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SMALE'S MEAN VALUE CONSTANT (1981) EQUALS 4

DAVE GOEL

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ABSTRACT. For a complex polynomial p of degree $d \geq 2$ with $p(0) = 0$ and $p'(0) \neq 0$, let $S(p)$ be the minimum of $|p(c)/(c\,p'(0))|$ over the critical points c of p, and let $K(d)$ be the supremum of $S(p)$ over all such p of degree d. In 1981 Smale proved that $K(d) \leq 4$ for every d, and asked whether 4 can be replaced by 1. We prove that Smale's mean value constant $\sup_d K(d)$ equals 4 and that $K(d) \to 4$ as $d \to \infty$, combining a reduction due to Jatar and Ng with an explicit construction. Each $K(d)$ is known to be below 4, so the supremum is not attained.[1] These results are verified in Lean.[2]

AI Disclosure. This work was done overwhelmingly by AI models, steered by the author in a continuous critical dialogue, variously with Google's Gemini, Anthropic's Claude and OpenAI's ChatGPT. The author takes full responsibility for the paper.

1. INTRODUCTION

Let p be a complex polynomial of degree $d \geq 2$ with $p(0) = 0$ and $p'(0) \neq 0$. A *critical point* of p is a zero of p'; since $p'(0) \neq 0$, no critical point is 0. Compare, at each critical point c, the slope $p(c)/c$ of the chord from 0 to c with the slope $p'(0)$ at 0, and put

$$S(p) := \min_{p'(c)=0} \left| \frac{p(c)}{c\,p'(0)} \right|.$$

Any point where the derivative does not vanish can be moved to 0 by a translation, and subtracting a constant then gives $p(0) = 0$, so the normalization loses nothing. Smale's mean value theorem [12] says that $S(p) \leq 4$ for every such p. For $d \geq 2$ let

$$K(d) := \sup\{S(p) : \deg p = d\}, \qquad K^* := \sup_{d \geq 2} K(d).$$

Thus $K(d)$ is the best constant in Smale's theorem for degree d, and K^*, *Smale's mean value constant*, is the best constant for all degrees. Jatar and Ng [9] call it M. The polynomial $z + z^d$ has $S = 1 - 1/d$, so $K^* \geq 1$, and Smale asked whether $K^* = 1$ [12, 14]. Smale listed this question again, as the Mean Value Problem, in the addenda to the second version of *Mathematical problems for the next century* [13, p. 21]. Moreover $K(d)$ does not decrease with d [4]. The idea is that adding εz^{d+1} to p, with ε small, moves the old critical points and the quotients $|p(c)/(c\,p'(0))|$ there only slightly, and creates one new critical point, of modulus of order $1/\varepsilon$, where this quotient is large. So $\lim_d K(d)$ exists and equals K^*.

Theorem 1.1. $K^* = \lim_{d \to \infty} K(d) = 4$.

So for every number $a < 4$ there is a polynomial with $|p(c)/(c\,p'(0))| > a$ at *every* critical point c.

The proof is not effective: for a given $a < 4$ it gives no bound on the least d with $K(d) > a$. The supremum is not attained: $K(d) \leq 4^{(d-2)/(d-1)} < 4$ [3], and $K(d) < 4 - 2.263/\sqrt{d}$ for $d \geq 8$ [5].

[1] Expanded version, with further results [6]: https://hexagonmath.org/2611.00283

[2] Theorem 1.1 is proved in Lean 4 from Lean's three standard axioms alone, and checked with `lake comparator`: https://github.com/david44hello/paper_smale-mean-value-constant-equals-4/tree/v3.10/lean

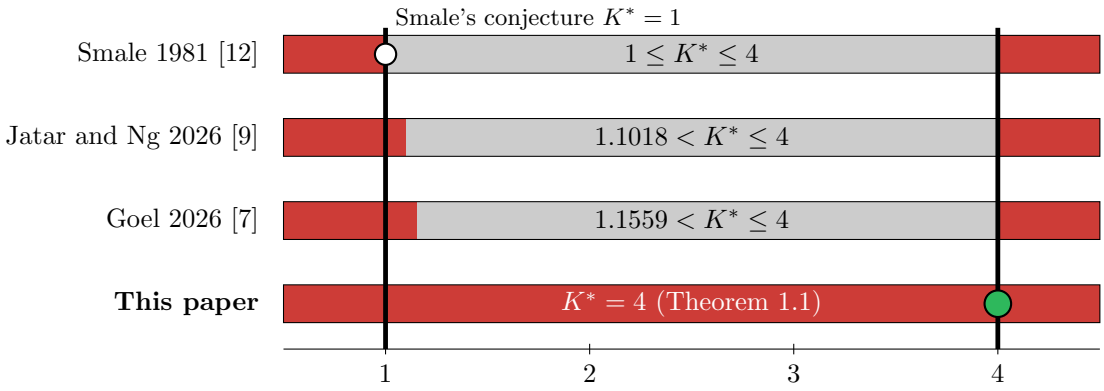


FIGURE 1. What is known about K^*, to scale. In each row the red parts are ruled out, and the grey part is where K^* may still lie; the black lines mark the interval $1 \le K^* \le 4$ of 1981. Smale proved $K^* \le 4$ in 1981 and conjectured $K^* = 1$ (white dot); that conjecture was disproved in 2026 [1, 2]. Theorem 1.1 shrinks the grey interval to the single point 4 (green dot).

Notation. $\mathbb{D}$ is the open unit disc and $\mathbb{T}$ the unit circle. A *Jordan domain* is a bounded domain whose boundary is a Jordan curve, and it is *analytic* if that curve is analytic. A function is *univalent* if it is holomorphic and injective. Koebe's quarter theorem says that if g is univalent on $\mathbb{D}$ and $g(0) = 0$, then $g(\mathbb{D})$ contains the disc of radius $|g'(0)|/4$ about 0; the growth theorem gives the lower bound $|g(t)| \ge |g'(0)|\,|t|/(1+|t|)^2$ on $\mathbb{D}$. The *Koebe function* $k(t) = t/(1+t)^2$ is univalent on $\mathbb{D}$ and maps it onto $\mathbb{C} \setminus [1/4, \infty)$. The boundary point of $k(\mathbb{D})$ nearest 0 is $1/4 = \lim_{t\to 1} k(t)$.

Earlier work. The conjecture $K^* = 1$ was disproved in 2026 [1, 2]. Jatar and Ng then proved $K^* > 1.1018$ [9], and the author proved $K^* > 1.1559$ [7] (Figure 1).

