of the arcs of $\mathbb{T}$ over the intervals $[\vartheta_j - \Delta_n/4, \vartheta_j + \Delta_n/4]$. So each mouth arc has length at most a constant times λ, with a constant that depends only on U, Υ, z_1 and n; hence $m_{\mathcal{O}} \to 0$ as $\lambda \to 0$, uniformly on the cube. Take λ so small that $m_{\mathcal{O}} \le \frac{1}{2}$.

(a) The function $\log(\hat{Z}(t)/t) =: x(t) + iy(t)$ is holomorphic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, with $y(0) = 0$ and $x(0) = \log \hat{Z}'(0) =: -x_*$, $x_* \ge 0$. Here $y(t) = \arg \hat{Z}(t) - \arg t$.

(b) On $\mathbb{T}$, $x = \log|\hat{Z}|$ vanishes off the mouth arcs. On the mouth arcs, $-\log 4 - x_* \le x < 0$, by Koebe's growth bound $|\hat{Z}(t)| \ge \hat{Z}'(0)|t|/(1+|t|)^2$.

(c) The mean of x over $\mathbb{T}$ is $-x_*$, so $x_* \le m_{\mathcal{O}}(\log 4 + x_*)$. Hence $x_* \le 2\, m_{\mathcal{O}} \log 4$.

(d) By Parseval's identity, since $y(0) = 0$, and with norms taken over the normalized measure on $\mathbb{T}$,

$$\|y\|_2 \;\le\; \|x\|_2 \;\le\; \sqrt{m_{\mathcal{O}}}\,(\log 4 + x_*) \;\le\; 2.8\sqrt{m_{\mathcal{O}}}.$$

(e) Let $t_1, t_2 \in \mathbb{T}$ lie off the open mouth arcs, let I be the positive arc of $\mathbb{T}$ from t_1 to t_2, and I' the positive arc from $\hat{Z}(t_1)$ to $\hat{Z}(t_2)$. Then

$$y(t_2) - y(t_1) = \operatorname{length}(I') - \operatorname{length}(I).$$

Indeed, $\arg \hat{Z}$ changes along I by exactly $\operatorname{length}(I')$: each crosscut $\hat{Z}(\mathcal{O}'_j)$, together with the arc $Z^{-1}(\partial \mathcal{C}_j \setminus \mathcal{O}_j)$ of $\mathbb{T}$ between its ends, bounds a region that does not contain 0, so the two are homotopic in $\overline{\mathbb{D}} \setminus \{0\}$. Further,

$$\frac{\operatorname{length}(I') - \operatorname{length}(I)}{2\pi} = (\text{excess}) + \varpi_\Omega(0, \text{channels in } I) - \varpi_U(0, \text{mouths in } I),$$

where $\varpi_\Omega(0, \text{channels in } I)$ is the harmonic measure in Ω, at 0, of the sets $\partial \mathcal{C}_j \setminus \mathcal{O}_j$ with $\mathcal{O}'_j \subset I$, the term $\varpi_U(0, \text{mouths in } I)$ is the harmonic measure in U of the corresponding mouths, and the excess is

$$\varpi_\Omega\big(0, G(I) \setminus \mathcal{O}\big) - \varpi_U\big(0, G(I) \setminus \mathcal{O}\big) \;\ge\; 0,$$

which is nonnegative by Carleman's principle [8], since $U \subset \Omega$ and $G(I) \setminus \mathcal{O} \subset \partial U \cap \partial\Omega$. Over the whole circle, the total excess plus $\varpi_\Omega(0, \text{all channels})$ equals $m_{\mathcal{O}}$. So the difference above lies in $[-m_{\mathcal{O}}, m_{\mathcal{O}}]$, and off the mouth arcs y oscillates by at most $2\pi m_{\mathcal{O}}$.

(f) The part of $\mathbb{T}$ off the mouth arcs has normalized measure at least $1 - m_{\mathcal{O}} \ge \frac{1}{2}$. Take t_* in it with $|y(t_*)|$ least. By (d), $|y(t_*)|^2(1 - m_{\mathcal{O}}) \le \|y\|_2^2$, so $|y(t_*)| \le 4\sqrt{m_{\mathcal{O}}}$. With (e), off the mouth arcs

$$|y| \;\le\; 2\pi m_{\mathcal{O}} + 4\sqrt{m_{\mathcal{O}}}.$$

(g) The centre $\hat{\vartheta}_j(\tau)$ lies in the arc between the images under $\hat{Z}$ of the two ends of the mouth arc $\mathcal{O}'_j$, an arc of length $2\pi\varpi_\Omega(0, \partial\mathcal{C}_j \setminus \mathcal{O}_j) \le 2\pi m_{\mathcal{O}}$; and τ_j lies in the mouth arc $\mathcal{O}'_j$, of length at most $2\pi m_{\mathcal{O}}$. So, if t_1 is an end of the mouth arc $\mathcal{O}'_j$, then

$$|\hat{\vartheta}_j(\tau) - \tau_j| \;\le\; |y(t_1)| + 4\pi m_{\mathcal{O}} \;\le\; 6\pi m_{\mathcal{O}} + 4\sqrt{m_{\mathcal{O}}}.$$

Take λ so small that this is at most $\Delta_n/8$, on the whole cube.