Our proof rests on a reduction due to Jatar and Ng [9, Proposition 4.3, Theorem 4.4] (Proposition 2.1 below). The reduction replaces polynomials by functions with nonvanishing derivative near the closure of a domain; it goes back to the construction of [1, 2] (see [2, Proposition 6.4]). Jatar and Ng show that K^* equals the supremum A of $\inf_{\partial\Omega} |f(z)/z|$ over such functions f on analytic Jordan domains $\Omega \ni 0$, normalized by $f(0) = 0$ and $f'(0) = 1$. In their notation our construction gives $A \ge 4$, so $A = M = 4$; this answers their question [9, Section 9] whether "one can get a larger lower bound of M through A". Theorem 5.1 of [9] shows that a different constant, the one in Smale's Problem 1A, equals 4; see Remark 6.3.

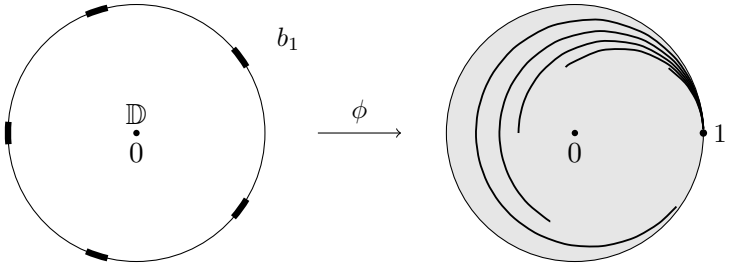
The idea of the proof. By Proposition 2.1, it is enough to find, for each $a < 4$, an analytic Jordan domain $\Omega \ni 0$ and a function f, holomorphic near $\overline{\Omega}$, with $f(0) = 0$, $f'(0) = 1$ and $f' \ne 0$ on $\overline{\Omega}$ such that $|f(z)/z| > a$ on $\partial\Omega$. For $a > 1$ the function f must have zeros other than 0 on $\overline{\Omega}$, so it is not injective (Remark 6.2). We look for f of the form $f = (W'(0)/F'(0))\, F \circ W^{-1}$, where W is univalent on $\mathbb{D}$ with $W(0) = 0$ and $\Omega = W(\rho\mathbb{D})$ for some $\rho < 1$ close to 1. If t_0 is a point of the circle $\rho\mathbb{T}$ where $|F|$ is least, then the value of $|f(z)/z|$ at $z = W(t_0)$ is

$$\frac{|F(t_0)|}{|F'(0)|} \cdot \frac{|W'(0)|}{|W(t_0)|}.$$

The first factor is at most ρ, by Schwarz's lemma: the branch of F^{-1} with value 0 at 0 extends to the disc $\{|v| < |F(t_0)|\}$ with values in $\rho\mathbb{D}$ (by analytic continuation of the inverse branch, using $F' \ne 0$; see Remark 6.1), so $|F(t_0)| \le \rho |F'(0)|$. The second is at most $(1+\rho)^2/\rho$, by Koebe's growth theorem. So this value is at most $(1+\rho)^2 < 4$. Our construction makes both bounds nearly sharp at the same point; this is possible because F and W can be chosen separately. The only link

between them is a set of n points of $\mathbb{T}$, the n-th roots $b_1, \dots, b_n$ of -1: near them $|F|$ dips, and there $|W|$ must be close to its smallest possible size.

We take $W = g \circ \phi - g(\phi(0))$, where g is a function close to the Koebe function and ϕ is an explicit univalent self-map of $\mathbb{D}$, with $\phi(0)$ close to 0 and $\phi'(0)$ close to 1, whose limit at *each* of the points b_j is the single point 1. The image $\phi(\mathbb{D})$ is the disc with n thin walls removed, and the channels between the walls all lead to the point 1. Since g sends 1 to the boundary point nearest 0, the second factor is close to 4 at each b_j. We construct F with $F'(0)$ close to 1 and with $|F|$ large near the whole circle, except near the points b_j, where $|F|$ drops to about 1. Away from these points $|F|$ is so large that $|f(z)/z|$ is large. Our function F is built from an entire function E with $E'(\xi) = \exp(\xi^n)(1 - \xi^n/c_\sigma)$, where c_σ is a large constant (Section 4). Figure 2 shows the two pieces.



$|F|$ dips to about 1 near the b_j $\phi(\mathbb{D})$, with its walls

FIGURE 2. The construction for $n = 5$, not to scale; the walls are drawn far apart. On the left is the unit disc; the thick arcs mark the neighbourhoods of the points b_j. On the right is $\phi(\mathbb{D})$, the disc minus 5 arcs of circles tangent to $\mathbb{T}$ at the point 1. The map ϕ sends points near each b_j into a channel between neighbouring walls, or between the outermost wall and $\mathbb{T}$, close to the point 1. The function g, close to the Koebe function, then sends the point 1 to the boundary point nearest 0.

The main step is the following theorem, which we prove in Section 5. For a function g on $\mathbb{D}$ we write $g(1) := \lim_{t \to 1,\, t \in \mathbb{D}} g(t)$, when this limit exists. We need the unrestricted limit, not only the radial or nontangential one, because $\phi(t)$ tends to 1 tangentially to $\mathbb{T}$ as $t \to b_j$.

Theorem 1.2. *Let g be a bounded univalent function on $\mathbb{D}$ with $g(0) = 0$ and $g'(0) = 1$, and suppose that g has a limit $g(1)$ at the point 1. Then $K^* \geq 1/|g(1)|$.*

The point $g(1)$ is not in $g(\mathbb{D})$, since g is open and injective, so it lies on $\partial g(\mathbb{D})$; hence $|g(1)| \geq 1/4$ by Koebe's theorem, and the lower bound $1/|g(1)|$ never exceeds 4.

2. From polynomials to pairs of functions on the disc

The following proposition is due to Jatar and Ng [9, Proposition 4.3]. For completeness we include the proof.

Proposition 2.1. *Let Ω be an analytic Jordan domain with $0 \in \Omega$. Let f be holomorphic on a neighbourhood of $\overline{\Omega}$, with $f(0) = 0$, $f'(0) = 1$ and $f' \neq 0$ on $\overline{\Omega}$. If $a > 0$ and $|f(z)/z| > a$ on $\partial\Omega$, then $K^* > a$.*

The proof uses the following form of Hilbert's lemniscate theorem [8, 10].