Continuity. The argument is that of Section 8: the boundary curves $\partial\Omega(\tau)$ move uniformly with τ, and Radó's theorem applies. Here we write out the uniform motion, by a flow, because the centres $\hat\vartheta_j(\tau)$ are defined through the ends $\mathcal{E}_j(\tau)$. Fix τ^0 in the cube. For τ near τ^0 take, in the disc of Ψ, the angular vector field

$$\sum_j \big(\hat\tau_j(\tau)-\hat\tau_j(\tau^0)\big)\,\chi_j\,\frac{\partial}{\partial \arg t},$$

where each χ_j is smooth and equal to 1 on a neighbourhood of the closed sector j; the χ_j have disjoint supports, which lie over a compact part of $\mathcal{J}$, and they taper to zero between the sectors and away from $\{\varepsilon_1\le|t|\le 1\}$. The field is tangent to the circles $|t|=1$ and $|t|=\varepsilon_1$. Carry it to the z-plane by Ψ, and across ∂U; this is possible because Ψ is analytic across $\mathcal{J}$, as ∂U is analytic. Its time-one flow $\mathcal{F}_\tau$ is a homeomorphism of the plane, equal to the identity on a fixed neighbourhood of 0, and preserving U; moreover $\mathcal{F}_\tau(\Omega(\tau^0))=\Omega(\tau)$ and $\mathcal{F}_\tau(\mathcal{E}_j(\tau^0))=\mathcal{E}_j(\tau)$. Indeed, in the coordinate t (with Ψ extended by reflection across a compact part of $\mathcal{J}$) the field is angular, so its flow preserves each circle $|t|=r$. Over the supports of the χ_j, the circles with r slightly above 1 lie in U (Ψ, extended by reflection across $\mathcal{J}$, maps them into U), the arcs of $\mathbb{T}$ go to arcs of ∂U, and the circle $|t|=\varepsilon_1$ is preserved. As the flow is the identity off a fixed compact set, $\mathcal{F}_\tau$ maps U onto U. On the set where $\chi_j=1$, the flow is the rotation $t\mapsto e^{i\gamma_j}t$, with $\gamma_j:=\hat\tau_j(\tau)-\hat\tau_j(\tau^0)$, as long as $|\gamma_j|$ is less than the margin of that set around the closed sector j. So it maps the closed sector j for τ^0 onto the one for τ, and so $\mathcal{C}_j$, $\mathcal{O}_j$ and $\mathcal{E}_j$ onto those for τ, and $\Omega(\tau^0)$ onto $\Omega(\tau)$. The supports of the χ_j depend on τ^0 only, and they stay at a positive distance from 0, so $\mathcal{F}_\tau$ is the identity on a fixed neighbourhood of 0. As the field is $\sum_j\gamma_j\mathcal{W}_j$, with fixed smooth fields $\mathcal{W}_j$ of compact support, and $\gamma_j\to 0$ as $\tau\to\tau^0$ (since $\hat\tau_j$ depends analytically on τ_j), each point moves by at most $\sup|\sum_j\gamma_j\mathcal{W}_j|$, so $\mathcal{F}_\tau\to\mathrm{id}$ uniformly as $\tau\to\tau^0$. So the Jordan curves $\partial\Omega(\tau)=\mathcal{F}_\tau(\partial\Omega(\tau^0))$ converge uniformly to $\partial\Omega(\tau^0)$, and by Radó's theorem [22] $Z_\tau\to Z_{\tau^0}$ uniformly on $\overline{\mathbb{D}}$. Put $\hat{\mathcal{F}}_\tau:=Z_\tau^{-1}\circ\mathcal{F}_\tau\circ Z_{\tau^0}$, a homeomorphism of $\overline{\mathbb{D}}$. Since $Z_{\tau^0}\circ\hat{\mathcal{F}}_\tau-Z_{\tau^0}=(\mathcal{F}_\tau-\mathrm{id})\circ Z_{\tau^0}+(Z_{\tau^0}-Z_\tau)\circ\hat{\mathcal{F}}_\tau$, and both terms tend to 0 uniformly, $Z_{\tau^0}\circ\hat{\mathcal{F}}_\tau\to Z_{\tau^0}$ uniformly; as $Z_{\tau^0}^{-1}$ is uniformly continuous on $\overline{\Omega(\tau^0)}$, $\hat{\mathcal{F}}_\tau\to\mathrm{id}$ uniformly on $\overline{\mathbb{D}}$. Hence $\hat\vartheta$ is continuous at τ^0, because $Z_\tau^{-1}(\mathcal{E}_j(\tau))=\hat{\mathcal{F}}_\tau\big(Z_{\tau^0}^{-1}(\mathcal{E}_j(\tau^0))\big)$.

Brouwer. By (g), the continuous map $\tau\mapsto\tau-\big(\hat\vartheta(\tau)-(\vartheta_1,\dots,\vartheta_n)\big)$, with $\hat\vartheta:=(\hat\vartheta_1,\dots,\hat\vartheta_n)$, maps the cube into itself. By Brouwer's fixed point theorem it has a fixed point, and there $\hat\vartheta(\tau)=(\vartheta_1,\dots,\vartheta_n)$. □

We fix this τ, the domain $\Omega:=\Omega(\tau)$ and its Riemann map $Z:=Z_\tau$, and we let $\lambda_*>0$ be such that Z maps the arcs of $\mathbb{T}$ of half-width λ_* about the points $e^{i\vartheta_j}$ into the ends $\mathcal{E}_j$. The number λ_* depends on n and λ, but not on H.

9.6. **Proof of Theorem 9.1.** The choices are made in the order U,ε; $\Upsilon,\tau_\infty,z_1,\varepsilon_1,z_{\max}$; R; n,ϑ_*; $\lambda,\tau,\Omega,\lambda_*$; H. Fix τ_∞, Υ, z_1, ε_1 and $z_{\max}$ as in Section 9.4. Take

$$R:=2+\frac{2z_{\max}}{d_U}.$$

Take n so large that $\Xi_n\ge 12$ and that Lemma 9.5 gives $r_{\mathcal{S}}\le 1+\varepsilon$; both hold for all H. Put $\vartheta_*:=\tau_\infty-\Delta_n/2$. Then choose λ, τ, Ω, Z and λ_* as in Section 9.5. Last, choose H, an odd multiple of π with $H\ge 2\Xi_n^2$, so large that Lemma 9.3(c) applies, that $e^{-\sqrt{H}/2}\le\varepsilon$, and that $\zeta(\mathcal{B}_j)$ lies within λ_* of $e^{i\vartheta_j}$ for every j; the last is possible by Lemma 9.4, since $\zeta(\mathcal{B}_j)$ is centred at $e^{i\vartheta_j}$ and has length $2\pi\varpi_{\mathcal{S}}(o,\mathcal{B}_j)$.

Let $f_{\mathcal{S}}$ be as in Section 9.2 and $f:=f_{\mathcal{S}}/f_{\mathcal{S}}'(0)$, so that $f(0)=0$ and $f'(0)=1$. Let $t\in\mathbb{T}$ and $z=Z(t)\in\partial\Omega$. By (13),

$$\Big|\frac{f(z)}{z}\Big|=\frac{r_\Omega}{r_{\mathcal{S}}}\cdot\frac{|\tilde p(\zeta^{-1}(t))|}{|Z(t)|},\qquad r_\Omega\ge G'(0),\quad r_{\mathcal{S}}\le 1+\varepsilon,$$

by Lemma 9.6 and the choice of n. There are two cases.

- If $\zeta^{-1}(t) \in \mathcal{B}_j$ for some j, then $|\tilde{p}| \geq 1-\varepsilon$ by Lemma 9.3(c), and $z \in \mathcal{E}_j$ by the choice of H, so that $|z| \leq d_U(1+\varepsilon)$ by Lemma 9.6. Hence
$$\Big|\frac{f(z)}{z}\Big| \;\geq\; \frac{G'(0)(1-\varepsilon)}{(1+\varepsilon)\,d_U(1+\varepsilon)} \;=\; \frac{G'(0)}{d_U}\,\frac{1-\varepsilon}{(1+\varepsilon)^2}.$$
- For every other t, the point $\zeta^{-1}(t)$ lies on a top or an exit, so $|\tilde{p}| \geq R-1$, and $|z| \leq z_{\max}$. Hence, by the choice of R,
$$\Big|\frac{f(z)}{z}\Big| \;\geq\; \frac{G'(0)(R-1)}{(1+\varepsilon)\,z_{\max}} \;\geq\; \frac{G'(0)}{d_U}.$$

As for continuity on the closure, Z is continuous on $\overline{\mathbb{D}}$ by Carathéodory's theorem [7], since $\partial\Omega$ is a Jordan curve. This theorem is for plane domains, and $\mathcal{S}$ is not one; but ζ also extends to a homeomorphism of the closures, as follows. Each point b_* of $\partial\mathcal{S}$ has a neighbourhood $\mathcal{N}_*$ in $\mathcal{S}$ which, in the chart $\tilde{p}$ or u, is a plane Jordan domain; we take it so small that $o \notin \overline{\mathcal{N}_*}$. Its boundary is a crosscut of $\mathcal{S}$ together with an open arc of $\partial\mathcal{S}$ through b_*: one analytic arc, two arcs meeting at the corner of an exit and an end, or two arcs joining tangentially where a top meets an exit. Let $\varphi_*\colon \mathbb{D} \to \mathcal{N}_*$ be a Riemann map; by Carathéodory's theorem it extends to a homeomorphism of the closures. Let $\mathcal{I}_*$ be the open arc of $\mathbb{T}$ that φ_* maps onto that open arc of $\partial\mathcal{S}$. The map $g_* := \zeta \circ \varphi_*$ is univalent and bounded. As $\zeta\colon \mathcal{S} \to \mathbb{D}$ is proper, $|g_*(t)| \to 1$ as t tends to $\mathcal{I}_*$. So $\log|g_*|$ is harmonic near $\mathcal{I}_*$ with boundary values 0 there, and by the reflection principle g_* extends holomorphically across $\mathcal{I}_*$, with $g_*(\mathcal{I}_*) \subset \mathbb{T}$. The extension is univalent near $\mathcal{I}_*$, since the reflected values lie outside the closed disc. Hence $\zeta = g_* \circ \varphi_*^{-1}$ extends continuously to that arc of $\partial\mathcal{S}$, as a local homeomorphism into $\mathbb{T}$. These local extensions give a continuous map of $\mathcal{S} \cup \partial\mathcal{S}$ onto $\overline{\mathbb{D}}$ that maps $\partial\mathcal{S}$ onto $\mathbb{T}$ by a local homeomorphism of degree 1: a curve in $\mathcal{S}$ near $\partial\mathcal{S}$ maps to a curve that winds once round 0. Since ζ is already a bijection of $\mathcal{S}$ onto $\mathbb{D}$, and a local homeomorphism of degree 1 of a circle onto a circle is a homeomorphism, this map is a homeomorphism. Hence $\tilde{p} \circ \zeta^{-1}$ is continuous on $\overline{\mathbb{D}}$, and f is continuous on $\overline{\Omega}$. Finally $f' \neq 0$ in Ω. This proves Theorem 9.1. □