Lemma 2.2. *In a closed topological annulus there is a nonconstant polynomial q such that the whole set $\{|q| = 1\}$ is an analytic Jordan curve in the interior of the annulus, separating its two boundary components. For such a q, $|q| < 1$ inside this curve and $|q| > 1$ outside it, and all zeros and all critical points of q lie inside it.*

Proof. We derive this single-curve form from the classical proof. Take an analytic Jordan curve in the interior of the annulus that separates its boundary components, and let X be the closed region bounded by this curve. Let G_X be the Green function of $\mathbb{C} \setminus X$ with pole at ∞; it has no critical point, since the complement of X in the Riemann sphere is simply connected, so that G_X is the logarithm of the modulus of a conformal map of $\mathbb{C} \setminus X$ onto $\{|w| > 1\}$. Fix $\gamma > 0$ so small that the set $\{0 < G_X \le 2\gamma\}$ lies in the interior of the annulus. Let Q_j be a monic Fekete polynomial of degree j of X, and put $q_j := (e^\gamma \operatorname{cap} X)^{-j} Q_j$, where $\operatorname{cap} X$ is the capacity of X. Put $u_j := j^{-1} \log|q_j| - (G_X - \gamma) = j^{-1} \log|Q_j| - \log \operatorname{cap} X - G_X$. The zeros of Q_j (the Fekete points) lie on ∂X, so each u_j is harmonic on $\mathbb{C} \setminus X$; it extends harmonically to ∞ with $u_j(\infty) = 0$, since Q_j is monic and $G_X(z) = \log|z| - \log \operatorname{cap} X + o(1)$. By the Bernstein–Walsh inequality, $u_j \le \varepsilon_j := j^{-1} \log \|Q_j\|_X - \log \operatorname{cap} X$ on $\mathbb{C} \setminus X$, and $\varepsilon_j \to 0$ [11, Theorems 5.5.2, 5.5.4 and 5.5.7]. So $\varepsilon_j - u_j$ is a nonnegative harmonic function on $(\mathbb{C} \setminus X) \cup \{\infty\}$ with value ε_j at ∞, and Harnack's inequality on the compact set $\{G_X \ge \gamma/4\} \cup \{\infty\}$ gives $0 \le \varepsilon_j - u_j \le V_0 \varepsilon_j$ there, with V_0 independent of j. Hence $u_j \to 0$ uniformly on $\{G_X \ge \gamma/4\}$, including near ∞, and, by the interior estimates for harmonic functions, its first derivatives tend to 0 uniformly on $\{\gamma/2 \le G_X \le 2\gamma\}$. Consider three regions, for large j.

- On the collar $\{\gamma/2 \le G_X \le 2\gamma\}$ the gradient of G_X does not vanish. The gradient lines of G_X make the collar a product of $\{G_X = \gamma\}$ with an interval. Along each of them, parametrized by arc length, $j^{-1} \log|q_j|$ increases (its derivative tends to $|\nabla G_X| > 0$, uniformly on the compact collar) from about $-\gamma/2$ to about γ, so it vanishes exactly once. Hence $\{|q_j| = 1\}$ meets the collar in a graph over $\{G_X = \gamma\}$ along the gradient lines, that is, in one Jordan curve close to $\{G_X = \gamma\}$; it is analytic, since $q_j \ne 0$ and $\nabla \log|q_j| \ne 0$ in the collar.
- On $\{G_X \ge 2\gamma\}$, up to ∞, we have $j^{-1} \log|q_j| \ge \gamma/2$, so $|q_j| > 1$.
- On $\{G_X \le \gamma/2\}$, which contains X, the Bernstein–Walsh inequality $|Q_j| \le \|Q_j\|_X \, e^{jG_X}$ [11, Theorem 5.5.7(a)] and $\|Q_j\|_X^{1/j} \to \operatorname{cap} X$ [11, Theorems 5.5.2 and 5.5.4] give $|q_j| < 1$.

So $q := q_j$ has the required property.

For such a q, $|q| < 1$ inside this curve and $|q| > 1$ outside it, by the maximum principle and connectedness. Since $|q| > 1$ outside the curve, all $\deg q$ zeros of q lie inside it; so q maps the inside of the curve onto $\mathbb{D}$ as a proper map of degree $\deg q$, and by the Riemann–Hurwitz formula it has $\deg q - 1$ critical points there, and these are all of them. □

Proof of Proposition 2.1. Step 1: a polynomial close to f. By continuity we can find $h > 0$ and the closure N of a Jordan domain, with $\overline{\Omega} \subset \operatorname{int} N$, such that f is holomorphic near N, $f' \ne 0$ on N, and $|f(z)/z| > a + 2h$ on $N \setminus \Omega$. Since $\mathbb{C} \setminus N$ is connected, Runge's theorem gives polynomials converging to f uniformly near N, and by Cauchy's estimates their derivatives converge on N. Subtracting the constant term, we obtain a polynomial p with $p(0) = 0$, with $p' \ne 0$ on N, with $|p(z)/z| > a + h$ on $N \setminus \Omega$, and with $|p'(0)| < (a + h)/(a + h/2)$.

Step 2: a lemniscate. Apply Lemma 2.2 to the closed annulus $N \setminus \Omega$. We obtain a polynomial q whose lemniscate $\{|q| = 1\}$ is a Jordan curve in the interior of this annulus. Then $\overline{\Omega}$ lies inside the curve, so $|q| < 1$ on $\overline{\Omega}$. Moreover $zq'(z) \ne 0$ on the curve and outside it, because 0 and all critical points of q lie inside. Fix $\nu \in (0, 1)$ so small that the compact set $A_\nu := \{1 - \nu \le |q| \le 1 + \nu\}$ lies in $\operatorname{int} N \setminus \overline{\Omega}$, and so that $\{|q| < 1 - \nu\}$ contains $\overline{\Omega}$ and every critical point of q.

Step 3: the polynomials H_m. Let $H_m := p + zq^m$, a polynomial of degree $m \deg q + 1$ once m is large. Then $H_m(0) = 0$, and

$$H_m' = p' + q^{m-1}(q + mzq').$$

We show that for all large m every critical point of H_m lies in A_ν.

(i) On $\{|q| \le 1 - \nu\}$, a compact subset of N, we have $m|q|^{m-1} \to 0$ uniformly, so $H_m' \to p'$ uniformly. Since $p' \ne 0$ on N, the polynomial H_m has no critical point there.

(ii) On $\{|q| \geq 1+\nu\}$ the polynomial zq' has no zero, and it has the same degree as q. Hence there are positive constants V_1, V_2, V_3 with $|zq'| \geq V_1(1+|z|)^{\deg q}$, with $|q| \geq V_2(1+|z|)^{\deg q}$ and with $|q/(zq')| \leq V_3$ on this set. So $|q+mzq'| \geq (m/2)|zq'|$ for large m. Let $d_1 := \deg p'$, so that $|p'| \leq V_4(1+|z|)^{d_1}$ on this set for some V_4, and let $m-1 \geq d_1$. Since $|q| \geq 1+\nu$ there, $|q|^{m-1} \geq (1+\nu)^{m-1-d_1}|q|^{d_1}$, and $|q|^{d_1} \geq V_2^{d_1}(1+|z|)^{d_1 \deg q} \geq V_2^{d_1}(1+|z|)^{d_1}$. With $|q+mzq'| \geq (m/2)V_1(1+|z|)^{\deg q} \geq (m/2)V_1$ we get

$$\left|q^{m-1}(q+mzq')\right| \;\geq\; (1+\nu)^{m-1-d_1}\, V_2^{d_1}(1+|z|)^{d_1} \cdot \frac{m}{2}\, V_1 \;>\; V_4(1+|z|)^{d_1} \;\geq\; |p'|$$

for large m, uniformly on this set. So there is no critical point there.