Remark 9.8 (Background). The surface $\mathcal{S}$, a disc with logarithmic ends glued along radial slits, is of a type studied by Volkovyskii [27, §§45 and 75]; see [12, Section 3]. For modern forms of Carathéodory's kernel theorem for Riemann surfaces see [4, 11]; Volkovyskii [26, p. 375] notes that conformal radii can grow without bound even when the kernel is hyperbolic, which is why R is fixed before n in Lemma 9.5.

Appendix A. The numbers $K(d)$ do not decrease

We prove in full the fact, sketched in the introduction, that $K(d+1) \geq K(d)$ for every $d \geq 2$ [9].

Dividing a polynomial p by $p'(0)$ changes neither its degree, nor its critical points, nor $S(p)$. So we may take $p'(0) = 1$, and then $S(p) = \min_{p'(c)=0} |p(c)/c|$.

Proposition A.1. *For every $d \geq 2$ we have $K(d+1) \geq K(d)$. Hence $K^* = \lim_{d\to\infty} K(d)$.*

Proof. Fix a polynomial $p(z) = \sum_{k=1}^{d} a_k z^k$ with complex coefficients, of degree exactly d, with $a_1 = p'(0) = 1$ and $a_d \neq 0$, and for $\varepsilon > 0$ put $p_\varepsilon := p + \varepsilon z^{d+1}$. Then p_ε has degree exactly $d+1$, because its top coefficient is $\varepsilon \neq 0$; also $p_\varepsilon(0) = 0$ and $p_\varepsilon'(0) = 1$, and $p_\varepsilon' = p' + (d+1)\varepsilon z^d$ has degree d. Put $p_1(z) := p(z)/z = \sum_{k=1}^{d} a_k z^{k-1}$, a polynomial.

Step 1: the critical points near those of p. All $d-1$ zeros of p', counted with multiplicity, are nonzero, because $p'(0) = 1$. Fix $\varrho > 0$ larger than the modulus of every zero of p'. On $|z| \leq \varrho$ we have $|p_\varepsilon'(z) - p'(z)| = (d+1)\varepsilon|z|^d \leq (d+1)\varepsilon\varrho^d$. Fix $r_1 > 0$ so small that the closed discs of radius r_1 about the zeros of p' lie in $\{|z| < \varrho\}$. The closed disc $\{|z| \leq \varrho\}$ minus the open discs of radius r_1 about the zeros of p' is compact, it contains the circle $|z| = \varrho$, and p' has no zero on it; so the

minimum ν_1 of $|p'|$ on it is positive. Let $0 < \varepsilon < \nu_1/((d+1)\varrho^d)$. By Rouché's theorem on the circle $|z| = \varrho$, the polynomial p'_ε has exactly $d-1$ zeros in $\{|z| < \varrho\}$, counted with multiplicity; and p'_ε has no zero on that compact set. Hence every zero c of p'_ε in $\{|z| < \varrho\}$ satisfies $|c - c_0| < r_1$ for some zero c_0 of p'. With $\nu_2 := \max_{|z|\le\varrho} |p'_1(z)|$ we get

$$\left|\frac{p_\varepsilon(c)}{c}\right| = |p_1(c) + \varepsilon c^d| \;\ge\; |p_1(c_0)| - \nu_2 r_1 - \varepsilon\varrho^d \;\ge\; S(p) - \nu_2 r_1 - \varepsilon\varrho^d.$$

Step 2: the escaping critical point. Since p'_ε has degree d and, by Step 1, exactly $d-1$ of its zeros lie in $\{|z| < \varrho\}$, exactly one zero, c_ε, lies in $\{|z| > \varrho\}$, and it is simple. Then $|c_\varepsilon| \to \infty$ as $\varepsilon \to 0^+$. Otherwise there are $\varepsilon_m \to 0^+$ with $c_{\varepsilon_m} \to c^*$ for some finite c^* with $|c^*| \ge \varrho$. Since $p'_\varepsilon \to p'$ uniformly on compact sets, $p'(c^*) = 0$, which contradicts $|c^*| \ge \varrho$.

From $p'_\varepsilon(c_\varepsilon) = 0$ we get $\varepsilon c_\varepsilon = -p'(c_\varepsilon)/((d+1)c_\varepsilon^{d-1})$. Since $p'(c)/c^{d-1} = da_d + \sum_{k=1}^{d-1} k a_k c^{k-d} \to da_d$ as $|c| \to \infty$, we get $\varepsilon c_\varepsilon \to -da_d/(d+1)$. Next,

$$\frac{p_\varepsilon(c)}{c} = c^{d-1}(a_d + \varepsilon c) + p_2(c), \qquad p_2(c) := \sum_{k=1}^{d-1} a_k c^{k-1},$$

where p_2 does not depend on ε and satisfies $|p_2(c)| \le \nu_3 |c|^{d-2}$ for $|c| \ge 1$, with $\nu_3 := \sum_{k=1}^{d-1} |a_k|$. Hence, once $|c_\varepsilon| \ge 1$,

$$\left|\frac{p_\varepsilon(c_\varepsilon)}{c_\varepsilon}\right| \;\ge\; |c_\varepsilon|^{d-2}\big(|c_\varepsilon|\,|a_d + \varepsilon c_\varepsilon| - \nu_3\big).$$

Here $|c_\varepsilon| \to \infty$, $|a_d + \varepsilon c_\varepsilon| \to |a_d|/(d+1) > 0$, and $d - 2 \ge 0$, so the right side tends to $+\infty$.

Step 3: conclusion. For $0 < \varepsilon < \nu_1/((d+1)\varrho^d)$, the critical points of p_ε are the $d-1$ zeros of p'_ε in $\{|z| < \varrho\}$ and c_ε. By Step 1, each of the first ones has $|p_\varepsilon(c)/c| \ge S(p) - \nu_2 r_1 - \varepsilon\varrho^d$. By Step 2, $|p_\varepsilon(c_\varepsilon)/c_\varepsilon| \to +\infty$. Hence $\liminf_{\varepsilon\to 0^+} S(p_\varepsilon) \ge S(p) - \nu_2 r_1$ for every small $r_1 > 0$, that is, $\liminf_{\varepsilon\to 0^+} S(p_\varepsilon) \ge S(p)$. Since p_ε has degree exactly $d+1$, $p_\varepsilon(0) = 0$ and $p'_\varepsilon(0) = 1$, we have $K(d+1) \ge S(p_\varepsilon)$ for every $\varepsilon > 0$, and so $K(d+1) \ge \liminf_{\varepsilon\to 0^+} S(p_\varepsilon) \ge S(p)$. Taking the supremum over p gives $K(d+1) \ge K(d)$.

A nondecreasing sequence in $[0, +\infty]$ converges to its supremum, so $K^* = \lim_d K(d)$. □

Appendix B. A rate for the conformal radius

This appendix gives a quantitative form of Lemma 9.5, with the notation of Section 9. It is not used elsewhere in the paper, and the symbols with a háček are local to it. For fixed R it shows that $\log r_{\mathcal{S}}$ is at most an absolute constant times $\Delta_n \log\log(1/\Delta_n)$, for all large n. Below, $\check{C}_1$, $\check{C}_2$, $\check{C}_3$ are numbers that depend on Δ_n and Ξ_n and are bounded absolutely for $\Delta_n \le 0.01$ and $\Xi_n \ge 12$.

Proposition B.1 (A rate in n)**.** *If $\Delta_n \le 0.01$ and $\Xi_n \ge 12$, then*

$$\log r_{\mathcal{S}} \;\le\; \Delta_n\Big(6 + \frac{2}{\pi}\log \Xi_n\Big),$$

uniformly in H.

Proof. Thomson's principle. Let $\check{r} > 0$ be small, and let $\check{O}$ be the component of $\{|\tilde{p}| \le \check{r}\}$ that contains o; the small discs around the other zeros of $\tilde{p}$ are not removed. Let ϖ_0 be the harmonic function on $\mathcal{S} \setminus \check{O}$ with $\varpi_0 = 1$ on $\partial\check{O}$ and $\varpi_0 = 0$ on $\partial\mathcal{S}$. Write $\|\cdot\|$ for the L^2 norm of a vector field, so that $\|\nabla\varpi_0\|^2$ is the Dirichlet energy of ϖ_0. The extremal distance between $\partial\check{O}$ and $\partial\mathcal{S}$ in $\mathcal{S} \setminus \check{O}$ is $1/\|\nabla\varpi_0\|^2$, and it equals $(2\pi)^{-1}\log(r_{\mathcal{S}}/\check{r}) + o(1)$ as $\check{r} \to 0$. If $\mathcal{V}$ is a divergence-free vector field on $\mathcal{S} \setminus \check{O}$ with unit flux out through $\partial\check{O}$, all of which leaves through $\partial\mathcal{S}$, then

$$1 = \left|\int \mathcal{V}\cdot\nabla\varpi_0\right| \;\le\; \|\mathcal{V}\|\,\|\nabla\varpi_0\|,$$

and so $\|\mathcal{V}\|^2$ is at least this extremal distance [20, Chapter II].

The flow. We now build such a field $\mathcal{V}$. It is invariant under the rotation symmetry, so the flux through each period is $1/n = \Delta_n/(2\pi)$. Energy is conformally invariant, so the energy of each piece below may be computed in its own chart.

(i) In $\{|\tilde p| < 1 - \Delta_n\}$ the field is radial. Its energy there is at most $(2\pi)^{-1}\log(1/\check r)$.