(iii) Therefore every critical point c of H_m lies in A_ν. On A_ν we have $|zq'| \geq V_5 > 0$, $|q| \leq 2$ and $|p'| \leq V_6$, so $H_m'(c) = 0$ gives

$$|q(c)|^{m-1} = \frac{|p'(c)|}{|q(c)+mcq'(c)|} \leq \frac{2V_6}{V_5 m}$$

for large m, and hence $|q(c)|^m = O(1/m)$. Since $H_m(c)/c = p(c)/c + q(c)^m$ and $c \in N \setminus \Omega$, we obtain $|H_m(c)/c| \geq a + h - O(1/m)$, uniformly over the critical points. Finally $H_m'(0) = p'(0) + q(0)^m \to p'(0)$, because $|q(0)| < 1$. Therefore

$$\liminf_{m\to\infty} S(H_m) \;\geq\; \frac{a+h}{|p'(0)|} \;>\; a + \frac{h}{2},$$

and so $K^* > a$. □

We use Proposition 2.1 in the following form.

Lemma 2.3. *Let W be univalent on $\mathbb{D}$ with $W(0) = 0$, and let F be holomorphic on $\mathbb{D}$ with $F(0) = 0$ and $F' \neq 0$ on $\mathbb{D}$. If $0 < \rho < 1$, $s > 0$, and*

$$|F(t)|\,|W'(0)| \;>\; s\,|F'(0)|\,|W(t)| \qquad \textit{for all } t \textit{ with } |t| = \rho,$$

then $K^* > s$.

Proof. The set $\Omega := W(\rho\mathbb{D})$ is an analytic Jordan domain whose closure lies in $W(\mathbb{D})$. The function $f := (W'(0)/F'(0))\, F \circ W^{-1}$ is holomorphic on $W(\mathbb{D})$, with $f(0) = 0$, $f'(0) = 1$ and $f' \neq 0$. On $\partial\Omega$, with $z = W(t)$ and $|t| = \rho$, we have $|f(z)/z| = |W'(0)/F'(0)|\,|F(t)/W(t)| > s$. Proposition 2.1 gives $K^* > s$. □

3. The map W

The map $\mu(t) := i(1+t)/(1-t)$ sends $\mathbb{D}$ conformally onto the upper half-plane $\mathbb{H}$, with $\mu(0) = i$ and $\mu'(0) = 2i$. Its inverse is $\mu^{-1}(w) = (w-i)/(w+i)$, so that $\mu^{-1}(w) - 1 = -2i/(w+i)$. Let $P \subset \mathbb{T} \setminus \{1\}$ be a finite nonempty set. For $b \in P$ the point $x_b := \mu(b)$ is real. For $\lambda > 0$ let

$$\Gamma_\lambda(z) := z + \lambda \sum_{b\in P} \log(z - x_b) \quad (z \in \mathbb{H}), \qquad \phi_\lambda := \mu^{-1} \circ \Gamma_\lambda \circ \mu,$$

with the principal branch of the logarithm.

Lemma 3.1. (a) *ϕ_λ is a univalent map of $\mathbb{D}$ into $\mathbb{D}$.*
(b) *For every $b \in P$, $\phi_\lambda(t) \to 1$ as $t \to b$ in $\mathbb{D}$.*
(c) *As $\lambda \to 0$, $\phi_\lambda(0) \to 0$ and $\phi_\lambda'(0) \to 1$.*

Proof. (a) We have $\operatorname{Im}\Gamma_\lambda(z) = \operatorname{Im} z + \lambda \sum_b \arg(z - x_b) > 0$, so Γ_λ maps $\mathbb{H}$ into $\mathbb{H}$. Further, $\Gamma_\lambda' = 1 + \lambda \sum_b 1/(z - x_b)$ has negative imaginary part on $\mathbb{H}$. Since $\mathbb{H}$ is convex, the difference quotient of Γ_λ between two points of $\mathbb{H}$, being the mean of Γ_λ' on the segment between them, has negative imaginary part, and so is not 0. Hence Γ_λ is injective.

(b) As $t \to b$, $z = \mu(t) \to x_b$ in $\mathbb{H}$. Then $\operatorname{Re}\Gamma_\lambda(z) = \lambda \log|z - x_b| + (\text{a term bounded near } x_b) \to -\infty$, so $\Gamma_\lambda(z) \to \infty$, and $\phi_\lambda(t) - 1 = -2i/(\Gamma_\lambda(z) + i) \to 0$. This holds however t approaches b.

(c) As $\lambda \to 0$, $\Gamma_\lambda(i) \to i$ and $\Gamma'_\lambda(i) \to 1$. So $\phi_\lambda(0) = \mu^{-1}(\Gamma_\lambda(i)) \to 0$, and $\phi'_\lambda(0) = -4\Gamma'_\lambda(i)/(\Gamma_\lambda(i) + i)^2 \to 1$. □

Figure 2 shows the shape of $\phi_\lambda(\mathbb{D})$. On each interval of the real axis whose right end is a point x_b, including the unbounded interval $(-\infty, \min_b x_b)$, the imaginary part of Γ_λ is constant and its real part runs from $-\infty$ up to a maximum and back. One checks that $\Gamma_\lambda(\mathbb{H})$ is $\mathbb{H}$ minus horizontal rays going to the left, and $\phi_\lambda(\mathbb{D})$ is $\mathbb{D}$ minus arcs of circles tangent to $\mathbb{T}$ at the point 1: the walls of Figure 2. This description is not used in the proofs.

Proposition 3.2. *Let g be as in Theorem 1.2, and let $B := \sup_{\mathbb{D}} |g|$. For every $\varepsilon \in (0,1)$ and every finite nonempty set $P \subset \mathbb{T} \setminus \{1\}$ we can find a univalent function W on $\mathbb{D}$ with $W(0) = 0$ and a number $\theta_0 > 0$ such that $|W'(0)| \geq 1 - \varepsilon$, $|W| \leq 2B$ on $\mathbb{D}$, and $|W(t)| < (1+\varepsilon)|g(1)|$ for every $t \in \mathbb{D}$ with $\operatorname{dist}(t, P) < \theta_0$.*

Note that the bound $2B$ does not depend on P. This is what allows us, in Section 5, to fix the size of $|F|$ before the number of points in P.

Proof. Let $W := g \circ \phi_\lambda - g(\phi_\lambda(0))$. This is univalent on $\mathbb{D}$ by Lemma 3.1(a), $W(0) = 0$, and $|W| \leq 2B$. By Lemma 3.1(c), as $\lambda \to 0$ we have $g(\phi_\lambda(0)) \to 0$ and $W'(0) = g'(\phi_\lambda(0))\phi'_\lambda(0) \to 1$. Fix λ so small that $|W'(0)| \geq 1 - \varepsilon$ and $|g(\phi_\lambda(0))| < (\varepsilon/2)|g(1)|$. By Lemma 3.1(b) and the limit of g at 1, for each $b \in P$ we have $|g(\phi_\lambda(t)) - g(1)| < (\varepsilon/2)|g(1)|$ for all $t \in \mathbb{D}$ near b. Since P is finite, this gives θ_0. □

4. The function F

Throughout this section $n \geq 8$ is an integer, $\sigma \geq 100$, $c_\sigma := 2\sigma$, and $R \geq 1$, with

$$200\,\sigma n \leq e^{(\sigma-1)/2}, \qquad R \leq e^{(\sigma-1)/2}. \tag{1}$$

Proposition 4.1. *For all $\varepsilon, \theta > 0$ we can find a function F, holomorphic on $\mathbb{D}$ with $F(0) = 0$ and $F' \neq 0$ on $\mathbb{D}$, and a number $r_0 \in (0,1)$, such that $0 < F'(0) \leq (2\sigma)^{1/n}$ and, for every t with $r_0 < |t| < 1$,*

$$|F(t)| \geq R, \qquad \text{or else both} \quad |F(t)| \geq \frac{n}{n+1} - \varepsilon \quad \text{and} \quad |t^n + 1| < \theta.$$

The plan of the proof is as follows (Figure 3). We put $F := E \circ \xi$, where E is the entire function of Step 1 and $\xi(t)^n = \eta(t^n)$, with η a conformal map of $\mathbb{D}$ onto the half strip $\Pi := \{y : \operatorname{Re} y < \sigma,\ |\operatorname{Im} y| < C\}$, where C is a large odd multiple of π, chosen last (Step 2; oddness is used in Step 4). Then $|F(t)| = |\Phi(y)|$ with $y = \eta(t^n)$, where $\Phi(y) := E(y^{1/n})$, and as $|t| \to 1$ the point y tends to the boundary of Π. In the band of width 1 along the right edge $\operatorname{Re} y = \sigma$ we show $|\Phi| \geq R$ (Step 3). In the bands of width $\frac{1}{2}$ along the two horizontal edges, and to the left of $\operatorname{Re} y = -C$, we show that $|\Phi|$ is at least about $n/(n+1)$ (Steps 4 and 5). The rest of Π is a compact set, so for $|t|$ close to 1 the point y lies in one of these regions; it lies outside the right band only when t^n is close to -1, that is, when t is close to one of the n-th roots of -1 (Step 6). On Π we have $\operatorname{Re} y < \sigma = c_\sigma/2$, so the factor $1 - y/c_\sigma$ (Step 1) does not vanish there, and it is bounded below in terms of $|y|$ (Step 3): $|1 - y/c_\sigma| \geq |y|/(6c_\sigma)$. This lower bound is what lets Step 3 bound $|\Phi|$ from below along the whole right edge, at every height up to C, although C is chosen last. The condition $n \geq 8$ is used in Step 4.

Proof. Step 1: the function E. Let

$$E(\xi) := \int_0^\xi \exp(\zeta^n)\Big(1 - \frac{\zeta^n}{c_\sigma}\Big)\, d\zeta.$$

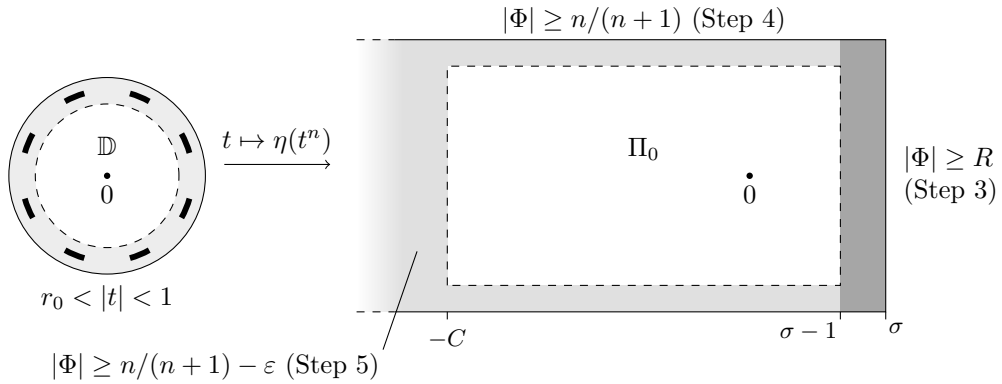


FIGURE 3. The plan of Section 4, for $n = 8$, not to scale. The map $t \mapsto \eta(t^n)$ sends $\mathbb{D}$ onto the half strip Π. Off the compact set Π_0, Steps 3–5 bound $|\Phi|$ from below (the band at the bottom edge is like the one at the top; in the far left, ε comes from the choice of C at the end of Step 6). For $r_0 < |t| < 1$ the point $\eta(t^n)$ lies outside Π_0, and it lies outside the dark band only when t is close to an n-th root of -1 (thick arcs; Step 6).

This is an entire function with $E'(\xi) = \exp(\xi^n)(1 - \xi^n/c_\sigma)$, and $E(\omega\xi) = \omega E(\xi)$ whenever $\omega^n = 1$. Integrating by parts, with $E_0(\xi) := \int_0^\xi \exp(\zeta^n)\, d\zeta$ and $\kappa := 1/(c_\sigma n)$,

$$E(\xi) = (1+\kappa)E_0(\xi) - \kappa\, \xi \exp(\xi^n). \tag{2}$$

On $\mathbb{C} \setminus (-\infty, 0]$, with principal branches and $u = \zeta^n$, put

$$\Phi(y) := E(y^{1/n}) = \int_0^y J(u)\, du, \qquad J(u) := \frac{1}{n}\, e^u \chi(u), \qquad \chi(u) := u^{1/n-1}\Big(1 - \frac{u}{c_\sigma}\Big).$$

The function Φ itself jumps across the negative axis, but since $\Phi(\bar{y}) = \overline{\Phi(y)}$, its modulus $|\Phi|$ extends continuously to all of $\mathbb{C}$. Every ξ with ξ^n off the negative axis has the form $\omega\, (\xi^n)^{1/n}$ with $\omega^n = 1$, so the rotation symmetry of E and continuity give

$$|E(\xi)| = |\Phi(\xi^n)| \qquad (\xi \in \mathbb{C}). \tag{3}$$

So it suffices to bound $|\Phi|$ on the closed upper half-plane, with boundary values from above on the negative axis.