(ii) In the cell $\{1 - \Delta_n < |\tilde p| < 1,\ |\arg\tilde p - \vartheta_j| < \Delta_n/2\}$, minus the half-disc where $|\tilde q| < \Delta_n/4$, the field carries the flux $1/n$ from the inner arc to the semicircle $|\tilde q| = \Delta_n/4$ inside $\mathbb{D}$, with zero flux through the arcs of the unit circle; so nothing enters the gaps between the slits. Rotating the $\tilde p$-plane, we may take $v_j = 1$. Take the rays $\tilde p = 1 - \check s\, e^{i\check\beta}$ from v_j, with $|\check\beta| \le \check\beta_0$, from $|\tilde q| = \Delta_n/4$ to the inner circle $|\tilde p| = 1 - \Delta_n$. The ray $\check\beta$ meets that circle at

$$\check s = \check\rho(\check\beta) := \cos\check\beta - \big(\cos^2\check\beta - 2\Delta_n + \Delta_n^2\big)^{1/2},$$

at the argument $-\check\phi(\check\beta)$, and $\check\beta_0$ is fixed by $\check\phi(\check\beta_0) = \Delta_n/2$. This cone lies in the cell: on its boundary rays $\check\beta = \pm\check\beta_0$, the function $|\tilde p|^2 = 1 - 2\check s\cos\check\beta_0 + \check s^2$ is convex in $\check s$ and decreasing at the inner end. Let the field be radial about v_j, with angular density

$$\check\psi(\check\beta) := \frac{1}{\Delta_n}\,\check\phi'(\check\beta)$$

(the flux in units of $1/n$). Then the inflow on the inner arc is uniform in $\arg\tilde p$, so it continues step (i) exactly. Nothing crosses the sides $\arg\tilde p = \pm\Delta_n/2$ of the cell or the unit circle, and the field leaves through the part $|\check\beta| \le \check\beta_0$ of the semicircle $|\tilde q| = \Delta_n/4$. Its energy is $\check C_1/n^2$, with

$$\check C_1 = \check C_1(\Delta_n) := \int \check\psi^2 \log\frac{4\check\rho}{\Delta_n}\, d\check\beta.$$

In the flux coordinate $\check w := \check\phi/\Delta_n \in [-\frac12, \frac12]$, elementary trigonometry in the triangle with vertices 0, v_j and $(1-\Delta_n)e^{-i\check\phi}$ gives $\tan\check\beta = \check N/\check D$ exactly, with

$$\check N := \frac{\sin(\Delta_n\check w)}{\Delta_n}, \qquad \check D := \frac{1}{1-\Delta_n} + \frac{2}{\Delta_n}\sin^2\frac{\Delta_n\check w}{2}.$$

Differentiating,

$$\frac{d\check\beta}{d\check w} = \frac{\cos(\Delta_n\check w)/(1-\Delta_n) - (2/\Delta_n)\sin^2(\Delta_n\check w/2)}{\check N^2 + \check D^2},$$

and the numerator is at least $\big(1 - (\Delta_n\check w)^2/2\big)(1+\Delta_n) - \Delta_n\check w^2/2 \ge 1$. As $\check\psi = d\check w/d\check\beta$, this gives $\check\psi \le \check N^2 + \check D^2$. By the law of sines, $\check\rho = \sin\check\phi/\sin(\check\beta + \check\phi) \le \sin\check\phi/\sin\check\beta = \Delta_n(\check N^2 + \check D^2)^{1/2}$, as $\check\beta + \check\phi < \pi/2$. Hence $\check\psi \le \check N^2 + \check D^2 \le 1 + \check w^2 + 2.285\,\Delta_n$ and $\check\rho/\Delta_n \le (\check N^2 + \check D^2)^{1/2}$. So, since $\check\psi^2\, d\check\beta = \check\psi\, d\check w$,

$$\check C_1(\Delta_n) \ \le\ \int_{-1/2}^{1/2} \check B\left(\log 4 + \frac12\log\check B\right) d\check w, \qquad \check B := 1.02285 + \check w^2,$$

and this is at most 1.60 for every $\Delta_n \le 0.01$.

(iii) In $\mathcal{T}_j$ we use the chart $\check\nu$ in the upper half-plane given by

$$u = i\pi + 2\big(\check\nu\sqrt{\check\nu^2 - 1} - \operatorname{arccosh}\check\nu\big), \qquad \frac{du}{d\check\nu} = 4\sqrt{\check\nu^2 - 1}.$$

This is the Schwarz–Christoffel map of the upper half-plane onto $\mathcal{P}$: the interval $(-1,1)$ goes to the window, the rays $\check\nu > 1$ and $\check\nu < -1$ to the edges, and $\check\nu = \pm1$ to the corners $u = \pm i\pi$, where $\mathcal{P}$ has the angle $3\pi/2$; and u is about $2\check\nu^2$ at infinity. In this chart the window and the edges lie on one line, so the field can be written down explicitly. The semicircle of (ii) is the part of the window with $|\operatorname{Im} u| < \arccos(\Delta_n/8)$, because on $|\tilde q| = \Delta_n/4$ we have $|\tilde p|^2 = 1 - (\Delta_n/2)\cos(\operatorname{Im} u) + \Delta_n^2/16$. In the chart it is a fixed interval of $(-1,1)$ as $\Delta_n \to 0$.