Step 2: the half strip and the definition of F. Let $C \geq 3\pi$ be an odd multiple of π, chosen last. Let $\beta := \pi/(2C)$, $\alpha := \sinh(\beta\sigma)$, and

$$\Pi := \{y : \operatorname{Re} y < \sigma,\ |\operatorname{Im} y| < C\}.$$

The function $w(y) := \sinh(\beta(y - \sigma))$ maps Π conformally onto the left half-plane: $y \mapsto \beta(y - \sigma)$ maps Π onto the half strip $\{\operatorname{Re} z < 0,\ |\operatorname{Im} z| < \pi/2\}$, which sinh maps conformally onto the left half-plane. So

$$\Theta(y) := \frac{\alpha + w(y)}{\alpha - w(y)}$$

maps Π conformally onto $\mathbb{D}$, with $\Theta(0) = 0$ since $w(0) = -\alpha$. Let $\eta := \Theta^{-1} : \mathbb{D} \to \Pi$. Then $\eta(0) = 0$, and from $w(\eta(t)) = -\alpha(1-t)/(1+t)$ we get

$$\eta'(0) = \frac{2\alpha}{\beta \cosh(\beta\sigma)} = \frac{2 \tanh(\beta\sigma)}{\beta} \in (0, 2\sigma].$$

The quotient $\eta(t)/t$ has no zero on $\mathbb{D}$, so

$$\xi(t) := t\left(\frac{\eta(t^n)}{t^n}\right)^{1/n},$$

with the branch that is positive at $t = 0$, is holomorphic on $\mathbb{D}$. It satisfies $\xi(t)^n = \eta(t^n)$ and $\xi(\omega t) = \omega\xi(t)$ for $\omega^n = 1$. These two identities show that ξ is injective: if $\xi(t_1) = \xi(t_2)$, then $\eta(t_1^n) = \eta(t_2^n)$, so $t_1^n = t_2^n$ and $t_2 = \omega t_1$ with $\omega^n = 1$; then $\xi(t_1) = \xi(t_2) = \omega\xi(t_1)$, so $\omega = 1$ or $\xi(t_1) = 0$, that is, $t_1 = t_2 = 0$. Hence $\xi' \neq 0$. Now let $F := E \circ \xi$. Then $F(0) = 0$, and

$$F'(t) = \exp(\eta(t^n))\Big(1 - \frac{\eta(t^n)}{c_\sigma}\Big)\xi'(t) \neq 0,$$

because $\operatorname{Re}\eta < \sigma < c_\sigma$. Also $F'(0) = \eta'(0)^{1/n} \leq (2\sigma)^{1/n}$. By (3),

$$|F(t)| = |\Phi(\eta(t^n))| \qquad (t \in \mathbb{D}). \tag{4}$$

Step 3: the right band, $\sigma - 1 \leq \operatorname{Re} y < \sigma$: here $|\Phi(y)| \geq R$. Let $y = x + i\tau$ with $\tau \geq 0$ and $\sigma - 1 \leq x < \sigma$, so that $|y| \geq x \geq 99$. Since $\operatorname{Re} y \leq c_\sigma/2$, we have $1 + |y|/c_\sigma \leq 6|1 - y/c_\sigma|$: if $|y| \leq 2c_\sigma$, then $|1 - y/c_\sigma| \geq 1 - \operatorname{Re} y/c_\sigma \geq \frac{1}{2}$ and $1 + |y|/c_\sigma \leq 3$; if $|y| > 2c_\sigma$, then $|1 - y/c_\sigma| \geq |y|/c_\sigma - 1 \geq |y|/(2c_\sigma)$ and $1 + |y|/c_\sigma \leq 3|y|/(2c_\sigma)$. Hence $|1 - y/c_\sigma| \geq |y|/(6c_\sigma)$. Let $T_0 := (1/n)e^x|\chi(y)|$, the modulus of the main term $(1/n)e^y\chi(y)$. Then

$$T_0 \;\geq\; \frac{e^x|y|^{1/n}}{6c_\sigma n}.$$

To bound the rest of $\Phi(y)$, note that the function $\Phi(u) - (1/n)e^u\chi(u)$ has derivative $-(1/n)e^u\chi'(u)$, since $\Phi' = J = (1/n)e^u\chi$. Integrate this derivative along the horizontal segment from $y_0 := x/2 + i\tau$ to y. For u on this segment, $|y|/2 \leq |u| \leq |y|$ and $|c_\sigma - u| \geq \sigma$. So $|\chi(u)| \leq 2(1 + |y|/c_\sigma)|y|^{1/n-1} \leq 12|\chi(y)|$ (the first step uses $|y|/2 \leq |u| \leq |y|$, the second $1 + |y|/c_\sigma \leq 6|1 - y/c_\sigma|$), and $|\chi'(u)/\chi(u)| \leq 1/|u| + 1/|c_\sigma - u| \leq 3/(\sigma - 1)$. Therefore

$$\Big|\Phi(y) - \frac{1}{n}e^y\chi(y)\Big| \;\leq\; |\Phi(y_0)| + \frac{1}{n}e^{x/2}|\chi(y_0)| + \frac{1}{n}\int_{x/2}^{x} e^v\,\frac{36|\chi(y)|}{\sigma - 1}\,dv.$$

We bound the three terms on the right in turn.

(i) Put $\xi_0 := y_0^{1/n}$, so that $\Phi(y_0) = E(\xi_0)$. On the segment from 0 to ξ_0 we have $|\exp(\zeta^n)| \leq e^{x/2}$, so $|E_0(\xi_0)| \leq |\xi_0|e^{x/2}$; also $|\xi_0\exp(\xi_0^n)| = |\xi_0|e^{x/2}$. By (2), $|y_0| \leq |y|$ and the lower bound for T_0,

$$|\Phi(y_0)| \leq (1 + 2\kappa)|y_0|^{1/n}e^{x/2} \leq 2|y|^{1/n}e^{x/2} \leq 12c_\sigma n\,e^{-x/2}\,T_0 \leq \frac{24}{200}\,T_0,$$

the last step because $12c_\sigma n\,e^{-x/2} = 24\sigma n\,e^{-x/2} \leq 24\sigma n\,e^{-(\sigma-1)/2} \leq 24/200$ by (1).

(ii) Since $|\chi(y_0)| \leq 12|\chi(y)|$, the second term is at most $12e^{-x/2}T_0 \leq 10^{-4}\,T_0$, as $x \geq 99$.

(iii) Since $\int_{x/2}^{x} e^v\,dv \leq e^x$, the third term is at most $36T_0/(\sigma - 1) \leq (36/99)\,T_0$.

Since $24/200 + 10^{-4} + 36/99 < 1/2$, the total is at most $T_0/2$, and

$$|\Phi(y)| \;\geq\; \frac{T_0}{2} \;\geq\; \frac{e^x}{24\sigma n} \;\geq\; \frac{e^{\sigma-1}}{24\sigma n} \;\geq\; \frac{200}{24}\,e^{(\sigma-1)/2} \;\geq\; R.$$

Step 4: the band $C - \frac{1}{2} \leq \operatorname{Im} y < C$: here $|\Phi(y)| \geq n/(n+1)$, if $10c_\sigma \leq C - \frac{1}{2}$. Fix τ in this range, and let $u = v + i\tau$ run along the horizontal line. Write

$$J(u) = \frac{1}{nc_\sigma}\,e^v\,\big(-e^{i\tau}\big)\,u^{1/n}\Big(1 - \frac{c_\sigma}{u}\Big).$$

Since C is an odd multiple of π, the factor $-e^{i\tau}$ has argument in $[-\frac{1}{2}, 0]$. The factor $e^{-i\pi/n}u^{1/n}$ has argument in $(-\pi/n, 0)$. And $|c_\sigma/u| \leq 1/10$, so $1 - c_\sigma/u$ has argument of modulus at most 0.11. Since $\frac{1}{2} + \pi/8 + 0.11 < \pi/2$, with $n \geq 8$ the total argument of $e^{-i\pi/n}J(u)$ has modulus below $\pi/2$,

so $\mathrm{Re}(e^{-i\pi/n}J) > 0$ along the whole line, and $\mathrm{Re}(e^{-i\pi/n}\Phi)$ increases along it as v increases. As $v \to -\infty$, $\Phi(v+i\tau) - \Phi(v) \to 0$, where $\Phi(v)$ is the boundary value from above. Indeed, on the vertical segment from v to $v+i\tau$, of length less than C, we have $|J| \le (1/n)e^v(1+(|v|+C)/c_\sigma)$ once $|v| \ge 1$, because then $|u|^{1/n-1} \le 1$ and $|1-u/c_\sigma| \le 1+(|v|+C)/c_\sigma$. So the integral of J over the segment is at most C times this bound, which tends to 0. On the negative axis $\Phi(-r^n) = e^{i\pi/n}\int_0^r \exp(-\zeta^n)(1+\zeta^n/c_\sigma)\,d\zeta$. So the limit of Φ at the far left of the line is $e^{i\pi/n}\Lambda_n$, where

$$\Lambda_n := \int_0^\infty \exp(-\zeta^n)\Big(1+\frac{\zeta^n}{c_\sigma}\Big)\,d\zeta \;\ge\; \int_0^1 (1-\zeta^n)\,d\zeta = \frac{n}{n+1}.$$

Hence $|\Phi(y)| \ge \mathrm{Re}(e^{-i\pi/n}\Phi(y)) \ge n/(n+1)$.

Step 5: the far left, $\mathrm{Re}\,y \le -C$, $0 \le \mathrm{Im}\,y < C$*: here* $|\Phi(y)| \ge n/(n+1) - (C/n)e^{-C}(1+2C/c_\sigma)$. Let $y = x+i\tau$. On the negative axis, $|\Phi(x)| \ge \int_0^1(1-\zeta^n)\,d\zeta = n/(n+1)$, because $|x|^{1/n} \ge 1$. On the vertical segment from x to y, of length at most C, we have $1 \le |u| \le |x|+C$, so $|J(u)| \le (1/n)e^x(1+(|x|+C)/c_\sigma) \le (1/n)e^{-C}(1+2C/c_\sigma)$, the last step because the middle expression decreases as $|x|$ grows. Integrating J along this segment gives $|\Phi(y)-\Phi(x)| \le (C/n)e^{-C}(1+2C/c_\sigma)$, and the bound follows.

Step 6: where η *goes, and the conclusion.* Let

$$\Pi_0 := \{y : -C \le \mathrm{Re}\,y \le \sigma - 1,\ |\,\mathrm{Im}\,y| \le C - \tfrac{1}{2}\},$$

a compact subset of Π, and let $r_0 := (\max_{\Pi_0}|\Theta|)^{1/n} < 1$. Let $r_0 < |t| < 1$ and $y := \eta(t^n)$. Then $|\Theta(y)| = |t|^n > r_0^n = \max_{\Pi_0}|\Theta|$, so $y \notin \Pi_0$. If $\mathrm{Re}\,y > \sigma - 1$, Step 3 and (4) give $|F(t)| \ge R$. Otherwise $\mathrm{Re}\,y < -C$ or $|\,\mathrm{Im}\,y| > C - \frac{1}{2}$, and Steps 4 and 5 bound $|F(t)|$ from below. In this case t^n is close to -1. Indeed $|\sinh(a_1+ia_2)|^2 = \sinh^2 a_1 + \sin^2 a_2$ shows that $|w(y)| \ge \sinh(\pi/2)$ if $\mathrm{Re}\,y < -C$, and $|w(y)| \ge \cos(\pi/(4C)) \ge \frac{1}{2}$ if $|\,\mathrm{Im}\,y| > C - \frac{1}{2}$. Since $1+t^n = 2\alpha/(\alpha - w)$ and $|\alpha - w| \ge |w|$ for $\mathrm{Re}\,w < 0$, we get $|t^n+1| \le 4\alpha$.

Given $\varepsilon, \theta > 0$, choose C so large that $10c_\sigma \le C - \frac{1}{2}$, that $(C/n)e^{-C}(1+2C/c_\sigma) \le \varepsilon$, and that $4\alpha = 4\sinh(\pi\sigma/(2C)) < \theta$. This proves the proposition. □

5. Proof of the theorems

Proof of Theorem 1.2. Let $B := \sup_{\mathbb{D}}|g|$, so that $B \ge |g(1)|$, and let $R := 2B/|g(1)| \ge 2$. Let $\varepsilon \in (0,1)$. For a large integer ℓ put $\sigma := \ell$ and $n := \ell^2$. Since R does not depend on ℓ, taking ℓ large we have $\sigma \ge 100$, $n \ge 8$, the conditions (1), the bound $(2\sigma)^{1/n} \le 1+\varepsilon$ (by Bernoulli's inequality, $(1+\varepsilon)^{\ell^2} \ge 1+\ell^2\varepsilon \ge 2\ell$), and $n/(n+1) \ge 1-\varepsilon/2$.

Let $P := \{b_1, \dots, b_n\}$ be the set of n-th roots of -1; it does not contain 1. Let W and θ_0 be as in Proposition 3.2. By compactness there is $\theta > 0$ with $|t^n+1| \ge \theta$ for all $t \in \overline{\mathbb{D}}$ with $\mathrm{dist}(t,P) \ge \theta_0$. Let F and r_0 be as in Proposition 4.1, for $\varepsilon/2$ and this θ, and fix $\rho \in (r_0, 1)$. Let $|t| = \rho$.

- If $|t^n+1| < \theta$, then $\mathrm{dist}(t,P) < \theta_0$, so $|W(t)| < (1+\varepsilon)|g(1)|$. Also $|F(t)| \ge 1-\varepsilon$, because $R \ge 2$ and $n/(n+1) - \varepsilon/2 \ge 1-\varepsilon$. So $|F(t)|/|W(t)| > (1-\varepsilon)/((1+\varepsilon)|g(1)|)$.
- If $|t^n+1| \ge \theta$, then $|F(t)| \ge R$ by Proposition 4.1, since its other case needs $|t^n+1| < \theta$. Also $|W(t)| \le 2B$, so $|F(t)|/|W(t)| \ge R/(2B) = 1/|g(1)| > (1-\varepsilon)/((1+\varepsilon)|g(1)|)$.