Matching at $|\tilde q| = \Delta_n/4$. There $\operatorname{Re} u = 0$ and $\operatorname{Im} u = \check\beta$. So the outflow of (ii) crosses the window at $\operatorname{Im} u = \check\beta$ with density $\check\psi(\check\beta)$, on $|\operatorname{Im} u| \le \check\beta_0$; here $\check\beta_0 < \arctan(1/2) < 0.46365 < \pi/2$. In the chart $\check\nu$, put $\check\nu = \sin\check\sigma$ on the window; then $\operatorname{Im} u = 2\check\sigma + \sin 2\check\sigma$. The flux enters through the interval $(-\check a, \check a)$, with $\check a \le 0.1162$.

The field in (iii). In the coordinate $\check\zeta := \log\check\nu = \check\xi + i\check\eta$, each half of the flux, $1/(2n)$, runs in a channel of height $\check h := \pi/8$ along $\check\eta = 0$ (resp. $\check\eta = \pi$) to $\check\xi = \log 2$. In the channel along $\check\eta = 0$, let $\check\mu(\check\xi)$ be the inflow per unit $\check\xi$ and $\check F(\check\xi)$ its integral, and take, in the coordinates $(\check\xi, \check\eta)$,

$$\mathcal{V} := \Big(\frac{\check F(\check\xi)}{\check h},\ \check\mu(\check\xi)\Big(1 - \frac{\check\eta}{\check h}\Big)\Big);$$

in the channel along $\check\eta = \pi$ the field is the mirror image, with $\pi - \check\eta$ in place of $\check\eta$. Then $\operatorname{div}\mathcal{V} = 0$. The normal flux is zero on the channel tops, on the real axis outside $(-\check a, \check a)$, and on the edges. So the field takes exactly the outflow of (ii), and it leaves through the two arcs $\{|\check\nu| = 2,\ 0 < \arg\check\nu < \pi/8\}$ and $\{|\check\nu| = 2,\ 7\pi/8 < \arg\check\nu < \pi\}$ with uniform angular density, which is exactly the inflow that the radial field of (iv) assumes. (The channels are the sectors $\{|\check\nu| < 2,\ 0 < \arg\check\nu < \pi/8\}$ and $\{|\check\nu| < 2,\ 7\pi/8 < \arg\check\nu < \pi\}$; on them $|u| \le 11.84 < \Xi_n$, so they lie in $\mathcal{T}_j$.) The energy of each channel is

$$\int \Big(\frac{\check F^2}{\check h} + \frac{\check\mu^2\,\check h}{3}\Big)\, d\check\xi,$$

so the energy in (iii) is $\check C_2/n^2$ with $\check C_2 \le 4.34$. Indeed, for $\check\xi < \log\check a$ we have $\check F \le \check K_F\, e^{\check\xi}$ with $\check K_F := 4.373$, and $\check\mu \le \check K_\mu\, e^{\check\xi}$ with $\check K_\mu := 5.0914$. These follow from $\check\psi \le 1.27285$, from $\check w \le 1.01136\tan(\operatorname{Im} u)$, from the bound of tan by 1.0785 times the identity on $[0, 0.46365]$, from $\operatorname{Im} u \le 4\check\sigma$, from $d(\operatorname{Im} u)/d\check\nu = 4\cos\check\sigma \le 4$, and from $\arcsin\check\nu \le 1.00227\,\check\nu$ on $[0, 0.1162]$. For $\check\xi > \log\check a$ we have $\check F = \frac12$ and $\check\mu = 0$. So each channel has energy at most

$$\frac{\log(2/\check a)}{4\check h} + \Big(\frac{\check K_F^2}{2\check h} + \frac{\check K_\mu^2\,\check h}{6}\Big)\check a^2 \ \le\ 1.8180 + 0.3516,$$

with $\check a \in [0.11504, 0.11618]$ (the first term is largest at the lower end, the second at the upper end). Hence $\check C_2 \le 2(1.8180 + 0.3516) = 4.3392 \le 4.34$.

(iv) From these arcs the field runs radially in the two sectors $\{|\check\nu| \ge 2,\ 0 \le \arg\check\nu \le \pi/8\}$ and $\{|\check\nu| \ge 2,\ 7\pi/8 \le \arg\check\nu \le \pi\}$, of total angle $\pi/4$, out to the line $\operatorname{Re} u = \Xi_n$. We show that each ray of these sectors meets this line exactly once, and not far out. The second sector is the mirror image of the first, so we consider the first. For $|\check\nu| \ge 2$ write $\sqrt{\check\nu^2 - 1} = \check\nu\sqrt{1 - \check\nu^{-2}}$. Then

$$\big|\arg\sqrt{1 - \check\nu^{-2}}\big| \le \tfrac12 \arcsin\tfrac14 < 0.13, \qquad \frac{\sqrt3}{2} \le \big|\sqrt{1 - \check\nu^{-2}}\big| \le \frac{\sqrt5}{2}.$$

So along each ray, as a function of $|\check\nu|$,

$$\frac{\partial \operatorname{Re} u}{\partial|\check\nu|} = 4|\check\nu|\operatorname{Re}\big(e^{2i\arg\check\nu}\sqrt{1 - \check\nu^{-2}}\big) \ \ge\ 4|\check\nu|\,\frac{\sqrt3}{2}\cos\Big(\frac{\pi}{4} + 0.13\Big) \ \ge\ 2|\check\nu|.$$