So Lemma 2.3 applies with $s = (1-\varepsilon)|W'(0)|/((1+\varepsilon)|g(1)|F'(0))$. Using $|W'(0)| \ge 1-\varepsilon$ and $F'(0) \le 1+\varepsilon$,

$$K^* \;>\; \Big(\frac{1-\varepsilon}{1+\varepsilon}\Big)^2 \frac{1}{|g(1)|}.$$

Letting $\varepsilon \to 0$ gives the theorem. □

Proof of Theorem 1.1. Let k be the Koebe function, and for $0 < \delta < 1$ let $g(t) := k(\delta t)/\delta = t/(1+\delta t)^2$. Then g is univalent on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, $|g| \le (1-\delta)^{-2}$, $g(0) = 0$, $g'(0) = 1$, and $g(1) = (1+\delta)^{-2}$. Theorem 1.2 gives $K^* \ge (1+\delta)^2$, and letting $\delta \to 1$ gives $K^* \ge 4$. Smale's

theorem gives $K^* \le 4$. So $K^* = 4$, and $\lim_{d\to\infty} K(d) = K^*$ because $K(d)$ does not decrease with d [4]. □

6. Remarks

Remark 6.1 (Why the method stops at 4). Let F, W, ρ, s be as in Lemma 2.3, normalized so that $F'(0) = W'(0) = 1$. Then $s < 4$. Let r be the largest radius for which the branch ψ of F^{-1} at 0 extends to the disc $\{|v| < r\}$ with values in $\rho\mathbb{D}$. The branch ψ cannot map $\{|v| < r\}$ into a disc $\{|t| \le \rho'\}$ with $\rho' < \rho$. Indeed, otherwise, since $F' \ne 0$, the inverse function theorem and compactness give one radius $r_* > 0$ such that for every z_0 with $|z_0| \le \rho'$, F has an inverse branch with values in $\rho\mathbb{D}$ on the disc of radius r_* about $F(z_0)$, taking $F(z_0)$ to z_0. Taking $z_0 = \psi(v_1)$ with $r - r_*/2 < |v_1| < r$, these branches continue ψ to discs that cover $\{r \le |v| < r + r_*/2\}$; two of these discs that meet do so in a convex set that meets $\{|v| < r\}$ (it contains the midpoint of their centres), where both continuations equal ψ. So ψ would extend to $\{|v| < r + r_*/2\}$ with values in $\rho\mathbb{D}$, against the choice of r. Hence there are points v_k with $|\psi(v_k)| \to \rho$; then $|v_k| \to r$, since ψ maps each disc $\{|v| \le r'\}$ with $r' < r$ onto a compact subset of $\rho\mathbb{D}$. Passing to a subsequence, $\psi(v_k)$ tends to a point t^* with $|t^*| = \rho$, and then $|F(t^*)| = \lim |v_k| = r$. In particular $r \ge \min_{|t|=\rho} |F(t)|$. Schwarz's lemma, applied to ψ, gives $r \le \rho$, so $\min_{|t|=\rho} |F(t)| \le \rho$: this is the bound on the first factor used in the introduction. The function $W \circ \psi$ is univalent on this disc, with derivative 1 at 0, and $W(t^*)$ is a boundary point of its image, so $|W(t^*)| \ge r/4$ by Koebe's theorem. The hypothesis gives $r = |F(t^*)| > s|W(t^*)| \ge sr/4$. This is Smale's argument, carried over to the pair (F, W). Jatar and Ng use the same two bounds in [9, Theorem 7.1, Remark 7.3].

Remark 6.2 (Why f cannot be injective). If f in Proposition 2.1 has no zero other than 0 on $\overline{\Omega}$, for instance if it is univalent, then z/f is holomorphic on $\overline{\Omega}$ with value 1 at 0, and the maximum principle gives $\min_{\partial\Omega} |f(z)/z| \le 1$. So $\min_{\partial\Omega} |f(z)/z| > 1$ requires f to have zeros other than 0 on $\overline{\Omega}$. The function F of Section 4 has such zeros t; the points $\eta(t^n)$ lie inside the compact set Π_0, since Steps 3–5 bound $|\Phi|$ from below off Π_0. By Jensen's formula for $F(t)/t$, the mean of $\log|F|$ over $\rho\mathbb{T}$ equals $\log(\rho|F'(0)|) + \sum_k \log(\rho/|t_k|)$, summed over the zeros $t_k \ne 0$ of F in $\rho\mathbb{D}$. For ρ close to 1, $|F|$ is large on most of $\rho\mathbb{T}$ and bounded below on the rest (Steps 3–6), so this mean is large, while $\log(\rho|F'(0)|)$ is near 0. The zeros supply the difference.

Remark 6.3 (The constant of Smale's Problem 1A). The constant of [9, Theorem 5.1], called K there, is the one in Smale's Problem 1A [12], in the form of [9, Section 5]:

$$K_{1A} := \sup_{j\ge 2} \sup_{p} \min_{p'(c)=0} \Big|\frac{p^{(j)}(0)}{j!}\Big|^{1/(j-1)} |p(c)| \qquad (p(0) = 0,\ p'(0) = 1),$$

in which $|p(c)|$ is not divided by $|c|$. The proof of [9, Theorem 5.1] applies a variant of the reduction of Proposition 2.1 [9, Proposition 5.2] to the inverse of the Koebe function. That argument does not give K^*, because for that function $|f(z)/z|$ is small at boundary points far from 0.

7. Discussion

Before this paper it was known that $1.1559 < K^* \le 4$ [7, 12]. Theorem 1.1 gives $K^* = 4$. We end with three questions.

How fast does $K(d)$ tend to 4? From above, $K(d) \le 4^{(d-2)/(d-1)}$ [3] and $K(d) < 4 - 2.263/\sqrt{d}$ for $d \ge 8$ [5]. From below, Theorem 1.1 gives only the limit: our proof is not effective.

Which is the least degree d with $K(d) > 1 - 1/d$? In its full strength, Smale's conjecture is $K(d) = 1 - 1/d$; this holds for $d \le 4$ [5, Section 1], and $K(d) > 1$ for every $d \ge 29{,}984$ [2, Section 1 and Theorem A.1]. So this degree lies between 5 and 29,984.

Blaschke products. In the notation of Jatar and Ng, $M = B$ [9, Theorem 1.2], where B is the supremum of constants B_d, $d \ge 1$, defined with Blaschke products of degree at most d; they prove

$B_1 < \sqrt{7/5}$ and $B_d < 4 - 529(\log 2)^2/(576\,d^2)$ [9, Theorem 1.3]. By Theorem 1.1, $B = 4$, so $B_1 < B$. This answers two of the three questions that [9, Section 9] calls open: whether $B_1 = B$, and whether $M = 4$. The third, whether B_d is strictly increasing in d, remains.

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