Hence $\operatorname{Re} u(\check\nu) \ge \operatorname{Re} u(2e^{i\arg\check\nu}) + |\check\nu|^2 - 4$, and $\operatorname{Re} u$ is strictly increasing along each ray. At $\check\nu = 2$ we have $u = i\pi + 2(2\sqrt3 - \operatorname{arccosh} 2)$, whose real part is $2(2\sqrt3 - \operatorname{arccosh} 2) = 4.29\ldots$, and on the circle $|\check\nu| = 2$ the derivative of u with respect to $\arg\check\nu$ has modulus at most $8\sqrt5$. So over the angle $\pi/8$ the real part of u on this circle stays in $(-3,\ 11.4)$. Therefore each ray meets $\operatorname{Re} u = \Xi_n$ exactly once, at a point with $|\check\nu|^2 \le \Xi_n + 7$. Since $\big|du/d|\check\nu|\big| \le 2\sqrt5\,|\check\nu|$, at that point

$$|u| \ \le\ \pi + 4.3 + 7.1 + 2.24\,|\check\nu|^2 \ \le\ 6\,\Xi_n.$$

As $H \geq 2\Xi_n^2 \geq 24\,\Xi_n$, the flow stops before $|u| = H$; so $\mathcal{V}$ does not depend on H. The energy in the two sectors is at most

$$\Big(\frac{1/n}{\pi/4}\Big)^2 \frac{\pi}{4} \log \frac{\sqrt{\Xi_n + 7}}{2} \;\leq\; \frac{2}{\pi}\frac{\log \Xi_n}{n^2} + \frac{\check{C}_3}{n^2},$$

with $\check{C}_3 := \frac{2}{\pi}\log\frac{1+7/\Xi_n}{4}$, which is negative for $\Xi_n \geq 12$.

The sum. The pieces (i)–(iv) are explicit fields whose normal fluxes match exactly across the curves they share, so Thomson's principle applies to the assembled field $\mathcal{V}$. Adding the n periods,

$$2\pi\Big(\|\mathcal{V}\|^2 - \frac{1}{2\pi}\log\frac{1}{\check{r}}\Big) \;\leq\; 2\pi n \cdot \frac{1}{n^2}\Big(\check{C}_1 + \check{C}_2 + \check{C}_3 + \frac{2}{\pi}\log\Xi_n\Big) \;\leq\; \Delta_n\Big(6 + \frac{2}{\pi}\log\Xi_n\Big),$$

since $2\pi n/n^2 = \Delta_n$, and since $\check{C}_1 + \check{C}_2 + \check{C}_3 \leq 6$ for $\Delta_n \leq 0.01$, as $\check{C}_1 \leq 1.60$, $\check{C}_2 \leq 4.34$ and $\check{C}_3 < 0$. As $\|\mathcal{V}\|^2 \geq (2\pi)^{-1}\log(r_{\mathcal{S}}/\check{r}) + o(1)$, letting $\check{r} \to 0$ proves the proposition. □

The numerical constants. We justify the numerical constants used above, for $\Delta_n \leq 0.01$. In (ii): $|\check{N}| \leq |\check{w}|$, and $\check{D} \leq 1/(1-\Delta_n) + \Delta_n\check{w}^2/2 \leq 1 + 1.1352\,\Delta_n$, so $\check{D}^2 \leq 1 + 2.2833\,\Delta_n$; this gives 2.285, and $1.27285 = 1 + 1/4 + 0.01 \cdot 2.285$. The bound for the numerator in (ii) follows from $\cos(\Delta_n\check{w}) \geq 1 - (\Delta_n\check{w})^2/2$, $1/(1-\Delta_n) \geq 1 + \Delta_n$ and $|\sin(\Delta_n\check{w}/2)| \leq \Delta_n|\check{w}|/2$; with $|\check{w}| \leq 1/2$ it gives at least $1 + \frac{7}{8}\Delta_n - (1+\Delta_n)\Delta_n^2/8 \geq 1$. Since $\log\check{B} \leq \check{B} - 1$, the integral for $\check{C}_1$ is at most $\log 4 \int \check{B} + \frac{1}{2}\int \check{B}(\check{B}-1) = 1.53350 + 0.06151 \leq 1.60$. In (iii): $\check{N} \geq \check{w}(1 - \Delta_n^2/24)$ for $0 \leq \check{w} \leq 1/2$, so $\tan\check{\beta} = \check{N}/\check{D}$ gives $\check{w} \leq 1.01136\tan\check{\beta}$. At $\check{w} = 1/2$ it gives $\tan\check{\beta}_0 \geq 0.494385$, so $\check{\beta}_0$ lies in $[0.459145, \arctan(1/2)]$. The functions tan and arcsin, divided by the identity, and $\check{\sigma} \mapsto 2\check{\sigma} + \sin 2\check{\sigma}$ increase. So 1.0785 and 1.00227 bound tan divided by the identity at 0.46365 and arcsin divided by the identity at 0.1162. Also, $2\check{\sigma} + \sin 2\check{\sigma} = 0.459143$ at $\check{\sigma} = \arcsin 0.11504$ and $= 0.463672$ at $\check{\sigma} = \arcsin 0.11618$, so $\check{a}$ lies in $[0.11504, 0.11618]$. Then $\check{K}_F = 4 \cdot 1.01136 \cdot 1.0785 \cdot 1.00227 = 4.3729\ldots$, rounded up, and $\check{K}_\mu = 4 \cdot 1.27285$.

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